RESEARCH PROJECT
ADVANCES IN INVERSE GALOIS THEORY

PIERRE DÈBES

Abstract. The prevailing theme of the project is *Inverse Galois Theory* (IGT): realizing certain algebraic systems with a given Galois structure. Our target is the situation over the field $\mathbb{Q}$ of rationals. Beyond the fundamental *Inverse Galois Problem* (IGP) – realize Galois extensions of $\mathbb{Q}$ with a given group G – and further classical problems: RIGP, Noether, Grunwald, we introduce a few more. Some are stronger than the classical ones, some are weaker; we wish to surround the main issues. The aim is to offer a structuring picture of IGT. Three coherent sub-topics from which the picture is built are revealed, also corresponding to three different sub-projects:
1. Lifting and parametrizing problems.
2. IGT with a Hilbert perspective.
3. Pre-Galois Theory.
The prospects and techniques may somehow differ but our plan in the end is that the combined achievements on these various facets contribute to some significant advances, including on the IGP. Another goal is to obtain a consistent and graded vision (even partially conjectural) of the distribution of groups w.r.t. to our IGT structured picture. Our IGT issues can be considered over other fields k than $\mathbb{Q}$, thereby inducing new problems, which for some are interesting for they own sake and in any case serve as a guidance for the fundamental situation over $\mathbb{Q}$. We consider local, Hilbertian, PAC fields, and although we mostly restrict to characteristic 0, the positive characteristic situation is a possible direction of our project. Realizing finite groups is our main focus, but solving embedding problems and moving to profinite groups are natural generalizations. Finally having a moduli space view is a further enriching and always rewarding approach.

Two hundred years ago, Galois drastically changed the world of polynomials. By providing a new viewpoint – Group Theory – he in fact opened a new era in mathematics. Yet, while Galois Theory has become ubiquitous, some fundamental mystery remains: to what extent can it be reversed? Notably, do all finite groups occur as Galois groups of polynomials with rational coefficients? This issue is the gist of the project, with the original situation over the field $\mathbb{Q}$ as our ultimate and resolute goal.

1. The territory of the project

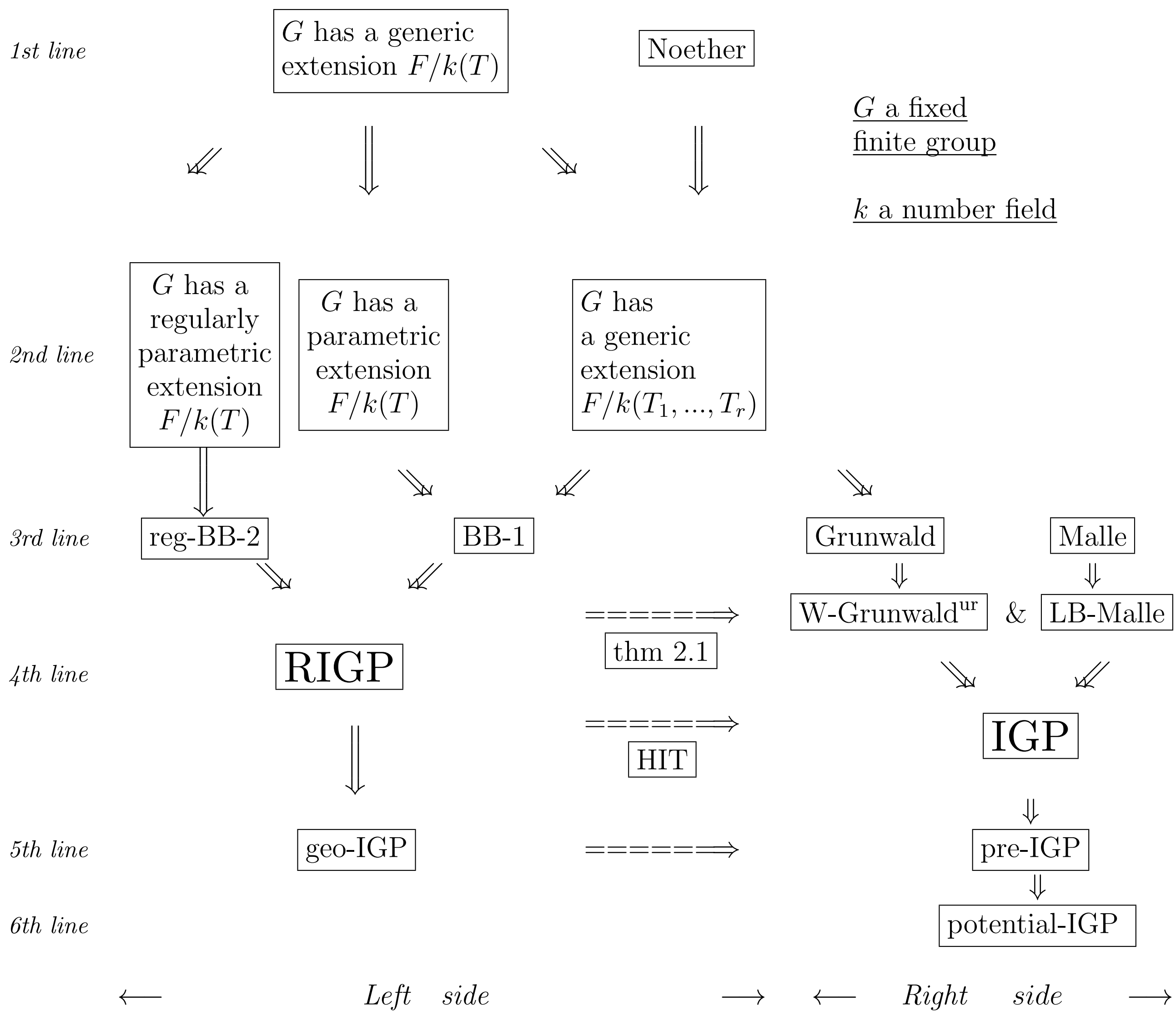
1.1. **Our IGT picture.** The diagram next page summarizes the target area and the milestones of our project. We refer to our glossary in §3 for a full definition of our abbreviated IGT statements (which experts will easily recognize). All implications shown on the picture are either classical or easy.

There are two main divisions in our picture: vertical and horizontal.

1.1.1. *Vertical division.* Left side consists of *geometric* statements while right side is concerned with *arithmetic* ones. Presence of indeterminates is the recognition sign of the former. *Specialization* connects the two sides. We specialize a k-regular Galois extension $F/k(T)$ or the corresponding k-cover $f : X \to \mathbb{P}^1_T$ in two ways[1]:
- for $t_0 \in k$, F_{t_0}/k, also denoted by f_{t_0}, is the classical specialized extension of F at t_0: the residue field extension at some prime ideal above t_0 in the extension $F/k(T)$.
- if $T_0 \in k(U) \setminus k$, $f_{T_0} : X_{T_0} \to \mathbb{P}^1_U$ is the pull-back of f along $T_0 : \mathbb{P}^1 \to \mathbb{P}^1$, which corresponds to specializing T to $T_0(U)$. We use the prefix G for this G(eometric) specialization, which stays on the left side.
Hilbert Irreducibility Theorem (HIT) is the fundamental specialization tool making the connection between geometric and arithmetic sides, notably proving R(egular)IGP $\Rightarrow$ IGP.

[1]Here the base space is the projective line $\mathbb{P}^1$. Considering more general ones: varieties with Hilbert property, with some approximation property, unirational varieties, is a possible direction of our project.

Our IGT picture



1.1.2. *Horizontal division.* The *4th line* implication RIGP $\Rightarrow$ IGP between the two Inverse Galois Problems (Regular and classical) is the dividing line. Statements above it are *strong forms* of the two historical statements. They may well be all too strong. Disproving them and thereby setting boundaries to the truth in IGT would be a first goal. For most of them, even very strong ones, it is a real issue. For all, a bigger goal is to investigate further (ideally classify) which groups satisfy the statements and which do not. Statements below RIGP $\Rightarrow$ IGP are *weak forms* of the problems, maybe more plausible and accessible. Proving them this time is a goal of the project.

1.2. **The three sub-projects.** The group $G = \mathbb{Z}/8\mathbb{Z}$ is a classical counter-example to (Grunwald) over $\mathbb{Q}$. Demarche-LucchiniArteche-Neftin have recently given new ones, over number fields with enough roots of unity [DGN17]. Thus (Grunwald) and the three statements above, which are old dreams of IGT, are false in general. They are however still investigated, for various types of groups. For example, by Buhler-Reichstein [BR97],

(*) *the only groups with a generic extension $F/\mathbb{Q}(T)$ are the four subgroups of S_3.*

Strikingly, *the status of all other problems was still open.* Making progress on them is a goal. We pursue it *via* 3 sub-projects, corresponding to a 3-part division of the global picture. They grew out of and rest on several seminal works: [Dèb17], [Dèb18], the PhD thesis of my student Legrand [Leg13] and a joint work in progress with Harbater [DH17].

1.2.1. Upper left: *Lifting and parametrizing problems.* Problems on *2nd Line* are variants of the idea that a group G could have a *universal Galois realization*, like $\mathbb{Q}(\sqrt{T})/\mathbb{Q}(T)$ for $G = \mathbb{Z}/2\mathbb{Z}$, *i.e.*, that would generate all Galois extensions E/k, or Galois covers $X \to \mathbb{P}^1_k$, of group G, by specialization, classical or G(eometric). The first two *parametric* statements, with one parameter, are introduced in [Dèb18] and [Leg13]. The third one allows several parameters, but requests genericity, *i.e.*, parametricity over every extension of k.

The first two statements on *3rd Line* are weaker *lifting properties*: they only request that for every set of N Galois extensions, or Galois covers, there is a Galois realization that generates them by (G-)specialization. (BB-N), short for *Beckmann-Black Arithmetic lifting property*, is fairly classical; the regular analog (reg-BB-N) is introduced in [Dèb18].

Disproving the whole *2nd Line* and (reg-BB-2) is a first achievement of our project (theorem 2.2). No counter-example is known yet for (BB-N) over $\mathbb{Q}$ [2], which is known to hold for $N = 1$ for S_n, A_n, abelian, odd dihedral groups and a few other sparse groups. *Proving or disproving* (BB-1) *over* $\mathbb{Q}$*, even for a single N, would be a breakthrough.*

1.2.2. Upper right: *IGT with a Hilbert perspective.* (Grunwald) on *3rd Line* is also a lifting property, of local extensions to some global extension of k. Its proof by Neukirch for solvable groups of order prime to the number of roots of 1 in k was a culminating point of the Abelian approach, initiated by Shafarevich. However (Grunwald) is false in general and wide open for non-solvable groups. Last statement (Malle) refers to the Malle conjecture, a celebrated quantitative version of (IGP), known to hold for nilpotent groups [KM04], again *via* the Abelian approach.

We offer a specialization approach (*4th Line*). As a weak (and more reasonable) form of (Grunwald), [DG12] introduces (UR-Grunwald$^\flat$) which limits the local extensions to be lifted to the UnRamified ones with big enough residue characteristic. As to (LB-Malle), it stands for the *Lower Bound part of the Malle conjecture* (which also has an upper bound part). *A base result for our project*, theorem 2.1 below, connects these statements.

1.2.3. Lower half: *Pre-Galois theory.* The weak versions of (R)IGP on *Last two Lines* come from a joint work in progress with D. Harbater [DH17]. Potentially Galois extensions are those field extensions E/k which become Galois after some linearly disjoint base change L/k. The pre-Galois variant requires further that the base change be Galois. Geometrically Galois extensions of $\kappa(T)$, for which $L = \overline{\kappa}(T)$, are special cases. Statements (geo-IGP), (pre-IGP) and (potential IGP) are the corresponding variants of (IGP).

Requiring that a geometrically Galois extension $F/k(T)$ is Galois over $\overline{k}$ but not necessarily over k avoids dealing with constant extension in the Galois closure, a main difficulty of RIGP. Others remain; (geo-IGP) *is an ambitious goal of the project, as is its consequence* (pre-IGP). An encouraging result is that (Potential-IGP) is true for every group over $\mathbb{Q}$ and over every Hilbertian field [DH17].

2. Methodology and expectations

Below we describe each sub-project: genesis, goals, approach, tools, and then discuss the expected convergence of the three sub-projects.

2.1. **IGT with a Hilbert perspective.** Theorem 2.1 improves on (RIGP) ⇒ (IGP).

Theorem 2.1. *Over a number field k,* (RIGP) *implies this unified version of* (LB-Malle) *and* (UR-Grunwald$^\flat$)*: there is a Malle lower bound* y^α *(for some specific* $\alpha > 0$*) for the number of Galois extensions* E/k *with group G, discriminant of norm* $|N_{k/\mathbb{Q}}(d_E)| \leq y$ (as in (Malle)) <u>*and*</u> *which in addition are solution of a given unramified Grunwald problem.*

After nilpotent groups, theorem 2.1 extends the set of groups satisfying (LB-Malle) to regular Galois groups over $\mathbb{Q}$, the second big category of groups known to be Galois groups over $\mathbb{Q}$. It is unclear whether the third big approach to realize groups over $\mathbb{Q}$, using Galois representations, can lead to results in the line of (Malle).

[2]Two are known, due to Colliot-Thélène, over *some* number field for $\mathbb{Z}/8\mathbb{Z}$ and for some p-group over some ample field.

Theorem 2.1 is already proved in [Dèb17] for $k = \mathbb{Q}$ and will be for a general number field in the PhD thesis of my student F. Motte. This generalization involves that of *a fundamental Diophantine result which provides fully explicit bounds for the number of integral points on a curve within a box of bounded size.* This result was proved, over $\mathbb{Q}$, by my former student Walkowiak [Wal05] by refining techniques of Heath-Brown.

Already exciting for itself, the generalization yields *an unconditional version of* LB-Malle*:*

(**) *for every group G, there is a number field k such that* (LB-Malle) *holds. One can take for k any field over which* (RIGP) *holds.*

Interestingly enough, it shows that the arithmetic statement (LB-Malle) behaves like geometric ones which become true after some base change.

The *production of solutions to Grunwald problems by specialization* is another development of our approach. An interesting point is that the groups involved are typically non-solvable groups, for which little is known regarding the Grunwald problem. Recent works of König-Legrand-Neftin even aim at extending the approach to the situation that the local extensions to be lifted may be ramified.

On the other hand, *the outcome of our approach may look too strong for certain* (*solvable*) *groups*, for example the possibility of constructing Galois extensions $E/\mathbb{Q}$ with "many" consecutive totally split primes ("many" compared to the discriminant size). *This may question* (RIGP) *for these groups.* Implementing techniques from analytic number theory, in the line of the Lagarias-Montgomery-Odlyzko work, is promising.

2.2. **Lifting and parametrizing problems.** The following results, which already show some significant progress, demonstrate the potential of our approach.

Theorem 2.2. (a) (Dèbes [Dèb18]) *Over any field $k \subset \mathbb{C}$, statement* (reg-BB-2) *does not hold for S_n, A_n, with $n > 6$, infinitely many* $\mathrm{PSL}_2(\mathbb{F}_p)$*, the Monster group, etc. Consequently, these groups do not have a k-regularly parametric cover.*

(b) (Koenig-Legrand-Neftin [KL18] [KLN]) *Over a number field k, none of these groups: S_n, A_n, $n > 3$, abelian groups of order neither a prime nor* 4*, dihedral groups D_{2n} with $n \neq 1, 9, p$ (p any prime), have a k-parametric extension $F/k(T)$.*

A big part of the basic material for this sub-project comes from [Dèb18] and Legrand's thesis [Leg13] (which has given rise to the papers [DL13], [DL12], [Leg16], [Leg15]), for example the general Legrand's non-parametricity criteria, which are used in [KL18].

[Dèb18] contains several other tools, which have become of daily use. First, an Invariant Comparizon Theorem which compares the invariants of a Galois cover $f : X \to \mathbb{P}^1$ with those of G-specializations $f_{T_0} : X_{T_0} \to \mathbb{P}^1$. For example it provides this key information when dealing with lifting and parametrizing problems:

(*) *the branch point number cannot drop under a rational pull-back $T_0 : \mathbb{P}^1 \to \mathbb{P}^1$ if f_{T_0} remains connected,*

which supplements the Grothendieck-Beckmann theorem on ramification in specializations.

[Dèb18] also contains the twisting lemma for families. The twisting lemma, which goes back to [DG12] (even to [Dèb99]), turns specialization questions into Diophantine ones. Starting from two Galois covers f and g of group G, the last *family version* produces a polynomial $\widetilde{Q} \in k[U, T, Y]$, obtained by "twisting f by g" and which has this property, for all but finitely $u_0 \in k$ and $t_0 \in k$:

(**) $f_{t_0} = g_{u_0}$ *if and only if* $\exists y_0 \in k$ *such that* $\widetilde{Q}(u_0, t_0, y_0) = 0$*,*

and this conclusion extends to analogous G-specialization issues.

[Dèb18] also introduces the idea of *investigating the questions on Hurwitz moduli spaces of covers* (instead of a single cover). This has led to an on-going development, joint with Koenig-Legrand-Neftin [DKLN18], around questions like whether a Hurwitz space $\mathcal{H}_{G,r}$ (for covers of group G and r branch points) can be *regularly parametric*, *i.e.*, whether all covers of group G can be obtained by rational pull-back from those in a given Hurwitz space. *We proved the following* which somehow settles the upper left part of the IGT picture over $\mathbb{C}$.

Theorem 2.3. *For $k=\mathbb{C}$, the following are equivalent.*
(a) $G \subset \mathrm{PGL}_2(\mathbb{C})$,
(b) *G has a regularly parametric cover $f : X \to \mathbb{P}^1_{\mathbb{C}}$,*
(c) *$\mathcal{H}_{G,\leq r_0} = \bigcup_{r\leq r_0} \mathcal{H}_{G,r}$ is regularly parametric for some $r_0 \geq 0$.*

Furthermore, among the groups from (a), the groups $\mathbb{Z}/n\mathbb{Z}$ ($n \geq 1$) and D_{2n} (n odd) are the only ones with a generic extension $F/\mathbb{C}(T)$. This last part was known from [BR97] and their *essential dimension* theory, but our tools, beside offering another proof, lead to more precise conclusions, giving the generic extensions on top of the groups.

Many questions remain of interest, like *studying systematically the ordered structure of the set of Galois covers $X \to \mathbb{P}^1$* with fixed group G, for this order:

(*) *$f \prec g$ if there exists $T_0 \in \mathbb{C}(U) \setminus \mathbb{C}$ such that g is isomorphic to f_{T_0}.*

Of interest is the surface $\widetilde{Q}(u_0, t_0, y_0) = 0$ of (**) which, under mild hypotheses, should be of general type. *The hyperbolicity conjecture of Green-Griffiths-Lang* on the rational curves on a surface and Miyaoka's results towards it could then be advantageously used *to prove some finiteness results* in our context, namely that two $\mathbb{C}$-Galois covers may only have finitely many minimal common pull-backs. Undoubtably this part over $\mathbb{C}$ has plenty to offer and seems *the* right way into the main situation over number fields (see §2.4).

2.3. **Pre-Galois theory.** A motivation is that while the main problems (RIGP) and (IGP) may seem out of reach, some weaker versions can be considered. (Potential-IGP) is that for every finite group G, there are linearly disjoint extensions E/k and L/k such that $G = \mathrm{Gal}(EL/L)$; (pre-IGP) requires further that L/k be Galois. As to (geo-IGP), it is the special case of (pre-IGP) for which $k = \kappa(T)$ and $L = \overline{\kappa}(T)$ for some field κ.

We already showed that (Potential-IGP) holds over $\mathbb{Q}$ (and over every Hilbertian field) for every group G [DH17]. But (geo-IGP) and (pre-IGP), though weaker than (RIGP) and (IGP), are open. Here are two statements that help evaluate the difference.

Both (RIGP) and (geo-IGP) essentially amount to finding k-rational points on some Hurwitz space $\mathcal{H}_{G,r}$ (for some r). But for (RIGP), the Hurwitz space is of G-Galois covers (with their automorphisms) while for (geo-IGP), it is of mere covers (without the automorphisms). The former covers the latter. Concerning (pre-IGP) *vs.* (IGP), we know that the former holds for G if and only if there exists a subgroup $A \subset \mathrm{Aut}(G)$ such that (IGP) holds for the semi-direct product $G \rtimes A$ [DH17].

The difference may not look so big. Yet we think that the weaker forms are way more accessible. We have already established that

(a) *every group G is a quotient of a group satisfying* (geo-IGP)*, and so of a group satisfying* (pre-IGP) *as well,* but at the moment we know that this implies (geo-IGP) (and (pre-IGP)) for G itself only if the quotient is characteristic,

(b) *if G is simple, then some power G^n satisfies* (geo-IGP) *and* (pre-IGP) *over $\mathbb{Q}$,*

(c) *a group G satisfies* (geo-IGP) *over $\mathbb{Q}^{\mathrm{ab}}$ if it possesses a weakly rigid tuple*, which seems much weaker than the rigid analog for (RIGP).

A number of tools are available: rigidity, Hurwitz spaces, Descent Theory, *etc.*, which have been sharpened and polished in the 80'-90'. We have adjusted them to pre-Galois theory and we feel that suitably combined with our own techniques, they could exactly fit our pre-Galois context *to show that* (geo-IGP) *holds over $\mathbb{Q}$, or at least over $\mathbb{Q}^{\mathrm{ab}}$, at least for simple groups.*

The whole (RIGP) would still be far away as we would have to control the new constants that appear when passing to the Galois closure $\widehat{F}/k(T)$ of some regular Galois extension $F/k(T)$. This new field of constants $\widehat{F} \cap \overline{\mathbb{Q}}$ is far from understood and (geo-IGP) and (pre-IGP) could be the best statements that could be obtained at this stage of knowledge. Still, *understanding better the constant extension phenomenon is another challenge.*

There is a further interesting *connection of our pre-Galois Theory with Hopf Algebra Theory* and the Hopf Galois extensions of Greither-Pareigis. This in particular sheds some light on a G-torsor viewpoint that we have developed for our pre-Galois theory.

2.4. **Convergence: the ultimate stage.** Our primary goal is the situation over number fields. Our tools are available in this environment and our approach implementable. *The* obstacle then is the Diophantine aspect of IGT, now well-understood but still seemingly insurmountable.

Considering easier Diophantine situations, *e.g.* over local or P(seudo) A(lgebraically) C(losed) fields, remains a good intermediate stage. Results on these fields not only help understand and solve the "other issues" – from Group and Galois Theory, Algebraic Geometry – but are also worthwhile for they own sake. Hilbertian PAC fields (*e.g.* $\mathbb{Q}^{\mathrm{tot}\mathbb{R}}(i)$ with $\mathbb{Q}^{\mathrm{tot}\mathbb{R}}$ the totally real algebraic number field), whose absolute Galois group is pro-free of countable rank, have some rich arithmetic. They show the way to number fields.

We insist however on considering these easier situations only *in fine* and on having *a more global approach*, over general fields k. The goal is to pinpoint the real Diophantine need. Here we take advantage of the explicitness of our tools, which makes it possible to
- closely follow our constructions and keep track of their definition field, and,
- unravel the various facets of the problem, and, thanks to some more focused upstream work on the group-theoretic and geometric ones, to reduce the Diophantine scope.

The remaining Diophantine issues then either can be solved (over number fields), or can be precisely connected to well-identified cases of standard Diophantine conjectures. Or, *in fine*, they vanish in easier environments.

Recent works illustrate this line of thought, *e.g.* [Dèb17] [Dèb18] where non-Diophantine preliminary considerations make it possible to use the twisting lemma over number fields, or provide decisive constraints for some lifting problems. [KL18] is another example.

3. Glossary

A finite group G and a field k of characteristic 0 are fixed.

IGP: *There exists a Galois extension E/k of group G.*

RIGP: *There exists a k-regular Galois extension $F/k(T)$ of group G.*

HIT: short for *Hilbert Irreducibility Theorem.*

Noether: *If $\mathbf{Y} = Y_1, \dots, Y_d$ are $d = |G|$ indeterminates, the fixed field $k(\mathbf{Y})^G$ of G in $k(\mathbf{Y})$, with $G \hookrightarrow S_d$ acting via its regular representation, is a pure transcendental extension of k.*

G **has a parametric extension** $F/k(T_1, \dots, T_r)$: *There is a Galois extension $F/k(T_1, \dots, T_r)$ of group G that is k-regular ($F \cap \overline{k} = k$) and k-parametric, i.e., every Galois extension E/k of group contained in G is the specialized extension $F_{\mathbf{t}_0}/k$ of $F/k(T_1, \dots, T_r)$ at some point $\mathbf{t}_0 \in k^r$, not in the branch locus of $F/k(T_1, \dots, T_r)$.*

G **has a parametric extension** $F/k(T)$: as above with $r = 1$.

G **has a generic extension** $F/k(T_1, \dots, T_r)$: *There is a k-regular Galois extension $F/k(T_1, \dots, T_r)$ of group G such that $FK/K(T_1, \dots, T_r)$ is K-parametric for every extension K/k.*

G **has a generic extension** $F/k(T)$: as above with $r = 1$.

G **has a regularly parametric extension** $F/k(T)$: *The corresponding k-regular Galois cover $f : X \to \mathbb{P}^1_k$ (of function field extension $F/k(T)$) has this property: every k-regular Galois cover $g : Y \to \mathbb{P}^1_k$ of group G is some rational pullback $f_{T_0} : X_{T_0} \to \mathbb{P}^1_k$ of f (for some T_0 in $k(U) \setminus k$). Equivalently, every k-regular Galois extension $L/k(U)$ of group G can be obtained from the k-regular Galois extension $F/k(T)$ by specializing $F(U)/k(U,T)$ at some T_0 in $k(U) \setminus k$.*

BB-N (short for Beckmann-Black Arithmetic lifting property): *For the given integer $N \geq 1$ and every N-tuple $(E_i/k)_{i=1,\dots,N}$ of Galois extensions of group contained in G, there exists a k-regular Galois extension $F/k(T)$ of group G and N unbranched points $t_1, \dots, t_N$ such that E_i/k is the specialized extension F_{t_i}/k of $F/k(T)$ at t_i, $i = 1, \dots, N$.*

reg-BB-N (short for Beckmann-Black Geometric lifting property): *For the given integer $N \geq 1$ and every N-tuple $(g_i : Y_i \to \mathbb{P}^1)_{i=1,\dots,N}$ of k-regular Galois covers of group G, there exists a k-regular Galois cover f of group G and N rational maps $T_i : \mathbb{P}^1_k \to \mathbb{P}^1_k$ such that g_i is isomorphic to the pull-back cover f_{T_I} of f along T_i, $i = 1, \dots, N$.*

geo-IGP: *There is a k-regular extension $F/k(T)$ such that $F\overline{k}/\overline{k}(T)$ is Galois of group G.*

potential-IGP: *There exists an extension E/k of degree $|G|$ and an extension L/k such that EL/L is Galois of group G.*

pre-IGP: *There exists an extension E/k of degree $|G|$ and a Galois extension L/k such that EL/L is Galois of group G.*

Below, k is a number field, k_v its v-completion w.r.t. a discrete valuation v and $N(G,y)$ the number of sub-Galois extensions E/k of $\overline{k}$ of group G and discriminant of norm $|N_{k/\mathbb{Q}}(d_E)| \leq y$.

Malle: *For $a(G) = (|G|(1-1/\ell)^{-1}$ with ℓ the smallest prime divisor of $|G|$, there exist constants $c_1, c_2 > 0$ such that for every $\varepsilon > 0$, we have*

$$c_1 y^{a(G)} \leq N(G,y) \leq c_2 y^{a(G)+\varepsilon} \quad \textit{for every } y \geq y_0(\varepsilon).$$

LB-Malle (short for Lower Bound part of **Malle** conjecture): *There exist constants $\alpha > 0$ and $y_0 > 0$ such that* $\quad N(G,y) \geq y^\alpha$ *for every* $y \geq y_0$.

Grunwald: *For any set $S = \{E_i/k_{v_i} | i = 1, \dots, N\}$ of local Galois extensions E_i/k_{v_i} of group $H_i \subset G$, there is a global extension E/k of group G such that $Ek_{v_i}/k_{v_i} = E_i/k_{v_i}$, $i = 1, \dots, N$.*

W-Grunwald$^{\mathrm{ur}}$: *There is a finite set $\mathcal{E}$ of discrete valuations of k for which this is true: for every set $S = \{E_i/k_{v_i} | i = 1, \dots, N\}$ of unramified Galois extensions of group $H_i \subset G$, with $v_i \notin \mathcal{E}$, there exists a Galois extension E/k of group G such that $Ek_{v_i}/k_{v_i} = E_i/k_{v_i}$, $i = 1, \dots, N$.*

(The symbol $\flat$ indicates that compared to **Grunwald**, an exceptional finite set $\mathcal{E}$ of discrete valuations is allowed).

LB-Malle & W-Grunwald$^{\mathrm{ur}}$: *There is a finite set $\mathcal{E}$ of discrete valuations of k and a constant $\alpha > 0$ for which this is true: for every finite set S as in* **W-Grunwald**$^{\mathrm{ur}}$, *the number $N(G,y,S)$ of those extensions E/k counted by $N(G,y)$ in* **LB-Malle** *which additionally fullfill the conclusion of* **W-Grunwald**$^{\mathrm{ur}}$ *satisfies*

$$N(G,y,S) \geq y^\alpha \quad \textit{for every } y \geq y_0(S).$$

References

[BR97] J. Buhler and Z. Reichstein. On the essential dimension of a finite group. *Compositio Math.*, 106(2):159–179, 1997.

[Dèb99] Pierre Dèbes. Galois covers with prescribed fibers: the Beckmann-Black problem. *Ann. Scuola Norm. Sup. Pisa,* Cl. Sci. (4), 28:273–286, 1999.

[Dèb17] Pierre Dèbes. On the Malle conjecture and the self-twisted cover. *Israel J. Math.*, 218:101–131, 2017.

[Dèb18] Pierre Dèbes. Groups with no parametric Galois realization. *Annales Sci. E.N.S.*, 51/1:143–179, 2018.

[DG12] Pierre Dèbes and Nour Ghazi. Galois covers and the Hilbert-Grunwald property. *Ann. Inst. Fourier*, 62/3:989–1013, 2012.

[DGN17] Cyril Demarche, Lucchini-Arteche Giancarlo, and Danny Neftin. The Grunwald problems and approximation properties for homogeneous spaces. *Ann. Inst. Fourier*, 67/3:1009–1033, 2017.

[DH17] Pierre Dèbes and David Harbater. Pre-Galois theory. *Preprint*, 2017.

[DKLN18] Pierre Dèbes, Joachim Koenig, François Legrand, and Danny Neftin. Rational pull-backs of Galois covers. *in progress*, 2018.

[DL12] Pierre Dèbes and François Legrand. Twisted covers and specializations. volume 63 of *Advanced Studies in Pure Math.*, pages 141–162. (H.Nakamura, F.Pop, L.Schneps, A.Tamagawa eds.), Mathematical Society of Japan, Tokyo, 2012.

[DL13] Pierre Dèbes and François Legrand. Specialization results in Galois theory. *Trans. Amer. Math. Soc.*, 365(10):5259–5275, 2013.

[KL18] Joachim Koenig and François Legrand. Non-parametric sets of regular realizations over number fields. *J. Algebra*, 497:302–336, 2018.

[KLN] Joachim Koenig, François Legrand, and Danny Neftin. On the local behaviour of specializations of function field extensions. IMRN. doi: 10.1093/imrn/rny016. to appear.

[KM04] Jürgen Klüners and Gunter Malle. Counting nilpotent Galois extensions. *J. Reine Angew. Math.*, 572:1–26, 2004.

[Leg13] François Legrand. Spécialisations de revêtements et théorie inverse de Galois. *PhD thesis - Université Lille 1*, 2013.

[Leg15] François Legrand. Parametric Galois extensions. *J. Algebra*, 422:187–222, 2015.

[Leg16] François Legrand. Specialization results and ramification conditions. *Israel J. Math.*, 214(2):621–650, 2016.

[Wal05] Y. Walkowiak. Théorème d'irréductibilité Hilbert effectif. *Acta Arith.*, 116/4:343–362, 2005.

E-mail address: `Pierre.Debes@math.univ-lille1.fr`

Laboratoire Paul Painlevé, Mathématiques, Université de Lille, 59655 Villeneuve d'Ascq Cedex, France