INTEGRALLY HILBERTIAN RINGS AND THE POLYNOMIAL SCHINZEL HYPOTHESIS

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Abstract. The Hilbert specialization property ensures that polynomial irreducibility over a field is preserved under specialization of some of the variables. We develop an integral analogue by introducing integrally Hilbertian rings, where specialization takes place inside a ring and irreducibility is required over the ring. Over Krull domains – including UFDs and Dedekind domains – we overcome new obstructions such as coefficient divisors and fixed divisors, leading to a general criterion and many examples, e.g. all rings of integers of number fields and polynomial rings. As an application, we prove a polynomial variant of the Schinzel Hypothesis on prime values of polynomials with integer coefficients: for any integrally Hilbertian ring $\mathcal{Z}$, the hypothesis becomes a true statement when $\mathbb{Z}$ is replaced by $\mathcal{Z}[U]$ and "prime" by "irreducible". This result fits in a unified framework for Schinzel-type phenomena, and yields noteworthy consequences for the classical Schinzel Hypothesis itself.

Résumé. (*Anneaux intégralement hilbertiens et hypothèse de Schinzel polynomiale*). La propriété de spécialisation de Hilbert assure que l'irréductibilité des polynômes sur un corps est préservée par spécialisation de certaines des variables. Nous développons un analogue intégral en introduisant la notion d'anneau intégralement hilbertien, pour laquelle la spécialisation s'effectue dans un anneau et l'irréductibilité est requise sur cet anneau. Sur les anneaux de Krull, notamment sur les anneaux factoriels et les anneaux de Dedekind, nous surmontons de nouvelles obstructions, telles que diviseurs de coefficients et diviseurs fixes. Cela conduit à un critère général et à de nombreux exemples, parmi lesquels les anneaux d'entiers de corps de nombres et les anneaux de polynômes. En application, nous démontrons une variante polynomiale de l'hypothèse de Schinzel sur les valeurs premières de polynômes à coefficients entiers: pour tout anneau intégralement hilbertien $\mathcal{Z}$, l'hypothèse est vraie si $\mathbb{Z}$ est remplacé par $\mathcal{Z}[U]$ et "premier" par "irréductible". Ce résultat s'inscrit dans un cadre unifié pour les phénomènes type Schinzel et a des conséquences notables pour l'hypothèse de Schinzel elle-même.

1. Introduction

The *Hilbert specialization property*, by which irreducibility of polynomials is preserved under specialization of some of the variables (Definition 1.1), is one of the few general and powerful tools in Arithmetic Geometry, making it possible, in certain situations, to specialize some parameters while keeping the algebraic structure unchanged. For example, when a field $\mathcal{Q}$ has the property, i.e., is a *Hilbertian field*, any finite separable field extension of $\mathcal{Q}(T)$ yields, via appropriate specializations in $\mathcal{Q}$ of the variable T, extensions of $\mathcal{Q}$ with the same degree and the same Galois group. Hilbert specialization can also be helpful in affine geometry to produce irreducible affine $\mathcal{Q}$-varieties from an algebraic $\mathcal{Q}(T)$-family of varieties parametrized by parameters T, notably when Bertini-type results, which require irreducibility of the family over the algebraic closure of $\mathcal{Q}(T)$, do not apply.

Hilbert specialization is a *field* process with polynomials defined over a *field* and irreducibility requested over that *field*. As initiated in [BDKN22], we aim to make it an *integral* tool with specialization *in a ring* and irreducibility required *over the ring*. The difficulty in passing from fields to rings is that irreducibility over a ring is sensitive to coefficient divisibility and "fixed divisors" phenomena (as illustrated before Definition 1.2); these have no analog over fields.

Our proposed integral version of Hilbertian field theory goes as follows. Definition 1.2 of *integrally Hilbertian rings* encapsulates the problem. We then have three main results. Theorem 1.3 is a criterion for a ring to be integrally Hilbertian, leading to many examples, e.g. all

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rings of integers of number fields (Corollary 1.4). Theorem 1.6 provides another fundamental example: any polynomial ring over an integral domain. Applications concern the celebrated Schinzel Hypothesis on prime values of polynomials with integer coefficients. Theorem 1.7 establishes a polynomial variant, typically for rings like $\mathbb{Z}[U]$ instead of $\mathbb{Z}$. This result improves on [BDN20, Theorem 1.1]; in particular, we have an additional assertion that has some noteworthy implications for the original Schinzel Hypothesis itself, over $\mathbb{Z}$ (Corollary 1.10).

We now present our work in more detail.

1.1. **Integrally Hilbertian rings.** Start with these classical definitions.

Definition 1.1. A domain $\mathcal{Z}$ with fraction field $\mathcal{Q}$, of characteristic 0 or imperfect (i.e. of characteristic $p > 0$ and $\mathcal{Z} \neq \mathcal{Z}^p$), is a *Hilbertian ring* if the following *Hilbert specialization property* holds. For any two tuples of indeterminates $\underline{T} = (T_1, \dots, T_k)$ (the *parameters*) and $\underline{Y} = (Y_1, \dots, Y_n)$ (the *variables*), with $k, n \geq 1$, and any irreducible polynomials $P_1, \dots, P_s \in \mathcal{Q}[\underline{T}, \underline{Y}]$, of degree ≥ 1 in $\underline{Y}$, the subset of $\mathcal{Z}^k$ consisting of all $\underline{t}$ such that $P_1(\underline{t}, \underline{Y}), \dots, P_s(\underline{t}, \underline{Y})$ are irreducible in $\mathcal{Q}[\underline{Y}]$ is Zariski-dense. A *Hilbertian field* is a field that is a Hilbertian ring.

Hilbertian fields include the rational field $\mathbb{Q}$, rational function fields $\kappa(u_1, \dots, u_r)$ in $r \geq 1$ variables over an arbitrary field κ, and their finite extensions. Hilbertian rings include all domains $\mathcal{Z}$ such that the fraction field $\mathcal{Q}$ is a field of the preceding type. [FJ23] is a classical reference on Hilbertian fields and rings. The *imperfectness assumption* at the beginning of Definition 1.1 is here for consistency with [FJ23] and is not restrictive (Remark 2.13).

A key obstruction in the intended integral setting is the existence of fixed divisors. Recall that given an integral domain $\mathcal{Z}$ with fraction field $\mathcal{Q}$ and a polynomial $P \in \mathcal{Z}[\underline{T}, \underline{Y}]$, a nonunit $a \in \mathcal{Z}$, $a \neq 0$, is called a *fixed divisor* of P w.r.t. $\underline{T}$ if $P(\underline{t}, \underline{Y}) \equiv 0 \pmod{a}$ for every $\underline{t} \in \mathcal{Z}^k$. For example $2 \in \mathbb{Z}$ is a fixed divisor of $P = (T^2 - T)Y + (T^2 - T - 2)$ w.r.t. T; consequently, no specialized polynomial $P(t, Y)$ with $t \in \mathbb{Z}$ is irreducible in $\mathbb{Z}[Y]$.

The following property of a ring is a central theme of the paper.

Definition 1.2. A domain $\mathcal{Z}$ is said to be *integrally Hilbertian* if the following condition is satisfied. For any polynomials $P_1, \dots, P_s \in \mathcal{Z}[\underline{T}, \underline{Y}]$, irreducible in $\mathcal{Q}[\underline{T}, \underline{Y}]$, of degree ≥ 1 in $\underline{Y}$, and such that the product $P_1 \cdots P_s$ has no fixed divisor w.r.t. $\underline{T}$, there is a Zariski-dense subset $H \subset \mathcal{Z}^k$ such that for every $\underline{t} \in H$,
(a) *the polynomials* $P_1(\underline{t}, \underline{Y}), \dots, P_s(\underline{t}, \underline{Y})$ *are irreducible in* $\mathcal{Q}[\underline{Y}]$,
(b) *the product* $\prod_{i=1}^s P_i(\underline{t}, \underline{Y})$ *has no nonunit divisor in* $\mathcal{Z}$.

This gives in particular that, as intended, for every $\underline{t} \in H$,
(c) *the polynomials* $P_1(\underline{t}, \underline{Y}), \dots, P_s(\underline{t}, \underline{Y})$ *are irreducible in* $\mathcal{Z}[\underline{Y}]$ *and in* $\mathcal{Q}[\underline{Y}]$.
Assertion (c) can in fact equivalently replace (a) & (b) if $s = 1$ *(one polynomial) or* $\mathcal{Z}$ *is a* UFD.[1]

Condition (b) shows the difference between the Hilbertian ring notion – only the field condition (a) is required, fixed divisors are irrelevant – and the integrally Hilbertian property – one must simultaneously control divisors (a) of positive degree and (b) of zero degree.

Remark 2.8 explains how the "no fixed divisor assumption" can concretely be guaranteed. Furthermore, under some common condition on $\mathcal{Z}$ (e.g. being a Krull domain as discussed below), one can get rid of fixed divisors of the product $P_1 \cdots P_s$ and so reduce to the assumption, at the cost of localizing $\mathcal{Z}$ to $\mathcal{Z}[1/\varphi]$ for some nonzero $\varphi \in \mathcal{Z}$ (Lemma 2.7).

As observed in Proposition 2.14, an integrally Hilbertian ring must be a Hilbertian ring. Conversely, the following result provides a condition for a Hilbertian ring to be integrally Hilbertian.

Theorem 1.3. *Let* $\mathcal{Z}$ *be a Hilbertian ring that is also a Krull domain. Then* $\mathcal{Z}$ *is an integrally Hilbertian ring.*

[1] As is usual, UFD stands for Unique Factorization Domain, and similarly, PID for Principal Ideal Domain.

Definition of Krull domains is recalled in §2.1.1. They include:
(a) Unique Factorization Domains,
(b) rings that are Noetherian and integrally closed (in particular Dedekind domains).

Special case (a) was established in [BDKN22]. Case (b) and the general Krull case are new.

An advantage of the Krull assumption is that it is preserved by localization (by any multiplicative subset), and by taking integral closures in finite extensions of the fraction field (Remark 2.2). The same is true with the Hilbertian ring property (Remark 2.13). Whence this complement:

Theorem 1.3 (continued) *If $\mathcal{Z}$ is a Hilbertian ring and a Krull domain, not only $\mathcal{Z}$ but also any localization of $\mathcal{Z}$, and any integral closure of $\mathcal{Z}$ in a finite extension of $\mathcal{Q}$, is integrally Hilbertian.*

In particular, for polynomials $P_1, \ldots, P_s$ *with* some fixed divisors, the integrally Hilbertian conclusion holds over $\mathcal{Z}[1/\varphi]$ for some nonzero $\varphi \in \mathcal{Z}$.

As the ring of integers $\mathbb{Z}$ and the polynomial ring $\kappa[\underline{x}]$, with κ any field and $\underline{x}$ any nonempty set of indeterminates, are Hilbertian rings, and are UFD, we immediately deduce from Theorem 1.3 these examples of integrally Hilbertian rings.

Corollary 1.4. *The ring of integers $\mathcal{Z}$ of any number field is integrally Hilbertian. If κ is any field, then the integral closure $\mathcal{Z}$ of $\kappa[\underline{x}]$ in any finite extension E of $\kappa(\underline{x})$ is integrally Hilbertian.*

Remark 1.5. For rings of integers of number fields, Corollary 1.4 improves on [BDKN22, Theorem 1.1] where $\mathcal{Z}$ is assumed to be of class number 1. There is a gap between the PID and the Dedekind cases, and, more generally, between the UFD and non UFD cases in Theorem 1.3. The traditional Dedekind idea to replace prime elements by prime ideals faces the following difficulty: a polynomial $P \in \mathcal{Z}[\underline{T}, \underline{Y}]$ may have no fixed divisor $a \in \mathcal{Z}$ but still have a fixed *prime ideal divisor* $\mathcal{P}$ w.r.t. $\underline{T}$ (i.e. $P(\underline{t}, \underline{Y}) \equiv 0 \pmod{\mathcal{P}}$ for every $\underline{t} \in \mathcal{Z}^k$); see Example 2.10.

The Dedekind idea (as discussed above) led to partial results in [BDKN22]. Here instead we identify two structural properties that are key in the UFD case and manage to extend them in some form – properties (NVA) and (SF) of §3.2 – to the Krull case. A central property of Krull rings established along the way is the so-called *coprime Schinzel Hypothesis* from [BDKN22], which is about coprime values of coprime polynomials (see Remark 2.16) and goes back to a fundamental result of Schinzel over $\mathbb{Z}$ [Sch02]. It appears here as a more flexible and stronger variant, which we call being *locally Schinzel* (Definition 2.15). The proof of Theorem 1.3 can then be summarized as follows: for a Hilbertian ring $\mathcal{Z}$,

$$\text{Krull} \Rightarrow \text{(NVA) \& (SF)} \Rightarrow \text{locally Schinzel} \Rightarrow \text{integrally Hilbertian.}$$

We refer to Lemmas 3.11 and 3.13 for first implication (true up to some equivalence), to Proposition 3.7 for the second one, and to Proposition 2.17 for the last one.[2]

Polynomial rings $\mathcal{Z} = \mathcal{R}[U]$ over a domain $\mathcal{R}$ are not Krull rings in general, and it is unclear whether they satisfy (NVA) and (SF). Yet we can show the following.

Theorem 1.6. *Let $\mathcal{R}$ be an arbitrary domain. Then the polynomial ring $\mathcal{R}[U]$ is locally Schinzel and integrally Hilbertian.*

Theorem 1.6 is reminiscent of the analogous result that $K(U)$ is a Hilbertian field for any field K. When $\mathcal{R}$ is not integrally closed, which implies that $\mathcal{R}[U]$ is not integrally closed either, Theorem 1.6 shows that implication "*integrally Hilbertian $\Rightarrow$ integrally closed*" does not hold.

1.2. **Schinzel type applications.** Our applications relate to the famous conjectural

Schinzel Hypothesis. *If $P_1, \ldots, P_s \in \mathbb{Z}[T]$ are irreducible polynomials such that the product $\prod_{i=1}^{s} P_i$ has no fixed divisor in $\mathbb{Z}$* (the local condition), *then there exist infinitely many integers $m \in \mathbb{Z}$ such that $P_1(m), \ldots, P_s(m)$ are simultaneously prime.*

[2] In the special case of ring of integers of number fields, passing from $\mathbb{Z}$ to the general non PID case can be achieved in a different manner, based on an alternate approach which evaluates the density of the subset H from Definition 1.2 of "good" specializations. See [BD23] for the "easy" case $\mathcal{Z} = \mathbb{Z}$ and [BDW26] for the general case for which more refined results from geometry of numbers are used.

We offer a generalizing definition in §5.1. Roughly speaking, a *Schinzel ring*[3] is defined to be a domain $\mathcal{Z}$, equipped with some *Weil heights*, for which, given polynomials $P_1, \dots, P_s \in \mathcal{Z}[\underline{T}]$, irreducible in $\mathcal{Q}[\underline{T}]$, the *local condition* – absence of fixed divisors w.r.t. $\underline{T}$ for $P_1 \cdots P_s$ – is the only obstruction to producing irreducible specializations $P_1(\underline{t}), \dots, P_s(\underline{t})$ at points $\underline{t} \in \mathcal{Z}^k$ of arbitrarily large heights; see Definition 5.1.

The original Schinzel Hypothesis corresponds to the special case $\mathcal{Z} = \mathbb{Z}$ and $k = 1$; see Theorem 5.2 for the equivalence between one and several parameters. The polynomial ring $\mathcal{Z}[Y_1, \dots, Y_n]$, equipped with the Weil heights induced by the partial degree $\deg_{Y_i}(\cdot)$ in each variable $Y_1, \dots, Y_n$, is another natural test for the Schinzel ring notion. We obtain the following.

Theorem 1.7. *Let $\mathcal{Z}$ be an integrally Hilbertian ring and $Y_1, \dots, Y_n$ be $n \geq 1$ variables. Then the polynomial ring $\mathcal{Z}[Y_1, \dots, Y_n]$ is a Schinzel ring w.r.t. the partial degree Weil heights $H_1, \dots, H_n$.*

More explicitly, this rephrases as follows:

(**) *given $s \geq 1$ polynomials $P_1, \dots, P_s \in \mathcal{Z}[\underline{Y}][\underline{T}]$, irreducible in $\mathcal{Q}(\underline{Y})[\underline{T}]$ and such that the product $P_1 \cdots P_s$ has no nonunit divisor in $\mathcal{Z}[\underline{Y}]$, the subset $\mathcal{S} \subset \mathcal{Z}[\underline{Y}]^k$ consisting of all k-tuples $(M_1(\underline{Y}), \dots, M_k(\underline{Y}))$ of polynomials*

- each of degree bigger than any prescribed constant $A > 0$ in each variable $Y_1, \dots, Y_n$,

- and such that $P_i(\underline{Y}, M_1(\underline{Y}), \dots, M_k(\underline{Y}))$ is irreducible in $\mathcal{Z}[\underline{Y}]$ ($i = 1, \dots, s$),

is Zariski-dense.

Theorem 5.3 shows a more precise version: the polynomials $M_i(\underline{Y})$ can be prescribed to be of degree in Y_j any given integer d_{ij} ($i = 1, \dots, k$, $j = 1, \dots, n$), provided that these integers d_{ij} are suitably large. Precise bounds will be given, and for fixed suitable d_{ij}, the set of tuples $(M_1(\underline{Y}), \dots, M_k(\underline{Y}))$, with these degrees, will be shown to be Zariski-dense, in some natural sense.

Remark 1.8. (a) Theorem 1.7 improves on [BDN20, Theorem 1.1] where $\mathcal{Z}$ is a UFD. The classical consequences of the Schinzel Hypothesis: the Dirichlet Theorem, the Twin Prime Problem, etc., as well as an analog of the Goldbach Problem hold for the ring $\mathcal{Z}[\underline{Y}]$ (as stated in [BDN20]) under our more general assumption that $\mathcal{Z}$ is integrally Hilbertian.

(b) There has been a parallel activity on the "polynomial Schinzel Hypothesis" in the case that $\mathcal{Z}$ is a finite field ([BW05] [Pol08] [BS12] [BSJ12] [Ent16]). Specific ingredients for finite fields, e.g. the Lang-Weil estimates, are used. Our Hilbertian environment allows a different specialization approach, which, thanks to our *integrally* Hilbertian property, can be performed *over the ring.*

It follows from Theorem 1.6 and Theorem 1.7 that if $\mathcal{Z}$ is an arbitrary domain, then the polynomial ring $\mathcal{Z}[Y_0, Y_1, \dots, Y_n]$ is a Schinzel ring w.r.t. the partial degree Weil heights $H_1, \dots, H_n$. More explicit forms of these results in fact show the following, which improves on [BDN20, Theorem 1.2] where $\mathcal{Z}$ is a field.

Corollary 1.9. *Let $\mathcal{Z}$ be an arbitrary domain. Then the polynomial ring $\mathcal{Z}[Y_0, Y_1, \dots, Y_n]$ with $n \geq 1$ is a Schinzel ring w.r.t. the partial degree Weil heights $H_0, H_1, \dots, H_n$.*

Obviously, the polynomial ring $\mathcal{Z}[Y_0]$ is not a Schinzel ring if $\mathcal{Z}$ is an algebraically closed field. So the assumption $n \geq 1$ cannot be relaxed. Another (subtler) counterexample is the polynomial ring $\mathbb{F}_2[Y_0]$; from an example of Swan [Swa62, pp. 1102-1103], the polynomial $M(Y_0)^8 + Y_0^3$ is reducible for all polynomials $M \in \mathbb{F}_2[Y_0]$.

Compared to [BDN20], beside the bigger generality of Theorem 1.7, we have the additional conclusion that the local condition can be preserved by specialization.

Theorem 1.7 (continued). *Retain assumptions and notation from (**) above. Assume further that $\mathcal{Z}$ is a Krull domain and that the product $P_1 \cdots P_s$ has no fixed divisor w.r.t. the $(k + n)$-tuple $(\underline{T}, \underline{Y})$. Then the subset $\mathcal{S}_0$ of the set $\mathcal{S}$ from statement (**) consisting*

[3] While the name "Schinzel ring" obviously comes from the motivating Schinzel Hypothesis, the name "locally Schinzel ring" that was previously introduced refers to the fact that, as we will see, the defining property of these rings is concerned with the *local condition* assumed in the Schinzel Hypothesis.

of k-tuples $(M_1(\underline{Y}), \dots, M_k(\underline{Y})) \in \mathcal{S}$ *satisfying the additional property that the polynomial* $(P_1 \cdots P_s)(\underline{Y}, M_1(\underline{Y}), \dots, M_k(\underline{Y}))$ *has no fixed divisor w.r.t.* $\underline{Y}$*, is still Zariski-dense in* $\mathcal{Z}[\underline{Y}]^k$.

Theorem 5.4 provides a more precise version, showing further that, *for* $k = n = 1$ (one parameter, one variable), *one can request that the polynomial* $M(Y)$ *be monic and of arbitrary degree* $d \geq 1$. A noteworthy consequence for the original Schinzel Hypothesis is the following.

Corollary 1.10. *Assume that the original Schinzel Hypothesis over* $\mathbb{Z}$ *is true. Let* $P_1, \dots, P_s \in \mathbb{Z}[T]$ *be some irreducible polynomials such that the product* $P_1 \cdots P_s$ *has no fixed divisor w.r.t.* T*. Then there exist monic polynomials* $M \in \mathbb{Z}[T]$ *of arbitrary degree* $d \geq 1$ *such that for infinitely many* $t \in \mathbb{N}$*, the integers* $P_1(M(t)), \dots, P_s(M(t))$ *are prime numbers.*

Namely, the special case $k = n = 1$ of Theorem 5.4, applied to the polynomials $P_1, \dots, P_s \in \mathbb{Z}[T] \subset \mathbb{Z}[Y][T]$ (of degree 0 in $\underline{Y}$) provides a monic polynomial $M(T) \in \mathbb{Z}[T]$ of arbitrary degree $d \geq 1$ such that the polynomials $P_1(M(T)), \dots, P_s(M(T))$ satisfy the assumptions of the Schinzel Hypothesis. Applying the Schinzel Hypothesis yields the required conclusion.

Corollary 1.10 shows in particular that, to prove the Schinzel Hypothesis, one may restrict to polynomials of arbitrarily large degree — a somewhat striking reduction.

The rest of the paper is organized as follows.

In Section 2, we introduce the general setup. Definitions of integrally Hilbertian rings and locally Schinzel rings are given and related (Proposition 2.17). We then state Theorem 2.18, which generalizes Theorem 1.3.

In Section 3, we prove Theorem 2.18. The proof is reduced to that of Lemma 3.1 that asserts that Krull rings (and other rings) are locally Schinzel. We then introduce the previously mentioned conditions (NVA) and (SF), and use them to prove Lemma 3.1.

Section 4 is devoted to the proof of Theorem 1.6 that polynomial rings are integrally Hilbertian.

In Section 5, we define Schinzel rings and discuss our main results in this context, mainly about polynomial rings being Schinzel rings.

Section 6 provides the proofs of the results of Section 5, including that of Theorem 1.7 and its corollaries.

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Fix for the whole paper a domain $\mathcal{Z}$ and denote its fraction field by $\mathcal{Q}$. Also fix two tuples of indeterminates: $\underline{T} = (T_1, \dots, T_k)$ $(k \geq 0)$, and $\underline{Y} = (Y_1, \dots, Y_n)$ $(n \geq 0)$, with $k = 0$ or $n = 0$ meaning that the corresponding tuple is empty. We call the T_i the *parameters*; they are to be specialized, unlike the Y_i which we call the *variables*.

2. Locally Schinzel rings and integrally Hilbertian rings

After some preliminary definitions in §2.1, integrally Hilbertian rings are introduced in §2.2, and locally Schinzel rings in §2.3. In §2.4, we state and comment on Theorem 2.18, which generalizes Theorem 1.3. §2.5 provides examples of Hilbertian but not integrally Hilbertian rings.

2.1. **Preliminaries.**

2.1.1. *Ring theory.* Our main criterion (Theorem 2.18) for a domain $\mathcal{Z}$ to be locally Schinzel or integrally Hilbertian includes $\mathcal{Z}$ being either a *Krull domain* or a *near UFD*. We recall below basic facts about these classes of rings.

Given a domain $\mathcal{Z}$, a proper ideal $\mathfrak{a} \subset \mathcal{Z}$ is *primary* if for every $x, y \in \mathcal{Z}$, $xy \in \mathfrak{a}$ implies $x \in \mathfrak{a}$ or y^m for some integer $m \geq 1$; and set

$$\mathrm{Spec}^{(1)}(\mathcal{Z}) = \{\mathfrak{p} \in \mathrm{Spec}\,\mathcal{Z} \mid \mathrm{ht}(\mathfrak{p}) = 1\},$$

where $\mathrm{ht}(\cdot)$ denotes the height.

Definition 2.1. A domain $\mathcal{Z}$ is called a *Krull domain* if the following conditions hold:

(1) $A_{\mathfrak{p}}$ is a discrete valuation ring for all $\mathfrak{p} \in \mathrm{Spec}^{(1)}(\mathcal{Z})$;
(2) Every non-zero principal ideal $\langle a \rangle$ of $\mathcal{Z}$ is the intersection of finitely many height-one primary ideals.

Remark 2.2. (a) Every UFD is a Krull domain: merely note that every height-one prime ideal in a UFD is generated by a prime element.

(b) Every integrally closed Noetherian domain is a Krull domain. This follows from [Mat80, §17.H, Theorem 37, Theorem 38].

(c) It is clear from the definition that being a Krull domain is preserved by localization (by any multiplicative subset). From the Mort–Nagata Integral Closure Theorem [Mat80, §41.B], so it is by taking integral closure in finite extensions of the fraction field $\mathcal{Q}$.

Assume that $\mathcal{Z}$ is a Krull domain. For each $\mathfrak{p} \in \mathrm{Spec}^{(1)}(\mathcal{Z})$, denote by $v_{\mathfrak{p}}$ the corresponding normalized valuation on $\mathcal{Z}_{\mathfrak{p}}$, which extends uniquely to $\mathrm{Frac}(\mathcal{Z})$. For $n \in \mathbb{N}$, let

$$\mathfrak{p}^{(n)} = \mathfrak{p}^n \mathcal{Z}_{\mathfrak{p}} \cap \mathcal{Z}$$

be the *n-th symbolic power* of $\mathfrak{p}$.

Lemma 2.3 (Weak Approximation Theorem)**.** *Let $\mathcal{Z}$ be a Krull domain. Let $\mathcal{S} = \{\mathfrak{p}_1, \dots, \mathfrak{p}_r\} \subset \mathrm{Spec}^{(1)}(\mathcal{Z})$ be a finite set, let $x_1, \dots, x_r \in \mathcal{Z}$, and let $n_1, \dots, n_r \in \mathbb{N}$. Then there exists $x \in \mathcal{Z}$ such that*

$$v_{\mathfrak{p}_i}(x - x_i) = n_i \quad \text{for all } 1 \leq i \leq r.$$

Proof. This result should be standard, but we include a proof due to the lack of a direct reference. By [Mat80, Page 289, V)], the valuations $v_{\mathfrak{p}_1}, \dots, v_{\mathfrak{p}_r}$ are independent. Classically then, an element x satisfying the required conclusion can be chosen in $\mathcal{Q}$. By [Jar91, Proposition 4.4] and the construction given in its proofs, one can indeed choose x to lie in $\mathcal{Z}$. □

See [Mat80, Section 41, Chapter 13] for more on Krull domains.

Definition 2.4. A *near UFD* is a domain such that

(2.4-1) *every nonzero element has finitely many prime divisors (modulo units), and*

(2.4-2) *every nonunit has at least one prime divisor* (in particular irreducibles are prime).

We also call *PDF ring* (for "Prime Divisor Finite") a domain that solely satisfies (2.4-1).

For more on near UFDs, see [BDKN22, §2.3] where this definition is introduced.

Remark 2.5. (a) Krull domains, Noetherian rings and near UFDs are PDF rings. It is obvious for near UFDs. For the first two, note first that both satisfy the Ascending Chain Condition on Principal ideals (ACCP) (obvious for Noetherian, whereas the Krull property is shown in [AAZ90, Proposition 2.2] to imply the so-called "bounded factorization" property, which in turn implies ACCP). Secondly, ACCP implies PDF: namely, ACCP easily gives that each nonzero element a is a product of finitely many irreducibles; any prime divisor of a then must divide one of these irreducibles.

(b) Clearly, UFDs are near UFDs. Conversely near UFDs satisfying ACCP must be UFD.

(c) *The two classes of Krull domains and near UFDs are not contained in each other.* Indeed it follows from (b) that rings that are both Krull and near UFDs are UFD. As there are Krull domains that are not UFD (every ring of integers of a number field of class number > 1), and near UFDs that are not UFD (e.g. [BDKN22, Example 2.4]), the claim follows. (Similarly the two classes of Noetherian rings and near UFDs are not contained in each other; and the same is true for Noetherian rings and Krull domains).

2.1.2. *Fixed divisor.*

Definition 2.6. Let $P \in \mathcal{Z}[\underline{T}, \underline{Y}]$ be a polynomial. A proper ideal $\mathfrak{p} \subset \mathcal{Z}$ is called a *fixed divisor* of P w.r.t. $\underline{T}$ if $P(\underline{t}, \underline{Y}) \equiv 0 \pmod{\mathfrak{p}}$ for every $\underline{t} \in \mathcal{Z}^k$. The set of all fixed divisors of P is denoted by $\mathcal{F}_{\underline{T}}(P)$.

When $k = 0$, a fixed divisor $\mathfrak{p}$ is merely an (ideal) divisor of $P(\underline{Y})$ in $\mathcal{Z}$ (in the sense that $P(\underline{Y}) \equiv 0 \pmod{\mathfrak{p}}$). For clarity, we use a different notation, $\mathrm{Div}_{\mathcal{Z}}(P)$, for the set of all proper *ideal divisors* of P. Also note that for $P \in \mathcal{Z}[\underline{T}, \underline{Y}]$, we have $\mathrm{Div}_{\mathcal{Z}}(P) \subset \mathcal{F}_{\underline{T}}(P)$.

Lemma 2.7. *Assume that $\mathcal{Z}$ is a Krull domain or a near UFD. Let $P(\underline{T}, \underline{Y}) \in \mathcal{Z}[\underline{T}, \underline{Y}]$ be a nonzero polynomial. Then there exists a nonzero element $\varphi \in \mathcal{Z}$ such that P has no fixed divisors w.r.t. $\underline{T}$ among the proper principal ideals of $\mathcal{Z}[1/\varphi]$.*

For near UFDs, the result is proved in [BDKN22, §4.3.1]. The case of Krull domains is handled in Remark 3.12.

Remark 2.8 (How to find fixed divisors)**.** Observe first that a nonunit $a \in \mathcal{Z}$ is a fixed divisor of some polynomial $P \in \mathcal{Z}[\underline{T}, \underline{Y}]$ w.r.t. $\underline{T}$ if and only if each coefficient $c(\underline{T})$ of P (viewed as a polynomial in $\underline{Y}$) does, i.e. satisfies $c(\underline{t}) \equiv 0 \pmod{a}$ for every $\underline{t} \in \mathcal{Z}^k$. Then, for example with $\mathcal{Z} = \mathbb{Z}$, one can reduce to the case $a = p$ is a prime; and the previous condition holds if and only if $c(\underline{T})$ lies in the ideal $\langle p, T_1^p - T_1, \ldots, T_k^p - T_k \rangle \subset \mathcal{Z}[\underline{T}]$ [Ter66, Lemme 1]. In particular, such primes p can be bounded in terms of the partial degrees of P [BDKN22, §3.1]. Evaluation of $c(\underline{T})$ at some points $\underline{t} \in \mathcal{Z}^k$ may then, in practice, yield the no fixed divisor assumption; an obvious situation is when $c(\underline{T})$ is a unit in $\mathcal{Z}$.

2.1.3. *Primality type.* In some situations, there may be some interest to obtain that a specialized polynomial $P(\underline{t}, \underline{Y})$, instead of having "no nonunit divisor in $\mathcal{Z}$", has "no *prime ideal divisor* $\mathfrak{p} \subset \mathcal{Z}$" – i.e., that $P(\underline{t}, \underline{Y}) \not\equiv 0$ *modulo every prime ideal* $\mathfrak{p} \subset \mathcal{Z}$. But then one needs to assume that P has no "*fixed prime ideal divisor*" – i.e., that *for no prime ideal* $\mathfrak{p} \subset \mathcal{Z}$*, we have* $P(\underline{t}, \underline{Y}) \equiv 0 \pmod{\mathfrak{p}}$ *for every* $\underline{t} \in \mathcal{Z}^k$. This creates a variant of the notion of integrally Hilbertian rings. Other possibly valuable variants appear from other choices of sets of ideals.

Our main variants will be for the following sets of ideals. Our original variant is the first one.

- $\mathrm{Nonunit}\,\mathcal{Z}$: the set of all *nonzero principal proper* ideals $\langle a \rangle \subset \mathcal{Z}$ (generated by nonunits a, $a \neq 0$).
- $\mathrm{Spec}^*\,\mathcal{Z}$: the set of all *nonzero prime* ideals of $\mathcal{Z}$.
- $\mathrm{Irred}\,\mathcal{Z}$: the set of all *principal* ideals $\langle a \rangle$ generated by *irreducible* elements $a \in \mathcal{Z}$.
- $\mathrm{Prime}\,\mathcal{Z}$: the set of all *nonzero principal prime* ideals $\langle a \rangle \subset \mathcal{Z}$ (generated by prime elements a).

When there is no risk of confusion on the ring $\mathcal{Z}$, we omit $\mathcal{Z}$ in the notation; for example, we write Prime for $\mathrm{Prime}\,\mathcal{Z}$. Also, as is usual, we often identify principal ideals $\langle a \rangle$ with elements $a \in \mathcal{Z}$ modulo units of $\mathcal{Z}$.

Remark 2.9. When $\mathcal{Z}$ is a PID, the three sets Spec^*, Irred and Prime are equal and every ideal in Nonunit uniquely factors as a nonempty product of ideals in Prime.

Example 2.10. Here is an example where $\mathcal{Z}$ is the ring of integers of a number field and $P(T, Y) \in \mathcal{Z}[T, Y]$ is a polynomial that has no fixed divisor in $\mathcal{P} = \mathrm{Nonunit}$ but has some in $\mathcal{P} = \mathrm{Spec}^*$. Take $\mathcal{Z} = \mathbb{Z}[\sqrt{-6}]$. It is the ring of integers of $\mathbb{Q}(\sqrt{-6})$. Consider the polynomial

$$P(T, Y) = (T^2 - T)Y + (T^2 - T + 2) \in \mathcal{Z}[T, Y].$$

Clearly $(2) = \mathfrak{p}^2$ for $\mathfrak{p} = (2, \sqrt{-6})$. Since the norm of $\mathfrak{p}$ is 2, $\mathfrak{p}$ is a fixed prime ideal divisor of P w.r.t. T. Assume on the contrary that π is a nonunit fixed divisor of P w.r.t. T. Then π divides $(-1)^2 - (-1) = 2$, and so π is associate with 2, as 2 is irreducible. However 2 does not divide

$$(1+\sqrt{-6})^2 - (1+\sqrt{-6}) = -6+\sqrt{-6},$$

a contradiction.

More generally, we call *primality type* any subset $\mathcal{P}$ of proper ideals of $\mathcal{Z}$. In the core Sections 2 and 3, our definitions and results are stated w.r.t. a given primality type $\mathcal{P}$. The primality type $\mathcal{P}$ = Nonunit remains the main case of interest. It is also $\mathcal{P}$ = Nonunit that is tacitly meant when there is no explicit reference to a primality type (as in Sections 1, 4, 5).

2.2. **Integrally Hilbertian ring.** Fix an integral domain $\mathcal{Z}$ and a primality type $\mathcal{P}$. Given s polynomials $P_1,\dots,P_s \in \mathcal{Z}[\underline{T},\underline{Y}]$, consider the following conditions, which serve both in the hypotheses and in the conclusions of our main results.

(Irred/$\mathcal{Q}(\underline{T})$) The polynomials $P_1,\dots,P_s$ are irreducible in $\mathcal{Q}(\underline{T})[\underline{Y}]$.

(Prim/$\mathcal{Q}[\underline{T}]$) The polynomials $P_1,\dots,P_s$ are primitive w.r.t. $\mathcal{Q}[\underline{T}]$.[4]

(NoFixDiv/$\mathcal{Z}[\underline{T}]$)$_{\mathcal{P}}$ The product $P_1\cdots P_s$ has no fixed divisor in $\mathcal{P}\mathcal{Z}$ w.r.t. $\underline{T}$.

The three depend on the k-tuple $\underline{T}$ of parameters. The third condition (NoFixDiv/$\mathcal{Z}[\underline{T}]$)$_{\mathcal{P}}$ also depends on the primality type $\mathcal{P}$. The answers to our main questions: is a ring $\mathcal{Z}$ integrally Hilbertian? a Schinzel ring? as subsequently defined, depend on $\mathcal{P}$.

Condition (Irred/$\mathcal{Q}(\underline{T})$) includes $\deg_{\underline{Y}}(P_i) \geq 1$, $i=1,\dots,s$. Conditions (Irred/$\mathcal{Q}(\underline{T})$) and (Prim/$\mathcal{Q}[\underline{T}]$) joined together are equivalent to the polynomials $P_1,\dots,P_s$ being irreducible in $\mathcal{Q}[\underline{T},\underline{Y}]$ and of degree ≥ 1 in $\underline{Y}$. When $k=0$, the second condition (Prim/$\mathcal{Q}$) merely means that the polynomials $P_1,\dots,P_s \in \mathcal{Q}[\underline{Y}]$ are nonzero.

Definition 2.11. The ring $\mathcal{Z}$ is said to be *integrally Hilbertian w.r.t. the primality type* $\mathcal{P}$ if for any integers $k,n,s\geq 1$ and any polynomials $P_1,\dots,P_s \in \mathcal{Z}[\underline{T},\underline{Y}]$ satisfying the three hypotheses (Irred/$\mathcal{Q}(\underline{T})$), (Prim/$\mathcal{Q}[\underline{T}]$) and (NoFixDiv/$\mathcal{Z}[\underline{T}]$)$_{\mathcal{P}}$, the subset

$$(2.11)\quad \mathcal{H}_{\mathcal{P}}(P_1,\dots,P_s) = \left\{ \underline{m}\in\mathcal{Z}^k \;\middle|\; \begin{array}{l}\text{the polynomials } P_1(\underline{m},\underline{Y}),\dots,P_s(\underline{m},\underline{Y})\in\mathcal{Z}[\underline{Y}]\\ \text{satisfy conditions (Irred/}\mathcal{Q}\text{) and (NoFixDiv/}\mathcal{Z}\text{)}_{\mathcal{P}}.\end{array}\right\}^{5}$$

is Zariski-dense in $\mathbb{A}^k(\mathcal{Q})$.

Remark 2.12 (Irreducibility in $\mathcal{Z}[\underline{Y}]$ of the specialized polynomials)**.** If a k-tuple $\underline{m}$ is in the set $\mathcal{H}_{\mathcal{P}}(P_1,\dots,P_s)$ and the primality type is $\mathcal{P}$ = Nonunit, then the specialized polynomials are irreducible in $\mathcal{Z}[\underline{Y}]$: indeed, from (Irred/$\mathcal{Q}$), they cannot have a nontrivial divisor in $\mathcal{Z}[\underline{Y}]$ of positive degree, and due to (NoFixDiv/$\mathcal{Z}$)$_{\mathcal{P}}$, they cannot have a nonunit divisor $a\in\mathcal{Z}$. The same conclusion holds with $\mathcal{P}$ = Irred if $\mathcal{Z}$ additionally assumed to be Noetherian (indeed, in a Noetherian ring, every nonunit is divisible by an irreducible). It does too for $\mathcal{P}$ = Prime if $\mathcal{Z}$ is a near UFD. The same question is not clear for $\mathcal{P} = \mathrm{Spec}^*$. If $\mathcal{Z}$ is a UFD, then irreducibility over $\mathcal{Z}$ is *equivalent* to irreducibility over $\mathcal{Q}$ joint with nonexistence of nonunit divisors in $\mathcal{Z}$.

The definition of an integrally Hilbertian ring given in Definition 1.2 corresponds to the situation of Definition 2.11 that the primality type is $\mathcal{P}$ = Nonunit, and in turn, this original definition corresponds to the condition from [BDKN22] that the so-called *Hilbert–Schinzel specialization property* holds for any integers $k,n,s\geq 1$.

Proposition 2.14 and Proposition 2.17 relate the notions of *integrally Hilbertian rings* and of *Hilbertian rings*. The preliminary Remark 2.13(a) is aimed at dissipating some possible confusion due to some subtelty in the literature. Remark 2.13(b) provides some details on a "well-known to experts" property of Hilbertian rings used in Section 1.

Remark 2.13 (Complements on Hilbertian rings and Hilbertian fields)**.**

[4]For a UFD $\mathcal{Z}$, a polynomial $P\in\mathcal{Z}[\underline{Y}]$ is *primitive* w.r.t. $\mathcal{Z}$ if its coefficients are coprime, i.e., if they have no nonunit common divisor.

[5]The polynomials $P_1(\underline{m},\underline{Y}),\dots,P_s(\underline{m},\underline{Y})$ are in $\mathcal{Z}[\underline{Y}]$; they have no more parameters T_i. As specified above, condition (NoFixDiv/$\mathcal{Z}$)$_{\mathcal{P}}$ means that they have no ideal divisor in $\mathcal{P}$.

(a) [FJ23, §13.1] defines the Hilbert specialization property satisfied by Hilbertian fields and Hilbertian rings as the special case of our definition (Definition 1.1) for which $n = 1$ and $P_1, \dots, P_s$ are separable as polynomials in Y. Our a priori stronger version of the property is equivalent under the *imperfectness assumption* that $\mathcal{Q}$ is of characteristic 0 or imperfect; for Hilbertian fields, we refer to [FJ23, Prop.13.4.3], and for Hilbertian rings to [BDN20, Prop.4.2]. For simplicity, we incorporated this assumption in our definition of Hilbertian rings and fields, thus avoiding any risk of confusion. There is no loss in doing so: Proposition 2.14 shows that integrally Hilbertian rings satisfy the imperfectness assumption.

(b) As asserted in Section 1, *if $\mathcal{Z}$ is a Hilbertian ring, then the same holds for*

(b-1) *any localization by some multiplicative subset $S \subset \mathcal{Z}$*, and,

(b-2) *any integral closure $\mathcal{Z}'_L$ of $\mathcal{Z}$ in a finite extension $\mathcal{L}/\mathcal{Q}$ of the fraction field.*

Assertion (b-1) is easy since $\mathrm{Frac}(S^{-1}\mathcal{Z}) = \mathcal{Q}$ and $\mathcal{Z} \subset S^{-1}\mathcal{Z}$. Concerning (b-2), it is clear if $\mathcal{L}/\mathcal{Q}$ is separable from [FJ23, Cor.13.2.3]. The case that $\mathcal{L}/\mathcal{Q}$ is inseparable can be deduced from an adjusted version of [FJ23, Prop.13.3.6(b)] that replaces the phrase "*Hilbert subsets of K (*resp. *of L) are non empty*" by "*Hilbert subsets of K (*resp. *of L) contain elements of $\mathcal{Z}$ (*resp. *of $\mathcal{Z}'_L$)*" in the assumption (resp. in the conclusion) of the statement. The main change in the proof is to observe that the element $\gamma \in \mathcal{L}$ from the proof of [FJ23, Lemma 13.3.5] can be found in $\mathcal{Z}'_L$. Other places of the proof of [FJ23, Prop.13.3.6(b)] use elements of Hilbert sets of K. These should be picked in $\mathcal{Z}$, as our modified assumption allows.

Proposition 2.14. *Let $\mathcal{Z}$ be an integrally Hilbertian ring w.r.t. a primality type $\mathcal{P}$. Then $\mathcal{Z}$ is of characteristic* 0 *or imperfect, and is a Hilbertian ring.*

Proof. If $p = \mathrm{char}(\mathcal{Z}) > 0$, the polynomial $Y^p - T$ is irreducible in $\mathcal{Q}[T, Y]$ and has no fixed divisor w.r.t. $\underline{T}$ (with respect to any primality type). Hence, by the integrally Hilbertian property, there exists $t \in \mathcal{Z}$ such that $t \notin \mathcal{Q}^p$. Hence $\mathcal{Z}$ is imperfect.

To show that $\mathcal{Z}$ is a Hilbertian ring, let $P_1, \dots, P_s \in \mathcal{Q}[\underline{T}, Y]$ be $s \geq 1$ irreducible polynomials of degrees ≥ 1 in the single variable Y, and separable in Y (Remark 2.13(a) explains why reduction to such polynomials is legitimate). Furthermore, from [FJ23, Lemma 13.1.6], one may assume that $s = 1$. Set $P = P_1$ and let $d = \deg_Y(P)$. Let $c(\underline{T}) \in \mathcal{Z}[\underline{T}]$ denote the leading coefficient of P when viewed as a polynomial in Y. Consider the polynomial

$$Q(\underline{T}, Y) = c^{d-1} P(\underline{T}, Y/c) \in \mathcal{Z}[\underline{T}, Y].$$

Clearly, Q is monic, and hence has no fixed divisors w.r.t $\underline{T}$. Moreover, Q remains irreducible in $\mathcal{Q}[\underline{T}, Y]$. Since $\mathcal{Z}$ is integrally Hilbertian, there exists a Zariski-dense set of $\underline{m} \in \mathcal{Z}^k$ such that $Q(\underline{m}, Y)$, and so $P(\underline{m}, Y)$ too, is irreducible in $\mathcal{Q}[Y]$. □

2.3. **Locally Schinzel ring.** We introduce a new property: being a locally Schinzel ring (Definition 2.15) and show that this property guarantees that a Hilbertian ring be integrally Hilbertian (Proposition 2.17). In the definition, we slightly abuse terminology and call *arithmetic progression* of $\mathcal{Z}$ every subset of $\mathcal{Z}$ of the form $\omega\mathcal{Z} + \alpha$ with $\omega, \alpha \in \mathcal{Z}$, $\omega \neq 0$.

Definition 2.15. The ring $\mathcal{Z}$ is said to be a *locally Schinzel ring w.r.t. the primality type* $\mathcal{P}$ if the following conclusion holds for any integers $k, n, s \geq 1$ and any polynomials $P_1, \dots, P_s \in \mathcal{Z}[\underline{T}, \underline{Y}]$ satisfying the two hypotheses (Prim/$\mathcal{Q}[\underline{T}]$) and (NoFixDiv/$\mathcal{Z}[\underline{T}]$)$_{\mathcal{P}}$ (from Section 2.2):

(2.15) Letting $\underline{T}' = (T_2, \dots, T_k)$, there is an arithmetic progression $\tau = (\omega\ell + \alpha)_{\ell \in \mathcal{Z}} \subset \mathcal{Z}$ ($\omega, \alpha \in \mathcal{Z}, \omega \neq 0$) such that, for all but finitely many $t_1 \in \tau$, the polynomials

$$P_1(t_1, \underline{T}', \underline{Y}), \dots, P_s(t_1, \underline{T}', \underline{Y}) \in \mathcal{Z}[\underline{T}', \underline{Y}]$$

satisfy (Prim/$\mathcal{Q}[\underline{T}']$) and (NoFixDiv/$\mathcal{Z}[\underline{T}']$)$_{\mathcal{P}}$.

Theorem 2.18 gives concrete examples of locally Schinzel rings.

Remark 2.16. Being a locally Schinzel ring relates to the following property called *coprime Schinzel Hypothesis* in [BDKN22]:

(*) *For any* $k \geq 1$, $\ell \geq 2$ *and any nonzero polynomials* $Q_1, \dots, Q_\ell \in \mathcal{Z}[\underline{T}]$, *coprime in* $\mathcal{Q}[\underline{T}]$ *and such that no nonunit* $p \in \mathcal{Z}$ *divides all values* $Q_1(\underline{m}), \dots, Q_\ell(\underline{m})$ *with* $\underline{m} \in \mathcal{Z}^k$, *there exists* $\underline{m} = (m_1, \dots, m_k) \in \mathcal{Z}^k$ *such that* $Q_1(\underline{m}), \dots, Q_\ell(\underline{m})$ *have no common divisor in* Nonunit $\mathcal{Z}$.

Locally Schinzel rings (resp. integrally Hilbertian rings) satisfy the coprime Schinzel Hypothesis: given $Q_1, \dots, Q_\ell$ as above, apply the locally Schinzel property one parameter after another (resp. the integrally Hilbertian property all parameters at once) to the polynomial $Q_1 Y_1 \cdots + Q_\ell Y_\ell$.

The point of the locally Schinzel property is that it links the Hilbertian ring and integrally Hilbertian ring notions.

Proposition 2.17. *Let* $\mathcal{Z}$ *be a domain and* $\mathcal{P}$ *be a primality type. If* $\mathcal{Z}$ *is both a locally Schinzel ring w.r.t.* $\mathcal{P}$ *and a Hilbertian ring, then it is integrally Hilbertian w.r.t.* $\mathcal{P}$.

Proof. Let $P_1, \dots, P_s \in \mathcal{Z}[\underline{T}, \underline{Y}]$ be s polynomials satisfying the three conditions (Irred/$\mathcal{Q}(\underline{T})$), (Prim/$\mathcal{Q}[\underline{T}]$) and (NoFixDiv/$\mathcal{Z}[\underline{T}]$)$_\mathcal{P}$. Set $\underline{T}' = (T_2, \dots, T_k)$. As $\mathcal{Z}$ is a locally Schinzel ring, there is an arithmetic progression $\tau = (\omega\ell + \alpha)_{\ell \in \mathcal{Z}} \subset \mathcal{Z}$ ($\omega, \alpha \in \mathcal{Z}, \omega \neq 0$) such that, for all $\ell \in \mathcal{Z}$ but in a finite set F, the polynomials

$$P_1(\omega\ell + \alpha, \underline{T}', \underline{Y}), \dots, P_s(\omega\ell + \alpha, \underline{T}', \underline{Y}) \in \mathcal{Z}[\underline{T}', \underline{Y}]$$

satisfy (Prim/$\mathcal{Q}[\underline{T}']$) and (NoFixDiv/$\mathcal{Z}[\underline{T}']$)$_\mathcal{P}$. Consider then the polynomials

$$P_1(\omega T_1 + \alpha, \underline{T}', \underline{Y}), \dots, P_s(\omega T_1 + \alpha, \underline{T}', \underline{Y}) \in \mathcal{Z}[\underline{T}, \underline{Y}].$$

They are irreducible in $\mathcal{Q}(\underline{T})[\underline{Y}]$. By [BDN20, Proposition 4.2], it follows from $\mathcal{Z}$ being a Hilbertian ring (including the imperfectness assumption) that there is an infinite subset $H \subset \mathcal{Z} \setminus F$, such that for each $\ell \in H$, the polynomials

$$P_1(\omega\ell + \alpha, \underline{T}', \underline{Y}), \dots, P_s(\omega\ell + \alpha, \underline{T}', \underline{Y}) \in \mathcal{Z}[\underline{T}', \underline{Y}]$$

are irreducible in $\mathcal{Q}(\underline{T}')[\underline{Y}]$, i.e., they satisfy condition (Irred/$\mathcal{Q}(\underline{T}')$). Thus they satisfy the three conditions (Irred/$\mathcal{Q}(\underline{T}')$), (Prim/$\mathcal{Q}[\underline{T}']$), (NoFixDiv/$\mathcal{Z}[\underline{T}']$)$_\mathcal{P}$ and the same argument can be applied to them. Repeating inductively the argument to each of the parameters $T_2, \dots, T_k$ leads to the requested condition that the set $\mathcal{H}_\mathcal{P}(P_1, \dots, P_s)$ is Zariski-dense. ☐

2.4. **Main results.** The following statement provides our main examples of locally Schinzel rings and of integrally Hilbertian rings.

Theorem 2.18. (a) *For a ring* $\mathcal{Z}$ *as in first row and a primality type* $\mathcal{P}$ *as in first column, we have the following answers to the question of whether* $\mathcal{Z}$ *is a locally Schinzel ring w.r.t.* $\mathcal{P}$:

	Krull	near UFD	PDF
Nonunit	*Yes*	*Yes*	*No*
Irred	*Yes*	*Yes*	*No*
Prime	*Yes*	*Yes*	*Yes*

(b) *Assume further that* $\mathcal{Z}$ *is a Hilbertian ring (of characteristic* 0 *or imperfect). Then the table above provides the same answers to the question of whether* $\mathcal{Z}$ *is integrally Hilbertian w.r.t.* $\mathcal{P}$.

Theorem 1.3 from Section 1 is the special case of Theorem 2.18 corresponding to the box "Krull-Nonunit" in statement (b). A Krull domain is in fact a locally Schinzel ring w.r.t. any of the 3 primality types Nonunit, Irred, Prime.

Remark 2.19. (a) It follows from Theorem 2.18 and Remark 2.16 that Krull domains satisfy the coprime Schinzel hypothesis. This is a progress compared to [BDN20] where, even in the special case of Dedekind domains, condition (*) from Remark 2.16 defining the coprime Schinzel property was only known for $k = 1$ (one parameter).

(b) Concerning the primality type Spec*, the answer to the question in (a) is unclear for our three types of rings (Krull, near UFD, PDF). From Proposition 3.7 and Example 3.6, the answer to (a) is "Yes" for Dedekind domains; hence, by Proposition 2.17, the answer to (b) is also "Yes".

2.5. **Rings that are Hilbertian but neither integrally Hilbertian nor locally Schinzel rings.** In [BDKN22, Proposition 2.10], the ring $\mathbb{Z}[\sqrt{5}]$ was identified as an example of a domain not satisfying the coprime Schinzel Hypothesis. Therefore it cannot be an integrally Hilbertian ring nor a locally Schinzel ring (Remark 2.16). The following shows that such rings, in fact, are abundant.

Lemma 2.20. *Let $\mathcal{Z}$ be a domain with non-associate irreducible elements p, q such that the ideal $(p) \cap (q)$ is contained in a unique maximal ideal. Then $\mathcal{Z}$ is neither integrally Hilbertian nor a locally Schinzel ring w.r.t. the primality type* Nonunit.

Proof. This follows immediately from [BDKN22, Proposition 2.8] and Remark 2.16. □

Corollary 2.21. *A local domain with more than one irreducible element (up to associates) cannot be integrally Hilbertian nor a locally Schinzel ring w.r.t. the primality type* Nonunit.

Corollary 2.22. *Every quadratic number field possesses subrings which are neither integrally Hilbertian nor locally Schinzel w.r.t. $\mathcal{P} =$* Nonunit*, but are PDF rings and Noetherian.*

Proof. Given any squarefree integer d and a prime p which is inert in $\mathbb{Q}(\sqrt{d})$, [CS12, Theorem 3.1] constructs, via localization of some order of $\mathbb{Q}(\sqrt{d})$, a subring of $\mathbb{Q}(\sqrt{d})$ which is local and has $p+1$ non-associate irreducible elements. From Corollary 2.21, such a ring is neither integrally Hilbertian nor a locally Schinzel ring. Obviously, they are also PDF rings, and they are Noetherian. □

3. Proof of Theorem 2.18

§3.1 is a first reduction stage, reducing the proof to that of Lemma 3.1 below. §3.2 introduces the strategy of the proof of this lemma. Definition 3.5 pins down two general properties of a primality type $\mathcal{P}$ of the integral domain $\mathcal{Z}$ that, first, guarantee that $\mathcal{Z}$ is a locally Schinzel ring w.r.t. $\mathcal{P}$ (Proposition 3.7, proved in §3.3), and second, are satisfied (up to equivalence) in all situations of Lemma 3.1 (Proposition 3.8, proved in §3.4).

3.1. **Reduction of the proof of Theorem 2.18.** Lemma 3.1 corresponds to the sole answers "Yes" in the table of statement (a) of Theorem 2.18.

Lemma 3.1. *A ring $\mathcal{Z}$ is a locally Schinzel ring w.r.t. $\mathcal{P}$ in each of the following situations:*

(1) $\mathcal{Z}$ is a Krull domain and $\mathcal{P} \in \{\text{Nonunit}, \text{Irred}, \text{Prime}\}$.
(2) $\mathcal{Z}$ is a PDF ring and $\mathcal{P} =$ Prime.
(3) $\mathcal{Z}$ is a near UFD and $\mathcal{P} \in \{\text{Nonunit}, \text{Irred}, \text{Prime}\}$.

Proof of Theorem 2.18 assuming Lemma 3.1. The situations from Lemma 3.1 correspond to the answers "Yes" from the table of Theorem 2.18. Thus Lemma 3.1 directly yields these answers "Yes" for statement (a) of Theorem 2.18, and also the answers "Yes" for statement (b), by Proposition 2.17.

The answers "No" concerning PDF rings w.r.t the primality type Nonunit follow directly from Corollary 2.22. As the rings there are also Noetherian, the same is equivalently true w.r.t. to the primality type $\mathcal{P} =$ Irred (the main point of the equivalence being that in a Noetherian ring, every nonunit is divisible by an irreducible). □

3.2. **Properties (NVA) and (SF).** Definition 3.5 requests some preliminary definitions. First, we say that two primality types $\mathcal{P}$ and $\mathcal{P}'$ of $\mathcal{Z}$ are *equivalent* if the following holds: every ideal in $\mathcal{P}$ is contained in some ideal in $\mathcal{P}'$, and every ideal in $\mathcal{P}'$ is contained in some ideal in $\mathcal{P}$. Note that, for such $\mathcal{P}$ and $\mathcal{P}'$, the ring $\mathcal{Z}$ is a locally Schinzel ring w.r.t. $\mathcal{P}$ if and only if $\mathcal{Z}$ is a locally Schinzel ring w.r.t. $\mathcal{P}'$; and the same holds for the integrally Hilbertian property.

Example 3.2. The primality types Nonunit $\mathcal{Z}$ and Irred $\mathcal{Z}$ are equivalent if $\mathcal{Z}$ is Noetherian or a Krull domain. Indeed in both these cases, the ascending chain condition for principal ideals is satisfied, which guarantees that every nonunit is divisible by an irreducible element.

Definition 3.3. The *support* of a nonzero ideal $\mathfrak{a}$ of $\mathcal{Z}$ is:

$$\mathrm{Supp}(\mathfrak{a}) = \{\mathfrak{p} \in \mathrm{Spec}^* \mathcal{Z} \mid \mathfrak{a} \subset \mathfrak{p}\}.$$

For a nonzero $a \in \mathcal{Z}$, we use the notation $\mathrm{Supp}(a)$ for $\mathrm{Supp}(\langle a \rangle)$. Given a subset $\mathcal{S} \subset \mathrm{Spec}^* \mathcal{Z}$, denote by $\Sigma(\mathcal{S})$ the set of all ideals $\mathfrak{a}$ of $\mathcal{Z}$ such that $\mathrm{Supp}(\mathfrak{a}) \subset \mathcal{S}$. More generally, for $\mathcal{S} \subset \mathcal{E} \subset \mathrm{Spec}^* \mathcal{Z}$, denote by $\Sigma(\mathcal{S})_{\mathcal{E}}$ the set of all ideals $\mathfrak{a}$ of $\mathcal{Z}$ such that $\mathrm{Supp}(\mathfrak{a}) \cap \mathcal{E} \subset \mathcal{S}$.

If $\mathcal{Z}$ is a Dedekind domain, $\Sigma(\mathcal{S})$ (resp., $\Sigma(\mathcal{S})_{\mathcal{E}}$) is the set of all ideals $\mathfrak{a}$ for which the prime ideals (resp., the prime ideals in $\mathcal{E}$) involved in the prime ideal factorization of $\mathfrak{a}$ are in $\mathcal{S}$.

Remark 3.4. For any subsets $\mathcal{S} \subset \mathrm{Spec}^* \mathcal{Z}$ and $\mathcal{E} \subset \mathrm{Spec}^* \mathcal{Z}$, we have $\Sigma(\mathcal{S}) \subset \Sigma(\mathcal{S} \cap \mathcal{E})_{\mathcal{E}}$.

Definition 3.5. A domain $\mathcal{Z}$ equipped with a primality type $\mathcal{P}$ is said to satisfy:

1. the *nonvanishing approximation property* – $(\mathrm{NVA})_{\mathcal{P}}$ for short, if for any nonzero polynomial $P \in \mathcal{Z}[\underline{T}, \underline{Y}]$, and any finite subset $\mathcal{S} \subset \mathcal{P}$, the following holds. If a family $(\underline{u}_{\mathfrak{a}})_{\mathfrak{a} \in \mathcal{S}}$ of k-tuples $\underline{u}_{\mathfrak{a}} \in \mathcal{Z}^k$ indexed by $\mathcal{S}$ satisfies:
$$P(\underline{u}_{\mathfrak{a}}, \underline{Y}) \not\equiv 0 \pmod{\mathfrak{a}} \text{ for all } \mathfrak{a} \in \mathcal{S},$$
then there exists $\underline{v} \in \mathcal{Z}^k$ such that:
$$P(\underline{v}, \underline{Y}) \not\equiv 0 \pmod{\mathfrak{a}} \text{ for all } \mathfrak{a} \in \mathcal{S},$$
2. the *support finiteness property* – $(\mathrm{SF})_{\mathcal{P}}$ for short, if there exists a subset $\mathcal{E} \subset \mathrm{Spec}^* \mathcal{Z}$ such that:
   - (a) for any nonzero $a \in \mathcal{Z}$, the set $\mathrm{Supp}(a) \cap \mathcal{E}$ is finite;
   - (b) for any finite subset $\mathcal{S} \subset \mathcal{E}$, the set $\Sigma(\mathcal{S})_{\mathcal{E}} \cap \mathcal{P}$ is finite.

Example 3.6. Let $\mathcal{Z}$ be a Dedekind domain. Using the Chinese remainder theorem, it is easily checked that $\mathcal{Z}$ satisfies $(\mathrm{NVA})_{\mathcal{P}}$ for $\mathcal{P} = \mathrm{Spec}^* \mathcal{Z}$; and property $(\mathrm{SF})_{\mathcal{P}}$ is also satisfied for $\mathcal{E} = \mathrm{Spec}^* \mathcal{Z}$. Property $(\mathrm{SF})_{\mathcal{P}}$ also holds for $\mathcal{P} = \mathrm{Irred}$ (and $\mathcal{E} = \mathrm{Spec}^* \mathcal{Z}$). But it does not for $\mathcal{P} = \mathrm{Nonunit}$: powers a^n of a nonunit a have the same support; hence condition (2)(b) fails.

We can state the two key intermediate results from which Lemma 3.1 follows.

Proposition 3.7. *Let $\mathcal{Z}$ be a domain equipped with a primality type $\mathcal{P}$. If properties* $(\mathrm{NVA})_{\mathcal{P}}$ *and* $(\mathrm{SF})_{\mathcal{P}}$ *are satisfied, then $\mathcal{Z}$ is a locally Schinzel ring w.r.t. $\mathcal{P}$.*

Proposition 3.8. *In each of the situations of Lemma 3.1, the primality type $\mathcal{P}$ is equivalent to a primality type $\mathcal{P}'$ of $\mathcal{Z}$ such that* $(\mathrm{NVA})_{\mathcal{P}'}$ *and* $(\mathrm{SF})_{\mathcal{P}'}$ *are satisfied.*

3.3. **Proof of Proposition 3.7.**

3.3.1. *Preliminary lemmas.* We establish some results needed for the proof of Proposition 3.7. For any integer $B \geq 1$, define:

$$\Gamma_B = \{\mathfrak{p} \in \mathrm{Spec}^* \mathcal{Z} \mid |\mathcal{Z}/\mathfrak{p}| \leq B\}.$$

Lemma 3.9. *For any integer $B \geq 1$, there exists a nonzero $a \in \mathcal{Z}$ such that $\Gamma_B \subset \mathrm{Supp}(a)$.*

Proof. The proof is the same as that of [BDKN22, Lemma 3.9] by replacing primes with prime ideals. For the sake of completeness, we reproduce the argument. Fix an integer $B \geq 1$. For every prime power $q = \ell^r \leq B$, take an element $m_q \in \mathcal{Z}$ such that $m_q^q - m_q \neq 0$. Let a be the product of all $m_q^q - m_q$ with q running over all prime powers $q \leq B$.

Consider now a prime ideal $\mathfrak{p} \in \Gamma_B$. The integral domain $\mathcal{Z}/\mathfrak{p}$, being finite, is a field. Hence $|\mathcal{Z}/\mathfrak{p}|$ is a prime power $q = \ell^r$, and $q \leq B$. Since $m_q^q - m_q \equiv 0 \pmod{\mathfrak{p}}$, the prime ideal $\mathfrak{p}$ contains a, i.e. $\mathfrak{p} \in \mathrm{Supp}(a)$. Therefore $\Gamma_B \subset \mathrm{Supp}(a)$. ☐

Lemma 3.10 locates the support of fixed divisors of polynomials. Recall that, for a nonzero polynomial $P \in \mathcal{Z}[\underline{T}, \underline{Y}]$, the set $\mathcal{F}_{\underline{T}}(P)$ denotes the set of all fixed ideal divisors of P w.r.t. $\underline{T}$. Moreover, for a nonzero $P \in \mathcal{Z}[\underline{Y}]$, the set $\mathrm{Div}_{\mathcal{Z}}(P)$ denotes the set of all ideal divisors of P.

Lemma 3.10. *Let $P \in \mathcal{Z}[\underline{T},\underline{Y}]$ be a nonzero polynomial. Set $\Delta = \max_{1\le i\le k} \deg_{T_i}(P)$ and $\Omega = \mathrm{Div}_{\mathcal{Z}}(P) \cap \mathrm{Spec}^* \mathcal{Z}$. Then we have these inclusions:*

(1) $(\mathscr{F}_{\underline{T}}(P) \cap \mathrm{Spec}^* \mathcal{Z}) \setminus \mathrm{Div}_{\mathcal{Z}}(P) \subset \Gamma_\Delta$.
(2) $\mathscr{F}_{\underline{T}}(P) \subset \Sigma(\Omega \cup \Gamma_\Delta)$.

Proof. (1) We wish to show that if $\mathfrak{p}$ is a fixed divisor of P w.r.t. $\underline{T}$ in $\mathrm{Spec}^* \mathcal{Z}$ such that $\mathfrak{p}$ does not divide P, then $|\mathcal{Z}/\mathfrak{p}| \le \Delta$. The proof follows from the same reasoning as in [BDKN22, Lemma 3.1 (a)], with prime elements replaced by prime ideals.

(2) Let $\mathfrak{a} \in \mathscr{F}_{\underline{T}}(P)$. We wish to show that $\mathrm{Supp}(\mathfrak{a}) \subset \Omega \cup \Gamma_\Delta$. Let $\mathfrak{p} \in \mathrm{Supp}(\mathfrak{a})$, i.e. $\mathfrak{p} \supset \mathfrak{a}$. If $\mathfrak{p} \in \mathrm{Div}_{\mathcal{Z}}(P)$, then $\mathfrak{p} \in \Omega$. Now assume that $\mathfrak{p} \notin \mathrm{Div}_{\mathcal{Z}}(P)$. Since $\mathfrak{a} \in \mathscr{F}_{\underline{T}}(P)$, we have $\mathfrak{p} \in \mathscr{F}_{\underline{T}}(P)$. By (1), we obtain $\mathfrak{p} \in \Gamma_\Delta$. □

3.3.2. *Proof of Proposition 3.7.* Let $\mathcal{E} \subset \mathrm{Spec}^* \mathcal{Z}$ be the subset associated with $(\mathrm{SF})_{\mathcal{P}}$.

Let $P_1, \dots, P_s \in \mathcal{Z}[\underline{T},\underline{Y}]$ be s polynomials satisfying $(\mathrm{Prim}/\mathcal{Q}[\underline{T}])$ and $(\mathrm{NoFixDiv}/\mathcal{Z}[\underline{T}])_{\mathcal{P}}$. Set $P = P_1 \cdots P_s$. Consider $P_1, \dots, P_s$ as polynomials in $\underline{T}' = (T_2, \dots, T_k)$ and $\underline{Y}$ and with coefficients in $\mathcal{Z}[T_1]$. Note that they are all primitive w.r.t. $\mathcal{Q}[T_1]$. If follows that P is primitive w.r.t. $\mathcal{Q}[T_1]$ as well. Thus, the coefficients $Q_1, \dots, Q_r \in \mathcal{Z}[T_1]$ $(r \ge 1)$ of P, considered as a polynomial in $\underline{T}', \underline{Y}$, are coprime in $\mathcal{Q}[T_1]$. As in [BDKN22, §2.1], choose a nonzero element

$$\delta \in \left(\sum_{j=1}^{r} Q_j \mathcal{Z}[T_1]\right) \cap \mathcal{Z}.$$

For $\Delta = \max_{1\le i\le k} \deg_{T_i}(P)$, set:

$$\mathcal{S} = \mathcal{P} \cap \Sigma\left(\left[\mathrm{Supp}(\delta) \cup \mathrm{Div}_{\mathcal{Z}}(P) \cup \Gamma_\Delta\right] \cap \mathcal{E}\right)_{\mathcal{E}}. \tag{1}$$

By the support finiteness property, $\mathrm{Supp}(\delta) \cap \mathcal{E}$ and $\mathrm{Div}_{\mathcal{Z}}(P) \cap \mathcal{E}$ are finite; for the latter, note that it is contained in $\mathrm{Supp}(a) \cap \mathcal{E}$ for any nonzero coefficient a of P in $\mathcal{Z}$. Moreover, by Lemma 3.9 and again the support finiteness property, $\Gamma_\Delta \cap \mathcal{E}$ is finite. Therefore $[\mathrm{Supp}(\delta) \cup \mathrm{Div}_{\mathcal{Z}}(P) \cup \Gamma_\Delta] \cap \mathcal{E}$ is finite too. It follows that $\mathcal{S}$ is finite. Choose a nonzero element $\omega \in \prod_{\mathfrak{p}\in\mathcal{S}} \mathfrak{p} \subset \bigcap_{\mathfrak{p}\in\mathcal{S}} \mathfrak{p}$ – we take $\omega = 1$ if $\mathcal{S} = \emptyset$.

Claim 1. *There exist arithmetic progressions*

$$\tau_1 = (\omega\ell + v_1)_{\ell\in\mathcal{Z}},\ \tau_2 = (\omega\ell + v_2)_{\ell\in\mathcal{Z}},\ \dots,\ \tau_k = (\omega\ell + v_k)_{\ell\in\mathcal{Z}} \subset \mathcal{Z}$$

such that

$$P(\underline{u},\underline{Y}) \not\equiv 0 \pmod{\mathfrak{p}}, \tag{2}$$

for all $\underline{u} = (u_1, \dots, u_k) \in \tau_1 \times \cdots \times \tau_k$ and all $\mathfrak{p} \in \mathcal{S}$.

Proof of Claim 1. From $(\mathrm{NoFixDiv}/\mathcal{Z}[\underline{T}])_{\mathcal{P}}$, for every $\mathfrak{p} \in \mathcal{S}$, there exists $\underline{u}_{\mathfrak{p}} \in \mathcal{Z}^k$ such that

$$P(\underline{u}_{\mathfrak{p}},\underline{Y}) \not\equiv 0 \pmod{\mathfrak{p}}. \tag{3}$$

By the nonvanishing approximation property, there exists $\underline{v} = (v_1, \dots, v_k) \in \mathcal{Z}^k$ such that

$$P(\underline{v},\underline{Y}) \not\equiv 0 \pmod{\mathfrak{p}} \text{ for all } \mathfrak{p} \in \mathcal{S}.$$

Set $\tau_1 \times \cdots \times \tau_k = (\omega\ell + v_1)_{\ell\in\mathcal{Z}} \times \cdots \times (\omega\ell + v_k)_{\ell\in\mathcal{Z}}$. Let $\underline{u} = (u_1, \dots, u_k) \in \tau_1 \times \cdots \times \tau_k$ and $\mathfrak{p} \in \mathcal{S}$. Since $\omega \in \bigcap_{\mathfrak{p}\in\mathcal{S}} \mathfrak{p}$, we have $\underline{u} \equiv \underline{v} \pmod{\mathfrak{p}}$, and thus

$$P(\underline{u},\underline{Y}) \equiv P(\underline{v},\underline{Y}) \not\equiv 0 \pmod{\mathfrak{p}}. \tag{4}$$

□

Consider the following polynomials, where V_1 is a new variable:

$$\widetilde{P}_i(V_1, \underline{T}', \underline{Y}) = P_i(\omega V_1 + v_1, \underline{T}', \underline{Y}) \in \mathcal{Z}[V_1, \underline{T}', \underline{Y}],\ \ (i = 1, \dots, s),$$

and set $\widetilde{P} = \prod_{i=1}^{s} \widetilde{P}_i$. To complete the proof of Proposition 3.7, it remains to establish the following.

Claim 2. *For all but finitely many* $\ell_1 \in \mathcal{Z}$, *the polynomials*

$$\widetilde{P}_1(\ell_1, \underline{T}', \underline{Y}), \ldots, \widetilde{P}_s(\ell_1, \underline{T}', \underline{Y})$$

satisfy (Prim/$\mathcal{Q}[\underline{T}']$) *and* (NoFixDiv/$\mathcal{Z}[\underline{T}']$)$_{\mathcal{P}}$.

Proof of Claim 2. As polynomials in $\mathcal{Q}[V_1, \underline{T}'][\underline{Y}]$, the polynomials $\widetilde{P}_1, \ldots, \widetilde{P}_s$ satisfy (Prim/$\mathcal{Q}[V_1, \underline{T}']$). We next use [BD23, Proposition 3.1], which is an analog for coprimality of the Bertini–Noether theorem (originally for irreducibility): for all but finitely many $\ell_1 \in \mathcal{Z}$, the polynomials

$$\widetilde{P}_1(\ell_1, \underline{T}', \underline{Y}), \ldots, \widetilde{P}_s(\ell_1, \underline{T}', \underline{Y})$$

satisfy (Prim/$\mathcal{Q}[\underline{T}']$).

Fix $\ell_1 \in \mathcal{Z}$ such that the above property holds. Assume on the contrary that

$$\widetilde{P}_1(\ell_1, \underline{T}', \underline{Y}), \ldots, \widetilde{P}_s(\ell_1, \underline{T}', \underline{Y})$$

do not satisfy (NoFixDiv/$\mathcal{Z}[\underline{T}']$)$_{\mathcal{P}}$, that is, there exists an ideal $\mathfrak{p}$ in $\mathcal{F}_{\underline{T}'}(\widetilde{P}(\ell_1, \underline{T}', \underline{Y})) \cap \mathcal{P}$. Then

$$\widetilde{P}(\ell_1, \underline{t}', \underline{Y}) \equiv 0 \pmod{\mathfrak{p}},$$

for all $\underline{t}' \in \mathcal{Z}^{k-1}$.

On the one hand, we claim that $\mathfrak{p} \notin \mathcal{S}$. Otherwise, since $\omega\ell_1 + v_1 \equiv v_1 \pmod{\mathfrak{p}}$, Claim 1 implies that:

$$0 \equiv \widetilde{P}(\ell_1, v_2, \ldots, v_k, \underline{Y}) \equiv P(\underline{v}, \underline{Y}) \not\equiv 0 \pmod{\mathfrak{p}},$$

a contradiction.

On the other hand, we claim that $\mathfrak{p} \in \mathcal{S}$. Indeed, with $\Delta' = \max_{2 \le i \le k} \deg_{T_i}(\widetilde{P}(\ell_1, \underline{T}', \underline{Y}))$, Lemma 3.10 and Remark 3.4 lead to

$$\mathscr{F}_{\underline{T}'}(\widetilde{P}(\ell_1, \underline{T}', \underline{Y})) \cap \mathcal{P} \subset \Sigma\left(\left(\mathrm{Div}_{\mathcal{Z}}(\widetilde{P}(\ell_1, \underline{T}', \underline{Y}) \cup \Gamma_{\Delta'}\right) \cap \mathcal{E}\right)_{\mathcal{E}} \cap \mathcal{P}. \tag{5}$$

If $\mathfrak{q} \in \mathrm{Div}_{\mathcal{Z}}(\widetilde{P}(\ell_1, \underline{T}', \underline{Y})) \cap \mathcal{E}$, then $Q_i(\omega\ell_1 + v_1) \equiv 0 \pmod{\mathfrak{q}}$ for $i = 1, \ldots, r$. Therefore $\mathfrak{q} \in \mathrm{Supp}(\delta)$ and so $\mathrm{Div}_{\mathcal{Z}}(\widetilde{P}(\ell_1, \underline{T}', \underline{Y})) \cap \mathcal{E} \subset \mathrm{Supp}(\delta) \cap \mathcal{E}$. Combining this with the inclusion $\Gamma_{\Delta'} \subset \Gamma_{\Delta}$, it now follows from (1) and (5) that

$$\mathscr{F}_{\underline{T}'}(\widetilde{P}(\ell_1, \underline{T}', \underline{Y})) \cap \mathcal{P} \subset \Sigma\left((\mathrm{Supp}(\delta) \cup \mathrm{Div}_{\mathcal{Z}}(P) \cup \Gamma_{\Delta}) \cap \mathcal{E}\right)_{\mathcal{E}} \cap \mathcal{P} = \mathcal{S}.$$

Thus $\mathfrak{p} \in \mathcal{S}$, thus contradicting the preceding paragraph.

This completes the proof of Claim 2 and of Proposition 3.7. □

3.4. **Proof of Proposition 3.8.**

3.4.1. *Preliminary lemmas.* To prove Proposition 3.8, we need the following two results.

Lemma 3.11. *Assume that* $\mathcal{Z}$ *is a Krull domain and let* $\mathcal{E} = \mathrm{Spec}^{(1)}(\mathcal{Z})$.

(1) The primality type $\mathcal{P} = \mathrm{Irred}$ *satisfies the support finiteness property for* $\mathcal{E}$.
(2) For any integer $B \geq 1$, *the set* $\Sigma(\Gamma_B \cap \mathcal{E})_{\mathcal{E}} \cap \mathrm{Irred}\,\mathcal{Z}$ *is finite.*

Proof. (1) Since $\mathcal{Z}$ is a Krull domain, by [Mat80, Page 289, III)], the set $\mathrm{Supp}(a) \cap \mathcal{E}$ is finite for every nonzero element $a \in \mathcal{Z}$. Hence condition (a) of Definition 3.5(2) holds. For condition (b), fix a finite subset $\mathcal{S} = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\} \subset \mathcal{E}$. We wish to show that the set

$$\Sigma(\mathcal{S})_{\mathcal{E}} \cap \mathcal{P} := \{(\theta) \in \mathcal{P} \mid \mathrm{Supp}(\theta) \cap \mathcal{E} \subset \mathcal{S}\}$$

is finite. Define

$$\mathcal{K} = \{v_{\mathcal{S}}((\theta)) := (v_{\mathfrak{p}_1}(\theta), \ldots, v_{\mathfrak{p}_r}(\theta)) \in \mathbb{N}^r \mid (\theta) \in \Sigma(\mathcal{S})_{\mathcal{E}} \cap \mathcal{P}\}.$$

By [Mat80, Page 289, III)], this is naturally in bijection with $\Sigma(\mathcal{S})_{\mathcal{E}} \cap \mathcal{P}$; Indeed, for any $(\theta) \in \Sigma(\mathcal{S})_{\mathcal{E}} \cap \mathcal{P}$, we have $(\theta) = \bigcap_{i=1}^{r} \mathfrak{p}_i^{(v_{\mathfrak{p}_i(\theta)})}$. Thus, it suffices to prove the following.

Claim 3. *The set* $\mathcal{K}$ *is finite.*

Proof of Claim 3. For any $\mathfrak{p} \in \mathcal{E} \setminus \mathcal{S}$ and any $(\theta) \in \Sigma(\mathcal{S})_{\mathcal{E}} \cap \mathcal{P}$, we have $v_{\mathfrak{p}}(\theta) = 0$. We equip $\mathcal{K}$ with the partial order defined by

$$(a_1, \ldots, a_r) \leq (b_1, \ldots, b_r) \quad \text{if and only if} \quad a_i \leq b_i \quad \text{for all } 1 \leq i \leq r.$$

By [Mat80, Page 288, I)], for $(\theta), (\tau) \in \Sigma(\mathcal{S})_{\mathcal{E}} \cap \mathcal{P}$, we have

$$v_{\mathcal{S}}(\theta) \leq v_{\mathcal{S}}(\tau) \text{ if and only if } \theta \mid \tau \text{ if and only if } (\tau) = (\theta). \tag{6}$$

Consequently, distinct elements of $\mathcal{K}$ are pairwise noncomparable; in this setting, $\mathbf{c}$ and $\mathbf{d}$ are *noncomparable* if $\mathbf{c} \not\leq \mathbf{d}$ and $\mathbf{d} \not\leq \mathbf{c}$.

On the other hand, with the partial order on $\mathbb{N}^r$:

$$\mathbf{c} = (c_1, \ldots, c_r) \leq \mathbf{d} = (d_1, \ldots, d_r) \text{ if and only if } c_1 \leq d_1, \ldots, c_r \leq d_r,$$

it is well-known that $(\mathbb{N}^r, \leq)$ does not contain an infinite family of pairwise noncomparable elements, see, e.g., [AA20, Theorem 1.1(1)].

Consequently $\mathcal{K}$ is finite. □

(2) Let $B \geq 1$ be an integer. By Lemma 3.9, there exists a nonzero $a \in \mathcal{Z}$ such that $\Gamma_B \cap \mathcal{E} \subset \mathrm{Supp}(a) \cap \mathcal{E}$, so $\Gamma_B \cap \mathcal{E}$ is finite. Combining it with (1), we deduce that $\Sigma(\Gamma_B \cap \mathcal{E})_{\mathcal{E}} \cap \mathrm{Irred}\,\mathcal{Z}$ is finite.

□

Remark 3.12. At this stage, we can provide a proof of the remaining Krull case of Proposition 2.7. Assume that $\mathcal{Z}$ is a Krull domain, and set $\mathcal{E} = \mathrm{Spec}^{(1)}(\mathcal{Z})$. Let $P \in \mathcal{Z}[\underline{T}, \underline{Y}]$ be a nonzero polynomial. One may assume that P has a fixed divisor in $\mathrm{Nonunit}\,\mathcal{Z}$ w.r.t. $\underline{T}$; otherwise, take $\varphi = 1$. Choose $\underline{m} \in \mathcal{Z}^k$ such that $P(\underline{m}, \underline{Y})$ is nonzero. Pick a nonzero coefficient a of $P(\underline{m}, \underline{Y})$ and consider the set $\mathcal{S} = \mathrm{Supp}(a) \cap \mathcal{E}$. By the support finiteness property (Lemma 3.11), the set $\mathcal{S}$ is finite, and so is the set

$$\Theta = \Sigma(\mathcal{S})_{\mathcal{E}} \cap \mathrm{Irred}\,\mathcal{Z}.$$

We claim that we can take φ to be a generator of the principal ideal $\prod_{\mathfrak{p} \in \Theta} \mathfrak{p}$. Indeed, assume on the contrary that P has a fixed divisor $\langle u/\varphi^\ell \rangle \in \mathrm{Nonunit}\,\mathcal{Z}[1/\varphi]$ w.r.t. $\underline{T}$, where $u \in \mathcal{Z}$ and $\ell \geq 0$. In particular,

$$P(\underline{m}, \underline{Y}) \equiv 0 \pmod{u/\varphi^\ell},$$

so u divides $a \cdot \varphi^\ell$. Then

$$\mathrm{Supp}(u) \cap \mathcal{E} \subset (\mathrm{Supp}(a) \cap \mathcal{E}) \bigcup \left(\left(\bigcup_{\mathfrak{p} \in \Theta} \mathrm{Supp}(\mathfrak{p}) \right) \cap \mathcal{E} \right) = \mathcal{S}.$$

Consider a decomposition $u = \pi_1 \cdots \pi_e$ into a product of irreducible elements. Since $\mathrm{Supp}(\langle \pi_i \rangle) \cap \mathcal{E}$ $(1 \leq i \leq e)$ is also contained in $\mathcal{S}$, we have $\langle \pi_1 \rangle, \ldots, \langle \pi_e \rangle \in \Theta$. Therefore, each element π_i divides φ $(i = 1, \ldots, e)$ and so u divides φ^e, implying that u is a unit in $\mathcal{Z}[1/\varphi]$ – a contradiction.

Consequently, P has no fixed divisor in $\mathrm{Nonunit}\,\mathcal{Z}[1/\varphi]$ w.r.t. $\underline{T}$.

Lemma 3.13. *Suppose that $\mathcal{Z}$ and $\mathcal{P}$ satisfy one of the following:*

(a) $\mathcal{Z}$ is a PDF ring and $\mathcal{P} = \mathrm{Prime}$.

(b) $\mathcal{Z}$ is a Krull domain and $\mathcal{P} = \mathrm{Irred}$.

Then $\mathcal{P}$ satisfies the nonvanishing approximation property.

Proof. Let $P \in \mathcal{Z}[\underline{T}, \underline{Y}]$ be a nonzero polynomial, let $\mathcal{S} \subset \mathcal{P}$ be a finite set, and let $(\underline{u}_{\mathfrak{a}})_{\mathfrak{a} \in \mathcal{S}} \subset \mathcal{Z}^k$ be such that:

$$P(\underline{u}_{\mathfrak{a}}, \underline{Y}) \not\equiv 0 \pmod{\mathfrak{a}} \quad \text{for all } \mathfrak{a} \in \mathcal{S}.$$

Case (a). The proof follows from that of [BDKN22, Lemma 4.2]. For the convenience of the reader, we repeat the argument. Denote by $\mathcal{S}_1 \subset \mathcal{S}$ the subset of prime ideals $\mathfrak{p}$ that are maximal ideals. We apply [BDKN22, Lemma 3.8]. From above, P is nonzero modulo each $\mathfrak{a} \in \mathcal{S} \setminus \mathcal{S}_1$, and we have $\mathfrak{a} \not\subset \mathfrak{a}'$ for any distinct $\mathfrak{a}, \mathfrak{a}' \in \mathcal{S}$ (as $\mathfrak{a}$ and $\mathfrak{a}'$ are principal ideals generated by prime elements). Thus, [BDKN22, Lemma 3.8] provides $\underline{v} = (v_1, \dots, v_k) \in \mathcal{Z}^k$ such that

$$\underline{v} \equiv \underline{u}_{\mathfrak{a}} \pmod{\mathfrak{a}} \quad \text{for all prime } \mathfrak{a} \in \mathcal{S}_1,$$

and

$$P(\underline{v}, \underline{Y}) \not\equiv 0 \pmod{\mathfrak{a}} \quad \text{for all } \mathfrak{a} \in \mathcal{S} \setminus \mathcal{S}_1.$$

These congruences imply that

$$P(\underline{v}, \underline{Y}) \not\equiv 0 \pmod{a} \quad \text{for all } \mathfrak{a} \in \mathcal{S}.$$

Case (b). Let $\mathcal{E} = \mathrm{Spec}^{(1)}(\mathcal{Z})$. Since $\mathcal{S}$ is finite, by [Mat80, Page 289, III)], the set

$$\mathcal{R} = \bigcup_{\mathfrak{a} \in \mathcal{S}} (\mathrm{Supp}(\mathfrak{a}) \cap \mathcal{E})$$

is finite, and writing $\mathcal{R} = \{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$ $(m \geq 1)$, we have, for each $\mathfrak{a} \in \mathcal{S}$

$$\mathfrak{a} = \mathfrak{p}_1^{(e_1(\mathfrak{a}))} \cap \dots \cap \mathfrak{p}_m^{(e_m(\mathfrak{a}))},$$

where

$$(e_1(\mathfrak{a}), \dots, e_m(\mathfrak{a})) = (v_{\mathfrak{p}_1}(\mathfrak{a}), \dots, v_{\mathfrak{p}_m}(\mathfrak{a})) \in \mathbb{N}^m.$$

For each $i = 1, \dots, m$, set

$$\mathcal{S}_i = \left\{ \mathfrak{a} \in \mathcal{S} \mid e_i(\mathfrak{a}) \geq 1 \text{ and } P(\underline{u}_{\mathfrak{a}}, \underline{Y}) \not\equiv 0 \pmod{\mathfrak{p}_i^{(e_i(\mathfrak{a}))}} \right\}.$$

Let

$$J = \{ j \in \{1, \dots, m\} \mid \mathcal{S}_j \neq \emptyset \}.$$

For each $j \in J$, choose $\mathfrak{a}_j \in \mathcal{S}_j$ such that

$$e_j(\mathfrak{a}_j) = \min_{\mathfrak{a} \in \mathcal{S}_j} (e_j(\mathfrak{a})) \geq 1.$$

By the weak approximation theorem (Lemma 2.3), there exists $\underline{v} \in \mathcal{Z}^k$ such that

$$\underline{v} \equiv \underline{u}_{\mathfrak{a}_j} \pmod{\mathfrak{p}_j^{(e_j(\mathfrak{a}_j))}} \quad \text{for all } j \in J.$$

To complete the proof, we show that $P(\underline{v}, \underline{Y}) \not\equiv 0 \pmod{\mathfrak{a}}$ for all $\mathfrak{a} \in \mathcal{S}$. Let $\mathfrak{a} \in \mathcal{S}$. Since $P(\underline{u}_{\mathfrak{a}}, \underline{Y}) \not\equiv 0 \pmod{\mathfrak{a}}$, there exists $j \in \{1, \dots, m\}$ with $e_j(\mathfrak{a}) \geq 1$ such that $P(\underline{u}_{\mathfrak{a}}, \underline{Y}) \not\equiv 0 \pmod{\mathfrak{p}_j^{(e_j(\mathfrak{a}))}}$; in other words, $\mathfrak{a} \in \mathcal{S}_j$. As

$$P(\underline{v}, \underline{Y}) \equiv P(\underline{u}_{\mathfrak{a}_j}, \underline{Y}) \not\equiv 0 \pmod{\mathfrak{p}_j^{(e_j(\mathfrak{a}_j))}}$$

and $e_j(\mathfrak{a}) \geq e_j(\mathfrak{a}_j)$, we obtain

$$P(\underline{v}, \underline{Y}) \not\equiv 0 \pmod{\mathfrak{p}_j^{(e_j(\mathfrak{a}))}}.$$

Therefore we can conclude that

$$P(\underline{v}, \underline{Y}) \not\equiv 0 \ (\mathrm{mod}\ \mathfrak{a}).$$

□

3.4.2. *End of proof of Proposition 3.8.* The three cases of Lemma 3.1 are cases (1),(2),(3) below. Lemma 3.13 ensures the nonvanishing approximation property for the (equivalent to $\mathcal{P}$) primality type $\mathcal{P}'$ as chosen below in each case. It remains to check the support finiteness property.

(1) Let $\mathcal{Z}$ be a Krull domain. For $\mathcal{P} = \mathrm{Prime}$, the support finiteness property holds since $\mathcal{Z}$ is a PDF ring by Remark 2.5. For $\mathcal{P} \in \{\mathrm{Irred}, \mathrm{Nonunit}\}$, take $\mathcal{P}' = \mathrm{Irred}$ and $\mathcal{E} = \mathrm{Spec}^{(1)}(\mathcal{Z})$. The support finiteness property then follows from Lemma 3.11.

(2) Let $\mathcal{Z}$ be a PDF ring and $\mathcal{P} = \mathrm{Prime}$. Take $\mathcal{P}' = \mathcal{P}$ and $\mathcal{E} = \mathcal{P}$. The support finiteness property holds from the definition of PDF ring.

(3) Let $\mathcal{Z}$ be a near UFD and $\mathcal{P} \in \{\mathrm{Irred}, \mathrm{Prime}, \mathrm{Nonunit}\}$. Take $\mathcal{P}' = \mathrm{Prime}$ and $\mathcal{E} = \mathrm{Prime}$. The support finiteness property holds from the definition of near UFDs.

4. Polynomial rings are locally Schinzel: Proof of Theorem 1.6

Here we show Theorem 1.6. We begin with a lemma linking fixed divisors and "actual" divisors over polynomial rings.

Lemma 4.1. *Let $\underline{T} = (T_1, \dots, T_k)$ and $\underline{Y} = (Y_1, \dots, Y_n)$ with $k, n \geq 1$. Let $\mathcal{Z} = \mathcal{R}[U]$ for an infinite domain $\mathcal{R}$. Then every fixed divisor $\pi \in \mathcal{Z}$ of a polynomial $f \in \mathcal{Z}[\underline{T}, \underline{Y}]$ w.r.t. $\underline{T}$ is a divisor of f in $\mathcal{Z}[\underline{T}]$.*

Proof. Without loss of generality, we may assume that f belongs to $\mathcal{Z}[\underline{T}]$. Indeed, $\pi \in \mathcal{Z}$ is a fixed divisor of $f \in \mathcal{Z}[\underline{T}, \underline{Y}]$ w.r.t. $\underline{T}$ if and only it is a fixed divisor w.r.t. $\underline{T}$ of each of its $\mathcal{Z}[\underline{T}]$-coefficients.

Step 1: The case $k = 1$. Write

$$f = a_d(U)T_1^d + \cdots + a_1(U)T_1 + a_0(U) \in \mathcal{Z}[T_1]$$

and suppose $\pi \in \mathcal{Z}$ is a fixed divisor of f w.r.t. T_1. Since $\mathcal{R}$ is infinite, there exists $s \in \mathcal{R}$ such that for the automorphism

$$\varphi_s : U \in \mathcal{Z} \mapsto U - s \in \mathcal{Z},$$

the polynomial $\varphi_s(\pi)$, viewed as a polynomial in U, has no root in

$$\{0\} \cup \{w \in \overline{\mathrm{Frac}(\mathcal{R})} \mid w^i = 1 \quad \text{for some } i = 1, \dots, d\}. \tag{7}$$

Clearly $\varphi_s(\pi)$ is a fixed divisor of $\varphi_s(f)$ w.r.t. T_1. Hence, we may assume without loss of generality that π itself has no root in (7). Since π is fixed divisor, we have

$$f(U^i) = a_d(U)U^{id} + \cdots + a_1(U)U^i + a_0(U) \equiv 0 \pmod{\pi} \quad i = 0, \dots, d.$$

Using Vandermonde determinant, this yields

$$\prod_{0 \leq i < j \leq d} (U^j - U^i)(a_0(U), \dots, a_d(U)) \equiv 0 \pmod{\pi}.$$

We claim that none of the factors $U^j - U^i$ is a zero divisor modulo π. Indeed, suppose

$$U^j(U^{j-i} - 1)g = (U^j - U^i)g = \pi r$$

for some $g, r \in \mathcal{Z}$. Since $0 \leq j - i \leq d$ and π has no root in (7), it follows that π and $U^j(U^{j-i} - 1)$ are coprime in $\mathrm{Frac}(\mathcal{R})[U]$. Hence $U^j(U^{j-i} - 1)$ divides r in $\mathrm{Frac}(\mathcal{R})[U]$, and therefore also in $\mathcal{Z} = \mathcal{R}[U]$ since $U^j(U^{j-i} - 1)$ is monic. Thus $U^j - U^i$ is a not a zero divisor modulo π. It follows that π divides $a_0(U), \dots, a_d(U)$ as desired.

Step 2: The general case. We proceed by induction on k. The case $k = 1$ has been established. Assume now that $k \geq 2$ and let $\underline{T}' = (T_2, \dots, T_k)$, and suppose the statement holds for $1, \dots, k - 1$ variables. Write

$$f = b_\ell(\underline{T}')T_1^\ell + \cdots + b_1(\underline{T}')T_1 + b_0(\underline{T}') \ \ (b_i(\underline{T}') \in \mathcal{Z}[\underline{T}']).$$

Fix $(v_2, \dots, v_k) \in \mathcal{Z}^{k-1}$. Then π is a fixed divisor of $f(T_1, v_2, \dots, v_d)$ w.r.t. T_1. By the one-variable case, π divides each coefficient

$$b_j(v_2, \dots, v_k), \quad j = 1, \dots, d$$

Since $(v_2, \dots, v_k)$ is arbitrary, the induction hypothesis applied to the polynomials $b_j(\underline{T}')$ implies that π divides each $b_j(\underline{T}')$. Hence π divides f. ☐

Building on Lemma 4.1, the following is the crucial ingredient in proving (a more explicit version of) Theorem 1.6.

Proposition 4.2. *Let $\mathcal{R}$ be an infinite domain, let $K := \mathrm{Frac}(\mathcal{R})$, and set $\mathcal{Z} = \mathcal{R}[U]$. Let $k, n, s \geq 1$, $\underline{T} := (T_1, \dots, T_k)$, $\underline{T}' := (T_2, \dots, T_k)$ and $\underline{Y} := (Y_1, \dots, Y_n)$.*
Let $\mathcal{P} \in \{\mathrm{Nonunit}\,\mathcal{Z}, \mathrm{Irred}\,\mathcal{Z}, \mathrm{Prime}\,\mathcal{Z}\}$ be a primality type, and let $P_1(\underline{T}, \underline{Y}), \dots, P_s(\underline{T}, \underline{Y}) \in \mathcal{Z}[\underline{T}, \underline{Y}]$ be polynomials satisfying the following.

i) *$P_1 \cdots P_s$ has no divisor in $\mathcal{P}$.*

ii) *When viewed as elements of* $K[U][\underline{T},\underline{Y}]$*, none of* $P_1,\dots,P_s$ *has a divisor in* $K[U,T_1]\setminus K[U]$.

Then there exists a polynomial $h(U)\in\mathcal{Z}\setminus\{0\}$ *such that for all sufficiently large integers* C *(with the bound depending on* $P_1,\dots,P_s$*), there is a finite set* $S\subset\mathcal{R}$ *fulfilling the following:*
For all $t_1(U)\in\mathcal{Z}$ *of the form*

$$t_1(U)=U^{2C}h(U)g(U)+U^C+\tau,$$

with $g(U)\in\mathcal{Z}$ *arbitrary and* $\tau\in\mathcal{R}\setminus S$*, the polynomial* $\prod_{i=1}^s P_i(t_1(U),\underline{T}',\underline{Y})\in\mathcal{Z}[\underline{T}',\underline{Y}]$ *does not have a divisor in* $\mathcal{P}$*.*

Proof. Let $P:=P_1\cdots P_n$. Let $c_1(U,T_1),\dots,c_N(U,T_1)\in\mathcal{R}[U,T_1]$ be the coefficients of P (viewed as a polynomial in $\underline{T}',\underline{Y}$). By Assumption i), there is no $d\in\mathcal{P}$ dividing all the c_i.

To show the assertion that $\prod_{i=1}^s P_i(t_1(U),\underline{T}',\underline{Y})$ does not have a divisor in $\mathcal{P}$ (for $t_1(U)$ as given above), we distinguish the two cases of $d\in\mathcal{P}\cap\mathcal{R}$ and of $d\in\mathcal{P}\cap(\mathcal{R}[U]\setminus\mathcal{R})$.

Case 1) To treat the case of divisors $d\in\mathcal{P}\cap\mathcal{R}$, choose an integer C larger than the maximal U-degree of $c_i(U,T_1)$, $i=1,\dots,N$. For arbitrary $g(U)\in\mathcal{Z}$ and $\tau\in\mathcal{R}$, set

$$t_1(U)=U^{2C}g(U)+U^C+\tau.$$

Up to replacing T_1 by $T_1':=T_1-\tau$, we may reduce to the case $\tau=0$, and will therefore assume this in the following argument. We claim that $c_1(U,t_1(U)),\dots,c_N(U,t_1(U))$ have no common (non-unit, resp. irreducible, resp. prime, depending on the primality type $\mathcal{P}$) divisor $d\in\mathcal{R}$. Assume on the contrary the existence of such a divisor d. Write $c_i(U,T_1)=\sum_{j=0}^{d_i}\gamma_{i,j}(U)T_1^j$ with $\gamma_{i,j}\in\mathcal{Z}$, and expand

$$c_i(U,t_1(U))=\sum_{j=0}^{d_i}\gamma_{i,j}(U)\cdot(U^{2C}g(U)+U^C)^j.$$

Note that, by choice of C, the terms of $\gamma_{i,0}(U)$ are exactly the ones of degree $<C$ in this expansion, whence d must divide $\gamma_{i,0}$. We may thus replace c_i by $c_i-\gamma_{i,0}(U)$ and now continue iteratively. I.e., we assume that $\gamma_{i,j}(U)$ is divisible by d for all j up to some bound $k<d_i$, and consider the remaining sum $\sum_{j=k+1}^{d_i}\gamma_{i,j}(U)\cdot(U^{2C}g(U)+U^C)^j$. Here, the lowest-degree terms form the sum $U^{C(k+1)}\gamma_{i,k+1}(U)$, allowing us to conclude that d must divide $\gamma_{i,k+1}(U)$ as well, and thus must eventually divide all the $\gamma_{i,j}(U)$. In conclusion, d divides all the c_i. But then d is a divisor of P, contradicting the assumptions of the theorem.

Note that this first case has required no exceptions among the values $\tau\in\mathcal{R}$ and no restrictions on the polynomial $h(U)$ from the assertion.

Case 2) To treat the case of divisors $d\in\mathcal{P}\cap(\mathcal{R}[U]\setminus\mathcal{R})$, denote by $C_1,\dots,C_N$ the affine plane curves over $\overline{K}$ given by the equations $c_1(U,T_1)=0,\dots,c_N(U,T_1)=0$, respectively. We will for the moment make the following extra assumption (*), to which the general case will be reduced:

(*) *The coefficients* $c_1(U,T_1),\dots,c_N(U,T_1)$ *have no non-unit common divisor in* $K[U]$.

Under Assumption (*), it follows that the curves $C_1,\dots,C_N$ do not have a common irreducible component. Indeed, otherwise $c_1(U,T_1),\dots,c_N(U,T_1)$ would have an irreducible common divisor in the UFD $K[U,T_1]$. But if such a divisor has positive T_1-degree, it would divide one of the P_i, contradicting Assumption ii); on the other hand, an irreducible divisor in $K[U]$ would contradict Assumption (*).

By Bézout's theorem, there are then only finitely many points $A_1,\dots,A_m\in\mathbb{A}^2(\overline{K})$ in which all the C_i intersect. Let $h(U)\in\mathcal{Z}\setminus\{0\}$ be such that $h(\alpha)=0$ for all α occurring as the U-coordinate of some A_i, $i=1,\dots,m$. Concretely, one may choose $h(U)$ as the product of minimal polynomials of these α over K, multiplied by a suitable constant in $\mathcal{R}$ to clear denominators. Next, given any $C\in\mathbb{N}$, let $S(=S(C))\subset\mathcal{R}$ be the finite set of values $\beta\in\mathcal{R}$ such that some A_i, $i=1,\dots,m$, has (U,T_1)-coordinates $(\alpha,\alpha^C+\beta)$, $\alpha\in\overline{K}$.

For $\tau\in\mathcal{R}\setminus S$ and for $\tilde{g}(U)\in\mathcal{Z}$ arbitrary, set $t_1(U)=\tilde{g}(U)h(U)+U^C+\tau$, and define the curve C_{N+1} via $C_{N+1}:T_1=t_1(U)$. By definition of S and τ, C_{N+1} does not pass through any of the points A_i, $i = 1, \dots, m$. This implies that $c_1(U, t_1(U)), \dots, c_N(U, t_1(U))$ have no common non-constant divisor $d \in \mathcal{R}[U] \setminus \mathcal{R}$, since indeed the roots u of such a divisor in $\overline{K}$ would by definition induce simultaneous intersection points $(u, t_1(u))$ of the curves $C_1, \dots, C_{N+1}$.

General situation of 2). Clearly, the set of values $t_1(U)$ given in the assertion of the proposition fulfills the requirements of both the sets given in Cases 1) and 2). To finish the proof of the proposition, it thus remains to treat the general situation of 2), i.e., without assuming the additional condition (*).

Assume to this end that $\delta(U) \in K[U]$ is the greatest common divisor (necessarily of T_1-degree 0 due to Assumption ii)) of the coefficients $c_1(U, T_1), \dots, c_N(U, T_1)$, viewed as elements of $K[U, T_1]$. Set $\tilde{c}_i := c_i/\delta \in K[U, T_1]$. Since the $\tilde{c}_i$ satisfy Assumption (*), we may conclude from the above (applied with K in place of $\mathcal{R}$) that the $\tilde{c}_i(U, t_1(U))$, $i = 1, \dots, N$, have no nonunit common divisor in $K[U]$ as soon as $t_1(U)$ is of the form

$$t_1(U) = U^{2C} g(U) h(U) + U^C + \tau \tag{8}$$

(for $g(U) \in K[U]$ arbitrary, $h(U) \in K[U]$ suitable, and $\tau \in K$ away from a suitable finite set depending on C). In particular, there is an infinite subfamily of values $t_1(U)$ satisfying (8) and additionally lying in $\mathcal{R}[U]$.[6]

Fix $t_1(U) \in \mathcal{R}[U]$ as above. Assume on the contrary that $c_1(U, t_1(U)), \dots, c_N(U, t_1(U))$ have a common divisor $d(U) \in \mathcal{R}[U] \setminus \mathcal{R}$. It follows from the $\tilde{c}_i(U, t_1(U))$, $i = 1, \dots, N$ having no nonunit common divisor in $K[U]$ that $d(U)$ divides $\delta(U)$ in $K[U]$. By Assumption i), there exists $1 \leq i_0 \leq N$ such that $c_{i_0}(U, T_1)$ is not divisible in $\mathcal{R}[U, T_1]$ by $d(U)$. As in Case 1), up to applying a shift in the variable T_1, we may and will assume $\tau = 0$ for the following. Then

$$c_{i_0}(U, t_1(U)) = \sum_{j=0}^{d_{i_0}} \gamma_{i_0,j}(U) \left(U^{2C} g(U) h(U) + U^C\right)^j.$$

For each $0 \leq j \leq d_{i_0}$, since $\delta(U)$ divides $\gamma_{i_0,j}(U)$ and $d(U)$ divides $\delta(U)$ in $K[U]$, we have

$$\tilde{\gamma}_{i_0,j}(U) := \frac{\gamma_{i_0,j}(U)}{d(U)} \in K[U].$$

Then

$$\sum_{j=0}^{d_{i_0}} \tilde{\gamma}_{i_0,j}(U) \left(U^{2C} g(U) h(U) + U^C\right)^j = \frac{c_{i_0}(U, t_1(U))}{d(U)} \in \mathcal{R}[U].$$

By the same argument as Case 1), one shows inductively that, for every $0 \leq j \leq d_{i_0}$, one has

$$\tilde{\gamma}_{i_0,j}(U) \in \mathcal{R}[U].$$

It follows that $d(U)$ divides $c_{i_0}(U, T_1)$, a contradiction. □

Proof of Theorem 1.6. By Proposition 2.17, since $\mathcal{Z} = \mathcal{R}[U]$ is a Hilbertian ring, the integrally Hilbertian property is implied by the locally Schinzel property, i.e., it suffices to prove the latter. When $\mathcal{R}$ is a finite domain, it is a finite field, and $\mathcal{Z} = \mathcal{R}[U]$ is a PID, hence a locally Schinzel domain as well. We may and will therefore assume that $\mathcal{R}$ is infinite. Let $\mathcal{Q} = \mathrm{Frac}(\mathcal{Z})$, let $\mathcal{P} \in \{\mathrm{Nonunit}\mathcal{Z}, \mathrm{Irred}\mathcal{Z}, \mathrm{Prime}\mathcal{Z}\}$ be a primality type, and $P_1(\underline{T}, \underline{Y}), \dots, P_s(\underline{T}, \underline{Y})$ be polynomials satisfying conditions (Prim/$\mathcal{Q}[\underline{T}]$) and (NoFixDiv/$\mathcal{Z}[\underline{T}]$)$_{\mathcal{P}}$ from §2.2. Let $P := P_1 \cdots P_s$. Due to Condition (NoFixDiv/$\mathcal{Z}[\underline{T}]$)$_{\mathcal{P}}$, P is in particular not divisible by any element in $\mathcal{P}$, and due to Condition (Prim/$\mathcal{Q}[\underline{T}]$), none of $P_1, \dots, P_s$ is divisible by any element in $\mathrm{Frac}(\mathcal{R})[T_1, U] \setminus \mathrm{Frac}(\mathcal{R})[U]$. By Proposition 4.2, there exist full arithmetic progressions $\omega\mathcal{Z} + \alpha \subset \mathcal{Z}$ of values $t_1(U)$ for which the polynomial $P(t_1(U), \underline{T}', \underline{Y}) \in \mathcal{Z}[\underline{T}', \underline{Y}]$ is not divisible by any element in $\mathcal{P}$. Concretely, Proposition 4.2 gives $\omega = U^{2C} h(U)$ for suitably large $C \in \mathbb{N}$ and suitable $h(U) \in \mathcal{Z} \setminus \{0\}$, and $\alpha = U^C + \tau$ for all $\tau \in \mathcal{R}$ away from a finite set (depending on C). By Lemma 4.1, this implies that $P_1(t_1(U), \underline{T}', \underline{Y}), \dots, P_s(t_1(U), \underline{T}', \underline{Y})$ satisfy (NoFixDiv/$\mathcal{Z}[\underline{T}']$)$_{\mathcal{P}}$.

[6] One should only multiply $h(U) \in K[U]$ by a suitable constant in $\mathcal{R}$ in order to replace it by a polynomial in $\mathcal{R}[U]$, which is possible due to $g(U)$ being arbitrary.

Moreover, condition (Prim/$\mathcal{Q}[\underline{T}]$) implies in particular that the coefficients w.r.t. $\underline{Y}$ of $P_1(\underline{T},\underline{Y}),\ldots,P_s(\underline{T},\underline{Y})$, viewed as elements of $\mathcal{Q}(T_1)[\underline{T}']$, are coprime, thus satisfying Condition ii) of [BD23, Proposition 3.1], applied with $\underline{a}=T_1$ and $\underline{x}=\underline{T}'$. Due to the equivalence ii)⇔iii) of [BD23, Proposition 3.1], there exists a finite subset $Z\subseteq\mathcal{Q}$ such that for all $t_1\in\mathcal{Q}\setminus Z$, the specializations of the above coefficients are coprime as elements of $\mathcal{Q}[\underline{T}']$, i.e., the polynomials $P_1(t_1,\underline{T}',\underline{Y}),\ldots,P_s(t_1,\underline{T}',\underline{Y})$ satisfy (Prim/$\mathcal{Q}[\underline{T}']$). In particular, up to increasing the lower bound on the exponent $C\in\mathbb{N}$ above, we may assume that this conclusion holds for all the values $t_1(U)\in\omega\mathcal{Z}+\alpha$ exhibited above. As a consequence, $\mathcal{Z}$ is locally Schinzel. □

5. Polynomial Schinzel rings

§5.1 introduces Schinzel rings (Definition 5.1). This notion recasts in a common environment the celebrated Schinzel Hypothesis (thanks to Theorem 5.2) and our results on polynomial rings. These are Theorem 5.3 and Theorem 5.4, which are more precise forms of Theorem 1.7. They are stated and commented on in §5.2 and in §5.3. Their proofs are postponed to Section 6. The primality type is tacitly taken to be $\mathcal{P}=$ Nonunit throughout the section.

5.1. **Schinzel rings.**

5.1.1. *Fields with a product formula.* Recall from [FJ23, §17.3] that a field $\mathcal{Q}$ equipped with a nonempty set S of primes $\mathfrak{p}$, with associated absolute value $|\ |_{\mathfrak{p}}$, is said to satisfy the *product formula* if for each $\mathfrak{p}\in S$, there exists $\beta_{\mathfrak{p}}>0$ such that:

(*) *For each $a\in\mathcal{Q}^\times$, the set $\{\mathfrak{p}\in S\mid |a|_{\mathfrak{p}}\neq 1\}$ is finite and $\prod_{\mathfrak{p}\in S}|a|_{\mathfrak{p}}^{\beta_{\mathfrak{p}}}=1$.*

The field $\mathbb{Q}$ is a typical example: the product formula is: $\prod_p |a|_p\cdot|a|=1$ for every $a\in\mathbb{Q}^*$, where p ranges over all prime numbers, $|\cdot|_p$ is the p-adic absolute value and $|\cdot|$ is the standard absolute value. Rational function fields $\kappa(u_1,\ldots,u_r)$ in $r\geq 1$ variables over an arbitrary field κ, and finite extensions of fields with a product formula are other examples [FJ23, §17.3].

From a result of Weissauer, fields with a product formula, of characteristic 0 or imperfect, are Hilbertian fields [FJ23, Theorem 17.3.3]. A domain $\mathcal{Z}$, of characteristic 0 or imperfect, such that the fraction field $\mathcal{Q}$ has a product formula is a Hilbertian ring [BDN20, Theorem 4.6].

Associated to a product formula, relative to a set S of primes, comes a natural height on $\mathcal{Q}$ – the *Weil height* – which we denote by H_S and is defined as follows: for every $a\in\mathcal{Q}^\times$,

$$H_S(a)=\prod_{\mathfrak{p}\in S}\max(1,|a|_{\mathfrak{p}})^{\beta_{\mathfrak{p}}}.$$

Clearly $H_S(a^n)=H_S(a)^n$ ($n\in\mathbb{N}$) and $H_S(1/a)=H_S(a)$ if $a\neq 0$. When $\mathcal{Q}=\mathbb{Q}$, H_S is the usual absolute value. On the field $\kappa(\underline{Y})$, each variable Y_i induces a partial degree Weil height: $H_i(\cdot)=2^{\deg_{u_i}(\cdot)}$, $i=1,\ldots,n$.

5.1.2. *Definition of Schinzel rings.*

Definition 5.1. Let $\mathcal{Z}$ be domain such that the fraction field $\mathcal{Q}$ is equipped with several sets of primes $S_1,\ldots,S_n$ satisfying the product formula. Let $H_1,\ldots,H_n$ be the corresponding Weil heights. The ring $\mathcal{Z}$ is called a *Schinzel ring w.r.t. the heights* $H_1,\ldots,H_n$ if the following holds. Let $\underline{P}$ be a finite set of polynomials $P_1,\ldots,P_s\in\mathcal{Z}[\underline{T}]$, irreducible in $\mathcal{Q}[\underline{T}]$ and such that the product $P_1\cdots P_s$ has no fixed divisor in $\mathcal{Z}$ w.r.t. $\underline{T}$. Let A be a positive real number. Then the following subset of $\mathcal{Z}^k$ is Zariski-dense:

$$\mathcal{S}_{\mathcal{Z}}(\underline{P},A)=\left\{\underline{t}=(t_1,\ldots,t_k)\in\mathcal{Z}^k\ \middle|\ \begin{array}{l}P_1(\underline{t}),\ldots,P_s(\underline{t})\text{ are irreducible in }\mathcal{Z},\text{and}\\ H_i(t_j)\geq A,\ \text{for all } j=1,\ldots,k\text{ and } i=1,\ldots,n.\end{array}\right\}.$$

The original Schinzel Hypothesis corresponds to the special case $\mathcal{Z}=\mathbb{Z}$ and $k=1$, but is in fact equivalent to the full definition of "$\mathbb{Z}$ is a Schinzel ring", as shown by Theorem 5.2 below, which already appeared in [KK25, Lemma 2.3] for $s=1$ and $\mathcal{Z}=\mathbb{Z}$, and here is proved in §6.4. Also note that for $\mathcal{Z}=\mathbb{Z}$, Definition 5.1 is unchanged if one drops the height condition in the definition of $\mathcal{S}_{\mathcal{Z}}(\underline{P},A)$. This is not the case however for $\mathcal{Z}=\kappa[Y_1,\ldots,Y_n]$.

Theorem 5.2. *Let $\mathcal{Z}$ be a Krull domain such that $\mathcal{Q}$ is equipped with several sets of primes $S_1, \ldots, S_n$ satisfying the product formula. Assume that $\mathcal{Z}$ satisfies the special $k = 1$ parameter case of Definition 5.1. Then $\mathcal{Z}$ is a Schinzel ring in the full sense, i.e. for several parameters.*

In this context of Schinzel rings, we have the following results:
(a) (stated as Theorem 1.7). *If $\mathcal{Z}$ is an integrally Hilbertian ring, then the polynomial ring $\mathcal{Z}[\underline{Y}]$ in $n \geq 1$ variables is a Schinzel ring w.r.t. the partial degrees Weil heights $\deg_{Y_i}(\cdot)$, $i = 1, \ldots, n$.*
(b) (stated as Corollary 1.9). *If $\mathcal{Z}$ is an arbitrary domain, the polynomial ring $\mathcal{Z}[Y_0, Y_1, \ldots, Y_n]$ in $n+1 \geq 2$ variables is a Schinzel ring w.r.t. the partial degree Weil heights $\deg_{Y_i}(\cdot)$, $i = 0, 1, \ldots, n$.*
Result (a) will be established as a weak form of Theorem 5.3, which is stated in §5.3 and proved in §6.1. Result (b) is proved in §6.3.

5.2. **Effective polynomial Schinzel Hypothesis.** Theorem 5.3 goes beyond the Schinzel ring property stated in Theorem 1.7. Its statement and proof are expressed in pure polynomial terms.

5.2.1. *Statement of Theorem 5.3.* Given an n-tuple $\underline{d} = (d_1, \ldots, d_n)$ of nonnegative integers, denote the number of monic monomials in $\underline{Y}$ of degree $\leq d_j$ in Y_j ($j = 1, \ldots, n$) by $\ell(\underline{d})$:

$$\ell(\underline{d}) = \prod_{j=1}^{n}(d_j + 1),$$

and denote the set of polynomials $M \in \mathcal{Z}[\underline{Y}]$ such that $\deg_{Y_j}(M) \leq d_j$ ($j = 1, \ldots, n$) by $\mathcal{P}ol_{\mathcal{Z},n,\underline{d}}$. We view it as the $\ell(\underline{d})$-dimensional affine space over $\mathcal{Z}$: the coordinates correspond to the coefficients.

Theorem 5.3. *Assume that $\mathcal{Z}$ is an integrally Hilbertian ring. Let $P_1, \ldots, P_s \in \mathcal{Z}[\underline{T}, \underline{Y}]$ be $s \geq 1$ polynomials, irreducible in $\mathcal{Q}[\underline{T}, \underline{Y}]$, and such that the product $P_1 \cdots P_s$ has no nonunit divisor in $\mathcal{Z}$. For each $i = 1, \ldots, k$, let $\underline{d}_i = (d_{i1}, \ldots, d_{in})$ be an n-tuple of nonnegative integers. Assume that*

(1) $\deg_{\underline{Y}}(P_l) \geq 1$ *for* $l = 1, \ldots, s$ *or* $\underline{d}_i \neq (0, \ldots, 0)$ *for* $i = 1, \ldots, k$.

Consider the subset

$$\mathcal{S}_{\mathcal{Z}[\underline{Y}]}(\underline{P}, \underline{d}_1, \ldots, \underline{d}_n) \subset \mathcal{P}ol_{\mathcal{Z},n,\underline{d}_1} \times \cdots \times \mathcal{P}ol_{\mathcal{Z},n,\underline{d}_k}$$

consisting of all k-tuples $(M_1(\underline{Y}), \ldots, M_k(\underline{Y}))$ such that

(2) $\quad P_i\,(M_1(\underline{Y}), \ldots, M_k(\underline{Y}), Y_1, \ldots, Y_n)$ *is irreducible in* $\mathcal{Z}[\underline{Y}]$, $i = 1, \ldots, s$.

Then we have the following three conclusions:

(a) *Assume further that*

(3.a) $\quad \ell(\underline{d}_i) > \sum_{j=1}^{s} \deg_{T_i}(P_j), \quad i = 1, \ldots, k.$

Then the subset $\mathcal{S}_{\mathcal{Z}[\underline{Y}]}(\underline{P}, \underline{d}_1, \ldots, \underline{d}_n)$ is Zariski-dense in $\mathcal{P}ol_{\mathcal{Z},n,\underline{d}_1} \times \cdots \times \mathcal{P}ol_{\mathcal{Z},n,\underline{d}_k}$.

(b) *If $\mathcal{Z}$ is a near UFD, then the same holds with* (3.a) *replaced by*

(3.b) $\quad 2^{\ell(\underline{d}_i)} > \sum_{j=1}^{s} \deg_{T_i}(P_j), \quad i = 1, \ldots, k.$

(c) *If $k = 1$ and $\underline{d}_1 = \underline{d} = (d_1, \ldots, d_n)$, the same holds with* (3.a) *replaced by*

(3.c) $\quad \sum_{j=1}^{n} d_j > \sum_{i=1}^{s} \deg_{\underline{Y}}(P_i).$

and, if additionally $\mathcal{Z}$ is a near UFD, by

(3.c') $\quad \sum_{j=1}^{n} d_j > \max_{1 \leq i \leq s} \deg_{\underline{Y}}(P_i).$

5.2.2. *Remarks.* (a) The original Schinzel Hypothesis situation is excluded by assumption (1). Indeed, in that situation, $P_1, \ldots, P_s$ are polynomials in T and $\underline{d} = (0, \ldots, 0)$, thus (1) fails.

(b) *The bounds in Theorem 5.3(3.b) are sharp.* Take $k = n = 1$ for simplicity. Given any integer $d \geq 0$, we construct below an irreducible polynomial $P \in \mathbb{Z}[T, Y]$ such that $2^{\ell(d)} = \deg_T(P)$ and $P(M(Y), Y)$ is reducible in $\mathbb{Z}[Y]$ for all polynomials $M \in \mathbb{Z}[Y]$ of degree d.

Denote the subset of all polynomials in $\mathbb{Z}[Y]$ of degree $\leq d$ and with coefficients 0 or 1 by $\mathcal{M}_d$. We have $\text{card}(\mathcal{M}_d) = 2^{d+1}$. Consider the following polynomial:

$$P_0(T,Y) = \prod_{p(Y)\in\mathcal{M}_d} (T - p(Y)).$$

Then take for P the polynomial

$$P(T,Y) = P_0(T,Y) + 2m,$$

where $m \in \mathbb{Z}$ is chosen such that P is irreducible in $\mathbb{Z}[T,Y]$. To see that such an integer m exists, apply Theorem 5.3(3.b) with $k=1$, $n=2$, $s=1$, with P_1 taken to be $Q(U,T,Y) = P_0(T,Y) + 2U$, with the parameter taken to be U and $\underline{Y} = (T,Y)$, and with $\underline{d} = (0,0)$. The polynomial Q is irreducible in $\mathbb{Z}[U,T,Y]$; condition (1) holds since $\deg_{T,Y}(Q) \geq 1$; and $\ell(0,0) = (0+1)(0+1) = 1$ so $2^{\ell(0,0)} = 2 > \deg_U(Q) = 1$. Therefore one can indeed take $m \in \mathbb{Z}[T,Y]$ such that $\deg_T(m) = \deg_Y(m) = 0$ and $Q(m,T,Y)$ irreducible in $\mathbb{Z}[T,Y]$.

We have $\deg_T(P) = 2^{d+1} = 2^{\ell(d)}$ and, by construction, for every polynomial $M(Y) \in \mathbb{Z}[Y]$ of degree d, we have $M(Y) \equiv p(Y) \pmod 2$ for some $p(Y) \in \mathcal{M}_d$ and so the polynomial $P(M(Y),Y)$ is divisible by 2.

(c) The bounds in Theorem 5.3(3.c) only depend on the degrees in $\underline{Y}$ of the polynomials $P_1,\ldots,P_s$. They can be interesting in certain situations, for example when $P_1,\ldots,P_s$ only depend on $\underline{T}$. The second bound (3.c') improves on the one given in [BDN20, Theorem 1.1].

5.2.3. *Principle of the proof.* The full proof of Theorem 5.3 is given in §6.1. The aimed explicitness of the statement may somehow hide the leading principle. Assume $k=1$ for simplicity. In each of the polynomials $P_1,\ldots,P_s$ of Theorem 5.3, replace the parameter T by the generic polynomial $M_{\underline{d}}(\underline{Y})$ of degree $\leq d_j$ in Y_j $(j=1,\ldots,n)$:

$$M_{\underline{d}}(\underline{Y}) = \sum_{\underline{r}=(r_1,\ldots,r_n)} \lambda_{\underline{r}}\, Y_1^{r_1}\cdots Y_n^{r_n};$$

where the sum ranges over all n-tuples $\underline{r} = (r_1,\ldots,r_n)$ with $r_j \leq d_j$ $(j=1,\ldots,n)$. The word "generic" means that the coefficients $\lambda_{\underline{j}}$ are indeterminates. The resulting polynomials

$$F_i = P_i\left(M_{\underline{d}}(\underline{Y}),\underline{Y}\right) \quad (i=1,\ldots,s)$$

are polynomials in $\mathcal{Z}[\underline{\Lambda},\underline{Y}]$, where $\underline{\Lambda}$ is the set consisting of all the indeterminates $\lambda_{\underline{j}}$. We will show that, if the integers $d_1,\ldots,d_n$ are suitably large, the polynomials $F_1,\ldots,F_s$ are irreducible in $\mathcal{Q}[\underline{\Lambda},\underline{Y}]$, of degree ≥ 1 in $\underline{Y}$ (Lemma 6.2), and that their product $\prod_{i=1}^s F_i$ has no fixed divisor w.r.t. $\underline{\Lambda}$ (Lemma 6.5). We will then be able to use the integrally Hilbertian property to specialize the indeterminates from $\underline{\Lambda}$ and conclude the proof.

5.3. **Local condition preservation.** As indicated in Section 1, our central Schinzel-type result – Theorem 1.7 – has a second part showing that, under appropriate assumptions, absence of fixed divisors can also be preserved by specialization. Theorem 5.4 is a more precise version.

Theorem 5.4. *Assume that $\mathcal{Z}$ is an integrally Hilbertian ring that is a near UFD or a Krull domain. Let $P_1(\underline{T},\underline{Y}),\ldots,P_s(\underline{T},\underline{Y}) \in \mathcal{Z}[\underline{T},\underline{Y}]$ be s polynomials, irreducible in $\mathcal{Q}[\underline{T},\underline{Y}]$, and such that $P_1\cdots P_s$ has no fixed divisor w.r.t. the $(k+n)$-tuple $(\underline{T},\underline{Y})$ in* Nonunit $\mathcal{Z}$.

(a) *Let A be a positive real number. Then the subset $\mathcal{S}^{\sharp}_{\mathcal{Z}[\underline{Y}]}(\underline{P},A) \subset \mathcal{Z}[\underline{Y}]^k$ of all k-tuples $\underline{M}(\underline{Y}) = (M_1(\underline{Y}),\ldots,M_k(\underline{Y}))$ such that*

(i) *$P_i(\underline{M}(\underline{Y}),\underline{Y})$ is irreducible in $\mathcal{Z}[\underline{Y}]$, $i=1,\ldots,s$,*

(ii) *the product $P_1(\underline{M}(\underline{Y}),\underline{Y})\cdots P_s(\underline{M}(\underline{Y}),\underline{Y})$ has no fixed divisor w.r.t. $\underline{Y}$ in* Nonunit $\mathcal{Z}$,

(iii) $\deg_{Y_j}(M_i) > A$, *$j=1,\ldots,n$, $i=1,\ldots,s$,*

is Zariski-dense.

(b) *For $k=n=1$,* (a) *holds with $M(Y)$ further required to be monic and* (iii) *replaced by*

(iii)' $\deg_Y(M_i) = d_i$, *$i=1,\ldots,s$, where $d_1,\ldots,d_s$ are arbitrary prescribed integers $d_i \geq 1$.*

6. Proofs of Schinzel-type results

§6.1, §6.2, §6.3, §6.4 respectively provide the proofs of Theorem 5.3, Theorem 5.4, Corollary 1.9, Theorem 5.2.

6.1. **Proof of Theorem 5.3.**

6.1.1. *Setup of the proof.* Fix s polynomials $P_1, \dots, P_s \in \mathcal{Z}[\underline{T}, \underline{Y}]$. For each index $i = 1, \dots, k$, let $\underline{Q_i} = (Q_{i0}, Q_{i1}, \dots, Q_{i\ell_i})$, with $Q_{i0} = 1$, be a $(\ell_i + 1)$-tuple of nonzero polynomials in $\mathcal{Z}[\underline{Y}]$, distinct up to multiplicative constants in $\mathcal{Q}^\times$ and let $\underline{\lambda_i} = (\lambda_{i0}, \lambda_{i1}, \dots, \lambda_{i\ell_i})$ $(\ell_i \geq 0)$ be a corresponding tuple of indeterminates. We consider here a more general than necessary to prove Theorem 5.3 for which one can take $\underline{Q_i}$ consisting of *all monic monomials* of degree $\leq d_{ij}$ in Y_j $(j = 1, \dots, n)$, and then $\ell_i + 1 = \ell(\underline{d_i})$, $i = 1, \dots, k$ (see Remark 6.4).

Assume that all the indeterminates $\lambda_{i,l}$, $i = 1, \dots, k$, $l = 0, \dots, \ell_i$, are algebraically independent and denote by $\underline{\Lambda}$ the tuple obtained by concatenating all tuples $\underline{\lambda_1}, \dots, \underline{\lambda_k}$. Consider then the polynomials:

$$M_{\underline{Q_i}}(\underline{\lambda_i}, \underline{Y}) = \sum_{l=0}^{\ell_i} \lambda_{i,l}\, Q_{i,l} \in \mathcal{Z}[\underline{\Lambda}, \underline{Y}], \quad i = 1, \dots, k.$$

Finally replace in each of the original polynomials $P_1, \dots, P_s$ each parameter T_i by $M_{\underline{Q_i}}(\underline{\lambda_i}, \underline{Y})$ $(i = 1, \dots, k)$. The resulting polynomials

$$F_i = P_i\left(M_{\underline{Q_1}}(\underline{\lambda_1}, \underline{Y}), \dots, M_{\underline{Q_k}}(\underline{\lambda_k}, \underline{Y}), Y_1, \dots, Y_n\right), \quad i = 1, \dots, s$$

are polynomials in $\mathcal{Z}[\underline{\Lambda}, \underline{Y}]$.

6.1.2. *Irreducibility of $F_1, \dots, F_s$.* The goal is Lemma 6.2. We start with Lemma 6.1. Fix $i \in \{1, \dots, s\}$ and, for simplicity, write $\underline{\lambda}$ for $\underline{\lambda_i}$, ℓ for ℓ_i and $\underline{Q}$ for $\underline{Q}_i$. Let $\underline{U}$ be a tuple of new variables and let $A = \mathcal{Q}[\underline{U}]$ (with $\underline{U}$ possibly empty in which case $A = \mathcal{Q}$).

Lemma 6.1. *Let*

$$P(T, \underline{Y}) = P_\rho(\underline{Y}) T^\rho + \cdots + P_1(\underline{Y}) T + P_0(\underline{Y}) \ \ (P_j(\underline{Y}) \in A[\underline{Y}], j = 1, \dots, \rho)$$

be an irreducible polynomial in $A[T, \underline{Y}]$ of degree ρ. Assume that either $\deg_{\underline{Y}}(P) \geq 1$ *or* ($\rho \geq 1$ *and* $\ell \geq 1$). *Then $G(\underline{\lambda}, \underline{Y}) = P(M_{\underline{Q}}(\underline{\lambda}, \underline{Y}), \underline{Y})$ is irreducible in $A[\underline{\lambda}, \underline{Y}]$ and of degree ≥ 1 in $\underline{Y}$.*

Proof. The irreducibility conclusion follows similarly to [BDN20, Lemma 2.1]. Consider the ring automorphism $\varphi : A[\underline{\lambda}, \underline{Y}] \to A[\underline{\lambda}, \underline{Y}]$ that is the identity on $A[\lambda_1, \dots, \lambda_\ell, \underline{Y}]$ and maps λ_0 to the polynomial $\lambda_0 + \sum_{i=1}^{\ell} \lambda_i Q_i(\underline{X})$. Since $P(\lambda_0, \underline{Y})$ is irreducible in $A[\lambda_0, \underline{Y}]$, it is also irreducible in $A[\underline{\lambda}, \underline{Y}]$. Hence $G(\underline{\lambda}, \underline{Y}) = \varphi(P(\lambda_0, \underline{Y}))$ is irreducible in $A[\underline{\lambda}, \underline{Y}]$. Regarding the degree conclusion, assume first $\ell \geq 1$. As a polynomial in λ_1, the leading coefficient of G is $P_\rho(\underline{Y}) Q_1(\underline{Y})^\rho$. If $\rho \geq 1$, it is of positive degree in $\underline{Y}$ since $Q_1(\underline{Y})$ is. If $\rho = 0$, then $\deg_{\underline{Y}}(P) \geq 1$, so $G(\underline{\lambda}, \underline{Y}) = P_0(\underline{Y}) = P(T, \underline{Y})$ is of positive degree in $\underline{Y}$. Finally consider the case $\ell = 0$. We then have $G(\underline{\lambda}, \underline{Y}) = P(\lambda_0, \underline{Y})$. The result follows since, in this case, we also have $\deg_{\underline{Y}}(P) \geq 1$. □

Fix now $l \in \{1, \dots, s\}$ and denote P_l and F_l by P and F in the following lemma.

Lemma 6.2. *Assume that P is irreducible in $\mathcal{Q}[\underline{T}, \underline{Y}]$ and satisfies*

(1) $\deg_{\underline{Y}}(P) \geq 1$ *or* $\ell_i \geq 1$ *for* $i = 1, \dots, k$.

Then the polynomial F is irreducible in $\mathcal{Q}[\underline{\Lambda}, \underline{Y}]$ and of degree ≥ 1 in $\underline{Y}$.

Proof. Set $\underline{T}' = (T_2, \dots, T_k)$ and view P as in $\mathcal{Z}[\underline{T}'][T_1, \underline{Y}]$ (with $\underline{T}'$ the empty tuple if $k = 1$). We distinguish two cases.

1st case: $\deg_{\underline{Y}}(P) \geq 1$. Apply Lemma 6.1 to the polynomial P, $T = T_1$, $\underline{Q} = \underline{Q_1}$ and A taken to be $\mathcal{Q}[\underline{T}']$ (i.e., $\underline{U} = \underline{T}'$). The resulting polynomial, say F_1, is a polynomial in $\mathcal{Z}[\underline{T}'][\underline{\lambda_1}, \underline{Y}]$, which is irreducible in $\mathcal{Q}[\underline{T}'][\underline{\lambda_1}, \underline{Y}]$ and such that $\deg_{\underline{Y}}(F_1) \geq 1$. The same lemma can in turn be applied to this polynomial F_1, viewed in $\mathcal{Q}[\underline{\lambda_1}, \underline{T}''][T_2, \underline{Y}]$, where $\underline{T}'' = (T_3, \dots, T_k)$, and with

$T = T_2$ and $\underline{Q} = \underline{Q_2}$. Proceeding inductively with each of the parameters $T_1, \ldots, T_k$ leads to the conclusion that F is irreducible in $\mathcal{Q}[\underline{\Lambda}, \underline{Y}]$ and of degree ≥ 1 in $\underline{Y}$.

2nd case: $\deg_{\underline{Y}}(P) = 0$. Due to assumption (1), we then have $\ell_i \geq 1$ for $i = 1, \ldots, k$. The polynomial P is in $\mathcal{Q}[\underline{T}]$ and is irreducible in $\mathcal{Q}[\underline{T}]$. View it in $\mathcal{Q}[\underline{T}'][T_1]$. If $\deg_{T_1}(P) = 0$, there is no specialization of T_1 to be performed, or, in other words, any specialization of T_1 to $M_{\underline{Q_1}}(\underline{\lambda_1}, \underline{Y})$ leaves the polynomial P unchanged. As P is not a constant polynomial, there is an index $i_0 \in \{1, \ldots, s\}$ such that $\deg_{T_{i_0}}(P) \geq 1$, and we may choose i_0 to be minimal. After specializations of $T_1, \ldots, T_{i_0-1}$ that have left P unchanged, we wish to specialize T_{i_0}. Lemma 6.1 can be applied to the polynomial P, viewed in $\mathcal{Q}[T_{i_0+1}, \ldots, T_k][T_{i_0}, \underline{Y}]$, with $T = T_{i_0}$, $\underline{Q} = \underline{Q_{i_0}}$ and $\ell = \ell_{i_0} \geq 1$, with as a result a polynomial, say $F_{i_0} \in \mathcal{Z}[T_{i_0+1}, \ldots, T_k][\underline{\lambda_{i_0}}, \underline{Y}]$, irreducible in $\mathcal{Q}[T_{i_0+1}, \ldots, T_k][\underline{\lambda_{i_0}}, \underline{Y}]$ and of degree ≥ 1 in $\underline{Y}$. Then, for the next parameter T_{i_0+1}, the argument of the 1st case can be applied to F_{i_0}, viewed in $\mathcal{Q}[T_{i_0+2}, \ldots, T_k, \lambda_{i_0}][T_{i_0+1}, \underline{Y}]$, and then again the same argument can be applied for the remaining parameters to finally lead to the desired conclusion that F is irreducible in $\mathcal{Q}[\underline{\Lambda}, \underline{Y}]$ and of degree ≥ 1 in $\underline{Y}$. □

6.1.3. *Partial conclusion.* Assume that all our polynomials $P_1, \ldots, P_s$ satisfy condition (1) from Lemma 6.2, i.e., we have:

(1/all) $\deg_{\underline{Y}}(P_l) \geq 1$ for $l = 1, \ldots, s$ or $\ell_i \geq 1$ for $i = 1, \ldots, k$.

Then, from Lemma 6.2, the polynomials $F_1, \ldots, F_s$ satisfy conditions (Irred/$\mathcal{Q}(\underline{\Lambda})$) and (Prim/$\mathcal{Q}[\underline{\Lambda}]$) from Section 2.2. By definition of "integrally Hilbertian", one obtains this conclusion.

Proposition 6.3. *Assume that $\mathcal{Z}$ is integrally Hilbertian. Let $P_1, \ldots, P_s \in \mathcal{Z}[\underline{T}, \underline{Y}]$, irreducible in $\mathcal{Q}[\underline{T}, \underline{Y}]$ and satisfying condition* (1/all). *If in addition, the polynomials $F_1, \ldots, F_s$ satisfy condition* (NoFixDiv/$\mathcal{Z}[\underline{\Lambda}]$) from Section 2.2*, then the set of k-tuples of polynomials*

$$M_{\underline{Q_i}}(\underline{\theta_i}, \underline{Y}) = \sum_{l=0}^{\ell_i} \theta_{i,l}\, Q_{i,l} \in \mathcal{Z}[\underline{Y}], \quad (i = 1, \ldots, k)$$

(corresponding to some $(\ell_i + 1)$-tuples $(\theta_{i,0}, \ldots, \theta_{i,\ell_i}) \in \mathcal{Z}^{\ell_i+1}$)

such that

each polynomial $P_i\,(M_{\underline{Q_1}}(\underline{\theta_1}, \underline{Y}), \ldots, M_{\underline{Q_k}}(\underline{\theta_k}, \underline{Y}), \underline{Y})$ is irreducible in $\mathcal{Z}[\underline{Y}]$, $i = 1, \ldots, s$,

is Zariski-dense in $\mathcal{P}ol_{\mathcal{Z},n,\underline{d}_1} \times \cdots \times \mathcal{P}ol_{\mathcal{Z},n,\underline{d}_k}$.

Remark 6.4. In §6.1.4, we show that condition (NoFixDiv/$\mathcal{Z}[\underline{\Lambda}]$) holds in the situation of Theorem 5.3 for which the lists $\underline{Q_1}, \ldots, \underline{Q_k}$ consist of all monomials of bounded degrees, and so can conclude the proof. It may be interesting to consider other situations corresponding to other lists of monomials; one needs then to check condition (NoFixDiv/$\mathcal{Z}[\underline{\Lambda}]$) for these lists. An extreme example of this is when $\underline{Q_i}$ consists of the single constant monomial $Q_{i0} = 1$ $(i = 1, \ldots, k)$. Condition (NoFixDiv/$\mathcal{Z}[\underline{\Lambda}]$) is then assumption (NoFixDiv/$\mathcal{Z}[\underline{T}]$) from §2.2 and Proposition 6.3 is Definition 2.11.

6.1.4. *The No Fixed Divisor condition* (NoFixDiv/$\mathcal{Z}[\underline{\Lambda}]$). The situation is that of Theorem 5.3. For each $i = 1, \ldots, k$, an n-tuple $\underline{d_i} = (d_{i1}, \ldots, d_{in})$ of nonnegative integers is given and $\underline{Q_i}$ is the tuple of all monic monomials of degree at most some given integer d_{ij} in Y_j (in some order with $Q_{i0} = 1$); we then have $\ell_i + 1 = \ell(\underline{d_i}) = \prod_{j=1}^{n}(d_{ij} + 1)$. In particular, $\ell_i \geq 1$ exactly corresponds to $\underline{d_i} \neq (0, \ldots, 0)$, so condition ((1/all)) from §6.1.3 corresponds to condition (1) from Theorem 5.3.

Set $\Pi = P_1 \cdots P_s$ and use the abbreviation $\underline{M_{\underline{Q}}}(\underline{\Lambda}, \underline{Y})$ for the k-tuple

$$\underline{M_{\underline{Q}}}(\underline{\Lambda}, \underline{Y}) = (M_{\underline{Q_1}}(\underline{\lambda_1}, \underline{Y}), \ldots, M_{\underline{Q_k}}(\underline{\lambda_k}, \underline{Y})).$$

Then we have $F_1 \cdots F_s = \Pi\,(\underline{M_{\underline{Q}}}(\underline{\Lambda}, \underline{Y}), \underline{Y})$. Similarly, given some ring A and some ring morphism $\mathcal{Z} \to A$, for every $\underline{\Theta} = (\underline{\theta_1}, \ldots, \underline{\theta_k}) \in \prod_{i=1}^{k} A^{\ell_i+1}$, we denote by $\underline{M_{\underline{Q}}}(\underline{\Theta}, \underline{Y})$ the k-tuple of polynomials in $A[\underline{Y}]$ obtained from $\underline{M_{\underline{Q}}}(\underline{\Lambda}, \underline{Y})$ by specializing the indeterminates in $\underline{\Lambda}$ to $\underline{\Theta}$.

Lemma 6.5. *Assume that* $P_1,\ldots,P_s \in \mathcal{Z}[\underline{T},\underline{Y}]$ *are irreducible in* $\mathcal{Q}[\underline{T},\underline{Y}]$*, the product* $P_1\cdots P_s$ *has no nonunit divisor in* $\mathcal{Z}$ *and condition* (1) *from Theorem 5.3 holds. Then, under the various assumptions of statements* (a),(b),(c) *of* Theorem 5.3*, the polynomials* $F_1,\ldots,F_s$ *satisfy condition* (NoFixDiv/$\mathcal{Z}[\underline{\Lambda}]$)*, i.e., the product* $F_1\cdots F_s$ *has no fixed divisor w.r.t.* $\underline{\Lambda}$ *in* Nonunit $\mathcal{Z}$*.*

Proof. Let p be a nonunit of $\mathcal{Z}$, $p\neq 0$. By assumption, the polynomial $\Pi(\underline{T},\underline{Y})$ is nonzero in $(\mathcal{Z}/p\mathcal{Z})[\underline{T},\underline{Y}]$. We start with case (b) of Theorem 5.3.

Case (b)*:* $\mathcal{Z}$ *is a near UFD and* (3.b) *holds:* $2^{\ell(\underline{d}_i)} > \sum_{j=1}^s \deg_{T_i}(P_j)$ $(i=1,\ldots,k)$. As $\mathcal{Z}$ is a near UFD, one may assume that p is a prime. For each $i=1,\ldots,k$, the subset

$$\{M_{\underline{Q_i}}(\underline{\overline{\theta_i}},\underline{Y})\,|\,\underline{\overline{\theta_i}}\in(\mathcal{Z}/p\mathcal{Z})^{\ell_i+1}\}\subset(\mathcal{Z}/p\mathcal{Z})[\underline{Y}],$$

is of cardinality $|\mathcal{Z}/p\mathcal{Z}|^{\ell_i+1}\geq 2^{\ell_i+1}$. Assumpiion (3.b) rewrites as $2^{\ell_i+1}>\deg_{T_i}(\Pi)$, $i=1,\ldots,k$. Since $\mathcal{Z}/p\mathcal{Z}$ is a domain, it follows that there exists a k-tuple

$$\underline{M_Q}(\underline{\overline{\Theta}},\underline{Y})=(M_{\underline{Q_1}}(\underline{\overline{\theta_1}},\underline{Y}),\ldots,M_{\underline{Q_k}}(\underline{\overline{\theta_k}},\underline{Y}))\in((\mathcal{Z}/p\mathcal{Z})[\underline{Y}])^k$$

such that $\Pi(\underline{M_Q}(\underline{\overline{\Theta}},\underline{Y}),\underline{Y})$ is nonzero in $(\mathcal{Z}/p\mathcal{Z})[\underline{Y}]$. Lifting the $\underline{\overline{\theta_i}}$ to some elements of $\mathcal{Z}$ provides a point $\underline{\Theta}=(\underline{\theta_1},\ldots,\underline{\theta_k})\in\prod_{i=1}^k\mathcal{Z}^{\ell_i+1}$ such that

$$\underline{M_Q}(\underline{\Theta},\underline{Y})=(M_{\underline{Q_1}}(\underline{\theta_1},\underline{Y}),\ldots,M_{\underline{Q_k}}(\underline{\theta_k},\underline{Y}))\in(\mathcal{Z}[\underline{Y}])^k$$

has the property that $\Pi(\underline{M_Q}(\underline{\Theta},\underline{Y}),\underline{Y})$ is not divisible by p. Conclude that $\underline{\Theta}$ is a special value of $\underline{\Lambda}$ that makes the polynomial $\Pi(\underline{M_Q}(\underline{\Lambda},\underline{Y}),\underline{Y})$ nonzero modulo p, i.e. p is not a fixed divisor of $F_1\cdots F_s$ w.r.t. $\underline{\Lambda}$.

Case (a)*:* $\mathcal{Z}$ *is integrally Hilbertian and* (3.a) *holds:* $\ell(\underline{d}_i)>\sum_{j=1}^s\deg_{T_i}(P_j)$ *$(i=1,\ldots,k)$.* The ring $\mathcal{Z}/p\mathcal{Z}$ is no longer integral in general. We will however be able to adjust the argument by working with a smaller subset of tuples of polynomials $\underline{M_Q}(\underline{\Theta},\underline{Y})$ than the one used in Case (b).

Specifically, for each $i=1,\ldots,k$, consider the following subset:

$$\mathcal{S}_i=\{Q_{il}\,|\,l=0,\ldots,\ell_i\}\subset\mathcal{Z}[\underline{Y}].$$

We claim that the corresponding elements, regarded in $(\mathcal{Z}/p\mathcal{Z})[\underline{Y}]$, have this property: any difference between two distinct such elements is not a zero divisor in $(\mathcal{Z}/p\mathcal{Z})[\underline{Y}]$. Namely, if $P\in(\mathcal{Z}/p\mathcal{Z})[\underline{Y}]$ is nonzero and Q is the largest monomial in P for the lexicographical order, then the largest monomial in the polynomial $Q_{il}P$ is $Q_{il}Q$. For two different $Q_{il},Q_{il'}\in\mathcal{S}_i$, we have $Q_{il}Q\neq Q_{il'}Q$ and so $Q_{il}P\neq Q_{il'}P$, which proves the claim. Deduce next that the product of all nonzero differences $(Q_{il}-Q_{il'})$ with $Q_{il},Q_{il'}\in\mathcal{S}_i$ is not a zero divisor in $(\mathcal{Z}/p\mathcal{Z})[\underline{Y}]$.

It follows from this conclusion that a nonzero polynomial in one variable of degree $\leq\ell_i$ and with coefficients in $(\mathcal{Z}/p\mathcal{Z})[\underline{Y}]$ cannot vanish at every one of the ℓ_i+1 elements Q_{il} with $l=0,\ldots,\ell_i$. Indeed the determinant of the linear system corresponding to the ℓ_i+1 vanishing conditions – a Van Der Monde determinant – is precisely the product of differences in $(\mathcal{Z}/p\mathcal{Z})[\underline{Y}]$ considered above. By construction it is not a zero divisor in $(\mathcal{Z}/p\mathcal{Z})[\underline{Y}]$, and so only the zero polynomial could be a solution to the system, which is excluded.

Conclude: as $\ell_i=\ell(\underline{d}_i)-1\geq\deg_{T_i}(\Pi)$ for each $i=1,\ldots,k$, there exist $l_i\in\{0,\ldots,\ell_i\}$ $(i=1,\ldots,k)$ such that the k-tuple $\underline{Q_l}=(Q_{1l_1},\ldots,Q_{kl_k})\in(\mathcal{Z}[\underline{Y}])^k$ satisfies the following.

$$\Pi(\underline{Q_l},\underline{Y})\text{ is not divisible by }p.$$

Conclude as in case (b) that p is not a fixed divisor of $F_1\cdots F_s$ w.r.t. $\underline{\Lambda}$.

Case (c)*:* $\mathcal{Z}$ *integrally Hilbertian,* $k=1$ *and* (3.c) *holds* $\sum_{j=1}^n d_j>\sum_{i=1}^s\deg_{\underline{Y}}(P_i)$*.* Write T for T_1, $\underline{\lambda}$ for $\underline{\lambda_1}$ and ℓ for $\ell(\underline{d})-1$. We may assume that $r=\deg_T(\Pi)\geq 1$. Set

$$\Pi=P_1\ldots P_s=Z_0(\underline{Y})+Z_1(\underline{Y})T+\cdots+Z_r(\underline{Y})T^r.$$

Assume on the contrary that $F_1\cdots F_s$ has a fixed divisor w.r.t. $\underline{\lambda}$, i.e., there is a nonunit $p\in\mathcal{Z}$, $p\neq 0$, such that

$$(F_1\cdots F_s)(\underline{m},\underline{Y})\equiv 0\pmod p\ \text{ for all }\underline{m}\in\mathcal{Z}^{\ell+1}.$$

In particular, for $\underline{m}$ corresponding to the monomial $Y_1^{d_1}\dots Y_n^{d_n}$, we obtain:

$$Z_0(\underline{Y}) + Z_1(\underline{Y})Y_1^{d_1}\dots Y_n^{d_n} + Z_2(\underline{Y})Y_1^{2d_1}\dots Y_n^{2d_n} + \cdots + Z_r(\underline{Y})Y_1^{rd_1}\dots Y_n^{rd_n} \equiv 0 \pmod p.$$

We claim that for every $0 \le i < j \le r$ such that $Z_i(\underline{Y})$ and $Z_j(\underline{Y})$ are nonzero, the nonzero monomials appearing in $Z_i(\underline{Y})Y_1^{id_1}\dots Y_n^{id_n}$ are different from the ones in $Z_j(\underline{Y})Y_1^{jd_1}\dots Y_n^{jd_n}$. Indeed for nonzero monomials A and B in $Z_i(\underline{Y})Y_1^{id_1}\dots Y_n^{id_n}$ and $Z_j(\underline{Y})Y_1^{jd_1}\dots Y_n^{jd_n}$ respectively, we have:

$$\begin{aligned}
\deg_{\underline{Y}}(A) &\le \deg_{\underline{Y}}(Z_i) + i\left(\sum_{i=1}^n d_i\right)\\
&\le \deg_{\underline{Y}}(\Pi) + i\left(\sum_{i=1}^n d_i\right)\\
&= \sum_{i=1}^s \deg_{\underline{Y}}(P_i) + i\left(\sum_{i=1}^n d_i\right)\\
&< \left(\sum_{i=1}^n d_i\right) + i\left(\sum_{i=1}^n d_i\right) \quad \text{(by assumption)}\\
&\le j\left(\sum_{i=1}^n d_i\right) \le \deg_{\underline{Y}}(B).
\end{aligned}$$

Thus $A \ne B$. Hence, for each $i = 1, \dots, r$, we have $Z_i(\underline{Y})Y_1^{i\cdot d_1}\dots Y_n^{i\cdot d_n} \equiv 0 \pmod p$, and so $Z_i(\underline{Y}) \equiv 0 \pmod p$. Therefore p divides Π, a contradiction.

Finally, assume as in the special case of Theorem 5.3(c) that $\mathcal{Z}$ *is a near UFD and that* (3.c') *holds:* $\sum_{j=1}^n d_j > \max_{1\le i\le s}\deg_{\underline{Y}}(P_i)$. If p is a fixed divisor of $F_1\cdots F_s$ w.r.t. $\underline{\lambda}$, then $\Pi(Y_1^{d_1}\dots Y_n^{d_n}, \underline{Y}) \equiv 0 \pmod p$. Without loss of generality, one may assume that p is prime. Since $(\mathcal{Z}/p\mathcal{Z})[\underline{Y}]$ is an integral domain, there exists $1 \le j \le s$ such that

$$P_j(Y_1^{d_1}\dots Y_n^{d_n}, \underline{Y}) \equiv 0 \pmod p.$$

In the second part of the argument above, with P_j replacing $P_1\cdots P_s$, we conclude that $P_j \equiv 0 \pmod p$, a contradiction. □

6.1.5. *End of proof of Theorem 5.3.* Lemmas 6.2 and 6.5 guarantee that Proposition 6.3 can be applied under the assumptions of Theorem 5.3. Its conclusion yields the requested conclusion of Theorem 5.3.

6.1.6. *Final comments.*

(a) For statement (b) and second part of statement (c) of Theorem 5.3, for which $\mathcal{Z}$ is assumed to be a near UFD, only the condition (2.4-2) – every nonunit has at least one prime divisor – is used in the proof. Thus this sole condition can more generally replace the near UFD assumption.

(b) The argument for Case (b) also works in the general case of an integrally Hilbertian ring *provided that* the local condition is strengthened to assume that the product $P_1\cdots P_s$ has no divisor in $\operatorname{Spec}^*\mathcal{Z}$ (instead of just $\operatorname{Nonunit}\mathcal{Z}$). Hence the better bound from Case (b) also holds in the general Case (a) under this stronger local condition.

The argument should be adjusted as follows. Starting from a non unit $p \in \mathcal{Z}$, $p \ne 0$, consider a maximal ideal $\mathfrak{p} \subset \mathcal{Z}$ containing p. By the stronger local condition, the polynomial $\Pi(\underline{T}, \underline{Y})$ is nonzero in $(\mathcal{Z}/\mathfrak{p})[\underline{T}, \underline{Y}]$. Then, one can work as we did in Case (b) with $\mathfrak{p}$ replacing $p\mathcal{Z}$ to construct a point $\underline{\Theta} = (\underline{\theta_1}, \dots, \underline{\theta_k}) \in \prod_{i=1}^k \mathcal{Z}^{\ell_i+1}$ such that

$$\underline{M}_{\underline{Q}}(\underline{\Theta}, \underline{Y}) = (M_{\underline{Q_1}}(\underline{\theta_1}, \underline{Y}), \dots, M_{\underline{Q_k}}(\underline{\theta_k}, \underline{Y})) \in (\mathcal{Z}[\underline{Y}])^k$$

$\Pi(\underline{M}_{\underline{Q}}(\underline{\Theta}, \underline{Y}), \underline{Y})$ is not divisible by $\mathcal{P}$, and so *a fortiori* not by p.

6.2. **Proof of Theorem 5.4.**

General setting. Since the primality type Nonunit $\mathcal{Z}$ is equivalent to:

$$\mathcal{P} = \begin{cases} \text{Irred}\,\mathcal{Z} & \text{if } \mathcal{Z} \text{ is a Krull domain,} \\ \text{Prime}\,\mathcal{Z} & \text{if } \mathcal{Z} \text{ is a near UFD,} \end{cases}$$

as before, we can carry out the proof using the primality $\mathcal{P}$. Recall from Proposition 3.8 that $\mathcal{Z}$ satisfies the properties (NVA) and (SF) w.r.t. $\mathcal{P}$ for the following choice of $\mathcal{E}$ in condition (2) of Definition 3.5:

$$\mathcal{E} = \begin{cases} \text{Spec}^{(1)}(\mathcal{Z}) & \text{if } \mathcal{Z} \text{ is a Krull domain,} \\ \text{Prime}\,\mathcal{Z} & \text{if } \mathcal{Z} \text{ is a near UFD.} \end{cases}$$

Without loss of generality, we may assume that $\deg_{\underline{T}}(P_1 \cdots P_s) \geq 1$. Write

$$P_1 \cdots P_s = a_0 \underline{T}^{\underline{w}_0} \underline{Y}^{\underline{z}_0} + \cdots + a_t \underline{T}^{\underline{w}_t} \underline{Y}^{\underline{z}_t},$$

with $a_0 \neq 0$, and

$$(\underline{w}_0, \underline{z}_0) > \cdots > (\underline{w}_t, \underline{z}_t)$$

in lexicographic order $T_1 \succ T_2 \succ \cdots \succ T_k \succ Y_1 \succ \cdots \succ Y_n$. Choose nonzero tuples $\underline{d}_1, \ldots, \underline{d}_k \in \mathbb{N}^n$ such that, for every $c = 1, \ldots, t$,

$$\sum_{i=1}^{n} \left(\sum_{r=1}^{k} w_{c,r} d_{r,i} + z_{c,i} \right) < \sum_{i=1}^{n} \left(\sum_{r=1}^{k} w_{0,r} d_{r,i} + z_{0,i} \right). \tag{9}$$

In case (b): $k = n = 1$ of Theorem 5.4, it is easy to see that $\underline{d}_1 \in \mathbb{N}$ can be any integer ≥ 1. In the general case (a), we choose $\underline{d}_1, \ldots, \underline{d}_k \in \mathbb{N}^n \cap]A, +\infty[^n$ so that this condition is satisfied (such a choice is easily seen to be possible).

Set $\Delta := \sum_{i=1}^{n} \left(\sum_{r=1}^{k} w_{0,r} d_{r,i} + z_{0,i} \right)$. Define

$$\begin{cases} \mathcal{S} = (\text{Supp}(a_0) \cap \mathcal{E}) \cup (\Gamma_\Delta \cap \mathcal{E}) \\ \Omega = \Sigma(\mathcal{S})_{\mathcal{E}} \cap \mathcal{P} \end{cases}$$

By Lemma 3.9 and the support finiteness property, $\mathcal{S}$ is finite; hence Ω is finite. Since $P_1 \cdots P_s$ has no fixed divisor w.r.t. $(\underline{T}, \underline{Y})$ in $\mathcal{P}$, by the nonvanishing approximation property, there exists $(\underline{\theta}, \underline{\alpha}) \in \mathcal{Z}^{k+n}$ such that

$$(P_1 \cdots P_s)(\underline{\theta}, \underline{\alpha}) \not\equiv 0 \pmod{\mathfrak{p}}, \tag{10}$$

for all $\mathfrak{p} \in \Omega$. Let $\omega \in \mathcal{Z}$ be a generator of the principal ideal $\prod_{\mathfrak{p} \in \Omega} \mathfrak{p}$, which is contained in $\bigcap_{\mathfrak{p} \in \Omega} \mathfrak{p}$.

Claim 4. *(1) If an ideal $\mathfrak{q} \in \mathcal{E}$ divides $a_0 \omega^\ell$, then $\mathfrak{q} \in \mathcal{S}$.*

(2) If an ideal $\mathfrak{p} \in \mathcal{P}$ divides $a_0 \omega^\ell$ for some $\ell \geq 0$, then $\mathfrak{p} \in \Omega$.

Proof of Claim 4. We have:

$$(\text{Supp}(a_0) \cup \text{Supp}(\omega)) \cap \mathcal{E} = (\text{Supp}(a_0) \cap \mathcal{E}) \bigcup \left(\bigcup_{\mathfrak{p} \in \Omega} \text{Supp}(\mathfrak{p}) \cap \mathcal{E} \right) \subset \mathcal{S}.$$

Assertions (1) and (2) follow then immediately from the definitions; for (2), note that

$$\text{Supp}(\mathfrak{p}) \cap \mathcal{E} \subset (\text{Supp}(a_0) \cup \text{Supp}(\omega)) \cap \mathcal{E} \subset \mathcal{S}.$$

□

For $r = 1, \ldots, k$, consider all monomials $Q_{r,0}, \ldots, Q_{r,\ell_r}$ of degree $\leq d_{r,j}$ in Y_j ($j = 1, \ldots, n$), where $Q_{r,0} = 1$ and $Q_{r,\ell_r} = Y_1^{d_{r,1}} \cdots Y_n^{d_{r,n}}$. For each $r = 1, \ldots, k$, fix a tuple $\underline{\lambda}_r = (\lambda_{r,0}, \ldots, \lambda_{r,\ell_r - 1})$ of indeterminates, and define

$$R_r(\underline{\lambda}_r, \underline{Y}) = Y_1^{d_{r,1}} Y_2^{d_{r,2}} \cdots Y_n^{d_{r,n}} + \sum_{i=0}^{\ell_r - 1} \lambda_{r,i} Q_{r,i}. \tag{11}$$

be the corresponding generic polynomial. Set

$$\underline{\lambda} = (\underline{\lambda}_1, \dots, \underline{\lambda}_k).$$

For each $i = 1, \dots, s$, define

$$F_i(\underline{\lambda}, \underline{Y}) = P_i\left(\theta_1 + \omega \cdot R_1(\underline{\lambda}_1, \underline{Y}), \dots, \theta_k + \omega \cdot R_k(\underline{\lambda}_k, \underline{Y}), \underline{Y}\right).$$

Claim 5. *The coefficient of* $\prod_{i=1}^n Y_i^{\sum_{r=1}^k w_{0,r} d_{r,i} + z_{0,i}}$ *in* $F_1 \cdots F_s$ *equals* $a_0 \omega^\Gamma$, *where* $\Gamma := \sum_{i=1}^n w_{0,i}$.

Proof. This easily follows from expanding $F_1 \cdots F_s$ using (9). □

Intermediate step. We wish to show that the polynomials $F_1, \dots, F_s$ satisfy the conditions (Irred/$\mathcal{Q}(\underline{\lambda})$), (Prim/$\mathcal{Q}[\underline{\lambda}]$) and (NoFixDiv/$\mathcal{Z}[\underline{\lambda}]_{\mathcal{P}}$).

Claim 6. *For every* $i = 1, \dots, s$, *the polynomial* F_i *is irreducible in* $\mathcal{Q}[\underline{\lambda}, \underline{Y}]$.

Proof of Claim 6. The proof is the same as that of Lemma 6.2. Fix $i \in \{1, \dots, s\}$. Consider the ring automorphism $\varphi : \mathcal{Q}[\underline{\lambda}, \underline{Y}] \to \mathcal{Q}[\underline{\lambda}, \underline{Y}]$ that is the identity on $\mathcal{Q}[\underline{\lambda} \setminus \{\lambda_{r,0} \mid r = 1, \dots, k\}, \underline{Y}]$ and, for $r = 1, \dots, k$, maps $\lambda_{r,0}$, to the polynomial $\theta_r + \omega \cdot R_r(\underline{\lambda}_r, \underline{Y})$. Since $P_i(\lambda_{1,0}, \dots, \lambda_{k,0})$ is irreducible in $\mathcal{Q}[\lambda_{1,0}, \dots, \lambda_{k,0}, \underline{Y}]$, it remains irreducible in $\mathcal{Q}[\underline{\lambda}, \underline{Y}]$. Therefore,

$$F_i(\underline{\lambda}, \underline{Y}) = \varphi(P_i(\lambda_{1,0}, \dots, \lambda_{k,0}))$$

is also irreducible in $\mathcal{Q}[\underline{\lambda}, \underline{Y}]$. □

Claim 7. *The product* $F_1 \cdots F_s$ *has no fixed divisor w.r.t.* $\underline{\lambda}$ *in* $\mathcal{P}$.

Proof of Claim 7. Assume on the contrary that $F_1 \cdots F_s$ has a fixed divisor $\mathfrak{p} \in \mathcal{P}$ w.r.t. $\underline{\lambda}$, that is,

$$(F_1 \cdots F_s)(\underline{m}, \underline{Y}) \equiv 0 \pmod{\mathfrak{p}},$$

for all $\underline{m} \in \mathcal{Z}^\ell$. In particular, $(F_1 \cdots F_s)(\underline{0}, \underline{Y}) \equiv 0 \pmod{\mathfrak{p}}$. By Claim 5, $\mathfrak{p}$ divides $a_0 \omega^\Gamma$. Therefore, by Claim 4, $\mathfrak{p} \in \Omega$ and $\mathfrak{p}$ divides ω. This implies that $(P_1 \dots P_s)(\underline{\theta}, \underline{\alpha}) \equiv (F_1 \dots F_s)(\underline{0}, \underline{\alpha}) \equiv 0 \pmod{\mathfrak{p}}$, contradicting (10). Consequently, $F = F_1 \cdots F_s$ has no fixed divisor w.r.t. $\underline{\lambda}$ in $\mathcal{P}$. □

End of proof of Theorem 5.4. Since $\mathcal{Z}$ is integrally Hilbertian, for

$$\underline{v} = (\underline{v}_1, \dots, \underline{v}_k)$$

in a Zariski-dense set $\mathcal{H} \subset \mathcal{Z}^{\ell_1 + \dots + \ell_k}$, the polynomials

$$G_i(\underline{Y}) = F_i(\underline{v}, \underline{Y}) = P_i(\theta_1 + \omega \cdot R_1(\underline{v}_1, \underline{Y}), \dots, \theta_k + \omega \cdot R_k(\underline{v}_k, \underline{Y}), \underline{Y}) \in \mathcal{Z}[\underline{Y}] \quad i = 1, \dots, s$$

are irreducible in $\mathcal{Z}[\underline{Y}]$. For $\underline{v} \in \mathcal{H}$, Let

$$\underline{M}_{\underline{v}}(\underline{Y}) = (M_{\underline{v},1}(\underline{Y}), \dots, M_{\underline{v},k}(\underline{Y})) = (\theta_1 + \omega \cdot R_1(\underline{v}, \underline{Y}), \dots, \theta_k + \omega \cdot R_k(\underline{v}, \underline{Y})) \in \mathcal{Z}[\underline{Y}]^k.$$

To finish the proof of Theorem 5.4, it remains to show the following.

Claim 8. *The product* $G_1 \cdots G_s$ *has no fixed divisor w.r.t.* $\underline{Y}$ *in* $\mathcal{P}$.

Proof of Claim 8. Assume on the contrary that $G_1 \cdots G_s$ has a fixed divisor $\mathfrak{p} \in \mathcal{P}$ w.r.t. $\underline{Y}$.

First we wish to show that $\mathrm{Supp}(\mathfrak{p}) \cap \mathcal{E} \subset \mathcal{S}$. Let $\mathfrak{q} \in \mathrm{Supp}(\mathfrak{p}) \cap \mathcal{E}$. Note that $\mathfrak{q}$ is a prime ideal and is a fixed divisor of $G_1 \cdots G_s$ w.r.t. $\underline{Y}$. By Lemma 3.10, either $|\mathcal{Z}/\mathfrak{q}| \le \max_{1 \le j \le n} \deg_{Y_j}(G_1 \cdots G_s)$ or $\mathfrak{q} \in \mathrm{Div}_{\mathcal{Z}}(G_1 \cdots G_s)$. In the first case,

$$|\mathcal{Z}/\mathfrak{q}| \le \max_{1 \le j \le n} \deg_{Y_j}(G_1 \cdots G_s) \le \sum_{i=1}^n \left(\sum_{r=1}^k m_{0,r} d_{r,i} + z_{0,i} \right) = \Delta,$$

so $\mathfrak{q} \in \mathcal{S}$. Suppose now we are in the second case, that is, $\mathfrak{q}$ divides $G_1 \cdots G_s$. By Claim 5, $\mathfrak{q}$ divides $a_0 \omega^\Gamma$. Therefore, by Claim 4, we have $\mathfrak{q} \in \mathcal{S}$. Consequently, $\mathrm{Supp}(\mathfrak{p}) \cap \mathcal{E} \subset \mathcal{S}$.

The conclusion above implies that $\mathfrak{p} \in \Omega$. In particular, $\mathfrak{p}$ divides ω. Therefore

$$(P_1 \cdots P_s)(\underline{\theta}, \underline{\alpha}) \equiv (G_1 \cdots G_s)(\underline{\alpha}) \equiv 0 \pmod{\mathfrak{p}},$$

contradicting (10). Consequently, $G_1 \cdots G_s$ has no fixed divisor in $\mathcal{P}$ w.r.t. $\underline{Y}$. □

6.3. **Proof of Corollary 1.9.** Consider the ring $\mathcal{Z}[Y_0, Y_1, \ldots, Y_n]$ ($n \geq 1$) equipped with the $(n+1)$ Weil heights induced by the partial degrees $\deg_{Y_i}(\cdot)$, $i = 0, 1, \ldots, n$. Set

$$\underline{Y}^+ = (Y_0, Y_1, \ldots, Y_n) \text{ and } \underline{Y} = (Y_1, \ldots, Y_n).$$

According to Definition 5.1, what is to be proved is this:

(**) *given $s \geq 1$ polynomials $P_1, \ldots, P_s \in \mathcal{Z}[\underline{Y}^+][\underline{T}]$, irreducible in $\mathcal{Q}(\underline{Y}^+)[\underline{T}]$ and such that the product $P_1 \cdots P_s$ has no nonunit divisor in $\mathcal{Z}[\underline{Y}^+]$, and given any constant $A > 0$, the subset $\mathcal{S}_{\mathcal{Z}}(\underline{P}, A) \subset \mathcal{Z}[\underline{Y}^+]^k$ consisting of all k-tuples $(M_1(\underline{Y}^+), \ldots, M_k(\underline{Y}^+))$ of polynomials such that*

(i) $P_i(\underline{Y}^+, M_1(\underline{Y}^+), \ldots, M_k(\underline{Y}^+))$ *is irreducible in* $\mathcal{Z}[\underline{Y}^+]$ *($i = 1, \ldots, s$),*

(ii) $\deg_{Y_i}(M_j) > A$, $i = 0, 1, \ldots, n$, $j = 1, \ldots, k$.

is Zariski-dense.

The idea is to apply Theorem 5.3 in the special case that the integrally Hilbertian ring there is $\mathcal{Z}[Y_0]$. Except for requirement (ii) with $i = 0$, Theorem 5.3 in this special case, gives a more precise conclusion than (**) for which the degrees $\deg_{Y_i}(M_j)$ ($i = 1, \ldots, n$, $j = 1, \ldots, k$) can be any integers d_{ij} provided that these integers are suitably large. The whole desired conclusion (**) follows then from a strengthening relative to the variable Y_0 of Theorem 5.3 for $\mathcal{Z}[Y_0]$.

More specifically, Theorem 5.3 asserts that the set, denoted there by

$$\mathcal{S}_{\mathcal{Z}[Y_0][\underline{Y}]}(\underline{P}, \underline{d}_1, \ldots, \underline{d}_n)$$

is Zariski-dense (under appropriate assumptions on $\underline{d}_1, \ldots, \underline{d}_n$). This subset is viewed as a subset of some affine space $\mathbb{A}^N$ of N-tuples (by viewing the polynomials $M_i(\underline{Y}^+)$ as the tuples of their coefficients). The strengthening we need is to be able to guarantee that, for any prescribed real number $A > 0$, if the coordinates (in $\mathcal{Z}[Y_0]$) of the N-tuples are also required to be of degree $> A$ in Y_0, the resulting subset of $\mathcal{S}_{\mathcal{Z}[Y_0][\underline{Y}]}(\underline{P}, \underline{d}_1, \ldots, \underline{d}_n)$ remains Zariski-dense.

As Proposition 6.3 recapitulates, the proof of Theorem 5.3 shows that the N-tuples in question (those in the set $\mathcal{S}_{\mathcal{Z}[Y_0][\underline{Y}]}(\underline{P}, \underline{d}_1, \ldots, \underline{d}_n)$) come from an application of the integral Hilbertian property to some polynomials. Thus, the question becomes whether one can guarantee that the integral Hilbertian property of $\mathcal{Z}[Y_0]$ (which is Theorem 1.7) is true in this more refined version — that the "good" specializations can be required to be of degree $> A$ in Y_0.

This final check should be done in two steps because integral Hilbertianity of $\mathcal{Z}[Y_0]$ is proved via the locally Schinzel property. Regarding the latter, the needed extra requirement is indeed guaranteed, by Proposition 4.2, which produces, as required for the locally Schinzel property, a certain arithmetic progression $(\omega\ell + \alpha)_{\ell \in \mathcal{Z}[Y_0]}$, with $\omega, \alpha \in \mathcal{Z}[Y_0]$, and shows further that ω can be chosen of degree $> A$ in Y_0. It remains to check that in the proof of Proposition 2.17 (deducing integral Hilbertianity from the locally Schinzel property), the produced "good" tuples can also be taken of degree $> A$ in Y_0: this is clear as each of their coordinates can be picked in an infinite subset of some arithmetic progression as above.

6.4. **Proof of Theorem 5.2.** Denote by $H_1, \ldots, H_n$ the Weil heights corresponding to the sets of primes $S_1, \ldots, S_n$ respectively. Let $P_1(\underline{T}), \ldots, P_s(\underline{T}) \in \mathcal{Z}[\underline{T}]$ be polynomials that are irreducible in $\mathcal{Q}[\underline{T}]$ and such that $P_1 \cdots P_s$ has no fixed divisor w.r.t. $\underline{T}$. As in the proof of Theorem 5.4, for each $r = 1, \ldots, k$, consider the tuple of variables $\underline{\lambda}_r = (\lambda_{r,0}, \ldots, \lambda_{r,\ell_r - 1})$ (with $\ell_r \geq 1$), and define the polynomial

$$M_r(\underline{\lambda}_r, Y) = \theta_r + \omega \cdot R_r(\underline{\lambda}_r, Y),$$

where R_r is as in (11) and ω as defined in the general setting, just before Claim 4. Set $\underline{\lambda} = (\underline{\lambda}_1, \ldots, \underline{\lambda}_k)$, and define

$$\underline{M}(\underline{\lambda}, Y) = (M_1(\underline{\lambda}_1, Y), \ldots, M_k(\underline{\lambda}_k, Y)).$$

As in the proof of Theorem 5.4, for $\underline{m} = (\underline{m}_1, \ldots, \underline{m}_k)$ in a Zariski-dense subset $\mathcal{H} \subset \mathcal{Z}^{\ell_1} \times \cdots \times \mathcal{Z}^{\ell_r} = \mathcal{Z}^{\ell_1 + \cdots + \ell_r}$, each polynomial $P_j(\underline{M}(\underline{m}, Y))$ is irreducible in $\mathcal{Q}[Y]$, and the product

$$\prod_{j=1}^{s} P_j(\underline{M}(\underline{m}, Y))$$

has no fixed divisor w.r.t. Y.

For each $\underline{m} \in \mathcal{H}$, there exists a constant $c_{\underline{m}} > 0$ such that

$$H_i(M_r(\underline{m}, x)) \geq c_{\underline{m}} H_i(x)^{\deg_Y(M_r(\underline{m},Y))}, \tag{12}$$

for all $r = 1, \ldots, k$, all $i = 1, \ldots, n$, and all $x \in \mathcal{Z}$.

Fix $A > 0$. Choose $B > 0$ such that $H_i(x) \geq B$ implies $c_{\underline{m}} H_i(x)^{\deg_Y(M_r(\underline{m},Y))} \geq A$ for all $r = 1, \ldots, k$ and all $i = 1, \ldots, n$. Since $\mathcal{Z}$ is is assumed to satisfy the 1-parameter case of the Schinzel property, for each $\underline{m} \in \mathcal{H}$ there exists an infinite set $K_{\underline{m},B} \subset \mathcal{Z} \cap \bigcap_{i=1}^n \{H_i \geq B\}$ such that for every $x \in K_{\underline{m},B}$, each polynomial $P_j(\underline{M}(\underline{m}, x))$ is irreducible in $\mathcal{Z}$.

For each $\underline{m} \in \mathcal{H}$, note that the set

$$L_{\underline{m},A} := \{x \in K_{\underline{m},B} \mid H_i(M_r(\underline{m}_r, x) \geq A, \quad \text{for all } r = 1, \ldots, k \text{ and } i = 1, \ldots, n\}$$

is infinite. It remains to prove that

$$R_A := \bigcup_{\underline{m} \in \mathcal{H}} E_{\underline{m},A} \subset \mathcal{Z}^k \cap \bigcap_{i=1}^n \{H_i \geq A\}$$

is Zariski-dense, where

$$E_{\underline{m},A} = \{\underline{M}(\underline{m}, x) \mid x \in L_{\underline{m},A}\}.$$

For that, let $Q(\underline{X}) \in \mathcal{Q}[\underline{X}]$ be a polynomial that vanishes on R_A. For every $\underline{m} \in \mathcal{H}$, since $L_{\underline{m},A}$ is infinite, it follows that

$$Q(\underline{M}(\underline{m}, Y))$$

is zero as a polynomial in Y. As $\mathcal{H}$ is Zariski-dense, we deduce that

$$Q(\underline{M}(\underline{\lambda}, Y)) = 0.$$

Since $0 = \varphi^{-1}(Q(\underline{M}(\underline{\lambda}, Y))) = Q(\lambda_{1,0}, \ldots, \lambda_{k,0})$–where φ is as in the proof of Claim 6)–we obtain $Q = 0$. Consequently R_A is Zariski-dense.

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