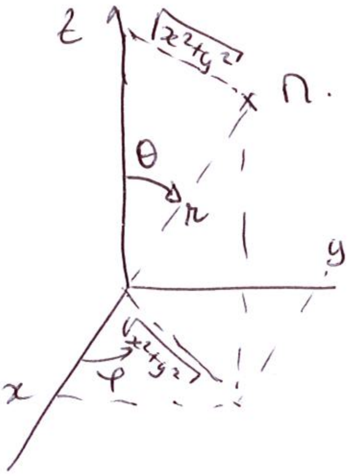
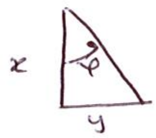
DS1 MPI 2022 - 2023

1. Question de Cours (2,5)

Q1 $r = \sqrt{x^2+y^2+z^2}$ 0,5





$$\theta = \arctan \frac{\sqrt{x^2+y^2}}{z} = \arcsin \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}} = \arccos \frac{z}{\sqrt{x^2+y^2+z^2}}$$ 0,5

$$\varphi = \arctan \frac{y}{x}$$ 0,5

Q2 Les 2 définitions sont telque

(i) $\vec{AG} = \frac{1}{m}\int \vec{AP}\,dm$

(ii) $\int \vec{GP}\,dm = \vec{0}$

(i) ⇒ (ii) On prend A = G ds la déf. (i) ⇒

$$\vec{GG} = \vec{0} = \frac{1}{m}\int \vec{GP}\,dm. \quad \text{CQFD.}$$ 0,5

(ii) ⇒ (i) Relat° de Chasles $\vec{GP} = \vec{GA} + \vec{AP}$

$$\int \vec{GA}\,dm + \int \vec{AP}\,dm = \vec{0}$$ 0,5

$$m\vec{GA} + \int \vec{AP}\,dm = \vec{0} \Rightarrow \vec{AG} = \frac{1}{m}\int \vec{AP}\,dm.$$

2. Exercice de cours (3,5)

Q3 $\{T_c(1/0)\} = \begin{Bmatrix} \omega_1 \vec{z_0} \\ \vec{0} \end{Bmatrix}_O$ 0,5

$\{T_c(2/0)\} = \begin{Bmatrix} \omega_2 \vec{z_0} \\ \vec{0} \end{Bmatrix}_B$. 0,5

Conditiᵒ de non glissement

$$\left.\begin{array}{l} \vec{V}(A \in 2/1) = \vec{0} \\ \text{i.e. } \vec{V}(A \in 2/0) = \vec{V}(A \in 1/0) \end{array}\right\} \text{ 0,5}$$

Or $\vec{V}(A \in 2/0) = \vec{V}(B \in 2/0) + \omega_2 \vec{z_0} \wedge \vec{BA}$

$= \vec{0} + \omega_2 \vec{z_0} \wedge R_2 \vec{y_0} = -\omega_2 R_2 \vec{x_0}$ 0,5

$\vec{V}(A \in 1/0) = \vec{V}(O \in 1/0) + \omega_1 \vec{z_0} \wedge \vec{OA}$

$= \vec{0} + \omega_1 \vec{z_0} \wedge R_1 \vec{y_0} = -\omega_1 R_1 \vec{x_0}$ 0,5

Au final $-\omega_2 R_2 \vec{x_0} = -\omega_1 R_1 \vec{x_0} \Rightarrow \dfrac{\omega_2}{\omega_1} = \dfrac{R_2}{R_1}$ 0,5

+0,5 Si tout nickel.

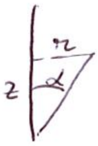
3. Calcul d'une surface (5 points) 3

Q4) $$S = \int_{r=0}^{Re} \int_{\varphi=0}^{2\pi} r \sin\alpha \, dr \, d\varphi$$

$$= \frac{Re^2}{\cancel{2}} \times \sin\alpha \times \cancel{2}\pi \quad \text{avec} \quad \sin\alpha = \frac{Rb}{Rl}$$

$$\boxed{S = \pi Rb Re}$$ 2

Q5) $$V = \int_{z=0}^{h} \int_{\theta=0}^{2\pi} \int_{r=0}^{z \tan\alpha} r \, dr \, d\theta \, dz$$



$$= 2\pi \times \int_{z=0}^{h} \frac{z^2 \tan^2\alpha}{2} dz$$

$\tan\alpha = \frac{r}{z}$

$= \frac{Rb}{h}$

$$= 2\pi \times \frac{h^3}{6} \tan^2\alpha.$$

$$= 2\pi \times \frac{h^3}{6} \times \frac{Rb^2}{h^2}$$

$$\boxed{V = \frac{\pi}{3} Rb^2 h.}$$ 3

Problème (9 points)

Q6 $\left\{T_c(1/0)\right\} = \left\{ \begin{matrix} \dot{\alpha}\,\vec{z_0} \\ \vec{0} \end{matrix} \right\}_{A_0=A_1}$ 0,5

$\left\{T_c(2/1)\right\} = \left\{ \begin{matrix} \dot{\beta}\,\vec{x_1} \\ \vec{0} \end{matrix} \right\}_{A_2}$ 0,5

$\left\{T_c(3/2)\right\} = \left\{ \begin{matrix} \dot{\gamma}\,\vec{x_1} \\ \vec{0} \end{matrix} \right\}_{A_3}$ 0,5

$\left\{T_c(4/3)\right\} = \left\{ \begin{matrix} 0 \\ -\dot{x}\,\vec{z_0} \end{matrix} \right\}_{A_3 \text{ ou } A_4}$ 1

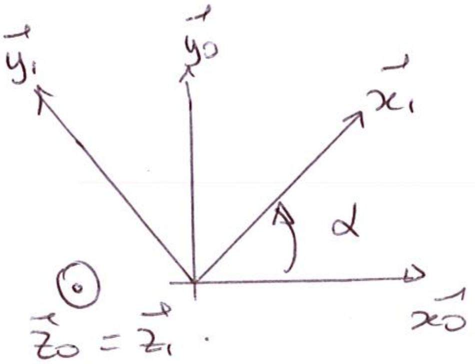
$-\dot{x}\,\vec{z_0} = \dot{x}\,\vec{z_3}$

Q7 $\beta + \gamma = \pi$

donc $\dot{\beta} + \dot{\gamma} = 0$

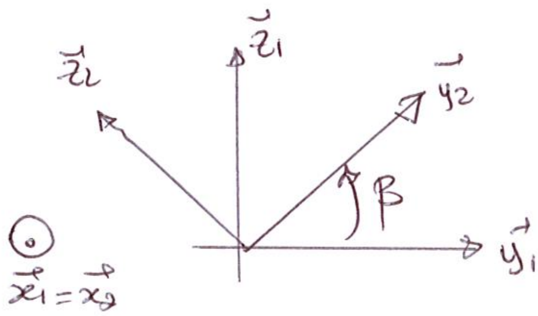
i.e. $\dot{\beta} = -\dot{\gamma}$ 0,5

Q8



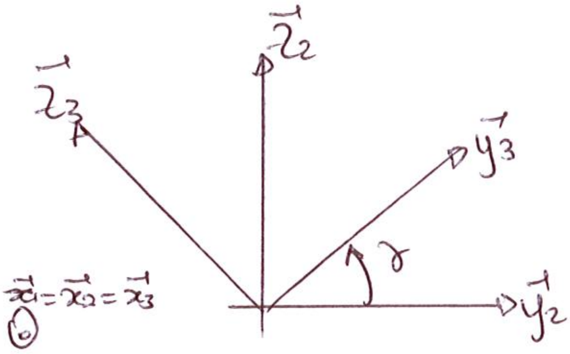
$$\left\{ \begin{aligned} \vec{x_1} &= \cos\alpha\,\vec{x_0} + \sin\alpha\,\vec{y_0} \\ \vec{y_1} &= -\sin\alpha\,\vec{x_0} + \cos\alpha\,\vec{y_0} \end{aligned} \right.$$

0,5



$$\left\{ \begin{aligned} \vec{y_2} &= \cos\beta\,\vec{y_1} + \sin\beta\,\vec{z_1} \\ \vec{z_2} &= -\sin\beta\,\vec{y_1} + \cos\beta\,\vec{z_1} \end{aligned} \right.$$

0,5



$$\begin{cases} \vec{y_3} = \cos\gamma\,\vec{y_2} + \sin\gamma\,\vec{z_2} \\ \vec{z_3} = -\sin\gamma\,\vec{y_2} + \cos\gamma\,\vec{z_2} \end{cases}$$ 0,5

Q9) $\vec{V}(A_4 \in 1/0) = \underbrace{\vec{V}(A_1 \in 1/0)}_{\vec{0}} + \underbrace{\vec{\Omega}(1/0)}_{\dot{\alpha}\vec{z_0}} \wedge \vec{A_1A_4}$

$\vec{A_1A_4} = d_1\vec{z_0} + d_2\vec{z_2} - X(t)\vec{z_0}$

$\dot{\alpha}\vec{z_0} \wedge (d_1\vec{z_0} + d_2\vec{z_2} - X(t)\vec{z_0}) = \dot{\alpha} d_2\, \vec{z_0} \wedge \vec{z_2}$

$= \dot{\alpha} d_2\, \vec{z_1} \wedge \vec{z_2}$

Comme $\vec{z_2} = -\sin\beta\,\vec{y_1} + \cos\beta\,\vec{z_1}$

$\vec{z_1} \wedge \vec{z_2} = \sin\beta\,\vec{x_1} = \sin\beta\,(\cos\alpha\,\vec{x_0} + \sin\alpha\,\vec{y_0})$

Au final

$$\boxed{\vec{V}(A_4 \in 1/0) = \dot{\alpha} d_2 \sin\beta\,(\cos\alpha\,\vec{x_0} + \sin\alpha\,\vec{y_0})}$$ 2

$\vec{V}(A_4 \in 2/1) = \vec{V}(A_2 \in 2/1) + \dot{\beta}\,\vec{x_1} \wedge (d_2\vec{z_2} - X(t)\underset{\vec{z_1}}{\underset{\parallel}{\vec{z_0}}})$

$\vec{x_1} \wedge \vec{z_2} = \vec{x_1} \wedge (-\sin\beta\,\vec{y_1} + \cos\beta\,\vec{z_1})$

$= -\sin\beta\,\vec{z_1} - \cos\beta\,\vec{y_1}$

$\vec{x_1} \wedge \vec{z_1} = -\vec{y_1}$.

Donc :

$$\vec{V}(A_4 \in 2/1) = \dot{\beta}\left[d_2 \left(-\sin\beta \underset{\vec{z_0}}{\underset{\parallel}{\vec{z_1}}} - \cos\beta\, \vec{y_1}\right) + X(t)\, \vec{y_1} \right]$$

$$\boxed{\begin{aligned}\vec{V}(A_4 \in 2/1) = \ & \dot{\beta}\left(X(t) - \cos\beta\, d_2\right)\left(-\sin\alpha\, \vec{x_0} + \cos\alpha\, \vec{y_0}\right) \\ & + \dot{\beta}\, d_2\, \vec{z_0}\end{aligned}} \quad (2)$$

$$\vec{V}(A_4 \in 3/2) = \dot{\gamma}\, \vec{x_1} \wedge -X(t) \underset{\vec{z_1}}{\underset{\parallel}{\vec{z_0}}}$$

$$= \dot{\gamma}\, X(t)\, \vec{y_1}$$

$$\boxed{\vec{V}(A_4 \in 3/2) = \dot{\gamma}\, X(t)\left(-\sin\alpha\, \vec{x_0} + \cos\alpha\, \vec{y_0}\right)} \quad (1)$$

$$\boxed{\vec{V}(A_4 \in 4/3) = -\dot{X}\, \vec{z_0}}$$

+0,5 Si tout ensemble.