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Mostow type rigidity theorems

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Abstract

This is a survey on rigidity theorems that rely on the quasi-conformal geometry of boundaries of hyperbolic spaces.

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1 Introduction

The Mostow celebrated rigidity theorem for rank-one symmetric spaces states that every isomorphism between fundamental groups of compact, negatively curved, locally symmetric manifolds, of dimension at least 3, is induced by an isometry. In his proof, Mostow exploits two major ideas: group actions on boundaries and regularity properties of quasi-conformal homeomorphisms. This set of ideas revealed itself very fruitful. It forms one of the bases of the theory of Gromov hyperbolic groups. It also serves as a motivation to develop quasi-conformal geometry on metric spaces.

The present text attempts to provide a synthetic presentation of the rigidity theorems that rely on the quasi-conformal geometry of boundaries of hyperbolic spaces. Previous surveys on the subject include [74, 34, 13, 102, 80]. The originality of this text lies more in its form. It has two objectives. The first one is to discuss and prove some classical results like Mostow's rigidity in rank one, Ferrand's solution of Lichnérowicz's conjecture, the Sullivan-Tukia and the Pansu

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quasi-isometry rigidity theorems. The second one is to present more briefly some of the numerous recent advances and results based on the quasi-conformal geometry of boundaries.

The paper starts with three preliminary sections. Section 2 is a survey on Gromov hyperbolic spaces, their boundaries and CAT(-1)-spaces. Section 3 concerns rank-one non-compact symmetric spaces. We describe in detail their boundaries in relation to the nilpotent structure. Section 4 presents the geometric analytic tools that serve in the sequel. This includes the convergence property, the Sullivan characterization of Möbius homeomorphisms, the notion of Loewner spaces and the regularity properties of quasi-conformal homeomorphisms. The heart of the paper is formed by Sections 5, 6, 7, 8. Each of them consists of a main part with major statements and sketches of proofs, and of a succinct exposition of several related results. Section 5 is devoted to the Mostow original theorem and its proof. We also state some generalizations, including Besson-Courtois-Gallot's theorem. In Section 6, Ferrand's theorem and Sullivan-Tukia's theorem are stated and proved by using the zoom method. We also present Tukia's proof of Mostow's theorem (for real hyperbolic spaces). Section 7 focuses on rigidity of quasi-isometries. A detailed sketch of the proof of Pansu's theorem is given. Right-angled Fuchsian buildings are also discussed. Finally, Section 8 presents some recent developments and perspectives, including Cannon's conjecture and the combinatorial Loewner property.

Clearly, several results presented in this paper would deserve a more detailed exposition. Moreover only very few aspects of rigidity are treated. In particular, infinitesimal rigidity, higher rank rigidity, superrigidity, harmonic maps and bounded cohomology, do not appear at all. This is due to my own limitations.

It is my pleasure to refer to P. Haïssinsky's survey [78], in this volume, for a complementary viewpoint of some of the subjects discussed here.

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Notation and convention

In a metric space X, we denote by $N_R(Y)$ the R-neighborhood of a subset $Y \subset X$, i.e. $N_R(Y) := \{x \in X \,|\, \mathrm{dist}(x, Y) \leq R\}$. The Hausdorff dimension of X is denoted by $\mathrm{Hausdim}(X)$. The homeomorphism and isometry groups of X are denoted by $\mathrm{Homeo}(X)$ and $\mathrm{Isom}(X)$ respectively. A map $f : X \to Y$ between metric spaces is a *homothety* if it is an isometry after possibly rescaling the metric of X or Y by a multiplicative positive constant.

Two real-valued functions f, g defined on a space X are said to be *comparable*, and we write $f \asymp g$, if there exists a constant $C > 0$ such that $C^{-1} f \leq g \leq C f$. We write $f \lesssim g$ if there is a constant $C > 0$ such that $f \leq C g$.

2 Gromov hyperbolic spaces

Hyperbolic spaces and hyperbolic groups were defined by Gromov [73] in the middle of the eighties. We give a brief survey on hyperbolic spaces and their boundaries, see e.g. [73, 56, 72, 37, 95, 40, 62] for more details.

2.1 First definitions and properties

Let (X, d) be a metric space. A *geodesic segment* in X is an isometrically embedded closed interval. A geodesic segment with endpoints $x, y \in X$ is denoted by $[x, y]$. Similarly are defined the geodesic rays and the bi-infinite geodesics in X. The space X is *geodesic* if every pair of points in X can be joined by a geodesic segment.

Definition 2.1. A geodesic metric space X is *hyperbolic* if there is a constant $\delta \geq 0$ such that for every points $x, y, z \in X$ and every geodesic segments $[x, y], [y, z], [z, x]$, one has $[x, y] \subset N_\delta([yz] \cup [zx])$.

A companion definition is :

Definition 2.2.

1. A map $F : X_1 \to X_2$ between metric spaces is a *quasi-isometric embedding*, if there are constants $\lambda \geq 1$ and $C \geq 0$ such that for every $x, y \in X_1$

$$\lambda^{-1} d(x, y) - C \leq d(F(x), F(y)) \leq \lambda d(x, y) + C.$$

   In the particular case where X_1 is an interval in $\mathbb{R}$, the map F is called a (λ, C)-*quasi-geodesic of* X_2.

2. The map F is a *quasi-isometry* if it is a quasi-isometric embedding and if there is a constant $R \geq 0$ such that $N_R(F(X_1)) = X_2$.

Examples are provided by the following situation. A group action is called *geometric* if it is an isometric, properly discontinuous and cocompact action. The *Cayley graph* of a finitely generated group (Γ, S), is the graph whose vertices are the elements of Γ, and whose edges are the pairs $\{g, gs\}$, with $g \in \Gamma$ and $s \in S \cup S^{-1}$. It is endowed with the path metric obtained by identifying every edge with the unit interval.

Proposition 2.3 (Svarc-Milnor)**.** *Suppose a group Γ acts geometrically on a geodesic metric space X. Let S be a finite generating set of Γ. Then for every $O \in X$, the orbit map $\Gamma \to X$, $g \mapsto g \cdot O$, extends to a quasi-isometry of the Cayley graph of (Γ, S) to X.*

The following result is called the Morse Lemma in [73].

Theorem 2.4. *Suppose X is a δ-hyperbolic space. There is a constant $R = R(\lambda, C, \delta)$, such that for every (λ, C)-quasi-geodesic $\gamma : I \to X$ there exists a geodesic $c \subset X$ with $\gamma(I) \subset N_R(c)$.*

As a consequence one obtains

Corollary 2.5. *Suppose $F : X_1 \to X_2$ is a quasi-isometry between geodesic metric spaces. Then X_1 is hyperbolic if and only if X_2 is.*

Since the Cayley graphs of Γ, for the various choices of generators, are pairwise quasi-isometric, the following definition does not depend on the finite generating set S.

Definition 2.6. A finitely generated group Γ is called *hyperbolic* if the Cayley graph of (Γ, S) is hyperbolic.

2.2 Boundary at infinity

The boundary at infinity of a hyperbolic space, equipped with a visual metric, is the main protagonist of the paper. The boundary at infinity is defined in Definition 2.7, the visual metrics are defined in the statement of Theorem 2.9.

Definition 2.7. Let X be a geodesic metric space. Geodesic rays $r_1, r_2 : [0{+}\infty) \to X$ are *asymptotic* if

$$\sup_{t \in [0,+\infty)} d(r_1(t), r_2(t)) < \infty.$$

The *boundary at infinity of X* is

$$\partial X := \{r : [0 + \infty) \to X \text{ geodesic ray}\}/\sim$$

where $r_1 \sim r_2$ when they are asymptotic. It is endowed with the topology induced by the topology of uniform convergence on the compact subsets of $[0, +\infty)$. The group $\mathrm{Isom}(X)$ acts on ∂X by homeomorphisms.

Recall that a metric space is *proper* if its closed balls are compact. Using Ascoli's theorem one has :

Proposition 2.8. *Suppose that X is a proper hyperbolic space.*

1. *Given an origin $O \in X$, every geodesic ray of X is asymptotic to a geodesic ray starting at O.*

2. *For every pair of non-asymptotic geodesic rays r_1, r_2, there is a geodesic $\gamma : \mathbb{R} \to X$ that is asymptotic to r_1 for $t \le 0$ and to r_2 for $t \ge 0$.*

3. *The boundary ∂X is compact.*

In the sequel, we denote by (z, w) every geodesic which is asymptotic to a pair of distinct points $z, w \in \partial X$. Note that (z, w) is not unique in general. But for two of such geodesics γ, η one has $\gamma \subset N_{10\delta}(\eta)$, thanks to δ-hyperbolicity.

Theorem 2.9 (Gromov)**.** *Suppose that X is a proper δ-hyperbolic space, and let $O \in X$ be an origin. Then:*

1. *There is a constant $a_0 > 1$ which depends only on δ, such that for every $a \in (1, a_0)$, there exists a metric d on ∂X with the following property: for every $z, w \in \partial X$, one has*

$$d(z,w) \asymp a^{-L}, \text{ where } L = \operatorname{dist}(O,(z,w)).$$

   A metric d on ∂X which satisfies the above property is called a visual metric.

2. *$X \cup \partial X$ is a natural metric compactification of X. More precisely, there is a metric d on $X \cup \partial X$ that enjoys the following property: for every $x, y \in X \cup \partial X$, one has*

$$d(x,y) \asymp a^{-L} \min\{1, |x-y|\}, \text{ where } L = \operatorname{dist}(O,(x,y)).$$

Observe that two visual metrics d, δ on ∂X, with parameters a and b respectively (and with possibly different origins) satisfy $\delta \asymp d^{\frac{\log a}{\log b}}$.

Sketch of proof. We sketch a proof of Part 1, and refer to [16] for Part 2. Suppose first that X is a tree. Then for every distinct $z, w \in \partial X$ the geodesic (z,w) is unique, and for every $a > 1$ the fonction

$$d(z,w) := a^{-\operatorname{dist}(O,(z,w))}$$

satisfies the ultrametric inequality

$$d(z_1,z_3) \le \max\{d(z_1,z_2), d(z_2,z_3)\}.$$

For a general δ-hyperbolic space X, the fonction

$$\varphi : (z,w) \in (\partial X)^2 \mapsto e^{-\operatorname{dist}(O,(z,w))}$$

satisfies

$$\varphi(z_1,z_3) \le e^{100\delta} \max\{\varphi(z_1,z_2), \varphi(z_2,z_3)\}.$$

Therefore part 1 of the theorem is a consequence of the following classical lemma (see e.g. [89] for a proof). □

Lemma 2.10. *Let Z be a set and $\varphi : Z \times Z \to [0, +\infty)$ be a* quasi-metric, *i.e. a function that satisfies the following properties:*

1. *$\varphi(z,w) = 0$ if and only if $z = w$.*
2. *$\varphi(w,z) = \varphi(z,w)$ for every $w, z \in Z$.*
3. *There is a $K > 0$ such that for every $z_1, z_2, z_3 \in Z$ one has*

$$\varphi(z_1,z_3) \le K \max\{\varphi(z_1,z_2), \varphi(z_2,z_3)\}.$$

Then there exists $\alpha_0 \in (0,1)$ which depends only on K, such that for every $\alpha \in (0, \alpha_0)$ there exists a metric d on Z with $d \asymp \varphi^\alpha$.

2.3 Cross-ratio and boundary extensions of quasi-isometries

In classical hyperbolic geometry, isometries extend to Möbius homeomorphisms of the boundary. This property generalizes, in some sense, to quasi-isometries of Gromov hyperbolic spaces.

Definition 2.11 (Väisälä [139])**.** Let (Z,d) be a metric space. The *cross-ratio* of four pairwise distinct points $z_1, z_2, z_3, z_4 \in Z$ is

$$[z_1, z_2, z_3, z_4] = \frac{d(z_1, z_3)d(z_2, z_4)}{d(z_1, z_4)d(z_2, z_3)}.$$

A map $f : Z_1 \to Z_2$ between metric spaces is *Möbius* if it preserves the cross-ratio. It is *quasi-Möbius* if there is a homeomorphism $\eta : [0, +\infty) \to [0, +\infty)$ such that for every four pairwise distinct points $z_1, z_2, z_3, z_4 \in Z_1$, one has

$$[f(z_1), f(z_2), f(z_3), f(z_4)] \leq \eta([z_1, z_2, z_3, z_4]).$$

The group of Möbius homeomorphisms of Z is denoted by $\mathrm{Moeb}(Z)$.

Note that switching z_1 and z_2 in the last inequality leads to the opposite inequality (with a different fonction η). Therefore inverses and compositions of quasi-Möbius homeomorphisms are quasi-Möbius as well.

Theorem 2.12 (Efremovich-Tihomirova [65])**.** *Let X_1 and X_2 be proper δ-hyperbolic spaces. Equip ∂X_1 and ∂X_2 with visual metrics. Then every quasi-isometry $F : X_1 \to X_2$ extends to a quasi-Möbius homeomorphism $f : \partial X_1 \to \partial X_2$. Moreover the distortion fonction η depends only on δ, the constants of the quasi-isometry F and the constants of the visual metrics.*

In particular $\mathrm{Isom}(X)$ *acts on ∂X by* uniform *quasi-Möbius homeomorphisms, i.e. with the same distortion function.*

Sketch of proof. Let O be an origin in X_1. In order to extend the quasi-isometry F to a map $f : \partial X_1 \to \partial X_2$, consider $z \in \partial X_1$ and a geodesic ray $[O, z)$. Its image by F is a quasi-geodesic ray. By the Morse lemma it lies within bounded distance from a geodesic ray $[F(O), w)$, with $w \in \partial X_2$. Define $f(z) = w$. It is easy to see that f is bijective.

To see that f is quasi-Möbius, one observes that there is a constant $C \geq 0$ which depends only on δ and the constants of the visual metric, such that for every pairwise distinct points $z_1, z_2, z_3, z_4 \in \partial X$, one has

$$|\,\mathrm{dist}\,((z_1, z_4), (z_2, z_3)) - \max\left\{0, \frac{\log[z_1, z_2, z_3, z_4]}{\log a}\right\}| \leq C.$$

These inequalities follow easily from the special case where X is a tree (in this case $C = 0$).

By combining the above inequalities with the Morse lemma one obtains that f is quasi-Möbius. □

We note that the converse of Theorem 2.12 is valid for *non-degenerate* hyperbolic spaces. A hyperbolic space X is *non-degenerate* if there is a constant $C \geq 0$ such that every $x \in X$ lies within distance at most C from all three sides of some ideal geodesic triangle Δ_x. According to [118, 25], every quasi-Möbius homeomorphism $f : \partial X_1 \to \partial X_2$, between boundaries of proper non-degenerate hyperbolic spaces, extends to a quasi-isometry $F : X_1 \to X_2$. Therefore the boundary is a full quasi-isometric invariant of the proper non-degenerate hyperbolic spaces.

2.4 Some dynamical properties

We give some properties of ∂X for the hyperbolic spaces X that admit a geometric action. The definition below of approximately self-similar spaces appears in [102].

Definition 2.13.

1. A metric space Z is *approximately self-similar* if there is a constant $L_0 \geq 1$ such that if $B(z, r) \subset Z$ is an open ball of radius $0 < r \leq \mathrm{diam}(Z)$, then there is an open subset $U \subset Z$ which is L_0-bi-Lipschitz homeomorphic to the rescaled ball $(B(z, r), \frac{1}{r}d)$.

2. A metric space Z is *Ahlfors Q-regular* (for some $Q \in (0, +\infty)$) if there is a measure ν on Z such that for every ball $B \subset Z$ of radius $0 < r \leq \mathrm{diam}(Z)$ one has $\nu(B) \asymp r^Q$.

Note that if Z is Ahlfors Q-regular, then Q is its Hausdorff dimension. Moreover the measure ν and the Q-Hausdorff measure are absolutely continuous with respect to each other, and their Radon-Nikodym derivatives are bounded. Observe also that an approximately self-similar space is Ahlfors Q-regular as soon as its Q-Hausdorff measure is finite and non-zero.

Theorem 2.14. *Let X be a proper hyperbolic space such that $|\partial X| \geq 3$, and suppose that a group Γ acts on X geometrically. Let d be a visual metric on ∂X, let a be its parameter, and let $O \in X$ be an origin. Then :*

1. *$(\partial X, d)$ is approximately self-similar, the partial bi-Lipschitz maps being restrictions of elements of Γ.*

2. *$(\partial X, d)$ is Ahlfors Q-regular with*

$$Q = \limsup_{R \to +\infty} \frac{\log |(\Gamma \cdot O) \cap B(O, R)|}{R \log a}.$$

3. *Let $\mathcal{H}$ be the Q-Hausdorff measure of $(\partial X, d)$. The diagonal action of Γ on $((\partial X)^2, \mathcal{H} \times \mathcal{H})$ is ergodic.*

Statement 2 is due to M. Coornaert [53, 55]. Statement 3 is proved in [2]. They generalize previous results of Patterson and Sullivan to hyperbolic spaces and groups. See e.g. [30] for a proof of 1.

Let again X be a proper hyperbolic space X, suppose that $|\partial X| \geq 3$, and consider the topological space

$$\partial^3 X := \{(z_1, z_2, z_3) \in (\partial X)^3 \mid z_i \neq z_j \text{ for } i \neq j\},$$

with the diagonal action of $\mathrm{Isom}(X)$ by homeomorphisms.

Proposition 2.15. *Suppose that a group Γ acts on X geometrically. Then Γ acts properly discontinuously and cocompactly on $\partial^3 X$.*

Conversely, suppose that X is a proper non-degenerate hyperbolic space (as defined in the last subsection). Then every group $\Gamma \subset \mathrm{Isom}(X)$ that acts properly discontinuously and cocompactly on $\partial^3 X$, acts geometrically on X.

Sketch of proof. To every $(z_1, z_2, z_3) \in \partial^3 X$, one associates an *ideal triangle* in X, i.e. the union of three bi-infinite geodesics (z_i, z_j), $i \neq j$, denoted by $\Delta(z_1, z_2, z_3)$. Then, by hyperbolicity, one constructs a *center* of the triangle $\Delta(z_1, z_2, z_3)$, i.e. a point $x \in X$ such that the maximal of its distances to the three sides is minimal. This defines a map $p : \partial^3 X \to X$, which is essentially proper and $\mathrm{Isom}(X)$-equivariant (see [53] for more details). □

2.5 CAT(−1)-spaces

A vast class of hyperbolic spaces is formed by the so-called CAT(−1)-spaces. We recall their definition and review some properties of their boundary.

Definition 2.16. Let $\mathbb{H}^2_{\mathbb{R}}$ be the real hyperbolic plane of curvature -1. Let X be a geodesic metric space, and let $\Delta \subset X$ be a geodesic triangle. A *comparison triangle* of Δ is a geodesic triangle $\overline{\Delta} \subset \mathbb{H}^2_{\mathbb{R}}$ whose edge lengths are the same as in Δ. Let $s : \overline{\Delta} \to \Delta$ be the natural map. Then X is a *CAT(−1)-space* if for every geodesic triangle $\Delta \subset X$, the map $s : \overline{\Delta} \to \Delta$ is 1-Lipschitz.

Thanks to the Alexandrov comparison theorem, any simply connected complete Riemannian manifold of sectional curvatures less than or equal to -1 is a CAT(−1)-space. Other examples include the negatively curved simply connected polyhedron complexes [3, 37], and among them Tits buildings associated to hyperbolic Coxeter groups [61].

A nice feature about CAT(−1)-spaces is the existence of canonical explicit visual metrics on their boundary and the relations between the groups $\mathrm{Moeb}(\partial X)$ and $\mathrm{Isom}(X)$. To present them, we start with some definitions.

Let X be a proper CAT(−1)-space and let $O \in X$ be an origin. Denote by $|x-y|$ the distance between $x, y \in X$. Since X is hyperbolic, its boundary appears as the frontier of the compactification $X \cup \partial X$ (Theorem 2.9.2). By using Definition 2.16 and some standard geometric properties of $\mathbb{H}^2_{\mathbb{R}}$, one sees that the function

$$(x, y) \in X^2 \mapsto \frac{1}{2}(|O-x| + |O-y| - |x-y|) \in \mathbb{R},$$

extends by continuity to a function on $\partial^2 X := \{(z, w) \in (\partial X)^2 \mid z \neq w\}$, denoted by $(z|w)$ and called the *Gromov product* of z, w. Similarly, the function

$$(x_1, x_2, x_3, x_4) \in X^4 \mapsto \frac{1}{2}(|x_1 - x_3| + |x_2 - x_4| - |x_1 - x_4| - |x_2 - x_3|) \in \mathbb{R},$$

extends by continuity to a function on

$$\partial^4 X := \{(z_1, z_2, z_3, z_4) \in (\partial X)^4 \mid z_i \neq z_j \text{ for } i \neq j\},$$

denoted by $(z_1|z_2|z_3|z_4)$.

Proposition 2.17 ([26]). *Let X be a proper CAT(-1)-space. Then*

$$d_{\mathrm{CAT}}(z, w) = e^{-(z|w)}$$

defines a visual metric of parameter e on ∂X. The associated cross-ratio (see Definition 2.11) satisfies $[z_1, z_2, z_3, z_4] = \exp(z_1|z_2|z_3|z_4)$.

Example 2.18. Let $\mathbb{H}^n_{\mathbb{R}}$ be the n-dimensional real hyperbolic space of constant curvature -1. The metric d_{CAT} on its boundary can be describe as follows. In the ball model $\mathbb{B}^n \subset \mathbb{R}^n$ of $\mathbb{H}^n_{\mathbb{R}}$, choose the origin O to be the center of $\mathbb{B}^n$. Then the metric d_{CAT} on $\partial\mathbb{H}^n_{\mathbb{R}}$ is half of the chordal distance on the boundary sphere of $\mathbb{B}^n$.

A description of d_{CAT} for the other rank-one symmetric spaces is given in Proposition 3.7.

As a consequence of Proposition 2.17, every isometry of X acts on ∂X as a Möbius homeomorphism with respect to the cross-ratio associated to the visual metric d_{CAT}. A basic inverse problem is the reconstruction of X from the boundary cross-ratio. The following result is an example of such a reconstruction.

Theorem 2.19 ([27]). *Let S be a rank-one non-compact symmetric space, and let X be a CAT(-1)-space. Assume the Riemannian metric of S is normalized so that the maximum of the sectional curvatures is equal to -1. Then every embedding $\partial S \to \partial X$, which preserves the cross-ratio, extends to a totally geodesic isometric embedding $S \to X$. In particular* $\mathrm{Isom}(S) = \mathrm{Moeb}(\partial S)$.

The idea of proof is the following. Every geodesic in S or X is determined by its two end-points in the boundary. The map $\partial S \to \partial X$ induces an injection from the set of geodesics in S to the set of geodesics in X. By using the metric d_{CAT} one can show that concurrent geodesics are preserved.

3 Rank-one symmetric spaces

Examples of CAT(-1)-spaces include the rank-one non-compact symmetric spaces. These are the hyperbolic spaces $\mathbb{H}^n_{\mathbb{R}}$, $\mathbb{H}^n_{\mathbb{C}}$, $\mathbb{H}^n_{\mathbb{Q}}$, $\mathbb{H}^2_{\mathbb{O}}$, where $n \geq 2$ and where $\mathbb{Q}$ and $\mathbb{O}$ denote the quaternions and the octonions, respectively. In the sequel $\mathbb{K}$ denotes one of the fields $\mathbb{R}, \mathbb{C}, \mathbb{Q}$, and we set $k := \dim_{\mathbb{R}} \mathbb{K}$.

This section gives a brief presentation the rank-one (non-compact) symmetric spaces and discusses some aspects of their boundaries. We start with a definition of the hyperboloid and parabolic models of $\mathbb{H}^n_{\mathbb{K}}$. Then we describe the nilpotent structure on $\partial\mathbb{H}^n_{\mathbb{K}} \setminus \{\infty\}$, and we compare the visual metric d_{CAT} with the Carnot-Carathéodory metric on $\partial\mathbb{H}^n_{\mathbb{K}}$. Althrough the material is standard – see [113, 116, 74, 37, 107] – we provide a detailed and self-contained exposition.

Because $\mathbb{O}$ is non-associative, the space $\mathbb{H}^2_{\mathbb{O}}$ requires a different treatment. For simplicity we consider only $\mathbb{H}^n_{\mathbb{R}}$, $\mathbb{H}^n_{\mathbb{C}}$, $\mathbb{H}^n_{\mathbb{Q}}$, and we refer to [113, 1] for the octonionic case.

3.1 Hyperboloid model

Let $n \geq 1$. We consider $\mathbb{K}^{n+1}$ as a right $\mathbb{K}$-module ([1]). Let $B : \mathbb{K}^{n+1} \times \mathbb{K}^{n+1} \to \mathbb{K}$ be the form

$$B(x,y) = \overline{x}_1 y_1 + ... + \overline{x}_n y_n - \overline{x}_{n+1} y_{n+1},$$

where $x \mapsto \overline{x}$ denotes the standard involution of $\mathbb{K}$. The associated quadratic form is

$$q(x) := B(x,x) = |x_1|^2 + ... + |x_n|^2 - |x_{n+1}|^2.$$

Let $H = \{x \in \mathbb{K}^{n+1} \ ; \ q(x) = -1\}$. The group $U := \{a \in \mathbb{K} \ ; \ |a| = 1\}$ acts on H on the right, and preserves q. Moreover, when restricted to every orthogonal tangent space of the U-orbits, q is positive definite.

The *hyperboloid model* of $\mathbb{H}^n_{\mathbb{K}}$ is the manifold H/U equipped with the Riemannian metric induced by q.

Observe that H/U lies in the projective space $\mathbb{P}^n_{\mathbb{K}}$. Indeed, it is the image of H by the projection map $\pi : \mathbb{K}^{n+1} \setminus \{0\} \to \mathbb{P}^n_{\mathbb{K}}$. The description of the Riemannian metric is the following.

Proposition 3.1. *Denote by g the Riemannian metric on $\mathbb{H}^n_{\mathbb{K}}$. Then for $x \in H$ and $v \in T_x H$ one has: $g_{\pi(x)}(\pi_*(v), \pi_*(v)) = q(v) + |B(x,v)|^2$.*

Proof. The tangent space to xU at x is $x\Im\mathbb{K}$. Its orthogonal in $T_x H$ is the set of $v \in \mathbb{K}^{n+1}$ such that $B(x,v) = 0$. Since $x \in H$ and $v \in T_x H$, one has $q(x) = -1$ and $B(x,v) \in \Im\mathbb{K}$. Thus the component of v along the orthogonal of xU is $w = v + xB(x,v)$. We obtain

$$g_{\pi(x)}(\pi_*(v), \pi_*(v)) = q(w) = B(w,w) = q(v) + |B(x,v)|^2,$$

as expected. □

In this model $\partial\mathbb{H}^n_{\mathbb{K}}$ is identified with the submanifold $\{q(x) = 0\}/\mathbb{K}^*$ of $\mathbb{P}^n_{\mathbb{K}}$. The equation $B(x,v) = 0$ defines a codimension $k-1$ distribution in the tangent space of $\{q(x) = 0\}$. Its leaves contain the tangent spaces of the $\mathbb{K}^*$-orbits. Therefore it induces in the quotient, a codimension $k-1$ distribution on $\partial\mathbb{H}^n_{\mathbb{K}}$ that we denote by T. We remark that T is the maximal $\mathbb{K}$-subbundle of the tangent bundle of $\partial\mathbb{H}^n_{\mathbb{K}}$.

The groups $PO(n,1)$, $PU(n,1)$, $PSp(n,1)$ act by isometries on $\mathbb{H}^n_{\mathbb{R}}$, $\mathbb{H}^n_{\mathbb{C}}$, $\mathbb{H}^n_{\mathbb{Q}}$ respectively, and they preserve T on the boundary. They are in fact equal to the whole isometry groups of the corresponding symmetric spaces.

[1] This convention is only relevant when $\mathbb{K}$ is non abelian. It allows $M(n+1,\mathbb{K})$ to act linearly on $\mathbb{K}^{n+1}$ by multiplication on the left.

3.2 Parabolic model

We now describe a model of $\mathbb{H}^n_{\mathbb{K}}$ where the point $[0, ..., 0, -1, 1] \in \partial\mathbb{H}^n_{\mathbb{K}}$ lies at infinity. It is called the *parabolic model* or *Siegel domain*.

We consider the $\mathbb{K}$-hyperplane P in $\mathbb{K}^{n+1}$ defined by $x_n + x_{n+1} = 1$. It contains the point $(0, ..., 0, 1)$ and is parallel to the vector $(0, ..., 0, -1, 1)$. We introduce the following coordinates on P and $\mathbb{P}^n_{\mathbb{K}}$:

$$(x', x_n) \in \mathbb{K}^n \mapsto (x', \frac{1}{2} - x_n, \frac{1}{2} + x_n) \in P \mapsto [x', \frac{1}{2} - x_n, \frac{1}{2} + x_n] \in \mathbb{P}^n_{\mathbb{K}}.$$

In these coordinates, the quadratic form q restricted to P is expressed as follows. For $(x', x_n) \in \mathbb{K}^n$:

$$q(x', x_n) = \|x'\|^2 - 2\Re x_n,$$

with $\|x'\|^2 := |x'_1|^2 + ... + |x'_{n-1}|^2$. Therefore, by setting $\infty := [0, ..., 0, -1, 1]$, one gets the following expressions

$$\mathbb{H}^n_{\mathbb{K}} = \{(x', x_n) \in \mathbb{K}^n \ ; \ \|x'\|^2 - 2\Re x_n < 0\},$$

$$\partial\mathbb{H}^n_{\mathbb{K}} \setminus \{\infty\} = \{(x', x_n) \in \mathbb{K}^n \ ; \ \|x'\|^2 - 2\Re x_n = 0\}.$$

A computation based on Proposition 3.1 shows that the Riemannian metric is expressed as follows. For $x = (x', x_n) \in \mathbb{H}^n_{\mathbb{K}}$ and $v = (v', v_n) \in \mathbb{K}^n$:

$$g_x(v, v) = \frac{\|v'\|^2}{2\Re x_n - \|x'\|^2} + \Big(\frac{|\{x', v'\} - v_n|}{2\Re x_n - \|x'\|^2}\Big)^2, \tag{3.2}$$

where $\{\cdot, \cdot\}$ denotes the form $\{x', y'\} = \overline{x}'_1 y'_1 + ... + \overline{x}'_{n-1} y'_{n-1}$. Another computation shows that the distribution T on $\partial\mathbb{H}^n_{\mathbb{K}} \setminus \{\infty\}$ is simply written as

$$\{x', v'\} = v_n. \tag{3.3}$$

3.3 Nilpotent structure on $\partial\mathbb{H}^n_{\mathbb{K}} \setminus \{\infty\}$

We use the coordinate system of the parabolic model. Let N be the Lie group $\mathbb{K}^{n-1} \times \Im\mathbb{K}$ with the following multiplication law

$$(z', z_n) \cdot (w', w_n) = (z' + w', \ z_n + w_n + \Im\{z', w'\}).$$

Its Lie algebra is written as $\mathfrak{n} = \mathbb{K}^{n-1} \oplus \Im\mathbb{K}$, with $\Im\mathbb{K}$ central, and for every $z', w' \in \mathbb{K}^{n-1}$:

$$[z', w'] = \Im\{z', w'\}. \tag{3.4}$$

Therefore when $\mathbb{K} \neq \mathbb{R}$ the group N is two-step nilpotent and otherwise it is abelian. It acts on $\mathbb{H}^n_{\mathbb{K}}$ as follows. For $(z', z_n) \in N$ and $(x', x_n) \in \mathbb{H}^n_{\mathbb{K}}$,

$$(z', z_n) \cdot (x', x_n) = (x' + z', \ x_n + z_n + \{z', x'\} + \frac{1}{2}\|z'\|^2).$$

By using the expression (3.2), one checks that it is an isometric action. Moreover N acts simply transitively on $\partial \mathbb{H}^n_{\mathbb{K}} \setminus \{\infty\}$. Therefore we get an identification

$$N \to \partial \mathbb{H}^n_{\mathbb{K}} \setminus \{\infty\}, \quad (z', z_n) \mapsto (z', z_n) \cdot 0,$$

In these coordinates, the distribution T is equal to the left invariant distribution on N generated by the subspace $\mathbb{K}^{n-1} \subset \mathfrak{n}$ (this follows from (3.3)). The Lie bracket expression (3.4) implies that

$$T \oplus [T, T] = T(\partial \mathbb{H}^n_{\mathbb{K}}). \tag{3.5}$$

Let $(\delta_t)_{t\in\mathbb{R}}$ be the 1-parameter subgroup of $\mathrm{Isom}(\mathbb{H}^n_{\mathbb{K}})$ defined by

$$\delta_t(x', x_n) = (e^t x', e^{2t} x_n).$$

Its elements are hyperbolic isometries with axis $(0, \infty)$. It normalizes N. More precisely we have for every $(z', z_n) \in N$:

$$\delta_t \circ (z', z_n) \circ \delta_t^{-1} = (e^t z', e^{2t} z_n).$$

The corresponding Lie algebra automorphisms are given by $e^{t\alpha}$, where α is the derivation of $\mathfrak{n}$, equal to id on $\mathbb{K}^{n-1}$ and to 2id on $\Im\mathbb{K}$.

The map δ_t preserves the distribution T. It acts as a similarity of ratio e^t on the leaves of T equipped with any N-invariant Riemannian metric. Indeed the distribution T is N-invariant, δ_t normalizes N, and its differential at the fixed point 0 is a homothety of ratio e^t when restricted to the leaf of T at 0.

The above properties show that N is a Carnot group with associated Carnot homotheties $(\delta_t)_{t\in\mathbb{R}}$. The required definitions are the following.

Definition 3.6. A *Carnot group* is a pair $(N, \mathfrak{v})$, where N is a simply connected Lie group, and $\mathfrak{v}$ is a linear subspace of the Lie algebra $\mathfrak{n}$ such that

$$\mathfrak{n} = \bigoplus_{r\geq 1} \mathfrak{v}^r,$$

with $\mathfrak{v}^1 := \mathfrak{v}$ and $\mathfrak{v}^{r+1} := [\mathfrak{v}, \mathfrak{v}^r]$. The linear subspace $\mathfrak{v}$ is called the *horizontal space*. The linear map α, whose restriction to every $\mathfrak{v}^r$ is $r \cdot \mathrm{id}$, is a derivation of $\mathfrak{n}$. The associated *Carnot homotheties* are the automorphisms of N induced by $e^{t\alpha} \in \mathrm{Aut}(\mathfrak{n})$.

3.4 Carnot-Carathéodory and visual metrics

The sectional curvature of $\mathbb{H}^n_{\mathbb{K}}$ is constant equal to -1 when $\mathbb{K} = \mathbb{R}$ and lies in $[-4, -1]$ otherwise. Thus $\mathbb{H}^n_{\mathbb{K}}$ is a CAT(-1)-space and its boundary carries the visual metrics d_{CAT}, see Proposition 2.17.

On the other hand, the relation (3.5) shows that the distribution T generates the tangent space of $\partial \mathbb{H}^n_{\mathbb{K}}$. Thus, according to Chow's theorem, any two boundary points can be joined by a smooth *horizontal* curve, i.e. whose tangent vectors lie in T. Choose an origin $O \in \mathbb{H}^n_{\mathbb{K}}$, and equip T with a Riemannian metric that is invariant under the isotropy group of O. The associated *Carnot-Carathéodory metric* d_{CC} is defined as follows. For every $z, w \in \partial\mathbb{H}^n_{\mathbb{K}}$, the distance $d_{\mathrm{CC}}(z, w)$ is the infimum of the lengths of the piecewise smooth horizontal curves joining z to w.

Proposition 3.7. *The metrics d_{CAT} and d_{CC} on $\partial\mathbb{H}^n_{\mathbb{K}}$ are Lipschitz equivalent. Their Hausdorff dimension is $kn + k - 2$.*

Proof. Observe that d_{CAT} and d_{CC} are both invariant under the isotropy group of O. Since this group acts transitively on $\partial\mathbb{H}^n_{\mathbb{K}}$, it is enough to prove the existence of a point $z_0 \in \partial\mathbb{H}^n_{\mathbb{K}}$, such that $d_{\mathrm{CAT}}(z_0, \cdot) \asymp d_{\mathrm{CC}}(z_0, \cdot)$ in a neighborhood of z_0.

Choose z_0 to be the common fixed point of the Carnot homotheties δ_t, see Subsection 3.3. As we saw in this subsection, δ_t multiplies by e^t the Carnot-Carathéodory metric on $\partial\mathbb{H}^n_{\mathbb{K}} \setminus \{\infty\}$ associated to any N-invariant Riemannian metric on T. Since this metric is Lipschitz equivalent to d_{CC} in a neighborhood of z_0, one has for z in a neighborhood of z_0 and $t \leq 0$:

$$d_{\mathrm{CC}}(z_0, \delta_t(z)) \asymp e^t d_{\mathrm{CC}}(z_0, z).$$

On the other hand, the map δ_t is a hyperbolic isometry of $\mathbb{H}^n_{\mathbb{K}}$ whose axis contains z_0 and whose translation length is $|t|$. Thus, with the definition of d_{CAT}, one has for z in a neighborhood of z_0 and $t \leq 0$:

$$d_{\mathrm{CAT}}(z_0, \delta_t(z)) \asymp e^t d_{\mathrm{CAT}}(z_0, z).$$

The expected property comes from these two homogeneity relations.

It remains to compute the Hausdorff dimension. Let d_N be the Carnot-Carathéodory metric on $N \simeq \partial\mathbb{H}^n_{\mathbb{K}} \setminus \{\infty\}$ associated to an N-invariant Riemannian metric on T. The metrics d_N and d_{CC} are locally Lipschitz equivalent, thus their Hausdorff dimensions are equal. The distance d_N is N-invariant and is multiplied by e^t under δ_t. On the other hand, the fonction

$$\phi(z', z_n) := \|z'\| + |z_n|^{1/2} \tag{3.8}$$

is positive on $N \setminus \{0\}$ and δ_t-homogeneous. Therefore the function $\Phi(z, w) := \phi(z^{-1}w)$ on N^2 is N-invariant and is multiplied by e^t under δ_t. It follows by homogeneity that d_N and Φ are Lipschitz equivalent. From definition (3.8) one sees that

$$\mathrm{Hausdim}(N, d_N) = \dim_{\mathbb{R}} \mathbb{K}^{n-1} + 2\dim_{\mathbb{R}} \Im\mathbb{K} = (n-1)k + 2(k-1) = nk + k - 2.$$

The statement follows. □

4 Some geometric analysis

This section describes some dynamical and analytic tools and results that will serve in the sequel. This includes the convergence property (Proposition 4.1), Sullivan's characterization of Möbius homeomorphisms (Proposition 4.3), some regularity properties of quasi-Möbius homeomorphisms between Loewner spaces (Theorem 4.7), and the relations between quasi-Möbius and quasi-conformal homeomorphisms of Loewner spaces (Theorem 4.10). We refer to the survey [78] for a complementary viewpoint and more details on some parts of this section.

4.1 Convergence property of quasi-Möbius maps

A standard and classical property is the following *convergence property* of sequences of uniform quasi-Möbius maps. It will serve repeatedly in Section 6.

Proposition 4.1. *Let Z_1, Z_2 be compact metric spaces. Let $f_k : Z_1 \to Z_2$ be a sequence of* uniform *quasi-Möbius homeomorphisms (i.e. we assume the distortion function η to be the same for every f_k, $k \in \mathbb{N}$). Then one of the following properties occurs:*

1. *Up to a subsequence, $(f_k)_{k\in\mathbb{N}}$ converges uniformly on Z_1 to a η-quasi-Möbius homeomorphism $f : Z_1 \to Z_2$.*
2. *Up to a subsequence, there exist $a \in Z_1$ and $b \in Z_2$, such that for every compact subset $K \subset Z \setminus \{a\}$, $(f_k)_{k\in\mathbb{N}}$ converges uniformly on K to the constant map b.*

The crucial lemma is the following.

Lemma 4.2. *Let $f_k : Z_1 \to Z_2$ be a sequence of uniform η-quasi-Möbius maps between compact metric spaces. Suppose there exists three pairwise distinct points $a_1, a_2, a_3 \in Z_1$ and three pairwise distinct points $b_1, b_2, b_3 \in Z_2$, such that for $i = 1, 2, 3$, $f_k(a_i) \to b_i$ when $k \to +\infty$. Then, up to a subsequence, $(f_k)_{k\in\mathbb{N}}$ converges uniformly on Z to a η-quasi-Möbius map $f : Z_1 \to Z_2$.*

Proof. Let η be the common distortion function for the f_k's. We prove that the sequence $(f_k)_{k\in\mathbb{N}}$ is equicontinuous, so the statement will follow from Ascoli's theorem. Let $D = \min\{d(a_i, a_j) \mid i \neq j,\ i, j = 1, 2, 3\}$, and let $z, w \in Z_1$ be such that $d(z, w) \leq D/4$. Then $\mathrm{dist}(\{z, w\}, a_i) \leq D/4$ for at most one of the a_i's. Suppose for example that $\mathrm{dist}(\{z, w\}, \{a_2, a_3\}) > D/4$. We have

$$\frac{d(z,w)d(a_2,a_3)}{d(z,a_3)d(a_2,w)} \asymp d(z,w).$$

Since $d(f_k(z), b_3)$ and $d(b_2, f_k(w))$ are bounded by above by $\mathrm{diam}(Z_2)$, we get for every $k \in \mathbb{N}$ large enough:

$$\begin{aligned} d(f_k(z), f_k(w)) &\lesssim \frac{d(f_k(z), f_k(w))d(b_2, b_3)}{d(f_k(z), b_3)d(b_2, f_k(w))} \\ &\lesssim \frac{d(f_k(z), f_k(w))d(f_k(a_2), f_k(a_3))}{d(f_k(z), f_k(a_3))d(f_k(a_2), f_k(w))} \\ &\lesssim \eta\Big(\frac{d(z,w)d(a_2,a_3)}{d(z,a_3)d(a_2,w)}\Big) \\ &\lesssim \eta(Cd(z,w)), \end{aligned}$$

for some constant $C \geq 1$ independent of k, z, w. Therefore $(f_k)_{k\in\mathbb{N}}$ is equicontinuous. □

Proof of the Proposition. Let a_1, a_2, a_3 be three distinct points in Z_1. Up to a subsequence we can assume that for $i = 1, 2, 3$, $f_k(a_i)$ converges in Z_2 when $k \to \infty$. Let b_i be their limits. If b_1, b_2, b_3 are pairwise distinct then the previous lemma applies.

Suppose $b_1 = b_2$ and write it b for simplicity. Let $c \in Z_2 \setminus \{b\}$ and consider $z_k = f_k^{-1}(c)$. Up to a subsequence we can assume that $(z_k)_{k\in\mathbb{N}}$ converges in Z_1; let a be its limit. We have $a \notin \{a_1, a_2\}$. Let $K \subset Z_1 \setminus \{a\}$ be a compact subset. For $k \in \mathbb{N}$ large enough and $w \in K$ and, we have

$$\begin{aligned}\frac{d(f_k(a_1), f_k(w))}{d(f_k(a_1), f_k(a_2))} &\lesssim \frac{d(f_k(a_1), f_k(w))d(f_k(z_k), f_k(a_2))}{d(f_k(a_1), f_k(a_2))d(f_k(z_k), f_k(w))} \\ &\lesssim \eta\Big(\frac{d(a_1, w)d(z_k, a_2)}{d(a_1, a_2)d(z_k, w)}\Big) \\ &\lesssim \eta\Big(\frac{C}{d(a, w)}\Big),\end{aligned}$$

for some constant $C \geq 1$ independent of K, k, w. Therefore

$$d(f_k(a_1), f_k(w)) \lesssim \eta\Big(\frac{C}{d(a, w)}\Big) \cdot d(f_k(a_1), f_k(a_2)),$$

and so f_k converges uniformly on K to the constant map b. □

4.2 Sullivan's characterization of Möbius homeomorphisms

The following result is a main ingredient in Sullivan's ergodic approach to Mostow's rigidity and its variants, that will be presented in Section 5. It will also serve repeatedly in the sequel.

Proposition 4.3 ([128]). *Suppose (Z_1, d_1) and (Z_2, d_2) are Q-regular metric spaces for some $Q > 0$. Let $\mathcal{H}_1, \mathcal{H}_2$ be the Q-Hausdorff measures of Z_1 and Z_2 respectively. For $i = 1, 2$ let μ_i be the measure*

$$\mu_i(z, w) = \frac{\mathcal{H}_i(z) \times \mathcal{H}_i(w)}{d_i(z, w)^{2Q}}$$

on Z_i^2. Then a homeomorphism $f : Z_1 \to Z_2$ is Möbius if and only if $(f \times f)^*\mu_2 = C\mu_1$ for some constant $C > 0$.

Its proof will use the

Lemma 4.4. *Let $f : (Z_1, d_1) \to (Z_2, d_2)$ be a Möbius homeomorphism between metric spaces with no isolated points. Then*

1. *For every $z \in Z_1$ the limit*

$$|f'(z)| := \lim_{w\to z} \frac{d_2(f(w), f(z))}{d_1(w, z)}$$

exists and belongs to $(0, +\infty)$.

2. For every $z, w \in Z_1$ one has

$$d_2(f(z), f(w))^2 = |f'(z)| \cdot |f'(w)| \cdot d_1(z,w)^2.$$

Proof of Lemma 4.4. Since $[f(z_1), f(z_2), f(w_1), f(w_2)] = [z_1, z_2, w_1, w_2]$, one gets

$$d_2(f(z_1), f(w_1))d_2(f(z_2), f(w_2)) = \\ = \frac{d_2(f(z_1), f(w_2))}{d_1(z_1, w_2)} \frac{d_2(f(z_2), f(w_1))}{d_1(z_2, w_1)} d_1(z_1, w_1) d_1(z_2, w_2).$$

By letting $z_i \to z$ and $w_i \to w$ we obtain the second and then the first statement. □

Proof of Proposition 4.3. Suppose that f is Möbius. Then, according to Lemma 4.4, we get $f^*\mathcal{H}_2 = |f'|^Q \mathcal{H}_1$. By substituting in

$$(f \times f)^* \mu_2(z,w) = \frac{f^*\mathcal{H}_2(z) \times f^*\mathcal{H}_2(w)}{d_2(f(z), f(w))^{2Q}},$$

it follows from Lemma 4.4.2. that $(f \times f)^*\mu_2 = \mu_1$.

Conversely if $(f \times f)^* \mu_2 = C\mu_1$ then $f^*\mathcal{H}_2 = \varphi \mathcal{H}_1$ for some measurable function φ. Thus

$$(f \times f)^* \mu_2(z,w) = \frac{\varphi(z)\mathcal{H}_2(z) \times \varphi(w)\mathcal{H}_2(w)}{d_2(f(z), f(w))^{2Q}}.$$

Therefore for almost all $(z, w) \in Z_1^2$ we have

$$\frac{\varphi(z) \cdot \varphi(w)}{d_2(f(z), f(w))^{2Q}} = \frac{C}{d_1(z,w)^{2Q}}.$$

By substituting in the cross-ratio expression, one obtains after cancellations that f preserves the cross-ratio almost everywhere in Z^4. Since the cross-ratio is continuous we get that f is Möbius. □

4.3 Loewner spaces

Loewner spaces have been introduced by J. Heinonen and P. Koskela [90]. Most of the classical Euclidean quasi-conformal analysis was generalized to the setting of Loewner spaces. We present only the material that will be useful for us. More discussions on the Loewner spaces can be found in [90, 89, 50, 91, 137, 100]. We refer to [89] for a gentle introduction to Loewner spaces. We will use the shorthand Loewner space for a metric space that is Q-regular and Q-Loewner for some $Q > 1$, in the sense of [90].

Let Z be a Q-regular metric space with $Q > 1$, and let $\mathcal{H}$ be its Hausdorff measure. Let $\mathcal{F}$ be a non-void family of continuous curves in Z. Its *Q-modulus* is defined by

$$\mathrm{Mod}_Q(\mathcal{F}) = \inf_\rho \int_Z (\rho)^Q \, d\mathcal{H},$$

where the infimum is over all $\mathcal{F}$-*admissible functions*, i.e. measurable functions $\rho : Z \to [0, +\infty]$ which satisfy $\int_\gamma \rho \geq 1$ for every rectifiable curve $\gamma \in \mathcal{F}$. If there is no rectifiable curve in $\mathcal{F}$, we set $\mathrm{Mod}_Q(\mathcal{F}) = 0$. The modulus is an outer measure on the full set of continuous curves in Z. Moreover if every curve in $\mathcal{F}_2$ contains a curve in $\mathcal{F}_1$ one has $\mathrm{Mod}_Q(\mathcal{F}_2) \leq \mathrm{Mod}_Q(\mathcal{F}_1)$.

We denote by $\mathcal{F}(A, B)$ the family of curves joining two subsets A and B of Z and by $\mathrm{Mod}_Q(A, B)$ its Q-modulus. When $A \subset B(z, r)$ and $B \subset Z \setminus B(z, R)$ with $0 < 2r < R$, then one has

$$\mathrm{Mod}_Q(A, B) \lesssim (\log \frac{R}{r})^{1-Q}.$$

Therefore $\mathrm{Mod}_Q(A, B)$ is small when $\mathrm{diam}(A)$ or $\mathrm{diam}(B)$ is small compared to $\mathrm{dist}(A, B)$. The Loewner spaces are the spaces for which a kind of a converse inequality occurs:

Definition 4.5. Denote by $\Delta(A, B)$ the *relative distance* between two disjoint compact connected subsets $A, B \subset Z$ i.e.

$$\Delta(A, B) = \frac{\mathrm{dist}(A, B)}{\min\{\mathrm{diam}\, A, \mathrm{diam}\, B\}}.$$

Then Z is a *Loewner space* if there exits a homeomorphism $\varphi : [0, +\infty) \to [0, +\infty)$ such that for every pair of disjoint compact connected subsets $A, B \subset Z$ one has

$$\mathrm{Mod}_Q(A, B) \geq \varphi(\Delta(A, B)^{-1}).$$

Example 4.6. Basic examples of Loewner spaces include the Euclidean spaces of dimension at least 2, and the Carnot groups (Definition 3.6) equipped with Carnot-Carathéodory metrics (see e.g. [84]). As a consequence, the boundaries of rank-one symmetric spaces different from $\mathbb{H}^2_\mathbb{R}$ are Loewner, see Proposition 3.7. Among the currently known examples of Loewner spaces, the only ones which arise as boundaries of hyperbolic groups are the boundaries of rank-one symmetric spaces (different from $\mathbb{H}^2_\mathbb{R}$), and the boundaries of Fuchsian buildings [32].

Just from the definitions, one sees that bi-Lipschitz homeomorphisms preserve the Loewner property. The situation for quasi-Möbius homeomorphisms is more subtle. The following theorem is a combination of results from [137] and [90].

Theorem 4.7. *Let $f : (Z_1, d_1, \mathcal{H}_1) \to (Z_2, d_2, \mathcal{H}_2)$ be a quasi-Möbius homeomorphism between Alhfors-regular metric spaces of dimensions Q_1 and Q_2 respectively. Suppose Z_1 is Loewner. Then $Q_2 \geq Q_1$. Moreover if $Q_2 = Q_1$, then*

1. *Z_2 is Loewner.*
2. *f is absolutely continuous with respect to $\mathcal{H}_1$ and $\mathcal{H}_2$.*
3. *f is absolutely continuous along almost curves $\gamma : [a, b] \to Z_1$. In other words, for γ lying outside of a set of curves of zero modulus, the maps $f \circ \gamma$ are absolutely continuous with respect to 1-dimensional Hausdorff measure in the target.*

As an illustration, we mention

Corollary 4.8. *Suppose S_1 and S_2 are quasi-isometric rank-one non-compact symmetric spaces. Then $S_1 = S_2$.*

Proof. By Theorem 2.12, the boundaries of S_1 and S_2 are homeomorphic. Thus the boundaries have the same topological dimension. Among rank-one symmetric spaces, $\mathbb{H}^2_{\mathbb{R}}$ is the only one whose boundary is a circle. Suppose now that S_1 and S_2 are different from $\mathbb{H}^2_{\mathbb{R}}$. Then their boundaries are Loewner spaces. Thus, Theorem 4.7 implies that the Hausdorff dimensions of ∂S_1 and ∂S_2 are equal. Since the topological and Hausdorff dimensions of $\partial \mathbb{H}^n_{\mathbb{K}}$ are respectively $kn-1$ and $kn+k-2$ (Proposition 3.7), the statement follows. □

4.4 Quasi-conformal homeomorphisms

We now discuss quasi-Möbius homeomorphisms in relation with the classical notion of quasi-conformal homeomorphisms. Let $f : Z_1 \to Z_2$ be a map between metric spaces. For $z \in Z_1$ and $0 < r < \mathrm{diam}(Z_1)$ define

$$L_f(z,r) = \sup\{d(f(z), f(w)) \mid w \in B(z,r)\},$$

$$l_f(z,r) = \inf\{d(f(z), f(w)) \mid w \in Z_1 \setminus B(z,r)\}.$$

Definition 4.9. A homeomorphism $f : Z_1 \to Z_2$ between metric spaces, is *quasi-conformal* if there exists a $H \geq 1$, such that for every $z \in Z_1$ one has $\limsup_{r\to 0} \frac{L_f(z,r)}{l_f(z,r)} \leq H$.

Examples of quasi-conformal homeomorphisms include bi-Lipschitz homeomorphisms. It is easy to see that quasi-Möbius homeomorphisms are quasi-conformal as well (with $H = \eta(1)$). For the compact Loewner spaces the notions of quasi-conformal and quasi-Möbius homeomorphisms are equivalent:

Theorem 4.10 (Heinonen-Koskela [90]). *Let Z_1 and Z_2 be compact Loewner spaces with the same Hausdorff dimension Q, and let $f : Z_1 \to Z_2$ be a homeomorphism. The following properties are equivalent.*

1. *f is H-quasi-conformal.*
2. *There is a constant $C \geq 1$ such that for every family $\mathcal{F}$ of curves in Z_1, one has $C^{-1}\,\mathrm{Mod}_Q(\mathcal{F}) \leq \mathrm{Mod}_Q(f(\mathcal{F})) \leq C\,\mathrm{Mod}_Q(\mathcal{F})$.*
3. *f is η-quasi-Möbius.*

Moreover the constants H, C and the distortion function η are quantitatively related just in terms of the geometric data of the spaces.

Here is an application to group actions, that will serve in Section 7.

Corollary 4.11 ([33]). *Suppose Z is a compact Loewner space of dimension Q and let $\mathcal{H}$ be its Q-Hausdorff measure. Let G be a group acting on Z by* uniform *quasi-conformal homeomorphisms (i.e. there is $H \geq 1$ such that every $g \in G$ is H-quasi-conformal). Suppose that G contains a subgroup Γ such that*

1. $\Gamma \subset \mathrm{Moeb}(Z)$.

2. The diagonal action of Γ *on* $(Z^2, \mathcal{H} \times \mathcal{H})$ *is ergodic.*

3. The diagonal action of Γ *on*

$$\mathcal{T}(Z) := \{(z_1, z_2, z_3) \in Z^3 \mid z_i \neq z_j \text{ for } i \neq j\}$$

is properly discontinuous and cocompact.

Then $G \subset \mathrm{Moeb}(Z)$.

Proof of the Corollary. Let $\overline{G}$ be the closure of G in $\mathrm{Homeo}(Z)$. According to Theorem 4.10, G acts on Z by *uniform* quasi-Möbius homeomorphisms (*i.e.* we can assume the distortion function to be the same for every $g \in G$). By continuity $\overline{G}$ also acts by uniform quasi-Möbius homeomorphisms. By Proposition 4.1 it acts properly on $\mathcal{T}(Z)$.

We will deduce the corollary from the Sullivan characterisation of Möbius homeomorphisms (Proposition 4.3). Since $\overline{G}$ acts properly on $\mathcal{T}(Z)$, our third assumption implies that Γ is a cocompact lattice in $\overline{G}$. In particular the Haar measure on $\overline{G}$ is bi-invariant. Let m be the finite measure on $\Gamma\backslash\overline{G}$ obtained by restricting the Haar measure to a fundamental domain of Γ in $\overline{G}$. Let μ be the measure on Z^2 defined in Proposition 4.3. Let $s : \Gamma\backslash\overline{G} \to \overline{G}$ be a measurable section. We define a new measure ν on Z^2 by

$$\nu(B) = \int_{\Gamma\backslash\overline{G}} \mu(s(\overline{g}) \cdot B)\ dm(\overline{g})$$

for every Borel set $B \subset Z^2$. By the first assumption and Proposition 4.3, the measure μ is Γ-invariant. Since m is invariant under the right action of $\overline{G}$, we get that ν is $\overline{G}$-invariant. Moreover ν is absolutely continuous with respect to μ. Indeed, the group $\overline{G}$ acts on Z by quasi-Möbius homeomorphisms, and quasi-Möbius homeomorphisms are absolutely continuous with respect to $\mathcal{H}$ (Proposition 4.7). With our second assumption we obtain that μ is equal to $C\nu$ for some $C > 0$. Therefore μ is G-invariant and we conclude by using Sullivan's characterization (Proposition 4.3). □

5 Mostow rigidity

This section is devoted to the proof of Mostow's theorem. We also present two generalizations: the Besson-Courtois-Gallot theorem (Theorem 5.2), and a generalization of Mostow's theorem to quasi-convex geometric actions (Theorem 5.4). The section ends with a survey of several related results.

5.1 Mostow theorem

We state Mostow's theorem (for rank-one non-compact symmetric spaces), and we explain how it can be deduced from results of the previous sections.

As usual we normalize the Riemannian metric of the rank-one symmetric spaces so that the maximum of the sectional curvatures is -1.

Theorem 5.1. *Let S_1 and S_2 be rank-one symmetric spaces different from $\mathbb{H}^2_{\mathbb{R}}$. For $i = 1, 2$ let Γ_i be a lattice in* $\mathrm{Isom}(S_i)$. *Then any group isomorphism $\varphi : \Gamma_1 \to \Gamma_2$ is the conjugacy by an isometry from S_1 to S_2.*

Mostow proved the cocompact lattice case [113]. G. Prasad extended the proof to finite covolume lattices [119].

Proof. Suppose first that Γ_1 and Γ_2 are cocompact lattices. Then Γ_i acts geometrically on S_i. By Svarc-Milnor (Proposition 2.3), the isomorphism φ induces a φ-equivariant quasi-isometry from S_1 to S_2. It extends to a φ-equivariant quasi-Möbius homeomorphism $f : \partial S_1 \to \partial S_2$ (see Theorem 2.12). Since S_1 and S_2 are rank-one symmetric spaces different from $\mathbb{H}^2_{\mathbb{R}}$, their boundaries (equipped with the visual metrics d_{CAT}) are Loewner spaces. Thus from Theorem 4.7, the Hausdorff dimensions of ∂S_i are equal and f is absolutely continuous with respect to the Hausdorff measures. Therefore $f \times f$ is absolutely continuous with respect to the measures μ_i defined in Proposition 4.3. Since μ_i is Γ_i-invariant and ergodic (Theorem 2.14), and since $f \times f$ is equivariant, we get that $(f \times f)^* \mu_2 = C\mu_1$ for some constant $C > 0$. Therefore Sullivan's characterization (Proposition 4.3) implies that f is a Möbius homeomorphism. Finally f is the boundary extension of an isometry by Theorem 2.19.

In the non-cocompact case, Prasad [119] proved that the equivariant quasi-isometry from S_1 to S_2 still exists, by using informations about the cusps. The rest of the proof is similar. □

This proof, arising from the Sullivan's ergodic approach [128], is similar in spirit to Mostow's original proof [113]. At the end, Mostow's argument is different, it relies – in a delicate way – on absolute continuity and ergodicity, to show that f preserves the $\mathbb{R}$-circles (i.e. boundaries of curvature -1 totally geodesic planes). Then he deduces from this property that f is the boundary extension of an isometry.

5.2 Besson-Courtois-Gallot theorem

A remarkable generalization of Mostow's theorem is due to Besson, Courtois, Gallot [6, 7].

Let M be a compact connected Riemannian manifold M and let $\tilde{M}$ be its universal cover. Pick $O \in \tilde{M}$ and define the *volume entropy of M* (independent of O) by

$$h(M) := \lim_{R \to +\infty} \frac{1}{R} \ln \mathrm{Vol}(B(O, R)),$$

where $\mathrm{Vol}(B(O, R))$ denotes the volume in $\tilde{M}$ of the ball $B(O, R)$.

Theorem 5.2. *Let M_0 and M be compact connected Riemannian manifolds of the same dimension n. Suppose M_0 is a locally symmetric manifold of negative sectional curvature. Then for every non-zero degree continuous map $f : M \to M_0$, one has the following inequality*

$$h(M)^n \,\mathrm{Vol}(M) \geq |\deg(f)| h(M_0)^n \,\mathrm{Vol}(M_0).$$

Moreover, for $n \geq 3$, the equality holds if and only if M is a locally symmetric space and f is homotopic to a homothetic covering.

The theorem admits several important applications in geometry, topology and dynamics. We refer to [6, 7] for more informations (see also subsection 5.4). It has been generalized to finite volume manifolds by Boland, Connell, Souto [12].

5.3 Quasi-convex geometric actions

We present a generalization of Mostow's theorem to quasi-convex geometric actions on CAT(-1)-spaces.

Definition 5.3. Let X be a proper hyperbolic space. A subset $Y \subset X$ is *quasi-convex* if there is a constant $R \geq 0$ such that for every pair of points $y_1, y_2 \in Y$ the geodesic segments $[y_1, y_2] \subset X$ lie in $N_R(Y)$. A finitely generated group Γ acts on X *quasi-convex geometrically* if it acts by isometries, properly discontinuously, and if its orbits are quasi-convex subsets of X. The *limit set* of Γ is the following subset of ∂X (independent of $O \in X$)

$$\Lambda = \overline{\Gamma \cdot O}^{X \cup \partial X} \cap \partial X.$$

Several properties of geometric actions generalize to quasi-convex geometric actions. In particular such groups are hyperbolic. The orbit map $g \in \Gamma \mapsto g \cdot O \in X$ is a quasi-isometric embedding. It extends canonically to a quasi-Möbius homeomorphism from $\partial\Gamma$ to Λ.

The following result was first proved by U. Hamenstädt [86] for geometric actions on simply connected Riemannian manifold of curvature at most equal to -1.

Theorem 5.4. *Let S be a rank-one symmetric space different from $\mathbb{H}^2_{\mathbb{R}}$. Suppose its Riemannian metric is normalized so that the maximum of the sectional curvatures is -1. Let Γ be a cocompact lattice in* $\mathrm{Isom}(S)$*. Assume that Γ acts quasi-convex geometrically on a CAT(-1)-space X and let $\Lambda \subset \partial X$ be its limit set. Then* $\mathrm{Hausdim}(\partial S) \leq \mathrm{Hausdim}(\Lambda)$*. Moreover the equality holds if and only if there exists a Γ-equivariant totally geodesic isometric embedding of S into X.*

The inequality is due to Pansu [117]. It is also a consequence of Theorem 4.7 since ∂S and Λ are quasi-Möbius homeomorphic. The equality case is proved in [27] (the case X is a rank-one symmetric space has been treated independently by C. Yue [144]). The proof is similar to the one we gave for the Mostow theorem, it relies on Theorem 4.7. Theorem 5.4 has been generalized in [141, 60] to geometrically finite actions of finite covolume lattices of $\mathrm{Isom}(S)$.

When $S = \mathbb{H}^2_{\mathbb{R}}$ and Γ is a lattice in $\mathrm{Isom}(\mathbb{H}^2_{\mathbb{R}})$, the same statement holds apart from the fact that the isometric embedding is not Γ-equivariant in general. This result is due to R. Bowen [36] when $X = \mathbb{H}^3_{\mathbb{R}}$, and to M. Bonk and B. Kleiner [19] for general CAT(-1)-spaces X.

5.4 Further results

1) One expects that some versions of Mostow's rigidity hold for quite general hyperbolic spaces. For instance, given "reasonable" hyperbolic spaces X_1 and X_2, and a group Γ acting on them geometrically, suppose that the canonical boundary map is absolutely continuous with respect to the Hausdorff measures (recall that the Hausdorff measure class is independent of the choice of a visual metric). Does this imply that there exists a Γ-equivariant homothety from X_1 to X_2 ?

When $X_1 = X_2 = \mathbb{H}^2_{\mathbb{R}}$, this was established by T. Kuusalo [104]. When X_1 and X_2 are metric trees, this is a result of Coornaert [54]. C. Croke [58] and J-P. Otal [115] proved the case X_1 and X_2 are simply connected Riemannian surfaces of negative curvature. Hersonsky-Paulin [93] generalized it to negatively curved Riemannian surfaces with singularities. When X_1 is a rank-one symmetric space, and X_2 is a simply connected negatively curved Riemannian manifold, this is due to Hamenstädt [88] (by using the Besson-Courtois-Gallot theorem).

We remark that the above problem admits several equivalent formulations (in terms of cross-ratio, marked length spectrum, geodesic flow, etc), see [106, 87].

In [111], I. Mineyev constructed $\mathrm{Isom}(X)$-invariant conformal structures and cross-ratios on boundaries of hyperbolic spaces X, that generalize the CAT(-1) setting.

K. Biswas [10] proved that every Möbius homeomorphism between the boundaries of proper geodesically complete CAT(-1)-spaces X_1 and X_2 extends to a $(1, \log 2)$-quasi-isometry from X_1 to X_2 with $\frac{1}{2}\log 2$-dense image in X_2.

2) P. Storm [126] proved a version of the Besson-Courtois-Gallot theorem for manifolds with boundary. As a consequence he solved the following conjecture of Bonahon. Let ρ_0, ρ_1 be quasi-convex geometric actions of a group Γ on $\mathbb{H}^n_{\mathbb{R}}$ with $n \geq 3$. Consider, for $i = 0, 1$, the convex hull $H_i \subset \mathbb{H}^n_{\mathbb{R}}$ of the limit set $\Lambda(\rho_i) \subset \mathbb{S}^{n-1}$, and suppose that the boundary of H_0 in $\mathbb{H}^n_{\mathbb{R}}$ is totally geodesic. Then $\mathrm{Vol}(H_0/\rho_0(\Gamma)) \leq \mathrm{Vol}(H_1/\rho_1(\Gamma))$. Moreover the equality holds if and only if ρ_0 and ρ_1 are conjugate in $\mathrm{Isom}(\mathbb{H}^n_{\mathbb{R}})$.

We note that for such ρ_0, ρ_1, it is commonly conjectured that

$$\mathrm{Hausdim}(\Lambda(\rho_0)) \leq \mathrm{Hausdim}(\Lambda(\rho_1)),$$

and that the equality holds if and only if ρ_0 and ρ_1 are conjugate in $\mathrm{Isom}(\mathbb{H}^n_{\mathbb{R}})$.

3) Divergence groups form a vast generalization of quasi-convex geometric actions on hyperbolic spaces X. A subgroup $\Gamma \subset \mathrm{Isom}(X)$ is a *divergence group* if it acts properly discontinuously on X, if the critical exponent

$$\delta(\Gamma) := \inf\{s > 0 \mid \sum_{g \in \Gamma} \exp(-sd(O, g \cdot O)) < +\infty\}$$

is finite, and if the sum $\sum_{g\in\Gamma} \exp(-sd(O, g \cdot O))$ diverges at $s = \delta(\Gamma)$. See [93] for several examples of such groups.

The limit set of a divergence group carries a natural finite measure called the *Patterson-Sullivan measure* (it coincides with the Hausdorff measure when Γ

acts quasi-convex geometrically). Its ergodic properties are well studied (see for instance [128], [129], [39], [122]).

Sullivan [128], Burger-Mozes [39] and Yue [145] obtained several rigidity theorems for such actions. In particular, the following result is established in [39]. Let X_1 and X_2 be proper CAT(-1)-spaces and let $\Gamma \in \mathrm{Isom}(X_1)$ be a divergence group. Let $\Lambda \subset \partial X_1$ be its limit set. Then, for every group homomorphism $\rho : \Gamma \to \mathrm{Isom}(X_2)$ with non-elementary image, there is a unique ρ-equivariant measurable map $f : \Lambda \to \partial X_2$, and almost all values of f belong to the limit set of $\rho(\Gamma)$.

Hersonsky-Paulin [93] obtained Mostow type theorems for divergence groups of CAT(-1)-spaces, under the assumption that the above map is absolutely continuous with respect to the Patterson-Sullivan measures.

4) In [128], Sullivan addressed the following problem. Suppose Γ is a finitely generated subgroup of $\mathrm{Isom}(\mathbb{H}^n_{\mathbb{R}})$ such that the topological dimension of its limit set Λ is equal to its Hausdorff dimension. Does it imply that Λ is a round sphere?

For quasi-convex subgroups, this was established by Yue [144] (in the statement he assumed that Λ is a topological sphere, but his argument does not need the latter assumption). M. Kapovich [97] proved it for geometrically finite groups, and T. Das, D. Simmons and M. Urbańsky [59] for even more general Kleinian groups.

Bonk and Kleiner obtained the following generalization [18, 19]. Suppose that a group Γ acts quasi-convex geometrically on a CAT(-1)-space X, and let $\Lambda \subset \partial X$ be its limit set. Let $n \geq 1$ be the topological dimension of Λ. Then $\mathrm{Hausdim}(\Lambda) \geq n$, and equality holds if and only if Γ acts geometrically on an isometric copy of $\mathbb{H}^{n+1}_{\mathbb{R}}$ in X.

K. Kinneberg [101] established a coarse version of this result. Suppose a group Γ acts geometrically on an $AC_u(-1)$-metric space X. These are the metric spaces with *asymptotic upper curvature* -1, a geometric property that is invariant by rough isometries, and that has been introduced by Bonk and Foertsch [15]. If ∂X is homeomorphic to $\mathbb{S}^n$ with $n \geq 2$, and if the volume entropy of X is at most equal to n, then Γ is virtually a cocompact lattice in $\mathrm{Isom}(\mathbb{H}^{n+1}_{\mathbb{R}})$ and X is roughly isometric to $\mathbb{H}^{n+1}_{\mathbb{R}}$.

Kapovich considered the following situation. Let Γ be a discrete virtually torsion free subgroup of $\mathrm{Isom}(\mathbb{H}^n_{\mathbb{R}})$ and let $\delta(\Gamma)$ be its critical exponent as defined above (when Γ is geometrically finite, $\delta(\Gamma)$ is equal to the Hausdorff dimension of the limit set [129]). Kapovich proved in [97] that $\delta(\Gamma) + 1$ is at least equal to the virtual homological dimension of the pair (Γ, Π), where Π denotes a set of representatives of the conjugacy classes of maximal virtually abelian subgroups of virtual rank at least two. Its proof uses Besson-Courtois-Gallot techniques. When $\Gamma \subset \mathrm{Isom}(\mathbb{H}^n_{\mathbb{R}})$ is discrete finitely generated, he conjectures that $\delta(\Gamma)+1$ is at least equal to the virtual cohomological dimension of the pair (Γ, Π), and that equality holds if and only if Γ is geometrically finite and its limit set is a round sphere of dimension $\delta(\Gamma)$.

6 The zoom method

The convergence property (Proposition 4.1) allows one to "zoom in" at some point of the space by using a sequence of uniformly quasi-Möbius homeomorphisms. Inspired by Mostow's theorem, J. Ferrand [66, 67] exploited this idea to solve the Lichnérowicz conjecture, which states that the conformal group of a compact manifold M is compact, unless M is conformally equivalent to an Euclidean sphere. Later, Tukia [133] used the zoom method to prove that every finitely generated group quasi-isometric to $\mathbb{H}^n_{\mathbb{R}}$ (with $n \geq 3$) is virtually isomorphic to a cocompact lattice in $\mathrm{Isom}(\mathbb{H}^n_{\mathbb{R}})$. This section is devoted to these results and their proofs. We also present a rather elementary proof of Mostow's theorem in the real case, based on the zoom method, and due to Tukia [132]. The section ends with a survey of several related results.

6.1 Ferrand theorem

The following theorem solves the generalized Lichnérowicz conjecture.

Theorem 6.1. *Let M be a Riemannian manifold of dimension $n \geq 2$. Denote by $\mathrm{Conf}(M)$ the group of conformal diffeomorphisms of M. Then $\mathrm{Conf}(M)$ acts properly on M unless M is conformally equivalent to the Euclidean n-sphere or n-space.*

The compact manifolds case was treated by Ferrand in 1969-71 in [66, 67] (Obata [114] proved it too, but only for $\mathrm{Conf}_0(M)$). Soon after, D. V. Alekseevskii proposed a proof for the non-compact manifolds. His proof was accepted for more than twenty years until R. Zimmer and K. Gutshera found a serious gap. Finally Ferrand solved the non-compact case in [68] by introducing new global conformal invariants. See [69] for a detailed story of the Lichnérowicz conjecture.

We will only prove Theorem 6.1 for compact manifolds, since in this case all the geometric tools have been already defined. For an alternative proof, which makes use of the Weyl tensor, see [70]. The generalized Lichnérowicz conjecture is discussed in [78].

Proof of the compact manifold case. A compact Riemannian manifold is a Loewner space, and a conformal diffeomorphism is a 1-quasi-conformal homeomorphism. Therefore Theorem 4.10 implies that $\mathrm{Conf}(M)$ acts by *uniform* quasi-Möbius homeomorphisms (*i.e.* we can assume the distortion function η to be the same for every $g \in \mathrm{Conf}(M)$).

Thus the convergence property (Proposition 4.1) in combination with the fact that $\mathrm{Conf}(M)$ is closed in $\mathrm{Homeo}(M)$ (see [67, 46]), implies the following characterization : $\mathrm{Conf}(M)$ is not compact if and only if there exists a sequence $(g_k)_{k\in\mathbb{N}}$ of elements in $\mathrm{Conf}(M)$, and points $a, b \in M$ (possibly $a = b$), such that for every compact subset $K \subset M \setminus \{a\}$ the sequence $(g_k)_{k\in\mathbb{N}}$ converges uniformly on K to the constant map b.

Assume now that $\mathrm{Conf}(M)$ is not compact and consider $(g_k)_{k\in\mathbb{N}}, a, b$ as above. Pick three pairwise distinct points $a_1, a_2, a_3 \in K \setminus \{a\}$. We claim that for every permutation α, β, γ of $1, 2, 3$ and k large enough:

$$d(g_k(a_\alpha), g_k(a_\beta)) \asymp d(g_k(a_\alpha), g_k(a_\gamma)).$$

Indeed, let $c \in M \setminus \{b\}$ and $c_k = g_k^{-1}(c)$. One has $c_k \to a$ when $k \to +\infty$. Since $g_k(a_i) \to b$ for $i = 1, 2, 3$, we have for k large enough,

$$\begin{aligned}\frac{d(g_k(a_\alpha), g_k(a_\beta))}{d(g_k(a_\alpha), g_k(a_\gamma))} &\asymp \frac{d(g_k(a_\alpha), g_k(a_\beta))d(c, g_k(a_\gamma))}{d(g_k(a_\alpha), g_k(a_\gamma))d(c, g_k(a_\beta))} \\ &\leq \eta\Big(\frac{d(a_\alpha, a_\beta)d(c_k, a_\gamma)}{d(a_\alpha, a_\gamma)d(c_k, a_\beta)}\Big) \asymp 1.\end{aligned}$$

Let $\exp : T_bM \to M$ be the Riemann exponential map. For every compact subset $K \subset M \setminus \{a\}$ the map $\exp^{-1} \circ g_k$ is well defined on K for k large enough. The previous discussion shows that for every permutation α, β, γ of $1, 2, 3$ and k large enough

$$\| \exp^{-1} \circ g_k(a_\alpha) - \exp^{-1} \circ g_k(a_\beta)\| \asymp \| \exp^{-1} \circ g_k(a_\alpha) - \exp^{-1} \circ g_k(a_\gamma)\|.$$

Thus there exist $\lambda_k \to +\infty$ when $k \to +\infty$, and affine homotheties h_k of T_bM, with ratios λ_k, so that for k large enough we have $h_k \circ \exp^{-1} \circ g_k(a_1) = 0$, and

$$\|h_k \circ \exp^{-1} \circ g_k(a_i) - h_k \circ \exp^{-1} \circ g_k(a_j)\| \asymp 1,$$

for every distinct $i, j \in \{1, 2, 3\}$. The sequence $(h_k \circ \exp^{-1} \circ g_k)$ is uniformly quasi-Möbius on K and normalized on the three points a_1, a_2, a_3. Therefore, by the convergence property (Lemma 4.2), up to a subsequence, it converges uniformly on K to a quasi-Möbius map $f_K : K \to \overline{T_bM}$, where $\overline{T_bM}$ denotes the one point compactification of T_bM. Moreover, since $(g_k)_{k\in\mathbb{N}}$ converges uniformly on K to the constant map b, and since the tangent map of $\exp^{-1}$ at b is the identity, f_K is a 1-quasi-conformal homeomorphism onto its image. By considering an exhaustion $(K_i)_{i\in\mathbb{N}}$ of $M \setminus \{a\}$, a diagonal argument shows that there is a quasi-Möbius map $f : M \setminus \{a\} \to \overline{T_bM}$, which is a 1-quasi-conformal homeomorphism onto its image.

We now establish that $f(M \setminus \{a\})$ is equal to the sphere $\overline{T_bM}$ minus a point. The subset $U := f(M \setminus \{a\})$ is open in $\overline{T_bM}$. Suppose by contradiction that U is different from $\overline{T_bM}$ minus a point. Then $\overline{U} \setminus U$ contains at least two distinct points z, w. Let $z_k, w_k \in U$ with $z_k \to z$ and $w_k \to w$ when $k \to +\infty$. The points $f^{-1}(z_k)$ and $f^{-1}(w_k)$ tend to a; thus $[a_1, f^{-1}(z_k), a_2, f^{-1}(w_k)]$ tends to 0. But $[f(a_1), z_k, f(a_2), w_k]$ tends to $[f(a_1), z, f(a_2), w] \neq 0$, which contradicts the fact that f is a quasi-Möbius map.

Therefore f extends to a quasi-Möbius homeomorphism from M to the Euclidean n-sphere, which is 1-quasi-conformal on $M \setminus \{a\}$. Such a map is a conformal diffeomorphism [67, 46]. The proof is complete. □

6.2 Sullivan-Tukia theorem

The following result characterizes the groups which are quasi-isometric to $\mathbb{H}^n_{\mathbb{R}}$.

Theorem 6.2 (Sullivan [127] for $n = 3$, Tukia [133] in general)**.** *Let* Γ *be a finitely generated group quasi-isometric to* $\mathbb{H}^n_{\mathbb{R}}$*, with* $n \geq 3$*. Then there exists a cocompact lattice* $\Phi \subset \mathrm{Isom}(\mathbb{H}^n_{\mathbb{R}})$ *and a surjective homomorphism of groups* $\Gamma \to \Phi$ *with finite kernel.*

The above statement holds for every rank-one non-compact symmetric space too. Its proof for the hyperbolic plane $\mathbb{H}^2_{\mathbb{R}}$ relies on works by Tukia [134], Gabai [71], Casson and Jungreis [49]. An alternative argument, using Perelman's proof of Thurston's geometrization conjecture, is given in [62]. In [109], V. Markovic gives a proof based on the recent work of Agol-Wise on cube complexes. The result for the hyperbolic complex spaces $\mathbb{H}^m_{\mathbb{C}}$ has been established by Chow [51]. For the remaining rank-one symmetric spaces, it follows from Pansu's theorem [116], see Theorem 7.1.

We remark that in addition to Tukia's original paper, there are several other expositions of Tukia's proof in the literature, e.g. [29], [98], [62].

Proof. Consider the isometric action of Γ on itself by left translations. Since Γ is quasi-isometric to $\mathbb{H}^n_{\mathbb{R}}$, every element of Γ induces a quasi-isometry of $\mathbb{H}^n_{\mathbb{R}}$, which is unique up to bounded distance, and with uniform quasi-isometry constants. Thus by Theorems 2.12 and 4.10, Γ acts on $\mathbb{S}^{n-1} = \partial\mathbb{H}^n_{\mathbb{R}}$ by uniform quasi-conformal homeomorphisms. The kernel of this action is finite. We still denote by Γ the quotient by the kernel.

We first search for a Γ-invariant structure on $\mathbb{S}^{n-1}$. A *measurable field of ellipsoids* on $\mathbb{S}^{n-1}$ is a measurable map which assigns to a.e. $z \in \mathbb{S}^{n-1}$ an $(n-2)$-ellipsoid centered at 0 in $T_z\mathbb{S}^{n-1}$.

We are only concerned with non-degenerate ellipsoids, up to homothety, and centered at 0. The space of such ellipsoids in $\mathbb{R}^{n-1}$ is the symmetric space

$$X := SL_{n-1}(\mathbb{R})/SO(n-1).$$

Since quasi-conformal homeomorphisms of $\mathbb{S}^{n-1}$ are differentiable a.e. (Rademacher-Stepanov's Theorem [138]) and absolutely continuous (Theorems 4.10 and 4.7), every quasi-conformal homeomorphism $f : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ acts on the space of measurable fields of ellipsoids, as follows. If $\xi = \{\xi_z\}_{z\in\mathbb{S}^{n-1}}$ is a measurable field of ellipsoids, then :

$$(f_*\xi)_z := D_{f^{-1}(z)}f(\xi_{f^{-1}(z)}).$$

Lemma 6.3. *There exists a bounded* Γ*-invariant measurable field of ellipsoids on* $\mathbb{S}^{n-1}$*.*

Proof of the lemma. For a.e. $z \in \mathbb{S}^{n-1}$, let

$$E_z = \big\{D_{g^{-1}(z)}g(S_{g^{-1}z}) \mid g \in \Gamma\big\},$$

where S_z is the unit sphere in $T_z\mathbb{S}^{n-1}$. By choosing a measurable trivialisation of the orthonormal frame bundle of $\mathbb{S}^{n-1}$, each set E_z can be identified with a subset of the symmetric space X defined above. In addition we have for $g \in \Gamma$, and a.e. $z \in \mathbb{S}^{n-1}$

$$E_{g(z)} = D_zg(E_z),$$

where $D_z g$ acts on X by isometry (indeed $SL_{n-1}(\mathbb{R})$ does). The eccentricities of the ellipsoids in E_z are bounded by the uniform quasi-conformal constant of the Γ-elements. Thus E_z is a bounded subset of X. Every bounded subset A in a Hadamard manifold, admits a well-defined canonical "center", namely the center of the unique smallest closed ball containing A. Define ξ_z to be the center of E_z. The field $\{\xi_z\}_{z\in Z}$ possesses the expected properties. □

Let $\xi = \{\xi_z\}_{z\in\mathbb{S}^{n-1}}$ be a bounded Γ-invariant measurable field of ellipsoids. Our goal is now to construct a quasi-conformal homeomorphism f of $\mathbb{S}^{n-1}$ such that

$$f\Gamma f^{-1} \subset \mathrm{Conf}(\mathbb{S}^{n-1}).$$

For $n = 3$, Sullivan observed that the measurable Riemann mapping theorem implies that there exists a quasi-conformal homeomorphism f of $\mathbb{S}^2$ such that $f_*\xi$ is a field of circles. Therefore $f\Gamma f^{-1}$ is a group of quasi-conformal homeomorphisms that are 1-quasi-conformal a.e. Such homeomorphisms are conformal diffeomorphisms (see e.g. [136]). Thus f admits the expected property.

When $n \geq 4$, the measurable Riemann mapping theorem is not valid. Instead, Tukia proposed the following argument based on the zoom method.

The field ξ is measurable, so it is *approximately continuous* a.e. In other words for a.e. $z \in \mathbb{S}^{n-1}$ and every $\varepsilon > 0$, we have

$$\lim_{r\to 0} \ \mathcal{H}\big\{w \in B(z,r) \mid d_X(\xi_z, \xi_w) < \varepsilon\big\}/\mathcal{H}(B(z,r)) = 1,$$

where $\mathcal{H}$ denotes the spherical measure on $\mathbb{S}^{n-1}$. Let $z_0 \in \mathbb{S}^{n-1}$ such that ξ is approximately continuous at z_0. By conjugating Γ by a projective isomorphism of $\mathbb{S}^{n-1}$ if necessary, we may assume that ξ_{z_0} is a round sphere. Let $O \in \mathbb{H}^n_{\mathbb{R}}$ be an origin, and let $(g_k)_{k\in\mathbb{N}}$ be a sequence in Γ so that $g_k \cdot O \to z_0$ when $k \to \infty$, and $\mathrm{dist}_{\mathbb{H}^n_{\mathbb{R}}}(g_k \cdot O, [O, z_0))$ is uniformly bounded. (The existence of $(g_k)_{k\in\mathbb{N}}$ comes from the fact that Γ and $\mathbb{H}^n_{\mathbb{R}}$ are quasi-isometric.) Let $h_k \in \mathrm{Isom}(\mathbb{H}^n_{\mathbb{R}})$ be a loxodromic element whose axis contains the ray $[O, z_0)$ and such that $(h_k \circ g_k)\cdot O$ is uniformly close to O. By considering an ideal triangle $\Delta(a, b, c) \subset \mathbb{H}^n_{\mathbb{R}}$ centered at O, and its images by the $h_k \circ g_k$'s, it follows from Lemma 4.2 that $(h_k \circ g_k)_{k\in\mathbb{N}}$ subconverges to a quasi-conformal homeomorphism f.

Since ξ is Γ-invariant, the group $(h_k \circ g_k)\Gamma(h_k \circ g_k)^{-1}$ leaves invariant the field $h_{k*}\xi$. Because ξ is approximately continuous at z_0 and ξ_{z_0} is a round sphere, the sequence $(h_{k*}\xi)_{k\in\mathbb{N}}$ converges in measure to the field of round spheres. Therefore for every $g \in \Gamma$, the eccentricity of the differential of $(h_k \circ g_k) \circ g \circ (h_k \circ g_k)^{-1}$ converges in measure to the constant function 1. This implies that the limit map $f \circ g \circ f^{-1}$ is a conformal diffeomorphism (see [133] Lemma B2 for more details). Thus we have $f\Gamma f^{-1} \subset \mathrm{Conf}(\mathbb{S}^{n-1})$.

We know that $\mathrm{Conf}(\mathbb{S}^{n-1}) = \mathrm{Moeb}(\mathbb{S}^{n-1}) = \mathrm{Isom}(\mathbb{H}^n_{\mathbb{R}})$. Indeed the first equality follows from Liouville's theorem. The second one is a standard result in classical hyperbolic geometry (it is also a particular case of Theorem 2.19). Thus we obtain that $f\Gamma f^{-1} \subset \mathrm{Isom}(\mathbb{H}^n_{\mathbb{R}})$.

It remains to prove that $f\Gamma f^{-1}$ is a cocompact lattice of $\mathrm{Isom}\,\mathbb{H}^n_{\mathbb{R}}$. By Proposition 2.15, Γ acts properly discontinuously and cocompactly on the space of triples of pairwise distinct points of $\partial\Gamma$. Thus $f\Gamma f^{-1}$ acts properly discontinuously and cocompactly on the space of triples of parwise distinct points of $\mathbb{S}^{n-1}$. Therefore it is a cocompact lattice in $\mathrm{Isom}(\mathbb{H}^n_{\mathbb{R}})$, thanks to Proposition 2.15. □

6.3 Another proof of Mostow's theorem for $\mathbb{H}^n_{\mathbb{R}}$

Mostow's theorem is stated in Subsection 5.1. The following proof (for $\mathbb{K} = \mathbb{R}$) is based on the zoom method. It is due to Tukia [132]. It appears also in [96].

Suppose $\varphi : \Gamma_1 \to \Gamma_2$ is an isomorphism between two cocompact lattices in $\mathrm{Isom}(\mathbb{H}^n_{\mathbb{R}})$ with $n \geq 3$. Let $f : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ be the induced φ-equivariant quasi-conformal homeomorphism. By Rademacher-Stepanov's theorem (see e.g. [138]) it is differentiable a.e. Since quasi-conformal homeomorphisms that are 1-quasiconformal a.e. are conformal diffeomorphims (see e.g. [136]), and since $\mathrm{Conf}(\mathbb{S}^{n-1}) = \mathrm{Moeb}(\mathbb{S}^{n-1}) = \mathrm{Isom}(\mathbb{H}^n_{\mathbb{R}})$ by Liouville's theorem, it is enough to prove that $D_z f$ is a conformal linear map for a.e. $z \in \mathbb{S}^{n-1}$.

Suppose f is differentiable at $z_0 \in \mathbb{S}^{n-1}$ and set $L := D_{z_0} f$. In the upper half-space model of $\mathbb{H}^n_{\mathbb{R}}$, let $0 = (0, ..., 0)$ and $O = (0, ..., 0, 1)$. We can assume that $z_0 = f(z_0) = 0$. Let $h_k \in \mathrm{Isom}(\mathbb{H}^n_{\mathbb{R}})$ be a linear homothety of ratio $k \in \mathbb{N}$. The maps $h_k \circ f \circ h_k^{-1}$ converge uniformly to L on compact subsets of $\mathbb{R}^{n-1}$ when $k \to \infty$.

Let $g_k \in \Gamma_1$ be such that $d_{\mathbb{H}^n_{\mathbb{R}}}(O, h_k \circ g_k^{-1}(O))$ is bounded independently of k. According to Lemma 4.2, the sequence $(h_k \circ g_k^{-1})_{k\in\mathbb{N}}$ subconverges uniformly on $\partial\mathbb{H}^n_{\mathbb{R}}$. Let $l_1 \in \mathrm{Moeb}(\mathbb{S}^{n-1}) = \mathrm{Isom}(\mathbb{H}^n_{\mathbb{R}})$ be the limit of a subsequence. Since we have

$$(h_k \circ f \circ h_k^{-1}) \circ (h_k \circ g_k^{-1}) = (h_k \circ \varphi(g_k)^{-1}) \circ f,$$

Lemma 4.2 implies that the sequence $(h_k \circ \varphi(g_k)^{-1})_{k\in\mathbb{N}}$ subconverges uniformly too. Let $l_2 \in \mathrm{Moeb}(\mathbb{S}^{n-1}) = \mathrm{Isom}(\mathbb{H}^n_{\mathbb{R}})$ be the limit of a subsequence.

The homeomorphism $h_k \circ f \circ h_k^{-1}$ conjugates $(h_k \circ g_k^{-1})\Gamma_1(h_k \circ g_k^{-1})^{-1}$ to the group $(h_k \circ \varphi(g_k)^{-1})\Gamma_2(h_k \circ \varphi(g_k)^{-1})^{-1}$. Therefore, by considering limits, the linear isomorphism L conjugates $l_1\Gamma_1 l_1^{-1}$ to $l_2\Gamma_2 l_2^{-1}$.

The field of round $(n-2)$-spheres in $\mathbb{R}^{n-1} = \partial\mathbb{H}^n_{\mathbb{R}} \setminus \{\infty\}$ is stabilized by the group $l_1\Gamma_1 l_1^{-1}$. Thus the group $l_2\Gamma_2 l_2^{-1}$ stabilizes the constant field of ellipsoids equal to $L(\mathbb{S}^{n-2})$ (measurable fields of ellipsoids are defined in Subsection 6.2).

Suppose by contradiction that L is not conformal. Then the linear subspace generated by the largest axes of $L(\mathbb{S}^{n-2})$ is a proper non-zero subspace E of $\mathbb{R}^{n-1}$. The constant Grassmannian field on $\mathbb{R}^{n-1}$ associated to E is invariant by $l_2\Gamma_2 l_2^{-1}$. Therefore the associated foliation of $\mathbb{R}^{n-1}$ is also invariant. But the induced foliation on $\mathbb{S}^{n-1} = \mathbb{R}^{n-1} \cup \{\infty\}$ is singular at ∞. Thus $l_2\Gamma_2 l_2^{-1}$ fixes ∞, which contradicts the fact that it is a lattice in $\mathrm{Isom}(\mathbb{H}^n_{\mathbb{R}})$. □

6.4 Further results

1) The zoom method is a main ingredient in the proof of the following result by R. Schwartz [123]. Let S_1 and S_2 be rank-one non-compact symmetric spaces different from $\mathbb{H}^2_{\mathbb{R}}$. Let Γ_1 and Γ_2 be non-cocompact lattices in $\mathrm{Isom}(S_1)$ and

Isom(S_2) respectively. Then, for any quasi-isometry $F : \Gamma_1 \to \Gamma_2$, there exists a homothety $h : S_1 \to S_2$ and finite-index subgroups $\Phi_i \subset \Gamma_i$ $(i = 1, 2)$, such that h conjugates Φ_1 to Φ_2 and lies within finite distance from F.

This result is in keeping with the following general problem called *pattern rigidity*. For $i = 1, 2$, let Γ_i be a finitely generated group and let $\mathcal{H}_i$ be a finite collection of quasi-convex subgroups of Γ_i. Suppose we are given a quasi-isometry $F : \Gamma_1 \to \Gamma_2$ and a constant $R \geq 0$ that enjoy the following properties :

1. For every $g_1 \in \Gamma_1$ and every $H_1 \in \mathcal{H}_1$ there exists $g_2 \in \Gamma_2$ and $H_2 \in \mathcal{H}_2$ such that $F(g_1 H_1)$ and $g_2 H_2$ lie within Hausdorff distance at most R.

2. For every $g_2 \in \Gamma_2$ and every $H_2 \in \mathcal{H}_2$ there exists $g_1 \in \Gamma_1$ and $H_1 \in \mathcal{H}_1$ such that $F(g_1 H_1)$ and $g_2 H_2$ lie within Hausdorff distance at most R.

Do there exist finite-index subgroups $\Phi_i \subset \Gamma_i$ and an isomophism $\varphi : \Phi_1 \to \Phi_2$ that lies at bounded distance from F ?

The above Schwartz theorem solves this problem in the case the Γ_i's are non-cocompact lattices in Isom($\mathbb{H}^n_{\mathbb{K}}$) and the $\mathcal{H}_i$'s are the patterns of the parabolic subgroups. The question of pattern rigidity has also been solved by Schwartz [124] for Γ a cocompact lattice in Isom($\mathbb{H}^n_{\mathbb{K}}$) and $\mathcal{H}$ a pattern associated to a finite collection of closed geodesics. In [11] Biswas and Mj generalize Schwartz's result to certain duality subgroups of Γ. Biswas [9] completely solved the pattern rigidity problem for Γ a cocompact lattice in Isom($\mathbb{H}^n_{\mathbb{R}}$) and H any infinite quasi-convex subgroup of infinite index. Mj [112] proved the following non-linear pattern rigidity result. Let Γ be a hyperbolic Poincaré duality group and let H be a quasi-convex codimension one filling subgroup. Suppose $\partial\Gamma$ carries a visual metric d such that

$$\mathrm{Hausdim}(\partial\Gamma, d) \leq \mathrm{Topdim}(\partial\Gamma) + 2.$$

Then the pattern-preserving quasi-isometry group of (Γ, H) is a finite extension of Γ.

2) The notion of zooming into the space at a point is exploited by Bonk and Kleiner in a somewhat different way in [18, 21].

Recall that a *weak tangent space* of a metric space (Z, d) is the Gromov-Hausdorff limit (as $k \to \infty$) of a sequence of pointed metric spaces of the form $(Z, z_k, \frac{1}{\varepsilon_k} d)$, where $z_k \in Z$ and $\varepsilon_k \to 0$. See [38] for more details.

Suppose that Γ is a hyperbolic group, and let d be a visual metric on $\partial\Gamma$. Bonk and Kleiner deduce from the convergence property (Subsection 4.1) that every weak tangent space of $(\partial\Gamma, d)$ is quasi-Möbius homeomorphic to $(\partial\Gamma \setminus \{w\}, d)$, for some $w \in \partial\Gamma$. In combination with previous works of B. Bowditch and G. Swarup [35, 130], they show that $\partial\Gamma$ is *linearly locally connected* as soon as it is connected. In other words there exists $C \geq 1$ such that

1. For every ball $B(z, r) \subset \partial\Gamma$ and every pair $\{w_1, w_2\} \subset B(z, r)$, there exists an arc $\gamma \subset B(z, Cr)$ that joins w_1 to w_2.

2. For every ball $B(z, r) \subset \partial\Gamma$ and every $\{w_1, w_2\} \subset \partial\Gamma \setminus B(z, r)$, there exists an arc $\gamma \subset \partial\Gamma \setminus B(z, \frac{r}{C})$ that joins w_1 to w_2.

Further developments on connectedness properties of $\partial\Gamma$ can be found in [18, 21, 108, 48].

3) The Sullivan-Tukia theorem can also be stated as follows. Suppose that Γ is a hyperbolic group whose boundary is quasi-Möbius homeomorphic to the Euclidean sphere $\mathbb{S}^{n-1}$, with $n \geq 3$. Then Γ is virtually a cocompact lattice in $\mathrm{Isom}(\mathbb{H}^n_{\mathbb{R}})$.

Bonk and Kleiner have considerably weakened the above assumption. They proved in [18] that if the boundary of a hyperbolic group Γ is quasi-Möbius homeomorphic to an $(n-1)$-regular metric space of topological dimension $n-1$, with $n \geq 3$, then Γ is virtually a cocompact lattice in $\mathrm{Isom}(\mathbb{H}^n_{\mathbb{R}})$.

4) P. Haïssinsky [81] recently proved the following generalization of Sullivan's theorem: every finitely generated group that admits a quasi-isometric embedding into $\mathbb{H}^3_{\mathbb{R}}$, contains a finite-index subgroup that acts quasi-convex geometrically on $\mathbb{H}^3_{\mathbb{R}}$ (see Definition 5.3).

5) T. Dymarz and X. Xie [64] established Sullivan-Tukia type theorems for actions of amenable groups on the boundary of certain (non symmetric) negatively curved homogeneous manifolds M. In particular, for $M = \mathbb{R}^n \rtimes_A \mathbb{R}$ where A is an expansive matrix diagonalisable over $\mathbb{C}$, they proved that every amenable group Γ that acts on M by uniform quasi-conformal homeomorphisms and cocompactly on $\partial^3 M$, is conjugate to a conformal group. They obtained applications to quasi-isometry rigidity of lattices in certain solvable Lie groups like cyclic extensions of abelian groups.

7 Rigidity of quasi-isometries

The proof of the Mostow theorem shows that a quasi-isometry of a rank-one symmetric space (different from $\mathbb{H}^2_{\mathbb{R}}$) which is equivariant with respect to a lattice, lies within bounded distance from an isometry. What about the non-equivariant quasi-isometries? This section gives examples of hyperbolic spaces where every quasi-isometry lies within bounded distance from an isometry. In particular we will give some ideas of the proof of the following theorem.

Theorem 7.1 (Pansu [116])**.** *Let $S = \mathbb{H}^n_{\mathbb{Q}}$ with $n \geq 2$, or $\mathbb{H}^2_{\mathbb{O}}$. Then any quasi-isometry of S lies within bounded distance from an isometry.*

We note that Theorem 7.1 is false for $\mathbb{H}^n_{\mathbb{R}}$ and $\mathbb{H}^n_{\mathbb{C}}$ (see [103] for the complex case). Observe that Theorem 7.1, in combination with Proposition 2.15, implies that the Sullivan-Tukia theorem (Theorem 6.2) holds for $\mathbb{H}^n_{\mathbb{Q}}$ and $\mathbb{H}^2_{\mathbb{O}}$.

Subsections 7.1 and 7.2 are devoted to the proof of Theorem 7.1. Subsection 7.3 discusses quasi-isometry rigidity of Fuchsian buildings. Subsection 7.4 contains a survey on several related results.

7.1 Differentiability in Carnot groups

The boundary of $\mathbb{H}^n_{\mathbb{K}}$ minus a point is modeled on a Carnot group (see Subsection 3.3). Pansu defined a notion of differentiability in Carnot groups by using the Carnot homotheties to zoom in at a point (Definition 7.2). He proved a Rademacher-Stepanov type theorem for quasi-conformal homeomorphisms of Carnot groups (Theorem 7.3). This result, with a compactness property (Proposition 7.4), form together the core of the proof of Theorem 7.1.

Let $(N, \mathfrak{v})$ be a Carnot group and $(\delta_t)_{t\in\mathbb{R}}$ its Carnot homotheties (see Definition 3.6). We fix a scalar product on $\mathfrak{v}$ and we propagate $\mathfrak{v}$ to a N-invariant distribution in the tangent space of N. These data determine a Carnot-Carathéodory metric d_N on N, which is N-invariant and multiplied by e^t under δ_t (see Subsection 3.4).

Definition 7.2 ([116])**.** Let N, N' be Carnot groups with Carnot homotheties δ_t, δ'_t respectively. A map $f : N \to N'$ is *δ-differentiable* at $z \in N$ if the maps

$$w \in N \mapsto \delta'_t(f(z)^{-1} f(z\delta_{-t}(w)) \in N'$$

converge as $t \to +\infty$, uniformly on compact subsets of N, to a group homomorphism $D_z f : N \to N'$, which commutes with δ_t and δ'_t.

We equip N with its Haar measure (which coincides with the Hausdorff measure of (N, d_N), see for instance [84]). The following version of Rademacher-Stepanov's theorem holds :

Theorem 7.3 ([116])**.** *Every quasi-conformal homeomorphism $f : (N, d_N) \to (N', d_{N'})$ is δ-differentiable almost everywhere.*

Recall from Subsection 3.3, that $\partial\mathbb{H}^n_{\mathbb{K}} \setminus \{\infty\}$ may be identified with a Carnot group $(N, \mathfrak{v})$. Its Lie algebra is $\mathfrak{n} = \mathbb{K}^{n-1} \oplus \Im\mathbb{K}$, with $\Im\mathbb{K}$ central, and for $x, y \in \mathbb{K}^{n-1}$:

$$[x, y] = \Im\{x, y\}.$$

The horizontal space is $\mathfrak{v} = \mathbb{K}^{n-1}$, and $[\mathfrak{v}, \mathfrak{v}] = \Im\mathbb{K}$ ([2]). Let α be the associated derivation of $\mathfrak{n}$. The key point in the proof of Theorem 7.1, which distinguishes between the case $\mathbb{R}, \mathbb{C}$ and the case $\mathbb{Q}, \mathbb{O}$, is the following compactness result :

Proposition 7.4 ([116])**.** *For $\mathbb{K} = \mathbb{Q}$ or $\mathbb{O}$, the subgroup of* $\mathrm{Aut}(\mathfrak{n})$ *which commutes with $(e^{t\alpha})_{t\in\mathbb{R}}$, is a semidirect product of a compact group with $(e^{t\alpha})_{t\in\mathbb{R}}$.*

The following proof of Proposition 7.4 is rather different from [116]. It relies on two lemmata. The first one, due to A. Kaplan and A. Tiraboschi, is of general interest.

Let $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ be a two-step nilpotent Lie algebra with center $\mathfrak{z}$. Let α be the associated derivation of $\mathfrak{n}$, and let $H \subset \mathrm{Aut}(\mathfrak{n})$ be the subgroup that commutes with $(e^{t\alpha})_{t\in\mathbb{R}}$. It can be written as

$$H = \{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a \in GL(\mathfrak{v}),\ b \in GL(\mathfrak{z}),\ [ax, ay] = b([x, y])\}.$$

Let $H_1 = \{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in H \mid \det(a) = \pm 1\}$ and $H_0 = \{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in H_1 \mid b = \mathrm{id}\}$.

[2] These expressions have been established in Subsection 3.3 for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{Q}$. They are also valid when $\mathbb{K} = \mathbb{O}$, see [113] p. 141.

We remark that H is the semidirect product of H_1 with $(e^{t\alpha})_{t\in\mathbb{R}}$, and that H_0 is closed and normal in H_1. Moreover the group H_1/H_0 is isomorphic to the image of H_1 by the projection map

$$\pi : \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in H \mapsto b \in GL(\mathfrak{z}).$$

Lemma 7.5 (Kaplan-Tiraboschi [94]). *Suppose that for every non-zero $x \in \mathfrak{v}$, the map* $\mathrm{ad}x : \mathfrak{n} \to \mathfrak{z}$ *is surjective. Then $\pi(H_1)$ is a compact subgroup of $GL(\mathfrak{z})$.*

Proof. We follow the proof in [94]. Fix arbitrary scalar products on $\mathfrak{v}$ and $\mathfrak{z}$. For $x, y \in \mathfrak{v}$ and $z \in \mathfrak{z}$, the relation

$$\langle x, T_z y\rangle_{\mathfrak{v}} = \langle [x, y], z\rangle_{\mathfrak{z}}$$

defines a linear map $z \mapsto T_z$ from $\mathfrak{z}$ to $\mathrm{End}(\mathfrak{v})$. By our hypothesis on $\mathrm{ad}x$, one has $T_z \in GL(\mathfrak{v})$ for $z \neq 0$. Set

$$P(z) := \det(T_z).$$

It is a homogeneous polynomial that is nonzero on $\mathfrak{z} \setminus \{0\}$. Therefore its level sets are bounded.

Let now $g = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in H$. Then we have

$$T_{b^* z} = a^* T_z a.$$

Indeed, $\langle x, T_{b^*z} y\rangle_{\mathfrak{v}} = \langle [x, y], b^* z\rangle_{\mathfrak{z}} = \langle b([x, y]), z\rangle_{\mathfrak{z}} = \langle [ax, ay], z\rangle_{\mathfrak{z}} = \langle ax, T_z ay\rangle_{\mathfrak{v}} = \langle x, a^* T_z ay\rangle_{\mathfrak{v}}$. Therefore $P(b^* z) = (\det a)^2 P(z)$, and so P is $\pi(H_1)^*$-invariant. Since the level sets of P are bounded, the group $\pi(H_1)^*$ is bounded in $GL(\mathfrak{z})$, and so is $\pi(H_1)$.

On the other hand, H_1 is a real algebraic group and π is a regular map. Thus $\pi(H_1)$ is open in its closure (see [146] 3.1.3). Since we know from above that the closure of $\pi(H_1)$ is compact, we obtain that $\pi(H_1)$ is compact too. □

Lemma 7.6. *Let $\mathfrak{n}$ be a Lie algebra as in Proposition 7.4. Then the group H_0 is compact.*

Proof. We prove it for $\mathbb{Q}$, and leave the reader adapt the proof to the octonions. Let $\langle \cdot, \cdot \rangle$ be the scalar product on $\mathfrak{v}$ defined by $\langle x, y\rangle = \Re\{x, y\}$. Let i, j, k be the standard basis of $\Im\mathbb{Q}$. For $\alpha = i, j, k$, denote by ω_α the symplectic form on $\mathfrak{v}$, so that

$$[x, y] = \omega_i(x, y)i + \omega_j(x, y)j + \omega_k(x, y)k.$$

These forms are given by

$$\omega_\alpha(x, y) = -\langle x, T_\alpha y\rangle,$$

where T_α is the right multiplication by α.

Let now $a \in GL(\mathfrak{v})$ such that $[ax, ay] = [x, y]$. For $\alpha = i, j, k$, we have $\omega_\alpha(ax, ay) = \omega_\alpha(x, y)$, and so

$$a^* T_\alpha a = T_\alpha.$$

This formula applied to a^{-1} yields by taking the inverses in both sides:

$$aT_\alpha a^* = T_\alpha,$$

because $T_\alpha^{-1} = -T_\alpha$. It follows that $aT_iT_j = T_i(a^*)^{-1}T_j = T_iT_ja$. Thus a commutes with $T_iT_j = -T_k$. Since we also have $a^*T_ka = T_k$, we obtain that $a^*a = \mathrm{id}$. Thus a preserves the scalar product. □

Proof of Proposition 7.4. From Lemma 7.5, the group H_1/H_0 is compact. According to Lemma 7.6, H_0 is compact too. Thus H_1 is compact. □

7.2 Proof of Theorem 7.1

Suppose that $S = \mathbb{H}^n_{\mathbb{Q}}$ or $\mathbb{H}^2_{\mathbb{O}}$. Recall from Proposition 3.7 that the metrics d_{CAT} and d_{CC} are Lipschitz equivalent on ∂S. Moreover we have $\mathrm{Moeb}(\partial S, d_{\mathrm{CAT}}) = \mathrm{Isom}(S)$, thanks to Theorem 2.19.

Let $G \subset \mathrm{Homeo}(\partial S)$ be the group of quasi-Möbius homeomorphisms. Theorem 7.3 and Proposition 7.4 show that there exists a $H \geq 1$ such that every $g \in G$ is a.e. H-quasi-conformal. This implies – because ∂S is Loewner – that G is a uniform quasi-conformal group (see Theorems 4.7 and 4.10).

Let $\Gamma \subset \mathrm{Isom}(S)$ be a cocompact lattice. It follows from above and from Subsection 2.4, that G and Γ satisfy the assumptions of Corollary 4.11. Therefore $G \subset \mathrm{Moeb}(\partial S, d_{\mathrm{CAT}}) = \mathrm{Isom}(S)$. Theorem 2.12 completes the proof. □

Pansu's original proof differs a little bit at the end from the one above. His argument is the following. Theorem 7.3 and Proposition 7.4 imply that every $g \in G$ is a 1-quasi-conformal homeomorphism of $(\partial S, d_{\mathrm{CC}})$. Pansu proves that every 1-quasi-conformal homeomorphism f of N is the boundary extension of an isometry of S. To do so, he shows that f can by written as a Carnot homothety composed with an isometry of N. The argument is based on the fact that 1-quasi-conformal homeomorphisms preserve moduli of curves (see subsection 4.3). To complete the proof, it remains to show that the isometries of N that fix the identity lie in $\mathrm{Aut}(N)$, and that $\mathrm{Aut}(N) \subset \mathrm{Isom}(S)$.

7.3 Right-angled Fuchsian buildings

Another family of hyperbolic spaces for which rigidity of quasi-isometries holds is provided by the so-called *Fuchsian buildings*. Fuchsian buildings are Tits buildings whose apartments are isomorphic to a Coxeter tiling of $\mathbb{H}^2_{\mathbb{R}}$. For simplicity we will consider only the right-angled ones. We refer to [61] and its references for Tits buildings, and to [28, 33] for right-angled Fuchsian buildings.

A nice way to define right-angled Fuchsian buildings uses complexes of groups. Let $r \geq 5$ be an integer, let R be a regular right angled r-gon in $\mathbb{H}^2_{\mathbb{R}}$, and let $(q_1, ..., q_r)$ be an r-tuple of integers with all $q_i \geq 2$. We label clockwise the edges of R by $\{1\}, ..., \{r\}$, and its vertices by $\{1, 2\}, ..., \{r-1, r\}, \{r, 1\}$ in a way compatible with the adjacence relation. Then we define a complex of groups as follows. To the face of R, we attach the group $\Gamma_\emptyset := \{1\}$; to the edge $\{i\}$, the group $\Gamma_{\{i\}} :=$

$\mathbb{Z}/(q_i+1)\mathbb{Z}$; to the vertice $\{i, i+1\}$, the group $\Gamma_{\{i,i+1\}} := \Gamma_{\{i\}} \times \Gamma_{\{i+1\}}$. An abstract theorem of Haefliger [37] shows that the complex of groups is *developable*, i.e.

1. There exits a contractible 2-cell complex Δ, called the *universal cover* of the complex of groups; its 0 and 1-cells are labelled by the same symbols as above, and its 2-cells are isomorphic to the labeled complex R.

2. There exists a group Γ, called the *fundamental group* of the complex of groups. It acts geometrically and label-preserving on Δ, with $\Gamma\backslash\Delta = R$, and in such a way that the stabilizer in Γ of a 2-cell, or a 1-cell $\{i\}$, or a 0-cell $\{i, i+1\}$, is isomorphic to the group $\Gamma_\emptyset$, $\Gamma_{\{i\}}$, $\Gamma_{\{i,i+1\}}$, respectively.

The group Γ admits the following presentation

$$\Gamma = \langle s_i, i \in \mathbb{Z}/r\mathbb{Z} \mid s_i^{q_i+1} = 1, [s_i, s_{i+1}] = 1 \rangle.$$

The link of a vertex $\{i, i+1\}$ of Δ is the complete bipartite graph with $(q_i+1) + (q_{i+1}+1)$ vertices ([3]). Since this graph is a spherical building of girth 4, and since $4 \times \pi/2 = 2\pi$, one obtains that Δ is a *building*. More precisely:

1. Its chambers are the 2-cells; its apartments are the subcomplexes that are isomorphic to the Coxeter tiling of $\mathbb{H}^2_{\mathbb{R}}$ by copies of R.

2. Any pair of chambers is contained in an apartment.

3. If A_1 and A_2 are apartments whose intersection is not empty, there exists an isomorphism of labelled cell complexes $\phi : A_1 \to A_2$ stabilizing pointwise $A_1 \cap A_2$.

We endow Δ with the length metric induced by its 2-cells. Then Δ is a CAT(-1)-space. It enjoys the following properties:

- Its boundary is homeomorphic to the Menger curve [4, 99, 63].

- The group Isom(Δ) is locally compact totally disconnected, and Γ is a co-compact lattice.

- The label-preserving isometries of Δ form a non-linear, simple, finite index subgroup of Isom(Δ) [77].

Theorem 7.7 ([33])**.** *Let Δ_1 and Δ_2 be right-angled Fuchsian buildings. Then any quasi-isometry $F : \Delta_1 \to \Delta_2$ lies within bounded distance from an isometry.*

As a consequence, the Mostow theorem and the Sullivan-Tukia theorem hold for the right-angled Fuchsian buildings too. Xie [140] generalized Theorem 7.7 to all Fuchsian buildings that admit a geometric action.

The strategy of the proof is similar in spirit to the one of Theorem 7.1. One proves that there exists an $H > 0$ such that every quasi-Möbius homeomorphism

[3] The link of a vertex v is the graph whose vertices are the edges of Δ that contain v, and whose edges are the pair of edges of Δ that are contained in a 2-cell.

$f : \partial\Delta_1 \to \partial\Delta_2$ is H-quasi-conformal. The analytic ingredients consist of a Loewner metric on $\partial\Delta$ and some reasonable differential properties of quasi-Möbius homeomorphisms $f : \partial\Delta_1 \to \partial\Delta_2$. More precisely, we show that for almost every $z \in \partial\Delta_1$ and every apartment $A \subset \Delta_1$ such that $z \in \partial A$, the limit

$$f'_{\partial A}(z) := \lim_{\substack{w \in \partial A \\ w \to z}} \frac{d(f(w), f(z))}{d(w, z)}$$

exists and belongs to $(0, +\infty)$. To prove that f is H-quasi-conformal for some uniform H, a key observation is that $f'_{\partial A}(z)$ depends only on z and not on ∂A. This follows from that fact that for every pair of apartments A_1, A_2 with $z \in \partial A_i$, there exists an apartment A_3 whose boundary contains z and locally coincides with ∂A_1 on the left side of z, and with ∂A_2 on the right side of z.

It is worth mentioning that, although Δ is a CAT(-1)-space, the metric d_{CAT} on $\partial\Delta$ is not Loewner. The Loewner metric on $\partial\Delta$ is associated to a chamber distance in Δ, see [32, 33].

7.4 Futher results

1) Kapovich and Kleiner [99] gave examples of hyperbolic groups Γ with connected boundary such that $\mathrm{Homeo}(\partial\Gamma) = \Gamma$. Therefore any quasi-isometry of Γ lies within bounded distance from the left multiplication by a $g \in \Gamma$. This situation contrasts with the case of rank-one symmetric spaces and Fuchsian buildings, where the homeomorphim group of the boundary is infinite dimensional.

2) Two subgroups Γ_1 and Γ_2 of a group G are said to be *commensurable* if there is $g \in G$ such that $g\Gamma_1 g^{-1} \cap \Gamma_2$ is of finite index in $g\Gamma_1 g^{-1}$ and in Γ_2. F. Haglund [76] worked out the commensurability question of cocompact lattices in the isometry group of certain negatively curved simplicial complexes. In particular, for right-angled Fuchsian buildings Δ (with $r \geq 5$ and $q_i = q \geq 3$) he showed that all cocompact lattices in $\mathrm{Isom}(\Delta)$ are commensurable and linear.

Note that commensurability and linearity fail for non-cocompact lattices in $\mathrm{Isom}(\Delta)$ [131, 121]. For a survey on recent developments on buildings and groups, see for instance [120].

3) ([4]) By Heintze's theorem [92], every negatively curved homogeneous manifold is isometric to a solvable Lie group (with a certain left invariant Riemannian metric) of the form $G = N \rtimes \mathbb{R}$, where N is a simply connected nilpotent Lie group , and $\mathbb{R}$ acts on N by expanding automorphisms. More precisely, the $\mathbb{R}$-action is given by $e^{t\alpha}$, where α is a derivation of the Lie algebra $\mathfrak{n}$ of N whose eigenvalues all have positive real parts. Such groups G are called *Heintze groups*. When all eigenvalues are real, G is said to be *purely real*. It is known that every Heintze group is bi-Lipschitz homeomorphic to a purely real Heintze group [57]. A Heintze group G is of *Carnot type*, if it is bi-Lipschitz homeomorphic to $N \rtimes_\alpha \mathbb{R}$, where N is a Carnot group and α is the derivation associated to the Carnot decomposition

[4] I owe Matias Carrasco several explanations about the material in this paragraph.

of $\mathfrak{n}$, see Definition 3.6. Observe that the boundary at infinity of every Heintze group G is identified canonically with the one point compactification $N \cup \{\infty\}$.

There are three major conjectures about Heintze groups G:

Conjecture 7.8 (Pointed sphere conjecture [57]). *Any quasi-isometry of G stabilizes ∞ unless G is bi-Lipschitz homeomorphic to a rank-one symmetric space.*

Conjecture 7.9 (Quasi-isometric classification [85, 57]). *Let G_1 and G_2 be purely real Heintze groups. If they are quasi-isometric, then they are isomorphic.*

Conjecture 7.10 (Quasi-isometric rigidity [143]). *Any quasi-isometry $F : G_1 \to G_2$ between Heintze groups lies at bounded distance from an almost similarity* ([5]) *unless one of the groups is bi-Lipschitz homeomorphic to a symmetric space.*

Conjecture 7.8 was proved by Pansu [117] for the non-Carnot type Heintze groups with α diagonalizable over $\mathbb{C}$. It was generalized by Carrasco [47] to all non-Carnot type Heintze groups. Conjecture 7.9 was established by Pansu [116] for the Carnot type Heintze groups, by using Theorem 7.3. The three conjectures hold when N is abelian (Shanmugalingam-Xie [125] and Xie [142]). They also hold when N is a Heisenberg group and α is diagonalizable over $\mathbb{R}$ [143]. Conjecture 7.10 was proved by Carrasco [47] for the non-Carnot type Heintze groups, and by Le Donne and Xie [105] for the reducible ([6]) Carnot type Heintze groups.

8 Some recent developments and perspectives

This section reports on recent progress on some open problems about quasi-conformal geometry of group boundaries. We discuss the Cannon conjecture in Subsection 8.1 and the combinatorial Loewner property in Subsection 8.2. As usual the section ends with a survey of several related results.

8.1 Cannon conjecture

A major problem in geometric group theory is the following conjecture.

Conjecture 8.1 (Cannon's conjecture [45]). *Suppose Γ is a hyperbolic group whose boundary is homeomorphic to the Euclidean 2-sphere, then Γ acts geometrically on $\mathbb{H}^3_{\mathbb{R}}$.*

Historically, this conjecture was motivated by Thurston's hyperbolization conjecture (recently solved by Perelman): *Every closed, aspherical, irreducible, atoroidal 3-manifold admits a Riemannian metric of constant curvature* -1. Cannon's conjecture also provides an approach to solve an open problem due to Wall: *Is every 3-dimensional Poincaré duality group a 3-manifold group?*

Sullivan-Tukia rigidity (Theorem 6.2) implies that Cannon's conjecture is equivalent to:

[5] An almost similarity is a quasi-isometry whose left and right handside multiplicative constants are equal.

[6] A Carnot group $(N, \mathfrak{v})$ is *reducible* if the horizontal subspace $\mathfrak{v}$ contains a proper non-trivial subspace that is invariant by the graded automorphisms of $(N, \mathfrak{v})$.

Conjecture 8.2 (Uniformization conjecture)**.** *Suppose that* Γ *is a hyperbolic group whose boundary is homeomorphic to the Euclidean sphere* $\mathbb{S}^2$*, then* $\partial\Gamma$ *is quasi-Möbius homeomorphic to* $\mathbb{S}^2$*.*

Hence Cannon's conjecture reduces to a problem of quasi-conformal geometry. We remark that the analogous problem for the circle is solved, since every approximately self-similar metric circle is quasi-Möbius homeomorphic to the Euclidean one [135]. It follows that the analog of Cannon's conjecture for $\mathbb{H}^2_{\mathbb{R}}$ holds (see the discussion after Theorem 6.2). On the other hand, there is no analog of Cannon's conjecture for $\mathbb{H}^n_{\mathbb{R}}$ when $n \geq 4$. For example $\partial\mathbb{H}^2_{\mathbb{C}}$ and $\partial\mathbb{H}^4_{\mathbb{R}}$ are topological 3-spheres, but $\mathbb{H}^2_{\mathbb{C}}$ and $\mathbb{H}^4_{\mathbb{R}}$ are not quasi-isometric (Corollary 4.8). See subsection 8.3 for more examples.

We describe in the sequel an approach to Conjecture 8.2 based on the so-called *combinatorial modulus* – a combinatorial variant of the analytic modulus defined in Subsection 4.3. Versions of the combinatorial modulus have been considered by several authors in connection with the Cannon conjecture [41, 45, 43, 17, 79], and in a more general context [117, 137]. Our presentation follows [30].

Let Z be a compact metric space, let $k \in \mathbb{N}$, and let $\kappa \geq 1$. A finite graph G_k is called a *κ-approximation of Z on scale k*, if it is the incidence graph of a covering of Z, such that for every vertex $v \in G^0_k$ there exists $z_v \in Z$ with

$$B(z_v, \kappa^{-1}2^{-k}) \subset v \subset B(z_v, \kappa 2^{-k}),$$

and for $v, w \in G^0_k$ with $v \neq w$:

$$B(z_v, \kappa^{-1}2^{-k}) \cap B(z_w, \kappa^{-1}2^{-k}) = \emptyset.$$

Note that we identify every vertex v of G_k with the corresponding subset in Z. A collection of graphs $\{G_k\}_{k\in\mathbb{N}}$ is called a *κ-approximation* of Z, if for each $k \in \mathbb{N}$ the graph G_k is a κ-approximation of Z on scale k.

Let $\gamma \subset Z$ be a continuous curve and let $\rho : G^0_k \to \mathbb{R}_+$ be any function. The *ρ-length* of γ is

$$L_\rho(\gamma) = \sum_{v\cap\gamma\neq\emptyset} \rho(v).$$

For $p \geq 1$ the *p-mass* of ρ is

$$M_p(\rho) = \sum_{v\in G^0_k} \rho(v)^p.$$

Let $\mathcal{F}$ be a non-empty family of (continuous) curves in Z. We define the *G_k-combinatorial p-modulus of $\mathcal{F}$* by

$$\mathrm{Mod}_p(\mathcal{F}, G_k) = \inf_\rho M_p(\rho),$$

where the infimum is over all *$\mathcal{F}$-admissible functions*, i.e. functions $\rho : G^0_k \to \mathbb{R}_+$ which satisfy $L_\rho(\gamma) \geq 1$ for every $\gamma \in \mathcal{F}$. It enjoys the following properties

1. The function $\mathrm{Mod}_p(\cdot, G_k)$ is non-decreasing and finitely subadditive.

2. If every curve in $\mathcal{F}_2$ contains a curve in $\mathcal{F}_1$, then one has $\mathrm{Mod}_p(\mathcal{F}_2, G_k) \leq \mathrm{Mod}_p(\mathcal{F}_1, G_k)$.

3. When Z is a doubling metric space ([7]) the combinatorial modulus does not depend on the choice of the graph approximation, up to a multiplicative constant.

4. If Z_1 is quasi-Möbius homeomorphic to a p-regular metric space Z_2, then the combinatorial p-modulus on Z_1 is comparable to the analytic p-modulus on Z_2 [79].

The following quasi-Möbius characterization of the Euclidean 2-sphere appears in [30]. It is a straightforward application of techniques and results developed in [43, 17].

Theorem 8.3. *Suppose that Z is an approximately self-similar metric 2-sphere. Let $\{G_k\}_{k\in\mathbb{N}}$ be a κ-approximation of Z. For $d_0 > 0$ denote by $\mathcal{F}_0$ the set of curves $\gamma \subset Z$ with $\mathrm{diam}(\gamma) \geq d_0$. Then Z is quasi-Möbius homeomorphic to the Euclidean 2-sphere, if and only if, for every $d_0 > 0$ small enough, there exists a constant $C \geq 1$ such that for every $k \in \mathbb{N}$ one has*

$$\mathrm{Mod}_2(\mathcal{F}_0, G_k) \leq C\,.$$

Sketch of proof. The direct implication is a consequence of Property 4 above. Our proof of the reverse implication combines several arguments from [17]. First, self-similarity allows one to improve the modulus control assumption as follows. There exists a positive increasing function ψ of $(0, +\infty)$ with $\lim_{t\to 0}\psi(t) = 0$, such that for every pair of disjoint non-degenerate compact connected subsets $A, B \subset Z$ and every integer k satisfying $2^{-k} \leq \min\{\mathrm{diam}(A), \mathrm{diam}(B)\}$, one has

$$\mathrm{Mod}_2(A, B, G_k) \leq \psi(\Delta(A, B)^{-1}), \tag{8.4}$$

where $\Delta(A, B)$ denotes the relative distance, defined in (4.5).

Secondly, Z being an approximately self-similar compact manifold, it is doubling and *linearly locally contractible* i.e. there exists a constant $\lambda \geq 1$ such that every ball $B(z, r) \subset Z$ with $0 < r < \mathrm{diam}(Z)/\lambda$ is contractible in $B(z, \lambda r)$. Bonk and Kleiner use the last properties to construct a κ-approximation $\{G_k\}_{k\in\mathbb{N}}$ of Z such that each graph G_k is (essentially) homeomorphic to the 1-skeleton of a triangulation of the 2-sphere. Therefore, according to the Andreev-Koebe-Thurston theorem, G_k is the incidence graph of a circle packing in the Euclidean sphere $\mathbb{S}^2$ (unique up to a homography). For every $k \in \mathbb{N}$ and $v \in G_k^0$, let $z_v \in Z$ be such that $B(z_v, \kappa^{-1}2^{-k}) \subset v \subset B(z_v, \kappa 2^{-k})$. We obtain a map $f_k : \{z_v \mid v \in G_k^0\} \to \mathbb{S}^2$

[7] A metric space is said to be *doubling* if there is a constant $n \in \mathbb{N}$ such that every ball $B(z, r)$ can be covered by at most n balls of radius $r/2$. For instance Ahlfors-regular metric spaces (Definition 2.13) are doubling.

which sends z_v to the center of the corresponding disc. Now the f_k's, when appropriately normalized, form an equicontinuous sequence that converges to a homeomorphism $f : Z \to \mathbb{S}^2$, which satisfies for every pair of disjoint non-degenerate compact connected subsets $A, B \subset Z$:

$$\Delta(f(A), f(B)) \geq \eta(\Delta(A, B)), \tag{8.5}$$

where η is a positive function of $(0, +\infty)$ that satisfies $\lim_{t \to +\infty} \eta(t) = 0$. To prove these properties, one uses the modulus bounds (8.4), and relates the modulus associated to circle packings with the analytic modulus of $\mathbb{S}^2$. Finally, inequality (8.5) implies that f is quasi-Möbius (Lemmata 2.10 and 3.3 in [17]). □

Theorem 8.3 provides a strategy to prove Cannon's conjecture. Let Γ be a 2-sphere boundary hyperbolic group. To show that $\partial\Gamma$ satisfies the assumptions of Theorem 8.3, one can try to use the action of Γ on $\partial\Gamma$. A crucial observation is the following. Say that two curve families $\mathcal{F}, \mathcal{G}$ in $\partial\Gamma$ *cross*, if every curve in $\mathcal{F}$ crosses every curve in $\mathcal{G}$. When $\mathcal{F}, \mathcal{G}$ cross, then one has:

$$\mathrm{Mod}_2(\mathcal{F}, G_k) \cdot \mathrm{Mod}_2(\mathcal{G}, G_k) \leq 1. \tag{8.6}$$

This is indeed a classical property of the analytic modulus in the Euclidean 2-sphere. It generalizes to metric 2-spheres and combinatorial 2-modulus [43, 79]. Now, suppose that we are given a curve family $\mathcal{F}$ in $\partial\Gamma$ and $g \in \Gamma$ such that $\mathcal{F}$ and $g(\mathcal{F})$ cross. Since $\mathrm{Mod}_2(\cdot, G_k)$ is invariant by bi-Lipschitz homeomorphism up to a multiplicative constant, we obtain from (8.6) that $\mathrm{Mod}_2(\mathcal{F}, G_k)$ is bounded by above independently of k.

This strategy, in a more elaborate form, can be applied to Coxeter groups. We obtain ([8]):

Theorem 8.7 ([30]). *Let Γ be a hyperbolic Coxeter group whose boundary is homeomorphic to the 2-sphere. Then Γ acts geometrically on $\mathbb{H}^3_{\mathbb{R}}$.*

8.2 Combinatorial Loewner property

Most of the rigidity results discussed so far rely on the Loewner property of the boundary. Unfortunately, among the currently known examples of Loewner spaces, the only ones which arise as boundaries of hyperbolic groups are the boundaries of rank-one symmetric spaces and Fuchsian buildings. In order to make a step toward improving this situation, Kleiner [102] introduced a combinatorial variant called the combinatorial Loewner property.

Suppose thet Z is an arcwise connected compact doubling metric space. Let $\{G_k\}_{k \in \mathbb{N}}$ be a κ-approximation of Z. Denote by $\mathcal{F}(A, B)$ the family of curves

[8] M. Davis pointed out to the author that Theorem 8.7 can be also established as follow. A theorem of Bestvina-Mess [8] and the boundary hypothesis show that Γ is a virtual 3-dimensional Poincaré duality group. Then Theorem 10.9.2 of [61] implies that Γ decomposes as $\Gamma = \Gamma_0 \times \Gamma_1$, where Γ_0 is a finite Coxeter group and where Γ_1 is a Coxeter group whose nerve is a 2-sphere. By applying Andreev's theorem to the dual polyhedron to the nerve, one obtains that Γ_1 acts on $\mathbb{H}^3_{\mathbb{R}}$ as a cocompact reflection group.

joining two subsets $A, B \subset Z$ and by $\operatorname{Mod}_p(A, B, G_k)$ its G_k-combinatorial p-modulus. Recall that $\Delta(A, B)$ denotes the relative distance defined in (4.5).

Definition 8.8. Suppose that $p > 1$. Then Z satisfies the *combinatorial p-Loewner property* if there exist two positive increasing functions ϕ, ψ on $(0, +\infty)$ with $\lim_{t\to 0} \psi(t) = 0$, such that

1. For every pair of disjoint compact connected subsets $A, B \subset Z$, and every k with $2^{-k} \leq \min\{\operatorname{diam} A, \operatorname{diam} B\}$, one has:
$$\phi(\Delta(A, B)^{-1}) \leq \operatorname{Mod}_p(A, B, G_k).$$
2. For every pair of concentric balls $B(z, r), B(z, R) \subset Z$ with $0 < 2r \leq R$, and every k with $2^{-k} \leq r$, one has:
$$\operatorname{Mod}_p(B(z, r), Z \setminus B(z, R), G_k) \leq \psi(r/R).$$

We say that Z satisfies the *combinatorial Loewner property* (CLP) if it satisfies the combinatorial p-Loewner property for some $p > 1$.

Theorem 8.9 ([30])**.**

1. *If Z is a compact Ahlfors p-regular Loewner space, then Z satisfies the combinatorial p-Loewner property.*
2. *If Z_1 satisfies the CLP and Z_2 is quasi-Möbius homeomorphic to Z_1, then Z_2 also satisfies the CLP (with the same exponent).*

Conversely, Kleiner made the following conjecture.

Conjecture 8.10 ([102])**.** *If Z satisfies the CLP and is approximately self-similar, then Z is quasi-Möbius homeomorphic to a Loewner space.*

Since the CLP is invariant under quasi-Möbius homeomorphisms — unlike the Loewner property (see Theorem 4.7) — it is in principle easier to verify that a given metric space admits the CLP, than to prove that it is quasi-Möbius homeomorphic to a Loewner space.

In addition to the already known Loewner spaces, examples of metric spaces that satisfy the CLP include the standard Sierpinski carpet and Menger curve [30], the boundaries of some hyperbolic Coxeter groups [30], and the boundaries of some hyperbolic buildings of dimension 3 and 4 [52]. See also [31] for examples of hyperbolic group boundaries which do not satisfy the CLP.

8.3 Further results

1) Gromov and Thurston [75] constructed, for every $n \geq 4$, examples of compact n-manifolds whose sectional curvatures are arbitrarily close to -1, but whose universal covers are not quasi-isometric to any symmetric space. Y. Benoist [5] constructed examples of compact locally CAT(-1) 4-manifold, whose universal cover is not quasi-isometric to any symmetric spaces, but whose fundamental groups admit a properly discontinuous cocompact projective action on a strictly convex open subset of the projective space $\mathbb{P}^4_{\mathbb{R}}$.

2) Bonk and Kleiner [17] obtained several quasi-Möbius characterizations of the Euclidean sphere $\mathbb{S}^2$. In particular

- Let Z be an Ahlfors 2-regular metric space homeomorphic to $\mathbb{S}^2$. Then Z is quasi-Möbius to $\mathbb{S}^2$ if and only if Z is linearly locally contractible (see the definition in the proof of Theorem 8.3).
- Let $Q \geq 2$ and Z be an Ahlfors Q-regular metric space homeomorphic to $\mathbb{S}^2$. If Z is Loewner, then $Q = 2$ and Z is quasi-Möbius to $\mathbb{S}^2$.

The techniques of proof are those at the origin of Theorem 8.3.

3) As a consequence of Theorem 4.7, the Hausdorff dimension of a Loewner space is minimal among all quasi-Möbius homeomorphic Ahlfors-regular metric spaces. Bonk and Kleiner [20] established a converse statement for boundaries of hyperbolic groups. Let $Q > 1$ and let Z be an Ahlfors Q-regular metric space quasi-Möbius homeomorphic to the boundary of a hyperbolic group. Suppose Q is minimal among all Ahlfors-regular metric spaces quasi-Möbius homeomorphic to Z. Then Z is a Loewner space.

4) Kapovich and Kleiner [99] conjectured that every hyperbolic group whose boundary is homeomorphic to the Sierpinski carpet, is virtually the fundamental group of a compact 3-manifold with constant curvature -1 and non-empty totally geodesic boundary. They observed that this conjecture is implied by Cannon's conjecture.

Bonk and Kleiner [13] announced the Kapovich-Kleiner conjecture under the additional assumption that the boundary is quasi-Möbius homeomorphic to a Q-regular metric space with $Q < 2$.

Haïssinsky [81] proved more generally that a hyperbolic group whose boundary is planar and quasi-Möbius homeomorphic to a Q-regular metric space with $Q < 2$, acts quasi-convex geometrically on $\mathbb{H}^3_{\mathbb{R}}$.

In [30] the Kapovich-Kleiner conjecture is established for Coxeter groups, as a consequence of Theorem 8.7.

5) Markovic [109] proved that Cannon's conjecture holds if one assumes in addition that every two distint points in $\partial\Gamma$ can be separated by the limit set of a quasi-convex surface subgroup of Γ. Haïssinsky [81] proved more generally that a hyperbolic group, whose boundary is planar and satisfies the above separation assumption, acts quasi-convex geometrically on $\mathbb{H}^3_{\mathbb{R}}$.

5) In addition to hyperbolic group boundaries, we remark that discrete quasiconformal techniques apply to several other dynamical situations. This includes finite subdivision rules [44, 42], rational maps and ramified covers [13, 82, 110, 83, 24], quasiconformal geometry on fractals like the Sierpinski carpets [13, 23, 14, 22, 78].

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