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Descent Theory for Algebraic Covers

Pierre Dèbes

ABSTRACT. In the sixties A. Grothendieck developed some conceptual tools to handle the general question of descent: fibered categories, gerbes, non-abelian cohomology, etc. In the seventies, M. Fried's moduli approach to the arithmetic of covers revealed two significant descent questions: descent to the field of moduli of a cover and existence of Hurwitz families above a moduli space of covers. Many works have since been devoted to these two questions. However it is only recently that Grothendieck's conceptual framework was (re-)introduced in this topic. Our goal in the paper is to join these two branches of the theory: that is, we wish to recast the work done about covers within Grothendieck's perspective, use this new light on the subject to measure the progress achieved and show how new developments already came out of this unified viewpoint.

1. Introduction

This paper is devoted to *descent theory*. This primarily refers to descent of fields of definition of algebraic objects. More generally, in a category where objects are defined over a certain base, descent is the opposite of base extension: given an object $\mathcal{O}$ over a base S, the question is to investigate models of $\mathcal{O}$, *i.e.*, objects $\mathcal{O}'$ defined over "smaller" bases S' (*i.e.*, with a map $S \to S'$) that are isomorphic to $\mathcal{O}$ after base extension from S' to S. A. Weil was the first one to consider this question in a systematic manner, in the context of algebraic varieties defined over fields. Weil's work was pursued by A. Grothendieck: his faithfully flat descent theory generalizes Weil's descent criterion to consider descent of objects that are defined over general schemes S.

We are interested in descent theory for *algebraic covers*. Our original motivation lies in the regular inverse Galois problem, which is, generally speaking, the construction of algebraic covers with small fields of definition. Furthermore, the case of covers is central in the theory as for many types (or categories) of objects X, descent questions reduce to that case through the consideration of the cover $X \to X/\mathrm{Aut}(X)$ (see [**DeEm**]).

We distinguish two main descent problems for covers. The first one is concerned with *fields of definition* and the associated models of a given cover. The second one

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deals with *Hurwitz families*, *i.e.*, families of covers of $\mathbb{P}^1$ with some invariants fixed (the group, the number of branch points, etc.): here the base S is the parameter space of the family. Some background is recalled in §2. For both problems, there is a best possible candidate to descend to: the *field of moduli* is the smallest possible field of definition, *if* it is a field a definition; and the best possible parameter space for a Hurwitz family is the associated *moduli space* — a Hurwitz space —; here "best" means that, given any Hurwitz family, its parameter space maps to the Hurwitz space. In general there is an *obstruction* to existence of a model defined over the field of moduli and of a family parametrized by the Hurwitz space. However this obstruction is thought as small in that in many circumstances it does not exist and in general it vanishes through a controlled base extension. We aim at results that make that precise.

There has been some progress in the recent years. This progress shows some analogy between the two problems, which, from now on, we refer to as the *cover problem* and the *Hurwitz family problem.* One goal of the paper is to revisit these problems with a tentatively unifying point of view. There is more than an analogy: through the moduli theory, a given cover (of $\mathbb{P}^1$) can be viewed as a point on a Hurwitz space and the cover problem then corresponds to some "specialization" to that point of the Hurwitz family problem. The theory of *gerbes* makes the relationship concrete.

For descent problems in general, Weil's and Grothendieck's criteria provide some theoretic description of the obstruction (§2.5). It is cohomological by nature. However, they do not provide any concrete measure of the obstruction. Based on [**DeDo3**] and [**DeDoEm**], §3.1 explains that Weil's and Grothendieck's conclusions essentially amount to saying that the obstruction can be viewed as a gerbe (under some suitable assumptions). The main achievement of [**DeDo3**] and [**DeDoEm**] was to show that, in the specific context of covers and Hurwitz families, it is possible to "compute" the gerbe and express it in terms of abelian cohomological classes.

Both the problems we have in mind can be handled simultaneously. The base category of the obstruction gerbe is the *moduli site* $\mathcal{S}_m$: the field of moduli in the cover problem and the moduli space in the Hurwitz family problem. Due to the moduli property, the gerbe comes equipped with *patching data.* The gerbe induces a class in $H^2(\mathcal{S}_m, \mathcal{L})$ with values in some *band* $\mathcal{L}$. This band is *a priori* locally representable by a *non-abelian* group. However this obstruction class can be reduced in $H^2(\pi_1(\mathcal{S}_m), Z(G))$ with values in the center of the group of the cover. This is more detailed in §3.2.

An alternate description of the obstruction was recently given [**DeDoMo**], which we refer to as the *diophantine* description in that it amounts to solving polynomial equations over certain fields. In terms of gerbes, this essentially corresponds to the obstruction gerbe being an *algebraic stack.* With this description, which we explain in §3.3, we could answer some open questions; this new viewpoint is still at a developing stage and we expect it will reveal to be even more fruitful.

Applications are collected in §4. This last section can thus serve as a survey of recent results on these topics. We also endeavour to parallel both problems there. A first series of basic applications include concrete vanishing criteria for the obstruction, bounds for the smallest extension of the moduli site over which the obstruction vanishes. A certain minimality property of the moduli site is also shown: in the cover problem (over $\overline{\mathbb{Q}}$), it is that the field of moduli is the intersection of all fields of definition, a property originally established by Coombes and Harbater for G-covers of $\mathbb{P}^1$.

§4.2 is then devoted to "local-global" results. The following statement, which conjoins several techniques from works of Douai, Emsalem, Harbater, Moret-Bailly and the author, may illustrate well the progress achieved. Assume for simplicity that f is a $\overline{\mathbb{Q}}$-cover of curves with field of moduli $\mathbb{Q}$. Call a prime p *good* if p does not divide the order of the group G of the cover and if the branch points remain distinct modulo p. For each prime p denote the subfield of $\overline{\mathbb{Q}}$ of all totally p-adic numbers by $\mathbb{Q}^{tp}$ (the biggest Galois extension of $\mathbb{Q}$ contained in $\mathbb{Q}_p$).

THEOREM. *If the field of moduli is $\mathbb{Q}$, then the cover f is defined over $\mathbb{Q}^{tp}$ (hence over $\mathbb{Q}_p$) for all good primes p. Furthermore, if f is a G-cover, local obstructions at bad primes are the only possible obstructions; that is, f is defined over $\mathbb{Q}$ if and only if f is defined over each $\mathbb{Q}_p$.*

In §4.3 we address another question, still rather mysterious though basic, which is whether covers are "often" defined over their field of moduli. Some first answers have recently been given [**DeDoMo**]: the subset of a Hurwitz space where the field of moduli *is* a field of definition is Zariski-dense; more precise conclusions can be drawn over "large" fields.

The theory of stacks and gerbes, which was developed by Grothendieck and Giraud [**Gi**] and which we use in this paper, seems to be the right set-up for the problems we consider. Their relevance for these questions had also been noted by M. Fried [**Fr**;p.58]. We will provide some background (§3.1) and introduce these tools gradually so that we hope that the reader who is not familiar with them can still follow the paper.

Moduli data appeared in the context of elliptic curves with the j*-modular* invariant. In [Sh], G. Shimura more generally "deals with certain systems of polarized abelian varieties parametrized by holomorphic functions and shows there exist meromorphic functions whose values are considered as "moduli" of the members of the system"; he also introduces a related notion of "field of moduli". For elliptic curves, the moduli space is the j-line and the field of moduli is $\mathbb{Q}(j)$; it is also a field of definition. Moduli problems are subtler for higher genus curves, for abelian varieties, for covers, and have been much studied, notably by Deligne-Mumford, Fried, Fulton, Shimura et al. Moduli spaces of covers go back to Hurwitz and Klein; the first general construction was given by M. Fried [**Fr**] (see [**De2**] for more references). Hurwitz spaces contain much information, including arithmetic information, about the covers they parametrize. Studying Hurwitz spaces has been a major program in the last 25 years. Questions discussed here are part of this moduli approach to the arithmetic of covers.

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2. Covers and Hurwitz families: descent problems

In this section we provide a parallel introduction to the two descent problems discussed in §1: main data, questions, the moduli property, Grothendieck's and Weil's results, etc. Notation, assumptions will be fixed for the rest of the paper.

2.1. General data. Let B be a regular geometrically integral variety defined over a field K; $B = \mathbb{P}^1$ over $K = \mathbb{Q}$ is the common situation. Fix a cover $f : X \to B$ defined over the separable closure $K^{\rm s}$ of K. Let $d = \deg(f)$ be the degree and $G \subset S_d$ be the (monodromy) group of f. We will work with algebraic branched covers $\mathcal{F} : \mathcal{X} \to S \times B$ such that the given cover $f : X \to B$ is a fiber of $\mathcal{F}$ for some closed point $s \in S$. Bases S are schemes and the *general problem* we consider is to find objects $\mathcal{F} : \mathcal{X} \to S \times B$, which we sometimes call *solutions*, and to study possible further descent of such objects along maps $S \to S'$. The two concrete problems we have in mind, which we more precisely present below, are special cases. In the sequel, we will discuss the problem in its general form first and then consider in a more concrete way each of the two special cases.

2.1.1. The cover problem. Here bases S are spectra $\mathrm{Spec}(\kappa)$ of fields κ with $K \subset \kappa \subset K^{\rm s}$ and the associated objects/solutions above are κ-models $f_\kappa : X_\kappa \to \mathrm{Spec}(\kappa) \times B$ of the cover f.

A more general *relative* situation is sometime considered in our papers where the given cover is *a priori* defined over a Galois extension F of K. To fix ideas we prefer to stick to the *absolute* situation where $F = K^{\rm s}$ is separably closed. Slight adjustments should be made to treat the more general case (replace $\mathrm{G}_K = \mathrm{Gal}(K^{\rm s}/K)$ by $\mathrm{Gal}(F/K)$, etc.).

2.1.2 The Hurwitz family problem. Here fields are of characteristic 0[1] and $B = \mathbb{P}^1$. Bases S are quasi-projective varieties $\mathcal{H}$ defined over fields k and the associated objects/solutions above are families parametrized by $\mathcal{H}$ of covers with the same degree, the same number of branch points and having f as a fiber. Formally, a Hurwitz family of covers (relative to integers r and d) is a map $\mathcal{F} : \mathcal{T} \to \mathcal{H} \times \mathbb{P}^1$ where

- $\mathcal{T}$ et $\mathcal{H}$ are quasi-projective varieties over k and the parameter space $\mathcal{H}$ is regular and geometrically irreducible,
- $\mathcal{F} : \mathcal{T} \to \mathcal{H} \times \mathbb{P}^1$ is a flat finite k-morphism such that, for all $h \in \mathcal{H}$, the fiber cover $\mathcal{F}_h : \mathcal{T}_h \to \mathbb{P}^1$ is a degree d cover with r branch points.
- the map $\mathrm{pr}_1 \circ \mathcal{F} : \mathcal{T} \to \mathcal{H}$ is a smooth projective morphism with an irreducible generic fiber (where pr_1 is the first projection $\mathcal{H} \times \mathbb{P}^1 \to \mathcal{H}$).

It classically follows [**DeFr1**;Lemma 1.5] from the conditions above that

- the (monodromy) group $G \subset S_d$, the genus of the covering space and some inertia invariant known as the Nielsen class are constant in the family,
- the map $\mathcal{H} \to \mathcal{U}_r$ (called the *branch point reference map*) which sends each $h \in \mathcal{H}$ to the branch point set of the fiber cover $\mathcal{F}_h$, is a morphism; here $\mathcal{U}_r$ is the variety of subsets of $\mathbb{P}^1$ of cardinality r (define $\mathcal{U}^r$ to be the subset of all r-tuples with r distinct entries in $\mathbb{P}^1$ and mod out by the action of S_r to get $\mathcal{U}_r$).

The variety $\mathcal{U}_r$ should be regarded as the *ground space* as the field K is the *ground field* in the cover problem.

[1] Using Wewers' work [**Wew**], Hurwitz families could also be developed in characteristic p for covers of groups of prime-to-p order.

2.1.3 Variants. There are two classical situations in the above problems:

G-cover situation: covers should be understood as G-covers, *i.e.*, Galois covers given with an isomorphism between the Galois group and a fixed group G. That is, automorphisms of the cover are part of the data. These are *a priori* the relevant objects to work with in the context of the regular inverse Galois problem.

Mere cover situation: covers should be understood as mere covers, *i.e.*, not necessarily Galois covers given without their automorphisms. There are many problems for which the covers involved are mere covers (see [**De2**]). Even for the regular inverse Galois problem, it might be more appropriate to work with mere covers first and to address the automorphism issue afterwards. As we will see, for descent issues, the mere cover situation is more complicated than the G-cover situation.

A more general notion is defined in [**De3**]: a SG-cover is a non necessarily Galois cover given with a "specified" subgroup S of the automorphism group. SG-covers generalizes both mere covers (take S trivial) and G-covers (take $S = G$).

Fixing an ordering on the branch points leads to other variants of the problems. A Hurwitz family *with ordered branch points* is given with a map $\mathcal{H} \to (\mathbb{P}^1)^r$ sending each point $h \in \mathcal{H}$ to the branch points, in some order, of the fiber cover $\mathcal{F}_h$. The branch point reference map is then a map $\mathcal{H} \to \mathcal{U}^r$.

We will implicitly consider the mere cover situation with unordered branch points. Up to slight adjustments, the other situations can be handled similarly.

2.2 Local study. Locally for the etale topology, both problems have *solutions*. That is, given any etale cover S of the ground space, one can find models of f (in the cover problem) and Hurwitz families (in the family problem) over a certain finite etale cover of S. Furthermore two such local solutions are locally isomorphic.

2.2.1. Cover problem. Local solutions are models over finite extensions of K. Local solvability merely means that f has a model over a suitably large finite extension of K and that two models of f are isomorphic on a suitably large finite extension of K.

2.2.2. Hurwitz family problem. Local solvability means that above a suitable finite etale cover $\mathcal{H}$ of $\mathcal{U}_r$, there exists a Hurwitz family (with the cover $\mathcal{H} \to \mathcal{U}_r$ as branch point reference map). This essentially follows from the fact that such Hurwitz families $\mathcal{F} : \mathcal{T} \to \mathcal{H} \times \mathbb{P}^1$ correspond to finite representations $\pi_1 \to S_d$. Namely, define $\mathcal{U}^{\bullet}_{r+1}$ to be the quotient of $\mathcal{U}^{r+1}$ by S_r acting on the first d entries; then π_1 should be understood as an open subgroup of the fundamental group $\pi_1(\mathcal{U}^{\bullet}_{r+1})$, which contains the $\overline{K}$-fundamental group $\pi_1(\mathbb{P}^1\text{-}\mathbf{t})_{\overline{K}}$ of $\mathbb{P}^1$ with the branch point set $\mathbf{t}$, as a closed normal subgroup. To prove local solvability, the main point is then that the representation $\pi_1(\mathbb{P}^1\text{-}\mathbf{t})_{\overline{K}} \to S_d$ corresponding to the given cover f extends to some open normal subgroup of $\pi_1(\mathcal{U}^{\bullet}_{r+1})$, thus yielding a Hurwitz family parametrized by an etale cover $\mathcal{H}$ of $\mathcal{U}_r$, and with the cover f as a fiber (see [**DeDoEm**;prop.3.10]). Similar considerations show that two Hurwitz families with f as a fiber become isomorphic over a suitable common etale cover of their parameter spaces: two representations $\pi_1 \to S_d$ are equal on some normal open subgroup [**DeDoEm**;prop.3.11].

Remark 2.1. Local solvability is also true with the etale topology replaced by the complex topology [**Fr**]. The reason is similar: there is a canonical notion of local deformation of covers. On the other hand this is not true for the Zariski topology; see however [**DeDoEm**;§4.4] and theorem 4.12 below for partial answers.

2.3. Moduli property. We consider now the *global* problems. That is, we wish to investigate and classify all solutions, and possibly find the best ones. There is a best candidate which we introduce now.

A base S (spectrum of field $\mathrm{Spec}(k)$ or parameter space $\mathcal{H}$) is said to have the *moduli property* if all *solutions* over bases S' lying above S (*via* a finite etale morphism $S' \to S$) can be equipped with *patching data* (relative to the map $S' \to S$). This roughly means that the action of $\pi_1(S)$ maps local solutions (models or Hurwitz families) to local solutions. We give specific definitions below for each problem; the general notion of patching data will be precisely given in §2.5. The moduli property of S is obviously a necessary condition for existence of a global solution over S. There is a best base S_{m} that can be equipped with patching data: it is the *field of moduli* in the cover problem and the *moduli space* in the family problem.

2.3.1. Field of moduli. Given a cover $f : X \to B$ over K^{s}, the field of moduli K_{m} of f is the fixed field in K^{s} of the subgroup

$$M(f) = \{\tau \in \mathrm{G}_K \mid f^\tau \underset{K^{\mathrm{s}}}{\simeq} f \quad \text{(as covers)}\}$$

That is, K_{m} is the smallest subfield of K^{s} such that $\mathrm{Gal}(K^{\mathrm{s}}/K_{\mathrm{m}})$ maps f to an isomorphic copy of itself, or in other words, maps every model of f to a model of f. As a consequence of the definition, the field of moduli is contained in every field of definition of f.

2.3.2. Moduli spaces. There exists a coarse moduli space, called Hurwitz space and denoted by $\mathcal{H}_{r,G}$, for the category of covers with r branch points and monodromy group $G \hookrightarrow S_d$ (in characteristic 0). The Hurwitz space $\mathcal{H}_{r,G}$ is a quasi-projective variety defined over $\mathbb{Q}$ (as a *reducible* variety). The gist of the moduli properties of $\mathcal{H}_{r,G}$ is the following:

Given any algebraically closed field $\overline{K}$ of characteristic 0, $\overline{K}$-rational points on $\mathcal{H}_{r,G}$ are in one-one correspondence with the isomorphism classes of covers of $\mathbb{P}^1$ with r branch points and monodromy group $G \hookrightarrow S_d$. Furthermore this correspondence is G_K-equivariant.

The Hurwitz space can be more formally defined by this property, which is analogous to the minimality property of the field of moduli:

Given any Hurwitz family $\mathcal{F} : \mathcal{T} \to \mathcal{H} \times \mathbb{P}^1$ defined over some field k, there exists a unique morphism $\gamma_{\mathcal{F}} : \mathcal{H} \to \mathcal{H}_{r,G}$, called structural morphism, such that, for each $h \in \mathcal{H}(\overline{k})$, the point $\gamma_{\mathcal{F}}(h)$ is the isomorphism class of the fiber cover $\mathcal{F}_h$.

The morphisms $\gamma_{\mathcal{F}}$ satisfy some further universal properties (functoriality, etc.), which are explicitly stated for example in [**DeDoEm**].

The moduli space $\mathcal{H}_{r,G}$ is not irreducible in general. In the sequel we work with the irreducible component of $\mathcal{H}_{r,G}$ containing the representative point $[f]$ of the given $\overline{K}$-cover f; we will denote it by $\mathcal{H}_{\mathrm{m}}$. We let k be a field of definition of $\mathcal{H}_{\mathrm{m}}$; k can be taken to be a number field. We will abuse terminology to still call $\mathcal{H}_{\mathrm{m}}$ a moduli space or a Hurwitz space.

We explain now that there is an action of $\pi_1(\mathcal{H}_{\mathrm{m}})$ that leaves invariant the set of local solutions, here Hurwitz families, above $\mathcal{H}_{\mathrm{m}}$. Consider a Hurwitz family $\mathcal{F}_1$ defined over some etale Galois cover $\gamma_{\mathcal{F}_1} : \mathcal{H}_1 \to \mathcal{H}_{\mathrm{m}}$. Each given element

$y \in \pi_1(\mathcal{H}_{\rm m})$ can be viewed as an automorphism of the cover $\gamma_{\mathcal{F}_1}$. Pull back the family $\mathcal{F}_1$ along the automorphism y to get a family $y^*\mathcal{F}_1$. This family $y^*\mathcal{F}_1$ is in fact another local solution; more precisely, the families $y^*\mathcal{F}_1$ and $\mathcal{F}_1$ are isomorphic over a suitable etale cover of $\mathcal{H}_1$. For that one should check the associated structural morphisms are equal: we have indeed $\gamma_{y^*\mathcal{F}_1} = \gamma_{\mathcal{F}_1} \circ y = \gamma_{\mathcal{F}_1}$; the first equality, which comes from the functoriality of the $\gamma_{\mathcal{F}}$, should be understood as the analog of the field of moduli condition $f^\tau \underset{K^{\rm s}}{\simeq} f$. See [**DeDoEm**;§3] for more details.

2.3.3. Connection. Let $[f]$ be the representative point of the cover f on $\mathcal{H}_{\rm m}$ and $K = k([f])$. Then each $\tau \in {\rm G}_K$ maps $[f]$ on $[f]^\tau = [f^\tau]$. Therefore $M(f)$ is the subgroup of all $\tau \in {\rm G}_K$ that fix the point $[f] \in \mathcal{H}_{\rm m}$. Hence the field of moduli of f is the field of definition of the point $[f] \in \mathcal{H}_{\rm m}$.

2.4. Main questions. From now on, $S_{\rm m}$ denotes the *moduli base*, that is, $S_{\rm m} = {\rm Spec}(K_{\rm m})$ in the cover problem and $S_{\rm m} = (\mathcal{H}_{\rm m})_k$ in the Hurwitz family problem. Action of $\pi_1(S_{\rm m})$ leaves invariant the set of local solutions above $S_{\rm m}$. A natural question is whether it follows that there exists a global solution over $S_{\rm m}$, *i.e.*, whether descent to $S_{\rm m}$ is possible. If so, descent to $S_{\rm m}$ is the best possible. Otherwise the problem becomes to find out on what etale covers of $S_{\rm m}$ one can descend.

2.4.1. Field of moduli versus field of definition. The question here is whether the field of moduli $K_{\rm m}$ is a field of definition, and more generally, it is to investigate on what extensions of $K_{\rm m}$ the cover can be defined. There is an *obstruction* in general: a cover need not be defined over its field of moduli; counter-examples are given in [**CoHa**] for G-covers and [**CouGr**], [**Cou**] for mere covers.

2.4.2. Hurwitz family above the Hurwitz space. The question here is that of existence of a Hurwitz family above the Hurwitz space $\mathcal{H}_{\rm m}$, or above controlled etale covers of $\mathcal{H}_{\rm m}$. There is an *obstruction* in general: examples (which use the counter-examples above) are given in [**DeDoEm**] where there is no Hurwitz family above a certain Hurwitz space.

2.4.3. Connection. If there is a Hurwitz family $\mathcal{F} : \mathcal{T} \rightarrow \mathcal{H}_{\rm m} \times \mathbb{P}^1$ defined over k, then f has a model defined over its field of moduli $k([f])$, namely the fiber cover $\mathcal{F}_{[f]} : \mathcal{T}_{[f]} \rightarrow \mathbb{P}^1$.

2.5. Theoretic descent criteria. Descent should be understood as some general *local-to-global patching problem.* A covering $(S_i)_i$ of a global base S is given with a corresponding collection $(\xi_i)_i$ of local data. The question is about the existence of a global object ξ over S whose restriction to each S_i is isomorphic to ξ_i. There are necessary conditions. First, for each pair (i,j), the restrictions of ξ_i and ξ_j to $S_i \cap S_j$ should be isomorphic, say *via* an isomorphism χ_{ij}. Such isomorphisms χ_{ij} are called *patching data.* When they exist, there is a next obvious necessary condition: for each triple (i,j,k), we should have $\chi_{ik} = \chi_{jk} \circ \chi_{ij}$ over $S_i \cap S_j \cap S_k$ for some choice of the isomorphisms χ_{ij}s. When this holds, patching data are called *descent data.* Descent data are called *effective* when these necessary conditions are the only obstructions to descent.

However these simple ideas look a little different in our context. The topology on the base S is the etale topology. Thus the maps $S_i \rightarrow S$ are finite etale morphisms; furthermore a single S' may be sufficient to cover all of S (think of ${\rm Spec}(K)$

which consists of a single point). Then the intersections $S_i \cap S_j$ [resp. $S_i \cap S_j \cap S_k$] should be understood as the *fiber products* $S' \times_S S'$ [resp. $S' \times_S S' \times_S S'$].

This viewpoint on descent goes back to Grothendieck [**Gr**]. More specifically, he fixes a base category $\mathcal{S}$ where fiber products exist (an etale site for example); descent problems are considered along maps $S' \to S$ from the category $\mathcal{S}$. He then gives himself a *fibered category* $\mathcal{C}$ of base $\mathcal{S}$. That is, for each object $S \in \mathcal{S}$, a category $\mathcal{C}(S)$ is given; and for each map $t : S' \to S$, there is a "base extension" functor $t^* : \mathcal{C}(S) \to \mathcal{C}(S')$; these data are required to satisfy some natural conditions (see [**Gr**]). A descent problem is the following: given a map $t : S' \to S$ and an object $\xi' \in \mathcal{C}(S')$, to find an object $\xi \in \mathcal{C}_S$ such that $t^*(\xi) = \xi'$.

As above there are natural obstructions. The first one is about existence of *patching data*. If p_1 and p_2 are the two projections $S' \times_S S' \to S'$, a patching data (for $\xi' \in \mathcal{C}(S')$ relative to the map $S' \to S$) is an isomorphism χ between $p_1^*(\xi')$ and $p_2^*(\xi')$ (see [**Gr**], [**Gi**] and also [**DeDo3**] and [**DeDoEm**]). Condition $p_1^*(\xi') \simeq p_2^*(\xi')$ is a generalization of $f^\tau \simeq f$ and $y^*\mathcal{F}_1 \simeq \mathcal{F}_1$ in the above contexts of covers and Hurwitz families respectively. Considering the 3 projections p_{ij} $(i,j = 1,2,3)$ from $S' \times_S S' \times_S S'$ to partial products of 2 copies of S', one obtains the second obstruction: some patching datas should be descent datas, that is

$$p_{31}^*(\chi) = p_{32}^*(\chi) p_{21}^*(\chi) \qquad \text{(compatiblity condition)}$$

2.5.1. Grothendieck's faithfully flat descent theorem. Grothendieck's result essentially asserts that, in certain circumstances, the necessary conditions from the previous paragraph are the only obstructions to descent. His result also addresses the descent of morphisms issue. A morphism $S' \to S$ is said to be a $\mathcal{C}$-*descent* morphism if given any two objects $\eta, \zeta \in \mathcal{C}(S)$, then $\mathrm{Hom}(\eta,\zeta)$ is a sheaf (relative to the map $S' \to S$). It is said to be a *strict* $\mathcal{C}$-*descent* morphism if in addition every descent data on an object $\xi' \in \mathcal{C}(S')$ is effective.

Theorem (Grothendieck [**Gr**]). *Let $\mathcal{S}$ be the category of pre-schemes and $t : S' \to S$ be a finite etale morphism (or more generally a faithfully flat and quasi-compact morphism) of the category $\mathcal{S}$. Then $t : S' \to S$ is a strict $\mathcal{C}$-descent morphism in the following cases:*
- $\mathcal{C}$ is the fibered category of coherent sheaves,
- $\mathcal{C}$ is the fibered category of affine schemes,
- $\mathcal{C}$ is the fibered category of quasi-projective schemes.

In particular, descent datas are effective in both our problems. Indeed $\mathcal{S}$ can be taken to be the category of spectra of fields in the cover problem and the category of quasi-projective varieties in the Hurwitz family problem. And the fibered category $\mathcal{C}$ of covers of fixed base space B falls into the first case and in the last case of Grothendieck's theorem (for the latter, view covers as graphs).

2.5.2. Weil's descent criterion. It is the special case of Grothendieck's theorem for which $\mathcal{S}$ is the category of spectra of fields. We state it below in the specific context of covers. By definition of field of moduli, for each $\tau \in \mathrm{Gal}(K^{\mathrm{s}}/K_{\mathrm{m}})$, we have $f \underset{K^{\mathrm{s}}}{\simeq} f^\tau$.

Theorem (Weil [**We**]). *The field of moduli K_{m} is a field of definition of f if and only if isomorphisms χ_τ between f and f^τ can be found such that*

$$\chi_{uv} = \chi_v^u \chi_u \quad (u,v \in \mathrm{G}_{K_{\mathrm{m}}}) \qquad \text{(cocycle condition)}$$

The cocycle condition is a special case of the general compatibility condition on the $p_{ij}^*(\chi)$s. Appearance of action of $\mathrm{G}_{K_{\mathrm{m}}}$ in the formula comes from the special form of the fiber products $S' \times_S S'$, $S' \times_S S' \times_S S'$. Here $S = \mathrm{Spec}(K_{\mathrm{m}})$, $S' = \mathrm{Spec}(K')$ with K'/K_{m} a finite Galois extension; then $S' \times_S S' = \mathrm{Spec}(K' \otimes_{K_{\mathrm{m}}} K')$ where

$$K' \otimes_{K_{\mathrm{m}}} K' \simeq \prod_{\tau \in \mathrm{Gal}(K'/K_{\mathrm{m}})} (K')^{\tau}$$

2.5.3. Classical consequences. Consider a local solution ξ' over a suitable etale cover S' of the moduli base S_{m}. From the moduli property of S_{m}, ξ' can be equipped with patching data. From Grothendieck's theorem, descent to S_{m} is then possible if such patching datas can be found satisfying the compatibility condition above. That is the case if objects in the category $\mathcal{C}$ have no non-trivial automorphism, which here amounts to assuming that the initial cover f has no non-trivial automorphism.

Corollary. *The field of moduli is a field of definition of f, and there is a Hurwitz family $\mathcal{F}_{\mathrm{m}} : \mathcal{T}_{\mathrm{m}} \to \mathcal{H}_{\mathrm{m}} \times \mathbb{P}^1$ above the Hurwitz space $\mathcal{H}_{\mathrm{m}}$, in each of the following situations:*

for mere covers:

- $\mathrm{Cen}_{S_d}(G) = \{1\}$ (Fried [**Fr**])

for G-covers:

- $Z(G) = \{1\}$ (Belyi [**Bel**]; [**De1**])

Furthermore, in that case $\mathcal{H}_{\mathrm{m}}$ is a *fine* moduli space, that is,

if $\mathcal{F} : \mathcal{T} \to \mathcal{H} \times \mathbb{P}^1$ is any given Hurwitz family, then it is isomorphic to the Hurwitz family $\gamma_{\mathcal{F}}^*(\mathcal{F}_{\mathrm{m}})$ obtained by pulling-back the Hurwitz family $\mathcal{F}_{\mathrm{m}}$ along the structural morphism $\gamma_{\mathcal{F}} : \mathcal{H} \to \mathcal{H}_{\mathrm{m}}$. Furthermore the isomorphism between $\mathcal{F}$ and $\gamma_{\mathcal{F}}^*(\mathcal{F}_{\mathrm{m}})$ is unique.

2.6. Towards concrete measures of the obstruction.

2.6.1. Cohomological nature of the obstruction. From Grothendieck's theorem, existence of local solutions $\xi' \in \mathcal{C}(S')$ (above S_{m}) equipped with patching datas that satisfy the compatibility condition is the only obstruction to descent to the moduli base S_{m}. Our next task will be to explain how to get a concrete measure of that obstruction. The isomorphism χ involved in the compatibility condition is well-defined up to automorphisms of the object ξ' (possibly after a base extension). Thus the question is whether χ can be modified by composing with automorphisms in such a way that the compatibility condition holds. This is typically a *cohomological obstruction* and the relevant cohomological set appears to be $H^2(S_{\mathrm{m}}, \mathrm{Aut}(f))$. We will explain in the next section how to treat this cohomological problem in general. We start with some classical situations where the obstruction can be easily handled.

2.6.2. The G-cover situation. Consider the 2-cochain $c_{u,v} = (\chi_{uv})^{-1}\chi_v^u\chi_u \in \mathrm{Aut}(f)$ involved in Weil's criterion. For G-covers $\mathrm{Aut}(f) \simeq Z(G)$ is abelian; it can then be straightforwardly checked that $(c_{u,v})_{u,v}$ induces a 2-cocycle $\Omega \in H^2(K_{\mathrm{m}}, Z(G))$, which represents the obstruction, that is:

$\Omega = 1$ *in $H^2(K_{\mathrm{m}}, Z(G))$ iff the field of moduli is a field of definition.*

This however does not carry over to the mere cover situation, which involves non-abelian cohomology (as $\mathrm{Aut}(f) \simeq \mathrm{Cen}_{S_d}(G)$ is not abelian in general). For mere covers, the expression $(\chi_{uv})^{-1}\chi_v^u\chi_u$ does not induce a 2-cocycle (not even in the sense of non-abelian $H^2(K_{\rm m}, \mathrm{Aut}(f))$). A complication in non-abelian cohomology is that $H^2(K_{\rm m}, \mathrm{Aut}(f))$ no longer has a single trivial class: there may be several or there may be none.

REMARK 2.3. The form of the cocycle $c_{u,v} = (\chi_{uv})^{-1}\chi_v^u\chi_u$ (for G-covers) suggests that it might be in the image of the connecting morphism $H^1(K_{\rm m}, G/Z(G)) \to H^2(K_{\rm m}, Z(G))$ associated with the exact sequence $Z(G) \hookrightarrow G \twoheadrightarrow G/Z(G)$. As a consequence the obstruction would vanish if the exact sequence splits. This is false: there are *abelian* G-covers that are *not defined* over their field of moduli [**DeEm**;§5]. The subtelty here is that identifying elements $\chi_u \in \mathrm{Aut}(f)$ with elements of G requires to fix a $K_{\rm m}$-rational point on the base space B; such points might not exist.

2.6.3. The Coombes-Harbater theorem. Assume here f is a mere cover of $\mathbb{P}^1$ that is Galois. The group $\mathrm{Aut}(f)$ acts freely and transitively on any unramified fiber $f^{-1}(t_o)$. Fix $t_o \in \mathbb{P}^1(K_{\rm m})$ not a branch point. One can then "rigidify" the choice of the χ_τ in such a way that for each $\tau \in \mathrm{G}_{K_{\rm m}}$, χ_τ maps a fixed point $y \in f^{-1}(t_o)$ to y^τ. This forces the cocycle condition to hold. We obtain this conclusion [**CoHa**]

a mere cover of $\mathbb{P}^1$ that is Galois is defined over its field of moduli.

An explicit form of this result was given in [**Sa**]. Due to the same difficulty as in remark 2.3 above, the Coombes-Harbater theorem does not extend to covers of a more general base space B: there exist Galois mere covers $f : X \to B$ that are not defined over their field of moduli [**DeEm**;§5].

3. Descriptions of the obstruction

Conclusions from §2 can be pleasantly summarized by using the language of stacks and gerbes: we explain in §3.1 that the studied obstructions can be viewed as gerbes. These gerbes are then "computed" in §3.2: the main result is theorem 3.1 which gives a description in terms of abelian $H^2(-,-)$. Another description is offered in §3.3, which we call *diophantine*. Notation from §2 is retained.

3.1. The obstruction as a gerbe. We recall some general definitions and then introduce the gerbe of models of f and the Hurwitz gerbe.

3.1.1. General definitions. Assume here that the base category $\mathcal{S}$ is an etale site: morphisms are finite etale covers of Zariski open subsets and there is a final object S_∞ in the category $\mathcal{S}$. Let $\mathcal{G}$ be a fibered category of base $\mathcal{S}$.

(i) $\mathcal{G}$ is called a *prestack* if the following *local-to-global condition on morphisms* holds: every morphism $S' \to S$ of the base category is a $\mathcal{G}$-descent morphism (as defined in §2.5.1).

(ii) $\mathcal{G}$ is called a *stack* if every morphism $S' \to S$ of the base category is a strict $\mathcal{G}$-descent morphism (as defined in §2.5.1).

If in addition the two following conditions hold, $\mathcal{G}$ is called a *gerbe*:

(iii) For each $S \in \mathcal{S}$, the category $\mathcal{G}(S)$ is locally non-empty, that is, there exists an open covering $(S_i)_{i\in I}$ of S such $\mathcal{G}(S_i) \neq \emptyset$ for all $i \in I$.

(iv) For each $S \in \mathcal{S}$, any two objects in $\mathcal{G}(S)$ are locally isomorphic, that is, on some suitable open covering $(S_i)_{i\in I}$ of S.

We have two more definitions:

(v) *Patching data* on a gerbe $\mathcal{G}$ consist of patching data for an object $\xi \in \mathcal{G}(S)$ above some $S \in \mathcal{C}$ (relative to a given map $S \to S_\infty$).

(vi) The gerbe $\mathcal{G}$ is *neutral* if global sections exist, *i.e.*, if $\mathcal{G}(S_\infty) \neq \emptyset$.

3.1.2. The gerbe $\mathcal{G}_f$ of models of f [**DeDo3**]. Here $\mathcal{C}$ is the etale site of finite separable algebra extensions of the field of moduli $K_{\rm m}$. Basic objects S of $\mathcal{C}$ are the finite Galois extensions $S = \mathrm{Spec}(E) \to \mathrm{Spec}(K_{\rm m})$. Above each such $S \in \mathcal{C}$, $\mathcal{G}_f(S)$ is defined to be the category of E-models of f (with E-isomorphisms).

3.1.3. The Hurwitz gerbe $\mathcal{G}_{\mathcal{H}_{\rm m}}$ [**DeDoEm**]. Here $\mathcal{C}$ is the etale site of finite etale covers $S = \mathcal{H} \to \mathcal{H}_{\rm m}$ of the moduli space $\mathcal{H}_{\rm m}$. Above each such $S \in \mathcal{C}$, $\mathcal{G}_{\mathcal{H}_{\rm m}}(S)$ is defined to be the category of Hurwitz families above $\mathcal{H}$.

3.1.4. Checks. From Grothendieck's faithfully flat descent theorem (§2.5.1), the fibered categories $\mathcal{G}_f$ and $\mathcal{G}_{\mathcal{H}_{\rm m}}$ are stacks (conditions (i) and (ii)); for $\mathcal{G}_f$ it is sufficient to invoke Weil's descent criterion. Conditions (iii) and (iv) are satisfied: indeed they correspond to the local solvability properties discussed in §2.2; furthermore, the open covering $(S_i)_{i\in I}$ of S involved in (iii) and (iv) can be taken to be a *single* finite etale cover S' of S. Thus $\mathcal{G}_f$ and $\mathcal{G}_{\mathcal{H}_{\rm m}}$ are gerbes. Patching data (condition (v)) come from the moduli property satisfied by the field of moduli $K_{\rm m}$ and the moduli space $\mathcal{H}_{\rm m}$ (see §2.3).

3.1.5. Main questions. The main questions from §2.4, that is, whether the field of moduli $K_{\rm m}$ is a field of definition and whether there exists a family above the moduli space $\mathcal{H}_m$ exactly correspond to whether the gerbes $\mathcal{G}_f$ and $\mathcal{G}_{\mathcal{H}_{\rm m}}$ are neutral (condition (vi)).

3.1.6. Connection. The gerbe language makes the connection between our two problems more concrete. Denote as above the representing point on $\mathcal{H}_{\rm m}$ of the cover f by $[f]$; its field of definition is $k([f]) = K_{\rm m}$. Pull back the Hurwitz gerbe $\mathcal{G}_{\mathcal{H}_{\rm m}}$ along the map $[f] : \mathrm{Spec}(k([f])) \to \mathcal{H}_{\rm m}$ to get a gerbe $[f]^*(\mathcal{G}_{\mathcal{H}_{\rm m}})$. The "specialized" gerbe $[f]^*(\mathcal{G}_{\mathcal{H}_{\rm m}})$ can be shown to be equivalent to the gerbe $\mathcal{G}_f$ of models of f ([**DeDoEm**;§4.5], [**Wew**]). We have this immediate consequence which rephrases §2.4.3: if $\mathcal{G}_{\mathcal{H}_{\rm m}}$ is neutral then so is $\mathcal{G}_f$.

3.1.7. Gerbes and cohomology. We will associate obstruction classes in some $H^2(S_{\rm m}, \mathcal{L})$ to the gerbes $\mathcal{G}_f$ and $\mathcal{G}_{\mathcal{H}_{\rm m}}$. We first briefly review the cohomological theory of gerbes. We refer to [**DeDo3**;§1.2] and [**Gi**;Chs.3&4] for more details.

Let $\mathcal{G}$ be a gerbe over an etale site $\mathcal{S}$. Given $S \in \mathcal{S}$ and any two objects $\xi, \xi' \in \mathcal{G}(S)$, isomorphisms γ between ξ and ξ', which exist locally, induce specific local isomorphisms $c_\gamma : \mathrm{Aut}(\xi) \to \mathrm{Aut}(\xi')$ (mapping each $g \in \mathrm{Aut}(\xi)$ to $\gamma g \gamma^{-1}$). There is a certain stack — the stack of *bands* — where $\mathrm{Aut}(\xi)$ and $\mathrm{Aut}(\xi')$ become equal, that is, they induce a single object over some suitable etale cover of S_∞. Furthermore, these "local" objects can be patched to provide an object over S_∞. This "global section" is called the band of the gerbe. Roughly speaking, it is the collection of all automorphisms groups $\mathrm{Aut}(\xi)$ of objects ξ in categories $\mathcal{G}(S)$ ($S \in \mathcal{S}$), given with all isomorphisms c_γ and identified accordingly.

The bands $\mathcal{L}_f$ and $\mathcal{L}_{\mathcal{H}_{\rm m}}$ of the gerbes $\mathcal{G}_f$ and $\mathcal{G}_{\mathcal{H}_{\rm m}}$ respectively are both *locally representable* by the constant group sheaf C where $C = \mathrm{Cen}_{S_d}(G)$ for mere covers and $C = Z(G)$ for G-covers. Here this comes down to the fact that for all $S \in \mathcal{S}$ and all objects ξ over S, $\mathrm{Aut}(\xi)$ becomes isomorphic to C after a suitable etale cover of S. The gerbes $\mathcal{G}_f$ and $\mathcal{G}_{\mathcal{H}_{\rm m}}$ are also said to be *locally bound* by the constant group sheaf C.

Two gerbes $\mathcal{G}$ and $\mathcal{G}'$ of base $\mathcal{S}$ and band $\mathcal{L}$ are said to be equivalent if there is an isomorphism of gerbes $\mathcal{G} \to \mathcal{G}'$ that induce the identity on the band $\mathcal{L}$. The set of all equivalence classes $[\mathcal{G}]$ is denoted by $H^2(\mathcal{S}, \mathcal{L})$ or $H^2(S_\infty, \mathcal{L})$.

A main difference with abelian cohomology is that there may be several neutral classes, *i.e.*, several non-isomorphic neutral gerbes, and there may be none. An *iff* condition for existence of neutral classes is that the band $\mathcal{L}$ is *globally* representable by a group sheaf $\underline{H}$ over $\mathcal{S}$: given $S \in \mathcal{S}$, we should have $\underline{H}(S) = \mathrm{Aut}(\xi)$ (as bands) over S itself (and not only over an etale cover). This is a first obstruction to neutrality of a gerbe $\mathcal{G}$ which generalizes the classical first obstruction to splitting of an exact sequence: the outer action on the kernel of the exact sequence should lift to an actual action. The set $H^2(S_\infty, \mathcal{L})'$ of all neutral classes can be shown to be a surjective image of $H^1(S_\infty, \mathrm{Inn}(\underline{H}))$.

3.2 Abelian cohomological description. From above, obstructions in both our problems can be regarded as two classes $[\mathcal{G}_f] \in H^2(K_{\mathrm{m}}, \mathcal{L}_f)$ and $[\mathcal{G}_{\mathcal{H}_{\mathrm{m}}}] \in H^2(\mathcal{H}_{\mathrm{m}}, \mathcal{L}_{\mathcal{H}_{\mathrm{m}}})$. For simplicity, denote them both by $[\mathcal{G}] \in H^2(S_{\mathrm{m}}, \mathcal{L})$. Based on [**DeDo1**], [**DeDo3**] and [**DeDoEm**], we now explain how to compute these classes in terms of more classical and manageable data in some abelian $H^2(-,-)$. There are two methods. The first one uses the theory of gerbes; it is more conceptual. The second one, which chronologically came first, rephrases the problem in terms of representations of π_1 to turn it into a pure group-theoretic problem, which can be tackled by means of classical abelian cohomology. The second method, which involves explicit cocycle calculations, is more elementary, and is self-contained: it implicitly reproves Weil's and Grothendieck's theorems.

3.2.1. First obstruction ([**DeDo1**;§3.1] and [**DeDoEm**;§3.3.1]). The obstruction is a two-step obstruction: as mentioned above, existence of neutral classes in $H^2(S_{\mathrm{m}}, \mathcal{L})$ (or, equivalently, representability of the band $\mathcal{L}$) is a first necessary condition for the obstruction to vanish. In our context, this first obstruction corresponds to existence of weak solutions to a certain embedding problem for $\pi_1(S_{\mathrm{m}})$. This condition is usually denoted by (λ/Lift): λ is the name of the map to be lifted in the embedding problem. Practical criteria for (λ/Lift) to hold can be given (see theorem 3.1 below) and also an *iff* criterion in terms of the vanishing of certain cohomological data [**DeDo1**;Th.4.7]. Furthermore when condition (λ/Lift) holds, the set $H^2(S_{\mathrm{m}}, \mathcal{L})'$ of all neutral classes, *i.e.*, of all liftings Λ of λ, is shown to correspond to the set $H^1(\pi_1(S_{\mathrm{m}}), C/Z(G))$ (for some action). As before C is $\mathrm{Cen}_{S_d}(G)$ for mere covers and $Z(G)$ for G-covers; for G-covers, there is a single neutral class.

This first obstruction can be interpreted in terms of *extension of constants in the Galois closure*. By definition, all K-models of a G-cover are regular and Galois over K: extension of constants is trivial. Unlikewise, a mere cover may have several models over K with different non trivial extensions of constants in the Galois closure over K. The first obstruction corresponds to existence of at least one "possible" extension of constants; and then neutral classes correspond to all possible extensions of constants.

3.2.2. Computation of the main obstruction.

1st method ([**DeDo3**] and [**DeDoEm**]). Assume that condition (λ/Lift) holds, *i.e.*, there is no first obstruction. Fix then a neutral class, *i.e.*, a lifting Λ of λ. Consider then the gerbe $\mathcal{G}_\Lambda$ whose objects are those from the gerbe $\mathcal{G}$ which induce Λ and whose morphisms are those which respect Λ. For example, in the cover problem, objects of $\mathcal{G}_\Lambda$ over an etale cover $\mathrm{Spec}(E) \to \mathrm{Spec}(K_{\mathrm{m}})$ are those E-models of f with a certain extension of constants in Galois closure (given by Λ).

It turns out that the gerbe $\mathcal{G}_\Lambda$ is locally bound by $C \cap G = Z(G)$. This group is abelian. It follows that the bound is globally representable, by the constant group sheaf $Z(G)$ and that the set $H^2(S_{\mathrm{m}}, Z(G))$ has a single neutral class [**DeDo2**;§1.2.7].

Furthermore the gerbe $\mathcal{G}_\Lambda$ is equipped with patching data (as the gerbe $\mathcal{G}$ is). This gives that the class $[\mathcal{G}_\Lambda]$ comes from an element $\check{\Omega}_\Lambda \in \check{H}^2((\mathcal{H}_{\mathrm{m}})_{\mathrm{et}}, Z(G))$ *via* the embedding

$$\check{H}^2(S_{\mathrm{m}}, Z(G)) \hookrightarrow H^2(S_{\mathrm{m}}, Z(G))$$

where $\check{H}^2(S_{\mathrm{m}}, Z(G))$ is the 2nd Cěch cohomology group.

In turn $\check{\Omega}_\Lambda$ corresponds to a cocycle $\Omega_\Lambda \in H^2(\pi_1(S_{\mathrm{m}}), Z(G))$ *via* the natural map

$$H^2(\pi_1(S_{\mathrm{m}}), Z(G)) \to \check{H}^2(S_{\mathrm{m}}, Z(G))$$

The reason is that there exist global etale coverings of S_{m} (and not just a covering family) that neutralize the gerbe $\mathcal{G}$.

2nd method ([**DeDo1**] and [**DeDoEm**]). As above assume condition (λ/Lift) and fix a lifting Λ of λ. The problem can be then stated as follows. Suppose first we are in the mere cover situation. Then f corresponds to some representation $\phi : \pi_1((\mathbb{P}^1\text{-}\mathbf{t})_{K^{\mathrm{s}}}) \to G \subset S_d$. Solutions to our problem are branched covers of $\mathbb{P}^1 \times S_{\mathrm{m}}$ (see §2.2.2 for notation). Denote the unbranched locus by $(\mathbb{P}^1 \times S_{\mathrm{m}})^*$; this is $(\mathbb{P}^1\text{-}\mathbf{t})_{K_{\mathrm{m}}}$ in the context of covers and it is $\mathcal{H}_{\mathrm{m}} \times_{\mathcal{U}_r} \mathcal{U}^{\bullet}_{r+1}$ in the context of Hurwitz families. We have an exact sequence

$$1 \to \pi_1(\mathbb{P}^1\text{-}\mathbf{t})_{K^{\mathrm{s}}}) \to \pi_1((\mathbb{P}^1 \times S_{\mathrm{m}})^*) \to \pi_1(S_{\mathrm{m}}) \to 1$$

with $\pi_1(S_{\mathrm{m}}) = \pi_1(\mathrm{Spec}(K_{\mathrm{m}})) = \mathrm{G}_{K_{\mathrm{m}}}$ in the cover problem and $\pi_1(S_{\mathrm{m}}) = \pi_1(\mathcal{H}_{\mathrm{m}})_k$ in the Hurwitz family problem. The question is to extend the representation $\phi : \pi_1((\mathbb{P}^1\text{-}\mathbf{t})_{K^{\mathrm{s}}}) \to G$ to a representation $\Phi : \pi_1((\mathbb{P}^1 \times S_{\mathrm{m}})^*) \to \mathrm{Nor}_{S_d}(G)$, that induces Λ on $\pi_1(S_{\mathrm{m}})$. There is an additional data, a morphism $\overline{\varphi} : \pi_1((\mathbb{P}^1 \times S_{\mathrm{m}})^*) \to \mathrm{Nor}_{S_d}(G)/\mathrm{Cen}_{S_d}(G)$, which comes from the moduli condition: values of $\overline{\varphi}$ essentially correspond to the patching data, *i.e.*, the isomorphisms between the objects and their conjugates under $\pi_1(S_{\mathrm{m}})$; these isomorphisms, *modulo* $\mathrm{Aut}(f) \simeq \mathrm{Cen}_{S_d}(G)$, are well-defined and satisfy the compatibility condition (from §2.5). A solution Φ should also induce the map $\overline{\varphi}$ modulo the centralizer $\mathrm{Cen}_{S_d}(G)$. Groups G and $Z(G)$ replace $\mathrm{Nor}_{S_d}(G)$ and $\mathrm{Cen}_{S_d}(G)$ in the G-cover situation.

This problem was studied in [**DeDo1**;§4] in general, that is, with a general exact sequence of groups replacing the exact sequence of π_1s above. Existence of solutions inducing the fixed map Λ is shown to correspond to the vanishing of some 2-cocycle $\Omega_\Lambda \in H^2(\pi_1(S_{\mathrm{m}}), Z(G))$, which is explicitly given in terms of the data.

Conclusion (of the two methods). The gerbe $\mathcal{G}_\Lambda$ is neutral if and only if the cocycle Ω_Λ is trivial in $H^2(\pi_1(S_{\mathrm{m}}), Z(G))$. The original gerbe $\mathcal{G}$ is trivial, *i.e.*, the original problem has solutions, if and only if there exists a lifting Λ of λ such that the gerbe $\mathcal{G}_\Lambda$ is neutral.

The last step consists in comparing the cocycles Ω_Λ for different liftings of Λ. For two such liftings Λ and Λ', we have

$$\Omega_{\Lambda'} - \Omega_\Lambda = \delta^1(\theta)$$

where $\theta = \Lambda'\Lambda^{-1} \in H^1(\pi_1(S_{\rm m}), C/Z(G))$ and δ^1 is the connecting morphism:

$$\delta^1 : H^1(\pi_1(S_{\rm m}), C/Z(G)) \to H^2(\pi_1(S_{\rm m}), Z(G))$$

This formula can be proved either from the gerbe viewpoint [**DeDo3**;§4.3] or by a direct calculation from the explicit form of Ω_Λ given by the second method [**DeDo1**;prop.4.5].

Final comment: what made the computation possible. Gerbes are conceptual objects that account for general situations. Here we were able to go further and actually compute the obstruction gerbes because we are in the more specific situation of covers. Covers correspond to *representations of* π_1 *in finite groups.* As a consequence the gerbes involved in our problems are locally bound by a *constant finite group sheaf.* Another particular feature is the following: the obstruction to the gerbe $\mathcal{G}$ being equivalent to a *specific* neutral class in $H^2(\mathcal{S}_m, \mathcal{L})$ happens to be a gerbe locally bound by an *abelian* group, namely $Z(G)$.

3.2.3. Statement of the main result. The following result, which is the main result of [**DeDo1**] (cover problem) and of [**DeDoEm**] (Hurwitz family problem), is the conclusion of the previous paragraph. Recall we are given a degree d cover f with monodromy group $G \subset S_d$. The result provides a practical cohomological description of obstruction to existence of "global solutions to our problems". Recall that what we call global solutions are

o models of f defined over its field of moduli $K_{\rm m}$ (cover problem), or

o Hurwitz families parametrized by the irreducible component $(\mathcal{H}_{\rm m})_k$[2] of the representing point $[f]$ of f on the Hurwitz space $\mathcal{H}_{r,G}$ (Hurwitz family problem).

The result can be stated for both problems together. As above $S_{\rm m}$ denotes $\mathrm{Spec}(K_{\rm m})$ in the cover problem and $\mathcal{H}_{\rm m}$ in the Hurwitz family problem; the group $\pi_1(S_{\rm m})$ is then $\mathrm{G}_{K_{\rm m}}$ and $\pi_1(\mathcal{H}_{\rm m})_k$ respectively. The result essentially says that there is an abelian characteristic class $\Omega \in H^2(\pi_1(S_{\rm m}), Z(G))$ that measures the obstruction. The obstruction however does not correspond to the vanishing of Ω but to the fact that Ω lies in a certain "small" subset $\Delta \subset H^2(\pi_1(S_{\rm m}), Z(G))$. This subset is trivial for G-covers: in that situation the obstruction *is* measured by the vanishing of Ω.

THEOREM 3.1. *There is a two-step obstruction to existence of global solutions.*
First obstruction. *A first necessary condition for existence of global solutions, called* (λ/Lift) *should be checked. This condition holds in particular in each of the following situations.*

- *(for mere covers):*

(a) *The fundamental group* $\pi_1(S_{\rm m})$ *is a projective profinite group, or*
(b) *The group* $\mathrm{Cen}_{S_d}(G)G/G$ *has a complement in the group* $\mathrm{Nor}_{S_d}(G)/G$*. This holds in particular for mere covers that are Galois of group G such that* $\mathrm{Inn}(G)$ *has a complement in* $\mathrm{Aut}(G)$*, or*
(c) *The group* $\mathrm{Cen}_{S_d}(G)/Z(G)$ *has no center and the group* $\mathrm{Inn}(\mathrm{Cen}_{S_d}(G)/Z(G))$ *has a complement in* $\mathrm{Aut}(\mathrm{Cen}_{S_d}(G)/Z(G))$ *(e.g.* $\mathrm{Cen}_{S_d}(G) = Z(G)$*).*

- *(for G-covers): always.*

Main obstruction. *Assume condition* (λ/Lift) *holds. Then there exists a* 2*-cocycle* $\Omega \in H^2(\pi_1(S_{\rm m}), Z(G))$[3] *with the following property.*

[2]Recall k is a field of definition of $\mathcal{H}_{\rm m}$.
[3]The 2-cocycle Ω and the action of $\pi_1(S_{\rm m})$ on $Z(G)$ are explicitely given in [**DeDo1**].

• *(for mere covers): there exists a global solution to the problem if and only if*

$$\Omega^{-1} \in \delta^1\left(H^1(\pi_1(S_{\mathrm{m}}), \mathrm{Cen}_{S_d}(G)/Z(G))\right)$$

where $\delta^1 : H^1(\pi_1(S_{\mathrm{m}}), \mathrm{Cen}_{S_d}(G)/Z(G)) \to H^2(\pi_1(S_{\mathrm{m}}), Z(G))$ *is the connecting morphism associated to the exact sequence* $Z(G) \hookrightarrow \mathrm{Cen}_{S_d}(G) \twoheadrightarrow \mathrm{Cen}_{S_d}(G)/Z(G)$

• *(for G-covers): the same holds with* $\mathrm{Cen}_{S_d}(G)$ *replaced by* $Z(G)$*. In particular the iff condition on* Ω *for existence of solutions is that* $\Omega = 1$ *in* $H^2(\pi_1(S_{\mathrm{m}}), Z(G))$.

3.2.4. The (Seq/Split) *condition.* Theorem 3.1 can be simplified if the following exact sequence splits

$$1 \to \pi_1(\mathbb{P}^1\text{-}\mathbf{t})_{K^{\mathrm{s}}}) \to \pi_1((\mathbb{P}^1 \times S_{\mathrm{m}})^*) \to \pi_1(S_{\mathrm{m}}) \to 1,$$

a condition denoted by (Seq/Split). Let $\overline{\varphi} : \pi_1((\mathbb{P}^1 \times S_{\mathrm{m}})^*) \to \mathrm{Nor}_{S_d}(G)/\mathrm{Cen}_{S_d}(G)$ be the map given by the moduli condition (see §3.2.2 (second method) above).

Theorem 3.2. *Assume condition* (Seq/Split) *and fix a section* $s : \pi_1(S_{\mathrm{m}}) \to \pi_1((\mathbb{P}^1 \times S_{\mathrm{m}})^*)$. *Then the problem has a solution iff the map* $\overline{\varphi} \circ s : \pi_1((\mathbb{P}^1 \times S_{\mathrm{m}})^*) \to \mathrm{Nor}_{S_d}(G)/\mathrm{Cen}_{S_d}(G)$ *can be lifted to map* $\varphi : \pi_1((\mathbb{P}^1 \times S_{\mathrm{m}})^*) \to \mathrm{Nor}_{S_d}(G)$.

We stated the result for mere covers. As usual the result for G-covers is obtained by replacing $\mathrm{Nor}_{S_d}(G)$ by G and $\mathrm{Cen}_{S_d}(G)$ by $Z(G)$.

In the cover context, *i.e.*, for $S_{\mathrm{m}} = \mathrm{Spec}(K_{\mathrm{m}})$, condition (Seq/Split) classically holds if the base space B has K-rational points off the branch point set. On the other hand, condition (Seq/Split) does not always hold: counter-examples are given in [**DeEm**;§5]. In the Hurwitz family context, *i.e.*, for $S_{\mathrm{m}} = (\mathcal{H}_{\mathrm{m}})_k$, condition (Seq/Split) holds in the situation of covers with ordered branch points and for k algebraically closed [**DeDoEm**;Lemme 3.9].

3.2.5. Structure on the set of solutions [**De3**;§2]. Assume there is at least one solution to the problem. Let Ω be the 2-cocycle from theorem 3.1.

Theorem 3.3. *The set of solutions* Σ *can be partitioned in a collection* $(\Sigma_\theta)_{\theta\in\Theta}$ *of subsets* Σ_θ *such that:*
- the index set Θ *is the subset of all* $\theta \in H^1(\pi_1(S_{\mathrm{m}}), \mathrm{Cen}_{S_d}(G)/Z(G))$ *such that* $\delta^1(\theta) = \Omega$,
- for each $\theta \in \Theta$, *the subset* Σ_θ *is in one-one correspondence with the* 1*-cochain set* $Z^1(\pi_1(S_{\mathrm{m}}), Z(G))$ *(for some action given in* [**De3**]*).*
Furthermore, if condition (Seq/Split) *holds, the set* Σ *can also be viewed as the* 1*-cochain set* $Z^1(\pi_1(S_{\mathrm{m}}), \mathrm{Cen}_{S_d}(G))$ *(for some action given in* [**De3**]*).*

The statement is for mere covers. For G-covers, the set Σ is in one-one correspondence with the 1-cochain set $Z^1(\pi_1(S_{\mathrm{m}}), Z(G))$. In the cover problem, the index set Θ consists of all actual extensions of constants in Galois closure of models of f over the field of moduli.

3.3. Diophantine description. Another description of the obstruction was recently given [**DeDoMo**]. We see it as diophantine as it amounts to solving polynomial equations.

3.3.1. Cover problem. The main result of [**DeDoMo**] is the construction of *descent varieties for algebraic covers.* For our given cover $f : X \to B$, these are parameter spaces of families of *models* of f satisfying the versal property: that is, each model of f is a fiber of the family, and so corresponds to points of V. Furthermore fields of definition of models and of their representative points correspond to one another. More specifically we have the following statement; the word "cover" can be understood as mere cover or G-cover.

THEOREM 3.4. *There exists an affine variety V with the following properties:*
(1) *V is smooth, geometrically irreducible and defined over the field of moduli $K_{\rm m}$.*
(2) *There exists a $K_{\rm m}$-family $\mathcal{F} : \mathcal{X} \to V \times B$ of covers of B parametrized by V, such that*
 (i) *For each $x \in V$, the fiber cover $\mathcal{F}_x : \mathcal{X}_x \to B_{K_{\rm m}(x)}$ is a $K_{\rm m}(x)$-model of f.*
 (ii) *If k is an extension of $K_{\rm m}$ and $\widetilde{f} : \widetilde{X} \to B_k$ a k-model of f, there exists $x \in V(k)$ such that $\widetilde{f}$ is isomorphic to the fiber cover $\mathcal{F}_x : \mathcal{X}_x \to B_k$.*
(3) *For every extension k of $K_{\rm m}$ for which $V(k) \neq \emptyset$, V is unirational over k.*

Thus we obtain this new approach to our cover problem:

the field of moduli $K_{\rm m}$ is a field of definition if and only if $V(K_{\rm m}) \neq \emptyset$.

For example, if $K_{\rm m}$ is Pseudo Algebraically Closed then "$V(K_{\rm m}) \neq \emptyset$" holds by definition; the field of moduli *is* a field of definition.

This approach led to new results, which had not been obtained through the cohomological approach. In particular, it made it possible to answer questions raised in [**DeHa**] about existence of *totally p-adic models* of covers at good primes of their field of moduli. More details are given in §4.2.2.

There might be other applications if one could describe more precisely these descent varieties. The descent varieties V we construct are unirational. A natural question is whether these varieties are rational and also whether the Hasse principle holds for these varieties. Answers to this question would yield some information about the Hasse principle for covers (see §4.2.3).

Two constructions of these descent varieties are offered in [**DeDoMo**]. The first one, which works for covers of $\mathbb{P}^1$, is quite explicit: basically, it is proven that affine equations $P(t, y) = 0$ for covers of $\mathbb{P}^1$ can be normalized in such a way that for models of a same cover they form a *family* of affine curves; for some "good" normalization, the parameter space can also be shown to be defined over the field of moduli. The second method, which is more general, uses the stack and gerbe approach. The gerbe $\mathcal{G}_f$ is noted to be an *algebraic stack* as it can be covered by affine schemes. General results about algebraic stacks are then used and refined to show that $\mathcal{G}_f$ can be in fact covered by an algebraic K-variety V as in theorem 3.4.

3.3.2. Global version. There is a version of theorem 3.4 above the Hurwitz space $\mathcal{H}_{\rm m}$.

THEOREM 3.5. *There exists a family $\mathcal{V} \to \mathcal{H}_{\rm m}$ parametrized by $\mathcal{H}_{\rm m}$ of smooth algebraic varieties such that for all $h \in \mathcal{H}_{\rm m}$, $\mathcal{V}_h$ is a descent variety for the $\overline{k(h)}$-cover $f_h : X_h \to \mathbb{P}^1_{\overline{k(h)}}$ corresponding to h. That is, we require that the fiber-variety $\mathcal{V}_h$ satisfies conditions* (1)–(3) *of theorem 3.4 with $K_{\rm m} = k(h)$ and $f = f_h$.*

Applications of this result are given in §4.3.1.

4. Applications

4.1. Basic applications.

4.1.1. Concrete criteria. The following criteria are straightforward consequences of theorems 3.1 and 3.2. The first one concerns the cover problem and the second one the Hurwitz family problem.

COROLLARY 4.1 [**DeDo1**;§3.4]. *The field of moduli is a field of definition in each of the following situations.*

(a) *The absolute Galois group* $\mathrm{G}_{K_\mathrm{m}}$ *is a projective profinite gp (e.g.* $\mathrm{cd}(K_\mathrm{m}) \leq 1$*).*

(b) $Z(G) = \{1\}$ *and condition* (λ/Lift) *holds (see theorem 3.1 for practical cases of* (λ/Lift)*). This holds in particular if f is a mere cover that is Galois of group G with trivial center and with a complement in* $\mathrm{Aut}(G)$*.*

(c) *Condition* (Seq/Split) *holds (e.g. there are unramified* K_m*-points on B) and*

• *(for mere covers):* $\mathrm{Cen}_{S_d}(G)$ *has a complement in* $\mathrm{Nor}_{S_d}(G)$*. This holds in particular if f is Galois.*

• *(for G-covers):* $Z(G)$ *has a complement in G (e.g. G is abelian).*

In the following statement, $\mathcal{H}$ is a geometrically irreducible variety with a morphism $\chi : \mathcal{H} \to \mathcal{H}_\mathrm{m}$, for example a subvariety of the Hurwitz space $\mathcal{H}_\mathrm{m}$.

COROLLARY 4.2 [**DeDoEm**;§4]. *There exists a Hurwitz family* $\mathcal{F} : \mathcal{T} \to \mathcal{H} \times \mathbb{P}^1$ *in each of the following situations.*

(a) *The fundamental group* $\pi_1(\mathcal{H})$ *is a projective profinite group. This holds in particular if* $\mathcal{H}$ *is an affine curve in* $\mathcal{H}_\mathrm{m}$ *over an algebraically closed field.*

(b) $\mathcal{H} = \mathcal{H}_\mathrm{m}$ *is the Hurwitz space,* $Z(G) = \{1\}$ *and condition* (λ/Lift) *holds. This holds in particular if f is a mere cover that is Galois of group G with trivial center and with a complement in* $\mathrm{Aut}(G)$*.*

(c) $\mathcal{H} = \mathcal{H}_\mathrm{m}$ *parametrizes covers with ordered branch points, the field k is algebraically closed and*

• *(for mere covers):* $\mathrm{Cen}_{S_d}(G)$ *has a complement in* $\mathrm{Nor}_{S_d}(G)$*. This holds in particular for Galois covers.*

• *(for G-covers):* $Z(G)$ *has a complement in G (e.g. G is abelian).*

Statement (b) in corollaries 4.1 and 4.2 improve on corollary 2.2: $\mathrm{Cen}_{S_d}(G) = \{1\}$ is a special case of (b). Corollary 4.1 (c) for mere covers generalizes the Coombes-Harbater theorem. Condition (Seq/Split) cannot be dropped in Corollary 4.1 (c) (and probably not either in Corollary 4.2 (c)): there exists mere Galois covers and abelian G-covers that are not defined over their field of moduli [**DeEm**]. Statements (a) and (c) from corollary 4.2 generalize statements (c) and (b) respectively of proposition 1.4 from [**CoHa**] where only the G-cover situation is considered. Corollary 1.4 (c) should also be compared to proposition 3 from [**Fr**] where G is supposed to be a centerless group.

4.1.2. Bounds for existence of solutions. The following result is proved in [**DeDo1**;§3.4] (for (a)) and in [**DeDoEm**;§4] (for (b)). Denote by D the integer

$$D = \begin{cases} \dfrac{|\mathrm{Nor}_{S_d} G|}{|\mathrm{Cen}_{S_d} G|} & \text{in the mere cover situation} \\ \dfrac{|G|}{|Z(G)|} & \text{in the G-cover situation} \end{cases}$$

COROLLARY 4.3. (a) *Assume there exists some unramified K_{m}-rational point on the base space B (or more generally that condition* (Seq/Split) *holds). Then the cover f has a field of definition of degree $\leq D$ over K_{m}.*
(b) *If $B = \mathbb{P}^1$ and if covers are given with an ordering of their branch points, then there exists a family above an étale cover of $(\mathcal{H}_{\mathrm{m}})_{\overline{\mathbb{Q}}}$ of degree $\leq D$.*

4.1.3. A minimality property of the moduli. The next statement extends some results originally proved in the G-cover situation by Coombes and Harbater [**CoHa**].

THEOREM 4.4. (a) *The field of moduli of a $\overline{\mathbb{Q}}$-cover is the intersection of all its fields of definition.*
(b) *The function field $\overline{\mathbb{Q}}(\mathcal{H}_{\mathrm{m}})$ of the Hurwitz space $\mathcal{H}_{\mathrm{m}}$ is the intersection of all function fields $\overline{\mathbb{Q}}(\mathcal{H})$ of irreducible $\overline{\mathbb{Q}}$-algebraic varieties $\mathcal{H}$ that parametrize Hurwitz families and are such that the associated structural morphism $\gamma_{\mathcal{H}}$ is dominant.*

(a) is proved in [**DeDo1**;§3.4] and (b) in [**DeDoEm**;§4]. In (a) the field of moduli K_{m} is a number field. The idea is then to write this number field as an intersection of fields κ with G_κ pro-cyclic (which implies projective) and to use corollary 4.1 (a). The principle of the proof is similar in (b): to write $\overline{\mathbb{Q}}(\mathcal{H}_{\mathrm{m}})$ as an intersection of function fields $\overline{\mathbb{Q}}(\mathcal{H})$ of varieties $\mathcal{H}$ that one can view as (families of) *curves* so as to apply corollary 4.2 (a).

4.2. Local-global results for fields of definition of covers. The theorem stated in the introduction is a combination of several results. A first one roughly says that, over complete (or more generally henselian) valued fields, the field of moduli of a curve cover *is* a field of definition, *unless* the residue characteristic p is a "bad" prime (§4.2.1). This result extends to fields that are existentially closed in complete valued fields, like the fields of totally p-adic numbers (§4.2.2). Over a global field, there can only be finitely many bad primes. Furthermore, in some circumstances, there is a Hasse principle (§4.2.3): if there is some global obstruction, it shows locally (at one of the bad primes).

4.2.1. Local result. In the next statement, we assume that the ground field K is the fraction field of a henselian discrete valuation ring $(O, \mathcal{P})$ whose residue field $O/\mathcal{P}$ is perfect, and that the base space B of the cover f is a geometrically irreducible smooth projective K-curve with a model over $\mathrm{Spec}(O)$ with good reduction at the prime $\mathcal{P}$.

THEOREM 4.5. *Assume further that the order of the group G of the cover f is not divisible by the residue characteristic $p = \mathrm{char}(O/\mathcal{P})$, and that no two branch points coalesce at any prime over $\mathcal{P}$ of the integral closure of O in K^{s}.*
(a) *The cover f has a model $\widetilde{f}$ over the unramified closure $K_{\mathrm{m}}^{\mathrm{ur}}$ of K_{m} that is stable, that is, K_{m} is still the field of moduli of this model (relative to the extension $K_{\mathrm{m}}^{\mathrm{ur}}/K$). Furthermore $K_{\mathrm{m}}^{\mathrm{ur}} = K(\mathbf{t})^{\mathrm{ur}}$ where $\mathbf{t}$ is the branch point set of f.*
(b) *If the residue field $O/\mathcal{P}$ is of cohomological dimension ≤ 1, then the field of moduli is a field of definition of the cover f (and also of the $K_{\mathrm{m}}^{\mathrm{ur}}$-model $\widetilde{f}$).*

This result is proved in [**DeHa**] for G-covers of $\mathbb{P}^1$. The second sentence of (a) is known as Beckmann's theorem [**Be1**]. The general case of theorem 4.5 is proved in [**Em**].

Existence of a $K_{\mathrm{m}}^{\mathrm{ur}}$-model of f follows from "$\mathrm{cd}(K_{\mathrm{m}}^{\mathrm{ur}}) \leq 1$" and corollary 4.1 (a). The point in (a) is to find a *stable* model over $K_{\mathrm{m}}^{\mathrm{ur}}$. In [**DeHa**], a good models result of Beckmann is used which shows that, under the assumptions, a $K_{\rm m}^{\rm ur}$-model can be twisted in another $K_{\rm m}^{\rm ur}$-model without vertical ramification [**Be2**]. As a consequence, this model has fibers consisting of $K_{\rm m}^{\rm ur}$-rational points. This ensures that the model is stable (stability criterion from [**De1**]). The method is different in [**Em**]. There the cover is first reduced modulo $\mathcal{P}$; from Fulton's theorem, there is good reduction under the assumptions. Grothendieck's specialization theorem is then used to lift the reduced cover to a $K_{\rm m}^{\rm ur}$-model of f. This model is stable, as automorphisms of the reduced cover also lift. If in addition $\mathrm{cd}(O/\mathcal{P}) \leq 1$ (and so $\mathrm{cd}(\mathrm{G}(K_{\rm m}^{\rm ur}/K_{\rm m})) \leq 1$), descent to $K_{\rm m}$ itself is possible, from corollary 4.1 (a) (or more exactly from a relative analog).

Corollary 4.6. *A $\overline{\mathbb{Q}}$-curve cover $f : X \to B$ is defined over all but finitely many completions $(K_{\rm m})_v$ of its field of moduli. More precisely f is defined over $(K_{\rm m})_v$ for every place v such that the residue characteristic is a good prime p, i.e., such that p does not divide $|G|$, the branch points do not coalesce modulo p and B has good reduction modulo p.*

In this result the places v are implicitly finite. The question of when an archimedean completion $(K_{\rm m})_v$ is a field of definition can be considered. This is trivial if $(K_{\rm m})_v = \mathbb{C}$ but interesting if $(K_{\rm m})_v = \mathbb{R}$. In this case the question can be decided by topological methods [**DeFr2**;§3.5].

4.2.2. Local-global result. The following result is a straightforward consequence of theorem 3.4 from §3.3 (the diophantine approach). Recall that a field κ is said to be existentially closed in a regular extension Ω if for each smooth geometrically irreducible κ-variety V, $V(\Omega) \neq \emptyset \Rightarrow V(\kappa) \neq \emptyset$.

Corollary 4.7. *Assume that the field of moduli $K_{\rm m}$ is existentially closed in some field of definition k_o of the cover f. Then f is defined over $K_{\rm m}$.*

The next corollary answers a question raised in [**DeHa**] (Question 5.3) about totally p-adic models of covers. Suppose $K_{\rm m}$ is a global field (*i.e.*, either a number field or a one-variable function field over a finite field). Given a nonempty finite set Σ of places of $K_{\rm m}$, denote the maximal extension of $K_{\rm m}$ in a fixed separable closure $K^{\rm s}$ which is totally split at each $v \in \Sigma$, by $K_{\rm m}^{\Sigma}$. For example, if $K_{\rm m} = \mathbb{Q}$ and $v = p$, $K_{\rm m}^{\Sigma}$ is the field $\mathbb{Q}^{{\rm t}p}$ of totally p-adic numbers.

Theorem 4.8. *Assume $f : X \to B$ is a curve cover. Let Σ be a finite set of good places of $K_{\rm m}$. Then f is defined over $K_{\rm m}^{\Sigma}$.*

Let V be a descent variety associated with the cover f as in theorem 3.4. From theorem 4.5, the cover f is defined over $(K_{\rm m})_v$ for all $v \in \Sigma$. Thus there are $(K_{\rm m})_v$-rational points on V for all $v \in \Sigma$. From a local-global principle for smooth algebraic varieties due to Moret-Bailly [**Mo1**] (see also [**Pop**]), it follows then that there are $K_{\rm m}^{\Sigma}$-points on V. These $K_{\rm m}^{\Sigma}$-points correspond to $K_{\rm m}^{\Sigma}$-models of f.

4.2.3. Hasse principle. The following local-to-global question was originally raised by E. Dew in his thesis [**Dew**]: if a cover $f : X \to B$ is defined over each completion of a number field K, then does it follow that it is defined over K? This was proved in [**De1**] and [**DeDo1**] for G-covers.

Theorem 4.9. *A G-cover $f : X \to B$ over $\overline{\mathbb{Q}}$ is defined over $\mathbb{Q}$ if and only if it is defined over each completion $\mathbb{Q}_p$ of $\mathbb{Q}$ (including $p = \infty$). More generally, the same conclusion holds with $\mathbb{Q}$ replaced by any number field K that is not an* *exception in the Grunwald-Wang theorem (see* [**DeDo1**;§3.5] *for a precise definition of the exceptional case).*

The proof uses the cohomological description of the obstruction (theorem 3.1). If f is defined over all $\mathbb{Q}_p$, then the field of moduli has to be $\mathbb{Q}$ (as it is contained in each $\mathbb{Q}_p$). Thus the obstruction to f being defined over $\mathbb{Q}$ is measured by a 2-cocycle $\Omega \in H^2(\mathbb{Q}, Z(G))$. This 2-cocycle vanishes in each $H^2(\mathbb{Q}_p, Z(G))$. Now, using the Tate-Poitou duality theorem conjoined with the Grunwald-Wang theorem, the natural map $H^2(\mathbb{Q}, Z(G)) \to \prod_p H^2(\mathbb{Q}_p, Z(G))$ can be shown to be injective.

The local-to-global principle has also been studied for mere covers. Some partial answers are known (*cf.* [**DeDo2**] and [**De3**]). For example, the local-to-global principle holds if $Z(G) = \mathrm{Cen}_{S_d}(G) \subset Z(\mathrm{Nor}_{S_d}(G))$. We conjecture however it does not hold in general (see [**DeDo2**;§3.3]. But we do not have any counter-example yet (see [**De1**;Remark 5.4]).

4.3. Specialization results on Hurwitz spaces.

4.3.1. Specialization results on Hurwitz spaces. The cover problem is a specialization of the Hurwitz family problem. More precisely, for each point $h = [f] \in \mathcal{H}_{\mathrm{m}}$, there is a specialization morphism Sp_h, which maps the Hurwitz gerbe $\mathcal{G}_{\mathcal{H}_{\mathrm{m}}}$ to the gerbe $\mathcal{G}_f$ of models of f. A natural question is whether the Hurwitz gerbe $\mathcal{G}_{\mathcal{H}_{\mathrm{m}}}$ is neutral if all gerbes $\mathcal{G}_h$ ($h \in \mathcal{H}_{\mathrm{m}}$) are neutral. Neither counter-examples nor results in this direction are known.

In the same vein, one may ask whether a cover is "often" defined over its field of moduli. To make the question precise, define the subset $\mathcal{H}_{\mathrm{m}}^{\mathrm{noob}} \subset \mathcal{H}_{\mathrm{m}}$ to be the set of closed points $h \in \mathcal{H}_{\mathrm{m}}$ such that the corresponding cover $f_h : X_h \to \mathbb{P}^1_{k(h)}$ is defined over its field of moduli $k(h)$ (*i.e.*, for which there is no ob(struction)). The next result, which follows from theorem 3.5, provides a description of $\mathcal{H}_{\mathrm{m}}^{\mathrm{noob}}$. Let $\mathcal{V} \to \mathcal{H}_{\mathrm{m}}$ be a family of descent varieties over the Hurwitz space as in theorem 3.5.

Theorem 4.10 [**DeDoMo**]. *The closed points $h \in \mathcal{H}_{\mathrm{m}}$ such that the corresponding cover $f_h : X_h \to \mathbb{P}^1_{k(h)}$ is defined over its field of moduli $k(h)$ are exactly those points for which $\mathcal{V}_h(k(h)) \neq \emptyset$.*

Combined with a result of Poonen [**Po**] this shows that the set $\mathcal{H}_{\mathrm{m}}^{\mathrm{noob}}$ is Zariski-dense. Furthermore, if k is large, the set $\mathcal{H}_{\mathrm{m}}^{\mathrm{noob}}(k) = \mathcal{H}_{\mathrm{m}}^{\mathrm{noob}} \cap \mathcal{H}_{\mathrm{m}}(k)$ is empty or Zariski-dense. Recall a field k is *large* if every smooth k-curve has infinitely many k-points provided it has at least one. In the last 10 years, many significant conjectures from geometric inverse Galois theory have been shown to hold when the base field is large. Essentially the reason is that one could reduce these conjectures to finding rational points on varieties. Theorem 4.10 is a new illustration of that. We have an exact description of the subset $\mathcal{H}_{\mathrm{m}}^{\mathrm{noob}}(k)$, which yields precise information if the base field is large; however, in general and in particular over $\mathbb{Q}$, giving a precise description of $\mathcal{H}_{\mathrm{m}}^{\mathrm{noob}}(\mathbb{Q})$ remains a difficult problem (as other problems from geometric inverse Galois theory).

The next result also uses theorem 3.5 but combines it with a stack-theoretic version of the Moret-Bailly local-global principle used in the proof of theorem 4.8 [**Mo2**]. Suppose K_{m} is a number field and fix a non-empty finite set Σ of places of K_{m}. Recall k is a field of definition of the Hurwitz space $\mathcal{H}_{\mathrm{m}}$; the field K_{m}^{Σ} appearing below was defined in §4.2.2.

Theorem 4.11 [**DeDoMo**]. *Assume that $k \subset K_{\rm m}^{\Sigma}$. Assume further that for each $v \in \Sigma$, there exists a $(K_{\rm m})_v$-cover $f_v : X_v \rightarrow \mathbb{P}^1$ corresponding to a point $h_v \in \mathcal{H}_{\rm m}((K_{\rm m})_v)$ (for each embedding $K_{\rm m}^{\Sigma} \hookrightarrow (K_{\rm m})_v$). Then there exists a point $[\widetilde{f}] \in \mathcal{H}_{\rm m}(K_{\rm m}^{\Sigma})$ such that the corresponding cover $\widetilde{f} : \widetilde{X} \rightarrow \mathbb{P}^1$ is defined over $K_{\rm m}^{\Sigma}$.*

4.3.2. Generic specialization. We end this paper with a specialization result at the generic point of $\mathcal{H}_{\rm m}$, proved in [**DeDoEm**].

Theorem 4.12. *If $\mathrm{Br}(\mathcal{H}_{\rm m})_n = 0$ for each divisor n of $|C|$ (with $C = \mathrm{Cen}_{S_d}(G)$ or $C = Z(G)$ as usual), then there exists a Hurwitz family above a non-empty Zariski open subset of $(\mathcal{H}_{\rm m})_{\overline{\mathbb{Q}}}$, or, equivalently, the cover corresponding to the generic point of $\mathcal{H}_{\rm m}$ is defined over its field of moduli (relative to the extension $\overline{\mathbb{Q}(\mathcal{H}_{\rm m})}/\overline{\mathbb{Q}}(\mathcal{H}_{\rm m})$).*

References

[Be1] S. Beckmann, *Ramified primes in the field of moduli of a branched covering of curves*, J. Algebra **125** (1989), 236–255.

[Be2] S. Beckmann, *On extensions of number fields obtained by specializing branched coverings*, J. reine angew. Math. **419** (1991), 27–53.

[Bel] G.V. Belyi, *On Galois extensions of a maximal cyclotomic field*, Math. USSR Izvestija **14** (1979), 247–256.

[CoHa] K. Coombes and D. Harbater, *Hurwitz families and arithmetic Galois groups*, Duke Math. J. **52** (1985), 821–839.

[Cou] J-M. Couveignes, *Quelques revêtements définis sur* $\mathbb{Q}$, Manuscripta Mathematica **94/4** (1997), 409–445.

[CouGr] J-M. Couveignes and L. Granboulan, *Dessins from a geometric point of view*, The Grothendieck theory of Dessins d'Enfants, L. Schneps ed., Camb. U. Press, 1995, pp. 79–113.

[De1] P. Dèbes, *Covers of* $\mathbb{P}^1$ *over the* p*-adics*, Recent developments in the Inverse Galois Problem, M. Fried ed., vol. 186, Contemporary Math., 1995, pp. 217–238.

[De2] P. Dèbes, *Arithmétique et espaces de modules de revêtements*, Number Theory in Progress, Proc. Number Theory conference in Zakopane, K. Gyory, H. Iwaniec and J. Urbanowicz ed., Walter de Gruyter, 1999, pp. 75–102.

[De3] P. Dèbes, *Some arithmetic properties of algebraic covers*, Aspects of Galois Theory, London Math. Soc. Lecture Note Series, H. Völklein, P. Mueller, D. Harbater and J. G. Thompson ed., vol. 256, Camb. U. Press, 1999, pp. 66–84.

[DeDo1] P. Dèbes and J-C. Douai, *Algebraic covers: field of moduli versus field of definition*, Annales Sci. E.N.S. **30** (1997), 303–338.

[DeDo2] P. Dèbes and J-C. Douai, *Local-global principles for algebraic covers*, Israel J. Math. **103** (1998), 237–257.

[DeDo3] P. Dèbes and J-C. Douai, *Gerbes and covers*, Comm. in Algebra **27/2** (1999), 577–594.

[DeDoEm] P. Dèbes, J-C. Douai and M. Emsalem, *Familles de Hurwitz et cohomologie non-abélienne*, Ann. Inst. Fourier **50/1** (2000), 1001–1037.

[DeDoMo] P. Dèbes, J-C. Douai and L. Moret-Bailly, *Descent varieties for algebraic covers*, preprint (2000).

[DeEm] P. Dèbes and M. Emsalem, *On fields of moduli of curves*, J. Algebra **211** (1999), 42–56.

[DeFr1] P. Dèbes and M. Fried, *Arithmetic variation of fibers in algebraic families of curves. Part 1: Criteria for existence of rational points*, J. fur die reine und angew. Math. **409** (1990), 106–137.

[DeFr2] P. Dèbes and M. Fried, *Non rigid situations in constructive Galois Theory*, Pacific J. Math. **163/1** (1994), 81–122.

[DeHa] P. Dèbes and D. Harbater, *Fields of definition of* p*-adic covers*, J. fur die reine und angew. Math. **498** (1998), 223–236.

[Dew] E. Dew, *Fields of moduli of arithmetic covers*, Thesis (1991).

[Em] M. Emsalem, *On reduction of covers of arithmetic surfaces*, Applications of Curves over Finite Fields, M. Fried ed., vol. 245, Contemporary Math., 1999, pp. 117–132.

[Fr] M. Fried, *Fields of definition of function fields and Hurwitz families, Groups as Galois groups*, Comm. in Alg. **1** (1977), 17–82.

[Gi] J. Giraud, *Cohomologie non abélienne*, Grundlehren Math. Wiss., vol. 179, Springer-Verlag, 1971.

[Gr] A. Grothendieck, *Technique de descente et théorèmes d'existence en géométrie algébrique. 1. Descente par morphisme fidèlement plats*, Séminaire Bourbaki, 1960.

[Mo1] L. Moret-Bailly, *Groupes de Picard et problèmes de Skolem II*, Annales Sci. E.N.S. **22** (1989), 181–194.

[Mo2] L. Moret-Bailly, *Problèmes de Skolem sur les champs algébriques*, to appear, Compositio Math..

[Po] B. Poonen, *Points having the same residue field as their image under a morphism*, preprint (2000).

[Pop] F. Pop, *Embedding problems over large fields*, Annals of Math. **144** (1996), 1–35.

[Sa] B. Sadi, *Descente effective du corps de définition des revêtements galoisiens*, J. Number Theory **77** (1999), 71–82.

[Sh] G. Shimura, *On the theory of automorphic functions*, Annals of Math. **70** (1959), 101–144.

[We] A. Weil, *The field of definition of a variety*, Oeuvres complètes (Collected papers) (II), Springer-Verlag, pp. 291–306.

[Wew] S. Wewers, *Construction of Hurwitz spaces*, Thesis (1998).

Mathématiques, Université Lille, 59655 Villeneuve d'Ascq Cedex, France
E-mail address: Pierre.Debes@univ-lille1.fr