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On Closed Ideals in Smooth Classes

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Abstract. We study closedness properties of ideals generated by real – analytic functions in some subrings $\mathcal{C}$ of $C^\infty(\Omega)$, where Ω is an open subset of $\mathbb{R}^n$. In contrast with the case $\mathcal{C} = C^\infty(\Omega)$, which has been clarified by famous works of HÖRMANDER, ŁOJASIEWICZ and MALGRANGE, it turns out that such ideals are generally not closed when $\mathcal{C}$ is an ultradifferentiable class. If $\mathcal{C}$ is sufficiently regular and non – quasianalytic, and under the assumption that the real zero locus of the ideal reduces to a single point, we obtain a sharp sufficient condition of closedness, expressed in terms of the geometry of common complex zeros for the germs of the generators at this point. This condition is shown to be also necessary in dimension 2, when the ideal is principal. Some related questions about rings of ultradifferentiable germs and about ultradistributions are discussed.

1. Introduction

Let Ω be a connected open subset of $\mathbb{R}^n$ and φ be a non – zero real – analytic function in Ω. A famous result asserts that any distribution T in Ω can be divided by φ, in other words there exists a distribution S in Ω such that $T = \varphi S$. This has been proved in the late fifties by HÖRMANDER [11] for a polynomial φ, and by ŁOJASIEWICZ [14] in the analytic situation. A possible approach to the division of distributions is to view it as a dual fact of closedness properties for the ideal generated by φ among C^∞ functions. It is indeed a particular case of a deep result of MALGRANGE [15], stating that any ideal generated in $C^\infty(\Omega)$ by a finite family of real – analytic functions is closed in $C^\infty(\Omega)$, endowed with its usual Fréchet space topology.

In this paper, we are concerned with the study of ideals generated by analytic functions in some special subrings of $C^\infty(\Omega)$, namely ultradifferentiable classes, a classical example of which is given by the Gevrey classes familiar in the theory of partial differential equations. The standard topology on ultradifferentiable classes is stronger than the C^∞ topology; however, in this context, ideals with analytic generators generally fail to be closed [8], even in the simplest situation of principal ideals $\varphi C^\infty(\Omega)$ with $\varphi^{-1}(0)$ reduced to a single point. It is then natural to ask for criteria to decide whether such ideals are closed: a purpose of the present work is to give a partial answer to this problem. Always assuming that the zero set of φ in Ω is an isolated point, say 0, we

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obtain a sufficient condition which is valid for any number of variables: if the local complexification of φ near 0 in $\mathbb{C}^n$ has a zero set which is transversal to $\mathbb{R}^n$ in the sense of regular separation, then the ideal generated by φ in any sufficiently regular non-quasianalytic Carleman class (like Gevrey ones) is closed. Moreover, in the case of two variables, some local geometry based on Puiseux expansions for φ allows us to show that the former condition is also necessary, giving thus a complete characterization. Note also that our sufficient condition extends, in a somewhat less pleasant form, to the situation of ideals with several analytic generators $\varphi_1, \ldots, \varphi_p$.

In fact, these results come here as byproducts of properties of ideals of germs at the origin in $\mathbb{R}^n$, namely ideals generated by a finite family of real-analytic germs in a given local ring $C_M(\mathbb{R}^n, 0)$ of ultradifferentiable germs (see Definition 2.1 below). The central question is to know whether this ideal is elliptic; in other words whether it contains all the germs of $C_M(\mathbb{R}^n, 0)$ which are flat at the origin.

The paper is organized as follows. In Section 2, we introduce some notations and state the main results concerning rings of ultradifferentiable germs. A proof of a sufficient condition of ellipticity is given. Section 3 is devoted to proving the necessity of this condition in the case of two variables and one generator φ. This requires to compute explicitly the Łojasiewicz exponent for the regular separation between $\mathbb{R}^n$ and the complex zeros of φ, in terms of Puiseux expansions. Such a computation, which may have an independent interest, has been inspired by a paper of Kuo [13]. In Section 4, we state and prove the results on closed ideals. We also apply them to the division of ultradistributions.

Throughout the paper, for any multi-index $L = (\ell_1, \ldots, \ell_n)$ in $\mathbb{N}^n$, we will denote by the corresponding lower case letter ℓ the length $\ell_1 + \cdots + \ell_n$ of L. We also put $L! = \ell_1! \ldots \ell_n!$ and denote by D^L the operator $\partial^\ell / \partial x_1^{\ell_1} \ldots \partial x_n^{\ell_n}$ associated to L.

2. Local results

2.1. Notations and prerequisites

Denote by $C^\infty(\mathbb{R}^n, 0)$ the ring of C^∞ function germs at the origin of $\mathbb{R}^n$, and $\underline{m}$ its maximal ideal. Let $\varphi = (\varphi_1, \ldots, \varphi_p)$ be a germ of real-analytic mapping from $(\mathbb{R}^n, 0)$ to $(\mathbb{R}^p, 0)$. In the sequel, it will always be assumed that one has

$$\{x \in (\mathbb{R}^n, 0) : \varphi(x) = 0\} = \{0\}. \tag{2.1}$$

Then it is well-known [15] that there exist real constants $a > 0$ and $r \geq 1$ such that the Łojasiewicz inequality

$$|\varphi(x)| \geq a\,|x|^r \tag{2.2}$$

holds for all x belonging to a suitable neighborhood of 0 in $\mathbb{R}^n$. It is now also classical (see e. g. [19]) that (2.2) is equivalent to the inclusion

$$\underline{m}^\infty \subset I_\varphi\,, \tag{2.3}$$

where $\underline{m}^\infty$ stands for $\bigcap_{k \geq 0} \underline{m}^k$, the ideal of germs which are flat at the origin, and I_φ is the ideal generated by $\varphi_1, \ldots, \varphi_p$ in $C^\infty(\mathbb{R}^n, 0)$. Property (2.3) is known under the terminology of *ellipticity* of the ideal I_φ, see [18].

We turn now to the study of ultradifferentiable germs. Let $M = (M_\ell)_{\ell \geq 0}$ be an increasing sequence of real numbers with $M_0 = 1$. The sequence M is said to be *strongly regular* if there exists a positive constant A such that the following assumptions are satisfied:

$$M \text{ is logarithmically convex}, \tag{2.4}$$

$$M_{j+k} \leq A^{j+k} M_j M_k \quad \text{for all} \quad (j,k) \in \mathbb{N}^2, \tag{2.5}$$

$$\sum_{j \geq \ell} \frac{M_j}{(j+1)M_{j+1}} \leq A \frac{M_\ell}{M_{\ell+1}} \quad \text{for all} \quad \ell \in \mathbb{N}. \tag{2.6}$$

Condition (2.4), which amounts to saying that $M_{\ell+1}/M_\ell$ increases with ℓ, implies

$$M_j M_k \leq M_{j+k} \quad \text{for all} \quad (j,k) \in \mathbb{N}^2. \tag{2.7}$$

Thus (2.5), which is in some sense converse to (2.7), means that the growth of M is not too wild. The role of condition (2.6) will be recalled later, after the definition of ultradifferentiable germs. To each strongly regular sequence M, one associates the function h_M defined by $h_M(t) = \inf_{\ell \geq 0} t^\ell M_\ell$ for all $t \in \mathbb{R}_+$. This function is continuous, non−decreasing; we have $h_M(0) = 0$ and $h_M(t) = 1$ for $t \geq 1$. In virtue of (2.4), the knowledge of h_M fully determines the sequence M since we have then

$$M_\ell = \sup_{t>0} t^{-\ell} h_M(t) \quad \text{for all} \quad \ell \in \mathbb{N}. \tag{2.8}$$

More precisely, for each $\ell \in \mathbb{N}$, there is a unique real $t(\ell) \in \,]0,1]$ such that $t \mapsto t^{-\ell} h_M(t)$ increases on $]0, t(\ell)[$ and decreases to zero on $]t(\ell), +\infty[$; thus the supremum is attained for $t = t(\ell)$. Easy considerations show that $t(\ell)$ tends to 0 as ℓ tends to infinity.

Now let s be a real number with $s \geq 1$. Obviously, the sequence $M^s = \left(M_\ell^s\right)_{\ell \geq 0}$ is also strongly regular, and we have $h_{M^s}(t^s) = (h_M(t))^s$. Moreover, as pointed out in [6], there exists a constant $\rho(s) \geq 1$, depending only on s and M, such that

$$h_M(t) \leq (h_M(\rho(s)t))^s \quad \text{for all} \quad t \in \mathbb{R}_+. \tag{2.9}$$

From (2.5), (2.8) and (2.9), it is straightforward to deduce that there exist positive constants $c(s)$ and $d(s)$, depending only on s and M, such that we have

$$c(s)^\ell M_\ell^s \leq M_{[s\ell]} \leq d(s)^\ell M_\ell^s \quad \text{for all} \quad \ell \in \mathbb{N}, \tag{2.10}$$

where the brackets denote the integer part. A most classical example of strongly regular sequence is given by the Gevrey sequences $M_\ell = \ell!^\alpha$ with $\alpha \in \mathbb{R}_+^*$.

Definition 2.1. For any strongly regular sequence M, let $C_M(\mathbb{R}^n, 0)$ be the class of germs f in $C^\infty(\mathbb{R}^n, 0)$ for which there exist a neighborhood U of 0 in $\mathbb{R}^n$ and a constant $C > 0$ such that the inequality

$$\left|D^L f(x)\right| \leq C^{\ell+1} \ell!\, M_\ell$$

holds for any $x \in U$ and any multi−index L.

By (2.4), (2.5) and standard results on ultradifferentiable functions [10], the class $C_M(\mathbb{R}^n, 0)$ is easily seen to be a local ring, stable under derivation. The sequence M

majorizes, in some sense, the defect of analyticity of its elements. Condition (2.6) is the so-called *strong non-quasianalyticity* condition ensuring the existence of "good" cutoff functions with C_M regularity [4] (see also [3], [6]). For a more detailed interpretation of (2.4), (2.5), (2.6) in terms of ultradifferentiable classes, we refer the reader to [4], [12]. Beware that the notations in [4], [12], are quite different, since M_ℓ plays there the role played by $\ell!\, M_\ell$ in the present paper. The latter notation, viewing M as a defect of analyticity, is more convenient here, since it seems, as a rule, that any loss of regularity involved by the various operations of differential analysis (division, etc.) in the setting of ultradifferentiable functions, acts only on their defect of analyticity.

Denote by $\underline{m}_M$ the maximal ideal of $C_M(\mathbb{R}^n, 0)$. Clearly, one has

$$\underline{m}_M^k \;=\; \underline{m}^k \cap C_M(\mathbb{R}^n, 0) \quad \text{for all} \quad k \in \mathbb{N} \cup \{\infty\}\,. \tag{2.11}$$

The following definition comes now in the same way as for the C^∞ case:

Definition 2.2. An ideal in $C_M(\mathbb{R}^n, 0)$ is said to be *elliptic* if it contains $\underline{m}_M^\infty$.

Since any ring $C_M(\mathbb{R}^n, 0)$ contains all the real-analytic germs, one can consider the ideal

$$I_{\varphi,M} \;=\; (\varphi_1, \ldots, \varphi_p) C_M(\mathbb{R}^n, 0)$$

generated in $C_M(\mathbb{R}^n, 0)$ by the germs $\varphi_1, \ldots, \varphi_p$ considered in Subsection 2.1 above. In contrast with the C^∞ case, we shall see below that $I_{\varphi,M}$ is not elliptic in general. When one looks, more generally, for conditions ensuring the inclusion of $\underline{m}_M^\infty$ in I_{φ,M^s} for some suitable real $s \geq 1$, it quickly turns out that the problem depends heavily on the geometry of the zero set of the complexification of φ; therefore it is necessary to introduce corresponding tools before going further.

2.2. Some Łojasiewicz exponents

Denote by $\mathcal{O}(\mathbb{R}^n, 0)$ (resp. $\mathcal{O}(\mathbb{C}^n, 0)$) the ring of real-analytic (resp. holomorphic) function germs at the origin in $\mathbb{R}^n$ (resp. $\mathbb{C}^n$). By means of the power series expansion, $\mathcal{O}(\mathbb{R}^n, 0)$ can obviously be viewed as the subset of elements of $\mathcal{O}(\mathbb{C}^n, 0)$ taking real values on $\mathbb{R}^n$. Thus, to each real-analytic germ γ one can associate the set of its complex zeros $Z_\gamma = \gamma^{-1}(0)$ as a germ of complex analytic set. Put

$$Z_\varphi \;=\; \bigcap_{1 \leq j \leq p} Z_{\varphi_j}\,.$$

The condition (2.1) on real zeros reads as

$$Z_\varphi \cap \mathbb{R}^n \;=\; \{0\}\,. \tag{2.12}$$

In view of (2.12), the regular separation [15] of the analytic sets Z_φ and $\mathbb{R}^n$ implies that there exist real numbers $c > 0$ and $\tau \geq 1$ such that

$$d\big(x, Z_\varphi\big) \;\geq\; c\,|x|^\tau \text{ for any } x \text{ close to 0 in } \mathbb{R}^n\,. \tag{2.13}$$

The Łojasiewicz exponent $\tau(\varphi)$ for the separation of Z_φ and $\mathbb{R}^n$ is defined as the infimum of all numbers τ for which there exists a constant $c > 0$ such that (2.13) holds.

Now denote by $\mathcal{K}_{\mathbb{R}}(\varphi)$ the ideal generated by $\varphi_1, \ldots, \varphi_p$ in $\mathcal{O}(\mathbb{R}^n, 0)$ and let $\mathcal{K}^0_{\mathbb{R}}(\varphi)$ be the subset of those $\gamma \in \mathcal{K}_{\mathbb{R}}(\varphi)$ such that $Z_\gamma \cap \mathbb{R}^n = \{0\}$. Then, just as for Z_φ, one gets a Łojasiewicz inequality

$$(2.14) \qquad d\big(x, Z_\gamma\big) \;\geq\; c\,|x|^\tau \text{ for any } x \text{ close to } 0 \text{ in } \mathbb{R}^n\,,$$

for some suitable constants $c > 0$ and $\tau \geq 1$. In the same way as precedingly, one defines the Łojasiewicz exponent $\tau(\gamma)$ for the regular separation of Z_γ and $\mathbb{R}^n$. In virtue of [2], the numbers $\tau(\varphi)$ (resp. $\tau(\gamma)$) are rational and (2.13) (resp. (2.14)) holds with $\tau = \tau(\varphi)$ (resp. $\tau = \tau(\gamma)$), for some suitable constant c. Now a first result can be stated as follows.

Theorem 2.3. *Let s be a real number with $s \geq 1$. Then, given any element γ of $\mathcal{K}^0_{\mathbb{R}}(\varphi)$, the inequality $s \geq \tau(\gamma)$ implies that for any strongly regular sequence M, one has the inclusion $\underline{m}^\infty_M \subset I_{\varphi,M^s}$.*

Proof. Applying the classical Łojasiewicz inequality [15] to the germ γ, one knows that there exist a neighborhood V of 0 in $\mathbb{C}^n$ and real constants $B_1 > 0$ and $\nu \geq 1$ such that

$$(2.15) \qquad |\gamma(z)| \;\geq\; B_1 d\big(z, Z_\gamma\big)^\nu \quad \text{for all} \quad z \in V\,.$$

Put $U = V \cap \mathbb{R}^n$ and, for any x in $U\backslash\{0\}$, consider the polydisc

$$\overline{P}_x \;=\; \left\{ z \in \mathbb{C}^n \,:\, |z_i - x_i| \leq \frac{1}{2\sqrt{n}}\, d\big(x, Z_\gamma\big) \text{ for } i = 1, \ldots, n \right\}.$$

Then, maybe after shrinking U, we get $d\big(x, Z_\gamma\big) \leq \frac{3}{2}\, d\big(z, Z_\gamma\big)$ for any x in $U\backslash\{0\}$ and z in $\overline{P}_x$. Besides, we know that (2.14) holds with $\tau = \tau(\gamma)$, as mentioned above. Thus, in view of (2.14) and (2.15), the Cauchy formula on $\overline{P}_x$ gives the estimate

$$(2.16) \qquad \left| D^J\left(\frac{1}{\gamma(x)}\right)\right| \;\leq\; B_2^{j+1} j!\, |x|^{-\tau(j+\nu)}$$

for any $x \in U\backslash\{0\}$, any multi−index J, and some suitable constant $B_2 > 0$ depending only on γ and n. Now let h be an element of $\underline{m}^\infty_M$. We may assume h to be defined in U, after shrinking it once more if necessary. Then, using the Taylor formula, the elementary estimate $k!\,p! \leq (k+p)! \leq 2^{k+p} k!\,p!$ and condition (2.5), it is easy to see that for any $x \in U$, any multi−index K of length k and any non−negative integer q, one has

$$(2.17) \qquad \big|D^K h(x)\big| \;\leq\; B_3^{k+1} k!\, M_k (B_3|x|)^q M_q$$

where B_3 is a constant depending only on M, h and n. Consider the function $g = h/\gamma$. It belongs clearly to $C^\infty(U\backslash\{0\})$. Moreover, with the help of (2.5) and (2.10), the application of (2.16) and (2.17) with $q = [\tau(j+\nu)] + 2$ yields

$$(2.18) \qquad \left| D^J\left(\frac{1}{\gamma(x)}\right) D^K h(x)\right| \;\leq\; B_4^{j+k+1} j!\, M_j^\tau\, k!\, M_k\, |x|$$

for some $B_4 > 0$. Thus the germ g actually belongs to $\underline{m}^\infty_{M^\tau}$ since each product $D^J(1/\gamma(x))D^K h(x)$ occuring with $J + K = L$ in the Leibniz formula for $D^L g(x)$ is, in virtue of (2.18), majorized by $B_5^{\ell+1}\ell!\,M_\ell^\tau|x|$ for some suitable B_5. In particular, h is equal to γg with $\gamma \in (\varphi_1, \ldots, \varphi_p)\mathcal{O}(\mathbb{R}^n, 0)$ and $g \in C_{M^\tau}(\mathbb{R}^n, 0)$. The desired result obviously follows. □

Corollary 2.4. *Assume that* $Z_\varphi = \{0\}$. *Then* $I_{\varphi,M}$ *is elliptic.*

P r o o f. Using the Nullstellensatz in the same way as in Proposition 3.4 of [17], one can find γ in $\mathcal{K}^0_{\mathbb{R}}(\varphi)$ such that $\tau(\gamma) = 1$; thus Theorem 2.3 applies with $s = 1$. The result follows. □

Corollary 2.5. *Assume that* Z_φ *has pure dimension* $n-1$. *Then the inclusion* $\underline{m}_M^\infty \subset I_{\varphi,M^s}$ *holds as soon as* $s \geq \tau(\varphi)$; *in particular, the ideal* $I_{\varphi,M}$ *is elliptic provided* $\tau(\varphi) = 1$.

P r o o f. As in Proposition 3.4 of [17], one can find a germ γ in $\mathcal{K}^0_{\mathbb{R}}(\varphi)$ such that $\tau(\gamma) = \tau(\varphi)$. □

Note that Corollary 2.5 applies to any ideal with one generator ($\varphi = \varphi_1$). In this situation, the special case of two variables is particularly interesting, as the following theorem shows.

Theorem 2.6. *Assume that* $n = 2$, $p = 1$ *and let* s *be a real number with* $s \geq 1$. *Then one has*

$$\underline{m}_M^\infty \subset I_{\varphi,M^s} \tag{2.19}$$

if and only if

$$s \geq \tau(\varphi)\,. \tag{2.20}$$

In particular, the ideal $I_{\varphi,M}$ *is elliptic if and only if* $\tau(\varphi) = 1$.

As remarked before, the "if" part of the result follows from Corollary 2.5. The "only if" part is more involved and the whole Section 3 of the paper will be devoted to its proof.

Problem 2.7. Does the converse part of Theorem 2.6 still hold for more than two variables? Since our proof relies heavily on Puiseux expansions, hence on a purely two-dimensional technique, the case $n \geq 3$ is not clear; but it is very natural to conjecture that the theorem should be valid in any dimension.

Remark 2.8. Note that all the results of the present paper are stated in the setting of strongly regular classes for sake of simplicity. Nevertheless, as it can be seen on the preceding proof, the particular statement of Theorem 2.3 doesn't need the full power of (2.6), but only the weaker Denjoy-Carleman condition of non-quasianalyticity

$$\sum_{j\geq 0} \frac{M_j}{(j+1)M_{j+1}} < +\infty\,,$$

ensuring that $\underline{m}_M^\infty$ is non-void.

3. Computation of a Łojasiewicz exponent

3.1. Preliminaries and notations

In this section, we turn to the proof of the "only if" part of Theorem 2.6 concerning the special case $n = 2$ and $p = 1$. The first step consists in finding how to relate the Łojasiewicz exponent $\tau(\varphi)$ to certain numbers arising in a Puiseux factorization of φ at the origin. From now on, we denote by $\omega(v)$ the order of any element v of $\mathcal{O}(\mathbb{C}, 0)$, that is the least degree of monomials occuring in the power series expansion of v, with the convention $\omega(0) = +\infty$.

After a linear change of coordinates with real matrix, one can clearly assume, without changing the fact that φ is real, that the z_1 – axis is transversal to Z_φ at 0. Using the factorization of φ in irreducible factors in $\mathcal{O}(\mathbb{C}^2, 0)$ and applying the Puiseux theorem for irreducible factors (a nice proof of which can be found in [1]), it is readily seen that there exist an integer m with $m \geq 1$, a unit u in $\mathcal{O}(\mathbb{C}^2, 0)$ and a finite family (v_j) in $\mathcal{O}(\mathbb{C}, 0)$, such that for all z near the origin in $\mathbb{C}^2$, we have $\varphi(z_1, z_2^m) = u(z_1, z_2^m) \prod_j (z_1 - v_j(z_2))^{m_j}$ for some suitable $m_j \in \mathbb{N}^*$. The transversality of the z_1 – axis to Z_φ ensures that $\omega(v_j) \geq m$ for all j. Since φ takes real values on $\mathbb{R}^2$ and has no non – trivial real zero, we deduce that for any real x_2, the roots $v_j(x_2)$ are all non – real and pairwise conjugate. Also, the germ $\tilde{u}(z) = u(z_1, z_2^m)$ is real – analytic too. One gets thus a factorization

$$\varphi(z_1, z_2^m) = \tilde{u}(z_1, z_2) \prod_{1 \leq p \leq p_0} (\psi_p(z_1, z_2))^{n_p} , \tag{3.1}$$

where p_0 and $n_1, \ldots, n_{p_0}$ are positive integers, and where each ψ_p can be written

$$\psi_p(z_1, z_2) = (z_1 - R_p(z_2) - iS_p(z_2))(z_1 - R_p(z_2) + iS_p(z_2)) \tag{3.2}$$

for some R_p and S_p belonging to $\mathcal{O}(\mathbb{R}, 0)$ and satisfying $m \leq \omega(R_p)$, $m \leq \omega(S_p) < +\infty$. Define now

$$\begin{aligned} \mu_p &= \omega(S_p) \quad \text{and} \quad \tau_p^+ = \mu_p/m \quad \text{for} \quad p = 1, \ldots, p_0 , \\ \tau^+(\varphi) &= \max \{\tau_p^+ : p = 1, \ldots, p_0\} , \\ \tau^-(\varphi) &= \tau^+(\check{\varphi}) \quad \text{where} \quad \check{\varphi}(z_1, z_2) = \varphi(z_1, -z_2) . \end{aligned}$$

Proposition 3.1. *With the notations above, one has* $\tau(\varphi) = \max(\tau^+(\varphi), \tau^-(\varphi))$.

P r o o f. Consider the principal determination of $z_2^{1/m}$ in $\mathbb{C} \backslash \mathbb{R}^-$. Then, for any $z = (z_1, z_2)$ with $\mathcal{R}e\, z_2 \geq 0$, one has $\varphi(z) = 0$ if and only if there exists $p \in \{1, \ldots, p_0\}$ such that

$$z_1 = R_p\left(z_2^{1/m}\right) \pm iS_p\left(z_2^{1/m}\right) .$$

We put $Z_\varphi^+ = \{z : \mathcal{R}e\, z_2 \geq 0 \text{ and } \varphi(z) = 0\}$ and

$$\begin{aligned} Z_p' &= \left\{z : \mathcal{R}e\, z_2 \geq 0 \text{ and } z_1 = R_p\left(z_2^{1/m}\right) + iS_p\left(z_2^{1/m}\right)\right\} \\ Z_p'' &= \left\{z : \mathcal{R}e\, z_2 \geq 0 \text{ and } z_1 = R_p\left(z_2^{1/m}\right) - iS_p\left(z_2^{1/m}\right)\right\} , \end{aligned}$$

in such a way that

$$Z_\varphi^+ \;=\; \bigcup_{1\le p\le p_0} \big(Z'_p \cup Z''_p\big)\,. \tag{3.3}$$

Put also $Z_\varphi^- = \{z\,:\,\mathcal{R}e\, z_2 \le 0 \text{ and } \varphi(z) = 0\}$; then Z_φ can be written as

$$Z_\varphi \;=\; Z_\varphi^+ \cup Z_\varphi^-\,. \tag{3.4}$$

Now we break the proof into a series of lemmas. The following notation will be useful: if $A(t)$ and $B(t)$ are two positive functions of a variable t belonging to a set T, one writes $A(t) \lesssim B(t)$ $\big($or equivalently $B(t) \gtrsim A(t)\big)$ to say that there exists a real C, not depending on t, such that the inequality $A(t) \le CB(t)$ holds for all $t \in T$. The notation $A(t) \approx B(t)$ means that one has $A(t) \lesssim B(t)$ and $B(t) \lesssim A(t)$ simultaneously.

Lemma 3.2. *For* $p = 1,\dots,p_0$ *and* $x \in \big(\mathbb{R}\times\mathbb{R}_+, 0\big)$, *one has* $d\big(x, Z'_p\big) \gtrsim |x|^{\tau_p^+}$.

Proof. For $z_1 \in \mathbb{C}$ and $\mathcal{R}e\, z_2 \ge 0$, we consider the complex−valued function

$$T_p(z) \;=\; z_1 - R_p\Big(z_2^{1/m}\Big) - iS_p\Big(z_2^{1/m}\Big)\,.$$

Since $\omega\big(R_p\big) \ge m$ and $\omega\big(S_p\big) \ge m$, the function T_p is of class C^1 up to the boundary in the half space $\mathbb{C}\times\{z_2\,:\,\mathcal{R}e\, z_2 \ge 0\}$; hence it extends to a C^1 function, still denoted by T_p, on the whole of $\mathbb{C}^2$. Moreover, the expansion $T_p(z) = z_1 + O(z_2)$ at the origin implies clearly that $T_p^{-1}(0)$ is a germ of (2−dimensional, real) submanifold of $\mathbb{C}^2$ at the origin. Thus we have

$$d\big(x, T_p^{-1}(0)\big) \;\approx\; \big|T_p(x)\big| \tag{3.5}$$

for x sufficiently close to 0 in $\mathbb{R}\times\mathbb{R}_+$. Besides, Z'_p is contained in $T_p^{-1}(0)$, hence

$$d\big(x, Z'_p\big) \;\ge\; d\big(x, T_p^{-1}(0)\big)\,. \tag{3.6}$$

Finally, since x_2 belongs to $\mathbb{R}_+$, we have $|T_p(x)| \approx \Big|x_1 - R_p\Big(x_2^{1/m}\Big)\Big| + \Big|S_p\Big(x_2^{1/m}\Big)\Big|$, with $\Big|R_p\Big(x_2^{1/m}\Big)\Big| \le A\,|x_2|$ for some suitable constant $A > 0$, and $\Big|S_p\Big(x_2^{1/m}\Big)\Big| \approx |x_2|^{\tau_p^+}$ in view of the definitions given in Subsection 3.1 above. There are now two possibilities.

First case: $|x_1| > 2A\,|x_2|$. We get $|x| \approx |x_1|$ and $\Big|x_1 - R_p\Big(x_2^{1/m}\Big)\Big| \ge \frac{1}{2}\,|x_1|$, thus $|T_p(x)| \gtrsim |x| \gtrsim |x|^{\tau_p^+}$.

Second case: $|x_1| \le 2A\,|x_2|$. We get $|x| \approx |x_2|$ and $|T_p(x)| \gtrsim |S_p(x)| \gtrsim |x_2|^{\tau_p^+} \gtrsim |x|^{\tau_p^+}$. Together with (3.5) and (3.6), this proves the lemma. □

Lemma 3.3. *For* $p = 1,\dots,p_0$ *and* $x \in (\mathbb{R}\times\mathbb{R}_-, 0)$, *one has* $d\big(x, Z'_p\big) \gtrsim |x|$.

Proof. Just as for Lemma 3.2, one has $d\big(x, Z'_p\big) \ge d\big(x, T_p^{-1}(0)\big) \approx |T_p(x)|$, but there is no explicit expression of $T_p(x)$ for $x_2 < 0$. Nevertheless, the expansion $T_p(x) = x_1 + O(x_2)$ still holds; so one can clearly find a constant $B > 0$ such that the inequality $|x_1| \ge B\,|x_2|$ implies $|T_p(x)| \approx |x_1| \approx |x|$, hence $d\big(x, Z'_p\big) \gtrsim |x|$. In the other case

$|x_1| < B\,|x_2|$, we get immediately $d(x, Z'_p) \geq d(x, \mathbb{R} \times \mathbb{R}_+) = |x_2| \approx |x|$. The lemma follows. □

Lemma 3.4. *For $x \in (\mathbb{R}^2, 0)$, one has $d(x, Z^+_\varphi) \gtrsim |x|^{\tau^+(\varphi)}$.*

Proof. Lemmas 3.2 and 3.3 show that $d(x, Z'_p) \gtrsim |x|^{\tau^+_p}$ holds for all $x \in (\mathbb{R}^2, 0)$. By the same arguments, one has also $d(x, Z''_p) \gtrsim |x|^{\tau^+_p}$, finally giving the estimate

$$d(x, Z'_p \cup Z''_p) \;\gtrsim\; |x|^{\tau^+_p} \;\gtrsim\; |x|^{\tau^+(\varphi)}\,.$$

Taking into account (3.3), the lemma is proved. □

Lemma 3.5. *For $x \in (\mathbb{R}^2, 0)$, one has $d(x, Z^-_\varphi) \gtrsim |x|^{\tau^-(\varphi)}$.*

Proof. For $z \in \mathbb{C}^2$, put $s(z) = (z_1, -z_2)$. Then it is obvious that $Z^-_\varphi = s(Z^+_{\check\varphi})$ and $d(x, Z^-_\varphi) = d(s(x), Z^+_{\check\varphi})$. Lemma 3.4 applies to $s(x)$ and $\check\varphi$, therefore we obtain $d(x, Z^-_\varphi) \gtrsim |s(x)|^{\tau^+(\check\varphi)}$ with $|s(x)| = |x|$ and $\tau^+(\check\varphi) = \tau^-(\varphi)$, hence the result. □

3.2. End of the proof of Proposition 3.1

In view of (3.4), Lemmas 3.4 and 3.5 show that for $x \in (\mathbb{R}^2, 0)$, we have

$$d(x, Z_\varphi) \;\gtrsim\; |x|^{\max(\tau^+(\varphi), \tau^-(\varphi))}\,.$$

The inequality $\tau(\varphi) \leq \max(\tau^+(\varphi), \tau^-(\varphi))$ follows.

The reverse inequality $\tau(\varphi) \geq \max(\tau^+(\varphi), \tau^-(\varphi))$ is simple and can be obtained as follows. For p fixed, $1 \leq p \leq p_0$, and for $\epsilon > 0$, put $x(\epsilon) = (R_p(\epsilon), \epsilon^m)$. Then $x(\epsilon)$ belongs to $\mathbb{R} \times \mathbb{R}_+$ and satisfies $|x(\epsilon)| \approx \epsilon^m$, since $R_p(\epsilon) = O(\epsilon^m)$. Moreover $x(\epsilon) + i(S_p(\epsilon), 0)$ belongs to Z_φ, hence $d(x(\epsilon), Z_\varphi) \leq |S_p(\epsilon)| \approx \epsilon^{\mu_p} \approx |x(\epsilon)|^{\tau^+_p}$. This implies $\tau(\varphi) \geq \tau^+_p$ for all $p = 1, \ldots, p_0$, thus $\tau(\varphi) \geq \tau^+(\varphi)$. The same argument applies to obtain $\tau(\varphi) \geq \tau^-(\varphi)$. □

We are now ready to prove the implication (2.19) ⇒ (2.20) of Theorem 2.6. The proof needs first a flat function in C_M satisfying some special estimates.

Lemma 3.6. *For any strongly regular sequence M, one can find a positive function η belonging to $C_M(\mathbb{R}, 0)$, which is even, vanishes at infinite order at the origin, and satisfies*

$$\eta(t) \;\geq\; h_M(b\,|t|) \quad \textit{for all} \quad t \in \mathbb{R}\,, \tag{3.7}$$

for some suitable constant $b > 0$ depending only on M.

Proof. Note that (3.7) means that η is, in some sense, extremal, because any function belonging to $C_M(\mathbb{R}, 0)$ and flat at 0 must be, by Taylor's formula, majorized by $C h_M(C'|t|)$ for some constants C, C' depending on the function. In the special case of Gevrey sequences $M_\ell = \ell!^{\,\alpha}$, one can take explicitely $\eta(t) = \exp(-|t|^{-1/\alpha})$. For general sequences, the construction of η uses more or less classical ideas, so we give only a sketch. First, let (t_j) and (r_j) be sequences of real numbers, with $r_j > 0$ for all j, chosen in such a way that for some constant $\delta > 1$, the families of intervals $I_j = \left]t_j - r_j, t_j + r_j\right[$ and $I_j^* = \left]t_j - \delta r_j, t_j + \delta r_j\right[$ enjoy the following properties: the I_j cover $\mathbb{R}_+^*$; each I_j^* intersects at most a fixed finite number of I_k^* with $k \neq j$; one has $t \approx r_j$ for all j and all $t \in I_j^*$. Using the cutoff functions of Bruna [4] (see also [3], [6]), for any j one can find a smooth positive function χ_j supported in I_j^*, taking its values in $[0,1]$, identically equal to 1 in I_j, and such that the estimate

$$\left|\chi_j^{(\ell)}(t)\right| \;\leq\; C^{\ell+1}\ell!\, M_\ell\big(h_M(r_j)\big)^{-1}$$

holds for all $(j,\ell) \in \mathbb{N}^2$ and $t \in \mathbb{R}$, with a constant C depending only on M and δ. Put now

$$\eta(t) \;=\; \sum_j \big(h_M(r_j)\big)^2 \chi_j(|t|)\,.$$

Playing with the properties of I_j, I_j^* and χ_j described above, and using also (2.9) (with $s = 2$), it is not difficult to see that η has all the required properties. □

3.3. Proof of the "only if" part of Theorem 2.6

Let η be the function of Lemma 3.6. Assuming (2.19), there exists a germ g belonging to $C_{M^s}\big(\mathbb{R}^2,0\big)$ and satisfying

$$\eta(x_2) \;=\; \varphi(x_1,x_2)g(x_1,x_2) \quad \text{for all} \quad x \in \big(\mathbb{R}^2,0\big)\,. \tag{3.8}$$

Since $\eta(x_2)$ does not depend on x_1 (!), this implies that the identity

$$\eta\big(x_2^m\big) \;=\; \varphi\big(x_1,x_2^m\big)g\big(x_1,x_2^m\big) \;=\; \varphi\big(x_1+R_p(x_2),x_2^m\big)g\big(x_1+R_p(x_2),x_2^m\big) \tag{3.9}$$

holds for all $p = 1,\dots,p_0$ and all $x \in \big(\mathbb{R}^2,0\big)$. Now consider the analytic germ $\tilde\varphi_p$ defined by

$$\tilde\varphi_p(x_1,x_2) \;=\; \tilde u\big(x_1+R_p(x_2),x_2\big)\left(x_1^2+(S_p(x_2))^2\right)^{n_p-1}\prod_{\ell\neq p}\big(\psi_\ell\big(x_1+R_p(x_2),x_2\big)\big)^{n_\ell}\,,$$

in such a way that we have, in view of (3.1) and (3.2),

$$\varphi\big(x_1+R_p(x_2),x_2^m\big) \;=\; \left(x_1^2+(S_p(x_2))^2\right)\tilde\varphi_p(x_1,x_2)\,. \tag{3.10}$$

Put also $\tilde g_p(x_1,x_2) = \tilde\varphi_p(x_1,x_2)g\big(x_1+R_p(x_2),x_2^m\big)$. Then $\tilde g_p$ belongs clearly to the ring $C_{M^s}\big(\mathbb{R}^2,0\big)$ and, by (3.9) and (3.10), one gets

$$\eta\big(x_2^m\big) \;=\; \left(x_1^2+(S_p(x_2))^2\right)\tilde g_p(x_1,x_2)\,. \tag{3.11}$$

We also know that there exists a constant $c > 0$ such that the inequality

$$|S_p(x_2)| \;\geq\; c\,|x_2|^{\mu_p} \tag{3.12}$$

holds for all $p = 1, \ldots, p_0$ and x_2 sufficiently close to 0. From (3.11) and (3.12) it is easy to deduce, for all x sufficiently close to 0 and satisfying $0 \le x_1 < c\,|x_2|^{\mu_p}$, the expansion

$$\tilde{g}_p(x_1, x_2) \;=\; \frac{\eta\big(x_2^m\big)}{(S_p(x_2))^2} \sum_{j=0}^{\infty} (-1)^j \left(\frac{x_1}{S_p(x_2)}\right)^{2j}.$$

This gives in particular, for all $j \in \mathbb{N}$ and $x_2 > 0$ small enough,

$$\frac{\partial^{2j} \tilde{g}_p}{\partial x_1^{2j}}(0, x_2) \;=\; (-1)^j (2j)!\, (S_p(x_2))^{-2j-2} \eta\big(x_2^m\big)\,,$$

hence, by (3.7) and (3.12),

$$\begin{aligned} \left|\frac{\partial^{2j} \tilde{g}_p}{\partial x_1^{2j}}(0, x_2)\right| \;&\ge\; \frac{(2j)!}{c^{2j+2}} \frac{h_M\big(b x_2^m\big)}{x_2^{(2j+2)\mu_p}} \\ &\ge\; \left(\frac{b^{\mu_p/m}}{c}\right)^{2j+2} (2j)! \left(\frac{h_M\big(b x_2^m\big)}{\big(b x_2^m\big)^{2j+2}}\right)^{\mu_p/m}. \end{aligned} \tag{3.13}$$

Now we use (3.13) with $x_2 = x_2(j) = (t(2j+2)/b)^{1/m}$, where the numbers $t(\ell)$ have been studied at the end of Subsection 2.1, ensuring in particular that $x_2(j)$ tends to 0 as j tends to infinity. Taking again (2.5) into account, we see that there exists a constant $d > 0$ such that we have

$$\left|\frac{\partial^{2j} \tilde{g}_p}{\partial x_1^{2j}}(0, x_2(j))\right| \;\ge\; d^{2j+1} (2j)!\, M_{2j}^{\mu_p/m} \quad \text{for all} \quad j \in \mathbb{N}\,.$$

Since each $\tilde{g}_p$ belongs to $C_{M^s}\big(\mathbb{R}^2, 0\big)$, this inequality implies $s \ge \mu_p/m = \tau_p^+$ for all $p = 1, \ldots, p_0$, hence

$$s \;\ge\; \tau^+(\varphi)\,. \tag{3.14}$$

Now note that since η is even, (3.8) implies also that $x \mapsto \eta(x_2)$ belongs to $I_{\check{\varphi}, M^s}$. All the previous work can therefore be done exactly in the same way with $\check{\varphi}$ instead of φ, giving then

$$s \;\ge\; \tau^-(\varphi)\,. \tag{3.15}$$

By Proposition 3.1, the estimates (3.14) and (3.15) yield the desired fact, namely that (2.19) implies (2.20). This completes our local results.

4. Conditions for closed ideals

4.1. Context

Let M be a given strongly regular sequence and Ω an open subset of $\mathbb{R}^n$. To each relatively compact, smoothly bounded, open subset X of Ω and each real $\sigma > 0$, we associate the space $C_{M,\sigma}(\overline{X})$ of those functions f belonging to $C^\infty(\overline{X})$ and such that there exists a constant $C_f > 0$ for which we have

$$(4.1) \qquad |D^L f(x)| \ \le\ C_f\, \sigma^\ell \ell!\, M_\ell \text{ for any } L \in \mathbb{N}^n \text{ and any } x \in \overline{X}\,.$$

In the sequel, the class under study will be the Roumieu – Carleman class $C_M(\Omega)$ defined as the set of functions f belonging to $C^\infty(\Omega)$ and such that for any X as before, there exists a constant σ for which the restriction of f to $\overline{X}$ belongs to $C_{M,\sigma}(\overline{X})$. This algebra of functions may be topologized as follows: denote by $\|f\|_{\overline{X},\sigma}$ the infimum of all constants C_f such that (4.1) holds. Then $C_{M,\sigma}(\overline{X})$ is a Banach space with respect to the norm $\|\cdot\|_{\overline{X},\sigma}$. Define the Carleman class $C_M(\overline{X})$ as the inductive limit of all spaces $C_{M,\sigma}(\overline{X})$ for $\sigma > 0$. Then $C_M(\Omega)$ is the projective limit of the spaces $C_M(\overline{X})$. We refer the reader to [12] for a detailed topological study of such classes.

Now let $\varphi = (\varphi_1, \dots, \varphi_p)$ be a real – analytic mapping from Ω to $\mathbb{R}^p$. As specified in the introduction, assume that 0 belongs to Ω and $\varphi^{-1}(0) = \{0\}$. We still denote by φ the germ of the mapping at the origin, and we shall use the notations of Sections 2 and 3 concerning germs. Let $I^\Omega_{\varphi,M}$ be the ideal generated by $\varphi_1, \dots, \varphi_p$ in $C_M(\Omega)$. The fact that $I^\Omega_{\varphi,M}$ is generally not closed in $C_M(\Omega)$ is pointed out in [8] by a simple example, with $n = 2$, $p = 1$ (note that in [8] the zero set of the ideal is the curve $x_1^2 + x_2^3 = 0$, but the computations work as well with $\varphi(x) = x_1^2 + x_2^4$ which has an isolated real zero at the origin). The following proposition allows us to relate now the closedness of ideals $I^\Omega_{\varphi,M}$ to the local results stated previously:

Proposition 4.1. *Under the above assumptions, the ideal $I^\Omega_{\varphi,M}$ is closed in $C_M(\Omega)$ if and only if the ideal of germs $I_{\varphi,M}$ is elliptic in $C_M(\mathbb{R}^n, 0)$.*

Proof. Multiplying by C_M cutoff functions, we can consider germs in $C_M(\mathbb{R}^n, 0)$ as functions in $C_M(\Omega)$. The proposition reduces then to a rather simple application of the C_M version of Whitney's spectral theorem [7]. □

Corollary 4.2. *Assume that $p = 1$. Then $I^\Omega_{\varphi,M}$ is closed as soon as $\tau(\varphi) = 1$. The condition is also necessary in the case $n = 2$.*

Proof. Immediate by Corollary 2.5, Theorem 2.6 and Proposition 4.1. □

Example 4.3. Let φ be a positive definite homogeneous polynomial. Then the ideal generated by φ in any strongly regular Roumieu – Carleman class $C_M(\mathbb{R}^n)$ is closed.

Example 4.4. For $k \in \mathbb{N}^*$ and $x \in \mathbb{R}^2$, put $\varphi(x) = x_1^2 + x_2^{2k}$. Then, for any strongly regular Roumieu – Carleman class $C_M(\mathbb{R}^2)$, the ideal generated by φ in $C_M(\mathbb{R}^2)$ is closed if and only if $k = 1$.

We state now an application of Corollary 4.2 to the division of ultradistributions. With the notations of Subsection 4.1, let $\mathcal{D}_{M,\sigma}(\overline{X})$ be the closed subspace of $C_{M,\sigma}(\overline{X})$ given by those elements which extend to C^∞ functions in $\mathbb{R}^n$, having compact support contained in $\overline{X}$; and let $\mathcal{D}_M(\overline{X})$ be the inductive limit of the spaces $\mathcal{D}_{M,\sigma}(\overline{X})$ for $\sigma > 0$. Denote finally by $\mathcal{D}_M(\Omega)$ the inductive limit of all spaces $\mathcal{D}_M(\overline{X})$ for X

relatively compact, smoothly bounded, open subset of Ω; and recall that Roumieu ultradistributions of class C_M are defined as the continuous linear forms on $\mathcal{D}_M(\Omega)$. It seems that the only result previously known about division of ultradistributions by functions is due to Droste [9], who proved that division of a Dirac mass at some point a by an ultradifferentiable function φ is possible in the corresponding class of ultradistributions only if φ has a zero of finite order at a. Assuming that Ω is connected, this necessary condition is obviously satisfied when φ is analytic, non identically zero. In this setting, we have now a sufficient condition, reading as follows.

Proposition 4.5. *Assume that $p = 1$ and $\tau(\varphi) = 1$. Then any ultradistribution T of class C_M in Ω can be divided by φ, in the sense that there exists another ultradistribution S of class C_M such that $T = \varphi S$.*

Proof. The basic scheme is the same as for distributions, but the more complicated topological structure of the test space $\mathcal{D}_M(\Omega)$ requires here some technical devices. We shall use a Beurling class version for the "if" part of Corollary 4.2. The Beurling class $B_M(\Omega)$ is the projective limit of all classes $B_M(\overline{X})$, being themselves defined as the projective limits of the Banach spaces $C_{M,\sigma}(\overline{X})$ for $\sigma > 0$. The nice feature of Beurling classes is their natural Fréchet topology. We claim also that in the case $p = 1$ and $\tau(\varphi) = 1$, the ideal $\varphi B_M(\Omega)$ generated by φ in $B_M(\Omega)$ is closed. To see this, the argument of Proposition 4.1 may be adapted as follows: if f belongs to the closure of this ideal in $B_M(\Omega)$, the spectral theorem of [7] shows that there exists f_0 belonging to $\varphi B_M(\Omega)$ and such that $f - f_0$ is flat at the origin. Pick a small ball X centered at 0. By Proposition 11 of [5] (see also [6]), one can find a strongly regular sequence M' such that $f - f_0$ belongs to $C_{M'}(\overline{X})$ and $C_{M'}(\overline{X}) \subset B_M(\overline{X})$. Using the ellipticity of $I_{\varphi,M'}$, it is readily seen that $f - f_0$ belongs to $\varphi C_{M'}(\overline{X})$, hence to $\varphi B_M(\overline{X})$, and the claim follows easily, by suitable truncations. Now let T be a given Roumieu ultradistribution of class C_M. By partitions of unity, we may assume that T has compact support in Ω. Following an idea of [7], we use Komatsu's first structure theorem (Section 8 of [12]) together with Proposition 14 of [5] (see also [6]) to construct a strongly regular sequence M'', depending on T, such that $\left(M_\ell/M''_\ell\right)^{1/\ell} \to 0$, which ensures the set inclusion $\mathcal{D}_M(\Omega) \subset B_{M''}(\Omega)$, and such that, moreover, T extends to a linear continuous form on $B_{M''}(\Omega)$ (this is not obvious, since the previous inclusion is not topological). Consider the mapping $\mathcal{P} : B_{M''}(\Omega) \to B_{M''}(\Omega)$ defined by $\mathcal{P}f = \varphi f$. It is obviously injective; and its range $\varphi B_{M''}(\Omega)$ is closed, as claimed just above. The open mapping theorem can be applied in $B_{M''}(\Omega)$ to derive that $\mathcal{P}$ has a continuous inverse $\mathcal{Q}$. The linear form $S = T\mathcal{Q}$ is thus continuous on $\varphi B_{M''}(\Omega)$, so it extends to the whole of $B_{M''}(\Omega)$ by the Hahn – Banach theorem. Remark now that the restriction of S to $\mathcal{D}_M(\Omega)$ is still continuous with respect to the corresponding (stronger) topology. This is enough to conclude, since we have $\langle S, \varphi f\rangle = \langle S, \mathcal{P}f\rangle = \langle T, \mathcal{QP}f\rangle = \langle T, f\rangle$ for any test function $f \in \mathcal{D}_M(\Omega)$. □

4.2. Concluding remarks and questions

(i) Although essentially optimal regarding the regularity of the functions, the previous approach of the problem of closed ideals is currently working well only in the setting of ideals with isolated real zeros. A completely different viewpoint on such problems is to be found in [8]: in this paper, desingularization techniques enable the authors to deal with zero sets of principal ideals in full generality; but in counterpart, most of the information on the regularity disappears, so that sufficiently good results can be obtained only in classes defined as, roughly speaking, intersections of all rings C_{M^s} for $s > 0$. Therefore it would be interesting to know how to extend our precise results to the case of ideals with general zero sets.

(ii) Another feature of [8] is that the classes under consideration may be quasianalytic. In contrast, Proposition 4.1 relies heavily on non – quasianalyticity, since the C_M version of BOREL's lemma [16], as well as considerations of flat functions, are hiding behind its proof. But does Corollary 4.2 still hold in the quasianalytic case ?

(iii) In the whole Section 4, we have mostly restricted ourselves to the consideration of a particular type of ultradifferentiable class (namely local Roumieu – Carleman classes) but, up to a lengthy discussion of various cases, similar results can be obtained in the same way for other ones, like Roumieu – Carleman up to the boundary of Ω, or Beurling classes, and so on. See e. g. [9], [12] for the definitions.

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Note added in proof: E. BIERSTONE (personal communication) has recently shown to the author how to reduce Problem 2.7 to the arguments of Section 3; therefore the answer to that question is: yes. Details will appear elsewhere.

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