The Malcev co-completion of Kummer rank one local systems

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Plan

Context

- The analytic approach
- The algebraic fundamental group of the projective line minus two points

K-potent fundamental group

- Algebraically trivial rank 1 local systems
- K-potent fundamental group
- Künneth formula

Stacky interpretation

- Torsion invertible sheaves on root stacks
- Local systems on root stacks

The unipotent fundamental group

k an algebraically closed field of characteristic zero.
U a smooth algebraic variety over k.
Favourite example: $U = \mathbb{P}^1 \backslash D$.
X a compactification (smooth proper variety such that $U = X \backslash D$, where D is simple normal crossings divisor).
(X, D) a "log-scheme".
Deligne introduced a pro-algebraic group over k denoted $\pi^{dR}(U)$ as the Tannaka group of the category $\mathrm{MIC}^{uni}(U)$.
Objects are pairs $(\mathcal{F}, \nabla)$ where $\mathcal{F}$ is a vector bundle and $\nabla : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_U} \Omega^1_{U/k}$ is a k-linear morphism satisfying Leibniz rule; moreover they must admit a finite filtration with successive quotients isomorphic to $(\mathcal{O}_U, d)$.
When $k = \mathbb{C}$, there is a morphism $\pi_1^{top}(U^{an}) \to \pi^{dR}(U)(\mathbb{C})$ which makes $\pi^{dR}(U)$ the pro-unipotent completion of $\pi_1^{top}(U^{an})$.
Nice properties of $\pi^{dR}(U)$: base change, Künneth formula.

Deligne's machinery

Definition (Del89 10.42)

To $F \subset \operatorname{obj}\mathrm{MIC}(U)$ associate $\mathcal{C} =< F, ext >_{\otimes}$ smallest Tannakian category containing F and stable by extensions.

Deligne gives a base change result and a Künneth formula for the associated fundamental groupoid $P(U/k, < F, ext >_{\otimes})$.
To cite Deligne, who only uses the case $F = \emptyset$: *'My reason for considering a more general case is that I hope that if F is a set of motives on X, $P(X/k, < F_{DR}, ext >_{\otimes})$ on $X \times X$ is motivic.'*
Aims of this talk:

- Give an example of a 'thin' set F that might be interesting,
- explain Deligne's machinery in terms of Malcev co-completion (Jacobsen).

A quasi-unipotent fundamental group

Rank 1 algebraically trivial local systems

When $k = \mathbb{C}$, one can define

$$H^1_{triv}(U^{an}, \mathbb{C}^*) = \ker \left(H^1(U^{an}, \mathbb{C}^*) \to H^1(U^{an}, \mathcal{O}_U^*) \right)$$

If M is an abelian group, let $M[\infty] = \varinjlim_n M[n]$ be its torsion. One then has a quasi-unipotent fundamental group as the Malcev completion:

$$\begin{tikzcd} 1 \arrow[r] & \mathcal{U}(\mathbb{C}) \arrow[r] & \pi_K^{dR}(U)(\mathbb{C}) \arrow[r] & \left(H^1_{triv}(U^{an}, \mathbb{C}^*)[\infty]\right)^\vee \\ & & \pi_1^{top}(U^{an}) \arrow[u] \arrow[ur] & \end{tikzcd}$$

When $D = \emptyset$, then $\pi_K^{dR}(U) = \pi^{dR}(U)$.

Riemann-Hilbert

$k = \mathbb{C}$
There are category equivalences:

$$MIC(U^{an}) \longrightarrow \mathrm{LocSys}(U^{an}, \mathrm{Vect}_k) \longrightarrow \mathrm{Rep}_k\, \pi_1^{top}(U^{an}, x)$$

$$(\mathcal{F}, \nabla) \longmapsto \mathcal{F}^{\nabla} \quad V \longmapsto V_x$$

This gives a Tannakian interpretation of the k-algebraic envelope of $\pi_1^{top}(U^{an}, x)$. For an abstract group G, the algebraic envelope $G^{k,alg}$ is the Tannaka group of $\mathrm{Rep}_k\, G$.
Moreover $\mathrm{MIC}(U^{an})$ has an algebraic incarnation: $\mathrm{MIC}^{\mathrm{reg}}(U)$.

Regular connections

A connection $(\mathcal{F}, \nabla)$ on U is called *regular* if it admits an extension $(\widetilde{\mathcal{F}}, \widetilde{\nabla})$ to X where $\widetilde{\nabla} : \widetilde{\mathcal{F}} \to \widetilde{\mathcal{F}} \otimes \Omega^1_{X/k}(\log D)$ has (at most) logarithmic poles.

Remark

1. *This is not the original definition,*
2. *there is no uniqueness of the extension.*

The corresponding category $\mathrm{MIC}^{\mathrm{reg}}(U)$ is Tannakian. We denote its Tannaka group by $\pi^{k,alg}(U)$.
When $k = \mathbb{C}$, there is an equivalence $\mathrm{MIC}(U^{an}) \simeq \mathrm{MIC}^{\mathrm{reg}}(U)$, so that $\pi^{k,alg}(U) \simeq \pi_1^{top}(U^{an})^{k,alg}$.

The algebraic fundamental group of $\mathbb{G}_m$

Proposition

If $U = \mathbb{G}_m = \mathbb{P}^1 \backslash \{0, \infty\}$, then $\pi^{k,alg}(U) \simeq \mathbb{G}_a \times D_k(\frac{k}{\mathbb{Z}})$.

Remark

1. *$\pi^{k,alg}(U)$ is not invariant under base change (alg. closed, char. 0), but its largest unipotent quotient $\pi^{dR}(U) \simeq \mathbb{G}_a$ is.*
2. *The algebraic envelope of $\mathbb{Z}$ ($\simeq \pi_1^{top}(U^{an})$ when $k = \mathbb{C}$) is $\mathbb{G}_a \times D_k(k^*)$.*
3. *We wish for an algebraic definition of $\pi_K^{dR}(U)$ such that for $U = \mathbb{G}_m$ we have $\pi_K^{dR}(U) \simeq \mathbb{G}_a \times D_k(\frac{\mathbb{Q}}{\mathbb{Z}})$.*

Details of the isomorphism $\pi^{k,alg}(U) \simeq \mathbb{G}_a \times D_k(\frac{k}{\mathbb{Z}})$

Denoting for $\omega \in H^0(U, \Omega^1_U)$ by $d + \omega : \mathcal{O}_U \to \Omega^1_U$ the connection on the trivial bundle given by $f \mapsto df + f\omega$, there is an isomorphism:

$$\begin{aligned} \frac{k}{\mathbb{Z}} &\longrightarrow \mathrm{Pic}(\mathrm{MIC}^{\mathrm{reg}}(U)) \\ \lambda &\longmapsto (\mathcal{O}_U, d + \lambda \tfrac{dt}{t}) \end{aligned}$$

On the other hand, a representation $\rho : \mathbb{G}_a \to \mathrm{GL}(V)$ is equivalent to giving a nilpotent endomorphism N_ρ, which defines:

$$\begin{aligned} \mathrm{Rep}(\mathbb{G}_a) &\longrightarrow \mathrm{MIC}^{uni}(U) \\ (V, \rho) &\longmapsto (\mathcal{O}_U \otimes_k V, d + N_\rho \tfrac{dt}{t}) \end{aligned}$$

The unipotent fundamental group of $\mathbb{P}^1\backslash\{0,1,\infty\}$

Proposition (Deligne)

Every vector bundle endowed with a unipotent integrable connection on $U = \mathbb{P}^1 \setminus \{0,1,\infty\}$ is canonically isomorphic to a bundle of the form

$$\left(\mathcal{O}_U \otimes V,\ d + \frac{dt}{t} \otimes A_0 + \frac{dt}{t-1} \otimes A_1\right),$$

where V is a finite-dimensional k-vector space, and $A_0, A_1 \in \mathrm{End}_k(V)$ are simultaneously nilpotent.

Remark

1. *This is a first step towards the description of $\mathcal{O}(\pi^{dR}(U))$.*
2. *For comparison, GAGA shows that $\pi_1^{et}(U) \simeq \widehat{F_2}$, but we do not have an explicit isomorphism, nor even an algebraic proof.*

Sketch of a proof

Theorem (Deligne-Manin)

Let τ be a set-theoretic section of $k \to \frac{k}{\mathbb{Z}}$. Every $(\mathcal{F}, \nabla)$ in $\mathrm{MIC}^{reg}(U)$ admits a unique logarithmic extension $(\widetilde{\mathcal{F}}, \widetilde{\nabla})$ on X satisfying $\forall x \in D$, $\mathrm{spec}\left(\mathrm{res}_x(\widetilde{\nabla})\right) \subset \mathrm{im}\,\tau$.

The restriction of this Deligne-Manin functor to $\mathrm{MIC}^{loc-uni}(U)$ is an exact tensor functor into $\mathrm{MIC}^{nil-res}(X(\log D))$.
So to every object $(\mathcal{F}, \nabla)$ of $\mathrm{MIC}^{uni}(U)$ we can associate its image $(\widetilde{\mathcal{F}}, \widetilde{\nabla})$ in $\mathrm{MIC}^{uni}(X(\log(D))$.
Now when $U = \mathbb{P}^1 \backslash \{0, 1, \infty\}$, Deligne's 'good condition' $H^1(X, \mathcal{O}_X) = 0$ implies that the vector bundle $\widetilde{\mathcal{F}}$ is canonically trivial:

$$H^0(X, \widetilde{\mathcal{F}}) \otimes_k \mathcal{O}_X \simeq \widetilde{\mathcal{F}}$$

Invertible regular connections

Let $\mathrm{Pic}^{\nabla}(U)$ be the Picard group of $\mathrm{MIC}^{\mathrm{reg}}(U)$.

Definition

$\mathrm{Pic}^{\nabla}_{triv}(U) = \ker\left(\mathrm{Pic}^{\nabla}(U) \to \mathrm{Pic}(U)\right)$

There is a natural exact sequence:

$$0 \to k^* \to H^0(U, \mathcal{O}_U^*) \xrightarrow{\mathrm{dlog}} H^0(X, \Omega^1_X(\log D))^{d=0} \to \mathrm{Pic}^{\nabla}(U) \to \mathrm{Pic}(U)$$

When $D = \emptyset$, $\mathrm{Pic}^{\nabla}_{triv}(U)[\infty] = 0$.

Proposition

If $n \in \mathbb{N}^*$ *then* $\mathrm{Pic}^{\nabla}_{triv}(U)[n] \simeq \frac{H^0(U,\mathcal{O}_U^*)}{H^0(U,\mathcal{O}_U^*)^n}$.

Explicitly, $f \in H^0(U, \mathcal{O}_U^*) \mapsto (\mathcal{O}_U, d + \frac{1}{n}\frac{df}{f})$.

Thinness

Proposition

$\mathrm{Pic}^{\nabla}_{triv}(U)[\infty]$ *is 'thin': if k'/k is an algebraically closed extension, then* $\mathrm{Pic}^{\nabla}_{triv}(U)[\infty] \simeq \mathrm{Pic}^{\nabla}_{triv}(U_{k'})[\infty]$

Proof sketch.

$$\begin{tikzcd}
0 \arrow[r] & \frac{H^0(U,\mathcal{O}_U^*)}{H^0(U,\mathcal{O}_U^*)^n} \arrow[r] \arrow[d, "l"] & H^1(U,\mu_n) \arrow[r] \arrow[d, "m"] & \mathrm{Pic}(U) \arrow[d, "r"] \\
0 \arrow[r] & \frac{H^0(U_{k'},\mathcal{O}_{U_{k'}}^*)}{H^0(U_{k'},\mathcal{O}_{U_{k'}}^*)^n} \arrow[r] & H^1(U_{k'},\mu_n) \arrow[r] & \mathrm{Pic}(U_{k'})
\end{tikzcd}$$

By smooth base change m is an isomorphism, and one shows that r is a monomorphism. □

Malcev completion, after Jacobsen

Let $\rho : P \to K$ be a faithfully flat morphism of affine group schemes over k. Denote by $\mathtt{Un}(\rho^*)$ the category of objects X of $\operatorname{Rep} P$ that admits a filtration $0 = X_0 \subset X_1 \subset \cdots \subset X_n = X$ such that X_{i+1}/X_i is in the (essential) image of ρ^*.

Definition (Jacobsen)

The Malcev completion of $\rho : P \to K$ is the canonical morphism $\tilde{\rho} : P \to \mathcal{G}$ where $\mathcal{G}$ is the Tannaka dual of $\mathtt{Un}(\rho^*)$.

There is a universal factorisation

$$\begin{tikzcd}
1 \arrow[r] & \mathcal{U} \arrow[r] & \mathcal{G} \arrow[r] & K \arrow[r] & 1 \\
 & & P \arrow[u, "\tilde{\rho}"] \arrow[ur, "\rho"'] & &
\end{tikzcd}$$

where the kernel $\mathcal{U}$ is unipotent.

Base change

Definition

The K-potent fundamental group is the (target of the) Malcev completion $\pi^{alg}(U) \to \pi_K^{dR}(U)$ of the natural morphism $\pi^{alg}(U) \to D_k(\mathrm{Pic}^{\nabla}_{triv}(U)[\infty])$.

Remark

The corresponding objects of $\mathrm{MIC}^{reg}(U)$ *are semi-finite in the sense of Otabe.*

Theorem (B.2024)

If k'/k is an algebraically closed extension, then the natural morphism

$$\pi_K^{dR}(U_{k'}) \to \pi_K^{dR}(U)_{k'}$$

is an isomorphism.

Proof of base change (sketch)

In the following diagram:

$$\begin{tikzcd}
\pi^{alg}(U)_{k'} \arrow[r, "\sigma"] & D_{k'}(\mathrm{Pic}^{\nabla}_{triv}(U)[\infty]) \\
\pi^{alg}(U_{k'}) \arrow[u] \arrow[ur, "\rho"] \arrow[r] & D_{k'}(\mathrm{Pic}^{\nabla}_{triv}(U_{k'})[\infty]) \arrow[u, "\wr"']
\end{tikzcd}$$

By thinness, the right vertical morphism is an isomorphism, so $\pi_K^{dR}(U_{k'})$ is the Malcev completion of ρ.
One shows that the left vertical morphism is Malcev-complete, and this implies that the Malcev completion of ρ identifies with the Malcev completion of σ.
Finally, one concludes by showing that Malcev-completion commutes with base change.

Künneth

Consequences of base change invariance

Theorem

Let, for $i = 1, 2$, U_i/k be smooth varieties. Then the natural morphism:

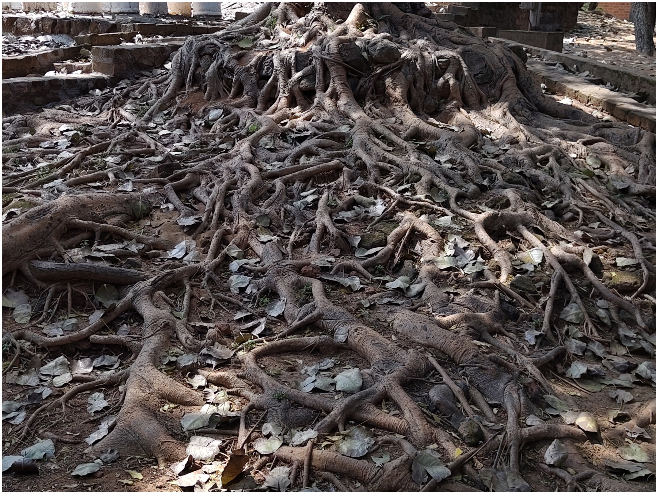
$$\pi_K^{dR}(U_1 \times U_2) \to \pi_K^{dR}(U_1) \times_k \pi_K^{dR}(U_2)$$

is an isomorphism.

Theorem

If $E \to U$ is a vector bundle, then $\pi_K^{dR}(E) \simeq \pi_K^{dR}(U)$.

Root stack



Root stack

X a smooth scheme over a field k.
D a smooth Cartier divisor (more generally: simple normal crossings divisor). (X, D) a "log-scheme".
$n \in \mathbb{N}^*$, invertible in k.
The root stack $\pi : \sqrt[n]{D/X} \to X$ is defined by:

$$
\begin{tikzcd}
\sqrt[n]{D/X} \arrow[r, "\mathfrak{D}"] \arrow[d, "\pi"] & \mathrm{Div} \arrow[r, equal] \arrow[d, "\times n"] & {[\mathbb{A}^1/\mathbb{G}_m]} \arrow[d, "\times n"] & {[\mathbb{A}^1/\mu_n]} \arrow[l] \arrow[d] \\
X \arrow[r, "D"] \arrow[rrr, dotted, bend right] & \mathrm{Div} \arrow[r, equal] & {[\mathbb{A}^1/\mathbb{G}_m]} & \mathbb{A}^1 \arrow[l]
\end{tikzcd}
$$

π is finite flat ramified, $\pi^* D = n\mathfrak{D}$.

Stacky torsion invertible sheaves with support

Let $\mathfrak{X}_n = \sqrt[n]{D/X}$ and let $\mathrm{Pic}_{triv}(\mathfrak{X}_n) = \ker(\mathrm{Pic}(\mathfrak{X}_n) \to \mathrm{Pic}(U))$.

Proposition

The morphism

$$\mathrm{Pic}^{\nabla}_{triv}(U)[n] \to \mathrm{Pic}_{triv}(\mathfrak{X}_n)[n]$$

sending $(\mathcal{O}_U, d + \frac{1}{n}\frac{df}{f})$ *to* $\mathcal{O}_{\mathfrak{X}_n}(\frac{1}{n}\operatorname{div}(f))$ *is an isomorphism.*

Remark

The de Rham and étale descriptions are in fact compatible.

Shift à la Esnault-Viehweg

(X, D) a "log-scheme".
We fix $B = \mu D$, $\mu \in \mathbb{Z}$.
There exists on $\mathcal{O}_X(B)$ a canonical logarithmic connection

$$\begin{array}{rcl} d(B) : \mathcal{O}_X(B) & \longrightarrow & \mathcal{O}_X(B) \otimes \Omega^1_X(\log D) \\ x^{-\mu} & \longmapsto & -\mu x^{-\mu} \frac{dx}{x} \end{array}$$

where $x = 0$ is a local equation of D.
There is an expression for the residue:

$$\mathrm{res}_D(d(B)) = -\mu \, \mathrm{id}$$

Extension of rank 1 local systems

Corollary

Let $\mathcal{O}_{\mathfrak{X}_n}(\frac{1}{n}\operatorname{div}(f))$ be an arbitrary element of $\mathrm{Pic}_{triv}(\mathfrak{X}_n)[n]$. There exists a unique holomorphic lifting to $\mathrm{Pic}^{\nabla}_{triv}(\mathfrak{X}_n)[n]$ which is

$$\left(\mathcal{O}_{\mathfrak{X}_n}(\frac{1}{n}\operatorname{div}(f)), d(\frac{1}{n}\operatorname{div}(f)) + \frac{1}{n}\frac{df}{f}\right) .$$

$$\begin{tikzcd} & \mathrm{Pic}^{\nabla}_{triv}(\mathfrak{X}_n)[n] \arrow[dl, "\sim"'] \arrow[dr, "\sim"] & \\ \mathrm{Pic}^{\nabla}_{triv}(U)[n] & & \mathrm{Pic}_{triv}(\mathfrak{X}_n)[n] \end{tikzcd}$$

Application to the K-potent fundamental group

Corollary

Every object of the Malcev co-completion of $\mathrm{Pic}^{\nabla}_{triv}(U)[n] - \mathrm{Vect}_k \to \mathrm{MIC}^{reg}(U)$ *extends canonically to an object of* $\mathrm{MIC}^{nil-res}(\mathfrak{X}_n)$.

The idea of using Deligne-Manin extension on the root stack is already present in Iyer-Simpson.

When $U = \mathbb{P}^1 \backslash \{0, 1, \infty\}$ the difference with Deligne's situation (case $n = 1$) are:

1. the underlying sheaves $\mathcal{O}_{\mathfrak{X}_n}(\frac{1}{n}\,\mathrm{div}(f))$ of the building blocks are not trivial,
2. even if we pull back to the finite torsor $Y_n \to \mathfrak{X}_\mathfrak{n}$ trivializing the underlying bundles of objects in $\mathrm{Pic}^{\nabla}_{triv}(\mathfrak{X}_n)[n]$, Deligne's 'good condition' need not apply for Y_n.

ありがとう！