A quasi-unipotent fundamental group

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Plan

The unipotent fundamental group

X a proper smooth scheme over a field k (alg. closed, char. 0).
D a smooth Cartier divisor (more generally: simple normal crossings divisor).
(X, D) a "log-scheme", $U = X \backslash D$. Favourite example: $U = \mathbb{P}^1 \backslash S$.
Deligne introduced a pro-algebraic group over k denoted $\pi^{dR}(U)$ as the Tannaka group of the category $\mathrm{MIC}^{uni}(U)$.
When $k = \mathbb{C}$, there is a morphism

$$\pi_1^{top}(U^{an}) \to \pi^{dR}(U)(\mathbb{C})$$

which makes $\pi^{dR}(U)$ the pro-unipotent completion of $\pi_1^{top}(U^{an})$.
This group behaves well under base change, satisfies the Künneth formula.

A quasi-unipotent fundamental group

Rank 1 algebraically trivial local systems

When $k = \mathbb{C}$, one can define

$$H^1_{triv}(U^{an}, \mathbb{C}^*) = \ker \left(H^1(U^{an}, \mathbb{C}^*) \to H^1(U^{an}, \mathcal{O}_U^*) \right)$$

One then has a quasi-unipotent fundamental group as the Malcev completion:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathcal{U}(\mathbb{C}) & \longrightarrow & \pi_{qu}^{dR}(U)(\mathbb{C}) & \longrightarrow & \left(H^1_{triv}(U^{an}, \mathbb{C}^*)[\infty]\right)^\vee \\ & & & & \uparrow & \nearrow & \\ & & & & \pi_1^{top}(U^{an}) & & \end{array}$$

When $D = \emptyset$, then $\pi_{qu}^{dR}(U) = \pi^{dR}(U)$.

Riemann-Hilbert

$k = \mathbb{C}$
There are category equivalences:

$$MIC(U^{an}) \longrightarrow \mathrm{LocSys}(U^{an}, \mathrm{Vect}_k) \longrightarrow \mathrm{Rep}_k\, \pi_1^{top}(U^{an}, x)$$

$$(\mathcal{F}, \nabla) \longmapsto \mathcal{F}^\nabla \quad V \longmapsto V_x$$

This gives a Tannakian interpretation of the k-algebraic envelope of $\pi_1^{top}(U^{an}, x)$. For an abstract group G, the algebraic envelope $G^{k,alg}$ is the Tannaka group of $\mathrm{Rep}_k\, G$.
Moreover $\mathrm{MIC}(U^{an})$ has an algebraic incarnation: $\mathrm{MIC}^{\mathrm{reg}}(U)$.

Regular connections

A connection $(\mathcal{F}, \nabla)$ on U is called *regular* if it admits an extension $(\widetilde{\mathcal{F}}, \widetilde{\nabla})$ to X where $\widetilde{\nabla} : \widetilde{\mathcal{F}} \to \widetilde{\mathcal{F}} \otimes \Omega^1_{X/k}(\log D)$ has (at most) logarithmic poles.

Remark

1. *This is not the original definition,*
2. *there is no uniqueness of the extension.*

The corresponding category $\mathrm{MIC}^{\mathrm{reg}}(U)$ is Tannakian. We denote its Tannaka group by $\pi^{k,alg}(U)$.
When $k = \mathbb{C}$, there is an equivalence $\mathrm{MIC}(U^{an}) \simeq \mathrm{MIC}^{\mathrm{reg}}(U)$, so that $\pi^{k,alg}(U) \simeq \pi_1^{top}(U^{an})^{k,alg}$.

The algebraic fundamental group of $\mathbb{G}_m$

Proposition

If $U = \mathbb{G}_m = \mathbb{P}^1 \setminus \{0, \infty\}$, then $\pi^{k,alg}(U) \simeq \mathbb{G}_a \times D_k(\frac{k}{\mathbb{Z}})$.

Remark

1. *$\pi^{k,alg}(U)$ is not invariant under base change (alg. closed, char. 0), but its largest unipotent quotient $\pi^{dR}(U) \simeq \mathbb{G}_a$ is.*
2. *The algebraic envelope of $\mathbb{Z}$ ($\simeq \pi_1^{top}(U^{an})$ when $k = \mathbb{C}$) is $\mathbb{G}_a \times D_k(k^*)$.*
3. *We wish for an algebraic definition of $\pi_{qu}^{dR}(U)$ such that for $U = \mathbb{G}_m$ we have $\pi_{qu}^{dR}(U) \simeq \mathbb{G}_a \times D_k(\frac{\mathbb{Q}}{\mathbb{Z}})$.*

Details of the isomorphism $\pi^{k,alg}(U) \simeq \mathbb{G}_a \times D_k(\frac{k}{\mathbb{Z}})$

Denoting for $\omega \in H^0(U, \Omega^1_U)$ by $d + \omega : \mathcal{O}_U \to \Omega^1_U$ the connection on the trivial bundle given by $f \mapsto df + f\omega$, there is an isomorphism:

$$\frac{k}{\mathbb{Z}} \longrightarrow \mathrm{Pic}(\mathrm{MIC}^{\mathrm{reg}}(U))$$

$$\lambda \longmapsto (\mathcal{O}_U, d + \lambda \frac{dt}{t})$$

On the other hand, a representation $\rho : \mathbb{G}_a \to \mathrm{GL}(V)$ is equivalent to giving a nilpotent endomorphism N_ρ, which defines:

$$\mathrm{Rep}(\mathbb{G}_a) \longrightarrow \mathrm{MIC}^{uni}(U)$$

$$(V, \rho) \longmapsto (\mathcal{O}_U \otimes_k V, d + N_\rho \frac{dt}{t})$$

The unipotent fundamental group of $\mathbb{P}^1 \backslash \{0, 1, \infty\}$

Theorem (Chen, Deligne, Fonseca-Matthes)

Every vector bundle endowed with a unipotent integrable connection on $U = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ is canonically isomorphic to a bundle of the form

$$\left(\mathcal{O}_U \otimes V,\ d + \frac{dt}{t} \otimes A_0 + \frac{dt}{t-1} \otimes A_1\right),$$

where V is a finite-dimensional k-vector space, and $A_0, A_1 \in \mathrm{End}_k(V)$ are nilpotent.

Remark

For comparison, GAGA shows that $\pi_1^{et}(U) \simeq \widehat{F_2}$, but we do not have an explicit isomorphism, nor even an algebraic proof.

Invertible regular connections

Let $\mathrm{Pic}^{dR,reg}(U)$ be the Picard group of $\mathrm{MIC}^{\mathrm{reg}}(U)$.

Definition

$\mathrm{Pic}^{dR,reg}_{triv}(U) = \ker\left(\mathrm{Pic}^{dR,reg}(U) \to \mathrm{Pic}(U)\right)$

There is a natural exact sequence:

$$0 \to k^* \to H^0(U, \mathcal{O}_U^*) \xrightarrow{\mathrm{dlog}} H^0(X, \Omega^1_X(\log D))^{d=0} \to \mathrm{Pic}^{\mathrm{dR,reg}}(U) \to \mathrm{Pic}(U)$$

When $D = \emptyset$, $\mathrm{Pic}^{dR,reg}_{triv}(U)[\infty] = 0$.

Proposition

If $n \in \mathbb{N}^*$ then $\mathrm{Pic}^{dR,reg}_{triv}(U)[n] \simeq \frac{H^0(U,\mathcal{O}_U^*)}{H^0(U,\mathcal{O}_U^*)^n}$.

Explicitly, $f \in H^0(U, \mathcal{O}_U^*) \mapsto (\mathcal{O}_U, d + \frac{1}{n}\frac{df}{f})$.

Rigidity

Theorem

$\mathrm{Pic}^{dR}_{triv}(U)[\infty]$ *is 'rigid': if* k'/k *is an algebraically closed extension, then* $\mathrm{Pic}^{dR}_{triv}(U)[\infty] \simeq \mathrm{Pic}^{dR}_{triv}(U_{k'})[\infty]$

Proof sketch.

$$\begin{array}{ccccccc}
0 & \longrightarrow & \dfrac{H^0(U,\mathcal{O}_U^*)}{H^0(U,\mathcal{O}_U^*)^n} & \longrightarrow & H^1(U,\mu_n) & \longrightarrow & \mathrm{Pic}(U) \\
 & & \Big\downarrow g & & \Big\downarrow m & & \Big\downarrow d \\
0 & \longrightarrow & \dfrac{H^0(U_{k'},\mathcal{O}_{U_{k'}}^*)}{H^0(U_{k'},\mathcal{O}_{U_{k'}}^*)^n} & \longrightarrow & H^1(U_{k'},\mu_n) & \longrightarrow & \mathrm{Pic}(U_{k'})
\end{array}$$

By smooth base change m is an isomorphism, and one shows that d is a monomorphism. □

Malcev completion, after Jacobsen

Let $\rho : P \to K$ be a morphism of affine group schemes over k. Denote by $\mathsf{Un}(\rho^*)$ the smallest Tannakian subcategory of $\mathsf{Rep}\, P$ that contains the image of ρ^* and is closed under extensions.

Definition (Jacobsen)

The Malcev completion of $\rho : P \to K$ is the canonical morphism $\tilde{\rho} : P \to \mathcal{G}$ where $\mathcal{G}$ is the Tannaka dual of $\mathsf{Un}(\rho^*)$.

There is a universal factorisation

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathcal{U} & \longrightarrow & \mathcal{G} & \longrightarrow & K \\ & & & & {\scriptstyle \tilde{\rho}}\uparrow & \nearrow {\scriptstyle \rho} & \\ & & & & P & & \end{array}$$

where the kernel $\mathcal{U}$ is unipotent.

Base change

Definition

The quasi-unipotent fundamental group is the (target of the) Malcev completion $\pi^{alg}(U) \to \pi^{dR}_{qu}(U)$ of the natural morphism $\pi^{alg}(U) \to D_k(\mathrm{Pic}^{dR,reg}_{triv}(U)[\infty])$.

Theorem

If k'/k is an algebraically closed extension, then the natural morphism

$$\pi^{dR}_{qu}(U)_{k'} \to \pi^{dR}_{qu}(U_{k'})$$

is an isomorphism.

Deligne's algebraic theory

Definition (Del89 10.26)

$\mathcal{C} \subset \mathrm{MIC}(U)$ a Tannakian subcategory defines $P(U/k, \mathcal{C})$ the algebraic fundamental groupoid.

To $F \subset \mathrm{obj}\,\mathrm{MIC}(U)$, Deligne associates $\mathcal{C} = < F, ext >_{\otimes}$

Corollary (Del89 10.42)

If k'/k is an algebraically closed extension, then $P(U_{k'}/k', < F', ext >_{\otimes}) \simeq P(U/k, < F, ext >_{\otimes}) \otimes_k k'$ where F' is the pullback of F along $U_{k'} \to U$.

To cite Deligne, who only uses the case $F = \emptyset$: *'Ma raison pour considérer un cas plus général est que j'espère que si F est un ensemble de motifs sur U, $P(U/k, < F_{DR}, ext >_{\otimes})$ sur $U \times U$ est motivique.'*

Künneth

Consequences of base change invariance

Theorem

Let, for $i = 1, 2$, X_i/k be proper smooth schemes, D_i a simple normal crossings divisor of X_i, and $U_i = X_i \backslash D_i$. Then the natural morphism:

$$\pi_{qu}^{dR}(U_1 \times U_2) \to \pi_{qu}^{dR}(U_1) \times_k \pi_{qu}^{dR}(U_2)$$

is an isomorphism.

Theorem

If $E \to U$ is a vector bundle, then $\pi_{qu}^{dR}(E) \simeq \pi(U)_{qu}^{dR}$.

Analytic proof of Künneth, beginning

- First step: use the Lefschetz principle to reduce to the case $k = \mathbb{C}$. By faithfully flat descent and the base change theorem, we see that the isomorphism is true if and only if it is true over an algebraically closed extension. Since the X_i are of finite type over k, they are defined over a field k_0 which is the algebraic closure of a finitely generated field over $\mathbb{Q}$. As $\mathbb{C}$ has infinite transcendence degree over $\mathbb{Q}$, there exists an embedding $k_0 \to \mathbb{C}$.
- Second step: use Riemann-Hilbert to identify $\pi(U_i)^{dR}_{qu}$ with the Malcev completion of the morphism:

$$\pi_1^{top}(U_i)^{k,alg} \to D_k\left(H^1_{triv}(U_i, \mathbb{C}^*)[\infty]\right)$$

for $i = 1, 2$, and similarly for the product $U_1 \times_k U_2$.

Analytic proof of Künneth, continued

- Third step: Künneth formula for algebraically trivial rank 1 local systems.

$$
\begin{array}{ccccccc}
0 \longrightarrow & H^1_{\mathrm{triv}}((U_1 \times U_2)^{\mathrm{an}}) & \longrightarrow & H^1((U_1 \times U_2)^{\mathrm{an}}, \mathbb{C}^*) & \longrightarrow & \mathrm{Pic}(U_1 \times U_2) \\
& \big\uparrow & & \big\uparrow \wr & & \big\uparrow \\
0 \rightarrowtail & H^1_{\mathrm{triv}}(U_1^{\mathrm{an}}) \times H^1_{\mathrm{triv}}(U_2^{\mathrm{an}}) & \rightarrowtail & H^1(U_1^{\mathrm{an}}, \mathbb{C}^*) \times H^1(U_2^{\mathrm{an}}, \mathbb{C}^*) & \rightarrowtail & \mathrm{Pic}(U_1) \times \mathrm{Pic}(U_2)
\end{array}
$$

The middle vertical arrow is an isomorphism because the Künneth formula holds for the topological fundamental group, and the right vertical arrow is a monomorphism.

Analytic proof of Künneth, end

- Fourth step: Künneth formula for algebraic fundamental groups. Using again that $\pi_1^{top}(U_1 \times U_2) \simeq \pi_1^{top}(U_1) \times \pi_1^{top}(U_2)$ and the fact that the k-algebraic completion commutes with products, we get that $\pi_1^{top}(U_1 \times U_2)^{k,alg} \simeq \pi_1^{top}(U_1)^{k,alg} \times \pi_1^{top}(U_2)^{k,alg}$.
- Last step: use the fact that Malcev completion commutes with products to conclude.

Remark

One should also be able to prove the Künneth formula algebraically.

Algebraic approach to the Künneth theorem

For $n \in \mathbb{N}^*$, the morphism $\pi_{qu}^{dR}(U) \to D_k(\mathrm{Pic}_{triv}^{dR,reg}(U)[n])$ defines a torsor $T_n \to U$ and setting $T = \varprojlim_n T_n$, we obtain an exact sequence:

$$1 \to \pi^{dR}(T) \to \pi_{qu}^{dR}(U) \to D_k(\mathrm{Pic}_{triv}^{dR,reg}(U)[\infty]) \to 1$$

We reduce to showing the Künneth formula for $\pi^{dR}(T)$ and for $D_k(\mathrm{Pic}_{triv}^{dR,reg}(U)[\infty])$.
For the second case, the solution is the same as in the analytic setting.
For the first case, one can follow a classical argument due to Nori using the notion of universal extension.

Root stack

Root stack

X a smooth scheme over a field k.
D a smooth Cartier divisor (more generally: simple normal crossings divisor). (X, D) a "log-scheme".
$n \in \mathbb{N}^*$, invertible in k.
The root stack $\pi : \sqrt[n]{D/X} \to X$ is defined by:

$$\begin{array}{ccccccc}
\sqrt[n]{D/X} & \xrightarrow{\mathfrak{D}} & \mathrm{Div} & = & [\mathbb{A}^1/\mathbb{G}_m] & \longleftarrow & [\mathbb{A}^1/\mu_n] \\
\downarrow \pi & & \downarrow \times n & & \downarrow \times n & & \downarrow \\
X & \xrightarrow{D} & \mathrm{Div} & = & [\mathbb{A}^1/\mathbb{G}_m] & \longleftarrow & \mathbb{A}^1
\end{array}$$

π is finite flat ramified, $\pi^* D = n\mathfrak{D}$.

Stacky torsion invertible sheaves with support

Let $\mathfrak{X}_n = \sqrt[n]{D/X}$ and let $\mathrm{Pic}_{triv}(\mathfrak{X}_n) = \ker(\mathrm{Pic}(\mathfrak{X}_n) \to \mathrm{Pic}(U))$.

Proposition

The morphism

$$\mathrm{Pic}^{dR,reg}_{triv}(U)[n] \to \mathrm{Pic}_{triv}(\mathfrak{X}_n)[n]$$

sending $(\mathcal{O}_U, d + \frac{1}{n}\frac{df}{f})$ *to* $\mathcal{O}_{\mathfrak{X}_n}(\frac{1}{n}\mathrm{div}(f))$ *is an isomorphism.*

Remark

The de Rham and étale descriptions are in fact compatible.

Shift à la Esnault-Viehweg

(X, D) a "log-scheme".
We fix $B = \mu D$, $\mu \in \mathbb{Z}$.
There exists on $\mathcal{O}_X(B)$ a canonical logarithmic connection

$$d(B) : \mathcal{O}_X(B) \longrightarrow \mathcal{O}_X(B) \otimes \Omega^1_X(\log D)$$

$$x^{-\mu} \longmapsto -\mu x^{-\mu} \frac{dx}{x}$$

where $x = 0$ is a local equation of D.
There is an expression for the residue:

$$\mathrm{res}_D(d(B)) = -\mu \,\mathrm{id}$$

Extension of rank 1 local systems

Corollary

Let $\mathcal{O}_{\mathfrak{X}_n}(\frac{1}{n}\operatorname{div}(f))$ be an arbitrary element of $\operatorname{Pic}_{triv}(\mathfrak{X}_n)[n]$. There exists a unique holomorphic lifting to $\operatorname{Pic}^{dR}_{triv}(\mathfrak{X}_n)[n]$ which is

$$\left(\mathcal{O}_{\mathfrak{X}_n}(\frac{1}{n}\operatorname{div}(f)), d(\frac{1}{n}\operatorname{div}(f)) + \frac{1}{n}\frac{df}{f}\right) .$$

$$\operatorname{Pic}^{dR,reg}_{triv}(U)[n] \xleftarrow{\sim} \operatorname{Pic}^{dR}_{triv}(\mathfrak{X}_n)[n] \xrightarrow{\sim} \operatorname{Pic}_{triv}(\mathfrak{X}_n)[n]$$

Thank you!

Questions

- Description of the kernel of $\pi_{qu}^{dR}(U) \to \pi^{dR}(U)$?
- Homotopy exact sequence for π_{qu}^{dR}?
- Does the Malcev completion commute with base change?
- Analogue of the Chen, Deligne, Fonseca-Matthes theorem for quasi-unipotent connections?