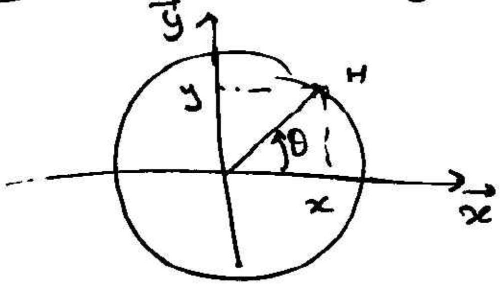
Corrigé DS1 2023/2024

1/

Q1) $r = \sqrt{x^2+y^2}$ (0,5)



$\theta = \arctan \dfrac{y}{x}$ (0,5) $\qquad z = z$.

Q2) $\vec{V}(B\in 1/0) = \vec{V}(A\in 1/0) + \vec{\Omega}(1/0) \wedge \vec{AB}$

$$\frac{d_{R_0}}{dt}\vec{V}(B\in 1/0) = \frac{d_{R_0}}{dt}\vec{V}(A\in 1/0) + \frac{d_{R_0}}{dt}\left(\vec{\Omega}(1/0)\right)\wedge\vec{AB} + \vec{\Omega}(1/0)\wedge\frac{d_{R_0}}{dt}\vec{AB} \quad (0,5)$$

$$\frac{d_{R_0}\vec{AB}}{dt} = \overset{\vec{0}}{\cancel{\frac{d_{R_1}}{dt}\vec{AB}}} + \vec{\Omega}(1/0)\wedge\vec{AB} \quad (1,5)$$

$$\vec{\Gamma}(B\in 1/0) = \vec{\Gamma}(A\in 1/0) + \frac{d_{R_0}}{dt}\vec{\Omega}(1/0)\wedge\vec{AB} + \vec{\Omega}(1/0)\wedge\left(\vec{\Omega}(1/0)\wedge\vec{AB}\right) \quad (0,5)$$

Q3) $$V = \int_{r=R_a}^{R_b}\int_{\theta=0}^{\pi/4}\int_{\varphi=0}^{\pi/2} r^2 \sin\theta \, d\theta \, d\varphi \, dr.$$

$$= \int_{r=R_a}^{R_b} r^2 dr \times \int_{\varphi=0}^{\pi/2} d\varphi \times \int_{\theta=0}^{\pi/4} \sin\theta \, d\theta$$

$$= \left[\frac{R_b^3 - R_a^3}{3}\right] \times \frac{\pi}{2} \times \left[1 - \underset{\text{i.e. } \frac{1}{\sqrt{2}}}{\frac{\sqrt{2}}{2}}\right] \quad (1,5)$$

2/

Q4) $\vec{R}(g \to S) = -\iiint \rho_v g \vec{z}\, dV$

$$= -\rho_v g \vec{z} \times V$$

$$= -\rho_v g \left[\frac{Rb^3 - Ra^3}{3}\right] \times \frac{\pi}{4} \times \left[1 - \frac{\sqrt{2}}{2}\right] \vec{z}.$$

0,5

Q5) $\vec{M}_o(\vec{g} \to S) = -\iiint \vec{OP} \wedge \rho_v g \vec{z}\, dV$

avec $\vec{OP} = r\vec{e_r}$

$$= \iiint \left(\vec{OP} \wedge \rho_v g \vec{z}\right) r^2 \sin\theta\, d\theta\, d\varphi\, dr.$$

$$= \rho_v g \int_{r=Ra}^{Rb} \int_{\theta=0}^{\pi/4} \int_{\varphi=0}^{\pi/2} \left(\vec{e_r} \wedge \vec{z}\right) r^3 \sin\theta\, d\theta\, d\varphi\, dr$$

avec $\vec{e_r} = \sin\theta\cos\varphi\, \vec{x} + \sin\theta\sin\varphi\, \vec{y} + \cos\theta\, \vec{z}$

$\vec{e_r} \wedge \vec{z} = -\sin\theta\cos\varphi\, \vec{y} + \sin\theta\sin\varphi\, \vec{x}.$

$$\int_0^{\pi/2} \cos\varphi\, d\varphi = \left[\sin\varphi\right]_0^{\pi/2} = 1$$

$$\int_0^{\pi/2} \sin\varphi\, d\varphi = \left[-\cos\varphi\right]_0^{\pi/2} = 1 .$$

$$\int_0^{\pi/4} \sin^2\theta\, d\theta = \frac{1}{8}(\pi - 2)$$

$$\int_{Ra}^{Rb} r^3\, dr = \frac{Rb^4 - Ra^4}{4}$$

3/ <u>Au final</u> :

$$\vec{M}_o(\vec{g} \to S) = \rho_v g \left[\frac{Rb^4 - Ra^4}{4}\right] \left[\frac{1}{8}(\pi - 2)\right] \left[-\vec{y} + \vec{x}\right]$$

4/ Problème

Q6) $\{T_c(1/0)\} = \left\{\begin{matrix} \dot{\varphi}\,\vec{z_0} \\ \dot{\lambda}(t)\,\vec{z_0} \end{matrix}\right\}_A$ 1

$\{T_c(2/1)\} = \left\{\begin{matrix} \dot{\theta}\,\vec{x_1} \\ \vec{0} \end{matrix}\right\}_B$ 0,5

$\{T_c(3/2)\} = \left\{\begin{matrix} \dot{\varphi}\,\vec{z_2} \\ \vec{0} \end{matrix}\right\}_C$ 0,5

Q7) $\vec{OA} = \lambda(t)\,\vec{z_0}$

$\vec{V}(A\in 1/0) = \dfrac{d_{R_0}}{dt}\left(\lambda(t)\,\vec{z_0}\right) = \dot{\lambda}\,\vec{z_0}$

$\vec{\Gamma}(A\in 1/0) = \dfrac{d^2_{R_0}}{dt^2}\left(\lambda(t)\,\vec{z_0}\right) = \ddot{\lambda}\,\vec{z_0}$ 0,5

Q8) $\vec{V}(B\in 1/0) = \vec{V}(A\in 1/0) + \vec{\Omega}(1/0) \wedge \vec{AB}$

$= \dot{\lambda}\,\vec{z_0} + \dot{\varphi}\,\underbrace{\vec{z_0}}_{\vec{z_1}} \wedge a\,\vec{y_1}$

$\vec{V}(B\in 1/0) = \dot{\lambda}\,\vec{z_0} - \dot{\varphi}\,\vec{x_1}$ 0,5

$\vec{\Gamma}(B\in 1/0) = \vec{\Gamma}(A\in 1/0) + \dfrac{d_{R_0}\vec{\Omega}(1/0)}{dt} \wedge \vec{AB}$

$+ \vec{\Omega}(1/0) \wedge \left(\vec{\Omega}(1/0) \wedge \vec{AB}\right)$

5/

$$= \ddot{\lambda}\,\vec{z_0} + \ddot{\varphi}\,\vec{z_1} \wedge a\vec{y_1} + \dot{\varphi}\,\vec{z_1} \wedge (\dot{\varphi}\,\vec{z_1} \wedge a\vec{y_1})$$

$$= \ddot{\lambda}\,\vec{z_0} - \ddot{\varphi}\,a\,\vec{x_1} - \dot{\varphi}^2\,a\,\vec{y_1} \quad 1{,}5$$

Q 9) $\vec{V}(D \in 3/2) = \vec{V}(C \in 3/2) + \vec{\Omega}(3/2) \wedge \vec{CD}$

$$= \vec{0} + \dot{\varphi}\,\vec{z_2} \wedge c\,\vec{x_3}$$

$$= \dot{\varphi}\,\vec{z_3} \wedge c\,\vec{x_3} = \dot{\varphi}\,c\,\vec{y_3} \quad 1$$

$\vec{V}(D \in 2/1) = \vec{V}(B \in 2/1) + \vec{\Omega}(2/1) \wedge \vec{BD}$

$$= \vec{0} + \dot{\theta}\,\vec{x_1} \wedge (-b\,\vec{z_2} + c\,\vec{x_3})$$

$$= \dot{\theta}\,\vec{x_2} \wedge (-b\,\vec{z_2} + c\,\vec{x_3})$$

$\vec{x_3} = \cos\varphi\,\vec{x_2} + \sin\varphi\,\vec{y_2}$

$\vec{x_2} \wedge \vec{z_2} = -\vec{y_2}$

$\vec{x_2} \wedge \vec{x_3} = \vec{x_2} \wedge (\cos\varphi\,\vec{x_2} + \sin\varphi\,\vec{y_2})$

$$= -\sin\varphi\,\vec{z_2}$$

Au final $\vec{V}(D \in 2/1) = \dot{\theta}\,b\,\vec{y_2} - c\,\dot{\theta}\sin\varphi\,\vec{z_2}$

1,5

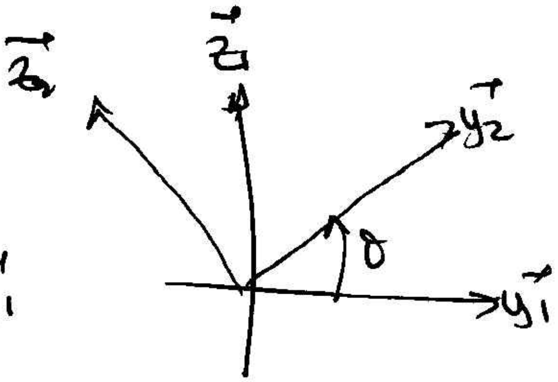
$$\vec V(D\in 1/0) = \vec V(A\in 1/0) + \vec\Omega(1/0)\wedge \vec{AD}$$

$$= \dot\lambda \vec z_0 + \dot\varphi\, \vec z_0 \wedge \left(\cancel{\lambda \vec z_0} + a\vec y_1 - b\vec z_2 + c\vec x_3\right)$$

$$\vec z_0\wedge\vec y_1 = \vec z_1\wedge \vec y_1 = -\vec x_1$$

$$\vec z_0\wedge \vec z_2 = \vec z_1\wedge \vec z_2$$

avec $\vec z_2 = \cos\theta\, \vec z_1 - \sin\theta\, \vec y_1$



donc

$$\vec z_0 \wedge \vec z_2 = \sin\theta\, \vec x_1$$

3

$$\vec z_0\wedge\vec x_3 = \vec z_1\wedge \vec x_3$$

avec $\vec x_3 = \cos\varphi\, \underset{\vec x_1}{\underset{\parallel}{\vec x_2}} + \sin\varphi\, \vec y_2$

$$\vec y_2 = \cos\theta\, \vec y_1 + \sin\theta\, \vec z_1$$

donc

$$\vec z_0\wedge \vec x_3 = \vec z_1 \wedge \left(\cos\varphi\, \vec x_1 + \sin\varphi\left(\cos\theta\, \vec y_1 + \cancel{\sin\theta\, \vec z_1}\right)\right)$$

$$= \cos\varphi\, \vec y_1 - \sin\varphi\cos\theta\, \vec x_1$$

Au final :

$$\vec V(D\in 1/0) = \dot\lambda \vec z_0 \cancel{+} - \dot\varphi a\, \vec x_1 - b\dot\varphi \sin\theta\, \vec x_1 + c\dot\varphi\left(\cos\varphi\, \vec y_1 - \sin\varphi\cos\theta\, \vec x_1\right)$$

Q10) $\vec{V}(D \in 3/0) = \dot{\varphi} c \vec{y_3} + \dot{\theta} b \vec{y_2} - c \dot{\theta} \sin\varphi \vec{z_2}$

$+ \dot{\lambda} \vec{z_0} - \dot{\varphi} a \vec{x_1} - b \dot{\varphi} \sin\theta \vec{x_1}$

$+ c \dot{\varphi} (\cos\varphi \vec{y_1} - \sin\varphi \cos\theta \vec{x_1})$ 1

Q11)

Q12)