ON THE NON-EXTENDABILITY OF QUASIANALYTIC GERMS

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Abstract. Let $\mathcal{E}_1(M)^+$ be the local ring of germs at 0 of functions belonging to a given Denjoy-Carleman quasianalytic class in a neighborhood of 0 in $[0,+\infty)$. We show that the ring $\mathcal{E}_1(M)^+$ contains elements that cannot be extended quasianalytically in a neighborhood of 0 in $\mathbb{R}$, unless it coincides with the ring of real-analytic germs.

Introduction

Let $\mathcal{E}_1(M)$ be the local ring of germs at 0 of functions that belong to the Denjoy-Carleman class associated with a suitable real sequence M in a neighborhood of 0 in $\mathbb{R}$ (see Section 1 for the definitions). Denote by $\mathcal{F}_1(M)$ the associated ring of all formal series $F = \sum_{j\geq 0} F_j x^j$ such that $\sup_{j\geq 0} \nu^{-j} M_j^{-1} |F_j| < \infty$ for some integer $\nu \geq 1$, so that the Borel map T_0, which to every smooth germ associates its Taylor series at 0, maps $\mathcal{E}_1(M)$ into $\mathcal{F}_1(M)$.

Assume that $\mathcal{E}_1(M)$ is quasianalytic and does not coincide with the ring $\mathcal{O}_1$ of real-analytic germs. Then, according to a classical theorem of Carleman ([1]; see [8] for a simple proof), the map $T_0 : \mathcal{E}_1(M) \to \mathcal{F}_1(M)$ is not surjective, in other words there is no Borel extension theorem from $\mathcal{F}_1(M)$ to $\mathcal{E}_1(M)$. Up to minor modifications, the proof given in [8] actually shows that there is no extension from $\mathcal{F}_1(M)$ to the ring $\mathcal{E}_1(M)^+$ of germs at 0 of functions belonging to the quasianalytic Denjoy-Carleman class associated with M in some interval $[0, b[$ with $b > 0$.

Thus, the Taylor series of elements of $\mathcal{E}_1(M)^+$ form a proper subspace of $\mathcal{F}_1(M)$, and it is now natural to ask whether the elements of this subspace can be extended to $\mathcal{E}_1(M)$. Equivalently, is it possible to obtain every element of $\mathcal{E}_1(M)^+$ as the restriction to the positive reals of an element of $\mathcal{E}_1(M)$ (or even of a larger quasianalytic local ring $\mathcal{E}_1(N)$)? The goal of this short paper is to establish a negative answer. Although this result could be expected, it doesn't seem to be explicitly stated, even less proven, in the literature[1]. Moreover, it cannot be obtained by the same arguments as Carleman's theorem: roughly speaking, non-extendability from $\mathcal{F}_1(M)$ to $\mathcal{E}_1(M)$, and to $\mathcal{E}_1(M)^+$ as well,

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[1]After this note was written, the author found out that a similar result, albeit with a slightly different proof, already appeared in a paper by M. Langenbruch (Manuscripta Math. 83 (1994), 123-143; see the remark after Corollary 2.4).

relies on the fact that a certain lacunarity condition on a given element F of $\mathcal{F}_1(M)$ is an obstruction for F to be the Taylor series of a germ in these rings, unless it is convergent (more precise versions of this phenomenon can be found in [5]). Therefore, when F is already supposed to be the Taylor series of an element of $\mathcal{E}_1(M)^+$, this condition is generally not fulfilled, and one has to find another idea to prove that it is not always possible to extend F as a quasianalytic function in some interval $]a,0]$ with $a<0$. We will obtain the desired result by means of a functional-analytic argument inspired by the proof of the failure of quasianalytic Weierstrass division by non-hyperbolic polynomials in [4].

1. Quasianalytic germs

1.1. **Local rings of ultradifferentiable germs.** We recall several basic facts on ultradifferentiable germs. For a more detailed account, we refer the reader to [8], for instance.

Let $M=(M_j)_{j\geq 0}$ be an increasing, logarithmically convex sequence of real numbers, with $M_0=1$. Let I be a bounded interval in $\mathbb{R}$. For any integer $\nu\geq 1$, we consider the vector space $\mathcal{E}_{1,\nu}(M,I)=\{f\in C^\infty(I):\|f\|_{M,\nu,I}<\infty\}$, where

$$\|f\|_{M,\nu,I}=\sup_{j\in\mathbb{N},\ x\in I}\frac{|f^{(j)}(x)|}{\nu^j j! M_j}.$$

Remark 1. As recalled in [8], the space $\mathcal{E}_{1,\nu}(M,I)$ is a Banach space, and for $\nu<\nu'$, the inclusion $\mathcal{E}_{1,\nu}(M,I)\hookrightarrow\mathcal{E}_{1,\nu'}(M,I)$ is compact.

We denote by $\mathcal{E}_1(M)$ the set of germs of C^∞ functions at 0 in $\mathbb{R}$ that belong to $\mathcal{E}_{1,\nu}(M,I)$ for some integer $\nu\geq 1$ and some open interval I containing 0. As explained in [8], the set $\mathcal{E}_1(M)$ is a local ring, with maximal ideal $\underline{m}_M=\{f\in\mathcal{E}_1(M):f(0)=0\}$. This local ring contains the ring $\mathcal{O}_1$ of real analytic function germs at 0, and we have

$$\mathcal{E}_1(M)=\mathcal{O}_1 \iff \sup_{j\geq 1}(M_j)^{1/j}<\infty. \tag{1}$$

More generally, if N denotes another increasing, logarithmically convex, sequence of real numbers with $N_0=1$, we have the equivalence

$$\mathcal{E}_1(M)\subseteq\mathcal{E}_1(N) \iff \sup_{j\geq 1}(M_j/N_j)^{1/j}<\infty.$$

A C^∞ function germ is said to be *flat* if it vanishes, together with all its derivatives, at the origin. The local ring $\mathcal{E}_1(M)$ is said to be *quasianalytic* if the only flat germ that it contains is 0. By the famous Denjoy-Carleman theorem, $\mathcal{E}_1(M)$ is quasianalytic if and only if

$$\sum_{j=0}^{+\infty}\frac{M_j}{(j+1)M_{j+1}}=+\infty. \tag{2}$$

Remark 2. In this case, it is easy to see that for any bounded intervals I and I' of $\mathbb{R}$, with $I' \subset I$, the natural restriction map $\mathcal{E}_{1,\nu}(M,I) \to \mathcal{E}_{1,\nu}(M,I')$ is injective.

We also define $\mathcal{E}_1(M)^+$ as the set of germs at 0 of C^∞ functions in $[0,+\infty[$ that belong to $\mathcal{E}_{1,\nu}(M,I)$ for some integer $\nu \geq 1$ and some interval $I = [0,b[$ with $b > 0$. Finally, if is N is another increasing, logarithmically convex, sequence of real numbers with $N_0 = 1$ and $\sup_{j\geq 1}(M_j/N_j)^{1/j} < \infty$, we denote by $\mathcal{E}_1(M,N)$ the set of elements of $\mathcal{E}_1(N)$ whose restriction to the positive reals belongs to $\mathcal{E}_1(M)^+$. We have

$$\mathcal{E}_1(M) \subseteq \mathcal{E}_1(M,N) \subseteq \mathcal{E}_1(N). \tag{3}$$

1.2. **Functional-analytic framework in the quasianalytic case.** For any integer $\nu \geq 1$, put $I_\nu =]-1/\nu, 1/\nu[$, $I_\nu^- =]-1/\nu, 0]$ and $I_\nu^+ = [0, 1/\nu[$. Let E_ν be the space of C^∞ functions on I_ν whose restrictions to I_ν^- and to I_ν^+ belong, respectively, to $\mathcal{E}_{1,\nu}(N, I_\nu^-)$ and to $\mathcal{E}_{1,\nu}(M, I_\nu^+)$. Clearly, E_ν is a Banach space for the norm $\|\cdot\|_\nu$ defined by

$$\|f\|_\nu = \max\left(\|f_{|I_\nu^-}\|_{N,\nu,I_\nu^-}\ ,\ \|f_{|I_\nu^+}\|_{M,\nu,I_\nu^+}\right).$$

Now, assume that N satisfies the quasianalyticity condition (2), and observe that (3) then implies that the same holds for M. From Remarks (1) and (2) with $\nu' = \nu + 1$, it is then easy to derive a compact injection $E_\nu \to E_{\nu+1}$. The space $\mathcal{E}_1(M,N)$ can be described as the inductive limit $\varinjlim E_\nu$ and is therefore endowed with a (DFS)-space topology (see e.g. Section 3.5 of [8] in the case $M = N$).

Analogously, the space $\mathcal{E}_1(M)^+$ is a (DFS)-space, since it can be written as the inductive limit of all Banach spaces $E_\nu^+ = \mathcal{E}_{1,\nu}(M, I_\nu^+)$ with $\nu \geq 1$ (indeed, an application of Remarks (1) and (2) with $\nu' = \nu+1$, $I = I_\nu^+$ and $I' = I_{\nu+1}^+$ provides a compact injection $E_\nu^+ \to E_{\nu+1}^+$). Finally, we set $\|\cdot\|_\nu^+ = \|\cdot\|_{M,\nu,I_\nu^+}$.

2. Non extendability

2.1. **The main result.** As in Subsection 1.1, we consider two sequences M and N such that $\mathcal{E}_1(M) \subseteq \mathcal{E}_1(N)$. We then have the following statement.

Theorem 1. *Assume that we have $\mathcal{O}_1 \subsetneq \mathcal{E}_1(M)$ and that $\mathcal{E}_1(N)$ is quasianalytic. Then there exist elements of $\mathcal{E}_1(M)^+$ that cannot be extended as elements of $\mathcal{E}_1(N)$.*

Proof. Consider the space $\mathcal{E}_1(M,N)$ defined in Subsection 1.1. The result amounts to proving that the restriction map $R : \mathcal{E}_1(M,N) \to \mathcal{E}_1(M)^+$, which to each element of $\mathcal{E}_1(M,N)$ associates its restriction to the positive reals, is not surjective. We proceed by contradiction. Assume that R is surjective. The quasianalyticity assumption on $\mathcal{E}_1(N)$ implies that it is injective as well. Since R is obviously continuous for the (DFS)-space topologies on $\mathcal{E}_1(M,N)$ and $\mathcal{E}_1(M)^+$, the Grothendieck version of the open mapping theorem ([6], Theorem 24.33) then implies that it is a topological isomorphism. The continuity of R^{-1}

is equivalent to the statement that for any integer $\nu \geq 1$, there is an integer $\lambda_\nu \geq 1$ and a positive constant C_ν such that we have $\|R^{-1}f\|_{\lambda_\nu} \leq C_\nu \|f\|_\nu^+$ for any $f \in E_\nu^+$. In particular this must be true when f is polynomial, and we therefore get $\|P\|_{\lambda_\nu} \leq C_\nu \|P\|_\nu^+$ for any $P \in \mathbb{C}[x]$. Set $\nu = 1$, $a = -1/\lambda_1$ and $A = C_1$. The preceding estimates then yield, in particular,

$$|P(a)| \leq A\|P\|_1^+ \quad \text{for any } P \in \mathbb{C}[x]. \tag{4}$$

We proceed to get a majorization of $\|P\|_1^+$. For any $x \in [0, 1[$, we apply the Cauchy formula on the closed disc D_x centered at x, with radius $|a|/2$. We obtain, for any integer $j \geq 0$, the bound $|P^{(j)}(x)| \leq (2/|a|)^j j! \sup_W |P|$ with $W = \{z \in \mathbb{C} : \mathrm{dist}(z, [0, 1]) \leq |a|/2\}$. This implies

$$\|P\|_1^+ \leq B \sup_W |P| \tag{5}$$

with $B = \sup_{j\geq 0}(2/|a|)^j/M_j$. Observe that, since we have assumed $\mathcal{O}_1 \subsetneq \mathcal{E}_1(M)$, property (1) implies $\sup_{j\geq 1}(M_j)^{1/j} = \infty$, hence B is finite. Now, gathering (4) and (5) and setting $C = AB$, we obtain

$$|P(a)| \leq C \sup_W |P| \quad \text{for any } P \in \mathbb{C}[x]. \tag{6}$$

It is now possible to conclude. The point a does not belong to W, and $W \cup \{a\}$ is a compact subset with connected complement in $\mathbb{C}$. By Runge's approximation theorem, there is a sequence of polynomials $(P_k)_{k\geq 0}$ converging uniformly to 0 on W, and such that $\lim_{k\to\infty} P_k(a) = 1$. This obviously contradicts (6).

2.2. **Extension to non-quasianalytic germs.** The conclusion of Theorem 1 is no longer true without the quasianalyticity assumption on the larger space $\mathcal{E}_1(N)$. For example, when $M_j = \ln(j+e)$, the local ring $\mathcal{E}_1(M)$ is quasianalytic but, according to a result of Carleson [2], every element of $\mathcal{E}_1(M)^+$ can be extended as an element of the non-quasianalytic ring $\mathcal{E}_1(N)$ with $N_j = (\ln(j+e))^2$. For more recent and more general results in this direction, we refer the reader to [3] and [7]. □

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