FUNCTIONS WITH ULTRADIFFERENTIABLE POWERS

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Abstract. We study the regularity of smooth functions f defined on an open subset of $\mathbb{R}^n$ and such that, for certain integers $p \geq 2$, the powers $f^p : x \mapsto (f(x))^p$ belong to a Denjoy-Carleman class $\mathcal{C}_M$ associated with a suitable weight sequence M. Our main result is a statement analogous to a classic theorem of H. Joris on $\mathcal{C}^\infty$ functions: if a function $f : \mathbb{R} \to \mathbb{C}$ is such that both functions f^p and f^q with $\gcd(p, q) = 1$ are of class $\mathcal{C}_M$ on $\mathbb{R}$, and if the weight sequence M satisfies a standard condition known as *moderate growth*, then f itself is of class $\mathcal{C}_M$. It is also shown that the result is no longer true without the moderate growth assumption. Various ancillary results, corollaries and examples are presented.

Introduction

Let f be a real or complex-valued function defined on an open subset of $\mathbb{R}^n$. Various classic problems involve, in one form or another, the recovery of regularity properties of the function f from regularity assumptions on some of its powers $f^p : x \mapsto (f(x))^p$ with $p \in \mathbb{N}$, $p \geq 2$. Such questions go back, at least, to Glaeser's work on square roots of nonnegative functions [8]; see also [3] and the references therein. Due to the nonlinear nature of the conditions on f, the task can be quite difficult, depending on the context and on the regularity properties under consideration.

However, in 1982, H. Joris proved the following striking result [14]: if a function $f : \mathbb{R} \to \mathbb{R}$ is such that both functions f^2 and f^3, or more generally f^p and f^q with $\gcd(p, q) = 1$, are of class $\mathcal{C}^\infty$ on $\mathbb{R}$, then f itself is of class $\mathcal{C}^\infty$. As pointed out in [5, 15], the result also holds for complex-valued functions. Various generalizations were subsequently established around the notion of pseudo-immersion [5, 15, 20].

In spite of its innocent-looking statement, Joris's theorem is not easy to establish. The original proof uses an elementary but intricate study of the vanishing of the derivatives of f at points of flatness, together with combinatorial relations arising from the Faà di Bruno formula.

A much simpler and shorter proof was given in 1989 by I. Amemyia and K. Masuda [1]. Its key argument is an algebraic lemma stating that the ring of formal power series with coefficient in a ring R inherits a suitable property of R relative to powers of its elements.

None of the aforementioned proofs seems to lend itself to explicit estimates suitable for other classes of smoothness than $\mathcal{C}^\infty$. However, in 2018, as Joris's theorem was discussed on the *MathOverflow* website, the anonymous contributor nicknamed "fedja" outlined a remarkable alternative proof based on complex analysis, using in particular a characterization of smooth functions on the real line by a property of holomorphic approximation. Fedja's argument [7] actually

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yields an even stronger result, as it also works for finite differentiability classes: roughly speaking, given p and q with $\gcd(p,q) = 1$, there is an integer m, depending only on p and q, such that for k large enough, the function f is of class $\mathcal{C}^k$ as soon as f^p and f^q are of class $\mathcal{C}^{mk}$, and the proof provides crude estimates for m.

The main goal of the present paper is to show that the property described by Joris's theorem holds in classical Denjoy-Carleman ultradifferentiable classes $\mathcal{C}_M$, provided the weight sequence M that defines the class satisfies the so-called *moderate growth* assumption. Our approach will follow the path of the aforementioned proof proposed by Fedja [7] for finite differentiability classes, while making suitable modifications in order to obtain estimates in the ultradifferentiable setting.

The paper is organized as follows.

Section 1 gathers the definitions and required material pertaining to weight sequences M and Denjoy-Carleman classes $\mathcal{C}_M$. We shall consider ultradifferentiable classes of Roumieu type on open subsets of $\mathbb{R}^n$, as well as corresponding rings of function germs. Whether those classes are quasianalytic or not will not be decisive, although we shall explain in Section 2 that the interest of the main result lies in the non-quasianalytic case.

Section 2 begins with a review of some known results on the regularity of $\mathcal{C}^\infty$ functions $f : \mathbb{R} \to \mathbb{R}$ such that f^p is of class $\mathcal{C}_M$ for a given integer $p \geq 2$. Incidentally, we show that, even for so-called *strongly regular* sequences M (for example, Gevrey sequences), such a function f need not be of class $\mathcal{C}_M$, which answers a question asked in [26]. This negative result can be viewed as a motivation for a $\mathcal{C}_M$ version of Joris's theorem (Theorem 2.2.1), which is then stated in the second part of Section 2. Various comments and corollaries follow. In particular, the case of functions of several variables is briefly discussed. We also provide an example showing that the conclusion generally fails without the moderate growth assumption on M.

Sections 3 and 4 are entirely devoted to the proof of Theorem 2.2.1. In Section 3, we gather the main technical ingredients required in the proof. In particular, a suitable approximation-theoretic characterization of $\mathcal{C}_M$ regularity on a real interval is established. This result (Proposition 3.3.2), which may be of independent interest, is partly based on Dynkin's theorem on $\bar{\partial}$-flat extensions of smooth functions [6]. In Section 4, the technical tools of Section 3 are finally used to complete the proof of Theorem 2.2.1, following the general pattern of Fedja's argument [7].

1. Denjoy-Carleman classes

1.1. **Some properties of sequences.** A sequence $M = (M_j)_{j\geq 0}$ of positive real numbers will be called a *weight sequence* if it satisfies the following assumptions:

$$M_0 = 1, \tag{1}$$

$$M \text{ is logarithmically convex,} \tag{2}$$

$$\lim_{j\to\infty}(M_j)^{1/j} = \infty. \tag{3}$$

Property (2) amounts to saying that the sequence $(M_{j+1}/M_j)_{j\geq 0}$ is nondecreasing. Together with (1), it implies

$$M_j M_k \leq M_{j+k} \text{ for any } (j,k)\in\mathbb{N}^2.$$

We say that a weight sequence M has *moderate growth* if there is a positive constant A such that we have

$$M_{j+k} \leq A^{j+k} M_j M_k \text{ for any } (j,k)\in\mathbb{N}^2. \tag{4}$$

We say that a weight sequence M satisfies the *strong non-quasianalyticity* condition if there is a positive constant A such that we have

$$\sum_{j\geq k}\frac{M_j}{(j+1)M_{j+1}} \leq A\frac{M_k}{M_{k+1}} \text{ for any } k\in\mathbb{N}. \tag{5}$$

Property (5) obviously implies the classical Denjoy-Carleman *non-quasianalyticity* condition

$$\sum_{j\geq 0}\frac{M_j}{(j+1)M_{j+1}} < \infty. \tag{6}$$

A weight sequence M is said to be *strongly regular* if it satisfies (4) and (5).

Example 1.1.1. Let α and β be real numbers, with $\alpha > 0$. One can define a strongly regular weight sequence M by setting $M_j = (j!)^\alpha(\ln j)^{\beta j}$ for j large enough and choosing suitable first terms. This is the case, in particular, for Gevrey sequences $M_j = (j!)^\alpha$.

Example 1.1.2. For any real $\beta > 0$, one can also define a weight sequence M with $M_j = (\ln j)^{\beta j}$ for j large enough. This sequence has moderate growth, and it satisfies the non-quasianalyticity property (6) if and only if $\beta > 1$. It does not satisfy the strong non-quasianalyticity property (5).

Example 1.1.3. For any real $\lambda > 0$, the weight sequence M^λ defined by $M_j^\lambda = \exp\left(\frac{\lambda}{4}j^2\right)$ satisfies (5) but it does not have moderate growth. The sequences M^λ will reappear in the examples of Section 2. They are usually called q-Gevrey sequences, with $q = \exp\left(\frac{\lambda}{4}\right)$.

With every weight sequence M, it is a standard procedure to associate the function h_M defined by $h_M(t) = \inf_{j\geq 0} t^j M_j$ for any real $t > 0$, and $h_M(0) = 0$. We briefly recall its properties. Using (1), (2) and (3), it is easy to see that $h_M(t) = t^j M_j$ for $j \geq 1$ and $\frac{M_j}{M_{j+1}} \leq t < \frac{M_{j-1}}{M_j}$, and $h_M(t) = 1$ for $t \geq 1/M_1$. In particular, h_M is continuous, nondecreasing and it fully determines M since it satisfies

$$M_j = \sup_{t>0} t^{-j} h_M(t) \text{ for any } j\in\mathbb{N}.$$

Setting $t_j = \frac{M_j}{M_{j+1}}$, we also have

$$M_j = t_j^{-j} h_M(t_j) \text{ with } \lim_{j\to\infty} t_j = 0. \tag{7}$$

It can be derived from [18, Proposition 3.6] that the moderate growth assumption (4) is equivalent to the existence, for any real $s \geq 1$, of a constant $\kappa_s \geq 1$ such that

$$h_M(t) \leq \big(h_M(\kappa_s t)\big)^s \text{ for any } t \geq 0. \tag{8}$$

As a consequence of (8) and of the definition of h_M, it is easy to see that if a weight sequence M has moderate growth, we have

$$t^{-j} h_M(t) \leq \kappa_2^j M_j h_M(\kappa_2 t) \text{ for any } t > 0 \text{ and any } j \in \mathbb{N}. \tag{9}$$

Other equivalent conditions for (4), or for the strong non-quasianalyticity property (5), can be found in the state-of-the-art study of weight sequences and weight functions carried out in the recent works [11, 12, 13], originating in J. Sanz's work on proximate orders [22].

Example 1.1.4. Let M be as in Example 1.1.1, and set $\eta(t) = \exp(-(t|\ln t|^\beta)^{-1/\alpha})$ for $t > 0$ small enough. Elementary computations show that there are constants $a > 0$, $b > 0$ such that $\eta(at) \leq h_M(t) \leq \eta(bt)$ as t tends to 0.

1.2. **Denjoy-Carleman classes.** In what follows, the length $j_1 + \cdots + j_n$ of a multi-index $J = (j_1, \ldots, j_n) \in \mathbb{N}^n$ will be denoted by the corresponding lower case letter j, and we put $\partial^J = \partial^j / \partial x_1^{j_1} \cdots \partial x_n^{j_n}$.

Let Ω be an open subset of $\mathbb{R}^n$, and let M be a weight sequence. We say that a $\mathcal{C}^\infty$ function $f : \Omega \to \mathbb{C}$ belongs to the *Denjoy-Carleman class* $\mathcal{C}_M(\Omega)$ if for any compact subset X of Ω, one can find a real number $\sigma > 0$ and a constant $C \geq 0$ such that

$$|\partial^J f(x)| \leq C\sigma^j j! M_j \ \text{ for any } \ J \in \mathbb{N}^n \ \text{ and } \ x \in X. \tag{10}$$

A germ of function at the origin in $\mathbb{R}^n$ is said to be of class $\mathcal{C}_M$ if it has a representative in $\mathcal{C}_M(\Omega)$ for some open neighborhood Ω of 0. We denote by $\mathcal{C}_M(\mathbb{R}^n, 0)$ the set of all such germs.

Corresponding definitions for functions on segments of $\mathbb{R}$ instead of an open set will be needed. Given a segment $[a, b]$ of $\mathbb{R}$, a real number $\sigma > 0$, and a $\mathcal{C}^\infty$ function $f : [a, b] \to \mathbb{C}$, we set

$$\|f\|_{[a,b],\sigma} = \sup_{x \in [a,b],\ j \in \mathbb{N}} \frac{|f^{(j)}(x)|}{\sigma^j j! M_j}.$$

We then say that the function f belongs to the space $\mathcal{C}_{M,\sigma}([a, b])$ if it satisfies $\|f\|_{[a,b],\sigma} < \infty$. It is easy to see that $\mathcal{C}_{M,\sigma}([a, b])$ is a Banach space for the norm $\|\cdot\|_{[a,b],\sigma}$. Finally, we define the *Denjoy-Carleman class* $\mathcal{C}_M([a, b])$ as the reunion of all spaces $\mathcal{C}_{M,\sigma}([a, b])$ for $\sigma > 0$. Given an open subset Ω of $\mathbb{R}$, it is clear that a function $f : \Omega \to \mathbb{C}$ belongs to $\mathcal{C}_M(\Omega)$ if and only if its restriction to every segment $[a, b]$ contained in Ω belongs to $\mathcal{C}_M([a, b])$.

We end this section with a brief review of the relationship between conditions on the sequence M and properties of the corresponding classes; we refer to [25] for details and references. Conditions (1) and (2) imply that $\mathcal{C}_M(\Omega)$, $\mathcal{C}_M(\mathbb{R}^n, 0)$ and $\mathcal{C}_M([a, b])$ are algebras, and that $\mathcal{C}_M$ regularity is stable under composition. Condition (3) ensures that $\mathcal{C}_M(\Omega)$ (resp. $\mathcal{C}_M(\mathbb{R}^n, 0)$) strictly contains the algebra of real-analytic functions in Ω (resp. real-analytic germs at the origin). The moderate growth assumption (4) can be interpreted in terms of stability of $\mathcal{C}_M$

regularity under the action of so-called ultradifferential operators; see [18]. It clearly implies the weaker condition

$$M_{j+1} \leq A^{j+1} M_j \text{ for any } j \in \mathbb{N} \tag{11}$$

which characterizes the stability of $\mathcal{C}_M$ classes under derivation. The non-quasianalyticity property (6) characterizes the existence of a non-trivial element of $\mathcal{C}_M(\mathbb{R}^n, 0)$ which is flat at 0, whereas the stronger condition (5) is a necessary and sufficient condition for a $\mathcal{C}_M$ version of Borel's extension theorem.

2. Functions with ultradifferentiable powers

2.1. **Background and known results.** Let M be a weight sequence and let f be a germ of complex-valued function of class $\mathcal{C}^\infty$ at the origin in $\mathbb{R}$. Assume that there is an integer $p \geq 2$ such that the germ $f^p : x \mapsto (f(x))^p$ belongs to $\mathcal{C}_M(\mathbb{R}, 0)$. As observed in [26, Remark 1], it is not difficult the check that if $\mathcal{C}_M(\mathbb{R}, 0)$ is stable under derivation and quasianalytic, then f also belongs to $\mathcal{C}_M(\mathbb{R}, 0)$. This is no longer true in the non-quasianalytic case: indeed, for any real $\lambda > 0$, set

$$g_\lambda(x) = \exp\left(-\frac{1}{\lambda}(\ln x)^2\right) \text{ for } x > 0 \text{ and } g_\lambda(x) = 0 \text{ for } x \leq 0. \tag{12}$$

The proof of [26, Lemma 1] shows that g_λ belongs to $\mathcal{C}_{M^\lambda}(\mathbb{R}, 0)$, where M^λ is defined in Example 1.1.3, but not to any strictly smaller ring $\mathcal{C}_M(\mathbb{R}, 0)$. In particular, for $f = g_{p\lambda}$, we see that f^p belongs to $\mathcal{C}_{M^\lambda}(\mathbb{R}, 0)$ whereas f does not. Thus, the result fails for the weight sequences M^λ, even though the associated classes are stable under derivation and strongly non-quasianalytic. Since M^λ does not have moderate growth, it was asked in [26] whether the result would hold for tamer sequences M, namely strongly regular ones. The answer is still negative, as shown by the following proposition.

Proposition 2.1.1. *Let M be a strongly regular weight sequence. For every integer $p \geq 2$, there is a smooth function germ f at the origin in $\mathbb{R}$ such that $f^p \in \mathcal{C}_M(\mathbb{R}, 0)$ and $f \notin \mathcal{C}_M(\mathbb{R}, 0)$.*

Proof. We start with a counter-example in two variables, slightly generalizing a construction of [23]. By [24, Lemma 3.6], there is an element η of $\mathcal{C}_M(\mathbb{R})$ which vanishes at infinite order at the origin and satisfies $\eta(t) \geq h_M(b|t|)$ for some suitable constant $b > 0$. Given an integer $m \geq 2$, we then define $F : \mathbb{R}^2 \to \mathbb{R}$ by

$$F(x, y) = (x^2 + y^{2m})\left(1 + \frac{x^2\eta(y)}{x^2 + y^{2m}}\right)^{1/p} \text{ for } (x, y) \neq (0, 0), \text{ and } F(0, 0) = 0.$$

Since η is flat at 0, the $\mathcal{C}^\infty$-smoothness of F is immediate. Moreover, we have $(F(x, y))^p = (x^2 + y^{2m})^p + x^2(x^2 + y^{2m})^{p-1}\eta(y)$, hence $F^p \in \mathcal{C}_M(\mathbb{R}^2, 0)$. Using the power series expansion of $(1 + t)^{1/p}$, we obtain, for (x, y) close enough to $(0, 0)$, the expansion

$$F(x, y) = x^2 + y^{2m} + \frac{1}{p}x^2\eta(y) + \sum_{j=1}^{+\infty}(-1)^j a_j \frac{x^{2j+2}}{y^{2mj}}\left(1 + \frac{x^2}{y^{2m}}\right)^{-j}(\eta(y))^{j+1}$$

with $a_j = \frac{(p-1)(2p-1)\cdots(jp-1)}{p^{j+1}(j+1)!}$ for $j \geq 1$. Assume $0 \leq x < y^m$. Expanding $\left(1+\frac{x^2}{y^{2m}}\right)^{-j}$ in power series, we then obtain the absolutely convergent expansion

$$F(x,y) = G(x,y) + \sum_{j=1}^{+\infty}\sum_{k=0}^{+\infty}(-1)^{j+k}a_j\binom{j+k-1}{j-1}\frac{x^{2(j+k)+2}}{y^{2m(j+k)}}(\eta(y))^{j+1} \tag{13}$$

with $G(x,y) = x^2(1+\frac{1}{p}\eta(y)) + y^{2m}$. We set $l = j+k$ and exchange the order of summation, so that (13) becomes

$$F(x,y) = G(x,y) + \sum_{l=1}^{+\infty}(-1)^l c_l(y)x^{2l+2} \text{ for } 0 \leq x < y^m, \tag{14}$$

with

$$c_l(y) = y^{-2ml}\sum_{j=1}^{l}a_j\binom{l-1}{j-1}(\eta(y))^{j+1} \text{ for } l \geq 1.$$

Clearly, (14) implies

$$\frac{\partial^{2l+2}F}{\partial x^{2l+2}}(0,y) = (-1)^l(2l+2)!\,c_l(y) \text{ for } y > 0 \text{ and } l \geq 1.$$

Observe that $c_l(y) \geq y^{-2ml}a_1(\eta(y))^2 \geq a_1(y^{-ml}h_M(by))^2$. Moreover, by (7), there is a sequence $(y_l)_{l\geq 0}$ of positive real numbers such that $\lim_{l\to\infty} y_l = 0$ and $h_M(by_l) = (by_l)^{ml}M_{ml}$, hence $c_l(y_l) \geq a_1 b^{2ml}(M_{ml})^2$. Using (2) and (4), we also have $(M_{ml})^2 \geq A^{-2ml}M_{2ml} \geq A^{-2ml}(M_{2l})^m \geq A^{-4ml-2m}M_2^{-m}(M_{2l+2})^m$. Thus, we finally see that there is a constant $C > 0$ such that

$$\left|\frac{\partial^{2l+2}F}{\partial x^{2l+2}}(0,y_l)\right| \geq C^{l+1}(2l+2)!(M_{2l+2})^m, \text{ with } \lim_{l\to\infty} y_l = 0, \tag{15}$$

which clearly implies $F \notin \mathcal{C}_M(\mathbb{R}^2,0)$. The existence of a similar counter-example in one variable is now a direct consequence of the results in [16, Section 3]: starting from (15), it is possible to construct a curve $\gamma : \mathbb{R} \to \mathbb{R}^2$, with components in $\mathcal{C}_M(\mathbb{R})$, such that $\gamma(0) = 0$ and $F \circ \gamma \notin \mathcal{C}_M(\mathbb{R},0)$. Thus, setting $f = F\circ\gamma$, we have $f^p = (F)^p \circ \gamma \in \mathcal{C}_M(\mathbb{R},0)$ and $f \notin \mathcal{C}_M(\mathbb{R},0)$. □

As in the classic $\mathcal{C}^\infty$ case of Joris's theorem, it turns out, however, that a positive result can be obtained with assumptions on two suitable powers of f.

2.2. **Joris's theorem for Denjoy-Carleman classes.** Due to the local nature of the problem, it is convenient to also state the main result of this article in terms of function germs.

Theorem 2.2.1. *Let M be a weight sequence that satisfies the moderate growth condition. Let f be a germ of complex-valued function at the origin in $\mathbb{R}$. Assume there is a couple (p,q) of non-zero natural integers with $\gcd(p,q) = 1$ such that both germs f^p and f^q belong to $\mathcal{C}_M(\mathbb{R},0)$. Then f belongs to $\mathcal{C}_M(\mathbb{R},0)$.*

Postponing the proof to Sections 3 and 4, we shall devote the rest of the present section to comments and corollaries.

Remark 2.2.2. Obviously, the above statement implies that if Ω is an open subset of $\mathbb{R}$ and $f : \Omega \to \mathbb{C}$ is a function such that f^p and f^q belong to $\mathcal{C}_M(\Omega)$, with $\gcd(p,q) = 1$, then f belongs to $\mathcal{C}_M(\Omega)$.

Remark 2.2.3. The result is no longer true without the moderate growth assumption. A counter-example is once again provided by the functions g_λ defined in (12). Indeed, assume for instance $p < q$ and set $f = g_{p\lambda}$. We then have $f^p = g_\lambda \in \mathcal{C}_{M^\lambda}(\mathbb{R},0)$ and $f^q = g_{\lambda'} \in \mathcal{C}_{M^{\lambda'}}(\mathbb{R},0)$ with $\lambda' = \frac{p}{q}\lambda < \lambda$, hence $f^q \in \mathcal{C}_{M^\lambda}(\mathbb{R},0)$. However f does not belong to $\mathcal{C}_{M^\lambda}(\mathbb{R},0)$.

Remark 2.2.4. The quasianalytic case does not require moderate growth, but only the much weaker assumption of stability under derivation. Indeed, the assumptions on f imply that it is of class $\mathcal{C}^\infty$ by virtue of Joris's theorem, and the $\mathcal{C}_M$ result can then be obtained by elementary arguments, as mentioned in Section 2.1. The interest of Theorem 2.2.1 therefore lies in the non-quasianalytic case, although non-quasianalyticity will not be used in the proof.

As noticed in the article of Joris [14], in the $\mathcal{C}^\infty$ case, a generalization to functions of several variables is immediate, thanks to the classical result of Boman [2] stating that $\mathcal{C}^\infty$ smoothness can be tested along curves. Analogously, for non-quasianalytic classes, the contents of [16, Section 3] immediately yield the following corollary of Theorem 2.2.1.

Corollary 2.2.5. *Let M be a weight sequence that satisfies the moderate growth and non-quasianalyticity conditions. Let f be a germ of complex-valued function at the origin in $\mathbb{R}^n$. Assume there is a couple (p,q) of non-zero natural integers with $\gcd(p,q) = 1$ such that both germs f^p and f^q belong to $\mathcal{C}_M(\mathbb{R}^n,0)$. Then f belongs to $\mathcal{C}_M(\mathbb{R}^n,0)$.*

The quasianalytic case is of a different nature and the results in [10] and [19] show that it cannot be treated directly by an argument of reduction to lower dimensions. The particular situation of quasianalytic classes obtained as intersections of non-quasianalytic ones as in [17] does not seem more immediately tractable, as the classes defining the intersections may not have suitable properties of logarithmic convexity or moderate growth.

Remark 2.2.6. Theorem 2.2.1 and Corollary 2.2.5 have been stated with an assumption on two functions f^p and f^q with $\gcd(p,q) = 1$ for the sake of simplicity. It should however be mentioned that, as in the classical $\mathcal{C}^\infty$ case [1, 14], their respective conclusions still hold under the slightly more general assumption that $f^{p_1}, \ldots, f^{p_r}$ are of class $\mathcal{C}_M$, where $p_1, \ldots, p_r$ are positive integers such that $\gcd(p_1, \ldots, p_r) = 1$. Indeed, the proof given in Section 4 is based on a reduction to a special case which extends immediately to this situation.

We now proceed with the proof of Theorem 2.2.1.

3. Preparations

3.1. **Uniform estimates for Cauchy-Riemann equations.** In what follows, for $1 \le p \le \infty$, we denote by $\|\cdot\|_p$ the usual norm on the space $L^p(\mathbb{C})$ associated with the standard Lebesgue measure λ. For $z \in \mathbb{C}$ and $r > 0$, we denote by $D(z,r)$ the open disk $\{\zeta \in \mathbb{C} : |z - \zeta| < r\}$. We write $\mathbb{1}_A$ for the indicator function of a set A.

Let $\mathcal{K}$ denote the Cauchy kernel in $\mathbb{C}$, that is, $\mathcal{K}(z) = \frac{1}{\pi z}$. Let U be a bounded open subset of $\mathbb{C}$. By elementary arguments, for any element w of $L^\infty(\mathbb{C})$ such that $w = 0$ in $\mathbb{C} \setminus U$, the convolution $v = \mathcal{K} * w$ defines a bounded continuous function in $\mathbb{C}$ that satisfies $\partial v/\partial \bar{z} = w$ in the sense of distributions in $\mathbb{C}$, and

$$\|v\|_\infty \leq C\|w\|_\infty \tag{16}$$

for some suitable constant C depending only on $\max_{\zeta\in U} |\zeta|$. In order to follow the pattern of [7], more subtle uniform estimates on v are needed. These estimates are described by the following lemma.

Lemma 3.1.1. *Let U, w and v be as above. Then for any real number $r \in (0, \frac{1}{2}]$ and any $z \in U$, we have*

$$|v(z)| \leq C\left(r\|w\|_\infty + (|\ln r|)^{1/2}\,\|w\|_2\right)$$

for some suitable constant C depending only on $\max_{\zeta\in U} |\zeta|$.

Proof. For the reader's convenience, we include the proof sketched in [7]. Choose $R \geq 1$ such that $U \subset D(0, \frac{R}{2})$. For $z \in U$ and $|\zeta| \geq R$ we have $|z - \zeta| > \frac{R}{2}$, hence $w(z - \zeta) = 0$. We can therefore write $v(z) = \int_{D(0,R)} \mathcal{K}(\zeta)w(z - \zeta)\,\mathrm{d}\lambda(\zeta) = \int_{D(0,r)} \mathcal{K}(\zeta)w(z-\zeta)\,\mathrm{d}\lambda(\zeta) + \int_{\{r\leq|\zeta|<R\}} \mathcal{K}(\zeta)w(z-\zeta)\,\mathrm{d}\lambda(\zeta)$. A crude majorization immediately yields $\left|\int_{D(0,r)} \mathcal{K}(\zeta)w(z - \zeta)\,\mathrm{d}\lambda(\zeta)\right| \leq \int_{D(0,r)} \frac{\mathrm{d}\lambda(\zeta)}{\pi|\zeta|}\|w\|_\infty = 2r\|w\|_\infty$. By the Cauchy-Schwarz inequality, we also have $\left|\int_{\{r\leq|\zeta|<R\}} \mathcal{K}(\zeta)w(z - \zeta)\,\mathrm{d}\lambda(\zeta)\right| \leq \left(\int_{\{r\leq|\zeta|<R\}} \frac{\mathrm{d}\lambda(\zeta)}{\pi^2|\zeta|^2}\right)^{1/2} \|w\|_2 = \left(\frac{2}{\pi} \ln(R/r)\right)^{1/2}\|w\|_2$. The result easily follows. □

3.2. **Technical estimates in ellipses.**

Definition 3.2.1. *For any $\varepsilon > 0$, we put $\Omega_\varepsilon = \varphi_\varepsilon(S)$, where S is the strip $\{z \in \mathbb{C} : |\Im z| < 1\}$ and φ_ε is the mapping of the complex plane defined by $\varphi_\varepsilon(z) = \sin(\varepsilon z)$.*

In other words, the open set Ω_ε is the interior of the ellipse with vertices $\pm\cosh\varepsilon$ and co-vertices $\pm i\sinh\varepsilon$. It contains the real interval $[-1, 1] = \varphi_\varepsilon(\mathbb{R})$, and shrinks to $[-1, 1]$ as ε tends to 0.

The following covering lemma is elementary.

Lemma 3.2.2. *For any real number ε with $0 < \varepsilon \leq 1$, there is a radius $\eta_\varepsilon > 0$ and a finite family of disks $D(z_{j,\varepsilon}, \eta_\varepsilon)$, $j = 1, \dots, N_\varepsilon$, with the following properties:*

$$\Omega_{\varepsilon/2} \subset \bigcup_{j=1}^{N_\varepsilon} D(z_{j,\varepsilon}, \eta_\varepsilon), \tag{17}$$

$$\overline{D(z_{j,\varepsilon}, 2\eta_\varepsilon)} \subset \Omega_\varepsilon \ \textit{ for } \ j = 1, \dots, N_\varepsilon, \tag{18}$$

$$N_\varepsilon \leq C\varepsilon^{-3} \ \textit{ for some absolute constant } C. \tag{19}$$

Proof. Basic arguments show that $\mathrm{dist}(\partial\Omega_{\varepsilon/2}, \partial\Omega_\varepsilon) \geq \frac{1}{4}\varepsilon^2$. Thus, any closed disk of radius $\frac{1}{8}\varepsilon^2$ that intersects $\Omega_{\varepsilon/2}$ is contained in Ω_ε. Set $\eta_\varepsilon = \frac{1}{16}\varepsilon^2$ and notice that $\Omega_{\varepsilon/2}$ is contained in a rectangle of length $2\cosh(\varepsilon/2)$ and width $2\sinh(\varepsilon/2)$. It is an easy exercise to check that such a rectangle can be covered by a family

$\mathcal{F}_\varepsilon$ of open disks of radius η_ε with card $\mathcal{F}_\varepsilon \le C\varepsilon^{-3}$ for some absolute constant C. Keeping only the elements of $\mathcal{F}_\varepsilon$ that intersect $\Omega_{\varepsilon/2}$, we obtain a family of disks having all the desired properties. □

We can now obtain technical estimates following closely a key statement in [7], with slight modifications required in our framework. For the reader's convenience, we give a complete proof.

Lemma 3.2.3. *Let ε be a real number with $0 < \varepsilon \le 1$, let g be a bounded holomorphic function in Ω_ε, and let K be a real number such that $|g| \le K$ in Ω_ε. For any real number $r > 0$, we have*

$$\int_{\Omega_{\varepsilon/2}} |g'|^2 \mathbb{1}_{\{|g|<r\}}\, \mathrm{d}\lambda \le C\frac{r^2}{\varepsilon^3} \ln\left(\frac{K^2}{r^2}+1\right)$$

for some absolute constant C.

Proof. For $j = 1, \ldots, N_\varepsilon$, consider the disk $D(z_{j,\varepsilon}, \eta_\varepsilon)$ of Lemma 3.2.2. It is easy to see that

$$\int_{D(z_{j,\varepsilon},\eta_\varepsilon)} |g'|^2 \mathbb{1}_{\{|g|<r\}}\, \mathrm{d}\lambda = \int_{D(0,\frac{1}{2})} |g'_{j,\varepsilon}|^2 \mathbb{1}_{\{|g_{j,\varepsilon}|<r\}}\, \mathrm{d}\lambda \tag{20}$$

where $g_{j,\varepsilon}$ is defined by

$$g_{j,\varepsilon}(\zeta) = g(z_{j,\varepsilon} + 2\eta_\varepsilon\zeta).$$

Property (18) and the assumptions on g ensure that the function $g_{j,\varepsilon}$ is holomorphic in a neighborhood of $\overline{D(0,1)}$. Set

$$\Psi_{j,\varepsilon} = \ln\left(|g_{j,\varepsilon}|^2 + r^2\right).$$

Then $\Psi_{j,\varepsilon}$ is a smooth subharmonic function in a neighborhood of $\overline{D(0,1)}$ and its Laplacian is

$$\Delta\Psi_{j,\varepsilon} = 4r^2 \frac{|g'_{j,\varepsilon}|^2}{(|g_{j,\varepsilon}|^2 + r^2)^2}.$$

In particular, we have $\Delta\Psi_{j,\varepsilon} \ge \frac{1}{r^2}|g'_{j,\varepsilon}|^2 \mathbb{1}_{\{|g_{j,\varepsilon}|<r\}}$. Thus, we get

$$\begin{aligned}\int_{D(0,\frac{1}{2})} |g'_{j,\varepsilon}|^2 \mathbb{1}_{\{|g_{j,\varepsilon}|<r\}}\, \mathrm{d}\lambda &\le r^2 \int_{D(0,\frac{1}{2})} \Delta\Psi_{j,\varepsilon}\, \mathrm{d}\lambda \\ &\le \frac{r^2}{\ln 2}\int_{D(0,\frac{1}{2})} \Delta\Psi_{j,\varepsilon}(\zeta) \ln\left(\frac{1}{|\zeta|}\right) \mathrm{d}\lambda(\zeta) \\ &\le \frac{r^2}{\ln 2}\int_{D(0,1)} \Delta\Psi_{j,\varepsilon}(\zeta) \ln\left(\frac{1}{|\zeta|}\right) \mathrm{d}\lambda(\zeta).\end{aligned} \tag{21}$$

Using Green's formula for the Laplacian, together with the obvious estimates $\Psi_{j,\varepsilon} \le \ln(K^2 + r^2)$ and $\Psi_{j,\varepsilon}(0) \ge \ln r^2$, we see that

$$\begin{aligned}\int_{D(0,1)} \Delta\Psi_{j,\varepsilon}(\zeta) \ln\left(\frac{1}{|\zeta|}\right) \mathrm{d}\lambda(\zeta) &= \int_0^{2\pi} \Psi_{j,\varepsilon}(e^{i\theta})\, \mathrm{d}\theta - 2\pi\Psi_{j,\varepsilon}(0) \\ &\le 2\pi(\ln(K^2+r^2) - \ln r^2).\end{aligned} \tag{22}$$

Gathering (20), (21) and (22), we obtain

$$\int_{D(z_{j,\varepsilon},\eta_\varepsilon)} |g'|^2 \mathbb{1}_{\{|g|<r\}}\, \mathrm{d}\lambda \le \frac{2\pi r^2}{\ln 2} \ln\left(\frac{K^2}{r^2}+1\right).$$

Together with (17) and (19), this implies the desired result. □

We end this section with a lemma which, roughly speaking, means that for bounded holomorphic functions in Ω_ε, a suitable property of "smallness" on the interval $[-1,1]$ still holds in $\Omega_{\varepsilon/2}$, up to constants.

Lemma 3.2.4. *Let ε be a positive real number and let g be a function holomorphic in Ω_ε and continuous up to the boundary. Assume that the weight sequence M satisfies the moderate growth property (4), and let L, a_1 and a_2 be positive numbers such that*

$$|g| \le L \text{ in } \Omega_\varepsilon \quad \text{and} \quad |g| \le a_1 h_M(a_2\varepsilon) \text{ on } [-1,1].$$

Then we have

$$|g| \le a_3 h_M(a_4\varepsilon) \text{ in } \Omega_{\varepsilon/2},$$

for suitable positive numbers a_3 and a_4 depending only on L, a_1, a_2 and on the sequence M.

Proof. With the notation of Definition 3.2.1, put $f = \frac{1}{a_1} g \circ \varphi_\varepsilon$. The function f is holomorphic in the strip S and continuous up to the boundary. Setting $K = \max(1, \frac{L}{a_1})$, we have $|f| \le K$ in S and $|f| \le h_M(a_2\varepsilon)$ on $\mathbb{R}$. Using Hadamard's three-lines theorem [21, pp. 33–34], we get $|f(z)| \le (h_M(a_2\varepsilon))^{1-|\Im z|} K^{|\Im z|}$ for every $z \in S$. Notice that $h_M(a_2\varepsilon) \le 1$ and $K \ge 1$. Since any point w in $\Omega_{\varepsilon/2}$ can be written $w = \varphi_\varepsilon(z)$ with $z \in S$ and $|\Im z| \le 1/2$, we therefore get the estimate $|g(w)| \le a_1(K h_M(a_2\varepsilon))^{1/2}$ for any such w. Since M has moderate growth, it then suffices to use (8) to obtain the desired result, with $a_3 = \max(a_1^{1/2}, L^{1/2})$ and $a_4 = \kappa_2 a_2$. □

3.3. An approximation-theoretic characterization of ultradifferentiable functions.

The approach of Joris's theorem in [7] relies on a characterization of $\mathcal{C}^k$ regularity of a function f on a bounded interval I in terms of the rate of approximation of f by uniformly bounded families of holomorphic functions in shrinking neighborhoods of I in $\mathbb{C}$. In this section, using the ellipses Ω_ε of Section 3.2 as neighborhoods of $[-1,1]$, we obtain, in the same spirit, a characterization of $\mathcal{C}_M$ regularity under the moderate growth assumption.

Definition 3.3.1. *Let M be a weight sequence. We shall say that a complex-valued function f defined on $[-1,1]$ satisfies property $(\mathcal{P}_M)$ if there are positive constants K, c_1, c_2 and a family $(f_\varepsilon)_{0<\varepsilon\le\varepsilon_0}$ of continuous functions in $\mathbb{C}$ such that, for any $\varepsilon \in (0, \varepsilon_0]$, the following conditions are satisfied:*

$$\text{the function } f_\varepsilon \text{ is holomorphic in } \Omega_\varepsilon, \tag{23}$$

$$|f_\varepsilon| \le K \text{ in } \Omega_\varepsilon, \tag{24}$$

$$|f - f_\varepsilon| \le c_1 h_M(c_2\varepsilon) \text{ on } [-1,1]. \tag{25}$$

Proposition 3.3.2. *Let M be a weight sequence that satisfies the moderate growth condition. Every element of $\mathcal{C}_M([-1,1])$ satisfies property $(\mathcal{P}_M)$. Conversely, if a complex-valued function defined on $[-1,1]$ satisfies $(\mathcal{P}_M)$, then it belongs to $\mathcal{C}_M([-b,b])$ for any real number b with $0 < b < 1$.*

Proof. Let f be an element of $\mathcal{C}_M([-1,1])$. By Dynkin's theorem on $\bar\partial$-flat extensions [6], there are positive constants c_1 and c_2, and a function g of class $\mathcal{C}^1$ with compact support in $\mathbb{C}$, such that $g = f$ on $[-1,1]$ and, for any $z \in \mathbb{C}$,

$$\left|\frac{\partial g}{\partial \bar z}(z)\right| \leq c_1 h_M(c_2 \operatorname{dist}(z,[-1,1])). \tag{26}$$

For every $\varepsilon \in (0,1]$, put

$$w_\varepsilon = \mathbb{1}_{\Omega_\varepsilon}\frac{\partial g}{\partial \bar z}.$$

Then w_ε is an element of $L^\infty(\mathbb{C})$, with $w_\varepsilon = 0$ in $\mathbb{C}\setminus\Omega_\varepsilon$. Besides, it is easy to see that for $z \in \Omega_\varepsilon$, we have $\operatorname{dist}(z,[-1,1]) \leq C\varepsilon$ for some absolute constant C. After multiplying c_2 by C, (26) implies

$$\|w_\varepsilon\|_\infty \leq c_1 h_M(c_2\varepsilon). \tag{27}$$

Now, set $v_\varepsilon = \mathcal{K} * w_\varepsilon$ where $\mathcal{K}$ is the Cauchy kernel. As explained in Section 3.1, v_ε is a continuous function in $\mathbb{C}$ such that $\partial v_\varepsilon/\partial\bar z = w_\varepsilon$ in the sense of distributions in $\mathbb{C}$, hence

$$\frac{\partial v_\varepsilon}{\partial \bar z} = \frac{\partial g}{\partial \bar z} \ \text{ in } \ \Omega_\varepsilon. \tag{28}$$

Moreover, by (16) and (27), it satisfies

$$\|v_\varepsilon\|_\infty \leq c_1 h_M(c_2\varepsilon) \tag{29}$$

after multiplying c_1 by a suitable absolute constant. Define $f_\varepsilon = g - v_\varepsilon$. Then f_ε is a bounded continuous function in $\mathbb{C}$ and we have $\|f_\varepsilon\|_\infty \leq \|g\|_\infty + c_1 h_M(c_2\varepsilon)$, hence (24) with $K = \|g\|_\infty + c_1 h_M(c_2)$. By (28), we have $\partial f_\varepsilon/\partial \bar z = 0$ in Ω_ε, hence (23). Finally, (29) implies (25) since f and g coincide on $[-1,1]$. Thus, property $(\mathcal{P}_M)$ is established, with $\varepsilon_0 = 1$.

Conversely, let $f : [-1,1] \to \mathbb{C}$ be a function that satisfies $(\mathcal{P}_M)$. For $0 < \varepsilon \leq \varepsilon_0/2$, it is readily seen that the function $f_\varepsilon - f_{2\varepsilon}$ meets the assumptions of Lemma 3.2.4 with $L = 2K$, $a_1 = 2c_1$ and $a_2 = 2c_2$. We therefore get

$$|f_\varepsilon - f_{2\varepsilon}| \leq a_3 h_M(a_4\varepsilon) \ \text{ in } \ \Omega_{\varepsilon/2}, \tag{30}$$

for some suitable constants a_3 and a_4 depending only on K, c_1 and c_2 Now, let b be a real number with $0 < b < 1$. By elementary geometric considerations, there is an absolute positive constant C such that for any $x \in [-b,b]$, the closed disk centered at x with radius $C(1-b)\varepsilon$ is contained in $\Omega_{\varepsilon/2}$. Using the Cauchy formula and (30), we therefore get $|(f_\varepsilon - f_{2\varepsilon})^{(j)}(x)| \leq a_3(C(1-b))^{-j} j!\varepsilon^{-j} h_M(a_4\varepsilon)$ for any $x \in [-b,b]$ and any $j \in \mathbb{N}$. Taking (9) into account, we get

$$\|f_\varepsilon - f_{2\varepsilon}\|_{[-b,b],\sigma} \leq a_3 h_M(a_5\varepsilon) \tag{31}$$

with $\sigma = \kappa_2 a_4 (C(1-b))^{-1}$ and $a_5 = \kappa_2 a_4$. Since $h_M(a_5\varepsilon) \leq a_5 M_1 \varepsilon$, this clearly implies the absolute convergence of the series $f_{\varepsilon_0} + \sum_{j\geq 1}\left(f_{\varepsilon_0 2^{-j}} - f_{\varepsilon_0 2^{-(j-1)}}\right)$ in the Banach space $\mathcal{C}_{M,\sigma}([-b,b])$. Let g denote its sum. For every integer $J \geq 1$, we have

$$g = f_{\varepsilon_0 2^{-J}} + \sum_{j\geq J+1}\left(f_{\varepsilon_0 2^{-j}} - f_{\varepsilon_0 2^{-(j-1)}}\right).$$

For $x \in [-b,b]$, we infer $|f(x)-g(x)| \le |f(x)-f_{\varepsilon_0 2^{-J}}(x)| + \sum_{j\ge J+1} |f_{\varepsilon_0 2^{-j}}(x) - f_{\varepsilon_0 2^{-(j-1)}}(x)| \le c_1 h_M(c_2\varepsilon_0 2^{-J}) + \sum_{j\ge J+1} \|f_{\varepsilon_0 2^{-j}} - f_{\varepsilon_0 2^{-(j-1)}}\|_{[-b,b],\sigma}$. Letting J tend to ∞, we obtain $f(x)=g(x)$, hence $f \in \mathcal{C}_M([-b,b])$. □

Remark 3.3.3. The moderate growth assumption is crucial in the proof of the converse part of Proposition 3.3.2, but the fact that the elements of $\mathcal{C}_M([-1,1])$ satisfy property $(\mathcal{P}_M)$ is still true under the weaker condition (11) of stability under derivation, which is required by Dynkin's result on $\bar\partial$-flat extensions.

4. Proof of the main result

4.1. **Reduction to a special case.** Consider two positive integers p and q such that $\gcd(p,q)=1$ and let f be a function germ at the origin in $\mathbb{R}$ such that f^p and f^q belong to $\mathcal{C}_M(\mathbb{R},0)$. Up to a linear change of variable, we can assume that f^p and f^q belong to $\mathcal{C}_M([-1,1])$. As pointed out in [14], if we set $m=pq$, then any integer $j\ge m$ can be written $j=pk+ql$ with $(k,l)\in\mathbb{N}^2$. Thus, we have $f^j=(f^p)^k(f^q)^l$ and, since $\mathcal{C}_M([-1,1])$ is an algebra, we see that f^j belongs to $\mathcal{C}_M([-1,1])$. In particular, we have

$$f^m \in \mathcal{C}_M([-1,1]) \text{ and } f^{m+1} \in \mathcal{C}_M([-1,1]). \tag{32}$$

In order to conclude that f belongs to $\mathcal{C}_M(\mathbb{R},0)$, it then suffices to prove that (32) implies $f\in\mathcal{C}_M([-b,b])$ for $0<b<1$.

4.2. **Construction of approximants.** By Proposition 3.3.2, there are constants $K\ge 1$, $c_1>0$, $c_2>0$ and families $(g_\varepsilon)_{0<\varepsilon\le\varepsilon_0}$ and $(h_\varepsilon)_{0<\varepsilon\le\varepsilon_0}$ of bounded continuous functions in $\mathbb{C}$ such that for $0<\varepsilon\le\varepsilon_0$, we have the following properties:

$$\text{the functions } g_\varepsilon \text{ and } h_\varepsilon \text{ are holomorphic in } \Omega_\varepsilon, \tag{33}$$

$$|g_\varepsilon|\le K \text{ and } |h_\varepsilon|_\infty \le K \text{ in } \Omega_\varepsilon, \tag{34}$$

$$|f^m-g_\varepsilon|\le c_1h_M(c_2\varepsilon) \text{ and } |f^{m+1}-h_\varepsilon|\le c_1h_M(c_2\varepsilon) \text{ on } [-1,1]. \tag{35}$$

In view of the above, the intuitive candidate for an holomorphic approximation of f on $[-1,1]$ is the quotient $h_\varepsilon/g_\varepsilon$, but it has to be modified to avoid small denominators. We therefore define

$$u_\varepsilon = \chi_\varepsilon \frac{\overline{g_\varepsilon}h_\varepsilon}{(\max(|g_\varepsilon|, r_\varepsilon))^2}$$

where r_ε is a positive real number, and $\chi_\varepsilon : \mathbb{C}\to[0,1]$ is a smooth cutoff function with $\chi_\varepsilon=1$ in $\Omega_{\varepsilon/2}$ and $\operatorname{supp}\chi_\varepsilon\subset\Omega_\varepsilon$. The function u_ε is well-defined, continuous with compact support in $\mathbb{C}$ and it coincides with $h_\varepsilon/g_\varepsilon$ in $\Omega_{\varepsilon/2}\cap\{|g_\varepsilon|>r_\varepsilon\}$, but it is obviously not holomorphic in a whole neighborhood of $[-1,1]$. In the rest of the proof, we shall however see that for a suitable choice of r_ε, this function satisfies uniform bounds and is "close enough" to f on $[-1,1]$, and we shall then recover a holomorphic approximant *via* a $\bar\partial$-problem.

Using (34), (35) and the elementary inequality $|z^j-\zeta^j|\le j\max(|z|,|\zeta|)^{j-1}|z-\zeta|$ with $j=m$ and with $j=m+1$, we see that there is a constant c_3 depending only on K, c_1 and m, such that $|h_\varepsilon^m-g_\varepsilon^{m+1}|\le c_3h_M(c_2\varepsilon)$ on $[-1,1]$. Moreover, $h_\varepsilon^m-g_\varepsilon^{m+1}$ is holomorphic in Ω_ε, continuous up to the boundary and we have

$|h_\varepsilon^m - g_\varepsilon^{m+1}| \le 2K^{m+1}$ in Ω_ε. Thus, applying Lemma 3.2.4 with $L = 2K^{m+1}$, $a_1 = c_3$ and $a_2 = c_2$, we obtain

$$|h_\varepsilon^m - g_\varepsilon^{m+1}| \le c_4 h_M(c_5\varepsilon) \ \text{ in } \ \Omega_{\varepsilon/2}, \tag{36}$$

where c_4 and c_5 depend only on K, c_1, c_2 and m. We shall now set

$$\delta_\varepsilon = c_4 h_M(c_5\varepsilon) \ \text{ and } \ r_\varepsilon = \delta_\varepsilon^{\frac{1}{m+1}}. \tag{37}$$

Since we can obviously assume $c_4 \ge c_1$ and $c_5 \ge c_2$, it is convenient to rewrite (35) and (36) as

$$\begin{aligned} &|f^{m+1} - h_\varepsilon| \le \delta_\varepsilon \ \text{ and } \ |f^m - g_\varepsilon| \le \delta_\varepsilon \ \text{ on } \ [-1,1], \\ &|h_\varepsilon^m - g_\varepsilon^{m+1}| \le \delta_\varepsilon \ \text{ in } \ \Omega_{\varepsilon/2}. \end{aligned} \tag{38}$$

Also, notice that we have $\delta_\varepsilon \le r_\varepsilon \le 1$ for ε small enough.

Lemma 4.2.1. *For any sufficiently small $\varepsilon > 0$, we have*

$$|u_\varepsilon| \le (2K)^{1/m} \ \text{ in } \ \Omega_{\varepsilon/2}.$$

Proof. By (38), in $\Omega_{\varepsilon/2}$, we have $|h_\varepsilon| \le (|g_\varepsilon^{m+1}| + |h_\varepsilon^m - g_\varepsilon^{m+1}|)^{1/m} \le (|g_\varepsilon|^{m+1} + r_\varepsilon^{m+1})^{1/m} \le 2^{1/m}(\max(|g_\varepsilon|, r_\varepsilon))^{\frac{m+1}{m}}$, hence $|u_\varepsilon| \le 2^{1/m}|g_\varepsilon|(\max(|g_\varepsilon|, r_\varepsilon))^{-1+\frac{1}{m}} \le 2^{1/m}(\max(|g_\varepsilon|, r_\varepsilon))^{\frac{1}{m}}$. The result then follows from (34). □

Lemma 4.2.2. *There is a constant c_6 depending only on K and m, such that, for any sufficiently small $\varepsilon > 0$, we have*

$$|f - u_\varepsilon| \le c_6 \delta_\varepsilon^{\frac{1}{m(m+1)}} \ \text{ on } \ [-1,1].$$

Proof. The estimate will be proved separately on the sets $F_\varepsilon = [-1,1] \cap \{|g_\varepsilon| \le r_\varepsilon\}$ and $G_\varepsilon = [-1,1] \cap \{|g_\varepsilon| > r_\varepsilon\}$. On the set F_ε, we have $f - u_\varepsilon = f - r_\varepsilon^{-2}\,\overline{g_\varepsilon} h_\varepsilon$, hence

$$|f - u_\varepsilon| \le |f| + r_\varepsilon^{-2}\,|g_\varepsilon||h_\varepsilon| \le |f| + r_\varepsilon^{-1}|h_\varepsilon|.$$

By (38), we also have $|f| \le (|g_\varepsilon| + |f^m - g_\varepsilon|)^{1/m} \le (r_\varepsilon + \delta_\varepsilon)^{1/m} \le (2r_\varepsilon)^{1/m}$ and $|h_\varepsilon| \le (|g_\varepsilon^{m+1}| + |h_\varepsilon^m - g_\varepsilon^{m+1}|)^{1/m} \le (r_\varepsilon^{m+1} + \delta_\varepsilon)^{1/m} = (2r_\varepsilon^{m+1})^{1/m} = r_\varepsilon(2r_\varepsilon)^{1/m}$. Setting $c_7 = 2^{1+\frac{1}{m}}$, we finally derive

$$|f - u_\varepsilon| \le c_7 r_\varepsilon^{1/m} = c_7 \delta_\varepsilon^{\frac{1}{m(m+1)}} \ \text{ on } \ F_\varepsilon. \tag{39}$$

On the set G_ε, we have

$$f - u_\varepsilon = f - \frac{h_\varepsilon}{g_\varepsilon} = \frac{f(g_\varepsilon - f^m) + f^{m+1} - h_\varepsilon}{g_\varepsilon}$$

with $|f| \le (|g_\varepsilon| + |f^m - g_\varepsilon|)^{1/m} \le (K + \delta_\varepsilon)^{1/m} \le (K+1)^{1/m}$. Thus, using (38), it is easy to obtain

$$|f - u_\varepsilon| \le c_8 \frac{\delta_\varepsilon}{r_\varepsilon} = c_8 \delta_\varepsilon^{\frac{m}{m+1}} \ \text{ on } \ G_\varepsilon, \tag{40}$$

with $c_8 = (K+1)^{1/m} + 1$. The lemma clearly follows from (39) and (40). □

Now we proceed to obtain a holomorphic modification of u_ε. As a starting point, we need basic information on $\partial u_\varepsilon / \partial \bar{z}$.

Lemma 4.2.3. *The distributional derivative $\partial u_\varepsilon/\partial\bar z$ is an element of $L^\infty(\mathbb{C})$ and we have*

$$\frac{\partial u_\varepsilon}{\partial \bar z} = \frac{1}{r_\varepsilon^2}\,\overline{g'_\varepsilon} h_\varepsilon \mathbb{1}_{\{|g_\varepsilon|<r_\varepsilon\}} \text{ in } \Omega_{\varepsilon/2}. \tag{41}$$

Proof. We introduce the sets $X_\varepsilon = \Omega_{\varepsilon/2} \cap \{|g_\varepsilon| < r_\varepsilon\}$, $Y_\varepsilon = \Omega_{\varepsilon/2} \cap \{|g_\varepsilon| > r_\varepsilon\}$ and $Z_\varepsilon = \Omega_{\varepsilon/2} \cap \{|g_\varepsilon| = r_\varepsilon\}$. Since g_ε is holomorphic in Ω_ε, either the set Z_ε has measure zero, or g_ε is constant. In the latter case, u_ε is a constant times h_ε and the conclusion of the lemma is immediate. We therefore focus on the general case of a non-constant g_ε. Since $\operatorname{supp}\chi_\varepsilon \subset \Omega_\varepsilon$ and $|g_\varepsilon|^2$ is smooth in Ω_ε, it is readily seen that the denominator $\max(|g_\varepsilon|^2, r_\varepsilon^2)$ is Lipschitz and bounded away from zero in a neighborhood of $\operatorname{supp}\chi_\varepsilon$. Taking into account the smoothness of $\overline{g_\varepsilon} h_\varepsilon$ in Ω_ε, we infer that u_ε is a bounded Lipschitz function in $\mathbb{C}$, hence it belongs to the Sobolev space $W^{1,\infty}(\mathbb{C})$ (see [4, Proposition 9.3] or [9, Theorem 6.12]). Thus, the distribution $\partial u_\varepsilon/\partial\bar z$ is an element of $L^\infty(\mathbb{C})$. Since $\Omega_{\varepsilon/2} = X_\varepsilon \cup Y_\varepsilon \cup Z_\varepsilon$ and Z_ε has measure zero, it then suffices to check (41) in each of the open sets X_ε and Y_ε, which boils down to an explicit computation using the holomorphicity of g_ε and h_ε in those sets. In X_ε, we have $u_\varepsilon = r_\varepsilon^{-2}\,\overline{g_\varepsilon} h_\varepsilon$, hence $\partial u_\varepsilon/\partial\bar z = r_\varepsilon^{-2}\,\overline{g'_\varepsilon} h_\varepsilon$. In Y_ε, we have $u_\varepsilon = h_\varepsilon/g_\varepsilon$, hence $\partial u_\varepsilon/\partial\bar z = 0$. The lemma is proved. ☐

We now set

$$w_\varepsilon = \mathbb{1}_{\Omega_{\varepsilon/2}} \frac{\partial u_\varepsilon}{\partial \bar z} \quad\text{and}\quad v_\varepsilon = \mathcal{K} * w_\varepsilon.$$

The function w_ε is an element of $L^\infty(\mathbb{C})$ with $w = 0$ in $\mathbb{C}\setminus\Omega_{\varepsilon/2}$. Thus, as explained in Section 3.1, v_ε is a bounded continuous function in $\mathbb{C}$ that satisfies $\partial v_\varepsilon/\partial\bar z = w_\varepsilon$ in the sense of distributions in $\mathbb{C}$, hence

$$\frac{\partial v_\varepsilon}{\partial \bar z} = \frac{\partial u_\varepsilon}{\partial \bar z} \text{ in } \Omega_{\varepsilon/2}. \tag{42}$$

The last ingredient of the proof will be an estimate for v_ε in $\Omega_{\varepsilon/2}$.

Lemma 4.2.4. *Let s be a real number, with $s > m(m+1)$. For $\varepsilon > 0$ small enough, we have*

$$|v_\varepsilon| \le c_9 \delta_\varepsilon^{1/s} \text{ in } \Omega_{\varepsilon/2},$$

where c_9 is a constant depending only on K, m and s.

Proof. By Lemma 3.1.1, there is a constant C such that for any $\varepsilon > 0$ small enough, we have

$$|v_\varepsilon| \le C\left(r_\varepsilon \|w_\varepsilon\|_\infty + (|\ln r_\varepsilon|)^{1/2}\,\|w_\varepsilon\|_2\right) \text{ in } \Omega_{\varepsilon/2}. \tag{43}$$

Using (38), we see that in the open set $\Omega_{\varepsilon/2}\cap\{|g_\varepsilon| < r_\varepsilon\}$, we have $|h_\varepsilon| \le (|g_\varepsilon|^{m+1} + \delta_\varepsilon)^{1/m} \le (r_\varepsilon^{m+1} + \delta_\varepsilon)^{1/m} = 2^{1/m} r_\varepsilon^{\frac{m+1}{m}}$. This implies

$$|w_\varepsilon| \le 2^{1/m} r_\varepsilon^{\frac{1}{m}-1} |g'_\varepsilon| \mathbb{1}_{\{|g_\varepsilon|<r_\varepsilon\}}. \tag{44}$$

Now recall that g_ε is holomorphic in Ω_ε, with $|g_\varepsilon| \le K$. Since any closed disk of radius $\frac{1}{8}\varepsilon^2$ centered in $\Omega_{\varepsilon/2}$ is contained in Ω_ε, the Cauchy formula then yields $|g'_\varepsilon| \le 8K\varepsilon^{-2}$ in $\Omega_{\varepsilon/2}$. Together with (44), this implies the uniform estimate

$$\|w_\varepsilon\|_\infty \le c_{10} \frac{r_\varepsilon^{\frac{1}{m}-1}}{\varepsilon^2}, \tag{45}$$

with $c_{10} = 8 \cdot 2^{1/m} K$. Using Lemma 3.2.3 and (44), we also get the L^2 estimate

$$\|w_\varepsilon\|_2 \leq c_{11} \frac{r_\varepsilon^{1/m}}{\varepsilon^{3/2}} \left(\ln \left(\frac{K^2}{r_\varepsilon^2} + 1 \right) \right)^{1/2} \tag{46}$$

for a positive constant c_{11} depending only on m. Since $r_\varepsilon = \delta_\varepsilon^{\frac{1}{m+1}}$ and $\delta_\varepsilon = o(\varepsilon^j)$ for every integer $j \geq 1$, the desired result follows from (43), (45) and (46). □

It is now possible to complete the proof of Theorem 2.2.1.

4.3. **End of the proof.** We consider $f_\varepsilon = u_{2\varepsilon} - v_{2\varepsilon}$ for $\varepsilon > 0$ small enough. The function f_ε is continuous in $\mathbb{C}$, and it is holomorphic in Ω_ε, since, by (42), we also have $\partial f_\varepsilon / \partial \bar{z} = 0$ in the sense of distributions in Ω_ε. Lemma 4.2.1 and Lemma 4.2.4 imply

$$|f_\varepsilon| \leq K' \ \text{ in } \ \Omega_\varepsilon,$$

with $K' = (2K)^{1/m} + c_9$. Finally, choose a real number s with $s > m(m+1)$. By Lemma 4.2.2 and Lemma 4.2.4, we have $|f - f_\varepsilon| \leq |f - u_{2\varepsilon}| + |v_{2\varepsilon}| \leq c_{12} \delta_{2\varepsilon}^{1/s}$ on $[-1, 1]$, for some suitable constant $c_{12} > 0$. Using (37) and the moderate growth property (8), we get $\delta_{2\varepsilon}^{1/s} \leq c_{13} h_M(c_{14}\varepsilon)$ with $c_{13} = c_4^{1/s}$ and $c_{14} = 2\kappa_s c_5$. Thus, we obtain

$$|f - f_\varepsilon| \leq c_1' h_M(c_2' \varepsilon) \ \text{ on } \ [-1, 1],$$

with $c_1' = c_{12} c_{13}$ and $c_2' = c_{14}$. We have therefore proved that, for ε_0' small enough, the family $(f_\varepsilon)_{0 < \varepsilon \leq \varepsilon_0'}$ meets the requirements of property $(\mathcal{P}_M)$. Thus, by Proposition 3.3.2, the function f belongs to $\mathcal{C}_M([-b, b])$ for any b with $0 < b < 1$, and Theorem 2.2.1 is now established.

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