LIE GROUPS AND QUASI-ISOMETRIES

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Abstract. These notes deal with some aspects of the large scale geometry of Lie groups, especially of the solvable ones.

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Introduction

These notes are issued from a mini-course that occurred during the "Summer School on Rigidity on Discrete Groups", at Mohali, the first week of July 2025. They are motivated by the following Gromov type problem:

Question 0.1. *Classify the (connected) Lie groups up to quasi-isometry.*

They are organized as follows. Section 1 is a presentation of the most classical quasi-isometric invariant: the volume growth. Gromov's Theorem on polynomial volume growth and Bass-Guivarc'h's formula are stated.

We focus in Section 2 on solvable Lie group, and present two of their invariants: the exponential radical and the rank.

Section 3 is about to the so-called *completely solvable Lie groups*, a class of groups introduced by Cornulier to approach Question 0.1. Some of their (nice) properties are discussed

Among the completely solvable Lie groups, the simplest ones are probably the so-called *Abelian-by-Abelian Lie groups*. They are defined in Section 4. We discuss their geometry, especially the existence of a left-invariant Riemannian metric of negative or non-positive curvature. Finally we present some results about their quasi-isometric classification.

These notes contain no original result, and they do not compose an exhaustive presentation of the quasi-isometric invariants of Lie groups. In particular, asymptotic cones, Dehn functions and other filling invariants are not discussed[1]. The aim here is more to provide an gentle introduction to the large scale geometry of Lie groups.

The reader is assumed to be familiar with some basic topics in geometric group theory, Lie groups and Riemannian geometry. References include [DK18, Chapter 8], [FH91, Chapters 7 to 9] and [GHL04, Chapters 2 and 3].

[1]Asymptotic invariants appears in the seminar work [Gro93], the book [DK18] contains detailed presentation of some of them, and [Cor14, Chapters 1 to 3] focus on the Lie groups case.

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Notation. The following classical notions and notation will serve repeatedly in the sequel:

- Two real valued functions f, g defined on a space X are said to be *comparable*, and then we write $f \asymp g$, if there exists a constant $C > 0$ such that $C^{-1} f \leq g \leq Cf$. We write $f \lesssim g$ if there exists a constant $C > 0$ such that $f \leq Cg$.
- Let G be an abstract group. Its identity element will be denoted by 1_G or more simply by 1.

  For $g \in G$, we denote respectively by L_g, R_g, $C_g = L_g \circ R_{g^{-1}}$, the left multiplication by g, the right one, and the conjugacy by g.

  For $g, h \in G$, let $[g, h] = ghg^{-1}h^{-1}$ be their commutator. Given subgroups $H, K \leq G$, we denote by $[H, K]$ the subgroup of G generated by the $[h, k]$, where $h \in H$, $k \in K$.

  The *central series* of G is

  $$G = C^1 G \geq C^2 G \geq C^3 G \geq \cdots \geq C^i G \geq \ldots,$$

  where $C^2 G = [G, G], C^{i+1} G = [G, C^i G]$. One has $[C^i G, C^j G] \leq C^{i+j} G$. Since $C_k[g, h] = [C_k g, C_k h]$, the subgroups $C^i G$ are normal in G. Moreover $[C^i G, C^i G] \leq C^{i+1} G$, thus the quotient groups $C^i G / C^{i+1} G$ are abelian.

  The *derived series* of G is

  $$G = D^0 G \geq D^1 G \geq D^2 G \geq \cdots \geq D^i G \geq \ldots,$$

  where $D^1 G = [G, G], D^{i+1} G = [D^i G, D^i G]$. Again the subgroups $D^i G$ are normal in G, and the quotients $D^i G / D^{i+1} G$ are abelian.

  Recall that G is said to be *nilpotent* (resp. *solvable*) if $C^i G$ (resp. $D^i G$) is trivial for i large enough.
- Let G be a Lie group and let $\mathfrak{g}$ be its Lie algebra. We denote by $\exp : \mathfrak{g} \to G$ the *exponential map*, and we recall that for $X \in \mathfrak{g}$, the 1-parameter subgroup $(\exp tX)_{t \in \mathbf{R}} \leq G$, is so that

the 1-parameter subgroup $(R_{\exp tX})_{t\in\mathbf{R}} \leq \mathrm{Diff}(G)$ is the flow of the left-invariant vector field on G generated by X.

The *adjoint representation of* G is $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$, defined by $\mathrm{Ad}(g) := d_{1_G} C_g$, for every $g \in G$ – where d_{1_G} denotes the differential at the identity. The *adjoint representation of* $\mathfrak{g}$ is $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$, defined by $\mathrm{ad} := d_{1_G}\mathrm{Ad}$.

Recall that for every $X \in \mathfrak{g}$ and $g \in G$, one has:

$$\mathrm{ad}X = [X, \cdot],\ C_g(\exp X) = \exp(\mathrm{Ad}(g)X),\ \mathrm{Ad}\exp X = e^{\mathrm{ad}X}.$$

1. Growth

1.1. **Generalities and examples.** We consider a group G that we assume to be either a finitely generated group, or a connected Lie group.

In the first case, we fix a finite set of generators S, and we equip G with the associated word length $|\cdot|_S$.

In the second case, we equip G with the Riemannian distance d associated to a given left-invariant Riemannian metric on G. We also denote by $\mathcal{H}$ a left-invariant Haar measure on G.

Definition 1.1. (1) If G is a finitely generated group, the *growth function* of (G, S) is $\rho = \rho_{G,S} : \mathbf{R}^+ \to \mathbf{R}^+$, defined by $\rho(R) = \mathrm{card} B_S(1_G, R) = \mathrm{card}\{g \in G\ ;\ |g|_S \leq R\}$.

(2) If G is a connected Lie group, the *growth function* of $(G, d, \mathcal{H})$ is $\rho = \rho_{G,d,\mathcal{H}} : \mathbf{R}^+ \to \mathbf{R}^+$, defined by $\rho(R) = \mathcal{H}(B_d(1_G, R))$.

Example 1.2. For $G = \mathbf{R}^n$, $\rho_G(R) = CR^n$ with $C = \mathrm{Vol}(B_d(0, 1))$.

Example 1.3. Let X be a bouquet of two circles, and let G be its fundamental group. Each of two circles induces a group element; and together they form a set of generators S of G. The group G is called the *free non-abelian group of two generators*. It acts simplicially on the universal cover of X denoted by T. The latter is a regular tree of valence 4. The action is simply transitive on the set of vertices of T (since X admits only one vertex). Therefore there is a bijective correspondence between the group elements of G and the vertices of T. The word length of (G, S) corresponds to the simplicial distance between the vertices of T. Thus, for $n \in \mathbf{N}$, one has $\rho_{G,S}(n) = 1 + 4\sum_{k=1}^{n} 3^{k-1}$.

Definition 1.4. (Equivalence of functions) Let $\rho_1, \rho_2 : \mathbf{R}^+ \to \mathbf{R}^+$ be arbitrary functions.

(1) We write $\rho_1 \precsim \rho_2$, if there exists constants $a, b > 0$ and $c, d \geq 0$, such that for every $R > 0$, one has $\rho_1(R) \leq a\rho_2(bR + c) + d$.
(2) We say that ρ_1 and ρ_2 are *equivalent*, and we write $\rho_1 \simeq \rho_2$, if $\rho_1 \precsim \rho_2$ and $\rho_2 \precsim \rho_1$.

It can be shown that $\simeq$ is an equivalence relation on the set of functions from $\mathbf{R}^+$ to $\mathbf{R}^+$.

Remark 1.5. (1) For $d, D > 0$, $R^d \simeq R^D$ if, and only if, $d = D$.
(2) A contrario, one has $e^{aR} \simeq e^{bR}$ for every $a, b > 0$.
(3) It is an exercise to show that $\rho_G(R) \precsim e^R$, for every finitely generated group or connected Lie group G.

This equivalence relation is relevant in geometric group theory, because of the following result:

Theorem 1.6. *Let G, H be finitely generated or connected Lie groups. If G is quasi-isometric to H, then $\rho_G \simeq \rho_H$.*

Idea of proof. The case where both groups G and H are finitely generated is straightforward. This case generalizes easily to quasi-isometric simplicial graphs of bounded valence. When one of the groups, say G, is a Lie group, then one can discretize it, as follow. Let $V \subset G$ be a maximum 1-separated subset, and let X be the incidence graph of the cover $\{B_G(x, 2)\}_{x \in V}$ of G. Then G and the simplicial graph X are quasi-isometric, and their growth functions can be shown to be equivalent. This completes the proof. □

Proposition 1.7. *Let G be a finitely generated group.*

(1) *If $H \leq G$ is a finitely generated subgroup, then $\rho_H \precsim \rho_G$.*
(2) *If $N \lhd G$, then $\rho_{G/H} \precsim \rho_G$.*

Proof. (1). Let $T \subset H$ be a finite generating set of H, and let $S \subset G$ be a finite generating set of G containing T. Then for every $h \in H$, one has $|h|_S \leq |h|_T$. Therefore for $R > 0$, one has $B_{H,T}(1_H, R) \subset B_{G,S}(1_G, R)$; which implies that $\rho_{H,T}(R) \leq \rho_{G,S}(R)$.

(2). Let $S \subset G$ be a finite generating set, and set $\overline{S} := \{sN \mid s \in S, s \notin N\}$. Then $|gN|_{\overline{S}} \leq |g|_S$, thus the quotient map $G \to G/N$ maps the ball $B_{G,S}(1, R)$ onto the ball $B_{G/N,\overline{S}}(1, R)$. Therefore $\rho_{G/N,\overline{S}}(R) \leq \rho_{G,S}(R)$. □

The following terminology is well-defined thanks to Theorem 1.6 and Remark 1.5.

Definition 1.8. Let G be a finitely generated or a connected Lie group. It is said to be of *polynomial volume growth* if $\rho_G(R) \precsim R^d$ for some $d \in \mathbf{N}$, and of *exponential volume growth* if $\rho_G(R) \simeq e^R$.

Proposition 1.9. *Let G be a connected Lie group.*

(1) *G is either of polynomial volume growth or of exponential volume growth.*
(2) *(Frank's lemma) Suppose G admits an expanding automorphism, i.e. an automorphism h, for which there is a constant $\lambda > 1$, such that for every $g, g' \in G$, one has $d(h(g), h(g')) \geq \lambda d(g, g')$. Then G has polynomial volume growth.*
(3) [Gui73, Lemme I.3] *Suppose that G is* **non***-unimodular, then G has exponential volume growth.*

Proof. (1). See e.g. [Je73, Corollary 2.1].

(2). For every $k \in \mathbf{N}$, one has $d(1_G, h^{-k}(g)) \leq \lambda^{-k} d(1_G, g)$. Thus $h^{-k}(B(1_G, \lambda^k)) \subset B(1_G, 1)$, and so $B(1_G, \lambda^k) \subset h^k(B(1_G, 1))$. Therefore one has $\mathcal{H}(B(1_G, \lambda^k) \leq (h^k)^* \mathcal{H}(B(1_G, 1))$.

In another hand, for every $g \in G$, one has

$$L_g^*(h^*\mathcal{H}) = (h \circ L_g)^*\mathcal{H} = (L_{h(g)} \circ h)^*\mathcal{H} = h^*(L_{h(g)}^*\mathcal{H}) = h^*\mathcal{H}.$$

Thus $h^*\mathcal{H}$ is a Haar measure on G, and so there is a constant $C > 0$ such that $h^*\mathcal{H} = C\mathcal{H}$. One gets that $(h^k)^*\mathcal{H} = C^k\mathcal{H}$, and so $\mathcal{H}(B(1_G, \lambda^k)) \leq C^k \mathcal{H}(B(1_G, 1))$, ie $\rho_G(\lambda^k) \leq C^k\mathcal{H}(B(1_G, 1))$. This implies that $\rho_G(R) \precsim R^d$.

(3). Let $g \in B(1_G, 1)$ be such that $R_g^*\mathcal{H} = C\mathcal{H}$, with $C > 1$. For every $h \in B(1_G, 1)$, one has $d(1_G, g^k h g^{-k}) \leq 2k+1$, which implies that $g^k B(1_G, 1) g^{-k} \subset B(1_G, 2k+1)$. By letting C_g be the conjugacy by g, we obtain

$$(C_g^k)^*\mathcal{H}\big(B(1_G, 1)\big) \leq \mathcal{H}\big(B(1_G, 2k+1)\big) = \rho_G(2k+1).$$

In another hand, by the choice of g, we have $(C_g^k)^*\mathcal{H} = C^k\mathcal{H}$. Therefore $\rho_G(2k+1) \geq C^k\mathcal{H}(B(1_G, 1))$, which implies that $\rho_G(R) \succsim e^R$, and then $\rho_G(R) \simeq e^R$, thanks to Remark 1.5(3). □

Example 1.10. The group of unipotent real $n \times n$ matrices

$$N_n := \{ \begin{pmatrix} 1 & * & \dots & * \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & 0 & 1 \end{pmatrix} \} \subset \mathrm{SL}_n(\mathbf{R}),$$

admits the following automorphism: $h_t(a_{ij}) = (b_{ij})$, with $b_{ij} = e^{t(j-i)}a_{ij}$. When $t > 0$, h_t is expanding. Therefore N_n has polynomial volume growth.

Example 1.11. Let $G = \mathrm{SL}_n(\mathbf{R})$, and let $G = KAN$ be its Iwazawa decomposition, where $K = \mathrm{SO}_n$ is compact, and AN is the real upper triangular group

$$AN = \{ \begin{pmatrix} a_{11} & * & \dots & * \\ 0 & a_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & 0 & a_{nn} \end{pmatrix} : \ a_{ii} > 0, \Pi_i a_{ii} = 1\}.$$

Since K is compact, G is quasi-isometric to AN, which is non-unimodular. Thus G has exponential volume growth.

Remark 1.12. According to Proposition 1.9(1) every connected Lie group is either of polynomial or exponential volume growth. This dichotomy holds also for finitely generated *linear groups* i.e. for finitely generated subgroups of a $\mathrm{GL}_n(\mathbf{R})$; this phenomenon is called the *Tits' alternative* – see [DK18, Chapter 15] for details. In 1968, Milnor asked whether there exists a finitely generated group G of *intermediate volume growth* i.e. whose growth function is larger than any polynomial and smaller than exponential. The first example of such a group has been given by Grigorchuk in 1983 –see [Gri14] for a survey on Milnor's problem.

1.2. **Nilpotent groups and Gromov's theorem.** As we saw in Example 1.10, the group N_n of $n \times n$ real unipotent matrices, is of polynomial growth. This result generalizes to every nilpotent group:

Theorem 1.13. *(Wolf 1968) Let G be a nilpotent group (finitely generated or Lie). Then G has polynomial volume growth.*

The proof follows from Proposition 1.7 (and from its variant for Lie groups), in combination with the following two classical theorems.

Theorem 1.14. *(Malcev 1949) Every finitely generated nilpotent group admits a finite index subgroup, which is isomorphic to a cocompact lattice in a simply connected nilpotent Lie group.*

Theorem 1.15. *(Ado-Engel) Every simply connected nilpotent Lie group is isomorphic to a closed subgroup of an N_n.*

Gromov's theorem is a converse of Theorem 1.13 for finitely generated nilpotent groups:

Theorem 1.16. [Gro81] *If G is a finitely generated group of polynomial volume growth, then it is virtually nilpotent, i.e. it admits a finite index nilpotent subgroup.*

There exists nowadays 3 different proofs of Gromov's theorem: the original one that uses Montgomery-Zippin's deep theorem (see also [DK18, Chapiter 16]), a cohomology/harmonic proof by Kleiner [Kl10], and a functional analysis proof by Ozawa [O18]. Gromov's theorem admits the following consequence.

Corollary 1.17. *A finitely generated group is quasi-isometric to a nilpotent group (finitely generated or Lie) if, and only if, it is virtually nilpotent.*

Remark 1.18. Gromov's theorem is false for Lie groups. For example[2], let $G = \mathbf{R} \ltimes_\varphi \mathbf{C}^2$, with $\varphi(t) = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{i\alpha t} \end{pmatrix}$ and $\alpha \in \mathbf{R} \setminus \mathbf{Q}$. This means that the group law of G is $(t, v) \cdot (s, w) = (t + s, v + \varphi(t)w)$. Then $\rho_G(R) \simeq R^5$, (in fact G is quasi-isometric to $\mathbf{R}^5$), but G is not nilpotent (since $C^k G = \mathbf{C}^2$ for every $k \geq 1$), nor admits a cocompact nilpotent subgroup.

Remark 1.19. It is known that a Lie group G has polynomial volume growth if, and only if, it is of *type R*, which means that for every $X \in \mathfrak{g} := \mathrm{Lie}(G)$, the spectrum of $\mathrm{ad}(X)$ is contained in $i\mathbf{R}$ – see [Je73]. Among them, the nilpotent ones are those for which the operators $\mathrm{ad}(X)$ are nilpotent (Engel's Theorem).

We also notice that every Lie group of polynomial volume growth is quasi-isometric to a nilpotent Lie group – see Theorem 3.2 and Proposition 3.7(2).

1.3. **Bass-Guivarc'h formula and abelian groups.** The following theorem of Bass and Guivarc'h provides the exact polynomial growth of nilpotent Lie groups.

Theorem 1.20. [Ba72, Theorem 2] [Gui73, Théorème II.1] *Let G be a simply connected nilpotent Lie group. Let N be the smallest $i \geq 1$ such that $C^{i+1}G = \{1\}$. For $i \leq N$, set $d_i := \dim(C^iG/C^{i+1}G)$. Then one has $\rho_G(R) \simeq R^d$ with $d = \sum_{i=1}^N i d_i$.*

In combination with Gromov's theorem, it can be used to classify the finitely generated groups which are quasi-isometric to an abelian one:

[2] I learned this example from Cornulier.

Corollary 1.21. *Let* Γ *be finitely generated group. Suppose it is quasi-isometric to* $\mathbf{Z}^n$*. Then* Γ *is virtually isomorphic to* $\mathbf{Z}^n$*.*

Proof. By Gromov's theorem Γ is virtually nilpotent, so we can assume it is nilpotent. By Malcev's theorem (Theorem 1.14), Γ is a cocompact lattice in a simply connected nilpotent Lie group G. Therefore it is enough to prove that G is isomorphic to $\mathbf{R}^n$.

By assumption, Γ is quasi-isometric to $\mathbf{Z}^n$, thus so is G, this in turn implies that $\rho_G(R) \simeq R^n$.

In another hand, one knows that for abstract connected Lie groups G_1, G_2, the following holds – see Remark 2.15: if G_1 and G_2 are quasi-isometric, then one has $\dim G_1/K_1 = \dim G_2/K_2$, where K_1 and K_2 are maximal compact subgroups of G_1 and G_2 respectively. By applying this result to the simply connected nilpotent Lie group G (for which $K = \{1\}$) and to $\mathbf{R}^n$, one gets that $\dim G = n$.

Now the Bass-Guivarc'h formula implies that $C^2G = \{1\}$, which means that G is abelian. Thus G is isomorphic to $\mathbf{R}^n$. □

An alternative proof of Corollary 1.21 (without using Gromov's theorem) was given by Shalom [Sh04].

Remark 1.22. We saw that finitely nilpotent groups enjoy two fundamental properties: they are quasi-isometrically rigid (Corollary 1.17), and linear (by Theorems 1.14 and 1.15, they virtually embed in a $\mathrm{GL}_n(\mathbf{R})$). These properties do not extend to the larger class of finitely generated solvable groups. Indeed:

- Dyubina-Erschler [Dy00] has given an example of two quasi-isometric finitely generated groups, such that one of them is solvable, while the other is non-virtually solvable.
- The *lamplighter group* $G := \mathbf{Z} \ltimes_\varphi \bigoplus_{i\in\mathbf{Z}} Z/2\mathbf{Z}$, where $\varphi : \mathbf{Z} \to \mathrm{Aut}(\bigoplus_{i\in\mathbf{Z}} Z/2\mathbf{Z})$ is the shift $\varphi_t((x_i)) = (x_i + t)$, is finitely generated (by $S = \{(1,0),(0,\delta_0)\}$), and is solvable (since $[G,G] = \bigoplus_{i\in\mathbf{Z}} Z/2\mathbf{Z}$ is abelian). It is **non**-linear; indeed every finitely generated subgroup of $\mathrm{GL}_n(\mathbf{R})$ is virtually torsion-free (by a theorem of Malcev).

2. Distortion, exponential radical, rank

We keep the same notations as in the previous section. For simplicity, we assume by convention that every Lie group is connected.

2.1. **Distortion.** The following definition appears in [Gro93, Chapter 3].

Definition 2.1. Let G be finitely generated group or a Lie group. An element $g \in G$ is said to be *distorted* if $\lim_{n\to+\infty} \frac{d_G(1,g^n)}{n} = 0$. It is said to be *exponentially distorted* if for every $n \in \mathbf{N}$ one has $d_G(1, g^n) \lesssim \log(1 + n)$; and *infinitely distorted* if for every $n \in \mathbf{N}$ one has $d_G(1, g^n) \lesssim 1$.

By triangular inequality and left-invariance, one sees that the sequence $\{d_G(1, g^n)\}_{n\in\mathbf{N}}$ is subadditive; and so $\lim_{n\to+\infty} \frac{d_G(1,g^n)}{n}$ always exists thanks to the subadditive theorem. Therefore, g *non-distorted* is equivalent to: $d_G(1, g^n) \asymp n$.

As a first example, we have:

Proposition 2.2. *Let $G = \mathrm{SL}_n(\mathbf{R})$. Let $\|\cdot\|$ be an arbitrary norm on $M_n(\mathbf{R})$, and define $\ell : G \to \mathbf{R}_+$ by letting $\ell(g) = \log\max\{\|g\|, \|g^{-1}\|\}$. Then there exists constant $C \geq 1$ and $D \geq 0$, such that for every $g \in G$, one has*

$$C^{-1}\ell(g) - D \leq d_G(1, g) \leq C\ell(g) + D.$$

In particular semisimple elements are non-distorted, unipotent ones are exponentially distorted, and the elements of $\mathrm{SO}_n(\mathbf{R})$ are infinitely distorted.

To prove the proposition, we will use the following lemma whose proof is left to the reader.

Lemma 2.3. (1) *Let $\varphi : G \to H$ be a group morphism. Then for every $g \in G$, one has $d_H(1, \varphi(g)) \lesssim d_G(1, g)$.*
(2) *Suppose that H is a subgroup of G and that there exists a group morphism $\varphi : G \to H$ such that $\varphi\big|_H = \mathrm{id}$. Then for every $h \in H$, one has $d_H(1, h) \asymp d_G(1, h)$.*

Proof of the Proposition. Let $g \in G$. By using the Cartan decomposition, one can write $g = kak'$, with $k, k' \in K := \mathrm{SO}_n(\mathbf{R})$ and $a \in A := \{\mathrm{diag}(a_{ii}) \mid a_{ii} > 0, \Pi_i a_{ii} = 1\}$.

Since $d_G(1, k) \lesssim 1$ and since $d_G(ka, g) = d_G(1, k') \lesssim 1$, one has $|d_G(1, g) - d_G(k, ka)| \lesssim 1$, and thus:

$$|d_G(1, g) - d_G(1, a)| \lesssim 1.$$

Since K is compact, the Iwasawa decomposition $G = KAN$ – where N is the group of unipotent real $n \times n$ matrices – yields to:

$$d_G(1, a) \asymp d_{AN}(1, a) \asymp d_A(1, a),$$

where the last relation comes from Lemma 2.3(2). Since A is isomorphic to $(\mathbf{R}_+^*)^{n-1}$, one gets that

$$d_A(1,a) \asymp \max_i |\log a_{ii}| \asymp \log \max\{\|a\|, \|a^{-1}\|\} = \ell(a).$$

Finally, $\|\cdot\|$ is Lipschitz equivalent to a submultiplicative norm, thus one has: $\|g\| = \|kak'\| \asymp \|a\|$. This in turn implies that $|\ell(g) - \ell(a)| \lesssim 1$. The statement follows. ☐

More generally one has

Proposition 2.4. *A Lie group G admits an exponentially non-infinitely distorted element if, and only if, G has exponential volume growth.*

Proof. Suppose that G has polynomial volume growth. Then G is of type R (see Remark 1.19); and so every element in G is at most polynomially distorted (a claim by Gromov later proved by Varopoulos [Va99], see also [Os02] for the exact distortion formula in nilpotent groups).

Conversely, suppose G is not of polynomial volume growth. Then, by Proposition 1.9(1) and Remark 1.19, G is of exponential volume growth, and that there exists $H \in \mathfrak{g} := \mathrm{Lie}(G)$ such that $\mathrm{ad}H \in \mathrm{End}(\mathfrak{g})$ admits an eigenvalue with non-zero real part. The following lemma concludes the proof. ☐

Lemma 2.5. *Let $H \in \mathfrak{g} := \mathrm{Lie}(G)$ be such that $\mathrm{ad}H \in \mathrm{End}(\mathfrak{g})$ admits an eigenvalue with non-zero real part. Let $X \in \mathfrak{g}$ be the real part of a generalized eigenvector for this eigenvalue. Then $\exp X$ is exponentially non-infinitely distorted in G.*

Proof. For simplicity, we assume that the eigenvalue is real (and non-zero). By considering a multiple of H, if necessary, we can suppose that the eigenvalue of X is $\log 2$. Set $h := \exp H$, and for $m, n \in \mathbf{N}$ consider the group elements $h^{-m} g^{2^n} h^m$. We claim that for $m = 2n$ and n large enough, one has $d_G(1, h^{-m} g^{2^n} h^m) \lesssim 1$. This implies that

$$\begin{aligned} d(1, g^{2^n}) &\leq d(1, h^m) + d(h^m, g^{2^n} h^m) + d(g^{2^n} h^m, g^{2^n}) \\ &= 2d(1, h^m) + d(1, h^{-m} g^{2^n} h^m) \\ &\lesssim 4n + 1, \end{aligned}$$

which means that g is exponentially distorted.

To establish the claim, we compute:

$$h^{-m}g^{2^n}h^m = h^{-m}(\exp 2^n X)h^m = \exp\big(\mathrm{Ad}(h^{-m})\cdot 2^n X\big)$$
$$= \exp\big(e^{-m\mathrm{ad}H}\cdot 2^n X\big).$$

The vector X is a generalized eigenvector of $e^{-\mathrm{ad}H}$ of eigenvalue $1/2$. Therefore one has

$$\|e^{-m\mathrm{ad}H}\cdot 2^n X\| = 2^n\|e^{-m\mathrm{ad}H}\cdot X\| \leq 2^n|P(m)|2^{-m}\|X\|,$$

where P is a polynomial function. The claim follows. □

2.2. **Exponential radical, rank.**

Definition 2.6. ([Os02]) Let G be Lie group. Its *exponential radical*, denoted by $\mathrm{R}_{\exp}G$, is the set of exponentially distorted elements in G.

Theorem 2.7. *(Guivarc'h* [Gui80], *Osin* [Os02]*) Suppose G is a simply connected solvable Lie group. Then* $\mathrm{R}_{\exp}G$ *is a characteristic subgroup of G. Moreover, it is the smallest closed normal subgroup H of G such that G/H has polynomial volume growth.*

Remark 2.8. The condition G solvable cannot be removed; e.g. in $\mathrm{SL}_2(\mathbf{R})$ the group elements $g_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $g_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ are exponentially distorted, but $g_1g_2 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ is not, since semisimple.

Remark 2.9. In any Lie group G, there is a unique smallest closed normal subgroup H such that G/H has polynomial volume growth. Indeed let $H_1, H_2 \lhd G$ with G/H_1 and G/H_2 of polynomial volume growth. Set $H_3 = H_1 \cap H_2 \lhd H$. Then G/H_3 is still of polynomial volume growth, since one has

$$1 \to H_1/H_3 \to G/H_3 \to G/H_1 \to 1,$$

with G/H_1 of polynomial volume growth, and with

$$H_1/H_3 = H_1/H_1 \cap H_2 \leq G/H_2$$

of polynomial volume growth too. When G is simply connected, the subgroup H belongs to the *stable term* of the central serie of G, i.e. to $S := \bigcap_{i\geq 1} C^iG$. Indeed S is a closed normal subgroup of G such that G/S is nilpotent – see the end of proof of Proposition 3.7(3) for details. In particular: $H \leq [G, G]$.

Elements of Proof. The fact that $\mathrm{R_{exp}}G$ is a closed normal subgroup is proved in [Os02, Theorem 1.1]. We notice that Osin's definition[3] of $\mathrm{R_{exp}}G$ differs a little bit from ours, however they coincide when G is a simply connected solvable Lie group, see the discussion right before [Os02, Lemma 3.1].

The smallest close normal subgroup $H < G$ such that G/H is of polynomial growth appears in [Gui80, Définition 8 and Proposition 5]. The equality between $\mathrm{R_{exp}}G$ and H is noticed in [Cor08, Theorem 6.1]. It can be established as follows:

- To show that $\mathrm{R_{exp}}G < H$, one notices from Lemma 2.3 that the image of every $g \in \mathrm{R_{exp}}G$ is exponentially distorted in G/H. Since G/H is of polynomial growth, its exponentially distorted elements are infinitely distorted (by Proposition 2.4). The minimality of H implies that G/H is simply connected. In a simply connected solvable Lie group S, every infinitely distorted element is trivial[4]; therefore the image of $\mathrm{R_{exp}}G$ in G/H is trivial.
- For the reverse inclusion, [Os02, Theorem 1.1(4)] implies that no element in $G/\mathrm{R_{exp}}G$ is strictly exponentially distorted. Therefore, by [Os02, Proposition 3.2] (which is a refinement of Proposition 2.4), $G/\mathrm{R_{exp}}G$ is of polynomial growth. Thus $H < \mathrm{R_{exp}}G$.

□

Corollary 2.10. *Let $G = A \ltimes N$ be a simply connected Lie group with A abelian and N nilpotent. Denote by $\mathfrak{a}$ and $\mathfrak{n}$ their Lie algebra. Suppose that there exists $\xi \in \mathfrak{a}$, such that the generalized eigenspaces of $\mathrm{ad}\xi\big|_{\mathfrak{n}} \in \mathrm{End}(\mathfrak{n})$, associated to the eigenvalues with non-zero real parts, generate $\mathfrak{n}$ as a Lie algebra. Then $\mathrm{R_{exp}}G = [G, G] = N$.*

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[3] In [Os02, Definition 3.2], $\mathrm{R_{exp}}G$ is defined as the union of $\{1_G\}$ with the set of *strictly* exponentially distorted element of G, i.e. the $g \in G$ such that $d(1, g^n) \asymp \log(1 + n)$.

[4] Indeed the image in $S/[S, S]$ of an infinitely distorted element of S is again infinitely distorted. But, since S is simply connected, $S/[S, S]$ is isomorphic to $\mathbf{R}^n$. (To see this, note that the projection map $\pi : S \to S/[S, S]$ factorises as $\pi = \tilde{\pi} \circ p$, where $p : \mathbf{R}^n \to S/[S, S]$ is the universal cover. $\mathrm{Ker}\,\tilde{\pi}$ is a subgroup of $[S, S]$ such that $S/\mathrm{Ker}\,\tilde{\pi}$ is abelian; therefore it contains $[S, S]$, which in turn implies that $p = \mathrm{id}$). Now $\mathbf{R}^n$ contains no non-trivial infinitely distorted element, thus every infinitely distorted element of S is contained in $[S, S]$. But the latter is a simply connected nilpotent Lie group, which again contains no non-trivial infinitely distorted element (by Theorem 1.15)

Proof. The assumptions, in combination with Lemma 2.5 and Theorem 2.7, show that $N \le \mathrm{R}_{\exp}G$. Moreover one has $\mathrm{R}_{\exp}G \le [G,G] \le N$ since G/N is abelian. The statement follows. □

Example 2.11. Let Heis_3 be the group of unipotent real 3×3 matrices (as in Example 1.10). Let $G = \mathbf{R} \ltimes_\varphi \mathrm{Heis}_3$, with

$$\varphi_t \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & e^t x & z \\ 0 & 1 & e^{-t} y \\ 0 & 0 & 1 \end{pmatrix}.$$

By applying the above corollary to any generator ξ of the $\mathbf{R}$ factor, and to the vectors $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ that generates the Lie algebra of Heis_3, we get that $\mathrm{R}_{\exp}G = \mathrm{Heis}_3$.

Definition 2.12. (Cornulier) The *rank* of a simply connected solvable Lie group G is $\mathrm{rank}(G) := \dim(G/\mathrm{R}_{\exp}G)$.

Theorem 2.13. *(Cornulier* [Cor08]*) Among simply connected solvable Lie groups G, the rank of G is a quasi-isometric invariant.*

Example 2.14. Let $G = KAN$ be the Iwasawa decomposition of a semisimple Lie group with finite center. Then Corollary 2.10, applied with a vector $\xi \in \mathfrak{a}$ that lies in the interior a Weyl chamber, shows that $\mathrm{R}_{\exp}(AN) = N$. Thus one has:

$$\mathrm{rank}(AN) = \dim A = \mathrm{rank}_{\mathbf{R}}(G).$$

More generally, the rank of a simply connected solvable Lie group that admits a left-invariant Riemannian metric of non-positive curvature, is equal to the maximal dimension of a totally geodesic Euclidean subspace [AW76].

Remark 2.15 (**Another quasi-isometric invariant**)**.** For every Lie group G and every maximal compact subgroup $K \subset G$, one knows that the homogeneous space G/K is diffeomorphic to $\mathbf{R}^d$, and that its dimension is a quasi-isometric invariant of the Lie group G [Roe93, Proposition 3.33 and Corollary 3.35]. More precisely, if G_1, G_2 are quasi-isometric Lie groups, and $K_i \subset G_i$ are maximal compact subgroups, then $\dim G_1/K_1 = \dim G_2/K_2$.

3. Completely solvable Lie groups

Again, by convention, any Lie group is assumed to be connected. For $n \in \mathbf{N}$, let T_n be the group of real upper triangular $n \times n$ matrices, with positive diagonal coefficients, that is

$$T_n = \{ \begin{pmatrix} a_{11} & * & \dots & * \\ 0 & a_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & 0 & a_{nn} \end{pmatrix} : \ a_{ii} > 0\} \subset \mathrm{GL}_n(\mathbf{R}).$$

3.1. **Definition and motivation.** The following definition appears in [Cor14].

Definition 3.1. A Lie group is said to be *completely solvable* if it isomorphic to a closed subgroup of a T_n.

Theorem 3.2. *(*[Cor08, Lemmata 2.4 and 6.7]*) Any Lie group is quasi-isometric to a completely solvable Lie group. Moreover the quasi-isometric equivalence can be obtained by a finite number of the following algebraic operations:*

- *extract a cocompact Lie subgroup,*
- *embed cocompactly into a Lie group,*
- *quotient by compact normal subgroup.*

Example 3.3. Let G be a semisimple Lie group with finite center. Consider its Iwazawa decomposition $G = KAN$. Then G is quasi-isometric to its cocompact subgroup AN which is completely solvable.

Example 3.4. Let $G = \mathbf{R} \ltimes_\varphi \mathbf{R}^2$, with $\varphi(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$. Let $H = (\mathbf{R} \times S^1) \ltimes_\rho \mathbf{R}^2$, with $\rho(t, e^{i\theta}) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. Then G embeds cocompactly in H, and H contains $\mathbf{R}^3$ as a cocompact subgroup. Therefore G is quasi-isometric to $\mathbf{R}^3$ which is completely solvable.

Example 3.5. Let Heis_3 be the Heisenberg group as defined in Example 2.11, and let $G := \mathrm{Heis}_3/\Gamma$, where Γ is a discrete subgroup of $Z(\mathrm{Heis}_3) \simeq \mathbf{R}$. Then the center of G is $Z(G) \simeq \mathbf{R}/\Gamma \simeq \mathbf{S}^1$. Therefore G is quasi-isometric to $G/Z(G) = \mathrm{Heis}_3/Z(\mathrm{Heis}_3)$, which is a completely solvable Lie group isomorphic to $\mathbf{R}^2$.

A motivation for studying completely solvable groups comes from the previous theorem in combination with the following conjecture of Cornulier.

Conjecture 3.6. [Cor18, Conjecture 19.113] *Any two completely solvable Lie groups are quasi-isomorphic if, and only if, they are isomorphic.*

3.2. **Basic properties of completely solvable Lie groups.** Completely solvable Lie groups enjoy nice properties:

Proposition 3.7. *Let G be a completely solvable Lie group and $\mathfrak{g}$ be its Lie algebra. Then:*

1. *The exponential map* $\exp : \mathfrak{g} \to G$ *is a diffeomorphism. In particular G is diffeomorphic to $\mathbf{R}^d$, and all its Lie subgroups are closed.*
2. *G is nilpotent if, and only if, it has polynomial volume growth.*
3. $\mathrm{R}_{\exp} G$ *is equal to the stable term in the central serie of G, that is* $\mathrm{R}_{\exp} G = \bigcap_{i \geq 1} C^i G$.
4. *Among completely solvable Lie groups G, the dimension of G is a quasi-isometric invariant.*

Proof. By definition there exists an $n \in \mathbf{N}$, such that G is isomorphic to a closed subgroup of T_n. For simplicity we will identify G with its image in T_n, and will denote T_n and its Lie algebra by T and $\mathfrak{t}$.

(1). First it is enough to show that $\exp : \mathfrak{t} \to T$ is a diffeomorphism. Indeed, if it is the case, then $\exp : \mathfrak{g} \to G$ is an injective open proper map. Therefore its image is open and closed in G. Since G is connected, $\exp : \mathfrak{g} \to G$ is onto and equal to the restriction of a diffeomorphism to a linear subspace of $\mathfrak{t}$; it is thus a diffeomorphism from $\mathfrak{g}$ to G.

Now, the fact that $\exp : \mathfrak{t} \to T$ is a diffeomorphism, follows from a general result of Dixmier and Saito, which states that the exponential map of a solvable simply connected Lie group, is a diffeomorphism if, and only if, the adjoint representation of its Lie algebra has no non-trivial purely imaginary roots[5]. The group T satisfies this last condition – see the argument in item (2) just below.

(2). Assume G is of polynomial growth. Then by Remark 1.19, it is of type R, and thus the eigenvalues of operators $\mathrm{ad}X$ ($X \in \mathfrak{g}$) belong to $i\mathbf{R}$. In another hand, $\mathrm{ad}X \in \mathrm{End}(\mathfrak{g})$ is the restriction to $\mathfrak{g}$ of $\mathrm{ad}X \in \mathrm{End}(\mathfrak{t})$. It is an exercice to show that in the canonical basis $(E_{ij})_{i \leq j}$ of $\mathfrak{t}$ – suitably ordered so that the diagonals $j - i = k$, $k \in \{0, \ldots, n-1\}$, appears successively – the matrix of $\mathrm{ad}X \in \mathrm{End}(\mathfrak{t})$ is triangular. In particular its eigenvalues are real. Therefore, for $X \in \mathfrak{g}$

[5]Elements of proof: According to [MS03, Theorem 5.1], the differential of $\exp : \mathfrak{g} \to G$ at $X \in \mathfrak{g}$ is equal to $dL_{\exp X} \circ \int_0^1 e^{-t\,\mathrm{ad}X} dt$. Thus the above condition on the adjoint representation is equivalent to the fact that the exponential map is everywhere a local diffeomorphism. When the target space is simply connected, a local diffeomorphism is a global one iff it is a proper map. In the special case of $\exp : \mathfrak{t} \to T$, properness is a consequence of the Dunford decomposition.

the operators $\mathrm{ad}X \in \mathrm{End}(\mathfrak{g})$ are nilpotent. Thus G is nilpotent, thanks to Engel's Theorem.

(3). Let $H = G/\mathrm{R}_{\exp}G$ and let $\mathfrak{h}$ be its Lie algebra. By Theorem 2.7, H has polynomial volume growth. As we saw in item (2) above, for every $X \in \mathfrak{g}$, the spectrum of the operator $\mathrm{ad}X \in \mathrm{End}(\mathfrak{g})$ is contained in $\mathbf{R}$. Thus the same holds for every operator $\mathrm{ad}Y \in \mathrm{End}(\mathfrak{h})$, with $Y \in \mathfrak{h}$. Therefore, by arguing as in item (2) above, we get that H is a nilpotent group. This implies that $C^i(G/\mathrm{R}_{\exp}G) = C^i(H) = \{1\}$ for i large enough. But for every $K \lhd G$, one has

$$C^i(G/K) = C^iG/C^iG \cap K, \tag{3.8}$$

– to see this, consider $C^iG \twoheadrightarrow C^i(G/K) \leq G/K$. Therefore, for i large enough, we have $C^i(G) \cap \mathrm{R}_{\exp}G = C^iG$, which means that $\mathrm{R}_{\exp}G$ contains the stable term of the central serie of G.

Conversely, let S be the stable term in the central serie of G. Then S is closed (by item (1) or by using a general result[6]); moreover the relation 3.8 implies that G/S is nilpotent, thus of polynomial volume growth. Since $\mathrm{R}_{\exp}G$ is the smallest closed normal subgroup K of G such that G/K is of polynomial growth, we finally obtain: $\mathrm{R}_{\exp}G = S$.

(4). This follows from Remark 2.15, since here we have $K = \{1\}$ thanks to item (1). □

4. Abelian-by-Abelian Lie groups

4.1. **Definition and first properties.** In these notes, *Abelian-by-Abelian Lie group*[7] will designate a group of the form $S_\alpha = \mathbf{R}^r \ltimes_\alpha \mathbf{R}^n$, where $\alpha : \mathbf{R}^r \to \mathrm{Diag}(\mathbf{R}^n)$ is a Lie group morphism with values into the diagonal subgroup of $\mathrm{GL}_n(\mathbf{R})$. Its multiplication law is thus

$$(u, x) \cdot (v, y) = (u + v, x + \alpha(u)y).$$

We can (and will) write $\alpha = \mathrm{diag}(e^{\varpi_1}, \ldots, e^{\varpi_n})$, with $\varpi_i \in (\mathbf{R}^r)^*$.

[6] In a simply connected Lie group G, every normal Lie subgroup H is closed. This follows from Lie's third Theorem: indeed with the obvious notations, the Lie algebra morphism $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ is induced by a group morphism $\varphi : G \to K$, where K is the simply connected Lie group whose Lie algebra is $\mathfrak{g}/\mathfrak{h}$; and thus H is equal to the identity component of $\mathrm{Ker}\,\varphi$.

[7] In [Pen11a, Pen11b] a more general definition of Abelian-by-Abelian Lie group is considered: the morphism α is allowed to take its values into the group T_n. Thefore the right terminology for the groups studied in this section is *diagonal Abelian-by-Abelian Lie group*; but for briefty we have choosen to omit the term "diagonal".

Some groups S_α can be written with several couples of exponents r, n (e.g. when they are abelian). In the sequel we will always assume that the dimension n of the second factor is minimal and non-zero. This assumption is equivalent to require every weight ϖ_i to be non-zero.

When α is injective, S_α is isomorphic to the following subgroup of T_{n+1}:

$$\{\begin{pmatrix} e^{\varpi_1(u)} & 0 & \dots & 0 & x_1 \\ & \ddots & \ddots & \vdots & \vdots \\ & & e^{\varpi_{n-1}(u)} & 0 & x_{n-1} \\ & & & e^{\varpi_n(u)} & x_n \\ & & & & 1 \end{pmatrix} : u \in \mathbf{R}^r, (x_1, \dots x_n) \in \mathbf{R}^n\}.$$

It is thus a completely solvable group. When α is non-injective, S_α decomposes as a product $S_\alpha = \mathbf{R}^\ell \times S_\beta$, with $\ell = \dim(\operatorname{Ker}\alpha)$ and $\beta : \mathbf{R}^{r-\ell} \to \operatorname{Diag}(\mathbf{R}^n)$ injective. Therefore, in any cases, S_α is completely solvable.

With Corollary 2.10 and Definition 2.12, we get:

Proposition 4.1. *One has:* $\mathrm{R}_{\exp} S_\alpha = [S_\alpha, S_\alpha] = \mathbf{R}^n$ *and* $\operatorname{rank} S_\alpha = r$.

It follows from the definition of $\mathrm{R}_{\exp} S_\alpha$, that every element in the subgroup $\mathbf{R}^n$ is exponentially distorted. A contrario, the subgroup $\mathbf{R}^r$ is totally geodesic in S_α. Indeed, it is an exercice to show that the projection map $(u, x) \mapsto (u, 0)$ is a contracting map.

Example 4.2. In rank one, i.e. when $r = 1$, every ϖ_i belongs to $(\mathbf{R})^*$, thus α can be written $\alpha(t) = \operatorname{diag}(e^{\lambda_1 t}, \dots, e^{\lambda_n t})$ for some $\lambda_i \in \mathbf{R} \setminus \{0\}$ (recall that by assumption: $\varpi_i \neq 0$ for every $i \in \{1, \dots n\}$).

When $\lambda_1 = \lambda_2 = \dots = \lambda_n$, then S_α is isomorphic to $\mathbf{R} \ltimes_{e^t \mathrm{id}_{\mathbf{R}^n}} \mathbf{R}^n$ which is – as a Riemannian manifold – isometric to $\mathbb{H}^{n+1}_{\mathbf{R}}$ the real hyperbolic space (of constant negative curvature) of dimension $n + 1$.

When $n = 2$ and $\lambda_1 = -\lambda_2$, then S_α is isomorphic to $\mathbf{R} \ltimes_\beta \mathbf{R}^2$, with $\beta(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$, i.e. to the group Sol – which is the unique simply connected solvable unimodular Lie group of dimension 3, of exponential volume growth.

4.2. **Curvature.** The Lie groups that admit a left-invariant Riemannian metric of negative (resp. non-positive) curvature, have been described algebraically by Heintze [He74] (resp. Azencott-Wilson [AW76]). This justifies the following terminology:

Definition 4.3. A Lie group that admits a left-invariant Riemannian metric of negative (resp. non-positive) curvature, is called a *Heintze group* (resp. an *Azencott-Wilson group*).

The next proposition complements the geometric description of the S_α's:

Proposition 4.4. *Let S_α be an Abelian-by-Abelian group.*

1. *S_α is a Heintze group if, and only if, it is of rank 1 and the expression of α is $\alpha(t) = \mathrm{diag}(e^{\lambda_1 t}, \dots, e^{\lambda_n t})$ with* **all** *the λ_i's of the same sign.*
2. *S_α is an Azencott-Wilson group if, and only if, 0 does* **not** *belong to the convex hull of $\{\varpi_1, \dots, \varpi_n\}$ in $(\mathbf{R}^r)^*$.*

Proof. (1). According to [He74], a Lie group S admits a left-invariant negatively curved Riemannian metric if, and only if, its Lie algebra $\mathfrak{s}$ enjoys the following properties:

(i) One can be write $\mathfrak{s} = \mathbf{R} \ltimes \mathfrak{n}$, with $\mathfrak{n}$ a nilpotent ideal.
(ii) There exists an element ξ in the factor $\mathbf{R}$, such that all the eigenvalues of $\mathrm{ad}\xi\big|_{\mathfrak{n}}$ have negative real parts.

Clearly, if $r = 1$ and the λ_i's are all of the same sign, then S_α satisfies the conditions (i) and (ii); thus it admits a left-invariant negatively curved Riemannian metric.

Conversely, if S_α satisfies the above conditions, then by Proposition 2.10, $\mathfrak{n}$ is the Lie algebra of $\mathrm{R}_{\exp} S_\alpha$ and $\mathrm{rank}(S_\alpha) = 1$. It is now an exercice to finish the proof of (1).

(2). According to [AW76], a Lie group S admits a left-invariant non-positively curved Riemannian metric if, and only if, its Lie algebra $\mathfrak{s}$ is an *NC algebra*. This means that $\mathfrak{s}$ enjoys the following properties:

(i) $\mathfrak{n} := [\mathfrak{s}, \mathfrak{s}]$ is a nilpotent ideal that is complemented in $\mathfrak{s}$ by an abelian subalgebra $\mathfrak{a}$.
(ii) There exists an element $\xi \in \mathfrak{a}$, such that all the eigenvalues of $\mathrm{ad}\xi\big|_{\mathfrak{n}}$ have negative real parts.
(iii) The action of $\mathfrak{a}$ on $\mathfrak{n}$ satisfies 3 additional conditions, which are automatically fulfilled when $\mathfrak{n}$ is abelian and the $\mathfrak{a}$-action is semisimple with real eigenvalues.

We refer to [AW76, Definition 6.2] for the precise definition of NC algebra, and to the paragraph right after it for a discussion of the special cases.

Clearly the Lie algebra of S_α satisfies Items (i) and (iii). It satisfies (ii) if, and only if, there exists an $u \in \mathbf{R}^r$ so that $\varpi_i(u) \leqslant -1$ for all $i \in I$. Let $v_i \in \mathbf{R}^r$ be such that $\varpi_i = \langle \cdot, v_i \rangle$, where $\langle \cdot, \cdot \rangle$ is a scalar product on $\mathbf{R}^r$. One has $\varpi_i(u) \leqslant -1$ for every $i \in I$ if, and only if, every v_i belongs to the subset defined by the inequality $\langle u, \cdot \rangle \leqslant -1$; i.e. to the affine half-space of $\mathbf{R}^r$, disjoint from 0, and delimited by the hyperplane orthogonal to u passing through $\frac{-u}{\|u\|^2}$. The proof of Item (2) is now complete. □

Observe (from their algebraic description recalled in the proof just above) that the Heintze groups coincide with the rank 1 Azencott-Wilson groups. In addition, except the Abelian ones, Azencott-Wilson groups are never unimodular.

4.3. **Quasi-isometric rigidity.** In this section we present several results about the quasi-isometric classification of the Abelian-by-Abelian groups.

According to Cornulier's Conjecture 3.6, any two quasi-isometric Abelian-by-Abelian Lie groups are isomorphic. However, even in the class of Abelian-by-Abelian Lie groups, Cornulier's conjecture is largely open. The only known results concern some few subclasses. Before we present them, let us make some few remarks:

- If $S_\alpha = \mathbf{R}^r \ltimes_\alpha \mathbf{R}^n$ and $S_\beta = \mathbf{R}^s \ltimes_\beta \mathbf{R}^m$ are quasi-isometric, then their ranks are equal, and also their dimensions – by Theorem 2.13 and Proposition 3.7(4). Therefore $r = s$ and $n = m$ by Proposition 4.1.
- It is an exercice[8] to show that $S_\alpha = \mathbf{R}^r \ltimes_\alpha \mathbf{R}^n$ and $S_\beta = \mathbf{R}^s \ltimes_\beta \mathbf{R}^m$ are isomorphic if, and only if, $r = s$, $n = m$, and there exists a linear isomorphism $\varphi : \mathbf{R}^r \to \mathbf{R}^s$ and a permutation σ of the diagonal entries of $\mathrm{Diag}(\mathbf{R}^m)$, such that $\sigma \circ \alpha_2 = \alpha_1 \circ \varphi$.
- In rank one, one has $\alpha(t) = \mathrm{diag}(e^{\lambda_1 t}, \ldots, e^{\lambda_n t})$ and $\beta(t) = \mathrm{diag}(e^{\mu_1 t}, \ldots, e^{\mu_n t})$ for some $\lambda_i, \mu_i \in \mathbf{R} \setminus \{0\}$. Thus, the above item implies that $S_\alpha = \mathbf{R} \ltimes_\alpha \mathbf{R}^n$ and $S_\beta = \mathbf{R} \ltimes_\beta \mathbf{R}^m$ are isomorphic if, and only if, there exists a constant $C \neq 0$ such that – up to a permutation of the indices of the μ_i's – one has $\mu_i = C\lambda_i$ for all i.

[8]Hints: the second factor of S_α is equal to its exponential radical, and the first factor is an undistorted abelian subgroup of maximal dimension.

Theorem 4.5. *Let S_1 and S_2 be Abelian-by-Abelian Lie groups. In each of the following disjoint situations, S_1 and S_2 are quasi-isometric if, and only if, they are isomorphic:*

(1) *(Pansu* [Pan99, Corollaire 2], [Pan08]*) S_1 and S_2 are Heintze groups.*
(2) *(Eskin-Fisher-Whyte* [EFW12]*) S_1 and S_2 belong to the family of the so-called Sol_λ groups; i.e. to the groups $\mathrm{Sol}_\lambda =: \mathbf{R} \ltimes_{\alpha_\lambda} \mathbf{R}^2$ with $\alpha_\lambda(t) := \begin{pmatrix} e^t & 0 \\ 0 & e^{-\lambda t} \end{pmatrix}$ and $\lambda > 0$.*
(3) *(Peng* [Pen11a, Pen11b]*) S_1 and S_2 are unimodular, (this is equivalent to require that the weights satisfy $\sum_i \varpi_i = 0$), with no non-trivial Abelian direct factor.*
(4) *(Bourdon-Rémy* [BR25]*) S_1 and S_2 are of the form $\mathbf{R}^2 \ltimes_\alpha \mathbf{R}^3$ and their weights $\varpi_1, \varpi_2, \varpi_3$ enjoy the following two properties:*
   - *they generate $(\mathbf{R}^2)^*$,*
   - *they are aligned (on a line necessarily disjoint from 0).*

   These groups are, in some sense, the most simple irreducible Azencott-Wilson groups of rank 2.

We notice that the groups in Items (2) and (3) are neither Heintze nor Azencott-Wilson (see Proposition 4.4).

About the proofs. The proof of Items (1) and (4) rely on L^p-cohomology, that is the cohomology of differential forms which, together with their differential, satisfy an L^p integrability condition with respect to the Riemannian metric. L^p-cohomology is a quasi-isometric invariant of contractible Lie groups – which is equivalent to be diffeomorphic to $\mathbf{R}^d$ – and more generally of *uniformly contractible* Riemannian manifolds. We denote the L^p-cohomology of a contractible Lie group G by $L^p\mathrm{H}^*_{\mathrm{dR}}(G)$.

In the proof of Items (1) and (4), the main objects of investigation are the so-called *critical exponents*: a number $\gamma > 1$ is a critical exponent in degree k if there exists $\varepsilon > 0$ such that $L^p\mathrm{H}^k_{\mathrm{dR}}(G) = 0$ for $p \in (\gamma, +\infty)$, and $\neq 0$ for $p \in (\gamma - \epsilon, \gamma)$ – or vice versa. Critical exponents (when they exist) provide numerical quasi-isometric invariants of G.

Let now G be as in Item (1). Up to isomorphism, we can (and will) assume that the λ_i's satisfy

$$1 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n. \tag{4.6}$$

Pansu uses contraction and negative curvature to show that for every $k \in \{2, \dots, n\}$, the number

$$\gamma_k := \frac{\sum_{i=1}^{n} \lambda_i}{\sum_{i=1}^{k-1} \lambda_i}$$

is a critical exponent in degree k. From that, one sees easily that the λ_i's (reorganized to satisfied 4.6) are quasi-isometric invariants of G. Item (1) follows.

In the proof of Item (4), we show the existence of a critical exponent in degree 2, and we compute it. Since the isomorphism classes of the groups appearing in Item (4) form a 1-parameter family, this invariant is enough to distinguish them up to quasi-isometry.

It must be mention that L^p-cohomology of solvable Lie groups – apart from the Heintze groups and in a smaller extent the Azencott-Wilson groups – is badly understood. For example there is no known L^p-cohomology proof of Item (2).

The proofs of Items (2) and (3) rely on a *coarse differentiation* method, see [EF10] for a survey and related results. □

Remark 4.7. Xie [Xie14] gave a different proof of Item (1) and also generalizations to other Heintze groups, by using a more geometric approach based on Gromov hyperbolic spaces and quasi-conformal geometry on their Gromov boundary.

Since Pansu's and Hamenstädt's pioneer works [Pan99, Ha87], the large scale geometry of Heintze groups has generated a lot of activities; see e.g. [Bo18, §7.4.3] for an (incomplete) list of conjectures and references.

A contrario, the large scale geometry of Azencott-Wilson groups has attracted much less attention so far (except in the case of Heintze groups and Riemannian symmetric spaces). To study them, it would be desirable to combine some geometric methods, like the simplicial structure of their Tits boundary [Heb93] – which should be a quasi-isometry invariant by [KL20] – with some analytic tools like L^p-cohomology or other functional spaces.

Remark 4.8. In [GP24], Grayevsky and Pallier ask for the quasi-isometric classification of all the 5-dim simply connected indecomposable solvable Lie groups of exponential volume growth. They provide the complete list of them and also of those which are in addition completely solvable. For each of them, the Lie algebra, the exponential radical, the rank, and the behaviour of the Dehn function, are determined. They also indicate those which are Heintze or Azencott-Wilson. In combination with some other methods, they obtain a partial quasi-isometric classification.

References

[AW76] R. Azencott, E.N. Wilson: Homogeneous manifolds with negative curvature. I. *Trans. Amer. Math. Soc.*, 215:323–362, 1976.

[Ba72] H. Bass: The degree of polynomial growth of finitely generated nilpotent groups. *Proceedings of the London Mathematical Society.*, 25(4):603–614, 1972.

[Bo18] M. Bourdon: Mostow type rigidity theorems. In *Handbook of Groups Action. Vol. IV,* Adv. Lect. Math. (ALM) 41, 139–188, Int. Press, Somerville, MA, 2018.

[BR25] M. Bourdon and B. Rémy: L^p-cohomology: critical exponents for higher rank spaces and groups, rigidity. *Geom. Topol.*, to appear, 2025.

[Cor08] Y. Cornulier: Dimension of asymptotic cones of Lie groups. *J. Topology*, 1(2):343–361, 2008.

[Cor14] ______: Aspects de la géométrie des groupes. Habilitation, 2014. Available at `https://arxiv.org/abs/2003.03993`

[Cor18] ______: On the quasi-isometric classification of locally compact groups. In *New directions in locally compact groups. London. Math. Soc. Lecture Note Ser.,* 447, 275–342, Cambridge Univ. Press, 2018.

[DK18] C. Drutu and M. Kapovich: Geometric group theory, *Amer. Math. Soc. Colloq. Publ.* 63, American Mathematical Society, RI, xx+819 pp., 2018.

[Dy00] A. Dyubina: Instability of the virtual solvability and the property of being virtually torsion-free. *Internat. Math. Res. Notices*, 21:1097–1101, 2000.

[EFW12] A. Eskin, D. Fisher and K. Whyte: Coarse differentiation of quasi-isometries I: Spaces not quasi-isometric to Cayley graphs. *Ann. of Math.*, 176(1): 221-260, 2012.

[EF10] A. Eskin and D. Fisher: Quasi-isometric rigidity of solvable groups. In *Proceedings of the international Congress of Mathematicians, Vol. III* , 1185–1208, Hindustan Book Agency, New Delhi, 2010.

[FH91] W. Fulton and J. Harris: Representation theory. A first course. *Graduate Texts in Mathematics,* 129, Springer-Verlag, New-York, xvi+551 pp., 1991.

[GHL04] S. Gallot, D. Hulin and J. Lafontaine: Riemannian Geometry, third edition. *Universitext,* Springer-Verlag, xv+321 pp., 2004.

[GP24] I. Grayevsky and G. Pallier: Sublinear Bilipschitz Equivalence and the Quasiisometric Classification of Solvable Lie Groups. `http://arXiv:2410.05042`, 2024.

[Gri14] R. Grigorchuk: Milnor's problem on the growth of groups and its consequences. In *Frontiers in complex dynamics, Princeton Math. Ser.* 51, 705–773, Princeton Univ. Press, 2014.

[Gro81] M. Gromov: Groups of polynomial growth and expanding maps. *Inst. Hautes Etudes Sci. Publ. Math.*, 53:53 – 73, 1981.

[Gro93] ———: Asymptotic invariants of infinite groups. In *Geometric group theory, Vol. 2 (Sussex, 1991)*, volume 182 of *London Math. Soc. Lecture Note Ser.*, pages 1–295. Cambridge Univ. Press, Cambridge, 1993.

[Gui73] Y. Guivarc'h: Croissance polynomiale et périodes des fonctions harmoniques. *Bull. Soc. Math. France*, 101:333–379, 1973.

[Gui80] ———: Sur la loi des grands nombres et le rayon spectral d'une marche aléatoire. *Astérisque*, 74:47–98, Soc. Math. France, 1980.

[Ha87] U. Hamenstädt: Theorie von Carnot-Carathéodory Metriken und ihren Anwendungen. *Bonner Math. Schriften* 180, 1987.

[He74] E. Heintze: On homogeneous manifolds of negative curvature. *Math. Ann.*, 211:23–34, 1974.

[Heb93] J. Heber: On the geometric rank of homogeneous spaces of nonpositive curvature. *Invent. Math.*, 112(1):151–170, 1993.

[Je73] J.E. Jenkins: Growth of Connected Locally Compact Groups. *Journal of Functional Analysis*, 12:113–127, 1973.

[Kl10] B. Kleiner: A new proof of Gromov's theorem on groups of polynomial growth. *J. Amer. Math. Soc.*, 23(3):815–829, 2010.

[KL20] B. Kleiner and U. Lang: Higher rank hyperbolicity. *Inventiones Math.*, 221:597–664, 2020.

[MS03] M. Moskowitz and R. Sacksteder: The Exponential Map and Differential Equations on Real Lie Groups. *Journal of Lie Theory* 13:291-306, 2003.

[Os01] D. Osin: Subgroup distortions in nilpotent groups. *Comm. Algebra.*29(12):5439–5463, 2001.

[Os02] ———: Exponential radicals of solvable Lie groups. *J. Algebra.*248(2):790–805, 2002.

[O18] N. Ozawa: A functional analysis proof of Gromov's polynomial growth theorem. *Ann. Sci. Ec. Norm. Supér.*51(4):549-556, 2018.

[Pan99] P. Pansu: Cohomologie L^p, espaces homogènes et pincement. Preprint 1999.

[Pan08] ———: Cohomologie L^p et pincement. *Comment. Math. Helv.*, 83(2):327–357, 2008.

[Pen11a] I. Peng: Coarse differentiation and quasi-isometries of a class of solvable Lie groups I. *Geom. Topol.*, 15(4):1883-1925, 2011.

[Pen11b] ———: Coarse differentiation and quasi-isometries of a class of solvable Lie groups II. *Geom. Topol.*, 15(4):1927-1981, 2011.

[Roe93] J. Roe: Coarse cohomology and index Theory on complete manifolds. *Mem. Amer. Math. Soc.* 104, Amer. Math. Soc., 1993.

[Sh04] Y. Shalom: Harmonic analysis, cohomology, and the large-scale geometry of amenable groups. *Acta Math.*, 192(2):119–185, 2004.

[Va99] N. T. Varopoulos: Distance distortion on Lie groups. In *Random Walks and Discrete Potential Theory*, (M. Picardello and W. Woess, Eds.) Cambridge Univ. Press, Cambridge, 1999.

[Xie14] X. Xie: Large scale geometry of negatively curved $\mathbf{R}^n \rtimes \mathbf{R}$. *Geom. Topol.*, 18(2):831–872, 2014.

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