L^p-COHOMOLOGY FOR HIGHER RANK SPACES AND GROUPS: CRITICAL EXPONENTS AND RIGIDITY

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ABSTRACT. We initiate the investigation of critical exponents (in degree equal to the rank) for the vanishing of L^p-cohomology of higher rank Lie groups and related manifolds. We deal with the rank 2 case and exhibit such phenomena for $\mathrm{SL}_3(\mathbf{R})$ and for a family of 5-dimensional solvable Lie groups. We use the critical exponents to compare the groups up to quasi-isometry. This leads us to exhibit a continuum of quasi-isometry classes of rank 2 irreducible solvable Lie groups of non-positive curvature. Along the proof, we provide a detailed description of the L^p-cohomology of the real and complex hyperbolic spaces. It is then combined with a spectral sequence argument, to derive our higher-rank results.

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CONTENTS

Introduction

Overview. L^p-cohomology, with $p \in (1, +\infty)$, provides a family of large scale geometry invariants for metric spaces and groups. It has many incarnations (such as asymptotic L^p-cohomology, or group L^p-cohomology via continuous cohomology of locally compact groups) which are all comparable to one another under suitable, not so demanding, conditions. Each variant brings its own insights: for instance asymptotic L^p-cohomology shows that L^p-cohomology is an invariant under quasi-isometry (in fact, under coarse isometry), and continuous cohomology allows one to use standard algebraic tools such as spectral sequences. In the present paper, we are interested in the de Rham L^p-cohomology, which we denote by $L^p\mathrm{H}^*_{\mathrm{dR}}$. Roughly speaking, we are dealing with forms satisfying, together with their differentials, L^p-integrability conditions with respect to measures given by suitable Riemannian metrics.

References for L^p-cohomology include [Gro93] for a general overview, [Pan95, Gen14, Seq24] for its invariance under quasi-isometry, [Ele98, CT11, SS18, BR20] for group L^p-cohomology, [BR20, BR23, LN23] for spectral sequences and applications, [Pan99, Pan07, Pan08, Pan09, Seq24] for de Rham cohomology of Lie groups. We also notice that [BR23, §6] contains a synthetic presentation of (some) of the several aspects of L^p-cohomology, as well as a description and comparison of their properties.

L^p-cohomology of a (connected) Lie group is better understood in rank 1 situations ([1]), when contractions and negative curvature arguments can be used to perform some computations. In particular, critical exponent phenomena with respect to p for vanishing vs non-vanishing of L^p-cohomology in degree 1, can be sometimes exhibited. An iconic example of this phenomenon is provided by the following family of solvable Lie groups: for $\lambda \geqslant 1$, let H_λ be the semi-direct product $H_\lambda = \mathbf{R} \ltimes_\lambda \mathbf{R}^2$, where $\mathbf{R}$ acts on $\mathbf{R}^2$ via the 1-parameter group of automorphisms $t \mapsto e^{-tA_\lambda}$, with $A_\lambda = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$. When $\lambda = 1$, the group

[1]The *rank* of a Lie group is the dimension of its asymptotic cones; it is therefore a quasi-isometric invariant. For semisimple Lie groups, this notion coincides with the $\mathbf{R}$-rank. When the group is simply connected and solvable, its rank is equal to the codimension of its exponential radical [Cor08, Corollary 1.3]. When the group admits a left-invariant Riemannian metric of non-positive curvature, its rank is the same as the maximal dimension of a totally geodesic Euclidean subspace [AW76]. We thank Yves Cornulier for providing us with this definition.

H_λ is naturally isometric to the real hyperbolic 3-space $\mathbb{H}^3_{\mathbf{R}}$. In general H_λ belongs to the family of the so-called *Heintze groups*, *i.e.* of the Lie groups that admit a left-invariant negatively curved Riemannian metric [He74]. Pansu's Theorem [Pan08] shows that $L^p\mathrm{H}^1_{\mathrm{dR}}(H_\lambda)$ vanishes for $p \in (1; 1+\lambda)$, and does not vanish for $p > 1+\lambda$. In other words, $1+\lambda$ is a critical exponent of the first L^p-cohomology of H_λ. As a consequence, the groups H_λ are pairwise non-quasiisometric. More generally, every Heintze group admits an explicit critical exponent in degree 1 [Pan07, CT11].

L^p-cohomology of higher rank Lie groups has attracted less attention so far. As a first step, it would be desirable to have a better understanding of L^p-cohomology in degree equal to the rank. Indeed, the first (reduced) L^p-cohomology of higher rank Lie groups is known to vanish for every p ([2]); and vanishing for every p is expected to remain true in any degree below the rank – at least for semisimple Lie groups [Gro93, p. 253] [LN23]. In degree equal to the rank, critical exponents are known to exist for several higher rank real Lie groups, including all the semisimple ones [BR23]; but their values has not been determined yet.

In the present paper, we study the second L^p-cohomology of solvable Lie groups of rank 2. More precisely, we exhibit, for some groups of this type, a critical exponent in degree 2. We then use these critical exponents to derive a quasi-isometric rigidity result.

A family of solvable Lie groups. We consider the solvable Lie groups of the form $S_\alpha = \mathbf{R}^2 \ltimes_\alpha \mathbf{R}^3$, where

$$\alpha : \mathbf{R}^2 \to \{\text{diagonal automorphisms of } \mathbf{R}^3\}$$

is a Lie group morphism. We denote by $\varpi_i \in (\mathbf{R}^2)^*$ $(i = 1, 2, 3)$ the *weights* associated to α, *i.e.* the linear forms such that $\alpha = e^{\mathrm{diag}(\varpi_1, \varpi_2, \varpi_3)}$.

We shall let $\mathcal{S}^{2,3}_{\mathrm{straight}}$ denote the set of the groups S_α whose weights enjoy the following two properties:

- they generate $(\mathbf{R}^2)^*$,
- they belong to an affine line (necessarily disjoint from 0).

[2] More precisely, apart from those which are quasi-isometric to an Heintze group, the reduced first L^p-cohomology of every Lie group vanihes for every p, and the non-reduced one vanishes if, and only, if the group is non-amenable or non-unimodular [Pan07, CT11]

Every group $S_\alpha \in \mathcal{S}^{2,3}_{\text{straight}}$ is of rank 2, admits a left-invariant Riemannian metric of non-positive curvature, and is irreducible provided its weights are pairwise distinct – see Proposition A.1 in the appendix for a proof of these properties (in a wider generality). Therefore, in some sense, $\mathcal{S}^{2,3}_{\text{straight}}$ appears as the simplest family of irreducible higher rank non-positively curved Lie groups.

Our first result exhibits a critical exponent of the second L^p-cohomology of the groups that belong to $\mathcal{S}^{2,3}_{\text{straight}}$. It answers partially a question of Cornulier.

Theorem A. *Let $S_\alpha \in \mathcal{S}^{2,3}_{\text{straight}}$. Its weights $\varpi_1, \varpi_2, \varpi_3$ belong to a line. Since permuting the coordinates of $\mathbf{R}^3$ preserves the isomorphism class of S_α, we can (and will) assume that the algebraic distances between the weights on the (suitably oriented) line satisfy: $0 \leqslant \varpi_2 - \varpi_3 \leqslant \varpi_1 - \varpi_2$. Set*

$$p_\alpha := 1 + \frac{\varpi_1 - \varpi_3}{\varpi_1 - \varpi_2} \in [2; 3].$$

Then $L^p\mathrm{H}^2_{\mathrm{dR}}(S_\alpha) = \{0\}$ for $p \in (1; p_\alpha) \setminus \{\frac{3}{2}\}$, and $L^p\mathrm{H}^2_{\mathrm{dR}}(S_\alpha) \neq \{0\}$ for $p \in (p_\alpha; +\infty) \setminus \{3\}$.

Observe that two triples $(\varpi_1, \varpi_2, \varpi_3)$ and $(\varpi_1', \varpi_2', \varpi_3')$ – of distinct elements in $(\mathbf{R}^2)^*$ that are aligned on lines disjoint from 0 – belong to the same $\mathrm{GL}_2(\mathbf{R})$-orbit if, and only, if

$$\frac{\varpi_1 - \varpi_3}{\varpi_1 - \varpi_2} = \frac{\varpi_1' - \varpi_3'}{\varpi_1' - \varpi_2'}.$$

Since precomposing α by an element of $\mathrm{GL}_2(\mathbf{R})$ preserves the isomorphism class of S_α, and since de Rham L^p -cohomology is a quasi-isometry invariant among Lie groups that are diffeomorphic to $\mathbf{R}^D$ [Pan95],[BR23, Appendice], Theorem A admits the following rigidity consequence:

Corollary B. *Among the groups in $\mathcal{S}^{2,3}_{\text{straight}}$, any two of them are quasi-isometric if, and only if, they are isomorphic.*

Theorem A also yields:

Corollary C. *There exists a continuum of quasi-isometry classes of rank 2 solvable irreducible non-positively curved Lie groups.*

The symmetric space $\mathrm{SL}_3(\mathbf{R})/\mathrm{SO}_3(\mathbf{R})$. Going back to semisimple groups, the same approach – which we describe below in more details – enables us to exhibit another critical exponent in degree 2:

Theorem D. *Let S be the symmetric space $\mathrm{SL}_3(\mathbf{R})/\mathrm{SO}_3(\mathbf{R})$, or equivalently the Borel subgroup of $\mathrm{SL}_3(\mathbf{R})$ consisting of upper triangular matrices with positive diagonal. Then $L^p\mathrm{H}^2_{\mathrm{dR}}(S)$ is zero for $p \in (1;2)\setminus\{\frac{4}{3}\}$, and non-zero for $p \in (2;+\infty) \setminus \{4\}$.*

By Iwasawa decomposition, the symmetric space S is naturally isometric to the Borel subgroup of $\mathrm{SL}_3(\mathbf{R})$; the latter is a specific rank 2 solvable group, namely the semidirect product of $\mathbf{R}^2$ and of the Heisenberg group in dimension 3.

About the proofs. In both instances of the solvable groups dealt with in the above theorems, we can decompose the action of $\mathbf{R}^2$ on the 3-dimensional subgroup $\mathbf{R}^3$ (resp. on the Heisenberg group in dimension 3, which we denote by Heis(3)) into two steps. In a first step, one factor $\mathbf{R}$ of $\mathbf{R^2}$ acts on $\mathbf{R}^3$ (resp. on Heis(3)) so that the intermediate (rank 1) semidirect product is a non-unimodular solvable group isometric to the real (resp. complex) hyperbolic space of real dimension 4. Then, as a second step, we consider the action of the second factor $\mathbf{R}$ of $\mathbf{R}^2$ and use a spectral sequence argument, together with the fact that we understand in detail the cohomology of the intermediate 4-dimensional group of the first step. Thus, at this stage, proving the non-vanishing of the considered L^p-cohomology amounts to showing that some de Rham classes on the rank 1 group satisfy a certain L^p-integrability condition (see Section 5.3, and Relation 7.7 in Section 7.4). The vanishing part requires to use a Poincaré duality argument in order to show the requested non-integrability of the relevant de Rham classes.

The main result about the rank 1 intermediate solvable groups above is Theorem 3.2. It provides a partial description of the L^p-cohomology of Lie groups containing a suitable 1-parameter subgroup of (semi) contractions acting on its complement. The obtained description complements some previous results of Pansu [Pan99, Sections 9 and 10]; we call it a *strip decomposition* since its hypotheses are stated as (double) inequalities that must be satisfied by the exponent p with respect to quantities depending on the degree k of the cohomology and on the infinitesimal eigenvalues of the contraction group. The conclusions deal with the following properties: vanishing, Hausdorff property, density of some explicit subspaces of closed forms, and finally Poincaré duality realized at infinity, *i.e.* on the group-theoretic complement of the contraction group (which is a Lie group seen as a boundary of the ambient group).

The statement of Theorem 3.2 is applied to the special semidirect products $\mathbf{R} \ltimes \mathbf{R}^n$ and $\mathbf{R} \ltimes \mathrm{Heis}(2m-1)$, that are isometric to the real hyperbolic space $\mathbb{H}^{n+1}_{\mathbf{R}}$ and to the complex one $\mathbb{H}^m_{\mathbf{C}}$. We obtain in this way a rather precise description of the L^p-cohomology of $\mathbb{H}^{n+1}_{\mathbf{R}}$. In the case of the complex hyperbolic space, the information given by Theorem 3.2 is fragmented, and a substantial additional amount of work dealing with Heisenberg groups of arbitrary dimension, elaborating on ideas due to Rumin [Rum94] and Pansu [Pan09], is required.

Again, from a technical point of view, the paper deals with de Rham cohomology only, and some of our results are valid for Riemannian manifolds endowed with a suitable contracting vector field, even though the main applications are relevant to the Lie group situation. This applies in particular to the main new technical result (Theorem 2.5) which translates the Poincaré duality in terms of currents on the "boundary".

Let us finish this introduction with some remarks.

Remark 0.1. Pansu has already used L^p-cohomology to show that the groups $H_\alpha := \mathbf{R} \ltimes_\alpha \mathbf{R}^n$, with

$$\alpha(t) = \mathrm{diag}(e^{-\alpha_1 t}, \dots, e^{-\alpha_n t}) \quad \text{and} \quad 1 = \alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_n,$$

form a continuous family of pairwise non-quasiisometric Heintze groups [Pan99, Corollary 2], [Seq24]. This result has been generalized by Xie [Xie14, Corollary 1.3] to non-diagonal automorphisms, by using more geometric methods.

Remark 0.2. It is a natural to ask whether the rigidity statement of Corollary B remains true for other family of groups, like the families $\mathcal{S}^{r,n}$ considered in the appendix. This question has been already studied in some special cases, *e.g.* for the family evoked just above in Remark 0.1. In [EFW12], the authors study the groups $\mathrm{Sol}_\lambda := \mathbf{R} \ltimes_\lambda \mathbf{R}^2$, where $\lambda \geqslant 1$, and where $\mathbf{R}$ acts on $\mathbf{R}^2$ via $t \mapsto e^{tB_\lambda}$, with $B_\lambda = \begin{pmatrix} 1 & 0 \\ 0 & -\lambda \end{pmatrix}$. They show that these groups are pairwise non-quasiisometric. Observe that the groups Sol_λ are rank 1 solvable Lie groups that do not carry any non-positively curved left-invariant Riemannian metric – see Proposition A.1(3).

In [Pen11b, Corollary 5.3.7], Peng establishes that if $G = \mathbf{R}^r \ltimes_\varphi \mathbf{R}^n$ and $G' = \mathbf{R}^{r'} \ltimes_{\varphi'} \mathbf{R}^{n'}$ are two quasi-isometric, *unimodular*, non-degenerate split abelian-by-abelian solvable Lie groups, then there exists an isomorphim $f : \mathbf{R}^r \to \mathbf{R}^{r'}$ such that φ and $\varphi' \circ f$ have the same Jordan form. Again, due to the unimodularity, these groups do not carry any non-positively curved left-invariant Riemannian metric.

Remark 0.3. Examples of non-positively curved, rank 2, *reducible*, solvable Lie groups, include the groups $\mathbf{R} \times H_\lambda$ (in dimension 4), and the groups $\mathbb{H}^2_{\mathbf{R}} \times H_\lambda$ (in dimension 5); where H_λ ($\lambda \geqslant 1$) is the Heintze group defined in the overview. As a consequence of a general quasi-isometric rigidity theorem for product metric spaces [KKL98, Theorem B], all these groups are pairwise non-quasiisometric.

Remark 0.4. Our results on de Rham L^p-cohomology of hyperbolic spaces can be compared with Borel's on L^2-cohomology of symmetric spaces [Bor85]. It turns out that for complex hyperbolic spaces our results are complementary in the sense that the exponent $p = 2$ is never contained in the interior of the strips we distinguish. Nevertheless, for $\mathbb{H}^m_{\mathbf{C}}$ it is in the closure (and in the middle) of the union $]2\frac{m}{m+1}; 2[\sqcup]2; 2\frac{m}{m-1}[$ of two critical strips. For p in the interior of each segment, our Theorem 6.1 says that $L^p\mathrm{H}^m_{\mathrm{dR}}(\mathbb{H}^m_{\mathbf{C}})$ is Hausdorff and non-zero, while Theorem A of [loc. cit.] says that $L^2\mathrm{H}^m_{\mathrm{dR}}(\mathbb{H}^m_{\mathbf{C}})$ is Hausdorff and non-zero. Moreover it describes the latter space in representation-theoretic terms. For real hyperbolic spaces $\mathbb{H}^{n+1}_{\mathbf{R}}$, we have to distinguish two cases according to the parity of n. When n is odd, our Theorem 4.1 recovers Theorem A(i) of [loc. cit.], saying that $L^2\mathrm{H}^\bullet_{\mathrm{dR}}(\mathbb{H}^{n+1}_{\mathbf{R}})$ is Hausdorff and concentrated in degree $\frac{n+1}{2}$. When n is even, Theorem B of [loc. cit.] complements our result, saying that $L^2\overline{\mathrm{H}^\bullet_{\mathrm{dR}}}(\mathbb{H}^{n+1}_{\mathbf{R}})$ is zero and $L^2\mathrm{H}^\bullet_{\mathrm{dR}}(\mathbb{H}^{n+1}_{\mathbf{R}})$ is not Hausdorff in degree $\frac{n}{2} + 1$.

Structure of the paper. Section 1 introduces currents in the context of L^p-cohomology; it also recalls Poincaré duality for the reduced variant of it. Section 2 introduces flows with suitable contraction properties on manifolds; it describes their effects on L^p-cohomology and introduces a version of Poincaré duality involving currents on the "boundary" of such a manifold. In Section 3, the situation is specialized to the case of Lie groups; the existence of a suitable 1-dimensional (semi) contracting group leads to the strip description of the L^p-cohomology of the groups under consideration. In Section 4, we apply the result of the previous section to deduce the description of the L^p-cohomology of real hyperbolic spaces. Section 5 focusses on the proof of Theorem A about the second L^p-cohomology of the groups S_α; this is where we determine our first critical exponent. Section 6 provides a description of the L^p-cohomology of complex hyperbolic spaces; this requires more care than for the real case, and in particular it leads to an intensive use of Heisenberg groups. In Section 7, using the same strategy as in Section 5, we determine our second higher-rank critical exponent, this time for the symmetric space $\mathrm{SL}_3(\mathbf{R})/\mathrm{SO}_3(\mathbf{R})$. At last, an appendix deals with the groups S_α in a wider generality, and establishes some of their basic properties.

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1. Currents and L^p-cohomology

In this section, we give a quick presentation of de Rham L^p-cohomology and related topics.

1.1. **Currents.** Currents play a central role in classical de Rham cohomology. We recall some of the basic definitions and properties useful for the L^p variant (see [DS05] for more informations).

Let M be a C^∞ orientable D-manifold without boundary. For $k \in \mathbf{Z}$, let $\Omega^k(M)$ be the space of C^∞ differential k-forms on M, and let $\Omega^k_c(M)$ be the space of compactly supported C^∞ differential k-forms, endowed with the C^∞ topology. As usual we set $\Omega^k(M) = \Omega^k_c(M) = \{0\}$ for $k < 0$.

A *k-current* on M is by definition a continuous real valued linear form on $\Omega^{D-k}_c(M)$. We denote by $\mathcal{D}'^k(M)$ the space of k-currents on M endowed with the weak*-topology.

To every $\omega \in \Omega^k(M)$, one associates the k-current T_ω defined by $T_\omega(\alpha) := \int_M \omega \wedge \alpha$. This defines an embedding of $\Omega^k(M)$ into $\mathcal{D}'^k(M)$, whose image is known to be dense. The *differential* of a k-current T is the $(k+1)$-current dT defined by $dT(\alpha) := (-1)^{k+1}T(d\alpha)$, for every $\alpha \in \Omega^{D-k-1}_c(M)$. The so-obtained map d satisfies $d \circ d = 0$. Since M is assumed to have no boundary, this definition is consistent with Stokes' formula:

$$\int_M d\omega \wedge \alpha = (-1)^{k+1} \int_M \omega \wedge d\alpha,$$

and gives in particular: $dT_\omega = T_{d\omega}$.

More generally, suppose we are given $\ell \in \mathbf{Z}$, and a continuous linear operator $L : \Omega^*(M) \to \Omega^{*-\ell}(N)$ – where M and N are orientable manifolds of dimension D_M and D_N respectively – such that there is a continuous operator $\tilde{L} : \Omega_c^{D_N-*+\ell}(N) \to \Omega_c^{D_M-*}(M)$, with

$$\int_N L(\omega) \wedge \alpha = \int_M \omega \wedge \tilde{L}(\alpha),$$

for every $\omega \in \Omega^*(M)$ and $\alpha \in \Omega_c^{D_N-*+\ell}(N)$. Then L extends by continuity to $\mathcal{D}'^*(M) \to \mathcal{D}'^{*-\ell}(N)$, by setting $(L(T))(\alpha) := T(\tilde{L}(\alpha))$. This applies *e.g.* to inner products $\iota_\xi : \Omega^*(M) \to \Omega^{*-1}(M)$ by a vector field ξ on M. One has $\tilde{\iota_\xi} = (-1)^{k+1}\iota_\xi$ on $\Omega_c^{D-k+1}(M)$, since ι_ξ is an anti-derivation (see *e.g.* [Tu08, Proposition 20.8]).

In local coordinates $(x_1, ..., x_D)$ on an open subset $U \subset M$, every k-current $T \in \mathcal{D}'^k(U)$ can be written $T = \sum_{|I|=k} T_I dx_I$, with $T_I \in \mathcal{D}'^0(U)$. For every $\alpha \in \Omega_c^{D-k}(U)$, one has $T(\alpha) = \sum_{|I|=k} T_I(dx_I \wedge \alpha)$.

1.2. **De Rham L^p-cohomology: definitions and notation.** We list and fix the definitions and notations for several objects that will appear repeatedly in the paper.

Let M be a C^∞ orientable manifold (without boundary), henceforth endowed with a Riemannian metric. We denote by $d$vol its Riemannian measure, and by $|v|$ the Riemannian length of a vector $v \in TM$.

- Let $p \in (1, +\infty)$. The *L^p-norm of* $\omega \in \Omega^k(M)$ is

$$\|\omega\|_{L^p\Omega^k} = \big(\int_M |\omega|_m^p \ d\mathrm{vol}(m)\big)^{1/p},$$

  where we set

$$|\omega|_m := \sup\{|\omega(m; v_1, \dots, v_k)| : v_1, \dots, v_k \in T_mM,\ |v_i| = 1\}.$$

- The space $L^p\Omega^k(M)$ is the norm completion of the normed space $\{\omega \in \Omega^k(M) : \|\omega\|_{L^p\Omega^k} < +\infty\}$, *i.e.* the Banach space of k-differential forms with measurable L^p coefficients.
- To every $\omega \in L^p\Omega^k(M)$, one associates the k-current T_ω defined by $T_\omega(\alpha) := \int_M \omega \wedge \alpha$. The *differential in the sense of currents* of $\omega \in L^p\Omega^k(M)$ is the $(k+1)$-current $d\omega := dT_\omega$. One says that *$d\omega$ belongs to* $L^p\Omega^{k+1}(M)$ if there exists $\theta \in L^p\Omega^{k+1}(M)$ such $d\omega = T_\theta$. In this case we set $\|d\omega\|_{L^p\Omega^{k+1}} := \|\theta\|_{L^p\Omega^{k+1}}$.

- For $\omega \in \Omega^k(M)$, we set
$$\|\omega\|_{\Omega^{p,k}} := \|\omega\|_{L^p\Omega^k} + \|d\omega\|_{L^p\Omega^{k+1}}.$$
The space $\Omega^{p,k}(M)$ is the norm completion of the normed space $\{\omega \in \Omega^k(M) : \|\omega\|_{\Omega^{p,k}} < +\infty\}$. It is a Banach space that coincides with the subspace of $L^p\Omega^k(M)$ consisting of the L^p k-forms whose differentials in the sense of currents belong to $L^p\Omega^{k+1}(M)$. Moreover the differential operator d on $\Omega^{p,*}(M)$ agrees with the differential in the sense of currents. (See *e.g.* [BR23, Lemma 1.5] for a proof).
- The *de Rham L^p-cohomology* of M is the cohomology of the complex
$$\Omega^{p,0}(M) \xrightarrow{d_0} \Omega^{p,1}(M) \xrightarrow{d_1} \Omega^{p,2}(M) \xrightarrow{d_2} \dots$$
It is denoted by $L^p\mathrm{H}^*_{\mathrm{dR}}(M)$. Its largest Hausdorff quotient is denoted by $L^p\overline{\mathrm{H}^*_{\mathrm{dR}}}(M)$ and is called the *reduced de Rham L^p-cohomology* of M. The latter is a Banach space; its (quotient) norm is denoted by $\|\cdot\|_{L^p\overline{\mathrm{H}^*}}$.
- Following Pansu, we also define $\Psi^{p,k}(M)$ to be the space of k-currents $\psi \in \mathcal{D}'^k(M)$ that can be written $\psi = \beta + d\gamma$, with $\beta \in L^p\Omega^k(M)$ and $\gamma \in L^p\Omega^{k-1}(M)$. In particular we have $\Psi^{p,0}(M) = L^p(M)$. Equipped with the norm
$$\|\psi\|_{\Psi^{p,k}} := \inf\Big\{\|\beta\|_{L^p\Omega^k} + \|\gamma\|_{L^p\Omega^{k-1}} : \psi = \beta + d\gamma, \text{ with } \beta \in L^p\Omega^k(M) \text{ and } \gamma \in L^p\Omega^{k-1}(M)\Big\},$$
the space $\Psi^{p,k}(M)$ is a Banach space, and the inclusion maps between differential complexes:
$$\Omega^{p,*}(M) \subset \Psi^{p,*}(M) \subset \mathcal{D}'^*(M)$$
are continuous (see [BR23, Lemma 1.3] for a proof).
- Suppose that M carries a C^∞ unit complete vector field ξ, and let $(\varphi_t)_{t\in\mathbf{R}}$ be its flow. Assume that $\varphi_t^* : L^p\Omega^k(M) \to L^p\Omega^k(M)$ is bounded for all $t \in \mathbf{R}$, $p \in (1, +\infty)$ and $k \in \mathbf{N}$. We set
$$\Psi^{p,k}(M,\xi) := \{\psi \in \Psi^{p,k}(M) : \varphi_t^*(\psi) = \psi \text{ for every } t \in \mathbf{R}\}.$$
The differential complex $\Psi^{p,*}(M,\xi)$ is a closed subcomplex of $\Psi^{p,*}(M)$. Let
$$\mathcal{Z}^{p,k}(M,\xi) := \mathrm{Ker}\big(d : \Psi^{p,k}(M,\xi) \to \Psi^{p,k+1}(M,\xi)\big)$$
be the space of k-cocycles.

1.3. **Poincaré duality.** Poincaré duality for de Rham L^p-cohomology takes the following form.

Proposition 1.1. *Let M be a complete oriented Riemannian manifold of dimension D. Let $p \in (1, +\infty)$, $q = p/(p-1)$ be its Hölder conjugate, and $k \in \{0, \ldots, D\}$. Then*

1. *$L^p\mathrm{H}^k_{\mathrm{dR}}(M)$ is Hausdorff if and only if $L^q\mathrm{H}^{D-k+1}_{\mathrm{dR}}(M)$ is.*
2. *$L^p\overline{\mathrm{H}^k_{\mathrm{dR}}}(M)$ and $L^q\overline{\mathrm{H}^{D-k}_{\mathrm{dR}}}(M)$ are dual Banach spaces, via the perfect pairing $L^p\overline{\mathrm{H}^k_{\mathrm{dR}}}(M) \times L^q\overline{\mathrm{H}^{D-k}_{\mathrm{dR}}}(M) \to \mathbf{R}$, defined by*
$$([\omega_1], [\omega_2]) \mapsto \int_M \omega_1 \wedge \omega_2.$$

Proof. See [Pan08, Corollaire 14] or [GT10]. □

The following terminology will be useful in the sequel.

Definition 1.2. Let $p, q \in (1, +\infty)$ and $k, \ell \in \{0, \ldots, D\}$. The couples (p, k) and (q, ℓ) are said to be *Poincaré dual* if p and q are Hölder conjugate and if $\ell = D - k$.

2. Flows and L^p-cohomology

This section exploits some dynamical properties of flows acting on forms to extract information on L^p-cohomology. The objects appearing in this section are defined in Section 1.2. In what follows, we keep M a C^∞ orientable manifold (without boundary) endowed with a Riemannian metric.

2.1. **Invariance, identification and vanishing.** We review several results due to Pansu, see [Pan08, Proposition 10] or [BR23, Section 1].

Let ξ be a C^∞ unit complete vector field on M, and denote by $(\varphi_t)_{t\in\mathbf{R}}$ its flow. We assume that $\varphi_t^* : L^p\Omega^k(M) \to L^p\Omega^k(M)$ is bounded for all $t \in \mathbf{R}$, $p \in (1, +\infty)$ and $k \in \mathbf{N}$. This happens *e.g.* when M is a manifold of *bounded geometry, i.e.* a manifold whose injectivity radius is bounded from below and whose sectional curvatures are bounded from above and from below.

Proposition 2.1. *Let $p \in (1, +\infty)$ and let $k \in \mathbf{N}$. Then for every $\omega \in \Omega^{p,k}(M) \cap \operatorname{Ker} d$ and $t \in \mathbf{R}$, the forms ω and $\varphi_t^*\omega$ are cohomologous in $L^p\mathrm{H}^k_{\mathrm{dR}}(M)$.*

Proof. See *e.g.* [BR23, Lemma 1.3]. □

Proposition 2.2. *Let $p \in (1, +\infty)$ and $k \in \mathbf{N}^*$. Suppose that there exist $C, \eta > 0$ such that for every $t \geqslant 0$, one has*

$$\|\varphi_t^*\|_{L^p\Omega^{k-1} \to L^p\Omega^{k-1}} \leqslant Ce^{-\eta t}.$$

Then:

(1) *Let $\omega \in \Omega^{p,k}(M) \cap \operatorname{Ker} d$. When $t \to +\infty$, the form $\varphi_t^*\omega$ converges in the Banach space $\Psi^{p,k}(M)$ (and so in the sense of currents); its limit ω_∞ is a closed current in $\mathcal{Z}^{p,k}(M, \xi)$.*

(2) *The map $\omega \mapsto \omega_\infty$ induces a canonical Banach isomorphism*

$$L^p\mathrm{H}^k_{\mathrm{dR}}(M) \simeq \mathcal{Z}^{p,k}(M, \xi).$$

In particular $L^p\mathrm{H}^k_{\mathrm{dR}}(M)$ is Hausdorff.

Proof. The statement is essentially contained in [Pan08, Proposition 10]. A proof also appears in [BR23, Proposition 1.9] under the stronger assumption that $\max_{i=k-2,k-1} \|\varphi_t^*\|_{L^p\Omega^i \to L^p\Omega^i} \leqslant Ce^{-\eta t}$. The extra assumption served only in parts (3) and (4) of the proof, to show that $\lim_{t\to+\infty} \|\varphi_t^*(d\theta)\|_{\Psi^{p,k}} = 0$ for every $\theta \in L^p\Omega^{k-1}(M)$. But the weaker hypothesis $\|\varphi_t^*\|_{L^p\Omega^{k-1} \to L^p\Omega^{k-1}} \leqslant Ce^{-\eta t}$ is enough to prove this property. Indeed, by combining the definition of $\|\cdot\|_{\Psi^{p,k}}$ with this assumption, one has

$$\|\varphi_t^*(d\theta)\|_{\Psi^{p,k}} = \|d(\varphi_t^*\theta)\|_{\Psi^{p,k}} \leqslant \|\varphi_t^*\theta\|_{L^p\Omega^{k-1}} \to 0$$

when $t \to \infty$. □

Corollary 2.3. *Let $p \in (1, +\infty)$ and $k \in \mathbf{N}^*$. Suppose that there exist $C, \eta > 0$ such that for every $t \geqslant 0$, one has*

$$\max_{i=k-1,k} \|\varphi_t^*\|_{L^p\Omega^i \to L^p\Omega^i} \leqslant Ce^{-\eta t}.$$

Then $L^p\mathrm{H}^k_{\mathrm{dR}}(M) = \{0\}$.

Proof. Our assumption implies that $\|\varphi_t^*\|_{\Psi^{p,k} \to \Psi^{p,k}} \leqslant Ce^{-\eta t}$; and also that $L^p\mathrm{H}^k_{\mathrm{dR}}(M) \simeq \mathcal{Z}^{p,k}(M, \xi)$ by Proposition 2.2. Since the elements of $\mathcal{Z}^{p,k}(M, \xi)$ are φ_t-invariant, one gets that $\mathcal{Z}^{p,k}(M, \xi) = \{0\}$. Therefore $L^p\mathrm{H}^k_{\mathrm{dR}}(M) = \{0\}$. □

2.2. **Boundary values, Poincaré duality revisited.** In this section, the oriented Riemanniann manifold M is supposed to be complete. We assume futhermore that M and the unit vector field ξ are such that the pair (M, ξ) is C^∞-diffeomorphic to a pair of the form $(\mathbf{R} \times N, \frac{\partial}{\partial t})$, where the vector field $\frac{\partial}{\partial t}$ is carried by the $\mathbf{R}$-factor.

We think of N as a "boundary" of M. Under some dynanical assumptions, we will represent the spaces $L^p\mathrm{H}^k_{\mathrm{dR}}(M)$ and the Poincaré duality on the boundary N (see Proposition 2.4 and Theorem 2.5 below).

Let $\pi : M \to N$ be the projection map and $\pi^* : \mathcal{D}'^i(N) \to \mathcal{D}'^i(M)$ be the continuous extension of the pull-back map $\pi^* : \Omega^i(N) \to \Omega^i(M)$. We set $n =: \dim N$ so that $D := \dim M = n+1$.

Proposition 2.4. *Let $p \in (1, +\infty)$ and $k \in \mathbf{N}^*$. Suppose that there exist $C, \eta > 0$ such that for $t \geqslant 0$:*

$$\|\varphi_t^*\|_{L^p\Omega^{k-1} \to L^p\Omega^{k-1}} \leqslant Ce^{-\eta t}.$$

Then for every $\psi \in \mathcal{Z}^{p,k}(M, \xi)$, there exists $T \in \mathcal{D}'^k(N) \cap \operatorname{Ker} d$ such that $\psi = \pi^*(T)$.

Proof. Recall that $\xi = \frac{\partial}{\partial t}$ and that the flow of ξ is denoted by φ_t. Every $\psi \in \mathcal{Z}^{p,k}(M, \xi)$ is φ_t-invariant; therefore showing that $\psi = \pi^*(T)$ is equivalent to proving that $\iota_\xi \psi = 0$. From Proposition 2.2, there exists $\omega \in \Omega^{p,k}(M) \cap \operatorname{Ker} d$ such that $\psi = \lim_{t\to+\infty} \varphi_t^*(\omega)$ in the sense of currents. Since the map $\iota_\xi : \mathcal{D}'^k(M) \to \mathcal{D}'^{k-1}(M)$ is continuous, one obtains that $\iota_\xi\psi = \lim_{t\to+\infty} \varphi_t^*(\iota_\xi\omega)$ in the sense of currents. But $\iota_\xi\omega \in L^p\Omega^{k-1}(M)$, and by assumption one has $\|\varphi_t^*\|_{L^p\Omega^{k-1}\to L^p\Omega^{k-1}} \to 0$ when $t \to +\infty$. Thus $\iota_\xi\psi = 0$.

Lastly, since $d\psi = 0$, one gets that $\pi^*(dT) = 0$, which in turn implies that $dT = 0$. Thus $T \in \mathcal{D}'^k(N) \cap \operatorname{Ker} d$, as expected. □

Let χ be a non-negative C^∞ function on M, depending only on the $\mathbf{R}$-variable, such that $\chi(t) = 0$ for $t \leqslant 0$ and $\chi(t) = 1$ for $t \geqslant 1$.

Theorem 2.5. *Let $p, q \in (1, +\infty)$ and $k, \ell \in \{1, \ldots, n\}$ be such that (p, k) and (q, ℓ) are Poincaré dual — see Definition 1.2. Suppose that there exist $C, \eta > 0$ such that for $t \geqslant 0$:*

(1) $\|\varphi_t^*\|_{L^p\Omega^{k-1}\to L^p\Omega^{k-1}} \leqslant Ce^{-\eta t}$,
(2) $\|\varphi_{-t}^*\|_{\operatorname{Ker}\iota_\xi \cap L^q\Omega^\ell \to \operatorname{Ker}\iota_\xi\cap L^q\Omega^\ell} \leqslant Ce^{-\eta t}$.

Then for every $\theta \in \Omega_c^{\ell-1}(N)$, the form $d(\chi \cdot \pi^*\theta)$ belongs to the space $\Omega^{q,\ell}(M) \cap \operatorname{Ker} d$; and for every $\omega \in \Omega^{p,k}(M) \cap \operatorname{Ker} d$, one has:

$$\int_M \omega \wedge d(\chi \cdot \pi^*\theta) = T(\theta),$$

where T is the closed k-current on N such that

$$\omega_\infty = \lim_{t\to+\infty} \varphi_t^*(\omega) = \pi^*(T),$$

as in Propositions 2.2 and 2.4.

As a consequence of Theorem 2.5, we will prove:

Corollary 2.6. *Suppose that the assumptions of Theorem 2.5 are satisfied. Then the classes of the* $d(\chi \cdot \pi^*\theta)$*'s (where* $\theta \in \Omega_c^{\ell-1}(N)$*) form a dense subspace in* $L^q\overline{\mathrm{H}_{\mathrm{dR}}^{\ell}}(M)$*. Moreover when* $\theta = d\alpha$ *is an exact form, with* $\alpha \in \Omega_c^{\ell-2}(N)$*, then* $[d(\chi \cdot \pi^*\theta)] = 0$ *in* $L^q\overline{\mathrm{H}_{\mathrm{dR}}^{\ell}}(M)$*.*

Recall that $L^p\overline{\mathrm{H}_{\mathrm{dR}}^{k}}(M)$ and $L^q\overline{\mathrm{H}_{\mathrm{dR}}^{\ell}}(M)$ are dual Banach spaces, via the pairing $([\omega_1],[\omega_2]) \mapsto \int_M \omega_1 \wedge \omega_2$ (see Proposition 1.1). In combination with Theorem 2.5 and Corollary 2.6 above, this yields immediately to the:

Corollary 2.7. *Suppose that the assumptions of Theorem 2.5 are satisfied. Let* $\omega \in \Omega^{p,k}(M) \cap \operatorname{Ker} d$ *and let* $T \in \mathcal{D}'^k(N) \cap \operatorname{Ker} d$ *be such that* $\lim_{t\to+\infty} \varphi_t^*(\omega) = \pi^*(T)$*. Then the norm of* $[\omega]$ *in* $L^p\mathrm{H}_{\mathrm{dR}}^k(M)$ *satisfies:*

$$\big\|[\omega]\big\|_{L^p\mathrm{H}^k} = \sup\Big\{T(\theta) : \theta \in \Omega_c^{\ell-1}(N),\ \big\|[d(\chi \cdot \pi^*\theta)]\big\|_{L^q\overline{\mathrm{H}^\ell}} \leqslant 1\Big\}.$$

Proof of Theorem 2.5. Step 1. We first show that the L^q-norm of the form $d(\chi \cdot \pi^*\theta)$ is finite. Set $\alpha := \pi^*\theta$ for simplicity. One has

$$d(\chi \cdot \alpha) = d\chi \wedge \alpha + \chi \cdot d\alpha.$$

The form $d\alpha$ belongs to $\operatorname{Ker} \iota_\xi$ and is φ_t-invariant. With the assumption (2) we obtain (since $s \geqslant 0$):

$$\begin{aligned}\|\chi \cdot d\alpha\|_{L^q\Omega^\ell} &\leqslant \|\mathbf{1}_{t\geqslant 0} \cdot d\alpha\ \|_{L^q\Omega^\ell} \\ &= \sum_{i=0}^{\infty} \|\mathbf{1}_{t\in[i,i+1]} \cdot d\alpha\|_{L^q\Omega^\ell} \\ &\leqslant C\sum_{i=0}^{\infty} e^{-\eta i}\|\mathbf{1}_{t\in[0,1]} \cdot d\alpha\|_{L^q\Omega^\ell} \\ &= \frac{C}{1-e^{-\eta}}\|\mathbf{1}_{t\in[0,1]} \cdot d\alpha\|_{L^q\Omega^\ell}.\end{aligned}$$

which is finite since $\mathbf{1}_{t\in[0,1]} \cdot d\alpha$ has compact support.

It remains to bound from above the L^q-norm of $d\chi \wedge \alpha$. One has $d\chi = \chi'(t)dt$, with χ' supported on $[0,1]$. Thus

$$\|d\chi \wedge \alpha\|_{L^q\Omega^\ell} \leqslant \|\chi' \cdot \alpha\|_{L^q\Omega^{\ell-1}} \leqslant C_1\|\mathbf{1}_{t\in[0,1]} \cdot \alpha\|_{L^q\Omega^{\ell-1}},$$

with $C_1 = \|\chi'\|_\infty$. Since $\mathbf{1}_{t\in[0,1]} \cdot \alpha$ has compact support, the L^q-norm of $d\chi \wedge \alpha$ is finite too. The statement follows.

Step 2. We now compute $\int \omega\wedge d(\chi\cdot\alpha)$. Since $\varphi_t^*(\omega)$ and ω are cohomologous (by Proposition 2.1), one has thanks to Proposition 1.1(2):

$$\int \omega\wedge d(\chi\cdot\alpha) = \int \varphi_t^*(\omega)\wedge d(\chi\cdot\alpha)$$
$$= \int \varphi_t^*(\omega)\wedge d\chi\wedge\alpha + \int \varphi_t^*(\omega)\wedge(\chi\cdot d\alpha).$$

Since the form $d\chi\wedge\alpha$ belongs to $\Omega_c^\ell(M)$, one has

$$\lim_{t\to\infty}\int \varphi_t^*(\omega)\wedge d\chi\wedge\alpha = (\pi^*T)(d\chi\wedge\alpha),$$

indeed $\varphi_t^*(\omega)$ tends to π^*T in the sense of currents thanks to assumption (1), Propositions 2.2 and 2.4.

One observes that the map $\pi^*:\mathcal{D}'^i(N)\to\mathcal{D}'^i(M)$ can be written as $(\pi^*T)(\beta)=T\big(j(\beta)\big)$ where $j:\Omega_c^{D-i}(M)\to\Omega_c^{D-1-i}(N)$ is defined by

$$j(\beta)=\int_{\mathbf{R}}(\iota_\xi\beta)_{(t,\cdot)}\ dt$$

(we recall that $\xi=\frac{\partial}{\partial t}$). Since the inner product is an anti-derivation (see *e.g.* [Tu08, Proposition 20.8]) and since $\iota_\xi\alpha=0$, one has

$$\iota_\xi(d\chi\wedge\alpha)=(\iota_\xi d\chi)\wedge\alpha - d\chi\wedge(\iota_\xi\alpha)=\chi'\cdot\pi^*\theta.$$

Therefore $j(d\chi\wedge\alpha)=\int_{\mathbf{R}}\chi'(t)\cdot\theta\ dt=\theta$, and we obtain

$$(\pi^*T)(d\chi\wedge\alpha)=T(\theta).$$

Step 3. According to the previous paragraph, it remains to prove that

$$\lim_{t\to+\infty}\int \varphi_t^*(\omega)\wedge(\chi\cdot d\alpha)=0.$$

For $s>0$, let $\chi_s:M\to\mathbf{R}$ be a C^∞-function depending only on the $\mathbf{R}$-variable, such that $\chi_s(t)=\chi(t)$ for $t\leqslant s$ and $\chi_s(t)=0$ for $t\geqslant s+1$. Observe that $\chi_s\cdot d\alpha$ is C^∞ with compact support. We claim that:

- For every $s>0$, one has $\lim_{t\to+\infty}\int\varphi_t^*(\omega)\wedge(\chi_s\cdot d\alpha)=0$,
- $\int\varphi_t^*(\omega)\wedge\big((\chi-\chi_s)\cdot d\alpha\big)$ tends to 0 uniformly in $t>0$ when $s\to+\infty$.

As explained above, the claim completes the proof of the theorem. The first item of the claim follows from the same type of argument that we used in Step 2. Note that here we have $\iota_\xi(\chi_s\cdot d\alpha)=\chi_s\cdot\iota_\xi\pi^*\theta=0$.

To prove the second item, recall from Proposition 2.2 that $\varphi_t^*(\omega)$ converges in $\Psi^{p,k}(M)$ when $t\to+\infty$. Therefore there exists $M>0$ such that $\|\varphi_t^*(\omega)\|_{\Psi^{p,k}}\leqslant M$ for every $t>0$. Write $\varphi_t^*(\omega)=\beta_t+d\gamma_t$

with $\|\beta_t\|_{L^p\Omega^k} + \|\gamma_t\|_{L^p\Omega^{k-1}} \leqslant 2M$. Observe that $(\chi - \chi_s)\cdot d\alpha$ belongs to $\Omega^{q,\ell}(M)$. Since M is complete, the space $\Omega^\ell_c(M)$ is dense in $\Omega^{q,\ell}(M)$ (see [GT10, Proof of Lemma 4]). Thus for every $t > 0$, one gets with Hölder:

$$\begin{aligned} &\Big| \int \varphi_t^*(\omega) \wedge \big((\chi - \chi_s)\cdot d\alpha\big)\Big| \\ &= \Big| \int \beta_t \wedge \big((\chi - \chi_s)\cdot d\alpha\big) + (-1)^k \int \gamma_t \wedge d(\chi - \chi_s) \wedge d\alpha \Big| \\ &\leqslant\ 2M\|(\chi - \chi_s)\cdot d\alpha\|_{L^q\Omega^\ell} + 2M\|d(\chi - \chi_s)\wedge d\alpha\|_{L^q\Omega^{\ell+1}}. \end{aligned}$$

By the same type of argument that we used in Step 1, one obtains that the last two norms tend to 0 when $s \to +\infty$. □

Proof of Corollary 2.6. Let $\omega \in \Omega^{p,k}(M) \cap \operatorname{Ker} d$ be such that

$$\int_M \omega \wedge d(\chi \cdot \pi^*\theta) = 0$$

for every $\theta \in \Omega^{\ell-1}_c(N)$. According to Poincaré duality (Proposition 1.1), it is enough to show that $[w] = 0$ in $L^p\mathrm{H}^k_{\mathrm{dR}}(M)$. By Propositions 2.2 and 2.4, this is equivalent to $T = 0$, where $T \in D'^k(N) \cap \operatorname{Ker} d$ is the k-current so that $\omega_\infty = \pi^*(T)$. From Theorem 2.5 and our assumption, one has for every $\theta \in \Omega^{\ell-1}_c(N)$:

$$T(\theta) = \int_M \omega \wedge d(\chi \cdot \pi^*\theta) = 0.$$

Thus $T = 0$.

Suppose now that $\theta = d\alpha$ is an exact form, with $\alpha \in \Omega^{\ell-2}_c(N)$. Then by using again Poincaré duality as above, we obtain that the class of $d(\chi \cdot \pi^*\theta)$ is null in $L^q\overline{\mathrm{H}^\ell_{\mathrm{dR}}}(M)$, since $T(\theta) = dT(\alpha) = 0$ for every $T \in D'^k(N) \cap \operatorname{Ker} d$. □

3. The Lie group case

We consider in this section a connected Lie group $G = \mathbf{R} \ltimes_\delta H$, whose law is $(t, x)\cdot(s, y) = (t + s, xe^{t\delta}(y))$, where $\delta \in \mathrm{Der}(\mathfrak{h})$ is a derivation of the Lie algebra $\mathfrak{h}$ of the closed subgroup H. **We will always assume that the eigenvalues of δ all have non-positive real parts, and that** $\mathrm{trace}(\delta) < 0$. In particular G is non-unimodular. We set $n := \dim H$ so that $D := \dim G = n + 1$. Equip G with a left-invariant Riemannian metric and with the associated Riemannian measure $d$vol.

3.1. **A strip decomposition.** We exhibit some regions of the set of parameters $(p,k) \in (1,+\infty)\times\{1,\dots,n\}$, where the results of the previous sections apply and give some informations on the spaces $L^p\mathrm{H}^k_{\mathrm{dR}}(G)$ — see Theorem 3.2 below. These regions form a kind of a "strip decomposition" of the set of parameters. Examples will be given in Sections 4 and 6.

We start with the following lemma which translates the norm assumptions that appeared repeatedly in the previous sections, into simple inequalities between the exponent p and the eigenvalues of $-\delta$.

Let $0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n$ be the ordered list of the real parts of the eigenvalues of $-\delta$, enumerated with their multiplicities in the generalized eigenspaces. We denote by $w_k = \sum_{i=1}^k \lambda_i$ the sum of the k first real part eigenvalues, and by $W_k = \sum_{j=0}^{k-1} \lambda_{n-j}$ the sum of the k last ones. We also set $w_0 = W_0 = 0$.

Note that we always have: $w_{k-1} \leqslant w_k \leqslant W_k$ and $w_{k-1} \leqslant W_{k-1} \leqslant W_k$, but the comparison between w_k and W_{k-1} is not automatic. This can be seen for instance by considering the example where $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = 1$ and $\lambda_n = a \geqslant 1$; then for $a > 2$ we have $W_{k-1} > w_k$, for $a = 2$ we have $W_{k-1} = w_k$ and for $a < 2$ we have $W_{k-1} < w_k$.

Let $h = \sum_{i=1}^n \lambda_i > 0$ be the trace of $-\delta$. If $w_k = 0$ (resp. $W_k = 0$), we put $\frac{h}{w_k} := +\infty$ (resp. $\frac{h}{W_k} := +\infty$). One has $w_k + W_{n-k} = h$ for every $k \in \{0,\dots,n\}$; therefore $\frac{h}{w_k}$ and $\frac{h}{W_{n-k}}$ are Hölder conjugated (even if w_k or W_{n-k} is 0).

Lemma 3.1. *Let $\xi = \frac{\partial}{\partial t}$ be the left-invariant vector field on G carried by the $\mathbf{R}$-factor, and let φ_t be its flow (it is just a translation along the $\mathbf{R}$-factor). Let $p, q \in (1,+\infty)$ and $k, \ell \in \{1,\dots,n\}$ be such that (p,k) and (q,ℓ) are Poincaré dual. The following properties are equivalent:*

(1) *There exist $C, \eta > 0$ such that for $t \geqslant 0$:*
$$\|\varphi_t^*\|_{L^p\Omega^{k-1}\to L^p\Omega^{k-1}} \leqslant Ce^{-\eta t}.$$
(2) *There exist $C, \eta > 0$ such that for $t \geqslant 0$:*
$$\|\varphi_{-t}^*\|_{\mathrm{Ker}\,\iota_\xi \cap L^q\Omega^\ell \to L^q\Omega^\ell \cap \mathrm{Ker}\,\iota_\xi} \leqslant Ce^{-\eta t}.$$
(3) *We have: $p < \frac{h}{W_{k-1}}$.*
(4) *We have: $q > \frac{h}{w_\ell}$.*

In particular conditions (1) and (2) in Theorem 2.5 are equivalent when $M = G$.

Proof. The equivalences (1)⇔(3) and (2)⇔(4) follow from the same line of arguments as in [BR23, Proof of Proposition 2.1]. To obtain (3)⇔(4) one notices that $\frac{h}{W_{k-1}}$ and $\frac{h}{w_\ell}$ are Hölder conjugated, since $w_{D-k} + W_{k-1} = h$. □

We can now summarize and specify the results of the previous sections, to obtain the following statement that complements results of Pansu [Pan99, Corollaire 53 and Proposition 57].

Theorem 3.2. *Let G be a Lie group as above. Let $p \in (1, +\infty)$ and $k \in \{1, \ldots, n\}$.*

1. [Vanishing] *If $p < \frac{h}{W_k}$ or $p > \frac{h}{w_{k-1}}$, then $L^p\mathrm{H}^k_{\mathrm{dR}}(G) = \{0\}$.*
2. [Hausdorff property] *If $\frac{h}{W_k} < p < \frac{h}{W_{k-1}}$, then $L^p\mathrm{H}^k_{\mathrm{dR}}(G)$ is Hausdorff and Banach-isomorphic to $\mathcal{Z}^{p,k}(G, \xi)$.*
3. [Density] *If $\frac{h}{w_k} < p < \frac{h}{w_{k-1}}$, then the classes of the $d(\chi \cdot \pi^*\theta)$'s (where $\theta \in \Omega^{k-1}_c(H)$) form a dense subspace in $L^p\overline{\mathrm{H}^k_{\mathrm{dR}}}(G)$.*
4. [Poincaré duality (on the boundary)] *Let (q, ℓ) be the Poincaré dual of (p, k). Then we have $\frac{h}{W_k} < p < \frac{h}{W_{k-1}}$ if and only if $\frac{h}{w_\ell} < q < \frac{h}{w_{\ell-1}}$, in which case for every $[\omega] \in L^p\mathrm{H}^k_{\mathrm{dR}}(G)$ and every $[d(\chi \cdot \pi^*\theta)] \in L^q\overline{\mathrm{H}^\ell_{\mathrm{dR}}}(G)$, we have*
$$\int_G \omega \wedge d(\chi \cdot \pi^*\theta) = T(\theta),$$
where T is the closed k-current on H such that $\lim_{t\to+\infty} \varphi_t^*(\omega) = \pi^*(T)$ (as in Propositions 2.2 and 2.4). Moreover, one has:
$$\big\|[\omega]\big\|_{L^p\mathrm{H}^k} = \sup\Big\{T(\theta) : \theta \in \Omega^{\ell-1}_c(H),\ \big\|[d(\chi \cdot \pi^*\theta)]\big\|_{L^q\overline{\mathrm{H}^\ell}} \leqslant 1\Big\}.$$

Proof. Item (2) follows from Proposition 2.2 and Lemma 3.1. Item (3) is a consequence of Corollary 2.6 and Lemma 3.1. One deduces Item (4) from Theorem 2.5, Corollary 2.7 and Lemma 3.1.

It remains to prove Item (1). Suppose first that $p < \frac{h}{W_k}$. Since $\frac{h}{W_k} \leqslant \frac{h}{W_{k-1}}$, Lemma 3.1 implies that $\max_{i=k-1,k} \|\varphi_t^*\|_{L^p\Omega^i \to L^p\Omega^i} \leqslant Ce^{-\eta t}$. Thus by Corollary 2.3, one has $L^p\mathrm{H}^k_{\mathrm{dR}}(G) = \{0\}$, and the first part of Item (1) is proved. The second part follows from the first one, in combination with Poincaré duality (Proposition 1.1), and the fact that $L^q\mathrm{H}^{D-k+1}_{\mathrm{dR}}(G)$ is Hausdorff follows from Lemma 3.1 and Proposition 2.2. □

Remark 3.3. In the special case where $H = \mathbf{R}^n$, Pansu [Pan08, Proposition 27] has complemented the picture seen in Theorem 3.2, by showing that the *torsion* in $L^p\mathrm{H}^k_{\mathrm{dR}}(G)$ – *i.e.* the kernel of the quotient map $L^p\mathrm{H}^k_{\mathrm{dR}}(G) \to L^p\overline{\mathrm{H}^k_{\mathrm{dR}}}(G)$ – is non-zero for $\frac{h}{W_{k-1}} < p < \frac{h}{w_{k-1}}$ (note that there is no such p for real hyperbolic spaces).

3.2. **Norm estimates.** We complement the norm expression obtained in Theorem 3.2(4). The following inequalities are not optimal, however they are often sufficient for our purposes.

Proposition 3.4. *Let $\ell \in \{1, \dots, n\}$ and $q > \frac{h}{w_\ell}$. There exists a constant $C > 0$ such that for every $\theta \in \Omega^{\ell-1}_c(H)$, the norm of the class of $d(\chi \cdot \pi^*\theta)$ in $L^q\overline{\mathrm{H}^\ell_{\mathrm{dR}}}(G)$ satisfies*

$$\big\|[d(\chi \cdot \pi^*\theta)]\big\|_{L^q\overline{\mathrm{H}^\ell}(G)} \leqslant C \inf_{t\in\mathbf{R}} \big\{ \|(e^{t\delta})^* d\theta\|_{L^q\Omega^\ell(H)} + \|(e^{t\delta})^*\theta\|_{L^q\Omega^{\ell-1}(H)} \big\}.$$

Proof. Since $q > \frac{h}{w_\ell}$, Lemma 3.1 shows that the assumptions of Theorem 2.5 are satisfied. By analysing Step 1 in the proof of Theorem 2.5, and by using the homogeneity of G, one sees that there exists a constant $C > 0$ such that for every $\theta \in \Omega^{\ell-1}_c(H)$:

$$\|d(\chi \cdot \pi^*\theta)\|_{\Omega^{q,\ell}(G)} \leqslant C\big(\|d\theta\|_{L^q\Omega^\ell(H)} + \|\theta\|_{L^q\Omega^{\ell-1}(H)}\big). \tag{3.5}$$

The left translation by $(t, 1_H)$, which we denote by $L_{(t,1_H)}$, is an isometry of G. Therefore it acts by isometry on $L^q\overline{\mathrm{H}^\ell_{\mathrm{dR}}}(G)$. One has:

$$L^*_{(t,1_H)}\big(d(\chi \cdot \pi^*\theta)\big) = d(\chi \circ \varphi_t \cdot \pi^*(e^{t\delta})^*\theta) = \varphi_t^*\big(d(\chi \cdot \pi^*(e^{t\delta})^*\theta)\big).$$

Thus, by Proposition 2.1, the forms $L^*_{(t,1_H)}(d(\chi\cdot\pi^*\theta))$ and $d(\chi\cdot\pi^*(e^{t\delta})^*\theta)$ are cohomologous in $L^q\overline{\mathrm{H}^\ell_{\mathrm{dR}}}(G)$. So the classes of $d(\chi\cdot\pi^*\theta)$ and of $d(\chi\cdot \pi^*(e^{t\delta})^*\theta)$ have equal norm. One obtains the proposition by applying inequality (3.5) to the $(e^{t\delta})^*\theta$'s. □

Proposition 3.4 provides upper bounds for norms of classes by means of norms of forms. These upper bounds on norms of forms can themself be obtained thanks to the following lemma:

Lemma 3.6. *Let H be a connected Lie group equipped with a left-invariant Riemannian metric, and let $\mathfrak{h}$ be its Lie algebra. Let $\delta \in$ $\mathrm{Der}(\mathfrak{h})$ be an $\mathbf{R}$-diagonalizable derivation of $\mathfrak{h}$. Then for every $k \in$ $\mathbf{N}$ the endomorphism $\delta^* : \Lambda^k\mathfrak{h}^* \to \Lambda^k\mathfrak{h}^*$ is diagonalizable too. Let $\{\omega_I\} \subset \Lambda^k\mathfrak{h}^*$ be a basis of eigenvectors, and denote by $\mu_I \in \mathbf{R}$ the* *corresponding eigenvalues. By identifying $\Lambda^k\mathfrak{h}^*$ with the space of left-invariant k-forms on H, every $\omega \in \Omega^k(H)$ decomposes uniquely as $\omega = \sum_I f_I\omega_I$, where $f_I \in \Omega^0(H)$. One has*

$$\|(e^\delta)^*\omega\|_{L^p\Omega^k(H)} \asymp_D \sum_I e^{\mu_I - \frac{h}{p}} \|f_I\|_{L^p(H)},$$

where h is the trace of δ, and $D > 0$ is a constant which depends only on p and the choice of $\{\omega_I\}$.

Proof. Since the norms on $\Lambda^k\mathfrak{h}^*$ are all equivalent, there exists a constant $C > 0$ such that for every $\omega = \sum_I f_I\omega_I \in \Omega^k(H)$ and $g \in H$:

$$|\omega|_g \asymp_C \big(\sum_I |f_I(g)|^p\big)^{\frac{1}{p}}.$$

On the other hand:

$$(e^\delta)^*\omega = \sum_I (f_I \circ e^\delta)\cdot(e^\delta)^*\omega_I = \sum_I (f_I \circ e^\delta)\cdot e^{\mu_I}\omega_I.$$

Therefore:

$$\begin{aligned}
\|(e^\delta)^*\omega\|^p_{L^p\Omega^k} &= \int_H |e^{\delta *}\omega|^p_g \ d\mathrm{vol}(g) \\
&\asymp_{C^p} \int_H \sum_I |e^{\mu_I}(f_I \circ e^\delta)(g)|^p \ d\mathrm{vol}(g) \\
&= \int_H \sum_I e^{p\mu_I}|f_I(g)|^p \mathrm{Jac}(e^{-\delta})(g) \ d\mathrm{vol}(g) \\
&= \int_H \sum_I e^{p(\mu_I - \frac{h}{p})}|f_I(g)|^p \ d\mathrm{vol}(g) \\
&= \sum_I e^{p(\mu_I - \frac{h}{p})}\|f_I\|^p_{L^p},
\end{aligned}$$

since the Jacobian of e^δ is e^h. Thus:

$$\|(e^\delta)^*\omega\|_{L^p\Omega^k} \asymp_C \big(\sum_I e^{p(\mu_I - \frac{h}{p})}\|f_I\|^p_{L^p}\big)^{\frac{1}{p}} \asymp_D \sum_I e^{\mu_I - \frac{h}{p}}\|f_I\|_{L^p},$$

where D depends only on p and $\{\omega_I\}$. □

4. Real hyperbolic spaces

We collect applications to a first series of concrete examples, namely real hyperbolic spaces. Let $R = \mathbf{R} \ltimes_\delta \mathbf{R}^n$ with $\delta = -\mathrm{id}_{\mathbf{R}^n} \in \mathrm{Der}(\mathbf{R}^n)$. Then R is a solvable Lie group isometric to the real hyperbolic space

$\mathbb{H}^{n+1}_{\mathbf{R}}$. Its cohomology admits the following rather simple description, which appears already in [Pan08] (apart from the density statement).

Theorem 4.1. *For every $k \in \{1, \dots, n\}$, one has:*

(1) *$L^p\mathrm{H}^k_{\mathrm{dR}}(R) = \{0\}$ for $1 < p < \frac{n}{k}$ or $p > \frac{n}{k-1}$.*
(2) *If $\frac{n}{k} < p < \frac{n}{k-1}$, then $L^p\mathrm{H}^k_{\mathrm{dR}}(R)$ is Hausdorff, and Banach isomorphic to $\mathcal{Z}^{p,k}(R,\xi)$. The space $\{\pi^* d\theta \mid \theta \in \Omega^{k-1}_c(\mathbf{R}^n)\}$ is dense in $\mathcal{Z}^{p,k}(R,\xi)$; in particular $L^p\mathrm{H}^k_{\mathrm{dR}}(R)$ is non-zero.*

Proof. For $k \in \{1, \dots, n\}$ one has $w_k = W_k = k$ and $h = n$. Item (1) comes from Theorem 3.2(1). Item (2) is an application of Theorem 3.2(2) and (3), in combination with Proposition 2.2. Indeed we have:

$$\lim_{t\to+\infty} \varphi_t^*\big(d(\chi \cdot \pi^*\theta)\big) = \lim_{t\to+\infty} d\big((\chi\circ\varphi_t)\cdot\pi^*\theta\big) = d\pi^*\theta = \pi^* d\theta,$$

in the sense of currents. □

We also obtain the following norm estimates. Recall from Proposition 2.4 that every $\psi \in \mathcal{Z}^{p,k}(R,\xi)$ can be written as $\psi = \pi^* T$ for some (unique) $T \in \mathcal{D}'^k(\mathbf{R}^n)$.

Proposition 4.2. *Let $k \in \{1, \dots, n\}$, $\frac{n}{k} < p < \frac{n}{k-1}$, and (q,ℓ) be the Poincaré dual of (p,k). There exists some constant $C > 0$ such that the norm of every current $\pi^*(T) \in \mathcal{Z}^{p,k}(R,\xi)$ satisfies*

$$\|\pi^* T\|_{\Psi^{p,k}(R)} \asymp_C \sup\Big\{T(\theta) : \theta \in \Omega^{\ell-1}_c(\mathbf{R}^n),\ \|\pi^* d\theta\|_{\Psi^{q,\ell}(R)} \leqslant 1\Big\}.$$

Moreover for $\theta \in \Omega^{k-1}_c(\mathbf{R}^n)$, the norm of the form $\pi^* d\theta \in \mathcal{Z}^{p,k}(R,\xi)$ satisfies

$$\|\pi^* d\theta\|_{\Psi^{p,k}(R)} \leqslant C \inf_{t\in\mathbf{R}} \big\{ e^{-(k-\frac{n}{p})t} \|d\theta\|_{L^p\Omega^k(\mathbf{R}^n)} + e^{(1-k+\frac{n}{p})t} \|\theta\|_{L^p\Omega^{k-1}(\mathbf{R}^n)} \big\}.$$

Observe that the exponents in the last inequality satisfy: $k - \frac{n}{p} > 0$ and $1 - k + \frac{n}{p} > 0$.

Proof. The spaces $\mathcal{Z}^{p,k}(R,\xi)$ and $L^p\mathrm{H}^k_{\mathrm{dR}}(R)$ are Banach isomorphic by the map $[\omega] \mapsto \omega_\infty$ of Prop. 2.2 (2). Thus the form $\pi^* d\theta \in \mathcal{Z}^{p,k}(R,\xi)$ and the class $[d(\chi\cdot\pi^*\theta)] \in L^p\mathrm{H}^k_{\mathrm{dR}}(R)$ have comparable norms. The inequalities follow then from Theorem 3.2(4), Proposition 3.4 and Lemma 3.6, applied with $H = \mathbf{R}^n$ and $\delta = -t\mathrm{id}_{\mathbf{R}^n}$. □

5. The groups $S_\alpha \in \mathcal{S}^{2,3}_{\mathrm{straight}}$

We prove Theorem A (stated in the introduction) that determines the second L^p-cohomology of the groups $S_\alpha \in \mathcal{S}^{2,3}_{\mathrm{straight}}$.

5.1. **Reduction and decomposition.** Let $S_\alpha = \mathbf{R}^2 \ltimes_\alpha \mathbf{R}^3 \in \mathcal{S}^{2,3}_{\mathrm{straight}}$. We provide a somewhat "canonical" presentation of the group S_α (see Proposition 5.2), which is then used to decompose S_α.

Recall that precomposing α with an element of $\mathrm{GL}_2(\mathbf{R})$, or postcomposing with an element of the permutation group S_3, does not affect the isomorphism class of S_α.

Denote by $\{\varepsilon_1, \varepsilon_2\}$ the canonical basis of $\mathbf{R}^2$ and by $\{\varepsilon_1^*, \varepsilon_2^*\}$ its dual basis. By definition of the family $\mathcal{S}^{2,3}_{\mathrm{straight}}$, the weights $\varpi_1, \varpi_2, \varpi_3$ of α generate $(\mathbf{R}^2)^*$ and belong to a line disjoint from 0. By precomposing α by an element of $\mathrm{GL}_2(\mathbf{R})$ and postcomposing by a permutation of the diagonal entries, if necessary, we can ensure that:

- the ϖ_i's belong to the vertical line Δ passing through $-\varepsilon_1^*$,
- they admit $-\varepsilon_1^*$ as center of mass,
- the algebraic distances between them on Δ (oriented by ε_2^*) satisfy: $0 \leqslant \varpi_2 - \varpi_3 \leqslant \varpi_1 - \varpi_2$; *i.e.* $\varpi_3, \varpi_2, \varpi_1$ lie in this order on Δ, and ϖ_2 is closer to ϖ_3 than to ϖ_1.

In other words, there exist $\mu_1, \mu_2, \mu_3 \in \mathbf{R}$, not all equal, so that the weights can be written $\varpi_i = -\varepsilon_1^* + \mu_i \varepsilon_2^*$, for $i = 1, 2, 3$, with:

$$\sum_{i=1}^{3} \mu_i = 0 \quad \text{and} \quad 0 \leqslant \mu_2 - \mu_3 \leqslant \mu_1 - \mu_2. \tag{5.1}$$

To sum up the above reduction, we have established:

Proposition 5.2. *There exists $D_\mu = \mathrm{diag}(\mu_1, \mu_2, \mu_3) \in \mathrm{Diag}(\mathbf{R}^3)$, a non-zero diagonal matrix, unique up to a positive multiplicative constant, enjoying the relations (5.1), such that, up to precomposition by an element of $\mathrm{GL}_2(\mathbf{R})$ and postcomposition by a permutation of the diagonal entries, the Lie group morphism α can be written:*

$$\alpha(t,s) = e^{-tI_3 + sD_\mu}, \text{ for every } (t,s) \in \mathbf{R}^2.$$

Moreover, with the notation of Theorem A, one has:

$$p_\alpha = 1 + \frac{\mu_1 - \mu_3}{\mu_1 - \mu_2}.$$

We assume from now on that the expression of α is as in the previous proposition. Let consider the following subgroups of S_α:

$$R := \mathbf{R} \ltimes_{-I_3} \mathbf{R}^3 \quad \text{and} \quad H_\mu := \mathbf{R} \ltimes_{D_\mu} \mathbf{R}^3.$$

The group R is naturally isometric to $\mathbb{H}^4_{\mathbf{R}}$. Let $\mathfrak{r}$ and $\mathfrak{h}_\mu$ be their Lie algebras. Let $(0, D_\mu)$ and $(0, -I_3)$ denote the derivations of $\mathfrak{r}$ and $\mathfrak{h}_\mu$ that trivially extend $-I_3$ and D_μ. Then S_α admits two decompositions, namely:

$$S_\alpha \;=\; \mathbf{R} \ltimes_{(0,D_\mu)} R \quad \text{and} \quad S_\alpha \;=\; \mathbf{R} \ltimes_{(0,-I_3)} H_\mu.$$

We denote again by ξ the left-invariant vector field on R carried by the $\mathbf{R}$-factor, and by π the projection map from R onto $\mathbf{R}^3$.

The proof of Theorem A will mainly rely on the decomposition $S_\alpha = \mathbf{R} \ltimes_{(0,D_\mu)} R$, in combination with the description of the L^p-cohomology of $R \simeq \mathbb{H}^4_{\mathbf{R}}$ given in Theorem 4.1 and Proposition 4.2. We will use the realization of de Rham L^p-cohomology by means of currents, which gives the Banach space isomorphism:

$$L^p\mathrm{H}^2_{\mathrm{dR}}(R) \simeq \mathcal{Z}^{p,2}(R, \xi),$$

where $\mathcal{Z}^{p,2}(R,\xi)$ is the space of closed 2-currents ψ on R, invariant under the flow (φ_t) of ξ and such that $\|\psi\|_{\Psi^{p,2}} < +\infty$. According to Proposition 2.4, every $\psi \in \mathcal{Z}^{p,2}(M,\xi)$ can be written $\psi = \pi^*(T)$ for some $T \in \mathcal{D}'^2(\mathbf{R}^3) \cap \operatorname{Ker} d$.

5.2. **First cohomological observations.** We derive from previous results some preliminary observations on $L^p\mathrm{H}^2_{\mathrm{dR}}(S_\alpha)$ whose statements do not depend on the morphism α. The notations are the same as in the previous section.

Proposition 5.3. *One has* $L^p\mathrm{H}^2_{\mathrm{dR}}(S_\alpha) = \{0\}$ *for* $p < \frac{3}{2}$.

Proof. One has $S_\alpha = \mathbf{R} \ltimes_\delta H_\mu$, with $\delta = (0, -I_3) \in \operatorname{Der}(\mathfrak{h}_\mu)$. The ordered list of eigenvalues of $-\delta$ enumerated with multiplicity, is

$$\lambda_1 = 0 < \lambda_2 = \lambda_3 = \lambda_4 = 1.$$

Thus, with the notations of Section 3, the trace of $-\delta$ is $h = 3$, and one has $W_2 = \lambda_3 + \lambda_4 = 2$. Therefore the statement follows from Theorem 3.2(1). □

Proposition 5.4. *For* $p \in (\frac{3}{2}; 3)$*, the Banach space* $\mathcal{Z}^{p,2}(R,\xi)$ *is non-zero, and there exists a linear isomorphism*

$$L^p\mathrm{H}^2_{\mathrm{dR}}(S_\alpha) \simeq \Big\{ \pi^* T \in \mathcal{Z}^{p,2}(R,\xi) : \int_{\mathbf{R}} \|\pi^* e^{sD_\mu *} T\|^p_{\Psi^{p,2}(R)} ds < +\infty \Big\}.$$

Proof. When $p \in (\frac{3}{2}; 3)$, Theorem 4.1 shows that $L^p\mathrm{H}^2_{\mathrm{dR}}(R)$ is non-zero and Hausdorff, and that $L^p\mathrm{H}^k_{\mathrm{dR}}(R) = \{0\}$ in all degrees $k \neq 2$. Since $S_\alpha = \mathbf{R} \ltimes_{(0,D_\mu)} R$, the above description of the cohomology of R, in combination with a Hochschild-Serre spectral sequence argument (see [BR23, Corollary 6.10]), yields the following linear isomorphism

$$L^p\mathrm{H}^2_{\mathrm{dR}}(S_\alpha) \simeq \Big\{[\omega] \in L^p\mathrm{H}^2_{\mathrm{dR}}(R) : \int_{\mathbf{R}} \|e^{s(0,D_\mu)^*}[\omega]\|^p_{L^p\mathrm{H}^2(R)} ds < +\infty\Big\}.$$

By Theorem 4.1(2), the Banach spaces $L^p\mathrm{H}^2_{\mathrm{dR}}(R)$ and $\mathcal{Z}^{p,2}(R,\xi)$ are isomorphic. Moreover every $\psi \in \mathcal{Z}^{p,2}(R,\xi)$ can be written $\psi = \pi^*(T)$ for some $T \in \mathcal{D}'^2(\mathbf{R}^3) \cap \operatorname{Ker} d$ (see Proposition 2.4). This leads to the desired linear isomorphism. □

Proposition 5.5. *For $p > 3$, the space $L^p\overline{\mathrm{H}^2_{\mathrm{dR}}}(S_\alpha)$ is non-zero.*

Proof. Consider again $\lambda_1 = 0 < \lambda_2 = \lambda_3 = \lambda_4 = 1$ the list of the eigenvalues of $-\delta = -(0, -I_3) \in \mathrm{Der}(\mathfrak{h}_\mu)$. The trace of $-\delta$ is $h = 3$, and one has $w_2 = \lambda_1 + \lambda_2 = 1$. Since the rank of S_α is equal to 2, it follows from [BR23, Theorem B and Corollary 3.4] that $L^p\overline{\mathrm{H}^2_{\mathrm{dR}}}(S_\alpha)$ is non-zero for $p > \frac{h}{w_2} = 3$. □

5.3. **Non-vanishing of the second L^p-cohomology.** We wish to establish the non-vanishing part of Theorem A. Thanks to Proposition 5.5, we just need to prove that $L^p\mathrm{H}^2_{\mathrm{dR}}(S_\alpha) \neq \{0\}$ for $p \in (p_\alpha; 3)$.

Assume $p \in (\frac{3}{2}; 3)$. By Proposition 5.4 there is a linear isomorphism

$$L^p\mathrm{H}^2_{\mathrm{dR}}(S_\alpha) \simeq \Big\{\pi^*T \in \mathcal{Z}^{p,2}(R,\xi) : \int_{\mathbf{R}} \|\pi^* e^{sD_\mu{}^*}T\|^p_{\Psi^{p,2}(R)} ds < +\infty\Big\}.$$

Recall from Theorem 4.1(2) that the space $\mathcal{Z}^{p,2}(R,\xi)$ contains the forms $\pi^*d\theta$, with $\theta \in \Omega^1_c(\mathbf{R}^3)$. Therefore in order to show that $L^p\mathrm{H}^2_{\mathrm{dR}}(S_\alpha)$ is non-zero, it is enough to exhibit a non-zero form $d\theta$ with $\theta \in \Omega^1_c(\mathbf{R}^3)$ and $\|\pi^*e^{sD_\mu{}^*}d\theta\|_{\Psi^{p,2}(R)} \to 0$ exponentially fast when $s \to \pm\infty$.

Let θ be a smooth compactly supported 1-form on $\mathbf{R}^3$. From Proposition 4.2, with θ replaced by $e^{sD_\mu{}^*}\theta$, we have:

$$\|\pi^*e^{sD_\mu{}^*}d\theta\|_{\Psi^{p,2}(R)} \leqslant C \inf_{t\in\mathbf{R}} \big\{e^{-(2-\frac{3}{p})t}\|e^{sD_\mu{}^*}d\theta\|_{L^p\Omega^2(\mathbf{R}^3)} + e^{(-1+\frac{3}{p})t}\|e^{sD_\mu{}^*}\theta\|_{L^p\Omega^1(\mathbf{R}^3)}\big\}.$$

Write $\theta = fdx + gdy + hdz$, so that $d\theta = Fdy\wedge dz + Gdx\wedge dz + Hdx\wedge dy$. Since the trace of D_μ is zero, Lemma 3.6 applied with $H = \mathbf{R}^3$ and $\delta = D_\mu = \mathrm{diag}(\mu_1, \mu_2, \mu_3)$ gives the following estimates:

$$\|e^{sD_\mu{}^*}d\theta\|_{L^p\Omega^2} \asymp e^{-s\mu_1}\|F\|_{L^p} + e^{-s\mu_2}\|G\|_{L^p} + e^{-s\mu_3}\|H\|_{L^p},$$

$$\|e^{sD_\mu{}^*}\theta\|_{L^p\Omega^1} \asymp e^{s\mu_1}\|f\|_{L^p} + e^{s\mu_2}\|g\|_{L^p} + e^{s\mu_3}\|h\|_{L^p}.$$

We denote by $\alpha_\pm$ the exponent of the leading term in the asymptotics of $\|e^{sD_\mu{}^*}d\theta\|$ when $s \to \pm\infty$, namely $\|e^{sD_\mu{}^*}d\theta\| \asymp_{s\to\pm\infty} e^{\alpha_\pm s}$; note that α_+ and α_- are opposites of diagonal coefficients of D_μ since $\mathrm{trace}(D_\mu) = 0$. Similarly, we denote by $\beta_\pm$ the exponent of the leading term in the asymptotics of $\|e^{sD_\mu{}^*}\theta\|$ when $s \to \pm\infty$, namely $\|e^{sD_\mu{}^*}\theta\| \asymp_{s\to\pm\infty} e^{\beta_\pm s}$; note that β_+ and β_- are diagonal coefficients of the matrix D_μ. One has:

Lemma 5.6. *Let $a, b > 0$ be positive real numbers and let $\alpha, \beta \in \mathbf{R}$. We assume that $A = A(s) \asymp e^{\alpha s}$ and $B = B(s) \asymp e^{\beta s}$ when $s \to +\infty$ (resp. when $s \to -\infty$). Then $\inf_{t\in\mathbf{R}}\{e^{-at}\ A + e^{bt}\ B\} \asymp e^{\frac{a\beta+b\alpha}{a+b}s}$, when $s \to +\infty$ (resp. when $s \to -\infty$). In particular, the infimum tends to 0 if and only if we have $(a\beta + b\alpha)s \to -\infty$, when $s \to +\infty$ (resp. when $s \to -\infty$); in which case the speed of convergence to 0 is exponential.*

Proof. Assume first that $A, B > 0$ are fixed and consider the function f defined by $f(t) = e^{-at}\ A + e^{bt}\ B$. We have $\lim_{t\to\pm\infty} f(t) = +\infty$ so f achieves its minimum at a point $t_{\min}$ such that $f'(t_{\min}) = 0$. Since $f'(t) = -ae^{-at}\ A + be^{bt}\ B$, we have $\frac{A}{B} = \frac{b}{a}e^{(a+b)t_{\min}}$ and therefore the minimal value of f is:

$$f(t_{\min}) = Be^{bt_{\min}}\left(\frac{A}{B}e^{-(a+b)t_{\min}} + 1\right) = Be^{bt_{\min}}\left(\frac{b}{a} + 1\right).$$

When $A \asymp e^{\alpha s}$ and $B \asymp e^{\beta s}$, we have $e^{(\alpha-\beta)s} \asymp \frac{A}{B} = \frac{b}{a}e^{(a+b)t_{\min}}$. Then $f(t_{\min}) \asymp e^{\beta s}e^{bt_{\min}} \asymp e^{(\beta + b\frac{\alpha-\beta}{a+b})s} = e^{\frac{a\beta+b\alpha}{a+b}s}$. $\square$

For $p \in (\frac{3}{2}; 3)$, with $a = 2 - \frac{3}{p}$ and $b = -1 + \frac{3}{p}$ in the above lemma, we obtain the following estimate when $s \to \pm\infty$:

$$\|\pi^* e^{sD_\mu{}^*}d\theta\|_{\Psi^{p,2}(R)} \lesssim e^{\frac{a\beta_\pm + b\alpha_\pm}{a+b}s} = e^{\{(2p-3)\beta_\pm + (-3+p)\alpha_\pm\}\frac{s}{p}}.$$

This shows that for the condition $\int_{\mathbf{R}} \|\pi^* e^{sD_\mu{}^*}T\|^p_{\Psi^{p,2}(R)}ds < +\infty$ to be satisfied by $T = d\theta$, it is sufficient to have:

$$(2p-3)\beta_+ + (3-p)\alpha_+ < 0 \quad\text{and}\quad (2p-3)\beta_- + (3-p)\alpha_- > 0. \tag{5.7}$$

To exhibit such a form $d\theta$, we will use the

Lemma 5.8. *There exist forms $\theta = fdx$ and $\Theta = gdy + hdz$ in $\Omega^1_c(\mathbf{R}^3)$, such that $d\theta = d\Theta = Gdx \wedge dz + Hdx \wedge dy \neq 0$.*

Proof. Let u be an arbitrary non-zero function in $\Omega^0_c(\mathbf{R}^3)$. Its differential is $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$. Set $\theta := \frac{\partial u}{\partial x}dx$ and $\Theta := -\frac{\partial u}{\partial y}dy - \frac{\partial u}{\partial z}dz$. Since $ddu = 0$, one has $d\theta = d\Theta = -\frac{\partial^2 u}{\partial x\partial z}dx \wedge dz - \frac{\partial^2 u}{\partial x\partial y}dx \wedge dy$. $\square$

Let $d\theta = d\Theta$ be as in the previous lemma. From the relations (5.1) we have $\alpha_+ = -\mu_3$ and $\alpha_- = -\mu_2$. Similarly we have $\beta_- = \beta_-(\theta) = \mu_1$ and $\beta_+ = \beta_+(\Theta) = \mu_2$. When $s \to +\infty$, the integrability conditions (5.7) and the relations (5.1), lead to the following condition

$$(2p-3)\mu_2 - (3-p)\mu_3 < 0,$$

hence $p(2\mu_2 + \mu_3) - 3\mu_2 - 3\mu_3 < 0$, that is $p(\mu_2 - \mu_1) + 3\mu_1 > 0$, amounting to

$$p > \frac{3\mu_1}{\mu_1 - \mu_2} = 1 + \frac{\mu_1 - \mu_3}{\mu_1 - \mu_2} = p_\alpha.$$

When $s \to -\infty$, they lead to

$$(2p-3)\mu_1 - (3-p)\mu_2 > 0,$$

hence $p(2\mu_1 + \mu_2) - 3\mu_1 - 3\mu_2 > 0$, that is $p(\mu_1 - \mu_3) + 3\mu_3 > 0$, amounting to

$$p > \frac{-3\mu_3}{\mu_1 - \mu_3} = 1 + \frac{\mu_2 - \mu_3}{\mu_1 - \mu_3}.$$

The latter condition is implied by the former one, since $1 + \frac{\mu_2 - \mu_3}{\mu_1 - \mu_3} \in [1; \frac{3}{2}]$ and $p_\alpha \in [2; 3]$. To sum up, we have shown that $L^p\mathrm{H}^2_{\mathrm{dR}}(S_\alpha) \neq \{0\}$ for $p \in (p_\alpha; 3)$, as expected.

5.4. **Vanishing of the second L^p-cohomology.** It remains to prove the vanishing statement in Theorem A. It will be obtained by using a Poincaré duality argument, together with some estimates similar to those from the non-vanishing part.

According to Proposition 5.3, it is enough to consider the case $p \in (\frac{3}{2}; 3)$. We start again from the identification given in Proposition 5.4:

$$L^p\mathrm{H}^2_{\mathrm{dR}}(S_\alpha) \simeq \Big\{\pi^*T \in \mathcal{Z}^{p,2}(R,\xi) : \int_{\mathbf{R}} \|\pi^* e^{sD_\mu{}^*} T\|^p_{\Psi^{p,2}(R)} ds < +\infty\Big\}.$$

To show that $L^p\mathrm{H}^2_{\mathrm{dR}}(S_\alpha)$ vanishes, it is enough to prove that every $\pi^*T \in \mathcal{Z}^{p,2}(R,\xi)$ satisfies $\|\pi^* e^{sD_\mu{}^*} T\|_{\Psi^{p,2}(R)} \to +\infty$, when $s \to +\infty$ or when $s \to -\infty$. Recall from Proposition 4.2, that:

$$\|\pi^*T\|_{\Psi^{p,2}(R)} \asymp \sup\{T(\theta) : \theta \in \Omega^1_c(\mathbf{R}^3),\ \|\pi^* d\theta)\|_{\Psi^{q,2}(R)} \leqslant 1\},$$

where q denotes the Hölder conjugate of p. When replacing π^*T by $\pi^* e^{sD_\mu{}^*} T$ with $s \in \mathbf{R}$, a change of variable provides:

$$\|\pi^* e^{sD_\mu{}^*} T\|_{\Psi^{p,2}(R)} \asymp \sup\{T(\theta) : \theta \in \Omega^1_c(\mathbf{R}^3),\ \|\pi^* e^{sD_\mu{}^*} d\theta)\|_{\Psi^{q,2}(R)} \leqslant 1\}.$$

By Proposition 4.2, with θ replaced by $e^{sD_\mu}{}^*\theta$, we obtain:

$$\|\pi^* e^{sD_\mu}{}^* d\theta\|_{\Psi^{q,2}(R)} \leqslant C \inf_{t\in\mathbf{R}} \big\{ e^{-(\frac{3}{p}-1)t} \|e^{sD_\mu}{}^* d\theta\|_{L^q\Omega^2(\mathbf{R}^3)} \\ + e^{(2-\frac{3}{p})t} \|e^{sD_\mu}{}^*\theta\|_{L^q\Omega^1(\mathbf{R}^3)} \big\},$$

where, in the upper bound, the coefficients in the exponentials in front of the norms come from the identity $2-\frac{3}{q}=\frac{3}{p}-1$ and $1+\frac{3}{q}=2-\frac{3}{p}$.

Using Lemma 5.6 with $a=\frac{3}{p}-1$ and $b=2-\frac{3}{p}$, and keeping the notation $\alpha_\pm$ and $\beta_\pm$ defined after replacing L^p norms by L^q norms, we obtain

$$\|\pi^* e^{sD_\mu}{}^* d\theta\|_{\Psi^{q,2}(R)} \lesssim e^{(a\beta_\pm + b\alpha_\pm)s}, \tag{5.9}$$

when $s\to\pm\infty$. These observations lead to the

Lemma 5.10. *Let $\pi^*T \in \mathcal{Z}^{p,2}(R,\xi)$ and let $T = T_1 dy\wedge dz + T_2 dx\wedge dz + T_3 dx\wedge dy$ be its writing in the canonical global coordinates of $\mathbf{R}^3$, with $T_i \in \mathcal{D}'^0(\mathbf{R}^3)$.*

(i) *If $T_1\neq 0$, then* $\lim_{s\to-\infty} \|\pi^* e^{sD_\mu}{}^* T\|_{\Psi^{p,2}(R)} = +\infty$.

(ii) *If $T_3\neq 0$ and if $p<p_\alpha$, then* $\lim_{s\to+\infty} \|\pi^* e^{sD_\mu}{}^* T\|_{\Psi^{p,2}(R)} = +\infty$.

Proof. (i). Let θ be of the form $\theta = f dx$ with f a smooth compactly supported function such that $T_1(f)=1$. Then $T(\theta)=T_1(f)=1$. We have $d\theta = -\frac{\partial f}{\partial y}\,dx\wedge dy - \frac{\partial f}{\partial z}\,dx\wedge dz$. Therefore with the relations (5.1), one has $\alpha_- = -\mu_2$ and $\beta_- = \mu_1$. We consider the quantity

$$p(a\beta_- + b\alpha_-) = (3-p)\mu_1 - (2p-3)\mu_2 = -p(\mu_2-\mu_3) - 3\mu_3.$$

Since $p\in(\frac{3}{2};3)$ and since $\mu_3\leqslant\mu_2\leqslant 0$, one has $-p(\mu_2-\mu_3)-3\mu_3 \geqslant -3\mu_2 \geqslant 0$. We deduce that $a\beta_- + b\alpha_- > 0$. Relation (5.9) then implies that $\lim_{s\to-\infty} \|\pi^* e^{sD_\mu}{}^* d\theta\|_{\Psi^{q,2}(R)} = 0$, hence that $\lim_{s\to-\infty} \|\pi^* e^{sD_\mu}{}^* T\|_{\Psi^{p,2}(R)} = +\infty$.

(ii). Let θ be of the form $\theta = h dz$ with h a smooth compactly supported function such that $T_3(h)=1$. Then $T(\theta)=T_3(h)=1$. We have $d\theta = \frac{\partial h}{\partial x}\,dx\wedge dz + \frac{\partial h}{\partial y}\,dy\wedge dz$. Thus $\alpha_+ = -\mu_2$ and $\beta_+ = \mu_3$. We consider the quantity

$$p(a\beta_+ + b\alpha_+) = (3-p)\mu_3 - (2p-3)\mu_2 = p(\mu_1-\mu_2) - 3\mu_1.$$

One has $a\beta_+ + b\alpha_+ < 0$ if and only if

$$p < \frac{3\mu_1}{\mu_1-\mu_2} = 1 + \frac{\mu_1-\mu_3}{\mu_1-\mu_2} = p_\alpha.$$

Thus for $p < p_\alpha$, we have $a\beta_+ + b\alpha_+ < 0$. Relation (5.9) then implies that $\lim_{s\to+\infty} \|\pi^* e^{sD_\mu}{}^* d\theta\|_{\Psi^{q,2}(R)} = 0$, hence that $\lim_{s\to+\infty} \|\pi^* e^{sD_\mu}{}^* T\|_{\Psi^{p,2}(R)} = +\infty$. □

We can now turn to the

Proof of $L^p\mathrm{H}^2_{\mathrm{dR}}(S_\alpha) = \{0\}$ for $p \in (\frac{3}{2}; p_\alpha)$. Let $\pi^*T \in \mathcal{Z}^{p,2}(R,\xi)$ be in correspondence with a non-zero class in $L^p\mathrm{H}^2_{\mathrm{dR}}(S_\alpha)$. By Item (i) in the previous lemma, we must have $T_1 = 0$, which implies that both T_2 and T_3 are $\neq 0$ because T is closed. By Item (ii) in the previous lemma, we must have $p \geqslant p_\alpha$. This proves the vanishing by contraposition. □

6. Complex hyperbolic spaces

Another family of concrete examples for which a strip decomposition as stated in Theorem 3.2 can be derived, is provided by the so-called complex hyperbolic spaces. Theorem 6.1 below describes the regions where the cohomology vanishes or not. It also states that the cohomology is Hausdorff. The proof of the (non-)vanishing statement relies on Theorem 3.2, in combination with some additional analysis on Heisenberg groups that is developed in Section 6.1. The Hausdorff statement is a deep result due to Pansu [Pan09, Théorème 1]. We refer to his paper for a proof. The section ends with some complementary results (Propositions 6.7 and 6.11) that provide a finer description of the cohomology.

Let $\mathrm{Heis}(2m-1)$ be the Heisenberg group of dimension $2m-1$ ($m \geqslant 2$), *i.e.* the simply connected nilpotent Lie group whose Lie algebra $\mathfrak{n}$ admits

$$X_1, \ldots, X_{m-1}, Y_1, \ldots, Y_{m-1}, Z$$

as a basis, and where the only non-trivial relations between the above generators are $[X_i, Y_i] = Z$, for all $i \in \{1, \ldots, m-1\}$. Let $R = \mathbf{R} \ltimes_\delta N$ with $N = \mathrm{Heis}(2m-1)$ and $\delta = -\mathrm{diag}(1, \ldots, 1, 2) \in \mathrm{Der}(\mathfrak{n})$. Then R is a solvable Lie group isometric to the complex hyperbolic space $\mathbb{H}^m_{\mathbf{C}}$.

Apart from the density statement and the (non-)vanishing statement in Items (2) and (3) – when $m \geqslant 3$ –, the following result already appears in [Pan99, Pan09].

Theorem 6.1. *Let $k \in \{1, \ldots, 2m-1\}$. One has:*

(1) $L^p\mathrm{H}^k_{\mathrm{dR}}(R) = \{0\}$ *for* $1 < p < \frac{2m}{k+1}$ *or* $p > \frac{2m}{k-1}$.

(2) *If* $\frac{2m}{k+1} < p < \frac{2m}{k}$*, then* $L^p\mathrm{H}^k_{\mathrm{dR}}(R)$ *is Hausdorff and Banach isomorphic to* $\mathcal{Z}^{p,k}(R,\xi)$*. Moreover* $L^p\mathrm{H}^k_{\mathrm{dR}}(R) \neq \{0\}$ *if and only if* $k \geqslant m$*.*

(3) *If* $\frac{2m}{k} < p < \frac{2m}{k-1}$*, then* $L^p\mathrm{H}^k_{\mathrm{dR}}(R)$ *is Hausdorff and the classes of the* $d(\chi \cdot \pi^*\theta)$*'s (where* $\theta \in \Omega_c^{k-1}(N)$*) form a dense subspace. Moreover* $L^p\mathrm{H}^k_{\mathrm{dR}}(R) \neq \{0\}$ *if and only if* $k \leqslant m$*.*

We notice that the above statement will be complemented in Section 6.2: Proposition 6.7 will describe the zero-elements among the classes $[d(\chi \cdot \pi^*\theta)]$'s, while Proposition 6.11 will exhibit a natural dense subspace in $\mathcal{Z}^{p,k}(R,\xi)$.

Beginning of proof of Theorem 6.1. For $k \in \{0,\dots,2m-2\}$, one has $h = 2m$, $w_k = k$, and $w_{2m-1} = h \geqslant 2m-1$. Similarly $W_k = k+1$ for $k \in \{1,\dots,2m-1\}$ and $W_0 = 0 \leq 1$. Item (1) comes from Theorem 3.2(1). The first part of Item (2) follows from Theorem 3.2(2). The Hausdorff statement in Item (3) is a deep theorem of Pansu [Pan09, Théorème 1], in combination with Poincaré duality (Proposition 1.1). The density property follows from Theorem 3.2(3).

It remains to establish the (non-)vanishing parts of Items (2) and (3). It will require more material, and the proof will be completed only at the end of the section. ☐

6.1. **Differential forms on the Heisenberg group.** We complete the proof of the (non-)vanishing statements in Theorem 6.1. They rely on two lemmata (Lemma 6.3 and 6.5 below). The material is inspired by Rumin's paper [Rum94].

Recall that $\mathfrak{n}$ denotes the Lie algebra of $N = \mathrm{Heis}(2m-1)$. It decomposes as $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$, where $\mathfrak{n}_1 := \mathrm{Span}(X_1,\dots,X_{m-1},Y_1,\dots,Y_{m-1})$ and $\mathfrak{n}_2 := \mathrm{Span}(Z)$ are respectively the eigenspaces of $-\delta$ corresponding to the eigenvalues 1 and 2.

Let $x_1,\dots,x_{m-1},y_1,\dots,y_{m-1},z$ be the coordinates on N induced by the exponential map $\mathfrak{n} \to N$. Let $\tau := dz - \frac{1}{2}\sum_{i=1}^{m-1}(x_i dy_i - y_i dx_i)$. We identify $\mathfrak{n}^*$ with the space of the left-invariant 1-forms on N. One has $\mathfrak{n}^* = \mathfrak{n}_1^* \oplus \mathfrak{n}_2^*$, where $\mathfrak{n}_1^* := \mathrm{Span}(dx_1,\dots,dx_{m-1},dy_1,\dots,dy_{m-1})$ and $\mathfrak{n}_2^* := \mathrm{Span}(\tau)$ are the eigenspaces of $-\delta$ of eigenvalues 1 and 2.

The form $d\tau = -\sum_{i=1}^{m-1} dx_i \wedge dy_i$ is a symplectic form when restricted to $\mathfrak{n}_1$. Therefore the Lefschetz map

$$L_k : \wedge^k \mathfrak{n}_1^* \to \wedge^{k+2}\mathfrak{n}_1^*, \ \alpha \mapsto \alpha \wedge d\tau, \tag{6.2}$$

is injective for $k \leqslant m-2$ and surjective for $k \geqslant m-2$, see [BBG03, Proposition 1.1].

The weight decomposition $\wedge^k \mathfrak{n}^* = \wedge^k \mathfrak{n}_1^* \oplus \wedge^{k-1}\mathfrak{n}_1^* \wedge \tau$ associated to $-\delta$, yields a decomposition

$$\Omega^k(N) = \Omega_1^k \oplus \Omega_2^k,$$

with $\Omega_2^k = \Omega_1^{k-1} \wedge \tau$. Therefore, every $\theta \in \Omega^k(N)$ decomposes uniquely as

$$\theta = \theta_1 + \theta_2 \wedge \tau, \text{ with } \theta_1 \in \Omega_1^k \text{ and } \theta_2 \in \Omega_1^{k-1}.$$

The form θ_1 is said to be *horizontal* and of *pure weight* k. The form $\theta_2 \wedge \tau$ is said to be *vertical* and of *pure weight* $k+1$. We will use the indices $_1$ and $_2$ to specify the components of differential forms according to the above direct sum.

When $\frac{2m}{k} < p < \frac{2m}{k-1}$, and according to what we have already proved for Theorem 6.1(3), the space $L^p\mathrm{H}^k_{\mathrm{dR}}(R)$ is Hausdorff and contains the dense subspace $\{[d(\chi \cdot \pi^*\theta)] : \theta \in \Omega_c^{k-1}(N)\}$. One has in addition:

Lemma 6.3. *Suppose $k \in \{2, \ldots, 2m-1\}$ and $\frac{2m}{k} < p < \frac{2m}{k-1}$.*

(1) *Let $\theta \in \Omega_c^{k-1}(N)$. If $\theta = \alpha \wedge d\tau + \beta \wedge \tau$, with $\alpha \in \Omega_c^{k-3}(N)$ and $\beta \in \Omega_c^{k-2}(N)$, then $[d(\chi \cdot \pi^*\theta)] = 0$ in $L^p\mathrm{H}^k_{\mathrm{dR}}(R)$.*
(2) *For $k \geqslant m+1$, one has $L^p\mathrm{H}^k_{\mathrm{dR}}(R) = \{0\}$.*

Proof. (1). Suppose first that $\theta = \beta \wedge \tau$. Since $p > \frac{2m}{k} \geqslant \frac{h}{w_k}$, Proposition 3.4 implies that

$$\big\|[d(\chi \cdot \pi^*\theta)]\big\|_{L^p\mathrm{H}^k(R)} \leqslant C \inf_{t\in\mathbf{R}} \big\{ \|(e^{t\delta})^*\theta\|_{L^p\Omega^{k-1}(N)} + \|(e^{t\delta})^* d\theta\|_{L^p\Omega^k(N)} \big\}.$$

Since θ is of pure weight k, one has $\|(e^{t\delta})^*\theta\|_{L^p\Omega^{k-1}} \asymp e^{(-k+\frac{2m}{p})t}$ by Lemma 3.6. The weight of $d\theta$ is at least k, thus for $t \geqslant 0$ one has $\|(e^{t\delta})^* d\theta\|_{L^p\Omega^k} \lesssim e^{(-k+\frac{2m}{p})t}$. Therefore $\big\|[d(\chi \cdot \pi^*\theta)]\big\|_{L^p\mathrm{H}^k} \to 0$ when $t \to +\infty$, and thus $[d(\chi \cdot \pi^*\theta)] = 0$ in $L^p\mathrm{H}^k_{\mathrm{dR}}(R)$.

Now suppose that $\theta = \alpha \wedge d\tau$. We claim that there exists $\gamma \in \Omega_c^{k-1}(N)$ such that $d\theta = d(\gamma \wedge \tau)$. By Corollary 2.6, this will imply that $[d(\chi \cdot \pi^*\theta)] = [d(\chi \cdot \pi^*(\gamma \wedge \tau))]$; which in turn implies that $[d(\chi \cdot \pi^*\theta)] = 0$ from the previous case. Let $\gamma := -(-1)^{d^\circ\alpha} d\alpha$. One has

$$d(\theta - \gamma \wedge \tau) = d\alpha \wedge d\tau + (-1)^{d^\circ\alpha+1}(-1)^{d^\circ\alpha} d\alpha \wedge d\tau = 0,$$

and so $d\theta = d(\gamma \wedge \tau)$.

(2). Let $\theta \in \Omega_c^{k-1}(N)$. Since the Lefschetz map L_i, defined in (6.2), is surjective for $i \geqslant m-2$, one can write the weight decomposition of θ as

$$\theta = \alpha \wedge d\tau + \theta_2 \wedge \tau.$$

Therefore Item (1) implies that $[d(\chi \cdot \pi^*\theta)] = 0$ in $L^p\mathrm{H}^k_{\mathrm{dR}}(R)$. This in turn implies that $L^p\mathrm{H}^k_{\mathrm{dR}}(R) = \{0\}$, thanks to the density of the $[d(\chi \cdot \pi^*\theta)]$'s. □

The weight decomposition of k-forms can be extended to k-currents. This induces a decomposition

$$\mathcal{D}'^k(N) = \mathcal{D}'^k_1 \oplus \mathcal{D}'^k_2,$$

with $\mathcal{D}'^k_2 = \mathcal{D}'^{k-1}_1 \wedge \tau$. Concretely, every $T \in \mathcal{D}'^k$ can be written uniquely as

$$T = \sum_{|I|=|J|=k} T_{IJ} dx_I \wedge dy_J + \sum_{|K|+|L|=k-1} T_{KL} dx_K \wedge dy_L \wedge \tau,$$

with $T_{IJ}, T_{KL} \in \mathcal{D}'^0(N)$. Its weight decomposition is then $T = T_1 + T_2 \wedge \tau$, with

$$T_1 = \sum_{|I|+|J|=k} T_{IJ} dx_I \wedge dy_J \quad \text{and} \quad T_2 = \sum_{|K|+|L|=k-1} T_{KL} dx_K \wedge dy_L.$$

The current T_1 is said to be *horizontal*, and $T_2 \wedge \tau$ to be *vertical.* For $\theta \in \Omega_c^{2m-1-k}(N)$, which weight decomposes as $\theta = \theta_1 + \theta_2 \wedge \tau$, one shows easily that

$$T(\theta) = T_1(\theta_2 \wedge \tau) + T_2(\tau \wedge \theta_1). \tag{6.4}$$

In Theorem 6.1(2), we have seen that $L^p\mathrm{H}^k_{\mathrm{dR}}(R)$ is Banach isomorphic to $\mathcal{Z}^{p,k}(R,\xi)$, for every $k \in \{1,\dots,2m-1\}$ and $\frac{2m}{k+1} < p < \frac{2m}{k}$. Moreover we know from Proposition 2.4 that every $\psi \in \mathcal{Z}^{p,k}(R,\xi)$ can be written as $\psi = \pi^*T$ for some (unique) $T \in \mathcal{D}'^k(N)$. One has futhermore:

Lemma 6.5. *Let $k \in \{1,\dots,2m-1\}$ and $\frac{2m}{k+1} < p < \frac{2m}{k}$.*

(1) *For every $\pi^*T \in \mathcal{Z}^{p,k}(R,\xi)$ the k-current T is vertical.*
(2) *Conversely, if $\varphi \in \Omega_c^{k-1}(N)$ is such that $d\varphi$ is vertical, then $\pi^*(d\varphi)$ belongs to $\mathcal{Z}^{p,k}(R,\xi)$.*
(3) *We have $\mathcal{Z}^{p,k}(R,\xi) \neq \{0\}$ for every $k \in \{m,\dots,2m-1\}$.*

Proof. (1). Every $T \in \mathcal{D}'^{2m-1}(N)$ is vertical, thus we can assume that $k \in \{1,\dots,2m-2\}$. Let (q,ℓ) be the Poincaré dual of (p,k) relatively to R; it satisfies $\ell \in \{2,\dots,2m-1\}$ and $\frac{2m}{\ell} < q < \frac{2m}{\ell-1}$. For every

$\pi^*T \in \mathcal{Z}^{p,k}(R,\xi)$ and every vertical $\theta \in \Omega_c^{\ell-1}(N)$, one has $T(\theta) = 0$, thanks to Theorem 3.2 and Lemma 6.3. According to Relation (6.4), this implies that the weight decomposition of T satisfies $T_1 = 0$. Thus T is vertical.

(2). Let $\varphi \in \Omega_c^{k-1}(N)$ be such that $d\varphi$ is vertical. First, we claim that $d(\chi \cdot \pi^*\varphi)$ belongs to $\Omega^{p,k}(R) \cap \operatorname{Ker} d$. One has $d(\chi \cdot \pi^*\varphi) = d\chi \wedge \pi^*\varphi + \chi \cdot \pi^* d\varphi$. Thus

$$\|d(\chi \cdot \pi^*\varphi)\|_{\Omega^{p,k}} = \|d(\chi \cdot \pi^*\varphi)\|_{L^p\Omega^k} \leqslant \|d\chi \wedge \pi^*\varphi\|_{L^p\Omega^k} + \|\chi \cdot \pi^* d\varphi\|_{L^p\Omega^k}.$$

The form $d\chi \wedge \pi^*\varphi$ is compactly supported, thus it belongs to $L^p\Omega^k(R)$. One has

$$\|\chi \cdot \pi^* d\varphi\|^p_{L^p\Omega^k(R)} \leqslant \|\mathbf{1}_{t\geq 0} \cdot \pi^* d\varphi\|^p_{L^p\Omega^k(R)} = \int_0^{+\infty} \|(e^{t\delta})^* d\varphi\|^p_{L^p\Omega^k(N)}\, dt.$$

Since $d\varphi$ is vertical, it is of pure weight $k+1$, and $\|(e^{t\delta})^* d\varphi\|_{L^p\Omega^k(N)} \asymp e^{(-k-1+\frac{2m}{p})t}$ by Lemma 3.6. Since $p > \frac{2m}{k+1}$, the above integral converges and the claim is proved.

One has $\pi^* d\varphi = \lim_{t\to+\infty} \varphi_t^*(d(\chi \cdot \pi^*\varphi))$ in the sense of currents; thus $\pi^*(d\varphi) \in \mathcal{Z}^{p,k}(R,\xi)$ by Proposition 2.2.

(3). It remains to show that there exists $\varphi \in \Omega_c^k(N)$ such that $d\varphi$ is vertical and non-zero. We distinguish the cases $k > m$ and $k = m$.

Suppose $k > m$. Then $\operatorname{Ker}(L : \wedge^{k-2}\mathfrak{n}_1^* \to \wedge^k\mathfrak{n}_1^*)$ is non-zero. Let $\alpha \in \Omega_1^{k-2}$ be non-zero, compactly supported and such that $\alpha \wedge d\tau = 0$. For every $f \in \Omega_c^0(N)$ consider the form $\varphi = \varphi_2 \wedge \tau$, with $\varphi_2 := f \cdot \alpha$. Then $d\varphi$ is vertical. Moreover $d\varphi \neq 0$ for generic f.

Assume now that $k = m$. Then the Lefschetz map $L : \wedge^{m-2}\mathfrak{n}_1^* \to \wedge^m\mathfrak{n}_1^*$ is an isomorphism. Pick any compactly supported $\varphi_1 \in \Omega_1^{m-1}$. Let $\varphi_2 \in \Omega_1^{m-2}$ be the unique solution of the equation:

$$(d\varphi_1)_1 = -(-1)^m \varphi_2 \wedge d\tau. \tag{6.6}$$

and set $\varphi := \varphi_1 + \varphi_2 \wedge \tau$. Then $d\varphi$ is vertical. We claim that for generic φ_1, one has $d\varphi \neq 0$. Indeed suppose that $d\varphi = 0$. Then there exists $\beta \in \Omega_c^{k-1}(N)$ so that $\varphi = d\beta$. Thus $\varphi_1 = (d\beta)_1$. Let $S \subset N$ be a complete horizontal submanifold of dimension $m-1$ (*e.g.* the boundary at infinity of an isometric copy of $\mathbb{H}^m_{\mathbf{R}}$ in $\mathbb{H}^m_{\mathbf{C}}$). Since S is horizontal, one has by Stokes' Theorem:

$$\int_S \varphi_1 = \int_S (d\beta)_1 = \int_S d\beta = 0.$$

The claim follows. $\square$

We can now conclude:

End of proof of Theorem 6.1. Note that we can use Poincaré duality (Proposition 1.1(2)) since we know at this stage that the cohomology spaces we consider are Hausdorff. The vanishing results in Items (2) and (3) follow from Lemma 6.3 and Poincaré duality. The non-vanishing ones are consequence of Lemma 6.5 and Poincaré duality. □

6.2. **Complement (on density) to Theorem 6.1.** We establish two results (Propsitions 6.7 and 6.11) that complement Theorem 6.1 and that could be useful in the future. The first one will serve partially in Section 7 to study the cohomology of $\mathrm{SL}_3(\mathbf{R})/\mathrm{SO}_3(\mathbf{R})$. The objects and notations are the same as in the previous section.

Recall from Theorem 6.1, that for $k \in \{1, \ldots, m\}$ and $\frac{2m}{k} < p < \frac{2m}{k-1}$, the space $L^p\mathrm{H}^k_{\mathrm{dR}}(R)$ is Hausdorff, non-zero, and admits the subspace $\{[d(\chi \cdot \pi^*\theta)] : \theta \in \Omega_c^{k-1}(N)\}$ as a dense subset. The first result of the section describes the classes $[d(\chi \cdot \pi^*\theta)]$ that are null in $L^p\mathrm{H}^k_{\mathrm{dR}}(R)$:

Proposition 6.7. *Let $k \in \{1, \ldots, m\}$ and $\frac{2m}{k} < p < \frac{2m}{k-1}$. For every $\theta \in \Omega_c^{k-1}(N)$, the following holds:*

(1) *When $k < m$, the class $[d(\chi \cdot \pi^*\theta)]$ is null in $L^p\mathrm{H}^k_{\mathrm{dR}}(R)$ if and only if $(d\theta)_1 = \gamma \wedge d\tau$, for some horizontal form $\gamma \in \Omega_c^{k-2}(N)$.*
(2) *When $k = m$, the class $[d(\chi \cdot \pi^*\theta)]$ is null in $L^p\mathrm{H}^k_{\mathrm{dR}}(R)$ if and only if*
$$d\Big(\theta - \mathcal{L}\big((d\theta)_1\big) \wedge \tau\Big) = 0,$$
where $\mathcal{L} : \Omega_1^m \to \Omega_1^{m-2}$ denotes the pointwise operator induced by the inverse of the Lefschetz isomorphism $L_{m-2} : \wedge^{m-2}\mathfrak{n}_1^* \to \wedge^m\mathfrak{n}_1^*$.

Proof. (1). Let $\theta \in \Omega_c^{k-1}(N)$ and suppose that $d\theta$ can be written $d\theta = \gamma \wedge d\tau + \delta \wedge \tau$, with γ and δ horizontal. When $k = 1$, such a relation is impossible unless $\theta = 0$. Namely the differential of a non-zero compactly supported function has always a non-zero horizontal component. Let us assume $k \geqslant 2$. We claim that there exists a horizontal form $\beta \in \Omega_c^{k-2}(N)$, such that $d\theta = d(\beta \wedge \tau)$. In combination with Corollary 2.6 and Lemma 6.3, this yields $[d(\chi \cdot \pi^*\theta)] = 0$.

Since β, γ and δ are horizontal forms, the equation $d\theta = d(\beta \wedge \tau)$ is equivalent to the following system of two equations
$$\gamma \wedge d\tau = (-1)^k \beta \wedge d\tau \quad \text{and} \quad \delta \wedge \tau = d\beta \wedge \tau.$$

Set $\beta := (-1)^k\gamma$. Then the first equation is satisfied. Moreover one has $d\beta \wedge \tau = (-1)^k d\gamma \wedge \tau = (-1)^k (d\gamma)_1 \wedge \tau$. Thus the second equation is satisfied if the relation $\delta = (-1)^k (d\gamma)_1$ holds. Since $dd\theta = 0$, one has $d\gamma \wedge d\tau + d\delta \wedge \tau - (-1)^k \delta \wedge d\tau = 0$. This implies that $((d\gamma)_1 - (-1)^k\delta) \wedge d\tau = 0$, which in turn implies that $(d\gamma)_1 - (-1)^k\delta = 0$ since the Lefschetz map L_{k-1} is injective (recall that $k < m$ by assumption). Therefore the second equation is satisfied and the claim is proved.

Conversely, let $\theta \in \Omega_c^{k-1}(N)$ be such that $[d(\chi \cdot \pi^*\theta)] = 0$. Denote by (q, ℓ) the Poincaré dual of (p, k) relatively to R. One has $\frac{2m}{\ell+1} < q < \frac{2m}{\ell}$ and $\ell > m$. In particular $L_{\ell-2}$ is not injective, and therefore the following subspace is non-trivial

$$\Gamma = \{\alpha \in \Omega_c^{\ell-2}(N) : \alpha \text{ is horizontal and } \alpha \wedge d\tau = 0\}.$$

Pick $\alpha \in \Gamma$ and consider the form $\varphi = \alpha \wedge \tau \in \Omega_c^{\ell-1}(N)$. One has $d\varphi = d\alpha \wedge \tau + (-1)^\ell \alpha \wedge d\tau = d\alpha \wedge \tau$. Thus $d\varphi$ is vertical, and so by Lemma 6.5(2) the form $\pi^*(d\varphi)$ belongs to $\mathcal{Z}^{q,\ell}(R, \xi)$. Our assumption $[d(\chi \cdot \pi^*\theta)] = 0$, in combination with Theorem 3.2(4), implies that $\int_N d\varphi \wedge \theta = 0$. With Stokes' formula and the definition of φ, it follows that $\int_N \alpha \wedge \tau \wedge d\theta = 0$, *i.e.* $\int_N \alpha \wedge \tau \wedge (d\theta)_1 = 0$. So far we have established that every $\theta \in \Omega_c^{k-1}(N)$ such that $[d(\chi \cdot \pi^*\theta)] = 0$ satisfies the following property:

$$\int_N \alpha \wedge \tau \wedge (d\theta)_1 = 0 \text{ for all } \alpha \in \Gamma. \tag{6.8}$$

When $k \geqslant 2$, we will show that Property (6.8) implies that $(d\theta)_1$ can be written $\gamma \wedge d\tau$. When $k = 1$, we will prove that (6.8) implies $(d\theta)_1 = 0$.

Suppose first that $k = 1$. One has $\ell = 2m - 1$, $L_{\ell-2} = 0$ and $\Gamma = \Omega_1^{\ell-2} \cap \Omega_c^{\ell-2}(N)$. Thus Property (6.8) yields that $\int_N \omega \wedge \tau \wedge (d\theta)_1 = 0$ for every $\omega \in \Omega_c^{\ell-2}(N)$. This in turn implies that $\tau \wedge (d\theta)_1 = 0$, *i.e.* $(d\theta)_1 = 0$.

Suppose now that $k \geqslant 2$. We will use the following lemma:

Lemma 6.9. *Let* $b : \wedge^{\ell-2}\mathfrak{n}_1^* \times \wedge^k \mathfrak{n}_1^* \to \mathbf{R}$, *be the non-degenerate bilinear form defined by* $b(u, v) = u \wedge v$. *Relative to* b, *one has* $(\operatorname{Ker} L_{\ell-2})^\perp = \operatorname{Im} L_{k-2}$.

Proof of Lemma 6.9. Since b is non-degenerate, the statement is equivalent to $(\operatorname{Im} L_{k-2})^\perp = \operatorname{Ker} L_{\ell-2}$. Let $u \in \wedge^{\ell-2}\mathfrak{n}_1^*$. It belongs to $(\operatorname{Im} L_{k-2})^\perp$ if and only if $u \wedge v \wedge d\tau = 0$ for all $v \in \wedge^{k-2}\mathfrak{n}_1^*$. This is equivalent to $u \wedge d\tau = 0$, *i.e.* to $u \in \operatorname{Ker} L_{\ell-2}$. □

Lemma 6.9 allows one to complete the proof of Item (1) as follows. By definition, Γ is the space of compactly supported smooth sections of the left-invariant vector bundle over N generated by $\operatorname{Ker} L_{\ell-2}$. Property (6.8) can be interpreted as saying that $(d\theta)_1$ is a smooth section of the left-invariant vector bundle generated by $(\operatorname{Ker} L_{\ell-2})^{\perp}$. By Lemma 6.9, this is equivalent to $(d\theta)_1 = \gamma \wedge d\tau$ for some horizontal form $\gamma \in \Omega_c^{k-2}(N)$.

(2). Suppose that $k = m$, and let (q, m) be the Poincaré dual of (p, m) relatively to R. One has $\frac{2m}{m+1} < q < \frac{2m}{m} = 2$. Let $\theta \in \Omega_c^{m-1}(N)$. According to Theorem 3.2 and Poincaré duality (Proposition 1.1), the class $[d(\chi \cdot \pi^*\theta)]$ vanishes in $L^p\mathrm{H}^m_{\mathrm{dR}}(R)$ if and only if $T(\theta) = 0$ for all $T \in \mathcal{D}'^m(N)$ such that $\pi^*T \in \mathcal{Z}^{q,m}(R, \xi)$.

Since L_{m-2} is an isomorphism, there exist unique horizontal forms $\alpha \in \Omega_c^{m-2}(N)$ and $\beta \in \Omega_c^{m-1}(N)$, such that $d\theta$ weight decomposes as $d\theta = \alpha \wedge d\tau + \beta \wedge \tau$.

Let $T \in \mathcal{D}'^m(N)$ be such that $\pi^*T \in \mathcal{Z}^{q,m}(R, \xi)$. Since N is contractible and T is closed, it admits a primitive, say $S \in \mathcal{D}'^{m-1}(N)$. Then $T(\theta)$ admits the following expression:

Lemma 6.10. *With the notation above, one has*

$$T(\theta) = (-1)^m S_1\Big(\big(\beta - (-1)^m d\alpha\big) \wedge \tau\Big).$$

Moreover:

$$\big(\beta - (-1)^m d\alpha\big) \wedge \tau = d\Big(\theta - (-1)^m \mathcal{L}\big((d\theta)_1\big) \wedge \tau\Big).$$

Assume for a moment that the lemma holds. Then the "if" part of item (2) follows immediately. To establish the "only if" part, we apply the lemma with some explicit currents T. Let $\varphi_1 \in \Omega_c^{m-1}(N)$ be an arbitrary horizontal form, let $\varphi_2 \in \Omega_c^{m-2}(N)$ be the horizontal form uniquely determined by the equation $(d\varphi_1)_1 = -(-1)^m \varphi_2 \wedge d\tau$. Set $\varphi := \varphi_1 + \varphi_2 \wedge \tau$. Then an easy computation shows that $d\varphi$ is vertical. Thus, by Lemma 6.5(2), the current $\pi^*(d\varphi)$ belongs to $\mathcal{Z}^{q,m}(R, \xi)$. Lemma 6.10 applied to $T = d\varphi$, yields that

$$\int_N \varphi_1 \wedge \big(\beta - (-1)^m d\alpha\big) \wedge \tau = 0,$$

for every horizontal $\varphi_1 \in \Omega_c^{m-1}(N)$. Therefore $\big(\beta - (-1)^m d\alpha\big) \wedge \tau = 0$, and the second part of the lemma completes the proof of item (2). $\square$

It remains to give the

Proof of Lemma 6.10. Since S is a primitive of T, we have $T(\theta) = dS(\theta) = (-1)^m S(d\theta)$. From Relation (6.4) and the expression $d\theta = \alpha \wedge d\tau + \beta \wedge \tau$, it follows that $S(d\theta) = S_1(\beta \wedge \tau) + S_2(\tau \wedge \alpha \wedge d\tau)$. By Lemma 6.5(1) the current T is vertical. This means (by a simple computation) that $(dS_1)_1 = -(-1)^m S_2 \wedge d\tau$. Therefore:

$$\begin{aligned} S_2(\tau \wedge \alpha \wedge d\tau) &= (-1)^m S_2(d\tau \wedge \alpha \wedge \tau) = -(dS_1)_1(\alpha \wedge \tau) \\ &= -dS_1(\alpha \wedge \tau) = -(-1)^m S_1\big(d(\alpha \wedge \tau)\big) \\ &= S_1\big(-(-1)^m d\alpha \wedge \tau\ \big). \end{aligned}$$

The expected formula for $T(\theta)$ follows. To establish the second formula, we compute

$$\begin{aligned} d\Big(\theta - (-1)^m \mathcal{L}\big((d\theta)_1\big) \wedge \tau\Big) &= d\big(\theta - (-1)^m \alpha \wedge \tau\big) \\ &= \alpha \wedge d\tau + \beta \wedge \tau - (-1)^m d\alpha \wedge \tau - \alpha \wedge d\tau \\ &= \big(\beta - (-1)^m d\alpha\big) \wedge \tau. \end{aligned}$$

The lemma is proved. □

The second result of the section deals with the Banach space $\mathcal{Z}^{p,k}(R,\xi)$ for $k \in \{m, 2m-1\}$ and $\frac{2m}{k+1} < p < \frac{2m}{k}$. According to Lemma 6.5(2), the forms $\pi^*(d\varphi)$, where $\varphi \in \Omega_c^{k-1}(N)$ and $d\varphi$ is vertical, belong to $\mathcal{Z}^{p,k}(R,\xi)$. A natural problem is to determine whether they form a dense subspace. For the norm topology, we do not know, but for the current topology this is indeed the case:

Proposition 6.11. *Let $k \in \{m, \dots, 2m-1\}$ and $\frac{2m}{k+1} < p < \frac{2m}{k}$. The set $\{d\varphi : \varphi \in \Omega_c^{k-1}(N),\ d\varphi \text{ is vertical}\}$ is a dense subspace in the sense of currents in $\{T \in \mathcal{D}'^k(N) : \pi^* T \in \mathcal{Z}^{p,k}(R,\xi)\}$.*

Proof. Set $F := \{d\varphi : \varphi \in \Omega_c^{k-1}(N),\ d\varphi \text{ is vertical}\}$ and $E := \{T \in \mathcal{D}'^k(N) : \pi^* T \in \mathcal{Z}^{p,k}(R,\xi)\}$ for simplicity. Let (q,ℓ) be the Poincaré dual of (p,k) relatively to R. The topology on E, which is induced by the weak*-topology of $\mathcal{D}'^k(N)$, is generated by the linear forms $\Lambda_\theta : E \to \mathbf{R}$ defined by $\Lambda_\theta(T) = T(\theta)$, where θ belongs to $\Omega_c^{\ell-1}(N)$. We shall let E_{top} denote E equipped with this topology.

According to the Hahn-Banach Theorem, showing that F is dense in E_{top}, is equivalent to proving the triviality of every $\Lambda \in E^*_{\text{top}}$ such that $\Lambda(F) = \{0\}$.

Since every element of E^*_{top} is a Λ_θ for some $\theta \in \Omega_c^{\ell-1}(N)$ (see [Ru74, Theorem 3.10]), we are led to showing the following: if $\theta \in \Omega_c^{\ell-1}(N)$ satisfies $\int_N d\varphi \wedge \theta = 0$ for every $d\varphi \in F$, then $\Lambda_\theta = 0$.

By analysing the "only if" parts of the proof of the previous Proposition 6.7, one sees that the θ's such that $\int_N d\varphi \wedge \theta = 0$ for every $d\varphi \in F$, are precisely those for which $[d(\chi \cdot \pi^*\theta)] = 0$ in $L^q\mathrm{H}^\ell_{\mathrm{dR}}(R)$. Therefore they satisfy $\Lambda_\theta = 0$, thanks to Theorem 3.2(4). □

7. The symmetric space $\mathrm{SL}_3(\mathbf{R})/\mathrm{SO}_3(\mathbf{R})$

We prove Theorem D (stated in the introduction) which describes the second L^p-cohomology of $\mathrm{SL}_3(\mathbf{R})/\mathrm{SO}_3(\mathbf{R})$. The strategy is similar to the one conducted in Section 5 to study the second L^p-cohomology of the groups S_α. It highly relies on the description of the cohomology of the complex hyperbolic plane discussed in Section 6.

7.1. **Notation and decomposition of** $\mathrm{SL}_3(\mathbf{R})$**.** At first we introduce the various subgroups of $\mathrm{SL}_3(\mathbf{R})$ we will be working with in the sequel.

The relevant Iwasawa decomposition here is $\mathrm{SL}_3(\mathbf{R}) = KAN$, with $K = \mathrm{SO}_3(\mathbf{R})$, $A = \mathrm{Diag}(\mathbf{R}^3) \cap \mathrm{SL}_3(\mathbf{R})$ and

$$N = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbf{R} \right\} \simeq \mathrm{Heis}(3).$$

Let $\mathfrak{a}$ and $\mathfrak{n}$ be the Lie algebras of A and N respectively. Every element of $\mathfrak{n}$ can be naturally denoted by a triple $(x, y, z) \in \mathbf{R}^3$.

Let $\xi, \eta \in \mathfrak{a}$ be defined by $\xi = \mathrm{diag}(-1, 0, 1)$ and $\eta = \frac{1}{3}\mathrm{diag}(1, -2, 1)$. They act on $\mathfrak{n}$ by:

$$\mathrm{ad}\xi \cdot (x, y, z) = (-x, -y, -2z) \text{ and } \mathrm{ad}\eta \cdot (x, y, z) = (x, -y, 0). \tag{7.1}$$

We consider the following subgroups of $\mathrm{SL}_3(\mathbf{R})$:

$$R := \{e^{t\xi}\}_{t\in\mathbf{R}} \ltimes N \simeq \mathbb{H}^2_{\mathbf{C}}, \quad H := \{e^{s\eta}\}_{s\in\mathbf{R}} \ltimes N \quad \text{and}$$

$$S := A \ltimes N = \{e^{s\eta}\}_{s\in\mathbf{R}} \ltimes R = \{e^{t\xi}\}_{t\in\mathbf{R}} \ltimes H.$$

The Lie group S is isometric to the symmetric space $\mathrm{SL}_3(\mathbf{R})/\mathrm{SO}_3(\mathbf{R})$.

7.2. **First observations.** We derive from previous results some preliminary observations on $L^p\mathrm{H}^2_{\mathrm{dR}}(S)$. The notations are the same as in the previous section.

Proposition 7.2. *One has* $L^p\mathrm{H}^2_{\mathrm{dR}}(S) = \{0\}$ *for* $p < \frac{4}{3}$.

Proof. Let $\mathfrak{h}$ denotes the Lie algebra of H. Set $\delta := \mathrm{ad}\xi\Big|_{\mathfrak{h}} \in \mathrm{Der}(\mathfrak{h})$, so that S can be written $S = \mathbf{R} \ltimes_\delta H$. The ordered list of eigenvalues of $-\delta$ enumerated with multiplicity, is

$$\lambda_1 = 0 < \lambda_2 = \lambda_3 = 1 < \lambda_4 = 2.$$

Thus, with the notation of Section 3, the trace of $-\delta$ is $h = 4$, and one has $W_2 = \lambda_3 + \lambda_4 = 3$. Therefore the statement follows from Theorem 3.2(1). □

Proposition 7.3. *For $p \in (\frac{4}{3}; 4) \setminus \{2\}$, the space $L^p\mathrm{H}^2_{\mathrm{dR}}(R)$ is non-zero and Hausdorff, and there exists a linear isomorphism*

$$L^p\mathrm{H}^2_{\mathrm{dR}}(S) \simeq \Big\{[\omega] \in L^p\mathrm{H}^2_{\mathrm{dR}}(R) : \int_{\mathbf{R}} \|e^{s\mathrm{ad}\eta^*}[\omega]\|^p_{L^p\mathrm{H}^2(R)} ds < +\infty\Big\}.$$

Proof. When $p \in (\frac{4}{3}; 4)\setminus\{2\}$, Theorem 6.1 shows that $L^p\mathrm{H}^2_{\mathrm{dR}}(R)$ is non-zero and Hausdorff, and that $L^p\mathrm{H}^k_{\mathrm{dR}}(R) = \{0\}$ in all degrees $k \neq 2$. Since $S = \{e^{s\eta}\}_{s\in\mathbf{R}} \ltimes R$, the above description of the cohomology of R, in combination with a Hochschild-Serre spectral sequence argument (see [BR23, Corollary 6.10]), yields the desired linear isomorphism. □

Proposition 7.4. *For $p > 4$, the space $L^p\overline{\mathrm{H}^2_{\mathrm{dR}}}(S)$ is non-zero.*

Proof. Consider again $\lambda_1 = 0 < \lambda_2 = \lambda_3 = 1 < \lambda_4 = 2$ the list of the eigenvalues of $-\delta = -\mathrm{ad}\xi\Big|_{\mathfrak{h}} \in \mathrm{Der}(\mathfrak{h})$. The trace of $-\delta$ is $h = 4$, and one has $w_2 = \lambda_1 + \lambda_2 = 1$. Since the rank of S is equal to 2, it follows from [BR23, Theorem C and Corollary 3.4] that $L^p\overline{\mathrm{H}^2_{\mathrm{dR}}}(S)$ is non-zero for $p > \frac{h}{w_2} = 4$. □

7.3. **Auxiliary results on $\mathbb{H}^2_{\mathbf{C}}$.** This section is devoted to auxiliary results (Lemmata 7.5 and 7.6) that will serve in the next section to prove Theorem D.

Recall that $R = \{e^{t\xi}\}_{t\in\mathbf{R}} \ltimes N \simeq \mathbb{H}^2_{\mathbf{C}}$. Let π_N be the projection map from R onto N.

Lemma 7.5. *Suppose that $p \in (2; 4)$ and let $\theta \in \Omega^1_c(N) \setminus \{0\}$ be of the form $\theta = fdx$ or gdy. Then:*

1. *The class $[d(\chi \cdot \pi_N^*\theta)]$ is non-zero in $L^p\mathrm{H}^2_{\mathrm{dR}}(R)$.*
2. *If $\theta = fdx$ (resp. gdy), one has*

$$\big\|[d(\chi \cdot \pi_N^* e^{s\mathrm{ad}\eta^*}\theta)]\big\|_{L^p\mathrm{H}^2(R)} \to 0$$

exponentially fast, when s tends to $-\infty$ (resp. $+\infty$).

Proof. (1). We apply the criterion in Proposition 6.7(2). Suppose first that $\theta = f dx$. With the notation of Sections 6.1 and 6.2, one has $d\theta = (Y\cdot f)dy\wedge dx + (Z\cdot f)\tau\wedge dx$. Since $d\tau = -dx\wedge dy$, we get that

$$d\Big(\theta - \mathcal{L}\big((d\theta)_1\big)\tau\Big) = d\big(\theta - (Y\cdot f)\tau\big)$$
$$= -(Z\cdot f + X\cdot Y\cdot f)dx\wedge\tau - (Y^2\cdot f)dy\wedge\tau.$$

The term $Y^2\cdot f$ is non-zero since the function f is non-zero and has compact support. Therefore $d(\theta - \mathcal{L}((d\theta)_1\tau)$ is non-zero, and the statement follows from Proposition 6.7(2). The case $\theta = g dy$ is similar.

(2). Suppose that $\theta = f dx$. For $(s,t)\in\mathbf{R}^2$, one has

$$e^{t\mathrm{ad}\xi *}(e^{s\mathrm{ad}\eta *}\theta) = (e^{t\mathrm{ad}\xi + s\mathrm{ad}\eta})^*\theta \text{ and } e^{t\mathrm{ad}\xi *}(de^{s\mathrm{ad}\eta *}\theta) = (e^{t\mathrm{ad}\xi + s\mathrm{ad}\eta})^* d\theta.$$

By Lemma 3.6 and relation (7.1), their L^p-norms satisfy

$$\big\|(e^{t\mathrm{ad}\xi + s\mathrm{ad}\eta})^*\theta\big\|_{L^p\Omega^1(N)} \asymp e^{(\frac{4}{p}-1)t+s}\|f\|_p,$$

$$\text{and } \big\|(e^{t\mathrm{ad}\xi + s\mathrm{ad}\eta})^* d\theta\big\|_{L^p\Omega^2(N)} \asymp e^{-(2-\frac{4}{p})t}\|Y\cdot f\|_p + e^{-(3-\frac{4}{p})t+s}\|Z\cdot f\|_p.$$

Set $a = \frac{4}{p}-1$, $b = 2-\frac{4}{p}$ and $c = 3-\frac{4}{p}$. One has $a,b,c>0$ since $2<p<4$. From Proposition 3.4, it follows that

$$\big\|[d(\chi\cdot\pi_N^* e^{s\mathrm{ad}\eta *}\theta)]\big\|_{L^p\mathrm{H}^2(R)} \leqslant C\inf_{t\in\mathbf{R}}\big\{(e^{at}+e^{-ct})e^s + e^{-bt}\big\}.$$

Suppose that $s\to-\infty$, and set $t = -\frac{s}{2a}$. Then

$$(e^{at}+e^{-ct})e^s + e^{-bt} = e^{\frac{s}{2}} + e^{(\frac{c}{2a}+1)s} + e^{\frac{b}{2a}s},$$

which tends to 0 exponentially fast. The case $\theta = g dy$ is similar. ☐

Lemma 7.6. *Suppose that $p\in(2;4)$. There exist non-zero forms $\theta = f dx$ and $\Theta = g dy$ in $\Omega^1_c(N)$, such that $[d(\chi\cdot\pi_N^*\theta)] = [d(\chi\cdot\pi_N^*\Theta)]$ in $L^p\mathrm{H}^2_{\mathrm{dR}}(R)$.*

Proof. Let u be an arbitrarily non-zero function in $\Omega^0_c(N)$. Its differential is $(X\cdot u)dx + (Y\cdot u)dy + (Z\cdot u)\tau$. Set $f := X\cdot u$, $g := -Y\cdot u$, $h := -Z\cdot u$, and let $\theta = f dx$ and $\Theta = g dy$. By Proposition 6.7(2), one has $[d(\chi\cdot\pi_N^*\theta)] = [d(\chi\cdot\pi_N^*\Theta)]$ in $L^p\mathrm{H}^2_{\mathrm{dR}}(R)$. Indeed:

$$d(\theta-\Theta) = d(h\tau) = dh\wedge\tau + hd\tau,$$

thus $\mathcal{L}((\theta-\Theta)_1) = h$, and we get that $d(\theta-\Theta-\mathcal{L}((\theta-\Theta)_1)\tau) = 0$. ☐

7.4. **Proof of Theorem D.** Thanks to Propositions 7.2 and 7.4, we can restrict ourselves to the region $p \in (\frac{4}{3}; 4)$. In this region, Proposition 7.3 shows that there is a linear isomorphism

$$(7.7)\quad L^p\mathrm{H}^2_{\mathrm{dR}}(S) \simeq \Big\{[\omega] \in L^p\mathrm{H}^2_{\mathrm{dR}}(R) : \int_{\mathbf{R}} \|e^{s\mathrm{ad}\eta^*}[\omega]\|^p_{L^p\mathrm{H}^2(R)} ds < \infty\Big\}.$$

We will use this representation to prove the theorem.

Step 1. $L^p\mathrm{H}^2_{\mathrm{dR}}(S) = \{0\}$ when $p \in (\frac{4}{3}; 2)$.

Let $p \in (\frac{4}{3}, 2)$ and let $(q, 2)$ be the Poincaré dual of $(p, 2)$ relatively to R. One has $q \in (2; 4)$. According to the relation (7.7), it is enough to show that for every non-trivial $[\omega]$, one has $\|e^{s\mathrm{ad}\eta^*}[\omega]\|_{L^p\mathrm{H}^2(R)} \to +\infty$, either when s tends to $+\infty$ or to $-\infty$.

So let $[\omega]$ be a non-trivial class in $L^p\mathrm{H}^2_{\mathrm{dR}}(R)$. By Theorem 3.2(4), it admits a boundary value $T \in \mathcal{D}'^2(N) \cap \operatorname{Ker} d$, so that

$$\|[\omega]\|_{L^p\mathrm{H}^2(R)} = \sup\Big\{T(\theta) : \theta \in \Omega^1_c(N), \big\|[d(\chi \cdot \pi_N^*\theta)]\big\|_{L^q\mathrm{H}^2(R)} \leqslant 1\Big\}.$$

In the group R, right multiplication by $\exp t\xi$ commutes with conjugacy by $\exp s\eta$. Therefore the boundary value of the class $e^{s\mathrm{ad}\eta^*}[\omega]$ is the current $e^{s\mathrm{ad}\eta^*}T$. With a change of variable, one gets

$$\|e^{s\mathrm{ad}\eta^*}[\omega]\|_{L^p\mathrm{H}^2(R)} = \sup\Big\{T(\theta) : \theta \in \Omega^1_c(N), \text{ with } \big\|[d(\chi \cdot \pi_N^* e^{s\mathrm{ad}\eta^*}\theta)]\big\|_{L^q\mathrm{H}^2(R)} \leqslant 1\Big\}.$$

By Lemma 6.5, the current T is vertical, thus it can be written as $T = F dy \wedge \tau + G dx \wedge \tau$, with $F, G \in \mathcal{D}'^0(N)$. If $F \neq 0$ (resp. $G \neq 0$), then for $\theta = f dx \in \Omega^1_c(N)$ (resp. $\theta = g dy$), one has $T(\theta) = F(f\mathrm{vol})$ (resp. $-G(g\mathrm{vol})$). In any case, there exists $\theta \in \Omega^1_c(N)$, of the form $f dx$ or $g dy$, such that $T(\theta) = 1$. The above equality in combination with Lemma 7.5(2), yields that

$$\|e^{s\mathrm{ad}\eta^*}[\omega]\|_{L^p\mathrm{H}^2(R)} \geqslant \big\|[d(\chi \cdot \pi_N^* e^{s\mathrm{ad}\eta^*}\theta)]\big\|^{-1}_{L^q\mathrm{H}^2(R)} \to +\infty,$$

either when s tends to $+\infty$ or to $-\infty$.

Step 2. $L^p\mathrm{H}^2_{\mathrm{dR}}(S) \neq \{0\}$ when $p \in (2; 4)$.

Let $p \in (2; 4)$. We will exhibit some non-trivial element in the right-hand side of (7.7).

Let $\theta = f dx$ and $\Theta = g dy$ be as in Lemma 7.6. Set $\omega := d(\chi \cdot \pi_N^*\theta)$ and $\Omega := d(\chi \cdot \pi_N^*\Theta)$. By Lemmata 7.6 and 7.5(1), their classes $[\omega]$ and $[\Omega]$ are equal and non-zero in $L^p\mathrm{H}^2_{\mathrm{dR}}(R)$.

Since ξ and η commute, one has $e^{s\mathrm{ad}\eta^*}\omega = d(\chi \cdot \pi_N^* e^{s\mathrm{ad}\eta^*}\theta)$. Thus by Lemma 7.5(2), the norm $\|e^{s\mathrm{ad}\eta^*}[\omega]\|_{L^p\mathrm{H}^2(R)}$ tends to 0 exponentially fast when s tends to $-\infty$, and similarly for $\|e^{s\mathrm{ad}\eta^*}[\Omega]\|_{L^p\mathrm{H}^2(S)}$ when s tends to $+\infty$. Since $[\omega] = [\Omega]$, the integral $\int_{\mathbf{R}} \|e^{s\mathrm{ad}\eta^*}[\omega]\|^p_{L^p\mathrm{H}^2(R)} ds$ converges. Thus $[\omega]$ provides a non-trivial element in the right-hand side of (7.7).

Appendix A. The groups $S_\alpha \in \mathcal{S}^{r,n}$: basic properties

Let $r, n \in \mathbf{N}$. We consider here the solvable Lie groups of the form $S_\alpha = \mathbf{R}^r \ltimes_\alpha \mathbf{R}^n$, where $\alpha : \mathbf{R}^r \to \{$diagonal automorphisms of $\mathbf{R}^n\}$ is a Lie group morphism. We denote by $\varpi_i \in (\mathbf{R}^r)^*$ $(i = 1, \dots, n)$ the *weights* associated to α, *i.e.* the linear forms so that $\alpha = e^{\mathrm{diag}(\varpi_1, \dots, \varpi_n)}$.

Some groups S_α can be written with several couples of exponents r, n – *e.g.* when they are abelian. In the sequel we will always assume that the dimension n of the second factor is minimal. This assumption is equivalent to require every weight ϖ_i to be non-zero; and it forces the rank of S_α to be equal to r.

Let denote by $\mathcal{S}^{r,n}$ the set of the groups S_α of exponents r, n (with the above convention).

We remark that the groups S_α appear as special cases of the so-called *abelian-by-abelian* solvable Lie groups. The latter ones are considered by Peng in [Pen11a, Pen11b]. For those which are in addition unimodular, she establishes several quasi-isometric rigidity results.

This Section is devoted to the following proposition, which establishes some of the basic properties of S_α. The first two are elementary. The third one follows from Azencott-Wilson's classification of Lie groups of non-positive curvature [AW76] (see also [Heb93] for a detailed account of their geometric properties).

Proposition A.1. *Let $S_\alpha \in \mathcal{S}^{r,n}$. Then:*

(1) *S_α admits* **no** *non-trivial abelian direct factor if, and only if, the weights $\varpi_1, \dots, \varpi_n$ generates the vector space $(\mathbf{R}^r)^*$.*
(2) *Suppose that S_α has no abelian direct factor. Then S_α is reductible – i.e. splits as a direct product of non-trivial closed subgroups – if and only if there exists a non-trivial partition $I_1 \sqcup I_2 = \{1, \dots, n\}$ such that*

$$\mathrm{Span}\{\varpi_i : i \in I_1\} \cap \mathrm{Span}\{\varpi_i : i \in I_2\} = \{0\}.$$

(3) *S_α admits a left-invariant Riemannian metric of non-positive curvature if, and only if,* 0 *does* **not** *belong to the convex hull of $\{\varpi_1, \dots, \varpi_n\}$ in $(\mathbf{R}^r)^*$. In this case the factor $\mathbf{R}^r$ is a totally geodesic Euclidean subspace of maximal dimension in S_α.*

(4) *There exists $\xi \in \mathbf{R}^r$, such that the associated subgroup $(\mathbf{R}\xi) \ltimes \mathbf{R}^n$ is quasi-isometric to the real hyperbolic space $\mathbb{H}^{n+1}_{\mathbf{R}}$ if, and only if, the weights $\varpi_1, \dots, \varpi_n$ are contained in an affine subspace of $(\mathbf{R}^r)^*$, disjoint from* 0*.*

It follows from Item (2) above, that the groups S_α which are irreducible, of higher rank, and of smallest dimension, belong to the family $\mathcal{S}^{2,3}$.

Proof. We start with some notations and preliminaries. Set $I := \{1, \dots, n\}$ and denote by $(e_i)_{i \in I}$ the canonical basis of $\mathbf{R}^n$. The multiplication law of S_α is

$$(u, x) \cdot (v, y) = \big(u + v, x + \alpha(u)y\big).$$

Let us denote the Lie algebra of S_α by $\mathfrak{s}_\alpha$, and set $\Pi := \mathrm{diag}(\varpi_1, \dots, \varpi_n) \in (\mathbf{R}^r)^* \otimes \mathrm{Diag}(\mathbf{R}^n)$, so that $\alpha = e^\Pi$. A standard computation gives the following expression for the Lie bracket in $\mathfrak{s}_\alpha$:

$$[(U, X), (V, Y)] = \big(0, \Pi(U)Y - \Pi(V)X\big). \tag{A.2}$$

Since every ϖ_i is non-zero, one checks that the center of $\mathfrak{s}_\alpha$ is

$$\mathfrak{z}(\mathfrak{s}_\alpha) = \{(U, 0) \in \mathfrak{s}_\alpha \mid \varpi_i(U) = 0 \text{ for all } i\},$$

and its derived subalgebra is $[\mathfrak{s}_\alpha, \mathfrak{s}_\alpha] = \{0_{\mathbf{R}^r}\} \times \mathbf{R}^n \simeq \mathbf{R}^n$.

Proof of (1) : Suppose that the weights $\varpi_1, \dots, \varpi_n$ do not generate $(\mathbf{R}^r)^*$. Then $\mathrm{Ker}\,\Pi = \cap_{i \in I} \mathrm{Ker}\,\varpi_i$ is non-trivial. Write $\mathbf{R}^r = \mathrm{Ker}\,\Pi \oplus E$, set $\mathfrak{h} := \mathrm{Ker}\,\Pi \times \{0_{\mathbf{R}^n}\}$ and $\mathfrak{k} := E \ltimes \mathbf{R}^n$. Then $\mathfrak{h}$ and $\mathfrak{k}$ are supplementary subalgebras in $\mathfrak{s}_\alpha$; moreover $\mathfrak{h}$ is central, thus S_α admits a direct abelian factor.

Conversely, suppose that S_α admits a direct abelian factor. Then the center of $\mathfrak{s}_\alpha$ is non-trivial. By the above description of $\mathfrak{z}(\mathfrak{s}_\alpha)$, there exists a non-zero $U \in \mathbf{R}^r$ that belongs to the kernel of every ϖ_i. Therefore the ϖ_i's do not generate $(\mathbf{R}^r)^*$.

Proof of (2) : Let $I = I_1 \sqcup I_2$ be a non-trivial partition as in the statement. One has

$$\bigcap_{i \in I_1} \mathrm{Ker}(\varpi_i) + \bigcap_{i \in I_2} \mathrm{Ker}(\varpi_i) = \mathbf{R}^r.$$

By assumption, every ϖ_i is non-zero. Therefore $\mathbf{R}^r$ admits proper supplementary subspaces E_1, E_2 such that for $j = 1, 2$:

$$\{0\} \neq E_j \subset \bigcap_{i \in I \setminus I_j} \mathrm{Ker}(\varpi_i).$$

Set $\mathfrak{h}_j := E_j \ltimes \mathrm{Span}\{e_i : i \in I_j\}$. Then $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are supplementary subalgebras in $\mathfrak{s}_\alpha$. They satisfy $[\mathfrak{h}_1, \mathfrak{h}_2] = 0$; indeed, for every $U \in E_1$ and $Y \in \mathrm{Span}\{e_i \ ; \ i \in I_2\}$, one has

$$\Pi(U)Y = \sum_{i \in I_2} Y_i \varpi_i(U) e_i = 0,$$

and similarly for $V \in E_2$ and $X \in \mathrm{Span}\{e_i : i \in I_1\}$. Thus S_α is reducible.

Conversely, suppose that $\mathfrak{s}_\alpha$ admits supplementary (non-abelian) subalgebras $\mathfrak{h}_1$, $\mathfrak{h}_2$ such that $[\mathfrak{h}_1, \mathfrak{h}_2] = 0$. One has

$$[\mathfrak{h}_1, \mathfrak{h}_1] \cap [\mathfrak{h}_2, \mathfrak{h}_2] \subset \mathfrak{h}_1 \cap \mathfrak{h}_2 = \{0\}$$
$$\text{and} \quad [\mathfrak{s}_\alpha, \mathfrak{s}_\alpha] = [\mathfrak{h}_1 + \mathfrak{h}_2, \mathfrak{h}_1 + \mathfrak{h}_2] = [\mathfrak{h}_1, \mathfrak{h}_1] + [\mathfrak{h}_2, \mathfrak{h}_2].$$

Therefore $[\mathfrak{s}_\alpha, \mathfrak{s}_\alpha] = [\mathfrak{h}_1, \mathfrak{h}_1] \oplus [\mathfrak{h}_2, \mathfrak{h}_2]$, which in turn implies that

$$\mathfrak{h}_j \cap [\mathfrak{s}_\alpha, \mathfrak{s}_\alpha] = [\mathfrak{h}_j, \mathfrak{h}_j], \tag{A.3}$$

for $j = 1, 2$. Let $\sigma : \mathbf{R}^r \times \mathbf{R}^n \to \mathbf{R}^r$ be the projection map on the first factor, and set $E_j := \sigma(\mathfrak{h}_j)$ for $j = 1, 2$. We claim that

$$\mathfrak{h}_j = E_j \ltimes [\mathfrak{h}_j, \mathfrak{h}_j]. \tag{A.4}$$

To see this, let $(U, X) \in \mathfrak{h}_j$. One has $U = \sigma(U, X)$, thus $U \in E_j$. Since $\mathfrak{h}_j$ is an ideal, relations (A.2) and (A.3) imply for every $V \in \mathbf{R}^r$:

$$\big(0, -\Pi(V)X\big) = [(U, X), (V, 0)] \in \mathfrak{h}_j \cap [\mathfrak{s}_\alpha, \mathfrak{s}_\alpha] = [\mathfrak{h}_j, \mathfrak{h}_j].$$

By choosing V such that $\Pi(V)$ is invertible, one has $X \in \Pi(V)^{-1}[\mathfrak{h}_j, \mathfrak{h}_j]$. On the other hand $\mathfrak{h}_j$ is an ideal, thus so is $[\mathfrak{h}_j, \mathfrak{h}_j]$. In combination with (A.2), this implies that $\Pi(V)[\mathfrak{h}_j, \mathfrak{h}_j] = [\mathfrak{h}_j, \mathfrak{h}_j]$. Therefore $X \in [\mathfrak{h}_j, \mathfrak{h}_j]$, and claim (A.4) follows now easily.

The subalgebras $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are supplementary subspaces in $\mathfrak{s}_\alpha$, thus so are E_1 and E_2 in $\mathbf{R}^r$ – thanks to (A.4). Since $\mathfrak{h}_j$ is non-abelian, E_j is non-zero. Set $I_1 := \{i \in I : \varpi_i\big|_{E_2} = 0\}$ and $I_2 := \{i \in I : \varpi_i\big|_{E_1} = 0\}$. Observe that $I_1 \cap I_2 = \varnothing$ since every ϖ_i is non-zero. We claim that I_1 and I_2 form a non-trivial partition of I. The proof of Item (2) follows then easily. Let $(U, X) \in \mathfrak{h}_1$ and $(V, Y) \in \mathfrak{h}_2$. Since $[\mathfrak{h}_1, \mathfrak{h}_2] = 0$, relation (A.2) implies that $\Pi(U)Y = \Pi(V)X$. By (A.4), one has $X \in [\mathfrak{h}_1, \mathfrak{h}_1]$ and $Y \in [\mathfrak{h}_2, \mathfrak{h}_2]$. Moreover, $[\mathfrak{h}_j, \mathfrak{h}_j]$ is an ideal, so it is invariant by $\Pi(\mathbf{R}^r)$. Therefore $\Pi(U)Y$ and $\Pi(V)X$ belong to $[\mathfrak{h}_j, \mathfrak{h}_j]$ for $j = 1, 2$.

Since the intersection of the latter subspaces is $\{0\}$, the vectors $\Pi(U)Y$ and $\Pi(V)X$ are 0. In other words, we have shown that

$$[\mathfrak{h}_1, \mathfrak{h}_1] \subset \bigcap_{V \in E_2} \operatorname{Ker} \Pi(V) \quad \text{and} \quad [\mathfrak{h}_2, \mathfrak{h}_2] \subset \bigcap_{U \in E_1} \operatorname{Ker} \Pi(U).$$

An easy computation shows that $\bigcap_{V \in E_2} \operatorname{Ker} \Pi(V) = \operatorname{Span}\{e_i : i \in I_1\}$, and similarly $\bigcap_{U \in E_1} \operatorname{Ker} \Pi(U) = \operatorname{Span}\{e_i : i \in I_2\}$. Since the subspaces $[\mathfrak{h}_1, \mathfrak{h}_1]$ and $[\mathfrak{h}_2, \mathfrak{h}_2]$ generate $\mathbf{R}^n$, we finally obtain that $I_1 \cup I_2 = I$.

Proof of (3) : According to [AW76], a connected Lie group S admits a left-invariant non-positively curved Riemannian metric if, and only if, its Lie algebra $\mathfrak{s}$ is an *NC algebra*. This means that $\mathfrak{s}$ enjoys the following properties:

(i) $\mathfrak{n} := [\mathfrak{s}, \mathfrak{s}]$ is a nilpotent ideal that is complemented in $\mathfrak{s}$ by an abelian subalgebra $\mathfrak{a}$.
(ii) There exists an element $\xi \in \mathfrak{a}$, such that all the eigenvalues of $\operatorname{ad}\xi\Big|_{\mathfrak{n}}$ have negative real parts.
(iii) The action of $\mathfrak{a}$ on $\mathfrak{n}$ satisfies 3 additional conditions, which are automatically fulfilled when $\mathfrak{n}$ is abelian and the $\mathfrak{a}$-action is semisimple with real eigenvalues.

We refer to [AW76, Definition 6.2] for the precise definition of NC algebra, and to the paragraph right after it for a discussion of the special cases.

Clearly the Lie algebra $\mathfrak{s}_\alpha$ satisfies Items (i) and (iii). It satisfies (ii) if and only if there exists a $U \in \mathbf{R}^r$ so that $\varpi_i(U) \leqslant -1$ for all $i \in I$. Let $v_i \in \mathbf{R}^r$ be such that $\varpi_i = \langle \cdot, v_i \rangle$, where $\langle \cdot, \cdot \rangle$ is a scalar product. One has $\varpi_i(U) \leqslant -1$ for every $i \in I$ if, and only if, every v_i belongs to the subset defined by the inequality $\langle U, \cdot \rangle \leqslant -1$; *i.e.* to the affine half-space of $\mathbf{R}^r$, disjoint from 0, and delimited by the hyperplane orthogonal to U passing through $\frac{-U}{\|U\|^2}$. The proof of Item (3) is now complete.

Proof of (4) : Let $\xi \in \mathbf{R}^r$ be a non-zero vector. The associated subgroup $(\mathbf{R}\xi) \ltimes \mathbf{R}^n$ is isomorphic to $\mathbf{R} \ltimes_\delta \mathbf{R}^n$, where $\delta \in \operatorname{Der}(\mathbf{R}^n)$ is the derivation $\operatorname{diag}(\varpi_1(\xi), \ldots, \varpi_n(\xi))$. Such a group is quasi-isometric to $\mathbb{H}^{n+1}_{\mathbf{R}}$ if and only if δ is a multiple of $-I_n$ [Pan07, CT11]. Therefore the existence of a ξ as in the statement, is equivalent to the existence of a $U \in \mathbf{R}^r$ such that $\varpi_i(U) = -1$ for every $i \in I$. Consider again the vectors $v_i \in \mathbf{R}^r$ so that $\varpi_i = \langle \cdot, v_i \rangle$, where $\langle \cdot, \cdot \rangle$ is a scalar product. One has $\varpi_i(U) = -1$ for every $i \in I$ if, and only if, every v_i belongs to the subset defined by the equation $\langle U, \cdot \rangle = -1$, *i.e.* to the affine hyperplane which is orthogonal to U and which passes through $\frac{-U}{\|U\|^2}$. □

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