

# INDECOMPOSABLE POLYNOMIALS AND THEIR SPECTRUM

ARNAUD BODIN, PIERRE DÈBES, AND SALAH NAJIB

ABSTRACT. We address some questions concerning indecomposable polynomials and their spectrum. How does the spectrum behave via reduction or specialisation, or via a more general ring morphism? Are the indecomposability properties equivalent over a field and over its algebraic closure? How many polynomials are decomposable over a finite field?

## 1. INTRODUCTION

Fix an integer  $n \geq 2$  and a  $n$ -tuple of indeterminates  $\underline{x} = (x_1, \dots, x_n)$ . A non-constant polynomial  $F(\underline{x}) \in k[\underline{x}]$  with coefficients in an algebraically closed field  $k$  is said to be *indecomposable* in  $k[\underline{x}]$  if it is not of the form  $u(H(\underline{x}))$  with  $H(\underline{x}) \in k[\underline{x}]$  and  $u \in k[t]$  with  $\deg(u) \geq 2$ . An element  $\lambda^* \in k$  is called a *spectral value* of  $F(\underline{x})$  if  $F(\underline{x}) - \lambda^*$  is reducible in  $k[\underline{x}]$ . It is well-known that

(1)  $F(\underline{x}) \in k[\underline{x}]$  is indecomposable if and only if  $F(\underline{x}) - \lambda$  is irreducible in  $\overline{k(\lambda)}[\underline{x}]$  (where  $\lambda$  is an indeterminate),

(2) if  $F(\underline{x}) \in k[\underline{x}]$  is indecomposable, then the subset  $\text{sp}(F) \subset k$  of all spectral values of  $F(\underline{x})$  — the spectrum of  $F(\underline{x})$  — is finite; and in the opposite case,  $\text{sp}(F) = k$ ,

(3) more precisely, if  $F(\underline{x}) \in k[\underline{x}]$  is indecomposable and for every  $\lambda^* \in k$ ,  $n(\lambda^*)$  is the number of irreducible factors of  $F(\underline{x}) - \lambda^*$  in  $k[\underline{x}]$ , then we have  $\rho(F) := \sum_{\lambda^* \in k} (n(\lambda^*) - 1) \leq \deg(F) - 1$ . In particular  $\text{card}(\text{sp}(F)) \leq \deg(F) - 1$ .

Statement (3), which is known as Stein's inequality, is due to Stein [13] in characteristic 0 and Lorenzini [10] in arbitrary characteristic (but for 2 variables); see [11] for the general case.

This paper offers some new results in this context.

In §2, given an indecomposable polynomial  $F(\underline{x})$  with coefficients in an integral domain  $A$  and a ring morphism  $\sigma : A \rightarrow k$  with  $k$  an algebraically closed field, we investigate the connection between the spectrum of  $F(\underline{x})$

---

*Date:* January 17, 2009.

*2000 Mathematics Subject Classification.* 12E05, 11C08.

*Key words and phrases.* Irreducible and indecomposable polynomials, Stein's theorem, spectrum of a polynomial.

and that of the polynomial  $F^\sigma(\underline{x})$  obtained by applying  $\sigma$  to the coefficients of  $F(\underline{x})$ . Theorem 2.1 provides a conclusion *à la Bertini-Noether*, which, despite its basic nature, does not seem to be available in the literature: under minimal assumptions on  $A$ , the connection is the expected one generically. For example if  $A = \mathbb{Z}$ , “spectrum” and “reduction modulo a prime  $p$ ” commute if  $p$  is suitably large (depending on  $F$ ). We give other typical applications, notably for a specialization morphism  $\sigma$ . Related results are given in [3].

For two variables, we can give in §3 an indecomposability criterion for a reduced polynomial modulo some prime  $p$  (theorem 3.1) that is more precise than theorem 2.1: the condition “for suitably large  $p$ ” is replaced by some explicit condition on  $F(x, y)$  and  $p$ , possibly satisfied for small primes. This criterion uses some results on good reduction of curves and covers due to Grothendieck, Fulton et al; we will follow here Zannier’s version [14]. Another criterion based on the Newton polygon of a polynomial is given in [4].

§4 is devoted to the connection between the indecomposability properties over a field  $K$  and over its algebraic closure  $\overline{K}$ . While it was known they are equivalent in many circumstances, for example in characteristic 0, it remained to handle the inseparable case to obtain a definitive conclusion. That is the purpose of proposition 4.1, which, conjoined with previous works, shows that the only polynomials  $F(\underline{x})$  indecomposable in  $K[\underline{x}]$  but decomposable in  $\overline{K}[\underline{x}]$  are  $p$ -th powers in  $\overline{K}[\underline{x}]$ , where  $p > 0$  is the characteristic of  $K$  (theorem 4.2).

§5 is aimed at counting the number of indecomposable polynomials of a given degree  $d$  with coefficients in the finite field  $\mathbb{F}_q$ . We show that most polynomials are indecomposable: the ratio  $I_d/N_d$  of indecomposables of degree  $d$  tends to 1 (as  $d \rightarrow \infty$  or as  $q \rightarrow \infty$ ), and we give some estimate for the error term  $1 - I_d/N_d$ . The constants involved in our estimates are explicit. For simplicity we mostly restrict to polynomials in two variables as calculations become more intricate when  $n > 2$ . We also consider the one variable situation (for which the definition of indecomposability is slightly different, see §4.3) with the restriction that  $q$  and  $d$  are relatively prime. A unified treatment of the general case  $n \geq 2$  is offered in a parallel work of von zur Gathen [8], who also considers the “wild case” ( $n = 1$  with  $(q, d) \neq 1$ ) in [9].

**Acknowledgments.** We wish to thank Joachim von zur Gathen for interesting discussions in Lille and in Bonn.

## 2. SPECTRUM AND MORPHISMS

**Notation:** If  $\sigma : A \rightarrow B$  is a ring morphism, we denote the image of elements  $a \in A$  by  $a^\sigma$ . For  $P(\underline{x}) \in A[\underline{x}]$ , we denote the polynomial obtained by applying  $\sigma$  to the coefficients of  $P$  by  $P^\sigma(\underline{x})$ . If  $V \subset \mathbb{A}_A^n$  is the Zariski closed subset associated with a family of polynomials  $P_i(\underline{x}) \in A[\underline{x}]$ , we denote by  $V^\sigma$  the Zariski closed subset of  $\mathbb{A}_B^n$  associated with the family of polynomials  $P_i^\sigma(\underline{x}) \in B[\underline{x}]$ .

If  $S \subset A$  is a multiplicative subset such that all elements from  $S^\sigma$  are invertible in  $B$ , we still denote by  $\sigma$  the natural extension  $S^{-1}A \rightarrow B$  of the original morphism  $\sigma$ .

Fix an integrally closed ring  $A$ , with a perfect fraction field  $K$ .

An effective divisor  $D = \sum_{i=1}^r n_i a_i$  of  $\overline{K}$  is said to be *K-rational* if the coefficients of the polynomial  $P(T) = \prod_{i=1}^r (T - a_i)^{n_i}$  are in  $K^1$ . A morphism  $\sigma : A \rightarrow k$  in an algebraically closed field  $k$  is then said to be *defined at D* if the coefficients of  $P(T)$  have a common denominator  $d \in A$  such that  $d^\sigma$  is non-zero in  $k$ <sup>2</sup>. In this case we denote by  $P^\sigma(T) \in k[T]$  the image polynomial of  $P(T)$  by the morphism  $\sigma$  (extended to the fraction field of  $A$  with denominators a power of  $d$ ) and by  $D^\sigma$  the effective divisor of  $k$  whose support is the set of roots of  $P^\sigma(T)$  and coefficients are the corresponding multiplicities.

**2.1. Statement.** For more precision, we use the *spectral divisor* rather than the spectrum: it is the divisor  $\text{spdiv}(F) = \sum_{\lambda^* \in k} (n(\lambda^*) - 1) \lambda^*$  of the affine line  $\mathbb{A}^1(k)$ . Its support is the spectrum of  $F$  and Stein's inequality rewrites:  $\deg(\text{spdiv}(F)) \leq \deg(F) - 1$ .

**Theorem 2.1.** *Let  $F(\underline{x}) \in A[\underline{x}]$  be indecomposable in  $\overline{K}[\underline{x}]$ . Then there exists a non-zero element  $h_F \in A$  such that the following holds. For every morphism  $\sigma : A \rightarrow k$  in an algebraically closed field  $k$ , if  $h_F^\sigma \neq 0$ , then  $F^\sigma(\underline{x})$  is indecomposable in  $k[\underline{x}]$ , the morphism  $\sigma : A \rightarrow k$  is defined at the divisor  $\text{spdiv}(F)$  and we have  $\text{spdiv}(F^\sigma) = (\text{spdiv}(F))^\sigma$ ; in particular  $\rho(F^\sigma) = \rho(F)$  and  $\text{sp}(F^\sigma) = (\text{sp}(F))^\sigma$ .*

The first stage of the proof will produce the spectrum as a Zariski closed subset of the affine line  $\mathbb{A}_A^1$  over the ring  $A$ . Specifically the following can

<sup>1</sup>which, under our hypothesis “ $K$  perfect”, amounts to the invariance of  $P(T)$ , or of  $D$ , under  $\text{Gal}(\overline{K}/K)$ .

<sup>2</sup>which, under our hypothesis “ $A$  integrally closed”, amounts to saying the elements  $a_i$  themselves have a common denominator  $d \in A$  (that is,  $da_i$  integral over  $A$ ,  $i = 1, \dots, r$ ) such that  $d^\sigma \neq 0$ .

be drawn from the proof: there is a proper<sup>3</sup> Zariski closed subset  $V_F \subset \mathbb{A}_A^1$  such that for every morphism  $\sigma : A \rightarrow k$  as above,

(\*) *the polynomial  $F^\sigma(\underline{x})$ , if it is of degree  $d$ , is indecomposable in  $k[\underline{x}]$  if and only if the Zariski closed subset  $V_F^\sigma \subset \mathbb{A}_k^1$  is proper, and in this case, we have  $\text{sp}(F^\sigma) = V_F^\sigma(k)$ .*

When applied to the inclusion morphism  $A \rightarrow \overline{K}$ , theorem 2.1 yields that the spectrum of  $F(\underline{x})$  is equal to the Zariski closed subset  $V_F(\overline{K})$ . In particular, it is  $K$ -rational. The same is true for the spectral divisor of  $F(\underline{x})$  as  $n(\lambda^\tau) = n(\lambda)$  for each  $\lambda \in \overline{K}$  and each  $\tau \in \text{Gal}(\overline{K}/K)$ .

Making the constant  $h_F$  from theorem 2.1 explicit is an interesting next goal. This requires to have good bounds for the ‘‘Noether forms’’ associated with the polynomial  $F(\underline{x}) - \lambda \in \overline{K(\lambda)}[\underline{x}]$  in §2.3.1. Some work of Busé, Chèze and Najib in this direction is in progress [3].

## 2.2. Typical applications.

2.2.1. *Situation 1.* For  $A = \mathbb{Z}$ , then  $h_F \in \mathbb{Z}$ ,  $h_F \neq 0$ . Theorem 2.1, applied with  $\sigma : \mathbb{Z} \rightarrow \overline{\mathbb{F}}_p$  the reduction morphism modulo a prime number  $p$  yields: *for all suitably large  $p$ , the reduced polynomial  $\overline{F}(\underline{x})$  modulo  $p$  is indecomposable in  $\overline{\mathbb{F}}_p[\underline{x}]$  and its spectral divisor is obtained by reducing that of  $F(\underline{x})$ , that is:  $\text{spdiv}(\overline{F}) = \overline{\text{spdiv}(F)}$ .*

2.2.2. *Situation 2.* Take  $A = k[\underline{t}]$  with  $k$  an algebraically closed field and  $\underline{t} = (t_1, \dots, t_r)$  some indeterminates. Denote in this situation by  $F(\underline{t}, \underline{x})$  the polynomial  $F(\underline{x})$  of the general statement. Theorem 2.1, applied with  $\sigma$  the specialisation morphism  $k[\underline{t}] \rightarrow k$  that maps  $\underline{t} = (t_1, \dots, t_r)$  on an  $r$ -tuple  $\underline{t}^* = (t_1^*, \dots, t_r^*) \in k^r$  yields:

*for all  $\underline{t}^*$  off a proper Zariski closed subset of  $k^r$ , the specialized polynomial  $F(\underline{t}^*, \underline{x})$  is indecomposable in  $k[\underline{x}]$  and its spectral divisor is obtained by specializing that of  $F(\underline{t}, \underline{x})$ .*

2.2.3. *Situation 3.*  $F(\underline{x})$  is the generic polynomial in  $n$  variables and of degree  $d$ . Take for  $A$  the ring  $\mathbb{Z}[a_i]$  generated by the indeterminates  $a_i$  corresponding to the coefficients of  $F(\underline{x})$ ; the multi-index  $\underline{i} = (i_1, \dots, i_n)$  ranges over the set  $I_{n,d}$  of all  $n$ -tuples of integers  $\geq 0$  such that  $i_1 + \dots + i_n \leq d$ .

Classically the polynomial  $F(\underline{x})$  is irreducible in  $\overline{\mathbb{Q}(a_i)}[\underline{x}]$ , hence it is indecomposable. Theorem 2.1, applied with  $\sigma : A \rightarrow k$  a specialization morphism of the  $a_i$ , yields that all polynomials  $f(\underline{x}) \in k[\underline{x}]$  of degree  $d$  are

<sup>3</sup>that is, distinct from the whole surrounding space (here the affine line  $\mathbb{A}_A^1$  over the ring  $A$ ); equivalently, there exists a non-zero polynomial in the associated ideal.

indecomposable but possibly those from the proper Zariski closed subset corresponding to the equation  $h_F = 0$  (with  $h_F$  viewed in  $k[a_i]$ ).

For polynomials  $f(\underline{x})$  outside the closed subset  $h_F = 0$ , the spectrum of  $f$  is obtained by specializing the generic spectrum. However we have:

**Proposition 2.2.** *For  $d > 2$  or  $n > 2$ , the generic spectrum is empty. For  $d = 2$ , it contains a single element, given by*

$$a_{00} - \frac{a_{02}a_{10}^2 + a_{20}a_{01}^2 - a_{01}a_{10}a_{11}}{4a_{02}a_{20} - a_{11}^2}$$

For  $d > 2$  or  $n > 2$ , polynomials with a non-empty spectrum lie in the Zariski closed subset  $h_F = 0$ .

*Proof.* Assume that the generic spectrum is not empty. If  $k$  is an algebraically closed field and  $\mathcal{R}_{n,d}$  (resp.  $\mathcal{P}_{n,d|a_0=0}$ ) denotes the set of polynomials  $P(\underline{x}) \in k[\underline{x}]$  of degree  $\leq d$  that are reducible in  $k[\underline{x}]$  (resp. whose constant term is zero), the correspondence  $P(\underline{x}) \rightarrow P(\underline{x}) - P(\underline{0})$  induces an algebraic morphism  $\mathcal{R}_{n,d} \rightarrow \mathcal{P}_{n,d|a_0=0}$  which is generically surjective (that is, surjective above a non-empty Zariski open subset of  $\mathcal{P}_{n,d|a_0=0}$ ). It follows that  $\mathcal{R}_{n,d}$  is of codimension  $\leq 1$  in the space  $\mathcal{P}_{n,d}$  of all polynomials in  $k[\underline{x}]$  of degree  $\leq d$ . This observation provides the desired conclusion in the case  $n = 2$  and  $d > 2$ : indeed we have  $\text{codim}_{\mathcal{P}_{2,d}}(\mathcal{R}_{2,d}) = d - 1$  [7, theorem 2].

For  $d = 2$ , the equation “ $(ux + ay + b)(vx + cy + d) = F(\underline{x})$  modulo the constant term” with unknowns  $u, a, b, v, c, d$  is readily solved: reduce to the case  $a_{20} = u = v = 1$ , find the unique solution for the 4-tuple  $(a, b, c, d)$  and compute  $bd$ ; the generic spectral value is then  $a_{00} - bd$ .

Finally assume that for  $d \geq 2$  and  $n > 2$ , there exists a generic spectral value  $\lambda \in \overline{K}$  (with  $K = \mathbb{Q}(a_i)$ ). Let  $F(\underline{x}) - \lambda = Q(\underline{x})R(\underline{x})$  be a non trivial factorization in  $\overline{K}[\underline{x}]$ . Specializing  $x_3, \dots, x_n$  to 0 gives a non trivial factorization in  $\overline{K}[x_1, x_2]$  of the generic polynomial of degree  $d$  in 2 variables. From the first part of the proof, we have  $d = 2$ . Furthermore, the above case provides the necessary value of  $\lambda$ . Now specializing  $x_2$  and  $x_4, \dots, x_n$  to 0 leads to a different value. Whence a contradiction.  $\square$

### 2.3. Proof of theorem 2.1.

2.3.1. *1st stage: elimination theory.* This stage is aimed at showing proposition 2.3 below, which generalizes the Bertini-Noether theorem [6, prop.9.4.3]. It is proved in the general situation

(Hyp) *a polynomial  $\mathcal{F}(\underline{\lambda}, \underline{x}) \in A[\underline{\lambda}, \underline{x}]$  is irreducible in  $\overline{K(\underline{\lambda})}[\underline{x}]$ , where  $\underline{\lambda} = (\lambda_1, \dots, \lambda_s)$  is an  $s$ -tuple of indeterminates ( $s \geq 0$ ).*

We will use it in the special case  $\mathcal{F}(\underline{\lambda}, \underline{x}) = F(\underline{x}) - \lambda$ . The hypotheses “ $A$  integrally closed” and “ $K$  perfect” are not necessary for this stage.

As in situation 3, consider some indeterminates  $(a_i)_{i \in I_{n,d}}$  corresponding to the coefficients of a polynomial of degree  $d$  in  $n$  variables. A polynomial with coefficients in a ring  $R$  corresponds to a morphism  $\phi : \mathbb{Z}[a_i] \rightarrow R$ ; denote by  $F(a_i^\phi)(\underline{x}) \in A[\underline{x}]$  the corresponding polynomial. Let  $\varphi_\lambda : \mathbb{Z}[a_i] \rightarrow A[\underline{\lambda}]$  be the morphism corresponding to the polynomial from statement (Hyp):  $\mathcal{F}(\underline{\lambda}, \underline{x}) = F(a_i^{\varphi_\lambda})(\underline{x})$ .

From Noether’s theorem [12, §3.1 theorem 32], there exist finitely many universal homogeneous forms  $\mathcal{N}_h(a_i)$  ( $1 \leq h \leq D = D(n, d)$ ) in the  $a_i$  and with coefficients in  $\mathbb{Z}$  such that:

(4) *for every morphism  $\phi : \mathbb{Z}[a_i] \rightarrow k$  in an algebraically closed field  $k$ , the polynomial  $F(a_i^\phi)(\underline{x})$ , if it is of degree  $d$ , is reducible in  $k[\underline{x}]$  if and only if  $\mathcal{N}_h(a_i^\phi) = 0$  for  $h = 1, \dots, D$ .*

For  $\phi$  taken to be the morphism  $\varphi_\lambda : \mathbb{Z}[a_i] \rightarrow A[\underline{\lambda}] \subset \overline{K(\underline{\lambda})}$ , the elements  $\mathcal{N}_h(a_i^{\varphi_\lambda}) \in A[\underline{\lambda}]$  are polynomials  $N_h(\underline{\lambda})$ . Let  $V_{\mathcal{F}} \subset \mathbb{A}_A^s$  be the Zariski closed subset corresponding to the ideal they generate; it is a proper closed subset. Indeed, as  $\mathcal{F}(\underline{\lambda}, \underline{x})$  is irreducible in  $\overline{K(\underline{\lambda})}[\underline{x}]$ , from (4), at least one of the polynomials  $N_h(\underline{\lambda})$ , say  $N_{h_0}(\underline{\lambda})$ , is non-zero. Denote by  $a_{\mathcal{F}} \in A$  the product of a non-zero coefficient of  $N_{h_0}(\underline{\lambda})$  and the non-zero coefficient of some monomial of  $\mathcal{F}(\underline{\lambda}, \underline{x})$  of degree  $d$  in  $\underline{x}$ .

If  $R$  is an integral domain and  $\Sigma : A[\underline{\lambda}] \rightarrow R$  a morphism, then (4), with  $\phi$  taken to be  $\Sigma \circ \varphi_\lambda : \mathbb{Z}[a_i] \rightarrow R \hookrightarrow \kappa$  and  $\kappa = \overline{\text{Frac}(R)}$ , yields that the polynomial  $\mathcal{F}^\Sigma \in R[\underline{x}]$ , if of degree  $d$ , is irreducible in  $\kappa[\underline{x}]$  if and only if at least one of the elements  $N_h^\Sigma \in R$  is non-zero (note that  $\mathcal{F}^\Sigma = F(a_i^{\Sigma \varphi_\lambda})(\underline{x})$  and  $\mathcal{N}_h(a_i^{\Sigma \varphi_\lambda}) = \mathcal{N}_h(a_i^{\varphi_\lambda})^\Sigma$ ), or, equivalently, if the corresponding Zariski closed subset of  $\text{Spec}(R)$  is proper.

Let  $\sigma : A \rightarrow k$  be a morphism with  $k$  algebraically closed. Apply the above first to the morphism  $\sigma \circ \varphi_\lambda : \mathbb{Z}[a_i] \rightarrow k[\underline{\lambda}]$  and then, for  $\underline{\lambda}^* \in k^s$ , to the morphism  $s_{\underline{\lambda}^*} \circ \sigma \circ \varphi_\lambda : \mathbb{Z}[a_i] \rightarrow k$  obtained by composing  $\sigma \circ \varphi_\lambda$  with the specialization morphism  $s_{\underline{\lambda}^*} : k[\underline{\lambda}] \rightarrow k$  to  $\underline{\lambda}^*$ . Conclude:

**Proposition 2.3** (Bertini-Noether generalized).

(a) *The polynomial  $\mathcal{F}^\sigma(\underline{\lambda}, \underline{x})$ , if it is of degree  $d$  in  $\underline{x}$ , is irreducible in  $\overline{k(\underline{\lambda})}[\underline{x}]$  if and only if the Zariski closed subset  $V_{\mathcal{F}^\sigma} \subset \mathbb{A}_k^s$  is proper. All these conditions are satisfied if  $a_{\mathcal{F}^\sigma}$  is non-zero in  $k$ .*

(b) *If the polynomial  $\mathcal{F}^\sigma(\underline{\lambda}^*, \underline{x})$  is of degree  $d$ , then it is reducible in  $k[\underline{x}]$  if and only if  $\underline{\lambda}^*$  is in the set  $V_{\mathcal{F}^\sigma}(k)$ .*

2.3.2. *2nd stage: implications for the spectrum of  $F(\underline{x})$ .* We return to the situation where  $\mathcal{F}(\underline{\lambda}, \underline{x}) = F(\underline{x}) - \lambda$ . Denote the Zariski closed subset

$V_{\mathcal{F}}$  from §2.3.1 by  $V_F$ ; it is a Zariski closed subset of the affine line  $\mathbb{A}_A^1$ . The preceding conclusions, conjoined with the connection, recalled in §1, between indecomposability of  $F(\underline{x})$  and irreducibility of  $F(\underline{x}) - \lambda$ , yield statement (\*) from §2.1.

Denote by  $s_F(\lambda)$  the g.c.d. of the polynomials  $N_h(\lambda)$  in the ring  $K[\lambda]$ . Write it as  $s_F(\lambda) = S_F(\lambda)/c_1$  with  $S_F(\lambda) \in A[\lambda]$  and  $c_1 \in A$  non-zero. The polynomial  $S_F(\lambda)$  is non-zero and its distinct roots in  $\overline{K}$ , say  $\lambda_1, \dots, \lambda_s$ , which are the common roots in  $\overline{K}$  of the polynomials  $N_h(\lambda)$ , are the spectral values of  $F(\underline{x})$  (note that  $F(\underline{x}) - \lambda^*$  is of degree  $d$  for all  $\lambda^* \in \overline{K}$ ). Thus we have  $S_F(\lambda) = c_2 \prod_{i=1}^s (\lambda - \lambda_i)^{n_i} \in A[\lambda]$  for some exponents  $n_i > 0$  and  $c_2 \in A$ ,  $c_2 \neq 0$ . It follows that the set  $\text{sp}(F) = \{\lambda_1, \dots, \lambda_s\}$  is  $K$ -rational. As already noted, the same is then true for the spectral divisor  $\text{spdiv}(F)$ .

*2.3.3. 3rd stage: invariance of the spectrum of  $F$  via morphisms.* Fix a morphism  $\sigma : A \rightarrow k$  with  $k$  algebraically closed. Denote by  $a_F$  the element  $a_{\mathcal{F}}$  from §2.3.1 for  $\mathcal{F} = F(\underline{x}) - \lambda$ . If  $a_F^\sigma \neq 0$ ,  $F^\sigma(\underline{x})$  is of degree  $d$  and indecomposable in  $k[\underline{x}]$ . Furthermore, its spectral values are the roots in  $k$  of the g.c.d. of the polynomials  $N_h^\sigma(\lambda)$ .

Note that the element  $c_2$  above is a common denominator of  $\lambda_1, \dots, \lambda_s$ ; if  $c_2^\sigma \neq 0$ , the morphism  $\sigma : A \rightarrow k$  is defined at  $\text{spdiv}(F)$ .

**Lemma 2.4.** *There exists  $c_3 \in A$ ,  $c_3 \neq 0$  such that, if  $a_F^\sigma c_1^\sigma c_2^\sigma c_3^\sigma \neq 0$ , the polynomial  $S_F^\sigma(\lambda) \in k[\lambda]$  equals (up to some non-zero multiplicative constant in  $k$ ) the g.c.d. in  $k[\lambda]$  of polynomials  $N_h^\sigma(\lambda)$  ( $1 \leq h \leq D$ ). In particular  $\text{sp}(F^\sigma) = (\text{sp}(F))^\sigma$ .*

*Proof.* The problem is whether the g.c.d. commutes with  $\sigma$ . The Euclidean algorithm provides the g.c.d. as the last non-zero remainder. To reach our goal, it suffices to guarantee that for each division  $a = bq + r$  in  $K[\lambda]$  involved in the algorithm, the identity  $a^\sigma = b^\sigma q^\sigma + r^\sigma$ , with  $\sigma$  suitably extended, be the division of  $a^\sigma$  by  $b^\sigma$  in  $k[\lambda]$ . For this, write  $a, b, q$  as  $r$  in the form  $n(\lambda)/m$  with  $n(\lambda) \in A[\lambda]$  and  $m \in A$ , consider the product  $\beta$  of denominators  $m$  of  $a, b, q$  and  $r$  with the coefficients of highest degree monomials in the numerators  $n(\lambda)$  of  $b$  and  $r$  and request that  $\beta^\sigma \neq 0$ . Multiplying all elements  $\beta$  for all divisions leading to the g.c.d. of two, then of all polynomials in question, leads to a non-zero element  $c_3 \in A$  which satisfies the desired statement.  $\square$

*Remark 2.5.* Morphisms and g.c.d. do not commute in general: for example  $\text{gcd}(\lambda, \lambda + a)$  is 1 generically, but equals  $\lambda$  if  $a = 0$ .

*2.3.4. 4th stage: invariance of  $\text{spdiv}(F)$  via morphisms.* It remains to extend the conclusion “ $\text{sp}(F^\sigma) = (\text{sp}(F))^\sigma$ ” to the spectral divisor  $\text{spdiv}(F)$ .

We will show how to guarantee that, *via* the morphism  $\sigma$ , the spectral values remain distinct and the associated decompositions of  $F(\underline{x}) - \lambda$  have the same numbers of distinct irreducible factors<sup>4</sup>.

Consider the discriminant of the polynomial  $\prod_{i=1}^s (\lambda - \lambda_i)$ ; it is a non-zero element of  $K$ . Write it as  $c_4/c_5$  with  $c_4, c_5 \in A$ , non-zero. If  $c_4^\sigma c_5^\sigma \neq 0$ , the polynomials  $S_F(\lambda)$  and  $S_F^\sigma(\lambda)$  have the same number of distinct roots, whence  $\text{card}(\text{sp}(F^\sigma)) = \text{card}((\text{sp}(F))^\sigma) = \text{card}(\text{sp}(F))$ .

For  $i = 1, \dots, s$ , let  $F(\underline{x}) - \lambda_i = \prod_{j=1}^{n(\lambda_i)} Q_{ij}(\underline{x})^{k_{ij}}$  be a factorization (into distinct irreducible polynomials) in  $\overline{K}[\underline{x}]$ . Let  $E/K$  be a finite Galois extension that contains the finite set  $\mathcal{C}$  of all coefficients involved in all above factorizations,  $c_6$  be a non-zero element of  $A$  such that  $c_6 c$  is integral over  $A$  for all  $c \in \mathcal{C}$  and  $c_7$  be the discriminant of a basis of  $E$  over  $K$  the elements of which are integral over  $A$ . Denote by  $B$  the fraction ring of  $A$  with denominator a power of  $c_6 c_7$  and by  $B'_E$  the integral closure of  $B$  in  $E$ . The ring  $B'_E$  is a free  $B$ -module of rank  $[E : K]$ . Assume that  $c_6^\sigma c_7^\sigma \neq 0$ . The morphism  $\sigma : A \rightarrow k$  extends to a morphism  $B \rightarrow k$ , and, as  $k$  is algebraically closed, this morphism  $\sigma : B \rightarrow k$  can in turn be extended to a morphism  $\tilde{\sigma} : B'_E \rightarrow k$ .

The polynomials  $Q_{ij}(\underline{x})$  are in the ring  $B'_E[\underline{x}]$  and are absolutely irreducible. The (classical) Bertini-Noether theorem provides a non-zero element  $\beta \in B'_E$  such that, if  $\beta^{\tilde{\sigma}} \neq 0$ , then each of the polynomials  $Q_{ij}^{\tilde{\sigma}}(\underline{x})$  is absolutely irreducible. Therefore the decomposition  $F^\sigma(\underline{x}) - \lambda_i^{\tilde{\sigma}} = \prod_{j=1}^{n(\lambda_i)} Q_{ij}^{\tilde{\sigma}}(\underline{x})$  obtained from the preceding one by applying  $\tilde{\sigma}$ , is the factorization of  $F^\sigma(\underline{x}) - \lambda_i^{\tilde{\sigma}}$  into irreducible polynomials in  $k[\underline{x}]$ .

It remains to assure that for  $i$  fixed, the polynomials  $Q_{ij}^{\tilde{\sigma}}(\underline{x})$  are different, even up to non-zero multiplicative constants. For any two (distinct) polynomials  $Q_{ij}(\underline{x}), Q_{ij'}(\underline{x})$ , the matrix with rows the tuples of coefficients of the two polynomials has a  $2 \times 2$ -block with a non-zero determinant. Denote the product of all such determinants for all possible couples  $(Q_{ij}(\underline{x}), Q_{ij'}(\underline{x}))$  by  $\delta$ ; it is a non-zero element of  $B'_E$ . Denote then by  $\nu$  the norm of  $\beta\delta$  relative to the extension  $E/K$ . As  $A$  is integrally closed, so is  $B$  and  $\nu \in B$ . Write it as  $\nu = c_8/(c_6 c_7)^\gamma$  with  $c_8 \in A$  and  $\gamma \in \mathbb{N}$ . Condition  $c_6^\sigma c_7^\sigma c_8^\sigma \neq 0$  implies  $\beta_F^{\tilde{\sigma}} \delta_F^{\tilde{\sigma}} \neq 0$ . Theorem 2.1 is finally established for  $h_F = a_F \prod_{i=1}^s c_i$ .

*Remark 2.6.* The same proof, with the polynomial  $\mathcal{F}(\underline{\lambda}, \underline{x})$  from §2.3.1 of the form  $F(\underline{x}) - \lambda G(\underline{x})$  with  $F(\underline{x}), G(\underline{x}) \in A[\underline{x}]$  and  $\deg G \leq \deg F$ , leads to the more general form of theorem 2.1 for which indecomposable polynomials are replaced by indecomposable rational functions (in this case,

<sup>4</sup>The argument will also show the degrees of these irreducible factors, say  $Q_{\lambda,j}$ , remain the same and thus so does the quantity  $\min_{\lambda \in \text{sp}(F)} (\sum_j \deg(Q_{\lambda,j}) - 1)$  which replaces  $\deg(F) - 1$  in Lorenzini's refined version [10] of Stein's inequality.



“indecomposable” means not of the form  $u(H(\underline{x}))$  with  $H(\underline{x})$  and  $u(t)$  rational functions and  $\deg(u) \geq 2$ <sup>5</sup>. A spectral value of a rational function  $F(\underline{x})/G(\underline{x})$  is an element  $\lambda$  such that the polynomial  $F(\underline{x}) - \lambda G(\underline{x})$  is reducible. Statements (1), (2) and (3) from §1 remain true, except that the bound in Stein’s inequality should be replaced by  $(\deg(F))^2 - 1$  [2] [10]. More generally one can take  $\mathcal{F}(\underline{\lambda}, \underline{x})$  of the form  $F(\underline{x}) - \lambda_1 G_1(\underline{x}) - \dots - \lambda_s G_s(\underline{x})$  with  $F(\underline{x}), G_1(\underline{x}), \dots, G_s(\underline{x}) \in A[\underline{x}]$  and handle other situations studied in the literature.

### 3. AN INDECOMPOSABILITY CRITERION MODULO $p$

In this section  $n = 2$ ,  $A$  is a Dedekind domain and its fraction field  $K$  is assumed to be of characteristic 0. Fix also a non-zero prime ideal  $\mathfrak{p}$  of  $A$  and assume its residue field  $k = A/\mathfrak{p}$  is of characteristic  $p > 0$ . Denote by  $\tilde{x}$  the image of an element  $x$  by the reduction morphism  $A \rightarrow k$ . The situation “ $A = \mathbb{Z}$  and  $\mathfrak{p} = p\mathbb{Z}$ ” is typical.

Let  $F(x, y) \in A[x, y]$  be an indecomposable polynomial in  $\overline{K}[x, y]$  of degree  $d \geq 1$ , monic in  $y$ .

Here is our strategy to guarantee indecomposability of  $F(x, y)$  modulo  $\mathfrak{p}$ . Pick  $\lambda^* \in A \setminus \text{sp}(F)$  (using Stein’s theorem, this can be done with  $\lambda^*$  not too big). Thus  $F(x, y) - \lambda^*$  is irreducible in  $\overline{K}[x, y]$ . It follows from the classical Bertini-Noether theorem that if “ $\mathfrak{p}$  is big enough”, then the reduced polynomial  $F(x, y) - \lambda^*$  modulo  $\mathfrak{p}$  is absolutely irreducible. Therefore  $F(x, y)$  is indecomposable modulo  $\mathfrak{p}$  (as there is at least one non spectral value). However the constants involved in the condition “ $\mathfrak{p}$  big enough” are too big for a practical algorithmic use. We will follow an alternate approach, based on good reduction criteria for covers, and more precisely Zannier’s criterion [14].

Consider the discriminant with respect to  $y$  of  $F(x, y) - \lambda$ :

$$\Delta_F(x, \lambda) = \text{disc}_y(F(x, y) - \lambda)$$

Denote then the product of all distinct irreducible factors of  $\Delta_F(x, \lambda)$  in  $K(\lambda)[x]$  by  $\Delta_F^{\text{red}}(x, \lambda)$ ; more precisely,  $\Delta_F^{\text{red}}(x, \lambda)$  is defined by the following formula, which is also algorithmically more practical:

$$\Delta_F^{\text{red}}(x, \lambda) = c(\lambda) \frac{\Delta_F(x, \lambda)}{\text{gcd}(\Delta_F(x, \lambda), (\Delta_F)'_x(x, \lambda))}$$

where the g.c.d. is calculated in the ring  $K(\lambda)[x]$  (using the Euclidean algorithm for example) and  $c(\lambda) \in K(\lambda)$  is the rational function, defined up to

---

<sup>5</sup>the degree of a rational function is the maximum of the degrees of its numerator and denominator.

some invertible element in  $A$ , that makes  $\Delta_F^{\text{red}}(x, \lambda)$  a primitive polynomial in  $A[\lambda][x]$ . Consider next the polynomial:

$$\Delta_F(\lambda) = \text{disc}_x(\Delta_F^{\text{red}}(x, \lambda)).$$

We have  $\Delta_F(\lambda) \in A[\lambda]$  and  $\Delta_F(\lambda) \neq 0$ . Finally let  $\Delta_0(\lambda) \in A[\lambda]$  be the coefficient of the highest monomial in  $\Delta_F(x, \lambda)$  (viewed in  $A[\lambda][x]$ ).

**Theorem 3.1.** *Assume, in addition to  $F(x, y)$  being indecomposable in  $\overline{K}[x, y]$ , that the reduced polynomial  $\tilde{\Delta}_0(\lambda)\tilde{\Delta}_F(\lambda)$  is non-zero in  $k[\lambda]$  and that  $p > \deg_y(F)$ . Then  $\tilde{F}(x, y)$  is indecomposable in  $\overline{k}[x, y]$ .*

The assumption  $p > \deg_y(F)$  can be replaced by the weaker condition that  $p$  does not divide the order of the Galois group of  $F(x, y) - \lambda$ , viewed as a polynomial in  $\overline{K(\lambda)}(x)$  (see footnote 8).

The assumptions of theorem 3.1 may not be sufficient to guarantee the extra conclusions  $\text{sp}(\tilde{F}) = \widetilde{\text{sp}(F)}$  and  $\text{spdiv}(\tilde{F}) = \widetilde{\text{spdiv}(F)}$  from theorem 2.1 (which may not even be well-defined). It is still true however that if  $V_F \subset \mathbb{A}_A^1$  is the Zariski closed subset from §2.1, then the reduced Zariski closed subset  $\widetilde{V}_F \subset \mathbb{A}_{\overline{k}}^1$  is proper and its points are the spectral values of  $\tilde{F}$ :  $\text{sp}(\tilde{F}) = \widetilde{V}_F(\overline{k})$ .

*Remark 3.2* (an indecomposability test). Theorem 3.1 provides the following procedure to decide whether a non-constant polynomial  $F(x, y) \in \mathbb{Q}[x, y]$  is indecomposable.

One may assume that  $F(x, y) \in \mathbb{Z}[x, y]$  and  $\deg_y(F) > 0$ . Up to changing  $y$  to  $ay$  for some  $a \in \mathbb{Z}$ , one may also reduce to the case  $F(x, y)$  is monic with respect to the reverse lexicographic order (for which  $y > x$ ). Observe that the polynomial  $\Delta_F(x, \lambda) = \text{disc}_y(F(x, y) - \lambda)$  is non-zero in general (whether  $F$  is indecomposable or not): indeed none of the roots  $y$  of  $(\partial F/\partial y)(x, y)$ , which are in  $\overline{\mathbb{Q}(x)}$ , can also be a root of  $F(x, y) - \lambda$ . Consequently the polynomial  $\Delta_0(\lambda)\Delta_F(\lambda)$  is non-zero in general. Pick a prime  $p$  satisfying the assumptions of theorem 3.1:  $p > \deg_y(F)$  and  $\tilde{\Delta}_0(\lambda)\tilde{\Delta}_F(\lambda)$  is non-zero in  $\mathbb{F}_p[\lambda]$ . Test for decomposability of  $\tilde{F}(x, y)$  (from §4, it is sufficient to only consider decompositions over  $\mathbb{F}_p$  (instead of  $\overline{\mathbb{F}_p}$ )). If  $\tilde{F}(x, y)$  is decomposable, then  $F(x, y)$  is decomposable by theorem 3.1. As we explain below the converse also holds: if  $\tilde{F}(x, y)$  is indecomposable, then  $F(x, y)$  is indecomposable.

Namely if  $F(x, y)$  is decomposable then  $F(x, y)$  has a non-trivial decomposition  $F(x, y) = u(H(x, y))$  with  $u$  and  $H$  with coefficients in  $\overline{\mathbb{Q}}$  and even in  $\mathbb{Q}$  (see §4). Furthermore  $F(x, y) \in \mathbb{Z}[x, y]$  being monic forces these coefficients to be in  $\mathbb{Z}$ . For one variable, this is explained in [5], and in general one can reduce to this case thanks to a Kronecker substitution. Specifically

one may assume that  $H(x, y)$  is monic (w.r.t. the same order), and then so is  $u$ , and that  $H(0, 0) = 0$ . Write  $F(x, x^m) = u(H(x, x^m))$  with  $m$  large enough to have  $F(x, x^m)$  and  $H(x, x^m)$  monic. From [5, theorem 2],  $u(t)$  and  $H(x, x^m)$  must have integral coefficients, and consequently so does  $H(x, y)$  (for  $m \gg 1$ ). Finally reduction modulo  $p$  of  $F(x, y) = u(H(x, y))$  provides a non-trivial decomposition of  $\tilde{F}(x, y)$ .

*Proof of theorem 3.1.* The prime ideal  $\mathfrak{p} \subset A$  determines a discrete valuation  $v$  of  $K$  whose valuation ring is the localized ring  $A_{\mathfrak{p}}$ ; the fraction field of  $A_{\mathfrak{p}}$  and its residue field remain equal to  $K$  and  $k$  respectively. Hypotheses and conclusions from theorem 3.1 are unchanged if  $A$  is replaced by  $A_{\mathfrak{p}}$ . The valued field  $(K, v)$  can then also be replaced by any finite extension of the completion  $K_v$  and  $A$  by the new valuation ring; the discrete valuation  $v$  uniquely extends, the residue field is replaced by some (finite) extension of  $k$ , the indecomposability properties of  $F(x, y)$  over  $K$  or over  $K_v$  are equivalent.

Thus we may and will assume that  $(K, v)$  is a complete discretely valued field, that  $A$  is its valuation ring (which is integrally closed) and that the field  $K$  and the residue field  $k$  contain as many (finitely many) algebraic elements over the original fields as necessary.

The polynomial  $\Delta_F(x, \lambda)$  is in  $A[x, \lambda]$  and its factorization into irreducible polynomials in  $K(\lambda)[x]$  can be written

$$\Delta_F(x, \lambda) = \delta_0(\lambda) \prod_{i=1}^s \Delta_i(x, \lambda)^{\alpha_i}$$

where the polynomials  $\Delta_i(x, \lambda)$  are in  $A[x, \lambda]$ , irreducible in  $K(\lambda)[x]$ , pairwise distinct (even up to some constant in  $K$ ) and are primitive in  $A[\lambda][x]$ , where  $\delta_0(\lambda) \in A[\lambda]$  and where the  $\alpha_i$  are positive integers. Then, up to some invertible element in  $A$ , we have

$$\Delta_F^{\text{red}}(x, \lambda) = \prod_{i=1}^s \Delta_i(x, \lambda)$$

Also note that the polynomial  $\Delta_0(\lambda)$  is a multiple in  $A[\lambda]$  of the product of  $\delta_0(\lambda)$  with the highest monomial coefficients  $\delta_1(\lambda), \dots, \delta_s(\lambda)$  of the polynomials  $\Delta_1(x, \lambda), \dots, \Delta_s(x, \lambda)$  (viewed in  $A(\lambda)[x]$ ).

Pick next  $\tilde{\lambda}^* \in k$  such that  $\tilde{\Delta}_0(\tilde{\lambda}^*)\tilde{\Delta}_F(\tilde{\lambda}^*) \neq 0$  in  $k$ , then lift it to some element  $\lambda^* \in A$  such that  $\lambda^* \notin \text{sp}(F)$ . This is possible in view of the preliminary remark.

The set of roots of  $\Delta_F(x, \lambda^*)$  contains the set of finite<sup>6</sup> branch points of the cover of  $\mathbb{P}_x^1$ <sup>7</sup> determined by the (absolutely irreducible) polynomial  $F(x, y) - \lambda^*$ . The preliminary remark makes it possible to assume that these roots are in  $K$ . Furthermore as  $\tilde{\delta}_i(\tilde{\lambda}^*) \neq 0$ , we have  $\delta_i(\lambda^*) \in A \setminus \mathfrak{p}$ ,  $i = 1, \dots, s$ ; therefore these roots are integral over  $A$  and so are in  $A$ .

As  $\Delta_F(\lambda^*) \neq 0$ , the roots of  $\Delta_F^{\text{red}}(x, \lambda^*)$  in  $\overline{K}$  are distinct and as  $\delta_0(\lambda^*) \neq 0$ , they are the roots of  $\Delta_F(x, \lambda^*)$ . As  $\tilde{\Delta}_0(\tilde{\lambda}^*) \neq 0$ ,  $\tilde{\Delta}_F(x, \tilde{\lambda}^*)$  is not the zero polynomial. As  $\tilde{\Delta}_F(\tilde{\lambda}^*) \neq 0$ , the roots of  $\tilde{\Delta}_F^{\text{red}}(x, \tilde{\lambda}^*)$ , which are those of the polynomial  $\tilde{\Delta}_F(x, \tilde{\lambda}^*)$ , are distinct. Thus we obtain that the distinct roots of the polynomial  $\Delta_F(x, \lambda^*)$ , and *a fortiori* the branch points of the cover considered above, have distinct reductions modulo the ideal  $\mathfrak{p}$ .

It follows from standard results on good reduction of covers, and more precisely here, from the main theorem of [14] that, under the assumption  $p > \deg_y(F)$ <sup>8</sup>,  $\tilde{F}(x, y) - \tilde{\lambda}^*$  is absolutely irreducible. Hence  $\tilde{F}(x, y)$  is indecomposable in  $\overline{k}[x, y]$ .  $\square$

#### 4. INDECOMPOSABILITY OVER $K$ VERSUS $\overline{K}$

**4.1. Statements** (for  $n \geq 2$  variables). The indecomposability property which we recalled the definition of in §1 over an algebraically closed field can in fact be defined over an arbitrary field: just require that the polynomials  $u(t)$  and  $H(\underline{x})$  involved have their coefficients in the field in question. The results below identify the only cases where the property is not the same over some field  $K$  and over some extension  $E$ . The following result handles the case that  $E/K$  is purely inseparable, which was missing in the literature.

**Proposition 4.1.** *Let  $E/K$  be a purely inseparable algebraic field extension of characteristic  $p > 0$  and  $F(\underline{x}) \in K[\underline{x}]$ . Assume  $F(\underline{x})$  is not of the form  $bG(\underline{x})^p + c$  with  $G(\underline{x}) \in E[\underline{x}]$  and  $b, c \in K$ . Then  $F(\underline{x})$  is indecomposable in  $K[\underline{x}]$  if and only if it is indecomposable in  $E[\underline{x}]$ .*

If  $E = \overline{K}$ , the assumption on  $F(\underline{x})$  rewrites to merely say that  $F(\underline{x})$  is not a  $p$ -th power in  $\overline{K}[\underline{x}]$ , which in turn is equivalent to at least one exponent in  $F(\underline{x})$  not being a multiple of  $p$ . Clearly this assumption cannot be removed: for example, if  $\alpha \in \overline{K} \setminus K$  but  $\alpha^p = a \in K$  then  $x^p + ay^p$  is indecomposable in  $K[\underline{x}]$  but decomposable in  $\overline{K}[\underline{x}]$ .

<sup>6</sup>*i.e.*, distinct from the point at infinity.

<sup>7</sup>The subscript “ $x$ ” indicates that the cover is induced by the correspondence  $(x, y) \rightarrow x$ . In fact the problem is symmetric in the variables  $x$  and  $y$  which can be switched in our statement.

<sup>8</sup>It suffices to assume that  $p$  does not divide the order of the Galois group of  $F(x, y) - \lambda^*$ , which divides the order of the Galois group of  $F(x, y) - \lambda$ , which itself divides  $(\deg_y(F))!$ .

In [1, proposition 1], Arzhantsev and Petravchuk show the equivalence from proposition 4.1 without any assumption on  $F(\underline{x})$ , but in the case of a separable extension  $E/K$  (possibly of positive transcendence degree). As any extension is a purely inseparable algebraic extension of some separable extension, conjoining their result with ours yields that, under the assumption on  $F(\underline{x})$  from proposition 4.1, the equivalence holds for an arbitrary extension  $E/K$ . We can be more precise.

**Theorem 4.2.** *Let  $E/K$  be a field extension and  $F(\underline{x}) \in K[\underline{x}]$  be a non-constant polynomial. Then the following are equivalent:*

- (i)  $F(\underline{x})$  is indecomposable in  $K[\underline{x}]$  but decomposable in  $E[\underline{x}]$ .
- (ii) (a)  $K$  is of characteristic  $p > 0$  and  $E/K$  is inseparable,
  - (b)  $F(\underline{x}) = bG(\underline{x})^p + c$  for some  $G(\underline{x}) \in E[\underline{x}]$  and  $b, c \in K$ , and
  - (c)  $G(\underline{x})^p$  is indecomposable in  $K[\underline{x}]$ .

Condition (ii) (c) implies that  $G(\underline{x})$  is not of the form  $u(H(\underline{x}))$  with  $u \in E[t]$ ,  $H(\underline{x}) \in E[\underline{x}]$ ,  $\deg(u) \geq 2$  and both  $u(t)^p \in K[t]$  and  $H(\underline{x})^p \in K[\underline{x}]$ . But there are other possible polynomials that should be excluded whose description is more intricate.

#### 4.2. Proofs.

*Proof of proposition 4.1.* The converse part is obvious. For the direct part, assume  $F(\underline{x})$  is decomposable in  $E[\underline{x}]$ . Then it is decomposable over some finite extension of  $K$  contained in  $E$ , which admits a finite system of generators  $\alpha_1, \dots, \alpha_s$  with irreducible polynomial over  $K$  of the form  $x^{p^n} - a$  with  $a \in K$ . The multiplicativity of the degree and of the separable degree imply that the extensions  $K(\alpha_1, \dots, \alpha_{j+1})/K(\alpha_1, \dots, \alpha_j)$  are purely inseparable,  $j = 1, \dots, s - 1$ . By induction one reduces to the case  $s = 1$ , and then a new induction reduces to the case  $E = K(\alpha)$  with  $\alpha^p = a \in K \setminus K^p$ .

Assume  $F(\underline{x}) = h(G(\underline{x}))$  with  $h(t) \in K(\alpha)[t]$  such that  $\deg(h) \geq 2$  and  $G(\underline{x}) \in K(\alpha)[\underline{x}]$ . We deduce

$$F(\underline{x})^p = {}^p h(G(\underline{x})^p)$$

where, if  $h(t) = \sum_{i=0}^{\deg(h)} h_i t^i$ , we set  ${}^p h(t) = \sum_{i=0}^{\deg(h)} h_i^p t^i$ . As  ${}^p h(t) \in K[t]$  and  $G(\underline{x})^p \in K[\underline{x}]$  (since  $y^p \in K$  for all  $y \in K(\alpha)$ ), this shows that the field  $K(F(\underline{x}), G(\underline{x})^p)$  is of transcendence degree 1 over  $K$ . From Gordan's theorem [12, §1.2, th.3], there exists  $\theta(\underline{x}) \in K(\underline{x})$  such that

$$K(F(\underline{x}), G(\underline{x})^p) = K(\theta(\underline{x}))$$

Furthermore from [12, §1.2, th.4], one may assume that  $\theta(\underline{x}) \in K[\underline{x}]$ . Thus we have

$$\begin{cases} F(\underline{x}) = u(\theta(\underline{x})) \text{ with } u(t) \in K[t] \\ G(\underline{x})^p = v(\theta(\underline{x})) \text{ with } v(t) \in K[t] \end{cases}$$

As  $F(\underline{x})$  and  $G(\underline{x})^p$  are polynomials,  $u(t), v(t)$  are necessarily in  $K[t]$ . It follows from the indecomposability of  $F(\underline{x})$  over  $K$  that  $\deg(u) = 1$ , which gives  $G(\underline{x})^p = w(F(\underline{x}))$  for some polynomial  $w \in K[t]$ . But then we obtain  $G(\underline{x})^p = w \circ h(G(\underline{x}))$ , which, since  $G(\underline{x})$  is non constant, amounts to  $T^p = w \circ h(T)$  where  $T$  is an indeterminate. As  $\deg(h) \geq 2$  and  $p$  is a prime, we have  $\deg(w) = 1$  and  $\deg(h) = p$ , which gives  $F(\underline{x}) = bG(\underline{x})^p + c$  for some  $b, c \in K$ .

Note that because of the inductive process, conclusion “ $b, c \in K$ ” should really be that  $b, c$  are in the first subfield of the initial reduction. But  $F(\underline{x})$  being in  $K[\underline{x}]$  then implies that  $b\gamma^p \in K$  for some non-zero  $\gamma \in E$  and  $bG(\underline{0})^p + c \in K$ . Up to changing  $G(\underline{x})$  to  $\gamma^{-1}G(\underline{x}) - \gamma^{-1}G(\underline{0})$ , one can indeed conclude that  $b, c \in K$  in the general situation.  $\square$

*Proof of theorem 4.2.* (i)  $\Rightarrow$  (ii): If  $K_s/K$  is the maximal separable extension contained in  $E$ , then, from the Arzhantsev-Petravchuk result,  $F(\underline{x})$  is indecomposable in  $K_s[\underline{x}]$ . In particular  $E \neq K_s$ , which gives (ii) (a). Proposition 4.1 then provides condition (ii) (b) except that  $b$  and  $c$  are *a priori* in  $K_s$ , but using again the final note of the proof of Proposition 4.1, one can indeed choose  $b, c \in K$ . Condition (ii) (c) then readily follows from (ii) (b) and the indecomposability of  $F(\underline{x})$  in  $K[\underline{x}]$ . The other implication (ii)  $\Rightarrow$  (i) is clear.  $\square$

**4.3. One variable.** In proposition 4.1,  $F(\underline{x})$  is a polynomial in two variables or more. In one variable, the indecomposability definition should be modified (for otherwise it is trivial): a polynomial  $F(x) \in k[x]$  is said to be indecomposable in  $k[x]$  if it is not of the form  $u(H(x))$  with  $H(x) \in k[x]$  and  $u \in k[t]$  with  $\deg(u) \geq 2$  and  $\deg(H) \geq 2$ .

**Proposition 4.3.** *Proposition 4.1 holds for one variable polynomials.*

*Proof.* The same proof can be used as for proposition 4.1. It leads to

$$\begin{cases} F(x) = u(\theta(x)) \text{ with } u(t) \in K[t] \\ G(x)^p = v(\theta(x)) \text{ with } v(t) \in K[t] \end{cases}$$

But from the indecomposability of  $F(\underline{x})$  over  $K$ , we now deduce that  $\deg(u) = 1$  or  $\deg(\theta) = 1$ .

The case  $\deg(u) = 1$  is handled as before. In the other case, we deduce from  $\deg(\theta) = 1$  that  $K(F(x), G(x)^p) = K(x)$ , which implies that  $K(\alpha)(h(G(x)), G(x)^p) = K(\alpha)(x)$  and so that

$$K(\alpha)(x) \subset K(\alpha)(G(x))$$

which forces  $\deg(G) = 1$  and contradicts the decomposability assumption in one variable made at the beginning of the proof.  $\square$

## 5. COUNTING INDECOMPOSABLE POLYNOMIALS OVER FINITE FIELDS

For each integer  $d \geq 1$ , denote the number of polynomials in  $\mathbb{F}_q[\underline{x}]$  ( $\underline{x} = (x_1, \dots, x_n)$ ) of degree  $d$  by  $N_d$ . We have

$$\begin{cases} N_d = \left( q^{\binom{n+d-1}{n-1}} - 1 \right) \cdot q^{\binom{n+d-1}{n}} & (\text{for general } n) \\ N_d = q^{\frac{1}{2}(d+1)(d+2)}(1 - q^{-d-1}) & (\text{for } n = 2) \\ N_d = (q - 1)q^d & (\text{for } n = 1) \end{cases}$$

Denote the number of those polynomials which are indecomposable (resp. decomposable) by  $I_d$  (resp.  $D_d$ ). We have  $N_d = I_d + D_d$ .

We will study separately the case of  $n \geq 2$  variables (§5.1 - §5.4) and the case  $n = 1$  (§5.5).

**5.1. Main result.** From §5.1 to §5.4, we assume  $n \geq 2$ .

**Theorem 5.1.** (a)  $I_d/N_d$  tends to 1 in the two situations where  $d \rightarrow \infty$  with  $q$  fixed, and where  $q \rightarrow \infty$  with  $d$  fixed.

(b) If  $d$  is a product of at most 2 prime numbers  $p \leq p'$ , then

- $d = p$  and  $D_d = q^d(q^n - 1)$ , or
- $d = p^2$  and  $D_d = q^{p-1}N_p + (q^d - q^{2p-1})(q^n - 1)$ , or
- $d = pp'$  with  $p < p'$  and  

$$D_d = q^{p-1}N_{p'} + q^{p'-1}N_p + (q^d - 2q^{p+p'-1})(q^n - 1).$$

(c) Assume  $n = 2$ . If  $d$  is the product of at least 3 prime numbers, then

$$\left| \frac{D_d}{N_d} - \alpha_d \right| \leq \alpha_d \beta_d \quad \text{where} \quad \begin{cases} \alpha_d = \frac{q^{\ell-1+\frac{1}{2}(\frac{d}{\ell}+1)(\frac{d}{\ell}+2)}}{q^{\frac{1}{2}(d+1)(d+2)}} \\ \beta_d = \frac{d}{q^{\frac{d}{\ell}}} \end{cases}$$

and  $\ell > 1$  is the first (hence prime) divisor of  $d$ .

A version of statement (c) in the general case of  $n$  variables is given in [8]. For  $n = 2$ , his first order estimate for  $D_d/N_d$  is the same as ours, that is  $\alpha_d$ ; his error term is improved by a factor  $O(q)$ .

**5.2. An induction formula.** Let  $K$  be an arbitrary field. Let  $F = u \circ H$  be a decomposition of  $F \in K[x]$  with  $u \in K[t]$ ,  $\deg u \geq 2$ , and  $H \in K[x]$ . We say that  $F = u \circ H$  is a *normalized decomposition* if  $H$  is indecomposable, monic (*i.e.* the coefficient of the leading term of a chosen order is 1) and its constant term equals zero. Given a decomposition  $F = u \circ H$ , there exists an associated normalized decomposition  $F = u' \circ H'$ . The following lemma shows it is unique.

**Lemma 5.2.** *Let  $F = u \circ H = u' \circ H'$  be two normalized decompositions of  $F \in K[x]$ . Then  $u = u'$  and  $H = H'$ .*

*Proof.* It follows from  $u(H) - u'(H') = 0$  that  $H$  and  $H'$  are algebraically dependent over  $K$ . By Gordan's theorem [12, §1.2, theorems 3 and 4] (already used in §4.2), there exists a polynomial  $\theta(x) \in K[x]$  such that  $K[\theta] = K[H, H']$ . That is, there exist  $v, v' \in K[t]$  such that  $H = v(\theta)$  and  $H' = v'(\theta)$ . As the two decompositions of  $F$  are normalized,  $H$  and  $H'$  are indecomposable, so  $\deg v = \deg v' = 1$ , and so using the other normalization conditions, we obtain  $H = H'$ . Finally it follows from  $u(H) = u'(H)$  that  $u = u'$ .  $\square$

**Corollary 5.3** (induction formula). *With notation as in §5.1, we have*

$$I_d = N_d - \sum_{d'|d, d' < d} q^{\frac{d}{d'} - 1} \times I_{d'}$$

*Proof.* Let  $d' \geq 1$  be a divisor of  $d$ . There are  $(q-1)q^{d/d'}$  polynomials  $u \in \mathbb{F}_q[t]$  of degree  $d/d'$  and  $I_{d'}/q(q-1)$  normalized indecomposable polynomials  $H \in \mathbb{F}_q[x]$  of degree  $d'$ . The formula follows as from lemma 5.2, every polynomial  $F$  counted by  $D_d$  can be uniquely written  $F = u \circ H$  with  $u$  and  $H$  as above for some integer  $d'$  such that  $d'|d$ ,  $d' < d$ .  $\square$

Conjoined with  $I_1 = N_1 = q(q^n - 1)$  this formula provides an algorithm to compute  $I_d$  and  $D_d$ , which is convenient for small  $d$ .

**5.3. Proof of theorem 5.1 (a) and (b).** The formulas in (b) straightforwardly follow from corollary 5.3. If  $d = p$  is a prime number, we have  $D_p = q^{p-1}I_1 = q^{p-1}N_1 = q^p(q^n - 1)$ . If  $d = p^2$  then

$$\begin{aligned} D_d &= q^{p-1}I_p + q^{p^2-1}I_1 \\ &= q^{p-1}(N_p - q^p(q^n - 1)) + q^{p^2}(q^n - 1). \end{aligned}$$

Computations are similar for  $d = pp'$ . To prove (a) we write

$$N_d - I_d = D_d = \sum_{d'|d, d' < d} q^{d/d'} I_{d'} \leq \sum_{d'|d, d' < d} q^{d/d'} N_{d'}$$

The sum has at most  $d$  terms and each is  $\leq q^d N_{d/2}$ , whence



$$1 - \frac{I_d}{N_d} \leq d q^d \frac{N_{d/2}}{N_d}$$

and the announced result as the right-hand side term tends to 0 in the two situations considered in the statement of theorem 5.1 (a).  $\square$

**5.4. Proof of theorem 5.1 (c).** In this subsection we assume that  $n = 2$  and that  $d$  has at least three prime divisors.

5.4.1. *A technical lemma.*

**Lemma 5.4.** *Let  $b(d) = \frac{1}{2}(d+1)(d+2)$ . Let  $\ell > 1$  be the first divisor of  $d$  and  $\ell' > \ell$  be the second divisor of  $d$ . Let  $\lambda \geq \ell'$  be a divisor of  $d$  and  $\ell'' > 1$  be the first divisor of  $d/\ell$ . Then we have*

- (1)  $b(d/\ell') + \ell' \geq b(d/\lambda) + \lambda$ .
- (2)  $b(d/\ell) + \ell - d/\ell \geq b(d/\ell') + \ell'$ .
- (3)  $b(d/\ell) + 1 - d/\ell \geq b(d/\ell'') + \ell''$ .

*Proof.* (1) We have

$$b(d/\ell') + \ell' - b(d/\lambda) - \lambda = \frac{1}{2} \left( \frac{d}{\ell'} - \frac{d}{\lambda} \right) \left( \frac{d}{\ell'} + \frac{d}{\lambda} + 3 - 2 \frac{\ell' \lambda}{d} \right) \geq 0$$

because  $d/\ell' - d/\lambda \geq 0$  and  $\frac{d}{\ell'} + \frac{d}{\lambda} + 3 - 2 \frac{\ell' \lambda}{d} \geq \frac{d}{\ell'} + 4 - 2\ell' \geq 0$  as  $d$  has at least 3 prime divisors.

(2) We have  $\ell \ell' \leq d$  so  $\ell' - \ell \leq \frac{d}{\ell}$ . Moreover we have  $\frac{d}{\ell'} \leq \frac{d}{\ell} - 2$  and for all  $d \geq 6$  we have  $b(d/\ell') \leq b(d/\ell - 2)$ . Hence

$$b(d/\ell) - b(d/\ell') + \ell - \ell' - \frac{d}{\ell} \geq b(d/\ell) - b(d/\ell - 2) - 2d/\ell = 1$$

(3) If we set  $\delta = d/\ell$  then

$$b(\delta) + 1 - \delta - b(\delta/\ell'') - \ell'' = \frac{1}{2} \left( \delta - \frac{\delta}{\ell''} \right) \left( \delta + \frac{\delta}{\ell''} - 2 \right) + \frac{1}{2} \left( 3\delta - 5 \frac{\delta}{\ell''} - 2\ell'' + 2 \right)$$

Now  $\delta - \frac{\delta}{\ell''} \geq 0$ ,  $\delta + \frac{\delta}{\ell''} - 2 \geq 0$  and as  $\delta$  has at least 2 prime divisors, then  $u(\ell'') = 3\delta - 5 \frac{\delta}{\ell''} - 2\ell'' + 2 \geq u(2) = \frac{\delta}{2} - 2 \geq 0$ .  $\square$

5.4.2. *An upper bound for  $D_d$ .* Using the notation of lemma 5.4, we have

$$\begin{aligned}
D_d &= q^{\ell-1} I_{d/\ell} + \sum_{\lambda|d, \lambda > \ell} q^{\lambda-1} I_{d/\lambda} \quad (\text{corollary 5.3}) \\
&\leq q^{\ell-1} N_{d/\ell} + \sum_{\lambda|d, \lambda > \ell} q^{\lambda-1} N_{d/\lambda} \\
&\leq q^{b(d/\ell)+\ell-1} \left(1 - \frac{1}{q^{\frac{d}{\ell}+1}}\right) + \sum_{\lambda|d, \lambda > \ell} q^{\lambda-1} q^{b(d/\lambda)} \quad (\text{explicit formula for } N_{d/\lambda}) \\
&\leq q^{b(d/\ell)+\ell-1} \left(1 - \frac{1}{q^{\frac{d}{\ell}+1}}\right) + (d-1)q^{b(d/\ell')+\ell'-1} \quad (\text{lemma 5.4 (1)}) \\
&\leq q^{b(d/\ell)+\ell-1} \left(1 - \frac{1}{q^{\frac{d}{\ell}+1}}\right) \left(1 + \frac{d}{q^{b(d/\ell)-b(d/\ell')+\ell-\ell'}}\right) \quad (\text{because } \frac{d-1}{1-q^{-\frac{d}{\ell}-1}} \leq d) \\
&\leq q^{b(d/\ell)+\ell-1} \left(1 - \frac{1}{q^{\frac{d}{\ell}+1}}\right) \left(1 + \frac{d}{q^{\frac{d}{\ell}}}\right) \quad (\text{lemma 5.4 (2)})
\end{aligned}$$

5.4.3. *A lower bound for  $D_d$ .* Start from  $D_d \geq q^{\ell-1} I_{d/\ell}$ . Then use §5.4.2 right above (or the formulas already proved from theorem 5.1 (b)) to bound  $I_{d/\ell} = N_{d/\ell} - D_{d/\ell}$  from below. We obtain

$$\begin{aligned}
D_d &\geq q^{\ell-1} \times \left( q^{b(\frac{d}{\ell})} \left(1 - \frac{1}{q^{\frac{d}{\ell}+1}}\right) - q^{b(\frac{d}{\ell\ell''})+\ell''-1} \left(1 - \frac{1}{q^{\frac{d}{\ell\ell''}+1}}\right) \left(1 + \frac{d/\ell}{q^{\frac{d}{\ell\ell''}}}\right) \right) \\
&\geq q^{\ell-1} \left(1 - \frac{1}{q^{\frac{d}{\ell}+1}}\right) \left( q^{b(\frac{d}{\ell})} - 2q^{b(\frac{d}{\ell\ell''})+\ell''-1} \right) \quad (\text{because } \frac{d/\ell}{q^{\frac{d}{\ell\ell''}}} \leq 1) \\
&= q^{\ell-1} \left(1 - \frac{1}{q^{\frac{d}{\ell}+1}}\right) q^{b(d/\ell)} \left(1 - \frac{2}{q^{b(d/\ell)-b(d/\ell\ell'')+1-\ell''}}\right) \\
&\geq q^{\frac{b}{\ell}+\ell-1} \left(1 - \frac{1}{q^{\frac{d}{\ell}+1}}\right) \left(1 - \frac{2}{q^{\frac{d}{\ell}}}\right) \quad (\text{lemma 5.4 (3)})
\end{aligned}$$

5.4.4. *Final estimate for  $D_d/N_d$ .* The upper and lower bounds for  $D_d$  yield the following inequalities

$$\frac{q^{b(\frac{d}{\ell})+\ell-1}}{q^{b(d)}} \times \frac{1 - q^{-\frac{d}{\ell}-1}}{1 - q^{-d-1}} \times \left(1 - \frac{2}{q^{\frac{d}{\ell}}}\right) \leq \frac{D_d}{N_d} \leq \frac{q^{b(\frac{d}{\ell})+\ell-1}}{q^{b(d)}} \times \frac{1 - q^{-\frac{d}{\ell}-1}}{1 - q^{-d-1}} \times \left(1 + \frac{d}{q^{\frac{d}{\ell}}}\right)$$

which are a little more precise than the announced statement.  $\square$

5.5. **One variable.** Here we assume  $n = 1$ . For polynomials in one variable, we use the definition of indecomposability given in §4.3.

5.5.1. *Main result.*

**Theorem 5.5.** *Assume  $q$  and  $d$  are relatively prime.*

(a) *If  $d$  is a product of at most 2 prime numbers  $p \leq p'$ , then*

- $d = p$  and  $D_d = 0$ , or
- $d = p^2$  and  $D_d = \frac{q-1}{q}q^{2p}$ , or
- $d = pp'$  with  $p < p'$  and

$$2\frac{q-1}{q}q^{p+p'} - q^5 \leq D_d \leq 2\frac{q-1}{q}q^{p+p'}$$

(b) Assume  $d$  is the product of at least 3 prime numbers. Let  $\ell > 1$  be the first divisor of  $d$  and  $\ell' > \ell$  be its second divisor. Then we have

$$\frac{d}{2\ell} \frac{1}{q^{\frac{d}{\ell} - \frac{d}{\ell^2} - \ell + 1}} \leq \frac{D_d}{N_d} - \alpha_d \leq \frac{d-2}{2q^{\ell + \frac{d}{\ell} - \ell' - \frac{d}{\ell}}} \quad \text{where} \quad \alpha_d = \frac{2}{q^{d - \ell - \frac{d}{\ell} + 1}}$$

As a consequence we have that  $I_d/N_d$  tends to 1 in the two situations where  $d \rightarrow \infty$  with  $q$  fixed, and  $q \rightarrow \infty$  with  $d$  fixed.

Theorem 5.5 fails if the assumption  $(q, d) = 1$  is removed. For example for  $q = 2$  and  $d$  even one can compute that  $D_d/N_d \sim 3.2^{-d/2}$  while  $\alpha_d = 4.2^{-d/2}$  in this case.

From now on we assume  $q$  and  $d$  are relatively prime. The rest of the paper is devoted to the proof of theorem 5.5. Our strategy is similar to the one used for  $n \geq 2$ . We view the set  $\mathcal{D}_d$  of all decomposable polynomials  $f(x) \in \mathbb{F}_q[x]$  of degree  $d$  as the union of smaller sets which we will estimate. More specifically we write

$$\mathcal{D}_d = \bigcup_{\lambda|d, \ell \leq \lambda \leq d/\ell} \mathcal{D}_{\lambda, d/\lambda}$$

where  $\mathcal{D}_{\lambda, d/\lambda} \subset \mathcal{D}_d$  is the subset of all  $f(x)$  which admit a decomposition  $f = u \circ v$  with  $u, v \in \mathbb{F}_q[x]$ ,  $\deg u = \lambda \geq 2$ ,  $\deg v = d/\lambda \geq 2$ ,  $v$  monic and of constant term equal to 0. A difference with the case  $n \geq 2$  is that we do *not* have a partition.

5.5.2. *1st stage: upper bounds.* (Assumption  $(q, d) = 1$  is not used in this paragraph). For every divisor  $\lambda \geq 1$  of  $d$ , denote the cardinality of  $\mathcal{D}_{\lambda, \frac{d}{\lambda}}$  by  $D_{\lambda, \frac{d}{\lambda}}$ . We have

$$D_{\lambda, \frac{d}{\lambda}} \leq N_\lambda \frac{N_{d/\lambda}}{q(q-1)} = \frac{q-1}{q} q^{\lambda + \frac{d}{\lambda}}$$

If  $\ell > 1$  is the first divisor of  $d$  and  $\ell' > \ell$  the second divisor, we have

$$D_d \leq \sum_{\lambda|d, \ell \leq \lambda \leq d/\ell} D_{\lambda, \frac{d}{\lambda}} \leq \frac{q-1}{q} \sum_{\lambda|d, \ell \leq \lambda \leq d/\ell} q^{\lambda + \frac{d}{\lambda}}$$

The idea is that the main contribution comes from  $D_{\ell, \frac{d}{\ell}}$  and  $D_{\frac{d}{\ell}, \ell}$ .

If  $d$  is the product of exactly 2 prime numbers  $\ell$  and  $d/\ell$ , then these are the only contributions and we have the desired upper bound. Otherwise we

write  $\lambda + \frac{d}{\lambda} \leq \ell' + \frac{d}{\ell'}$  to bound the extra terms and obtain

$$D_d \leq \frac{q-1}{q} q^{\ell + \frac{d}{\ell}} \left( 2 + \frac{d-2}{q^{\ell + \frac{d}{\ell} - \ell' - \frac{d}{\ell'}}} \right)$$

which yields all announced upper bounds in theorem 5.5. We also deduce this practical bound:  $D_d \leq d \frac{q-1}{q} q^{\ell + \frac{d}{\ell}}$  (as  $\ell + \frac{d}{\ell} - \ell' - \frac{d}{\ell'} \geq 1$ ).

5.5.3. *2nd stage: uniqueness results.* We will use Ritt's theorems to control the number of possible decompositions of a given polynomial.

**Proposition 5.6.** *Let  $K$  be a field and  $f \in K[x]$  be a polynomial of degree  $d > 0$ . Assume the characteristic  $p$  of  $K$  does not divide  $d$ . Suppose we have two decompositions  $f = u \circ v = u' \circ v'$  of  $f$  with*

- $u, v, u', v'$  indecomposable,
- $\deg u = \deg u' \geq 2$ ,  $\deg v = \deg v' \geq 2$ ,
- with  $v, v'$  monic with a zero constant term.

Then  $u = u'$  and  $v = v'$ .

*Proof.* This follows from the first Ritt theorem [12, §1.3 theorem 7] which more generally describes in which cases an equality  $G_1 \circ \dots \circ G_r = H_1 \circ \dots \circ H_s$  with  $G_i, H_j$  indecomposable of degree  $> 1$  may hold.  $\square$

As an immediate consequence, we obtain the case  $d = p^2$  of theorem 5.5 (a): namely we have  $D_{p^2} = D_{p,p} = \frac{q-1}{q} q^{2p}$ .

5.5.4. *3rd stage: lower bounds for  $D_{\frac{d}{\ell}, \ell}$  and  $D_{\ell, \frac{d}{\ell}}$ .*

**Lemma 5.7.** *Assume  $d$  is not a prime number. Then we have*

$$D_{\ell, \frac{d}{\ell}} \geq \frac{q-1}{q} q^{\ell + \frac{d}{\ell}} \left( 1 - \frac{d/\ell}{q^{\frac{d}{\ell} - \frac{d}{\ell^2} - \ell + 1}} \right).$$

And the same inequality holds for  $D_{\ell, \frac{d}{\ell}}$  replaced by  $D_{\frac{d}{\ell}, \ell}$ .

*Proof.* We only give the proof for  $D_{\frac{d}{\ell}, \ell}$  as computations for  $D_{\ell, \frac{d}{\ell}}$  are the same. In  $\mathcal{D}_{\ell, \frac{d}{\ell}}$  we will only count those polynomials  $f$  which decompose as  $f = u \circ v$  with  $u$  and  $v$  as in proposition 5.6. Then we obtain

$$\begin{aligned} D_{\ell, \frac{d}{\ell}} &\geq \frac{1}{q(q-1)} I_\ell \cdot I_{\frac{d}{\ell}} \\ &\geq \frac{1}{q(q-1)} N_\ell (N_{\frac{d}{\ell}} - D_{\frac{d}{\ell}}) \quad (D_\ell = 0 \text{ as } \ell \text{ is prime}) \\ &= \frac{1}{q(q-1)} (q-1) q^\ell \left( (q-1) q^{\frac{d}{\ell}} - D_{\frac{d}{\ell}} \right) \\ &= \frac{q-1}{q} q^{\ell + \frac{d}{\ell}} \left( 1 - \frac{D_{\frac{d}{\ell}}}{(q-1) q^{\frac{d}{\ell}}} \right) \end{aligned}$$

If  $d$  is the product of exactly 2 primes then  $D_{\frac{d}{\ell}} = 0$  and

$$(*) \quad D_{\ell, \frac{d}{\ell}} \geq \frac{q-1}{q} q^{\ell + \frac{d}{\ell}}$$

which in this case is better than the announced result.

If  $d$  is the product of at least 3 primes, use the practical upper bound for  $D_d$  obtained in §5.5.2 to write  $D_{\frac{d}{\ell}} \leq \frac{d}{\ell} \frac{q-1}{q} q^{\ell + \frac{d}{\ell^2}}$  and deduce

$$D_{\ell, \frac{d}{\ell}} \geq \frac{q-1}{q} q^{\ell + \frac{d}{\ell}} \left( 1 - \frac{(d/\ell) \frac{q-1}{q} q^{\ell + \frac{d}{\ell^2}}}{(q-1)q^{\frac{d}{\ell}}} \right) = \frac{q-1}{q} q^{\ell + \frac{d}{\ell}} \left( 1 - \frac{d/\ell}{q^{\frac{d}{\ell} - \frac{d}{\ell^2} - \ell + 1}} \right)$$

□

5.5.5. *Estimating the multiple decompositions.* Next we write

$$D_d \geq \text{card}(\mathcal{D}_{\ell, \frac{d}{\ell}} \cup \mathcal{D}_{\frac{d}{\ell}, \ell}) = D_{\ell, \frac{d}{\ell}} + D_{\frac{d}{\ell}, \ell} - \text{card}(\mathcal{D}_{\ell, \frac{d}{\ell}} \cap \mathcal{D}_{\frac{d}{\ell}, \ell})$$

In order to estimate  $D_d$  we need to estimate the intersection.

**Lemma 5.8.** *We have*

$$\left\{ \begin{array}{l} \text{card}(\mathcal{D}_{\ell, \frac{d}{\ell}} \cap \mathcal{D}_{\frac{d}{\ell}, \ell}) \leq \frac{d}{\ell} q^{\frac{d}{\ell^2} + 2\ell - 1} \\ D_d \geq 2 \frac{q-1}{q} q^{\ell + \frac{d}{\ell}} \left( 1 - \frac{2d}{\ell} \frac{1}{q^{\frac{d}{\ell} - \frac{d}{\ell^2} - \ell + 1}} \right) \end{array} \right.$$

The lower bound for  $D_d$  is the remaining inequality to be proved in theorem 5.5 (b). The more precise inequality (\*\*\*) in the proof below will complete the proof of theorem 5.5 (a) in the special case  $d = pp'$ .

*Proof of lemma 5.8.* (a) If  $\text{gcd}(\ell, d/\ell) = 1$  then  $\text{card}(\mathcal{D}_{\ell, \frac{d}{\ell}} \cap \mathcal{D}_{\frac{d}{\ell}, \ell}) \leq q^5$ .

Indeed let  $f \in \mathcal{D}_{\ell, \frac{d}{\ell}} \cap \mathcal{D}_{\frac{d}{\ell}, \ell}$  and let  $f = u \circ v$  be a decomposition with  $\deg u = \ell$  and  $\deg v = d/\ell$ . We follow Ritt's second theorem (see [12, §1.4, theorem 8] and the notation there). The hypotheses of that result are satisfied because the derivative  $u'$  of  $u$  is non zero; otherwise  $f' = 0$ , and so  $f \in \mathbb{F}_q[x^p]$  and the characteristic  $p$  of  $\mathbb{F}_q$  divides  $d = \deg f$ . In first case of Ritt's second theorem we have  $L_1 \circ u = x^r P(x)^n$  and  $v \circ L_2 = x^n$  (where  $r \geq 0$ ,  $P \in \mathbb{F}_q[x]$  and  $L_1, L_2$  are linear functions). In our situation we get  $n = \frac{d}{\ell}$  and  $\ell = r + \frac{d}{\ell} \deg P$ . Then  $\deg P = \frac{\ell^2 - \ell r}{d} \leq \frac{\ell^2}{d} < 1$  so  $\deg P = 0$ ,  $L_1 \circ u = x^\ell$  and  $v \circ L_2 = x^{\frac{d}{\ell}}$ . Considering all possible linear functions yield at most  $(q-1)^2 q^2$  such decompositions. In second case of Ritt's second theorem we have  $L_1 \circ u = D_m(x, a^n)$  and  $v \circ L_2 = D_n(x, a)$ ,  $a \in \mathbb{F}_q$  (where  $D_n(x, a)$  denote Dickson's polynomials). We here obtain  $m = \ell$  and  $n = \frac{d}{\ell}$ . Considering all possible linear functions and all  $a \in \mathbb{F}_q$  yield at most  $(q-1)^2 q^3$  such decompositions. Finally we obtain

$$(*) \quad \text{card}(\mathcal{D}_{\ell, \frac{d}{\ell}} \cap \mathcal{D}_{\frac{d}{\ell}, \ell}) \leq (q-1)^2 q^2 + (q-1)^2 q^3 \leq q^5$$

(b) If  $\gcd(\ell, d/\ell) \neq 1$  then we have  $\text{card}(\mathcal{D}_{\ell, \frac{d}{\ell}} \cap \mathcal{D}_{\frac{d}{\ell}, \ell}) \leq \frac{d}{\ell} q^{\frac{d}{\ell^2} + 2\ell - 1}$ .

Indeed let  $f \in \mathcal{D}_{\ell, \frac{d}{\ell}} \cap \mathcal{D}_{\frac{d}{\ell}, \ell}$  and let  $f = u \circ v$  be a decomposition with  $\deg u = \ell$  and  $\deg v = d/\ell$ . By Ritt's first theorem and because  $\gcd(\ell, d/\ell) \neq 1$  either  $u$  or  $v$  is decomposable. But as  $\ell$  is a prime  $\mathcal{D}_\ell$  is empty and so  $v \in \mathcal{D}_{\frac{d}{\ell}}$ . Thus we obtain

$$\begin{aligned} \text{card}(\mathcal{D}_{\ell, \frac{d}{\ell}} \cap \mathcal{D}_{\frac{d}{\ell}, \ell}) &\leq N_\ell \frac{1}{q(q-1)} D_{\frac{d}{\ell}} \\ &\leq \frac{1}{q(q-1)} (q-1) q^\ell \frac{d}{\ell} q^{\ell + \frac{d}{\ell^2}} \quad (\text{end of §5.5.2}) \\ &\leq \frac{d}{\ell} q^{\frac{d}{\ell^2} + 2\ell - 1} \end{aligned}$$

The proof follows as for all  $d > 6$  we have  $\frac{d}{\ell^2} + 2\ell - 1 \geq 5$ . □

#### REFERENCES

- [1] I. V. Arzhantsev, A. P. Petravchuk, *Closed polynomials and saturated subalgebras of polynomial algebras*. Ukrain. Mat. Zh. 59/12 (2007), 1783–1790.
- [2] A. Bodin, *Reducibility of rational fractions in several variables*, Israel J. Math. 164 (2008), 333–348.
- [3] L. Busé, G. Chèze, S. Najib, *Noether's forms for the study of indecomposable rational functions and their spectrum*, preprint.
- [4] G. Chèze, S. Najib, *Indecomposable polynomials via jacobian matrix*, preprint.
- [5] A. Dujella, I. Gusić, *Indecomposability of polynomials and related diophantine equations*, Quart. J. Math. Oxford Ser. 498, (2005), 173–199.
- [6] M. Fried, M. Jarden, *Field Arithmetic*, Ergebnisse der Mathematik und ihrer Grenzgebiete, **11**, Springer-Verlag, 2004.
- [7] J. von zur Gathen, *Counting reducible and singular bivariate polynomials*, Finite Fields Appl. 14 (2008), 944–978.
- [8] J. von zur Gathen, *Counting decomposable multivariate polynomials*, preprint.
- [9] J. von zur Gathen, *Counting decomposable univariate polynomials*, preprint.
- [10] D. Lorenzini, *Reducibility of polynomials in two variables*, J. Algebra 156 (1993), 65–75.
- [11] S. Najib, *Une généralisation de l'inégalité de Stein-Lorenzini*, J. Algebra 292 (2005), 566–573.
- [12] A. Schinzel, *Polynomials with special regard to reducibility*, Encyclopedia of Mathematics and its Applications 77. Cambridge University, 2000.
- [13] Y. Stein, *The total reducibility order of a polynomial in two variables*, Israel J. Math. 68 (1989), 109–122.
- [14] U. Zannier, *On the reduction modulo  $p$  of an absolutely irreducible polynomial  $f(x, y)$* , Arch. Math. 68 (1997), 129–138.

*E-mail address:* Arnaud.Bodin@math.univ-lille1.fr

*E-mail address:* Pierre.Debes@math.univ-lille1.fr

*E-mail address:* Salah.Najib@math.univ-lille1.fr