Harbater-Mumford Components and Towers of Moduli Spaces

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Abstract. A method of choice for realizing finite groups as regular Galois groups over $\mathbb{Q}(T)$ is to find $\mathbb{Q}$-rational points on Hurwitz moduli spaces of covers. In another direction, the use of the so-called patching techniques has led to the realization of all finite groups over $\mathbb{Q}_p(T)$. Our main result shows that, under some conditions, these p-adic realizations lie on some special irreducible components of Hurwitz spaces (the so-called Harbater-Mumford components), thus connecting the two main branches of the area. As an application, we construct, for every projective system $(G_n)_{n\geq 0}$ of finite groups, a tower of corresponding Hurwitz spaces $(\mathcal{H}_{G_n})_{n\geq 0}$, geometrically irreducible and defined over some cyclotomic extension of $\mathbb{Q}$, which admits projective systems of $\mathbb{Q}_p^{\mathrm{ur}}$-rational points for all primes p not dividing the orders $|G_n|$ $(n\geq 0)$.

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Introduction

Let $(G_n)_{n\geq 0}$ be a projective system of finite groups, given with surjective morphisms $s_n : G_n \twoheadrightarrow G_{n-1}$ $(n > 0)$. In [DeDes2] was investigated the problem, given a field k, of realizing the projective system $(G_n)_n$ by a regular tower $K_0 \subset \cdots \subset K_n \subset K_{n+1} \subset \cdots$ of extensions $K_n/k(T)$: that is, $\mathrm{Gal}(K_n/k(T)) \simeq G_n$, compatibly with the s_n and K_n/k is regular $(n \geq 0)$. Constructions of such towers were then notably performed in the case that k is a henselian field containing all roots of 1 of order prime to the residue characteristic $p \geq 0$ of k, under the only assumption that each group G_n is of order prime to p, *i.e.*, is a p'-group $(n \geq 0)$. As an application, the free profinite group $\widehat{F}_\omega$ with countably many generators can be regularly realized as the Galois group of an extension of $\mathbb{Q}^{\mathrm{ab}}((x))(T)$; and similarly, its prime-to-p quotient $\widehat{F}_\omega^{(p')}$ over $\mathbb{Q}_p^{\mathrm{ur}}(T)$ (see [DeDes2] for more examples).

Using moduli spaces of covers, these problems and results interpret as those of existence of projective systems of k-rational points on certain towers $(\mathcal{H}_n)_{n\geq 0}$ of algebraic varieties (given with maps $\mathcal{H}_{n+1} \to \mathcal{H}_n$). However the varieties $\mathcal{H}_n$ of [DeDes2] — some Hurwitz spaces — are reducible in general. Our motivation in the current paper was to obtain a similar result but with the $\mathcal{H}_n$ independent of p, geometrically irreducible and defined over $\mathbb{Q}$ or some controlled cyclotomic extension of $\mathbb{Q}$ ($n \geq 0$).

The key is to use the Harbater-Mumford components of Hurwitz spaces, which have been introduced by Fried [Fr1]. Their definition, of topological nature, is recalled in section 1. We prove the following fact, which is a main ingredient of our final construction: the p-adic covers constructed by Harbater's patching methods [Ha] or by its rigid variants [Li] [Po1] lie on HM-components (under some assumptions). How we pass from p-adic to complex objects, is of course a crucial point. A main idea, already present in [Fr1], is that HM-components can be characterized by the way the covers they carry degenerate; our theorem 1.4 is a precise form of this. A consequence is that HM-components are permuted by $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, which was proved in [Fr1] under some conditions. We use Wewers' compactification of Hurwitz spaces [We1], [We2] to handle degeneration of covers. The key part of our approach (how components can be recovered from their boundary) consists in some deformation argument. We offer two versions. One (from $\mathbb{C}$ to $\mathbb{C}\{\{t\}\}$) is based on a general "comparison theorem" (proved in [Em2]) expressing the fundamental group of a semi-stable curve in terms of those of the components of the special fiber. The second one is *ad hoc* and purely topological (over $\mathbb{C}$).

Our original goal is reached in the final section. To any system $(G_n)_{n\geq 0}$ can be attached a tower $(\mathcal{H}_n)_{n\geq 0}$ of algebraic varieties $\mathcal{H}_n$, geometrically irreducible and defined over some controlled cyclotomic extension of $\mathbb{Q}$, and which has the following properties (see theorem 4.1 for a full statement):

- each $\mathcal{H}_n$ is a component of some moduli space of Galois covers of group G_n ($n \geq 0$).
- there exist projective systems of $\mathbb{Q}_p^{\mathrm{ur}}$-points, for every p such that all G_n are p'-groups,
- there exist projective systems of $\mathbb{Q}^{\mathrm{ab}}((x))$-points,
- there exist projective systems of $\mathbb{R}$-points.

The paper is organized as follows. Section 1 presents the main results. Section 2 provides the main tools. Section 3 gives the proofs of the main results. Section 4 is devoted to the motivating application: we show the above result, improving on [DeDes2].

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Throughout this paper, we assume a copy of the complex number field $\mathbb{C}$ has been fixed, along with an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ of the field of algebraic numbers.

1. Main results

1.1. HM-components of Hurwitz spaces. For every integer $r \geq 2$, denote as usual the configuration space for finite subsets of $\mathbb{P}^1$ of cardinality r by $\mathcal{U}_r$. It is a scheme over $\mathbb{Z}$. Given a subset $\mathbf{t} \in \mathcal{U}_r(\mathbb{C})$, define a *topological bouquet* for $\mathbb{P}^1 \setminus \mathbf{t}$ to be a r-tuple $\underline{\Gamma} = (\Gamma_1, \ldots, \Gamma_r)$ of homotopy classes of paths $\gamma_1, \ldots, \gamma_r$ based at some point $t_o \notin \mathbf{t}$ of the form $\gamma_i = \vartheta_i \delta_i \vartheta_i^{-1}$ where, for $i, j = 1, \ldots, r$,

(i) δ_i clockwise bounds a disc Δ_i containing a unique point $t_i \in \mathbf{t}$,

(ii) ϑ_i starts at t_o and ends at some point on δ_i,

(iii) excluding their beginning and end points, the paths γ_i and γ_j never meet if $i \neq j$,

(iv) the first intersection points of $\gamma_1, \ldots, \gamma_r$ with a small circle centered at t_o are clockwise ordered according to their subscript numbering.

Following Fried [Fr2] we call the γ_is *sample loops* around the t_is. It follows from these conditions that $\Gamma_1, \ldots, \Gamma_r$ generate the topological fundamental group $\pi_1^{\mathrm{top}}(\mathbb{P}^1(\mathbb{C}) \setminus \mathbf{t}, t_0)$ with the unique relation $\Gamma_1 \cdots \Gamma_r = 1$ ([Fr2] chapter 4 theorem 1.8).

Given $\mathbf{t} \in \mathcal{U}_r(\mathbb{C})$ and a topological bouquet $\underline{\Gamma}$ for $\mathbb{P}^1 \setminus \mathbf{t}$, the map sending every complex branched cover $f : X \to \mathbb{P}^1_{\mathbb{C}}$ with branch point set $\mathbf{t}$ to the r-tuple whose entries are the monodromy permutations of $f^{-1}(t_0)$ associated with $\Gamma_1, \ldots, \Gamma_s$, will be denoted by $\mathrm{BCD}_{\underline{\Gamma}}$ (where BCD stands for "branch cycle description"). We recall the notion of Harbater-Mumford type for covers of $\mathbb{P}^1$, which was introduced by M. Fried [Fr1].

Definition 1.1 — *A cover f with branch point set $\mathbf{t}$ is said to be of Harbater-Mumford type (a HM-cover for short) if $r = 2s$ is even and there exists a topological bouquet $\underline{\Gamma}$ for $\mathbb{P}^1 \setminus \mathbf{t}$ such that $\mathrm{BCD}_{\underline{\Gamma}}(f)$ is of the form $(g_1, g_1^{-1}, \ldots, g_s, g_s^{-1})$.*

Fried was interested in the connected components of HM-covers in the associated *Hurwitz spaces.* Generally speaking, Hurwitz spaces are moduli spaces of covers of $\mathbb{P}^1$ with fixed monodromy group G and with a fixed number $r \geq 3$ of branch points. The basic notation for it is $\mathcal{H}_{r,G}$ and a point representing a cover f, or more exactly its equivalence class, is denoted by $[f]$.

There are two variants of Hurwitz spaces, depending on whether one is interested

- in *mere covers*, in which case, the covers are not necessarily Galois and G is the monodromy group, given as a subgroup of the symmetric group S_d (with d the degree of the covers) and isomorphisms between two covers $f : X \to \mathbb{P}^1$ and $g : Y \to \mathbb{P}^1$ are isomorphisms $\chi : X \to Y$ of algebraic curves such that $g \circ \chi = f$, or,

- in G-*covers*, in which case, the covers are Galois covers given with an isomorphism between their automorphism group and the group G and isomorphisms between two G-covers are those isomorphisms between the associated mere covers which in addition are compatible with the action of G.

For simplicity, we will not distinguish the notation in these different situations, which, unless otherwise specified, are both covered in this paper.

At this beginning stage, covers are considered over the complex field $\mathbb{C}$. The corresponding moduli space is then a complex smooth quasi-projective variety, which we denote by $\mathcal{H}_{r,G}^{\infty}$. We will freely use the Hurwitz space theory in this context; we refer to [Fr2], [Vo], see also [De], [Em1].

Due to smoothness of $\mathcal{H}_{r,G}^{\infty}$, its connected components also are its irreducible components (in the sequel, we just say components). Given an (unordered) r-tuple $\mathbf{C} = (C_1, \ldots, C_r)$ of conjugacy classes of G, we let $\mathcal{H}_{r,G}^{\infty}(\mathbf{C})$ be the union of those components of $\mathcal{H}_{r,G}^{\infty}$ whose points correspond to covers with inertia canonical invariant $\mathbf{C}$: recall that this invariant is the collection $(C_t)_t$ of conjugacy classes C_t of distinguished generators of inertia groups[1] above t as t ranges over the branch points of the cover.

For each $\tau \in \mathrm{Aut}(\mathbb{C})$, the conjugate space $\mathcal{H}_{r,G}^{\infty}(\mathbf{C})^{\tau}$ is still a Hurwitz space, which only depends on the restriction $\tau|_{\mathbb{Q}^{\mathrm{ab}}} \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$; namely it is $\mathcal{H}_{r,G}^{\infty}(\mathbf{C}^{\chi(\tau)})$ (where χ is the cyclotomic character and $\mathbf{C}^{\chi(\tau)} = (C_1^{\chi(\tau)}, \ldots, C_r^{\chi(\tau)})$). Thus the (generally reducible) varieties $\mathcal{H}_{r,G}^{\infty}$ and $\mathcal{H}_{r,G}^{\infty}(\mathbf{C})$ can be defined over $\mathbb{Q}$ and $\mathbb{Q}^{\mathrm{ab}}$ respectively, in the sense that their (geometric) components are permuted transitively by $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}^{\mathrm{ab}})$ respectively. Furthermore, the Hurwitz space $\mathcal{H}_{r,G}^{\infty}(\mathbf{C})$ is itself defined over $\mathbb{Q}$ if $\mathbf{C}$ is a *rational union of conjugacy classes* of G, *i.e.*, if for every integer m prime to $|G|$, there exists $\sigma \in S_r$ such that $C_i^m = C_{\sigma(i)}$. More generally, given a field $k \subset \mathbb{Q}^{\mathrm{ab}}$, we say $\mathbf{C}$ is a k-rational union of conjugacy classes of G if the same property holds for all integers $m \equiv \chi(\tau)$ modulo $|G|$ with $\tau \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/k)$. Under this condition, the Hurwitz space $\mathcal{H}_{r,G}^{\infty}(\mathbf{C})$ is defined over k. For example, the field generated by all roots of unity of order $|G|$ is a rationality field for $\mathbf{C}$.

We denote by $\Psi_r : \mathcal{H}_{r,G}^{\infty} \to \mathcal{U}_r \otimes_{\mathbb{Z}} \mathbb{C}$ the étale cover mapping each point $[f] \in \mathcal{H}_{r,G}^{\infty}(\mathbb{C})$ to the branch point set $\mathbf{t} \in \mathcal{U}_r(\mathbb{C})$ of the isomorphism class of the cover f. For any choice of a topological bouquet $\underline{\Gamma}$ for $\mathbb{P}^1 \setminus (\mathbf{t}\}$ (with base point $t_0 \notin \mathbf{t}$), the map $\mathrm{BCD}_{\underline{\Gamma}}$ provides a one-one correspondence between the fiber $\Psi_r^{-1}(\mathbf{t})$ and the set

[1] We assume throughout the paper we have fixed a coherent system $(\zeta_n)_{n>0}$ of roots of unity; the distinguished generator of some inertia group I, say of order e, is the generator that corresponds to ζ_e in the natural isomorphism between I and the group μ_e of e-th roots of 1.

$$\mathrm{ni}(\mathbf{C})^\bullet = \left\{ (g_1,\dots,g_r)\in G^r \;\middle|\; \begin{array}{l} g_1\cdots g_r = 1 \\ <g_1,\dots,g_r> = G \\ g_i\in C_{\sigma(i)}, i=1,\dots,r \text{ for some } \sigma\in S_r \end{array} \right\} \;/\; \sim$$

where, by "$/ \sim$", we mean that the tuples $(g_1,\dots,g_r)$ are regarded up to componentwise conjugation by elements of G for G-covers, and, by elements of the normalizer $\mathrm{Nor}_{S_d}(G)$ for mere covers (in which case $\mathrm{ni}(\mathbf{C})^\bullet$ is usually denoted by $\mathrm{ni}(\mathbf{C})^{\mathrm{in}}$ or $\mathrm{ni}(\mathbf{C})^{\mathrm{ab}}$ respectively).

There is a classical outer action of the *Hurwitz braid group* $\pi_1^{\mathrm{top}}(\mathcal{U}_r,\mathbf{t})$ on $\pi_1^{\mathrm{top}}(\mathbb{P}^1(\mathbb{C})\setminus \mathbf{t},t_0)$, which induces an action on the fiber $\Psi_r^{-1}(\mathbf{t})$, and on $\mathrm{ni}(\mathbf{C})^\bullet$ *via* maps $\mathrm{BCD}_{\underline{\Gamma}}$. This induced action on $\Psi_r^{-1}(\mathbf{t})$ is the monodromy action corresponding to the topological cover $\Psi_r : \mathcal{H}_{r,G}^\infty(\mathbb{C})\to\mathcal{U}_r(\mathbb{C})$. It can be explicitly determined: $\pi_1(\mathcal{U}_r,\mathbf{t})$ has generators $Q_1,\dots,Q_{r-1}$ whose action on $\Psi_r^{-1}(\mathbf{t})$, when computed relative to some suitable topological bouquet $\underline{\Gamma}$, corresponds to the following action on $\mathrm{ni}(\mathbf{C})^\bullet$:

$$(g_1,\dots,g_r)\xrightarrow{\;Q_i\;}(g_1,\dots,g_{i-1},g_ig_{i+1}g_i^{-1},g_i,g_{i+2},\dots,g_r),\qquad i=1,\dots,r-1.$$

Components of $\mathcal{H}_{r,G}^\infty(\mathbf{C})$ correspond to orbits of the Hurwitz braid group action. More precisely, fix $\mathbf{t}_0\in\mathcal{U}_r(\mathbb{C})$ and a topological bouquet $\underline{\Gamma}_0$ for $\mathbb{P}^1\setminus\mathbf{t}_0$. Then, *via* $\mathrm{BCD}_{\underline{\Gamma}_0}$, each component $X\subset\mathcal{H}_{r,G}^\infty(\mathbf{C})$ corresponds to some orbit $\mathcal{O}\subset\mathrm{ni}(\mathbf{C})^\bullet$, and we have:

(*) *X is the set of those points $[f]$ which have this property: for any $\mathbf{g}\in\mathcal{O}$, there exists a topological bouquet $\underline{\Gamma}$ for $\mathbb{P}^1\setminus\mathbf{t}$ where $\mathbf{t}=\Psi_r([f])$ such that the branch cycle description* $\mathrm{BCD}_{\underline{\Gamma}}(f)$ *of the cover f is $\mathbf{g}$; and $\mathcal{O}$ is then the set of all* $\mathrm{BCD}_{\underline{\Gamma}}(g)$ *with $[g]\in X\cap\Psi_r^{-1}(\mathbf{t})$.*

(**) *Given any $\mathbf{t}\in\mathcal{U}_r(\mathbb{C})$ and any topological bouquet $\underline{\Gamma}$ for $\mathbb{P}^1\setminus\mathbf{t}$, the orbit $\mathcal{O}$ is exactly the set of all branch cycle descriptions* $\mathrm{BCD}_{\underline{\Gamma}}(f)$ *with $[f]\in X\cap\Psi_r^{-1}(\mathbf{t})$.*

Assertion (*) is part of general theory of topological covers; (**) uses in addition the fact that the Hurwitz braid group acts transitively on topological bouquets up to conjugation[2].

Suppose $r=2s$ and $\mathbf{C}$ consists of s pairs (C_i,C_i^{-1}), $i=1,\dots,s$. Let $\mathrm{HM}(\mathbf{C})$ be the set of all r-tuples in $\mathrm{ni}(\mathbf{C})^\bullet$ of the form $\mathbf{g}=(g_1,g_1^{-1},\dots,g_s,g_s^{-1})$. These tuples are called H(arbater-)M(umford) representatives of $\mathrm{ni}(\mathbf{C})^\bullet$ in [Fr1].

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2 In fact, the Hurwitz braid action comes from the natural outer action of the mapping class group $M_{0,r}$ of the r-marked sphere (a canonical quotient of $\pi_1^{\mathrm{top}}(\mathcal{U}_r,\mathbf{t})$ by its center) on $\pi_1^{\mathrm{top}}(\mathbb{P}^1(\mathbb{C})\setminus\mathbf{t},t_0)$. Given a topological bouquet $\underline{\Gamma}_0$, $M_{0,r}$ has the following description [MclHar]: $M_{0,r}\simeq\mathrm{Aut}^\star(\pi_1^{\mathrm{top}}(\mathbb{P}^1(\mathbb{C})\setminus\mathbf{t},t_0))\,/\,\mathrm{Inn}(\pi_1^{\mathrm{top}}(\mathbb{P}^1(\mathbb{C})\setminus\mathbf{t},t_0))$ where $\mathrm{Aut}^\star(\pi_1^{\mathrm{top}}(\mathbb{P}^1(\mathbb{C})\setminus\mathbf{t},t_0))$ is the subgroup of $\mathrm{Aut}(\pi_1^{\mathrm{top}}(\mathbb{P}^1(\mathbb{C})\setminus\mathbf{t},t_0))$ of those automorphisms φ for which there exists $\sigma\in S_r$ such that $\varphi(\Gamma_i)$ is conjugate to $\Gamma_{\sigma(i)}$, $i=1,\dots,r$, and $\mathrm{Inn}(\pi_1^{\mathrm{top}}(\mathbb{P}^1(\mathbb{C})\setminus\mathbf{t},t_0))$ denotes the group of inner automorphisms of $\pi_1^{\mathrm{top}}(\mathbb{P}^1(\mathbb{C})\setminus\mathbf{t},t_0)$.

Definition 1.2 — *A H(arbater-)M(umford) component of the Hurwitz space $\mathcal{H}_{r,G}^{\infty}(\mathbf{C})$ is the component of some HM-cover. Equivalently, it is a component that corresponds to the orbit of some HM-representative under the action of the Hurwitz braid group.*

All points in a HM-component correspond to HM-covers but in general there may be several HM-components. However, Fried proved the following [Fr1] theorem 3.21. He defines first the notion of *g-complete* and *HM-g-complete* tuples $\mathbf{C}$. A tuple $\mathbf{C}$ is g-complete if it satisfies "$g_i \in C_i,\ i = 1, \ldots, r \Rightarrow < g_1, \ldots, g_r >= G$". A tuple $\mathbf{C}$ with the shape $(C_1, C_1^{-1}, \ldots, C_s, C_s^{-1})$ is HM-g-complete if it has this property: if any pair C_i, C_i^{-1} is removed then what remains is g-complete. He then proves that if $\mathbf{C}$ is HM-g-complete, then all HM-representatives are in the same orbit of the Hurwitz braid group. Consequently, there is then a unique HM-component. Furthermore, if $Z(G) = \{1\}$ and if $\mathbf{C}$ is a rational union of conjugacy classes, then this HM-component is defined over $\mathbb{Q}$. We will re-establish this fact, as a consequence of theorem 1.4, without assuming $Z(G) = \{1\}$ (corollary 1.5).

1.2. The Wewers' compactification. Fix a finite group G and an integer $r \geq 3$. In his thesis [We1] which is our main reference for this paragraph, S. Wewers gives a more general construction of Hurwitz spaces, which leads to a definition of $\mathcal{H}_{r,G}$ and of some compactification $\overline{\mathcal{H}_{r,G}}$ as schemes over $\mathrm{Spec}(\mathbb{Z}[1/|G|])$ (see also [We2]). For each prime p not dividing $|G|$, we denote the corresponding fibers above p by $\mathcal{H}_{r,G}^p$ and $\overline{\mathcal{H}_{r,G}^p}$. This includes the case of the prime at infinity for which one recovers the space $\mathcal{H}_{r,G}^{\infty}$ of §1.1.

There is good reduction of $\mathcal{H}_{r,G}$ at those primes $p \not| |G|$: the fiber $\mathcal{H}_{r,G}^p$ is a (reducible) smooth variety defined over $\overline{\mathbb{F}}_p$ and its components correspond to those of $\mathcal{H}_{r,G}^{\infty}$ through the reduction process. Furthermore, each $\mathcal{H}_{r,G}^p$ is a moduli space, for covers of $\mathbb{P}^1$ with r branch points and monodromy group G, over algebraically closed fields of characteristic p.

Consider next the compactification $\overline{\mathcal{H}_{r,G}}$. Locally $\overline{\mathcal{H}_{r,G}}$ is the quotient of a smooth variety by a finite group and two distinct components of $\mathcal{H}_{r,G}$ have disjoint boundaries in $\overline{\mathcal{H}_{r,G}}$; so components in $\overline{\mathcal{H}_{r,G}}$ are closures of components in $\mathcal{H}_{r,G}$. The natural étale morphism $\Psi_r : \mathcal{H}_{r,G} \to \mathcal{U}_r$ extends to a ramified cover $\overline{\mathcal{H}_{r,G}} \to \overline{\mathcal{U}}_r$. Points on the boundary $\overline{\mathcal{U}}_r \setminus \mathcal{U}_r$ represent stable marked curves of genus 0 with a root, *i.e.* trees of curves of genus 0 with a distinguished component T_0 — the *root* — equipped with an isomorphism $\mathbb{P}^1 \simeq T_0$ and at least three marked points (including the double points) on any component but the root. Typical examples are the combs defined below. Points on the boundary $\overline{\mathcal{H}_{r,G}} \setminus \mathcal{H}_{r,G}$ represent *admissible covers* of stable marked curves B of genus 0 with root; see [We1], [We2]. A key notion in this paper will be that of HM-admissible cover.

Definition 1.3 — *Given an algebraically closed field κ, a comb over κ is a κ-stable curve of genus 0 marked by $r = 2s$ points consisting of a genus 0 root T_0 attached to s other κ-genus 0-curves $T_1, \dots, T_s$, called the end components, each of them marked by two points. A HM-admissible cover is an admissible cover of a comb that is unramified at the singular points (which are the intersection points of $T_1, \dots, T_s$ with T_0).*

We summarize some properties of admissible covers we will use in the rest of this paper. Let $\mathcal{O}$ be a henselian discrete valuation ring, k its quotient field and κ its residue field. Let $\widetilde{P}$ be a $\mathcal{O}$-curve of genus 0 marked by r sections $\widetilde{x}_1, \dots, \widetilde{x}_r$ with smooth generic fiber and a r-marked stable special fiber $\bar{P}$.

(1) given an admissible cover $\bar{X} \to \bar{P}$ tamely ramified at the marked points and possibly at the singular points, there are deformations $\widetilde{X} \to \widetilde{P}$ to covers of $\widetilde{P}$ ramified along the sections $\widetilde{x}_1, \dots, \widetilde{x}_r$.

(2) in the case where the special fiber $\bar{P}$ is a comb, and $\bar{X} \to \bar{P}$ is an HM-admissible cover, the deformation is unique.

(3) in the other direction, if $X \to P_\eta$ is a p'-cover of the generic fiber ramified at the marked points, after a possible finite extension of k, it extends uniquely to a cover $\widetilde{X} \to \widetilde{P}_{\mathcal{O}}$ ramified along the sections $\widetilde{x}_1, \dots, \widetilde{x}_r$, with special fiber an admissible cover of the special fiber $\bar{P}$ of $\widetilde{P}_{\mathcal{O}}$ ramified at $\bar{x}_1, \dots, \bar{x}_r$ and possibly at the singular points of $\bar{P}$.

1.3. Characterization of HM-components. Fix an even integer $r = 2s$, a finite group G and an r-tuple $\mathbf{C} = (C_1, C_1^{-1}, \dots, C_s, C_s^{-1})$ of conjugacy classes of G. The following statement is one goal of this paper: it will be established in §3.

Theorem 1.4 — *The HM-components of $\mathcal{H}^{\infty}_{r,G}(\mathbf{C})$ are those components whose induced component in $\overline{\mathcal{H}_{r,G}}$ contains points representing HM-admissible covers over some algebraically closed field (possibly of positive characteristic).*

As a first consequence of theorem 1.4, we obtain:

Corollary 1.5 — *Each $\tau \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ maps the HM-components of $\mathcal{H}^{\infty}_{r,G}(\mathbf{C})$ on those of $\mathcal{H}^{\infty}_{r,G}(\mathbf{C})^{\tau}$. In particular, given a field $k \subset \mathbb{Q}^{\mathrm{ab}}$, if $\mathbf{C}$ is a k-rational union of conjugacy classes of G, then action of $\mathrm{Gal}(\overline{k}/k)$ permutes the HM-components of $\mathcal{H}^{\infty}_{r,G}(\mathbf{C})$. If in addition, there is a unique HM-component $\mathcal{H} \subset \mathcal{H}^{\infty}_{r,G}(\mathbf{C})$, it is defined over k.*

Proof. Let $\mathcal{H}$ be some HM-component of $\mathcal{H}^{\infty}_{r,G}(\mathbf{C})$. Denote its closure in $\overline{\mathcal{H}_{r,G}}$ by $\overline{\mathcal{H}}$. From the direct part of theorem 1.4 the boundary of $\overline{\mathcal{H}}$ contains a point representing a

HM-admissible cover f defined over some algebraically closed field κ, which, as the proof will show, may be assumed to be of characteristic 0.

As recalled in §1.2, the cover f extends to some cover $\widetilde{f}$ of $\mathbb{P}^1_{\kappa((x))}$ over the field $\kappa((x))$ of Laurent series with coefficients in κ. The representative point $[\widetilde{f}]$ still lies in $\overline{\mathcal{H}}$.

Let $\tau \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Then $\mathcal{H}^\tau$ is a component of the Hurwitz space $\mathcal{H}^\infty_{r,G}(\mathbf{C})^\tau$; furthermore $\overline{\mathcal{H}^\tau} = \overline{\mathcal{H}}^{\,\tau}$. Extend τ to a $\mathbb{Q}$-automorphism of $\kappa((x))$ fixing x. Then $[\widetilde{f}^{\,\tau}] = [\widetilde{f}]^\tau \in \overline{\mathcal{H}}^{\,\tau}$ and the reduction of $\widetilde{f}^{\,\tau}$ modulo the maximal ideal of $\kappa[[x]]$ is f^τ, which is a HM-admissible cover. Conclude from theorem 1.4 that $\mathcal{H}^\tau$ is a HM-component of $\mathcal{H}^\infty_{r,G}(\mathbf{C})^\tau$.

The rest of corollary 1.5 is straightforward. □

Remark 1.6. Classically constructing a Hurwitz space $\mathcal{H}_{r,G}(\mathbf{C})$ for a given group G with some component defined over $\mathbb{Q}$ can alternatively be done as follows: choose for $\mathbf{C}$ a rational union of conjugacy classes of G of the form $\mathbf{C} = (C_1, C_1^{-1}, \ldots, C_s, C_s^{-1})$ and use patching methods to construct a Galois cover over $\mathbb{Q}((t))$ with group G and inertia canonical invariant $\mathbf{C}$; the component of the representative point in $\mathcal{H}_{r,G}(\mathbf{C})$ is then defined over $\mathbb{Q}((t)) \cap \overline{\mathbb{Q}} = \mathbb{Q}$. Furthermore, this component has p-adic points for each prime p (including $p = \infty$) (*e.g.* [DeDes1] §4.2).

There is however some advantage in working with the somewhat more intrinsic (when unique) HM-components of corollary 1.5. In the final section, given a projective system $(G_n)_{n\geq 0}$ of finite groups, we will construct a tower of such components carrying, for each prime p not dividing the orders $|G_n|$, projective systems of rational points over some appropriate p-adic field. The alternate argument recalled above also provides towers of components but it is unclear to us that projective systems of points over p-adic fields that can be constructed all belong to the same tower when p varies.

2. Tools

2.1. Comparison theorem of fundamental groups. A main tool in the proof of theorem 1.4 is a comparison theorem between the fundamental groups of the generic fiber and of the components of the special fiber of a stable marked curve. We only state the topological version. The general version and the proof are given in [Em2] (see also [Sa]).

The situation is the following. We are given a stable marked curve Z over the ring $\mathbb{C}\{\{\epsilon\}\}$ of convergent power series with coefficients in $\mathbb{C}$. We only consider here the special case where Z is of genus 0 and its special fiber is a comb. We denote its root by T_0, its end components by $T_1, \ldots, T_s$, the intersection point of T_0 with T_i by $\bar{a}_i$ and the marked points on T_i by $\bar{x}_i, \bar{y}_i$, $i = 1, \ldots, s$. We also denote by $\{x_1, y_1, \ldots x_s, y_s\}$ the marked points on the generic fiber Z_η, which extend to sections $\{\widetilde{x}_1, \widetilde{y}_1, \ldots, \widetilde{x}_s, \widetilde{y}_s\}$ on Z specializing in $\{\bar{x}_1, \bar{y}_1, \ldots, \bar{x}_s, \bar{y}_s\}$.

Choose a base point $\bar{\xi}_i$ in $T_i \setminus \{\bar{x}_i, \bar{y}_i, \bar{a}_i\}$ and a base point ξ_i in the geometric generic fiber $Z_{\bar{\eta}} \setminus \{x_1, y_1, \ldots, x_s, y_s\}$ which specializes in $\bar{\xi}_i$, $i = 1, \ldots, s$. The natural restriction functors from the category of covers of the geometric generic fiber $Z_{\bar{\eta}}$ to the category of covers of T_i induce morphisms of fundamental groups[3]

$$\widetilde{\theta}_i : \pi_1^{\mathrm{top}}(T_i \setminus \{\bar{x}_i, \bar{y}_i, \bar{a}_i\}, \bar{\xi}_i) \to \pi_1^{\mathrm{top}}(Z_{\bar{\eta}} \setminus \{x_1, y_1, \ldots, x_s, y_s\}, \xi_i) \qquad (i = 1, \ldots, s)$$
$$\widetilde{\theta}_0 : \pi_1^{\mathrm{top}}(T_0 \setminus \{\bar{a}_1, \ldots, \bar{a}_s\}, \bar{\xi}_0) \to \pi_1^{\mathrm{top}}(Z_{\bar{\eta}} \setminus \{x_1, y_1, \ldots, x_s, y_s\}, \xi_0)$$

The base points $\xi_0, \xi_1, \ldots \xi_s$ in the right-hand side terms can be changed to a common base point $\xi \in Z_{\bar{\eta}} \setminus \{x_1, y_1, \ldots, x_s, y_s\}$ but then the morphisms, which we denote by $\theta_0, \theta_1, \ldots \theta_s$ (*i.e.*, we remove the tilde), are only defined up to conjugation depending on the choice of a path δ_i from ξ to ξ_i, $i = 1, \ldots, s$. Clearly the images of sample loops based at $\bar{\xi}_i$ by $\widetilde{\theta}_i$ (resp. θ_i) are sample loops based at ξ_i (resp. ξ), $i = 0, 1, \ldots, s$.

Theorem 2.1 — *There exist*
- a topological bouquet $\underline{\Gamma}^{(0)} = \{\Gamma_1^{(0)}, \ldots, \Gamma_s^{(0)}\}$ *for* $T_0 \setminus \{\bar{a}_1, \ldots, \bar{a}_s\}$ *based at* $\bar{\xi}_0$,
- a topological bouquet $\underline{\Gamma}^{(i)} = \{\Gamma_0^{(i)}, \Gamma_1^{(i)}, \Gamma_2^{(i)}\}$ *for* $T_i \setminus \{\bar{a}_i, \bar{x}_i, \bar{y}_i\}$ *based at* $\bar{\xi}_i$, $i = 1, \ldots, s$, *and*
- elements $\sigma_i \in \pi_1^{\mathrm{top}}(Z_{\bar{\eta}} \setminus \{x_1, y_1, \ldots, x_s, y_s\}, \xi)$, $i = 1, \ldots, s$,
such that $\pi_1^{\mathrm{top}}(Z_{\bar{\eta}} \setminus \{x_1, y_1, \ldots, x_s, y_s\}, \xi)$ *is generated by the elements*

- $\theta_0(\Gamma_1^{(0)}), \ldots, \theta_0(\Gamma_s^{(0)})$, *and*
- $\theta_i(\Gamma_0^{(i)}), \theta_i(\Gamma_1^{(i)}), \theta_i(\Gamma_2^{(i)})$, $i = 1, \ldots, s$

with the only relations $\theta_0(\Gamma_i^{(0)}) \cdot \theta_i(\Gamma_0^{(i)})^{\sigma_i} = 1$, $i = 1, \ldots, s$. *Moreover the* σ_i *can be chosen in such a way that* $\theta_1(\Gamma_1^{(1)})^{\sigma_1}, \theta_1(\Gamma_2^{(1)})^{\sigma_1}, \ldots, \theta_s(\Gamma_1^{(s)})^{\sigma_s}, \theta_s(\Gamma_2^{(s)})^{\sigma_s}$ *form a topological bouquet for* $Z_{\bar{\eta}} \setminus \{x_1, y_1, \ldots, x_s, y_s\}$. [4]

---

[3] For $r>0$ small enough the fiber of $Z \to \mathrm{Spec}\mathbb{C}\{\{\epsilon\}\}$ over $\{0<|\epsilon|<r\}$ is an analytic variety and the topological fundamental group $\pi_1^{\mathrm{top}}(Z_\epsilon^{\mathrm{an}} \setminus \{x_1^\epsilon, y_1^\epsilon, \ldots, x_s^\epsilon, y_s^\epsilon\})$ is constant for ϵ real in $]0,r[$. This fundamental group is by definition the topological fundamental group of the geometric generic fiber, and is denoted by $\pi_1^{\mathrm{top}}(Z_{\bar{\eta}} \setminus \{x_1, y_1, \ldots, x_s, y_s\})$.

[4] In order to get this last conclusion from theorem 4.3 of [Em2], note that for $i=1,\ldots,s$, one can always choose a path δ_i from ξ to ξ_i in such a way that $\delta_1, \ldots, \delta_s$ do not intersect and that their intersection with a small circle around ξ are clockwise ordered according to their subscript numbering. Then the two loops representing $\theta_i(\Gamma_1^{(i)})^{\sigma_i}, \theta_i(\Gamma_2^{(i)})^{\sigma_i}$ can be separated into two sample loops $\omega_1^{(i)}$, $\omega_2^{(i)}$ around x_i and y_i, $i=1,\ldots,r$, in such a way that $\omega_1^{(1)}, \omega_2^{(1)}, \ldots, \omega_1^{(s)}, \omega_2^{(s)}$ satisfy all conditions (i)-(iv) from §1.1 and so that their homotopy classes $\theta_1(\Gamma_1^{(1)})^{\sigma_1}, \theta_1(\Gamma_2^{(1)})^{\sigma_1}, \ldots, \theta_s(\Gamma_1^{(s)})^{\sigma_s}, \theta_s(\Gamma_2^{(s)})^{\sigma_s}$ form a topological bouquet.

2.2. HM-covers degenerating to HM-admissible covers. The general construction below, which shows some HM-covers degenerate to HM-admissible covers (over $\mathbb{C}$), will be used in the proof of theorem 1.4.

Let $S^2 \subset \mathbb{R}^3$ be the unit sphere (identified with $\mathbb{P}^1(\mathbb{C})$) and let $\mathbf{t} = \{x_1, y_1, \ldots, x_s, y_s\} \subset S^2$ be a subset of $r = 2s$ distinct points. Suppose also given s open disks $U_1, \ldots, U_s$ such that $\overline{U_i} \cap \overline{U_j} = \emptyset$ and $x_i, y_i \in U_i$, and pick a point a_i on the line segment $[x_i, y_i]$, $(i, j = 1, \ldots, s$ and $i \neq j)$.

Consider the continuous deformation $\mathbf{t}^\theta = \{x_1^\theta, y_1^\theta, \ldots, x_s^\theta, y_s^\theta\}$ parametrized by $\theta \in [0,1]$ of the marking $\mathbf{t} = \mathbf{t}_0$ given by

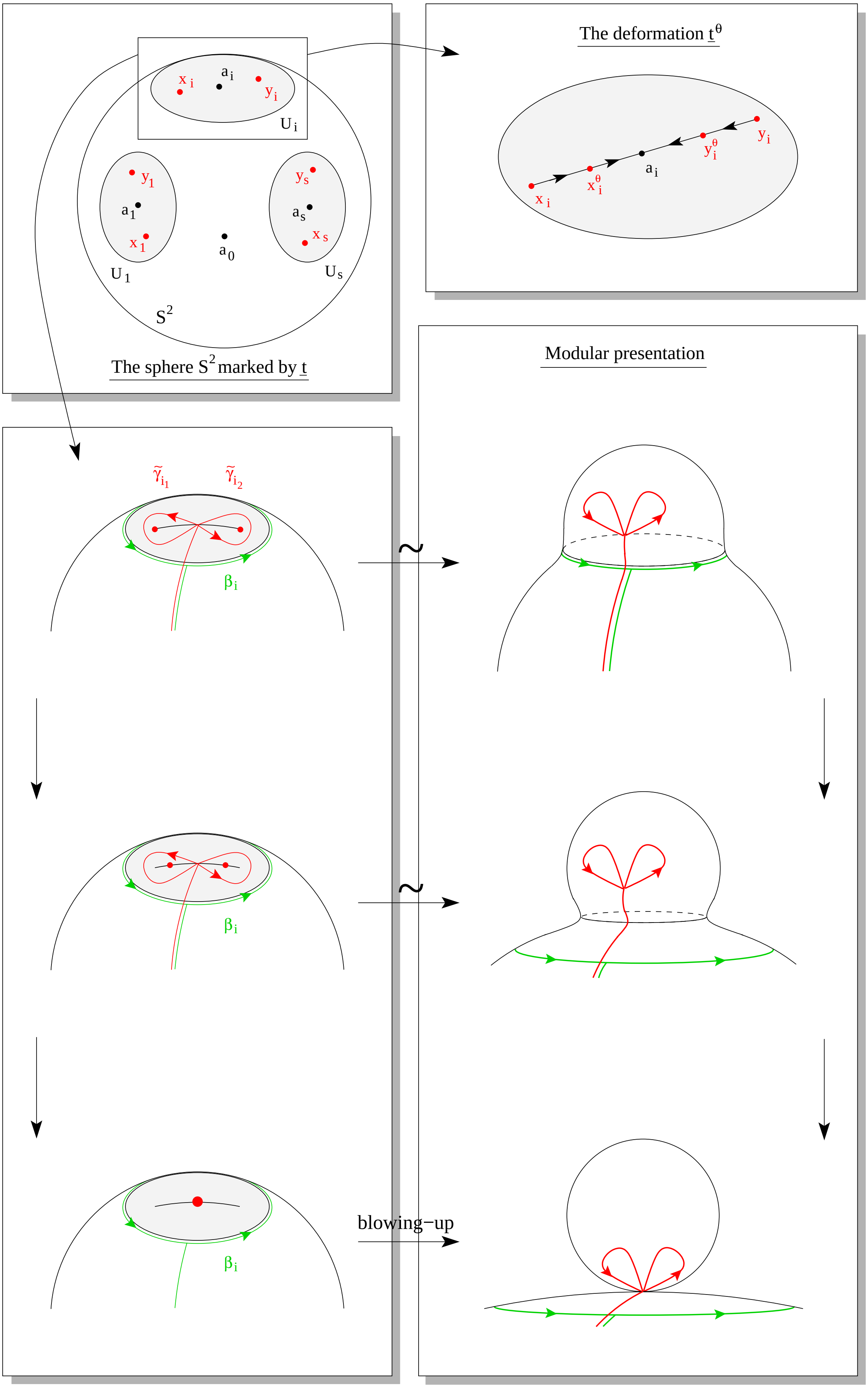
$$\begin{cases} x_i^\theta = (1-\theta)x_i + \theta a_i \\ y_i^\theta = (1-\theta)y_i + \theta a_i \end{cases} \qquad (i = 1, \ldots, s)$$

This deformation induces a continuous path between the representing points on the moduli space $\overline{\mathcal{U}_r}$. In Wewers' modular compactification of $\mathcal{U}_r$, the limit point (for $\theta = 1$) represents a *comb*. This comb is obtained by blowing up the deformation space $\mathbf{t}^\theta$ $(\theta \in [0,1])$ at each double point $x_i^1 = y_i^1$, $i = 1, \ldots, s$ (see figures for a topological representation of this process). Denote the resulting comb by $\mathcal{C}$, which is the union of the sphere S^2 with s "small" spheres $\Sigma_1, \ldots, \Sigma_s$, pairwise disjoint, attached to S^2 at the points $a_1, \ldots, a_s$ respectively and marked by two distinct points (distinct from $a_1, \ldots, a_s$).

For each $i = 1, \ldots, s$, let $\gamma_{i,1}, \gamma_{i,2}$ be closed paths based at a_i, revolving around the segment line $[x_i, a_i]$ and $[a_i, y_i]$; for each $\theta \in [0,1[$, their homotopy classes $\Gamma_{i,1}, \Gamma_{i,2}$ freely generate $\pi_1^{\rm top}(U_i \setminus \{x_i^\theta, y_i^\theta\}, a_i)$. Fix a point $a_0 \in S^2 \setminus \bigcup_{1 \leq i \leq s} \overline{U_i}$ and a set of paths $\delta_1, \ldots, \delta_s$, pairwise disjoint and connecting a_0 to $a_1, \ldots, a_s$ respectively, in such a way that, setting $\widetilde{\gamma}_{i,j} = \delta_i \gamma_{i,j} \delta_i^{-1}$ $(i = 1, \ldots, s,\ j = 1, 2)$, the corresponding homotopy classes $\widetilde{\Gamma}_{1,1}, \widetilde{\Gamma}_{1,2}, \ldots, \widetilde{\Gamma}_{s,1}, \widetilde{\Gamma}_{s,2}$ constitute a topological bouquet $\underline{\widetilde{\Gamma}}$ for each base space $S^2 \setminus \mathbf{t}^\theta$ based at a_0 $(\theta \in [0,1[)$[5].

Next let $d \geq 1$ be an integer and $G \subset S_d$ be a subgroup of S_d given with a generating system $\{g_1, \ldots, g_s\}$. For every $\theta \in [0,1[$, let $\phi_\theta : \pi_1^{\rm top}(S^2 \setminus \mathbf{t}^\theta, a_0) \to G \subset S_d$ be the epimorphism mapping $\widetilde{\Gamma}_{i,1}$ to g_i and $\widetilde{\Gamma}_{i,2}$ to g_i^{-1}, $i = 1, \ldots, s$. Denote the associated $\mathbb{C}$-cover by f_θ and the corresponding representing point on $\mathcal{H}_{r,G}^\infty$ by h_θ. By construction, the covers f_θ are those obtained from f_0 by the deformation $\mathbf{t}^\theta$ $(\theta \in [0,1[)$. The first part of the lemma below also follows by construction.

[5] As above in footnote 4, the paths $\widetilde{\gamma}_{1,1}, \widetilde{\gamma}_{1,2}, \ldots, \widetilde{\gamma}_{s,1}, \widetilde{\gamma}_{s,2}$ themselves do not satisfy conditions (i)-(iv) of §1.1 but they are homotopic to paths which do.



The deformation $\underline{t}^\theta$
x_i
x_i^θ
a_i
y_i^θ
y_i
a_i
U_i
y_1
a_1
x_1
U_1
a_0
y_s
a_s
x_s
U_s
S^2
The sphere S^2 marked by $\underline{t}$
Modular presentation
$\tilde{\gamma}_{i_1}$
$\tilde{\gamma}_{i_2}$
β_i
$\sim$
β_i
$\sim$
β_i
blowing−up

Lemma 2.2 — *The covers f_θ are HM-covers ($\theta \in [0,1[$). Furthermore, the collection of points $h_\theta = [f_\theta]$ converges in $\overline{\mathcal{H}_{r,G}}(\mathbb{C})$ as $\theta \to 1$ and the limit point h_1 corresponds to the isomorphism class of a HM-admissible cover f_1 of the comb $\mathcal{C}$ with cyclic restriction of inertia canonical invariant $\{g_i, g_i^{-1}\} \subset S_d$ above each sphere Σ_i, $i = 1, \ldots, s$.*

Proof. For the second part, let $(\theta_n)_{n>0}$ be a sequence of elements in $[0,1[$ such that $\theta_n \to 1$ and $(h_{\theta_n})_{n>0}$ converges in $\overline{\mathcal{H}_{r,G}}(\mathbb{C})$ as $n \to \infty$. Due to the continuity of $\overline{\mathcal{H}_{r,G}}(\mathbb{C}) \to \overline{\mathcal{U}_r}(\mathbb{C})$, the limit point h_1 corresponds to the isomorphism class of a cover f_1 of the comb $\mathcal{C}$.

Set $\mathcal{B}' = S^2 \setminus \bigcup_{i=1}^s \overline{U_i}$ and $\beta_i = \check{\delta}_i u_i \check{\delta}_i^{-1}$ where $\check{\delta}_i$ is the part of δ_i from a_o to the first intersection point, say b_i, with the disk U_i and u_i is the loop based at b_i that clockwise bounds the disk U_i, $i = 1, \ldots, s$; the homotopy classes $[\beta_1], \ldots, [\beta_s]$ generate $\pi_1^{\rm top}(\mathcal{B}', a_0)$ with the single relation $[\beta_1] \cdots [\beta_s] = 1$. For every $\theta \in [0,1]$, denote by ϕ'_θ the representation $\pi_1^{\rm top}(\mathcal{B}', a_0) \to S_d$ associated with the restriction f'_θ to $\mathcal{B}'$ of the cover f_θ ($\theta \in [0,1]$).

For $\theta \in [0,1[$, ϕ'_θ is the restriction of ϕ_θ to $\pi_1^{\rm top}(\mathcal{B}', a_0)$. As in $\pi_1^{\rm top}(S^2 \setminus \mathbf{t}^\theta, a_0)$ we have $[\beta_i] = [\widetilde{\gamma}_{i,1}][\widetilde{\gamma}_{i,2}]$, the definition of ϕ_θ yields $\phi_\theta([\beta_i]) = 1$, $i = 1, \ldots, s$, for all $\theta \in [0,1[$. It follows that $\phi'_\theta([\beta_i]) = 1$ in $\pi_1^{\rm top}(\mathcal{B}', a_0)$, $i = 1, \ldots, s$, and so, $\phi'_\theta = 1$, for all $\theta \in [0,1[$.

Now the assumption $\lim_{n\to\infty} h_{\theta_n} = h_1$ implies that $\phi'_1 = \phi'_{\theta_n} = 1$ (for all $n > 0$). Therefore the restriction of f_1 to $S^2 \setminus \mathbf{t}^1$ is unramified at the points $a_1, \ldots, a_s$: f_1 restricts to a trivial cover above the root S^2 of the comb. This also shows that the restriction of f_1 to each sphere Σ_i (each end component) is unramified at a_i, hence is a cyclic cover branched at two points, $i = 1, \ldots, s$. More precisely, this cover is determined by the monodromy action along the paths $\gamma_{i,1}$ and $\gamma_{i,2}$ (based at a_i) viewed on the comb $\mathcal{C}$; as f_1 is trivial above the root S^2, it is the same as the monodromy action along the paths $\widetilde{\gamma}_{i,1}$ and $\widetilde{\gamma}_{i,2}$ (based at a_0); by construction it is given by g_i and g_i^{-1}, $i = 1, \ldots, s$. $\square$

Addendum to lemma 2.2. In the proof of lemma 3.2, we will have to use that
(*) *the construction above can be achieved with the extra constraint that the comb $\mathcal{C}$ and the HM-admissible cover f_1 are prescribed in advance.*

That is, the following will be given: the comb $\mathcal{C}$ given as a root sphere Σ_0 attached to s spheres $\Sigma_1, \ldots, \Sigma_s$ at given points $a_1, \ldots, a_s$ respectively, the group $G \subset S_d$, and, for $i = 1, \ldots, s$, the (not necessarily transitive) representation $\pi_1^{\rm top}(\Sigma_i \setminus \{2 \text{ pts}\}, a_i) \to S_d$ corresponding to the restriction of f_1 to Σ_i, where $\{1, \ldots, d\}$ is the fiber of some fixed point $a_0 \in \Sigma_0$ in the cover f_1, which the fibers of $a_1, \ldots, a_s$ are identified to as the restriction of the cover to Σ_0 is trivial. This last part of the data readily provides an r-tuple $(g_1, g_1^{-1}, \ldots, g_s, g_s^{-1})$: take for g_i, g_i^{-1} the images of two standard generators of $\pi_1^{\rm top}(\Sigma_i \setminus \{2 \text{ pts}\}, a_i)$, $i = 1, \ldots, s$. From this, one easily forms an r-tuple $\mathbf{t}$, a deformation

$\mathbf{t}^\theta$ and a cover f_0 as above such that the corresponding specialization of f_0 for $\theta = 1$ is the prescribed cover f_1.

2.3. Construction of HM-covers from patching methods. This paragraph is aimed at reinterpreting the notion of HM-covers in the rigid viewpoint and will be used in §4 to show that the rigid covers that are used in [DeDes2] can be constructed to be HM-covers. In §4.3, we will provide an alternate formal approach to the result of [DeDes2], which does not use this paragraph.

Fix an even integer $r = 2s$ and a discrete valuation ring $\mathcal{O}$, assumed to be complete, with fraction field k and algebraically closed residue field κ of characteristic $p \geq 0$. Let $\mathbf{t} = \{x_1, y_1, \ldots, x_s, y_s\} \in \mathcal{U}_r(k)$ with $\mathbf{t} \subset \mathbb{P}^1(k)$. Assume further that, modulo the maximal ideal $\mathcal{P}$ of $\mathcal{O}$,

(*) x_i and y_i are in the same coset, $i = 1, \ldots, r$, and,
$x_1, \ldots, x_s$ lie in pairwise distinct cosets.

(For points a, b in k identified with $\mathbb{P}^1(k) \setminus \{\infty\}$, being in the same coset modulo $\mathcal{P}$ more explicitly means that either $|a| \leq 1$, $|b| \leq 1$ and $|a - b| < 1$, or, $|a| > 1$ and $|b| > 1$).

Classically $\mathbb{P}^1_k$ marked by the r-points $x_1, y_1, \ldots, x_s, y_s$ has a unique stable model $\widetilde{P}_\mathbf{t}$ over $\mathcal{O}$ such that the points $x_1, y_1, \ldots, x_s, y_s$ extends to sections $\widetilde{x}_1, \widetilde{y}_1 \ldots, \widetilde{x}_s, \widetilde{y}_s$ specializing at distinct points $\bar{x}_1, \bar{y}_1, \ldots, \bar{x}_s, \bar{y}_s$ of the special fiber. The special fiber $\overline{P}_\mathbf{t}$ is a comb over κ with root T_0 attached to s end components $T_1, \ldots, T_s$. Denote its singular points by $\bar{a}_1, \ldots, \bar{a}_s$. The model $\widetilde{P}_\mathbf{t}$ induces an $\mathcal{O}$-model of the rigid analytic space $P_{\mathbf{t},\mathrm{rig}}$, which is the maximal spectrum of the generic fiber of the formal completion of $\widetilde{P}_\mathbf{t}$ along the special fiber (*e.g.* [Ga]).

For each $i = 1, \ldots, r$, pick a point ω_i such that $|x_i - \omega_i| = |y_i - \omega_i| = |x_i - y_i| = r_i$ and denote by D_i the open disk of center ω_i and radius 1 and by ∂D_i the subset of D_i of all points x such that $1 > |x - \omega_i| > r_i$. Points x verifying $|x - \omega_i| = r_i$ specialize on T_i; those for which $|x - \omega_j| = 1$ for all $j = 1, \ldots, s$, specialize on T_0; and points of ∂D_i specialize at $\bar{a}_i$, $i = 1, \ldots, s$.

Fix a finite group G of prime-to-p order and an r-tuple $\mathbf{C} = (C_1, C_1^{-1}, \ldots, C_s, C_s^{-1})$ of conjugacy classes of G. Suppose given a k-cover $f : X \to \mathbb{P}^1_k$ of group G, with branch point set $\mathbf{t}$ and inertia canonical invariant $\mathbf{C}$. After some finite extension of k, the cover $f : X \to \mathbb{P}^1_k$ uniquely extends to a cover $\widetilde{f} : \widetilde{X} \to \widetilde{P}_\mathbf{t}$ (§1.2).

Denote also by f_i, $i = 1, \ldots, s$, the restricted rigid cover f above the disk D_i.

Proposition 2.3 — *The following conditions are equivalent.*
(i) *each restricted cover* f_i *is trivial above* ∂D_i, $i = 1, \dots, s$,
(ii) *each restricted cover* f_i *extends to a cover* $g_i : Y_i \to \mathbb{P}^1_{\mathrm{rig}}$ *with only two branch points*[6] *(*x_i *and* y_i*)* , $i = 1, \dots, s$,
(iii) *the special fiber* $\bar{f}$ *of* $\widetilde{f}$ *is unramified at the singular points* $\bar{a}_1, \dots, \bar{a}_s$ *of the comb* $\bar{P}_{\mathbf{t}}$, *that is,* $\bar{f}$ *is a HM-admissible cover.*

Proof. Fix some index $i \in \{1, \dots, s\}$.

(i)⇒(iii). The restriction $\bar{f}_i$ of the admissible cover $\bar{f}$ to T_i can be viewed as the reduction modulo an uniformizing parameter of f_i. The restriction of f_i to ∂D_i is supposed to be trivial, and the fiber of $\bar{a}_i$ in the restriction of $\widetilde{f}$ to T_i corresponds to the fiber of ∂D_i in f_i. This shows that this restriction is unramified at $\bar{a}_i$.

(iii)⇒(ii). We suppose the restriction $\bar{f}_i$ of $\bar{f}$ to T_i is unramified at $\bar{a}_i$. So $\bar{f}_i$ extends to a cover of $\mathbb{P}^1_{\mathcal{O}}$ unramified outside two points x_i, y_i. The generic fiber of this cover induces a rigid analytic cover $g_i : Y_i \to \mathbb{P}^1_{\mathrm{rig}}$ unramified outside $\{x_i, y_i\}$, whose restriction to D_i can be identified to f_i.

(ii) ⇒(i). Suppose that f_i extends to a cover $g_i : Y_i \to \mathbb{P}^1_{\mathrm{rig}}$ unramified outside $\{x_i, y_i\}$. The restriction of g_i to any disk containing neither x_i nor y_i is trivial. Then the restriction of f_i to any closed annulus contained in ∂D_i is trivial. So the same is true for the restriction of f_i to ∂D_i. □

Assume further that k is of characteristic 0. Let $\mathbb{Q}(\mathbf{t}) \subset k$ be the subfield generated by the branch point set $\mathbf{t} = \{x_1, y_1, \dots, x_s, y_s\}$ of the cover f and $\overline{\mathbb{Q}(\mathbf{t})} \subset \overline{k}$ be its algebraic closure inside $\overline{k}$. It classically follows (from Riemann's existence theorem or Grothendieck's specialization theorem) that $f \otimes_k \overline{k}$ has a model $\widetilde{f}_{\overline{\mathbb{Q}(\mathbf{t})}}$ over $\overline{\mathbb{Q}(\mathbf{t})}$. Next fix an embedding $i : \overline{\mathbb{Q}(\mathbf{t})} \hookrightarrow \mathbb{C}$. *Via* this embedding, the cover $\widetilde{f}_{\overline{\mathbb{Q}(\mathbf{t})}}$ induces a $\mathbb{C}$-cover $f^i : X^i \to \mathbb{P}^1_{\mathbb{C}}$ of group G, with branch point set $\mathbf{t}^i$ and with inertia canonical invariant $\mathbf{C}^{\chi(i)}$. Denote the corresponding complex point in $\mathcal{H}^{\infty}_{r,G}(\mathbf{C}^{\chi(i)})$ by $[f]^i$. It is the image *via* i of the k-point $[f] \in \mathcal{H}^{\infty}_{r,G}(\mathbf{C})$. As a consequence of corollary 1.5 we obtain

Corollary 2.4 — *If the cover* f *satisfies the equivalent conditions of proposition 2.3, then the point* $[f]^i$ *lies in a HM-component of* $\mathcal{H}^{\infty}_{r,G}(\mathbf{C}^{\chi(i)})$.

---
[6] g_i is then necessarily a cyclic cover of $\mathbb{P}^1$ by a curve Y_i of genus 0.

3. Proof of theorem 1.4

3.1. Direct part. Fix s open disks $U_1,\ldots,U_s$ in $\mathbb{P}^1(\mathbb{C})$, pairwise disjoint, and pick distinct points x_i, y_i in U_i, $i=1,\ldots,s$. Set $\mathbf{t}=\{x_1,y_1\ldots,x_s,y_s\}$ and fix a topological bouquet $\underline{\Gamma}$ for $\mathbb{P}^1\setminus\mathbf{t}$ as in §2.2. From assertion (**) of §1.1, if $\mathcal{H}$ is any HM-component of $\mathcal{H}^{\infty}_{r,G}(\mathbf{C})$, there exists an isomorphism class of cover $[f_o]\in\mathcal{H}$ with branch point set $\mathbf{t}$ and with branch cycle description relative to $\underline{\Gamma}$ of the form $(g_1,g_1^{-1},\ldots,g_s,g_s^{-1})$. The construction given in §2.2 applies to show that $\mathcal{H}$ has HM-admissible covers in its boundary.

Alternatively theorem 2.1 can be used to prove this direct part.

3.2. Converse. Suppose given a component $\mathcal{H}$ of $\mathcal{H}^{\infty}_{r,G}(\mathbf{C})$ whose boundary $\overline{\mathcal{H}}$ in $\overline{\mathcal{H}_{r,G}}$ contains a point representing a HM-admissible cover φ defined over some algebraically closed field κ. We will describe "a path in the closure $\overline{\mathcal{H}}$" from the point representing φ to a complex point representing a HM-cover. If κ is of characteristic 0, the first stage can be skipped.

3.2.1. First stage. Suppose the field κ is of characteristic $p>0$. Let k be a henselian field of characteristic 0 and of residue field κ. Call $\mathcal{O}$ the ring of integers of k.

Lemma 3.1 — *The κ-cover φ lifts to a $\overline{k}$-HM-admissible cover $\bar{f}$ of a comb with s end components $\bar{T}_1,\ldots,\bar{T}_s$, each of them being a copy of $\mathbb{P}^1_{\overline{k}}$, and satisfying the following:*

- *the restricted cover $\bar{f}$ above $\bar{T}_i$ is a (not necessarily connected) cyclic cover branched at two points and unramified at the intersection of $\bar{T}_i$ and the root $\bar{T}_0$; its inertia canonical invariant is $\{g_i,g_i^{-1}\}$ for some $g_i\in C_i$, $i=1,\ldots,s$,*
- *the restricted cover $\bar{f}$ above the root $\bar{T}_0$ is trivial,*
- *$g_1,\ldots,g_r$ generate the group G,*
- *the point $[\bar{f}]$ lies on $\overline{\mathcal{H}}$.*

Proof. The base space of the cover φ is a comb τ defined over κ, which consists in a root $\tau_0\simeq\mathbb{P}^1_\kappa$ with s marked distinct points $\alpha_1,\ldots,\alpha_s$, and s end components $\tau_1,\ldots,\tau_s$ attached to the root at $\alpha_1,\ldots,\alpha_s$ respectively, each of them marked by two points. Choose a deformation $\widetilde{\tau}_0$ of the marked curve τ_0 over $\mathcal{O}$: $\mathbb{P}^1_{\mathcal{O}}$ marked by $\widetilde{\alpha}_1,\ldots,\widetilde{\alpha}_s$. At each section $\widetilde{\alpha}_i$ of $\widetilde{\tau}_0$ attach a copy $\widetilde{\tau}_i$ of $\mathbb{P}^1_{\mathcal{O}}$ marked by two points ($i=1,\ldots,s$). Denote the resulting space over $\mathcal{O}$ by $\widetilde{\tau}$.

The restriction of φ to each component of τ_i is a cyclic cover branched at two points, $i=1,\ldots,s$. For every given integer $d\geq 1$, there is, up to isomorphism, a unique connected cyclic cover of $\mathbb{P}^1$ of degree d branched at two points. Thus each component of $\varphi|_{\tau_i}$ has a unique deformation to a $\mathcal{O}$-cover of $\widetilde{\tau}_i$ branched at two sections. The trivial cover given by the restriction of φ to τ_0 obviously extends to a trivial $\mathcal{O}$-cover of $\widetilde{\tau}_0$. The patching datas above α_i between the restrictions of φ to τ_0 and τ_i uniquely extend to patching datas over $\mathcal{O}$ $(i = 1, \ldots, s)$. This follows from the fact that the points of the fiber of $\tilde{\alpha}_i$ are defined over $\mathcal{O}$. As a result we obtain a cover $\widetilde{\varphi}$ of $\widetilde{\tau}$. Denote its geometric generic fiber by $\bar{f}$; it is a $\overline{k}$-HM-admissible cover. From Wewers' work, the representative point is on $\overline{\mathcal{H}_{r,G}}$. As it reduces to $[\varphi]$ modulo the maximal ideal of $\mathcal{O}$, it has to be on $\overline{\mathcal{H}}$. The rest of lemma 3.1 readily follows. □

3.2.2. Second stage. If κ is of characteristic $p > 0$, retain the notation of §3.2.1. If κ is of characteristic 0, set $k = \kappa$ and $\bar{f} = \varphi$. In both cases, $\bar{f}$ is a HM-admissible cover of a comb over k. In fact $\bar{f}$ can be defined over the algebraic closure $\overline{k_0} \subset \overline{k}$ of the field of definition k_0 of the branch points. Its representing point $[\bar{f}]$ on the moduli space $\overline{\mathcal{H}_{r,G}}$ is a $\overline{k_0}$-point on $\overline{\mathcal{H}}$. This in particular provides an embedding $F \hookrightarrow \overline{k_0}$ of the field of definition F of $\overline{\mathcal{H}}_{\mathbb{Q}}$ into $\overline{k_0}$; F is a number field contained in $\mathbb{C}$. Extend the inclusion $F \subset \mathbb{C}$ to an embedding $\iota : \overline{k_0} \hookrightarrow \mathbb{C}$. The $\mathbb{C}$-cover $\bar{f}^\iota$ obtained *via* this embedding corresponds to a complex point in $\overline{\mathcal{H}}$.

By construction, $\bar{f}^\iota$ is a complex HM-admissible cover of a comb T: it is trivial above the root $T_0 \simeq \mathbb{P}^1_{\mathbb{C}}$, has s end components $T_1, \ldots, T_s$ isomorphic to $\mathbb{P}^1_{\mathbb{C}}$, and each of the restrictions of $\bar{f}^\iota$ to some connected component above T_i is a $\mathbb{C}$-cyclic cover of group with inertia canonical invariant $\{g_i, g_i^{-1}\}$ for some $g_i \in C_i$ and with group $< g_i > \subset G$, $i = 1, \ldots, s$. Furthermore, the elements $g_1, \ldots, g_s$ generate the group G.

Lemma 3.2 — *The $\mathbb{C}$-cover $\bar{f}^\iota$ is in the topological closure of some HM-component of the Hurwitz space $\mathcal{H}^\infty_{r,G}(\mathbf{C})$.*

Theorem 1.4 will then follow immediately. Indeed the representing points of the covers φ and $\bar{f}^\iota$ are in the same component $\overline{\mathcal{H}}$ of $\overline{\mathcal{H}_{r,G}(\mathbf{C})}$; hence they are in the boundary of the same component $\mathcal{H}$ of $\mathcal{H}^\infty_{r,G}(\mathbf{C})$, which from lemma 3.2 is a HM-component.

We give two proofs of lemma 3.2. The first one uses §2.1 and the second one §2.2.

1st proof. The complex comb T can be deformed over the ring $\mathbb{C}\{\{t\}\}$ of Taylor series of positive radius of convergence to a stable curve $\widetilde{P}_{\mathbf{t}}$ of genus 0 with $2s$ sections $\widetilde{x}_1, \widetilde{y}_1, \ldots, \widetilde{x}_s, \widetilde{y}_s$ and whose generic fiber is a $\mathbb{P}^1$ marked by $\mathbf{t} = \{x_1, y_1, \ldots, x_s, y_s\}$. Using §1.2 again, extend the HM-admissible cover $\bar{f}^\iota$ to a $\mathbb{C}\{\{t\}\}$-cover $\widetilde{f}$ with generic fiber a smooth cover of $\mathbb{P}^1$ branched at $x_1, y_1, \ldots, x_s, y_s$. As all the varieties we consider are of finite type over $\mathbb{C}\{\{t\}\}$, there exists a real number $\rho > 0$ such that $\widetilde{f}$ induces an analytic family of covers $\widetilde{f}_\theta$ $(0 < \theta \leq \rho)$ of $\mathbb{P}^1$ defined over $\mathbb{C}$ ramified at $2s$ points $x^\theta_1, y^\theta_1, \ldots, x^\theta_s, y^\theta_s$ (the specializations of $\widetilde{x}_1, \widetilde{y}_1, \ldots, \widetilde{x}_s, \widetilde{y}_s$ at $t = \theta$).

We now apply theorem 2.1. The topological fundamental group of the fiber at θ of $\widetilde{P}_{\mathbf{t}}$ which we denote below by $(\mathbb{P}^1)^\theta$, is constant. With the notation of theorem 2.1, we have homotopy classes $\theta_1(\Gamma_1^{(1)})^{\sigma_1}, \theta_1(\Gamma_2^{(1)})^{\sigma_1}, \ldots, \theta_s(\Gamma_1^{(s)})^{\sigma_s}, \theta_s(\Gamma_2^{(s)})^{\sigma_s}$ which constitute a topological bouquet of $(\mathbb{P}^1)^\theta \setminus \{x_1^\theta, y_1^\theta, \ldots, x_s^\theta, y_s^\theta\}$.

As the cover $\bar{f}^\iota$ is unramified at each point $\bar{a}_i$, $i = 1, \ldots, s$ ($\bar{f}^\iota$ is HM-admissible), the Branch Cycle Description of the cover $\widetilde{f}_\theta$ with respect to this topological bouquet is of the form $g_1^{h_1}, (g_1^{-1})^{h_1}, \ldots, g_s^{h_s}, (g_s^{-1})^{h_s}$. The cover $\widetilde{f}_\theta$ is the unique deformation of $\bar{f}^\iota$ along the path $\{x_1^\theta, y_1^\theta, \ldots, x_s^\theta, y_s^\theta\}$ ($\theta \in]0, 1[$) and hence is a connected cover of monodromy group G. Thus the cover $\widetilde{f}_\theta$ is a complex HM-cover corresponding to some point in $\mathcal{H}_{r,G}^\infty(\mathbf{C})$, which proves lemma 3.2. □

2nd proof. §2.2 explains how to construct a family of HM-covers degenerating to a complex HM-admissible cover f_1. From the *addendum* to lemma 2.2, there is no restriction on the degenerate cover f_1; we can take it to be $\bar{f}^\iota$. The HM-covers f_θ $(0 < \theta < 1)$ provided by the construction have then $2s$ branch points, their group is the group G generated by $g_1, \ldots, g_s$ and the inertia canonical invariant is the tuple $\mathbf{C}$ consisting of the s pairs of conjugacy classes C_i, C_i^{-1} of g_i and g_i^{-1}, $i = 1, \ldots, s$. This shows indeed that $\bar{f}^\iota$ is in the topological closure of some HM-component of $\mathcal{H}_{r,G}^\infty(\mathbf{C})$. □

4. Application to Hurwitz towers

This section is devoted to our application to inverse Galois theory; the previous sections are used in the special context of G-covers.

4.1. Statement of the main result.

Theorem 4.1 — *Suppose given a projective system* $\mathbf{G} = (G_n)_{n\geq 0}$ *of finite groups with surjective morphisms* $s_n : G_n \twoheadrightarrow G_{n-1}$ $(n > 0)$*. Consider the field generated over* $\mathbb{Q}$ *by all roots of unity of order* $|G_n|$ $(n \geq 0)$ *and denote its maximal real subfield by* $\mathbb{Q}(\mu_{\mathbf{G}})^c$*. Then one can construct a projective system (a tower)* $(\mathcal{H}_n)_{n\geq 0}$ *of varieties* $\mathcal{H}_n$*, geometrically irreducible, with algebraic morphisms* $\psi_n : \mathcal{H}_n \to \mathcal{H}_{n-1}$*, defined over* $\mathbb{Q}(\mu_{\mathbf{G}})^c$ *and with the following properties:*

(i) *For each* $n \geq 0$*, the variety* $\mathcal{H}_n$ *is the unique HM-component of some Hurwitz space* $\mathcal{H}_{r_n,G_n}^\infty(\mathbf{C}_n)$ *(for some integer* $r_n > 0$ *and some* r_n*-tuple of conjugacy classes of* G_n*).*

(ii) *As a consequence of* (i)*, recall that if* k *is any field containing* $\mathbb{Q}(\mu_{\mathbf{G}})^c$*, existence of* k*-rational points on* $\mathcal{H}_n$ *implies the group* G_n *can be realized as the automorphism group of a* $\overline{k}$*-G-cover of* $\mathbb{P}^1$ *of field of moduli* k*.*

(iii) *If k be a henselian field of characteristic* 0*, of residue characteristic either $p=0$ or $p>0$ not dividing any of the orders $|G_n|$ $(n\geq 0)$, and containing all roots of* 1 *of prime-to-p order, there exist projective systems of k-points on the tower $(\mathcal{H}_n)_{n\geq 0}$. For example, there exist projective systems of $\mathbb{Q}^{\rm ab}((x))$-rational points and there exist projective systems of $\mathbb{Q}_p^{\rm ur}$-rational points, for each prime p such that all G_n $(n\geq 0)$ are of prime-to-p order*[7]*. Furthermore, there also exist projective systems of real points.*

(iv) *In* (iii)*, the projective systems of rational points have the extra property that at each level $n\geq 0$, the point lies in the no-obstruction locus of $\mathcal{H}_n$, that is, where the field of moduli is a field of definition. Consequently, the projective systems of k-rational points in question in* (iii) *correspond to projective systems of k-G-covers $X_n\to\mathbb{P}^1$, or equivalently, to towers of k-regular extensions $K_n/k(T)$, realizing the system $(G_n)_{n\geq 0}$.*

Remarks 4.2. (a) In general, Hurwitz spaces are coarse moduli spaces and so k-rationality of their points $[f]$ only corresponds to f being of field of moduli k but not necessarily defined over k. We do have conclusions about fields of definition. So some information is lost in stating the results in terms of rational points on moduli spaces as in (iii). Assertion (iv) compensates this loss. We could have instead stated the result in terms of stacks rather than moduli spaces. However in this refined version, the stack-theoretic H_n counterpart of $\mathcal{H}_n$ would not be an algebraic variety anymore.

(b) Recall that presence of roots of 1 in the base field k in (iii) is not just a technical assumption due to the method employed. The result would be false otherwise: for example, the group $\mathbb{Z}_p=\varprojlim\mathbb{Z}/p^n\mathbb{Z}$ is not a regular Galois group over $\mathbb{Q}_\ell(T)$ [Se].

4.2. Proof of theorem 4.1. The first stage consists in constructing a sequence $(r_n)_{n\geq 0}$ of integers $r_n\geq 3$ and a sequence $(\mathbf{C}_n)_{n\geq 0}$ of r_n-tuples $\mathbf{C}_n$ of conjugacy classes of G_n with the following property: if k is a henselian field as in (iii) or if $k=\mathbb{R}$, then there exists a projective system $(f_n)_{n\geq 0}$ of G-covers f_n defined over k with group G_n, with r_n branch points and with inertia canonical invariant $\mathbf{C}_n$. Such a construction is the main result of [DeDes2] (see théorème 4.1). Over henselian fields it was performed using rigid patching techniques; formal techniques can be used alternatively; we explain how in §4.3.

This construction thus yields a tower $(\mathcal{H}^\infty_{r_n,G_n}(\mathbf{C}_n))_{n\geq 0}$ satisfying the desired assertions (iii) and (iv) of theorem 4.1 (with $\mathcal{H}^\infty_{r_n,G_n}(\mathbf{C}_n)$ replacing $\mathcal{H}_n$). However the varieties $\mathcal{H}^\infty_{r_n,G_n}(\mathbf{C}_n)$ are not geometrically irreducible as the $\mathcal{H}_n$ are claimed to be in theorem 4.1.

[7] The fields $\mathbb{Q}^{\rm ab}((x))$ and $\mathbb{Q}_p^{\rm ur}$ can even be replaced by the smaller henselian subfields $\mathbb{Q}^{\rm ab}((x))^{\rm alg}$ and $(\mathbb{Q}_p^{\rm ur})^{\rm alg}$ of all elements algebraic over $\mathbb{Q}(x)$ and $\mathbb{Q}$ respectively.

In order to get the full statement of theorem 4.1, we will use theorem 1.4 to show that the realizing covers f_n can be taken to be HM-covers.

In the formal setting, we refer to theorem 4.4 for this point. In the rigid setting this can be justified as follows. The first stage of the method of [DeDes2] is to construct, at each level $n \geq 0$, some cover f_n as required, over the completion of k first, thanks to the Serre-Liu-Pop rigid patching method ($n \geq 0$) [Se; §8.4.4], [Li], [Po1]. By construction, this f_n satisfies condition (i) from proposition 2.3 and the following condition on the branch point set $\mathbf{t} = \{x_1, y_1 \ldots, x_s, y_s\} \subset \mathbb{P}^1(k)$:

(**) $|x_i - y_i| < |x_i - x_j|\, |p|^{\frac{1}{p-1}}$, $j \neq i, i = 1, \ldots, s$ (with $|p|^{\frac{1}{p-1}} = 1$ if $p = 0$).

If one picks the points $x_1, y_1 \ldots, x_s, y_s \in \mathbb{P}^1(k)$ satisfying both conditions (*) from §2.3 and (**) above (*e.g.* $|x_i| = 1$, $|x_i - x_j| = 1$ and $|x_i - y_i| < |p|^{\frac{1}{p-1}}$ $(i, j \in \{1, \ldots, s\}, i \neq j)$), then the construction leads to a cover f_n satisfying the conclusions of proposition 2.3, that is, thanks to theorem 1.4, to a HM-cover. Follow then [DeDes2] (§3.1 and théorème 3.4) to show that there exists a projective system of such HM-covers, and that their field of definition can be descended to k (from its completion) as the branch points are in $\mathbb{P}^1(k)$.

Finally, as explained in [DeDes2], the sequences $(r_n)_{n\geq 0}$ and $(\mathbf{C}_n)_{n\geq 0}$ can be initially chosen in such a way that $\mathbf{C}_n$ is HM-g-complete and so there is a single HM-component on $\mathcal{H}^{\infty}_{G_n, r_n}(\mathbf{C}_n)$, and it is defined over $\mathbb{Q}(\mu_{\mathbf{G}})^c$ $(n \geq 0)$[3]. Define $\mathcal{H}_n$ to be this HM-component $(n \geq 0)$. The projective system $(\mathcal{H}_n)_{n\geq 0}$ fulfills all conclusions of theorem 4.1. □

Remark 4.3. The tower $(\mathcal{H}_n)_{n\geq 0}$ constructed above can more precisely be defined over any field $k \subset \mathbb{Q}^{\mathrm{ab}}$ over which $\mathbf{C}_n$ is k-rational for all $n \geq 0$. In many interesting cases, it is possible to construct tuples $\mathbf{C}_n$ as above with the extra property that they are all $\mathbb{Q}$-rational: that is the case for example if $\mathcal{G} = \varprojlim G_n$ is generated by finitely many elements of finite order, and in particular in the situation of modular towers (see §4.4). However this is not possible in general. Indeed assume that elements of finite order together with elements with trivial image in G_0 do not generate the group $\mathcal{G} = \varprojlim G_n$ (think of $\mathcal{G} = \mathbb{Z}_p$) and suppose given a projective system $(\mathbf{C}_n)_{n\geq 0}$ as above. Then one may assume that $C_{n,1}$ is of order ν_n $(n \geq 0)$ with $\nu_n \to \infty$ and $C_{0,1} \neq \{1\}$. For each $n \geq 0$, $\mathbb{Q}$-rationality of $\mathbf{C}_n$ implies it must contain $\phi(\nu_n)$ distinct prime-to-ν_n powers $C^{\mu}_{n,1}$ of $C_{n,1}$ (where ϕ is the Euler function). Now these classes map to non trivial classes of G_0 to provide as many entries in $\mathbf{C}_0$ (vith possible repetitions): a contradiction as $\phi(\nu_n)$ tends to ∞.

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3 We point out that there is a mistake in the second part of statement (a) of théorème 4.1 of [DeDes2], which can be rectified as follows: the field of definition of the component $\mathcal{H}^{\infty}_n$ in question is not $\mathbb{Q}$ as asserted but is equal to the field of definition of the whole Hurwitz space $\mathcal{H}_{G_n, r_n}(\mathbf{C}_n)$ (it is $\mathbb{Q}$ if it is assumed further $\mathbf{C}_n$ is a rational union of conjugacy classes).

4.3. Formal approach. We give here the alternate proof of théorème 4.1 of [DeDes2] using formal geometry, thus providing a complete formal approach to theorem 4.1.

Theorem 4.4 — *Let* $(s_n : G_n \twoheadrightarrow G_{n-1})_{n>0}$ *be a projective system of finite groups. There exists a sequence of even integers* $r_n = 2q_n$ $(n \geq 0)$ *and for each* $n \geq 0$ *a* r_n*-tuple* $\mathbf{C}_n$ *of conjugacy classes* $C_{n1}, C_{n1}^{-1} \ldots, C_{nq_n}, C_{nq_n}^{-1}$ *in* G_n *for which the following holds:*
For any henselian field k *of residue characteristic* $p \geq 0$ *not dividing any of the orders of* G_n *and containing all roots of* 1 *of prime-to-p order, there exists a projective system* $(f_n)_{n\geq 0}$ *of HM-Galois covers of* $\mathbb{P}^1$ *defined over* k *and with Galois groups* $(G_n)_{n\geq 0}$*.*

Proof. Choose a non-decreasing sequence $(q_n)_{n\geq 0}$ of positive integers and for each $n \geq 0$ a generating system $\underline{g}^{(n)} = (g_1^{(n)}, \ldots, g_{q_n}^{(n)})$ of G_n such that

$$\begin{cases} s_{n+1}(g_j^{(n+1)}) = g_j^{(n)} & j = 1, \ldots, q_n \\ s_{n+1}(g_j^{(n+1)}) = 1 & \text{for all } j > q_n \end{cases}$$

We denote by C_{nj} the conjugacy class in G_n of $g_j^{(n)}$ $(j = 1, \ldots, q_n,\, n \geq 0)$.

On the other hand one can construct an infinite set of points $\{x_1, y_1, x_2, y_2, \ldots\}$ of $\mathbb{P}^1(k)$ and a projective system $(T^{(n)})_{n\geq 0}$ of stable marked curves $T^{(n)}$ over the valuation ring $\mathcal{O}$ of k, whose generic fiber is $\mathbb{P}^1$ marked by the set $\mathbf{t}_n = \{x_1, y_1, \ldots, x_{q_n}, y_{q_n}\}$ and the special fiber is a comb with roots $T_0^{(n)}$ and end components $T_j^{(n)}$ $(j = 1, \ldots, q_n)$, x_j, y_j specializing on $T_j^{(n)}$, $1 \leq j \leq q_n$, with morphisms $t_{n+1} : T^{(n+1)} \to T^{(n)}$ $(n \geq 0)$ inducing the identity map Id on the generic fiber and inducing the following map on the special fiber:

$$\begin{cases} \mathrm{Id} : T_0^{(n+1)} \to T_0^{(n)} \\ \mathrm{Id} : T_j^{(n+1)} \to T_j^{(n)},\ j = 1, \ldots, q_n \\ \qquad T_j^{(n+1)} \to \bar{a}_j,\ \text{for all } j > q_n \end{cases}$$

where $\bar{a}_j$ denotes the intersection point of $T_j^{(n+1)}$ with $T_0^{(n+1)}$.

For every $n \geq 0$ the restriction functors from the generic fiber to the components of the special fiber induce morphisms of fundamental groups $\theta_j^{(n)} : \pi_1(T_j^{(n)} \setminus \{\bar{a}_j, \bar{x}_j, \bar{y}_j\}) \to \pi_1(T_{\bar{\eta}}^{(n)} \setminus \mathbf{t}_n)$ (defined up to conjugation), and similarly with $j = 0$, making the following diagrams commutative (up to conjugation)

$$\begin{array}{ccc} \pi_1(T_j^{(n+1)} \setminus \{\bar{a}_j, \bar{x}_j, \bar{y}_j\}) & \to & \pi_1(T_{\bar{\eta}}^{(n+1)} \setminus \mathbf{t}_{n+1}) \\ \downarrow & & \downarrow \psi_n \\ \pi_1(T_j^{(n)} \setminus \{\bar{a}_j, \bar{x}_j, \bar{y}_j\}) & \to & \pi_1(T_{\bar{\eta}}^{(n)} \setminus \mathbf{t}_n) \end{array}$$

(similarly with $j = 0$).

Given $\mathbf{t} \in \mathcal{U}_r$, we call *product-one distinguished generating system* for the algebraic fundamental group $\pi_1(\mathbb{P}^1 \setminus \mathbf{t})$ every generating system $(\Gamma_1, \ldots, \Gamma_r)$ of $\pi_1(\mathbb{P}^1 \setminus \mathbf{t})$ such that $\Gamma_1 \cdots \Gamma_r = 1$ and Γ_i is an inertia distinguished generator at some point $t_i \in \mathbf{t}$, $i = 1, \ldots, r$.

For every $n \geq 0$, the algebraic comparison theorem from [Em2] provides a product-one distinguished generating system $(G_1^{n,1}, G_2^{n,1}, \ldots, G_1^{n,q_n}, G_2^{n,q_n})$ for $\pi_1(T_{\bar{\eta}}^{(n)} \setminus \mathbf{t}_n)$ with

$$\begin{cases} G_1^{n,j} = \theta_j^{(n)}(\Gamma_1^j)^{\tau_j^n} \\ G_2^{n,j} = \theta_j^{(n)}(\Gamma_2^j)^{\tau_j^n} \end{cases}$$

where Γ_1^j (resp. Γ_2^j) is the generator attached to $\bar{x}_j$ (resp. to $\bar{y}_j$) in a product-one distinguished generating system for $\pi_1(T_j^{(n)} \setminus \{\bar{a}_j, \bar{x}_j, \bar{y}_j\})$, and $\tau_j^n \in \pi_1(T_{\bar{\eta}}^{(n)} \setminus \mathbf{t}_n)$, $j = 1, \ldots, q_n$. For each $n \geq 0$ there exist elements $\omega_n \in \pi_1(T_{\bar{\eta}}^{(n)} \setminus \mathbf{t}_n)$ such that

$$\begin{cases} \psi_n(G_1^{n+1,j}) = (G_1^{n,j})^{\omega_n}, \ j = 1, \ldots, q_n \\ \psi_n(G_2^{n+1,j}) = (G_2^{n,j})^{\omega_n}, \ j = 1, \ldots, q_n \\ \psi_n(G_i^{n+1,j}) = 1, \ \text{for all } j > q_n \quad i = 1, 2 \end{cases}$$

Consider an integer $n \geq 0$. Starting from the (non connected) cyclic covers of the end components corresponding to the morphisms $\pi_1(T_j^{(n)} \setminus \{\bar{a}_j, \bar{x}_j, \bar{y}_j\}) \to G_n$ mapping Γ_1^j to $g_1^{(n)}$ and Γ_2^j to $(g_1^{(n)})^{-1}$, $j = 1, \ldots, q_n$, and from a trivial cover of the root $T_0^{(n)}$, build a HM-admissible cover of the special fiber of $T^{(n)}$. The generic fiber of a deformation of this HM-admissible cover gives a p'-cover $f_n : Z_n \to T_{\bar{\eta}}^{(n)}$ of the geometric generic fiber $T_\eta^{(n)}$ of group G_n branched at the $r_n = 2q_n$ marked points, corresponding to a morphism $\pi_1(T_{\bar{\eta}}^{(n)} \setminus \mathbf{t}_n) \to G_n$ mapping $G_1^{n,j}$ to some conjugate $h_j^{(n)}$ of $g_j^{(n)}$ and $G_2^{n,j}$ to $(h_j^{(n)})^{-1}$, $j = 1, \ldots, q_n$. From theorem 1.4 the representing point $[f_n]$ belongs to some HM-component of the Hurwitz space $\mathcal{H}_{r_n,G_n}^{\infty}(\mathbf{C}_n)$. Moreover one can require that this cover, which is defined over k, has a totally rational fiber (*i.e.* consisting of k-rational points) over some fixed k-rational point of the basis; this follows for instance from the fact that the special fiber of the cover is trivial over the root, and so has many totally κ-rational fibers, which extend to totally k-rational fibers. A consequence of this property is that two such covers which are isomorphic over $\bar{k}$ already are over k.

Let $\mathcal{S}_n$ be the set of k-isomorphism classes of such HM-covers of $T_\eta^{(n)}$ $(n \geq 0)$. It is a non-empty finite set. Moreover, if $Z_{n+1} \to T_\eta^{(n+1)}$ is a representative of an element of $\mathcal{S}_{n+1}$, the cover $Z_{n+1}/\mathrm{Ker}(s_n) \to T_\eta^{(n+1)}$ is unramified at the $2q_{n+1} - 2q_n$ marked points which specialize on $T_j^{(n)}$, $q_n < j \leq q_{n+1}$, and it induces a G_n-cover $Z_{n+1}/\mathrm{Ker}(s_n) \to T_\eta^{(n)}$ ramified at the $2q_n$ points from $\mathbf{t}_n$. The isomorphism class of this cover belongs to $\mathcal{S}_n$.

We have constructed a map from $\mathcal{S}_{n+1}$ to $\mathcal{S}_n$ $(n \geq 0)$, and the projective limit of the non-empty finite sets $\mathcal{S}_n$ is non-empty. An element of this projective limit is a coherent system of HM-covers of groups $(G_n)_{n\geq 0}$. □

4.4. Application to modular towers. Suppose given a finite group G and a prime number ℓ dividing $|G|$ and assume G has a set of generators of order ρ prime to ℓ. Denote the ℓ-universal Frattini cover of G by ${}_\ell\widetilde{G}$; it is naturally the inverse limit of some projective system $({}^n_\ell\widetilde{G} \rightarrow G)_{n\geq 0}$ of finite Frattini covers (of groups) (see [BaFr], [Fr1]). A typical example is this: G is the dihedral group D_ℓ of order 2ℓ, $\rho = 2$ and the projective system $({}^n_\ell\widetilde{G})_{n\geq 0}$ is the sequence of dihedral groups $(D_{\ell^{n+1}})_{n\geq 0}$, which converges to $D_{\ell^\infty} = \mathbb{Z}_\ell \times^s \mathbb{Z}/2$. Suppose now given a henselian field k of characteristic 0; it is not assumed here that k contains roots of 1. Then the general construction of [DeDes2] applies to yield a realization of ${}_\ell\widetilde{G}$ by a tower of regular Galois extensions of $k(T)$; furthermore the inertia canonical invariant $\mathbf{C}_n$ of the realizing cover at level n consists of a fixed number, say r, of conjugacy classes of order ρ $(n \geq 0)$. Again this interprets as the existence of a projective system of k-rational points on a certain tower of Hurwitz spaces, namely the tower $(\mathcal{H}^\infty_{r,{}^n_\ell\widetilde{G}}(\mathbf{C}_n))_{n\geq 0}$. This tower is a *modular tower*, as constructed by M. Fried [Fr1] [BaFr]. As before, the results of this paper show the covers used to realize all finite levels ${}^n_\ell\widetilde{G}$ $(n \geq 0)$ can be taken to be of Harbater-Mumford type. If in addition, $\mathbf{C}_0$ is HM-g-complete and is a rational union of conjugacy classes of G, then so are all $\mathbf{C}_n$ — a consequence of the Frattini property of ${}^n_\ell\widetilde{G} \rightarrow G$ — and so each space $\mathcal{H}^\infty_{r,{}^n_\ell\widetilde{G}}(\mathbf{C}_n)$ has a unique HM-component, defined over $\mathbb{Q}$ $(n \geq 0)$. Conclude as before that the projective system of k-rational points mentioned above can be found on a tower of algebraic varieties, geometrically irreducible and defined over $\mathbb{Q}$; furthermore these varieties are here all of the same dimension, namely r.

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