ALMOST RINGS AND PERFECTOID SPACES

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CONTENTS

0. Introduction

Both the focus of this monograph and its subject matter have evolved considerably in the last few years. On the one hand, the insistence on making the text self-contained (aside from a reduced canon of basic references, which should ideally contain only EGA and some parts of Bourbaki's *Éléments*), has resulted in a rather weighty mass of material of independent interest, that is applied to, but is completely separate from almost ring theory, and whose relationship to p-adic Hodge theory is thus even more indirect. Rather than stemming from a well-thought-out plan, this part is the outcome of a haphazard process, lumbering between alternating phases of accretion and consolidation, with new topics piled up as dictated by need, or occasionally by whim, when we just branched out from the main flow to pursue a certain line of thought to its logical destination. Nevertheless, a few themes have spontaneously emerged, around which the originally amorphous magma has been able to settle, to the point where by now a distinct shape is finally discernible, and it is perhaps time to pause and take stock of its broad outlines.

Now then, we may distinguish :

• First of all, a rather thorough exposition of the foundations of logarithmic algebraic geometry, comprising chapters 6, 12 and 13. Inevitably, our treatment owes a lot to the works of Kato and his school : our contribution is foremost that of gathering and tidying up the subject, which until now was scattered in a disparate number of research articles, many of them still unpublished and even unfinished. A closer scrutiny would also reveal a few technical innovations that we hope will become standard issue of the working algebraic log-geometer : we may mention the systematic use of pointed monoids and pointed modules, the projective fan associated with a graded monoid, or a definition of α-flatness for log structures which refines and generalizes an older notion of “toric flatness”. Furthermore, we took the occasion to repair a few small (and not so small) mistakes and inaccuracies that we detected in the literature.

• Two other chapters are dedicated to local cohomology and Grothendieck’s duality theory. Early on, the emphasis here was on generalizations : especially, we were interested in removing from the theory the pervasive noetherian assumptions, to pave the way for our recasting of Faltings’s almost purity in the framework of valuation theory. Applications of local cohomology to non-archimedean analytic geometry furnished another influential motivation, though one that has remained, so far, hidden from view. More recently, the noetherian aspects have also become relevant to our project, and this latest release contains a detailed account of the most important properties of noetherian rings endowed with a dualizing complex. The latter, in turn, could be dealt with satisfactorily only after a thorough revisitation of the general theory of the dualizing complex, so that our chapters 10 and 11 can also be regarded as complementary to Conrad’s book [50] (dedicated to the trace morphism and the deeper aspects of duality) : totaling our respective efforts, it should eventually become possible to bypass entirely Hartshorne’s notes [87] which, as is well known, are wanting in many ways.

• Chapters 7 and 9 present (for the time being, anyway) a looser structure : a miscellanea of self-contained units devoted to more or less independent topics. However, there is at least one thread running through several sections, and whose stretch can be traced all the way back to the earliest beginnings of almost ring theory; it connects sections 7.4 and 7.5 – on simplicial homotopy theory – to a section 7.10 dedicated to homotopical algebra, then on to sections 9.6 and 9.7, which make extensive use of the cotangent complex to derive important characterizations of regular and excellent rings, including an up-to-date presentation of classical results due to André, extracted from his monograph [3], and from his paper [4] on localization of formal smoothness. This homotopical algebraic thread resurfaces again in section 14.1, but there we are already squarely into almost ring theory proper.

On the other hand, two recent notable developments are compelling a revision of our understanding of almost ring theory itself, and of its situation within commutative algebra and algebraic geometry at large :

• The first is Scholze’s PhD thesis [149] on *perfectoid spaces*, that contains both a maximal generalization of the almost purity theorem, and a major simplification of its proof, based on his “tilting” technique (and completely different from Faltings’s). However, the range of Scholze’s theory transcends the domain of p-adic Hodge theory (which was not even his original motivation) : to drive the point home, his thesis concludes with a clever application to the long standing weight monodromy conjecture, thus affording the unusual spectacle of a tool which was fashioned out of purely p-adic concerns, and ends up playing a crucial role in the solution of a purely ℓ-adic problem.

• The second spectacular development is Yves André's proof of the direct summand conjecture ([5]); the latter is a deceptively simple assertion that has been a central problem in commutative algebra for the last thirty years : it asserts that every finite injective ring homomorphism $f : A \to B$ from a regular local ring A, admits a A-linear splitting. The relevance of almost purity to this question was first surmised by Paul Roberts in 2001 (after a talk by the second author at the University of Utah), and has been widely advertised by him ever since. André's solution uses perfectoid techniques, and builds on earlier work by Bhargav Bhatt, who in [22] proved the conjecture in the case where A is essentially smooth over a mixed characteristic discrete valuation ring and $f \otimes_{\mathbb{Z}} \mathbb{Q}$ is étale outside a relative normal crossings divisor of $\mathrm{Spec}\, A$. Moreover, Bhatt has subsequently simplified some of André's arguments and shown how the same method yields a more general "derived version" of the conjecture, for proper schemes over any regular ring : see [23].

We see then, that almost ring theory has emancipated itself from its former ancillary role in the exclusive service of p-adic Hodge theory, and is now elbowing out a niche in the wider ecosystem of algebraic geometry.

The present release completes the project announced in the introduction of the 6th release :

• First we introduce a class of topological rings that generalize the perfectoid rings of [149]; it is very easy to say what a (generalized) perfectoid $\mathbb{F}_p$-algebra is : namely, it is just a perfect and complete topological $\mathbb{F}_p$-algebra whose topology is linear, defined by an ideal of finite type. The general definition is somewhat more involved, but we prove the following characterization. For any perfectoid $\mathbb{F}_p$-algebra E, we consider the ring of Witt vectors $W(E)$, and we endow it with a natural topology, induced from that of E; then every perfectoid ring is a topological quotient of the form $A := W(E)/\underline{a}W(E)$, for such a suitable E, and where $\underline{a} \in W(E)$ is what we call a *distinguished element* : see definition 16.1.6. Moreover, just as in Scholze's work, the perfectoid $\mathbb{F}_p$-algebra E can be recovered from A via a *tilting functor* that establishes, more precisely, an equivalence between the category of all perfectoid rings and that of pairs $(E, \mathscr{I})$ consisting of a perfectoid $\mathbb{F}_p$-algebra E and a principal ideal $\mathscr{I} \subset W(E)$ generated by a distinguished element (as it is well known, this construction is rooted in Fontaine and Winterberger's theory of the *field of norms*). The distinguished ideal $\mathscr{I}$ represents an extra parameter that remains hidden in Scholze's original approach : the reason is that he fixes from the start a base perfectoid field K, thereby implicitly fixing as well a distinguished element $\underline{a}$ in the ring of Witt vectors of the tilt of K, and then every perfectoid ring in his work is supposed to be a K-algebra, which – from our viewpoint – amounts to restricting to perfectoid rings whose associated distinguished ideal is generated by $\underline{a}$. Having thus removed the parameter $\mathscr{I}$, he can then also do away with Witt vectors altogether, and the inverse to the tilting construction is obtained in [149] via a more abstract deformation theoretic argument. This route is precluded to us, so we rely instead on direct and rather concrete Witt vectors calculations. A similar strategy has been proposed in [114], and our viewpoint can indeed be described fairly as an interpolation of those of Scholze and Kedlaya-Liu, though we only studied [149] in detail.

• The first three sections of chapter 16 are devoted to exploring this new class of perfectoid rings and its manifold remarkable features. The rest of the chapter then merges the theory of perfectoid rings with Huber's adic spaces, to forge the perfectoid spaces that are the main tool for our proof of almost purity, whose most general form is given by theorem 16.8.44 and applies to *formally perfectoid rings*, *i.e.* to topological rings whose completion is perfectoid. The proof proceeds via several preliminary reductions : first, to the case of a *perfectoid quasi-affinoid ring*, covered by theorem 16.8.32, then – by exploiting the local geometry of adic spaces – to the case of a perfectoid valuation ring, which was treated already in our monograph [75]. What enables here this localization argument is a basic feature of the étale topology of arbitrary adic spaces : the fibred category of finite étale coverings of the affinoid subsets of an adic space is a stack. The latter result is in turn a special case of our theorem 15.7.6.

• We also include a detailed treatment of the foundations of the theory of adic spaces, that essentially follows [98], but contains some modest improvement : notably, the systematic use of *analytically noetherian rings* (borrowed from [71]) allows us to unify the two classes of topological rings that Huber dealt with separately in his work (the strongly noetherian rings and the f-adic rings with a noetherian ring of definition). We also point out a henselian variant of the structure sheaf that is available on the adic spectrum of any f-adic ring, with no restriction whatsoever.

• The last chapter proposes a few applications : in section 17.1 we introduce a class of *model algebras* over any rank one valuation ring K^+ of mixed characteristic $(0, p)$, and we show that when K^+ is deeply ramified, such algebras are formally perfectoid rings for their p-adic topology; hence the theory of chapter 16 immediately yields an almost purity theorem for model algebras. Likewise, section 17.2 proves an almost purity theorem for certain very ramified towers of log-regular rings; again, after some preliminary reductions, the proof amounts to the observation that the inductive limits of such towers are formally perfectoid for their p-adic topology. These instances of almost purity were already contained in a previous draft of our work (Release 6), where they were proven by an extension of Faltings's method, that relied on deep results from logarithmic algebraic geometry, and also entailed the construction of certain *normalized lengths* for torsion modules over model algebras, and respectively over the rings occurring in section 17.2. Neither of these two ingredients intervenes any longer in the new proofs; however, we have found worthwhile to explain how model algebras arise from suitable very ramified towers of log-smooth K^+-algebras, and we have also retained the construction of normalized lengths for model algebras and for limits of towers log-regular rings, since they are sufficiently interesting in their own right, and might be useful for other applications (normalized lengths for torsion K^+-modules are exploited in [150]).

Section 17.3 contains our account of André's work on the direct summand conjecture, which we generalize to the case of a finite injective ring homomorphism $A \to B$ where A is *log-regular* : see theorem 17.3.12, whose proof relies on a refinement of André's *perfectoid Abhyankar's lemma*, which is the main result of section 16.9. In a future release we shall present a corresponding logarithmic generalization of Bhatt's derived version of the direct summand conjecture.

Section 17.4 considers a finite injective map $f : A \to B$ of noetherian rings, where A is again log-regular; if f is étale, $B \otimes_A B$ has a canonical *diagonal idempotent* e_f, so called because its support is the open and closed diagonal in $\mathrm{Spec}\, B \times_{\mathrm{Spec}\, A} \mathrm{Spec}\, B$. When f is only generically étale, e_f is only well defined on some localization $B \otimes_A B[a^{-1}]$ (for some $a \in A$ whose zero locus is nowhere dense in $\mathrm{Spec}\, A$), and the problem we address, is to exhibit an explicit $d \in B \otimes_A B$ such that de_f is *integral*, *i.e.* lies in the image of $B \otimes_A B$. This question has an easy answer when B is a projective A-module : indeed, in this case we have a well defined trace map $B \to A$, whence a *different ideal* $\mathscr{D} \subset B$, and a standard calculation shows that $(b \otimes 1) \cdot e_f$ is integral for every $b \in \mathscr{D}$. If B is not projective, the question seems to be much more difficult : our solution makes an essential use of perfectoid techniques, and especially the perfectoid Abhyankar's lemma of section 16.9 : see theorem 17.4.19 and remark 17.4.44.

Section 17.5 proves the existence and weak functoriality of *big Cohen-Macaulay algebras*, over any local noetherian ring A. The case where A is equicharacteristic was already known before André's work, and is due to Hoschster and Huneke, but our construction is different from theirs, and yields actual functors on suitable subcategories : see remark 17.5.36(ii) and theorem 17.5.45. For the mixed characteristic case, we rely, as in [7], on the perfectoid Abhyankar's lemma, but again, our construction differs from the one given in the latter work; also here we get good functorial properties, but the proofs are much harder and longer than for equicharacteristic rings. A crucial ingredient for both cases, is a simple construction, introduced in (17.5.2) – and inspired by some ultrafilter techniques – whose remarkable property is to transform *almost Cohen-Macaulay* (resp. *almost faithfully flat*) algebras into actual Cohen-Macaulay (resp. faithfully flat) algebras. Lastly, weak functoriality is given by theorem 17.5.90.

The final section 17.6 explains how to apply the existence and weak functoriality of Cohen-Macaulay algebras, to deduce the monomial conjecture, first in its original form due to Hochster (theorem 17.6.1), and then in its more recent strong form (theorem 17.6.6).

1. Basic category theory

The purpose of this chapter is to fix some notation that shall stand throughout this work, and to collect, for ease of reference, a few well known generalities on categories and functors that are frequently used. Our main reference on general nonsense is the treatise [28], and another good reference is the more recent [110].

Sooner or later, any honest discussion of categories and topoi gets tangled up with some foundational issues revolving around the manipulation of large sets. For this reason, to be able to move on solid ground, it is essential to select from the outset a definite set-theoretical framework (among the several currently available), and stick to it unwaveringly.

Thus, *throughout this work we will accept the so-called Zermelo-Fraenkel system of axioms for set theory*. (In this version of set theory, everything is a set, and there is no primitive notion of class, in contrast to other axiomatisations.)

Additionally, following [8, Exp.I, §0], we shall assume that, for every set S, there exists a *universe* V such that $S \in \mathsf{V}$. (For the notion of universe, the reader may also see [28, §1.1].)

Throughout this chapter, we fix some universe U such that $\mathbb{N} \in \mathsf{U}$ (where $\mathbb{N}$ is the set of natural numbers; the latter condition is required, in order to be able to perform some standard set-theoretical operations without leaving U). A set S is U-*small* (resp. *essentially* U-*small*), if $S \in \mathsf{U}$ (resp. if S has the cardinality of a U-small set). If the context is not ambiguous, we shall just write small, instead of U-small.

1.1. **Categories, functors and natural transformations.** A *category* $\mathscr{C}$ is the datum of a set $\mathrm{Ob}(\mathscr{C})$ of *objects* and, for every $A, B \in \mathrm{Ob}(\mathscr{C})$, a set of *morphisms* from A to B, denoted :

$$\mathrm{Hom}_{\mathscr{C}}(A, B)$$

and as usual we write $f : A \to B$ to signify $f \in \mathrm{Hom}_{\mathscr{C}}(A, B)$. Furthermore, we set

$$\mathrm{Morph}(\mathscr{C}) := \{(A, B, f) \mid A, B \in \mathrm{Ob}(\mathscr{C}),\ f \in \mathrm{Hom}_{\mathscr{C}}(A, B)\}.$$

For any $\underline{f} := (A, B, f) \in \mathrm{Morph}(\mathscr{C})$, the object A is called the *source* of $\underline{f}$, and B is the *target* of $\underline{f}$. We also often use the notation

$$\mathrm{End}_{\mathscr{C}}(A) := \mathrm{Hom}_{\mathscr{C}}(A, A)$$

and the elements of $\mathrm{End}_{\mathscr{C}}(A)$ are called the *endomorphisms* of A in $\mathscr{C}$. We say that a pair of elements $(\underline{f}, \underline{g})$ of $\mathrm{Morph}(\mathscr{C})$ is *composable* if the target of $\underline{f}$ equals the source of $\underline{g}$. Moreover, for every $A, B, C \in \mathrm{Ob}(\mathscr{C})$ we have a *composition law*

$$\mathrm{Hom}_{\mathscr{C}}(A, B) \times \mathrm{Hom}_{\mathscr{C}}(B, C) \to \mathrm{Hom}_{\mathscr{C}}(A, C) \quad : \quad (f, g) \mapsto g \circ f$$

fulfilling the following two standard axioms :

- For every $A \in \mathrm{Ob}(\mathscr{C})$ there exists an *identity endomorphism* $\mathbf{1}_A$ of A, such that

$\mathbf{1}_A \circ f = f \quad g \circ \mathbf{1}_A = g$ for every $B, C \in \mathrm{Ob}(\mathscr{C})$ and every $f : B \to A$ and $g : A \to C$.

- The composition law is *associative*, *i.e.* we have
$$(h \circ g) \circ f = h \circ (g \circ f)$$
for every $A, B, C, D \in \mathrm{Ob}(\mathscr{C})$ and every $f : A \to B$, $g : B \to C$ and $h : C \to D$.

Clearly, it follows that $(\mathrm{End}_{\mathscr{C}}(A), \circ, \mathbf{1}_A)$ is a monoid, and we get a group :
$$\mathrm{Aut}_{\mathscr{C}}(A) \subset \mathrm{End}_{\mathscr{C}}(A)$$
of invertible endomorphisms, *i.e.* the *automorphisms* of the object A.

1.1.1. We say that the category $\mathscr{C}$ is U-*small* (or just *small*), if both $\mathrm{Ob}(\mathscr{C})$ and $\mathrm{Morph}(\mathscr{C})$ are small sets. We say that $\mathscr{C}$ *has small* Hom-*sets* if $\mathrm{Hom}_{\mathscr{C}}(A, B) \in \mathsf{U}$ for every $A, B \in \mathrm{Ob}(\mathscr{C})$.

A *subcategory* of $\mathscr{C}$ is a category $\mathscr{B}$ with $\mathrm{Ob}(\mathscr{B}) \subset \mathrm{Ob}(\mathscr{C})$ and $\mathrm{Morph}(\mathscr{B}) \subset \mathrm{Morph}(\mathscr{C})$.

The *opposite category* $\mathscr{C}^o$ is the category with $\mathrm{Ob}(\mathscr{C}^o) = \mathrm{Ob}(\mathscr{C})$, and such that :
$$\mathrm{Hom}_{\mathscr{C}^o}(A, B) := \mathrm{Hom}_{\mathscr{C}}(B, A) \qquad \text{for every } A, B \in \mathrm{Ob}(\mathscr{C})$$
(with composition law induced by that of $\mathscr{C}$, in the obvious way). Given $A \in \mathrm{Ob}(\mathscr{C})$, sometimes we denote by A^o the same object, viewed as an element of $\mathrm{Ob}(\mathscr{C}^o)$; likewise, given a morphism $f : A \to B$ in $\mathscr{C}$, we write f^o for the corresponding morphism $B^o \to A^o$ in $\mathscr{C}^o$.

1.1.2. A morphism $f : A \to B$ in $\mathscr{C}$ is said to be a *monomorphism* if the induced map
$$\mathrm{Hom}_{\mathscr{C}}(X, f) : \mathrm{Hom}_{\mathscr{C}}(X, A) \to \mathrm{Hom}_{\mathscr{C}}(X, B) \qquad g \mapsto f \circ g$$
is injective, for every $X \in \mathrm{Ob}(\mathscr{C})$. Dually, we say that f is an *epimorphism* if f^o is a monomorphism in $\mathscr{C}^o$. Also, f is an *isomorphism* if there exists a morphism $g : B \to A$ such that $g \circ f = \mathbf{1}_A$ and $f \circ g = \mathbf{1}_B$. Obviously, an isomorphism is both a monomorphism and an epimorphism. The converse does not necessarily hold, in an arbitrary category.

Two monomorphisms $f : A \to B$ and $f' : A' \to B$ are *equivalent*, if there exists an isomorphism $h : A \to A'$ such that $f = f' \circ h$. A *subobject* of B is defined as an equivalence class of monomorphisms $A \to B$. Dually, a *quotient* of B is a subobject of B^o in $\mathscr{C}^o$.

One says that $\mathscr{C}$ is *well-powered* if, for every $A \in \mathrm{Ob}(\mathscr{C})$, the set :
$$\mathrm{Sub}(A)$$
of all subobjects of A is essentially small. Dually, $\mathscr{C}$ is *co-well-powered*, if $\mathscr{C}^o$ is well-powered.

1.1.3. Let $\mathscr{A}$ and $\mathscr{B}$ be any two categories; a *functor* $F : \mathscr{A} \to \mathscr{B}$ is a pair of maps
$$\mathrm{Ob}(\mathscr{A}) \to \mathrm{Ob}(\mathscr{B}) \qquad \mathrm{Morph}(\mathscr{A}) \to \mathrm{Morph}(\mathscr{B})$$
both denoted also by F, such that

- F assigns to any morphism $f : A \to A'$ in $\mathscr{A}$, a morphism $Ff : FA \to FA'$ in $\mathscr{B}$
- $F\mathbf{1}_A = \mathbf{1}_{FA}$ for every $A \in \mathrm{Ob}(\mathscr{A})$
- $F(g \circ f) = Fg \circ Ff$ for every $A, A', A'' \in \mathrm{Ob}(\mathscr{A})$ and every pair of morphisms $f : A \to A'$, $g : A' \to A''$ in $\mathscr{A}$.

If $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{C}$ are any two functors, we get a composition
$$G \circ F : \mathscr{A} \to \mathscr{C}$$
which is the functor whose maps on objects and morphisms are the compositions of the respective maps for F and G. We denote by
$$\mathrm{Fun}(\mathscr{A}, \mathscr{B})$$
the set of all functors $\mathscr{A} \to \mathscr{B}$. Moreover, any such F induces a functor $F^o : \mathscr{A}^o \to \mathscr{B}^o$ with $F^oA^o := (FA)^o$ and $F^of^o := (Ff)^o$ for every $A \in \mathrm{Ob}(\mathscr{A})$ and every $f \in \mathrm{Morph}(\mathscr{A})$.

Definition 1.1.4. Let $F : \mathscr{A} \to \mathscr{B}$ be a functor.

(i) We say that F is *faithful* (resp. *full*, resp. *fully faithful*), if for every $A, A' \in \mathrm{Ob}(\mathscr{A})$ it induces injective (resp. surjective, resp. bijective) maps :

$$\mathrm{Hom}_{\mathscr{A}}(A, A') \to \mathrm{Hom}_{\mathscr{B}}(FA, FA') \quad : \quad f \mapsto Ff.$$

(ii) We say that F *reflects monomorphisms* (resp. *reflects epimorphisms*, resp. *is conservative*) if the following holds. For every morphism $f : A \to A'$ in $\mathscr{A}$, if the morphism Ff of $\mathscr{B}$ is a monomorphism (resp. epimorphism, resp. isomorphism), then the same holds for f.

(iii) If $\mathscr{A}$ is a subcategory of $\mathscr{B}$, and F is the natural inclusion functor, then F is obviously faithful, and we say that $\mathscr{A}$ is a *full subcategory* of $\mathscr{B}$, if F is fully faithful.

(iv) The *essential image of* F is the full subcategory of $\mathscr{B}$ whose objects are the objects of $\mathscr{B}$ that are isomorphic to an object of the form FA, for some $A \in \mathrm{Ob}(\mathscr{A})$. We say that F is *essentially surjective* if its essential image is $\mathscr{B}$.

(v) We say that F is an *equivalence*, if it is fully faithful and essentially surjective.

Remark 1.1.5. For later use, it is convenient to introduce the notion of n*-faithful functor*, for all integers $n \leq 2$. Namely : if $n < 0$, every functor is n-faithful; a functor $F : \mathscr{A} \to \mathscr{B}$ (between any two categories $\mathscr{A}$ and $\mathscr{B}$) is 0-faithful, if it is faithful; F is 1-faithful, if it is fully faithful; finally, we say that F is 2-faithful, if it is an equivalence.

Example 1.1.6. (i) The collection of all small categories, together with the functors between them, forms a category

$$\mathsf{U}\text{-}\mathbf{Cat}.$$

Unless we have to deal with more than one universe, we shall usually omit the prefix U, and write just $\mathbf{Cat}$. It is easily seen that $\mathbf{Cat}$ is a category with small Hom-sets.

(ii) The category of all small sets shall be denoted U-$\mathbf{Set}$ or just $\mathbf{Set}$, if there is no need to emphasize the chosen universe. There is a natural fully faithful embedding :

$$\mathbf{Set} \to \mathbf{Cat}.$$

Indeed, to any set S one may assign its *discrete category* also denoted S, *i.e.* the unique category such that $\mathrm{Ob}(S) = S$ and $\mathrm{Morph}(S) = \{(s, s, \mathbf{1}_s) \mid s \in S\}$. If S and S' are two discrete categories, the datum of a functor $S \to S'$ is clearly the same as a map of sets $\mathrm{Ob}(S) \to \mathrm{Ob}(S')$.

Notice also the natural functor

$$\mathrm{Ob} : \mathbf{Cat} \to \mathbf{Set} \qquad \mathscr{C} \mapsto \mathrm{Ob}(\mathscr{C})$$

that assigns to each functor $F : \mathscr{C} \to \mathscr{D}$ the underlying map $\mathrm{Ob}(\mathscr{C}) \to \mathrm{Ob}(\mathscr{D}) : C \mapsto FC$.

(iii) Recall that a *preordered set* is a pair $(I, \leq)$ consisting of a set I and a binary relation $\leq$ on I which is reflexive and transitive. In this case, we also say that $\leq$ is a *preordering on* I. We say that $(I, \leq)$ is a *partially ordered set*, if $\leq$ is also *antisymmetric*, *i.e.* if we have

$$(x \leq y \text{ and } y \leq x) \Rightarrow x = y \qquad \text{for every } x, y \in I.$$

We say that $(I, \leq)$ is a *totally ordered set*, if it is partially ordered and any two elements are comparable, *i.e.* for every $x, y \in I$ we have either $x \leq y$ or $y \leq x$. An *order-preserving map* $f : (I, \leq) \to (J, \leq)$ between preordered sets is a mapping $f : I \to J$ such that

$$x \leq y \;\Rightarrow\; f(x) \leq f(y) \qquad \text{for every } x, y \in I.$$

We denote by $\mathbf{Preorder}$ (resp. $\mathbf{POSet}$) the category of small preordered (resp. partially ordered) sets, with morphisms given by the order-preserving maps. To any preordered set $(I, \leq)$ one assigns a category whose set of objects is I, and whose morphisms are given as follows. For every $i, j \in I$, the set of morphisms $i \to j$ contains exactly one element when $i \leq j$, and is empty otherwise. Clearly, this rule defines a fully faithful functor

$$\mathbf{Preorder} \to \mathbf{Cat}.$$

Notice that if a category C lies in the essential image of this functor, then the same holds for C^o. Indeed, if C corresponds to the preordered set $(I, \leq)$, then C^o corresponds to the preordered set $(I^o, \leq)$ with $I^o := I$ and $x \leq y$ in I^o if and only if $y \leq x$ in I, for every $x, y \in I$. Clearly $(I, \leq)$ is a partially ordered set if and only if the same holds for $(I^o, \leq)$.

1.1.7. Let $\mathscr{A}$, $\mathscr{B}$ be two categories, $F, G : \mathscr{A} \to \mathscr{B}$ two functors. A *natural transformation*

$$\alpha : F \Rightarrow G \tag{1.1.8}$$

from F to G is a family of morphisms $(\alpha_A : FA \to GA \mid A \in \mathrm{Ob}(\mathscr{A}))$ of $\mathscr{B}$ such that, for every morphism $f : A \to B$ in $\mathscr{A}$, the diagram :

$$\begin{array}{ccc} FA & \xrightarrow{\alpha_A} & GA \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ FB & \xrightarrow{\alpha_B} & GB \end{array} \tag{1.1.9}$$

commutes. If α_A is an isomorphism for every $A \in \mathrm{Ob}(\mathscr{A})$, we say that α is a *natural isomorphism* of functors. For instance, the rule that assigns to any object A the identity morphism $\mathbf{1}_{FA} : FA \to FA$, defines a natural isomorphism $\mathbf{1}_F : F \Rightarrow F$. A natural transformation (1.1.8) is also indicated by a diagram of the type :

$$\mathscr{A} \underset{G}{\overset{F}{\rightrightarrows}} \mathscr{B}. \quad (\Downarrow \alpha)$$

1.1.10. The natural transformations between functors $\mathscr{A} \to \mathscr{B}$ can be composed; namely, if $\alpha : F \Rightarrow G$ and $\beta : G \Rightarrow H$ are two such transformations, we obtain a natural transformation

$$\beta \odot \alpha : F \Rightarrow H \qquad \text{by the rule :} \qquad A \mapsto \beta_A \circ \alpha_A \qquad \text{for every } A \in \mathrm{Ob}(\mathscr{A}).$$

With this composition, $\mathrm{Fun}(\mathscr{A}, \mathscr{B})$ is the set of objects of a category which we shall denote

$$\mathsf{Fun}(\mathscr{A}, \mathscr{B}).$$

There is also a second composition law for natural transformations : if $\mathscr{C}$ is another category, and we have a diagram of functors and natural transformations

$$\mathscr{A} \underset{G}{\overset{F}{\rightrightarrows}} \mathscr{B} \underset{G'}{\overset{F'}{\rightrightarrows}} \mathscr{C} \quad (\Downarrow \alpha,\ \Downarrow \alpha')$$

we get a natural transformation

$$\alpha' * \alpha : F' \circ F \Rightarrow G' \circ G \quad : \quad A \mapsto \alpha'_{GA} \circ F'(\alpha_A) = G'(\alpha_A) \circ \alpha'_{FA} \qquad \text{for every } A \in \mathrm{Ob}(\mathscr{A})$$

called the *Godement product* of α and α' ([28, Prop.1.3.4]). Especially, if $H : \mathscr{B} \to \mathscr{C}$ (resp. $H : \mathscr{C} \to \mathscr{A}$) is any functor, we write $H * \alpha$ (resp. $\alpha * H$) instead of $\mathbf{1}_H * \alpha$ (resp. $\alpha * \mathbf{1}_H$).

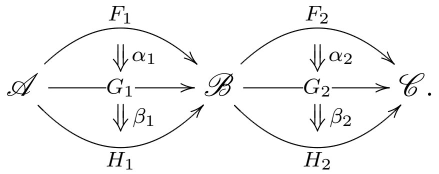
Both composition laws are associative, *i.e.*, if we have additional natural transformations

$$\mathscr{A} \underset{K}{\overset{H}{\rightrightarrows}} \mathscr{B} \quad (\Downarrow \gamma) \qquad \mathscr{C} \underset{G''}{\overset{F''}{\rightrightarrows}} \mathscr{D} \quad (\Downarrow \alpha'')$$

then we get the identities

$$\gamma \odot (\beta \odot \alpha) = (\gamma \odot \beta) \odot \alpha \qquad \alpha'' * (\alpha' * \alpha) = (\alpha'' * \alpha') * \alpha.$$

Moreover, the composition laws are related as follows. Suppose that $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ are three categories, and we have a diagram of six functors and four natural transformations :



Then we have the identity :

(1.1.11) $$(\beta_2 * \beta_1) \odot (\alpha_2 * \alpha_1) = (\beta_2 \odot \alpha_2) * (\beta_1 \odot \alpha_1).$$

The proofs are left as exercises for the reader (see [28, Prop.1.3.5]).

Remark 1.1.12. (i) In the situation of (1.1.10), if $\mathscr{A}$ and $\mathscr{B}$ are small categories, the same holds for $\mathsf{Fun}(\mathscr{A}, \mathscr{B})$.

(ii) Also, if $\mathscr{A}$ is small, $\mathscr{B}$ has small Hom-sets and $\mathrm{Ob}(\mathscr{B}) \subset \mathsf{U}$, then $\mathsf{Fun}(\mathscr{A}, \mathscr{B})$ has small Hom-sets, and $\mathrm{Ob}(\mathsf{Fun}(\mathscr{A}, \mathscr{B})) \subset \mathsf{U}$.

(iii) Assertion (ii) depends on our choices on how to encode arbitrary maps of sets : according to our (implicit) convention, *a map of sets* $f : S \to S'$ *is the graph* $\Gamma(f) \subset S \times S'$. This does not agree, *e.g.* with the definition found in Bourbaki's treatise [36], where such a map f is the triple $(S, S', \Gamma(f))$. With Bourbaki's convention, assertion (ii) fails. Other references are not so explicit about their choices for encoding maps, but for instance the SGA4 treatise ([8], [9], [10]) appears to follow Bourbaki's conventions, in view of [8, Exp.I, Rem.1.1.2], which states that $\mathrm{Fun}(\mathscr{A}, \mathscr{B})$ is not necessarily a subset of U, and $\mathsf{Fun}(\mathscr{A}, \mathscr{B})$ does not necessarily have small Hom-sets, under the assumptions of (ii). On the other hand, under the same assumptions, it is stated in [76, Ch.II, Prop.1] that $\mathsf{Fun}(\mathscr{A}, \mathscr{B})$ has small Hom-sets, so the set-theoretic conventions of the latter are not compatible with those of SGA4.

1.1.13. *Adjoint pairs of functors.* Let $\mathscr{A}$ and $\mathscr{B}$ be two categories, $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{A}$ two functors. We say that G is *left adjoint* to F if there exist bijections

$$\vartheta_{A,B} : \mathrm{Hom}_{\mathscr{A}}(GB, A) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{B}}(B, FA) \qquad \text{for every } A \in \mathrm{Ob}(\mathscr{A}) \text{ and } B \in \mathrm{Ob}(\mathscr{B})$$

and these bijections are natural in both A and B, *i.e.*

$$\vartheta_{A'B'}(g \circ f \circ Gh) = Fg \circ \vartheta_{AB}(f) \circ h \quad \text{for all morphisms } GB \xrightarrow{f} A \xrightarrow{g} A' \text{ in } \mathscr{A} \text{ and } B' \xrightarrow{h} B \text{ in } \mathscr{B}.$$

Then one also says that F is *right adjoint* to G, that (G, F) is an *adjoint pair of functors*, and that $\vartheta_{\bullet\bullet}$ is an *adjunction* for the pair (G, F).

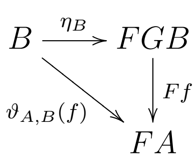
Especially, to any object B of $\mathscr{B}$ (resp. A of $\mathscr{A}$), the adjunction $\vartheta_{\bullet\bullet}$ assigns a morphism $\vartheta_{GB,B}(\mathbf{1}_{GB}) : B \to FGB$ (resp. $\vartheta^{-1}_{A,FA}(\mathbf{1}_{FA}) : GFA \to A$), whence a natural transformation

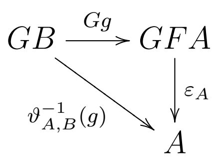
(1.1.14) $$\eta : \mathbf{1}_{\mathscr{B}} \Rightarrow F \circ G \qquad (\text{resp. } \varepsilon : G \circ F \Rightarrow \mathbf{1}_{\mathscr{A}})$$

called the *unit* (resp. *counit*) of the adjunction. The naturality of η follows from the calculation:

$$FG(f) \circ \eta_B = FG(f) \circ \vartheta_{GB,B}(\mathbf{1}_{GB}) = \vartheta_{GB,B'}(Gf \circ \mathbf{1}_{GB}) = \vartheta_{GB,B'}(\mathbf{1}_{GB'} \circ Gf) = \eta_{B'} \circ f$$

for every morphism $f : B \to B'$ in $\mathscr{B}$. A similar computation shows the naturality of ε. The naturality of $\vartheta_{\bullet\bullet}$ implies that we have commutative diagrams





for every morphism $f : GB \to A$ in $\mathscr{A}$ and $g : B \to FA$ in $\mathscr{B}$. Taking $f = \varepsilon_A$ and $g = \eta_B$, we see that the unit and counit are related by the so-called *triangular identities* expressed by the commutative diagrams :

$$\begin{array}{ccc} F & \xRightarrow{\eta * F} & FGF \\ & {}_{\mathbf{1}_F}\searrow\!\!\searrow & \Downarrow{\scriptstyle F*\varepsilon} \\ & & F \end{array} \qquad \begin{array}{ccc} G & \xRightarrow{G * \eta} & GFG \\ & {}_{\mathbf{1}_G}\searrow\!\!\searrow & \Downarrow{\scriptstyle \varepsilon * G} \\ & & G. \end{array}$$

Conversely, we have ([28, Th.3.1.5] or [110, Prop.1.5.4]) :

Proposition 1.1.15. *Let $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{A}$ be two functors.*

(i) *If η, ε are natural transformations as in* (1.1.14)*, fulfilling the triangular identities of* (1.1.13)*, then there is a unique adjunction for the pair (G,F), with unit η and counit ε.*

(ii) *Suppose that (G, F) is an adjoint pair, and η is the unit (resp. ε is the counit) of an adjunction for (G, F), then there exists a unique natural transformation ε (resp. η) as in* (1.1.14)*, fulfilling the triangular identities of* (1.1.13).

Proof. (i): Let $A \in \mathrm{Ob}(\mathscr{A})$ and $B \in \mathrm{Ob}(\mathscr{B})$ be any two objects; in light of the discussion of (1.1.13), we see that the sought natural bijection $\vartheta_{A,B}$ must be given by the rule :

$$f \mapsto Ff \circ \eta_B \qquad \text{for every } f \in \mathrm{Hom}_{\mathscr{A}}(GB, A)$$

and its inverse must be the mapping

$$g \mapsto \varepsilon_A \circ Gg \qquad \text{for every } g \in \mathrm{Hom}_{\mathscr{B}}(B, FA).$$

So we come down to checking that the triangular identities imply that these rules do induce mutually inverse bijections on the respective Hom-sets. We leave the verification to the reader.

(ii) is clear from the explicit construction of $\vartheta_{\bullet\bullet}$ in (i). □

Example 1.1.16. To every preordered set $(\mathscr{F}, \leq)$ we may attach its *partially ordered quotient*

$$(\mathscr{F}/\!\sim, \leq)$$

where $\sim$ denotes the equivalence relation such that $x \sim y$ if and only if $x \leq y$ and $y \leq x$, for every $x, y \in \mathscr{F}$. The ordering on $\mathscr{F}/\!\sim$ is the unique one such that the quotient map $\mathscr{F} \to \mathscr{F}/\!\sim$ defines a morphism

$$q_{\mathscr{F}} : (\mathscr{F}, \leq) \to (\mathscr{F}/\!\sim, \leq)$$

of preordered sets, and it is easily seen that the rule $(\mathscr{F}, \leq) \mapsto (\mathscr{F}/\!\sim, \leq)$ defines a left adjoint to the inclusion functor

$$\mathbf{POSet} \to \mathbf{Preorder}.$$

Moreover, the rule $(\mathscr{F}, \leq) \mapsto q_{\mathscr{F}}$ is a unit for this adjunction.

The following observations are borrowed from, and are further developed in [81, §I.6].

Remark 1.1.17. (i) Consider two adjoint pairs (G_1, F_1) and (G_2, F_2) :

$$\mathscr{A} \underset{G_1}{\overset{F_1}{\rightleftarrows}} \mathscr{B} \underset{G_2}{\overset{F_2}{\rightleftarrows}} \mathscr{C}$$

and suppose that for $i = 1, 2$ we are given adjunctions $\vartheta_{i,\bullet\bullet}$ for the pair (G_i, F_i). Then clearly $(G_1 \circ G_2, F_2 \circ F_1)$ is an adjoint pair, and we get an induced adjunction for this pair, by the composition :

$$\mathrm{Hom}_{\mathscr{A}}(G_1G_2C, A) \xrightarrow{\vartheta_{1,A,G_2C}} \mathrm{Hom}_{\mathscr{B}}(G_2C, F_1A) \xrightarrow{\vartheta_{2,F_1A,C}} \mathrm{Hom}_{\mathscr{B}}(C, F_2F_1A)$$

for every $A \in \mathrm{Ob}(\mathscr{A})$ and $C \in \mathrm{Ob}(\mathscr{C})$. We denote this adjunction by

$$(\vartheta_2 \circ \vartheta_1)_{\bullet\bullet}$$

and we call it the *composition* of the adjunctions ϑ_1 and ϑ_2. If (η_i, ε_i) are the units and counits of $\vartheta_{i,\bullet\bullet}$ (for $i = 1, 2$), then the unit and counit of $(\vartheta_2 \circ \vartheta_1)_{\bullet\bullet}$ are respectively :

$$(F_2 * \eta_1 * G_2) \odot \eta_2 \qquad \text{and} \qquad \varepsilon_1 \odot (G_1 * \varepsilon_2 * F_1).$$

Moreover, suppose that $\mathscr{C} \underset{G_3}{\overset{F_3}{\rightleftarrows}} \mathscr{D}$ is another adjoint pair of functors, and $\vartheta_{3,\bullet\bullet}$ an adjunction for this pair; with this notation, it is also then clear that

$$(\vartheta_3 \circ (\vartheta_2 \circ \vartheta_1))_{\bullet\bullet} = (\vartheta_3 \circ (\vartheta_2 \circ \vartheta_1))_{\bullet\bullet}.$$

(ii) Suppose that we have two pairs of adjoint functors and two natural transformations

$$\mathscr{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{B} \qquad \mathscr{A} \underset{G'}{\overset{F'}{\rightleftarrows}} \mathscr{B} \qquad \tau : F \Rightarrow F' \qquad \mu : G' \Rightarrow G$$

and let us fix units and counits (η, ε) (resp. (η', ε')) for the adjoint pair (G, F) (resp. (G', F')). Then we obtain *adjoint transformations*

$$\tau^\dagger : G' \Rightarrow G \qquad \mu^\dagger : F \Rightarrow F'$$

given by the compositions :

$$G'B \xrightarrow{G'(\eta_B)} G'FGB \xrightarrow{G'(\tau_{GB})} G'F'GB \xrightarrow{\varepsilon'_{GB}} GB \qquad \text{for every } B \in \mathrm{Ob}(\mathscr{B})$$
$$FA \xrightarrow{\eta'_{FA}} F'G'FA \xrightarrow{F'(\mu_{FA})} F'GFA \xrightarrow{F'(\varepsilon_A)} F'A \qquad \text{for every } A \in \mathrm{Ob}(\mathscr{A}).$$

We claim that $(\tau^\dagger)^\dagger = \tau$ and $(\mu^\dagger)^\dagger = \mu$. Indeed, let $\vartheta_{\bullet\bullet}$ (resp. $\vartheta'_{\bullet\bullet}$) be the adjunctions corresponding to (η, ε) (resp. to (η', ε')), and notice that

$$\tau^\dagger_B = \vartheta'^{-1}_{GB,FGB}(\tau_{GB}) \circ G'(\eta_B) \qquad \mu^\dagger_A = F'(\varepsilon_A) \circ \vartheta'_{GFA,FA}(\mu_{FA})$$

so we may compute :

$$\begin{aligned}
(\tau^\dagger)^\dagger_A &= F'(\varepsilon_A) \circ \vartheta'_{GFA,FA}(\vartheta'^{-1}_{GFA,FGFA}(\tau_{GFA}) \circ G'(\eta_{FA})) \\
&= \vartheta'_{A,FA}(\varepsilon_A \circ \vartheta'^{-1}_{GFA,FGFA}(\tau_{GFA}) \circ G'(\eta_{FA})) \\
&= \vartheta'_{A,FA}(\vartheta'^{-1}_{A,FGFA}(F'(\varepsilon_A) \circ \tau_{GFA}) \circ G'(\eta_{FA})) \\
&= \vartheta'_{A,FA}(\vartheta'^{-1}_{A,FGFA}(\tau_A \circ F(\varepsilon_A)) \circ G'(\eta_{FA})) \\
&= \vartheta'_{A,FA}(\vartheta'^{-1}_{A,FA}(\tau_A) \circ G'F(\varepsilon_A) \circ G'(\eta_{FA})) \\
&= \vartheta'_{A,FA}(\vartheta'^{-1}_{A,FA}(\tau_A)) \\
&= \tau_A
\end{aligned}$$

where the second, third and fifth identities follow from the naturality of $\vartheta'_{\bullet\bullet}$, the fourth from the naturality of τ, and the sixth from the triangular identities of (1.1.13). We leave to the reader the similar calculation which gives the second identity. Hence the rule

$$\tau \mapsto \tau^\dagger := (\tau, \vartheta, \vartheta')^\dagger$$

establishes a natural bijection from the set of natural transformations $F \Rightarrow F'$, to the set of natural transformations $G' \Rightarrow G$. Notice that this correspondence depends not only on (G, F) and (G', F'), but also on (η, ε) and (η', ε'). Sometimes we denote this adjoint transformation also by $(\tau, \eta, \eta')^\dagger$.

(iii) Moreover, using the triangular identities of (1.1.13), it is easily seen that the diagrams :

$$\begin{array}{ccc} G'\circ F & \xRightarrow{G'*\tau} & G'\circ F' \\ {\scriptstyle \tau^\dagger * F}\Downarrow & & \Downarrow{\scriptstyle \varepsilon'} \\ G\circ F & \xRightarrow{\ \varepsilon\ } & \mathbf{1}_{\mathscr{A}} \end{array} \qquad\qquad \begin{array}{ccc} \mathbf{1}_{\mathscr{B}} & \xRightarrow{\ \eta\ } & F\circ G \\ {\scriptstyle \eta'}\Downarrow & & \Downarrow{\scriptstyle \tau * G} \\ F'\circ G' & \xRightarrow{F'*\tau^\dagger} & F'\circ G \end{array}$$

commute. Also, $(\tau,\vartheta,\vartheta')^\dagger$ is characterized as the unique natural transformation $G'\Rightarrow G$ such that the following diagram commutes for every $A\in\mathrm{Ob}(\mathscr{A})$ and $B\in\mathrm{Ob}(\mathscr{B})$:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathscr{A}}(GB,A) & \xrightarrow{\ \vartheta_{A,B}\ } & \mathrm{Hom}_{\mathscr{B}}(B,FA) \\ {\scriptstyle \mathrm{Hom}_{\mathscr{A}}(\tau^\dagger_B,A)}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}_{\mathscr{B}}(B,\tau_A)} \\ \mathrm{Hom}_{\mathscr{A}}(G'B,A) & \xrightarrow{\ \vartheta'_{A,B}\ } & \mathrm{Hom}_{\mathscr{B}}(B,F'A) \end{array} \tag{1.1.18}$$

Indeed, letting $A:=GB$ and recalling that $\vartheta_{GB,B}(\mathbf{1}_{GB})=\eta_B$, we see easily that the commutativity of (1.1.18) determines uniquely $\tau^\dagger$ (details left to the reader). Conversely, if $\tau^\dagger$ is defined as in (i), we may compute, for every morphism $f:GB\to A$ in $\mathscr{A}$:

$$\begin{aligned} \vartheta'_{A,B}(f\circ\tau^\dagger_B) &= \vartheta'_{A,B}(f\circ\vartheta'^{-1}_{GB,FGB}(\tau_{GB})\circ G'(\eta_B)) \\ &= \vartheta'_{A,FGB}(f\circ\vartheta'^{-1}_{GB,FGB}(\tau_{GB}))\circ\eta_B \\ &= \vartheta'_{A,FGB}(\vartheta'^{-1}_{A,FGB}(F'f\circ\tau_{GB}))\circ\eta_B \\ &= F'f\circ\tau_{GB}\circ\eta_B \\ &= \tau_A\circ Ff\circ\eta_B \\ &= \tau_A\circ\vartheta_{A,B}(f). \end{aligned}$$

(iv) Furthermore, suppose we have a third pair of adjoint functors

$$\mathscr{A}\ \underset{G''}{\overset{F''}{\rightleftarrows}}\ \mathscr{B} \qquad \text{and a natural transformation} \qquad \omega : F'\Rightarrow F''$$

and let us fix an adjunction $\vartheta''_{\bullet\bullet}$ for the pair (G'',F''). Then we have :

$$(\omega\odot\tau,\vartheta,\vartheta'')^\dagger = (\tau,\vartheta,\vartheta')^\dagger\odot(\omega,\vartheta',\vartheta'')^\dagger.$$

Indeed, this identity follows easily from the characterization of $\tau^\dagger$, $\omega^\dagger$ and $(\omega\circ\tau)^\dagger$ given in (iii) (details left to the reader).

(v) Lastly, in the situation of (i), suppose moreover that we have two other adjoint pairs

$$\mathscr{A}\ \underset{G'_1}{\overset{F'_1}{\rightleftarrows}}\ \mathscr{B}\ \underset{G'_2}{\overset{F'_2}{\rightleftarrows}}\ \mathscr{C} \qquad \text{and natural transformations} \qquad \tau_1 : F_1\Rightarrow F'_1 \qquad \tau_2 : F_2\Rightarrow F'_2$$

and for $i=1,2$, let us fix an adjunction $\vartheta'_{i,\bullet\bullet}$ for (G'_i,F'_i). Then we get as in (ii) the natural transformations $\tau^\dagger_1 : G'_1\Rightarrow G_1$ and $\tau^\dagger_2 : G'_2\Rightarrow G_2$, and we have the identity

$$(\tau_2*\tau_1,\vartheta_2\circ\vartheta_1,\vartheta'_2\circ\vartheta'_1)^\dagger = (\tau_1,\vartheta_1,\vartheta'_1)^\dagger * (\tau_2,\vartheta_2,\vartheta'_2)^\dagger.$$

Indeed, taking into account (iii) we get the commutative diagram

$$\begin{array}{ccccc}
\mathrm{Hom}_{\mathscr{A}}(G_1G_2C,A) & \xrightarrow{\vartheta_{1,A,G_2C}} & \mathrm{Hom}_{\mathscr{B}}(G_2C,F_1A) & \xrightarrow{\vartheta_{2,F_1A,C}} & \mathrm{Hom}_{\mathscr{C}}(C,F_2F_1A) \\
{\scriptstyle \mathrm{Hom}_{\mathscr{A}}(\tau^\dagger_{1,G_2C},A)}\downarrow & & {\scriptstyle \mathrm{Hom}_{\mathscr{B}}(G_2C,\tau_{1,C})}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}_{\mathscr{C}}(C,F_2(\tau_{1,C}))} \\
\mathrm{Hom}_{\mathscr{A}}(G'_1G_2C,A) & \xrightarrow{\vartheta'_{1,A,G_2C}} & \mathrm{Hom}_{\mathscr{B}}(G_2C,F'_1A) & \xrightarrow{\vartheta_{2,F'_1A,C}} & \mathrm{Hom}_{\mathscr{C}}(C,F_2F'_1A) \\
{\scriptstyle \mathrm{Hom}_{\mathscr{A}}(G'_1(\tau^\dagger_{1,C}),A)}\downarrow & & {\scriptstyle \mathrm{Hom}_{\mathscr{B}}(\tau^\dagger_{2,C},F'_1A)}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}_{\mathscr{C}}(C,\tau_{2,F'_1C})} \\
\mathrm{Hom}_{\mathscr{A}}(G'_1G'_2C,A) & \xrightarrow{\vartheta'_{1,A,G'_2C}} & \mathrm{Hom}_{\mathscr{B}}(G'_2C,F'_1A) & \xrightarrow{\vartheta'_{2,F'_1A,C}} & \mathrm{Hom}_{\mathscr{C}}(C,F'_2F'_1A)
\end{array}$$

for every $A \in \mathrm{Ob}(\mathscr{A})$ and $C \in \mathrm{Ob}(\mathscr{C})$. The sought identity follows after invoking again (iii).

(vi) Especially, taking into account the triangular identities of (1.1.13), it is easily seen that :

$$\begin{aligned}
(\mathbf{1}_{F_2},\vartheta_2,\vartheta_2)^\dagger &= \mathbf{1}_{G_2} \\
(F_2 * \tau_1, \vartheta_2\circ\vartheta_1, \vartheta_2\circ\vartheta'_1)^\dagger &= (\tau_1,\vartheta_1,\vartheta'_1)^\dagger * G_2 \\
(\tau_2 * F_1, \vartheta_2\circ\vartheta_1, \vartheta'_2\circ\vartheta_1)^\dagger &= G_1 * (\tau_2,\vartheta_2,\vartheta'_2)^\dagger.
\end{aligned}$$

Remark 1.1.19. (i) For any two categories $\mathscr{C},\mathscr{D}$ we have a natural isomorphism of categories

$$\mathsf{Fun}(\mathscr{C},\mathscr{D})^o \xrightarrow{\sim} \mathsf{Fun}(\mathscr{C}^o,\mathscr{D}^o)$$

that assigns to any functor $H : \mathscr{C} \to \mathscr{D}$ the opposite functor $H^o : \mathscr{C}^o \to \mathscr{D}^o$, and to any natural transformation $\tau : H \Rightarrow K$ the *opposite transformation* $\tau^o : K^o \Rightarrow H^o$ such that $\tau^o_{C^o} := (\tau_C)^o$ for every $C \in \mathrm{Ob}(\mathscr{C})$. Also, in the situation of (1.1.10), notice the identities

$$(\beta\odot\alpha)^o = \alpha^o \odot \beta^o \qquad \text{and} \qquad (\alpha' * \alpha)^o = \alpha'^o * \alpha^o.$$

(ii) Let $\mathscr{B}$ and $\mathscr{E}$ be two other categories, $f : \mathscr{B} \to \mathscr{C}$ and $g : \mathscr{D} \to \mathscr{E}$ two functors; we get an induced functor

$$\mathsf{Fun}(f,g) : \mathsf{Fun}(\mathscr{C},\mathscr{D}) \to \mathsf{Fun}(\mathscr{B},\mathscr{E}) \qquad H \mapsto g\circ H\circ f \qquad (\alpha : H \Rightarrow K) \mapsto g * \alpha * f.$$

Likewise, if $f' : \mathscr{B} \to \mathscr{C}$ and $g' : \mathscr{D} \to \mathscr{E}$ are any two other functors, every pair of natural transformations $\gamma : f \Rightarrow f'$ and $\delta : g \Rightarrow g'$ induces a transformation

$$\mathsf{Fun}(\gamma,\delta) : \mathsf{Fun}(f,g) \Rightarrow \mathsf{Fun}(f',g') \qquad H \mapsto \delta * H * \gamma.$$

We shall usually write $\mathsf{Fun}(f,\mathscr{D})$ (resp. $\mathsf{Fun}(\mathscr{C},g)$) instead of $\mathsf{Fun}(f,\mathbf{1}_{\mathscr{D}})$ (resp. of $\mathsf{Fun}(\mathbf{1}_{\mathscr{C}},g)$). Furthermore, in the situation of (1.1.10), notice the identities

$$\begin{aligned}
\mathsf{Fun}(\beta,\mathscr{D}) \odot \mathsf{Fun}(\alpha,\mathscr{D}) &= \mathsf{Fun}(\beta\odot\alpha,\mathscr{D}) \\
\mathsf{Fun}(\alpha,\mathscr{D}) * \mathsf{Fun}(\alpha',\mathscr{D}) &= \mathsf{Fun}(\alpha' * \alpha,\mathscr{D}).
\end{aligned}$$

(iii) In the situation of (ii), suppose that the functor f admits a left adjoint $g : \mathscr{C} \to \mathscr{B}$. Then $f^* := \mathsf{Fun}(f,\mathscr{D})$ is left adjoint to $g^* := \mathsf{Fun}(g,\mathscr{D})$. More precisely, let η be a unit and ε a counit for the adjoint pair (g,f). From the triangular identities (1.1.13) for (g,f), and taking into account (ii), we deduce commutative diagrams :

$$\begin{array}{ccc}
g^* & \xrightarrow{\mathsf{Fun}(\eta,\mathscr{D})*g^*} & g^*f^*g^* \\
& \searrow{=} & \Downarrow{\scriptstyle g^**\mathsf{Fun}(\varepsilon,\mathscr{D})} \\
& & g^*
\end{array}
\qquad\qquad
\begin{array}{ccc}
f^* & \xrightarrow{f^**\mathsf{Fun}(\eta,\mathscr{D})} & f^*g^*f^* \\
& \searrow{=} & \Downarrow{\scriptstyle \mathsf{Fun}(\varepsilon,\mathscr{D})*f^*} \\
& & f^*
\end{array}$$

which, in light of proposition 1.1.15(i), says that $\mathsf{Fun}(\eta,\mathscr{D})$ is a unit and $\mathsf{Fun}(\varepsilon,\mathscr{D})$ a counit for the adjoint pair (f^*,g^*).

(iv) Let $F : \mathscr{C} \to \mathscr{D}$ be a functor, and $G : \mathscr{D} \to \mathscr{C}$ a left adjoint to F. Then F^o is left adjoint to G^o. More precisely, let η be a unit and ε a counit for the adjoint pair (G,F); then it follows easily from (i) and proposition 1.1.15(i) that ε^o is a unit and η^o is a counit for the adjoint pair (F^o, G^o) : details left to the reader.

Proposition 1.1.20. *Let $F : \mathscr{A} \to \mathscr{B}$ be a functor.*

(i) *The following conditions are equivalent :*
   (a) *F is fully faithful and has a fully faithful left adjoint.*
   (b) *There exist a functor $G : \mathscr{B} \to \mathscr{A}$ and isomorphisms of functors*

$$G \circ F \xrightarrow{\sim} \mathbf{1}_{\mathscr{A}} \qquad \mathbf{1}_{\mathscr{B}} \xrightarrow{\sim} F \circ G.$$

   (c) *F is an equivalence.*

(ii) *Suppose that F admits a left adjoint $G : \mathscr{B} \to \mathscr{A}$, and let $\eta : \mathbf{1}_{\mathscr{B}} \Rightarrow F \circ G$ and $\varepsilon : G \circ F \Rightarrow \mathbf{1}_{\mathscr{A}}$ be a unit and respectively a counit for the adjoint pair (G, F). Then F (resp. G) is faithful if and only if ε_X is an epimorphism for every $X \in \mathrm{Ob}(\mathscr{A})$ (resp. η_Y is a monomorphism for every $Y \in \mathrm{Ob}(\mathscr{B})$).*

(iii) *In the situation of* (ii)*, the following conditions are equivalent :*
   (a) *F (resp. G) is fully faithful.*
   (b) *The counit ε (resp. the unit η) is an isomorphism of functors.*
   (c) *There exists an isomorphism of functors $\varepsilon' : G \circ F \xrightarrow{\sim} \mathbf{1}_{\mathscr{A}}$ (resp. $\eta' : \mathbf{1}_{\mathscr{B}} \xrightarrow{\sim} F \circ G$).*

   Moreover, if (c) *holds, there exists a unique adjunction ϑ' for the pair (G, F) whose counit is ε' (resp. whose unit is η').*

(iv) *Suppose that F admits both a left adjoint $G : \mathscr{B} \to \mathscr{A}$ and a right adjoint $H : \mathscr{B} \to \mathscr{A}$. Then G is fully faithful if and only if H is fully faithful.*

Proof. (ii): In view of remark 1.1.19(iv), it suffices to consider the assertion relative to $\varepsilon_\bullet$. Thus, suppose that $Ff_1 = Ff_2$ for two morphisms $f_1, f_2 : X \to X'$. The naturality of $\varepsilon_\bullet$ yields the identity : $\varepsilon_{X'} \circ GFf_i = f_i \circ \varepsilon_X$ for $i = 1, 2$, and if ε_X is an epimorphism, we deduce $f_1 = f_2$. Conversely, suppose F is faithful and $f_1 \circ \varepsilon_X = f_2 \circ \varepsilon_X$; from the triangular identities we get :

$$Ff_i = F(\varepsilon_{X'}) \circ FGF(f_i) \circ \eta_{FX} = F(f_i \circ \varepsilon_X) \circ \eta_{FX}$$

so $Ff_1 = Ff_2$ and therefore $f_1 = f_2$ which shows that ε_X is an epimorphism.

(iii): Again, remark 1.1.19(iv) reduces to considering the assertion for $\varepsilon_\bullet$. We check first that (iii.a)⇒(iii.b) : indeed, if F is fully faithful, for every $X \in \mathrm{Ob}(\mathscr{A})$, there exists a morphism $\beta_X : X \to GFX$ such that $F\beta_X = \eta_{FX} : FX \to FGFX$. From the triangular identities of (1.1.13) we deduce that $F(\varepsilon_X \circ \beta_X) = F(\varepsilon_X) \circ \eta_{FX} = \mathbf{1}_{FX}$, whence $\varepsilon_X \circ \beta_X = \mathbf{1}_X$, since F is faithful. Next, let ϑ be the unique adjunction for the pair (G, F) whose unit and counit are η and ε; by inspection of the explicit description of ϑ in the proof of proposition 1.1.15(i) we get

$$\vartheta_{GFX,FX}(\beta_X \circ \varepsilon_X) = F(\beta_X \circ \varepsilon_X) \circ \eta_{FX} = \eta_{FX} \circ F(\varepsilon_X) \circ \eta_{FX} = \eta_{FX} = \vartheta_{GFX,FX}(\mathbf{1}_{GFX})$$

whence $\beta_X \circ \varepsilon_X = \mathbf{1}_{GFX}$. Obviously (iii.b)⇒(iii.c). Lastly, suppose that (iii.c) holds; for every $X, Y \in \mathrm{Ob}(\mathscr{A})$ we deduce a bijection

$$\xi_{X,Y} : \mathrm{Hom}_{\mathscr{A}}(X, Y) \xrightarrow{\varphi_{X,Y}} \mathrm{Hom}_{\mathscr{A}}(GFX, Y) \xrightarrow{\vartheta_{GX,Y}} \mathrm{Hom}_{\mathscr{B}}(FX, FY)$$

where $\varphi_{X,Y}(f) := f \circ \varepsilon'_X$ for every morphism $f : X \to Y$ in $\mathscr{A}$. It is easily seen that $\xi_{X,Y}$ is natural in both X and Y; especially, for every $f \in \mathrm{Hom}_{\mathscr{A}}(X, Y)$ we have :

$$\xi_{Y,Y}(\mathbf{1}_Y) \circ Ff = \xi_{X,Y}(\mathbf{1}_Y \circ f) = \xi_{X,Y}(f \circ \mathbf{1}_X) = Ff \circ \xi_{X,X}(\mathbf{1}_X).$$

Letting $X = Y$, and taking $f : X \to X$ with $\xi_{X,X}(f) = \mathbf{1}_{FX}$, we deduce that $\xi_{X,X}(\mathbf{1}_X)$ is an isomorphism in $\mathscr{B}$, for every $X \in \mathrm{Ob}(\mathscr{A})$. Since $\xi_{X,Y}(f) = Ff \circ \xi_{X,X}(\mathbf{1}_X)$ for every $f \in \mathrm{Hom}_{\mathscr{A}}(X, Y)$, we conclude that the rule $f \mapsto Ff$ is a bijection $\mathrm{Hom}_{\mathscr{A}}(X, Y) \xrightarrow{\sim}$

$\mathrm{Hom}_{\mathscr{B}}(FX, FY)$, whence (iii.a). Lastly, let us check the existence and uniqueness of the adjunction ϑ' whose counit is ε'. To this aim, let $\omega : \mathbf{1}_{\mathscr{A}} \stackrel{\sim}{\to} \mathbf{1}_{\mathscr{A}}$ be the automorphism such that $\omega \circ \varepsilon = \varepsilon'$, and set $\eta' := (F * \omega^{-1} * G) \circ \eta$. We compute :

$$\begin{aligned}(F * \varepsilon') \odot (\eta' * F) &= (F * \omega) \odot (F * \varepsilon) \odot (F * \omega^{-1} * GF) \odot (\eta * F) \\ &= (F * \omega) \odot (F * \omega^{-1}) \odot (F * \varepsilon) \odot (\eta * F) = \mathbf{1}_F.\end{aligned}$$

Likewise we check that $(\varepsilon' * G) \odot (G * \eta') = \mathbf{1}_G$, whence the contention, by proposition 1.1.15.

(i): The equivalence (i.a)⇔(i.b) follows immediately from (iii). Moreover, if (i.b) holds, then for every $Y \in \mathrm{Ob}(\mathscr{B})$ we have an isomorphism $Y \stackrel{\sim}{\to} FGY$, so F is essentially surjective. Since we have just seen that (i.b) implies that F is fully faithful, we deduce as well that (i.b)⇒(i.c).

(i.c)⇒(i.a): We construct a left adjoint $G : \mathscr{B} \to \mathscr{A}$ to F as follows. Since F is essentially surjective, for every $B \in \mathrm{Ob}(\mathscr{B})$ we may find an object $A \in \mathrm{Ob}(\mathscr{A})$ with an isomorphism $\eta_B : B \stackrel{\sim}{\to} FA$, and we set $GB := A$. Next, since F is fully faithful, for every morphism $g : B \to B'$ in $\mathscr{B}$ there exists a unique morphism $f : GB \to GB'$ in $\mathscr{A}$ which makes commute the diagram

$$\begin{array}{ccc} B & \xrightarrow{g} & B' \\ {\scriptstyle \omega_B}\downarrow & & \downarrow{\scriptstyle \omega_{B'}} \\ FGB & \xrightarrow{Ff} & FGB' \end{array}$$

and we let $Gg := f$. It is easily seen that these rules yield a well defined functor G as sought, and η is then a natural isomorphism $\mathbf{1}_{\mathscr{B}} \Rightarrow FG$. Moreover from the full faithfulness of F we deduce that G is fully faithful as well (details left to the reader). To conclude we remark, more generally :

Claim 1.1.21. Let $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{A}$ be two functors, $\eta : \mathbf{1}_{\mathscr{B}} \Rightarrow FG$ an isomorphism of functors, and suppose that F is fully faithful. Then there exists a unique adjunction ϑ for the pair (G, F) whose unit is η.

Proof of the claim. From the proof of proposition 1.1.15 we know that η determines ϑ, by the rule : $\vartheta_{A,B}(f) := Ff \circ \eta_B$ for every $A \in \mathrm{Ob}(\mathscr{A})$, $B \in \mathrm{Ob}(\mathscr{B})$ and every morphism $f : GB \to A$ in $\mathscr{A}$. Conversely, our assumptions easily imply that this rule does yield a natural bijection $\mathrm{Hom}_{\mathscr{A}}(GB, A) \stackrel{\sim}{\to} \mathrm{Hom}_{\mathscr{B}}(B, FA)$, whence the contention (details left to the reader). ◊

(iv): Let $\eta : \mathbf{1}_{\mathscr{B}} \Rightarrow FG$ (resp. $\eta' : \mathbf{1}_{\mathscr{A}} \Rightarrow HF$) be the unit and $\varepsilon : GF \Rightarrow \mathbf{1}_{\mathscr{A}}$ (resp. $\varepsilon' : FH \Rightarrow \mathbf{1}_{\mathscr{B}}$) the counit of a given adjunction for the adjoint pair (G, F) (resp. (F, H)). By remark 1.1.19(iv) we may assume that H is fully faithful, and we show that the same holds for G. By (iii), this is the same as assuming that ε' is an isomorphism, and we need to check that the same holds for η. To this aim, denote $\gamma : FG \Rightarrow \mathbf{1}_{\mathscr{B}}$ the composition

$$FG \xrightarrow{(FG)*\varepsilon'^{-1}} FGFH \xrightarrow{F*\varepsilon*H} FH \xrightarrow{\varepsilon'} \mathbf{1}_{\mathscr{B}}.$$

We show that γ is inverse to η. Indeed we have

$$\gamma \odot \eta = \varepsilon' \odot (F * \varepsilon * H) \odot (\eta * FH) \odot \varepsilon'^{-1} = \varepsilon' \odot \varepsilon'^{-1} = \mathbf{1}_{\mathscr{B}}$$

where the first identity holds by the naturality of η, and the second follows from the triangular identites of (1.1.13). Likewise, we have

$$\begin{aligned}\eta\odot\gamma &= (\varepsilon'*FG)\odot(FH*\eta)\odot(F*\varepsilon*H)\odot(FG*\varepsilon'^{-1})\\ &= (\varepsilon'*FG)\odot(F*\varepsilon*HFG)\odot(FGFH*\eta)\odot(FG*\varepsilon'^{-1})\\ &= (\varepsilon'*FG)\odot(F*\varepsilon*HFG)\odot(FG*\varepsilon'^{-1}*FG)\odot(FG*\eta)\\ &= (\varepsilon'*FG)\odot(F*\varepsilon*HFG)\odot(FGF*\eta'*G)\odot(FG*\eta)\\ &= (\varepsilon'*FG)\odot(F*\eta'*G)\odot(F*\varepsilon*G)\odot(FG*\eta)\\ &= (\mathbf{1}_F*G)\odot(F*\mathbf{1}_G)=\mathbf{1}_{FG}\end{aligned}$$

where the first and third identities follow from the naturality of ε', the second and fifth from that of ε, the fourth and sixth from the triangular identities for the pairs (η',ε') and (η,ε). □

Definition 1.1.22. Let $F:\mathscr{A}\to\mathscr{B}$ be any equivalence of categories. A *quasi-inverse* for F is the datum of a functor $G:\mathscr{B}\to\mathscr{A}$ and an adjunction for the pair (G,F). Then we also say that F is a *quasi-inverse* for G.

Remark 1.1.23. It follows easily from proposition 1.1.20(i,iii) and claim 1.1.21, that a quasi-inverse for an equivalence $F:\mathscr{A}\to\mathscr{B}$ is the same as the datum of a functor $G:\mathscr{B}\to\mathscr{A}$ and an isomorphism of functors $\mathbf{1}_{\mathscr{B}}\xrightarrow{\sim}FG$, and moreover any such G is also an equivalence. Taking into account remark 1.1.19(iv), we see that a quasi-inverse for F is also the same as the datum of such a G and an isomorphism of functors $GF\xrightarrow{\sim}\mathbf{1}_{\mathscr{A}}$.

1.1.24. *Slice categories.* A standard construction attaches to any $X\in\mathrm{Ob}(\mathscr{C})$ a category :

$$\mathscr{C}/X$$

as follows. The objects of $\mathscr{C}/X$ are all the pairs (A,f) where $A\in\mathrm{Ob}(\mathscr{C})$ and $f:A\to X$ is any morphism of $\mathscr{C}$. For any two such objects (A,f), (B,g), the set $\mathrm{Hom}_{\mathscr{C}/X}((A,f),(B,g))$ consists of all the commutative diagrams of morphisms of $\mathscr{C}$:

$$\begin{array}{ccc} A & \xrightarrow{\ h\ } & B \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle g} & \\ & X & \end{array}$$

with composition of morphisms induced by the composition law of $\mathscr{C}$. We denote sometimes such a morphism of $\mathscr{C}/X$ by

$$h/X:(A,f)\to(B,g).$$

An object (resp. a morphism) of $\mathscr{C}/X$ is also called an X*-object* (resp. an X*-morphism*) of $\mathscr{C}$. Dually, one defines

$$X/\mathscr{C}:=(\mathscr{C}^o/X^o)^o$$

i.e. the objects of $X/\mathscr{C}$ are the pairs (A,f) with $A\in\mathrm{Ob}(\mathscr{C})$ and $f:X\to A$ any morphism of $\mathscr{C}$. We have an obvious faithful *source* functor

$$\mathsf{s}_X:\mathscr{C}/X\to\mathscr{C}\qquad(A,f)\mapsto A\qquad((A,f)\xrightarrow{h/X}(B,g))\mapsto(A\xrightarrow{h}B)$$

and likewise one obtains a *target* functor

$$\mathsf{t}_X:=\mathsf{s}^o_{X^o}:X/\mathscr{C}\to\mathscr{C}.$$

Moreover, any morphism $f:X\to Y$ in $\mathscr{C}$ induces functors :

$$\begin{array}{lcl} f_*:\mathscr{C}/X\to\mathscr{C}/Y & : & (A,g:A\to X)\mapsto(A,f_*g:=f\circ g:A\to Y)\\ f^*:Y/\mathscr{C}\to X/\mathscr{C} & : & (B,h:Y\to B)\mapsto(B,f^*h:=h\circ f:X\to B).\end{array}\tag{1.1.25}$$

Furthermore, given a functor $F : \mathscr{C} \to \mathscr{B}$, any $X \in \mathrm{Ob}(\mathscr{C})$ induces functors :

$$(1.1.26)\qquad \begin{array}{rcl} F_{|X} : \mathscr{C}/X \to \mathscr{B}/FX & : & (A,g) \mapsto (FA, Fg) \\ {}_{X|}F : X/\mathscr{C} \to FX/\mathscr{B} & : & (B,h) \mapsto (FB, Fh). \end{array}$$

1.1.27. The categories $\mathscr{C}/X$ and $X/\mathscr{C}$ are special cases of the following more general construction. Let $F : \mathscr{A} \to \mathscr{B}$ be any functor. For any $B \in \mathrm{Ob}(\mathscr{B})$, we define

$$F\mathscr{A}/B$$

as the category whose objects are all the pairs (A, f), where $A \in \mathrm{Ob}(\mathscr{A})$ and $f : FA \to B$ is a morphism in $\mathscr{B}$. The morphisms $g : (A, f) \to (A', f')$ are the morphisms $g : A \to A'$ in $\mathscr{A}$ such that $f' \circ Fg = f$. There are well-defined functors :

$$F/B : F\mathscr{A}/B \to \mathscr{B}/B \ : \ (A,f) \mapsto (FA, f) \qquad \text{and} \qquad \mathsf{s}_B : F\mathscr{A}/B \to \mathscr{A} \ : \ (A,f) \mapsto A.$$

Dually, we define :

$$B/F\mathscr{A} := (F^o\mathscr{A}^o/B^o)^o$$

and likewise one has natural functors :

$$B/F : B/F\mathscr{A} \to B/\mathscr{B} \qquad \text{and} \qquad \mathsf{t}_B : B/F\mathscr{A} \to \mathscr{A}.$$

Any morphism $g : B' \to B$ induces functors :

$$\begin{array}{rcl} g/F\mathscr{A} : B/F\mathscr{A} \to B'/F\mathscr{A} & : & (A,f) \mapsto (A, f \circ g) \\ F\mathscr{A}/g : F\mathscr{A}/B' \to F\mathscr{A}/B & : & (A,f) \mapsto (A, g \circ f). \end{array}$$

Obviously, the category $\mathscr{C}/X$ (resp. $X/\mathscr{C}$) is the same as $\mathbf{1}_\mathscr{C}\mathscr{C}/X$ (resp. $X/\mathbf{1}_\mathscr{C}\mathscr{C}$).

1.1.28. In the situation of (1.1.27), the categories of the form $F\mathscr{A}/B$ can be faithfully embedded in a single category $F\mathscr{A}/\mathscr{B}$. The latter is the category whose objects are all the triples (A, B, f), where $A \in \mathrm{Ob}(\mathscr{A})$, $B \in \mathrm{Ob}(\mathscr{B})$ are any two objects, and $f : FA \to B$ is any morphism of $\mathscr{B}$. If $f : FA \to B$ and $f' : FA' \to B'$ are any two objects, the set

$$\mathrm{Hom}_{F\mathscr{A}/\mathscr{B}}((A,B,f),(A',B',f'))$$

consists of all pairs (g, g') where g is a morphism in $\mathscr{A}$ and g' a morphism in $\mathscr{B}$, that make commute the diagram :

$$(1.1.29)\qquad \begin{array}{ccc} FA & \xrightarrow{f} & B \\ {\scriptstyle Fg}\downarrow & & \downarrow{\scriptstyle g'} \\ FA' & \xrightarrow{f'} & B' \end{array}$$

with composition of morphisms induced by the composition laws of $\mathscr{A}$ and $\mathscr{B}$, in the obvious way. There are two natural *source* and *target* functors :

$$\mathscr{A} \xleftarrow{\mathsf{s}} F\mathscr{A}/\mathscr{B} \xrightarrow{\mathsf{t}} \mathscr{B}$$

such that $\mathsf{s}(FA \to B) := A$, $\mathsf{t}(FA \to B) := B$ for any object $FA \to B$ of $F\mathscr{A}/\mathscr{B}$, and $\mathsf{s}(g, g') = g$, $\mathsf{t}(g, g') = g'$. Dually, we let

$$\mathscr{B}/F\mathscr{A} := (F^o\mathscr{A}^o/\mathscr{B}^o)^o$$

and the corresponding source and target functors are switched :

$$\mathscr{A} \xleftarrow{\mathsf{t}} \mathscr{B}/F\mathscr{A} \xrightarrow{\mathsf{s}} \mathscr{B}.$$

1.1.30. For the special case of the identity endofunctor $\mathbf{1}_{\mathscr{C}}$ of any category $\mathscr{C}$, we obtain the *category of arrows of* $\mathscr{C}$

$$\mathsf{Morph}(\mathscr{C}) := \mathbf{1}_{\mathscr{C}}\mathscr{C}/\mathscr{C}.$$

So, the set of objects of $\mathsf{Morph}(\mathscr{C})$ is $\mathrm{Morph}(\mathscr{C})$ (notation of (1.1)) and the morphisms are the commutative square diagrams in $\mathscr{C}$. The functor s_X of (1.1.24) is the restriction of $\mathsf{s} : \mathsf{Morph}(\mathscr{C}) \to \mathscr{C}$ to the subcategory $\mathscr{C}/X$, and likewise for t_X. Likewise, in the situation of (1.1.28), the target functor on $F\mathscr{A}/\mathscr{B}$ and the source functor on $\mathscr{B}/F\mathscr{A}$ factor through functors

$$F\mathscr{A}/\mathscr{B} \xrightarrow{\mathsf{T}} \mathsf{Morph}(\mathscr{B}) \xleftarrow{\mathsf{S}} \mathscr{B}/F\mathscr{A}$$

where $\mathsf{T}(A, B, f) := (FA, B, f)$ for every object (A, B, f) of $F\mathscr{A}/\mathscr{B}$, and T assigns to any morphism $(g, g') : (A, B, f) \to (A', B', f')$ the commutative square (1.1.29), regarded as a morphism $(FA, B, f) \to (FA', B', f')$ in $\mathsf{Morph}(\mathscr{B})$. Likewise one describes the functor S.

Notice also the natural transformation

$$\mathsf{Morph}(\mathscr{C}) \underset{\mathsf{t}}{\overset{\mathsf{s}}{\rightrightarrows}} \mathscr{C} \qquad \Downarrow \mathsf{m}$$

where $\mathsf{m}(A, B, f) := f$ for every $(A, B, f) \in \mathrm{Morph}(\mathscr{C})$. Furthermore, every functor $F : \mathscr{B} \to \mathscr{C}$ induces a functor

$$\mathsf{Morph}(F) : \mathsf{Morph}(\mathscr{B}) \to \mathsf{Morph}(\mathscr{C})$$

which maps $(A, B, f) \in \mathrm{Morph}(\mathscr{B})$ to $(FA, FB, Ff) \in \mathrm{Morph}(\mathscr{C})$ and which sends every commutative square diagram D in $\mathscr{B}$ to the commutative square diagram FD in $\mathscr{C}$.

Notice that a natural transformation α as in (1.1.8) is equivalent to the datum of a functor

$$\widetilde{\alpha} : \mathscr{A} \to \mathsf{Morph}(\mathscr{B}) \qquad \text{such that} \qquad \mathsf{s} \circ \widetilde{\alpha} = F \qquad \text{and} \qquad \mathsf{t} \circ \widetilde{\alpha} = G.$$

Namely, one defines $\widetilde{\alpha}$ by the rule : $A \mapsto (FA, GA, \alpha_A)$ for every $A \in \mathrm{Ob}(\mathscr{A})$, and for every morphism $f : A \to B$ in $\mathscr{A}$, one lets $\widetilde{\alpha}(f)$ be the commutative square diagram (1.1.9).

1.1.31. Let $\mathscr{A}, \mathscr{B}$ be two categories, $F : \mathscr{A} \to \mathscr{B}$ a functor, $G : \mathscr{B} \to \mathscr{A}$ a left adjoint for F, and ϑ an adjunction for the pair (G, F). Then for every $A \in \mathrm{Ob}(\mathscr{A})$ the functor $F_{|A} : \mathscr{A}/A \to \mathscr{B}/FA$ of (1.1.26) admits a left adjoint that we denote

$$G_{|A} : \mathscr{B}/FA \to \mathscr{A}/A.$$

Namely, to every $(f : B \to FA) \in \mathrm{Ob}(\mathscr{B}/FA)$ we assign the object $(\vartheta^{-1}_{AB}(f) : GB \to A) \in \mathrm{Ob}(\mathscr{A}/A)$, and to every morphism $h/FA : (f : B \to FA) \to (f' : B' \to FA)$ in $\mathscr{B}/FA$ we assign the morphism $Gh/A : \vartheta^{-1}_{AB}(f) \to \vartheta^{-1}_{AB'}(f')$ in $\mathscr{A}/A$. Indeed, ϑ induces an adjunction $\vartheta_{|A}$ for the pair $(G_{|A}, F_{|A})$: to every $(f : B \to FA) \in \mathrm{Ob}(\mathscr{B}/FA)$ and $(g : A' \to A) \in \mathrm{Ob}(\mathscr{A}/A)$ we assign the bijection

$$(\vartheta_{|A})_{g,f} : \mathrm{Hom}_{\mathscr{A}/A}(\vartheta^{-1}_{AB}(f), g) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{B}/FA}(f, Fg) \qquad h/A \mapsto \vartheta_{GB,A'}(h)/FA.$$

Dually, since F^o is left adjoint to G^o (remark 1.1.19(iv)), we see that for every $B \in \mathrm{Ob}(\mathscr{B})$ the functor ${}_{B|}G : B/\mathscr{B} \to GB/\mathscr{A}$ of (1.1.26) admits the right adjoint

$${}_{B|}F : GB/\mathscr{A} \to B/\mathscr{B} \qquad (GB \xrightarrow{f} A) \mapsto (B \xrightarrow{\vartheta_{AB}(f)} FA) \qquad GB/h \mapsto B/Fh.$$

The detailed verifications shall be left to the reader. See example 1.2.28 for a related result.

1.1.32. In the situation of (1.1.31) we get furthermore for every $A \in \mathrm{Ob}(\mathscr{A})$ an isomorphism of categories :

$$\vartheta^A : G\mathscr{B}/A \xrightarrow{\sim} \mathscr{B}/FA \qquad (f : GB \to A) \mapsto (\vartheta_{AB}(f) : B \to FA)$$

that assigns to every morphism $g : (f : GB \to A) \to (f' : GB' \to A)$ of $G\mathscr{B}/A$ the morphism $g : (\vartheta_{AB}(f) : B \to FA) \to (\vartheta_{AB'}(f') : B' \to FA)$ of $\mathscr{B}/FA$. Also, for every morphism $h : A \to A'$ of $\mathscr{A}$, we get a commutative diagram of categories :

$$\begin{array}{ccc} G\mathscr{B}/A & \xrightarrow{\vartheta^A} & \mathscr{B}/FA \\ {\scriptstyle G\mathscr{B}/h}\downarrow & & \downarrow{\scriptstyle (Fh)_*} \\ G\mathscr{B}/A' & \xrightarrow{\vartheta^{A'}} & \mathscr{B}/FA'. \end{array}$$

Clearly, the isomorphisms ϑ^A are restrictions of a single isomorphism of categories :

$$G\mathscr{B}/\mathscr{A} \xrightarrow{\sim} \mathscr{B}/F\mathscr{A}.$$

The detailed verifications shall be again left to the reader.

1.2. **Presheaves and limits.** A very important construction associated with every category $\mathscr{C}$ is the category

$$\mathscr{C}^\wedge_{\mathsf{U}} := \mathsf{Fun}(\mathscr{C}^o, \mathsf{U}\text{-}\mathbf{Set})$$

whose objects are called the U-*presheaves* on $\mathscr{C}$ (notation of (1.1.10)). We usually drop the subscript U, unless we have to deal with more than one universe. If U' is another universe with $\mathsf{U} \subset \mathsf{U}'$, the natural inclusion of categories :

$$\mathscr{C}^\wedge_{\mathsf{U}} \to \mathscr{C}^\wedge_{\mathsf{U}'} \tag{1.2.1}$$

is fully faithful (verification left to the reader). For every functor $F : \mathscr{B} \to \mathscr{C}$ and every natural transformation $\alpha : F \Rightarrow G$ we set

$$F^\wedge_{\mathsf{U}} := \mathsf{Fun}(F^o, \mathbf{Set}) : \mathscr{C}^\wedge_{\mathsf{U}} \to \mathscr{B}^\wedge_{\mathsf{U}} \qquad \alpha^\wedge_{\mathsf{U}} := \mathsf{Fun}(\alpha^o, \mathbf{Set}) : G^\wedge_{\mathsf{U}} \Rightarrow F^\wedge_{\mathsf{U}} \tag{1.2.2}$$

(notation of remark 1.1.19(i,ii)). Again, we shall drop the subscript and write simply $F^\wedge$ and $\alpha^\wedge$, unless the omission of U may be a source of ambiguities. Clearly, for every pair of functors $F_1 : \mathscr{A} \to \mathscr{B}$ and $F_2 : \mathscr{B} \to \mathscr{C}$ we have

$$(F_2 \circ F_1)^\wedge = F_1^\wedge \circ F_2^\wedge.$$

Moreover, in the situation of (1.1.10), remark 1.1.19(i,iii) yields the identities

$$(\beta \odot \alpha)^\wedge = \alpha^\wedge \odot \beta^\wedge \qquad \text{and} \qquad (\alpha' * \alpha)^\wedge = \alpha^\wedge * \alpha'^\wedge. \tag{1.2.3}$$

1.2.4. If $\mathscr{C}$ has small Hom-sets (see (1.1.1)), there is a natural functor

$$h_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}^\wedge$$

called the *Yoneda embedding*, which assigns to every $X \in \mathrm{Ob}(\mathscr{C})$ the functor

$$h_X : \mathscr{C}^o \to \mathbf{Set} \qquad Y \mapsto \mathrm{Hom}_{\mathscr{C}}(Y, X) \quad \text{for every } Y \in \mathrm{Ob}(\mathscr{C})$$

and to any morphism $f : X \to X'$ in $\mathscr{C}$, the natural transformation $h_f : h_X \Rightarrow h_{X'}$ such that

$$h_{f,Y}(g) := f \circ g \qquad \text{for every } Y \in \mathrm{Ob}(\mathscr{C}) \text{ and every } g \in \mathrm{Hom}_{\mathscr{C}}(Y, X).$$

Definition 1.2.5. Let $\mathscr{C}$ be a category, and V any universe such that $\mathscr{C}$ has V-small Hom-sets. We say that an object F of $\mathscr{C}^\wedge_{\mathsf{V}}$ is *representable in* $\mathscr{C}$, if there exists an isomorphism

$$h_X \xrightarrow{\sim} F \qquad \text{in } \mathscr{C}^\wedge_{\mathsf{V}}$$

for some $X \in \mathrm{Ob}(\mathscr{C})$, in which case we also say that X *represents* F. From the full faithfulness of (1.2.1), it follows that the representability of a presheaf is independent of the universe V.

Proposition 1.2.6 (Yoneda's lemma)**.** *With the notation of* (1.2.4)*, we have :*

(i) *The functor $h_{\mathscr{C}}$ is fully faithful.*
(ii) *Moreover, for every $F \in \mathrm{Ob}(\mathscr{C}^\wedge)$ and every $X \in \mathrm{Ob}(\mathscr{C})$ there is a natural bijection*

$$F(X) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{C}^\wedge}(h_X, F)$$

functorial in both X and F.

Proof. Clearly it suffices to check (ii). However, the sought bijection is obtained explicitly as follows. To a given $a \in F(X)$, we assign the natural transformation $\tau_a : h_X \Rightarrow F$ such that

$$\tau_{a,Y}(f) := Ff(a) \qquad \text{for every } Y \in \mathrm{Ob}(\mathscr{C}) \text{ and every } f \in h_X(Y).$$

Conversely, to a given natural transformation $\tau : h_X \Rightarrow F$ we assign $\tau_X(\mathbf{1}_X) \in F(X)$. It is easily seen that these rules establish mutually inverse bijections. The functoriality in F is immediate, and the functoriality in the argument X amounts to the commutativity of the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathscr{C}^\wedge}(h_{X'}, F) & \xrightarrow{\sim} & F(X') \\ {\scriptstyle \mathrm{Hom}_{\mathscr{C}^\wedge}(h_\varphi, F)} \downarrow & & \downarrow {\scriptstyle F(\varphi)} \\ \mathrm{Hom}_{\mathscr{C}^\wedge}(h_X, F) & \xrightarrow{\sim} & F(X) \end{array}$$

for every morphism $\varphi : X \to X'$ in $\mathscr{C}$: the verification shall also be left to the reader. □

Example 1.2.7. (i) Every representable presheaf on the category $\mathbf{Set}$ admits a natural representing object. Indeed, let $\mathbb{1}$ denote any set with one element (a canonical choice for $\mathbb{1}$ is the set $\{\varnothing\}$); then for every representable $F \in \mathrm{Ob}(\mathbf{Set}^\wedge)$ we get a natural isomorphism

$$F \xrightarrow{\sim} h_{F(\mathbb{1})}$$

as follows. For any set S, we have a natural bijection $S \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Set}}(\mathbb{1}, S)$ that assigns to every $s \in S$ the unique map $\mathbb{1}_s : \mathbb{1} \to S$ whose image is $\{s\}$. We deduce a map

$$S \times F(S) \to F(\mathbb{1}) \qquad (s, a) \mapsto F(\mathbb{1}_s)(a)$$

which is the same as a map $F(S) \to \mathrm{Hom}_{\mathbf{Set}}(S, F(\mathbb{1}))$ that realizes the sought isomorphism.

(ii) We may apply Yoneda's lemma to prove the uniqueness of the (left or right) adjoint for a given functor. Indeed, let $G : \mathscr{B} \to \mathscr{A}$ be a functor between any two categories, suppose that F and F' are both right adjoint to G, and fix adjunctions $\vartheta_{\bullet\bullet}$ and $\vartheta'_{\bullet\bullet}$ for F and respectively F'. Choose also a universe U such that both $\mathscr{A}$ and $\mathscr{B}$ are U-small. Then ϑ and ϑ' can be regarded as isomorphisms from the functor $G^\wedge \circ h_{\mathscr{A}}$ to $h_{\mathscr{B}} \circ F$ and respectively to $h_{\mathscr{B}} \circ F'$. In this situation, proposition 1.2.6(i) implies that there exists a unique isomorphism $\omega : F \xrightarrow{\sim} F'$ which makes commute the diagram

$$\begin{array}{ccc} & G^\wedge \circ h_{\mathscr{A}} & \\ {\scriptstyle \vartheta} \swarrow & & \searrow {\scriptstyle \vartheta'} \\ h_{\mathscr{B}} \circ F & \xrightarrow{h_{\mathscr{B}} * \omega} & h_{\mathscr{B}} \circ F'. \end{array}$$

Taking into account remark 1.1.19(iv), a dual argument yields a corresponding uniqueness assertion for left adjoints.

Remark 1.2.8. (i) Let $\mathscr{C}$ be any category, and F_1, F_2 two representable presheaves on $\mathscr{C}$, and pick $X_1, X_2 \in \mathrm{Ob}(\mathscr{C})$ with isomorphisms $\omega_i : h_{X_i} \xrightarrow{\sim} F_i$ ($i = 1, 2$). It follows from proposition 1.2.6(i) that for every morphism $f : F_1 \to F_2$ in $\mathscr{C}^\wedge$ there exists a unique morphism $\varphi : X_1 \to X_2$ in $\mathscr{C}$ that makes commute the diagram

$$\begin{array}{ccc} h_{X_1} & \xrightarrow{h_\varphi} & h_{X_2} \\ \downarrow & & \downarrow \\ F_1 & \xrightarrow{f} & F_2. \end{array}$$

In this situation, we say that φ *represents* the morphism f.

(ii) Likewise, suppose that $G : \mathscr{B} \to \mathscr{C}^\wedge$ is any functor. We say that G is *representable* if there exists a functor $\gamma : \mathscr{B} \to \mathscr{C}$ with an isomorphism $G \xrightarrow{\sim} h_\mathscr{C} \circ \gamma$. It follows easily from (i) that G is representable if and only if $G(B)$ is representable in $\mathscr{C}$ for every $B \in \mathrm{Ob}(\mathscr{B})$. Moreover, any two representatives for G are isomorphic in $\mathsf{Fun}(\mathscr{B}, \mathscr{C})$. Furthermore, if $G, G' : \mathscr{B} \to \mathscr{C}$ are any two representable functors, and $\gamma, \gamma' : \mathscr{B} \to \mathscr{C}$ two corresponding representing functors, then any natural transformation $t : G \Rightarrow G'$ is *represented* uniquely by a natural transformation $\tau : \gamma \Rightarrow \gamma'$, by which we mean that $t = h_\mathscr{C} * \tau$.

1.2.9. *Limits and colimits.* We wish to explain some standard constructions of presheaves that are used pervasively throughout this work. Namely, let I be a small category, $\mathscr{C}$ a category and X any object of $\mathscr{C}$. We denote by $c_X : I \to \mathscr{C}$ the *constant functor* associated with X :

$$c_X(i) := X \quad \text{for every } i \in \mathrm{Ob}(I) \qquad c_X(\varphi) := \mathbf{1}_X \quad \text{for every } \varphi \in \mathrm{Morph}(I).$$

Any morphism $f : X' \to X$ induces a natural transformation

$$c_f : c_{X'} \Rightarrow c_X \qquad \text{by the rule} \ : \ (c_f)_i := f \text{ for every } i \in I.$$

If $F : I \to \mathscr{C}$ is any functor, a *cone of vertex X and basis F* is any natural transformation $c_X \Rightarrow F$. Dually, a *cocone with vertex X and basis F* is a natural transformation $F \Rightarrow c_X$.

Definition 1.2.10. With the notation of (1.2.9), let V be any universe with $\mathsf{U} \subset \mathsf{V}$, such that $\mathscr{C}$ has V-small Hom-sets, and let $F : I \to \mathscr{C}$ be any functor.

(i) The *limit* of F is the V-presheaf on $\mathscr{C}$ denoted

$$\lim_I F : \mathscr{C}^o \to \mathsf{V}\text{-}\mathbf{Set}$$

that assigns to every $X \in \mathrm{Ob}(\mathscr{C})$, the set $\lim_I F(X)$ of all cones $c_X \Rightarrow F$. Any morphism $f : X' \to X$ of $\mathscr{C}$ induces the map

$$\lim_I F(f) : \lim_I F(X) \to \lim_I F(X') \qquad \tau \mapsto \tau \odot c_f \quad \text{for every } \tau : c_X \Rightarrow F.$$

Then we also say that I is the *indexing category* for the limit of F.

(ii) Dually, the *colimit* of F is the V-presheaf on $\mathscr{C}^o$

$$\operatorname*{colim}_I F := \lim_{I^o} F^o.$$

Remark 1.2.11. Let $\mathscr{C}, \mathscr{C}'$ be any two categories with V-small Hom-sets for some universe V such that $\mathsf{U} \subset \mathsf{V}$, and I, I' two small categories.

(i) Any diagram of functors

$$I' \xrightarrow{\varphi} I \xrightarrow{F} \mathscr{C} \xrightarrow{H} \mathscr{C}'$$

induces a natural morphism

$$\lim_\varphi H : \lim_I F \to H^\wedge(\lim_{I'} H \circ F \circ \varphi) \qquad \text{in } \mathscr{C}^\wedge_\mathsf{V}$$

(notation of (1.2.2)) by ruling that

$$\lim_\varphi H(X)(\tau) := H * \tau * \varphi \qquad \text{for every } X \in \mathrm{Ob}(\mathscr{C}) \text{ and every cone } \tau : c_X \Rightarrow F$$

(notation of (1.1.10)). If $\psi : I'' \to I'$ and $K : \mathscr{C}' \to \mathscr{C}''$ are any two other functors, with I'' also small and $\mathscr{C}''$ with V-small Hom-sets, the resulting diagram in $\mathscr{C}^\wedge_\mathsf{V}$ commutes :

(1.2.12)
$$\begin{array}{ccc}
\lim_I F & \xrightarrow{\lim_{\varphi\circ\psi} K\circ H} & H^\wedge \circ K^\wedge(\lim_{I''} K \circ H \circ F \circ \varphi \circ \psi) \\
{\scriptstyle \lim_\varphi H}\downarrow & \nearrow {\scriptstyle H^\wedge(\lim_\psi K^\wedge)} & \\
H^\wedge(\lim_{I'} H \circ F \circ \varphi). & &
\end{array}$$

The reader may spell out the corresponding assertions for colimits.

(ii) Any diagram of functors and natural transformations

$$I' \underset{\varphi_2}{\overset{\varphi_1}{\rightrightarrows}} I \underset{F_2}{\overset{F_1}{\rightrightarrows}} \mathscr{C} \qquad (\Downarrow\alpha,\ \Downarrow g)$$

induces commutative diagrams

$$\begin{array}{ccc}
\lim_I F_1 & \xrightarrow{\lim_I g} & \lim_I F_2 \\
{\scriptstyle \lim_{\varphi_1} \mathbf{1}_\mathscr{C}}\downarrow & & \downarrow{\scriptstyle \lim_{\varphi_2} \mathbf{1}_\mathscr{C}} \\
\lim_{I'}(F_1\circ\varphi_1) & \xrightarrow{\lim_{I'}(g*\alpha)} & \lim_{I'}(F_2\circ\varphi_2)
\end{array}
\qquad
\begin{array}{ccc}
\mathrm{colim}_I F_2 & \xrightarrow{\mathrm{colim}_I g} & \mathrm{colim}_I F_1 \\
\downarrow{\scriptstyle \mathrm{colim}_{\varphi_2} \mathbf{1}_\mathscr{C}} & & \downarrow{\scriptstyle \mathrm{colim}_{\varphi_1} \mathbf{1}_\mathscr{C}} \\
\mathrm{colim}_{I'}(F_2\circ\varphi_2) & \xrightarrow{\mathrm{colim}_{I'}(g*\alpha)} & \mathrm{colim}_{I'}(F_1\circ\varphi_1)
\end{array}$$

where the top (resp. bottom) arrow of the left diagram is given by the rule :

$$\tau \mapsto g \odot \tau \qquad (\text{ resp. } \tau \mapsto (g * \alpha) \odot \tau\ )$$

for every $X \in \mathrm{Ob}(\mathscr{C})$ and every cone $\tau : c_X \Rightarrow F_1$ (resp. $\tau : c_X \Rightarrow F_1 \circ \varphi_1$). The top (resp. bottom) arrow of the right diagram is given by the rule

$$\tau \mapsto \tau \odot g \qquad (\text{ resp. } \tau \mapsto \tau \odot (g * \alpha)\ )$$

for every $X \in \mathrm{Ob}(\mathscr{C})$ and every cocone $\tau : F_2 \Rightarrow c_X$ (resp. $\tau : F_2 \circ \varphi_1 \Rightarrow c_X$).

(iii) Suppose that $\lim_I F$ is representable by $L \in \mathrm{Ob}(\mathscr{C})$, so we have an isomorphism

(1.2.13)
$$\omega : h_L \xrightarrow{\sim} \lim_I F \qquad \text{in } \mathscr{C}^\wedge_\mathsf{V}.$$

Notice that the cone $\tau := \omega_L(\mathbf{1}_L)$ determines ω : indeed the latter assigns, to every $X \in \mathrm{Ob}(\mathscr{C})$, the bijection :

$$\omega_X : \mathrm{Hom}_\mathscr{C}(X, L) \xrightarrow{\sim} \lim_I F(X) \qquad f \mapsto \tau \odot c_f.$$

Conversely, we say that a given $\tau : c_L \Rightarrow F$ is a *universal cone*, if the induced map ω_X is a bijection for every $X \in \mathrm{Ob}(\mathscr{C})$, in which case the limit of F is representable by L. Clearly, the universal property for a cone is independent of the choice of auxiliary universe V.

Likewise, if $\mathrm{colim}_I F$ is representable by $C^o \in \mathrm{Ob}(\mathscr{C}^o)$, the choice of an isomorphism

(1.2.14)
$$h_{C^o} \xrightarrow{\sim} \mathrm{colim}_I F \qquad \text{in } (\mathscr{C}^o)^\wedge_\mathsf{V}$$

induces a *universal cocone* $\mu : F \Rightarrow c_C$, which in turns determines (1.2.14) by the rule :

$$\mathrm{Hom}_\mathscr{C}(C, X) \xrightarrow{\sim} \mathrm{colim}_I F(X^o) \qquad f \mapsto c_f \odot \mu.$$

Moreover, with the notation of (ii), suppose that the limits (resp. colimits) of F_1 and F_2 are representable by objects L_1 and L_2 of $\mathscr{C}$ (resp. C_1^o and C_2^o of $\mathscr{C}^o$), and let us fix isomorphisms as (1.2.13) and (1.2.14) for F_1 and F_2. Then :

- The limit of g is represented by a morphism $L_1 \to L_2$ in $\mathscr{C}$.
- The colimit of g is represented by a morphism $C_2^o \to C_1^o$ in $\mathscr{C}^o$, *i.e.* by a morphism $C_1 \to C_2$ in $\mathscr{C}$ (see remark 1.2.8(ii)).

(iv) Every $i \in I$ induces a morphism of presheaves

$$t_i : \lim_I F \to h_{Fi}$$

that assigns to every $X \in \mathrm{Ob}(\mathscr{C})$ and every cone $\tau : c_X \Rightarrow F$ the morphism $\tau_i : X \to Fi$, regarded as an element of $h_{Fi}(X)$. We obtain in this way a natural cone

(1.2.15) $$c_{\lim_I F} \Rightarrow h_{\mathscr{C}} \circ F \qquad i \mapsto t_i$$

where $h_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}^{\wedge}_{\mathsf{V}}$ is the Yoneda embedding. We call (1.2.15) the *tautological cone* associated with F. We claim that the tautological cone is universal. Indeed, say that G is any presheaf on $\mathscr{C}$, and $\tau : c_G \Rightarrow h_{\mathscr{C}} \circ F$ any cone. We define a morphism $f_\tau : G \to \lim_I F$ as follows. To every $i \in \mathrm{Ob}(I)$, the cone τ attaches a morphism of presheaves $\tau_i : G \to h_{Fi}$, which is the datum, for every $X \in \mathrm{Ob}(\mathscr{C})$ of a map $\tau_i^X : G(X) \to \mathrm{Hom}_{\mathscr{C}}(X, Fi)$, and for fixed $s \in G(X)$, the system

$$\tau_\bullet^X(s) := (\tau_i^X(s) : X \to Fi \mid X \in \mathrm{Ob}(\mathscr{C}))$$

is a cone $c_X \Rightarrow F$. Then we let $f_\tau(X) : G(X) \to \lim_I F(X)$ be the map such that $s \mapsto \tau_\bullet^X(s)$ for every $X \in \mathrm{Ob}(\mathscr{C})$ and every $s \in G(X)$. The rule $\tau \mapsto f_\tau$ yields an inverse for the map

$$\omega_G : \mathrm{Hom}_{\mathscr{C}^\wedge}(G, \lim_I F) \to \lim_I h_{\mathscr{C}} \circ F$$

associated, as in (iii), with the cone (1.2.15), whence the claim : details left to the reader.

Dually, with F we may also associate a *tautological cocone*

$$h_{\mathscr{C}^o} \circ F^o \Rightarrow h_{\mathrm{colim}_I F} \qquad \text{in } (\mathscr{C}^o)^\wedge_{\mathsf{V}}$$

which is a universal cocone.

Example 1.2.16. Let V be a universe containing U, and $\mathscr{C}$ any category with V-small Hom-sets.

(i) For $i = 1, 2$, let $f_i : A \to B_i$ be two morphisms in $\mathscr{C}$; the *push-out* or *amalgamated sum* of f_1 and f_2 is the colimit of the functor $F : I \to \mathscr{C}$, defined as follows. The set $\mathrm{Ob}(I)$ consists of three objects s, t_1, t_2 and $\mathrm{Morph}(I)$ consists of two morphisms

$$t_1 \xleftarrow{\varphi_1} s \xrightarrow{\varphi_2} t_2$$

(in addition to the identity morphisms of the objects of I); the functor is given by the rule : $Fs := A$, $Ft_i := B_i$ and $F\varphi_i := f_i$ (for $i = 1, 2$). For any $C \in \mathrm{Ob}(\mathscr{C})$, a cocone $F \Rightarrow c_C$ amounts to a pair of morphisms $f'_i : B_i \to C$ ($i = 1, 2$) such that $f'_1 \circ f_1 = f'_2 \circ f_2$. If the cocone is universal (and thus, C represents the push-out of f_1 and f_2), we say that the resulting commutative diagram :

$$\begin{array}{ccc} A & \xrightarrow{f_1} & B_1 \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle f'_1} \\ B_2 & \xrightarrow{f'_2} & C \end{array}$$

is *cocartesian*. Dually one defines the *fibre product* or *pull-back* of two morphisms $g_i : A_i \to B$ ($i = 1, 2$). If $D \in \mathrm{Ob}(\mathscr{C})$ represents this fibre product, a cone with vertex D is given by a pair of morphisms $g'_i : D \to A_i$ ($i = 1, 2$) such that $g_1 \circ g'_1 = g_2 \circ g'_2$, and we say that the diagram :

$$\begin{array}{ccc} D & \xrightarrow{g'_1} & A_1 \\ {\scriptstyle g'_2}\downarrow & & \downarrow{\scriptstyle g_1} \\ A_2 & \xrightarrow{g_2} & B \end{array}$$

is *cartesian* if this cone is universal. The push-out of f_1 and f_2 is a V-presheaf, and is usually just called the amalgamated sum of B_1 and B_2 over A, denoted

$$B_1 \amalg_{(f_1, f_2)} B_2 \qquad \text{or simply} \qquad B_1 \amalg_A B_2$$

unless the notation gives rise to ambiguities. Likewise one writes

$$A_1 \times_{(g_1,g_2)} A_2 \qquad \text{or just} \qquad A_1 \times_B A_2$$

for the fibre product of g_1 and g_2, which is a V-presheaf as well.

(ii) Similarly, consider the category I' with $\mathrm{Ob}(I') = \{s, t\}$, and with $\mathrm{Morph}(I')$ consisting of two morphisms

$$s \overset{\psi_1}{\underset{\psi_2}{\rightrightarrows}} t$$

(in addition to $\mathbf{1}_s$ and $\mathbf{1}_t$). If $A, B \in \mathrm{Ob}(\mathscr{C})$ are any two objects and $f_1, f_2 : A \to B$ any two morphisms, we get a functor $F' : I' \to \mathscr{C}$ by the rule : $s \mapsto A$, $t \mapsto B$ and $\psi_i \mapsto f_i$ for $i = 1, 2$. Then, the colimit of F' is also called the *coequalizer* of f_1 and f_2, and is sometimes denoted

$$\mathrm{Coequal}(f_1, f_2).$$

Dually, the limit of F' is also called the *equalizer* of f_1 and f_2, sometimes denoted

$$\mathrm{Equal}(g_1, g_2).$$

(iii) Let $I \in \mathsf{U}$ be any small set, and $\underline{B} := (B_i \mid i \in I)$ any family of objects of $\mathscr{C}$. We may regard I as a discrete category (see example 1.1.6(ii)), and then the rule $i \mapsto B_i$ yields a functor $I \to \mathscr{C}$, whose limit (resp. colimit) is called the *product* (resp. *coproduct*) of the family $\underline{B}$, and is denoted

$$\prod_{i \in I} B_i \qquad (\text{resp. } \coprod_{i \in I} B_i\ ).$$

If $I = \{1, 2\}$ is a set with exactly two elements, we also write $B_1 \times B_2$ (resp. $B_1 \amalg B_2$) for this limit (resp. colimit), and we call it sometimes a *binary product* (resp. a *binary coproduct*).

(iv) Let $\mathscr{B}$ be any category, and suppose that a given $X \in \mathrm{Ob}(\mathscr{B})$ represents the *empty product* in $\mathscr{B}$, *i.e.* the product of an empty family of objects of $\mathscr{B}$. This means precisely that $\mathrm{Hom}_{\mathscr{B}}(Y, X)$ consists of exactly one element, for every $Y \in \mathrm{Ob}(\mathscr{B})$. Any such X is called a *final object of $\mathscr{B}$*.

Dually, we say that X is an *initial object of $\mathscr{B}$*, if $\mathrm{Hom}_{\mathscr{B}}(X, Y)$ consists of a single element, for every $Y \in \mathrm{Ob}(\mathscr{B})$. Then an object of $\mathscr{B}$ is initial if and only if it represents the *empty coproduct* in $\mathscr{B}$.

Moreover, we shall say that an object X of $\mathscr{B}$ is *disconnected*, if there exist $A, B \in \mathrm{Ob}(\mathscr{B})$, neither of which is an initial object of $\mathscr{B}$, and such that X represents the coproduct $A \amalg B$ (which, again, is well defined in any sufficiently large universe V). We say that X is *connected*, if X is not disconnected and is not an initial object of $\mathscr{B}$.

(v) For instance, the initial object of $\mathbf{Set}$ is the empty set, and any set $\mathbb{1}$ with one element is a final object of $\mathbf{Set}$. More generally, if $\mathscr{B}$ is any category, the initial object of $\mathscr{B}^\wedge$ is the presheaf $\varnothing_{\mathscr{B}}$ such that $\varnothing_{\mathscr{B}}(X) = \varnothing$ for every $X \in \mathrm{Ob}(\mathscr{B})$, and the presheaf $\mathbb{1}_{\mathscr{B}}$ such that $\mathbb{1}_{\mathscr{B}}(X) = \mathbb{1}$ for every $X \in \mathrm{Ob}(\mathscr{B})$ is a final object.

1.2.17. Let $\mathscr{C}$ be a category, $f : A \to B$ a morphism in $\mathscr{C}$; as explained in example 1.2.16(i), the pair of morphisms $B \xleftarrow{\mathbf{1}_B} B \xrightarrow{\mathbf{1}_B} B$ can be regarded as an element $\tau \in (B \amalg_{(f,f)} B)(B)$, which, by Yoneda's lemma (proposition 1.2.6(ii)) corresponds to a morphism

$$\pi_f^o : h_B \to B \amalg_A B \qquad \text{in } (\mathscr{C}^o)^\wedge.$$

If $B \amalg_A B$ is representable in $\mathscr{C}$, this is the same as a morphism

$$\pi_f : B \amalg_A B \to B \qquad \text{in } \mathscr{C}$$

called the *codiagonal of f*. Dually, we have a natural morphism :

$$\iota_f : h_A \to A \times_B A \qquad \text{in } \mathscr{C}^\wedge$$

which – in case $A \times_B A$ is representable in $\mathscr{C}$ – is the same as a *diagonal morphism of f*

$$\iota_f : A \to A \times_B A \qquad \text{in } \mathscr{C}.$$

Proposition 1.2.18. (i) *With the notation of* (1.2.17)*, the following conditions are equivalent :*

(a) *$f : A \to B$ is a monomorphism in $\mathscr{C}$*
(b) *$\iota_f : h_A \to A \times_B A$ is an isomorphism in $\mathscr{C}^\wedge$.*

In case $A \times_B A$ is representable in $\mathscr{C}$, denote by $A \xleftarrow{p_1} A \times_B A \xrightarrow{p_2} A$ the universal cone of this fibre product. Then (a) *and* (b) *are moreover equivalent to :*

(c) *$\iota_f : A \to A \times_B A$ is an isomorphism in $\mathscr{C}$.*
(d) *$\iota_f : A \to A \times_B A$ is an epimorphism in $\mathscr{C}$.*
(e) $p_1 = p_2$.

(ii) *Dually, the following conditions are equivalent :*

(a) *$f : A \to B$ is an epimorphism in $\mathscr{C}$.*
(b) *$\pi^o_f : h_B \to B \amalg_A B$ is an isomorphism in $(\mathscr{C}^o)^\wedge$.*

In case $B \amalg_A B$ is representable in $\mathscr{C}$, denote by $B \xrightarrow{e_1} B \amalg_A B \xleftarrow{e_2} B$ the universal cocone of this amalgamated sum. Then (a) *and* (b) *are moreover equivalent to :*

(c) *$\pi_f : B \amalg_A B \to B$ is an isomorphism in $\mathscr{C}$.*
(d) *$\pi_f : B \amalg_A B \to B$ is a monomorphism in $\mathscr{C}$.*
(e) $e_1 = e_2$.

Proof. (i.a)⇔(i.b): For every $X \in \mathrm{Ob}(\mathscr{C})$ the map $\iota_{f,X} : h_A(X) \to A \times_B A(X)$ is given by the rule : $(X \xrightarrow{g} A) \mapsto (A \xleftarrow{g} X \xrightarrow{g} A)$. Hence, ι_f is an isomorphism if and only if for every $X \in \mathrm{Ob}(\mathscr{C})$ and every pair of morphisms $g_1, g_2 : X \to A$ such that $f \circ g_1 = f \circ g_2$, we have $g_1 = g_2$. This means precisely that f is a monomorphism.

(i.b)⇔(i.c) is clear, since the Yoneda imbedding $h_\mathscr{C}$ is fully faithful.

(i.c)⇒(i.d) is trivial.

(i.d)⇒(i.e): By definition, $p_1 \circ \iota_f = \mathbf{1}_A = p_2 \circ \iota_f$. If ι_f is an epimorphism, then $p_1 = p_2$.

(i.e)⇒(i.c): Set $p := p_1$; since $p \circ \iota_f = \mathbf{1}_A$, it suffices to show that $\iota_f \circ p = \mathbf{1}_{A \times_B A}$. Due to the universality of the cone $A \xleftarrow{p_1} A \times_B A \xrightarrow{p_2} A$, we are then reduced to checking that $p_1 \circ \iota_f \circ p = p_1$ and $p_2 \circ \iota_f \circ p = p_2$, which is clear, since by assumption $p = p_2$.

Assertion (ii) follows from (i) by duality. □

Some special kinds of limits occur frequently in applications; we gather some of them in the following :

Definition 1.2.19. Let I be a category.

(i) We say that I is *finite* if both $\mathrm{Ob}(I)$ and $\mathrm{Morph}(I)$ are finite sets.

(ii) We say that I is *connected*, if $\mathrm{Ob}(I) \neq \varnothing$ and every two objects i, j can be connected by a finite sequence of morphisms in I :

$$i \to k_1 \leftarrow k_2 \to \cdots \leftarrow k_n \to j. \tag{1.2.20}$$

(iii) We say that I is *directed*, if for every $i, j \in \mathrm{Ob}(I)$ there exist $k \in \mathrm{Ob}(\mathscr{C})$ and morphisms $i \to k \leftarrow j$ in I. We say that I is *codirected*, if I^o is directed.

(iv) We say that I is *locally directed* if, for every $i \in \mathrm{Ob}(I)$, the category i/I is directed. We say that I is *locally codirected*, if I^o is locally directed.

(v) We say that I is *pseudo-filtered* if it is locally directed, and the following *coequalizing condition* holds. For any $i, j \in \mathrm{Ob}(I)$, and any two morphisms $f, g : i \to j$, there exist $k \in \mathrm{Ob}(I)$ and a morphism $h : j \to k$ such that $h \circ f = h \circ g$.

(vi) We say that I is *filtered*, if it is pseudo-filtered and connected. We say that I is *cofiltered* if I^o is filtered.

(vii) Let $F : I \to \mathscr{C}$ be a functor to any category $\mathscr{C}$. Then $\lim_I F$ is a well defined V-presheaf, for any sufficiently large universe V, and we say that *the limit of* F *is small* (resp. *connected*, resp. *codirected*, resp. *locally codirected*, resp. *cofiltered*, resp. *finite*), if I is small (resp. connected, resp. codirected, resp. locally codirected, resp. cofiltered, resp. finite). Dually, one defines *small*, *connected*, *directed*, *locally directed*, *filtered* and *finite* colimits.

(viii) We say that a category $\mathscr{C}$ is *complete* (resp. *finitely complete*) if every small limit (resp. finite limit) of $\mathscr{C}$ is representable in $\mathscr{C}$. We say that $\mathscr{C}$ is *cocomplete* (resp. *finitely cocomplete*) if every small colimit (resp. finite colimit) of $\mathscr{C}$ is representable in $\mathscr{C}^o$.

Remark 1.2.21. Let I be any category.

(i) If I is connected and locally directed, then I is directed. Indeed, let i, j be any two objects of I; we have to show that there exist $k \in \mathrm{Ob}(I)$ and morphisms $a : i \to k$ and $b : j \to k$. However, by assumption there exists a sequence (1.2.20), for some $n \in \mathbb{N}$ and objects $k_1, \dots, k_n$ of I. Then, a simple induction reduces the assertion to the case where $n = 2$. But since I is locally directed, we may then find $k \in \mathrm{Ob}(I)$ and morphisms $c : k_1 \to k$, $b : j \to k$, so it suffices to take $a := c \circ d$, where $d : i \to k_1$ is the morphism appearing in (1.2.20). Dually, if I is connected and locally codirected, then I is codirected.

It follows that a category I is filtered if and only if it is directed, satisfies the coequalizing condition of definition 1.2.19(v), and its set of objects is non-empty.

(ii) Let I be a small category. It is easily seen that I is connected if and only if it is a connected object of the category $\mathbf{Cat}$ (see example 1.2.16(iv)). In general, there is a natural decomposition in $\mathbf{Cat}$:

$$I \stackrel{\sim}{\to} \coprod_{s \in S} I_s$$

where each I_s is a connected category, and S is a small set. We call $\{I_s \mid s \in S\}$ the *set of connected components* of I, and we denote it

$$\pi_0(I).$$

Especially, every small pseudo-filtered category is a coproduct of a (small) family of filtered categories. This decomposition induces natural isomorphisms in $\mathscr{A}^\wedge$:

$$\operatorname*{colim}_I F \stackrel{\sim}{\to} \coprod_{s \in S} \operatorname*{colim}_{I_s} F \circ e_s \qquad \lim_I F \stackrel{\sim}{\to} \prod_{s \in S} \lim_{s \in S} F \circ e_s$$

for any functor $F : I \to \mathscr{A}$, where $e_s : I_s \to I$ is the natural inclusion functor, for every $s \in S$. The details shall be left to the reader.

Proposition 1.2.22. *For any category* $\mathscr{C}$ *we have :*

- (i) $\mathscr{C}$ *is complete (resp. finitely complete) if and only if the following conditions hold :*
  - (a) *The product of every small (resp. finite) family of objects is representable in* $\mathscr{C}$.
  - (b) *All equalizers are representable in* $\mathscr{C}$.
- (ii) *Dually,* $\mathscr{C}$ *is cocomplete (resp. finitely cocomplete) if and only if the following holds :*
  - (a) *The coproduct of any small (resp. finite) family of objects is representable in* $\mathscr{C}$.
  - (b) *All coequalizers are representable in* $\mathscr{C}$.

Proof. Clearly it suffices to show (i). Now, conditions (a) and (b) obviously hold if $\mathscr{C}$ is complete. Conversely, suppose that the conditions hold, let I be any small (resp. finite) category, and $F : I \to \mathscr{C}$ any functor. We regard $\mathrm{Ob}(I)$ (resp. $\mathrm{Morph}(I)$) as a discrete subcategory of I (resp. of $\mathsf{Morph}(I)$: see example 1.1.6(ii)), and we consider the diagram

$$\mathrm{Morph}(I) \underset{\mathrm{t}}{\overset{\mathrm{s}}{\rightrightarrows}} \mathrm{Ob}(I) \xrightarrow{\mathrm{ob}} I$$

where s and t are respectively the restrictions of the functors s and t from (1.1.30), and ob is the inclusion functor. We let as well $\mathrm{m} : \mathrm{ob} \circ \mathrm{s} \Rightarrow \mathrm{ob} \circ \mathrm{t}$ be the restriction of the natural transformation m of (1.1.30). By assumption, we may find objects $P, Q \in \mathrm{Ob}(\mathscr{C})$ with isomorphisms

$$\omega_P : h_P \xrightarrow{\sim} L_1 := \lim_{\mathrm{Ob}(I)} F \circ \mathrm{ob} \qquad \omega_Q : h_Q \xrightarrow{\sim} L_2 := \lim_{\mathrm{Morph}(I)} F \circ \mathrm{ob} \circ \mathrm{t} \qquad \text{in } \mathscr{C}^\wedge_{\mathsf{V}}$$

for any universe V such that $\mathscr{C}$ has V-small Hom-sets. We define two morphisms of presheaves $a, b : L_1 \to L_2$ by the rules :

$$a(\mu) := \mu * \mathrm{t} \qquad \text{and} \qquad b(\mu) := (F * \mathrm{m}) \odot (\mu * \mathrm{s})$$

for every $X \in \mathrm{Ob}(\mathscr{C})$ and every cone $\mu : c_X \Rightarrow F \circ \mathrm{ob}$. Then a and b are represented by unique morphisms $\alpha, \beta : P \to Q$ in $\mathscr{C}$ (see example 1.2.8(i)) such that

$$a \circ \omega_P = \omega_Q \circ h_\alpha \qquad \text{and} \qquad b \circ \omega_P = \omega_Q \circ h_\beta.$$

Lastly, let $E := \mathrm{Equal}(\alpha, \beta)$ (notation of example 1.2.16(ii)), and notice that the tautological cone for E yields a monomorphism $E \to h_P$, so we may regard E as a subobject of h_P; it follows easily that ω_P restricts to an isomorphism between $E \subset h_P$ and the presheaf $E' \subset L_1$ such that, for every $X \in \mathrm{Ob}(\mathscr{C})$, the set $E'(X)$ consists of all the cones $\mu : c_X \Rightarrow F \circ \mathrm{ob}$ with $a(\mu) = b(\mu)$. A simple inspection shows that the latter condition describes precisely the cones $c_X \Rightarrow F$; on the other hand, by assumption E is representable in $\mathscr{C}$, so the same holds for the limit of F. □

Example 1.2.23. (i) The category $\mathbf{Set}$ is complete and cocomplete. More precisely, for any small family of sets $S_\bullet := (S_i \mid i \in I)$ we have natural representatives for the product and coproduct of $S_\bullet$: namely, for the product one may take the usual cartesian product of the sets S_i, and for the coproduct one may take the disjoint union

$$\coprod_{i \in I} S_i := \bigcup_{i \in I} (\{i\} \times S_i).$$

Likewise, we have natural representatives for the equalizer and coequalizer of any pair of maps $f, g : S \to S'$: namely, the equalizer of f and g is represented by the subset

$$\mathrm{Equal}(f, g) := \{s \in S \mid f(s) = g(s)\}$$

and the coequalizer is represented by the quotient $S'/\sim$, where $\sim$ is the smallest equivalence relation on S' whose graph contains the subset $\{(f(s), g(s)) \mid s \in S\}$. Taking into account proposition 1.2.22(i,ii), we get natural representatives for all limits and colimits in $\mathbf{Set}$. For instance, by inspecting the proof of proposition 1.2.22(i), we see that the colimit of a functor $F : I \to \mathbf{Set}$ (for any small category I) is represented by the quotient

$$C(F) := \Big(\coprod_{i \in \mathrm{Ob}(I)} Fi \Big) / \sim$$

where $\sim$ is the smallest equivalence relation such that $(i, s) \sim (j, F(\varphi)(s))$ for every morphism $\varphi : i \to j$ in I and every $s \in Fi$. In the special case where I is filtered, this equivalence relation can be described more explicitly : namely $(i, s) \sim (i', s')$ if and only if there exist $j \in \mathrm{Ob}(I)$ and morphisms $\varphi : i \to j$, $\varphi' : i' \to j$ such that $F(\varphi)(s) = F(\varphi')(s')$.

For a general small category I, to every $i \in \mathrm{Ob}(I)$ we may attach a natural map $\tau_i : Fi \to C(F)$: namely, $\tau_i(s)$ is the class of (i, s) in $C(F)$, for every $s \in Fi$; then the system $(\tau_i \mid i \in \mathrm{Ob}(I))$ yields a universal cocone $\tau : F \Rightarrow c_{C(F)}$, in the sense of remark 1.2.11(iii).

(ii) In view of (i) and example 1.2.7, we can also give a more compact description of the limit of any functor $F : I \to \mathbf{Set}$: namely, if $\mathbb{1}$ is any fixed set with one element, the set

$L(F) := \lim_I F(\mathbb{1})$, consisting of all cones $c_{\mathbb{1}} \Rightarrow F$, is another natural representative for the limit of F. More precisely, we get a natural isomorphism

$$h_{L(F)} \xrightarrow{\sim} \lim_I F \qquad \text{in } \mathbf{Set}^\wedge$$

as follows. A map of sets $S \to L(F)$ amounts to a system of cones $\{\tau_s : c_{\mathbb{1}} \Rightarrow F \mid s \in S\}$, and the latter is naturally identified with a cone $c_S \Rightarrow F$ (details left to the reader). We easily deduce the following natural universal cone $\tau : c_{L(F)} \Rightarrow F$. For every $i \in \mathrm{Ob}(I)$, the map $\tau_i : L(F) \to Fi$ assigns to any cone $\mu : c_{\mathbb{1}} \Rightarrow F$ the image of $\mu_i : \mathbb{1} \to Fi$.

Example 1.2.24. (i) More generally, for every category $\mathscr{C}$, the category $\mathscr{C}^\wedge$ is complete and cocomplete. Indeed, let I be any small category, and $F : I \to \mathscr{C}^\wedge$ any functor. For every $X \in \mathrm{Ob}(\mathscr{C})$ we obtain a functor $F_X : I \to \mathbf{Set}$ such that $i \mapsto Fi(X)$ for every $i \in \mathrm{Ob}(I)$ and $(\varphi : i \to j) \mapsto (F\varphi(X) : Fi(X) \to Fj(X))$ for every morphism φ of I; we let LX be the natural representative for $\lim_I F_X$, and $\tau^X : c_{LX} \Rightarrow F_X$ the corresponding universal cone, as described in example 1.2.23(ii). Every morphism $f : X \to Y$ in $\mathscr{C}$ induces a natural transformation $F_f : F_Y \Rightarrow F_X$ by the rule : $i \mapsto Fi(f) : F_Y(i) \to F_X(i)$, whence a natural transformation $\lim_I F_f : \lim_I F_Y \Rightarrow \lim_I F_X$ (remark 1.2.11(ii)) which is in turn represented by a unique map of sets $Lf : LY \to LX$ such that

$$\tau^X \odot c_{Lf} = F_f \odot \tau^Y$$

(remark 1.2.8(ii)). It is then easily seen that the rules : $X \mapsto LX$ and $f \mapsto Lf$ yield a well defined presheaf L on $\mathscr{C}$. Also, for every $i \in \mathrm{Ob}(I)$ the rule $X \mapsto \tau_i^X$ yields a morphism of presheaves $\tau_i^\bullet : L \to Fi$, and the rule $i \mapsto \tau_i^\bullet$ yields a cone $\tau : c_L \Rightarrow F$. Now, let $\eta : c_G \Rightarrow F$ be any cone with vertex $G \in \mathrm{Ob}(\mathscr{C}^\wedge)$. For every $X \in \mathrm{Ob}(\mathscr{C})$ we deduce a cone $\eta^X : c_{G,X} \Rightarrow F_X$ by the rule : $i \mapsto \eta_i(X) : GX = G_X(i) \to F_X(i)$, whence a unique morphism $t_X : GX \to LX$ such that $\tau^X \odot c_{t_X} = \eta^X$. With this notation, a direct inspection shows that the diagram

$$\begin{array}{ccc} c_{G,Y} & \xRightarrow{\eta^Y} & F_Y \\ {\scriptstyle c_{G,f}} \Downarrow & & \Downarrow {\scriptstyle F_f} \\ c_{G,X} & \xRightarrow{\eta^X} & F_X \end{array}$$

commutes for every morphism $f : X \to Y$ of $\mathscr{C}$. Thus, $\tau^X \odot c_{t_X \circ G,f} = F_f \odot \tau^Y \odot c_{t_Y} = \tau^X \odot c_{Lf \circ t_Y}$, whence $t_X \circ Gf = Lf \circ t_Y$, by the universality of τ^X. In other words, the rule : $X \mapsto t_X$ yields a morphism $t : G \to L$ in $\mathscr{C}^\wedge$, and by construction, it is clear that t is the unique such morphism with $\tau \odot c_t = \eta$. This shows that τ is a universal cone, whence the completeness of $\mathscr{C}^\wedge$. A similar argument proves the cocompleteness of $\mathscr{C}^\wedge$: the details are left to the reader.

(ii) Furthermore, let $F : I \to \mathscr{C}$ be a functor from a small category I, whose limit $\mathscr{L}$ (which is an object of $\mathscr{C}^\wedge_{\mathsf{V}}$ for a suitable universe V) is representable by some $L \in \mathrm{Ob}(\mathscr{C})$. Let $\tau : c_L \Rightarrow F$ be a universal cone and denote by $\omega : h_L \xrightarrow{\sim} \mathscr{L}$ the isomorphism corresponding to τ, as in remark 1.2.11(iii).. Then the limit of $h_{\mathscr{C}} \circ F : I \to \mathscr{C}^\wedge_{\mathsf{V}}$ is represented by h_L and $h_{\mathscr{C}} * \tau : c_{h_L} \Rightarrow h_{\mathscr{C}} \circ F$ is a universal cone. Indeed, let $t : c_{\mathscr{L}} \Rightarrow h_{\mathscr{C}} \circ F$ be the tautological cone (remark 1.2.11(iv)); a simple inspection shows that the resulting diagram

$$\begin{array}{ccccc} & & c_{h_L} & & \\ & {\scriptstyle c_\omega} \swarrow & & \searrow {\scriptstyle h_{\mathscr{C}} * \tau} & \\ c_{\mathscr{L}} & & \xRightarrow{\quad t \quad} & & h_{\mathscr{C}} \circ F \end{array}$$

commutes. On the other hand, we know that t is universal, so the same follows for $h_{\mathscr{C}} * \tau$.

Example 1.2.25. (i) Also the category $\mathbf{Cat}$ is complete and cocomplete. Indeed :

• If $F : I \to \mathbf{Cat}$ is any functor from a small category I, the limit of F is representable by the category $\mathscr{C}$ with $\mathrm{Ob}(\mathscr{C}) := \lim_I \mathrm{Ob} \circ F$, where $\mathrm{Ob} : \mathbf{Cat} \to \mathbf{Set}$ is the functor of example 1.1.6(ii). Hence, an object of $\mathscr{C}$ is a family $(C_i \mid i \in \mathrm{Ob}(I))$ where $C_i \in \mathrm{Ob}(\mathscr{C}_i)$ for every $i \in I$, and $F(\varphi)(C_i) = C_j$ for every $\varphi : i \to j$ in I. For any two objects $C_\bullet := (C_i \mid i \in I)$ and $C'_\bullet := (C'_i \mid i \in I)$, notice that the rule $i \mapsto \mathrm{Hom}_{\mathscr{C}_i}(C_i, C'_i)$ defines a functor $H_{C_\bullet, C'_\bullet} : I \to \mathbf{Set}$; namely, to every $\varphi : i \to j$ in I we assign the map $H_{C_\bullet,C'_\bullet}(i) \to H_{C_\bullet,C'_\bullet}(j)$ given by $F(\varphi) : \mathrm{Morph}(\mathscr{C}_i) \to \mathrm{Morph}(\mathscr{C}_j)$. Then we set

$$\mathrm{Hom}_{\mathscr{C}}(C_\bullet, C'_\bullet) := \lim_I H_{C_\bullet,C'_\bullet}.$$

The composition of morphisms in $\mathscr{C}$ is induced by the composition laws of the categories $\mathscr{C}_i$, in the obvious way. The obvious projection functors $\mathscr{C} \to \mathscr{C}_i$ yield a universal cone for $\lim_I F$.

• Especially, for any pair of functors

$$\mathscr{A} \xrightarrow{F} \mathscr{C} \xleftarrow{G} \mathscr{B}$$

the fibre product (in the category $\mathbf{Cat}$) of F and G is represented by the category :

$$\mathscr{A} \times_{(F,G)} \mathscr{B}$$

whose set of objects is $\mathrm{Ob}(\mathscr{A}) \times_{\mathrm{Ob}(\mathscr{C})} \mathrm{Ob}(\mathscr{B})$; the morphisms $(A, B) \to (A', B')$ are the pairs (f, g) where $f : A \to A'$ (resp. $g : B \to B'$) is a morphism in $\mathscr{A}$ (resp. in $\mathscr{B}$), such that $Ff = Gg$. If the notation is not ambiguous, we may also denote this category by $\mathscr{A} \times_{\mathscr{C}} \mathscr{B}$. In case $\mathscr{B}$ is a subcategory of $\mathscr{C}$ and G is the natural inclusion functor, we also write $F^{-1}\mathscr{B}$ instead of $\mathscr{A} \times_{\mathscr{C}} \mathscr{B}$. The two obvious functors $\mathscr{A} \leftarrow \mathscr{A} \times_{\mathscr{C}} \mathscr{B} \to \mathscr{B}$ provide a universal cone.

• If $(\mathscr{C}_i \mid i \in I)$ is any small family of categories, the coproduct $\mathscr{C} := \amalg_{i\in I}\mathscr{C}_i$ is the category whose set of objects is the disjoint union $\amalg_{i\in I}\mathrm{Ob}(\mathscr{C}_i)$ (see example 1.2.23(i)). Hence, an object of $\mathscr{C}$ is a pair (i, C), with $i \in I$ and $C \in \mathrm{Ob}(\mathscr{C}_i)$. For any two such objects (i, C) and (i', C'), we have

$$\mathrm{Hom}_{\mathscr{C}}((i, C), (i', C')) = \begin{cases} \mathrm{Hom}_{\mathscr{C}_i}(C, C') & \text{if } i = i' \\ \varnothing & \text{otherwise.} \end{cases}$$

The obvious inclusion functors $\mathscr{C}_i \to \mathscr{C}$ provide a universal cocone. The proof of the cocompleteness of $\mathbf{Cat}$ shall be postponed to example 1.5.20 (see [28, Prop.5.1.7] for a more constructive proof, and also example 1.5.10).

(ii) For any category $\mathscr{A}$, the objects of the fibre product category $\mathsf{Morph}(\mathscr{A}) \times_{(\mathsf{t},\mathsf{s})} \mathsf{Morph}(\mathscr{A})$ are just the composable pairs of morphisms of $\mathscr{A}$ (notation of (1.1.30)), and the morphisms are the pairs of commutative square diagrams that have one vertical edge in common. Thus, we have a natural *composition functor*

$$\mathsf{c} : \mathsf{Morph}(\mathscr{A}) \times_{(\mathsf{t},\mathsf{s})} \mathsf{Morph}(\mathscr{A}) \to \mathsf{Morph}(\mathscr{A})$$

such that $\mathsf{c}(f, g) := g \circ f$ for every composable pair of morphisms of $\mathscr{A}$, and where c assigns to every pair of commutative square diagrams

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ {\scriptstyle h_1}\downarrow & & {\scriptstyle h_2}\downarrow & & \downarrow{\scriptstyle h_3} \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' \end{array}$$

their composition, which is the square diagram whose top (resp. bottom) horizontal arrow is $g \circ f$ (resp. $g' \circ f'$) and whose vertical arrows are h_1 and h_3.

(iii) We can use the constructions of (ii) to rephrase the definition of the composition laws for natural transformations. Indeed, let us return to the situation of (1.1.10), and denote by

$$\widetilde{\alpha}, \widetilde{\beta} : \mathscr{A} \to \mathrm{Morph}(\mathscr{B})$$

the functors attached to α and β as in (1.1.30). Clearly $\mathsf{t} \circ \widetilde{\alpha} = \mathsf{s} \circ \widetilde{\beta}$, whence a unique functor

$$(\widetilde{\alpha}, \widetilde{\beta}) : \mathscr{A} \to \mathsf{Morph}(\mathscr{B}) \times_{(\mathsf{t},\mathsf{s})} \mathsf{Morph}(\mathscr{B})$$

whose composition with the first (second) projection to $\mathsf{Morph}(\mathscr{B})$ equals $\widetilde{\alpha}$ (resp. $\widetilde{\beta}$). Then it is easily seen that $\mathsf{c} \circ (\widetilde{\alpha}, \widetilde{\beta}) : \mathscr{A} \to \mathsf{Morph}(\mathscr{A})$ is the (unique) functor associated with $\beta \odot \alpha$.

The Godement product of two natural transformations α and α' as in (1.1.10) can be similarly described : to this aim, notice that

$$(G' * \alpha) \odot (\alpha' * F) = \alpha' * \alpha = (\alpha' * G) \odot (F' * \alpha)$$

so we are reduced to exhibit the functors $\mathscr{A} \to \mathsf{Morph}(\mathscr{C})$ associated with $G' * \alpha$ and $\alpha' * F$. However, it is easily seen that the latter are respectively $\mathsf{Morph}(G') \circ \widetilde{\alpha}$ and $\widetilde{\alpha}' \circ F$ (details left to the reader). These considerations shall be amplified in section 2.1, in order to construct analogous composition laws for pseudo-natural transformations : see remark 2.2.5.

Example 1.2.26. Let $\mathscr{C}$ be any category, and $\varphi : X \to Y$ any morphism of $\mathscr{C}$. The *image* of φ

$$\mathrm{Im}(\varphi)$$

is defined the smallest of the subobjects $Y' \to Y$ such that φ factors through a (necessarily unique) morphism $X \to Y'$. If the image of φ exists, clearly a subobject $Z' \to Y$ contains $\mathrm{Im}(\varphi)$ if and only if φ factors through Z'.

(i) In this generality, the image of φ does not necessarily exists; however, suppose that all equalizers in $\mathscr{C}$ are representable, and the same for the limit (*i.e.* the intersection) of any system of subobjects of Y in $\mathscr{C}$ (these limits are well defined U-presheaves, for a sufficiently large universe U). Then we claim that φ admits an image $i : Z \to Y$, and the unique morphism $p : X \to Z$ such that $\varphi = i \circ p$ is an epimorphism. Indeed, let $\mathscr{F}$ be the family of subobjects $Y' \to Y$ such that φ factors through Y'; choose – for every equivalence class $c \in \mathscr{F}$ – a representing monomorphism $Y_c \to Y$, and let I be the full subcategory of $\mathscr{C}/Y$ such that $\mathrm{Ob}(I) = \{Y_c \to Y \mid c \in \mathscr{F}\}$. Then I is a U-small category, and we let i (any representing object for) the limit of the inclusion functor $\iota : I \to \mathscr{C}/Y$. The resulting system of morphisms $(X \to Y_c \mid c \in \mathscr{F})$ is a cone on the indexing category I, whence a unique morphism $p : X \to Z$, as required. To show that p is an epimorphism, consider any two morphisms $f, g : Z \to Z'$ (for any $Z' \in \mathrm{Ob}(\mathscr{C})$) such that $f \circ p = g \circ p$, and let $E \in \mathrm{Ob}(\mathscr{C})$ be any object representing the equalizer of f and g. Thus, the induced morphism $\psi : E \to Z$ a monomorphism, hence E is a subobject of Y, and φ still factors through E. By the minimality of Z, we deduce that ψ is an isomorphism, and therefore $f = g$, whence the claim.

(ii) Dually, if all coequalizers in $\mathscr{C}$ are representable, and the same for the colimit of any system of quotients of X in $\mathscr{C}$, then the family of all quotients of X through which φ factors admits a smallest element $\psi' : X \to Z$, so that a quotient Z' of X maps to Z if and only if φ factors through Z'. Arguing as in (i), we see that the resulting morphism $Z' \to Y$ is also a monomorphism. We call Z the *coimage* of φ, and we denote it $\mathrm{Coim}(\varphi)$.

(iii) If both the image and the coimage of φ exist in $\mathscr{C}$, we get a natural factorization of φ

$$X \to \mathrm{Im}(\varphi) \xrightarrow{\omega} \mathrm{Coim}(\varphi) \to Y.$$

But ω is not necessarily an isomorphism, in this generality.

Remark 1.2.27. (i) If all equalizers and all amalgamated sums are representable in the category $\mathscr{C}$, then one can alternatively define the image of any morphism $\varphi : X \to Y$ of $\mathscr{C}$ as the equalizer of the natural morphisms $e_1, e_2 : Y \to Y \amalg_X Y$. This definition does not necessarily agree with that of example 1.2.26(i), but if every monomorphism $i : Z \to Y$ of $\mathscr{C}$ is *regular*, *i.e.* represents the equalizer of a pair of morphisms $f, g : Y \to Y'$, then one can show that the two definitions coincide.

(ii) Dually, if all coequalizers and all fibre products are representable in $\mathscr{C}$, then one has a (dual) alternative definition of coimages, which does not necessarily agree with that of example 1.2.26(ii), unless all the epimorphisms of $\mathscr{C}$ are *regular*, *i.e.* unless all the monomorphisms of $\mathscr{C}$ are regular.

(iii) Moreover, if $\mathscr{C}$ is both finitely complete and finitely cocomplete, then these alternative images $\mathrm{Im}^*(\varphi)$ and coimages $\mathrm{Coim}^*(\varphi)$ as in (i) and (ii) are well defined for every morphism $\varphi : X \to Y$, and we get a natural factorization of φ :

$$X \to \mathrm{Coim}^*(\varphi) \xrightarrow{\omega^*} \mathrm{Im}^*(\varphi) \to Y$$

where, again, ω^* is not necessarily an isomorphism.

Example 1.2.28. (i) In the situation of (1.1.31), let $\eta_\bullet : \mathbf{1}_{\mathscr{B}} \Rightarrow FG$ and $\varepsilon_\bullet : GF \Rightarrow \mathbf{1}_{\mathscr{A}}$ be the unit and counit of the adjunction ϑ, and suppose that *all the fibre products of $\mathscr{B}$ are representable*; then for every $B \in \mathrm{Ob}(\mathscr{B})$ the functor $G_{|B} : \mathscr{B}/B \to \mathscr{A}/GB$ admits a right adjoint, denoted

$$F_{|B} : \mathscr{A}/GB \to \mathscr{B}/B.$$

Indeed, for every object $(g : A \to GB)$ of $\mathscr{A}/GB$ let us fix a cartesian diagram :

$$\begin{array}{ccc} F^*A & \xrightarrow{F^*g} & B \\ {\scriptstyle \eta^*_A}\downarrow & & \downarrow{\scriptstyle \eta_B} \\ FA & \xrightarrow{Fg} & FGB. \end{array}$$

If $h/GB : (g : A \to GB) \to (g' : A' \to GB)$ is a morphism in $\mathscr{A}/GB$, the universal property of the fibre product yields a unique morphism $F^*h : F^*A \to F^*A'$ such that $F^*g' \circ F^*h = F^*g$ and $F^*g' \circ \eta^*_{A'} = Fh \circ \eta^*_A$. With this notation, we set

$$F_{|B}(A, g) := (F^*A, F^*g) \qquad \text{and} \qquad F_{|B}(h/GB) := F^*h/B : F_{|B}(A, g) \to F_{|B}(A', g')$$

for every such (A, g) and h/GB. It is easily seen that these rules define a functor as sought. In order to check that $F_{|B}$ is right adjoint to $G_{|B}$, consider any morphism

$$h/GB : G_{|B}(B' \xrightarrow{f} B) \to (A \xrightarrow{g} GB) \qquad \text{in } \mathscr{A}/GB.$$

Hence, $h : GB' \to A$ is a morphism in $\mathscr{A}$ with $g \circ h = Gf$, and we notice that :

$$Fg \circ \vartheta(h) = Fg \circ Fh \circ \eta_{B'} = FGf \circ \eta_{B'} = \eta_B \circ f.$$

It follows that the pair $(\vartheta_{AB'}(h), f)$ determines a unique morphism $k : B' \to F^*A$ such that $\eta^*_A \circ k = \vartheta_{AB'}(h)$ and $F^*g \circ k = f$. With this notation, we set $(\vartheta_{|B})_{g,f}(h/GB) := k/B : f \to F^*g$ in $\mathscr{B}/B$. Conversely, to every morphism $k/B : (f : B' \to B) \to F_{|B}(g : A \to GB)$ in $\mathscr{B}/B$, we attach the morphism $h := \vartheta^{-1}_{AB'}(\eta^*_A \circ k) : GB' \to A$. Let us show that $h/GB : Gf \to g$ is a morphism in $\mathscr{A}/GB$; recalling that $F^*g \circ k = f$, it suffices to compute :

$$g \circ h = g \circ \varepsilon_A \circ G(\eta^*_A \circ k) = \varepsilon_{GB} \circ GFg \circ G(\eta^*_A \circ k) = \varepsilon_{GB} \circ G(\eta_B \circ F^*g \circ k) = Gf$$

where the last equality follows from the triangular identities of (1.1.13). Hence the map

$$(\vartheta_{|B})_{g,f} : \mathrm{Hom}_{\mathscr{A}/GB}(G_{|B}f, g) \to \mathrm{Hom}_{\mathscr{B}/B}(f, F_{|B}g)$$

is a bijection with inverse given by the rule : $k/B \mapsto \vartheta^{-1}_{AB'}(\eta^*_A \circ k)/GB$. Indeed, by definition $(\vartheta_{|B})_{g,f}(\vartheta^{-1}_{AB'}(\eta^*_A \circ k)/GB)$ is the morphism $f \to F_{|B}g$ of $\mathscr{B}/B$ determined by the pair $(\vartheta_{AB'}\vartheta^{-1}_{AB'}(\eta^*_A \circ k), f) = (\eta^*_A \circ k), f)$, which is just k/B, and on the other hand, $\vartheta^{-1}_{AB'}(\eta^*_A \circ (\vartheta_{|B})_{g,f}(h/GB)) = \vartheta^{-1}_{AB'}\vartheta_{AB'}(h)/GB = h/GB$, whence the contention. The naturality of $(\vartheta_{|B})_{g,f}$ with respect to g and f follows by a simple inspection : details left to the reader.

(ii) Dually, if *all the coproducts of $\mathscr{A}$ are representable*, then the functor ${}_{A|}F : A/\mathscr{A} \to FA/\mathscr{B}$ admits a left adjoint ${}_{A|}G : FA/\mathscr{B} \to A/\mathscr{A}$, which the reader is invited to spell out.

(iii) Keep the situation of (i), and let $F' : \mathscr{B} \to \mathscr{C}$ be another functor with a left adjoint $G' : \mathscr{C} \to \mathscr{B}$. Then we have an isomorphism of functors :

$$F'_{|C} \circ F_{|G'C} \xrightarrow{\sim} (F'F)_{|C} \qquad \text{for every } C \in \mathrm{Ob}(\mathscr{C}).$$

Indeed, clearly we have $G_{|G'C} \circ G'_C = (GG')_{|C}$, so the assertion follows from example 1.2.7(ii).

1.3. **Adjunctions and Kan extensions.** Let $\varphi : I \to J$ be a functor between small categories. The calculation of limits indexed by I can be sometimes simplified by a Fubini-style technique of "integration along the fibres of φ", which replaces any given functor $F : I \to \mathscr{C}$ (to any category $\mathscr{C}$) with another functor $J \to \mathscr{C}^\wedge$ whose limit is isomorphic to that of F. Namely, let V be a universe containing U, such that $\mathscr{C}$ has V-small Hom-sets; notice first that by virtue of remark 1.2.11(ii), the limit construction yields a well defined functor

$$\lim_I : \mathsf{Fun}(I, \mathscr{C}) \to \mathscr{C}^\wedge_\mathsf{V} \qquad F \mapsto \lim_I F.$$

Moreover, if $\mathscr{C}$ is complete (resp. if I is finite and $\mathscr{C}$ is finitely complete), this functor is naturally isomorphic to the composition of the Yoneda embedding $h_\mathscr{C}$ and a functor

$$\mathop{\mathrm{Lim}}_I : \mathsf{Fun}(I, \mathscr{C}) \to \mathscr{C}$$

which is right adjoint to the *constant functor*

(1.3.1) $$c : \mathscr{C} \to \mathsf{Fun}(I, \mathscr{C}) \qquad X \mapsto c_X \qquad (f : X \to Y) \mapsto (c_f : c_X \Rightarrow c_Y)$$

(notation of (1.2.9)), and especially, it is well defined up to isomorphism (example 1.2.7(ii)). We define now as follows a functor

$$\int_\varphi^\wedge : \mathsf{Fun}(I, \mathscr{C}) \to \mathsf{Fun}(J, \mathscr{C}^\wedge_\mathsf{V}).$$

For every functor $F : I \to \mathscr{C}$, every $j \in \mathrm{Ob}(J)$, and every morphism $g : j \to j'$ in J we set

$$\int_\varphi^\wedge F(j) := \lim_{j/\varphi I} F \circ \mathsf{t}_j \qquad \int_\varphi^\wedge F(g) := \lim_{g/\varphi I} \mathbf{1}_\mathscr{C}$$

(notation of (1.1.27) and remark 1.2.11(i)). We may then state :

Proposition 1.3.2. *With the notation of* (1.3)*, the diagram*

$$\begin{array}{ccc} \mathsf{Fun}(I, \mathscr{C}) & \xrightarrow{\lim_I} & \mathscr{C}^\wedge_\mathsf{V} \\ {\scriptstyle \int_\varphi^\wedge} \downarrow & \nearrow_{\mathrm{Lim}_J} & \\ \mathsf{Fun}(J, \mathscr{C}^\wedge_\mathsf{V}) & & \end{array}$$

is essentially commutative, i.e. *there is a natural isomorphism of functors :* $\lim_I \xrightarrow{\sim} \mathrm{Lim}_J \circ \int_\varphi^\wedge$.

Proof. Notice that the functor Lim_J is well defined by virtue of example 1.2.24. Let $F : I \to \mathscr{C}$ be any functor; unwinding the definitions, we see that $\mathrm{Lim}_J \circ \int_\varphi^\wedge F$ is isomorphic to the presheaf that assigns to every $X \in \mathrm{Ob}(\mathscr{C})$ the set of all compatible systems

$$\tau_\bullet^\bullet := (\tau_\bullet^j : c_X \Rightarrow F \circ \mathsf{t}_j \mid j \in \mathrm{Ob}(J))$$

such that $\tau_\bullet^{j'} = \tau_\bullet^j * (g/\varphi I)$ for every morphism $g : j \to j'$ in J. Explicitly, $\tau_\bullet^\bullet$ is the datum, for every $(h : j' \to \varphi i) \in \mathrm{Ob}(J/\varphi I)$, of a morphism $\tau_h^j : X \to Fi$ in $\mathscr{C}$, such that

$$\tau_{h \circ g}^j = \tau_h^{j'} \quad \text{and} \quad Ff \circ \tau_h^j = \tau_{\varphi(f) \circ h}^j$$

for every g as above, and every morphism $f : i \to i'$ in I. Clearly, every such system is determined by the family $(\overline{\tau}_i := \tau_{\mathbf{1}_{\varphi i}}^{\varphi i} \mid i \in \mathrm{Ob}(I))$, and the latter is a cone $\overline{\tau}_\bullet : c_X \Rightarrow F$. Conversely, given any cone $\mu_\bullet : c_X \Rightarrow F$, the family $\overline{\mu}_\bullet^\bullet := (\mu_\bullet * \mathsf{t}_j \mid j \in \mathrm{Ob}(J))$ is a compatible system of the foregoing type. A simple inspection shows that the rules $\tau^\bullet_\bullet \mapsto \overline{\tau}_\bullet$ and $\mu_\bullet \mapsto \overline{\mu}^\bullet_\bullet$ are mutually inverse, whence the proposition. $\square$

Remark 1.3.3. (i) In the situation of (1.3), suppose moreover that $\mathscr{C}$ is complete. Then $\int_\varphi^\wedge$ is isomorphic to the composition of the functor $\mathsf{Fun}(J, h_\mathscr{C})$ (notation of remark 1.1.19(ii) and (1.2.4)) and a functor

$$\int_\varphi : \mathsf{Fun}(I, \mathscr{C}) \to \mathsf{Fun}(J, \mathscr{C})$$

called the *right Kan extension along* φ. Taking into account example 1.2.24(ii), proposition 1.3.2 implies that the resulting diagram

$$\begin{array}{ccc} \mathsf{Fun}(I,\mathscr{C}) & \xrightarrow{\mathrm{Lim}_I} & \mathscr{C} \\ {\scriptstyle \int_\varphi}\downarrow & \nearrow_{\mathrm{Lim}_J} & \\ \mathsf{Fun}(J,\mathscr{C}) & & \end{array}$$

is also essentially commutative (details left to the reader).

(ii) By remark 1.2.11(ii), we may also define a colimit functor

$$\operatorname*{colim}_I : \mathsf{Fun}(I, \mathscr{C}) \to \mathscr{C}^{o\wedge o}_\mathsf{V} \qquad F \mapsto \operatorname*{colim}_I F$$

and notice that the isomorphism of remark 1.1.19(i) identifies this functor with the opposite of the functor $\lim_{I^o}$ defined on $\mathsf{Fun}(I^o, \mathscr{C}^o_\mathsf{V})$. Consequently, if $\mathscr{C}$ is cocomplete (resp. if I is finite and $\mathscr{C}$ is finitely cocomplete), colim_I is isomorphic to the composition of $h^o_{\mathscr{C}^o}$ and of a functor

$$\operatorname*{Colim}_I : \mathsf{Fun}(I, \mathscr{C}) \to \mathscr{C}$$

which is likewise identified with the opposite of the functor $\mathrm{Lim}_{I^o} : \mathsf{Fun}(I^o, \mathscr{C}^o) \to \mathscr{C}^o$. Notice as well that Colim_I is left adjoint to the constant functor (1.3.1). Moreover, the opposite of the functor $\int^\wedge_{\varphi^o}$ on $\mathsf{Fun}(I^o, \mathscr{C}^o)$ is identified with a functor

$$\int_\wedge^\varphi : \mathsf{Fun}(I, \mathscr{C}) \to \mathsf{Fun}(J, \mathscr{C}^{o\wedge o}_\mathsf{V}).$$

Explicitly, we have

$$\int_\wedge^\varphi F(j) = \operatorname*{colim}_{\varphi I/j} F \circ \mathsf{s}_j \qquad \text{for every functor } F : I \to \mathscr{C} \text{ and every } j \in \mathrm{Ob}(J)$$

(where s_j is the source functor) and we get an essentially commutative diagram :

$$\begin{array}{ccc} \mathsf{Fun}(I,\mathscr{C}) & \xrightarrow{\mathrm{colim}_I} & \mathscr{C}^{o\wedge o}_\mathsf{V} \\ {\scriptstyle \int^\varphi_\wedge}\downarrow & \nearrow_{\mathrm{Colim}_J} & \\ \mathsf{Fun}(J,\mathscr{C}^{o\wedge o}_\mathsf{V}) & & \end{array}$$

Notice the natural identification : $\mathscr{C}^{o\wedge o}_\mathsf{V} \xrightarrow{\sim} \mathsf{Fun}(\mathscr{C}^o, \mathsf{V}\text{-}\mathbf{Set}^o)$, again due to remark 1.1.19(i). If $\mathscr{C}$ is cocomplete, the opposite of the functor $\int_{\varphi^o}$ is naturally identified with a functor

$$\int^\varphi : \mathsf{Fun}(I, \mathscr{C}) \to \mathsf{Fun}(J, \mathscr{C})$$

called the *left Kan extension along* φ, fitting into the essentially commutative diagram :

$$\begin{array}{ccc} \mathsf{Fun}(I,\mathscr{C}) & \xrightarrow{\mathrm{Colim}_I} & \mathscr{C} \\ {\scriptstyle \int^\varphi}\downarrow & \nearrow_{\mathrm{Colim}_J} & \\ \mathsf{Fun}(J,\mathscr{C}). & & \end{array}$$

(iii) Notice that the functors $\int_\varphi^\wedge$ and $\int_\wedge^\varphi$ are defined more generally whenever I is small and J has small Hom-sets. Likewise, if $\mathscr{C}$ is complete (resp. cocomplete) the right (resp. left) Kan extension along φ is well defined under the same weakened assumptions.

(iv) In the same vein, if the category $j/\varphi I$ (resp. $\varphi I/j$) is finite for every $j \in \mathrm{Ob}(J)$, for the right (resp. left) Kan extension along φ to be well defined it suffices that $\mathscr{C}$ is finitely complete (resp. finitely cocomplete).

Theorem 1.3.4. *Let I, J and $\mathscr{C}$ be categories fulfilling the conditions of either part* (iii) *or part* (iv) *of remark* 1.3.3*, and $\varphi : I \to J$ any functor. Then the right (resp. left) Kan extension along $\varphi : I \to J$ is right (resp. left) adjoint to the functor*

$$\mathsf{Fun}(\varphi,\mathscr{C}) : \mathsf{Fun}(J,\mathscr{C}) \to \mathsf{Fun}(I,\mathscr{C}) \qquad G \mapsto G \circ \varphi.$$

Proof. (Cp. [28, Th.3.7.2] or [110, Th.2.3.3]). Pick a universe V containing U, and such that $\mathscr{C}$ has V-small Hom-sets. In light of Yoneda's lemma (proposition 1.2.6(ii)) we get for every pair of functors $F : I \to \mathscr{C}$ and $G : J \to \mathscr{C}$ a natural bijection

$$\mathrm{Hom}_{\mathsf{Fun}(J,\mathscr{C})}\Big(G, \int_\varphi F\Big) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{Fun}(J,\mathscr{C}^\wedge_{\mathsf{V}})}\Big(h_\mathscr{C} \circ G, \int_\varphi^\wedge F\Big)$$

where $h_\mathscr{C} : \mathscr{C} \to \mathscr{C}^\wedge_{\mathsf{V}}$ is the Yoneda embedding of (1.2.4). Hence, in order to prove the assertion for right Kan extensions, it suffices to exhibit a natural bijection :

$$\mathrm{Hom}_{\mathsf{Fun}(I,\mathscr{C})}(G \circ \varphi, F) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{Fun}(J,\mathscr{C}^\wedge_{\mathsf{V}})}\Big(h_\mathscr{C} \circ G, \int_\varphi^\wedge F\Big). \tag{1.3.5}$$

Notice that, by virtue of proposition 1.2.6(ii), if $X \in \mathrm{Ob}(\mathscr{C})$ and $j \in \mathrm{Ob}(J)$ are any two objects, a morphism $h_X \to \int_\varphi^\wedge F(j)$ in $\mathscr{C}^\wedge_{\mathsf{V}}$ is the same as the datum of a cone $c_X \Rightarrow F \circ \mathsf{t}_j$, where $\mathsf{t}_j : j/\varphi I \to I$ is the target functor. Now, let $\tau : G \circ \varphi \Rightarrow F$ be any natural transformation; for every $j \in \mathrm{Ob}(J)$ and every $(i, f : j \to \varphi(i)) \in \mathrm{Ob}(j/\varphi I)$ we set

$$\tau'_{j,(i,f)} := \tau_i \circ Gf : Gj \to Fi.$$

It is easily seen that, for fixed $j \in \mathrm{Ob}(J)$, the system $(\tau'_{j,(i,f)} \mid (i,f) \in \mathrm{Ob}(j/\varphi I))$ gives a cone $\tau'_j : c_{Gj} \Rightarrow F \circ \mathsf{t}_X$, *i.e.* a morphism $h_{Gj} \to \int_\varphi^\wedge F(j)$ in $\mathscr{C}^\wedge_{\mathsf{V}}$, and the rule $j \mapsto \tau'_j$ for every $j \in \mathrm{Ob}(J)$ yields a natural transformation $\tau' : h_\mathscr{C} \circ G \Rightarrow \int_\varphi^\wedge F$. Conversely, if $\mu : h_\mathscr{C} \circ G \Rightarrow \int_\varphi^\wedge F$ is any such transformation, every $j \in \mathrm{Ob}(J)$ yields a cone $\mu_j : c_{Gj} \Rightarrow F \circ \mathsf{t}_j$, and we define a natural transformation $\mu' : G \circ \varphi \Rightarrow F$ by setting

$$\mu'_i := \mu_{\varphi(i),(i,\mathbf{1}_{\varphi(i)})} : G \circ \varphi(i) \to Fi \qquad \text{for every } i \in \mathrm{Ob}(I).$$

The reader can check that $(\tau')' = \tau$ and $(\mu')' = \mu$ for every τ and μ as above, and clearly the resulting bijections (1.3.5) are functorial in both F and G. The assertion for left Kan extensions, in case $\mathscr{C}$ is cocomplete, is dual to the foregoing, by virtue of remark 1.1.19(i). □

Remark 1.3.6. (i) Let $F : \mathscr{B} \to \mathscr{C}$ be a functor from a small category $\mathscr{B}$ to any category $\mathscr{C}$ with small Hom-sets; from theorem 1.3.4 we obtain both a left and a right adjoint for $F^\wedge_{\mathsf{U}} = \mathsf{Fun}(F^o, \mathsf{U}\text{-}\mathbf{Set})$, denoted respectively

$$F_{\mathsf{U}!} : \mathscr{B}^\wedge_{\mathsf{U}} \to \mathscr{C}^\wedge_{\mathsf{U}} \qquad \text{and} \qquad F_{\mathsf{U}*} : \mathscr{B}^\wedge_{\mathsf{U}} \to \mathscr{C}^\wedge_{\mathsf{U}}.$$

As usual, we drop the subscript U, unless the omission may cause ambiguities.

(ii) With the notation of (i), notice that the diagram of functors :

$$\begin{array}{ccc} \mathscr{B} & \xrightarrow{h_{\mathscr{B}}} & \mathscr{B}^\wedge \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F_!} \\ \mathscr{C} & \xrightarrow{h_{\mathscr{C}}} & \mathscr{C}^\wedge \end{array} \tag{1.3.7}$$

(whose horizontal arrows are the Yoneda embeddings) is *essentially commutative*, *i.e.* the two compositions $F_! \circ h_{\mathscr{B}}$ and $h_{\mathscr{C}} \circ F$ are isomorphic functors. Indeed – by proposition 1.2.6(ii) – for every $B \in \mathrm{Ob}(\mathscr{B})$, the objects $F_! h_B$ and h_{FB} both represent the functor

$$\mathscr{C}^\wedge \to \mathbf{Set} \quad : \quad \varphi \mapsto \varphi(FB)$$

(the latter is a V-presheaf on $\mathscr{C}^{\wedge o}$, for any universe V such that $\mathscr{C}^\wedge$ has small Hom-sets).

(iii) Moreover, taking into account example 1.2.23 we see that there are natural choices for the left and right adjoints of the functor $F^\wedge$. Namely :

- For any presheaf f on $\mathscr{B}$, the presheaf $F_! f$ on $\mathscr{C}$ can be given canonically by the rule :

$$C \mapsto \operatorname*{colim}_{(C/F\mathscr{B})^o} f \circ \mathsf{t}^o_C \qquad \text{for every } C \in \mathrm{Ob}(\mathscr{C}) \tag{1.3.8}$$

where the colimits in this expression denote (by abuse of notation) the natural representatives for the corresponding presheaves, as described in example 1.2.23(i). Therefore, every element of $F_! f(C)$ is the equivalence class of a pair $(\psi : C \to FB, s)$, where ψ is any object of $C/F\mathscr{B}$ and $s \in fB$ is any element. If $\varphi : C' \to C$ is any morphism of $\mathscr{C}$, the induced map $F_! f(\varphi) : F_! f(C) \to F_! f(C')$ assigns to any such pair the equivalence class of the pair $(\psi \circ \varphi : C' \to FB, s)$. Furthermore, the adjunction for the pair $(F_!, F^\wedge)$ provided by the proof of theorem 1.3.4 comes down to the natural bijection

$$\mathrm{Hom}_{\mathscr{B}^\wedge}(f, F^\wedge g) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{C}^\wedge}(F_! f, g) \qquad \text{for every } f \in \mathrm{Ob}(\mathscr{B}^\wedge) \text{ and } g \in \mathrm{Ob}(\mathscr{C}^\wedge)$$

which, to any morphism $\tau : f \to F^\wedge g$ in $\mathscr{B}^\wedge$ and to any $C \in \mathrm{Ob}(\mathscr{C})$ assigns the map of sets $F_! f(C) \to gC$ given by the rule :

$$(\psi : C \to FB, s) \mapsto g(\psi) \circ \tau_B(s) \qquad \text{for every } (B, \psi) \in \mathrm{Ob}(C/F\mathscr{B}) \text{ and every } s \in fB.$$

The inverse to this bijection assigns to any morphism $\mu : F_! f \to g$ and any $B \in \mathrm{Ob}(\mathscr{B})$ the map $f(B) \to g(FB)$ given by the rule :

$$s \mapsto \mu_{FB}(\mathbf{1}_{FB}, s) \qquad \text{for every } s \in f(B).$$

- And the presheaf $F_* f$ can be chosen canonically by the rule :

$$C \mapsto \lim_{(F\mathscr{B}/C)^o} f \circ \mathsf{s}^o_C(\mathbb{1}) \qquad \varphi \mapsto \lim_{(F\mathscr{B}/\varphi)^o} \mathbf{1}_{\mathscr{B}}(\mathbb{1})$$

for every $C \in \mathrm{Ob}(\mathscr{C})$ and every morphism $\varphi : C' \to C$ in $\mathscr{C}$ (notation of (1.1.27), and $\mathbb{1}$ is any fixed set with one element). More explicitly, under this identification, the map $F_* f(\varphi) : F_* f(C) \to F_* f(C')$ is given by the rule :

$$(\tau : c_{\mathbb{1}} \Rightarrow f \circ \mathsf{s}^o_C) \mapsto (\tau * (F\mathscr{B}/\varphi)^o : c_{\mathbb{1}} \Rightarrow f \circ \mathsf{s}^o_{C'}).$$

(iv) Let V be a universe such that $\mathsf{U} \subset \mathsf{V}$. Notice that, with the canonical choices of (iii), we get a commutative diagram of categories

$$\begin{array}{ccccc} \mathscr{C}^\wedge_{\mathsf{U}} & \xleftarrow{F_{\mathsf{U}*}} & \mathscr{B}^\wedge_{\mathsf{U}} & \xrightarrow{F_{\mathsf{U}!}} & \mathscr{C}^\wedge_{\mathsf{U}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathscr{C}^\wedge_{\mathsf{V}} & \xleftarrow{F_{\mathsf{V}*}} & \mathscr{B}^\wedge_{\mathsf{V}} & \xrightarrow{F_{\mathsf{V}!}} & \mathscr{C}^\wedge_{\mathsf{V}} \end{array}$$

whose vertical arrows are the inclusion functors.

(v) Lastly, with the canonical choices in (iii), we can make (ii) more precise : we get a natural identification

$$F_! h_B \xrightarrow{\sim} h_{FB} \qquad \text{for every } B \in \mathrm{Ob}(\mathscr{B}).$$

Namely, to any pair $(\psi' : Y \to FB', s : B' \to B)$ consisting of an object of $Y/F\mathscr{B}$ and an element $s \in h_B(B')$, we attach the element $\beta(\psi', s) := F(s) \circ \psi' \in h_{FB}(Y)$, and it is easily seen that $\beta(\psi', s)$ depends only on the class of (ψ', s) in $F_! h_B(Y)$, whence the required functorial bijection (details left to the reader). Notice as well that – under these identifications – the unit of the adjunction for the pair $(F_!, F^\wedge)$ explicited in (iii) becomes the natural morphism

$$h_B \to F^\wedge h_B \quad : \quad (s : B' \to B) \mapsto (Fs : FB' \to FB) \qquad \text{for every } B \in \mathrm{Ob}(B).$$

Example 1.3.9. Let $\mathbb{1}_{\mathscr{B}}$ be a final object of $\mathscr{B}^\wedge$ (see example 1.2.16(v)); by specializing the explicit description of $F_!$ provided by remark 1.3.6(iii), we get a natural isomorphism

$$F_!(\mathbb{1}_{\mathscr{B}}) \xrightarrow{\sim} \pi_0(\bullet/F\mathscr{B}) \qquad \text{in } \mathscr{C}^\wedge$$

where $\pi_0(\bullet/F\mathscr{B})$ is the presheaf that assigns to any $C \in \mathrm{Ob}(\mathscr{C})$ the set $\pi_0(C/F\mathscr{B})$, and to any morphism $\varphi : C' \to C$ in $\mathscr{C}$ the map $\pi_0(\varphi/F\mathscr{B}) : \pi_0(C/F\mathscr{B}) \to \pi_0(C'/F\mathscr{B})$ induced by $\varphi/F\mathscr{B}$ in the obvious way (notation of (1.1.27) : we leave the verification to the reader).

1.3.10. For many questions, it is useful to know whether a given functor commutes with a certain limit, or more generally with a prescribed class of limits (or colimits), in the sense explained by the following :

Definition 1.3.11. Let I, $\mathscr{C}$ and $\mathscr{D}$ be any three categories, and $F : I \to \mathscr{C}$, $f : \mathscr{C} \to \mathscr{D}$ two functors. Pick also a universe V such that I, $\mathscr{C}$ and $\mathscr{D}$ are V-small.

(i) According to remark 1.2.11(i) we have a natural morphism in $\mathscr{C}^\wedge_{\mathsf{V}}$

$$\lim_I F \to f^\wedge_{\mathsf{V}} \lim_I (f \circ F) \tag{1.3.12}$$

whence – by the explicit adjunction of remark 1.3.6(iii) – a morphism in $\mathscr{D}^\wedge_{\mathsf{V}}$

$$f_{\mathsf{V}!}(\lim_I F) \to \lim_I (f \circ F). \tag{1.3.13}$$

Then, we say that f *commutes with the limit of* F, if (1.3.13) is an isomorphism.

(ii) We say that f *commutes with the colimit of* F, if the dual condition holds, *i.e.* if f^o commutes with the limit of F^o, so the natural morphism

$$f^o_{\mathsf{V}!}(\operatorname*{colim}_I F) \to \operatorname*{colim}_I (f \circ F)$$

is an isomorphism.

(iii) We say that the functor f is *left exact*, if f commutes with all finite limits. Dually, we say that f is *right exact* if it commutes with all finite colimits. Finally, we say that f is *exact* if it is both left and right exact.

It follows easily from remark (1.3.6)(iv) that these definitions depend only on f and F, and not on the choice of the universe V.

Remark 1.3.14. (i) Keep the situation of definition 1.3.11. The most interesting case is that where the limit of F is representable, say by an object L of $\mathscr{C}$, so we have an isomorphism $\beta : h_L \xrightarrow{\sim} \lim_I F$. Indeed, in this case, by remark 1.3.6(v) we deduce a morphism

$$\gamma : h_{fL} \xrightarrow{\sim} f_! h_L \xrightarrow{\sim} f_! \lim_I F \to \lim_I (f \circ F)$$

that is an isomorphism if and only if f commutes with the limit of F, in which case we see that the limit of $f \circ F$ is representable by fL. More precisely, let $\tau := \beta_L(\mathbf{1}_L) : c_L \Rightarrow F$ be the universal cone deduced from β; we claim that

$$\gamma_{fL}(\mathbf{1}_{fL}) = f * \tau : c_{fL} \Rightarrow f \circ F$$

so f commutes with the limit of F if and only if $f * \tau$ is a universal cone for $f \circ F$. Indeed, notice that the composition $h_L \to f^\wedge_\vee \lim_I(f \circ F)$ of β and (1.3.12) assigns, to any $C \in \mathrm{Ob}(\mathscr{C})$ and any morphism $s : C \to L$ in $\mathscr{C}$, the cone $(f * \tau) \odot c_{fs} : c_{fC} \Rightarrow f \circ F$. In light of remark 1.3.6(iii), it follows that the adjoint morphism $f_! h_L \to \lim_I f \circ F$ assigns, to every $D \in \mathrm{Ob}(\mathscr{D})$ and every pair $(\psi : D \to fC, s : C \to L)$ the cone $(f * \tau) \odot c_{fs} \odot c_\psi : c_D \Rightarrow f \circ F$. Lastly, notice that the isomorphism $f_! h_L \xrightarrow{\sim} h_{fL}$ provided from remark 1.3.6(v) maps the pair $(\mathbf{1}_{fL}, \mathbf{1}_L)$ to $\mathbf{1}_{fL} \in h_{fL}(fL)$; we conclude that $\gamma_{fL}(\mathbf{1}_{fL}) = (f * \tau) \odot c_{f\mathbf{1}_L} \odot c_{\mathbf{1}_{fL}} = f * \tau$, as claimed.

(ii) Dually, suppose that the colimit of F is representable by an object C^o of $\mathscr{C}^o$, and pick any universal cocone $\mu : F \Rightarrow c_C$. Then f commutes with the colimit of F if and only if $f * \mu$ is a universal cocone for $f \circ F$.

Example 1.3.15. (i) Let $\mathscr{C}$ be any category, V a universe such that $\mathscr{C}$ has V-small Hom-sets, and $f : \mathscr{C}^o \to \mathsf{V}\text{-}\mathbf{Set}$ a representable functor. Then it is easily seen that for every small category I, every functor $F : I \to \mathscr{C}$ and every universal cocone $\tau : F \Rightarrow c_X$, the induced cone $f * \tau^o : c_{fX} \Rightarrow f \circ F^o$ is universal. In view of remark 1.3.14(i), we conclude that f commutes with all the representable limits of $\mathscr{C}^o$ (which are the representable colimits of $\mathscr{C}$). On the other hand, if $\mathscr{C}$ has small Hom-sets and all small products of $\mathscr{C}$ are representable, then f commutes even with non-representable colimits of $\mathscr{C}$: see example 1.4.4(i).

(ii) Dually, if $f : \mathscr{C} \to \mathbf{Set}$ is a functor representable by some object Y^o of the opposite category $\mathscr{C}^o$, then f commutes with all representable limits of $\mathscr{C}$, and even with non-representable limits, if all small coproducts of $\mathscr{C}$ are representable and $\mathscr{C}$ has small Hom-sets.

Example 1.3.16. (i) Let $\mathscr{A}$ be any finitely complete category, and $F : \mathscr{A} \to \mathscr{B}$ any left exact functor. Then the category $B/F\mathscr{A}$ is cofiltered, for every $B \in \mathrm{Ob}(\mathscr{B})$. Indeed, if $\varphi_i : B \to FA_i$ $(i = 1, 2)$ are any two objects of $B/F\mathscr{A}$, the product $A_1 \times A_2$ is representable by some object A of $\mathscr{A}$, and a universal cone is given by a pair of morphisms $(p_i : A \to A_i \mid i = 1, 2)$; by remark 1.3.14(i), it follows that the pair $(Fp_i : FA \to FA_i \mid i = 1, 2)$ is still a universal cone for the product $FA_1 \times FA_2$, whence a unique morphism $\varphi : B \to FA$ such that $Fp_i \circ \varphi = \varphi_i$ for $i = 1, 2$. This shows that $B/F\mathscr{A}$ is codirected. Moreover, if A is any final object of $\mathscr{A}$, then FA is a final object of $\mathscr{B}$, hence $\mathrm{Ob}(B/F\mathscr{A})$ is non-empty. By remark 1.2.21(i), it then remains only to check that $B/F\mathscr{A}$ satisfies the equalizing condition dual to that of definition 1.2.19(v). Namely, let φ_1 and φ_2 as in the foregoing, and suppose we have a pair of morphisms $\psi, \psi' : A_1 \to A_2$ such that $F\psi \circ \varphi_1 = \varphi_2 = F\psi' \circ \varphi_1$. The equalizer of ψ and ψ' is representable by some object E of $\mathscr{A}$, and a universal cone for E is given by a morphism $\beta : E \to A_1$ such that $\psi \circ \beta = \psi' \circ \beta$; by remark 1.3.14(i), FE represents the equalizer of $F\psi$ and $F\psi'$, and $F\beta : FE \to FA_1$ still yields a universal cone. There follows a unique morphism $\gamma : B \to FE$ in $\mathscr{B}$ such that $F\beta \circ \gamma = \varphi_1$, whence the claim.

(ii) Dually, if $\mathscr{A}$ is finitely cocomplete and F is right exact, then the category $F\mathscr{A}/B$ is filtered for every $B \in \mathrm{Ob}(\mathscr{B})$.

Proposition 1.3.17. *Let I, $\mathscr{C}$, $\mathscr{D}$ and $\mathscr{E}$ be four categories, and $F : I \to \mathscr{C}$, $f : \mathscr{C} \to \mathscr{D}$ and $g : \mathscr{D} \to \mathscr{E}$ any three functors. The following holds :*

- (i) *If f commutes with the limit of F, and g commutes with the limit of $f \circ F$, then $g \circ f$ commutes with the limit of F.*
- (ii) *Especially, if both f and g are left (resp. right) exact, then the same holds for $g \circ f$.*

Proof. Clearly it suffices to show (i), and after replacing U by a larger universe, we may assume that the four categories of the proposition are small. Set $L := \lim_I F$, $L' := \lim_I(f \circ F)$, $L'' := \lim_I(g \circ f \circ F)$, and denote

$$\omega_{f,F} : L \to f^\wedge L' \qquad \text{and} \qquad \omega_{g,f\circ F} : L' \to g^\wedge L''$$

the morphisms provided by remark 1.2.11(i). Let also (ε^f, η^f) (resp. (ε^g, η^g)) be the units and counits associated with the explicit adjunction ϑ_f (resp ϑ_g) for the pair $(f_!, f^\wedge)$ (resp. $(g_!, g^\wedge)$) provided by remark 1.3.6(iii); by assumption, the adjoint morphisms

$$\omega^*_{f,F} := \varepsilon^f_{L'} \circ (f_!\omega_{f,F}) : f_!L \to L' \qquad \text{and} \qquad \omega^*_{g,f\circ F} := \varepsilon^g_{L''} \circ (g_!\omega_{g,f\circ F}) : g_!L' \to L''$$

are isomorphisms, so the same holds for $\omega^{**}_{g\circ f,F} := \omega^*_{g,f\circ F} \circ g_!\omega^*_{f,F} : g_! \circ f_!L \to L''$. However :

$$\omega^{**}_{g\circ f,F} = \varepsilon^g_{L''} \circ g_!(\omega_{g,f\circ F} \circ \omega^*_{f,F})$$

corresponds, under the adjunction ϑ_g, to the morphism $\omega_{g,f\circ F} \circ \omega^*_{f,F} : f_!L \to g^\wedge L''$. The latter, in turns, corresponds – under the adjunction ϑ_f – to the morphism

$$\omega_{g\circ f,F} := f^\wedge(\omega_{g,f\circ F} \circ \omega^*_{f,F}) \circ \eta^f_L : L \to f^\wedge \circ g^\wedge L'' = (g \circ f)^\wedge L''$$

and notice that $f^\wedge(\omega^*_{f,F}) \circ \eta^f_L = \omega_{f,F}$. Taking into account (1.2.12), we conclude that $\omega_{g\circ f,F}$ is precisely the natural morphism denoted $\lim_{\mathbf{1}_I}(g \circ f)$ in remark 1.2.11(i).

Summing up, we have shown that, under the adjunction $\vartheta_g \circ \vartheta_f$ for the pair $(g_! \circ f_!, (g \circ f)^\wedge)$ (notation of remark 1.1.17(i)), the morphism $\omega_{g\circ f,F}$ corresponds to the isomorphism $\omega^{**}_{g\circ f,F}$. Then the assertion follows from example 1.2.7(ii). □

Proposition 1.3.18. *Let $\mathscr{C}, \mathscr{D}$ be two categories, $F : \mathscr{C} \to \mathscr{D}$ a functor, and $\varphi : X \to Y$ a morphism in $\mathscr{C}$. The following holds :*

(i) *If F commutes with pull backs and φ is a monomorphism, then $F\varphi$ is a monomorphism.*
(ii) *If F commutes with push-outs and φ is an epimorphism, then $F\varphi$ is an epimorphism.*

Proof. (i): After replacing U by a larger universe, we may assume that $\mathscr{C}$ and $\mathscr{D}$ have small Hom-sets. Notice then that the tautological cone of $X \times_Y X$ amounts to a pair of morphisms

$$p_1, p_2 : X \times_Y X \to h_X \qquad \text{in } \mathscr{C}^\wedge$$

such that $h_\varphi \circ p_1 = h_\varphi \circ p_2$ (see remark 1.2.11(iv)), and the diagonal morphism $\iota_\varphi : h_X \to X \times_Y X$ (see (1.2.17)) is characterized as the unique morphism in $\mathscr{C}^\wedge$ such that $p_1 \circ \iota_\varphi = \mathbf{1}_{h_X} = p_2 \circ \iota_\varphi$. Likewise we can characterize $\iota_{F\varphi}$. Now, by assumption there exist unique morphisms j and q_i for $i = 1, 2$, fitting into a commutative diagram

(1.3.19)
$$\begin{array}{ccccc} F_!h_X & \xrightarrow{F_!\iota_\varphi} & F_!(X \times_Y X) & \xrightarrow{F_!p_i} & F_!h_X \\ \downarrow & & \downarrow & & \downarrow \\ h_{FX} & \xrightarrow{j} & FX \times_{FY} FX & \xrightarrow{q_i} & h_{FX} \end{array}$$

whose leftmost and rightmost vertical arrows are the natural identifications of remark 1.3.6(v), and whose central vertical arrow is the isomorphism given by definition 1.3.11(i). Taking into account proposition 1.2.18(i), we see that the assertion will follow, once we know that $j = \iota_{F\varphi}$. However, the composition of the two top horizontal arrows equals $\mathbf{1}_{F_!h_X}$, so the composition of the bottom horizontal arrows is $\mathbf{1}_{h_{FX}}$, and thus we come down to showing :

Claim 1.3.20. q_1 and q_2 are the two morphisms defining the tautological cone for $FX \times_{FY} FX$.

Proof of the claim. Notice that, for any $Z \in \mathrm{Ob}(\mathscr{C})$, an element of $X \times_Y X(Z)$ is the same as the datum of a pair of morphisms $\alpha, \beta : Z \to X$ such that $\varphi \circ \alpha = \varphi \circ \beta$. The natural morphism $X \times_Y X \to F^\wedge(FX \times_{FY} FX)$ of (1.3.12) sends such a pair (α, β) to the pair $(F\alpha, F\beta)$.

Therefore, the central vertical arrow of (1.3.19) is the map that assigns to any $W \in \mathrm{Ob}(\mathscr{D})$ and any pair $(\psi : W \to FZ, (\alpha, \beta : Z \to X))$ the pair $(F\alpha \circ \psi, F\beta \circ \psi : W \to FX)$. Then, q_1 (resp. q_2) sends such a pair to $F\alpha \circ \psi$ (resp. to $F\beta \circ \psi$). On the other hand, $F_! p_1$ (resp. $F_! p_2$) sends $(\psi, (\alpha, \beta))$ to the pair (ψ, α) (resp. to (ψ, β)). Now the claim follows by inspecting the explicit definition of the rightmost vertical arrow of (1.3.19). ◊

(ii) follows by dualizing the foregoing argument (details left to the reader). □

Proposition 1.3.21. (i) *Let $\mathscr{C}, \mathscr{D}$ be two categories, $F : \mathscr{C} \to \mathscr{D}$ a left exact (resp. right exact) functor; suppose that $\mathscr{C}$ is finitely complete (resp. finitely cocomplete), and consider the following conditions :*

(a) *F is conservative (see definition* 1.1.4(ii)*)*
(b) *F is faithful*
(c) *F reflects epimorphisms (resp. monomorphisms)*
(d) *F reflects monomorphisms (resp. epimorphisms).*

Then (a)⇒(b)⇒(c)⇒(d).

(ii) *Suppose moreover that the isomorphisms in $\mathscr{C}$ are the morphisms that are both monomorphisms and epimorphisms. Then* (c)⇒(a).

Proof. We prove the assertions in case f is left exact and $\mathscr{C}$ is finitely complete; the assertions for the case where F is right exact and $\mathscr{C}$ is finitely cocomplete will follow by considering $F^o : \mathscr{C}^o \to \mathscr{D}^o$.

(i.a)⇒(i.b): Let $f; g : A \to B$ be two morphisms in $\mathscr{C}$; by assumption, the equalizer of f and g is representable in $\mathscr{C}$ by some object E, and the universal cone is the datum of a morphism $h : E \to A$ such that $f \circ h = g \circ h$. Moreover, FE represents the equalizer of Ff and Fg, and $Fh : FE \to FA$ is a universal cone for this equalizer, by remark 1.3.14(i). Now, suppose that $Ff = Fg$; then Fh is an isomorphism, hence the same holds for h, and therefore $f = g$.

(i.b)⇒(i.c): Let $f : A \to B$ and $g, h : B \to C$ be three morphisms in $\mathscr{C}$ with $g \circ f = h \circ f$, and suppose that Ff is an epimorphism; since $Fg \circ Ff = Fh \circ Ff$, it follows that $Fg = Fh$, and then $g = h$, since F is faithful. This shows that f is an epimorphism.

(i.c)⇒(i.d): Let $f : A \to B$ be a morphism in $\mathscr{C}$ such that $Ff : FA \to FB$ is a monomorphism; by assumption the fibre product $A \times_B A$ is representable in $\mathscr{C}$, and if $p_1, p_2 : A \times_B A \to A$ are the natural projections, then $F(A \times_B A)$ represents $FA \times_{FB} FA$, and $Fp_1, Fp_2 : F(A \times_B A) \to FA$ provide a universal cone for this fibre product. It follows easily that if $\iota_f : A \to A \times_B A$ is the diagonal of f, then $F\iota_f : FA \to F(A \times_B A)$ is the diagonal of Ff, hence $F\iota_f$ is an isomorphism in $\mathscr{D}$, by proposition 1.2.18(i). By assumption, ι_f is then an epimorphism, so f is a monomorphism, again by proposition 1.2.18(i).

(ii): Let $f : A \to B$ be a morphism such that Ff is an isomorphism; in particular, Ff is an epimorphism, so by assumption f is an epimorphism. But Ff is also a monomorphism, hence the same holds for f, because we know already that (i.c)⇒(i.d). Then f is an isomorphism, under our assumption on $\mathscr{C}$. □

Proposition 1.3.22. *Let $\mathscr{C}, \mathscr{D}$ be any two categories, and $f : \mathscr{C} \to \mathscr{D}$ any functor.*

(i) *Suppose that*
  (a) *f commutes with equalizers and with products (resp. and with finite products)*
  (b) *all small products (resp. all finite products) are representable in $\mathscr{C}$.*
  Then f commutes with all limits of $\mathscr{C}$ (resp. f is left exact).

(ii) *Dually, suppose that*
  (a) *f commutes with equalizers and with coproducts (resp. and with finite coproducts)*
  (b) *all small coproducts (resp. all finite coproducts) are representable in $\mathscr{C}$.*
  Then f commutes with all colimits of $\mathscr{C}$ (resp. f is right exact).

Proof. (i): With the notation of the proof of proposition 1.2.22(i), we have shown that the rule

$$(\mu : c_X \Rightarrow F) \mapsto \omega_P^{-1}(\mu * \mathrm{ob}) \qquad \text{for every } X \in \mathrm{Ob}(\mathscr{C}) \text{ and } \mu \in \lim_I F(X)$$

defines an isomorphism $g : \lim_I F \xrightarrow{\sim} E \subset h_P$. Also, since by assumption f commutes with products (resp. with finite products), the isomorphisms ω_P and ω_Q induce isomorphisms

$$\gamma_P : h_{fP} \xrightarrow{\sim} \lim_{\mathrm{Ob}(I)} f \circ F \circ \mathrm{ob} \qquad \gamma_Q : h_{fQ} \xrightarrow{\sim} \lim_{\mathrm{Morph}(I)} f \circ F \circ \mathrm{ob} \circ \mathrm{t}$$

as explained in remark 1.3.14(i). We set $E' := \mathrm{Equal}(f\alpha, f\beta)$, and we identify E' as well with a subobject of h_{fP}, via the monomorphism $E' \to h_{fP}$ provided by the tautological cone. Then we define likewise a morphism $g' : \lim_I(f \circ F) \to E'$ by the rule :

$$(\mu : c_Y \Rightarrow f \circ F) \mapsto \gamma_P^{-1}(\mu * \mathrm{ob}) \qquad \text{for every } Y \in \mathrm{Ob}(\mathscr{D}) \text{ and } \mu \in \lim_I (f \circ F)(Y).$$

Consider now the diagram of presheaves on $\mathscr{D}$:

(1.3.23)
$$\begin{array}{ccc} f_! \lim_I F & \xrightarrow{f_! g} & f_! E \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle e} \\ \lim_I f \circ F & \xrightarrow{g'} & E' \end{array}$$

whose vertical arrows are the morphisms (1.3.13).

Claim 1.3.24. Diagram (1.3.23) commutes.

Proof of the claim. Let $\tau : c_P \Rightarrow F \circ \mathrm{ob}$ be the universal cone arising from the isomorphism ω_P; according to remark 1.3.14(i), for every $X \in \mathrm{Ob}(\mathscr{C})$ and every cone $\mu : c_X \Rightarrow F$ (resp. every $Y \in \mathrm{Ob}(\mathscr{D})$ and every cone $\mu' : c_Y \Rightarrow f \circ F \circ \mathrm{ob}$), the morphism $g(\mu)$ (resp. $g'(\mu')$) is characterized as the unique one such that

$$\tau \odot c_{g(\mu)} = \mu * \mathrm{ob} \qquad (\text{resp. } (f * \tau) \odot c_{g'(\mu')} = \mu' * \mathrm{ob}\ ).$$

Now, let $(\psi : Y \to fX, \mu : c_X \Rightarrow F)$ be the representative of any given element $s \in f_! \lim_I F(Y)$. Then $f_! g(s)$ is represented by the pair $(\psi, g(\mu))$, and

$$d(s) = (f * \mu) \odot c_\psi \qquad e(f_! g(s)) = f(g(\mu)) \circ \psi$$

(remark 1.3.6(iii)), and the foregoing yields the identities

$$\begin{aligned} (f * \tau) \odot c_{g'((f*\mu)\circ c_\psi)} &= ((f * \mu) \odot c_\psi) * \mathrm{ob} \\ &= (f * \mu * \mathrm{ob}) \odot c_\psi \\ &= (f * (\tau \odot c_{g(\mu)})) \odot c_\psi \\ &= (f * \tau) \odot (f * c_{g(\mu)}) \odot c_\psi \\ &= (f * \tau) \odot c_{f(g(\mu))\circ\psi} \end{aligned}$$

whence $e(f_! g(s)) = g'(d(s))$, as claimed. ◇

Now, by assumption both $f_! g$ and e are isomorphisms; in light of claim 1.3.24, it then suffices to check that g' is an isomorphism. By unwinding the definitions, we see that the latter assertion amounts to the following. For every $Y \in \mathrm{Ob}(\mathscr{D})$ he rule $\mu \mapsto \mu * \mathrm{ob}$ yields a bijection from the set of all cones $\mu : c_Y \Rightarrow f \circ F$ onto the set of all cones $\lambda : c_Y \Rightarrow f \circ F \circ \mathrm{ob}$ such that $\lambda * \mathrm{t} = ((f \circ F) * \mathrm{m}) \odot (\lambda * \mathrm{s})$. In turns, this follows by a direct inspection. □

Proposition 1.3.25. *Let $\mathscr{A}, \mathscr{B}$ be two categories, $F : \mathscr{A} \to \mathscr{B}$ a functor that admits a left adjoint $G : \mathscr{B} \to \mathscr{A}$. The following holds :*

(i) *If $\mathscr{A}$ and $\mathscr{B}$ are small, we have natural isomorphisms of functors*

$$F^\wedge \xrightarrow{\sim} G_* \qquad F_! \xrightarrow{\sim} G^\wedge.$$

(ii) *If $\mathscr{A}$ and $\mathscr{B}$ have small* Hom-*sets, then for every small category I and every functor $\varphi : I \to \mathscr{A}$, there is a natural isomorphism of presheaves*

$$G^\wedge(\lim_I \varphi) \stackrel{\sim}{\to} \lim_I (F \circ \varphi).$$

(iii) *F commutes with all limits of $\mathscr{A}$.*
(iv) *Dually, G commutes with all colimits of $\mathscr{B}$.*
(v) *F transforms monomorphisms into monomorphisms, and G transforms epimorphisms into epimorphisms.*

Proof. (i): In view of example 1.2.7(ii), the assertion means that $G^\wedge$ is left adjoint to $F^\wedge$; however F^o is left adjoint to G^o (remark 1.1.19(iv)), so it suffices to apply remark 1.1.19(iii), which shows more precisely that $\varepsilon^\wedge$ and $\eta^\wedge$ are respectively the unit and counit of a unique adjunction for the pair $(G^\wedge, F^\wedge)$.

(ii): We are easily reduced to the case where $\mathscr{A}$ and $\mathscr{B}$ are small. Set $L := \lim_I F \circ \varphi$. According to remark 1.2.11(i), we have a natural morphism $\omega : \lim_I \varphi \to F^\wedge L$, to which the adjunction exhibited in the proof of (i) attaches a unique morphism $\omega^* : G^\wedge(\lim_I \varphi) \to L$. We shall show more precisely that ω^* is an isomorphism of presheaves. Indeed, let $B \in \mathrm{Ob}(\mathscr{B})$ be any object; by unwinding the definitions, we see that ω^*_B is given by the rule

$$\mu \mapsto \mu^* := ((F * \mu) \odot c_{\eta_B} : c_B \Rightarrow F \circ \varphi) \qquad \text{for every cone } \mu : c_{GB} \Rightarrow \varphi$$

(notation of (1.2.9)). We provide an explicit inverse for the rule $\mu \mapsto \mu^*$, by setting

$$\tau^* := (\varepsilon * \varphi) \odot (G * \tau) : c_{GB} \Rightarrow \varphi \qquad \text{for every cone } \tau : c_B \Rightarrow F \circ \varphi.$$

Clearly the rule $\tau \mapsto \tau^*$ defines a morphism of presheaves $L \to G^\wedge(\lim_I \varphi)$. Then, for every such τ the triangular identities of (1.1.13) give :

$$\begin{aligned}(\tau^*)^* &= (F * ((\varepsilon * \varphi) \odot (G * \tau))) \odot c_{\eta_B} \\ &= (F * \varepsilon * \varphi) \odot (F * G * \tau) \odot c_{\eta_B} \\ &= (F * \varepsilon * \varphi) \odot (\eta * F * \varphi) \odot \tau \\ &= \tau.\end{aligned}$$

Likewise one sees that $(\mu^*)^* = \mu$ for every cone $\mu : c_{GB} \Rightarrow \varphi$, as needed.

(iii): Again, we can assume that $\mathscr{A}$ and $\mathscr{B}$ are small. We have just seen that the adjunction for the pair $(G^\wedge, F^\wedge)$ exhibited in (i) transforms the morphism ω into an isomorphism ω^*. By example 1.2.7(ii), it follows that any adjunction for the pair $(F_!, F^\wedge)$ will also transform ω into an isomorphism, whence the contention. (Notice that $F_!$ is – *a priori* – well defined only as a functor on V-presheaves, for some universe V such that $\mathscr{A}$ and $\mathscr{B}$ are small V-categories, but (i) implies that actually it is – up to natural isomorphism – already defined on U-presheaves.)

(iv) follows as usual, by considering the opposite categories and taking into account remark 1.1.19(iv). Lastly, (v) follows from (iii),(iv) and proposition 1.3.18. □

Let $\mathscr{A}$ be a subcategory of a category $\mathscr{B}$, and $\varphi : I \to \mathscr{A}$ a functor from a small category I, whose limit is representable in $\mathscr{A}$ by an object A. In this situation – and even in case $\mathscr{A}$ is a full subcategory of $\mathscr{B}$ – it is not necessarily true that A represents the limit of the composition $I \to \mathscr{B}$ of φ with the inclusion functor $\mathscr{A} \to \mathscr{B}$. We have nevertheless the following result :

Corollary 1.3.26. *Let $\mathscr{A}, \mathscr{B}$ be two categories, $F : \mathscr{A} \to \mathscr{B}$ a fully faithful functor admitting a left adjoint $G : \mathscr{B} \to \mathscr{A}$, and $\varphi : I \to \mathscr{A}$ a functor from a small category I. We have :*

(i) *If the limit of $F \circ \varphi$ is representable by some $L \in \mathrm{Ob}(\mathscr{B})$, then the limit of φ is representable by some object $M \in \mathrm{Ob}(\mathscr{A})$.*
(ii) *In the situation of* (i), *there exist isomorphisms $L \stackrel{\sim}{\to} FM$ in $\mathscr{B}$ and $M \stackrel{\sim}{\to} GL$ in $\mathscr{A}$.*
(iii) *Especially, if $\mathscr{B}$ is complete, the same holds for $\mathscr{A}$.*

Proof. Let $\eta : \mathbf{1}_{\mathscr{B}} \Rightarrow F \circ G$ be a unit and $\varepsilon : G \circ F \Rightarrow \mathbf{1}_{\mathscr{A}}$ a counit for the pair (G, F), and $\tau : c_L \Rightarrow F \circ \varphi$ a universal cone; set $\tau^* := (\varepsilon * \varphi) \odot (G * \tau) : c_{GL} \Rightarrow \varphi$. There exists a unique morphism $f : FGL \to L$ in $\mathscr{B}$ such that $\tau \odot c_f = F * \tau^*$. We have :

$$\tau \odot c_{f \circ \eta_L} = \tau \odot c_f \odot c_{\eta_L} = (F * \tau^*) \odot (\eta * c_L) = (F * \varepsilon * \varphi) \odot (FG * \tau) \odot (\eta * c_L)$$
$$= (F * \varepsilon * \varphi) \odot (\eta * F\varphi) \odot \tau = \tau$$

whence $f \circ \eta_L = \mathbf{1}_L$, by the universality of τ. On the other hand, since F is fully faithful, there exists a morphism $h : GL \to GL$ in $\mathscr{A}$ such that $Fh = \eta_L \circ f : FGL \to FGL$. Now, the adjunction of the pair (G, F) assigns to h the morphism $Fh \circ \eta_L = \eta_L \circ f \circ \eta_L = F(\mathbf{1}_{GL}) \circ \eta_L$ (see (1.1.13)), so we must have $h = \mathbf{1}_{GL}$, therefore f is an isomorphism. Lastly, since F is fully faithful, we deduce easily that GL represents the limit of F, whence the corollary. □

1.4. **Special properties of the categories of presheaves.** In this section we gather some results concerning the structure of categories of presheaves and the exactness properties of various natural functors between such categories. We begin with a corollary of proposition 1.3.25 :

Corollary 1.4.1. *Let $\mathscr{A}$ and $\mathscr{B}$ be two categories. We have :*

(i) *If $\mathscr{A}$ is small and $\mathscr{B}$ has small* Hom*-sets, the category* $\mathsf{Fun}(\mathscr{A}, \mathscr{B})$ *has small* Hom*-sets.*

(ii) *Suppose that $\mathscr{B}$ is complete or cocomplete. Then, for every $A \in \mathrm{Ob}(\mathscr{A})$ the functor*

$$\varepsilon_A : \mathsf{Fun}(\mathscr{A}, \mathscr{B}) \to \mathscr{B} \qquad F \mapsto FA \qquad (\eta : F \Rightarrow G) \mapsto (\eta_A : FA \to GA)$$

commutes with all colimits and with all limits.

(iii) *If $\mathscr{B}$ is finitely complete (resp. finitely cocomplete) then ε_A commutes with finite limits (resp. with finite colimits), for every $A \in \mathrm{Ob}(\mathscr{A})$.*

(iv) *If $\mathscr{B}$ is complete (resp. cocomplete, resp. finitely complete, resp. finitely cocomplete), the same holds for* $\mathsf{Fun}(\mathscr{A}, \mathscr{B})$.

Proof. (i): If $F, G : \mathscr{A} \to \mathscr{B}$ are any two functors, then $\mathrm{Hom}_{\mathsf{Fun}(\mathscr{A},\mathscr{B})}(F, G)$ is a subset of the product $\prod_{A \in \mathrm{Ob}(\mathscr{A})} \mathrm{Hom}_{\mathscr{B}}(FA, GA)$, hence it is a small set.

(iv): Let I be any small category, and

$$F : I \to \mathsf{Fun}(\mathscr{A}, \mathscr{B}) \qquad i \mapsto (F_i : \mathscr{A} \to \mathscr{B}) \qquad (\varphi : i \to j) \mapsto (F_\varphi : F_i \Rightarrow F_j)$$

any functor. For every $A \in \mathrm{Ob}(\mathscr{A})$ we obtain a functor $F_A : I \to \mathscr{B}$ by the rule

$$F_A(i) := F_i A \qquad F_A(\varphi) := (F_\varphi)_A : F_i A \to F_j A$$

for every $i, j \in \mathrm{Ob}(I)$ and every $\varphi \in \mathrm{Hom}_I(i, j)$. If $\mathscr{B}$ is complete, for every such A we may find $LA \in \mathrm{Ob}(\mathscr{B})$ and a universal cone $\gamma_A : c_{LA} \Rightarrow F_A$. Next, if $g : A \to A'$ is any morphism in $\mathscr{A}$, we obtain a natural transformation $F_g : F_A \Rightarrow F_{A'}$ by the rule

$$(F_g)_i := F_i(g) : F_i A \to F_i A' \qquad \text{for every } i \in \mathrm{Ob}(I)$$

and the limit of F_g is represented by a morphism $Lg : LA \to LA'$ in $\mathscr{B}$ such that

$$F_g \circ \gamma_A = \gamma_{A'} \circ c_{Lg}.$$

It is then easily seen that the rules $A \mapsto LA$ and $g \mapsto Lg$ amount to a well defined functor $\mathscr{A} \to \mathscr{B}$ that represents the limit of F : details left to the reader. Lastly, if $\mathscr{B}$ is cocomplete (resp. finitely cocomplete), $\mathscr{B}^o$ is complete (resp. finitely complete), so the same holds for $\mathsf{Fun}(\mathscr{A}^o, \mathscr{B}^o)$, by the foregoing; then $\mathsf{Fun}(\mathscr{A}, \mathscr{B})$ is cocomplete (resp. finitely complete), by remark 1.1.19(i).

(ii): Suppose that $\mathscr{B}$ is complete (resp. cocomplete); in this case, the proof of (iv) already shows that ε_A commutes with all limits (resp. colimits). Hence, it remains only to check that ε_A commutes as well with all colimits (resp. limits). To this aim, we may assume that $\mathscr{A}$ and $\mathscr{B}$ are both small, and we denote by $\mathbf{1}$ the category with one object and one morphism; we let

$i_A : \mathbf{1} \to \mathscr{A}$ be the (unique) functor that sends the object of $\mathbf{1}$ to A. We have a commutative diagram of functors

$$\begin{array}{ccc} \mathsf{Fun}(\mathscr{A}, \mathscr{B}) & \xrightarrow{\mathsf{Fun}(i_A, \mathscr{B})} & \mathsf{Fun}(\mathbf{1}, \mathscr{B}) \\ & \searrow^{\varepsilon_A} & \downarrow \\ & & \mathscr{B} \end{array}$$

whose vertical arrow is an obvious isomorphism of categories. It then suffices to check that the functor $\mathsf{Fun}(i_A, \mathscr{B})$ commutes with all colimits (resp. limits). However, by theorem 1.3.4, the functor $\mathsf{Fun}(i_A, \mathscr{B})$ admits a right (resp. left) adjoint, so it suffices to invoke proposition 1.3.25(ii,iii).

(iii) follows likewise from the proof of (iv). □

Remark 1.4.2. (i) By inspecting the proof of theorem 1.3.4, it is easily seen that, more generally, the functors ε_A of corollary 1.4.1 commute with limits when $\mathscr{B}$ is a category with representable coproducts. Dually, if all products of $\mathscr{B}$ are representable, ε_A commutes with colimits.

(ii) In the same vein, suppose moreover that $\mathrm{Hom}_{\mathscr{A}}(A, A')$ is a finite set for every $A, A' \in \mathrm{Ob}(\mathscr{A})$. Then, again by inspection of the proof of theorem 1.3.4 we see that for the functors ε_A to commute with limits (resp. colimits) it suffices that all finite coproducts (resp. finite products) of $\mathscr{B}$ are representable.

(iii) Furthermore, the proof of corollary 1.4.1(iv) shows more precisely the following. If $F : I \to \mathrm{Fun}(\mathscr{A}, \mathscr{B})$ is a functor from a small category I, and the limit (resp. colimit) of $\varepsilon_A \circ F$ is representable in $\mathscr{B}$ for every $A \in \mathrm{Ob}(\mathscr{A})$, then the limit (resp. colimit) of F is representable in $\mathrm{Fun}(\mathscr{A}, \mathscr{B})$, and ε_A commutes with this limit (resp. colimit), for every $A \in \mathrm{Ob}(\mathscr{A})$.

Corollary 1.4.3. *Let $\mathscr{C}$ be any small category, and $f : F \to G$ a morphism in $\mathscr{C}^\wedge$.*

(i) *$\mathscr{C}^\wedge$ has small* Hom*-sets and is both well-powered and co-well-powered (see* (1.1.2)*).*

(ii) *For every $X \in \mathrm{Ob}(\mathscr{C})$, the functor*

$$\mathscr{C}^\wedge \to \mathbf{Set} \quad : \quad F \mapsto FX \qquad (f : F \to G) \mapsto (f_X : FX \to GX)$$

commutes with all limits and all colimits (in other words, the limits and colimits in $\mathscr{C}^\wedge$ are computed argumentwise).

(iii) *f is a monomorphism (resp. an epimorphism) if and only if the map $f_X : FX \to GX$ is injective (resp. surjective) for every $X \in \mathrm{Ob}(\mathscr{C})$.*

(iv) *f is an isomorphism if and only if it is both a monomorphism and an epimorphism.*

(v) *Let $f(F) \in \mathrm{Ob}(\mathscr{C}^\wedge)$ be the subobject of F with $f(F)(X) := \mathrm{Im}(f_X : FX \to GX)$ for every $X \in \mathrm{Ob}(\mathscr{C})$. Then $f(F) = \mathrm{Im}(f) = \mathrm{Coim}(f)$ (see example* 1.2.26*).*

(vi) *The Yoneda embedding $h_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}^\wedge$ commutes with all representable limits of $\mathscr{C}$.*

(vii) *If $\mathscr{D}$ is any category with small* Hom*-sets and $\varphi : \mathscr{C} \to \mathscr{D}$ any functor, then $\varphi^\wedge$ commutes with all limits and all colimits of $\mathscr{D}^\wedge$, and $\varphi_!$ (resp. φ_*) commutes with all colimits (resp. with all limits) of $\mathscr{C}^\wedge$.*

Proof. (i): The first assertion is a special case of corollary 1.4.1(i). Next, to see that $\mathscr{C}^\wedge$ is well-powered notice that for every presheaf F on $\mathscr{C}$, and every $X \in \mathrm{Ob}(\mathscr{C})$, the set of subsets of $F(X)$ is small, and a subobject of F is just a compatible system of subsets $F'(X) \subset F(X)$, for X ranging over the small set of objects of $\mathscr{C}$. Likewise one sees that $\mathscr{C}^\wedge$ is co-well-powered.

(ii) is a special case of corollary 1.4.1(ii), but it is also already clear from example 1.2.24(i).

(iii): It is easily seen that if f_X is injective (resp. surjective) for every $X \in \mathrm{Ob}(\mathscr{C})$, then f is a monomorphism (resp. an epimorphism). The converse follows from (ii) and proposition 1.3.18.

(iv) is an immediate consequence of (iii).

(v): It follows from (iii) that the induced morphism $F \to f(F)$ is a quotient of F, and the sequence $F \to f(F) \to G$ is the unique factorization of f (up to unique isomorphism) as a composition of an epimorphism followed by a monomorphism, whence the assertion.

(vi) follows directly from example 1.2.24(ii) and remark 1.3.14(i).

(vii) is a special case of proposition 1.3.22(iii,iv). □

Example 1.4.4. Let $\mathscr{C}$ be a category with small Hom-sets, and $Y \in \mathrm{Ob}(\mathscr{C})$ any object.

(i) Suppose that all the products of small families of objects of $\mathscr{C}$ are representable, and consider the functor

$$Y^\bullet : \mathbf{Set}^o \to \mathscr{C}^\wedge$$

which to any (U-small) set S assigns the product in $\mathscr{C}$ of S copies of Y, *i.e.* the limit of the functor $\varphi_Y^S : S \to \mathscr{C}$ such that $\varphi_Y(s) := Y$ for every $s \in S$ (where S is regarded as a discrete category), and to any map of sets $f : T \to S$, attaches the morphism of presheaves $Y^f := \lim_f \varphi_Y$. The assumption on $\mathscr{C}$ implies that $Y^\bullet$ is representable by a functor which we denote as well

$$Y^\bullet : \mathbf{Set}^o \to \mathscr{C}$$

(see remark 1.2.8(ii)). On the other hand, we have the functor $h_Y : \mathscr{C}^o \to \mathbf{Set}$ such that $h_Y(X^o) := \mathrm{Hom}_{\mathscr{C}}(X, Y)$ for every $X \in \mathrm{Ob}(\mathscr{C})$, and notice the natural identifications :

$$\mathrm{Hom}_{\mathscr{C}^o}((Y^S)^o, X^o) = \mathrm{Hom}_{\mathscr{C}}(X, Y^S) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Set}}(S, h_Y(X^o))$$

which say that h_Y is right adjoint to the functor $(Y^\bullet)^o$. By proposition 1.3.25(iii) it follows that h_Y commutes with all small limits of $\mathscr{C}^o$.

(ii) Likewise, if all the small coproducts of $\mathscr{C}$ are representable, we may apply (i) to the functor $h_{Y^o} : \mathscr{C} \to \mathbf{Set}$. We find that h_{Y^o} commutes with all small limits of $\mathscr{C}$.

(iii) Especially, let $I, \mathscr{C}$ be two small categories, $F : I \to \mathscr{C}$ a functor, and G any presheaf on $\mathscr{C}$. From (i),(ii), example 1.2.24(i) and corollary 1.4.3(vi) we get natural isomorphisms :

$$(1.4.5) \qquad \lim_{I^o} \mathrm{Hom}_{\mathscr{C}^\wedge}(h_{\mathscr{C}}^o \circ F^o, G) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{C}^\wedge}(\operatorname*{colim}_I h_{\mathscr{C}} \circ F, G)$$

$$(1.4.6) \qquad \lim_{I} \mathrm{Hom}_{\mathscr{C}^\wedge}(G, h_{\mathscr{C}} \circ F) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{C}^\wedge}(G, \lim_I F)$$

(more precisely, the limit on the left is represented by the set on the right, in both cases). Furthermore, since the tautological cone τ (resp. the tautological cocone μ) for F is universal, we get the universal cone $\mathrm{Hom}_{\mathscr{C}^\wedge}(\tau, G)$ (resp. $\mathrm{Hom}_{\mathscr{C}^\wedge}(G, \mu)$) for the limit appearing in (1.4.5) (resp. in (1.4.6) : see remarks 1.2.11(iv)) and 1.3.14(i)).

1.4.7. It is an important fact that *every presheaf on a small category is the colimit of a system of representable presheaves*; more generally, for any category $\mathscr{C}$ and every $F \in \mathrm{Ob}(\mathscr{C}^\wedge)$, let us consider the *category of elements of* F :

$$\mathscr{F}ib(F)$$

whose objects are the pairs (X, s) where $X \in \mathrm{Ob}(\mathscr{C})$ and $s \in FX$ (the notation shall be explained in (3.1.15)). The morphisms $f : (X, s) \to (Y, t)$ in $\mathscr{F}ib(F)$ are the morphisms $f : X \to Y$ of $\mathscr{C}$ such that $Ff(t) = s$. We have an obvious *source functor*

$$\mathsf{s}_F : \mathscr{F}ib(F) \to \mathscr{C} \qquad (X, s) \mapsto X \qquad ((X, s) \xrightarrow{f} (Y, t)) \mapsto (X \xrightarrow{f} Y).$$

Notice that if $\mathscr{C}$ is small, the same holds for $\mathscr{F}ib(F)$. Also, if $\mathscr{C}$ has small Hom-sets, Yoneda's lemma naturally identifies $\mathscr{F}ib(F)$ with the category $h_{\mathscr{C}}\mathscr{C}/F$ (notation of (1.1.27) and (1.2.4)); namely, a pair (X, s) corresponds to the unique morphism of presheaves

$$h_{(X,s)} : h_X \to F \qquad \text{such that} \qquad h_{(X,s),X}(\mathbf{1}_X) = s.$$

Moreover, in this case s_F is identified with the source functor $\mathsf{s}_F : h_{\mathscr{C}}\mathscr{C}/F \to \mathscr{C}$ of (1.1.27). We have a natural cocone

$$h^F : h_{\mathscr{C}} \circ \mathsf{s}_F \Rightarrow c_F \qquad (X, s) \mapsto h_{(X,s)}.$$

Lemma 1.4.8. *With the notation of* (1.4.7)*, for every category* $\mathscr{C}$ *and every presheaf* F *on* $\mathscr{C}$*, the cocone* h^F *is universal.*

Proof. Let G be any presheaf on $\mathscr{C}$. A natural transformation $\tau : h_{\mathscr{C}} \circ \mathsf{s}_F \Rightarrow c_G$ is a rule that assigns to every $(X, s) \in \mathrm{Ob}(h_{\mathscr{C}}\mathscr{C}/F)$ a morphism of presheaves $h_X \to G$; the latter is naturally identified with an element $\tau_{(X,s)} \in GX$. Thus, τ induces a map $\tau_{(X,\bullet)} : FX \to GX$ for every $X \in \mathrm{Ob}(\mathscr{C})$, by the rule : $(X, s) \mapsto \tau_{(X,s)}$. Moreover, the naturality of τ implies that

$$Gf(\tau_{(Y,s)}) = \tau_{(X,s)} \qquad \text{for every morphism } f : (X, s) \to (Y, t) \text{ in } h_{\mathscr{C}}\mathscr{C}/F$$

(details left to the reader); *i.e.* the rule $X \mapsto \tau_{(X,\bullet)}$ for every $X \in \mathrm{Ob}(\mathscr{C})$ yields a morphism of presheaves $h^\tau : F \to G$, and it is easily seen that $\tau = c_{h^\tau} \odot h^F$, whence the contention. □

Example 1.4.9. (i) Let $\mathscr{C}$ be a small category, and $\mathbb{1}_{\mathscr{C}}$ a final object of $\mathscr{C}^\wedge$ (see example 1.2.16(v)); since the source functor $\mathsf{s}_{\mathbb{1}_{\mathscr{C}}} : h_{\mathscr{C}}\mathscr{C}/\mathbb{1}_{\mathscr{C}} \to \mathscr{C}$ is an isomorphism of categories, we deduce from lemma 1.4.8 a natural isomorphism

$$\operatorname*{colim}_{\mathscr{C}} h_{\mathscr{C}} \xrightarrow{\sim} \mathbb{1}_{\mathscr{C}} \qquad \text{in } \mathscr{C}^\wedge.$$

(ii) Let $\mathscr{C}$ be a category with small Hom-sets, $X \in \mathrm{Ob}(\mathscr{C})$, and $h_X : \mathscr{C}^o \to \mathbf{Set}$ the corresponding presheaf (notation of (1.2.4)). By inspecting the definition, we find that

$$\mathscr{F}ib(h_X) = \mathscr{C}/X$$

and under this identification, the source functor s_{h_X} of (1.4.7) corresponds to the source functor s_X of (1.1.24). So, lemma 1.4.8 in this case shows that X represents in $\mathscr{C}$ the colimit of $h_{\mathscr{C}} \circ \mathsf{s}_X$.

(iii) More generally, if $\mathscr{A}$ and $\mathscr{B}$ are two categories, $F : \mathscr{A} \to \mathscr{B}$ any functor, we have

$$\mathscr{F}ib(h_B \circ F) = F\mathscr{A}/B \qquad \text{for every } B \in \mathrm{Ob}(\mathscr{B})$$

(notation of (1.1.27)) and again, the respective source functors agree, under this identification. Hence, if $\mathscr{A}$ is small and $\mathscr{B}$ has small Hom-sets, then the presheaf $h_B \circ F$ on $\mathscr{A}$ is isomorphic to the colimit of the functor $h_{\mathscr{A}} \circ \mathsf{s}_B : F\mathscr{A}/B \to \mathscr{A}^\wedge$.

1.4.10. We conclude this section with a few observations concerning presheaves on the slice categories of (1.1.24). Thus, let $\mathscr{C}$ be a category, X any object of $\mathscr{C}$, denote by $\mathsf{s}_X : \mathscr{C}/X \to \mathscr{C}$ the source functor of (1.1.24), and suppose that $\mathscr{C}$ is V-small, for some universe V. To begin with, a direct inspection of (1.3.8) yields a natural isomorphism :

(1.4.11) $$(\mathsf{s}_X)_{\mathsf{V}!}F(Y) \xrightarrow{\sim} \{(\varphi, a) \mid \varphi \in \mathrm{Hom}_{\mathscr{C}}(Y, X),\ a \in F(\varphi)\}$$

for every V-presheaf F on $\mathscr{C}/X$ and every $Y \in \mathrm{Ob}(\mathscr{C})$. Indeed, from *loc.cit.* we see that every element of $(\mathsf{s}_X)_{\mathsf{V}!}F(Y)$ is the equivalence class of a datum

$$(\psi : Y \to Z, \alpha : Z \to X, s \in F(\alpha))$$

and such a datum is equivalent to $(\mathbf{1}_Y, \alpha \circ \psi : Y \to X, F(\psi)(s) \in F(\alpha \circ \psi))$. If $\psi : Z \to Y$ is any morphism of $\mathscr{C}$, the corresponding map $(\mathsf{s}_X)_{\mathsf{V}!}F(Y) \to (\mathsf{s}_X)_{\mathsf{V}!}F(Z)$ is given by the rule :

(1.4.12) $$(\varphi, a) \mapsto (\varphi \circ \psi, F(\psi)(a)) \qquad \text{for every } \varphi : Y \to X \text{ and } a \in F(\varphi).$$

Moreover, under this natural identification the adjoint pair $((\mathsf{s}_X)_{\mathsf{V}!}, \mathsf{s}_X^\wedge)$ assigns to any morphism $t : F \to \mathsf{s}_X^\wedge G$ of presheaves on $\mathscr{C}/X$ the morphism $t^* : (\mathsf{s}_X)_{\mathsf{V}!}F \to G$ in $\mathscr{C}^\wedge$ given by the rule :

$$(\varphi, a) \mapsto t(\varphi)(a) \qquad \text{for every } Y \in \mathrm{Ob}(\mathscr{C}), \varphi : Y \to X \text{ and } a \in F(\varphi).$$

Proposition 1.4.13. *Let* $\mathscr{C}$ *be any category and* X *any object of* $\mathscr{C}$*.*

(i) *The source functor* $\mathsf{s}_X : \mathscr{C}/X \to \mathscr{C}$ *commutes with all connected limits and with the colimit of every functor* $F : I \to \mathscr{C}/X$ *such that* $\mathrm{colim}_I\, \mathsf{s}_X \circ F$ *is representable in* $\mathscr{C}$.
(ii) *Dually, the target functor* $\mathsf{t}_X : X/\mathscr{C} \to \mathscr{C}$ *commutes with all connected colimits and with the limit of every functor* $F : I \to X/\mathscr{C}$ *such that* $\lim_I \mathsf{t}_X \circ F$ *is representable in* $\mathscr{C}$.
(iii) *If all the binary products of* $\mathscr{C}$ *are representable,* s_X *admits a right adjoint, and therefore it commutes with all colimits of* $\mathscr{C}/X$.
(iv) *Dually, if all the binary coproducts of* $\mathscr{C}$ *are representable,* t_X *admits a left adjoint, and therefore it commutes with all limits of* $X/\mathscr{C}$.
(v) *If* $\mathscr{C}$ *is complete (resp. cocomplete, resp. finitely complete, resp. finitely cocomplete), then the same holds for both* $\mathscr{C}/X$ *and* $X/\mathscr{C}$.
(vi) *Suppose that* $\mathscr{C}$ *has small* Hom*-sets. Then the following holds :*
  (a) *The functor* $\mathsf{s}_X^\wedge$ *admits a left adjoint* $\mathsf{s}_{X!} : (\mathscr{C}/X)^\wedge \to \mathscr{C}^\wedge$.
  (b) *Moreover* $\mathsf{s}_{X!}$ *factors as the composition of an equivalence of categories*

$$e_X : (\mathscr{C}/X)^\wedge \to \mathscr{C}^\wedge/h_X$$

  and the source functor $\mathsf{s}_{h_X} : \mathscr{C}^\wedge/h_X \to \mathscr{C}^\wedge$ *(notation of* (1.2.4)*).*
  (c) $\mathsf{s}_{X!}$ *commutes with fibre products and preserves monomorphisms.*

Proof. (v): Suppose that $\mathscr{C}$ is complete (resp. finitely complete); to show that the same holds for $\mathscr{C}/X$, we may apply the criterion of proposition 1.2.22(i). Indeed, suppose that $p_\bullet := (p_i : Y_i \to X \mid i \in I)$ is any family (resp. any finite family) of objects of $\mathscr{C}/X$, indexed by a small set I, which we regard as a discrete category. We consider the category I_X whose set of objects is the disjoint union of I and $\{X\}$, and with

$$\mathrm{Morph}(I_X) = \mathrm{Morph}(I) \cup \{(X, X, \mathbf{1}_X)\} \cup \{(i, X, p_i) \mid i \in I\}$$

so that X is the unique final object of I_X. Next, we define a functor $F : I_X \to \mathscr{C}$, by letting

$$FX := X \qquad \text{and} \qquad Fi := Y_i \quad Fp_i := p_i \qquad \text{for every } i \in I.$$

Any universal cone $\tau : c_L \Rightarrow F$ yields a morphism $\tau_X : L \to X$, and it is easily seen that the object τ_X of $\mathscr{C}/X$ represents the product of the family $p_\bullet$. Lastly, let $p_i : Y_i \to X$ ($i = 1, 2$) be two objects of $\mathscr{C}/X$, and $f, g : p_1 \to p_2$ two morphisms in $\mathscr{C}/X$. Denote by E any object of $\mathscr{C}$ representing the equalizer of $\mathsf{s}_X(f), \mathsf{s}_X(g) : Y_1 \to Y_2$ in $\mathscr{C}$, where $\mathsf{s}_X : \mathscr{C}/X \to \mathscr{C}$ is the source functor. The corresponding universal cone yields a morphism $h : E \to Y_1$ such that $\mathsf{s}_X(f) \circ h = \mathsf{s}_X(g) \circ h$, and it is easily seen that the object $p_1 \circ h : E \to X$ of $\mathscr{C}/X$ represents the equalizer of f and g (details left to the reader).

Next, suppose that $\mathscr{C}$ is cocomplete (resp. finitely cocomplete), and let $F : I \to \mathscr{C}/X$ be a functor from a small (resp. finite) category I. Let also C be any object of $\mathscr{C}$ representing the colimit of $\mathsf{s}_X \circ F$; the functor F can be regarded as a cocone $\mathsf{s}_X \circ F \Rightarrow c_X$, which corresponds to a morphism $\tau : C \to X$, and it is easily seen that $\tau \in \mathrm{Ob}(\mathscr{C}/X)$ represents the colimit of F.

The assertions for $X/\mathscr{C}$ follows as usual, by considering the opposite categories.

(vi.a): The discussion of (1.4.10) yields a left adjoint $(\mathsf{s}_X)_{\mathsf{V}!}$ to $(\mathsf{s}_X)^\wedge_{\mathsf{V}}$, for any universe V such that $\mathscr{C}$ has V-small Hom-sets. However, by inspecting (1.4.11), we see that if $\mathscr{C}$ has small Hom-sets and F is a U-presheaf on $\mathscr{C}/X$, then $(\mathsf{s}_X)_{\mathsf{V}!}F$ is a U-presheaf. Since the inclusion $(\mathscr{C}/X)^\wedge_{\mathsf{U}} \to (\mathscr{C}/X)^\wedge_{\mathsf{V}}$ is fully faithful, we deduce that the target of (1.4.11) can be used to define the sought left adjoint $(\mathsf{s}_X)_! : (\mathscr{C}/X)^\wedge_{\mathsf{U}} \to \mathscr{C}^\wedge_{\mathsf{U}}$ to $(\mathsf{s}_X)^\wedge_{\mathsf{U}}$.

(vi.b): Notice that the presheaf $\mathbb{1}_{\mathscr{C}/X} := h_{\mathbf{1}_X}$ is a final object of $(\mathscr{C}/X)^\wedge$: indeed, for every object $f : Y \to X$ of $\mathscr{C}/X$ we have $\mathbb{1}_{\mathscr{C}/X}(f) = \{f/X\}$ (notation of (1.1.24)). Then remark 1.3.6(v) yields a natural isomorphism :

$$\mathsf{s}_{X!}(\mathbb{1}_{\mathscr{C}/X}) \xrightarrow{\sim} h_X$$

that identifies the unit of adjunction with the morphism of presheaves :

$$(1.4.14)\qquad \mathbb{1}_{\mathscr{C}/X} \to \mathsf{s}_X^\wedge(h_X) \qquad \text{such that} \qquad f/X \mapsto f \qquad \text{for every } f \in \mathrm{Ob}(\mathscr{C}/X).$$

Now, every presheaf F on $\mathscr{C}/X$ admits a unique morphism $F \to \mathbb{1}_{\mathscr{C}/X}$, whence a natural morphism $\mathsf{s}_{X!}F \to h_X$, so we see already that $\mathsf{s}_{X!}$ factors through s_{h_X} and a functor e_X as required; more precisely, (1.4.12), says that the morphism $e_X(F) : \mathsf{s}_{X!}F \to h_X$ is given by the rule : $(f, a) \mapsto f$ for every $Y \in \mathrm{Ob}(\mathscr{C})$ and every $(f, a) \in \mathsf{s}_{X!}F(Y)$. To check that e_X is an equivalence, we exhibit an explicit quasi-inverse : namely, to every object $\varphi : F \to h_X$ of $\mathscr{C}^\wedge/h_X$ we assign the presheaf

$$e'_X(\varphi) := \mathsf{s}_X^\wedge(F) \times_{\mathsf{s}_X^\wedge(h_X)} \mathbb{1}_{\mathscr{C}/X}$$

defined via (1.4.14) and the morphism $\mathsf{s}_X^\wedge(\varphi) : \mathsf{s}_X^\wedge(F) \to \mathsf{s}_X^\wedge(h_X)$. Now, let F be any presheaf on $\mathscr{C}/X$, and $f : Y \to X$ any object of $\mathscr{C}/X$; by unwinding the definitions, we see that

$$(e'_X \circ e_X(F))(f) = \mathsf{s}_{X!}F(Y) \times_{h_X(Y)} \{f/X\}$$

which is the set of all pairs (f, a) with $a \in Ff$. Thus we get an obvious isomorphism of presheaves $e'_X \circ e_X(F) \xrightarrow{\sim} F$, natural with respect to F. Likewise, for every object $\varphi : G \to h_X$ of $\mathscr{C}^\wedge/h_X$ and every $Y \in \mathrm{Ob}(\mathscr{C})$ we get the map

$$(e_X \circ e'_X(\varphi))_Y : \mathsf{s}_{X!}(\mathsf{s}_X^\wedge(F) \times_{\mathsf{s}_X^\wedge(h_X)} \mathbb{1}_{\mathscr{C}/X})(Y) \to h_X(Y)$$

which is described as follows. First, $\mathsf{s}_{X!}(\mathsf{s}_X^\wedge(F) \times_{\mathsf{s}_X^\wedge(h_X)} \mathbb{1}_{\mathscr{C}/X})(Y)$ is the set of all pairs (f, a) with $f : Y \to X$ a morphism of $\mathscr{C}$, and $a \in GY \times_{h_X(Y)} \{f/X\}$; *i.e.* the pairs (f, a) with $\varphi_Y(a) = f$. Then, the map $(e_X \circ e'_X(\varphi))_Y$ is given by the rule : $(f, a) \mapsto f$ for every such pair (f, a). Again, we deduce an obvious isomorphism $e_X \circ e'_X(\varphi) \xrightarrow{\sim} \varphi$ of h_X-objects of $\mathscr{C}^\wedge$, natural with respect to φ, as required.

(vi.c) follows from the explicit description of $\mathsf{s}_{X!}$ given in (1.4.10), together with corollary 1.4.3(ii,iii).

(i): The assertion for representable colimits follows from the proof of (v) (together with remark 1.3.14(i)). Next, let $F : I \to \mathscr{C}/X$ be any functor from a connected small category I; taking into account examples 1.2.7(ii) and (1.4.10) it suffices to show that the natural morphism

$$((\mathsf{s}_X)_{\mathsf{V}!} \lim_I F)(Z) \to (\lim_I \mathsf{s}_X \circ F)(Z) \qquad (\varphi : Z \to X, \tau : c_\varphi \Rightarrow F) \mapsto (\mathsf{s}_X * \tau : c_Z \Rightarrow \mathsf{s}_X \circ F)$$

is an isomorphism for every $Z \in \mathrm{Ob}(\mathscr{C})$ and any sufficiently large universe V. Now, let $\tau : c_Z \Rightarrow \mathsf{s}_X \circ F$ be any cone (with vertex an arbitrary $Z \in \mathrm{Ob}(\mathscr{C})$); we remark :

Claim 1.4.15. For every $i, j \in \mathrm{Ob}(I)$ we have $Fi \circ \tau_i = Fj \circ \tau_j : Z \to X$.

Proof of the claim. Since I is connected, a simple induction argument reduces to the case where there exists $k \in \mathrm{Ob}(I)$ with morphisms $a : k \to i$ and $b : k \to j$. In this case, we may compute directly : $Fi \circ \tau_i = Fi \circ \mathsf{s}_X(Fa) \circ \tau_k = F_k \circ \tau_k = Fj \circ \mathsf{s}_X(Fb) \circ \tau_k = Fj \circ \tau_j$. ◊

Hence, pick any $i \in \mathrm{Ob}(I)$ (recall that this set is non-empty), and set $\varphi := Fi \circ \tau_i$; claim 1.4.15 says that τ lifts uniquely to a cone $\tau_{/X} : c_\varphi \Rightarrow F$, such that $\mathsf{s}_X * \tau_{/X} = \tau$, whence the contention.

(iv): Pick a universe V such that $\mathscr{C}$ has V-small Hom-sets. For every $Y \in \mathrm{Ob}(\mathscr{C})$, the functor

$$\mathscr{C} \to \mathscr{C}_{\mathsf{V}}^\wedge \qquad Y \mapsto X \amalg Y$$

is then representable by a functor $\mathscr{C} \to \mathscr{C}$, and we abuse notation, by denoting both functors in the same way. Moreover, the universal cocone for the coproduct consists of a pair of morphisms $X \xrightarrow{i_{X,Y}} X \amalg Y \xleftarrow{i'_{X,Y}} Y$ whence a functor

$$i_X : \mathscr{C} \to X/\mathscr{C} \quad : \quad Y \mapsto i_{X,Y} \qquad \text{for every } Y \in \mathrm{Ob}(\mathscr{C}).$$

It is easily seen that i_X is left adjoint to t_X (details left to the reader). Thus, the assertion follows from proposition 1.3.25(i,iii).

(iii): Notice that the source functor $\mathsf{s}_X : \mathscr{C}/X \to \mathscr{C}$ equals $\mathsf{t}^o_{X^o}$ and the coproducts of $\mathscr{C}^o$ are representable; then the assertion follows from (iv) and remark 1.1.19(iv). The same argument shows that (i) implies (ii). □

Remark 1.4.16. Let $\mathscr{C}$ and X be as in proposition 1.4.13(vi).

(i) The functor $\mathsf{s}_{X!}$ does not generally preserve final objects, hence it is not generally exact.

(ii) The quasi-inverse to e_X constructed in the proof of proposition 1.4.13(vi.b), can be described more compactly as the functor that assigns to any object $g : F \to h_X$ of $\mathscr{C}^\wedge/h_X$ the presheaf given by the rule :

$$(Y \xrightarrow{\varphi} X) \mapsto \mathrm{Hom}_{\mathscr{C}^\wedge/h_X}((h_Y \xrightarrow{h_\varphi} h_X), g).$$

Example 1.4.17. (i) Let $\mathscr{C}$ be a category whose binary products are representable, and pick a universe V such that $\mathscr{C}$ has V-small Hom-sets. For every $X \in \mathrm{Ob}(\mathscr{C})$, we have a functor

$$p_X : \mathscr{C} \to \mathscr{C}^\wedge_{\mathsf{V}} \qquad Y \mapsto X \times Y$$

and the assumption on $\mathscr{C}$ implies that p_X is representable by a functor $\pi_X : \mathscr{C} \to \mathscr{C}$ (remark 1.2.8(ii)). Let now $F : I \to \mathscr{C}$ be a functor from a small category I; we say that *the colimit of F commutes with products*, if π_X commutes with the colimit of F, for every $X \in \mathrm{Ob}(\mathscr{C})$.

(ii) A more restricted class of colimits plays a role in applications. Namely, suppose now that all fibre products of $\mathscr{C}$ are representable. Let F be as in (i), and consider any cone $\tau : F \Rightarrow c_X$, for any $X \in \mathrm{Ob}(\mathscr{C})$. Notice that τ is the same as the datum of a functor $F_\tau : I \to \mathscr{C}/X$ which *lifts F, i.e.* such that $\mathsf{s}_X \circ F_\tau = F$, where $\mathsf{s}_X : \mathscr{C}/X \to \mathscr{C}$ is the source functor. Moreover, the assumption on $\mathscr{C}$ implies that the products are representable in $\mathscr{C}/X$, for every such X. In this situation, we shall then say that the colimit of F is *universal*, if the colimit of F_τ commutes with products, for every X and τ as above.

(iii) Suppose moreover that the colimit of F is representable by an object C^o of $\mathscr{C}^o$, and fix a universal cocone $\mu : F \Rightarrow c_C$. Then any τ as in (ii) corresponds to a unique morphism $f : C \to X$, and μ lifts to a universal cocone $\mu_\tau : F_\tau \Rightarrow c_f$. For every morphism $g : Y \to X$ in $\mathscr{C}$ we may consider the functor

$$F \times_{(\tau,g)} Y := \mathsf{s}_X \circ \pi_g \circ F_\tau : I \to \mathscr{C}$$

(that is, $F \times_{(\tau,g)} Y(i)$ represents $Fi \times_{(\tau_i,g)} Y$ for every $i \in \mathrm{Ob}(I)$). Likewise, we abuse notation and write $C \times_{(f,g)} Y$ to denote the representative $\mathsf{s}_X \circ \pi_g(f)$ for this fibre product; taking into account proposition 1.4.13(v), we see that the colimit of F is universal if and only if the cocone

$$(\mathsf{s}_X \circ \pi_g) * \mu_\tau : F \times_{(\tau,g)} Y \Rightarrow c_{C \times_{(f,g)} Y}$$

is universal for every such X, τ and g.

(iv) For instance, from the explicit descriptions of example 1.2.23(i) one may deduce that all colimits in $\mathbf{Set}$ are universal. In view of example 1.2.24(i) and corollary 1.4.3(ii), it follows easily that all colimits in $\mathscr{B}^\wedge$ are universal, for any small category $\mathscr{B}$.

1.4.18. Let $F : \mathscr{A} \to \mathscr{B}$ be a functor from a small category $\mathscr{A}$ to a category $\mathscr{B}$ with small Hom-sets. By remark 1.3.6(ii), for every object $A \in \mathrm{Ob}(\mathscr{A})$ we have a natural isomorphism $h_{FA} \xrightarrow{\sim} F_!A$; combining with example 1.2.28(i), we deduce that the adjoint pair of functors $F^\wedge : \mathscr{B}^\wedge \to \mathscr{A}^\wedge$ and $F_! : \mathscr{A}^\wedge \to \mathscr{B}^\wedge$ induces an adjoint pair :

$$(F^\wedge)_{|h_A} : \mathscr{B}^\wedge/h_{FA} \xrightarrow{\sim} \mathscr{B}^\wedge/F_!h_A \to \mathscr{A}^\wedge/h_A \qquad (F_!)_{|h_A} : \mathscr{A}^\wedge/h_A \to \mathscr{B}^\wedge/F_!h_A \xrightarrow{\sim} \mathscr{B}^\wedge/h_{FA}.$$

For every category I, let $I_\circ$ be the category such that

- $\mathrm{Ob}(I_\circ)$ is the disjoint union of $\mathrm{Ob}(I)$ with a set $\{i_\circ\}$

- $i_\circ$ is the unique final object of $I_\circ$
- I is a full subcategory of $I_\circ$ (more precisely, the full subcategory of $I_\circ$ whose set of objects is $\mathrm{Ob}(I)$ coincides with I).

It is clear that these conditions determine uniquely $I_\circ$. Moreover, for every category $\mathscr{C}$ and every $C \in \mathrm{Ob}(\mathscr{C})$, every functor $\varphi : I \to \mathscr{C}/C$ extends uniquely to a well defined functor $\varphi_\circ : I_\circ \to \mathscr{C}/C$ such that $\varphi_\circ(i_\circ) = \mathbf{1}_C$.

Proposition 1.4.19. (i) *With the notation of* (1.4.18)*, for every* $A \in \mathrm{Ob}(\mathscr{A})$ *we have* essentially commutative *diagrams :*

$$\begin{array}{ccc} (\mathscr{B}/FA)^\wedge & \xrightarrow{(F_{|A})^\wedge} & (\mathscr{A}/A)^\wedge \\ {\scriptstyle e_{FA}}\downarrow & & \downarrow{\scriptstyle e_A} \\ \mathscr{B}^\wedge/h_{FA} & \xrightarrow{(F^\wedge)_{|h_A}} & \mathscr{A}^\wedge/h_A \end{array} \qquad \begin{array}{ccc} (\mathscr{A}/A)^\wedge & \xrightarrow{(F_{|A})_!} & (\mathscr{B}/FA)^\wedge \\ {\scriptstyle e_A}\downarrow & & \downarrow{\scriptstyle e_{FA}} \\ \mathscr{A}^\wedge/h_A & \xrightarrow{(F_!)_{|h_A}} & \mathscr{B}^\wedge/h_{FA} \end{array}$$

where e_A *and* e_{FA} *are the equivalences of proposition* 1.4.13(vi.b).

(ii) *Let moreover* $\varphi : I \to \mathscr{A}/A$ *be any functor. Then* $F_{|A}$ *commutes with the limit of* φ *if and only if* F *commutes with the limit of* $\mathsf{s}_A \circ \varphi_\circ : I_\circ \to \mathscr{A}$.

Proof. (i): The assertion means that there exists an isomorphism of functors $e_A \circ (F_{|A})^\wedge \xrightarrow{\sim} (F^\wedge)_{|h_A} \circ e_{FA}$, and likewise for the second diagram. Let $\varphi : K \to h_{FA}$ be any object of $\mathscr{B}^\wedge/h_{FA}$; with e'_A and e'_{FA} as in the proof of proposition 1.4.13(vi.b), we get :

$$\begin{aligned} (F_{|A})^\wedge \circ e'_{FA}(\varphi) &= (F_{|A})^\wedge(\mathsf{s}^\wedge_{FA} K \times_{\mathsf{s}^\wedge_{FA} h_{FA}} \mathbb{1}_{\mathscr{B}/FA}) \\ &\xrightarrow{\sim} (\mathsf{s}_{FA} \circ F_{|A})^\wedge K \times_{(\mathsf{s}_{FA}\circ F_{|A})^\wedge h_{FA}} \mathbb{1}_{\mathscr{A}/A} \\ &\xrightarrow{\sim} \mathsf{s}^\wedge_A \circ F^\wedge K \times_{\mathsf{s}^\wedge_A \circ F^\wedge h_{FA}} \mathbb{1}_{\mathscr{A}/A} \\ &\xrightarrow{\sim} e'_A(u^\wedge K \times_{F^\wedge h_{FA}} h_A \to h_A) \\ &\xrightarrow{\sim} e'_A \circ (F^\wedge)_{|h_A}(\varphi) \end{aligned}$$

whence the essential commutativity of the left square diagram. Since e_A and e_{FA} are equivalences, and since $(F_!)_{|h_A}$ is left adjoint to $(F^\wedge)_{|h_A}$, we deduce the essential commutativity for the right square diagram as well.

(ii): Let $j : I \to I_\circ$ be the inclusion functor; notice first that for every category $\mathscr{C}$, every $C \in \mathrm{Ob}(\mathscr{C})$ and every functor $\psi : I \to \mathscr{C}$ we have $\psi = \psi_\circ j$, and the morphism of presheaves

$$\lim_j \mathbf{1}_{\mathscr{C}} : \lim_{I_\circ} \psi_\circ \to \lim_I \psi$$

is an isomorphism (notation of remark 1.2.11(i)). Moreover, it is clear that $F_{|A} \circ \varphi_\circ = (F_{|A} \circ \varphi)_\circ$. There follows a commutative diagram of presheaves :

$$\begin{array}{ccc} (F_{|A})_!(\lim_{I_\circ} \varphi_\circ) & \longrightarrow & \lim_{I_\circ} F_{|A} \circ \varphi_\circ \\ \downarrow & & \downarrow \\ (F_{|A})_!(\lim_I \varphi) & \longrightarrow & \lim_I F_{|A} \circ \varphi. \end{array}$$

whose vertical arrows are isomorphisms, and whose horizontal arrows are the morphisms of definition 1.3.11(i). Especially, $F_{|A}$ commutes with the limit of φ if and only if it commutes with the limit of $\varphi_\circ$. We are thus reduced to checking the following more general :

Claim 1.4.20. Let J be a connected category, and $\psi : J \to \mathscr{A}/A$ a functor. Then $F_{|A}$ commutes with the limit of ψ if and only if F commutes with the limit of $\mathsf{s}_A \circ \psi$.

Proof of the claim. Recall first that $\mathsf{s}_{FA!} = \mathsf{s}_{h_{FA}} \circ e_{FA}$. Since $\mathsf{s}_{h_{FA}}$ is obviously conservative, and e_{FA} is an equivalence, it follows that $\mathsf{s}_{FA!}$ is conservative. Recall moreover that s_A and s_{FA} commute with every connected limit (proposition 1.4.13(i)); summing up, we see that $F_{|A}$ commutes with the limit of ψ if and only if the morphism

$$\mathsf{s}_{FA!} \circ (F_{|A})_!(\lim_J \psi) \xrightarrow{\mathsf{s}_{FA!}(\omega)} \mathsf{s}_{FA!}(\lim_J F_{|A} \circ \psi) \xrightarrow{\omega'} \lim_J \mathsf{s}_{FA} \circ F_{|A} \circ \psi$$

is an isomorphism, where $\omega : (F_{|A})_!(\lim_J \psi) \to \lim_J F_{|A} \circ \psi$ and ω' are the morphisms of definition 1.3.11(i). But this composition is, up to isomorphism, the same as the morphism

$$(F \circ \mathsf{s}_A)_!(\lim_J \psi) \to \lim_J F \circ \mathsf{s}_A \circ \psi \qquad (1.4.21)$$

of definition 1.3.11(i), relative to the functor $\mathsf{s}_{FA} \circ F_{|A} = F \circ \mathsf{s}_A$ and the limit of ψ (see the proof of proposition 1.3.17(i)). By the same token, (1.4.21) is also the composition $\omega''' \circ F_!(\omega'')$ of the same type, where $\omega'' : \mathsf{s}_{A!}(\lim_J \psi) \to \lim_J \mathsf{s}_A \circ \psi$ is an isomorphism (since J is connected), and $\omega''' : F_!(\lim_J \psi) \to \lim_J F \circ \psi$ is an isomorphism if and only if F commutes with J. □

1.5. **Coinitial and cofinal functors.** Let I be a small category, $\mathscr{C}$ a category, and $F : I \to \mathscr{C}$ any functor. For the computation of the limit or colimit of F, it may sometimes be desirable to replace the indexing category I by simpler ones. That is, we would like to be able to detect whether a given functor $\varphi : J \to I$ induces an isomorphism from the colimit of F to that of $F \circ \varphi$, and if possible, to construct useful functors of this type, to aid with the calculation of limits or colimits. Concerning the first aim, one has a general criterion, for which we shall need the following :

Definition 1.5.1. Let I, J be any two categories, and $\varphi : J \to I$ any functor.

(i) We say that φ is *cofinal* if the category $i/\varphi J$ is connected, for every $i \in \mathrm{Ob}(I)$ (see definition 1.2.19(ii)).
(ii) We say that φ is *coinitial* if $\varphi^o : J^o \to I^o$ is cofinal.
(iii) If φ is the inclusion functor of a subcategory J of I, and φ is cofinal (resp. coinitial) we also say that J is *cofinal in* I (resp. *coinitial in* I).
(iv) We say that the category I is *coinitially small* (resp. *cofinally small*), if there exists a coinitial (resp. cofinal) functor $\varphi : J \to I$ with J a small category. (See corollary 1.6.5.)

Proposition 1.5.2. *Let $\varphi : J \to I$ be any functor between small categories. Then the following conditions are equivalent :*

(a) *For every category $\mathscr{C}$ with small* Hom*-sets, and every functor $F : I \to \mathscr{C}$, the functor φ induces an isomorphism of presheaves on I^o (see remark* 1.2.11(i)*) :*

$$\operatorname*{colim}_{\varphi} \mathbf{1}_{\mathscr{C}} : \operatorname*{colim}_I F \xrightarrow{\sim} \operatorname*{colim}_J F \circ \varphi. \qquad (1.5.3)$$

(b) *The functor φ is cofinal.*

Proof. (a)⇒(b): We fix $i \in \mathrm{Ob}(I)$ and we apply the definition to the functor

$$h_{i^o} : I \to \mathbf{Set} \qquad i' \mapsto \mathrm{Hom}_I(i, i')$$

whose colimit is a presheaf on the category $\mathbf{Set}^o$. Then the assertion is an immediate consequence of the following :

Claim 1.5.4. For every $i \in \mathrm{Ob}(I)$ we have :

(i) The colimit of h_{i^o} is representable by a set with one element.
(ii) The colimit of $h_{i^o} \circ \varphi$ is representable by $\pi_0(i/\varphi J)$ (notation of remark 1.2.21(ii)).

Proof of the claim. Notice that (i) is the special case of (ii) for $\varphi = \mathbf{1}_I$, since the category i/I admits an initial object, and hence is connected.

(ii): Let $h_I : I \to I^\wedge$ be the Yoneda embedding and choose a representative $L \in \mathrm{Ob}(I^\wedge)$ for the colimit of the functor $h_I \circ \varphi$. By corollary 1.4.3(ii) and remark 1.3.14(i), the set $L(i)$ represents the colimit of $h_{i^o} \circ \varphi$. On the other hand, from remark 1.3.6(ii) we see that L represents as well the colimit of $\varphi_! \circ h_J$. Moreover, recall that the colimit of h_J is represented by the coinitial object $\mathbb{1}_J$ of $J^\wedge$ (example 1.4.9(i)); by corollary 1.4.3(vii) and remark 1.3.14(i) we deduce that L is isomorphic to $\varphi_!(\mathbb{1}_J)$. Then the assertion follows from example 1.3.9. ◊

(b)⇒(a): Let $F : I \to \mathscr{C}$ be any functor; recall that the colimit of F is the functor $L : \mathscr{C} \to \mathbf{Set}$ that assigns to any $X \in \mathrm{Ob}(\mathscr{C})$ the set $L(X)$ of all cocones $F \Rightarrow c_X$, *i.e.* all the compatible systems of morphisms $(\tau_i : Fi \to X \mid i \in \mathrm{Ob}(I))$ in $\mathscr{C}$, such that

$$\tau_{i'} \circ Fg = \tau_i \qquad \text{for every morphism } g : i \to i' \text{ in } I.$$

Likewise, the colimit of $F \circ \varphi$ is the functor $L' : \mathscr{C} \to \mathbf{Set}$ that assigns to any $X \in \mathrm{Ob}(\mathscr{C})$ the set $L'(X)$ of all compatible systems $(\mu_j : F\varphi(j) \to X \mid j \in \mathrm{Ob}(J))$ in $\mathscr{C}$, such that

$$\mu_{j'} \circ F\varphi(h) = \mu_j \qquad \text{for every morphism } h : j \to j' \text{ in } J.$$

Under these identifications, (1.5.3) is the map that assigns to a given τ as above, the compatible system $\tau * \varphi$, such that $(\tau * \varphi)_j := \tau_{\varphi(j)}$ for every $j \in \mathrm{Ob}(J)$. We have to check that the rule $\tau \mapsto \tau * \varphi$ is a bijection $\beta_X : L(X) \xrightarrow{\sim} L'(X)$. However, say that $\tau * \varphi = \tau' * \varphi$ for given $\tau, \tau' \in L(X)$, and consider any $i \in \mathrm{Ob}(I)$; since $i/\varphi J$ is non-empty, we may find $j \in \mathrm{Ob}(J)$ and a morphism $g : i \to \varphi(j)$ in I, whence

$$\tau_i = \tau_{\varphi(j)} \circ Fg = \tau'_{\varphi(j)} \circ Fg = \tau'_i$$

hence β_X is injective. Next, let $\mu \in L'(X)$ be any compatible system; we consider the map

$$\tau^* : \mathrm{Ob}(I/\varphi J) \to \mathrm{Morph}(\mathscr{C}) \qquad (g : i \to \varphi(j)) \mapsto \mu_j \circ Fg$$

(notation of (1.1.28) and (1.1.30)). We claim that τ^* factors through the map

$$\mathrm{Ob}(I/\varphi J) \to \mathrm{Ob}(I) \quad : \quad (g : i \to \varphi(j)) \mapsto i.$$

Indeed, say that $g : i \to \varphi(j)$ and $g' : i \to \varphi(j')$ are any two morphisms in I; we have to show that $\tau^*_g = \tau^*_{g'}$. To this aim, since $i/\varphi J$ is connected, we may assume that there exists a morphism $h : g \to g'$ in $i/\varphi J$; then

$$\tau^*_g = \mu_j \circ Fg = \mu_{j'} \circ F\varphi(h) \circ Fg = \mu_{j'} \circ Fg' = \tau^*_{g'}$$

as stated. Thus, let $\tau : \mathrm{Ob}(I) \to \mathrm{Morph}(\mathscr{C})$ be the resulting map; we claim that τ is an element of $L(X)$. Indeed, say that $h : i \to i'$ is any morphism in I, and pick any $j, j' \in \mathrm{Ob}(J)$ with morphisms $g : i \to \varphi(j)$, $g' : i' \to \varphi(j')$ in I; we have just seen that $\tau^*_{g' \circ h} = \tau^*_g$, which translates as the identity : $\tau_i = \tau_{i'} \circ Fh$, whence the claim. Lastly, by construction we have $\beta_X(\tau) = \mu$, so β_X is also surjective. □

Remark 1.5.5. (i) If $\varphi : I \to J$ and $\psi : J \to K$ are cofinal functors, the same holds for $\psi \circ \varphi$. For the proof, we may replace our universe U by a larger one, after which we may assume that I, J, K are small categories, and then the assertion follows directly from proposition 1.5.2.

(ii) Let $\varphi : J \to I$ be a functor between small categories. Then φ is coinitial if and only if it induces isomorphisms of presheaves on I

$$\lim_I F \xrightarrow{\sim} \lim_J F \circ \varphi \tag{1.5.6}$$

for every category $\mathscr{C}$ with small Hom-sets, and every functor $F : I \to \mathscr{C}$.

(iii) Let $\varphi : J \to I$ and $F : I \to \mathscr{C}$ be any two functors; suppose that the induced morphism of presheaves $\mathrm{colim}_\varphi \mathbf{1}_\mathscr{C} : \mathrm{colim}_I F \to \mathrm{colim}_J F \circ \varphi$ is an isomorphism, and let $\tau : F \Rightarrow c_L$ be a cocone. Then τ is universal if and only if the same holds for $\tau * \varphi : F \circ \varphi \Rightarrow c_L$. Indeed, τ and $\tau * \varphi$ determine morphisms of presheaves $t : h_{L^o} \to \mathrm{colim}_I F$ and $t' : h_{L^o} \to \mathrm{colim}_J F \circ \varphi$ on $\mathscr{C}^o$ that assign to every $X \in \mathrm{Ob}(\mathscr{C})$ and every section $f : L \to X$ of $h_{L^o}(X^o)$ the cocone $\tau \odot c_f \in \mathrm{colim}_I F(X^o)$ and respectively the cocone $(\tau * \varphi) \odot c_f \in \mathrm{colim}_J F \circ \varphi(X)$, and t (resp. t') is an isomorphism if and only if τ (resp. $\tau * \varphi$) is universal. However, $(\mathrm{colim}_\varphi \mathbf{1}_{\mathscr{C}}) \circ t$ is the morphism of presheaves $h_{L^o} \xrightarrow{\sim} \mathrm{colim}_J F \circ \varphi$ that assigns to every such X and f the cone $(\tau \odot c_f) * \varphi = (\tau * \varphi) \odot c_f$, i.e. $t' = (\mathrm{colim}_\varphi \mathbf{1}_{\mathscr{C}}) \circ t$. So t is an isomorphism if and only if the same holds for t', whence the contention. Likewise, if (1.5.6) is an isomorphism, a cone $\tau' : c_{L'} \Rightarrow F$ is universal if and only if the same holds for $\tau' * \varphi : c_{L'} \Rightarrow F \circ \varphi$.

Lemma 1.5.7. *Let I, J be two categories, and $\varphi : J \to I$ a functor. We have :*

- (i) *If J is filtered, then φ is cofinal if and only if the following two conditions hold :*
  - (a) *For every $i \in \mathrm{Ob}(I)$ there exist $j \in \mathrm{Ob}(J)$ and a morphism $i \to \varphi(j)$ in I.*
  - (b) *For every $i \in \mathrm{Ob}(I)$, every $j \in \mathrm{Ob}(J)$ and every pair of morphisms $f, g : i \to \varphi(j)$ in I, there exists a morphism $h : j \to j'$ in J such that $\varphi(h) \circ f = \varphi(h) \circ g$.*
- (ii) *If J is filtered and φ is cofinal, then I is filtered.*
- (iii) *Suppose that I is filtered, and condition* (i.a) *is fulfilled. Then the following holds :*
  - (a) *If φ is full, then φ is cofinal and J is directed.*
  - (b) *If φ is fully faithful, J is filtered.*

Proof. (i): Indeed, suppose that the categories $i/\varphi J$ are connected for every $i \in \mathrm{Ob}(I)$; then (a) clearly holds. To check that (b) holds as well, notice that, since J is filtered, $i/\varphi J$ is locally directed for every $i \in \mathrm{Ob}(I)$ (details left to the reader), and therefore it is directed, by remark 1.2.21(i). This implies that, for given f, g as in (b), there exist $j' \in \mathrm{Ob}(J)$ and morphisms $h_1, h_2 : j \to j'$ such that $\varphi(h_1) \circ f = \varphi(h_2) \circ g$. But since J is filtered, we may then find a morphism $h' : j' \to j''$ in J such that $h' \circ h_1 = h' \circ h_2$, so (b) holds with $h := h' \circ h_1$.

Conversely, if (a) holds, then $i/\varphi J$ is non-empty for every $i \in \mathrm{Ob}(I)$. Next, let $g : i \to \varphi(k)$ and $g' : i \to \varphi(k')$ be any two objects of $i/\varphi J$; since J is directed we may find morphisms $h : k \to j$ and $h' : k' \to j$ for some $j \in \mathrm{Ob}(J)$, whence objects $\varphi(h) \circ g, \varphi(h') \circ g' : i \to \varphi(j)$ of $i/\varphi J$, and using (b) we deduce that there exist an object $g'' : i \to \varphi(j')$ and morphisms $g \to g''$, $g' \to g''$, *i.e.* $i/\varphi J$ is directed.

(ii): Let us check that I is directed : if i and i' are two objects of I, in view of condition (i.a), we may find $j, j' \in \mathrm{Ob}(J)$ and morphisms $f : i \to \varphi(j)$, $f' : i' \to \varphi(j')$ in I, and since J is directed, we have as well morphisms $g : j \to j''$ and $g' : j' \to j''$ in J, for some $j'' \in \mathrm{Ob}(J)$, whence morphisms $\varphi(g) \circ f : i \to \varphi(j'')$ and $\varphi(g') \circ f' : i' \to \varphi(j'')$ in I. By remark 1.2.21(i) it remains to check the coequalizing condition of definition 1.2.19(v), but the latter is an immediate consequence of conditions (i.a) and (i.b).

(iii.a): It has already been remarked that condition (i.a) says that the category $i/\varphi J$ is non-empty for every $i \in \mathrm{Ob}(I)$. Next, let $g : i \to \varphi(k)$ and $g' : i \to \varphi(k')$ be two objects of $i/\varphi J$. Since I is filtered, we may find $i' \in \mathrm{Ob}(I)$ and morphisms $h : \varphi(k) \to i'$ and $h' : \varphi(k') \to i'$ of I such that $f := h \circ g = h' \circ g'$, and since (i.a) holds, we may find $k'' \in \mathrm{Ob}(J)$ with a morphism $l : i' \to \varphi(k'')$ in I; then, since φ is full, there exist morphisms $t : k \to k''$ and $t' : k' \to k''$ in J such that $\varphi(t) = l \circ h$ and $\varphi(t') = l \circ h'$. Then $l \circ f : i \to \varphi(k'')$ is an object of $i/\varphi J$ with morphisms $t : g \to l \circ f$ and $t' : g' \to l \circ f$, so $i/\varphi J$ is directed, and this shows that φ is cofinal.

Next, let $j, j' \in \mathrm{Ob}(J)$; since I is filtered, there exists $i \in \mathrm{Ob}(I)$ with morphisms $g : \varphi(j) \to i$ and $g' : \varphi(j') \to i$ in I. Since (i.a) holds, there exists a morphism $h : i \to \varphi(k)$ in I, and since φ is full, there exist morphisms $t : j \to k$ and $t' : j' \to k$ in J with $\varphi(t) = h \circ g$ and $\varphi(t') = h \circ g'$; this shows that J is directed.

(iii.b): Clearly the assumptions show that $\mathrm{Ob}(J) \neq \varnothing$, and we know already that J is directed, by (iii.a). Next, let $g, g' : j \to j'$ be two morphisms in J; since I is filtered, we have a morphism $h : \varphi(j') \to i$ in I such that $h \circ \varphi(g) = h \circ \varphi(g')$, and since (i.a) holds, there exists $j'' \in \mathrm{Ob}(J)$ with a morphism $l : i \to \varphi(j'')$ in I. Since φ is full, there exists a morphism $t : j' \to j''$ such that $\varphi(t) = l \circ h$, so that $\varphi(t \circ g) = \varphi(t \circ g')$, and since φ is faithful, we must then have $t \circ g = t \circ g'$, so J is filtered, by remark 1.2.21(i). □

Remark 1.5.8. In light of lemma 1.5.7(i), we see that if I is a filtered (resp. cofiltered) preordered set and J is a subset, then J is cofinal (resp. coinitial) in I in the sense of [36, Ch.III, §8, n.7] if and only if the inclusion functor $J \to I$ is cofinal (resp. coinitial) in the sense of our definition 1.5.1(i,ii).

Example 1.5.9. (i) For instance, if I is a filtered category, and i is any object of I, the category i/I is again filtered, and the target functor $\mathsf{t}_i : i/I \to I$ is cofinal; indeed, one checks easily that t_i fulfills conditions (a) and (b) of lemma 1.5.7(i). Dually, if I is cofiltered, the category I/i is again cofiltered, and the source functor $\mathsf{s}_i : I/i \to I$ is coinitial.

(ii) If I admits a final object i_0, then the (full) subcategory J of I with $\mathrm{Ob}(J) = \{i_0\}$ fulfills conditions (a) and (b) of lemma 1.5.7(i), so the inclusion functor $J \to I$ is cofinal (and I is trivially filtered). Then, let $F : I \to \mathscr{C}$ be any functor, and $\tau : F \Rightarrow c_L$ a universal cocone; taking into account remark 1.5.5(iii), we see that Fi_0 represents the colimit of F, and $\tau_{i_0} : Fi \to L$ is an isomorphism in $\mathscr{C}$. Dually, if i'_0 is any initial object of I, and $\tau' : c_{L'} \Rightarrow F$ any universal cone, then Fi'_0 represents the limit of F and $\tau_{i'_0} : L' \to Fi'_0$ is an isomorphism.

Example 1.5.10. As an application, we give an explicit construction of the filtered colimits of $\mathbf{Cat}$. Thus, let I be a small filtered category, and consider a functor

$$\mathscr{C}_\bullet : I \to \mathbf{Cat} \qquad i \mapsto \mathscr{C}_i \qquad (\varphi : i \to j) \mapsto (\mathscr{C}_\varphi : \mathscr{C}_i \to \mathscr{C}_j).$$

We deduce a functor $\mathrm{Ob}(\mathscr{C}_\bullet) : I \to \mathbf{Set}$ that assigns to every $i \in \mathrm{Ob}(I)$ the set $\mathrm{Ob}(\mathscr{C}_i)$, and to every morphism $\varphi : i \to j$ in I the map $\mathrm{Ob}(\mathscr{C}_i) \to \mathrm{Ob}(\mathscr{C}_j)$ defined by $\mathscr{C}_\varphi$. We set

$$L := \operatorname*{colim}_I \mathrm{Ob}(\mathscr{C}_\bullet).$$

According to example 1.2.23(i) – and since I is filtered – the elements of L are the equivalence classes $[i, X]$ of pairs (i, X), where $i \in \mathrm{Ob}(I)$ and $X \in \mathrm{Ob}(\mathscr{C}_i)$, for the equivalence relation such that $(i, X) \sim (j, Y)$ if and only if there exist $k \in \mathrm{Ob}(I)$ and morphisms $\varphi_1 : i \to k$ and $\varphi_2 : j \to k$ such that $\mathscr{C}_{\varphi_1} X = \mathscr{C}_{\varphi_2} Y$. For every $i \in \mathrm{Ob}(I)$, let $\mathsf{t}_i : i/I \to I$ be the target functor (see (1.1.24)); notice that for every pair of objects $i, j \in \mathrm{Ob}(I)$, the objects of the category $(i, j)/I := i/I \times_{(\mathsf{t}_i,\mathsf{t}_j)} j/I$ are the pairs $i \xrightarrow{\varphi_1} k \xleftarrow{\varphi_2} j$ of morphisms of I, and the morphisms

$$(1.5.11) \qquad (i, j)/\nu : (i \xrightarrow{\varphi_1} k \xleftarrow{\varphi_2} j) \to (i \xrightarrow{\varphi'_1} k' \xleftarrow{\varphi'_2} j)$$

are the morphisms $\nu : k \to k'$ of I such that $\nu \circ \varphi_r = \varphi'_r$ for $r = 1, 2$. For every couple of pairs $((i, X), (j, Y))$ as in the foregoing, we get a functor $h(i, X, j, Y) : (i, j)/I \to \mathbf{Set}$ that assigns to every $(i \xrightarrow{\varphi_1} k \xleftarrow{\varphi_2} j) \in \mathrm{Ob}((i, j)/I)$ the set $\mathrm{Hom}_{\mathscr{C}_k}(\mathscr{C}_{\varphi_1} X, \mathscr{C}_{\varphi_2} Y)$, and to every morphism $(i, j)/\nu$ as in (1.5.11) the induced map

$$\mathrm{Hom}_{\mathscr{C}_k}(\mathscr{C}_{\varphi_1} X, \mathscr{C}_{\varphi_2} Y) \to \mathrm{Hom}_{\mathscr{C}_{k'}}(\mathscr{C}_{\varphi'_1} X, \mathscr{C}_{\varphi'_2} Y) \qquad f \mapsto \mathscr{C}_\nu(f)$$

and we set

$$H(i, X, j, Y) := \operatorname*{colim}_{(i,j)/I} h(i, X, j, Y).$$

Let also $\mathsf{t}_{i,j} : (i, j)/I \to I$ be the target functor given by the rules : $(i \xrightarrow{\varphi_1} k \xleftarrow{\varphi_2} j) \mapsto k$, and $(i, j)/\nu \mapsto \nu$; for every $i, j, k \in \mathrm{Ob}(I)$ we set as well $(i, j, k)/I := (i, j)/I \times_{(\mathsf{t}_{i,j},\mathsf{t}_k)} k/I$. Explicitly, the objects of this category are all the systems $\varphi_\bullet := (\varphi_1 : i \to t, \varphi_2 : j \to t, \varphi_3 : k \to t)$ of morphisms of I with a common target t that we call *the target of* $\varphi_\bullet$. The morphisms

$$(i, j, k)/\nu : \varphi_\bullet \to \varphi'_\bullet$$

are the morphisms ν of I with $\nu \circ \varphi_r = \varphi'_r$ for $r = 1, 2, 3$. With this notation, for every third pair (k, Z) as in the foregoing, we get a system of composition maps

$$h(i,X,j,Y)(\varphi_\bullet) \times h(j,Y,k,Z)(\varphi_\bullet) \to h(i,X,k,Z)(\varphi_\bullet) \qquad \text{for every } \varphi_\bullet \in \mathrm{Ob}((i,j,k)/I)$$

given by the composition law of the category $\mathscr{C}_t$, where t is the target of $\varphi_\bullet$. Clearly this system of maps is natural with respect to morphisms of $(i,j,k)/I$; moreover we have obvious projection functors $(i,k)/I \leftarrow (i,j,k)/I \to (i,j)/I$ and $(i,j,k)/I \to (j,k)/I$, and using the criterion of lemma 1.5.7(i) it is easily seen all that these three functors are cofinal. Therefore, taking colimits over $(i,j,k)/I$ of the foregoing compatible system of composition maps yields a well defined map

$$H(i,X,j,Y) \times H(j,Y,k,Z) \to H(i,X,k,Z) \qquad (f,g) \mapsto g \circ f$$

and a simple inspection of the construction shows that this composition law is associative : if $h \in H(k,Z,t,W)$ is any other element, we have $h \circ (g \circ f) = (h \circ g) \circ f$ (details left to the reader); likewise, the class of $\mathbf{1}_X$ in $H(i,X,i,X)$ yields a left and right unit for this composition law. Hence, let us choose for every $A \in L$ a representative (i_A, X_A); we obtain a well defined category $\mathscr{L}$ with $\mathrm{Ob}(\mathscr{L}) = L$ and with $\mathrm{Hom}_{\mathscr{L}}(A,B) := H(i_A, X_A, i_B, X_B)$ for every $A, B \in L$, with the composition law thus defined. Moreover, for every $i \in I$ we obtain a well defined functor $G_i : \mathscr{C}_i \to \mathscr{L}$ by setting $GX := [i, X]$ for every $X \in \mathrm{Ob}(\mathscr{C}_i)$, and by assigning to every morphism $X \to Y$ of $\mathscr{C}_i$ its class in $H(i,X,i,Y)$. Lastly, let $F : \mathscr{C}_\bullet \Rightarrow \mathscr{D}$ be a cocone with basis $\mathscr{C}_\bullet$ and whose vertex is any category $\mathscr{D}$. The induced cocone $\mathrm{Ob}(\mathscr{C}_\bullet) \Rightarrow \mathrm{Ob}(\mathscr{D})$ yields a well defined map $L \to \mathrm{Ob}(\mathscr{D})$. Moreover, for every two pairs (i,X) and (j,Y) and every object $(i \xrightarrow{\varphi_1} k \xleftarrow{\varphi_2} j)$ of $(i,j)/I$, the functor F_k yields a map

$$\mathrm{Hom}_{\mathscr{C}_k}(\mathscr{C}_{\varphi_1}X, \mathscr{C}_{\varphi_2}Y) \to \mathrm{Hom}_{\mathscr{D}}(F_k \circ \mathscr{C}_{\varphi_1}X, F_k \circ \mathscr{C}_{\varphi_2}Y) = \mathrm{Hom}_{\mathscr{D}}(F_iX, F_jY)$$

which defines a cocone with vertex $\mathrm{Hom}_{\mathscr{D}}(F_iX, F_jY)$ and for basis the functor $h(i,X,j,Y)$, whence a well defined map

$$F_{i,X,j,Y} : H(i,X,j,Y) \to \mathrm{Hom}_{\mathscr{D}}(F_iX, F_jY).$$

A simple inspection shows that $F_{i,X,k,Z}(g \circ f) = F_{j,Y,k,Z}(g) \circ F_{i,X,j,Y}(f)$ for every f, g as in the foregoing (details left to the reader). We obtain therefore a well defined functor $F : \mathscr{L} \to \mathscr{D}$ by the rule : $A \mapsto F_{i_A}X_A$ and $F_{AB}f := F_{i_A,X_A,i_B,X_B}(f)$ for every $A, B \in L$ and every $f \in \mathrm{Hom}_{\mathscr{L}}(A,B)$. By construction, we have $F \circ G_i = F_i$ for every $i \in I$, and F is clearly the unique functor fulfilling this system of identities, so the proof is concluded.

Remark 1.5.12. (i) Let I be a coinitially small category, $\mathscr{C}$ a category with small Hom-sets, and $F : I \to \mathscr{C}$ any functor. By definition, there exists a coinitial functor $\varphi : J \to I$ with J a small category, and therefore we may define the presheaf

$$L_{F,\varphi} := \lim_J F \circ \varphi.$$

We claim that $L_{F,\varphi}$ is independent - up to natural isomorphism - of the choice of φ. To see this, let us choose a universe U' such that $\mathsf{U} \subset \mathsf{U}'$ and such that I is a small U'-category. By remark 1.5.5(i), the induced morphism of U'-presheaves

$$\omega_\varphi : \lim_I F \to L_{F,\varphi}$$

is an isomorphism. Therefore, if $\varphi' : J' \to I$ is any other coinitial functor with J' also small, we get an isomorphism of U'-presheaves $\omega_{\varphi'} \circ \omega_\varphi^{-1} : L_{F,\varphi} \xrightarrow{\sim} L_{F,\varphi'}$. Since the inclusion (1.2.1) is fully faithful, the latter is actually an isomorphism of U-presheaves, as required.

(ii) In view of (i), we may define the *limit of* F as $L_{F,\varphi}$, for any choice of φ. This presheaf is therefore well defined, up to natural isomorphism, and we denote it as usual by

$$\lim_I F.$$

We also say that $\lim_I F$ is a *coinitially small limit* of $\mathscr{C}$. Of course, if I is already small, we can choose $\varphi := \mathbf{1}_I$, so the above notation is compatible with that of definition 1.2.10(i).

(iii) Dually, if I is cofinally small, we can define the *colimit of* F

$$\operatorname*{colim}_I F := \operatorname*{colim}_J F \circ \varphi$$

for any choice of cofinal functor $\varphi : J \to I$ with J small, and again the resulting presheaf is well defined up to natural isomorphism, and we also say that it is a *cofinally small colimit* of $\mathscr{C}$. Clearly, if $\mathscr{C}$ is complete (resp. cocomplete), then every coinitially small (resp. cofinally small) limit is representable in $\mathscr{C}$. As an application, we may prove Freyd's *adjoint functor theorem*, which is the following partial converse of proposition 1.3.25(iii) :

Theorem 1.5.13. *Let $\mathscr{A}$ be a complete category, $F : \mathscr{A} \to \mathscr{B}$ a functor, and suppose that $\mathscr{A}$ and $\mathscr{B}$ have small* Hom*-sets (see* (1.1.1)*). The following conditions are equivalent :*

(a) *F admits a left adjoint.*

(b) *F commutes with all the limits of $\mathscr{A}$, and every object B of $\mathscr{B}$ admits a* solution set, i.e. *an essentially small subset $S_B \subset \mathrm{Ob}(\mathscr{A})$ such that, for every $A \in \mathrm{Ob}(\mathscr{A})$, every morphism $f : B \to FA$ admits a factorization of the form*

$$f = Fh \circ g$$

for $h : A' \to A$ a morphism in $\mathscr{A}$ with $A' \in S_B$, and $g : B \to FA'$ a morphism in $\mathscr{B}$.

Proof. If F admits a left adjoint G, then F commutes with all limits of $\mathscr{A}$, by proposition 1.3.25(iii), and it is easily seen that $\{GB\}$ is a solution set for every object $B \in \mathrm{Ob}(\mathscr{B})$.

Conversely, fix any $B \in \mathrm{Ob}(\mathscr{B})$; under the stated assumptions, example 1.3.16(i) says that the category $B/F\mathscr{A}$ is cofiltered. Denote by $\mathscr{E}_B$ the full subcategory of $B/F\mathscr{A}$ whose objects are all the morphisms $B \to FA$ with $A \in S_B$. Then condition (b) means that for every object X of $B/F\mathscr{A}$ there exists an object $X' \in \mathscr{E}_B$ with a morphism $X' \to X$ of $B/F\mathscr{A}$. It follows easily that $\mathscr{E}_B$ is cofiltered as well; then the inclusion functor $\mathscr{E}_B \to B/F\mathscr{A}$ is coinitial, by lemma 1.5.7(i), and $\mathscr{E}_B$ is small, hence $B/F\mathscr{A}$ is coinitially small. Following remark 1.5.12(ii), we may thus define

$$G'B := \lim_{B/F\mathscr{A}} \mathsf{t}_B \qquad G'f := \lim_{f/F\mathscr{A}} \mathbf{1}_{\mathscr{A}}$$

for any $B \in \mathrm{Ob}(\mathscr{B})$ and any morphism $f : B' \to B$ (notation of (1.1.27)). Next, since $\mathscr{A}$ is complete, G' is representable by some functor $G : \mathscr{B} \to \mathscr{A}$ (remark 1.2.8(ii)), and we claim that G is the sought left adjoint. To check this assertion, we may replace the universe U by a larger one, after which we may assume that $\mathscr{A}$ and $\mathscr{B}$ are small, and therefore $G'B$ is the usual limit of definition 1.2.10(i). We shall then exhibit explicit unit and counit for the adjoint pair (G, F), as follows. Let $\tau^B : c_{GB} \Rightarrow \mathsf{t}_B$ be a universal cone; since F commutes with limits, the cone $F * \tau^B : c_{FGB} \Rightarrow F \circ \mathsf{t}_B$ is still universal (remark 1.3.14(i)); on the other hand, the functor $B/F : B/F\mathscr{A} \to B/\mathscr{B}$ can be regarded as a cone $B/F : c_B \Rightarrow F \circ \mathsf{t}_B$, so there exists a unique morphism $\eta_B : B \to FGB$ such that $B/F = (F * \tau^B) \odot c_{\eta_B}$. Likewise, for any $A \in \mathrm{Ob}(\mathscr{A})$ we get a morphism $\varepsilon_A := \tau^{FA}_{(A, \mathbf{1}_{FA})} : GFA \to A$. It is easily seen that η (resp. ε) is a natural transformation $\mathbf{1}_{\mathscr{B}} \Rightarrow FG$ (resp. $GF \Rightarrow \mathbf{1}_{\mathscr{A}}$), and the identity $(F * \varepsilon) \odot (\eta * F) = \mathbf{1}_F$ is immediate from the construction. Lastly, we fix $B \in \mathrm{Ob}(\mathscr{B})$ and we check that $(\varepsilon * G)_B \circ (G * \eta)_B = \mathbf{1}_{G(B)}$; to this aim, and due the universality of τ^B, it suffices to show that

$$\tau^B_\psi = \tau^B_\psi \circ \varepsilon_{GB} \circ G(\eta_B) \qquad \text{for every object } \psi : B \to FA \text{ of } B/F\mathscr{A}.$$

However, notice that the construction of Gf yields the identity

$$(1.5.14) \qquad \tau^{B'}_\psi \circ Gf = \tau^B_{\psi \circ f} \qquad \text{for all morphisms } f : B \to B' \text{ and } \psi : B' \to FA \text{ in } \mathscr{C}.$$

Thus, we may compute :

$$\tau^B_\psi \circ \varepsilon_{GB} \circ G(\eta_B) = \tau^B_\psi \circ \tau^{FGB}_{\mathbf{1}_{FGB}} \circ G(\eta_B) = \tau^{FGB}_{F(\tau^B_\psi)} \circ G(\eta_B) = \tau^B_{F(\tau^B_\psi)\circ\eta_B} = \tau^B_\psi$$

where the second equality holds by the naturality of τ^{FGB}, the third follows from (1.5.14), and the fourth comes from the construction of η_B. □

Of course, the "dual" of theorem 1.5.13 yields a criterion for the existence of right adjoints.

Example 1.5.15. (i) Let J and K be any two small categories; set $I := J \times K$ and let $\pi : I \to J$ be the projection functor. For every $j \in \mathrm{Ob}(J)$ we have an induced functor

$$\iota_j : K \to j/\pi I \qquad k \mapsto ((j,k), \mathbf{1}_j) \qquad (g : k \to k') \mapsto (\mathbf{1}_j, g)$$

and we claim that ι_j is coinitial (definition 1.5.1(ii)). Indeed, for any $((j_0,k_0), f_0 : j \to j_0) \in \mathrm{Ob}(j/\pi I)$, the category $\iota_j K/((j_0,k_0), f_0)$ is isomorphic to K/k_0, which is obviously connected, whence the claim.

(ii) Moreover, every morphism $f : j \to j'$ in J induces a natural transformation

$$\beta^f : \iota_j \Rightarrow (f/\pi I) \circ \iota_{j'}$$

which assigns to every $k \in \mathrm{Ob}(K)$ the morphism $(f, \mathbf{1}_k) : ((j,k), \mathbf{1}_j) \to ((j',k), f)$. Then, $\mathtt{t}_j * \beta^f$ is a natural transformation $\mathtt{t}_j \circ \iota_j \Rightarrow \mathtt{t}_j \circ (f/\pi I) \circ \iota_{j'} = \mathtt{t}_{j'} \circ \iota_{j'}$ and a direct inspection of the definitions yields a commutative diagram for every functor $F : I \to \mathscr{C}$

$$\begin{array}{ccc}
\int^\wedge_\pi F(j) & \xrightarrow{\int^\wedge_\pi F(f)} & \int^\wedge_\pi F(j') \\
\downarrow & & \downarrow \\
\lim_K F \circ \mathtt{t}_j \circ \iota_j & \xrightarrow{\lim_K (F * \mathtt{t}_j * \beta^f)} & \lim_K F \circ \mathtt{t}_{j'} \circ \iota_{j'}
\end{array}$$

whose vertical arrows are the natural isomorphisms provided by (i) and remark 1.5.5(ii) and whose bottom arrow is given by remark 1.2.11(ii). In other words, the functor $\int^\wedge_\pi$ is naturally isomorphic to the one that assigns to every such F the functor

$$\int^\wedge_K F : J \to \mathscr{C}^\wedge \qquad j \mapsto \lim_K F \circ \mathtt{t}_j \circ \iota_j \qquad (f : j \to j') \mapsto \lim_K (F * \mathtt{t}_j * \beta^f).$$

Lastly, proposition 1.3.2 can be restated in this case more synthetically, as the existence of a natural isomorphism in $\mathscr{C}^\wedge$

$$\lim_{J\times K} F \xrightarrow{\sim} \operatorname*{Lim}_J \int^\wedge_K F \qquad \text{for every functor } F : J \times K \to \mathscr{C}.$$

(iii) On the other hand, we may also apply the same considerations to the second projection functor $I \to K$. Summing up, we deduce a natural isomorphism

$$\operatorname*{Lim}_J \int^\wedge_K F \xrightarrow{\sim} \operatorname*{Lim}_K \int^\wedge_J F \qquad \text{for every functor } F : J \times K \to \mathscr{C}$$

which expresses the well known *interchange property* for double limits.

(iv) Dually, we obtain as well an interchange property for double colimits, that states the existence of natural isomorphisms

$$\operatorname*{Colim}_J \int^K_\wedge F \xrightarrow{\sim} \operatorname*{Colim}_{J\times K} F \xrightarrow{\sim} \operatorname*{Colim}_K \int^J_\wedge F \qquad \text{for every functor } F : J \times K \to \mathscr{C}$$

where $\int^K_\wedge$ is the opposite of the functor $\int^\wedge_{K^o}$ (and likewise for $\int^J_\wedge$: details left to the reader).

Remark 1.5.16. (i) Keep the situation of example 1.5.15, and suppose moreover that $\mathscr{C}$ is complete (resp. cocomplete). Then the functor $\int_K^\wedge$ (resp. $\int_\wedge^K$) is isomorphic to the composition of $\mathsf{Fun}(J, h_\mathscr{C})$ (resp. of $\mathsf{Fun}(J, h^o_{\mathscr{C}^o})$) and a functor

$$\int_K : \mathsf{Fun}(I, \mathscr{C}) \to \mathsf{Fun}(J, \mathscr{C}) \qquad \text{(resp. } \int^K : \mathsf{Fun}(I, \mathscr{C}) \to \mathsf{Fun}(J, \mathscr{C}))$$

that is well defined up to isomorphism, and is isomorphic to $\int_\pi$ (resp. to $\int^\pi$), so the interchange properties of example 1.5.15(iii,iv) can, in this case, be expressed also in terms of these new functors (details left to the reader).

(ii) Furthermore, if $\mathscr{C}$ is both complete and cocomplete, we get a natural morphism

$$\text{(1.5.17)} \qquad \operatorname*{Colim}_J \int_K F \to \operatorname*{Lim}_K \int^J F \qquad \text{for every functor } F : J \times K \to \mathscr{C}$$

as follows. First, for every $(j, k) \in \mathrm{Ob}(J \times K)$ we have a natural morphism

$$\omega_{j,k} : \int_K F(j) \to F(j, k) \to \int^J F(k)$$

given by the choice of a universal cone and cocone for the limit and colimit that are represented respectively by the source and target of this morphism. By inspecting the constructions, we then check easily that, for fixed $j \in \mathrm{Ob}(J)$, the system $(\omega_{j,k} \mid k \in \mathrm{Ob}(K))$ is a cone with vertex $\int_K F(j)$; the choice of a universal cone for the functor $\int^K F$ then determines a unique morphism

$$\omega_j : \int_K F(j) \to \operatorname*{Lim}_J \int^K F \qquad \text{for every } j \in \mathrm{Ob}(J).$$

In turns, the system $(\omega_j \mid j \in \mathrm{Ob}(J))$ is a cocone which determines (1.5.17), after fixing a universal cocone for the functor $\int_K F$. Of course, (1.5.17) depends on the choices of all these universal cones and cocones, but two different sets of such choices will modify the morphism by left and right composition with uniquely determined isomorphisms.

(iii) The question rises, whether (1.5.17) is an isomorphism. This is not always the case. If (1.5.17) is an isomorphism for every functor $F : J \times K \to \mathscr{C}$, we say that in the category $\mathscr{C}$ *the limits indexed by K commute with the colimits indexed by J.*

(iv) For instance, if K is finite and J is filtered, then the limits indexed by K commute with the colimits indexed by J in the category $\mathbf{Set}$ ([28, Th.2.13.4]). We say briefly that *the finite limits commute with the filtered colimits* in $\mathbf{Set}$.

Taking into account example 1.2.24(i), we deduce more generally that, for any category $\mathscr{C}$, the finite limits in $\mathscr{C}^\wedge$ commute with all filtered colimits.

Here is another useful application to Kan extensions :

Proposition 1.5.18. *In the situation of theorem* 1.3.4*, suppose that φ is fully faithful. Then the same holds for the right (resp. left) Kan extension along φ.*

Proof. Consider the right Kan extension along φ (and hence, we assume that the relevant completeness conditions for $\mathscr{C}$ as in remark 1.3.3(iii,iv) are fulfilled). Let $F : I \to \mathscr{C}$ be any functor, and to ease notation set $K := \int_\varphi F$; by proposition 1.1.20(iii), it suffices to exhibit an isomorphism of functors $\varepsilon : K \circ \varphi \xrightarrow{\sim} F$. Now, recall that $Kj \in \mathscr{C}$ represents the limit of the functor $F \circ \mathsf{t}_j : j/\varphi I \to \mathscr{C}$, for every $j \in \mathrm{Ob}(J)$, and the construction of K involves the choice of a universal cone $\tau^j : c_{Kj} \Rightarrow F \circ \mathsf{t}_j$ for every such j. However, notice that $(i, \mathbf{1}_{\varphi(i)})$ is an initial object of $\varphi(i)/\varphi I$: indeed, let $(i', f : \varphi(i) \to \varphi(i'))$ be any other object; since φ is fully faithful, there exists a unique morphism $g : i \to i'$ in J such that $\varphi(g) = f$, and then $(g, \mathbf{1}_f)$ is the unique morphism $(i, \mathbf{1}_{\varphi(i)}) \to (i', f)$ in $\varphi(i)/\varphi I$. By example 1.5.9(ii), it follows that

$$\varepsilon_i := \tau^{\varphi(i)}_{(i,\mathbf{1}_{\varphi(i)})} : K \circ \varphi(i) \to Fi$$

is an isomorphism for every $i \in \mathrm{Ob}(I)$. It remains to check the naturality of the rule : $i \mapsto \varepsilon_i$. Hence, let $g : i \to i'$ be any morphism of I; by construction, for every $X \in \mathrm{Ob}(\mathscr{C})$ we have a commutative diagram of sets :

$$\begin{array}{ccc} h_{K\varphi(i)}(X) & \longrightarrow & \lim_{\varphi(i)/\varphi I} F \circ \mathsf{t}_{\varphi(i)}(X) \\ {\scriptstyle h_{K\varphi(g)}}\downarrow & & \downarrow \\ h_{K\varphi(i')}(X) & \longrightarrow & \lim_{\varphi(i')/\varphi I} F \circ \mathsf{t}_{\varphi(i')}(X) \end{array}$$

where h denotes as usual the Yoneda embedding, and where the top and bottom horizontal arrows are the bijections induced by τ^i and $\tau^{i'}$ respectively. The right vertical arrow is the map that sends every cone $\beta : c_X \Rightarrow F \circ \mathsf{t}_{\varphi(i)}$ to the cone $\beta * (\varphi g/\varphi I) : c_X \Rightarrow F \circ \mathsf{t}_{\varphi(i')}$, where $\varphi g/\varphi I : \varphi(i')/\varphi I \to \varphi(i)/\varphi I$ is the functor defined as in (1.1.27). Taking $X := K\varphi(i)$, we see that the top horizontal (resp. left vertical) arrow maps $\mathbf{1}_{K\varphi(i)}$ to $\tau^{\varphi(i)}$ (resp. to $K\varphi(g)$); then we get the identity :

$$\tau^{\varphi(i')} \odot c_{K\varphi(g)} = \tau^{\varphi(i)} * (\varphi g/\varphi I)$$

whence $\varepsilon^{i'} \circ K\varphi(g) = \tau^{\varphi(i)}_{(i',\varphi(g))} : K\varphi(i) \to Fi'$. Lastly, the morphism g induces a morphism $\varphi(i)/g : (i, \mathbf{1}_{\varphi(i)}) \to (i', \varphi(g))$ of $\varphi(i)/\varphi I$, whence the identity : $\tau^{\varphi(i)}_{(i',\varphi(g))} = Fg \circ \tau^{\varphi(i)}_{(i,\mathbf{1}_{\varphi(i)})}$. Summing up, we conclude that

$$\varepsilon^{i'} \circ K\varphi(g) = Fg \circ \varepsilon^i$$

as required. The assertion for left Kan extensions admits the dual proof. □

Corollary 1.5.19. *Let $\mathscr{B}$, $\mathscr{C}$ be two small categories, and $F : \mathscr{B} \to \mathscr{C}$ a functor. We have :*

(i) *If $\mathscr{B}$ is finitely complete and F is left exact, then $F_!$ is exact.*

(ii) *F is fully faithful $\Leftrightarrow$ $F_!$ is fully faithful $\Leftrightarrow$ F_* is fully faithful.*

Proof. (i): Under these assumptions, the category $Y/F\mathscr{B}$ is cofiltered for every $Y \in \mathrm{Ob}(\mathscr{C})$ (example 1.3.16(i)), hence $(Y/F\mathscr{B})^o$ is filtered. However, the filtered colimits in the category $\mathbf{Set}$ commute with all finite limits (see remark 1.5.16(iv)), so the assertion follows from remark 1.3.6(iii) and proposition 1.3.25(iv).

(ii): By proposition 1.5.18 we know that if F is fully faithful, the same holds for $F_!$ and F_*. Also, by proposition 1.1.20(iv), $F_!$ is fully faithful if and only if the same holds for F_*. Lastly, if $F_!$ is fully faithful, then (1.3.7) and the full faithfulness of the Yoneda embeddings, imply that F is fully faithful. □

Example 1.5.20. Let I be any small category; we wish to apply Freyd's adjoint theorem 1.5.13 in order to exhibit a left adjoint

$$\mathop{\mathrm{Colim}}_{I} : \mathsf{Fun}(I, \mathbf{Cat}) \to \mathbf{Cat}$$

for the corresponding constant functor c for the category $\mathbf{Cat}$. In particular, this will prove that all the colimits indexed by I are representable in $\mathbf{Cat}$, and since I is arbitrary, we will conclude that $\mathbf{Cat}$ is cocomplete. Now, we know already that $\mathbf{Cat}$ is complete (example 1.2.25(i)), and c commutes with all limits of $\mathbf{Cat}$, by virtue of corollary 1.4.1(ii). It remains to check that every functor $\mathscr{C}_\bullet : I \to \mathbf{Cat}$ admits a solution set $S \subset \mathrm{Ob}(\mathbf{Cat})$. To this aim, let $\mathscr{D} := \amalg_{i\in\mathrm{Ob}(I)}\mathscr{C}_i$ be the coproduct of the family of categories $(\mathscr{C}_i \mid i \in \mathrm{Ob}(I))$, as in example 1.2.25(i); for every small category $\mathscr{B}$ and every co-cone $F_\bullet : \mathscr{C}_\bullet \Rightarrow \mathscr{B}$, let $F : \mathscr{D} \to \mathscr{B}$ be the resulting functor, and denote by $\mathrm{Im}(F) \subset \mathscr{B}$ the image of F, defined as in example 1.2.26 (it is easily seen that $\mathbf{Cat}$ is well-powered, so the image is well defined). The system of categories

$$\mathscr{F} := (\mathrm{Im}(F) \mid \mathscr{B} \in \mathrm{Ob}(\mathbf{Cat}), F_\bullet : \mathscr{C}_\bullet \to \mathscr{B})$$

is not small, but if we pick in $\mathscr{F}$ a representative for each isomorphism class, we do get a small family (details left to the reader); it is easily seen that any such choice of representatives yields a solution set for $\mathscr{C}_\bullet$.

In applications, often the indexing category of a limit (or colimit) is a partially ordered set, and usually such limits are easier to handle than limits over general indexing categories. Now, given such a general indexing category I, one may try to find a cofinal functor $J \to I$ from a partially ordered set J. The following proposition – whose part (ii) is due to Deligne ([8, Exp.I, Prop.8.1.6]) – says that this can always be achieved.

Proposition 1.5.21. *Let I be any category. The following holds :*

(i) *There exists a cofinal functor $e : J \to I$, with J a partially ordered set.*
(ii) *If moreover I is filtered, we can find a cofinal functor e as in* (i)*, with J filtered as well.*
(iii) *In the situation of* (i) *or* (ii)*, if furthermore I is small, we can take J to be also small.*

Proof. (i): Let $\mathbb{N}$ be the filtered category associated with the totally ordered set of natural numbers, and $\mathscr{P}_0(\mathbb{N})$ the set of finite non-empty subsets of $\mathbb{N}$; we endow each $\Sigma \in \mathscr{P}_0(\mathbb{N})$ with the restriction of the ordering of $\mathbb{N}$, so that Σ is a full subcategory of $\mathbb{N}$. If $\Sigma' \subset \Sigma$ are two elements of $\mathscr{P}_0(\mathbb{N})$, and $F : \Sigma \to I$ is any functor, we let $F_{|\Sigma'} : \Sigma' \to I$ be the restriction of F. We endow

$$J := \bigcup_{\Sigma \in \mathscr{P}_0(\mathbb{N})} \mathrm{Fun}(\Sigma, I)$$

with the partial ordering such that :

$$(F' : \Sigma' \to I) \leq (F : \Sigma \to I) \Leftrightarrow \Sigma' \subset \Sigma \text{ and } F' = F_{|\Sigma'}.$$

We define a functor $e : J \to I$ as follows. For every $(F : \Sigma \to I) \in J$, we let

$$e(F) := F(s) \qquad \text{where } s \text{ is the largest element of } \Sigma.$$

For every pair $(F' : \Sigma' \to I) \leq (F : \Sigma \to I)$ of elements of J, let $s \in \Sigma$ and $s' \in \Sigma'$ be the respective maximal elements, and $\psi_{s',s} : s' \to s$ (resp. $\varphi_{F',F} : F' \to F$) the corresponding unique morphism of Σ (resp. of J); then we let

$$e(\varphi_{F',F}) := F(\psi_{s',s}).$$

Let now $i \in \mathrm{Ob}(I)$; we need to check that the category i/eJ is connected. To this aim, let $F_i : \{0\} \to I$ be the unique functor such that $F_i(0) := i$; clearly $(\mathbf{1}_i : i \to e(F_i)) \in \mathrm{Ob}(i/eJ)$, so $\mathrm{Ob}(i/eJ)$ is non-empty. Next, let $(F : \Sigma \to I) \in J$ and $\beta : i \to i' := e(F)$ be any morphism of I; we construct step-by-step a chain of morphisms in i/eJ connecting (β, F) with $(\mathbf{1}_i, F_i)$. First, let s be the largest element of Σ; if $s = 0$, we take $\Delta := \{0, 1\}$, and consider the functor $G : \Delta \to I$ such that $G(0) = G(1) := i'$, and $G(\psi_{0,1}) := \mathbf{1}_{i'}$. Thus, $F \leq G$, and $(\beta : i \to e(G)) \in \mathrm{Ob}(i/eJ)$, so we may replace F by G, and assume from start that $s > 0$. Then we have $F' := F_{|\{s\}} \leq F$, and $(\beta : i \to e(F')) \in \mathrm{Ob}(i/eJ)$. Define $F'' : \{0, s\} \to I$ by : $F''(0) = i$, $F''(s) := i'$, and $F''(\psi_{0,s}) := \beta$; again, $F' \leq F''$, and $(\beta : i \to e(F'')) \in \mathrm{Ob}(i/eJ)$. Lastly, $F_i \leq F''$, and this completes the sought chain of morphisms.

(ii): Let us say that I admits a *largest object*, if there exists $i \in \mathrm{Ob}(I)$ such that $\mathrm{Hom}_I(i', i) \neq \varnothing$ for every $i' \in \mathrm{Ob}(I)$. We notice :

Claim 1.5.22. Let $j \in \mathrm{Ob}(I)$ be any object, and $(f_t : i_t \to j \mid t = 1, \dots, n)$ any finite system of morphisms in I. If I does not have a largest object, there exists a morphism $j \to i$ in I such that $i \neq i_t$ for every $t = 1, \dots, n$.

Proof of the claim. Suppose the claim fails, in which case we may assume that $j = i_1$. Since I is filtered, for every $k \in \mathrm{Ob}(I)$ we may find morphisms $g : k \to k'$ and $j \to k'$, and the assumption implies that $k' = i_{n(k)}$ for some $n(k) \in \{1, \dots, n\}$. Set $h := f_{n(k)} \circ g$, so h is a morphism $k \to i_1$, and especially $\mathrm{Hom}_I(k, i_1) \neq \varnothing$ for every $k \in \mathrm{Ob}(I)$, which is absurd, since I does not have a largest object. ◊

Next, define the filtered category $\mathbb{N}$ as in the proof of (i); we remark that if I is any filtered category, then $\mathbb{N} \times I$ is still filtered, and it does not admit a largest object; moreover, one checks easily that the projection functor $\mathbb{N} \times I \to I$ fulfills conditions (a) and (b) of lemma 1.5.7(i), so it is cofinal. In light of remark 1.5.5(i), we may then replace I by $\mathbb{N} \times I$, and assume from start that I does not have a largest object.

Let us say that a *diagram* of I is any pair $D := (A, B)$ with $A \subset \mathrm{Ob}(I)$, $B \subset \mathrm{Morph}(I)$, and such that, for every $f \in B$, the source and target of f lie in A. We say that A (resp. B) is the set of *objects* (resp. *morphisms*) of D. An element e of A is said to be *final* in (A, B) if the following holds :

- for every $i \in \mathrm{Ob}(I)$, the set $\mathrm{Hom}_I(i, e) \cap B$ contains exactly one element f_i^D
- $f_e^D = \mathbf{1}_e$, and for every morphism $g : i \to j$ in B we have $f_j^D \circ g = f_i^D$.

Denote by J the set consisting of all diagrams (A, B) which admit a unique final element $e(A, B)$, and such that A and B are finite sets. We order J by inclusion, *i.e.* $(A, B) \leq (A', B')$ if and only if $A \subset A'$ and $B \subset B'$. Then, if $D, D' \in J$ and $D \leq D'$, there exists a unique morphism $e(D, D') : e(D) \to e(D')$ in the set of morphisms of D', and if $D \leq D' \leq D''$, then clearly $e(D, D'') = e(D', D'') \circ e(D, D')$. Thus, the rule $D \mapsto e(D)$ yields a well defined functor $e : J \to I$, and we have to prove that e is cofinal.

To this aim, we check that conditions (a) and (b) of lemma 1.5.7(i) hold for e. Now, if i is any object of I, the diagram $D_i := (\{i\}, \{\mathbf{1}_i\})$ lies in J, and $\mathbf{1}_i : i \to e(D_i)$ is a morphism in I, so condition (a) holds. Next, let $D = (A, B) \in J$ be any diagram, $i \in \mathrm{Ob}(I)$ any object, and $f, g : i \to e(D)$ any two morphisms in I; we notice :

Claim 1.5.23. There exist $i' \in \mathrm{Ob}(I) \setminus A$ and a morphism $h : e(D) \to i'$ such that $h \circ f = h \circ g$.

Proof of the claim. Since I is filtered, we can find a morphism $h : e(D) \to i'$ that coequalizes f and g, and claim 1.5.22 says that we may choose $i \notin A$. ◊

Now, let i' be as in claim 1.5.23, and D' the diagram of I whose set of objects is $A \cup \{i'\}$, and whose set of morphisms is $B \cup \{h \circ f_j \mid j \in A\}$. It is easily seen that $D' \in J$, and by construction, $e(D') = i'$ and $e(D, D') = h$, so (b) holds.

It remains to show that J is filtered. However, say that $D = (A, B)$ and $D' = (A', B')$ are any two elements of J; we notice :

Claim 1.5.24. There exist $i \in \mathrm{Ob}(I) \backslash (A \cup A')$ and morphisms $f : e(D) \to i$ and $f' : e(D') \to i$ in I.

Proof of the claim. This is similar to the proof of claim 1.5.23 : we can always find $f : e(D) \to i$ and $f' : e(D') \to i$, and by claim 1.5.22 we may arrange that $i \notin A \cup A'$. ◊

Hence, let f, f' be as in claim 1.5.24, and denote $D(f, f')$ the diagram whose set of objects is $A \cup A' \cup \{i\}$, and whose set of morphisms is $B \cup B' \cup \{f \circ f_j^D \mid j \in A\} \cup \{f' \circ f_{j'}^{D'} \mid j' \in A'\}$. Clearly, the assertion will follow from :

Claim 1.5.25. We can choose f and f' so that $D(f, f') \in J$.

Proof of the claim. It amounts to checking that, for suitable f and f', we have $f \circ f_j^D = f' \circ f_j^D$ for every $j \in A \cap A'$. However, if f and f' as in claim 1.5.24, have been picked, a simple induction on the cardinality of $A \cap A'$ shows that we may find $h : i \to i'$ such that

$$h \circ f \circ f_j^D = h \circ f' \circ f_j^D \qquad \text{for every } j \in A \cap A'$$

and by claim 1.5.22, we may arrange that $i' \notin A \cup A'$. Then the diagram $D(h \circ f, h \circ f')$ will do. ◊

Lastly, if I is small, clearly the same holds for J as exhibited in the proofs of (i) and (ii). □

Example 1.5.26. Let us say that a category I is *countable* if both $\mathrm{Ob}(I)$ and $\mathrm{Morph}(I)$ are countable sets. A simple inspection of the proof of proposition 1.5.21(ii) shows that if I is a filtered countable category, there exist a countable filtered partially ordered set $J := \{j_n \mid n \in \mathbb{N}\}$ and a cofinal functor $J \to I$. But it is easy to construct a cofinal functor $f : \mathbb{N} \to J$ (where $\mathbb{N}$ is endowed with its standard total ordering) : indeed, we may choose $f(0) := j_0$, and then pick inductively for every $n \in \mathbb{N}$ an element $f(n+1) \in J$ larger than j_{n+1} and $f(n)$. Summing up, we see that for every countable filtered category I there exists a cofinal functor $\mathbb{N} \to I$.

1.6. **Localizations of categories.** Given a category $\mathscr{C}$ and an arbitrary set of morphisms $\Sigma \subset \mathrm{Morph}(\mathscr{C})$, we wish to describe a general procedure that adds formally to $\mathscr{C}$ an inverse f^{-1} for every $f \in \Sigma$. More details can be found in [28, Ch.5]. We begin with the following :

Definition 1.6.1. (i) A *graph* $\mathscr{G}$ is the datum of

- a set $V(\mathscr{G})$, whose elements are called the *vertices* of $\mathscr{G}$
- for every $A, B \in V(\mathscr{G})$ a set $\mathscr{G}(A, B)$, whose elements are called the *arrows* from A to B. If $f \in \mathscr{G}(A, B)$, we say also that A is the *source* and B is the *target* of f.

(ii) A *morphism* of graphs $F : \mathscr{G} \to \mathscr{G}'$ consists of

- A map of sets $F : V(\mathscr{G}) \to V(\mathscr{G}')$
- For every $A, B \in V(\mathscr{G})$, a map of sets $\mathscr{G}(A, B) \to \mathscr{G}'(FA, FB) : f \mapsto Ff$.

(iii) A graph is *small* if $V(\mathscr{G})$ is a small set, and $\mathscr{G}(A, B)$ is a small set for all $A, B \in V(\mathscr{G})$.

Obviously, morphisms of graphs can be composed. so we have a category

$$\mathsf{U}\text{-}\mathbf{Graph}$$

whose objects are all the small graphs. As usual, we shall omit the prefix U, unless we have to deal with more than one universe. Clearly, there is a natural functor

$$\mathbf{Cat} \to \mathbf{Graph} \tag{1.6.2}$$

which assigns to any category $\mathscr{C}$ its underlying graph, whose vertices are the objects of $\mathscr{C}$, and whose arrows are the morphisms of $\mathscr{C}$, and simply forgets the composition law of $\mathscr{C}$. We are going to construct an explicit left adjoint for the functor (1.6.2). To this aim, let $\mathscr{G}$ be any graph and $n \in \mathbb{N}$ any integer; a *path of length* n in $\mathscr{G}$ is a sequence

$$p := (A_0, f_1, A_1, f_2, \dots, A_n)$$

alternating vertices $A_0, \dots, A_n$ of $\mathscr{G}$ and arrows $f_1, \dots, f_n$, such that the source and target of f_i are respectively A_{i-1} and A_i for every $i = 1, \dots, n$. Then A_0 is called the *source* and A_n the *target* of p. Given paths $p := (A_0, f_1, A_1, \dots, A_n)$ and $p' := (A_n, f_{n+1}, A_{n+1}, \dots, A_{n+m})$ of lengths respectively n and m, and such that the target of p equals the source of p', we obtain a path of length $n + m$, by the rule :

$$p' \circ p := (A_0, f_1, \dots, A_n, f_n, \dots, A_{n+m}).$$

Denote by $P(\mathscr{G})$ the set of all paths of $\mathscr{G}$ (of arbitrary length); with this composition law, clearly $P(\mathscr{G})$ is the set of morphisms of a *path category* whose set of objects is $V(\mathscr{G})$ (for any $A \in V(\mathscr{G})$, the path (A) of length zero serves as the identity endomorphism of A). Also, if $F : \mathscr{G} \to \mathscr{G}'$ is any morphism of graphs, we have an induced functor

$$P(F) : (V(\mathscr{G}), P(\mathscr{G})) \to (V(\mathscr{G}'), P(\mathscr{G}'))$$

whose map on objects is F, and such that $P(F)p := (FA_0, Ff_1, \dots, FA_n)$ for every path $p := (A_0, f_1, \dots, A_n)$ of $\mathscr{G}$. We have thus defined a functor

$$\textbf{Graph} \to \textbf{Cat} \qquad \mathscr{G} \mapsto (V(\mathscr{G}), P(\mathscr{G})). \tag{1.6.3}$$

Proposition 1.6.4. *The functor* (1.6.3) *is left adjoint to the forgetful functor* (1.6.2).

Proof. Indeed, given a graph $\mathscr{G}$, a category $\mathscr{C}$ and a morphism of graphs $F : \mathscr{G} \to \mathscr{C}$, define a functor $G : (V(\mathscr{G}), P(\mathscr{G})) \to \mathscr{C}$ by the rule

$$GA := FA \qquad Gp := Ff_n \circ \cdots Ff_1$$

for every $A \in V(\mathscr{G})$ and every $p := (A_0, f_1, \dots, A_n) \in P(\mathscr{G})$. Clearly G is the unique functor whose restriction to $\mathscr{G}$ agrees with F, whence the contention. □

Corollary 1.6.5. *A category $\mathscr{C}$ is coinitially (resp. cofinally) small if and only if it admits a small coinitial (resp. cofinal) subcategory.*

Proof. Indeed, the condition is obviously sufficient. Conversely, suppose that $\varphi : I \to \mathscr{C}$ is a coinitial (resp. cofinal) functor, with I small; the datum $\mathscr{G} := (\varphi(\mathrm{Ob}(I)), \varphi(\mathrm{Morph}(I)))$ is a subgraph of the category $\mathscr{C}$ (see (1.6)), and the inclusion morphism into $\mathscr{C}$ extends to a functor $\varphi' : I' \to \mathscr{C}$, where I' is the path category of $\mathscr{G}$ (proposition 1.6.4). It is then easily seen that the graph $(\varphi(\mathrm{Ob}(I')), \varphi'(\mathrm{Morph}(I')))$ is actually a coinitial (resp. cofinal) small subcategory of $\mathscr{C}$ (details left to the reader). □

The path category of a graph $\mathscr{G}$ is a sort of free category generated by $\mathscr{G}$. We explain next how to define equivalence relations on such a path category, and how to form the corresponding quotient categories. To this aim, let $\mathscr{G}$ be any graph; we shall call a *commutativity condition* on $\mathscr{G}$ any pair of paths of $\mathscr{G}$ with the same sources and targets. A *conditional graph* is a pair $(\mathscr{G}, C)$ consisting of a graph $\mathscr{G}$ and a set C of commutativity conditions on $\mathscr{G}$. A *morphism* $F : (\mathscr{G}, C) \to (\mathscr{G}', C')$ of conditional graphs is a morphism $F : \mathscr{G} \to \mathscr{G}'$ of graphs, which induces a map $C \to C' : (p_1, p_2) \mapsto (P(F)p_1, P(F)p_2)$. With this composition rule, the conditional graphs form also a category, and we say that a conditional graph is *small* if its underlying graph is small. We denote

$$\mathsf{U}\text{-}\textbf{CondGraph}$$

the category of all small conditional graphs, and as usual we drop the subscript U, unless ambiguities are likely to arise from this omission.

1.6.6. We may attach to any small category $\mathscr{C}$ a natural set of commutativity conditions; namely, one takes the set $C_\mathscr{C}$ consisting of all pairs of paths $((A_0, f_1, \dots, A_n), (B_0, g_1, \dots, B_m))$ in $\mathscr{C}$ such that $A_0 = B_0$, $A_n = B_m$, and $f_n \circ \cdots \circ f_1 = g_m \circ \cdots \circ g_1$. (Especially, for $n = 1$ and $m = 0$, we get the pairs $((A, \mathbf{1}_A, A), (A))$, for A ranging over all the objects of $\mathscr{C}$).

Clearly, every functor $F : \mathscr{C} \to \mathscr{C}'$ induces a map $C_\mathscr{C} \to C_{\mathscr{C}'}$, whence a forgetful functor

$$\textbf{Cat} \to \textbf{CondGraph} \qquad \mathscr{C} \mapsto (\mathscr{C}, C_\mathscr{C}). \tag{1.6.7}$$

Proposition 1.6.8. *The forgetful functor* (1.6.7) *admits a left adjoint.*

Proof. Given a conditional graph $(\mathscr{G}, C)$, denote by $\mathscr{P}$ the path category of $\mathscr{G}$, and by $\mathscr{Q}$ the category such that

- $\mathrm{Ob}(\mathscr{Q}) = V(\mathscr{G})$
- for every $A, B \in \mathrm{Ob}(\mathscr{Q})$, the set $\mathrm{Hom}_\mathscr{Q}(A, B)$ consists of all pairs $(p_1, p_2) \in P(\mathscr{G}) \times P(\mathscr{G})$ such that the source (resp. the target) of both p_1 and p_2 is A (resp. is B)
- the composition law of $\mathscr{Q}$ is given by the rule : $(p_1, p_2) \circ (p'_1, p'_2) := (p_1 \circ p'_1, p_2 \circ p'_2)$ for every pair of composable morphisms $(p_1, p_2), (p'_1, p'_2) \in \mathrm{Morph}(\mathscr{Q})$

and notice that $C \subset \mathrm{Morph}(\mathscr{Q})$. Consider the family S_C of all subcategories $\mathscr{S}$ of $\mathscr{Q}$ such that

- $\mathrm{Ob}(\mathscr{S}) = \mathrm{Ob}(\mathscr{Q})$ and $C \subset \mathrm{Morph}(\mathscr{S})$
- $\mathrm{Morph}(\mathscr{S})$ is an equivalence relation on $P(\mathscr{G}) = \mathrm{Morph}(\mathscr{P})$.

Clearly $\mathscr{Q} \in S_C$, so $S_C \neq \varnothing$, and S_C admits a smallest element, namely the subcategory $\mathscr{R}$ whose set of objects is $\mathrm{Ob}(\mathscr{Q})$ and whose set of morphisms is $\bigcap_{\mathscr{S}\in S_C} \mathrm{Morph}(\mathscr{S})$. We may then form a category

$$\mathscr{P}/C$$

whose set of objects is $\mathrm{Ob}(\mathscr{P}) = V(\mathscr{G})$, and such that

$$\mathrm{Hom}_{\mathscr{P}/C}(A, B) := \mathrm{Hom}_{\mathscr{P}}(A, B)/\sim_C$$

where $\sim_C$ is the equivalence relation induced by $\mathscr{R}$ on $\mathrm{Hom}_{\mathscr{P}}(A, B)$. The composition law for morphisms is the unique one such that the resulting pair of maps

$$\mathrm{Ob}(\mathscr{P}) \to \mathrm{Ob}(\mathscr{P}/C) \qquad \mathrm{Morph}(\mathscr{P}) \to \mathrm{Morph}(\mathscr{P}/C)$$

(which is the identity map on objects, and the natural projection on morphisms) is a functor

$$\mathscr{P} \to \mathscr{P}/C.$$

To check that this law is well defined, say that $p, q : A \to A'$ and $p', q' : A' \to A''$ are two pairs of morphisms in $\mathscr{P}$, such that $p \sim_C q$ and $p' \sim_C q'$; since $\mathscr{R}$ is a subcategory, it follows easily that $p' \circ p \sim_C q' \circ p \sim_C q' \circ q$, whence the claim. Lastly, let $\mathscr{C}$ be any small category, and $F : (\mathscr{G}, C) \to (\mathscr{C}, C_{\mathscr{C}})$ a given morphism of conditional graphs; by proposition 1.6.4, the morphism F extends to a functor $G : \mathscr{P} \to \mathscr{C}$. Let $\mathscr{S} \subset \mathscr{Q}$ be the subcategory such that $\mathrm{Ob}(\mathscr{S}) = \mathrm{Ob}(\mathscr{Q})$ and whose morphisms are all the pairs of paths (p, p') such that $(P(F)p, P(F)p') \in C_{\mathscr{C}}$; by construction, $\mathscr{S} \in S_C$, so $\mathscr{S}$ contains $\mathscr{R}$; it follows that G factors uniquely through $\mathscr{P}/C$, whence the proposition. □

Theorem 1.6.9. *Let $\mathscr{C}$ be any category and $\Sigma \subset \mathrm{Morph}(\mathscr{C})$ any subset. Then there exist a category $\mathscr{C}[\Sigma^{-1}]$ and a* localization functor

$$L : \mathscr{C} \to \mathscr{C}[\Sigma^{-1}]$$

with the following properties :

(i) *Lf is an isomorphism in $\mathscr{C}[\Sigma^{-1}]$, for every $f \in \Sigma$.*

(ii) *For every category $\mathscr{B}$ and any functor $G : \mathscr{C} \to \mathscr{B}$ such that Gf is an isomorphism in $\mathscr{B}$ for every $f \in \Sigma$, there exists a unique functor $G' : \mathscr{C}[\Sigma^{-1}] \to \mathscr{B}$ such that $G = G' \circ L$.*

Proof. Pick any universe U' containing U, and such that $\mathscr{C}$ is U'-small; after replacing U by U', we may assume that $\mathscr{C}$ is small. We consider the graph $\mathscr{G}$ such that

- $V(\mathscr{G}) = \mathrm{Ob}(\mathscr{C})$
- $\mathscr{G}(A, B)$ is the disjoint union of $\mathrm{Hom}_{\mathscr{C}}(A, B)$ and $\mathrm{Hom}_{\mathscr{C}}(B, A) \cap \Sigma$, for every $A, B \in V(\mathscr{G})$.

For every $f \in \mathrm{Hom}_{\mathscr{C}}(B, A) \cap \Sigma$, we write $f^{-1} : A \to B$ for the corresponding arrow of $\mathscr{G}$. Next, we endow $\mathscr{G}$ with the set Θ consisting of the following commutativity conditions :

- For every $A \in \mathrm{Ob}(\mathscr{C})$, the pair $((A, \mathbf{1}_A, A), (A))$
- for every $A, B, C \in \mathrm{Ob}(\mathscr{C})$, every $f \in \mathrm{Hom}_{\mathscr{C}}(A, B)$ and every $g \in \mathrm{Hom}_{\mathscr{C}}(B, C)$, the pair $((A, f, B, g, C), (A, g \circ f, C))$
- the pairs $((A, f^{-1}, B, f, A), (A, \mathbf{1}_A, A))$ and $((B, f, A, f^{-1}, B), (B, \mathbf{1}_B, B))$, for every $A, B \in \mathrm{Ob}(\mathscr{C})$ and every $f \in \mathrm{Hom}_{\mathscr{C}}(B, A) \cap \Sigma$.

By proposition 1.6.8, the left adjoint of (1.6.7) maps the conditional graph $(\mathscr{G}, \Theta)$ to a category which we denote $\mathscr{C}[\Sigma^{-1}]$. Now, suppose that $G : \mathscr{C} \to \mathscr{B}$ is a functor such that Gf is an isomorphism in $\mathscr{B}$, for every $f \in \Sigma$. We define a morphism of graphs $H : \mathscr{G} \to \mathscr{B}$ by the rule :

- $H(A) := GA$ for every $A \in V(\mathscr{G})$
- $H(f) := Gf$ for every $A, B \in V(\mathscr{G})$ and every $f \in \mathrm{Hom}_{\mathscr{C}}(A, B)$

- $H(f) := (Gf)^{-1}$ for every $A, B \in V(\mathscr{G})$ and every $f \in \mathrm{Hom}_{\mathscr{C}}(B, A) \cap \Sigma$.

It is easily seen that H induces a morphism of conditional graphs

$$(\mathscr{G}, \Theta) \to (\mathscr{B}, C_{\mathscr{B}})$$

which, under the adjunction of proposition 1.6.8, corresponds to a unique functor $\mathscr{C}[\Sigma^{-1}] \to \mathscr{B}$ with the sought properties. □

Remark 1.6.10. (i) Clearly conditions (i) and (ii) of theorem 1.6.9 characterize $\mathscr{C}[\Sigma^{-1}]$ up to isomorphism of categories. We call a *localization of* $\mathscr{C}$ *relative to* Σ the datum of a functor $G : \mathscr{C} \to \mathscr{D}$ and an equivalence of categories $H : \mathscr{C}[\Sigma^{-1}] \to \mathscr{D}$ such that $H \circ L = G$, where $L : \mathscr{C} \to \mathscr{C}[\Sigma^{-1}]$ is the localization functor of theorem 1.6.9.

(ii) For any subset $\Sigma \subset \mathrm{Morph}(\mathscr{C})$, let us set $\Sigma^o := \{s^o \mid s \in \Sigma\} \subset \mathrm{Morph}(\mathscr{C}^o)$ (notation of (1.1.1)). In view of (i), we see that the localization functor $\mathscr{C}^o \to \mathscr{C}^o[\Sigma^{o-1}]$ extends uniquely to an isomorphism of categories :

$$\mathscr{C}[\Sigma^{-1}]^o \xrightarrow{\sim} \mathscr{C}^o[\Sigma^{o-1}].$$

Corollary 1.6.11. *In the situation of theorem* 1.6.9*, the localization functor* L *induces a fully faithful functor*

$$\mathsf{Fun}(L, \mathscr{B}) : \mathsf{Fun}(\mathscr{A}[\Sigma^{-1}], \mathscr{B}) \to \mathsf{Fun}(\mathscr{A}, \mathscr{B}) \qquad \textit{for every category } \mathscr{B}.$$

Proof. Let $F, G : \mathscr{A}[\Sigma^{-1}] \to \mathscr{B}$ be two functors; recall that a natural transformation $\alpha : F \Rightarrow G$ is the same as the datum of a functor $\widetilde{\alpha} : \mathscr{A}[\Sigma^{-1}] \to \mathsf{Morph}(\mathscr{B})$ with $\mathsf{s} \circ \widetilde{\alpha} = F$ and $\mathsf{t} \circ \widetilde{\alpha} = G$ (see (1.1.30)). Notice also that a morphism (g, g') of $\mathsf{Morph}(\mathscr{B})$ is invertible if and only if both $\mathsf{s}(g, g') := g$ and $\mathsf{t}(g, g') := g'$ are invertible in $\mathscr{B}$. Combining with the universal property of F from theorem 1.6.9, we conclude that the datum of α is equivalent to that of a functor $\widetilde{\beta} : \mathscr{A} \to \mathsf{Morph}(\mathscr{B})$ such that $\mathsf{s} \circ \widetilde{\beta} = F \circ L$ and $\mathsf{t} \circ \widetilde{\beta} = G \circ L$; but the latter is the same as a natural transformation $F \circ L \Rightarrow G \circ L$, whence the contention. □

Example 1.6.12. Let $\mathscr{C}$ be a category that admits either a final or an initial object, and let Σ be the set of all morphisms of $\mathscr{C}$. Then $\mathscr{C}[\Sigma^{-1}]$ is (isomorphic to) the category $\overline{\mathscr{C}}$ whose set of objects is $\mathrm{Ob}(\mathscr{C})$ and such that for every $X, Y \in \mathrm{Ob}(\mathscr{C})$ the set $\mathrm{Hom}_{\overline{\mathscr{C}}}(X, Y)$ contains exactly one element. Indeed, there exists a unique functor $p : \mathscr{C} \to \overline{\mathscr{C}}$ which is the identity on objects; since every morphism of $\overline{\mathscr{C}}$ is an isomorphism, p factors uniquely through a functor $\mathscr{C}[\Sigma^{-1}] \to \overline{\mathscr{C}}$ that is still the identity on objects. On the other hand, say that X_0 is a final object for $\mathscr{C}$, and for every $X \in \mathrm{Ob}(\mathscr{C})$, let $t_X : X \to X_0$ be the unique morphism; denote by $[t_X]$ the class of t_X in $\mathscr{C}[\Sigma^{-1}]$. Then we have also a functor $q : \overline{\mathscr{C}} \to \mathscr{C}[\Sigma^{-1}]$ that is identity on objects, and sends every morphism $X \to Y$ of $\overline{\mathscr{C}}$ to the morphism $\tau_{XY} := [t_Y]^{-1} \circ [t_X] : X \to Y$ of $\mathscr{C}[\Sigma^{-1}]$. Clearly $p \circ q = \mathbf{1}_{\overline{\mathscr{C}}}$, and to conclude, it suffices to check that if $f : Y \to X$ is any morphisms of $\mathscr{C}$, then the class $[f] : X \to Y$ of f in $\mathscr{C}[\Sigma^{-1}]$ coincides with τ_{XY}. But the latter assertion is clear, since $t_Y \circ f = t_X$ in $\mathscr{C}$. One argues likewise in case $\mathscr{C}$ admits an initial object.

Proposition 1.6.13. *Let* $\mathscr{C}, \mathscr{D}$ *be two categories,* $F : \mathscr{C} \to \mathscr{D}$, $G : \mathscr{D} \to \mathscr{C}$ *two functors, and set*

$$\Sigma_F := \{f \in \mathrm{Morph}(\mathscr{C}) \mid Ff \textit{ is an isomorphism}\}.$$

Suppose that G *is fully faithful, and is either right or left adjoint to* F*. Then* F *factors uniquely through the localization* $L : \mathscr{C} \to \mathscr{C}[\Sigma_F^{-1}]$ *and an equivalence*

$$F' : \mathscr{C}[\Sigma_F^{-1}] \xrightarrow{\sim} \mathscr{D}.$$

Proof. To ease notation, set $\mathscr{C}' := \mathscr{C}[\Sigma_F^{-1}]$, and notice that $\Sigma_{F^o} = \Sigma_F^o$ (notation of remark 1.6.10(ii)), whence a natural isomorphism of categories :

$$\mathscr{C}'^o \xrightarrow{\sim} \mathscr{C}^o[\Sigma_{F^o}^{-1}]$$

that identifies $L^o : \mathscr{C}^o \to \mathscr{C}'^o$ with the localization $\mathscr{C}^o \to \mathscr{C}^o[\Sigma_{F^o}^{-1}]$; moreover, recall that (F, G) is an adjoint pair if and only if the same holds for the pair (G^o, F^o) (remark 1.1.19(iv)). Thus, we may assume that G is right adjoint to F. Now, the existence and uniqueness of F' is clear from the universal property of the localization; it remains to check that F' is an equivalence. To this aim, set as well $G' := L \circ G : \mathscr{D} \to \mathscr{C}'$, and let (η, ε) be the unit and counit of an adjunction for the pair (F, G). Since $F = F' \circ L$, we get $F' \circ G' = F \circ G$, hence ε is also a natural transformation $F'G' \Rightarrow \mathbf{1}_{\mathscr{D}}$. On the other hand, by corollary 1.6.11 there exists a unique natural transformation

$$\eta' : \mathbf{1}_{\mathscr{C}'} \Rightarrow G'F' \qquad \text{such that} \qquad \eta' * L = L * \eta.$$

Using the triangular identities of (1.1.13), we compute :

$$(G' * \varepsilon) \odot (\eta' * G') = (LG * \varepsilon) \odot (L * \eta * G) = L * ((G * \varepsilon) \odot (\eta * G)) = L * \mathbf{1}_G = \mathbf{1}_{G'}.$$

Likewise, let us show that $(\varepsilon * F') \odot (F' * \eta') = \mathbf{1}_{F'}$; to this aim, again by virtue of corollary 1.6.11, it suffices to check that

$$((\varepsilon * F') \odot (F' * \eta')) * L = \mathbf{1}_{F'} * L = \mathbf{1}_F.$$

However, the left-hand side equals $(\varepsilon * F'L) \odot (F' * \eta' * L) = (\varepsilon * L) \odot (F'L * \eta)$, so the sought identity follows again from (1.1.13). By proposition 1.1.15(i), it follows that (F', G') is an adjoint pair of functors, with unit η' and counit ε. Lastly, since G is fully faithful, ε is an isomorphism of functors (proposition 1.1.20(i)); invoking again the triangular identities (1.1.13), we deduce that $F * \eta$ is an isomorphism of functors, hence $\eta_X \in \Sigma_F$ for every $X \in \mathrm{Ob}(\mathscr{C})$. Since $\mathrm{Ob}(\mathscr{C}) = \mathrm{Ob}(\mathscr{C}')$, it follows easily that η' is an isomorphism of functors, whence the proposition. □

Definition 1.6.14. Let $\mathscr{C}$ be a category, $\Sigma \subset \mathrm{Morph}(\mathscr{C})$ a set of morphisms.

(i) We say that Σ *admits a right calculus of fractions* if the following conditions hold :

(CF1) $\mathbf{1}_A \in \Sigma$, for every $A \in \mathrm{Ob}(\mathscr{C})$.

(CF2) For every $s : A \to B$ and $t : B \to C$ with $s, t \in \Sigma$, we have $t \circ s \in \Sigma$ as well.

(CF3) For every morphism $f : A \to B$ in $\mathscr{C}$ and every $s : C \to B$ in Σ, there exist $g : D \to C$ in $\mathscr{C}$ and $t : D \to A$ in Σ such that $f \circ t = s \circ g$.

(CF4) If $f, g : A \to B$ are any two morphisms in $\mathscr{C}$ such that $s \circ f = s \circ g$ for some $s : B \to C$ in Σ, then there exists $t : D \to A$ in Σ such that $f \circ t = g \circ t$.

(ii) We say that Σ *admits a left calculus of fractions* if the subset $\Sigma^o := \{s^o \mid s \in \Sigma\}$ admits a right calculus of fractions (notation of remark 1.6.10(ii)).

(iii) For every $A \in \mathrm{Ob}(\mathscr{C})$, let Σ_A be the full subcategory of $\mathscr{C}/A$ whose objects are the elements of Σ with target equal to A (notation of (1.1.24)). We say that Σ is *right cofinally small*, if Σ_A is cofinally small for every $A \in \mathrm{Ob}(\mathscr{C})$ (see definition 1.5.1(iv)).

1.6.15. Let now $\mathscr{C}$ and Σ be as in definition 1.6.14, and suppose that $\mathscr{C}$ has small Hom-sets, and Σ admits a right calculus of fractions. For every $A, B \in \mathrm{Ob}(\mathscr{C})$, let us consider the functor

$$H_{A,B} : \Sigma_A^o \to \mathbf{Set} \qquad (s : I \to A) \mapsto \{(s, f) \mid f \in \mathrm{Hom}_{\mathscr{C}}(I, B)\}$$

where, for every morphism $h/A : s \to t$ in Σ_A, the mapping $H_{A,B}(h/A)$ is given by the rule

$$H_{A,B}(t) \to H_{A,B}(s) \quad : \quad (t, f) \mapsto (s, f \circ h).$$

Furthermore, we define a mapping

$$c_s^{A,B} : H_{A,B}(s) \to \mathrm{Hom}_{\mathscr{C}[\Sigma^{-1}]}(A, B) \qquad (s, f) \mapsto f \circ s^{-1} \qquad \text{for every } s \in \mathrm{Ob}(\Sigma_A).$$

Proposition 1.6.16. *In the situation of* (1.6.15)*, the following holds :*

(i) *For every $A \in \mathrm{Ob}(\mathscr{C})$, the category Σ_A^o is filtered.*

(ii) *For every* $A, B \in \mathrm{Ob}(\mathscr{C})$*, the rule :*

$$s \mapsto c_s^{A,B} \qquad \text{for every } s \in \mathrm{Ob}(\Sigma_A)$$

defines a universal cocone with basis $H_{A,B}$ *and vertex* $\mathrm{Hom}_{\mathscr{C}[\Sigma^{-1}]}(A, B)$*.*

(iii) *If* Σ *is right cofinally small, The category* $\mathscr{C}[\Sigma^{-1}]$ *has small* Hom*-sets.*

Proof. (i): First, Σ_A^o is not empty, due to (CF1). Next, say that $s, s' \in \mathrm{Ob}(\Sigma_A)$, and denote by I and I' the sources of s and respectively s'; due to (CF3), we may find $I'' \in \mathrm{Ob}(\mathscr{C})$, a morphism $f : I'' \to I$ in $\mathscr{C}$, and an element $g : I'' \to I'$ of Σ such that $s \circ f = s' \circ g$. Then, (CF2) says that $s' \circ g$ lies in Σ, and therefore it defines an object s'' of Σ_A, with morphisms $f : s \to s''$ and $g : s' \to s''$ in Σ_A^o. Lastly, say that $f, f' : s \to s'$ are any two morphisms in Σ_A^o, and denote by I' the source of s'. By (CF4), we may find $g : I'' \to I'$ in Σ such that $f \circ g = f' \circ g$; clearly $s' \circ g$ defines a morphism $h : s' \to s''$ in Σ_A^o such that $h \circ f = h \circ f'$. Now the assertion follows from remark 1.2.21(i).

(ii): Let $s : I \to A$ and $s' : I' \to A$ be two objects of Σ_A, and $g : I' \to I$ a morphism $s \to s'$ in Σ_A^o. Given $f : I \to B$, set $f' := f \circ g$; we notice :

Claim 1.6.17. $f \circ s^{-1} = f' \circ s'^{-1}$ in $\mathscr{C}[\Sigma^{-1}]$.

Proof of the claim. Due to (CF3), we may find $I'' \in \mathrm{Ob}(\mathscr{C})$, an element $t : I'' \to I$ of Σ and a morphism $t' : I'' \to I'$ such that $s \circ t = s' \circ t'$. We compute :

$$s \circ g \circ t' = s' \circ t' = s \circ t$$

whence, by (CF4), an element $h : I''' \to I''$ of Σ, such that $g \circ t' \circ h = t \circ h$. It follows that $g \circ t' = t$ in $\mathscr{C}[\Sigma^{-1}]$ and therefore

$$f \circ t = f \circ g \circ t' = f' \circ t' \qquad \text{in } \mathscr{C}[\Sigma^{-1}].$$

Consequently :

$$f \circ s^{-1} = f' \circ t' \circ t^{-1} \circ s^{-1} = f' \circ s'^{-1} \circ s \circ t \circ t^{-1} \circ s^{-1} = f' \circ s'^{-1} \qquad \text{in } \mathscr{C}[\Sigma^{-1}]$$

as stated. ◊

Claim 1.6.17 says already that $c_\bullet^{A,B}$ is a well defined cocone. To check the universality property, we define a new category $\mathscr{D}$ with $\mathrm{Ob}(\mathscr{D}) = \mathrm{Ob}(\mathscr{C})$ and such that $\mathrm{Hom}_{\mathscr{D}}(A, B)$ is the colimit of the functor $H_{A,B}$, for every $A, B \in \mathrm{Ob}(\mathscr{C})$. From (i) and example 1.2.23(i) we see that this is the set of equivalence classes $[s, f]$ of pairs $(s, f) \in H_{A,B}(s)$ with s ranging over all objects of Σ_A; two such pairs (s, f) and (s', f') are equivalent, if there exist $t \in \mathrm{Ob}(\Sigma_A)$ and morphisms $h : t \to s$, $h' : t \to s'$ in Σ_A such that $f \circ h = f' \circ h'$. The composition of morphisms in $\mathscr{D}$ is defined as follows. Let $A, B, C \in \mathrm{Ob}(\mathscr{C})$ be any three objects, $s : I \to A$, $t : J \to B$ any elements of Σ, and $(s, f) \in H_{A,B}(s)$, $(t, g) \in H_{B,C}(t)$; by (CF3) we find $D \in \mathrm{Ob}(\mathscr{C})$, a morphism $f' : D \to J$ in $\mathscr{C}$ and an element $t' : D \to I$ of Σ such that $t \circ f' = f \circ t'$. We define

$$(t, g) \circ (s, f) := [s \circ t', g \circ f'] \in \mathrm{Hom}_{\mathscr{D}}(A, C). \tag{1.6.18}$$

Let us check that this class does not depend on the choice of D, f' and t'. Indeed, suppose that $D' \in \mathrm{Ob}(\mathscr{C})$ is another object and $t'' : D' \to I$ in Σ, $f'' : D' \to J$ in $\mathscr{C}$ satisfy the condition $t \circ f'' = f \circ t''$; by (CF3) we may then find $D'' \in \mathrm{Ob}(\mathscr{C})$, $h : D'' \to D$ in Σ and $h' : D'' \to D'$ in $\mathscr{C}$ such that $t' \circ h = t'' \circ h'$. We compute :

$$t \circ f' \circ h = f \circ t' \circ h = f \circ t'' \circ h' = t \circ f'' \circ h'.$$

By (CF4), it follows that there exist $D''' \in \mathrm{Ob}(\mathscr{C})$ and an element $u : D''' \to D''$ of Σ such that $f' \circ h \circ u = f'' \circ h' \circ u$. By (CF2), we may then replace D'' by D''' and h, h' with $h \circ u$ and respectively $h' \circ u$, after which we may assume as well that $f'' \circ h' = f' \circ h$. Finally, we find :

$$[s \circ t', g \circ f'] = [s \circ t' \circ h, g \circ f' \circ h] = [s \circ t'' \circ h', g \circ f'' \circ h'] = [s \circ t'', g \circ f'']$$

as required. It also follows that $[s \circ t', g \circ f']$ depends only on $[s, f]$ and $[t, g]$; indeed, say that $u : I' \to I$ is any element of Σ, and set $(s', f'') := (s \circ u, f \circ u)$. Suppose also that $f'' \circ t'' = t \circ f'$ for some $t'' : D \to I'$ in Σ; then $(s' \circ t'', g \circ f') = (s \circ u \circ t'', g \circ f')$, and the foregoing shows that the latter pair is equivalent to $(s \circ t', g \circ f')$, which shows the independence on the chosen representative for $[s, f]$; similarly one checks the independence on the representative for $[t, g]$.

We next check the associativity of the composition law thus obtained. To this aim, say that $A, B, C, D \in \mathrm{Ob}(\mathscr{C})$ are any four objects, $s : I \to A$, $t : J \to B$, $u : K \to C$ any three elements of Σ, and $f : I \to B$, $g : J \to C$, $h : K \to D$ any three morphisms of $\mathscr{C}$. Choose $E \in \mathrm{Ob}(\mathscr{C})$ with an element $t' : E \to I$ of Σ and a morphism $f' : E \to J$ of $\mathscr{C}$ such that $f \circ t' = t \circ f'$, and therefore

$$[t, g] \circ [s, f] = [s \circ t', g \circ f'].$$

Then, choose $F \in \mathrm{Ob}(\mathscr{C})$ with $u' : F \to J$ in Σ and $g' : F \to K$ in $\mathscr{C}$ such that $g \circ u' = u \circ g'$, and therefore

$$[u, h] \circ [t, g] = [t \circ u', h \circ g'].$$

By (CF3) we may find $G \in \mathrm{Ob}(\mathscr{C})$, $u'' : G \to E$ in Σ and $f'' : G \to F$ in $\mathscr{C}$ such that $u' \circ f'' = f' \circ u''$. By (CF2), $t' \circ u''$ lies in Σ, and therefore it is easily seen that the pair $(s \circ t' \circ u'', h \circ g' \circ f'')$ represents both $[t \circ u', h \circ g'] \circ [s, f]$ and $[u, h] \circ [s \circ t', g \circ f']$, whence the assertion. Let us also remark that – due to (CF1) – for every $A \in \mathrm{Ob}(\mathscr{C})$, the class $[\mathbf{1}_A, \mathbf{1}_A]$ is the identity endomorphism of A, seen as an object of $\mathscr{D}$. This completes the construction of the category $\mathscr{D}$. Next, we claim that the system of cocones $c_\bullet^{\bullet\bullet}$ yields a functor

$$F : \mathscr{D} \to \mathscr{C}[\Sigma^{-1}]$$

whose map on objects is the identity mapping of $\mathrm{Ob}(\mathscr{C})$, and such that

$$F[s, f] := f \circ s^{-1} \qquad \text{for every morphism } [s, f] \text{ in } \mathscr{D}.$$

Indeed, claim 1.6.17 implies that the foregoing rule yields a well defined map

$$\mathrm{Hom}_{\mathscr{D}}(A, B) \to \mathrm{Hom}_{\mathscr{C}[\Sigma^{-1}]}(A, B) \qquad \text{for every } A, B \in \mathrm{Ob}(\mathscr{C}).$$

It is also clear that $F\mathbf{1}_A = \mathbf{1}_A$ for every $A \in \mathrm{Ob}(\mathscr{C})$, hence it remains only to check that F respects the composition laws of the two categories; however, say that s, f, t, g are as in (1.6.18); then we have

$$(g \circ f') \circ (s \circ t')^{-1} = (g \circ t^{-1} \circ f \circ t') \circ t'^{-1} \circ s^{-1} = (g \circ t^{-1}) \circ (f \circ s^{-1}) \qquad \text{in } \mathscr{C}[\Sigma^{-1}]$$

whence the contention. Likewise, we have a functor

$$G : \mathscr{C} \to \mathscr{D}$$

which is the identity mapping on objects, and such that $Gf := [\mathbf{1}_A, f]$ for every $A, B \in \mathrm{Ob}(\mathscr{C})$ and every morphism $f : A \to B$ in $\mathscr{C}$. Moreover, the identities

$$[\mathbf{1}_I, s] \circ [s, \mathbf{1}_I] = [\mathbf{1}_A, \mathbf{1}_A] \qquad [s, \mathbf{1}_I] \circ [\mathbf{1}_I, s] = [\mathbf{1}_I, \mathbf{1}_I] \qquad \text{for every } s : I \to A \text{ in } \Sigma$$

show that Fs is invertible in $\mathscr{D}$, for every such s. By theorem 1.6.9, it follows that G factors through a unique functor $G' : \mathscr{C}[\Sigma^{-1}] \to \mathscr{D}$, and a simple inspection shows that $F \circ G'$ is the identity endofunctor of $\mathscr{C}[\Sigma^{-1}]$. Likewise, it is easily seen that $G' \circ F = \mathbf{1}_{\mathscr{D}}$, so F establishes an isomorphism of categories, and (ii) follows.

(iii) is an immediate consequence of (ii). □

Remark 1.6.19. Proposition 1.6.16(ii) means that $c_\bullet^{A,B}$ induces a natural identification

$$\operatorname*{colim}_{\Sigma_A^o} H_{A,B} \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{C}[\Sigma^{-1}]}(A, B) \qquad \text{for every } A, B \in \mathrm{Ob}(\mathscr{C}).$$

The dual of this assertion provides a corresponding computation of the morphisms in $\mathscr{C}[\Sigma^{-1}]$ in terms of left fractions. Namely, suppose that Σ admits a left calculus of fractions; for every

$A \in \mathrm{Ob}(\mathscr{C})$ we let Σ^A be the full subcategory of $A/\mathscr{C}$ whose objects are the elements of Σ with source equal to A, and we say that Σ is *left cofinally small* if Σ^A is cofinally small for every such A. For every other object B of $\mathscr{C}$ we consider the functor

$$H_{B,A} : \Sigma^A \to \mathbf{Set} \qquad (s : A \to I) \mapsto \{(f, s) \mid f \in \mathrm{Hom}_{\mathscr{C}}(B, I)\}$$

where, for every morphism $h : s \to t$ in Σ^A, the mapping $H_{B,A}(h)$ is given by the rule

$$H_{B,A}(s) \to H_{B,A}(t) \quad : \quad (f, s) \mapsto (h \circ f, t).$$

Then Σ^A is a filtered category, and if Σ is left cofinally small, $\mathscr{C}[\Sigma^{-1}]$ has small Hom-sets; more precisely, we have a natural identification :

$$\operatorname*{colim}_{\Sigma^A} H_{B,A} \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{C}[\Sigma^{-1}]}(B, A) \qquad (f, s) \mapsto s^{-1} \circ f.$$

1.6.20. Let $\mathscr{A}$ be a category, $\mathscr{A}_0$ a full subcategory of $\mathscr{A}$, $\Sigma \subset \mathrm{Ob}(\mathscr{A})$ any subset, and set

$$\Sigma_0 := \Sigma \cap \mathrm{Morph}(\mathscr{A}_0).$$

Clearly, the inclusion functor $i : \mathscr{A}_0 \to \mathscr{A}$ extends uniquely to a functor

$$i[\Sigma_0^{-1}] : \mathscr{A}_0[\Sigma_0^{-1}] \to \mathscr{A}[\Sigma^{-1}].$$

Proposition 1.6.21. *In the situation of* (1.6.20)*, suppose that :*

(a) *For every $A \in \mathrm{Ob}(\mathscr{A})$, $A_0 \in \mathrm{Ob}(\mathscr{A}_0)$ and every morphism $f : A \to iA_0$ that lies in Σ, there exist $A'_0 \in \mathrm{Ob}(\mathscr{A}_0)$ and a morphism $g : iA'_0 \to A$ such that $f \circ g \in \Sigma$.*

(b) *Σ admits a right calculus of fractions.*

Then the set Σ_0 admits a right calculus of fractions in $\mathscr{A}_0$, and the functor $i[\Sigma_0^{-1}]$ is fully faithful.

Proof. As usual, after replacing our universe U by a larger one, we may assume that $\mathscr{A}$ is small. Let us check that Σ_0 admits a right calculus of fractions. (CF1) and (CF2) are obviously fulfilled by Σ_0. Next, suppose that we have $A, B, C \in \mathrm{Ob}(\mathscr{A}_0)$, $D \in \mathrm{Ob}(\mathscr{A})$ and a commutative diagram in $\mathscr{A}$

$$\begin{array}{ccc} D & \xrightarrow{g} & iC \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle is} \\ iA & \xrightarrow{if} & iB \end{array} \qquad \text{with } s, t \in \Sigma. \tag{1.6.22}$$

By (a), it follows that there exists a morphism $u : iD_0 \to D$ for some $D_0 \in \mathrm{Ob}(\mathscr{A}_0)$, such that $t' := t \circ u : iD_0 \to iA$ lies in Σ_0, $g' := g \circ u : iD_0 \to iC$ lies in $\mathrm{Hom}_{\mathscr{A}_0}(D_0, C)$, and clearly $f \circ t' = s \circ g'$, whence (CF3). Lastly, suppose that $A, B \in \mathrm{Ob}(\mathscr{A}_0)$, $D \in \mathrm{Ob}(\mathscr{A})$, and we have morphisms $f, g : A \to B$ and $t : D \to iA$ such that $i(f) \circ t = i(g) \circ t$ and with $t \in \Sigma$. Pick $u : iD_0 \to D$ as in the foregoing, and let again $t' := t \circ u$; then $f \circ t' = g \circ t'$, and we have just seen that $t' \in \Sigma_0$, whence (CF4).

Next, let us check that $i[\Sigma_0^{-1}]$ is fully faithful. In light of propositions 1.6.16(ii) and 1.5.2, it suffices to show that the inclusion functor

$$\Sigma^o_{0,A} \to \Sigma^o_{iA} \tag{1.6.23}$$

is cofinal for every $A \in \mathrm{Ob}(\mathscr{A}_0)$ (notation of (1.6.15)). To this aim, in view of proposition 1.6.16(i), it suffices to check that the functor (1.6.23) fulfills conditions (a) and (b) of lemma 1.5.7(i). However, condition (a) of *loc.cit.* translates directly as the assumption (a) of the proposition. Next, consider any pair of morphisms $f, g : iD_0 \to D$ and any element $s : D \to iA$ in Σ such that $s \circ f = s \circ g$. By (CF4), there exists $t : E \to iD_0$ in Σ such that $f \circ t = g \circ t$, and then assumption (a) yields a morphism $u : iE_0 \to E$ such that $t' := t \circ u$ lies in Σ_0, and $f \circ i(t') = g \circ i(t')$, which shows that condition (b) of lemma 1.5.7(iv) holds as well. $\square$

Remark 1.6.24. Taking into account remarks 1.6.10(ii) and 1.6.19, we see that the dual of proposition 1.6.21 holds as well; *i.e.* in the situation of (1.6.20), suppose that

(a) For every $A \in \mathrm{Ob}(\mathscr{A})$, $A_0 \in \mathrm{Ob}(\mathscr{A}_0)$ and every morphism $f : iA_0 \to A$ that lies in Σ, there exist $A'_0 \in \mathrm{Ob}(\mathscr{A})$ and a morphism $g : A \to iA'_0$ such that $g \circ f \in \Sigma$.

(b) Σ admits a left calculus of fractions.

Then the set Σ_0 admits a left calculus of fractions in $\mathscr{A}_0$, and the functor $i[\Sigma_0^{-1}]$ is fully faithful.

Proposition 1.6.25. *Let $\mathscr{C}$ be a category with small* Hom*-sets, $\Sigma \subset \mathrm{Morph}(\mathscr{C})$ a set of morphisms admitting a right calculus of fractions, $F : \mathscr{C} \to \mathscr{C}[\Sigma^{-1}]$ the localization. We have:*

(i) *For every $X \in \mathrm{Ob}(\mathscr{C}[\Sigma^{-1}])$ the category $X/F\mathscr{C}$ is cofiltered.*

(ii) *If $\mathscr{C}$ is small, the functor $F_! : \mathscr{C}^\wedge \to \mathscr{C}[\Sigma^{-1}]^\wedge$ is exact (notation of remark* 1.3.6(i)*).*

Proof. (i): Let $((Y_i, f_i : X \to FY_i) \mid i = 1, 2)$ be a pair of objects of $X/F\mathscr{C}$. For $i = 1, 2$, we may write $f_i = g_i \circ s_i^{-1}$ with some morphism $s_i : Z_i \to X$ in Σ and some morphism $g_i : Z_i \to Y_i$ in $\mathscr{C}$. By condition (CF3) of definition 1.6.14 we may then find a morphism $t_1 : Z \to Z_1$ in Σ and a morphism $t_2 : Z \to Z_2$ in $\mathscr{C}$ such that $s := s_1 \circ t_1 = s_2 \circ t_2$, and notice that $s \in \Sigma$, by condition (CF2). For $i = 1, 2$, the composition $t_i \circ g_i$ then yields a morphism $(Z, s^{-1} : X \to FZ) \to (Y_i, f_i)$ in $X/F\mathscr{C}$. Next, let $h_1, h_2 : (Y_1, f_1) \to (Y_2, f_2)$ be two morphisms in $X/F\mathscr{C}$; this means that $h_i : Y_1 \to Y_2$ is a morphism in $\mathscr{C}$ for $i = 1, 2$, and $Fh_1 \circ f_1 = Fh_2 \circ f_1 = f_2$ in $\mathscr{C}[\Sigma^{-1}]$. Thus, $F(h_1 \circ g_1) = F(h_2 \circ g_1)$, and therefore there exists $(u : Z' \to Z_1) \in \Sigma$ such that $h_1 \circ g_1 \circ u = h_2 \circ g_1 \circ u$ in $\mathscr{C}$. Then the composition $g := g_1 \circ u$ yields a morphism $g : (Z', (s_1 \circ u)^{-1} : X \to FZ') \to (Y_1, f_1)$ in $X/F\mathscr{C}$ such that $h_1 \circ g = h_2 \circ g$. The assertion then follows from remark 1.2.21(i).

(ii): The assertion follows from (i), arguing as in the proof of corollary 1.5.19(i). □

2. 2-Category theory

In dealing with categories, the notion of equivalence is much more central than the notion of isomorphism. On the other hand, equivalence of categories is usually not preserved by the standard categorical operations discussed thus far. For instance, consider the following :

Example 2.0.1. Let $\mathscr{C}$ be the category with $\mathrm{Ob}(\mathscr{C}) = \{a, b\}$, and whose only morphisms are $\mathbf{1}_a$, $\mathbf{1}_b$ and $u : a \to b$, $v : b \to a$. Then necessarily $u \circ v = \mathbf{1}_b$ and $v \circ u = \mathbf{1}_a$. Let $\mathscr{C}_a$ (resp. $\mathscr{C}_b$) be the unique subcategory of $\mathscr{C}$ with $\mathrm{Ob}(\mathscr{C}_a) = \{a\}$ (resp. $\mathrm{Ob}(\mathscr{C}_b) = \{b\}$). Clearly both inclusion functors $\mathscr{C}_a \to \mathscr{C} \leftarrow \mathscr{C}_b$ are equivalences. However, $\mathscr{C}_a \times_{\mathscr{C}} \mathscr{C}_b$ is the empty category; especially, this fibre product is not equivalent to $\mathscr{C} = \mathscr{C} \times_{\mathscr{C}} \mathscr{C}$.

It is therefore natural to seek a new framework for the manipulation of categories and functors "up to equivalences", and thus more consonant with the very spirit of category theory. Precisely such a framework is provided by the theory of 2-categories, the subject of this chapter.

2.1. 2-Categories and pseudo-functors.

The category $\mathbf{Cat}$, together with the category structure on the sets $\mathrm{Fun}(-, -)$ (as in (1.1.10)), provides the first example of a 2*-category*. The latter is the datum of :

- A set $\mathrm{Ob}(\mathscr{A})$, whose elements are called the *objects of* $\mathscr{A}$.
- For every $A, B \in \mathrm{Ob}(\mathscr{A})$, a category $\mathscr{A}(A, B)$. The objects of $\mathscr{A}(A, B)$ are called 1*-cells* or *arrows*, and are designated by the usual arrow notation $f : A \to B$. Given $f, g \in \mathrm{Ob}(\mathscr{A}(A, B))$, we shall write $f \Rightarrow g$ to denote a morphism from f to g in $\mathscr{A}(A, B)$. Such morphisms are called 2*-cells*. The composition of 2-cells $\alpha : f \Rightarrow g$ and $\beta : g \Rightarrow h$ shall be denoted by $\beta \odot \alpha : f \Rightarrow h$.
- For every $A, B, C \in \mathrm{Ob}(\mathscr{A})$, a *composition bifunctor* :

$$c_{ABC} : \mathscr{A}(A, B) \times \mathscr{A}(B, C) \to \mathscr{A}(A, C).$$

Given 1-cells $A \xrightarrow{f} B \xrightarrow{g} C$, we write $g \circ f := c_{ABC}(f, g)$.
Given two 2-cells $\alpha : f \Rightarrow g$ and $\beta : h \Rightarrow k$, respectively in $\mathscr{A}(A, B)$ and $\mathscr{A}(B, C)$, we use the notation

$$\beta * \alpha := c_{ABC}(\alpha, \beta) : h \circ f \Rightarrow k \circ g.$$

Also, if h is any 1-cell of $\mathscr{A}(B, C)$, we usually write $h * \alpha$ instead of $\mathbf{1}_h * \alpha$. Likewise, we set $\beta * f := \beta * \mathbf{1}_f$, for every 1-cell f in $\mathscr{A}(A, B)$.

- For every element $A \in \mathrm{Ob}(\mathscr{A})$, a *unit functor* :

$$u_A : \mathbf{1} \to \mathscr{A}(A, A)$$

where $\mathbf{1} := (*, \mathbf{1}_*)$ is the final object of $\mathbf{Cat}$. Hence u_A is the datum of an object:

$$\mathbf{1}_A \in \mathrm{Ob}(\mathscr{A}(A, A))$$

and its identity endomorphism, which we shall denote by $i_A : \mathbf{1}_A \to \mathbf{1}_A$.

The bifunctors c_{ABC} are required to satisfy an *associativity axiom*, which says that the diagram:

$$\begin{array}{ccc}
\mathscr{A}(A,B) \times \mathscr{A}(B,C) \times \mathscr{A}(C,D) & \xrightarrow{\mathbf{1} \times c_{BCD}} & \mathscr{A}(A,B) \times \mathscr{A}(B,D) \\
{\scriptstyle c_{ABC} \times \mathbf{1}} \downarrow & & \downarrow {\scriptstyle c_{ABD}} \\
\mathscr{A}(A,C) \times \mathscr{A}(C,D) & \xrightarrow{c_{ACD}} & \mathscr{A}(A,D)
\end{array}$$

commutes for every $A, B, C, D \in \mathrm{Ob}(\mathscr{A})$. Likewise, the functor u_A is required to satisfy a *unit axiom*; namely, the diagram :

$$\begin{array}{ccccc}
\mathbf{1} \times \mathscr{A}(A,B) & \xleftarrow{\sim} & \mathscr{A}(A,B) & \xrightarrow{\sim} & \mathscr{A}(A,B) \times \mathbf{1} \\
{\scriptstyle u_A \times \mathbf{1}_{\mathscr{A}(A,B)}} \downarrow & & \| & & \downarrow {\scriptstyle \mathbf{1}_{\mathscr{A}(A,B)} \times u_B} \\
\mathscr{A}(A,A) \times \mathscr{A}(A,B) & \xrightarrow{c_{AAB}} & \mathscr{A}(A,B) & \xleftarrow{c_{ABB}} & \mathscr{A}(A,B) \times \mathscr{A}(B,B)
\end{array}$$

commutes for every $A, B \in \mathrm{Ob}(\mathscr{A})$.

Remark 2.1.1. (i) In any 2-category, consider a diagram with two 2-cells :

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \underset{g'}{\overset{f'}{\rightrightarrows}} C. \qquad (\Downarrow\alpha,\ \Downarrow\alpha')$$

In this situation, we have the following commutation identity :

$$(\alpha' * g) \odot (f' * \alpha) = \alpha' * \alpha = (g' * \alpha) \odot (\alpha' * f)$$

which is sometimes useful, to perform certain verifications. To see this, notice that $\alpha' * \alpha = (\alpha' \odot \mathbf{1}_{f'}) * (\mathbf{1}_g \odot \alpha)$, whence the first stated identity, by functoriality of the composition for 2-cells. Likewise, we get $\alpha' * \alpha = (\mathbf{1}_{g'} \odot \alpha') * (\alpha \odot \mathbf{1}_f)$, whence the second identity.

(ii) As already announced, the category $\mathbf{Cat}$ becomes naturally a 2-category, by letting

$$\mathbf{Cat}(\mathscr{A}, \mathscr{B}) := \mathsf{Fun}(\mathscr{A}, \mathscr{B}) \qquad \text{for every } \mathscr{A}, \mathscr{B} \in \mathrm{Ob}(\mathbf{Cat})$$

with composition bifunctor defined as in (1.1.10). Indeed, by spelling out the definitions, we see that the functoriality of the Godement product boils down precisely to the identity (1.1.11).

On the other hand, from every 2-category $\mathscr{A}$ we obtain a category, simply by forgetting the 2-cells : the set of objects of this *underlying category of* $\mathscr{A}$ is $\mathrm{Ob}(\mathscr{A})$, and the morphisms are the 1-cells of $\mathscr{A}$, with composition law induced by the composition bifunctor of $\mathscr{A}$.

(iii) We shall say that a 2-category $\mathscr{A}$ *has* U*-small* Hom*-categories* if $\mathscr{A}(A, B)$ is a U-small category for every $A, B \in \mathrm{Ob}(\mathscr{A})$. We shall say that $\mathscr{A}$ is U*-small* if it has U-small Hom-categories and $\mathrm{Ob}(\mathscr{A})$ is U-small. As usual, the universe U will be dropped from the notation, unless there is a danger of ambiguities. For instance, the 2-category $\mathbf{Cat}$ has small Hom-categories.

2.1.2. If $\mathscr{A}$ is a 2-category, we have three distinct constructions of *opposite* 2*-categories* associated with $\mathscr{A}$. Namely we have :

- The 2-category $\mathscr{A}^o$ such that $\mathrm{Ob}(\mathscr{A}^o) := \{A^o \mid A \in \mathrm{Ob}(\mathscr{A})\}$ (cp. (1.1.1)) and with
$$\mathscr{A}^o(A^o, B^o) := \mathscr{A}(B, A) \qquad \text{for every } A^o, B^o \in \mathrm{Ob}(\mathscr{A}^o).$$
For any 1-cell $f : B \to A$ in $\mathscr{A}$, we denote $f^o : A^o \to B^o$ the corresponding 1-cell in $\mathscr{A}^o$. The composition bifunctor $c^o_{\bullet\bullet\bullet}$ and unit functors $u^o_\bullet$ of $\mathscr{A}^o$ are given by the rules :
$$c^o_{A^oB^oC^o} := c_{CBA} \qquad \mathbf{1}_{A^o} := \mathbf{1}^o_A \qquad \text{for every } A^o, B^o, C^o \in \mathrm{Ob}(\mathscr{A}^o).$$
Notice that to any 2-cell $\alpha : f \Rightarrow g$ in $\mathscr{A}$ there corresponds a 2-cell $\alpha^o : f^o \Rightarrow g^o$ in $\mathscr{A}^o$. The associativity and unit axioms for $\mathscr{A}^o$ follow formally from the corresponding properties for $\mathscr{A}$.
- The 2-category ${}^o\mathscr{A}$ such that $\mathrm{Ob}({}^o\mathscr{A}) := \{{}^oA \mid A \in \mathrm{Ob}(\mathscr{A})\}$ and with
$${}^o\mathscr{A}({}^oA, {}^oB) := \mathscr{A}(A, B)^o \qquad \text{for every } {}^oA, {}^oB \in \mathrm{Ob}({}^o\mathscr{A}).$$
For any 1-cell $f : A \to B$ in $\mathscr{A}$, we denote ${}^of : {}^oA \to {}^oB$ the corresponding 1-cell in ${}^o\mathscr{A}$. The composition bifunctor ${}^oc_{\bullet\bullet\bullet}$ and unit functors ${}^ou_\bullet$ of ${}^o\mathscr{A}$ are given by the rules :
$${}^oc_{{}^oA{}^oB{}^oC} := c^o_{ABC} \qquad \mathbf{1}_{{}^oA} := {}^o\mathbf{1}_A \qquad \text{for every } {}^oA, {}^oB, {}^oC \in \mathrm{Ob}({}^o\mathscr{A}).$$
Then to any 2-cell $\alpha : f \Rightarrow g$ in $\mathscr{A}$ there corresponds a 2-cell ${}^o\alpha : {}^og \Rightarrow {}^of$ in ${}^o\mathscr{A}$. The associativity and unit axioms for ${}^o\mathscr{A}$ are also immediately deduced from those of $\mathscr{A}$.
- Lastly, we may combine the two foregoing constructions to get the 2-category ${}^o\mathscr{A}^o$, which inverts the direction of both 1-cells and 2-cells.

Definition 2.1.3. Let $\mathscr{A}$ be any 2-category, and $f : A \to B$ any 1-cell of $\mathscr{A}$.

(i) A 2-cell $\beta : f \Rightarrow f'$ of $\mathscr{A}$ is *invertible*, if it is an isomorphism in the category $\mathscr{A}(A, B)$.

(ii) We say that the 1-cell $g : B \to A$ of $\mathscr{A}$ is *right adjoint* to f, if there exist 2-cells $\eta : \mathbf{1}_A \Rightarrow g \circ f$ and $\varepsilon : f \circ g \Rightarrow \mathbf{1}_B$ fulfilling the *triangular identities*
$$(g * \varepsilon) \odot (\eta * g) = \mathbf{1}_g \qquad (\varepsilon * f) \odot (f * \eta) = \mathbf{1}_f.$$
(see [28, Def.7.1.2]). In this case, we also say that g is *right adjoint* to f, and that (f, g) is an *adjoint pair of* 1*-cells*; moreover, we say that (η, ε) is an *adjunction* for (f, g). Then η is called the *unit* and ε the *counit* of the adjunction (η, ε).

(iii) We say that a 1-cell $f : A \to B$ of $\mathscr{A}$ is an *equivalence* if there exist 1-cells $g, h : B \to A$ and invertible 2-cells $\mathbf{1}_A \Rightarrow g \circ f$ and $f \circ h \Rightarrow \mathbf{1}_B$. In this case, we also say that g and h are *quasi-inverse* 1-cells for f.

Remark 2.1.4. (i) Let f, g and h be as in definition 2.1.3(iii), so we have invertible 2-cells $\eta : \mathbf{1}_A \Rightarrow g \circ f$ and $\varepsilon : f \circ h \Rightarrow \mathbf{1}_B$. Then we have the invertible 2-cell $\beta := (g * \varepsilon) \odot (h * \eta) : h \Rightarrow g$, whence the invertible 2-cell $\varepsilon \circ (f * \beta^{-1}) : f \circ g \Rightarrow \mathbf{1}_B$. In other words, we may as well assume that $g = h$ in definition 2.1.3(iii).

(ii) A composition of equivalences is an equivalence. Indeed, let $f_1 : A_1 \to A_2$ and $f_2 : A_2 \to A_3$ be two equivalences, $g_1 : A_2 \to A_1$ (resp. $g_2 : A_3 \to A_2$) a quasi-inverse for f_1 (resp. for f_2) so that we have invertible 2-cells $\alpha_i : \mathbf{1}_{A_i} \Rightarrow g_i \circ f_i$ and $\beta_i : f_i \circ g_i \Rightarrow \mathbf{1}_{A_{i+1}}$ for $i = 1, 2$. Then $g := g_1 \circ g_2$ is a quasi-inverse for $f := f_2 \circ f_1$, since we have the invertible 2-cells $(g_1 * \alpha_2 * f_1) \odot \alpha_1 : \mathbf{1}_{A_1} \Rightarrow g \circ f$ and $\beta_2 \odot (f_2 * \beta_1 * g_2) : f \circ g \Rightarrow \mathbf{1}_{A_3}$.

(iii) Let $f, f' : A \to B$ be two 1-cells of $\mathscr{A}$, and $\beta : f \xrightarrow{\sim} f'$ an invertible 2-cell. Then f is an equivalence if and only if the same holds for f'. Indeed, suppose we have a 1-cell $g : B \to A$ with an invertible 2-cells $\alpha : \mathbf{1}_A \xrightarrow{\sim} g \circ f$ and $\alpha' : f \circ g \xrightarrow{\sim} \mathbf{1}_B$; then we get the invertible 2-cells $(g * \beta) \odot \alpha : \mathbf{1}_A \xrightarrow{\sim} g \circ f'$ and $\alpha' \odot (\beta^{-1} * g) : f' \circ g \xrightarrow{\sim} \mathbf{1}_B$, so f' is an equivalence.

(iv) Likewise, if (f, g) is an adjoint pair of 1-cells of $\mathscr{A}$ and we have invertible 2-cells $\alpha : g \stackrel{\sim}{\to} g'$ and $\beta : f \stackrel{\sim}{\to} f'$, then (f', g') is an adjoint pair of 1-cells. Indeed, by assumption we have 2-cells $\eta : \mathbf{1}_A \Rightarrow g \circ f$ and $\varepsilon : f \circ g \Rightarrow \mathbf{1}_B$ fulfilling the triangular identities of definition 2.1.3(ii); it suffices to check that $\eta' := (\alpha * \beta) \odot \eta$ and $\varepsilon' := \varepsilon \odot (\beta * \alpha)^{-1}$ are a unit and a counit for the pair (f', g'). However, we have :

$$\begin{aligned}(g' * \varepsilon') \odot (\eta' * g') &= (g' * (\varepsilon \odot (\beta * \alpha)^{-1})) \odot (((\alpha * \beta) \odot \eta) * g') \\ &= (g' * \varepsilon) \odot (g' * \beta * \alpha)^{-1} \odot (\alpha * \beta * g') \odot (\eta * g') \\ &= (g' * \varepsilon) \odot (\alpha * f * \alpha^{-1}) \odot (\eta * g') \\ &= (g' * \varepsilon) \odot (\alpha * f * g) \odot (g * f * \alpha^{-1}) \odot (\eta * g') \\ &= \alpha \odot (g * \varepsilon) \odot (\eta * g) \odot \alpha^{-1} \\ &= \alpha \odot \alpha^{-1} = \mathbf{1}_{g'}\end{aligned}$$

where the third, fourth and fifith equalities follow from remark 2.1.1(i). Likewise one checks the other required triangular identity.

(v) Furthermore, if $f : A \to B$ is a 1-cell in $\mathscr{A}$ and $g, g' : B \to A$ are both right adjoint to f, then we have an invertible 2-cell $\alpha : g \stackrel{\sim}{\to} g'$. Indeed, let (η, ε) (resp. (η', ε')) be an adjunction for the pair (f, g) (resp. for the pair (f, g')). We set

$$\alpha := (g' * \varepsilon) \odot (\eta' * g) : g \Rightarrow g' \qquad \beta := (g * \varepsilon') \odot (\eta * g') : g' \Rightarrow g.$$

Applying repeatedly remark 2.1.1(i), we compute :

$$\begin{aligned}\beta \odot \alpha &= (g * \varepsilon') \odot (g * f * g' * \varepsilon) \odot (\eta * g' * f * g) \odot (\eta' * g) \\ &= (g * \varepsilon) \odot (g * \varepsilon' * f * g) \odot (g * f * \eta' * g) \odot (\eta * g) \\ &= (g * \varepsilon) \odot (g * ((\varepsilon' * f) \odot (f * \eta')) * g) \odot (\eta * g) \\ &= (g * \varepsilon) \odot (\eta * g) = \mathbf{1}_g.\end{aligned}$$

Likewise one checks that $\alpha \odot \beta = \mathbf{1}_{g'}$, whence the contention.

2.1.5. Let $\mathscr{A}$ be any 2-category; an *oriented square in $\mathscr{A}$* is a (not necessarily commutative) diagram of the type :

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & \swarrow_{\alpha} & \downarrow{\scriptstyle h} \\ A' & \xrightarrow[f']{} & B' \end{array} \tag{2.1.6}$$

where $f, f'g, h$ are any four 1-cells and $\alpha : h \circ f \Rightarrow f' \circ g$ is any 2-cell of $\mathscr{A}$. If α is an invertible 2-cell, we also say that the square (2.1.6) is *essentially commutative*. We say that α *orients* the square (2.1.6), and often we refer to such a square just by naming its orienting 2-cell; when dealing with complex diagrams, this shorthand may lead to unacceptable ambiguities, but in these cases usually one may resolve such ambiguities just by specifying two of the opposite sides of the square, for which we shall employ a fractional notation : thus, the square (2.1.6) shall be denoted, depending on the context, in either of the following three manners :

$$\alpha \qquad \frac{\alpha}{f|f'} \qquad \frac{\alpha}{g|h}.$$

One basic operation consists in combining two squares that share one edge, to obtain a new square, essentially by omitting the common edge of the squares. Namely, suppose we have two squares α and β as in the diagram :

(2.1.7)
$$\begin{array}{ccccc} A & \xrightarrow{g} & A' & \xrightarrow{k} & A'' \\ {\scriptstyle f}\downarrow & {}^{\alpha}\Nearrow & \downarrow{\scriptstyle f'} & {}^{\beta}\Nearrow & \downarrow{\scriptstyle f''} \\ B & \xrightarrow[h]{} & B' & \xrightarrow[i]{} & B'' \end{array}$$

Then we say that α and β are *vertically composable*, and we define the square

$$\beta \boxbar \alpha \qquad : \qquad \begin{array}{ccc} A & \xrightarrow{k\circ g} & A'' \\ {\scriptstyle f}\downarrow & {}^{\gamma}\Nearrow & \downarrow{\scriptstyle f''} \\ B & \xrightarrow[i\circ h]{} & B'' \end{array} \qquad \text{where } \gamma := (\beta * g) \odot (i * \alpha).$$

In this operation, we have joined to the square α of (2.1.6) another square β that shares the edge f'. Clearly, if β' is another square that shares with α the edge f, the same operation can be performed to obtain the square $\alpha \boxbar \beta'$ which omits the edge f. On the other hand, the rule to join to α a square that shares one of the two remaining edges g or h, is slightly different. Namely, suppose we have the diagram of two squares :

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{i} & C \\ {\scriptstyle g}\downarrow & \Swarrow_{\alpha} & \downarrow{\scriptstyle h} & \Swarrow_{\gamma} & \downarrow{\scriptstyle k} \\ A' & \xrightarrow[f']{} & B' & \xrightarrow[i']{} & C'. \end{array}$$

Then we say that α and γ are *horizontally composable*, and we set

$$\frac{\gamma}{i|i'} \boxminus \frac{\alpha}{f|f'} := \frac{(i' * \alpha) \odot (\gamma * f)}{i \circ f | i' \circ f'}.$$

However, notice that the second join operation is turned into the first one, when we replace $\mathscr{A}$ by its opposite 2-categories; namely, we have the identities :

(2.1.8)
$$(\gamma \boxminus \alpha)^o = \alpha^o \boxbar \gamma^o \qquad {}^o(\gamma \boxminus \alpha) = {}^o\gamma \boxbar {}^o\alpha$$

The following proposition establishes the two basic rules of the algebra of oriented squares.

Proposition 2.1.9. *With the notation of* (2.1.5)*, the following holds :*

(i) *The join operation is associative,* i.e. *for any diagram of three squares*

$$\begin{array}{ccccccc} A & \xrightarrow{g} & A' & \xrightarrow{k} & A'' & \xrightarrow{l} & A''' \\ {\scriptstyle f}\downarrow & {}^{\alpha}\Nearrow & \downarrow{\scriptstyle f'} & {}^{\beta}\Nearrow & \downarrow{\scriptstyle f''} & {}^{\gamma}\Nearrow & \downarrow{\scriptstyle f'''} \\ B & \xrightarrow[h]{} & B' & \xrightarrow[i]{} & B'' & \xrightarrow[m]{} & B''' \end{array}$$

we have

$$\gamma \boxbar (\beta \boxbar \alpha) = (\gamma \boxbar \beta) \boxbar \alpha.$$

(ii) *For every (not necessarily commutative) diagram*

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ {\scriptstyle h}\downarrow & \swarrow_{\alpha} & \downarrow{\scriptstyle k} & \swarrow_{\beta} & \downarrow{\scriptstyle i} \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' \\ {\scriptstyle h'}\downarrow & \swarrow_{\alpha'} & \downarrow{\scriptstyle k'} & \swarrow_{\beta'} & \downarrow{\scriptstyle i'} \\ A'' & \xrightarrow[f'']{} & B'' & \xrightarrow[g'']{} & C''. \end{array}$$

we have

$$(\beta' \boxbar \beta) \boxminus (\alpha' \boxbar \alpha) = (\beta' \boxminus \alpha') \boxbar (\beta \boxminus \alpha).$$

Proof. (i): We compute :

$$\begin{aligned} \gamma \boxbar (\beta \boxbar \alpha) &= \gamma \boxbar ((\beta * g) \odot (i * \alpha)) \\ &= ((\gamma * (k \circ g)) \odot (m * ((\beta * g) \odot (i * \alpha))) \\ &= ((\gamma * (k \circ g)) \odot (m * (\beta * g)) \odot (m * (i * \alpha)) \\ &= (((\gamma * k) \odot (m * \beta)) * g) \odot ((m \circ i) * \alpha) \\ &= (\gamma \boxbar \beta) \boxbar \alpha. \end{aligned}$$

(ii): We compute :

$$\begin{aligned} (\beta' \boxbar \beta) \boxminus (\alpha' \boxbar \alpha) &= (g'' * ((\alpha' * h) \odot (k' * \alpha))) \odot (((\beta' * k) \odot (i' * \beta)) * f) \\ &= (g'' * \alpha' * h) \odot (g'' * k' * \alpha) \odot (\beta' * k * f) \odot (i' * \beta * f) \\ &= (g'' * \alpha' * h) \odot (\beta' * f * h) \odot (i' * g' * \alpha) \odot (i' * \beta * f) \\ &= ((\beta' \boxminus \alpha') * h) \odot (i' * (\beta \boxminus \alpha)) \\ &= (\beta' \boxminus \alpha') \boxbar (\beta \boxminus \alpha) \end{aligned}$$

where the fourth equality follows from the commutation rule of remark 2.1.1(i). □

Remark 2.1.10. Proposition 2.1.5(i) establishes explicitly only the associativity of the join operation $\boxbar$, but in light of (2.1.8) we also get immediately the associativity of $\boxminus$.

Example 2.1.11. (i) Let $\mathscr{A}$ be any 2-category. We may construct a new 2-category

$$\mathsf{2\text{-}Morph}(\mathscr{A})$$

which is the 2-categorical counterpart of the category of arrows (see (1.1.30)). Namely :

- the objects of $\mathsf{2\text{-}Morph}(\mathscr{A})$ are the 1-cells of $\mathscr{A}$.
- If $f : A \to B$ and $f' : A' \to B'$ are any two 1-cells of $\mathscr{A}$, the arrows $f \to f'$ in $\mathsf{2\text{-}Morph}(\mathscr{A})$ are all the oriented squares (2.1.6).
- For 1-cells $(g_1, h_1, \alpha_1), (g_2, h_2, \alpha_2) : f \to f'$, the 2-cells $(g_1, h_1, \alpha_1) \Rightarrow (g_2, h_2, \alpha_2)$ are all the pairs

$$(\beta : g_1 \Rightarrow g_2, \gamma : h_1 \Rightarrow h_2) \qquad \text{such that} \qquad \frac{\alpha_1}{h_1|g_1} \boxminus \frac{\beta}{g_1|g_2} = \frac{\gamma}{h_1|h_2} \boxminus \frac{\alpha_2}{h_2|g_2}.$$

- For f as in the foregoing, the unit functor $\mathbf{1} \to \mathsf{2\text{-}Morph}(f, f)$ is given by the 2-cell

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle \mathbf{1}_A}\downarrow & \swarrow_{\mathbf{1}_f} & \downarrow{\scriptstyle \mathbf{1}_B} \\ A & \xrightarrow[f]{} & B \end{array} \tag{2.1.12}$$

and its identity endomorphism (i_A, i_B).

For f and f' as in the foregoing, the composition law in 2-$\mathsf{Morph}(\mathscr{A})(f, f')$ is given by the rule

$$(\beta', \gamma') \odot (\beta, \gamma) := (\beta' \odot \beta, \gamma' \odot \gamma)$$

for every composable pair $((\beta, \gamma), (\beta', \gamma'))$ of 2-cells in 2-$\mathsf{Morph}(\mathscr{A})$. The composition bifunctor is given by the composition law for squares; *i.e.*, given 1-cells $f \to f$' and $f' \to f''$ in 2-$\mathsf{Morph}(\mathscr{A})$ as in (2.1.7), we set

$$(\beta, k, i) \circ (\alpha, g, h) := (\beta \boxdot \alpha, k \circ g, i \circ h).$$

Lastly, given 2-cells (β, γ) in 2-$\mathsf{Morph}(\mathscr{A})(f, f')$ and (β', γ') in 2-$\mathsf{Morph}(\mathscr{A})(f', f'')$ we set

$$(\beta', \gamma') * (\beta, \gamma) := (\beta' * \beta, \gamma' * \gamma).$$

The functoriality and associativity properties of the composition laws for 2-cells are then immediate from the definitions. The associativity of the composition law for 1-cells holds by proposition 2.1.9(i).

(ii) Just as in (1.1.24), for every $X \in \mathrm{Ob}(\mathscr{A})$ we may consider the 2-subcategories $\mathscr{A}/X$ and $X/\mathscr{A}$ of 2-$\mathsf{Morph}(\mathscr{A})$. Explicitly, the objects of $\mathscr{A}/X$ are the 1-cells $A \to X$ of $\mathscr{A}$. Given two such objects f, f', the 1-cells $f \to f'$ of $\mathscr{A}/X$ are all the diagrams in $\mathscr{A}$ of the type

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle g}\downarrow & \swarrow_{\alpha} & \| \\ A' & \xrightarrow[f']{} & X \end{array}$$

and for 1-cells $(g_1, \alpha_1), (g_2, \alpha_2) : f \to f'$, the 2-cells of $\mathscr{A}/X$ are the 2-cells $\beta : g_1 \Rightarrow g_2$ of $\mathscr{A}$ such that $(f' * \beta) \odot \alpha_1 = \alpha_2$. A similar description applies to $X/\mathscr{A}$.

(iii) To every 1-cell $h : X \to Y$ in $\mathscr{A}$ we attach functors

$$h^Z_* : \mathscr{A}(Z, X) \to \mathscr{A}(Z, Y) \qquad \text{for every } Z \in \mathrm{Ob}(\mathscr{A})$$

by ruling that $h^Z_*(k) := h \circ k$ and $h^Z_*(\beta) := h * \beta$ for every 1-cell $k : Z \to X$ of $\mathscr{A}$ and every morphism $\beta : k \Rightarrow k'$ in $\mathscr{A}(Z, X)$. Let $\nu : h \Rightarrow h'$ be any 2-cell in $\mathscr{A}$; we claim that the rule

$$k \mapsto (\nu^Z_{*k} := \nu * k : h^Z_*(k) \Rightarrow h'^Z_*(k)) \qquad \text{for every } (k : Z \to X) \in \mathrm{Ob}(\mathscr{A}(Z, X))$$

defines a natural transformation

$$\nu^Z_* : h^Z_* \Rightarrow h'^Z_* \qquad \text{for every } Z \in \mathrm{Ob}(\mathscr{A}).$$

Indeed, let $k, k' : Z \to X$ be any two objects of $\mathscr{A}(Z, X)$, and $\mu : k \Rightarrow k'$ a 2-cell; we need to show that the resulting diagram commutes in $\mathscr{A}$:

$$\begin{array}{ccc} h^Z_*(k) & \xrightarrow{\nu * k} & h'^Z_*(k) \\ {\scriptstyle h*\mu}\Downarrow & & \Downarrow{\scriptstyle h'*\mu} \\ h^Z_*(k') & \xrightarrow{\nu * k'} & h'^Z_*(k') \end{array}$$

But this follows directly from remark 2.1.1(i). Likewise, we may attach to f the functors

$$h^*_Z : \mathscr{A}(Y, Z) \to \mathscr{A}(X, Z) \qquad k \mapsto k \circ h \qquad \beta \mapsto \beta * h \qquad \text{for every } Z \in \mathrm{Ob}(\mathscr{A})$$

and then ν induces as well a natural transformation

$$\nu^*_Z : h^*_Z \Rightarrow h'^*_Z \qquad k \mapsto k * \nu \qquad \text{for every } Z \in \mathrm{Ob}(\mathscr{A}).$$

We apply these constructions to prove the following result, which generalizes propositions 1.1.15(ii) and 1.1.20(i) to arbitrary 2-categories :

Lemma 2.1.13. *Let $\mathscr{A}$ be a 2-category, $f : A \to B$, $g : B \to A$ two 1-cells in $\mathscr{A}$. We have :*

(i) *Suppose that f is an equivalence, and g a quasi-inverse for f. Then, for every invertible 2-cell $\eta : \mathbf{1}_A \Rightarrow g \circ f$ there exists a unique invertible 2-cell $\varepsilon : f \circ g \Rightarrow \mathbf{1}_B$ such that (η, ε) is an adjunction for the pair (f, g).*

(ii) *Suppose that (f, g) is an adjoint pair, and that (η, ε) and (η', ε') are two adjunctions for (f, g). Then $\eta = \eta'$ if and only if $\varepsilon = \varepsilon'$.*

(iii) *The following conditions are equivalent :*
  (a) *f is an equivalence.*
  (b) *For every $Z \in \mathrm{Ob}(\mathscr{A})$ the functor $f^*_Z : \mathscr{A}(B, Z) \to \mathscr{A}(A, Z)$ is an equivalence.*
  (c) *For every $Z \in \mathrm{Ob}(\mathscr{A})$ the functor $f^Z_* : \mathscr{A}(Z, A) \to \mathscr{A}(Z, B)$ is an equivalence.*

Proof. (i): By assumption, there exists an invertible 2-cell $\varepsilon' : f \circ g \Rightarrow \mathbf{1}_B$. Now, since η and ε' are invertible 2-cells, it is clear that η^Z_* and ε'^Z_* are isomorphisms of functors for every $Z \in \mathrm{Ob}(\mathscr{A})$, so f^Z_* and g^Z_* are equivalences of categories (proposition 1.1.20(i)); therefore there exists a unique adjunction for the pair (f^Z_*, g^Z_*) whose unit is η^Z_* (claim 1.1.21) and we denote by ε^Z_* the unique counit for this adjunction. Then also ε^Z_* is an isomorphism of functors (proposition 1.1.20(iii)), and we define the invertible 2-cell $\varepsilon := \varepsilon^B_{\mathbf{1}_B} : f \circ g \Rightarrow \mathbf{1}_B$. Lastly, the triangular identities (1.1.13) for the pair $(\eta^Z_*, \varepsilon^Z_*)$ immediately imply the corresponding identities for (η, ε).

(ii): As in the foregoing, we obtain for every $Z \in \mathrm{Ob}(\mathscr{A})$ an adjoint pair of functors (f^Z_*, g^Z_*), and two adjunctions $(\eta^Z_*, \varepsilon^Z_*)$, $(\eta'^Z_*, \varepsilon'^Z_*)$ for (f^Z_*, g^Z_*). Then proposition 1.1.15(ii) implies that $\eta^Z_* = \eta'^Z_*$ if and only if $\varepsilon^Z_* = \varepsilon'^Z_*$. The assertion is an immediate consequence.

(iii.a)⇒(iii.b),(iii.c): Indeed, let $g : B \to A$ be a quasi-inverse for f; it is easily seen that the functor g^*_Z is a quasi-inverse for f^*_Z and g^Z_* is a quasi-inverse for f^Z_*.

(iii.b)⇒(iii.a): Since f^*_A is an equivalence, we find a 1-cell $g : B \to A$ of $\mathscr{A}$ with an isomorphism $g \circ f = f^*_A(g) \xrightarrow{\sim} \mathbf{1}_A$. There follows an isomorphism $f^*_B(f \circ g) = f \circ g \circ f \xrightarrow{\sim} f = f^*_B(\mathbf{1}_B)$, and since f^*_B is an equivalence, we deduce an isomorphism $f \circ g \xrightarrow{\sim} \mathbf{1}_B$, whence the contention. Likewise one shows that (iii.c)⇒(iii.a), or alternatively one can deduce this implication from the foregoing one, by considering the opposite 2-categories. □

Definition 2.1.14. ([28, Def.7.5.1]) Let $\mathscr{A}$ and $\mathscr{B}$ be two 2-categories.

(i) A *pseudo-functor* $F : \mathscr{A} \to \mathscr{B}$ is the datum of :

- For every $A \in \mathrm{Ob}(\mathscr{A})$, an object $FA \in \mathrm{Ob}(\mathscr{B})$.
- For every $A, B \in \mathrm{Ob}(\mathscr{A})$, a functor :

$$F_{AB} : \mathscr{A}(A, B) \to \mathscr{B}(FA, FB).$$

  We shall often omit the subscript, and write only Ff instead of $F_{AB}f : FA \to FB$, for a 1-cell $f : A \to B$, and likewise for 2-cells.

- For every $A, B, C \in \mathrm{Ob}(\mathscr{A})$, a natural isomorphism γ_{ABC} between two functors $\mathscr{A}(A, B) \times \mathscr{A}(B, C) \to \mathscr{B}(FA, FC)$ as indicated by the (not necessarily commutative) diagram :

$$\begin{array}{ccc} \mathscr{A}(A,B) \times \mathscr{A}(B,C) & \xrightarrow{c_{ABC}} & \mathscr{A}(A,C) \\ {\scriptstyle F_{AB}\times F_{BC}}\downarrow & \gamma_{ABC} \nearrow & \downarrow{\scriptstyle F_{AC}} \\ \mathscr{B}(FA,FB) \times \mathscr{B}(FB,FC) & \xrightarrow[c_{FA,FB,FC}]{} & \mathscr{B}(FA,FC). \end{array} \tag{2.1.15}$$

  To ease notation, for every $(f, g) \in \mathscr{A}(A, B) \times \mathscr{A}(B, C)$, we shall write $\gamma_{f,g}$ instead of $(\gamma_{ABC})_{(f,g)} : c_{FA,FB,FC}(F_{AB}f, F_{BC}g) \Rightarrow F_{AC}(c_{ABC}(f, g))$.

- For every $A \in \mathrm{Ob}(\mathscr{A})$, a natural isomorphism δ_A between functors $\mathbf{1} \to \mathscr{B}(FA, FA)$, as indicated by the (not necessarily commutative) diagram :

$$\begin{array}{ccc} \mathbf{1} & \xrightarrow{u_A} & \mathscr{A}(A,A) \\ \| & \delta_A \nearrow & \downarrow{\scriptstyle F_{AA}} \\ \mathbf{1} & \xrightarrow[u_{FA}]{} & \mathscr{B}(FA,FA). \end{array} \tag{2.1.16}$$

The system $(\delta_\bullet, \gamma_{\bullet\bullet\bullet})$ is called the *coherence constraint* for F. This datum is required to satisfy:

- A *composition axiom*, which says that the diagram

$$\begin{array}{ccc} Fh \circ Fg \circ Ff & \xRightarrow{Fh*\gamma_{f,g}} & Fh \circ F(g \circ f) \\ {\scriptstyle \gamma_{g,h}*Ff}\Downarrow & & \Downarrow{\scriptstyle \gamma_{g\circ f,h}} \\ F(h \circ g) \circ Ff & \xRightarrow{\gamma_{f,h\circ g}} & F(h \circ g \circ f) \end{array}$$

  commutes for every sequence of arrows $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ in $\mathscr{A}$. We will write

$$\gamma_{f,g,h} : Fh \circ Fg \circ Ff \Rightarrow F(h \circ g \circ f)$$

  for the common composition of these two pairs of cells.

- A *unit axiom*, which says that the diagrams :

$$\begin{array}{ccc} Ff \circ \mathbf{1}_{FA} & \xRightarrow{Ff*\delta_A} & Ff \circ F\mathbf{1}_A \\ {\scriptstyle \mathbf{1}_{Ff}}\Downarrow & & \Downarrow{\scriptstyle \gamma_{\mathbf{1}_A,f}} \\ Ff & \xRightarrow{\mathbf{1}_{Ff}} & F(f \circ \mathbf{1}_A) \end{array} \qquad \begin{array}{ccc} \mathbf{1}_{FB} \circ Ff & \xRightarrow{\delta_B*Ff} & F(\mathbf{1}_B) \circ Ff \\ {\scriptstyle \mathbf{1}_{Ff}}\Downarrow & & \Downarrow{\scriptstyle \gamma_{f,\mathbf{1}_B}} \\ Ff & \xRightarrow{\mathbf{1}_{Ff}} & F(\mathbf{1}_B \circ f) \end{array}$$

  commute for every arrow $f : A \to B$ (where, to ease notation, we have written δ_A instead of $(\delta_A)_* : \mathbf{1}_{FA} \Rightarrow F_{AA}\mathbf{1}_A$, and likewise for δ_B).

(ii) A pseudo-functor F as in (i) is called *strict* if (2.1.15) and (2.1.16) both commute, and γ_{ABC} and δ_A are the identity natural transformations for every $A, B, C \in \mathrm{Ob}(\mathscr{A})$.

Remark 2.1.17. With the notation of definition 2.1.14, we have :

(i) The functoriality of F_{AB}, for every $A, B \in \mathrm{Ob}(\mathscr{A})$, comes down to the identities

$$F(\mathbf{1}_f) = \mathbf{1}_{Ff} \qquad F(\beta \odot \alpha) = F\beta \odot F\alpha \qquad \text{for every 1-cell } f \text{ and 2-cells } \alpha, \beta \text{ of } \mathscr{A}(A,B).$$

(ii) Likewise, the naturality of γ_{ABC} boils down to the commutativity of the diagram

$$\begin{array}{ccc} Fg \circ Ff & \xRightarrow{\gamma_{f,g}} & F(g \circ f) \\ {\scriptstyle F\beta*F\alpha}\Downarrow & & \Downarrow{\scriptstyle F(\beta*\alpha)} \\ Fg' \circ Ff' & \xRightarrow{\gamma_{f',g'}} & F(g' \circ f') \end{array}$$

for every 1-cells $f, f' : A \to B$, $g, g' : B \to C$ and 2-cells $\alpha : f \Rightarrow f'$, $\beta : g \Rightarrow g'$.

(iii) If $h : C \to D$ and $h' : C' \to D'$ are any two other 1-cells, and $\mu : h \Rightarrow h'$ is any 2-cell, it follows easily that the diagram

$$\begin{array}{ccc} Fh \circ Fg \circ Ff & \xRightarrow{\gamma_{f,g,h}} & F(h \circ g \circ f) \\ {\scriptstyle F\mu*F\beta*F\alpha}\Downarrow & & \Downarrow{\scriptstyle F(\mu*\beta*\alpha)} \\ Fh' \circ Fg' \circ Ff' & \xRightarrow{\gamma_{f',g',h'}} & F(h' \circ g' \circ f') \end{array}$$

commutes : details left to the reader.

(iv) The pseudo-functor F induces *opposite pseudo-functors*

$$F^o : \mathscr{A}^o \to \mathscr{B}^o \qquad \text{and} \qquad {}^oF : {}^o\mathscr{A} \to {}^o\mathscr{B}$$

(as well as the functor ${}^oF^o$, by combining the two operations). Namely, F^o is given by the rules

$$F^o(A^o) := (FA)^o \qquad F^o(f^o) := (Ff)^o \qquad F^o(\beta^o) := (F\beta)^o$$

for every $A \in \mathrm{Ob}(\mathscr{A})$, every 1-cell f and every 2-cell β of $\mathscr{A}$. The coherence constraint of F^o is the pair (γ^o, δ^o) such that

$$\gamma^o_{g^o,f^o} := (\gamma_{f,g})^o \qquad \text{and} \qquad \delta^o_{A^o} := (\delta_A)^o$$

for every composable pair of 1-cells f, g of $\mathscr{A}$, and every $A \in \mathrm{Ob}(\mathscr{A})$. The composition and unit axioms for F^o follow immediately from the same for F. Likewise, ${}^oF$ is given by the rules

$${}^oF({}^oA) := {}^o(FA) \qquad {}^oF({}^of) := {}^o(Ff) \qquad {}^oF({}^o\beta) := {}^o(F\beta)$$

for every A, f, β as in the foregoing. The coherence constraint for ${}^oF$ is the pair $({}^o\gamma, {}^o\delta)$ with

$${}^o\gamma_{{}^of,{}^og} := {}^o(\gamma_{f,g})^{-1} \qquad \text{and} \qquad {}^o\delta_{{}^oA} := {}^o(\delta_A)^{-1}$$

whose composition and unit axioms are again derived straightforwardly from the same for F.

(v) Let $\mathscr{C}$ be a third 2-category, and $G : \mathscr{B} \to \mathscr{C}$ another pseudo-functor. We may define a composition $G \circ F : \mathscr{A} \to \mathscr{B}$, which is the pseudo-functor such that

$$G \circ F(A) := G(FA) \qquad \text{and} \qquad (G \circ F)_{AB} := G_{FA,FB} \circ F_{AB} \qquad \text{for every } A, B \in \mathrm{Ob}(\mathscr{A}).$$

Denote by (δ^F, γ^F) and (δ^G, γ^G) the coherence constraints of F and respectively G; the coherence constraint of $G \circ F$ is then the pair $(\delta^{G\circ F}, \gamma^{G\circ F})$ such that

$$\delta^{G\circ F}_A := G(\delta^F_A) \odot \delta^G_{FA} \qquad \gamma^{G\circ F}_{f,g} := G(\gamma^F_{f,g}) \odot \gamma^G_{Ff,Fg}$$

for every $A, B, C \in \mathrm{Ob}(\mathscr{A})$ and every 1-cell f of $\mathscr{A}(A, B)$ and g of $\mathscr{A}(B, C)$. Let us verify the composition axiom for $\gamma^{G\circ F}$: given a composable sequence of 1-cells f, g, h, we have

$$\begin{aligned}
\gamma^{G\circ F}_{g\circ f,h} \odot (GFh * \gamma^{G\circ F}_{f,g}) &= G(\gamma^F_{g\circ f,h}) \odot \gamma^G_{F(g\circ f),Fh} \odot (GFh * (G(\gamma^F_{f,g}) \odot \gamma^G_{Ff,Fg})) \\
&= G(\gamma^F_{g\circ f,h}) \odot \gamma^G_{F(g\circ f),Fh} \odot (GFh * G(\gamma^F_{f,g})) \odot (GFh * \gamma^G_{Ff,Fg}) \\
&= G(\gamma^F_{g\circ f,h}) \odot G(Fh * \gamma^F_{f,g}) \odot \gamma^G_{Fg\circ Ff,Fh} \odot (GFh * \gamma^G_{Ff,Fg}) \\
&= G(\gamma^F_{g\circ f,h} \odot (Fh * \gamma^F_{f,g})) \odot \gamma^G_{Ff,Fh\circ Fg} \odot (\gamma^G_{Fg,Fh} * GFf) \\
&= G(\gamma^F_{f,h\circ g} \odot (\gamma^F_{g,h} * Ff)) \odot \gamma^G_{Ff,Fh\circ Fg} \odot (\gamma^G_{Fg,Fh} * GFf) \\
&= G(\gamma^F_{f,h\circ g}) \odot G(\gamma^F_{g,h} * Ff) \odot \gamma^G_{Ff,Fh\circ Fg} \odot (\gamma^G_{Fg,Fh} * GFf) \\
&= G(\gamma^F_{f,h\circ g}) \odot \gamma^G_{Ff,F(h\circ g)} \odot (G(\gamma^F_{g,h}) * GFf) \odot (\gamma^G_{Fg,Fh} * GFf) \\
&= \gamma^{G\circ F}_{f,h\circ g} \odot ((G(\gamma^F_{g,h}) \odot \gamma^G_{Fg,Fh}) * GFf) \\
&= \gamma^{G\circ F}_{f,h\circ g} \odot (\gamma^{G\circ F}_{g,h} * GFf)
\end{aligned}$$

where the third and seventh equalities follow from (ii) (applied to G), the fourth from (i) and the composition axiom for G, the fifth from the composition axiom for F, the sixth from (i).

Next, to check the unit axiom, we compute :

$$\begin{aligned}\gamma^{G\circ F}_{\mathbf{1}_A,f} \odot (GFf * \delta^{G\circ F}_A) &= G(\gamma^F_{\mathbf{1}_A,f}) \odot \gamma^G_{F\mathbf{1}_A,Ff} \odot (GFf * (G(\delta^F_A) \odot \delta^G_{FA})) \\ &= G(\gamma^F_{\mathbf{1}_A,f}) \odot \gamma^G_{F\mathbf{1}_A,Ff} \odot (GFf * G(\delta^F_A)) \odot (GFf * \delta^G_{FA}) \\ &= G(\gamma^F_{\mathbf{1}_A,f}) \odot G(Ff * \delta^F_A) \odot \gamma^G_{\mathbf{1}_{FA},Ff} \odot (GFf * \delta^G_{FA}) \\ &= G(\gamma^F_{\mathbf{1}_A,f} \odot (Ff * \delta^F_A)) \\ &= G(\mathbf{1}_{Ff}) \\ &= \mathbf{1}_{GFf}\end{aligned}$$

where the third equality follows from (ii), the fourth from (i) and the unit axiom for G, and the fifth from the unit axiom for F. Similarly one verifies the commutativity of the remaining diagram required for the unit axiom : details left to the reader.

(vi) Also, if $H : \mathscr{C} \to \mathscr{D}$ is any other pseudo-functor, then $H \circ (G \circ F) = (H \circ G) \circ F$. Indeed, taking into account (i) we may compute :

$$\begin{aligned}\gamma^{H\circ(G\circ F)}_{f,g} &= H(\gamma^{G\circ F}_{f,g}) \odot \gamma^H_{G\circ Ff,G\circ Fg} \\ &= H\circ G(\gamma^F_{f,g}) \odot H(\gamma^G_{Ff,Gf}) \odot \gamma^H_{G\circ Ff,G\circ Fg} \\ &= H\circ G(\gamma^F_{f,g}) \odot \gamma^{H\circ G}_{Ff,Gf} \\ &= \gamma^{(H\circ G)\circ F}_{f,g}.\end{aligned}$$

Similarly one checks that $\delta^{H\circ(G\circ F)}_A = \delta^{(H\circ G)\circ F}_A$ for every $A \in \mathrm{Ob}(\mathscr{A})$, whence the assertion.

(vii) Notice as well that if F and G are strict, then the same holds for $G \circ F$.

Remark 2.1.18. (i) Remark 2.1.17(v) shows that the set of all U-small 2-categories together with all pseudo-functors between them forms a category with small Hom-sets that we denote

$$\mathsf{U}\text{-}2\text{-}\mathbf{Cat}$$

(and as always, we shall usually drop the universe U from the notation). We do not know whether 2-$\mathbf{Cat}$ is a complete category : for instance, if $F : \mathscr{A} \to \mathscr{B}$ and $F' : \mathscr{A}' \to \mathscr{B}$ are any two pseudo-functors, it is not clear whether the fibre product of F and F' is always representable in 2-$\mathbf{Cat}$. However, in case F and F' are both strict, this fibre product is indeed representable by a 2-category that we denote

$$\mathscr{A} \times_{(F,F')} \mathscr{A}'$$

(or simply $\mathscr{A} \times_{\mathscr{B}} \mathscr{A}'$, if the notation is not ambiguous). Namely, the objects of $\mathscr{A} \times_{\mathscr{B}} \mathscr{A}'$ are all the pairs (A, A') consisting of objects A of $\mathscr{A}$ and A' of $\mathscr{A}'$ such that $FA = F'A'$, and the Hom-categories are given by the rule :

$$\mathscr{A} \times_{\mathscr{B}} \mathscr{A}'((A_1, A'_1), (A_2, A'_2)) := \mathscr{A}(A_1, A_2) \times_{\mathscr{B}(FA_1,FA_2)} \mathscr{A}(A_2, A'_2)$$

for every pair of objects $(A_1, A'_1), (A_2, A'_2)$ of $\mathscr{A} \times_{\mathscr{B}} \mathscr{A}'$ (see example 1.2.25(i)). The composition laws and unit functors for $\mathscr{A} \times_{\mathscr{B}} \mathscr{A}'$ are deduced from those of $\mathscr{A}$ and $\mathscr{A}'$, in the obvious fashion. A universal cone for this fibre product is provided by the two projections $\mathscr{A} \leftarrow \mathscr{A} \times_{\mathscr{B}} \mathscr{A}' \to \mathscr{A}'$, *i.e.* the strict pseudo-functors defined in the obvious fashion.

(ii) For instance, just as in (1.1.30), for every 2-category $\mathscr{A}$ we have two natural *source* and *target* strict pseudo-functors :

$$\mathscr{A} \xleftarrow{\mathsf{s}} 2\text{-}\mathsf{Morph}(\mathscr{A}) \xrightarrow{\mathsf{t}} \mathscr{A}$$

(notation of example 2.1.11(i)). Namely, s assigns to every 1-cell $f : A \to B$ the object A of $\mathscr{A}$, to every diagram (2.1.6) the 1-cell $g : \mathsf{s}(f) \to \mathsf{s}(f')$ of $\mathscr{A}$, and to every 2-cell (β, γ) of

$\mathsf{2\text{-}Morph}(\mathscr{A})$, the 2-cell β of $\mathscr{A}$. The reader may likewise spell out the definition of t. Then we have as well a natural *composition* strict pseudo-functor

$$\mathsf{c} : \mathsf{2\text{-}Morph}(\mathscr{A}) \times_{(\mathsf{t},\mathsf{s})} \mathsf{2\text{-}Morph}(\mathscr{A}) \to \mathsf{2\text{-}Morph}(\mathscr{A})$$

which assigns :

- to every pair of 1-cells (f, g) such that $\mathsf{t}(f) = \mathsf{s}(g)$, the composition $\mathsf{c}(f, g) := g \circ f$
- to every pair of 1-cells of $\mathsf{2\text{-}Morph}(\mathscr{A})$:

(2.1.19)
$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ {\scriptstyle h}\big\downarrow & \not\Swarrow_{\alpha} & \big\downarrow{\scriptstyle k} & \not\Swarrow_{\beta} & \big\downarrow{\scriptstyle i} \\ A' & \xrightarrow[f']{} & B' & \xrightarrow[g']{} & C' \end{array}$$

  the 1-cell of $\mathsf{2\text{-}Morph}(\mathscr{A})$

$$\mathsf{c}((\alpha, h, k), (\beta, k, i)) := (\beta \boxminus \alpha, h, i)$$

- for $(f, g), (f', g')$ as in (2.1.19), and every 2-cells $(\varphi_1, \varphi_2) : (\alpha_1, h_1, k_1) \Rightarrow (\alpha_2, h_2, k_2)$ in $\mathsf{2\text{-}Morph}(f, f')$ and $(\varphi_2, \varphi_3) : (\beta_1, k_1, i_1) \Rightarrow (\beta_2, k_2, i_2)$ in $\mathsf{2\text{-}Morph}(g, g')$, the pair

$$\mathsf{c}((\varphi_1, \varphi_2), (\varphi_2, \varphi_3)) := (\varphi_1, \varphi_3)$$

  which is a well defined 2-cell $\mathsf{c}((\alpha_1, h_1, k_1), (\beta_1, k_1, i_1)) \Rightarrow \mathsf{c}((\alpha_2, h_2, k_2), (\beta_2, k_2, i_2))$ in $\mathsf{2\text{-}Morph}(g \circ f, g' \circ f')$, since by assumption we have

$$\frac{\alpha_1}{h_1|g_1} \boxminus \frac{\varphi_1}{g_1|g_2} = \frac{\varphi_2}{h_1|h_2} \boxminus \frac{\alpha_2}{h_2|g_2} \qquad \frac{\beta_1}{k_1|h_1} \boxminus \frac{\varphi_2}{h_1|h_2} = \frac{\varphi_3}{k_1|k_2} \boxminus \frac{\beta_2}{k_2|h_2}$$

  and therefore

$$\begin{aligned}(\beta_1 \boxminus \alpha_1) \boxminus \varphi_1 &= \beta_1 \boxminus (\alpha_1 \boxminus \varphi_1) \\ &= \beta_1 \boxminus (\varphi_2 \boxminus \alpha_2) \\ &= (\beta_1 \boxminus \varphi_2) \boxminus \alpha_2 \\ &= (\varphi_3 \boxminus \beta_2) \boxminus \alpha_2 \\ &= \varphi_3 \boxminus (\beta_2 \boxminus \alpha_2).\end{aligned}$$

With these rules, the functoriality of c for 2-cells is immediate. To check the functoriality of c for 1-cells, consider a diagram as in proposition 2.1.9(ii), and to ease notation, set $\underline{\alpha} := (\alpha, h, k)$ and define likewise $\underline{\alpha}'$, $\underline{\beta}$ and $\underline{\beta}'$. Then proposition 2.1.9(ii) translates as the identity

$$\mathsf{c}(\underline{\alpha}' \circ \underline{\alpha}, \underline{\beta}' \circ \underline{\beta}) = \mathsf{c}(\underline{\alpha}', \underline{\beta}') \circ \mathsf{c}(\underline{\alpha}, \underline{\beta})$$

as required.

Example 2.1.20. (i) Let $\mathscr{A}$, $\mathscr{B}$ be two 2-categories, and $F : \mathscr{A} \to \mathscr{B}$ any pseudo-functor, with coherence constraint (γ^F, δ^F). Then F induces a pseudo-functor

$$\mathsf{2\text{-}Morph}(F) : \mathsf{2\text{-}Morph}(\mathscr{A}) \to \mathsf{2\text{-}Morph}(\mathscr{B})$$

by the following rule. To every 1-cell $f : A \to B$ in $\mathscr{A}$ we assign the 1-cell $Ff : FA \to FB$. To every diagram (2.1.6) we assign the diagram

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ {\scriptstyle Fg}\big\downarrow & \not\Swarrow_{\alpha^F} & \big\downarrow{\scriptstyle Fh} \\ FA' & \xrightarrow[Ff']{} & FB' \end{array} \qquad \text{where } \alpha^F := (\gamma^F_{g,f'})^{-1} \odot F\alpha \odot \gamma^F_{f,h}$$

and to any 2-cell $(\beta, \delta) : (g_1, h_1, \alpha_1) \Rightarrow (g_2, h_2, \alpha_2)$ in 2-$\mathsf{Morph}(f, f')$, we assign the pair $(F\beta, F\delta)$. Indeed, let us check that the latter is a well defined 2-cell in 2-$\mathsf{Morph}(Ff, Ff')$; the assertion boils down to the identity :

$$(\gamma^F_{g_2,f'})^{-1} \odot F\alpha_2 \odot \gamma^F_{f,h_2} \odot (F\delta * Ff) = (Ff' * F\beta) \odot (\gamma_{g_1,f'})^{-1} \odot F\alpha_1 \odot \gamma^F_{f,h_1}.$$

However, from remark 2.1.17(ii) we get

$$(Ff' * F\beta) \odot (\gamma^F_{g_1,f'})^{-1} = (\gamma^F_{g_2,f'})^{-1} \odot F(f' * \beta)$$

hence it suffices to check that

$$F\alpha_2 \odot \gamma^F_{f,h_2} \odot (F\delta * Ff) = F(f' * \beta) \odot F\alpha_1 \odot \gamma^F_{f,h_1}.$$

But by assumption, we know that $(f' * \beta) \odot \alpha_1 = \alpha_2 \odot (\delta * f)$, so – taking into account remark 2.1.17(i) – we are further reduced to showing that

$$\gamma^F_{f,h_2} \odot (F\delta * Ff) = F(\delta * f) \odot \gamma^F_{f,h_1}$$

which follows again from remark 2.1.17(ii). Lastly, the coherence constraint of 2-$\mathsf{Morph}(F)$ assigns to any diagram (2.1.12) (resp. (2.1.7)) the pair

(2.1.21) $$(\delta^F_A, \delta^F_B) \qquad \text{(resp. } (\gamma^F_{g,k}, \gamma^F_{h,i}) \text{).}$$

Indeed, let us check that (δ^F_A, δ^F_B) is a well defined 2-cell in 2-$\mathsf{Morph}(Ff, Ff)$: the assertion boils down to the identity

$$(Ff * \delta^F_A) \odot \mathbf{1}^F_f = \mathbf{1}^F_f \odot (\delta^F_B * Ff) \qquad \text{where} \qquad \mathbf{1}^F_f := (\gamma^F_{\mathbf{1}_A,f})^{-1} \odot \gamma^F_{f,\mathbf{1}_B}$$

which in turns follows easily from the unit axiom for δ^F. Likewise, the assertion that $(\gamma^F_{g,k}, \gamma^F_{h,i})$ is a well defined 2-cell in 2-$\mathsf{Morph}(Ff, Ff'')$ comes down to the identity

(2.1.22) $$\gamma^F_{g,k} \boxminus (\beta^F \boxed{} \alpha^F) = (\beta \boxed{} \alpha)^F \boxminus \gamma^F_{h,i}.$$

However, by definition, the left-hand side of (2.1.22) equals

$$(Ff'' * \gamma^F_{g,k}) \odot ((\gamma^F_{k,f''})^{-1} * Fg) \odot (F\beta * Fg) \odot (\gamma^F_{f',i} * Fg) \odot (Fi * (\gamma^F_{g,f'})^{-1}) \odot (Fi * (F\alpha \odot \gamma^F_{f,h}))$$

and the composition axiom for F yields the identities :

$$(Ff'' * \gamma^F_{g,k}) \odot ((\gamma^F_{k,f''})^{-1} * Fg) = (\gamma^F_{k\circ g,f''})^{-1} \odot \gamma^F_{g,f''\circ k}$$
$$(\gamma^F_{f',i} * Fg) \odot (Fi * (\gamma^F_{g,f'})^{-1}) = (\gamma^F_{g,i\circ f'})^{-1} \odot \gamma^F_{f'\circ g,i}.$$

So the left-hand side of (2.1.22) also equals

$$(\gamma^F_{k\circ g,f''})^{-1} \odot \gamma^F_{g,f''\circ k} \odot (F\beta * Fg) \odot (\gamma^F_{g,i\circ f'})^{-1} \odot \gamma^F_{f'\circ g,i} \odot (Fi * (F\alpha \odot \gamma^F_{f,h})).$$

Next, remark 2.1.17(ii) yields the identities

$$(F\beta * Fg) \odot (\gamma^F_{g,i\circ f'})^{-1} = (\gamma^F_{g,f''\circ k})^{-1} \odot F(\beta * g)$$
$$\gamma^F_{f'\circ g,i} \odot (Fi * F\alpha) = F(i * \alpha) \odot \gamma^F_{h\circ f,i}.$$

So we have to show that the right-hand side of (2.1.22) equals

$$(\gamma^F_{k\circ g,f''})^{-1} \odot F(\beta * g) \odot F(i * \alpha) \odot \gamma^F_{h\circ f,i} \odot (Fi * \gamma^F_{f,h}).$$

But by unwinding the definition, we see that the right-hand side of (2.1.22) equals

$$(\gamma^F_{k\circ g,f''})^{-1} \odot F(\beta \boxed{} \alpha) \odot \gamma^F_{f,i\circ h} \odot (\gamma^F_{h,i} * Ff) = (\gamma^F_{k\circ g,f''})^{-1} \odot F(\beta \boxed{} \alpha) \odot \gamma^F_{h\circ f,i} \odot (Fi * \gamma^F_{f,h})$$

(where the equality holds by the composition axiom). Thus, we come down to checking that

$$F(\beta * g) \odot F(i * \alpha) = F(\beta \boxed{} \alpha)$$

which follows from remark 2.1.17(i). With these rules, the functoriality of 2-$\mathsf{Morph}(F)$ is clear from remark 2.1.17(i), and the naturality of (2.1.21), as well as the composition and unit axioms for the latter, follow from the respective properties of (γ^F, δ^F), by a direct inspection.

(ii) In the situation of (i), let $f : A \to B$ be any 1-cell of $\mathscr{A}$. Then the unit axiom for the coherence constraint (γ^F, δ^F) implies the identity :

$$\mathbf{1}^F_f = (Ff * \delta^F_A) \odot ((\delta^F_B)^{-1} * Ff).$$

(iii) Let $\mathscr{C}$ be a third 2-category, and $G : \mathscr{B} \to \mathscr{G}$ another pseudo-functor; then we have

$$\mathsf{2\text{-}Morph}(G \circ F) = \mathsf{2\text{-}Morph}(G) \circ \mathsf{2\text{-}Morph}(F)$$

(details left to the reader).

Example 2.1.23. Let $\mathscr{A}$ be any 2-category.

(i) We have a natural strict isomorphism of 2-categories

$${}^o(\mathsf{2\text{-}Morph}(\mathscr{A}))^o \xrightarrow{\sim} \mathsf{2\text{-}Morph}({}^o\mathscr{A}^o)$$

that assigns :

- to every object $f : A \to B$ of $\mathsf{2\text{-}Morph}(\mathscr{A})$ the object ${}^of^o : {}^oB^o \to {}^oA^o$
- to every oriented square (2.1.6) the corresponding oriented square ^{o}(2.1.6)^o, which is a 1-cell ${}^o\alpha^o : {}^of'^o \to {}^of^o$ in $\mathsf{2\text{-}Morph}({}^o\mathscr{A}^o)$
- to every 2-cell (β, γ) in $\mathsf{2\text{-}Morph}(\mathscr{A})(f, f')$ the pair $({}^o\gamma^o, {}^o\beta^o)$ which is a 2-cell in $\mathsf{2\text{-}Morph}({}^o\mathscr{A}^o)({}^of'^o, {}^of^o)$.

(ii) We deduce from (i) a strict isomorphism of 2-categories

$$(\mathsf{2\text{-}Morph}(\mathscr{A}^o))^o \xrightarrow{\sim} {}^o(\mathsf{2\text{-}Morph}({}^o\mathscr{A})).$$

These latter 2-categories can be described as follows. The objects are the same as those of $\mathsf{2\text{-}Morph}(\mathscr{A})$, and the 1-cells $f \to f'$ are the oriented squares just like (2.1.6), but *with reversed direction of the orienting arrow*, *i.e.* α is replaced by a 2-cell $f' \circ g \Rightarrow h \circ f$. Given two 1-cells $(g_1, h_1, \alpha_1), (g_2, h_2, \alpha_2) : f \to f'$ in $\mathsf{2\text{-}Morph}(\mathscr{A}^o)^o(f, f')$, the 2-cells $(g_1, h_1, \alpha_1) \Rightarrow (g_2, h_2, \alpha_2)$ are still all the pairs $(\beta : g_1 \Rightarrow g_2, \gamma : h_1 \Rightarrow h_2)$ such that $\alpha_1 \boxminus \beta = \gamma \boxminus \alpha_2$, and the composition rule for such 2-cells is the same as in $\mathsf{2\text{-}Morph}(\mathscr{A})$. However, the composition law for 1-cells in $\mathsf{2\text{-}Morph}(\mathscr{A}^o)^o$ is given by the join operation $\boxminus$ instead of $\boxbar$: details left to the reader.

Example 2.1.24. We have a natural strict isomorphism of 2-categories :

$$(-)^o : {}^o\mathbf{Cat} \xrightarrow{\sim} \mathbf{Cat}$$

that assigns to every small category $\mathscr{C}$ the opposite category $\mathscr{C}^o$, to every functor $F : \mathscr{B} \to \mathscr{C}$ the opposite functor $F^o : \mathscr{B}^o \to \mathscr{C}^o$, and to every natural transformation $\beta : F \Rightarrow G$ the opposite transformation $\beta^o : G^o \Rightarrow F^o$.

2.2. **Pseudo-natural transformations and their modifications.** The following definition introduces the 2-categorical analogue of a natural transformation between functors.

Definition 2.2.1. Consider two pseudo-functors $F, G : \mathscr{A} \to \mathscr{B}$ between 2-categories $\mathscr{A}$, $\mathscr{B}$.

(i) A *lax-natural transformation* $\alpha : F \Rightarrow G$ is the datum of :

- For every object A of $\mathscr{A}$, a 1-cell $\alpha_A : FA \to GA$.
- For every $A, B \in \mathrm{Ob}(\mathscr{A})$, a natural transformation τ_{AB} between two functors $\mathscr{A}(A, B) \to \mathscr{B}(FA, GB)$, as shown by the (not necessarily commutative) diagram

(2.2.2)
$$\begin{array}{ccc} \mathscr{A}(A,B) & \xrightarrow{F_{AB}} & \mathscr{B}(FA,FB) \\ {\scriptstyle G_{AB}}\downarrow & \overset{\tau_{AB}}{\nearrow} & \downarrow{\scriptstyle H_{\mathscr{B}}(\mathbf{1}_{FA},\alpha_B)} \\ \mathscr{B}(GA,GB) & \xrightarrow[H_{\mathscr{B}}(\alpha_A,\mathbf{1}_{GB})]{} & \mathscr{B}(FA,GB) \end{array}$$

where $H_{\mathscr{B}}(\mathbf{1}_{FA}, \alpha_B)$ and $H_{\mathscr{B}}(\alpha_A, \mathbf{1}_{GB})$ are as in example 2.2.7. The datum $\tau_{\bullet\bullet}$ is called the *coherence constraint* for α, and is required to satisfy the following *coherence axioms* (in which we denote by (δ^F, γ^F) and (δ^G, γ^G) the coherence constraints for F and respectively G) :

- For every $A \in \mathrm{Ob}(\mathscr{A})$, the following diagram commutes :

$$\begin{array}{ccccc} \alpha_A & \xRightarrow{\mathbf{1}_{\alpha_A}} & \mathbf{1}_{GA} \circ \alpha_A & \xRightarrow{\delta^G_A * \alpha_A} & G(\mathbf{1}_A) \circ \alpha_A \\ {\scriptstyle \mathbf{1}_{\alpha_A}} \Downarrow & & & & \Downarrow {\scriptstyle \tau_{\mathbf{1}_A}} \\ \alpha_A \circ \mathbf{1}_{FA} & & \xRightarrow{\alpha_A * \delta^F_A} & & \alpha_A \circ F(\mathbf{1}_A). \end{array}$$

- For each pair of 1-cells $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathscr{A}$, the following diagram commutes :

$$\begin{array}{ccccc} Gg \circ Gf \circ \alpha_A & \xRightarrow{Gg * \tau_f} & Gg \circ \alpha_B \circ Ff & \xRightarrow{\tau_g * Ff} & \alpha_C \circ Fg \circ Ff \\ {\scriptstyle \gamma^G_{f,g} * \alpha_A} \Downarrow & & & & \Downarrow {\scriptstyle \alpha_C * \gamma^F_{f,g}} \\ G(g \circ f) \circ \alpha_A & & \xRightarrow{\tau_{g \circ f}} & & \alpha_C \circ F(g \circ f). \end{array}$$

(ii) A lax-natural transformation α as in (i) is called a *pseudo-natural transformation* (resp. *strict*) if τ_{AB} is an isomorphism of functors (resp. if (2.2.2) commutes and τ_{AB} is the identity natural transformation) for every $A, B \in \mathrm{Ob}(\mathscr{A})$.

(iii) A pseudo-natural transformation α as in (i) is called a *pseudo-natural isomorphism* if α_A is an invertible 1-cell, for every $A \in \mathrm{Ob}(\mathscr{A})$.

Example 2.2.3. For any 2-category $\mathscr{C}$, denote by

$$p_{\mathscr{C}}, p'_{\mathscr{C}} : \text{2-Morph}(\mathscr{C}) \times_{(\mathsf{t},\mathsf{s})} \text{2-Morph}(\mathscr{C}) \to \text{2-Morph}(\mathscr{C})$$

the projections, *i.e.* the strict pseudo-functors that yield a universal cone for this fibre product, where t and s are the target and source pseudo-functors for 2-Morph$(\mathscr{C})$ as in remark 2.1.18(ii). Now, let $\mathscr{A}, \mathscr{B}$ be two 2-categories, and $F : \mathscr{A} \to \mathscr{B}$ a pseudo-functor; it is easily seen that

$$\mathsf{t} \circ \text{2-Morph}(F) \circ p_{\mathscr{A}} = \mathsf{s} \circ \text{2-Morph}(F) \circ p'_{\mathscr{A}}$$

where 2-Morph(F) is defined as in example 2.1.20(i). There follows a unique pseudo-functor

$$\text{2-Morph}(F, F) : \text{2-Morph}(\mathscr{A}) \times_{(\mathsf{t},\mathsf{s})} \text{2-Morph}(\mathscr{A}) \to \text{2-Morph}(\mathscr{B}) \times_{(\mathsf{t},\mathsf{s})} \text{2-Morph}(\mathscr{B})$$

such that

$$p_{\mathscr{B}} \circ \text{2-Morph}(F, F) = \text{2-Morph}(F) \circ p_{\mathscr{A}} \qquad \text{and} \qquad p'_{\mathscr{B}} \circ \text{2-Morph}(F, F) = \text{2-Morph}(F) \circ p'_{\mathscr{A}}$$

and we claim that the coherence constraint γ^F of F can be viewed as a strict pseudo-natural isomorphism

$$\begin{array}{ccc} \text{2-Morph}(\mathscr{A}) \times_{(\mathsf{t},\mathsf{s})} \text{2-Morph}(\mathscr{A}) & \xrightarrow{\mathsf{c}} & \text{2-Morph}(\mathscr{A}) \\ {\scriptstyle \text{2-Morph}(F,F)} \downarrow & \gamma^F \nearrow\!\!\!\nearrow & \downarrow {\scriptstyle \text{2-Morph}(F)} \\ \text{2-Morph}(\mathscr{B}) \times_{(\mathsf{t},\mathsf{s})} \text{2-Morph}(\mathscr{B}) & \xrightarrow[\mathsf{c}]{} & \text{2-Morph}(\mathscr{B}). \end{array}$$

Namely, for every pair of composable 1-cells (f, g) of $\mathscr{A}$, we regard $\gamma^F_{f,g} : Fg \circ Ff \Rightarrow F(g \circ f)$ as a 1-cell of 2-Morph$(\mathscr{B})$. Then the strictness of γ^F boils down to the identity

$$\gamma^F_{f',g'} \boxplus (\beta^F \boxminus \alpha^F) = (\beta \boxminus \alpha)^F \boxplus \gamma^F_{f,g}$$

for every diagram (2.1.19) in $\mathscr{A}$. For the verification, notice that the associativity constraint for the functor $(\text{2-Morph}(F^o))^o$ assigns to (2.1.19) the pair $(\gamma^o_{f,g}, \gamma^o_{f',g'})^o = (\gamma_{f,g}, \gamma_{f',g'})$ (see remark 2.1.17(iv) and example 2.1.20(i)), and then the foregoing identity just translates the assertion that $(\gamma_{f,g}, \gamma_{f',g'})$ is a well defined 2-cell in $\mathsf{2\text{-}Morph}(F^o h^o, F^o i^o)^o$ (see example 2.1.23(ii)). The coherence axioms for γ^F follow easily, by unwinding the definitions.

Example 2.2.4. (i) Any category $\mathscr{A}$ can be regarded as a 2-category in a natural way : namely, for any two objects A and B of $\mathscr{A}$ one lets $\mathscr{A}(A, B)$ be the discrete category $\mathrm{Hom}_{\mathscr{A}}(A, B)$; hence the only 2-cells of $\mathscr{A}$ are the identities $\mathbf{1}_f : f \Rightarrow f$, for every morphism $f : A \to B$. The composition bifunctor c_{ABC} is of course given (on 1-cells) by the composition law for morphisms of $\mathscr{A}$. Likewise, the functor u_A assigns to every object A its identity endomorphism.

(ii) In the same vein, every functor between usual categories is a strict pseudo-functor between the corresponding 2-categories as in (i). Finally, every natural transformation of usual functors can be regarded naturally as a strict pseudo-natural transformation between the corresponding strict pseudo-functors.

Remark 2.2.5. With the notation of definition 2.2.1, we have :

(i) The coherence constraint of α can be regarded as a system of oriented squares

$$\begin{array}{ccc} FA & \xrightarrow{\alpha_A} & GA \\ {\scriptstyle Ff}\downarrow & \swarrow\!\!\!\!/ \, \tau_f & \downarrow{\scriptstyle Gf} \\ FB & \xrightarrow[\alpha_B]{} & GB \end{array} \qquad \text{for every 1-cell } f : A \to B \text{ of } \mathscr{A}.$$

Then α is a pseudo-natural transformation if and only if the orientation τ_f is invertible for every 1-cell f of $\mathscr{A}$. The naturality of $\tau_{\bullet\bullet}$ comes down to the commutativity of the diagram

$$\begin{array}{ccc} Gf \circ \alpha_A & \xRightarrow{\tau_f} & \alpha_B \circ Ff \\ {\scriptstyle G\beta * \alpha_A}\Downarrow & & \Downarrow{\scriptstyle \alpha_B * F\beta} \\ Gf' \circ \alpha_A & \xRightarrow{\tau_{f'}} & \alpha_B \circ Ff' \end{array}$$

for every $A, B \in \mathrm{Ob}(\mathscr{A})$ and every 2-cell $\beta : f \Rightarrow f'$ in $\mathscr{A}(A, B)$. The latter can be also be written as the identity

$$\tau_f \boxminus F\beta = G\beta \boxminus \tau_{f'}.$$

Likewise, the coherence axioms can be written as the identities

$$\delta^G_A \boxminus \tau_{\mathbf{1}_A} = \mathbf{1}_{\alpha_A} \boxminus \delta^F_A \qquad (\tau_g \boxbar \tau_f) \boxminus \gamma^F_{f,g} = \gamma^G_{f,g} \boxminus \tau_{g\circ f}$$

for every $A \in \mathrm{Ob}(\mathscr{A})$ and every composable pair (f, g) of 1-cells of $\mathscr{A}$.

(ii) The foregoing list of requirements becomes more intelligible, when we observe that a lax-natural transformation $\alpha : F \Rightarrow G$ is equivalent to the datum of a pseudo-functor

$$\widetilde{\alpha} : \mathscr{A} \to \mathsf{2\text{-}Morph}(\mathscr{B}) \qquad \text{such that} \qquad \mathsf{s} \circ \widetilde{\alpha} = F \qquad \text{and} \qquad \mathsf{t} \circ \widetilde{\alpha} = G$$

and α is a pseudo-natural transformation if and only if $\widetilde{\alpha}(f)$ is an invertible 2-cell for every 1-cell f of $\mathscr{A}$. Namely, in light of (i) we obtain such $\widetilde{\alpha}$ by setting

$$\widetilde{\alpha}(A) := \alpha_A \qquad \widetilde{\alpha}(f) := \tau_f \qquad \widetilde{\alpha}(\beta) := (F\beta, G\beta)$$

for every $A \in \mathrm{Ob}(\mathscr{A})$, every 1-cell f of $\mathscr{A}$, and every 2-cell β of $\mathscr{A}$. The coherence constraint (δ, γ) of $\widetilde{\alpha}$ are defined by setting

$$\delta_A := (\delta^F_A, \delta^G_A) \qquad \gamma_{f,g} := (\gamma^F_{f,g}, \gamma^G_{f,g})$$

for every $A \in \mathrm{Ob}(\mathscr{A})$ and every composable pair (f, g) of 1-cells of $\mathscr{A}$. Especially, $\widetilde{\alpha}$ is a strict pseudo-functor if and only if the same holds for both F and G.

(iii) If $\alpha : F \Rightarrow G$ and $\beta : G \Rightarrow H$ are two lax-natural transformations, let $\widetilde{\alpha}, \widetilde{\beta} : \mathscr{A} \to$ 2-$\mathsf{Morph}(\mathscr{B})$ be the associated pseudo-functors, as in (ii); then we may proceed as in example 1.2.25(iii) to define the composition

$$\beta \odot \alpha : F \Rightarrow H.$$

Namely, it shall be the lax-natural transformation associated with the pseudo-functor

$$\mathscr{A} \xrightarrow{(\widetilde{\alpha},\widetilde{\beta})} \text{2-}\mathsf{Morph}(\mathscr{A}) \times_{(\mathsf{t},\mathsf{s})} \text{2-}\mathsf{Morph}(\mathscr{A}) \xrightarrow{\mathsf{c}} \text{2-}\mathsf{Morph}(\mathscr{A})$$

(notation of remark 2.1.18(ii)). Unwinding the definitions, we see that this is the lax-natural transformation given by the rule

$$A \mapsto \beta_A \circ \alpha_A \qquad \text{for every } A \in \mathrm{Ob}(\mathscr{A})$$

and with coherence constraint $\tau^{\beta\odot\alpha}_{\bullet\bullet}$ given by the rule (notation of remark 2.1.18(ii)) :

$$(f : A \to B) \mapsto \tau^{\beta}_f \boxminus \tau^{\alpha}_f \qquad \text{for every 1-cell } f \text{ of } \mathscr{A}$$

where $\tau^{\alpha}_{\bullet\bullet}$ (resp. $\tau^{\beta}_{\bullet\bullet}$) denotes the coherence constraint of α (resp. of β). Hence, if both α and β are pseudo-natural (resp. strict), the same holds for $\beta \odot \alpha$. Since the composition law $\boxminus$ is associative (proposition 2.1.9(i) and remark 2.1.10), we deduce that

$$\lambda \odot (\beta \odot \alpha) = (\lambda \odot \beta) \odot \alpha$$

for every lax-natural transformation $\lambda : H \Rightarrow K$ (and any pseudo-functor $K : \mathscr{A} \to \mathscr{B}$).

(iv) If $\mathscr{C}$ is a third 2-category, $F', G' : \mathscr{B} \to \mathscr{C}$ two other pseudo-functors, and $\alpha' : F' \Rightarrow G'$ another lax-natural transformation, we may likewise define the *Godement products*

$$\alpha' * F : F' \circ F \Rightarrow G' \circ F \qquad G' * \alpha : G' \circ F \Rightarrow G' \circ G.$$

Namely, they shall be the pseudo-natural transformations attached to the pseudo-functors

$$\widetilde{\alpha}' \circ F \qquad \text{and respectively} \qquad \text{2-}\mathsf{Morph}(G') \circ \widetilde{\alpha}$$

where $\widetilde{\alpha}' : \mathscr{B} \to$ 2-$\mathsf{Morph}(\mathscr{C})$ is the pseudo-functor associated with α', as in (i), and we define 2-$\mathsf{Morph}(G')$ as in example 2.1.20(i). If α (resp. α') is pseudo-natural, the same holds for $G' * \alpha$ (resp. for $\alpha' * F$). In view of example 2.1.20(iii) and remark 2.1.17(vi), it is then clear that

$$\alpha' * (F \circ K) = (\alpha' * F) * K \qquad \text{and} \qquad (K' \circ G') * \alpha = K' * (G' * \alpha)$$

for any other 2-category $\mathscr{D}$, and every pseudo-functors $K : \mathscr{D} \to \mathscr{A}$ and $K' : \mathscr{C} \to \mathscr{D}$. Furthermore, if β is as in (ii), and $\beta' : G' \Rightarrow H'$ is another lax-natural transformation (to a target pseudo-functor $H' : \mathscr{B} \to \mathscr{C}$), then we get straightforwardly :

$$(G' * \alpha) * K = G' * (\alpha * K) \qquad \text{and} \qquad (\beta' \odot \alpha') * F = (\beta' * F) \odot (\alpha' * F).$$

On the other hand, example 2.2.3 says that the coherence constraint $\gamma^{G'}$ of G' induces a pseudo-natural isomorphism from the pseudo-functor associated with $(G' * \beta) \odot (G' * \alpha)$ to the pseudo-functor associated with $G' * (\beta \odot \alpha)$. Thus, if G' is a strict pseudo-functor, then these two lax-natural transformations coincide, but in general they will be different.

(v) In the same vein, it is tempting to mimick example 1.2.25(iii), in order to define a full Godement product $\alpha' * \alpha$ of any two lax-natural transformations as in (iv) : namely, one could for example declare this product to be $(G' * \alpha) \odot (\alpha' * F)$. However, notice that this lax-natural transformation shall be, in general, different from $(\alpha' * G) \odot (F' * \alpha)$, which would be another plausible definition. More precisely, we have a 2-cell

$$\tau^{\alpha'}_{\alpha_A} : ((G' * \alpha) \odot (\alpha' * F))_A \Rightarrow ((\alpha' * G) \odot (F' * \alpha))_A \qquad \text{for every } A \in \mathrm{Ob}(\mathscr{A})$$

where $\tau^{\alpha'}_{\bullet\bullet}$ denotes the coherence constraint of α'. Thus, the two possible definition will in general coincide only in case α' is a strict pseudo-natural transformation.

(vi) Taking into account remark 2.1.1(i) and the observations in (iv) and (v), we conclude that the datum of all small 2-categories, all pseudo-functors between such 2-categories, and all lax-natural (or pseudo-natural) transformations, does *not* form a 2-category, at least not with Godement products defined as in the foregoing. Rather, the pseudo-natural isomorphisms exhibited in (iv) hint at a higher order structure, whose full explication would require the introduction of a 3-categorical formalism that lies beyond the bounds of our treatise. Instead, we will venture only as far as needed to reach the notion of *modification* of pseudo-natural transformations (see definition 2.2.9), which will leave us just outside the threshold of 3-category theory, but still will provide a workable language for expressing the foregoing higher order compatibilities, and others as well of a similar nature, that will be encountered in this section.

(vii) Let us also mention that the datum of all small 2-categories, all strict pseudo-functors between such categories, and all strict pseudo-natural transformations of such pseudo-functors, does carry a natural 2-category structure, with the composition laws for pseudo-functors and pseudo-natural transformations defined above : the easy verification shall be left to the reader.

Example 2.2.6. (i) Consider two pseudo-functors $F, G : \mathscr{A} \to \mathscr{B}$ between 2-categories, and $\beta : F \Rightarrow G$ a pseudo-natural transformation. Denote by (γ^F, δ^F) (resp. (γ^G, δ^G), resp. τ^β) the coherence constraint of F (resp. G, resp. β). We deduce from β pseudo-natural transformations

$$\beta^o : G^o \Rightarrow F^o \qquad {}^o\beta : {}^oF \Rightarrow {}^oG$$

with coherence constraints given by the rules

$$(f^o : A^o \to A'^o) \mapsto \tau^{\beta^o}_{f^o} := (\tau^{\beta\ -1}_f)^o : F^o f^o \circ \beta^o_{A'^o} \Rightarrow \beta^o_{A^o} \circ G^o f^o$$
$$({}^of : {}^oA \to {}^oA') \mapsto \tau^{{}^o\beta}_{f^o} := {}^o(\tau^{\beta\ -1}_f) : {}^oG^o f \circ {}^o\beta_{{}^oA} \Rightarrow {}^o\beta_{{}^oA'} \circ {}^oF^o f.$$

Indeed, the coherence axioms for τ^β imply the identities

$$(\tau^{\beta\ -1}_{\mathbf{1}_A})^o \boxminus (\delta^{G\ -1}_A)^o = (\delta^{F\ -1}_A)^o \boxminus \mathbf{1}^o_{\beta_A} \qquad (\gamma^{F\ -1}_{f,f'})^o \boxminus ((\tau^{\beta\ -1}_f)^o \boxdot (\tau^{\beta\ -1}_{f'})^o) = (\tau^{\beta\ -1}_{f'\circ f})^o \boxminus (\gamma^{G\ -1}_{f,f'})^o$$

whence :

$$(\delta^F_A)^o \boxminus (\tau^{\beta\ -1}_{\mathbf{1}_A})^o = \mathbf{1}^o_{\beta_A} \boxminus (\delta^{G\ -1}_A)^o \qquad ((\tau^{\beta\ -1}_f)^o \boxdot (\tau^{\beta\ -1}_{f'})^o) \boxminus (\gamma^G_{f,f'})^o = (\gamma^F_{f,f'})^o \boxminus (\tau^{\beta\ -1}_{f'\circ f})^o$$

which in view, of remark 2.1.17(iv), shows that $\tau^{\beta^o}_\bullet$ fulfills the required coherence axioms. Likewise one verifies easily the naturality condition, whence the assertion. The corresponding verifications for ${}^o\beta$ are similar, and shall be left to the reader.

(ii) By virtue of (i), one can say that – just as for usual natural transformation between functors – the datum of a pseudo-natural transformation $\beta : F \Rightarrow G$ is equivalent to that of a pseudo-natural transformation $\beta^o : F^o \Rightarrow G^o$, and also to that of a pseudo-natural transformation ${}^o\beta : {}^oF \Rightarrow {}^oG$. On the other hand, these equivalences *does not extend to general lax-natural transformations*. By inspecting the definition, we see that a lax-natural transformation $\alpha : {}^oF \Rightarrow {}^oG$ is the datum of a system of 1-cells $(\alpha_A : FA \to GA \mid A \in \mathrm{Ob}(\mathscr{A}))$ together with oriented squares as in remark 2.2.5(i), but *whose orientation is reversed*, hence τ_f is a 2-cell $\alpha_B \circ Ff \Rightarrow Gf \circ \alpha_A$ in $\mathscr{B}$. The naturality and coherence conditions for such a system of oriented squares must be likewise suitably reoriented : the details shall be left to the reader. In some literature, such data $(\alpha_\bullet, \tau_{\bullet\bullet})$ are called *oplax-natural transformations*, but we shall not use this terminology.

Example 2.2.7. (i) Let $\mathscr{C}$ be any 2-category, and pick a universe U such that $\mathscr{C}$ is U-small; according to remark 2.1.18(i) the product $\mathscr{C}^o \times \mathscr{C}$ is representable in U-2-$\mathbf{Cat}$. Then we get the strict pseudo-functor

$$H_{\mathscr{C}} : \mathscr{C}^o \times \mathscr{C} \to \mathsf{U}\text{-}\mathbf{Cat}$$

(for the natural 2-category structure on U-$\mathbf{Cat}$ discussed in remark 2.1.1(ii)) that assigns :

- to every $C, C' \in \mathrm{Ob}(\mathscr{C})$, the category $H_{\mathscr{C}}(C, C') := \mathscr{C}(C, C')$

- to every pair of 1-cells $f : C_2 \to C_1$ and $f' : C'_1 \to C'_2$ of $\mathscr{C}$, the functor
$$H_{\mathscr{C}}(f, f') : \mathscr{C}(C_1, C'_1) \to \mathscr{C}(C_2, C'_2) \qquad g \mapsto f' \circ g \circ f \quad \alpha \mapsto f' * \alpha * f$$
where $g : C_1 \to C'_1$ is any 1-cell and α any 2-cell in $\mathscr{C}(C_1, C'_1)$
- to every pair of 2-cells $\beta : f \Rightarrow g$, $\beta' : f' \Rightarrow g'$ (with 1-cells $f, g : C_2 \to C_1$ and $f', g' : C'_1 \to C'_2$), the natural transformation
$$H_{\mathscr{C}}(\beta, \beta') : H_{\mathscr{C}}(f, f') \Rightarrow H_{\mathscr{C}}(g, g') \qquad (h : C_1 \to C'_1) \mapsto \beta' * h * \beta.$$

(ii) If $F : \mathscr{A} \to \mathscr{C}$ and $G : \mathscr{B} \to \mathscr{C}$ are any two pseudo-functors, we also define
$$H_{\mathscr{B}}(F, G) := H_{\mathscr{B}} \circ (F^o \times G) : \mathscr{A}^o \times \mathscr{B} \to \mathsf{U}\text{-}\mathbf{Cat}.$$
Likewise, if $F' : \mathscr{A} \to \mathscr{C}$ and $G' : \mathscr{B} \to \mathscr{C}$ is another pair of pseudo-functors, and $\alpha : F \Rightarrow F'$, $\beta : G \Rightarrow G'$ two pseudo-natural transformations, we let
$$H_{\mathscr{B}}(\alpha, \beta) := H_{\mathscr{B}} * (\alpha^o \times \beta) : H_{\mathscr{B}}(F, G) \Rightarrow H_{\mathscr{B}}(F', G').$$

Example 2.2.8. (i) For every 2-category $\mathscr{A}$ and every $X \in \mathrm{Ob}(\mathscr{A})$, the restriction $\mathsf{s}_X : \mathscr{A}/X \to \mathscr{A}$ (resp. $\mathsf{t}_X : X/\mathscr{A} \to \mathscr{A}$) of the pseudo-functor s (resp. t) of remark 2.1.18(ii) is a well defined strict pseudo-functor on the 2-category $\mathscr{A}/X$ (resp. $X/\mathscr{A}$ of example 2.1.11(ii)). Every 1-cell $h : X \to X'$ induces a strict pseudo-functor
$$h_* : \mathscr{A}/X \to \mathscr{A}/X' \qquad (A \xrightarrow{g} X) \mapsto (A \xrightarrow{h \circ g} X').$$
To every pair of objects $f : A \to X$ and $f' : A' \to X$ and every 1-cell $(g, \alpha) : f \Rightarrow f'$ of $\mathscr{A}/X$, the pseudo-functor h_* assigns the 1-cell $(g, h * \beta) : h_* f \Rightarrow h_* f'$ of $\mathscr{A}/X'$; lastly, for every pair of 1-cells $(g_1, \alpha_1), (g_2, \alpha_2) : f \to f'$ and every 2-cell $\beta : g_1 \Rightarrow g_2$, we have $h_*\beta := h * \beta$.

Additionally, every 2-cell $\beta : h \Rightarrow h'$ between 1-cells $h, h' : X \to X'$ induces a strict pseudo-natural transformation
$$\beta_* : h_* \Rightarrow h'_*$$
which assigns to every $(A \xrightarrow{g} X) \in \mathrm{Ob}(\mathscr{A}/X)$, the 1-cell $(\mathbf{1}_A, \beta * g) : h \circ g \to h' \circ g$ of $\mathscr{A}/X'$.

Lastly, notice that ${}^o({}^o\mathscr{A}^o/{}^oX^o)^o = X/\mathscr{A}$; we get therefore a strict pseudo-functor
$$h^* := {}^o(({}^oh^o)_*)^o : X'/\mathscr{A} \to X/\mathscr{A} \qquad (X' \xrightarrow{g} A) \mapsto (X \xrightarrow{g \circ h} A)$$
and the 2-cell β as in the foregoing induces as well a strict pseudo-natural transformation
$$\beta^* := {}^o(({}^o\beta^o)_*)^o : h^* \Rightarrow h'^*.$$

(ii) More generally, if $F : \mathscr{A} \to \mathscr{B}$ is any pseudo-functor with coherence constraint (δ^F, γ^F), and $B \in \mathrm{Ob}(\mathscr{B})$ any object, we may define a 2-category
$$F\mathscr{A}/B$$
whose objects are the pairs $(A, f : FA \to B)$ where $A \in \mathrm{Ob}(\mathscr{A})$ and f is a 1-cell of $\mathscr{B}$. The 1-cells $(g, \alpha) : (A, f) \to (A', f')$ consist of a 1-cell $g : A \to A'$ of $\mathscr{A}$ and a 2-cell $\alpha : f \Rightarrow f' \circ Fg$. For 1-cells $(g_1, \alpha_1), (g_2, \alpha_2) : (A, f) \to (A', f')$, the 2-cells $\beta : (g_1, \alpha_1) \Rightarrow (g_2, \alpha_2)$ are the 2-cells $\beta : g_1 \Rightarrow g_2$ such that $(f' * F\beta) \odot \alpha_1 = \alpha_2$. The composition law for 1-cells assigns to every pair of 1-cells $(A, f) \xrightarrow{(g,\alpha)} (A', f') \xrightarrow{(g',\alpha')} (A'', f'')$ the 1-cell
$$(g', \alpha') \circ (g, \alpha) := \Big(g' \circ g, \Big(\frac{\alpha'}{f'|f''} \boxminus \frac{\alpha}{f|f'}\Big) \boxminus \frac{\gamma^F_{g,g'}}{\mathbf{1}_{FA}|\mathbf{1}_{FA''}}\Big).$$

Let us check the associativity property : let $(A'', f'') \xrightarrow{(g'',\alpha'')} (A''', f''')$ be another 1-cell; then

$$\begin{aligned}(g'',\alpha'')\circ((g',\alpha')\circ(g,\alpha)) &= (\alpha''\boxminus((\alpha'\boxminus\alpha)\boxminus\gamma^F_{g,g'}))\boxminus\gamma^F_{g'\circ g,g''}\\ &=((\alpha''\boxminus\mathbf{1}_{Fg''})\boxminus((\alpha'\boxminus\alpha)\boxminus\gamma^F_{g,g'}))\boxminus\gamma^F_{g'\circ g,g''}\\ &=(\alpha''\boxminus\alpha'\boxminus\alpha)\boxminus(\mathbf{1}_{Fg''}\boxminus\gamma^F_{g,g'})\boxminus\gamma^F_{g'\circ g,g''}.\end{aligned}$$

On the other hand, a similar calculation yields :

$$((g'',\alpha'')\circ(g',\alpha'))\circ(g,\alpha) = (\alpha''\boxminus\alpha'\boxminus\alpha)\boxminus(\gamma^F_{g',g''}\boxminus\mathbf{1}_{Fg})\boxminus\gamma^F_{g,g''\circ g'}$$

and the coherence axiom for F translates as : $(\mathbf{1}_{Fg''}\boxminus\gamma^F_{g,g'})\boxminus\gamma^F_{g'\circ g,g''} = (\gamma^F_{g',g''}\boxminus\mathbf{1}_{Fg})\boxminus\gamma^F_{g,g''\circ g'}$, whence the contention. Then it is easily seen that the identity 1-cell of any object (A, f) is given by the pair $(\mathbf{1}_A, \delta^F_A)$: details left to the reader. The composition laws for 2-cells in $F\mathscr{A}/B$ are induced by those of $\mathscr{A}$, in the obvious way; then the required associativity and functoriality properties follow straightforwardly. Lastly we define :

$$B/F\mathscr{A} := {}^o({}^oF^o({}^o\mathscr{A}^o)/{}^oB^o)^o.$$

The source and target pseudo-functors of (i) generalize as well : we get strict pseudo-functors

$$\begin{aligned}\mathsf{s}_B &: F\mathscr{A}/B\to\mathscr{A} \quad : \quad (A, f: FA\to B)\mapsto A\\ \mathsf{t}_B &: B/F\mathscr{A}\to\mathscr{A} \quad : \quad (A, f: B\to FA)\mapsto A\end{aligned}$$

and every 1-cell $h : B \to B'$ of $\mathscr{B}$ induces strict pseudo-functors

$$\begin{aligned}h_* &: F\mathscr{A}/B\to F\mathscr{A}/B' \quad : \quad (A, f: FA\to B)\mapsto (A, h\circ f)\\ h^* &: B'/F\mathscr{A}\to B/F\mathscr{A} \quad : \quad (A, f: B'\to FA)\mapsto (A, f\circ h).\end{aligned}$$

Likewise, if $h' : B \to B'$ is another 1-cell of $\mathscr{B}$, every 2-cell $\beta : h' \Rightarrow h'$ of $\mathscr{B}'$ induces strict pseudo-natural transformations :

$$\begin{aligned}\beta_* &: h_*\Rightarrow h'_* \quad : \quad (A, f: FA\to B)\mapsto (\mathbf{1}_A, \beta * f)\\ \beta^* &: h^*\Rightarrow h'^* \quad : \quad (A, f: B\to FA)\mapsto (\mathbf{1}_A, f * \beta).\end{aligned}$$

Definition 2.2.9. Consider two pseudo-functors $F, G : \mathscr{A} \to \mathscr{B}$ between 2-categories $\mathscr{A}$, $\mathscr{B}$, and two lax-natural transformations $\alpha, \beta : F \Rightarrow G$. A *modification* $\Xi : \alpha \rightsquigarrow \beta$ is a family

$$\Xi_A : \alpha_A \Rightarrow \beta_A \qquad \text{for every } A\in \mathrm{Ob}(\mathscr{A})$$

of 2-cells of $\mathscr{B}$ that satisfies the *compatibility condition* :

$$(\Xi_{A'} * Ff)\odot\tau^\alpha_f = \tau^\beta_f\odot(Gf*\Xi_A) \qquad \text{every 1-cell } f: A\to A' \text{ of } \mathscr{A}$$

where $\tau^\alpha_{\bullet\bullet}$ (resp. $\tau^\beta_{\bullet\bullet}$) denotes the coherence constraint for α (resp. for β).

Remark 2.2.10. In the situation of definition 2.2.9, let $\Xi : \alpha \rightsquigarrow \beta$ be any modification.

(i) For any pair of 1-cells $f, g : A \to A'$ of $\mathscr{A}$, and any 2-cell $\gamma : f \Rightarrow g$ we have the identity:

$$(\Xi_{A'} * F\gamma)\odot\tau^\alpha_f = \tau^\beta_g\odot(G\gamma*\Xi_A).$$

Indeed, we may compute :

$$\begin{aligned}(\Xi_{A'} * F\gamma)\odot\tau^\alpha_f &= (\beta_{A'} * F\gamma)\odot(\Xi_{A'}*Ff)\odot\tau^\alpha_f\\ &=(\beta_{A'}*F\gamma)\odot\tau^\beta_f\odot(Gf*\Xi_A)\\ &=\tau^\beta_g\odot(G\gamma*\beta_A)\odot(Gf*\Xi_A)\\ &=\tau^\beta_g\odot(G\gamma*\Xi_A)\end{aligned}$$

where the first and the last identities follow from remark 2.1.1(i), and the third one from remark 2.2.5(i). Notice also that the condition of definition 2.2.9 can be restated as the identity

$$\frac{\Xi_{A'}}{\alpha_{A'}|\beta_{A'}} \boxbar \frac{\tau_f^\alpha}{\alpha_A|\alpha_{A'}} = \frac{\tau_f^\beta}{\beta_A|\beta_{A'}} \boxbar \frac{\Xi_A}{\alpha_A|\beta_A}. \tag{2.2.11}$$

(ii) Let $\widetilde{\alpha}, \widetilde{\beta} : \mathscr{A} \to 2\text{-}\mathsf{Morph}(\mathscr{B})$ be the pseudo-functors attached to α and β as in remark 2.2.5(ii). By unwinding the definitions, it is easily seen that Ξ is equivalent to the datum of a strict pseudo-natural transformation

$$\widetilde{\widetilde{\Xi}} : \widetilde{\alpha} \Rightarrow \widetilde{\beta} \qquad \text{such that } \mathsf{s} * \widetilde{\widetilde{\Xi}} = \mathbf{1}_F \text{ and } \mathsf{t} * \widetilde{\widetilde{\Xi}} = \mathbf{1}_G.$$

Namely, we obtain such a $\widetilde{\widetilde{\Xi}}$ by assigning, to every object A of $\mathscr{A}$, the 1-cell of $2\text{-}\mathsf{Morph}(\mathscr{B})$:

$$\begin{array}{ccc} FA & \xrightarrow{\alpha_A} & GA \\ \| & \Swarrow_{\Xi_A} & \| \\ FA & \xrightarrow[\beta_A]{} & GA. \end{array}$$

(iii) If $\gamma : F \Rightarrow G$ is any other lax-natural transformation, and $\Theta : \beta \rightsquigarrow \gamma$ any other modification, we can define the composition

$$\Theta \odot \Xi : \alpha \rightsquigarrow \gamma \quad : \quad A \mapsto \Theta_A \odot \Xi_A \quad \text{for every } A \in \mathrm{Ob}(\mathscr{A}).$$

In order to check that $\Theta \odot \Xi$ is indeed a well defined modification, we may notice that if $\widetilde{\widetilde{\Xi}}$ and $\widetilde{\Theta}$ are defined as in (ii), then $\Xi \odot \Theta$ is the modification corresponding to the pseudo-natural transformation

$$\widetilde{\Theta} \odot \widetilde{\widetilde{\Xi}} : \mathscr{A} \to 2\text{-}\mathsf{Morph}(\mathscr{B})$$

with the composition rule $\odot$ for pseudo-natural transformations given by remark 2.2.5(iii). It is clear that this composition law is associative for any triple of composable modifications.

(iv) Likewise, if $H : \mathscr{A} \to \mathscr{B}$ is another pseudo-functor, $\alpha', \beta' : G \Rightarrow H$ two other lax-natural transformations, and $\Xi' : \alpha' \rightsquigarrow \beta'$ any modification, we can define the modification

$$\Xi' * \Xi : \alpha' \odot \alpha \rightsquigarrow \beta' \odot \beta \qquad A \mapsto \Xi'_A * \Xi_A \qquad \text{for every } A \in \mathrm{Ob}(\mathscr{A}).$$

To see that $\Xi' * \Xi$ is well defined, consider – for any 1-cell $f : A \to A'$ in $\mathscr{A}$ – the diagram

$$\begin{array}{ccccc} FA & \xrightarrow{\alpha_A} & GA & \xrightarrow{\alpha'_A} & HA \\ {\scriptstyle Ff}\downarrow & \Swarrow_{\tau_f^\alpha} & \downarrow{\scriptstyle Gf} & \Swarrow_{\tau_f^{\alpha'}} & \downarrow{\scriptstyle Hf} \\ FA' & \xrightarrow{\alpha_{A'}} & GA' & \xrightarrow{\alpha'_{A'}} & HA' \\ \| & \Swarrow_{\Xi_{A'}} & \| & \Swarrow_{\Xi'_{A'}} & \| \\ FA' & \xrightarrow[\beta_{A'}]{} & GA' & \xrightarrow[\beta'_{A'}]{} & HA'. \end{array}$$

We compute :

$$\begin{aligned} (\Xi'_{A'} * \Xi_{A'} * Ff) \odot \tau_f^{\alpha' \circ \alpha} &= (\Xi'_{A'} \boxminus \Xi_{A'}) \boxbar (\tau_f^{\alpha'} \boxminus \tau_f^\alpha) \\ &= (\Xi'_{A'} \boxbar \tau_f^{\alpha'}) \boxminus (\Xi_{A'} \boxbar \tau_f^\alpha) \\ &= (\tau_f^{\beta'} \boxbar \Xi'_A) \boxminus (\tau_f^\beta \boxbar \Xi_A) \\ &= (\tau_f^{\beta'} \boxminus \tau_f^\beta) \boxbar (\Xi'_A \boxminus \Xi_A) \\ &= \tau_f^{\beta' \circ \beta} \odot (Hf * \Xi'_A * \Xi_A) \end{aligned}$$

where the second and fourth equalities follow from proposition 2.1.9(ii), and the third follows from (2.2.11). The contention follows. This operation is clearly associative. As usual, we write $\alpha * \Xi$ and $\Xi * \alpha'$ instead of $\mathbf{1}_\alpha * \Xi$ and respectively $\Xi * \mathbf{1}_{\alpha'}$, for any lax-natural transformations $\alpha : G \Rightarrow H$ and $\alpha' : E \Rightarrow F$.

(v) There is a third type of operation for modifications : namely, let $\mathscr{C}$ be another 2-category and $I : \mathscr{C} \to \mathscr{A}$ any pseudo-functor; then we obtain the modification

$$\Xi \circ I : \alpha * I \Rightarrow \beta * I \qquad C \mapsto \Xi_{IC} \qquad \text{for every } C \in \mathrm{Ob}(\mathscr{C}).$$

To check that $\Xi \circ I$ is indeed a modification, it suffices to notice that the coherence constraint of $\alpha * I$ is given by the system of 2-cells τ^α_{If}, for f ranging over the 1-cells of $\mathscr{C}$, and correspondingly for the coherence constraint of $\beta * I$.

(vi) Likewise, if $K : \mathscr{B} \to \mathscr{C}$ is any pseudo-functor, we get the modification

$$K \circ \Xi : K * \alpha \rightsquigarrow K * \beta \qquad A \mapsto K(\Xi_A) \qquad \text{for every } A \in \mathrm{Ob}(\mathscr{A}).$$

Indeed, notice that the coherence constraints of $K*\alpha$ and $K*\beta$ attach to every 1-cell $f : A \to A'$ in $\mathscr{A}$ the 2-cells

$$\tau^{K*\alpha}_f := \gamma^{K\ -1}_{Ff,\alpha_{A'}} \odot K(\tau^\alpha_f) \odot \gamma^K_{\alpha_A,Gf} \qquad \tau^{K*\beta}_f := \gamma^{K\ -1}_{Ff,\beta_{A'}} \odot K(\tau^\beta_f) \odot \gamma^K_{\beta_A,Gf}.$$

Hence we may compute :

$$\begin{aligned}
(K(\Xi_{A'}) * KFf) \odot \tau^{K*\alpha}_f &= (K(\Xi_{A'}) * KFf) \odot \gamma^{K\ -1}_{Ff,\alpha_{A'}} \odot K(\tau^\alpha_f) \odot \gamma^K_{\alpha_A,Gf} \\
&= \gamma^{K\ -1}_{Ff,\beta_{A'}} \odot K(\Xi_{A'} * Ff) \odot K(\tau^\alpha_f) \odot \gamma^K_{\alpha_A,Gf} \\
&= \gamma^{K\ -1}_{Ff,\beta_{A'}} \odot K((\Xi_{A'} * Ff) \odot \tau^\alpha_f) \odot \gamma^K_{\alpha_A,Gf} \\
&= \gamma^{K\ -1}_{Ff,\beta_{A'}} \odot K(\tau^\beta_f \odot (\Xi_A * Gf)) \odot \gamma^K_{\alpha_A,Gf} \\
&= \gamma^{K\ -1}_{Ff,\beta_{A'}} \odot K(\tau^\beta_f) \odot K(\Xi_A * Gf) \odot \gamma^K_{\alpha_A,Gf} \\
&= \gamma^{K\ -1}_{Ff,\beta_{A'}} \odot K(\tau^\beta_f) \odot \gamma^K_{\beta_A,Gf} \odot (K(\Xi_A) * KGf) \\
&= \tau^{K*\beta}_f \odot (K(\Xi_A) * KGf)
\end{aligned}$$

where the second and sixth equality follows from remark 2.1.17(ii), and the third and fifth follow from remark 2.1.17(i).

(vii) Let Θ and Ξ be as in (iii), and I and K like in (v) and (vi); then it is easily seen that

$$(\Theta \odot \Xi) \circ I = (\Theta \circ I) \odot (\Xi \circ I) \qquad K \circ (\Theta \odot \Xi) = (K \circ \Theta) \odot (K \circ \Xi).$$

Indeed, the first identity is immediate, and the second follows directly from remark 2.1.17(i). Furthermore, if Ξ and Ξ' are as in (iv), we have the identities

$$(\Xi' * \Xi) \circ I = (\Xi' \circ I) * (\Xi \circ I) \qquad K \circ (\Xi' * \Xi) \odot \gamma^K_{\alpha,\alpha'} = \gamma^K_{\beta,\beta'} \odot ((K \circ \Xi') * (K \circ \Xi)).$$

Indeed, the first identity is immediate, and the second follows directly from remark 2.1.17(ii).

Definition 2.2.12. Let $\mathscr{A}$ and $\mathscr{B}$ be two 2-categories.

(i) For every pair of pseudo-functors $F, G : \mathscr{A} \to \mathscr{B}$ we have a category

$$\mathsf{PsNat}(F, G)$$

whose objects are the pseudo-natural transformations $F \Rightarrow G$, and whose morphisms are the modifications $\alpha \rightsquigarrow \beta$ between them, with the composition rule $\odot$ given by remark 2.2.10(iii).

(ii) The pseudo-functors $\mathscr{A} \to \mathscr{B}$ are the objects of a 2-category :

$$\mathsf{PsFun}(\mathscr{A}, \mathscr{B})$$

whose 1-cells $F \to G$ are the pseudo-natural transformations $F \Rightarrow G$ of, and whose 2-cells are the modifications. For fixed $F, G : \mathscr{A} \to \mathscr{B}$, the category structure on the set of 1-cells $F \to G$ is the one of $\mathsf{PsNat}(F, G)$. The composition functor

$$\mathsf{PsNat}(F, G) \times \mathsf{PsNat}(G, H) \to \mathsf{PsNat}(F, H)$$

assigns, to any two modifications $\Xi : \alpha \rightsquigarrow \beta$ and $\Xi' : \alpha' \rightsquigarrow \beta'$, the modification $\Xi' * \Xi$. The associativity and unit axioms for this composition functor follow by a simple inspection.

(iii) The strict pseudo-functors $\mathscr{A} \to \mathscr{B}$ are the objects of a 2-category :

$$\mathsf{stPsFun}(\mathscr{A}, \mathscr{B})$$

defined as the sub-2-category of $\mathsf{PsFun}(\mathscr{A}, \mathscr{B})$ whose 1-cells are the strict pseudo-natural transformations, and whose 2-cells are all the modifications.

2.2.13. In the situation of definition 2.2.9, suppose that α and β are pseudo-natural; in view of example 2.2.6(i), the modification Ξ induces modifications :

$$\begin{aligned} \Xi^o &: \alpha^o \rightsquigarrow \beta^o & A^o &\mapsto ((\Xi^o_A : \alpha^o_{A^o} \Rightarrow \beta^o_{A^o}) \\ {}^o\Xi &: {}^o\beta \rightsquigarrow {}^o\alpha & {}^oA &\mapsto ({}^o(\Xi_A) : {}^o\beta_{{}^oA} \Rightarrow {}^o\alpha_{{}^oA}). \end{aligned}$$

The detailed verification shall be left to the reader. With the notation of definition 2.2.12, it follows that the rules : $(F : \mathscr{A} \to \mathscr{B}) \mapsto (F^o : \mathscr{A}^o \to \mathscr{B}^o)$, $(\beta : F \Rightarrow G) \mapsto (\beta^o : F^o \Rightarrow G^o)$ and $(\Xi : \alpha \rightsquigarrow \beta) \mapsto (\Xi^o : \alpha^o \rightsquigarrow \beta^o)$ yield a strict isomorphism of 2-categories :

$$\mathsf{PsFun}(\mathscr{A}, \mathscr{B})^o \xrightarrow{\sim} \mathsf{PsFun}(\mathscr{A}^o, \mathscr{B}^o).$$

Likewise, the rules $F \mapsto {}^oF$, $\beta \mapsto {}^o\beta$ and $\Xi \mapsto {}^o\Xi$ yield a strict isomorphism of 2-categories :

$${}^o\mathsf{PsFun}(\mathscr{A}, \mathscr{B}) \xrightarrow{\sim} \mathsf{PsFun}({}^o\mathscr{A}, {}^o\mathscr{B}).$$

Remark 2.2.14. (i) The isomorphisms of the category $\mathsf{PsFun}(\mathscr{A}, \mathscr{B})$ are precisely the pseudo-natural isomorphisms of pseudo-functors. Indeed, if $\alpha : F \Rightarrow G$ is such a pseudo-natural isomorphism, and τ is its coherence constraint, we obtain a pseudo-natural isomorphism $\alpha^{-1} : G \Rightarrow F$ by the rule that assigns to every $A \in \mathrm{Ob}(\mathscr{A})$ the 1-cell $(\alpha_A)^{-1} : GA \to FA$, and to every 1-cell $f : A \to A'$ the oriented square

$$\begin{array}{ccc} GA & \xrightarrow{\alpha_A^{-1}} & FA \\ {\scriptstyle Gf}\downarrow & \swarrow\!\!\!\!\!/ \, \tau'_f & \downarrow{\scriptstyle Ff} \\ GA' & \xrightarrow[\alpha_{A'}^{-1}]{} & FA' \end{array} \qquad \text{where } \tau'_f := \alpha_{A'}^{-1} * \tau_f^{-1} * \alpha_A^{-1}.$$

Indeed, let us check the coherence axioms for α^{-1}; first, we compute, for every $A \in \mathrm{Ob}(\mathscr{A})$:

$$\begin{aligned} \tau'_{\mathbf{1}_A} \odot (\delta^F_A * \alpha_A^{-1}) = (\alpha_A^{-1} * \tau_{\mathbf{1}_A}^{-1} * \alpha_A^{-1}) \odot (\delta^F_A * \alpha_A^{-1}) &= ((\alpha_A^{-1} * \tau_{\mathbf{1}_A}^{-1}) \odot \delta^F_A) * \alpha_A^{-1} \\ &= \alpha_A^{-1} * (\tau_{\mathbf{1}_A}^{-1} \odot (\alpha_A * \delta^F_A)) * \alpha_A^{-1} \\ &= \alpha_A^{-1} * (\delta^F_A * \alpha_A) * \alpha_A^{-1} \\ &= \alpha_A^{-1} * \delta^F_A \end{aligned}$$

which proves the first axiom. Next, for every pair of 1-cells $A \xrightarrow{f} A' \xrightarrow{g} A''$ of $\mathscr{A}$ we have :

$$\begin{aligned} (\tau'_g \boxplus \tau'_f) \boxminus \gamma^G_{f,g} &= (\alpha_{A''}^{-1} * \gamma^G_{f,g}) \odot (\alpha_{A''}^{-1} * \tau_g^{-1} * \alpha_{A'}^{-1} * Gf) \odot (Fg * \alpha_{A'}^{-1} * \tau_f^{-1} * \alpha_A^{-1}) \\ &= (\alpha_{A''}^{-1} * \gamma^G_{f,g}) \odot (\alpha_{A''}^{-1} * Gg * \tau_f^{-1} * \alpha_A^{-1}) \odot (\alpha_{A''}^{-1} * \tau_g^{-1} * Ff * \alpha_A^{-1}) \\ &= \alpha_{A''}^{-1} * (\gamma^G_{f,g} \odot (Gg * \tau_f^{-1} * \alpha_A^{-1}) \odot (\tau_g^{-1} * Ff * \alpha_A^{-1})) \\ &= \alpha_{A''}^{-1} * ((\gamma^G_{f,g} * \alpha_A) \odot (Gg * \tau_f^{-1}) \odot (\tau_g^{-1} * Ff)) * \alpha_A^{-1} \end{aligned}$$

and on the other hand

$$\gamma^F_{f,g} \boxminus \tau'_{g\circ f} = (\alpha^{-1}_{A''} * \tau^{-1}_{g\circ f} * \alpha^{-1}_A) \odot (\gamma^F_{f,g} * \alpha^{-1}_A) = \alpha^{-1}_{A''} * (\tau^{-1}_{g\circ f} \odot (\alpha_{A''} * \gamma^F_{f,g})) * \alpha^{-1}_A$$

so we are reduced to showing that

$$(\gamma^G_{f,g} * \alpha_A) \odot (Gg * \tau^{-1}_f) \odot (\tau^{-1}_g * Ff) = \tau^{-1}_{g\circ f} \odot (\alpha_{A''} * \gamma^F_{f,g})$$

which is equivalent to the second coherence axiom for α. Conversely, it is clear that every isomorphism in $\mathsf{PsFun}(\mathscr{A}, \mathscr{B})$ must be a pseudo-natural isomorphism.

(ii) In the same vein, with the notation of definition 2.2.12(i), it is easily seen that the isomorphisms in the category $\mathsf{PsNat}(F, G)$ are the modifications $\Xi : \alpha \rightsquigarrow \beta$ such that $\Xi_A : \alpha_A \Rightarrow \beta_A$ is an isomorphism in $\mathscr{B}(FA, GA)$ for every $A \in \mathrm{Ob}(\mathscr{A})$: the details are left to the reader.

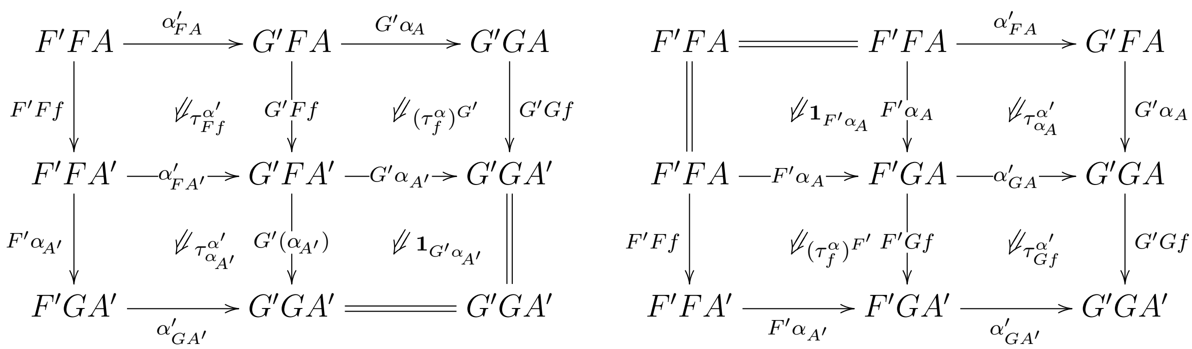
Example 2.2.15. (i) With the notation of remark 2.2.5(iv), we may now say that the coherence constraint $\gamma^{G'}$ furnishes an invertible modification

$$\gamma^{G'}_{\alpha,\beta} : (G' * \beta) \odot (G' * \alpha) \rightsquigarrow G' * (\beta \odot \alpha).$$

(ii) Likewise, let us check that the 2-cell of remark 2.2.5(v) yields a modification

$$\tau^{\alpha'}_\alpha : (G' * \alpha) \odot (\alpha' * F) \rightsquigarrow (\alpha' * G) \odot (F' * \alpha)$$

which is invertible if α' is pseudo-natural. To this aim, for any 1-cell $f : A \to A'$ in $\mathscr{A}$ we consider the diagrams :



(notation of example 2.1.20(i)) and notice that

$$\begin{aligned}
(\tau^{\alpha'}_{\alpha_{A'}} * F'Ff) \odot \tau^{(G'*\alpha)\odot(\alpha'*F)}_f = \lambda &:= (\mathbf{1}_{G'\alpha_{A'}} \boxminus \tau^{\alpha'}_{\alpha_{A'}}) \boxbar ((\tau^\alpha_f)^{G'} \boxminus \tau^{\alpha'}_{Ff}) \\
&= (\mathbf{1}_{G'\alpha_{A'}} \boxbar (\tau^\alpha_f)^{G'}) \boxminus (\tau^{\alpha'}_{\alpha_{A'}} \boxbar \tau^{\alpha'}_{Ff}) \\
\tau^{(\alpha'*G)\odot(F'*\alpha)}_f \odot (G'Gf * \tau^{\alpha'}_{\alpha_A}) = \mu &:= (\tau^{\alpha'}_{Gf} \boxminus (\tau^{\alpha'}_f)^{F'}) \boxbar (\tau^{\alpha'}_{\alpha_A} \boxminus \mathbf{1}_{F'\alpha_A}) \\
&= (\tau^{\alpha'}_{Gf} \boxbar \tau^{\alpha'}_{\alpha_A}) \boxminus ((\tau^{\alpha'}_f)^{F'} \boxbar \mathbf{1}_{F'\alpha_A}).
\end{aligned}$$

So we have only to check that

$$\lambda \boxminus \frac{\mathbf{1}_{F'\alpha_{A'}\circ F'Ff}}{\mathbf{1}_{F'FA}|F'\alpha_{A'}} = \frac{\mathbf{1}_{G'Gf\circ G'\alpha_A}}{G'\alpha_A|\mathbf{1}_{G'GA}} \boxminus \mu.$$

However, the definition of $2\text{-}\mathsf{Morph}(G')$ and the coherence axiom for α' yield respectively

$$\mathbf{1}_{G'\alpha_{A'}} \boxbar (\tau^\alpha_f)^{G'} = \frac{\gamma^{G'}_{\alpha_A,Gf}}{G'(\alpha_A)|\mathbf{1}_{G'GA'}} \boxminus \frac{G'(\tau^\alpha_f)}{\mathbf{1}_{G'FA}|\mathbf{1}_{G'GA'}} \boxminus \frac{(\gamma^{G'}_{Ff,\alpha_{A'}})^{-1}}{\mathbf{1}_{G'FA}|\mathbf{1}_{G'GA'}}$$

$$\tau^{\alpha'}_{\alpha_{A'}} \boxbar \tau^{\alpha'}_{Ff} = \frac{\gamma^{G'}_{Ff,\alpha_{A'}}}{\mathbf{1}_{G'FA}|\mathbf{1}_{G'GA'}} \boxminus \frac{\tau^{\alpha'}_{\alpha_{A'}\circ Ff}}{\alpha'_{FA}|\alpha'_{GA'}} \boxminus \frac{(\gamma^{F'}_{Ff,\alpha_{A'}})^{-1}}{\mathbf{1}_{F'FA}|\mathbf{1}_{F'GA'}}$$

therefore :

$$\begin{aligned}\lambda &= \gamma^{G'}_{\alpha_A,Gf} \boxminus G'(\tau^\alpha_f) \boxminus \tau^{\alpha'}_{\alpha_{A'}\circ Ff} \boxminus (\gamma^{F'}_{Ff,\alpha_{A'}})^{-1} \\ &= \gamma^{G'}_{\alpha_A,Gf} \boxminus \frac{\tau^{\alpha'}_{Gf\circ\alpha_A}}{\alpha'_{FA}|\alpha'_{GA'}} \boxminus \frac{F'(\tau^\alpha_f)}{\mathbf{1}_{F'FA}|\mathbf{1}_{F'GA'}} \boxminus (\gamma^{F'}_{Ff,\alpha_{A'}})^{-1} \\ &= \frac{\mathbf{1}_{G'Gf\circ G'\alpha_A}}{G'\alpha_A|\mathbf{1}_{G'GA'}} \boxminus (\tau^{\alpha'}_{Gf} \boxplus \tau^{\alpha'}_{\alpha_A}) \boxminus \frac{(\gamma^{F'}_{\alpha_A,Gf})}{\mathbf{1}_{F'FA}|\mathbf{1}_{F'GA'}} \boxminus \frac{(\gamma^{F'}_{\alpha_A,Gf})^{-1}}{\mathbf{1}_{F'FA}|\mathbf{1}_{F'GA'}} \boxminus \frac{(\tau^\alpha_f)^{F'}}{\mathbf{1}_{F'FA}|\mathbf{1}_{F'GA'}} \\ &= \frac{\mathbf{1}_{G'Gf\circ G'\alpha_A}}{G'\alpha_A|\mathbf{1}_{G'GA'}} \boxminus (\tau^{\alpha'}_{Gf} \boxplus \tau^{\alpha'}_{\alpha_A}) \boxminus \frac{(\tau^\alpha_f)^{F'}}{\mathbf{1}_{F'FA}|\mathbf{1}_{F'GA'}}\end{aligned}$$

where the second equality follows from remark 2.2.5(i) and the coherence axiom for α'. The sought identity follows straightforwardly.

(iii) For every 2-category $\mathscr{A}$, we denote by $i_\mathscr{A}$ the identity (strict) pseudo-natural transformation of $\mathbf{1}_\mathscr{A}$. In other words, $i_\mathscr{A}$ assigns to every $A \in \mathrm{Ob}(\mathscr{A})$ the 1-cell $\mathbf{1}_A$. Let $\mathscr{B}, \mathscr{C}$ be any other 2-categories, and $F : \mathscr{A} \to \mathscr{B}$, $G : \mathscr{C} \to \mathscr{A}$ any two pseudo-functors; clearly

$$i_\mathscr{A} * G = \mathbf{1}_G.$$

On the other hand, $F * i_\mathscr{A}$ is a pseudo-natural transformation that assigns to every $A \in \mathrm{Ob}(\mathscr{A})$ the 1-cell $F\mathbf{1}_A : FA \to FA$. If (δ^F, γ^F) is the coherence constraint for F, the coherence constraint of $F * i_\mathscr{A}$ assigns to every 1-cell $f : A \to A'$ of $\mathscr{A}$ the 2-cell $\mathbf{1}^F_f := \gamma^{F\,-1}_{f,\mathbf{1}_{A'}} \odot \gamma^F_{\mathbf{1}_A,f}$. Then, the unit axiom for F implies that the system of 2-cells $(\delta^F_A \mid A \in \mathrm{Ob}(\mathscr{A}))$ defines an invertible modification

$$\delta^F : \mathbf{1}_F \rightsquigarrow F * i_\mathscr{A}.$$

Remark 2.2.16. As already pointed out (remark 2.2.5(vi)), the datum of the category 2-$\mathbf{Cat}$ and the system of categories $(\mathsf{PsFun}(\mathscr{A}, \mathscr{B}) \mid \mathscr{A}, \mathscr{B} \in \mathrm{Ob}(2\text{-}\mathbf{Cat}))$ does not amount to a 2-category. However, example 2.2.15 suggests a straightforward variation that does result in an interesting 2-category. Namely, for every pair of categories $\mathscr{A}$ and $\mathscr{B}$, let us denote by

$$\overline{\mathsf{PsFun}}(\mathscr{A}, \mathscr{B})$$

the category whose objects are the pseudo-functors $\mathscr{A} \to \mathscr{B}$ and whose morphisms are the equivalence classes $[\beta]$ of pseudo-natural transformations β between such pseudo-functors, where we declare that two pseudo-natural transfomations $\alpha, \beta : F \Rightarrow G$ are equivalent if and only if there exists an invertible modification $\alpha \rightsquigarrow \beta$. The composition law for such equivalence classes is given by the rule

$$[\beta] \odot [\beta'] := [\beta \odot \beta']$$

for every two pseudo-natural transformations $\beta : F \Rightarrow G$, $\beta' : G \Rightarrow H$ of pseudo-functors $F, G, H : \mathscr{A} \to \mathscr{B}$. Then, if $\mathscr{C}$ is another 2-category, $F', G' : \mathscr{B} \to \mathscr{C}$ and $\alpha : F' \Rightarrow G'$ another pseudo-natural transformation, we set

$$[\alpha] * [\beta] := [(G' * \beta) \odot (\alpha * F)].$$

By virtue of example 2.2.15, this operation yields then a well defined functor

$$\overline{\mathsf{PsFun}}(\mathscr{A}, \mathscr{B}) \times \overline{\mathsf{PsFun}}(\mathscr{B}, \mathscr{C}) \to \overline{\mathsf{PsFun}}(\mathscr{A}, \mathscr{C})$$

that satisfies the associativity axiom for the composition bifunctor in a 2-category (see (2.1)). Thus, for every universe U', the U'-small 2-categories are the objects of a 2-category

$$\mathsf{U}'\text{-}\overline{2\text{-}\mathbf{Cat}}$$

that we call the *reduced* 2*-category of* U'*-small* 2*-categories*, whose 1-cells are the pseudo-functors and whose Hom-categories are the foregoing categories $\overline{\mathsf{PsFun}}(-, -)$. As usual, when $\mathsf{U}' = \mathsf{U}$ we write simply $\overline{2\text{-}\mathbf{Cat}}$ for this 2-category.

Remark 2.2.17. (i) Let $\mathscr{A}$ and $\mathscr{B}$ be two 2-categories. Every pair of pseudo-functors $F : \mathscr{A}' \to \mathscr{A}$ and $G : \mathscr{B} \to \mathscr{B}'$ induces a pseudo-functor

$$\mathsf{PsFun}(F, G) : \mathsf{PsFun}(\mathscr{A}, \mathscr{B}) \to \mathsf{PsFun}(\mathscr{A}', \mathscr{B}') \qquad (H : \mathscr{A} \to \mathscr{B}) \mapsto G \circ H \circ F$$

that assigns to every pseudo-natural transformation $\beta : H \Rightarrow H'$ between pseudo-functors $H, H' : \mathscr{A} \to \mathscr{B}$ the pseudo-natural transformation $G * \beta * F : G \circ H \circ F \Rightarrow G \circ H' \circ F$, and to every modification $\Xi : \beta \rightsquigarrow \beta'$ between such pseudo-natural transformations β, β', the modification $G \circ \Xi \circ F : G * \beta * F \rightsquigarrow G * \beta' * F$. The coherence constraint of $\mathsf{PsFun}(F, G)$ is the pair $(\delta^{(F,G)}_\bullet, \gamma^{(F,G)}_{\bullet\bullet})$ defined as follows :

$$\delta^{(F,G)}_H := \delta^G \circ H \circ F : \mathbf{1}_{GHF} \rightsquigarrow G * \mathbf{1}_{HF} \qquad \text{for every pseudo-functor } H : \mathscr{A} \to \mathscr{B}$$

where $\delta^G : \mathbf{1}_G \rightsquigarrow G * i_{\mathscr{A}}$ is the invertible modification provided by example 2.2.15(iii), and

$$\gamma^{(F,G)}_{\beta,\beta'} := \gamma^G_{\beta,\beta'} \circ F : (G * \beta' * F) \odot (G * \beta * F) \rightsquigarrow G * (\beta' \odot \beta) * F$$

for every pair of pseudo-natural transformations $\beta : H \Rightarrow H'$ and $\beta' : H' \Rightarrow H'$, where $\gamma^G \beta, \beta'$ is the invertible modification as in example 2.2.15(i). The verification is straightforward, in view of remark 2.2.10(vii). In case $F = \mathbf{1}_{\mathscr{A}}$ (resp. $G = \mathscr{B}$), we also write $\mathsf{PsFun}(\mathscr{A}, G)$ (resp. $\mathsf{PsFun}(F, \mathscr{B})$) for this pseudo-functor. Notice that $\mathsf{PsFun}(F, \mathscr{B})$ is strict for every pseudo-functor F, whereas $\mathsf{PsFun}(\mathscr{A}, G)$ is strict only if G is strict.

(ii) Every pseudo-natural transformation $\alpha : F \Rightarrow F'$ between pseudo-functors $F, F' : \mathscr{A}' \to \mathscr{A}$ induces a pseudo-natural transformation

$$\mathsf{PsFun}(\alpha, \mathscr{B}) : \mathsf{PsFun}(F, \mathscr{B}) \Rightarrow \mathsf{PsFun}(F', \mathscr{B}) \qquad (H : \mathscr{A} \to \mathscr{B}) \mapsto (H * \alpha : H \circ F \Rightarrow H \circ F')$$

whose coherence constraint assigns to every pseudo-natural transformation $\beta : H \Rightarrow H'$ the oriented square :

$$\begin{array}{ccc} H \circ F & \xRightarrow{H * \alpha} & H \circ F' \\ {\scriptstyle \beta * F} \Downarrow & \swarrow\!\!\!\!\!\! /_{(\tau^\beta_\alpha)^{-1}} & \Downarrow {\scriptstyle \beta * F'} \\ H' \circ F & \xRightarrow[H' * \alpha]{} & H' \circ F' \end{array}$$

where τ^β_α is the invertible modification provided by example 2.2.15(ii). Indeed, the naturality of the rule $\beta \mapsto (\tau^\beta_\alpha)^{-1}$ with respect to modifications $\Xi : \beta \rightsquigarrow \beta'$ follows directly from the compatibility condition for Ξ. To check the coherence axioms, notice first that $\tau^{\mathbf{1}_H}_\alpha = \mathbf{1}_{H * \alpha}$ for every pseudo-functor $H : \mathscr{A} \Rightarrow \mathscr{B}$. Lastly, if $\beta : H \Rightarrow H'$ and $\beta' : H' \Rightarrow H''$ are two pseudo-natural transformations, we need to show the identity

$$(\tau^{\beta'}_\alpha)^{-1} \boxbar (\tau^\beta_\alpha)^{-1} = (\tau^{\beta' \odot \beta}_\alpha)^{-1}$$

or equivalently : $\tau^{\beta'}_\alpha \boxminus \tau^\beta_\alpha = \tau^{\beta' \odot \beta}_\alpha$, which follows by direct inspection.

(iii) Likewise, every pseudo-natural transformation $\lambda : G \Rightarrow G'$ between pseudo-functors $G, G' : \mathscr{B} \to \mathscr{B}'$ induces a pseudo-natural transformation

$$\mathsf{PsFun}(\mathscr{A}, \lambda) : \mathsf{PsFun}(\mathscr{A}, G) \Rightarrow \mathsf{PsFun}(\mathscr{A}, G') \qquad (H : \mathscr{A} \to \mathscr{B}) \mapsto (\lambda * H : G \circ F \Rightarrow G' \circ F)$$

whose coherence constraint assigns to every pseudo-natural transformation $\beta : H \Rightarrow H'$ the oriented square τ^λ_β provided by example 2.2.15(ii). Indeed, the naturality of the rule : $\beta \mapsto \tau^\lambda_\beta$ with respect to modifications $\Xi : \beta \rightsquigarrow \beta'$ follows from the naturality of τ^λ, applied to the 2-cells $\Xi_A : \beta_A \to \beta'_A$, for every $A \in \mathrm{Ob}(\mathscr{A})$. The coherence condition for $\tau^\lambda_{\mathbf{1}_H}$ follows from the coherence condition for τ^λ_{HA}, for every pseudo-functor $H : \mathscr{A} \to \mathscr{B}$ and every $A \in \mathrm{Ob}(\mathscr{A})$. Likewise, the coherence condition relative to a composable pair of pseudo-natural transformations $\beta : H \Rightarrow H'$, $\beta' : H' \Rightarrow H''$ follows from the coherence condition for τ^λ, relative to the composable pair of 1-cells β_A, β'_A, for every $A \in \mathrm{Ob}(\mathscr{A})$.

(iv) Furthermore, for every F, F' as in (ii), every modification $\Xi : \alpha \rightsquigarrow \beta$ between pseudo-natural transformations $\alpha, \beta : F \Rightarrow F'$ induces a modification

$$\mathsf{PsFun}(\Xi, \mathscr{B}) : \mathsf{PsFun}(\alpha, \mathscr{B}) \rightsquigarrow \mathsf{PsFun}(\beta, \mathscr{B}) \qquad (G : \mathscr{A} \to \mathscr{B}) \mapsto (G \circ \Xi : G * \alpha \rightsquigarrow G * \beta).$$

Indeed, the compatibility condition for this modification amounts to the identity :

$$(G'(\Xi_X) * \gamma_{FX}) \odot (\tau^\gamma_{\alpha_X})^{-1} = (\tau^\gamma_{\beta_X})^{-1} \odot (\gamma_{F'X} * G(\Xi_X)) \qquad \text{for every } X \in \mathrm{Ob}(\mathscr{A})$$

which follows easily from the naturality of τ^γ.

(v) Suppose moreover that $\mathscr{B}$ is small; then we deduce a strict pseudo-functor

$$\mathsf{PsFun}(-, \mathscr{B}) : \overline{2\text{-}\mathbf{Cat}}^o \to \overline{2\text{-}\mathbf{Cat}} \qquad \mathscr{A} \mapsto \mathsf{PsFun}(\mathscr{A}, \mathscr{B})$$

acting on the 1-cells and 2-cells of $\overline{2\text{-}\mathbf{Cat}}^o$ as explicited in (i) and (ii). Indeed, if $F : \mathscr{A}' \to \mathscr{A}$ and $F' : \mathscr{A}'' \to \mathscr{A}'$ are two pseudo-functors between small 2-categories, a simple inspection shows that $\mathsf{PsFun}(F', \mathscr{B}) \circ \mathsf{PsFun}(F, \mathscr{B}) = \mathsf{PsFun}(F \circ F', \mathscr{B})$. Moreover, (iv) implies that if $\alpha, \beta : F \Rightarrow F'$ are pseudo-natural transformations with $[\alpha] = [\beta]$ in $\overline{2\text{-}\mathbf{Cat}}$, then $[\mathsf{PsFun}(\alpha, \mathscr{B})] = [\mathsf{PsFun}(\beta, \mathscr{B})]$. Next, for every three pseudo-functors $F, F', F'' : \mathscr{A}' \to \mathscr{A}$ and every pseudo-natural transformations $\alpha : F \Rightarrow F'$ and $\alpha' : F' \Rightarrow F''$, with the notation of remark 2.2.16, we need to check that :

$$(2.2.18) \qquad [\mathsf{PsFun}(\alpha', \mathscr{B}) \odot \mathsf{PsFun}(\alpha, \mathscr{B})] = [\mathsf{PsFun}(\alpha' \odot \alpha, \mathscr{B})].$$

Now, to every pseudo-functor $G : \mathscr{A} \to \mathscr{B}$, the left-hand side of (2.2.18) assigns the class of $(G*\alpha')\odot(G*\alpha)$, whereas the right-hand side assigns the class of $G*(\alpha'\odot\alpha)$. These two pseudo-natural transformations are related by the invertible modification $\gamma^G_{\alpha,\alpha'}$ of example 2.2.15(i), so we come down to checking that the rule : $G \mapsto \gamma^G_{\alpha,\alpha'}$ defines an invertible modification $\mathsf{PsFun}(\alpha', \mathscr{B}) \odot \mathsf{PsFun}(\alpha, \mathscr{B}) \rightsquigarrow \mathsf{PsFun}(\alpha' \odot \alpha, \mathscr{B})$. The latter assertion is an immediate consequence of the coherence axioms for τ^β : details left to the reader. Lastly, let us check that for every two pairs of pseudo-functors $F_1, F'_1 : \mathscr{A}' \to \mathscr{A}$, $F_2, F'_2 : \mathscr{A}'' \to \mathscr{A}'$ and every pseudo-natural transformations $\alpha_1 : F_1 \Rightarrow F'_1$, $\alpha_2 : F_2 \Rightarrow F'_2$ we have :

$$(2.2.19) \qquad [\mathsf{PsFun}(\alpha_2, \mathscr{B})] * [\mathsf{PsFun}(\alpha_1, \mathscr{B})] = [\mathsf{PsFun}([\alpha_1] * [\alpha_2], \mathscr{B})].$$

We may consider separately the case where $F_2 = F'_2$ and $\alpha_2 = \mathbf{1}_{F_2}$, and the case where $F_1 = F'_1$ and $\alpha_1 = \mathbf{1}_{F_1}$. In the first case, the left-hand side assigns to every $G : \mathscr{A} \to \mathscr{B}$ the pseudo-natural transformation $(G * \alpha_1) * F_2$, whereas the right-hand side assigns $G * (\alpha_1) * F_2$; but these coincide, and likewise, the coherence constraints of both sides is given by the rule : $\beta \mapsto (\tau^\beta_{\alpha_1 * F_2})^{-1}$. One argues similarly in the second case : details left to the reader.

2.3. **The formalism of base change.** Let $\mathscr{A}$ be a 2-category, and A, B two objects of $\mathscr{A}$. A *link from A to B in $\mathscr{A}$* is a datum

$$\mathscr{L} := (F, G, \eta, \varepsilon) : A \to B$$

consisting of :

- 1-cells $F : A \to B$ and $G : B \to A$
- 2-cells $\eta : \mathbf{1}_B \Rightarrow F \circ G$ and $\varepsilon : G \circ F \Rightarrow \mathbf{1}_A$

such that η and ε are related by the *triangular identities* :

$$(F * \varepsilon) \odot (\eta * F) = \mathbf{1}_F \qquad \text{and} \qquad (\varepsilon * G) \odot (G * \eta) = \mathbf{1}_G.$$

Especially, (G, F) is an adjoint pair of 1-cells of $\mathscr{A}$. We say that η and ε are the *unit* and *counit* of the link $\mathscr{L}$. We define a composition law for links as follows. With $\mathscr{L}$ as in the foregoing, a third object C of $\mathscr{A}$ and a second link $\mathscr{L}' := (F', G', \eta', \varepsilon') : B \to C$ we let

$$\mathscr{L}' \circ \mathscr{L} := (F' \circ F, G \circ G', \eta^{\mathscr{L}' \circ \mathscr{L}}, \varepsilon^{\mathscr{L}' \circ \mathscr{L}}) : A \to C$$

where $\eta^{\mathscr{L}'\circ\mathscr{L}} := (F' * \eta * G') \odot \eta'$ and $\varepsilon^{\mathscr{L}'\circ\mathscr{L}} := \varepsilon \odot (G * \varepsilon' * F)$. We need to check that $\alpha := (F'F * (\varepsilon \odot G * \varepsilon' * F)) \odot ((F' * \eta * G' \odot \eta') * F'F) = \mathbf{1}_{F'F}$. We compute :

$$\begin{aligned}\alpha &= (F' * ((F * \varepsilon) \odot (FG * \varepsilon' * F))) \odot (((F' * \eta * G'F') \odot (\eta' * F')) * F) \\ &= (F' * ((F * \varepsilon) \odot (\eta * F))) \odot (((F' * \varepsilon') \odot (\eta' * F')) * F) \\ &= (F' * \mathbf{1}_F) \odot (\mathbf{1}_{F'} * F) \\ &= \mathbf{1}_{F'F}\end{aligned}$$

where the second equality holds by remark 2.1.1(i) and the third follows from the triangular identities for (η, ε) and (η', ε'). Likewise one checks the second triangular identity for (η'', ε'') (the details are left to the reader).

Remark 2.3.1. (i) Notice that the composition law for links is associative; namely, in the situation of (2.3), if $\mathscr{L}'' := (F'', G'', \eta'', \varepsilon'') : C \to D$ is another link, we have $(\mathscr{L}'' \circ \mathscr{L}') \circ \mathscr{L} = \mathscr{L}'' \circ (\mathscr{L}' \circ \mathscr{L})$. Indeed, the assertion boils down to the identities :

$$\begin{aligned}(F'' * ((F' * \eta * G') \odot \eta') * G'') \odot \eta'' &= (F''F' * \eta * G'G'') \odot (F'' * \eta' * G'') \odot \eta'' \\ \varepsilon \odot (G * \varepsilon' * F) \odot (GG' * \varepsilon'' * F'F) &= \varepsilon \odot (G * (\varepsilon' \odot (G' * \varepsilon'' * F')) * F)\end{aligned}$$

which are both clear.

(ii) Notice also that every link $\mathscr{L} := (F, G, \eta, \varepsilon) : A \to B$ induces two *opposite links*

$$\begin{aligned}\mathscr{L}^o &:= (G^o, F^o, \eta^o, \varepsilon^o) : A^o \to B^o && \text{in } \mathscr{A}^o \\ {}^o\mathscr{L} &:= ({}^oG, {}^oF, {}^o\varepsilon, {}^o\eta) : {}^oB \to {}^oA && \text{in } {}^o\mathscr{A}.\end{aligned}$$

2.3.2. Let $\mathscr{L}, \mathscr{L}' : A \to B$ be two links, with $\mathscr{L} = (F, G, \eta, \varepsilon)$ and $\mathscr{L}' = (F', G', \eta', \varepsilon')$; a *transformation from $\mathscr{L}$ to $\mathscr{L}'$*, denoted

$$\beta : \mathscr{L} \Rightarrow \mathscr{L}'$$

is defined as a 2-cell $\beta : F \Rightarrow F'$. The standard operations on 2-cells can be upgraded easily to transformations of links :

- If $\mathscr{L}'' : A \to B$ is another link and $\beta' : \mathscr{L}' \Rightarrow \mathscr{L}''$ a second transformation, we get the transformation

$$\beta' \odot \beta : \mathscr{L} \Rightarrow \mathscr{L}''.$$

- For a diagram of links and transformations of links :

$$(2.3.3) \qquad A \underset{\mathscr{L}'}{\overset{\mathscr{L}}{\rightrightarrows}} \!\!\Downarrow\!\beta\; B \underset{\mathscr{P}'}{\overset{\mathscr{P}}{\rightrightarrows}} \!\!\Downarrow\!\alpha\; C$$

we get the transformation

$$\alpha * \beta : \mathscr{P} \circ \mathscr{L} \Rightarrow \mathscr{P}' \circ \mathscr{L}'.$$

For $\alpha = \mathbf{1}_{\mathscr{P}}$ (resp. for $\beta = \mathbf{1}_{\mathscr{L}}$) we also denote this transformation by $\mathscr{P} * \beta$ (resp. by $\alpha * \mathscr{L}$). Thus, the links of $\mathscr{A}$ are the 1-cells of a 2-category

$$\mathsf{Link}(\mathscr{A})$$

whose objects are the objects of $\mathscr{A}$ and whose 2-cells are the transformations of links, with the compositions laws for 1-cells and 2-cells given in the foregoing.

- For a transformation $\beta : \mathscr{L} \Rightarrow \mathscr{L}'$, we define an *adjoint* 2-cell of $\mathscr{A}$:

$$\beta^\dagger := (\varepsilon' * G) \odot (G' * \beta * G) \odot (G' * \eta) : G' \Rightarrow G.$$

The operations of remark 2.3.1(ii) can be combined with this adjoint construction to produce natural isomorphisms of 2-categories; indeed, we have :

Proposition 2.3.4. *For every 2-category $\mathscr{A}$ we have strict isomorphisms of 2-categories :*

$$^o\mathsf{Link}(\mathscr{A}) \stackrel{\sim}{\to} \mathsf{Link}(\mathscr{A}^o) \qquad \mathsf{Link}(\mathscr{A})^o \stackrel{\sim}{\to} \mathsf{Link}(^o\!\mathscr{A}).$$

Proof. Indeed, the first strict pseudo-functor is given by the rules :

$$A \mapsto A^o \qquad (\mathscr{L} : A \to B) \mapsto (\mathscr{L}^o : A^o \to B^o) \qquad (\beta : \mathscr{L} \Rightarrow \mathscr{L}') \mapsto (\beta^{\dagger o} : \mathscr{L}'^o \Rightarrow \mathscr{L}^o)$$

and the second one is given by the rules :

$$A \mapsto {}^o\!A \qquad (\mathscr{L} : A \to B) \mapsto ({}^o\mathscr{L} : {}^oB \to {}^o\!A) \qquad (\beta : \mathscr{L} \Rightarrow \mathscr{L}') \mapsto ({}^o\beta^\dagger : {}^o\mathscr{L} \Rightarrow {}^o\mathscr{L}')$$

for every object A, every 1-cell $\mathscr{L}$ and every 2-cell β of $\mathsf{Link}(\mathscr{A})$. The verification that these rules yield strict pseudo-functors comes down to the following :

Claim 2.3.5. (i) Let $\mathscr{L}, \mathscr{L}', \mathscr{L}'' : A \to B$ be three links of $\mathscr{A}$, and $\beta : \mathscr{L} \Rightarrow \mathscr{L}'$, $\beta' : \mathscr{L}' \Rightarrow \mathscr{L}''$ two transformations. Then we have $\beta^\dagger \odot \beta'^\dagger = (\beta' \odot \beta)^\dagger$.
(ii) In the situation of diagram (2.3.3) we have $(\alpha * \beta)^\dagger = \beta^\dagger * \alpha^\dagger$.

Proof of the claim. Say that $\mathscr{L} = (F, G, \eta, \varepsilon)$, $\mathscr{L}' = (F', G', \eta', \varepsilon')$, $\mathscr{L}'' = (F'', G'', \eta'', \varepsilon'')$. Recall that every 1-cell $f : X \to Y$ of $\mathscr{A}$ induces functors

$$f_*^Z : \mathscr{A}(Z, X) \to \mathscr{A}(Z, Y) \qquad (h : Z \to X) \mapsto f \circ h \qquad (\nu : h \Rightarrow h') \mapsto f * \nu$$

for every $Z \in \mathrm{Ob}(\mathscr{A})$, and every 2-cell $\lambda : f \Rightarrow g$ of $\mathscr{A}$ induces natural transformations

$$\lambda_*^Z : f_*^Z \Rightarrow g_*^Z \qquad (h : Z \to X) \mapsto (\lambda * h : f_*^Z(h) \Rightarrow g_*^Z(h))$$

(see the proof of lemma 2.1.13(i)). Then it is clear that for every $Z \in \mathrm{Ob}(\mathscr{A})$, every link $\mathscr{P} := (H, K, \eta^{\mathscr{P}}, \varepsilon^{\mathscr{P}})$ induces an adjoint pair of functors (H_*^Z, K_*^Z) with unit $(\eta^{\mathscr{P}})^Z$ and counit $(\varepsilon^{\mathscr{P}})^Z$. With this notation, it is easily seen that

$$(2.3.6) \qquad (\beta^\dagger)_*^Z = (\beta_*^Z, \eta_*^Z, \eta'^Z_*)^\dagger$$

where the right-hand side is the adjoint of the natural transformation β_*^Z, defined as in remark 1.1.17(ii) : the details are left to the reader. Then both (i) and (ii) follow straightforwardly from remark 1.1.17(iv,v). ◊

Lastly, in order to see that these pseudo-functors are isomorphisms of 2-categories, it suffices to show more precisely that for every transformation of links $\beta : \mathscr{L} \Rightarrow \mathscr{L}'$ we have :

$$(\beta^\dagger)^\dagger = \beta.$$

Again, by virtue of (2.3.6), the assertion is reduced to the corresponding identity for adjoints of natural transformations, and the latter is known by virtue of remark 1.1.17(ii). □

2.3.7. Notice that every 1-cell $F : A \to B$ of the 2-category $\mathscr{A}$ induces a strict pseudo-functor

$$F_* : \mathscr{A}/A \to \mathscr{A}/B \qquad (g : X \to A) \mapsto (F \circ g : X \to B)$$

that maps every 1-cell $(h, \alpha) : (X \xrightarrow{g} A) \to (Y \xrightarrow{g'} A)$ of $\mathscr{A}/A$ to the 1-cell of $\mathscr{A}/B$

$$F * (h, \alpha) := (h, F * \alpha)$$

and every 2-cell $\beta : (h_1, \alpha_1) \Rightarrow (h_2, \alpha_2)$ of $\mathscr{A}/A$ to the 2-cell $\beta : (h_1, F * \alpha_1) \Rightarrow (h_2, F * \alpha_2)$ of $\mathscr{A}/B$ (notation of example 2.1.11(ii)). With this notation, we claim that every link $\mathscr{L} := (F, G, \eta, \varepsilon) : A \to B$ of $\mathscr{A}$ induces an adjunction for the pair of functors $F_* : \mathscr{A}/A \to \mathscr{A}/B$ and $G_* : \mathscr{A}/B \to \mathscr{A}/A$. Namely, we have a natural bijection :

$$\vartheta^{\mathscr{L}}_{g,f} : \mathscr{A}/B(Y \xrightarrow{g} B, X \xrightarrow{F \circ f} B) \stackrel{\sim}{\to} \mathscr{A}/A(Y \xrightarrow{G \circ g} A, X \xrightarrow{f} A) \quad (h, \alpha) \mapsto (h, (\varepsilon * f) \odot (G * \alpha))$$

whose inverse $\mu^{\mathscr{L}}_{g,f} : \mathscr{A}/A(G \circ g, f) \stackrel{\sim}{\to} \mathscr{A}/B(g, F \circ f)$ is given by the rule :

$$(k, \beta) \mapsto (k, (F * \beta) \odot (\eta * g)).$$

The verification of the naturality of $\vartheta^{\mathscr{L}}_{f,g}$ with respect to f and g shall be left to the reader. Let us check that $\mu^{\mathscr{L}}_{g,f} \circ \vartheta^{\mathscr{L}}_{g,f}$ is the identity map on the set $\mathscr{A}/B(g, F \circ f)$: indeed, for every 1-cell (h, α), this composition is computed as the composition of the following oriented squares :

$$\begin{array}{ccccc}
Y & \xrightarrow{h} & X & = & X \\
{\scriptstyle g}\downarrow & & {\scriptstyle f}\downarrow & \mathbf{1}\Nearrow & \downarrow{\scriptstyle f} \\
B & \alpha\Nearrow & A & = & A \\
\| & & {\scriptstyle F}\downarrow & \varepsilon\Nearrow & \| \\
B & = & B & \xrightarrow{G} & A \\
\| & \mathbf{1}\Nearrow & \| & \eta\Nearrow & \downarrow{\scriptstyle F} \\
B & = & B & = & B.
\end{array}$$

Then the assertion follows easily, by applying the triangular identities for the pair (η, ε) and invoking as usual proposition 2.1.9. Likewise we check that $\vartheta^{\mathscr{L}}_{g,f} \circ \mu^{\mathscr{L}}_{g,f}$ is the identity on $\mathscr{A}/A(G \circ g, f)$: the details shall be left to the reader.

2.3.8. Consider now an *oriented square* $\mathscr{D}$ of links, together with its *adjoint squares* :

$$\mathscr{D}\ :\ \begin{array}{ccc}
A' & \xrightarrow{\mathscr{L}'} & B' \\
{\scriptstyle \mathscr{M}'}\downarrow & \swarrow\!\!\!\!\!\!/\ {\scriptstyle\beta} & \downarrow{\scriptstyle \mathscr{M}} \\
A & \xrightarrow[\mathscr{L}]{} & B
\end{array}
\qquad
\mathscr{D}^\dagger\ :\ \begin{array}{ccc}
A'^o & \xrightarrow{\mathscr{L}'^o} & B'^o \\
{\scriptstyle \mathscr{M}'^o}\downarrow & {\scriptstyle\beta^{\dagger o}}\Nearrow & \downarrow{\scriptstyle \mathscr{M}^o} \\
A^o & \xrightarrow[\mathscr{L}^o]{} & B^o
\end{array}
\qquad
{}^\dagger\mathscr{D}\ :\ \begin{array}{ccc}
{}^oB & \xrightarrow{{}^o\mathscr{L}} & {}^oA \\
{\scriptstyle {}^o\mathscr{M}}\downarrow & {\scriptstyle {}^o\beta^{\dagger}}\Nearrow & \downarrow{\scriptstyle {}^o\mathscr{M}'} \\
{}^oB' & \xrightarrow[{}^o\mathscr{L}']{} & {}^oA'
\end{array}$$

i.e. the orientation $\beta : \mathscr{M} \circ \mathscr{L}' \Rightarrow \mathscr{L} \circ \mathscr{M}'$ is a transformation of links, so that ${}^o\beta^\dagger$ is the adjoint transformation ${}^o\mathscr{L}' \circ {}^o\mathscr{M} \Rightarrow {}^o\mathscr{M}' \circ {}^o\mathscr{L}$, and likewise for $\beta^{\dagger o}$. Say that $\mathscr{L} = (L_*, L^*, \eta^L, \varepsilon^{\mathscr{L}})$, $\mathscr{L}' = (L'_*, L'^*, \eta^{\mathscr{L}'}, \varepsilon^{\mathscr{L}'})$, $\mathscr{M} = (M_*, M^*, \eta^M, \varepsilon^{\mathscr{M}})$ and $\mathscr{M}' = (M'_*, M'^*, \eta^{\mathscr{M}'}, \varepsilon^{\mathscr{M}'})$. Notice that for horizontally composable oriented squares of links we have the identities :

$$(2.3.9)\qquad (\mathscr{D}' \boxminus \mathscr{D})^\dagger = \mathscr{D}'^\dagger \boxplus \mathscr{D}^\dagger \qquad {}^\dagger(\mathscr{D}' \boxminus \mathscr{D}) = {}^\dagger\mathscr{D} \boxplus {}^\dagger\mathscr{D}'.$$

Thus, the operation $\mathscr{D} \mapsto \mathscr{D}^\dagger$ for oriented squares of links differs in this respect from the analogous operation for oriented squares in the 2-category $\mathscr{A}$: see (2.1.8).

With this notation, we attach to the oriented square $\mathscr{D}$ the *base change oriented square* in $\mathscr{A}$:

$$\Upsilon(\mathscr{D})\ :\ \begin{array}{ccc}
B' & \xrightarrow{M_*} & B \\
{\scriptstyle L'^*}\downarrow & \swarrow\!\!\!\!\!\!/\ {\scriptstyle\Upsilon(\beta)} & \downarrow{\scriptstyle L^*} \\
A' & \xrightarrow[M'_*]{} & A
\end{array}$$

whose orientation is the *base change transformation*

$$\Upsilon(\beta) := (\varepsilon^{\mathscr{L}} * M'_* L'^*) \odot (L^* * \beta * L'^*) \odot (L^* M_* * \eta^{\mathscr{L}'}) : L^* M_* \Rightarrow M'_* L'^*.$$

Proposition 2.3.10. *With the notation of* (2.3.8)*, we have :*

$$\Upsilon(\mathscr{D})^o = \Upsilon(\mathscr{D}^\dagger) \qquad {}^o\Upsilon(\mathscr{D}) = \Upsilon({}^\dagger\mathscr{D}).$$

Proof. The sought identity for $\Upsilon({}^\dagger\mathscr{D})$ will be proven by inspecting the following diagram :

$$\begin{array}{ccccccccccc}
B' & \xrightarrow{M_*} & B & = & B & = & B & = & B & = & B \\
\| & \Swarrow_{\varepsilon^{\mathscr{M}}} & \downarrow{\scriptstyle M^*} & \Swarrow_{\mathbf{1}_{M^*}} & \downarrow{\scriptstyle M^*} & \Swarrow_{\mathbf{1}_{M^*}} & \downarrow{\scriptstyle M^*} & \Swarrow_{\eta^{\mathscr{M}}} & \| & \Swarrow_{i_B} & \| \\
B' & = & B' & = & B' & = & B' & \xrightarrow{M_*} & B & = & B \\
{\scriptstyle L'^*}\downarrow & \Swarrow_{\mathbf{1}_{L'^*}} & \downarrow{\scriptstyle L'^*} & \Swarrow_{\mathbf{1}_{L'^*}} & \downarrow{\scriptstyle L'^*} & \Swarrow_{\eta^{\mathscr{L}'}} & \| & \Swarrow_{\mathbf{1}_{M_*}} & \| & \Swarrow_{i_B} & \| \\
A' & = & A' & = & A' & \xrightarrow{L'_*} & B' & \xrightarrow{M_*} & B & = & B \\
\| & \Swarrow_{i_{A'}} & \| & \Swarrow_{\mathbf{1}_{M'_*}} & \downarrow{\scriptstyle M'_*} & \Swarrow_{\beta} & \downarrow{\scriptstyle M_*} & \Swarrow_{\mathbf{1}_{M_*}} & \| & \Swarrow_{i_B} & \| \\
A' & = & A' & \xrightarrow{M'_*} & A & \xrightarrow{L_*} & B & = & B & = & B \\
\| & \Swarrow_{i_{A'}} & \| & \Swarrow_{\mathbf{1}_{M_*}} & \| & \Swarrow_{\varepsilon^{\mathscr{L}}} & \downarrow{\scriptstyle L^*} & \Swarrow_{\mathbf{1}_{L^*}} & \downarrow{\scriptstyle L^*} & \Swarrow_{\mathbf{1}_{L^*}} & \downarrow{\scriptstyle L^*} \\
A' & = & A' & \xrightarrow{M'_*} & A & = & A & = & A & = & A \\
\| & \Swarrow_{i_{A'}} & \| & \Swarrow_{\varepsilon^{\mathscr{M}'}} & \downarrow{\scriptstyle M'^*} & \Swarrow_{\mathbf{1}_{M'^*}} & \downarrow{\scriptstyle M'^*} & \Swarrow_{\mathbf{1}_{M'^*}} & \downarrow{\scriptstyle M'^*} & \Swarrow_{\eta^{\mathscr{M}'}} & \| \\
A' & = & A' & = & A' & = & A' & = & A' & \xrightarrow[M'_*]{} & A.
\end{array}$$

Indeed, since the unit $\eta^{{}^o\mathscr{M}}$ of ${}^o\mathscr{M}$ is ${}^o(\varepsilon^{\mathscr{M}})$ and the counit $\varepsilon^{{}^o\mathscr{M}'}$ of ${}^o\mathscr{M}'$ is ${}^o(\eta^{\mathscr{M}'})$, we have :

$$\begin{aligned}
\Upsilon({}^\dagger\mathscr{D}) &= (\varepsilon^{{}^o\mathscr{M}'} * {}^oL^*\,{}^oM_*) \odot ({}^oM'_* * {}^o\beta^\dagger * {}^oM_*) \odot ({}^oM'_*\,{}^oL'^* * \eta^{{}^o\mathscr{M}}) \\
&= {}^o((M'_*L'^* * \varepsilon^{\mathscr{M}}) \odot (M'_* * \beta^\dagger * M_*) \odot (\eta^{\mathscr{M}'} * L^*M_*))
\end{aligned}$$

and if $\eta^{\mathscr{M}\circ\mathscr{L}'}$ is the unit of $\mathscr{M}\circ\mathscr{L}'$ and $\varepsilon^{\mathscr{L}\circ\mathscr{M}'}$ is the counit of $\mathscr{L}\circ\mathscr{M}'$, we have

$$\eta^{\mathscr{M}\circ\mathscr{L}'} = (M_* * \eta^{\mathscr{L}'} * M^*) \odot \eta^{\mathscr{M}} \qquad \varepsilon^{\mathscr{L}\circ\mathscr{M}'} = \varepsilon^{\mathscr{M}'} \odot (M'^* * \varepsilon^{\mathscr{L}} * M'_*)$$
$$\beta^\dagger = (\varepsilon^{\mathscr{L}\circ\mathscr{M}'} * L'^*M^*) \odot (M'^*L^* * \beta * L'^*M^*) \odot (M'^*L^* * \eta^{\mathscr{M}\circ\mathscr{L}'}).$$

Thus, if in the foregoing diagram we initially disregard the first and last columns, we find that the composition of the six remaining squares in the uppermost two rows equals precisely $\eta^{\mathscr{M}\circ\mathscr{L}'}$, and the composition of the six squares in the bottom two rows equals $\varepsilon^{\mathscr{L}\circ\mathscr{M}'}$. Therefore, if we further compose these two blocks of six squares with the central row of three squares, we get precisely $\beta^\dagger$. And if we finally compose the result with the two omitted left and right columns, we then get ${}^o\Upsilon({}^\dagger\mathscr{D})$. But according to proposition 2.1.9 we may compose these squares in a different order : notice then that the composition of the five squares in the uppermost row equals $\mathbf{1}_{M_*}$, and the composition of the five square in the bottom row equals $\mathbf{1}_{M'_*}$. So, we may disregard these two rows; but in the remaining three central rows, we may likewise disregard everything except the central column, since the squares outside the central column are all identities. So, finally we are left with the vertical composition of the squares in the central column, and that gives precisely $\Upsilon(\mathscr{D})$, as required. Lastly, we have :

$$\begin{aligned}
\Upsilon(\mathscr{D}^\dagger) &= (\varepsilon^{\mathscr{M}^o} * L'^{*o}\,M'^o_*) \odot (M^o_* * \beta^{\dagger o} * M'^o_*) \odot (M^o_*\,L^{*o} * \eta^{\mathscr{M}'^o}) \\
&= ((M'_*L'^* * \varepsilon^{\mathscr{M}}) \odot (M'_* * \beta^\dagger * M_*) \odot (\eta^{\mathscr{M}'} * L^*M_*))^o
\end{aligned}$$

so the sought identity for $\Upsilon(\mathscr{D}^\dagger)$ follows from that for $\Upsilon({}^\dagger\mathscr{D})$. □

Remark 2.3.11. In the situation of (2.3.8), suppose that β is an invertible 2-cell of $\mathscr{A}$ and that both L_* and L'_* (resp. both M_* and M'_*) are equivalences in $\mathscr{A}$. Then it follows easily from the definition (resp. and from proposition 2.3.10) that also $\Upsilon(\beta)$ is an invertible 2-cell.

Proposition 2.3.12. *Consider two oriented squares of links :*

$$\mathscr{D}\ :\quad \begin{array}{ccc} A' & \xrightarrow{\mathscr{L}'} & B' \\ {\scriptstyle \mathscr{M}'}\big\downarrow & \not\swarrow_{\beta} & \big\downarrow{\scriptstyle \mathscr{M}} \\ A & \xrightarrow[\mathscr{L}]{} & B \end{array} \qquad\qquad \mathscr{D}'\ :\quad \begin{array}{ccc} C' & \xrightarrow{\mathscr{P}'} & D' \\ {\scriptstyle \mathscr{Q}'}\big\downarrow & \not\swarrow_{\alpha} & \big\downarrow{\scriptstyle \mathscr{Q}} \\ C & \xrightarrow[\mathscr{P}]{} & D. \end{array}$$

(i) *Suppose that $A = C'$, $B = D'$ and $\mathscr{L} = \mathscr{P}'$, so that $\mathscr{D}' \boxminus\!\!\!\!\mid\ \mathscr{D}$ is well defined. Then :*

$$\Upsilon(\mathscr{D}' \,\square\!\!\!\!\mid\,\, \mathscr{D}) = \Upsilon(\mathscr{D}') \boxminus \Upsilon(\mathscr{D}).$$

(ii) *Suppose that $B = C$, $B' = C'$ and $\mathscr{M} = \mathscr{Q}'$, so that $\mathscr{D}' \boxminus \mathscr{D}$ is well defined. Then :*

$$\Upsilon(\mathscr{D}' \boxminus \mathscr{D}) = \Upsilon(\mathscr{D}) \,\square\!\!\!\!\mid\,\, \Upsilon(\mathscr{D}').$$

Proof. (i): Say that $\mathscr{L} = (L_*, L^*, \eta^{\mathscr{L}}, \varepsilon^{\mathscr{L}})$, $\mathscr{P} = (P_*, P^*, \eta^{\mathscr{P}}, \varepsilon^{\mathscr{P}})$, $\mathscr{Q} = (Q_*, Q^*, \eta^{\mathscr{Q}}, \varepsilon^{\mathscr{Q}})$, $\mathscr{M}' = (M'_*, M'^*, \eta^{\mathscr{M}'}, \varepsilon^{\mathscr{M}'})$, $\mathscr{M} = (M_*, M^*, \eta^{\mathscr{M}}, \varepsilon^{\mathscr{M}})$ and $\mathscr{Q}' = (Q'_*, Q'^*, \eta^{\mathscr{Q}'}, \varepsilon^{\mathscr{Q}'})$. We consider the following diagram :

$$\begin{array}{ccccccccccc}
B' & \xrightarrow{L'^*} & A' & \xrightarrow{M'_*} & A & = & A & \xrightarrow{Q'^*} & C & = & C \\
\| & \eta^{\mathscr{L}'}\Uparrow & \big\downarrow{\scriptstyle L'_*} & \beta\Uparrow & \big\downarrow{\scriptstyle L_*} & \varepsilon^{\mathscr{L}}\Uparrow & \| & \mathbf{1}_{Q'^*}\Uparrow & \| & i_C\Uparrow & \| \\
B' & = & B' & \xrightarrow{M_*} & B & \xrightarrow{L^*} & A & \xrightarrow{Q'^*} & C & = & C \\
\| & i_{B'}\Uparrow & \| & \mathbf{1}_{M_*}\Uparrow & \| & \eta^{\mathscr{L}}\Uparrow & {\scriptstyle L_*}\big\downarrow & \alpha\Uparrow & {\scriptstyle P_*}\big\downarrow & \varepsilon^{\mathscr{P}}\Uparrow & \| \\
B' & = & B' & \xrightarrow[M_*]{} & B & = & B & \xrightarrow[Q_*]{} & D & \xrightarrow[P^*]{} & C
\end{array}$$

and notice that the composition of the five squares of the top (resp. bottom) row equals $\Upsilon(\mathscr{D})$ (resp. $\Upsilon(\mathscr{D}')$), so the composition of all the square in the diagram is precisely $\Upsilon(\mathscr{D}') \boxminus \Upsilon(\mathscr{D})$. On the other hand, the composition of the two squares in the first (resp. second, resp. third, resp. fourth, resp. fifth) column from the left equals $\eta^{\mathscr{L}'}$ (resp. β, resp. $\mathbf{1}_{L_*}$, resp. α, resp. $\varepsilon^{\mathscr{P}}$), and then the composition of these five columns also equals $\Upsilon(\mathscr{D}' \,\square\!\!\!\!\mid\,\, \mathscr{D})$, as stated.

(ii): We compute :

$$\begin{aligned}
\Upsilon(\mathscr{D}' \boxminus \mathscr{D}) = \Upsilon(\mathscr{D}'^{\dagger\dagger} \boxminus \mathscr{D}^{\dagger\dagger}) = \Upsilon((\mathscr{D}'^{\dagger} \,\square\!\!\!\!\mid\,\, \mathscr{D}^{\dagger})^{\dagger}) &= \Upsilon(\mathscr{D}'^{\dagger} \,\square\!\!\!\!\mid\,\, \mathscr{D}^{\dagger})^{o} \\
&= (\Upsilon(\mathscr{D}'^{\dagger}) \boxminus \Upsilon(\mathscr{D}^{\dagger}))^{o} \\
&= \Upsilon(\mathscr{D}^{\dagger})^{o} \boxminus \Upsilon(\mathscr{D}'^{\dagger})^{o} \\
&= \Upsilon(\mathscr{D}^{\dagger\dagger}) \boxminus \Upsilon(\mathscr{D}'^{\dagger\dagger}) \\
&= \Upsilon(\mathscr{D}) \boxminus \Upsilon(\mathscr{D}')
\end{aligned}$$

where the second equality follows from (2.3.9), the third and sixth follow from proposition 2.3.10, and the fourth follows from (i), and the fifth follows from (2.1.8). □

2.3.13. In the situation of (2.3.8), the links $\mathscr{L}$ and $\mathscr{L}'$ induce adjunctions $\vartheta^{\mathscr{L}}$ and $\vartheta^{\mathscr{L}'}$ for the pairs of functors $((L^*)_*, (L_*)_*)$ and respectively $((L'^*)_*, (L'_*)_*)$, as in (2.3.7). The base change transformation $\Upsilon(\beta)$ relates these adjunctions, as explained by the following :

Proposition 2.3.14. *With the notation of* (2.3.13)*, for every $(X \xrightarrow{f} A') \in \mathrm{Ob}(\mathscr{A}/A')$, every $(Y \xrightarrow{g} B') \in \mathrm{Ob}(\mathscr{A}/B')$, and every 1-cell $(h, \alpha) : g \to L'_* \circ f$ of $\mathscr{A}/B'$ we have the identity :*

$$(M'_* * \vartheta^{\mathscr{L}'}_{f,g}(h, \alpha)) \circ (\mathbf{1}_Y, \Upsilon(\beta) * g) = \vartheta^{\mathscr{L}}_{M'_* \circ f, M_* \circ g}((\mathbf{1}_X, \beta * f) \circ (h, M_* * \alpha)) \qquad \text{in } \mathscr{A}/A.$$

Proof. The left-hand side of the stated identity is computed as the composition of the following oriented squares :

$$\begin{array}{ccccccccc}
Y & \xrightarrow{h} & X & = & X & = & & = & X \\
{\scriptstyle g}\downarrow & & {\scriptstyle f}\downarrow & \mathbf{1}\nearrow & \downarrow{\scriptstyle f} & & & & \| \\
B' & \alpha\nearrow & A' & = & A' & & \mathbf{1}\nearrow & & \| \\
\| & & {\scriptstyle L'_*}\downarrow & \varepsilon^{\mathscr{L}'}\nearrow & \| & & & & \| \\
B' & = & B' & \xrightarrow{L'^*} & A' & \xrightarrow{M'_*} & A & = & A \\
\| & \mathbf{1}\nearrow & \| & \eta^{\mathscr{L}'}\nearrow & {\scriptstyle L'_*}\downarrow & \beta\nearrow & {\scriptstyle L_*}\downarrow & \varepsilon^{\mathscr{L}}\nearrow & \| \\
B' & = & B' & = & B' & \xrightarrow{M_*} & B & \xrightarrow{L^*} & A.
\end{array}$$

By applying as usual proposition 2.1.9, and the triangular identities for the pair $(\eta^{\mathscr{L}'}, \varepsilon^{\mathscr{L}'})$, we easily see that the same composition of squares yields as well the right-hand side of the sought identity : the details are left to the reader. □

2.3.15. Recall that the links of $\mathscr{A}$ are the objects of the 2-category $2\text{-}\mathsf{Morph}(\mathsf{Link}(\mathscr{A}))$: see example 2.1.11(i). An oriented square $\mathscr{D}$ as in (2.3.8) yields a 1-cell in this 2-category

$$\mathscr{D}(\beta, \mathscr{M}, \mathscr{M}') : \mathscr{L}' \to \mathscr{L}.$$

Given a pair of oriented squares $\mathscr{D}_1(\beta_1, \mathscr{M}_1, \mathscr{M}'_1), \mathscr{D}_2(\beta_2, \mathscr{M}_2, \mathscr{M}'_2) : \mathscr{L}' \to \mathscr{L}$, a 2-cell $(\alpha, \alpha') : \mathscr{D}_1 \Rightarrow \mathscr{D}_2$ is the datum of a pair of 2-cells of the 2-category $\mathsf{Link}(\mathscr{A})$

$$(2.3.16) \qquad \alpha : \mathscr{M}_1 \Rightarrow \mathscr{M}_2 \quad \alpha' : \mathscr{M}'_1 \Rightarrow \mathscr{M}'_2 \quad \text{such that} \quad \beta_2 \odot (\alpha * \mathscr{L}') = (\mathscr{L} * \alpha') \odot \beta_1.$$

Theorem 2.3.17. *The rules :*

$$\mathscr{L} := (L_*, L^*, \eta, \varepsilon) \mapsto L^{*o} \qquad \mathscr{D} \mapsto \Upsilon(\mathscr{D})^o \qquad (\alpha, \alpha') \mapsto (\alpha^o, \alpha'^o)$$

for every link $\mathscr{L}$, every oriented square $\mathscr{D}$ of $\mathsf{Link}(\mathscr{A})$, and every 2-cell (α, α') as in (2.3.15) *define a strict pseudo-functor :*

$$\Upsilon_{\mathscr{A}} : 2\text{-}\mathsf{Morph}(\mathsf{Link}(\mathscr{A}))^o \to 2\text{-}\mathsf{Morph}(\mathscr{A}^o).$$

Proof. The functoriality of Υ for composition of 1-cells is already clear from proposition 2.3.12. Let us next check that for every 2-cell $(\alpha, \alpha') : \mathscr{D}_1 \Rightarrow \mathscr{D}_2$ as in (2.3.15), the pair (α^o, α'^o) is a 2-cell $\Upsilon(\mathscr{D}_1)^o \Rightarrow \Upsilon(\mathscr{D}_2)^o$ in $2\text{-}\mathsf{Morph}(\mathscr{A}^o)$. We come down to showing the identity

$$(2.3.18) \qquad \Upsilon(\beta_2)^o \odot (\alpha^o * L'^{*o}) = (L'^{*o} * \alpha') \odot \Upsilon(\beta_1)^o.$$

To this aim, for every $X \in \mathrm{Ob}(\mathscr{A})$ let $\mathbb{1}_X : X \to X$ be the link $(\mathbf{1}_X, \mathbf{1}_X, i_X, i_X)$ (where i_X is the identity automorphism of $\mathbf{1}_X$); also, for $i = 1, 2$ let $\mathscr{P}_i := (P_{i*} : X \to Y, P_i^*, \eta_i, \varepsilon_i)$ be a link of $\mathscr{A}$, and $\tau : P_{1*} \Rightarrow P_{2*}$ a 2-cell of $\mathscr{A}$. We consider the following oriented squares of $\mathsf{Link}(\mathscr{A})$ and of $\mathscr{A}^o$:

$$\mathscr{D}_\tau \ : \ \begin{array}{ccc} X & \xrightarrow{\mathbb{1}_X} & X \\ {\scriptstyle \mathscr{P}_2}\downarrow & \swarrow_\tau & \downarrow{\scriptstyle \mathscr{P}_1} \\ Y & \xrightarrow[\mathbb{1}_Y]{} & Y \end{array} \qquad\qquad \mathscr{E}_\tau \ : \ \begin{array}{ccc} Y^o & \xrightarrow{\mathbf{1}_{Y^o}} & Y^o \\ {\scriptstyle P^o_{2*}}\downarrow & \swarrow_{\tau^o} & \downarrow{\scriptstyle P^o_{1*}} \\ X^o & \xrightarrow[\mathbf{1}_{X^o}]{} & X^o. \end{array}$$

With this notation, by applying proposition 2.3.10 it is easily seen that

$$\Upsilon(\mathscr{D}_\tau)^o = \mathscr{E}_\tau$$

and (2.3.18) is equivalent to the identity :

$$\Upsilon(\mathscr{D}_1)^o \boxminus \mathscr{E}_{\alpha'} = \mathscr{E}_\alpha \boxminus \Upsilon(\mathscr{D}_2)^o.$$

In view of (2.3.9) and proposition 2.3.12 we are then reduced to checking that

$$\mathscr{D}_1 \boxminus \mathscr{D}_{\alpha'} = \mathscr{D}_\alpha \boxminus \mathscr{D}_2$$

which follows immediately from (2.3.16). Lastly, the functoriality of Υ for the two types of compositions of 2-cells is obvious from the definition. □

2.3.19. Resume the situation of (2.3.8), and notice that the units and counits for $\mathscr{M}$ and $\mathscr{M}'$ do not play any role in the definition of $\Upsilon(\beta)$, and neither do the 1-cells M^* and M'^*. This leads to the following variant of theorem 2.3.17. We replace $2\text{-}\mathsf{Morph}(\mathsf{Link}(\mathscr{A}))$ by the 2-category

$$\mathsf{wLink}(\mathscr{A})$$

of *weak links in* $\mathscr{A}$, whose objects are still the links of $\mathscr{A}$; for any two links $\mathscr{L} := (L_* : A \to B, L^*, \eta^{\mathscr{L}}, \varepsilon^{\mathscr{L}})$ and $\mathscr{L}' := (L'_* : A' \to B', L'^*, \eta^{\mathscr{L}'}, \varepsilon^{\mathscr{L}'})$, the 1-cells $\mathscr{L} \to \mathscr{L}'$ are the 1-cells $(M_*, M'_*, \beta) : L_* \to L'_*$ of the 2-category $2\text{-}\mathsf{Morph}(\mathscr{A})$, *i.e.* the oriented squares

$$\mathscr{D} \ : \quad \begin{array}{ccc} A' & \xrightarrow{L'_*} & B' \\ {\scriptstyle M'_*}\downarrow & \swarrow_{\beta} & \downarrow{\scriptstyle M_*} \\ A & \xrightarrow[L_*]{} & B. \end{array}$$

The 2-cells of $\mathsf{wLink}(\mathscr{A})$, and the composition laws for 1-cells and 2-cells are then the same as for $2\text{-}\mathsf{Morph}(\mathscr{A})$. For any such 1-cell $\mathscr{D}$, we may then define the 2-cell $\Upsilon(\beta)$ and the oriented square $\Upsilon(\mathscr{D})$ as in (2.3.8). A direct inspection shows that part (i) of proposition 2.3.12 still holds for weak links, with the same proof. The proof of part (ii) of proposition 2.3.12 does not apply to weak links, but we may check the assertion directly. Indeed, consider another 1-cell $\mathscr{P}' := (P'_*, P'^*, \eta^{\mathscr{P}'}, \varepsilon^{\mathscr{P}'}) \to \mathscr{P} := (P_*, P^*, \eta^{\mathscr{P}}, \varepsilon^{\mathscr{P}})$ of $\mathsf{wLink}(\mathscr{A})$:

$$\mathscr{D}' \ : \quad \begin{array}{ccc} B' & \xrightarrow{M_*} & C' \\ {\scriptstyle Q_*}\downarrow & \swarrow_{\alpha} & \downarrow{\scriptstyle Q_*} \\ B & \xrightarrow[P_*]{} & C. \end{array}$$

Set :

$$X := \varepsilon^{\mathscr{L}} * M'_* L'^* P'^* \qquad\qquad Y := L^* P^* Q_* * \eta^{\mathscr{P}'}$$

$$\delta := (P_* * \beta) \odot (\alpha * L'_*) \qquad \alpha' := P^* * \alpha * P'^* \qquad \beta' := L^* * \beta * L'^*.$$

By definition we have :

$$\Upsilon(\mathscr{D}' \boxminus \mathscr{D}) = X \odot (L^* * \varepsilon^{\mathscr{P}} * L_* M'_* L'^* P'^*) \odot (L^* P^* * \delta * L'^* P'^*) \odot (L^* P^* Q_* P'_* * \eta^{\mathscr{L}'} * P'^*) \odot Y$$

$$\Upsilon(\mathscr{D}) \boxdot \Upsilon(\mathscr{D}') = X \odot ((\beta' \odot (L^* M_* * \eta^{\mathscr{L}'})) * P'^*) \odot (L^* * ((\varepsilon^{\mathscr{P}} * M_* P'^*) \odot \alpha')) \odot Y.$$

Hence we are reduced to checking that

$$Z := (\varepsilon^{\mathscr{P}} * L_* M'_* L'^*) \odot (P^* * \delta * L'^*) \odot (P^* Q_* P'_* * \eta^{\mathscr{L}'})$$

equals

$$Z' := ((\beta * L'^*) \odot (M_* * \eta^{\mathscr{L}'})) \odot (\varepsilon^{\mathscr{P}} * M_*) \odot (P^* * \alpha).$$

The latter follows by applying repeatedly remark 2.1.1(i) : the details shall be left to the reader. With these preliminaries, we may then also repeat the proof of theorem 2.3.17 : summing up, we obtain a strict pseudo-functor :

$$\Upsilon : \mathsf{wLink}(\mathscr{A})^o \to 2\text{-}\mathsf{Morph}(\mathscr{A}^o)$$

given by the same rules as for the previously defined one on $2\text{-}\mathsf{Morph}(\mathsf{Link}(\mathscr{A}))^o$.

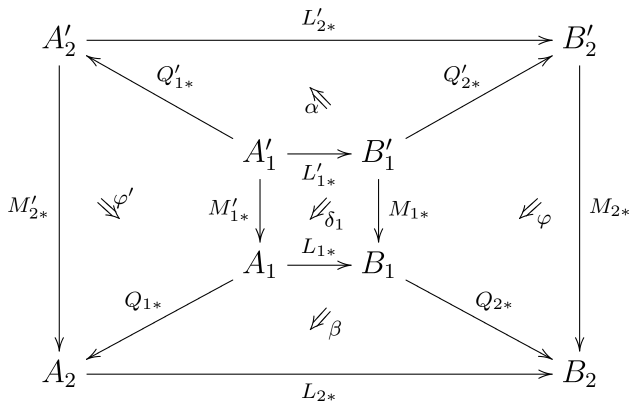
Remark 2.3.20. In the situation of (2.3.8), notice that the base change oriented square $\Upsilon(\mathscr{D})$ can be regarded as a 1-cell $\mathscr{M} \to \mathscr{M}'$ in the 2-category of weak links. By the discussion of (2.3.19), the oriented square $\Upsilon(\Upsilon(\mathscr{D}))$ is then well defined. A straightforward computation that we shall leave to the reader yields the identity :

$$\Upsilon(\Upsilon(\mathscr{D})) = \beta^\dagger.$$

2.3.21. Let us consider two 1-cells in the 2-category of weak links in $\mathscr{A}$:

$$(M'_{1*}, M_{1*}, \delta_1) : \mathscr{L}'_1 := (L'_{1*}, L'^*_1, \eta^{\mathscr{L}'_1}, \varepsilon^{\mathscr{L}'_1}) \to \mathscr{L}_1 := (L_{1*}, L^*_1, \eta^{\mathscr{L}_1}, \varepsilon^{\mathscr{L}_1})$$
$$(M'_{2*}, M_{2*}, \delta_2) : \mathscr{L}'_2 := (L'_{2*}, L'^*_2, \eta^{\mathscr{L}'_2}, \varepsilon^{\mathscr{L}'_2}) \to \mathscr{L}_2 := (L_{2*}, L^*_2, \eta^{\mathscr{L}_2}, \varepsilon^{\mathscr{L}_2})$$

and a diagram of 2-cells :



so that α and β are 1-cells $\mathscr{L}'_1 \to \mathscr{L}'_2$ and $\mathscr{L}_1 \to \mathscr{L}_2$ in $\mathsf{wLink}(\mathscr{A})$. We complete the diagram by adding as well the 2-cell δ_2 (the reader should picture this as a cubical diagram whose faces are oriented by the given 2-cells), and we suppose moreover that the resulting diagram *commutes on* 2*-cells, i.e.* we have :

$$(\delta_2 \boxdot \alpha) \boxminus \varphi' = \varphi \boxminus (\beta \boxdot \delta_1).$$

We may then regard α and β as 1-cells of $\mathsf{wLink}(\mathscr{A})$:

$$(Q'_{1*}, Q'_{2*}, \alpha) : \mathscr{L}'_1 \to \mathscr{L}'_2 \qquad (Q_{1*}, Q_{2*}, \beta) : \mathscr{L}_1 \to \mathscr{L}_2$$

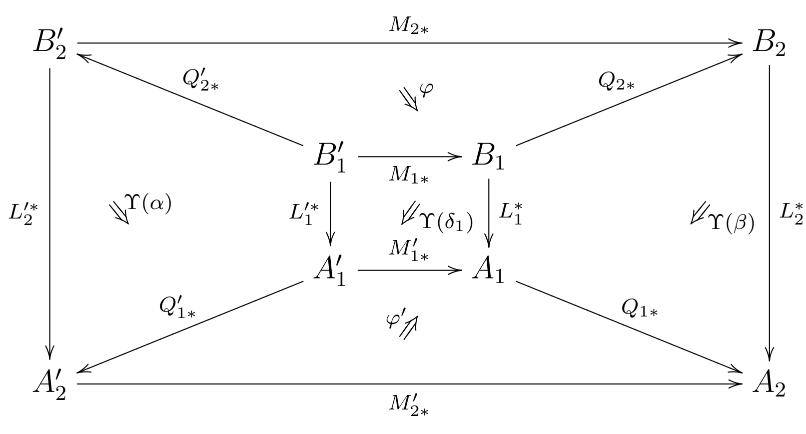
and the pair (φ', φ) as a 2-cell of $\mathsf{wLink}(\mathscr{A})$:

$$(\varphi', \varphi) : (M'_{2*}, M_{2*}, \delta_2) \circ (Q'_{1*}, Q'_{2*}, \alpha) \Rightarrow (Q_{1*}, Q_{2*}, \beta) \circ (M'_{1*}, M_{1*}, \delta_1).$$

The pseudo-functor Υ of (2.3.19) then yields the 2-cell :

$$(\varphi'^o, \varphi^o) : \Upsilon(M'_{2*}, M_{2*}, \delta_2)^o \circ \Upsilon(Q'_{1*}, Q'_{2*}, \alpha)^o \Rightarrow \Upsilon(Q_{1*}, Q_{2*}, \beta)^o \circ \Upsilon(M'_{1*}, M_{1*}, \delta_1)^o$$

which in turns translates as the diagram of 2-cells



completed to a cubical diagram, by adding in the 2-cell $\Upsilon(M'_{2*}, M_{2*}, \delta_2)$, and commuting again on 2-cells, *i.e.* verifying the identity :

$$(\Upsilon(\beta) \boxminus \Upsilon(\delta_1)) \boxbar \varphi = \varphi' \boxbar (\Upsilon(\delta_2) \boxminus \Upsilon(\alpha)).$$

Remark 2.3.22. (i) In the situation of (2.3.21), suppose that $\Upsilon(\alpha)$, $\Upsilon(\beta)$, $\Upsilon(\delta_1)$, φ and φ' are invertible 2-cells of $\mathscr{A}$. In this case, we deduce that the same holds for $\Upsilon(\delta_2) * Q'_2$, and if Q'_2 is an equivalence in $\mathscr{A}$, we conclude easily that also $\Upsilon(\delta_2)$ is an invertible 2-cell.

(ii) For instance, consider the case where $L_{1*} = L_{2*}$, $L'_{1*} = L'_{2*}$, $M_{1*} = M_{2*}$, $M'_{1*} = M'_{2*}$ and where all the 1-cells $Q_{1*}, Q_{2*}, Q'_{1*}, Q'_{2*}$ and all the 2-cells α, β, φ and φ' are identities (and then $\delta_1 = \delta_2$). In this case, applying (i) we conclude that $\Upsilon(\delta_1 : \mathscr{L}'_1 \to \mathscr{L}_1)$ is invertible if and only if the same holds for $\Upsilon(\delta_2 : \mathscr{L}'_2 \to \mathscr{L}_2)$.

(iii) We may also consider the variant where we invert the orientation of the 2-cells φ and φ' (and keep unchanged the others); then the commutativity on 2-cells becomes the condition :

$$\varphi \boxminus (\delta_2 \boxbar \alpha) = (\beta \boxbar \delta_1) \boxminus \varphi'$$

so that the pair (φ, φ') can be regarded as a 1-cell of $\mathsf{wLink}(\mathscr{A})$ as in the foregoing, but with reversed direction. After applying Υ, we deduce the identity

$$\varphi' \boxbar (\Upsilon(\beta) \boxminus \Upsilon(\delta_1)) = (\Upsilon(\delta_2) \boxminus \Upsilon(\alpha)) \boxbar \varphi.$$

Especially, (i) applies *verbatim* to this variant as well.

2.3.23. Lastly, let $\mathscr{A}$ and $\mathscr{B}$ be two 2-categories, and $\varphi : \mathscr{A} \to \mathscr{B}$ a pseudo-functor with coherence constraints $(\delta_\bullet, \gamma_\bullet)$. Then φ induces a pseudo-functor :

$$\mathsf{Link}(\varphi) : \mathsf{Link}(\mathscr{A}) \to \mathsf{Link}(\mathscr{B}) \qquad A \mapsto \varphi A$$

that assigns to every link $\mathscr{L} := (F, G, \eta^{\mathscr{L}}, \varepsilon^{\mathscr{L}}) : A \to B$ of $\mathscr{A}$ the datum

$$\varphi(\mathscr{L}) := (\varphi F, \varphi G, \eta^{\varphi(\mathscr{L})}, \varepsilon^{\varphi(\mathscr{L})})$$

where $\eta^{\varphi(\mathscr{L})}$ and $\varepsilon^{\varphi(\mathscr{L})}$ are the unique 2-cells that make commute the diagrams

$$\begin{array}{ccc} \mathbf{1}_{\varphi B} & \xRightarrow{\eta^{\varphi(\mathscr{L})}} & \varphi F \circ \varphi G \\ {\scriptstyle \delta_B}\Downarrow & & \Downarrow{\scriptstyle \gamma_{F,G}} \\ \varphi \mathbf{1}_B & \xRightarrow{\varphi(\eta^{\mathscr{L}})} & \varphi(F \circ G) \end{array} \qquad \begin{array}{ccc} \varphi G \circ \varphi F & \xRightarrow{\varepsilon^{\varphi(\mathscr{L})}} & \mathbf{1}_{\varphi A} \\ {\scriptstyle \gamma_{G,F}}\Downarrow & & \Downarrow{\scriptstyle \delta_A} \\ \varphi(G \circ F) & \xRightarrow{\varphi(\varepsilon^{\mathscr{L}})} & \varphi \mathbf{1}_A. \end{array}$$

We leave to the reader the verification that $\varphi(\mathscr{L})$ is indeed a link $\varphi A \to \varphi B$ in $\mathscr{B}$. To every transformation of links $\beta : \mathscr{L} \Rightarrow \mathscr{L}'$, the pseudo-functor $\mathsf{Link}(\varphi)$ assigns the transformation $\varphi(\beta) : \varphi(\mathscr{L}) \Rightarrow \varphi(\mathscr{L}')$. The associativity constraint of $\mathsf{Link}(\varphi)$ assigns to every composable pair of morphisms of links $\mathscr{L} := (F, G, \eta^{\mathscr{L}}, \varepsilon^{\mathscr{L}}) : A \to B$, $\mathscr{L}' := (F', G', \eta^{\mathscr{L}'}, \varepsilon^{\mathscr{L}'}) : B \to C$ the 2-cell $\gamma^*_{\mathscr{L},\mathscr{L}'} := \gamma_{F,F'} : \varphi(\mathscr{L}') \circ \varphi(\mathscr{L}) \Rightarrow \varphi(\mathscr{L}' \circ \mathscr{L})$. Likewise, the unit constraint of $\mathsf{Link}(\varphi)$ assigns to every object A of $\mathscr{A}$ the 2-cell $\delta^*_A := \delta_A$.

Proposition 2.3.24. *In the situation of* (2.3.23)*, the following diagram commutes :*

$$\begin{array}{ccc} \mathsf{2\text{-}Morph}(\mathsf{Link}(\mathscr{A}))^o & \xrightarrow{\Upsilon_{\mathscr{A}}} & \mathsf{2\text{-}Morph}(\mathscr{A}^o) \\ {\scriptstyle \mathsf{2\text{-}Morph}(\mathsf{Link}(\varphi))^o}\downarrow & & \downarrow{\scriptstyle \mathsf{2\text{-}Morph}(\varphi^o)} \\ \mathsf{2\text{-}Morph}(\mathsf{Link}(\mathscr{B}))^o & \xrightarrow{\Upsilon_{\mathscr{B}}} & \mathsf{2\text{-}Morph}(\mathscr{B}^o). \end{array}$$

Proof. The commutativity on objects and 2-cells of $\mathsf{2\text{-}Morph}(\mathsf{Link}(\mathscr{A}))^o$ is immediate from the definition. To check commutativity on 1-cells, consider a diagram $\mathscr{D}$ as in (2.3.8); we come down to verifying that the 2-cell of $\mathscr{B}$:

$$X := (\varepsilon^{\varphi(\mathscr{L})} * \varphi M'_* \varphi L'^*) \odot (\varphi L^* * (\gamma^{-1}_{M'_*, L_*} \odot \varphi(\beta) \odot \gamma_{L'_*, M_*}) * \varphi L'^*) \odot (\varphi L^* \varphi M_* * \eta^{\varphi(\mathscr{L}')})$$

equals the 2-cell :

$$Y := \gamma^{-1}_{M'_*,L'^*} \odot \varphi((\varepsilon^{\mathscr{L}} * M'_*L'^*) \odot (L^* * \beta * L'^*) \odot (L^*M_* * \eta^{\mathscr{L}'})) \odot \gamma_{L^*,M_*}.$$

To this aim, set

$$Z := ((\delta_A^{-1} \odot \varphi(\varepsilon^{\mathscr{L}}) \odot \gamma_{L^*,L_*}) * \varphi M'_*) \odot (\varphi L^* * \gamma^{-1}_{M'_*,L_*})$$
$$Z' := (\gamma_{L'_*,M_*} * \varphi L'^*) \odot (\varphi M_* * (\gamma^{-1}_{L'_*,L'^*} \odot \varphi(\eta^{\mathscr{L}'}) \odot \delta_{B'}))$$

so that $X = (Z * \varphi L'^*) \odot X' \odot (\varphi L^* * Z')$, with $X' := \varphi L^* * \varphi(\beta) * \varphi L'^*$. We compute :

$$\begin{aligned} Z &= \gamma_{\mathbf{1}_A,M'_*} \odot (\varphi(\varepsilon^{\mathscr{L}}) * \varphi M'_*) \odot (\gamma_{L^*,L_*} * \varphi M'_*) \odot (\varphi L^* * \gamma^{-1}_{M'_*,L_*}) \\ &= \gamma_{\mathbf{1}_A,M'_*} \odot (\varphi(\varepsilon^{\mathscr{L}}) * \varphi M'_*) \odot \gamma^{-1}_{L^*L_*,M'_*} \odot \gamma_{L^*,L_*M'_*} \\ &= \varphi(\varepsilon^{\mathscr{L}} * M'_*) \odot \gamma_{L^*,L_*M'_*} \\ Z' &= (\gamma_{L'_*,M_*} * \varphi L'^*) \odot (\varphi M_* * \gamma^{-1}_{L'_*,L'^*}) \odot (\varphi M_* * \varphi(\eta^{\mathscr{L}'})) \odot \gamma^{-1}_{M_*,\mathbf{1}_B} \\ &= \gamma^{-1}_{M_*L'_*,L'^*} \odot \gamma_{M_*,L'_*L'^*} \odot (\varphi M_* * \varphi(\eta^{\mathscr{L}'})) \odot \gamma^{-1}_{M_*,\mathbf{1}_B} \\ &= \gamma^{-1}_{M_*L'_*,L'^*} \odot \varphi(M_* * \eta^{\mathscr{L}'}). \end{aligned}$$

On the other hand, we have :

$$\gamma^{-1}_{M'_*,L'^*} \odot \varphi(\varepsilon^{\mathscr{L}} * M'_*L'^*) = (\varphi(\varepsilon^{\mathscr{L}} * M'_*) * \varphi L'^*) \odot \gamma^{-1}_{L^*L_*M'_*,L'^*}$$
$$\varphi(L^*M_* * \eta^{\mathscr{L}'}) \odot \gamma_{L^*,M_*} = \gamma_{L^*,M_*L'_*L'^*} \odot (\varphi L^* * \varphi(M_* * \eta^{\mathscr{L}'}))$$

so we are reduced to showing the identity :

$$(\gamma_{L^*,L_*M'_*} * \varphi L'^*) \odot X' \odot (\varphi L^* * \gamma^{-1}_{M_*L'_*,L'^*}) = \gamma^{-1}_{L^*L_*M'_*,L'^*} \odot \varphi(L^* * \beta * L'^*) \odot \gamma_{L^*,M_*L'_*L'^*}.$$

The latter is the same as the identity : $\gamma_{L^*,L_*M'_*,L'^*} \odot X' = \varphi(L^* * \beta * L'^*) \odot \gamma_{L^*,M_*L'_*,L'^*}$, which holds by virtue of remark 2.1.17(iii). □

2.3.25. We conclude with a further construction derived from the formalism of base change, that shall help us in the following section. First, to every 2-category $\mathscr{A}$ we attach the 2-category

$$\mathsf{Equiv}(\mathscr{A})$$

defined as the sub-2-category of 2-$\mathsf{Morph}(\mathscr{A})$ whose objects are the equivalences in $\mathscr{A}$ (see definition 2.1.3(iii)); for any two equivalences $f : X \to Y$ and $f' : X' \to Y'$ in $\mathscr{A}$, the 1-cells $f \to f'$ in $\mathsf{Equiv}(\mathscr{A})$ are the essentially commutative oriented squares :

(2.3.26)
$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g'}\downarrow & \swArrow_{\beta} & \downarrow{\scriptstyle g} \\ X' & \xrightarrow[f']{} & Y' \end{array}$$

and the 2-cells between such 1-cells are as in 2-$\mathsf{Morph}(\mathscr{A})$. We define a pseudo-functor

$$\mathsf{L} : \mathsf{Equiv}(\mathscr{A}) \to \mathsf{wLink}(\mathscr{A})$$

as follows. For any $(f : X \to Y) \in \mathrm{Ob}(\mathsf{Equiv}(\mathscr{A}))$ we choose a quasi-inverse $f^\dagger : Y \to X$ (see definition 2.1.3(iii)) and the unit η^f and counit ε^f of an adjunction for $(f^\dagger, f)$, and we set

$$\mathsf{L}(f) := (f^\dagger, f, \eta^f, \varepsilon^f) \in \mathrm{Ob}(\mathsf{wLink}(\mathscr{A})).$$

Then every 1-cell $(g, g', \beta) : f \to f'$ as in (2.3.26) yields a 1-cell $\mathsf{L}(f) \to \mathsf{L}(f')$ in $\mathsf{wLink}(\mathscr{A})$ which we denote $\mathsf{L}(g, g', \beta)$; likewise, L is the identity on 2-cells. Then of course the coherence constraints of L are given by identities, but L *is not* necessarily strict, since we do not necessarily have $(g \circ f)^\dagger = f^\dagger \circ g^\dagger$ for every $f, g \in \mathrm{Ob}(\mathsf{Equiv}(\mathscr{A}))$.

2.3.27. Moreover, we have a natural strict isomorphism of 2-categories :

$$\mathsf{Equiv}(\mathscr{A}) \stackrel{\sim}{\to} \mathsf{Equiv}(\mathscr{A}^o)^o.$$

Namely, to every equivalence $f : X \to Y$ of $\mathscr{A}$ we assign the equivalence $f^o : Y^o \to X^o$ of $\mathscr{A}^o$, and to every 1-cell (2.3.26) we assign the following 1-cell $f'^o \to f^o$ of $\mathsf{Equiv}(\mathscr{A}^o)^o$:

$$\begin{array}{ccc} Y'^o & \xrightarrow{f'^o} & X'^o \\ {\scriptstyle g^o}\downarrow & \Downarrow_{(\beta^o)^{-1}} & \downarrow{\scriptstyle g'^o} \\ Y^o & \xrightarrow[f^o]{} & X^o. \end{array}$$

Lastly, let $(h, h', \beta') : f \to f'$ be another 1-cell of $\mathsf{Equiv}(\mathscr{A})$, and $(\alpha_1, \alpha_2) : (g, g', \beta) \Rightarrow (h, h', \beta')$ any 2-cell; recall that the latter is a pair of 2-cells $\alpha_1 : g' \Rightarrow h'$ and $\alpha_2 : g \Rightarrow h$ of $\mathscr{A}$ such that $\alpha_1 \odot \beta = \beta' \odot \alpha_2$. Then our strict isomorphism assigns to (α_1, α_2) the pair (α_2^o, α_1^o), which is easily seen to be a 2-cell $(g'^o, g^o, (\beta^o)^{-1}) \Rightarrow (h'^o, h^o, (\beta'^o)^{-1})$ of $\mathsf{Equiv}(\mathscr{A}^o)^o$.

Next, by composing the pseudo-functors L of (2.3.25) and Υ of (2.3.19) we get a pseudo-functor $\Upsilon^o \circ \mathsf{L} : \mathsf{Equiv}(\mathscr{A}) \to 2\text{-}\mathsf{Morph}(\mathscr{A}^o)^o$, and taking into account remark 2.3.11 we see that $\Upsilon^o \circ \mathsf{L}$ factors through the inclusion strict pseudo-functor $\mathsf{Equiv}(\mathscr{A}^o)^o \to 2\text{-}\mathsf{Morph}(\mathscr{A}^o)^o$. Therefore, we can further compose with the foregoing isomorphism to deduce a pseudo-functor

$$\widetilde{\Upsilon} : \mathsf{Equiv}(\mathscr{A}) \to \mathsf{Equiv}(\mathscr{A}) \qquad f \mapsto f^\dagger \qquad (g, g', \beta) \mapsto \Upsilon(g, g', \beta)^{-1}.$$

2.4. **Adjunctions in 2-categories.** We begin by introducing two constructions that will often allow us to reduce the proof of assertions concerning general pseudo-functors, to the special case of strict, or at least *unital* pseudo-functors, in the sense of definition 2.4.1.

Definition 2.4.1. Let $\mathscr{A}$ and $\mathscr{B}$ be two 2-categories, and $F : \mathscr{A} \to \mathscr{B}$ a pseudo-functor with coherence constraint $(\delta^F_\bullet, \gamma^F_{\bullet\bullet})$. We say that F is *unital* if $F\mathbf{1}_A = \mathbf{1}_{FA}$ and $\delta^F_A = i_{FA}$ for every $A \in \mathrm{Ob}(\mathscr{A})$. Since the component $\delta^F_\bullet$ is determined by the these conditions, we shall usually say that the coherence constraint of a unital pseudo-functor F is the component $\gamma^F_{\bullet\bullet}$.

Remark 2.4.2. Let $F : \mathscr{A} \to \mathscr{B}$ be a unital pseudo-functor with coherence constraint γ^F.

(i) Directly from the composition axiom we see that for every pair of 1-cells $A \xrightarrow{f} B \xrightarrow{g} C$ of $\mathscr{A}$, the 2-cell $\gamma^F_{f,g}$ equals $\mathbf{1}_{Fg}$ if $f = \mathbf{1}_A$, and equals $\mathbf{1}_{Ff}$ if $g = \mathbf{1}_B$.

(ii) Let $G : \mathscr{A} \to \mathscr{B}$ be another unital pseudo-functor, and $\alpha : F \Rightarrow G$ any pseudo-natural transformation with coherence constraint τ. From the first coherence axiom for α it is clear that $\tau_{\mathbf{1}_A} = \mathbf{1}_{\alpha_A}$ for every $A \in \mathrm{Ob}(\mathscr{A})$.

Proposition 2.4.3. *For every pseudo-functor $F : \mathscr{A} \to \mathscr{B}$ there exists a unital pseudo-functor $F^u : \mathscr{A} \to \mathscr{B}$ with a pseudo-natural isomorphism $F \stackrel{\sim}{\to} F^u$.*

Proof. Let (δ^F, γ^F) be the coherence constraint of F; for every 1-cell $f : A \to B$ of $\mathscr{A}$, we define the 2-cell of $\mathscr{B}$

$$\omega_f := \begin{cases} \mathbf{1}_{Ff} & \text{if } f \neq \mathbf{1}_A \\ \delta^F_A & \text{if } f = \mathbf{1}_A \end{cases} \qquad F^u f \Rightarrow Ff.$$

We shall show more precisely, that there exists a unique well-defined unital pseudo-functor $F^u : \mathscr{A} \to \mathscr{B}$ with coherence constraint γ^{F^u} such that :

- $F^u A = FA$ for every $A \in \mathrm{Ob}(\mathscr{A})$
- $F^u f = Ff$ for every $A, B \in \mathrm{Ob}(\mathscr{A})$ every 1-cell $f : A \to B$ such that $f \neq \mathbf{1}_A$
- for every pair of 1-cells $A \xrightarrow{f} B \xrightarrow{g} C$ of $\mathscr{A}$ we have

$$\gamma^{F^u}_{f,g} = \omega^{-1}_{g \circ f} \odot \gamma^F_{f,g} \odot (\omega_g * \omega_f)$$

- for every pair of 1-cells $f, f' : A \to B$ of $\mathscr{A}$ and every 2-cell $\beta : f \Rightarrow f'$, we have
$$F^u\beta = \omega_{f'}^{-1} \odot F\beta \odot \omega_f.$$

Indeed, the uniqueness of such F^u is obvious; it is also clear that for every $A, B \in \mathrm{Ob}(\mathscr{A})$ the stated rules yield a well-defined functor $F^u_{AB} : \mathscr{A}(A, B) \to \mathscr{B}(F^uA, F^uB)$. For the naturality of γ^{F^u} consider 1-cells $f, f' : A \to B$ and $g, g' : B \to C$ in $\mathscr{A}$ and 2-cells $\beta_1 : f \Rightarrow f'$ and $\beta_2 : g \Rightarrow g'$; we notice that

$$\begin{aligned} F^u\beta_2 * F^u\beta_1 &= (\omega_{g'}^{-1} \odot F\beta_2 \odot \omega_g) * (\omega_{f'}^{-1} \odot F\beta_1 \odot \omega_f) \\ &= (\omega_{g'}^{-1} * \omega_{f'}^{-1}) \odot (F\beta_2 * F\beta_1) \odot (\omega_g * \omega_f) \end{aligned}$$

whence

$$\begin{aligned} F^u(\beta_2 * \beta_1) \odot \gamma^{F^u}_{f,g} &= \omega^{-1}_{g'\circ f'} \odot F(\beta_2 * \beta_1) \odot \gamma^F_{f,g} \odot (\omega_g * \omega_f) \\ &= \omega^{-1}_{g'\circ f'} \odot \gamma^F_{f',g'} \odot (F\beta_2 * F\beta_1) \odot (\omega_g * \omega_f) \\ &= \gamma^{F^u}_{f',g'} \odot (F^u\beta_2 * F^u\beta_1) \end{aligned}$$

which is the contention, by remark 2.1.17(ii). Next, let us check the composition axiom for γ^{F^u}; we consider three 1-cells $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ of $\mathscr{A}$, and we need to show the identity

$$X := \gamma^{F^u}_{f,h\circ g} \odot (\gamma^{F^u}_{g,h} * F^uf) = Y := \gamma^{F^u}_{g\circ f,h} \odot (F^uh * \gamma^{F^u}_{f,g}).$$

We compute, on the one hand :

$$\begin{aligned} X &= \omega^{-1}_{h\circ g\circ f} \odot \gamma^F_{f,h\circ g} \odot (\omega_{h\circ g} \odot \omega_f) \odot ((\omega^{-1}_{h\circ g} \odot \gamma^F_{g,h} * (\omega_h * \omega_g)) * F^uf) \\ &= \omega^{-1}_{h\circ g\circ f} \odot \gamma^F_{f,h\circ g} \odot (\omega_{h\circ g} * \omega_f) \odot (\omega^{-1}_{h\circ g} * F^uf) \odot ((\gamma^F_{g,h} \odot (\omega_h * \omega_g)) * F^uf) \\ &= \omega^{-1}_{h\circ g\circ f} \odot \gamma^F_{f,h\circ g} \odot (F(h\circ g) * \omega_f) \odot ((\gamma^F_{g,h} \odot (\omega_h * \omega_g)) * F^uf) \\ &= \omega^{-1}_{h\circ g\circ f} \odot \gamma^F_{f,h\circ g} \odot (F(h\circ g) * \omega_f) \odot (\gamma^F_{g,h} * F^uf) \odot (\omega_h * \omega_g * F^uf) \\ &= \omega^{-1}_{h\circ g\circ f} \odot \gamma^F_{f,h\circ g} \odot (\gamma^F_{g,h} * Ff) \odot (Fh * Fg * \omega_f) \odot (\omega_h * \omega_g * F^uf) \\ &= \omega^{-1}_{h\circ g\circ f} \odot \gamma^F_{f,h\circ g} \odot (\gamma^F_{g,h} * Ff) \odot (\omega_h * \omega_g * \omega_f) \\ &= \omega^{-1}_{h\circ g\circ f} \odot \gamma^F_{g\circ f,h} \odot (Fh * \gamma^F_{f,g}) \odot (\omega_h * \omega_g * \omega_f) \end{aligned}$$

and on the other hand :

$$\begin{aligned} Y &= \omega^{-1}_{h\circ g\circ f} \odot \gamma^F_{g\circ f,h} \odot (\omega_h * \omega_{g\circ f}) \odot (F^uh * (\omega^{-1}_{g\circ f} \odot \gamma^F_{f,g} \odot (\omega_g * \omega_f))) \\ &= \omega^{-1}_{h\circ g\circ f} \odot \gamma^F_{g\circ f,h} \odot (\omega_h * \omega_{g\circ f}) \odot (F^uh * \omega^{-1}_{g\circ f}) \odot (F^uh * (\gamma^F_{f,g} \odot (\omega_g * \omega_f))) \\ &= \omega^{-1}_{h\circ g\circ f} \odot \gamma^F_{g\circ f,h} \odot (\omega_h * F(g\circ f)) \odot (F^uh * \gamma^F_{f,g}) \odot (F^uh * \omega_g * \omega_f) \\ &= \omega^{-1}_{h\circ g\circ f} \odot \gamma^F_{g\circ f,h} \odot (Fh * \gamma^F_{f,g}) \odot (\omega_h * Fg * Fh) \odot (F^uh * \omega_g * \omega_f) \\ &= \omega^{-1}_{h\circ g\circ f} \odot \gamma^F_{g\circ f,h} \odot (Fh * \gamma^F_{f,g}) \odot (\omega_h * \omega_g * \omega_f) \end{aligned}$$

whence the contention. Lastly, the unit axiom can be verified by a simple inspection.

The sought pseudo-isomorphism is given by the rule that assigns $\alpha_A := \mathbf{1}_{FA} : FA \to F^uA$ to every $A \in \mathrm{Ob}(\mathscr{A})$, and the oriented square

$$\begin{array}{ccc} FA & \xrightarrow{\alpha_A} & F^uA \\ {\scriptstyle Ff}\downarrow & \swarrow_{\omega_f} & \downarrow{\scriptstyle F^uf} \\ FB & \xrightarrow[\alpha_B]{} & F^uB \end{array} \qquad \text{to every 1-cell } f : A \to B \text{ of } \mathscr{A}.$$

The naturality of ω_f and the coherence axioms follow by a simple inspection. □

Remark 2.4.4. (i) For any two 2-categories $\mathscr{A}$ and $\mathscr{B}$, let us denote by

$$\mathsf{uniPsFun}(\mathscr{A},\mathscr{B})$$

the sub-2-category of $\mathsf{PsFun}(\mathscr{A},\mathscr{B})$ whose objects are the unital pseudo-functors $\mathscr{A} \to \mathscr{B}$, and whose Hom-categories are given by the categories $\mathsf{PsNat}(F,G)$, for any two such unital pseudo-functors F,G. For every pseudo-functor $F : \mathscr{A} \to \mathscr{B}$, let also $\alpha^F : F \xrightarrow{\sim} F^u$ be the pseudo-natural isomorphism furnished by proposition 2.4.3. It is easily seen that the rules $F \mapsto F^u$ for every pseudo-functor $F : \mathscr{A} \to \mathscr{B}$, $(\beta : F \Rightarrow G) \mapsto \beta^u := \alpha^G \odot \beta \odot (\alpha^F)^{-1}$ for every pseudo-natural transformation β, and $(\Xi : \beta \rightsquigarrow \beta') \mapsto \Xi^u := \alpha^G * \Xi * (\alpha^F)^{-1}$ for every modification Ξ, define a strict 2-equivalence of 2-categories

$$(-)^u : \mathsf{PsFun}(\mathscr{A},\mathscr{B}) \xrightarrow{\sim} \mathsf{uniPsFun}(\mathscr{A},\mathscr{B})$$

(details left to the reader). Likewise, if $i : \mathsf{uniPsFun}(\mathscr{A},\mathscr{B}) \to \mathsf{PsFun}(\mathscr{A},\mathscr{B})$ denotes the inclusion strict pseudo-functor, the rule : $F \mapsto \alpha^F$ for every pseudo-functor $F : \mathscr{A} \to \mathscr{B}$ yields a strict pseudo-natural isomorphism of strict pseudo-functors :

$$\mathbf{1}_{\mathsf{PsFun}(\mathscr{A},\mathscr{B})} \xrightarrow{\sim} i \circ (-)^u. \tag{2.4.5}$$

(ii) Let $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{C}$ be two pseudo-functors, $\alpha^F : F \xrightarrow{\sim} F^u$, $\alpha^G : G \xrightarrow{\sim} G^u$, $\alpha^{G\circ F} : G \circ F \xrightarrow{\sim} (G \circ F)^u$ the corresponding pseudo-natural isomorphisms as in (i). Since α^F and α^G are not strict, we have distinct pseudo-natural isomorphisms

$$(\alpha^G * F^u) \odot (G * \alpha^F), (G^u * \alpha^F) \odot (\alpha^G * F) : G \circ F \xrightarrow{\sim} G^u \circ F^u$$

that are related by an invertible modification (see example 2.2.15(ii)). By composing with $(\alpha^{G\circ F})^{-1}$, we get therefore two pseudo-natural isomorphisms of unital pseudo-functors

$$(G \circ F)^u \xrightarrow{\sim} G^u \circ F^u$$

but in general these pseudo-functors are different.

Proposition 2.4.6. *For every pseudo-functor $F : \mathscr{A} \to \mathscr{B}$ there exist :*

(a) *A 2-category $\mathscr{A}^F$ and strict pseudo-functors $\mathscr{A} \xleftarrow{\pi^F} \mathscr{A}^F \xrightarrow{F^\flat} \mathscr{B}$.*
(b) *With an isomorphism of pseudo-functors $\omega^F : F \circ \pi^F \xrightarrow{\sim} F^\flat$.*
(c) *A pseudo-functor $\sigma^F : \mathscr{A} \to \mathscr{A}^F$ such that :*

$$\pi^F \circ \sigma^F = \mathbf{1}_{\mathscr{A}} \qquad F^\flat \circ \sigma^F = F \qquad \omega^F * \sigma^F = \mathbf{1}_F.$$

(d) *A pseudo-natural equivalence $\mu^F : \sigma^F \circ \pi^F \xrightarrow{\sim} \mathbf{1}_{\mathscr{A}^F}$.*

Proof. We let $\mathscr{A}^F$ be the 2-category with $\mathrm{Ob}(\mathscr{A}^F) = \mathrm{Ob}(\mathscr{A})$, and whose 1-cells

$$(f,g,\lambda) : X \to Y \qquad \text{for every } X,Y \in \mathrm{Ob}(\mathscr{A})$$

are all the data consisting of a 1-cell $f : X \to Y$ of $\mathscr{A}$, a 1-cell $g : FX \to FY$ of $\mathscr{B}$, and an invertible 2-cell $\lambda : g \xrightarrow{\sim} Ff$. Given two such 1-cells $(f,g,\lambda),(f',g',\lambda') : X \to Y$, the 2-cells

$$(f,g,\lambda) \Rightarrow (f',g',\lambda')$$

are the pairs (α,β) where $\alpha : f \Rightarrow f'$ (resp. $\beta : g \Rightarrow g'$) is a 2-cell of $\mathscr{A}$ (resp. of $\mathscr{B}$), such that

$$\lambda' \odot \beta = F\alpha \odot \lambda.$$

The composition of two 1-cells $X \xrightarrow{(f_1,g_1,\lambda_1)} Y \xrightarrow{(f_2,g_2,\lambda_2)} Z$ is the 1-cell

$$(f_2,g_2,\lambda_2) \circ (f_1,g_1,\lambda_1) := (f_2 \circ f_1, g_2 \circ g_1, \gamma^F_{f_1,f_2} \odot (\lambda_2 * \lambda_1))$$

where (δ^F,γ^F) denotes the coherence constraint of the pseudo-functor F. The associativity of this composition law follows easily from the composition axiom for γ^F : the details are left to the reader. Especially, for every $X \in \mathrm{Ob}(\mathscr{A})$ the triple $(\mathbf{1}_X, \mathbf{1}_{FX}, \delta^F_X)$ is the identity 1-cell of X in $\mathscr{A}^F$. Given three 1-cells $(f,g,\lambda),(f',g',\lambda'),(f'',g'',\lambda''):X\to Y$, and two 2-cells $(\alpha,\beta):(f,g,\lambda)\Rightarrow(f',g',\lambda')$ and $(\alpha',\beta'):(f',g',\lambda')\Rightarrow(f'',g'',\lambda'')$, we let

$$(\alpha',\beta')\odot(\alpha,\beta):=(\alpha'\odot\alpha,\beta'\odot\beta)$$

which is a well defined 2-cell $(f,g,\lambda)\Rightarrow(f'',g'',\lambda'')$. Lastly, if $(f_1,g_1,\lambda_1),(f'_1,g'_1,\lambda'_1):X\to Y$ and $(f_2,g_2,\lambda_2),(f'_2,g'_2,\lambda'_2):Y\to Z$ are four given 1-cells, and $(\alpha_1,\beta_1):(f_1,g_1,\lambda_1)\Rightarrow(f'_1,g'_1,\lambda'_1)$, $(\alpha_2,\beta_2):(f_2,g_2,\lambda_2)\Rightarrow(f'_2,g'_2,\lambda'_2)$ are two 2-cells, we set :

$$(\alpha_2,\beta_2)*(\alpha_1,\beta_1):=(\alpha_2*\alpha_1,\beta_2*\beta_1)$$

which is a well defined 2-cell $(f_2,g_2,\lambda_2)\circ(f_1,g_1,\lambda_1)\Rightarrow(f'_2,g'_2,\lambda'_2)\circ(f'_1,g'_1,\lambda'_1)$. The associativity of these two composition laws are obvious, and it is then easily seen that $\mathscr{A}^F$ is a well defined 2-category with such laws for 1-cells and 2-cells : the details are left to the reader.

The sought strict pseudo-functors $F^\flat$ and π^F are given by the rules :

$$\begin{array}{rlrll}
F^\flat X & :=FX & \pi^F X & :=X & \text{for every } X\in\mathrm{Ob}(\mathscr{A}^F)\\
F^\flat(f,g,\lambda) & :=g & \pi^F(f,g,\lambda) & :=f & \text{for every 1-cell } (f,g,\lambda)\\
F^\flat(\alpha,\beta) & :=\beta & \pi^F(\alpha,\beta) & :=\alpha & \text{for every 2-cell } (\alpha,\beta).
\end{array}$$

The isomorphism ω^F is given by the rule : $X\mapsto\mathbf{1}_{FX}$ for every $X\in\mathrm{Ob}(\mathscr{A}^F)$, and its coherence costraint assigns to every 1-cell $(f,g,\lambda):X\to Y$ the oriented square

$$\begin{array}{ccc}
FX & = & FX\\
{\scriptstyle Ff}\downarrow & \swarrow_{\lambda} & \downarrow{\scriptstyle g}\\
FY & = & FY.
\end{array}$$

Next, the pseudo-functor σ^F is given by the rules :

$$X\mapsto X\qquad f\mapsto(f,Ff,\mathbf{1}_{Ff})\qquad \alpha\mapsto(\alpha,F\alpha)$$

for every $X\in\mathrm{Ob}(\mathscr{A})$, every 1-cell f, and every 2-cell α of $\mathscr{A}$. The coherence constraints of σ^F are given by the rules :

$$X\mapsto(i_X,\delta^F_X)\qquad (f,f')\mapsto(\mathbf{1}_{f'\circ f},\gamma^F_{f,f'})$$

for every $X\in\mathrm{Ob}(\mathscr{A})$ and every composable pair of 1-cells (f,f') of $\mathscr{A}$. The required coherence axioms are easily verified, and the desired identities as in (c) follow straightforwardly : details left to the reader. Lastly, we define $\mu^F_X:=\mathbf{1}_{\sigma^F X}=(\mathbf{1}_X,\mathbf{1}_{FX},\delta^F_X)$ for every $X\in\mathrm{Ob}(\mathscr{A}^F)$. For every 1-cell $(f,g,\lambda):X\to Y$ of $\mathscr{A}^F$, notice that $\sigma^F\circ\pi^F(f,g,\lambda)=(f,Ff,\mathbf{1}_{Ff})$; then the coherence constraint of μ^F assigns to (f,g,λ) the oriented square in $\mathscr{A}^F$:

$$\begin{array}{ccc}
X & \xrightarrow{(\mathbf{1}_X,\mathbf{1}_{FX},\delta^F_X)} & X\\
{\scriptstyle (f,Ff,\mathbf{1}_{Ff})}\downarrow & \swarrow_{(\mathbf{1}_f,\lambda)} & \downarrow{\scriptstyle (f,g,\lambda)}\\
Y & \xrightarrow[(\mathbf{1}_Y,\mathbf{1}_{FY},\delta^F_Y)]{} & Y.
\end{array}$$

A little diagram chase that we leave to the reader shows that these rules yield the sought pseudo-natural equivalence. □

Proposition 2.4.7. (i) *Let $\mathscr{A},\mathscr{B}$ be two 2-categories, $F:\mathscr{A}\to\mathscr{B}$ and $G:\mathscr{B}\to\mathscr{A}$ two pseudo-functors. Then there exist :*

(a) *Two 2-categories $\mathscr{A}^{F,G}$ and $\mathscr{B}^{G,F}$, and a diagram of strict pseudo-functors :*

$$\mathscr{A}\xleftarrow{\pi^{F,G}}\mathscr{A}^{F,G}\underset{G^\flat}{\overset{F^\flat}{\rightleftarrows}}\mathscr{B}^{G,F}\xrightarrow{\pi^{G,F}}\mathscr{B}.$$

(b) *With two isomorphisms of pseudo-functors :*

$$\omega^{F,G} : F \circ \pi^{F,G} \xrightarrow{\sim} \pi^{G,F} \circ F^\flat \qquad \textit{and} \qquad \omega^{G,F} : G \circ \pi^{G,F} \xrightarrow{\sim} \pi^{F,G} \circ G^\flat.$$

(c) *Two pseudo-functors* $\sigma^{F,G} : \mathscr{A} \to \mathscr{A}^{F,G}$ *and* $\sigma^{G,F} : \mathscr{B} \to \mathscr{B}^{G,F}$ *such that :*

$$\begin{aligned} \pi^{F,G} \circ \sigma^{F,G} &= \mathbf{1}_{\mathscr{A}} & F^\flat \circ \sigma^{F,G} &= \sigma^{G,F} \circ F & \omega^{F,G} * \sigma^{F,G} &= \mathbf{1}_F \\ \pi^{G,F} \circ \sigma^{G,F} &= \mathbf{1}_{\mathscr{B}} & G^\flat \circ \sigma^{G,F} &= \sigma^{F,G} \circ G & \omega^{G,F} * \sigma^{G,F} &= \mathbf{1}_G. \end{aligned}$$

(d) *With two isomorphisms of pseudo-functors :*

$$\psi^{F,G} : \sigma^{F,G} \circ \pi^{F,G} \xrightarrow{\sim} \mathbf{1}_{\mathscr{A}^{F,G}} \quad \textit{and} \quad \psi^{G,F} : \sigma^{G,F} \circ \pi^{G,F} \xrightarrow{\sim} \mathbf{1}_{\mathscr{B}^{G,F}}$$

(ii) *Suppose moreover that* F *and* G *are unital. Then we have as well :*

$$G^\flat * \psi^{G,F} = (\psi^{F,G} * G^\flat) \odot (\sigma^{F,G} * \omega^{G,F}) \qquad F^\flat * \psi^{F,G} = (\psi^{G,F} * F^\flat) \odot (\sigma^{G,F} * \omega^{F,G}).$$

Proof. (i): To begin with, for every $k, n \in \mathbb{N}$ let us set :

$$H_{2k} := G \qquad H_{2k+1} := F \qquad K_0 := \mathbf{1}_{\mathscr{A}} \qquad K_{n+1} := H_{n+1} \circ H_{n-1} \circ \cdots \circ H_1.$$

Then we let $\mathscr{A}^{F,G}$ be the 2-category with $\mathrm{Ob}(\mathscr{A}^{F,G}) = \mathrm{Ob}(\mathscr{A})$, and whose 1-cells

$$(f_\bullet, \lambda_\bullet) : X \to Y \qquad \text{for every } X, Y \in \mathrm{Ob}(\mathscr{A})$$

are all the systems of 1-cells $(f_n : K_n X \to K_n Y \mid n \in \mathbb{N})$, and of invertible 2-cells $(\lambda_n : f_{n+1} \xrightarrow{\sim} H_{n+1} f_n \mid n \in \mathbb{N})$. Given such 1-cells $(f_\bullet, \lambda_\bullet), (f'_\bullet, \lambda'_\bullet) : X \to Y$, the 2-cells

$$\alpha_\bullet : (f_\bullet, \lambda_\bullet) \Rightarrow (f'_\bullet, \lambda'_\bullet)$$

are the systems of 2-cells $(\alpha_n : f_n \to f'_n \mid k \in \mathbb{N})$ such that :

$$H_{n+1}(\alpha_n) \odot \lambda_n = \lambda'_n \odot \alpha_{n+1} \qquad \text{for every } n \in \mathbb{N}. \tag{2.4.8}$$

The composition of two 1-cells $X \xrightarrow{(f_{1,\bullet}, \lambda_{1,\bullet})} Y \xrightarrow{(f_{2,\bullet}, \lambda_{2,\bullet})} Z$ is the 1-cell

$$(f_{2,\bullet}, \lambda_{1,\bullet}) \circ (f_{1,\bullet}, \lambda_{1,\bullet}) := (f_{2,n} \circ f_{1,n}, \gamma^{H_{n+1}}_{f_{1,n}, f_{2,n}} \odot (\lambda_{2,n} * \lambda_{1,n}) \mid n \in \mathbb{N})$$

where $(\delta^{H_n}, \gamma^{H_n})$ denotes the coherence constraint of the pseudo-functor H_n, for every $n \in \mathbb{N}$. Just as in the proof of proposition 2.4.6, the associativity of this rule follows easily from the composition axioms for γ^{H_n}, and for every $X \in \mathrm{Ob}(\mathscr{A})$, the identity 1-cell of X in $\mathscr{A}^{F,G}$ is the datum $\mathbf{1}^{F,G}_X := (\mathbf{1}_{K_n X}, \delta^{H_{n+1}}_{K_n X} \mid n \in \mathbb{N})$. Given three 1-cells $(f_\bullet, \lambda_\bullet), (f'_\bullet, \lambda'_\bullet), (f''_\bullet, \lambda''_\bullet) : X \to Y$, and two 2-cells $\alpha_\bullet : (f_\bullet, \lambda_\bullet) \Rightarrow (f'_\bullet, \lambda'_\bullet)$ and $\alpha'_\bullet : (f'_\bullet, \lambda'_\bullet) \Rightarrow (f''_\bullet, \lambda''_\bullet)$, we let

$$\alpha'_\bullet \odot \alpha_\bullet := (\alpha'_n \odot \alpha_n \mid n \in \mathbb{N})$$

which is a well defined 2-cell $(f_\bullet, \lambda_\bullet) \Rightarrow (f''_\bullet, \lambda''_\bullet)$. Lastly, if $(f_{1,\bullet}, \lambda_{1,\bullet}), (f'_{1,\bullet}, \lambda'_{1,\bullet}) : X \to Y$ and $(f_{2,\bullet}, \lambda_{2,\bullet}), (f'_{2,\bullet}, \lambda'_{2,\bullet}) : Y \to Z$ are four given 1-cells, and $\alpha_{1,\bullet} : (f_{1,\bullet}, \lambda_{1,\bullet}) \Rightarrow (f'_{1,\bullet}, \lambda'_{1,\bullet})$, $\alpha_{2,\bullet} : (f_{2,\bullet}, \lambda_{2,\bullet}) \Rightarrow (f'_{2,\bullet}, \lambda'_{2,\bullet})$ are two 2-cells, we set :

$$\alpha_{2,\bullet} * \alpha_{1,\bullet} := (\alpha_{2,n} * \alpha_{1,n} \mid n \in \mathbb{N})$$

which is a well defined 2-cell $(f_{2,\bullet}, \lambda_{2,\bullet}) \circ (f_{1,\bullet}, \lambda_{1,\bullet}) \Rightarrow (f'_{2,\bullet}, \lambda'_{2,\bullet}) \circ (f'_{1,\bullet}, \lambda'_{1,\bullet})$. The associativity of these two composition laws are obvious, and it is then easily seen that $\mathscr{A}^{F,G}$ is a well defined 2-category with such laws for 1-cells and 2-cells : the details are left to the reader.

We define $\mathscr{B}^{G,F}$ by exactly the same rules, after swapping $\mathscr{A}$ and F with respectively $\mathscr{B}$ and G. Then the sought strict pseudo-functors $F^\flat$ and $\pi^{F,G}$ are given by the rules :

$$\begin{aligned} F^\flat X &:= FX & \pi^{F,G} X &:= X & &\text{for every } X \in \mathrm{Ob}(\mathscr{A}^{F,G}) \\ F^\flat(f_\bullet, \lambda_\bullet) &:= (f_{n+1}, \lambda_{n+1} \mid n \in \mathbb{N}) & \pi^{F,G}(f_\bullet, \lambda_\bullet) &:= f_0 & &\text{for every 1-cell } (f_\bullet, \lambda_\bullet) \\ F^\flat(\alpha_\bullet) &:= (\alpha_{n+1} \mid n \in \mathbb{N}) & \pi^{F,G}(\alpha_\bullet) &:= \alpha_0 & &\text{for every 2-cell } \alpha_\bullet \end{aligned}$$

and again, $G^\flat$ and $\pi^{G,F}$ are defined by the same rules, after swapping F and G.

The isomorphism $\omega^{F,G}$ is given by the rule : $X \mapsto \mathbf{1}_{FX}$ for every $X \in \mathrm{Ob}(\mathscr{A}^F)$, and its coherence costraint assigns to every 1-cell $(f_\bullet, \lambda_\bullet) : X \to Y$ the oriented square

$$\begin{array}{ccc} FX & = & FX \\ {\scriptstyle Ff_0}\downarrow & \swarrow_{\lambda_0} & \downarrow{\scriptstyle f_1} \\ FY & = & FY. \end{array}$$

Lastly, the pseudo-functor $\sigma^{F,G}$ is given by the rules :

$$X \mapsto X \qquad f \mapsto (K_n f, \mathbf{1}_{K_{n+1}f} \mid n \in \mathbb{N}) \qquad \alpha \mapsto (K_n\alpha \mid n \in \mathbb{N})$$

for every $X \in \mathrm{Ob}(\mathscr{A})$, every 1-cell f, and every 2-cell α of $\mathscr{A}$. For every $n \in \mathbb{N}$, let $(\delta^{K_n}, \gamma^{K_n})$ be the coherence constraint of K_n; the coherence constraint of $\sigma^{F,G}$ is given by the rules :

$$X \mapsto (\delta^{K_n}_X \mid n \in \mathbb{N}) \qquad (f, f') \mapsto (\gamma^{K_n}_{f,f'} \mid n \in \mathbb{N})$$

for every $X \in \mathrm{Ob}(\mathscr{A})$ and every composable pair of 1-cells (f, f') of $\mathscr{A}$, where H. The required coherence axioms are verified as in the proof of the corresponding assertions of the proposition 2.4.6, and shall again be left to the reader. Likewise, one defines $\omega^{G,F}$ and $\sigma^{G,F}$ by the same rules, after swapping the roles of F and G. Then the desired identities in (c) follow straightforwardly. It remains to exhibit an isomorphism of pseudo-functors $\psi^{F,G} : \sigma^{F,G} \circ \pi^{F,G} \xrightarrow{\sim} \mathbf{1}_{\mathscr{A}^{F,G}}$. To this aim, notice that $\sigma^{F,G} \circ \pi^{F,G}$ is given by the rules :

$$X \mapsto X \qquad (f_\bullet, \lambda_\bullet) \mapsto (K_n f_0, \mathbf{1}_{K_{n+1}f_0} \mid n \in \mathbb{N}) \qquad \alpha_\bullet \mapsto (K_n\alpha_0 \mid n \in \mathbb{N})$$

for every object X, every 1-cell $(f_\bullet, \lambda_\bullet)$ and every 2-cell $\alpha_\bullet$ of $\mathscr{A}^{F,G}$; then we let $\psi^{F,G}_X := \mathbf{1}^{F,G}_X$ for every such X, and to every such $(f_\bullet, \lambda_\bullet)$ we attach the coherence constraint

$$\tau^{\psi^{F,G}}_{(f_\bullet,\lambda_\bullet),\bullet} : (f_\bullet, \lambda_\bullet) \to (K_n f_0, \mathbf{1}_{K_{n+1}f_0} \mid n \in \mathbb{N})$$

defined inductively by the rules :

$$\tau^{\psi^{F,G}}_{(f_\bullet,\lambda_\bullet),0} := \mathbf{1}_{f_0} \qquad \tau^{\psi^{F,G}}_{(f_\bullet,\lambda_\bullet),n+1} := H_{n+1}(\tau^{\psi^{F,G}}_{(f_\bullet,\lambda_\bullet),n}) \odot \lambda_n \qquad \text{for every } n \in \mathbb{N}.$$

The coherence axioms for τ^ψ amount to the identities :

$$\left.\begin{array}{r} \tau^{\psi^{F,G}}_{\mathbf{1}^{F,G}_X,n} = \delta^{K_n}_X \\ A_n := \gamma^{K_n}_{f_0,f'_0} \odot (\tau^{\psi^{F,G}}_{(f'_\bullet,\lambda'_\bullet),n} * \tau^{\psi^{F,G}}_{(f_\bullet,\lambda_\bullet),n}) = B_n := \tau^{\psi^{F,G}}_{(f''_\bullet,\lambda''_\bullet),n} \end{array}\right. \qquad \text{for every } n \in \mathbb{N}$$

for every object X of $\mathscr{A}$ and every composable pair of 1-cells $(f_\bullet, \lambda_\bullet)$ and $(f'_\bullet, \lambda'_\bullet)$, with $(f''_\bullet, \lambda''_\bullet) := (f'_\bullet, \lambda'_\bullet) \circ (f_\bullet, \lambda_\bullet)$. The first identity is checked easily by induction on n, recalling that $\delta^{K_{n+1}}_X = H_{n+1}(\delta^{K_n}_X) \odot \delta^{H_{n+1}}_{K_n X}$ for every $n \in \mathbb{N}$. The second identity is obvious for $n = 0$. Suppose then that the second identity is known for some $n \in \mathbb{N}$; we compute :

$$\begin{aligned} A_{n+1} &= H_{n+1}(\gamma^{K_n}_{f_0,f'_0}) \odot \gamma^{H_{n+1}}_{K_n f_0, K_n f'_0} \odot ((H_{n+1}(\tau^\psi_{(f'_\bullet,\lambda'_\bullet),n}) \odot \lambda'_n) * (H_{n+1}(\tau^\psi_{(f_\bullet,\lambda_\bullet),n}) \odot \lambda_n)) \\ &= H_{n+1}(\gamma^{K_n}_{f_0,f'_0}) \odot \gamma^{H_{n+1}}_{K_n f_0, K_n f'_0} \odot (H_{n+1}(\tau^\psi_{(f'_\bullet,\lambda'_\bullet),n}) * H_{n+1}(\tau^\psi_{(f_\bullet,\lambda_\bullet),n})) \odot (\lambda'_n * \lambda_n) \\ &= H_{n+1}(\gamma^{K_n}_{f_0,f'_0}) \odot H_{n+1}(\tau^\psi_{(f'_\bullet,\lambda'_\bullet),n} * \tau^\psi_{(f_\bullet,\lambda_\bullet),n}) \odot \gamma^{H_{n+1}}_{f_0,f'_0} \odot (\lambda'_n * \lambda_n) \\ &= H_{n+1}(A_n) \odot \lambda''_n \\ &= H_{n+1}(B_n) \odot \lambda''_n \\ &= B_{n+1} \end{aligned}$$

where the first equality follows after recalling that $\gamma^{K_{n+1}}_{f_0,f'_0} = H_{n+1}(\gamma^{K_n}_{f_0,f'_0}) \odot \gamma^{H_{n+1}}_{K_n f_0, K_n f'_0}$. The same rules yield, *mutatis mutandis*, the sought isomorphism $\psi^{G,F} : \sigma^{G,F} \circ \pi^{G,F} \xrightarrow{\sim} \mathbf{1}_{\mathscr{B}^{G,F}}$.

(ii): Let us set as well :

$$H'_{2k} := F \qquad H'_{2k+1} := G \qquad K'_0 := \mathbf{1}_{\mathscr{B}} \qquad K'_{n+1} := H'_{n+1} \circ H'_{n-1} \circ \cdots \circ H'_1.$$

With this notation, since F are unital, we have :

$$\begin{aligned}(G^\flat * \psi^{G,F})_X &= G^\flat(\mathbf{1}^{G,F}_X) = (\mathbf{1}_{K'_{n+1}X}, i_{K'_{n+2}X} \mid n \in \mathbb{N}) \\ (\psi^{F,G} * G^\flat)_X &= \psi^{F,G}_{GX} = (\mathbf{1}_{K_n GX}, i_{K_{n+1}GX} \mid n \in \mathbb{N}) \\ (\sigma^{F,G} * \omega^{G,F})_X &= \sigma^{F,G}_{GX} = (\mathbf{1}_{K_n GX}, i_{K_{n+1}GX} \mid n \in \mathbb{N}).\end{aligned}$$

Noticing that $K'_{n+2} = K_{n+1}G$ and $\gamma^{H_{n+1}}_{\mathbf{1}_{K_nGX},\mathbf{1}_{K_nGX}} = \delta^{H_{n+1}}_{K_nGX} = i_{K_nGX}$, we deduce :

$$(\psi^{F,G} * G^\flat)_X \circ (\sigma^{F,G} * \omega^{G,F})_X = (G^\flat * \psi^{G,F})_X \qquad \text{for every } X \in \mathrm{Ob}(\mathscr{B}^{G,F}).$$

Next, the coherence constraint of $\sigma^{F,G} * \omega^{G,F}$ is given by the rule :

$$(X \xrightarrow{(f_\bullet,\lambda_\bullet)} Y) \mapsto ((\gamma^{K_n}_{Gf_0,\mathbf{1}_{GY}})^{-1} \odot K_n\lambda_0 \odot \gamma^{K_n}_{\mathbf{1}_{GX},f_1} \mid n \in \mathbb{N}) = (K_n\lambda_0 \mid n \in \mathbb{N})$$

whereas the ones of $\psi^{F,G} * G^\flat$ and respectively $G^\flat * \psi^{G,F}$ are defined by the rules :

$$(f_\bullet,\lambda_\bullet) \mapsto (\tau^{\psi^{F,G}}_{(f_{n+1},\lambda_{n+1} \mid n\in\mathbb{N}),n} \mid n \in \mathbb{N}) \qquad (f_\bullet,\lambda_\bullet) \mapsto (\tau^{\psi^{G,F}}_{(f_\bullet,\lambda_\bullet),n+1} \mid n \in \mathbb{N}).$$

Hence we are reduced to showing that

$$K_n\lambda_0 \odot \tau^{\psi^{F,G}}_{(f_{n+1},\lambda_{n+1} \mid n\in\mathbb{N}),n} = \tau^{\psi^{G,F}}_{(f_\bullet,\lambda_\bullet),n+1} \qquad \text{for every } n \in \mathbb{N}.$$

The latter follows by an easy induction on n : details left to the reader. □

Definition 2.4.9. Let $\mathscr{A}$, $\mathscr{B}$ be two 2-categories, and $F, F' : \mathscr{A} \to \mathscr{B}$ two pseudo-functors.

(i) We say that a pseudo-natural transformation $F \Rightarrow F'$ is a *pseudo-natural equivalence* if it is an equivalence in the 2-category $\mathsf{PsFun}(\mathscr{A},\mathscr{B})$ (in the sense of definition 2.1.3(iii)).

(ii) We say that F is *fully faithful* (resp. *strongly faithful*) if for every $A, A' \in \mathrm{Ob}(\mathscr{A})$, the functor $F_{AA'} : \mathscr{A}(A,A') \to \mathscr{B}(FA,FA')$ is an equivalence (resp. is an isomorphism of categories).

(iii) We say that $\mathscr{A}$ is a *full sub-2-category* (resp. a *strong sub-2-category*) of $\mathscr{B}$, if it is a sub-2-category and the inclusion pseudo-functor $\mathscr{A} \to \mathscr{B}$ is fully (resp. strongly) faithful.

(iv) We say that F is a *2-equivalence* (resp. a *strong 2-equivalence*) from $\mathscr{A}$ to $\mathscr{B}$ if it is fully faithful (resp. strongly faithful), and for every $B \in \mathrm{Ob}(\mathscr{B})$ there exists $A \in \mathrm{Ob}(\mathscr{A})$ with an equivalence (resp. an isomorphism) $FA \to B$.

Remark 2.4.10. Consider a square diagram of 2-categories and pseudo-functors

$$\mathscr{D} \quad : \quad \begin{array}{ccc} \mathscr{A} & \xrightarrow{F} & \mathscr{B} \\ {\scriptstyle H}\downarrow & & \downarrow{\scriptstyle G} \\ \mathscr{C} & \xrightarrow{K} & \mathscr{D}. \end{array}$$

We shall say that $\mathscr{D}$ is *essentially commutative* (resp. *pseudo-commutative*) if there exists an isomorphism of pseudo-functors (resp. a pseudo-natural equivalence) $\alpha : G \circ F \Rightarrow K \circ H$. Even though the set of (small) 2-categories does not form a 2-category, the datum $(\mathscr{D}, \alpha)$ is a type of oriented square analogous to those contemplated in (2.1.5). Especially, two such data $(\mathscr{D}, \alpha)$ and $(\mathscr{D}', \alpha')$ can be composed if they share a side, in exactly the same way as for usual oriented square. Then, it is clear that a horizontal or vertical composition of essentially commutative (resp. pseudo-commutative) squares is again essentially commutative (resp. pseudo-commutative). On the other hand, the discussion of remark 2.2.5(iv,v) shows that the identities of proposition 2.1.9 hold only up to isomorphism of pseudo-natural transformations, *i.e.* up to invertible modifications.

Lemma 2.4.11. *Let $\mathscr{A}, \mathscr{B}$ be two 2-categories, $F : \mathscr{A} \to \mathscr{B}$ a pseudo-functor, and $f : A \to B$ a 1-cell in $\mathscr{A}$. We have :*

(i) *If f admits a right adjoint g, then Ff admits the right adjoint Fg.*
(ii) *If f is an equivalence in $\mathscr{A}$, then Ff is an equivalence in $\mathscr{B}$, and if g is a quasi-inverse for f, then Fg is a quasi-inverse for Ff.*
(iii) *If F is fully faithful, then f is an equivalence if and only if Ff is an equivalence in $\mathscr{B}$.*

Proof. (i): Indeed, let $\varepsilon : f \circ g \Rightarrow \mathbf{1}_B$ and $\eta : \mathbf{1}_A \Rightarrow g \circ f$ be the unit and counit of an adjunction for the pair (f, g). We consider the 2-cells

$$\varepsilon' := \delta_B^{F\ -1} \odot F(\varepsilon) \odot \gamma^F_{g,f} : Ff \circ Fg \Rightarrow \mathbf{1}_{FB} \qquad \eta' := \gamma^{F\ -1}_{f,g} \odot F(\eta) \odot \delta^F_A : \mathbf{1}_{FA} \Rightarrow Fg \circ Ff$$

and we compute :

$$\begin{aligned}(Fg * F\varepsilon') \odot (F\eta' * Fg) &= (Fg * \delta^F_B)^{-1} \odot (Fg * F\varepsilon) \odot \gamma^{F\ -1}_{f\circ g, g} \odot \gamma^F_{g, g\circ f} \odot (F\eta * Fg) \odot (\delta^F_A * Fg) \\ &= (Fg * \delta^F_B)^{-1} \odot \gamma^{F\ -1}_{\mathbf{1}_B, g} \odot F(g * \varepsilon) \odot F(\eta * g) \odot \gamma^F_{g, \mathbf{1}_B} \odot (\delta^F_A * Fg) \\ &= \mathbf{1}_{Fg}.\end{aligned}$$

Likewise we check that $(F\varepsilon * Ff) \odot (Ff * F\eta) = \mathbf{1}_{Ff}$: the details are left to the reader.

(ii): It is clear from the proof of (i) that if ε and η are invertible 2-cells, the same holds for η' and ε', whence the contention.

(iii): Due to (ii), we may assume that Ff is an equivalence in $\mathscr{B}$, and we check that f is an equivalence in $\mathscr{A}$. To this aim, let $g : FB \to FA$ be a quasi-inverse for Ff, and $\eta : \mathbf{1}_{FB} \to Ff \circ g$ and $\varepsilon : g \circ Ff \Rightarrow \mathbf{1}_{FA}$ two invertible 2-cells. Since F_{AB} is an equivalence, we have a 1-cell $h : B \to A$ and an invertible 2-cell $\beta : Fh \Rightarrow g$, whence the invertible 2-cell

$$\begin{aligned}\varepsilon' &:= \delta^F_A \odot \varepsilon \odot (\beta * Ff) \odot \gamma^{F\ -1}_{f,h} : F(h \circ f) \Rightarrow F\mathbf{1}_A \\ \eta' &:= \eta' := 1\gamma^F_{h,f} \odot (Ff * \beta^{-1}) \odot \eta \odot \delta_B^{F\ -1} : F\mathbf{1}_B \Rightarrow F(f \circ h).\end{aligned}$$

Again, since F_{AB} is an equivalence, we deduce invertible 2-cells $\varepsilon'' : h \circ f \Rightarrow \mathbf{1}_A$ and $\eta'' : \mathbf{1}_B \Rightarrow f \circ h$ such that $F\varepsilon'' = \varepsilon'$ and $F\eta'' = \eta'$, whence the contention. □

Theorem 2.4.12. *Let $\mathscr{A}, \mathscr{B}$ be two 2-categories, $F, G : \mathscr{A} \to \mathscr{B}$ two pseudo-functors, and $\alpha : F \Rightarrow G$ a pseudo-natural transformation. The following conditions are equivalent :*

(a) *α is a pseudo-natural equivalence of pseudo-functors.*
(b) *The 1-cell $\alpha_A : FA \to GA$ is an equivalence in $\mathscr{B}$, for every $A \in \mathrm{Ob}(\mathscr{A})$.*

Proof. Let $\mathscr{A}^F$ be the 2-category associated with F as in proposition 2.4.6, together with the strict pseudo-functors $F^\flat : \mathscr{A}^F \to \mathscr{B}$ and $\pi^F : \mathscr{A}^F \to \mathscr{A}$, the pseudo-functor $\sigma^F : \mathscr{A} \to \mathscr{A}^F$ and the pseudo-natural isomorphism $\omega^F : F \circ \pi^F \xrightarrow{\sim} F^\flat$. Set $\alpha' := (\alpha * \pi^F) \odot \omega^{F\ -1} : F^\flat \Rightarrow G \circ \pi^F$, and suppose that there exists a pseudo-natural transformation $\beta' : G \circ \pi^F \Rightarrow F^\flat$ with invertible modifications $\Xi_1 : \mathbf{1}_{F^\flat} \rightsquigarrow \beta' \odot \alpha'$ and $\Xi_2 : \mathbf{1}_{G\circ\pi^F} \rightsquigarrow \alpha' \odot \beta'$. Then notice that $\alpha' * \sigma^F = \alpha$, and set $\beta := \beta' * \pi^F$; we deduce invertible modifications $\Xi_1 \circ \pi^F : \mathbf{1}_F \rightsquigarrow \beta \odot \alpha$ and $\Xi_2 \circ \pi^F : \mathbf{1}_G \rightsquigarrow \alpha \odot \beta$. Thus, we may replace $\mathscr{A}$, F, G and α by $\mathscr{A}^F$, $F^\flat$, $G \circ \pi^F$ and α', and assume from start that F is strict. Arguing likewise with $\mathscr{A}^G$ and $G^\flat$, we may then reduce to the case where both F and G are strict.

(b)⇒(a): Notice that the pseudo-functor $\widetilde{\alpha} : \mathscr{A} \to \text{2-}\mathsf{Morph}(\mathscr{B})$ associated with α (see remark 2.2.5(ii)) factors through the inclusion pseudo-functor $\mathsf{Equiv}(\mathscr{B}) \to \text{2-}\mathsf{Morph}(\mathscr{B})$ (notation of (2.3.25)); then define $\widetilde{\beta}$ as the composition

$$\mathscr{A} \xrightarrow{\widetilde{\alpha}} \mathsf{Equiv}(\mathscr{B}) \xrightarrow{\widetilde{\Upsilon}} \mathsf{Equiv}(\mathscr{B})$$

(notation of (2.3.27)). Notice as well that the coherence constraints of $\widetilde{\Upsilon}$ are given by identities, and $\widetilde{\alpha}$ is strict (since F and G are strict), therefore the coherence constraints of $\widetilde{\beta}$ are given as well by identities, so $\widetilde{\beta}$ is strict. Furthermore, a simple inspection shows that $\mathsf{s} \circ \widetilde{\beta} = G$ and $\mathsf{t} \circ \widetilde{\beta} = F$, where s and t are the restrictions to $\mathsf{Equiv}(\mathscr{A})$ of the source and target pseudo-functors of remark 2.1.18(ii). We conclude that $\widetilde{\beta}$ is the pseudo-functor associated as in remark 2.2.5(ii) with a pseudo-natural transformation $\beta : G \Rightarrow F$. We can describe the coherence constraint τ^β of β as follows. Let τ^α be the coherence constraint of α, which assigns to every 1-cell $f : A \to B$ of $\mathscr{A}$ the oriented square τ^α_f as in remark 2.2.5(i). Recall that the pseudo-functor L of (2.3.25) assigns to every 1-cell $g : X \to Y$ of $\mathscr{B}$ a link $\mathsf{L}(g) := (g^\dagger, g, \eta^g, \varepsilon^g) : X \to Y$; then τ^α_f yields the 1-cell $\mathsf{L}(\tau^\alpha_f) : \mathsf{L}(\alpha_A) \to \mathsf{L}(\alpha_B)$ in the 2-category $\mathsf{wLink}(\mathscr{B})$. With this notation, we have $\beta_A := (\alpha_A)^\dagger$ for every $A \in \mathrm{Ob}(\mathscr{A})$, and τ^β assigns to f the oriented square

$$\tau^\beta_f \quad : \quad \begin{array}{ccc} GA & \xrightarrow{\ \beta_A\ } & FA \\ {\scriptstyle Gf}\downarrow & \swArrow \Upsilon(\mathsf{L}(\tau^\alpha_f))^{-1} & \downarrow{\scriptstyle Ff} \\ GB & \xrightarrow[\ \beta_B\ ]{} & FB. \end{array}$$

To conclude, it remains to check that the systems of 2-cells :

$$\lambda_A := (\varepsilon^{\alpha_A})^{-1} \qquad \text{and} \qquad \mu_A := (\eta^{\alpha_A})^{-1} \qquad \text{for every } A \in \mathrm{Ob}(\mathscr{A})$$

amount to two invertible modifications

$$\lambda_\bullet : \mathbf{1}_F \rightsquigarrow \beta \odot \alpha \qquad \mu_\bullet : \alpha \odot \beta \rightsquigarrow \mathbf{1}_G.$$

However, if $\tau^{\alpha\odot\beta}$ denotes the coherence constraint of $\alpha \odot \beta$, the compatibility condition for $\mu_\bullet$ comes down to the identity :

$$(2.4.13) \qquad (\mu_B * Gf) \odot \tau^{\alpha\odot\beta}_f = Gf * \mu_A \qquad \text{for every 1-cell } f : A \to B \text{ of } \mathscr{A}.$$

Let us then consider the diagram :

$$\begin{array}{ccccccccc}
GA & \xrightarrow{\beta_A} & FA & = & FA & = & FA & \xrightarrow{\alpha_A} & GA \\
\| & \swArrow \mathbf{1}_{\beta_A} & \| & \swArrow \mathbf{1}_{Ff} & \downarrow{\scriptstyle Ff} & \swArrow \mathbf{1}_{Ff} & \downarrow{\scriptstyle Ff} & \swArrow \tau^\alpha_f & \downarrow{\scriptstyle Gf} \\
GA & \xrightarrow{\beta_A} & FA & \xrightarrow{Ff} & FB & = & FB & \xrightarrow{\alpha_B} & GB \\
\| & \swArrow \mu_A & \downarrow{\scriptstyle \alpha_A} & \swArrow \mathsf{L}(\tau^\alpha_f)^{-1} & \downarrow{\scriptstyle \alpha_B} & \swArrow \lambda_B & \| & \swArrow \mathbf{1}_{\alpha_B} & \| \\
GA & = & GA & \xrightarrow{Gf} & GB & \xrightarrow{\beta_B} & FB & \xrightarrow{\alpha_B} & GB \\
{\scriptstyle Gf}\downarrow & \swArrow \mathbf{1}_{Gf} & \downarrow{\scriptstyle Gf} & \swArrow \mathbf{1}_{Gf} & \| & \swArrow \mu_B & \downarrow{\scriptstyle \alpha_B} & \swArrow \mathbf{1}_{\alpha_B} & \| \\
GB & = & GB & = & GB & = & GB & = & GB.
\end{array}$$

We see that the composition of the squares of the top row equals $\tau^\alpha_f * \beta_A$, and the composition of the squares of the middle row equals $\alpha_B * \tau^\beta_f$. Hence the composition of the squares of the top and middle rows equals $\tau^{\alpha\odot\beta}_f$, and by further composing with the squares of the bottom row we get the left hand-side of (2.4.13). On the other hand, by proposition 2.1.9 we can also compose the squares column by column : then notice that the composition of the squares in the third column from the left equals the identity of $\alpha_B \circ Ff$. Thus, we may disregard this column, and then notice that the composition of the squares of the second column equals the inverse of the composition of the squares of the fourth column. So we may disregard all columns except the first from the left; but the composition of the squares of the latter equals the right-hand side of (2.4.13), as required. The verification for $\lambda_\bullet$ is similar, and shall be left to the reader.

(a)⇒(b) is immediate from the definitions. □

Corollary 2.4.14. *Let $\mathscr{A}, \mathscr{B}$ be two 2-categories, $F, F' : \mathscr{A} \to \mathscr{B}$ two pseudo-functors, and $\alpha : F \Rightarrow F'$ a pseudo-natural equivalence. We have :*

(i) *For every* 2*-category* $\mathscr{A}'$ *and every pseudo-functor* $G : \mathscr{A}' \to \mathscr{A}$*, the composition* $\alpha * G : F \circ G \Rightarrow F' \circ G$ *is a pseudo-natural equivalence (notation of remark* 2.2.5(iv)*).*
(ii) *For every* 2*-category* $\mathscr{B}'$ *and every pseudo-functor* $H : \mathscr{B} \to \mathscr{B}'$*, the composition* $H * \alpha : H \circ F \Rightarrow H \circ F'$ *is a pseudo-natural equivalence.*
(iii) *If* $F' : \mathscr{A} \to \mathscr{B}$ *is any other pseudo-functor, and* $\alpha' : F' \Rightarrow F''$ *any other pseudo-natural equivalence, then the composition* $\alpha' \odot \alpha : F \Rightarrow F''$ *is a pseudo-natural equivalence as well (notation of remark* 2.2.5(iii)*).*

Proof. Assertions (i) and (ii) follow immediately from theorem 2.4.12 and lemma 2.4.11(ii). Assertion (iii) follows from theorem 2.4.12 and lemma 2.1.4(ii) : details left to the reader. □

Definition 2.4.15. Let $\mathscr{A}$, $\mathscr{B}$ be two 2-categories, $F : \mathscr{A} \to \mathscr{B}$, $G : \mathscr{B} \to \mathscr{A}$ two pseudo-functors. We say that G is a *right* 2*-adjoint to* F (resp. a *strong right* 2*-adjoint to* F) if there exists a pseudo-natural equivalence (resp. a pseudo-natural isomorphism) of pseudo-functors

$$\vartheta : H_{\mathscr{B}}(F, \mathbf{1}_{\mathscr{B}}) \Rightarrow H_{\mathscr{A}}(\mathbf{1}_{\mathscr{A}}, G)$$

(definition 2.4.9(i)). In this case, we say that (F, G) is a 2*-adjoint pair* (resp. a *strong* 2*-adjoint pair*) of pseudo-functors, and ϑ is a 2*-adjunction* (resp. a *strong* 2*-adjunction*) for (F, G).

Remark 2.4.16. According to theorem 2.4.12 and remark 2.2.5(i), the datum of a 2-adjunction as in definition 2.4.15 is equivalent to that of :

- a system of equivalences of categories

$$\vartheta_{AB} : \mathscr{B}(FA, B) \to \mathscr{A}(A, GB) \qquad \text{for every } A \in \mathrm{Ob}(\mathscr{A}) \text{ and } B \in \mathrm{Ob}(\mathscr{B})$$

- and for every pair of 1-cells $f : A' \to A$ in $\mathscr{A}$, $g : B \to B'$ in $\mathscr{B}$, a natural isomorphism of functors

$$\begin{array}{ccc} \mathscr{B}(FA, B) & \xrightarrow{\vartheta_{AB}} & \mathscr{A}(A, GB) \\ {\scriptstyle \mathscr{B}(Ff,g)}\downarrow & \swarrow \tau^{\vartheta}_{(f,g)} & \downarrow{\scriptstyle \mathscr{A}(f,Gg)} \\ \mathscr{B}(FA', B') & \xrightarrow[\vartheta_{A'B'}]{} & \mathscr{A}(A', GB') \end{array}$$

- such that, for every 2-cells $\beta : f \Rightarrow f'$ in $\mathscr{A}$ and $\lambda : g \Rightarrow g'$ in $\mathscr{B}$, we have the identity

$$\tau^{\vartheta}_{(f,g)} \boxminus H_{\mathscr{B}}(F\beta, \lambda) = H_{\mathscr{A}}(\beta, G\lambda) \boxminus \tau^{\vartheta}_{(f',g')} \tag{2.4.17}$$

- and for every composable pairs of 1-cells $A'' \xrightarrow{f'} A' \xrightarrow{f} A$ in $\mathscr{A}$, and $B \xrightarrow{g} B' \xrightarrow{g'} B''$ in $\mathscr{B}$, we have the identities

$$(\tau^{\vartheta}_{(f',g')} \boxdot \tau^{\vartheta}_{(f,g)}) \boxminus H_{\mathscr{B}}(\gamma^F_{f',f}, \mathbf{1}_{g' \circ g}) = H_{\mathscr{A}}(\mathbf{1}_{f \circ f'}, \gamma^G_{g,g'}) \boxminus \tau^{\vartheta}_{(f \circ f', g' \circ g)}$$
$$H_{\mathscr{A}}(i_A, \delta^G_B) \boxminus \tau^{\vartheta}_{(\mathbf{1}_A, \mathbf{1}_B)} = \mathbf{1}_{\vartheta_{AB}} \boxminus H_{\mathscr{B}}(\delta^F_A, i_B)$$

where (δ^F, γ^F) (resp. (δ^G, γ^G)) denote by coherence constraint for F (resp. for G).

We wish to attach to every 2-adjunction suitable units and counits, as for usual adjoint pairs of functors; we shall see that the triangular identities will have to be replaced by certain invertible modifications. We begin with a few auxiliary lemmata :

Lemma 2.4.18. *Let* $\mathscr{A}$,$\mathscr{B}$ *be any two* 2*-categories,* $F, F' : \mathscr{A} \to \mathscr{B}$ *two strict pseudo-functors,* $\beta : F \Rightarrow F'$ *a pseudo-natural transformation. We have :*

(i) F *induces a strict pseudo-natural transformation (notation of example* 2.2.7(ii)*) :*

$$H_F : H_{\mathscr{A}} \Rightarrow H_{\mathscr{B}}(F, F) \qquad (A, A') \mapsto (F_{AA'} : \mathscr{A}(A, A') \to \mathscr{B}(FA, FA')).$$

(ii) *The coherence constraint τ^β of β induces an invertible modification*

$$H_\beta : H_{\mathscr{B}}(\beta, \mathbf{1}_{F'}) \odot H_{F'} \rightsquigarrow H_{\mathscr{B}}(\mathbf{1}_F, \beta) \odot H_F$$

that assigns to every object (A_1, A_2) of $\mathscr{A}^o \times \mathscr{A}$ the natural transformation

$$\tau^\beta_{A_1A_2} : H_{\mathscr{B}}(\beta_{A_1}, \mathbf{1}_{F'A_2}) \circ F'_{A_1A_2} \Rightarrow H_{\mathscr{B}}(\mathbf{1}_{FA_1}, \beta_{A_2}) \circ F_{A_1A_2}.$$

(iii) *If F is fully faithful, H_F is a pseudo-natural equivalence.*

Proof. Assertion (i) is straightforward, and (iii) follows immediately from theorem 2.4.12.

(ii): For every 1-cell $(f_1, f_2) : (A_1, A_2) \to (A'_1, A'_2)$ of $\mathscr{A}^o \times \mathscr{A}$ we consider the diagrams

$$\begin{array}{ccccc}
\mathscr{A}(A_1, A_2) & \xrightarrow{H_{F'}} & \mathscr{B}(F'A_1, F'A_2) & \xrightarrow{\mathscr{B}(\beta_{A_1}, \mathbf{1}_{F'A_2})} & \mathscr{B}(FA_1, F'A_2) \\
{\scriptstyle \mathscr{A}(f_1,f_2)}\downarrow & \Downarrow \mathbf{1} & \downarrow{\scriptstyle \mathscr{B}(F'f_1, F'f_2)} & \Downarrow \mathscr{B}((\tau^\beta_{f_1})^{-1}, \mathbf{1}_{F'f_2}) & \downarrow{\scriptstyle \mathscr{B}(Ff_1, F'f_2)} \\
\mathscr{A}(A'_1, A'_2) & \xrightarrow[H_{F'}]{} & \mathscr{B}(F'A'_1, F'A'_2) & \xrightarrow[\mathscr{B}(\beta_{A'_1}, \mathbf{1}_{F'A'_2})]{} & \mathscr{B}(FA'_1, F'A'_2) \\
\| & & \Downarrow \tau^\beta_{A'_1A'_2} & & \| \\
\mathscr{A}(A'_1, A'_2) & \xrightarrow[H_F]{} & \mathscr{B}(FA'_1, FA'_2) & \xrightarrow[\mathscr{B}(\mathbf{1}_{FA'_1}, \beta_{A_2})]{} & \mathscr{B}(FA'_1, F'A'_2)
\end{array}$$

$$\begin{array}{ccccc}
\mathscr{A}(A_1, A_2) & \xrightarrow{H_{F'}} & \mathscr{B}(F'A_1, F'A_2) & \xrightarrow{\mathscr{B}(\beta_{A_1}, \mathbf{1}_{F'A_2})} & \mathscr{B}(FA_1, F'A_2) \\
\| & & \Downarrow \tau^\beta_{A_1A_2} & & \| \\
\mathscr{A}(A_1, A_2) & \xrightarrow{H_F} & \mathscr{B}(FA_1, FA_2) & \xrightarrow{\mathscr{B}(\mathbf{1}_{FA_1}, \beta_{A_2})} & \mathscr{B}(FA_1, F'A_2) \\
{\scriptstyle \mathscr{A}(f_1,f_2)}\downarrow & \Downarrow \mathbf{1} & \downarrow{\scriptstyle \mathscr{B}(Ff_1, Ff_2)} & \Downarrow \mathscr{B}(\mathbf{1}_{Ff_1}, \tau^\beta_{f_2}) & \downarrow{\scriptstyle \mathscr{B}(Ff_1, F'f_2)} \\
\mathscr{A}(A'_1, A'_2) & \xrightarrow[H_F]{} & \mathscr{B}(FA'_1, FA'_2) & \xrightarrow[\mathscr{B}(\mathbf{1}_{FA'_1}, \beta_{A'_2})]{} & \mathscr{B}(FA'_1, F'A'_2)
\end{array}$$

and notice that the coherence constraint of $H_{\mathscr{B}}(\beta, \mathbf{1}_{F'}) \odot H_{F'}$ is given by the compositions of oriented squares on the top row of the first diagram (see example 2.2.6). Likewise, the coherence constraint of $H_{\mathscr{B}}(\mathbf{1}_F, \beta) \odot H_F$ is given by the composition of the oriented squares on the bottom row of the second diagram. Therefore, we come down to checking that the composition of the oriented squares of the first diagram equals the composition of the oriented squares of the second diagram. The latter translates as the identity :

$$\tau^\beta_{f_2 \circ t \circ f_1} \odot (F'(f_2 \circ t) * \tau^{\beta\,-1}_{f_1}) = (\tau^\beta_{f_2} * F(t \circ f_1)) \odot (F'f_2 * \tau^\beta_t * Ff_1)$$

for every 1-cell $t : A_1 \to A_2$ in $\mathscr{A}$. However, the coherence axiom for β yields the identities :

$$\begin{aligned}
(\tau^\beta_t * Ff_1) \odot (F't * \tau^\beta_{f_1}) &= \tau^\beta_{t \circ f_1} \\
(\tau^\beta_{f_2} * F(t \circ f_1)) \odot (F'f_2 * \tau^\beta_{t \circ f_1}) &= \tau^\beta_{f_2 \circ t \circ f_1}.
\end{aligned}$$

The sought identity is an immediate consequence. □

2.4.19. Conversely, consider now two strict pseudo-functors $F, G : \mathscr{A} \to \mathscr{B}$, and a pseudo-natural transformation $\lambda : H_{\mathscr{A}} \Rightarrow H_{\mathscr{B}}(F, G)$ with coherence constraint $\tau_\bullet$. We set

$$\lambda^\vee_A := \lambda_{(A,A)}(\mathbf{1}_A) : FA \to GA \qquad \text{for every } A \in \mathrm{Ob}(\mathscr{A})$$

and for every 1-cell $f : A \to A'$ in $\mathscr{A}$ we let

$$\tau^\vee_f : Gf \circ \lambda^\vee_A \Rightarrow \lambda^\vee_{A'} \circ Ff$$

be the unique 2-cell of $\mathscr{B}$ that fits into the commutative diagram

$$\begin{array}{ccccc} Gf \circ \lambda^\vee_A & & \xRightarrow{\tau^\vee_f} & & \lambda^\vee_{A'} \circ Ff \\ \| & & & & \| \\ Gf \circ \lambda^\vee_A \circ F\mathbf{1}_A & \xRightarrow{\tau_{(\mathbf{1}_A,f),\mathbf{1}_A}} & \lambda_{(A,A')}(f) & \xLeftarrow{\tau_{(f,\mathbf{1}_{A'}),\mathbf{1}_{A'}}} & G\mathbf{1}_{A'} \circ \lambda^\vee_{A'} \circ Ff. \end{array}$$

Lemma 2.4.20. *With the notation of* (2.4.19)*, the following holds :*

(i) *The system of* 1*-cells* $(\lambda^\vee_A \mid A \in \mathrm{Ob}(\mathscr{A}))$ *defines a pseudo-natural transformation*

$$\lambda^\vee : F \Rightarrow G$$

with coherence constraint given by the system of 2*-cells* $\tau^\vee_\bullet$.

(ii) $H^\vee_F = \mathbf{1}_F$.

(iii) *Let* $\mu : H_{\mathscr{A}} \Rightarrow H_{\mathscr{B}}(F,G)$ *be another pseudo-natural transformations, and* $\Theta : \lambda \rightsquigarrow \mu$ *any modification. Then we have a modification* $\Theta^\vee : \lambda^\vee \rightsquigarrow \mu^\vee$ *given by the rule :*

$$A \mapsto (\Theta^\vee_A := \Theta_{(A,A),\mathbf{1}_A} : \lambda^\vee_A \Rightarrow \mu^\vee_A) \qquad \textit{for every } A \in \mathrm{Ob}(\mathscr{A}).$$

Proof. (i): Let $f, f' : A \to A'$ be two 1-cells in $\mathscr{A}$; the naturality of $\tau^\vee_\bullet$ amounts to the commutativity of the diagram

$$\begin{array}{ccccc} Gf \circ \lambda^\vee_A & \xRightarrow{\tau_{(\mathbf{1}_A,f),\mathbf{1}_A}} & \lambda_{(A,A')}(f) & \xLeftarrow{\tau_{(f,\mathbf{1}_{A'}),\mathbf{1}_{A'}}} & \lambda^\vee_{A'} \circ Ff \\ {\scriptstyle G\alpha * \lambda^\vee_A}\Downarrow & & \Downarrow{\scriptstyle \lambda_{(A,A')}(\alpha)} & & \Downarrow{\scriptstyle \lambda^\vee_{A'} * F\alpha} \\ Gf' \circ \lambda^\vee_A & \xRightarrow{\tau_{(\mathbf{1}_A,f'),\mathbf{1}_A}} & \lambda_{(A,A')}(f') & \xLeftarrow{\tau_{(f',\mathbf{1}_{A'}),\mathbf{1}_{A'}}} & \lambda^\vee_{A'} \circ Ff' \end{array}$$

for every 2-cell $\alpha : f \Rightarrow f'$ in $\mathscr{A}$. However, the commutativity of the two square subdiagrams follows by applying the naturality of τ to the 2-cells (i_A, α) and $(\alpha, i_{A'})$ of $\mathscr{A}^o \times \mathscr{A}$.

Let us check the coherence axioms for $\tau^\vee_\bullet$. First, since F and G are strict, we need to show :

$$\tau^\vee_{\mathbf{1}_A} = \mathbf{1}_{\lambda^\vee_A} \qquad \text{for every } A \in \mathrm{Ob}(\mathscr{A})$$

which follows by a simple inspection. Next, let $A \xrightarrow{f} A' \xrightarrow{g} A''$ be a composable pair of 1-cell of $\mathscr{A}$; we need to show that

$$(\tau^\vee_g * Ff) \odot (Gg * \tau^\vee_f) = \tau^\vee_{g \circ f}.$$

Now, the coherence axiom for $\tau_\bullet$ yields the identity

$$\tau_{(\mathbf{1}_A,g),f} \odot (Gg * \tau_{(\mathbf{1}_A,f),\mathbf{1}_A}) = \tau_{(\mathbf{1}_A,g\circ f),\mathbf{1}_A}.$$

Hence we are reduced to showing that

$$(\tau^\vee_g * Ff) \odot (Gg * \tau^{-1}_{(f,\mathbf{1}_{A'}),\mathbf{1}_{A'}}) = \tau^{-1}_{(g\circ f,\mathbf{1}_{A''}),\mathbf{1}_{A''}} \odot \tau_{(\mathbf{1}_A,g),f}.$$

Next, again by the coherence axiom for $\tau_\bullet$ we see that

$$\tau_{(g\circ f,\mathbf{1}_{A''}),\mathbf{1}_{A''}} = \tau_{(f,\mathbf{1}_{A''}),g} \odot (\tau_{(g,\mathbf{1}_{A''}),\mathbf{1}_{A''}} * Ff).$$

Hence we are further reduced to checking the identity :

$$\tau_{(f,\mathbf{1}_{A''}),g} \odot (\tau_{(\mathbf{1}_{A'},g),\mathbf{1}_{A'}} * Ff) = \tau_{(\mathbf{1}_A,g),f} \odot (Gg * \tau_{(f,\mathbf{1}_{A'}),\mathbf{1}_{A'}}). \tag{2.4.21}$$

However, applying twice the coherence axiom for $\tau_\bullet$ to the identities in $\mathscr{A}^o \times \mathscr{A}$:

$$(f, \mathbf{1}_{A''}) \circ (\mathbf{1}_{A'}, g) = (f, g) = (\mathbf{1}_A, g) \circ (f, \mathbf{1}_{A'})$$

we see that both sides of (2.4.21) equal $\tau_{(f,g),\mathbf{1}_{A'}}$.

(ii) follows by a direct inspection of the definitions.

(iii): The assertion follows from the commutativity of the diagram :

$$\begin{array}{ccccc} Gf\circ\lambda^\vee_A & \xrightarrow{\tau^\lambda_{(\mathbf{1}_A,f),\mathbf{1}_A}} & \lambda_{(A,A')}(f) & \xleftarrow{\tau^\lambda_{(f,\mathbf{1}_{A'}),\mathbf{1}_{A'}}} & \lambda^\vee_{A'}\circ Ff. \\ {\scriptstyle Gf*\Theta^\vee_A}\Downarrow & & {\scriptstyle \Theta_{(A,A'),f}}\Downarrow & & \Downarrow{\scriptstyle \Theta^\vee_{A'}*Ff} \\ Gf\circ\mu^\vee_A & \xrightarrow{\tau^\mu_{(\mathbf{1}_A,f),\mathbf{1}_A}} & \mu_{(A,A')}(f) & \xleftarrow{\tau^\mu_{(f,\mathbf{1}_{A'}),\mathbf{1}_{A'}}} & \mu^\vee_{A'}\circ Ff \end{array}$$

which in turn follows directly from the definition of Θ. □

2.4.22. Let now $F:\mathscr{A}\to\mathscr{B}$ be a pseudo-functor, and $G:\mathscr{B}\to\mathscr{A}$ a right 2-adjoint for F, and ϑ a 2-adjunction for the pair (F,G) as in definition 2.4.15. We suppose moreover that F and G are strict; by lemma 2.1.13(i) we may find a pseudo-natural equivalence

$$\psi : H_{\mathscr{A}}(\mathbf{1}_{\mathscr{A}},G)\Rightarrow H_{\mathscr{B}}(F,\mathbf{1}_{\mathscr{B}})$$

as well as invertible modifications

$$\Xi:\mathbf{1}_{H_{\mathscr{B}}(F,\mathbf{1}_{\mathscr{B}})}\rightsquigarrow\psi\odot\vartheta \qquad \Theta:\vartheta\odot\psi\rightsquigarrow\mathbf{1}_{H_{\mathscr{A}}(\mathbf{1}_{\mathscr{A}},G)}$$

such that

$$(\psi*\Theta)\odot(\Xi*\psi)=\mathbf{1}_\psi \qquad\text{and}\qquad (\Theta*\vartheta)\odot(\vartheta*\Xi)=\mathbf{1}_\vartheta.$$

To ease notation, we shall drop the subscripts when referring to $\vartheta_{AB}(h)$ or $\vartheta_{AB}(\mu)$ for any 1-cell $h:FA\to B$ or 2-cell $\mu:h\Rightarrow h'$ in $\mathscr{B}$, and write simply $\vartheta(h)$ and $\vartheta(\mu)$. Likewise, we shall write simply $\psi(k)$ and $\psi(\nu)$ for any 1-cell $k:A\to GB$ and any 2-cell $\nu:k\Rightarrow k'$ in $\mathscr{A}$. Now, for every $A\in\mathrm{Ob}(\mathscr{A})$ and $B\in\mathrm{Ob}(\mathscr{B})$ let us set

$$\eta_A:=\vartheta(\mathbf{1}_{FA}):A\to GFA \qquad \varepsilon_B:=\psi(\mathbf{1}_{GB}):FGB\to B.$$

For every 1-cell $f:A\to A'$ in $\mathscr{A}$, and every 1-cell $g:B\to B'$ in $\mathscr{B}$ we let

$$\tau^\eta_f:GFf\circ\eta_A\Rightarrow\eta_{A'}\circ f \qquad\text{and}\qquad \tau^\varepsilon_g:g\circ\varepsilon_B\Rightarrow\varepsilon_{B'}\circ FGg$$

be the unique 2-cells in $\mathscr{A}$ and respectively in $\mathscr{B}$ that make commute the diagrams :

$$\begin{array}{ccc} GFf\circ\eta_A & \xRightarrow{\tau^\eta_f} & \eta_{A'}\circ f \\ {\scriptstyle\tau^\vartheta_{(\mathbf{1}_A,Ff),\mathbf{1}_{FA}}}\searrow & & \swarrow{\scriptstyle\tau^\vartheta_{(f,\mathbf{1}_{FA'}),\mathbf{1}_{FA'}}} \\ & \vartheta(Ff) & \end{array} \qquad \begin{array}{ccc} g\circ\varepsilon_B & \xRightarrow{\tau^\varepsilon_g} & \varepsilon_{B'}\circ FGg \\ {\scriptstyle\tau^\psi_{(\mathbf{1}_{GB},g),\mathbf{1}_{GB}}}\searrow & & \swarrow{\scriptstyle\tau^\psi_{(Gg,\mathbf{1}_{B'}),\mathbf{1}_{GB'}}} \\ & \psi(Gg) & \end{array}$$

where, as usual, τ^ψ denotes the coherence constraint of ψ.

Lemma 2.4.23. *With the notation of* (2.4.22)*, we have :*

(i) *The system* $(\eta_A\mid A\in\mathrm{Ob}(\mathscr{A}))$ *defines a pseudo-natural transformation*

$$\eta:\mathbf{1}_{\mathscr{A}}\Rightarrow GF$$

whose coherence constraint is given by the system of 2*-cells* $\tau^\eta_\bullet$.

(ii) *The system* $(\varepsilon_B\mid B\in\mathrm{Ob}(\mathscr{B}))$ *defines a pseudo-natural transformation*

$$\varepsilon:FG\Rightarrow\mathbf{1}_{\mathscr{B}}$$

whose coherence constraint is given by the system of 2*-cells* $\tau^\varepsilon_\bullet$.

Proof. (i): Define the pseudo-natural transformation $H_F:H_{\mathscr{A}}\Rightarrow H_{\mathscr{B}}(F,F)$ as in lemma 2.4.18(i), and set

$$\lambda:=(\vartheta*(\mathbf{1}_{\mathscr{A}^o}\times F))\odot H_F:H_{\mathscr{A}}\Rightarrow H_{\mathscr{A}}(\mathbf{1}_{\mathscr{A}},G\circ F).$$

Lemma 2.4.20(i) yields a pseudo-natural transformation

$$\lambda^\vee:\mathbf{1}_{\mathscr{A}}\Rightarrow G\circ F$$

and it is easily seen that $\lambda^\vee_A = \eta_A$ for every $A \in \mathrm{Ob}(\mathscr{A})$. It remains to check that τ^η agrees with the coherence constraint $\tau^{\lambda^\vee}$ of $\lambda^\vee$. However, denote by τ^λ, τ^{H_F} and $\tau^{\vartheta*(\mathbf{1}_\mathscr{A}\times F)}$ the coherence constraints of respectively λ, H_F and $\vartheta * (\mathbf{1}_\mathscr{A} \times F)$; by inspecting the definitions we find that

$$\tau^\lambda_{(\mathbf{1}_A,f),\mathbf{1}_A} = (\vartheta * (\mathbf{1}_\mathscr{A} \times F))_{A,A'}(\tau^{H_F}_{(\mathbf{1}_A,f),\mathbf{1}_A}) \odot \tau^{\vartheta*(\mathbf{1}_\mathscr{A}\times F)}_{(\mathbf{1}_A,f),\mathbf{1}_{FA}} = \tau^\vartheta_{(\mathbf{1}_A,Ff),\mathbf{1}_{FA}}$$

and likewise : $\tau^\lambda_{(f,\mathbf{1}_{A'}),\mathbf{1}_{A'}} = \tau^\vartheta_{(f,\mathbf{1}_{FA'}),\mathbf{1}_{FA'}}$. The assertion follows.

(ii): A similar calculation shows that $\varepsilon = ((\psi * (G \times \mathbf{1}_\mathscr{B})) \odot H_G)^\vee$. □

We now turn to the case of an arbitrary 2-adjoint pair of pseudo-functors :

Theorem 2.4.24. *Let $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{A}$ be two pseudo-functors. We have :*

(i) *If (F, G) is a 2-adjoint pair, there exist pseudo-natural transformations*

$$\eta : \mathbf{1}_\mathscr{A} \Rightarrow GF \qquad \varepsilon : FG \Rightarrow \mathbf{1}_\mathscr{B}$$

related by a pair of invertible modifications

$$\Sigma : (G * \varepsilon) \odot (\eta * G) \rightsquigarrow \mathbf{1}_G \qquad \Sigma' : (\varepsilon * F) \odot (F * \eta) \rightsquigarrow \mathbf{1}_F.$$

We call η a unit *and ε a* counit *for the 2-adjunction ϑ, and Σ and Σ' the* triangular modifications *associated with (η, ε).*

(ii) *Conversely, the existence of pseudo-natural transformations ε, η and invertible modifications Σ, Σ' as in* (i)*, implies that G is right 2-adjoint to F.*

Proof. After replacing U by a larger universe, we may assume that $\mathscr{A}$ and $\mathscr{B}$ are small; in this case, let us remark that the existence of pseudo-natural transformations η and ε related by triangular modifications as in (i) means precisely that (F, G) is an adjoint pair of 1-cells of the 2-category $\overline{2\text{-}\mathbf{Cat}}$ of remark 2.2.16.

Claim 2.4.25. We may assume that F and G are unital pseudo-functors.

Proof of the claim. Let F^u and G^u be the unital pseudo-functors associated with F and G, with pseudo-natural isomorphisms $\alpha^F : F \Rightarrow F^u$ and $\alpha^G : G \Rightarrow G^u$, as in proposition 2.4.3. If ϑ is a 2-adjunction for (F, G), we get a unique pseudo-natural equivalence ϑ^* fitting into the commutative diagram :

$$\begin{array}{ccc} H_\mathscr{B}(F, \mathbf{1}_\mathscr{B}) & \stackrel{\vartheta}{\Longrightarrow} & H_\mathscr{A}(\mathbf{1}_\mathscr{A}, G) \\ {\scriptstyle H_\mathscr{B}(\alpha^F, i_\mathscr{B})}\Downarrow & & \Downarrow{\scriptstyle H_\mathscr{A}(i_\mathscr{A}, \alpha^G)} \\ H_\mathscr{B}(F^u, \mathbf{1}_\mathscr{B}) & \stackrel{\vartheta^*}{\Longrightarrow} & H_\mathscr{A}(\mathbf{1}_\mathscr{A}, G^u) \end{array}$$

(notation of example 2.2.7(ii); here $i_\mathscr{A}$ is the identity transformation of the pseudo-functor $\mathbf{1}_\mathscr{A}$, and likewise for $i_\mathscr{B}$). Then ϑ^* is a 2-adjunction for the pair (F^u, G^u). By the same token, from a given 2-adjunction for the pair (F^u, G^u) we easily deduce a 2-adjunction for (F, G). Also, (F, G) is an adjoint pair of 1-cells in $\overline{2\text{-}\mathbf{Cat}}$ if and only if the same holds for the pair (F^u, G^u) (remark 2.1.4(iv)). Thus, it suffices to show the theorem for F^u and G^u. ◊

Thus, henceforth we assume that F and G are unital. Next we consider the 2-categories $\mathscr{A}^{F,G}$ and $\mathscr{B}^{G,F}$ of proposition 2.4.7, as well as the strict pseudo-functors $F^\flat, G^\flat, \pi^{F,G}$ and $\pi^{G,F}$, the pseudo-functors $\sigma^{F,G}$ and $\sigma^{G,F}$, and the isomorphisms of pseudo-functors

$$\psi^{F,G} : \sigma^{F,G} \circ \pi^{F,G} \xrightarrow{\sim} \mathbf{1}_{\mathscr{A}^{F,G}} \qquad \psi^{G,F} : \sigma^{G,F} \circ \pi^{G,F} \xrightarrow{\sim} \mathbf{1}_{\mathscr{B}^{G,F}}.$$

Claim 2.4.26. $\pi^{F,G}$ and $\pi^{G,F}$ are 2-equivalences.

Proof of the claim. Since $\pi^{F,G}$ is the identity on objects, it suffices to check that it is fully faithful. The latter is clear, since $\pi^{F,G} \circ \sigma^{F,G} = \mathbf{1}_\mathscr{A}$ and $\sigma^{F,G} \circ \pi^{F,G}$ is isomorphic to $\mathbf{1}_{\mathscr{A}^{F,G}}$ (details left to the reader). The same argument applies to $\pi^{G,F}$. ◊

Claim 2.4.27. We may assume that F and G are strict pseudo-functors.

Proof of the claim. Let $\vartheta : H_{\mathscr{B}}(F, \mathbf{1}_{\mathscr{B}}) \stackrel{\sim}{\to} H_{\mathscr{A}}(\mathbf{1}_{\mathscr{A}}, G)$ be a 2-adjunction for the pair (F, G). By claim 2.4.26 and lemma 2.4.18(iii), $\pi^{F,G}$ and $\pi^{G,F}$ induce pseudo-natural equivalences

$$H_{\pi^{F,G}} : H_{\mathscr{A}^{F,G}} \stackrel{\sim}{\to} H_{\mathscr{A}}(\pi^{F,G}, \pi^{F,G}) \qquad \text{and} \qquad H_{\pi^{G,F}} : H_{\mathscr{B}^{G,F}} \stackrel{\sim}{\to} H_{\mathscr{B}}(\pi^{G,F}, \pi^{G,F}).$$

Then we may find a pseudo-natural equivalence $\vartheta^\flat : H_{\mathscr{B}^{G,F}}(F^\flat, \mathbf{1}_{\mathscr{B}^{G,F}}) \stackrel{\sim}{\to} H_{\mathscr{A}^{F,G}}(\mathbf{1}_{\mathscr{A}^{F,G}}, G^\flat)$ fitting into the essentially commutative diagram of pseudo-functors :

$$\begin{array}{ccc}
H_{\mathscr{B}^{G,F}}(F^\flat, \mathbf{1}_{\mathscr{B}^{G,F}}) & \xrightarrow{\quad \vartheta^\flat \quad} & H_{\mathscr{A}^{F,G}}(\mathbf{1}_{\mathscr{A}^{F,G}}, G^\flat) \\
{\scriptstyle H_{\pi^{G,F}}*(F^{\flat o} \times \mathbf{1}_{\mathscr{B}^{G,F}})}\downarrow & & \downarrow{\scriptstyle H_{\pi^{F,G}}*(\mathbf{1}_{\mathscr{A}^{F,G}} \times G^\flat)} \\
H_{\mathscr{B}}(\pi^{G,F} \circ F^\flat, \pi^{G,F}) & & H_{\mathscr{A}}(\pi^{F,G}, \pi^{F,G} \circ G^\flat) \\
{\scriptstyle H_{\mathscr{B}}*(\omega^{F,G} \times \mathbf{1}_{\pi^{G,F}})}\downarrow & & \uparrow{\scriptstyle H_{\mathscr{A}}*(\mathbf{1}_{\pi^{F,G}} \times \omega^{G,F})} \\
H_{\mathscr{B}}(F \circ \pi^{F,G}, \pi^{G,F}) & \xrightarrow{\vartheta*(\pi^{F,G} \times \pi^{G,F})} & H_{\mathscr{A}}(\pi^{F,G}, G \circ \pi^{G,F}).
\end{array}$$

Conversely, if $\vartheta^\flat$ is a given 2-adjunction for the pair $(F^\flat, G^\flat)$, then we deduce a pseudo-natural equivalence ϑ fitting into the essentially commutative diagram :

$$\begin{array}{ccc}
H_{\mathscr{B}}(F, \mathbf{1}_{\mathscr{B}}) & \xrightarrow{\quad \vartheta \quad} & H_{\mathscr{A}}(\mathbf{1}_{\mathscr{A}}, G) \\
{\scriptstyle H_{\pi^{G,F}}*((\sigma^{G,F} \circ F)^o \times \sigma^{G,F})}\uparrow & & \uparrow{\scriptstyle H_{\pi^{F,G}}*(\sigma^{F,G} \times (\sigma^{F,G} \circ G))} \\
H_{\mathscr{B}^{G,F}}(\sigma^{G,F} \circ F, \sigma^{G,F}) & & H_{\mathscr{A}^{F,G}}(\sigma^{F,G}, \sigma^{F,G} \circ G) \\
\| & & \| \\
H_{\mathscr{B}^{G,F}}(F^\flat \circ \sigma^{F,G}, \sigma^{G,F}) & \xrightarrow{\vartheta^\flat*(\sigma^{F,G} \times \sigma^{G,F})} & H_{\mathscr{A}^{F,G}}(\sigma^{F,G}, G^\flat \circ \sigma^{G,F}).
\end{array}$$

Thus (F, G) is a 2-adjoint pair if and only if the same holds for $(F^\flat, G^\flat)$.

Likewise, if $\eta : \mathbf{1}_{\mathscr{A}} \Rightarrow GF$ and $\eta : FG \Rightarrow \mathbf{1}_{\mathscr{B}}$ are given pseudo-natural transformations fulfilling the triangular identities $(G * \varepsilon) \odot (\eta * G) = \mathbf{1}_G$ and $(\varepsilon * F) \odot (F * \eta) = \mathbf{1}_F$ in $\overline{\mathbf{2\text{-}Cat}}$, there follow pseudo-natural transformations :

$$\begin{array}{ccccc}
\eta^\flat : \mathbf{1}_{\mathscr{A}^{F,G}} & \xRightarrow{\psi^{F,G\,-1}} & \sigma^{F,G} \circ \pi^{F,G} & \xRightarrow{\sigma^{F,G}*\eta*\pi^{F,G}} & \sigma^{F,G} \circ GF \circ \pi^{F,G} \\
 & & & & \| \\
G^\flat F^\flat & \xLeftarrow{G^\flat F^\flat * \psi^{F,G}} & G^\flat F^\flat \circ \sigma^{F,G} \circ \pi^{F,G} & = & G^\flat \circ \sigma^{G,F} \circ F \circ \pi^{F,G} \\
\varepsilon^\flat : F^\flat G^\flat & \xRightarrow{F^\flat G^\flat * \psi^{G,F\,-1}} & F^\flat G^\flat \circ \sigma^{G,F} \circ \pi^{G,F} & = & F^\flat \circ \sigma^{F,G} \circ G \circ \pi^{G,F} \\
 & & & & \| \\
\mathbf{1}_{\mathscr{B}^{F,G}} & \xLeftarrow[\psi^{G,F}]{} & \sigma^{G,F} \circ \pi^{G,F} & \xLeftarrow[\sigma^{G,F}*\varepsilon*\pi^{G,F}]{} & \sigma^{G,F} \circ FG \circ \pi^{G,F}
\end{array}$$

Set $X := G^\flat * \psi^{G,F}$ and $Y := \psi^{F,G} * G^\flat$. By proposition 2.4.7(ii) we have :

$$Z := G^\flat F^\flat * ((G^\flat * \psi^{G,F})^{-1} \odot (\psi^{F,G} * G^\flat)) = (G^\flat F^\flat \sigma^{F,G} * \omega^{G,F})^{-1} = (\sigma^{F,G} GF * \omega^{G,F})^{-1}$$

whence : $Z \odot (\sigma^{F,G} * \eta * \pi^{F,G} * G^\flat) = \sigma^{F,G} * ((\eta * G\pi^{G,F}) \odot \omega^{G,F\,-1})$. We deduce :

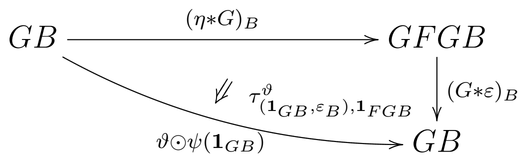
$$\begin{aligned}
(G^\flat * \varepsilon^\flat) \odot (\eta^\flat * G^\flat) &= X \odot (G^\flat \sigma^{G,F} * \varepsilon * \pi^{G,F}) \odot Z \odot (\sigma^{F,G} * \eta * \pi^{F,G} * G^\flat) \odot Y^{-1} \\
&= X \odot (G^\flat \sigma^{G,F} * \varepsilon * \pi^{G,F}) \odot (\sigma^{F,G} * ((\eta * G\pi^{G,F}) \odot \omega^{G,F\,-1})) \odot Y^{-1} \\
&= X \odot (\sigma^{F,G} G * \varepsilon * \pi^{G,F}) \odot (\sigma^{F,G} * ((\eta * G\pi^{G,F}) \odot \omega^{G,F\,-1})) \odot Y^{-1} \\
&= X \odot (\sigma^{F,G} * \omega^{G,F\,-1}) \odot Y^{-1} \\
&= \mathbf{1}_{G^\flat}
\end{aligned}$$

where the last identity follows again from proposition 2.4.7(ii). A similar computation that we leave to the reader shows that $(\varepsilon^\flat * F^\flat) \odot (F^\flat * \eta^\flat) = \mathbf{1}_{F^\flat}$ in $\overline{2\text{-}\mathbf{Cat}}$. Conversely, if given pseudo-natural transformations $\eta^\flat : \mathbf{1}_{\mathscr{A}^{F,G}} \Rightarrow G^\flat F^\flat$ and $\varepsilon^\flat : F^\flat G^\flat \Rightarrow \mathbf{1}_{\mathscr{B}^{G,F}}$ satisfy the triangular identities, then we get pseudo-natural transformations

$$\eta := \pi^{F,G} * \eta^\flat * \sigma^{F,G} : \mathbf{1}_{\mathscr{A}} \Rightarrow GF \qquad \varepsilon := \pi^{G,F} * \varepsilon^\flat * \sigma^{G,F} : FG \Rightarrow \mathbf{1}_{\mathscr{B}}$$

that also fulfill the corresponding triangular identities (details left to the reader). Thus, it suffices to prove the theorem for $F^\flat$ and $G^\flat$. $\Diamond$

Thus, henceforth we assume that F and G are strict. In order to exhibit Σ, we mimick the discussion of (1.1.13) : first, for every $B \in \mathrm{Ob}(\mathscr{B})$ we have an invertible 2-cell



where τ^ϑ denotes the coherence constraint of ϑ, and ψ is the pseudo-natural equivalence as in (2.4.22) used to define ε. Then, with the notation of (2.4.22), we set

$$\Sigma_B := \Theta_{GB,B} \odot \tau^{\vartheta}_{(\mathbf{1}_{FGB},\varepsilon_B),\mathbf{1}_{FGB}} \qquad \text{for every } B \in \mathrm{Ob}(\mathscr{B}).$$

We have to check that the system $(\Sigma_B \mid B \in \mathrm{Ob}(\mathscr{B}))$ is a modification $(G * \varepsilon) \odot (\eta * G) \rightsquigarrow \mathbf{1}_G$. However, set (notation of remark 2.2.10(iv,v) and lemma 2.4.18(i))

$$\mu := ((\vartheta \odot \psi) * (G \times \mathbf{1}_{\mathscr{B}})) \odot H_G \qquad \text{and} \qquad \Theta' := (\Theta \circ (G \times \mathbf{1}_{\mathscr{B}})) * H_G : \mu \rightsquigarrow H_G.$$

Unwinding the definitions, we see that $\Theta'_{B,B'} = \Theta_{GB,B'} * G_{BB'}$ for every $(B, B') \in \mathrm{Ob}(\mathscr{B}^o \times \mathscr{B})$; according to lemma 2.4.20(ii,iii) we then obtain an invertible modification

$$\Theta'^\vee : \mu^\vee \rightsquigarrow \mathbf{1}_G \qquad B \mapsto \Theta_{GB,B}.$$

We are thus reduced to showing that the system $(\tau^{\vartheta}_{(\mathbf{1}_{GB},\varepsilon_B),\mathbf{1}_{FGB}} \mid B \in \mathrm{Ob}(\mathscr{B}))$ yields an invertible modification

$$\Theta'' : (G * \varepsilon) \odot (\eta * G) \rightsquigarrow \mu^\vee$$

since in this case we shall have $\Sigma = \Theta'^\vee \odot \Theta''$. However, by unwinding the definitions we see that the coherence constraint of $\mu^\vee$ is the system of 2-cells

$$\tau^\mu_g := (\tau^{\vartheta}_{(Gg,\mathbf{1}_{B'}),\varepsilon_{B'}})^{-1} \odot \vartheta(\tau^\varepsilon_g) \odot \tau^{\vartheta}_{(\mathbf{1}_{GB},g),\varepsilon_B} \qquad \text{for every 1-cell } g : B \to B' \text{ of } \mathscr{B}$$

whereas that of $(G * \varepsilon) \odot (\eta * G)$ is given by the system of 2-cells

$$\tau^{(G*\varepsilon)\odot(\eta*G)}_g := (G(\varepsilon_{B'}) * \tau^\eta_{Gg}) \odot (G(\tau^\varepsilon_g) * \eta_{GB}) \qquad \text{for every } g : B \to B'$$

and we need to show the identities :

$$(\tau^{\vartheta}_{(\mathbf{1}_{GB'},\varepsilon_{B'}),\mathbf{1}_{FGB'}} * Gg) \odot \tau^{(G*\varepsilon)\odot(\eta*G)}_g = X := \tau^\mu_g \odot (Gg * \tau^{\vartheta}_{(\mathbf{1}_{FGB},\varepsilon_B),\mathbf{1}_{FGB}})$$

for every 1-cell $g : B \to B'$ of $\mathscr{B}$. We compute :

$$\begin{aligned} X &= (\tau^{\vartheta}_{(Gg,\mathbf{1}_{B'}),\varepsilon_{B'}})^{-1} \odot \vartheta(\tau^\varepsilon_g) \odot \tau^{\vartheta}_{(\mathbf{1}_{GB},g\circ\varepsilon_B),\mathbf{1}_{FGB}} \\ &= (\tau^{\vartheta}_{(Gg,\mathbf{1}_{B'}),\varepsilon_{B'}})^{-1} \odot \tau^{\vartheta}_{(\mathbf{1}_{GB},\varepsilon_{B'}\circ FGg),\mathbf{1}_{FGB}} \odot (G(\tau^\varepsilon_g) * \eta_{GB}) \end{aligned}$$

where the first equality follows from the coherence axioms for τ^ϑ, and the second follows from (2.4.17). So, we are further reduced to checking the identities :

$$\tau^{\vartheta}_{(Gg,\mathbf{1}_{B'}),\varepsilon_{B'}} \odot (\tau^{\vartheta}_{(\mathbf{1}_{GB'},\varepsilon_{B'}),\mathbf{1}_{FGB'}} * Gg) \odot (G(\varepsilon_{B'}) * \tau^\eta_{Gg}) = \tau^{\vartheta}_{(\mathbf{1}_{GB},\varepsilon_{B'}\circ FGg),\mathbf{1}_{FGB}}.$$

However, by applying again the coherence axiom for τ^ϑ we see that

$$\tau^\vartheta_{(\mathbf{1}_{GB},\varepsilon_{B'}\circ FGg),\mathbf{1}_{FGB}} = \tau^\vartheta_{\mathbf{1}_{GB},\varepsilon_{B'},FGg} \odot (G(\varepsilon_{B'}) * \tau^\vartheta_{(\mathbf{1}_{GB},FGg),\mathbf{1}_{FGB}})$$
$$\tau^\vartheta_{(Gg,\mathbf{1}_{B'}),\varepsilon_{B'}} \odot (\tau^\vartheta_{(\mathbf{1}_{GB'},\varepsilon_{B'}),\mathbf{1}_{FGB'}} * Gg) = \tau^\vartheta_{(Gg,\varepsilon_{B'}),\mathbf{1}_{FGB'}}$$

so we are further reduced to showing :

$$\tau^\vartheta_{(Gg,\varepsilon_{B'}),\mathbf{1}_{FGB'}} = \tau^\vartheta_{\mathbf{1}_{GB},\varepsilon_{B'},FGg} \odot (G(\varepsilon_{B'}) * \tau^\vartheta_{(Gg,\mathbf{1}_{FGB'}),\mathbf{1}_{FGB'}}).$$

But the latter follows by yet another application of the coherence axioms for τ^ϑ. This concludes the construction of Σ. Concerning Σ', notice that for every $A \in \mathrm{Ob}(\mathscr{A})$ and $B \in \mathrm{Ob}(\mathscr{B})$

$$\mathscr{A}^o(G^oB^o, A^o) = \mathscr{A}(A, GB) \qquad \text{and} \qquad \mathscr{B}^o(B^o, F^oA^o) = \mathscr{B}(FA, B)$$

and ψ can thus be regarded as a system of equivalences of categories $\psi^o_{B^oA^o} : \mathscr{A}^o(G^oB^o, A^o) \to \mathscr{B}^o(B^o, F^oA^o)$. The latter amounts then to a pseudo-natural equivalence

$$\psi^o : \mathscr{A}^o(G^o, \mathbf{1}_{\mathscr{A}^o}) \Rightarrow \mathscr{B}^o(\mathbf{1}_{\mathscr{B}^o}, F^o)$$

with coherence constraint given by the system of isomorphisms of functors $\tau^{\psi^o}_{(g^o,f^o)} := \tau^\psi_{(f,g)}$, for every 2-cell f of $\mathscr{A}$ and g of $\mathscr{B}$. Likewise, we may define the 2-adjunction ϑ^o for the pair (G^o, F^o), and a simple inspection shows that performing the foregoing constructions on this new pair of pseudo-natural equivalences amounts to swapping the roles of ε and η; then the modification Σ^o for the new pair of unit and counit will give the sought Σ' for the original pair.

(ii): Given such η and ε, we set

$$\vartheta := H_{\mathscr{A}}(\eta, \mathbf{1}_G) \odot (H_G * (F^o \times \mathbf{1}_{\mathscr{B}})) : H_{\mathscr{B}}(F, \mathbf{1}_{\mathscr{B}}) \Rightarrow H_{\mathscr{A}}(\mathbf{1}_{\mathscr{A}}, G)$$
$$\psi := H_{\mathscr{B}}(\mathbf{1}_F, \varepsilon) \odot (H_F * (\mathbf{1}^o_{\mathscr{A}} \times G)) : H_{\mathscr{A}}(\mathbf{1}_{\mathscr{A}}, G) \Rightarrow H_{\mathscr{B}}(F, \mathbf{1}_{\mathscr{B}}).$$

Now, example 2.2.15(ii) yields the following identity in $\overline{2\text{-}\mathbf{Cat}}$:

$$\Xi : (H_F * (\mathbf{1}^o_{\mathscr{A}} \times G)) \odot H_{\mathscr{A}}(\eta, \mathbf{1}_G) = (H_{\mathscr{B}}(F, F) * (\eta^o \times \mathbf{1}_G)) \odot (H_F * ((GF)^o \times G))$$

and on the other hand, it is easily seen that

$$(H_F*((GF)^o \times G)) \odot (H_G*(F^o \times \mathbf{1}_{\mathscr{B}})) = H_{FG}*(F^o \times \mathbf{1}_{\mathscr{B}}) : H_{\mathscr{B}}(F, \mathbf{1}_{\mathscr{B}}) \Rightarrow H_{\mathscr{B}}(FGF, FG)$$
$$(H_{\mathscr{B}}(\mathbf{1}_F, \varepsilon) \odot (H_{\mathscr{B}}(F, F) * (\eta^o \times \mathbf{1}_G)) = H_{\mathscr{B}}(F * \eta, \varepsilon) : H_{\mathscr{B}}(FGF, FG) \Rightarrow H_{\mathscr{B}}(F, \mathbf{1}_{\mathscr{B}})$$

whence the identity in $\overline{2\text{-}\mathbf{Cat}}$:

$$\psi \odot \vartheta = H_{\mathscr{B}}(F*\eta, \varepsilon) \odot (H_{FG}*(F^o \times \mathbf{1}_{\mathscr{B}})) = H_{\mathscr{B}}(F*\eta, i_{\mathscr{B}}) \odot ((H_{\mathscr{B}}(\mathbf{1}_{FG}, \varepsilon) \odot H_{FG})*(F^o \times \mathbf{1}_{\mathscr{B}})).$$

But we have $H_{\mathscr{B}}(\mathbf{1}_{FG}, \varepsilon) \odot H_{FG} = H_{\mathscr{B}}(\varepsilon, i_{\mathscr{B}})$ in $\overline{2\text{-}\mathbf{Cat}}$, by lemma 2.4.18(ii), so that :

$$\begin{aligned}
\psi \odot \vartheta &= H_{\mathscr{B}}(F * \eta, i_{\mathscr{B}}) \odot (H_{\mathscr{B}}(\varepsilon, i_{\mathscr{B}}) * (F^o \times \mathbf{1}_{\mathscr{B}})) \\
&= H_{\mathscr{B}}(F * \eta, i_{\mathscr{B}}) \odot H_{\mathscr{B}}(\varepsilon * F, i_{\mathscr{B}}) \\
&= H_{\mathscr{B}}((\varepsilon * F) \odot (F * \eta), i_{\mathscr{B}}) \\
&= H_{\mathscr{B}}(\mathbf{1}_F, i_{\mathscr{B}}) \\
&= \mathbf{1}_{H_{\mathscr{B}}(F, \mathbf{1}_{\mathscr{B}})}.
\end{aligned}$$

Arguing likewise, we obtain as well the identity $\vartheta \odot \psi = \mathbf{1}_{H_{\mathscr{A}}(\mathbf{1}_{\mathscr{A}}, G)}$ in $\overline{2\text{-}\mathbf{Cat}}$: the details shall be left to the reader. Summing up, we have shown that ϑ is an equivalence in the 2-category $\mathsf{PsFun}(\mathscr{A}^o \times \mathscr{B}, \mathbf{Cat})$ with quasi-inverse ψ, and the proof of the theorem is concluded. □

Remark 2.4.28. (i) Let $(F : \mathscr{A} \to \mathscr{B}, G : \mathscr{B} \to \mathscr{A})$ be a 2-adjoint pair of pseudo-functors, and denote by η and ε the unit and the counit of a 2-adjunction for this pair, and Σ,Σ' the associated triangular modifications as in theorem 2.4.24. There follow pseudo-natural transformations

$$\eta^o : G^oF^o \Rightarrow \mathbf{1}_{\mathscr{A}^o} \qquad \varepsilon^o : \mathbf{1}_{\mathscr{B}^o} \Rightarrow F^oG^o$$

and invertible modifications :

$$\Sigma^o : (\eta^o * G^o) \odot (G^o * \varepsilon^o) \rightsquigarrow \mathbf{1}_{G^o} \qquad \Sigma'^o : (F^o * \eta^o) \odot (\varepsilon^o * F^o) \rightsquigarrow \mathbf{1}_{F^o}.$$

In light of theorem 2.4.24, it follows that (G^o, F^o) is a 2-adjoint pair of pseudo-functors, and ε^o and η^o are respectively a unit and a counit of a 2-adjunction for this pair.

(ii) Likewise, we see that $({}^oF, {}^oG)$ is a 2-adjoint pair of pseudo-functors, and ${}^o\eta$ and ${}^o\varepsilon$ are the unit and respectively the counit of a 2-adjunction for this pair.

(iii) Moreover, the right 2-adjoint G of F is unique up to pseudo-natural equivalence of pseudo-functors. Indeed, we have already observed that the pair (F, G) is 2-adjoint if and only if it is an adjoint pair of 1-cells of the 2-category $\mathsf{U}'\text{-}\overline{2\text{-}\mathbf{Cat}}$ (for a suitable universe U'); on the other hand, the pseudo-natural equivalences of pseudo-functors are precisely the invertible 2-cells of $\mathsf{U}'\text{-}\overline{2\text{-}\mathbf{Cat}}$, hence the assertion follows from remark 2.1.4(v).

Corollary 2.4.29. *Let $F : \mathscr{A} \to \mathscr{B}$ be a pseudo-functor and $G : \mathscr{B} \to \mathscr{A}$ a right 2-adjoint for F. Let also $\eta : \mathbf{1}_{\mathscr{A}} \Rightarrow GF$ and $\varepsilon : FG \Rightarrow \mathbf{1}_{\mathscr{B}}$ be the unit and counit of a 2-adjunction ϑ for the pair (F, G). Then the following conditions are equivalent :*

(a) *F (resp. G) is fully faithful.*

(b) *η (resp. ε) is a pseudo-natural equivalence.*

(c) *There exists a pseudo-natural equivalence $\eta' : \mathbf{1}_{\mathscr{A}} \xrightarrow{\sim} GF$ (resp. $\varepsilon' : FG \xrightarrow{\sim} \mathbf{1}_{\mathscr{B}}$).*

Moreover, if (c) *holds, there exists a pseudo-natural transformation ε' (resp. η') with invertible modifications $(G * \varepsilon') \odot (\eta' * G) \rightsquigarrow \mathbf{1}_G$ and $(\varepsilon' * F) \odot (F * \eta') \rightsquigarrow \mathbf{1}_F$.*

Proof. By inspecting the construction of η and ε, we are easily reduced to the case where $\mathscr{A}$ and $\mathscr{B}$ are small and F and G are strict. Also, by remark 2.4.28(i) it suffices to check the assertions for F, η and η'. Suppose now that (a) holds for F. Then the composition

$$\lambda_{A'A} : \mathscr{A}(A', A) \xrightarrow{F_{A'A}} \mathscr{B}(FA', FA) \xrightarrow{\vartheta_{A',FA}} \mathscr{A}(A', GFA) \qquad \text{for every } A, A' \in \mathrm{Ob}(\mathscr{A})$$

is an equivalence of categories, and $\eta_A = \lambda_{AA}(\mathbf{1}_A)$ for every $A \in \mathrm{Ob}(\mathscr{A})$. Taking $A' := GFA$, we find a 1-cell $f : GFA \to A$ with an invertible 2-cell $\lambda_{GFA,A}(f) \xrightarrow{\sim} \mathbf{1}_{GFA}$. Moreover, as explained in the proof of lemma 2.4.23(i), the rule : $(A', A) \mapsto \lambda_{A'A}$ is pseudo-natural in both A and A'; thus, for every 1-cell $h : A' \to A$ of $\mathscr{A}$, the coherence constraint τ^λ of λ yields an isomorphism of functors :

$$\tau^\lambda_{(h,\mathbf{1}_A)} : \mathscr{A}(h, \mathbf{1}_{GFA}) \circ \lambda_{AA} \xrightarrow{\sim} \lambda_{GFA,A} \circ \mathscr{A}(h, \mathbf{1}_A)$$

and especially, we get as well an invertible 2-cell $\eta_A \circ f \xrightarrow{\sim} \lambda_{GFA,A}(f)$; summing up, we get an invertible 2-cell $\eta_A \circ f \xrightarrow{\sim} \mathbf{1}_{GFA}$. On the other hand, the invertible modification $(\varepsilon * F) \odot (F * \eta) \rightsquigarrow \mathbf{1}_F$ provided by theorem 2.4.24(i) yields an invertible 2-cell $\varepsilon_{FA} \circ F\eta_A \xrightarrow{\sim} \mathbf{1}_{FA}$. We conclude that $F\eta_A$ is an equivalence, and then the same holds for η_A, by virtue of lemma 2.4.11(iii). Combining with theorem 2.4.12, we see that (b) holds for η.

Next, obviously (b)⇒(c). Suppose then that $\eta' : \mathbf{1}_{\mathscr{A}} \xrightarrow{\sim} GF$ is a pseudo-natural equivalence, and denote also by $i_{\mathscr{A}^o}$ the identity automorphism of $\mathbf{1}_{\mathscr{A}^o}$; we choose a pseudo-natural equivalence $\psi : H_{\mathscr{A}}(\mathbf{1}_{\mathscr{A}}, G) \xrightarrow{\sim} H_{\mathscr{B}}(F, \mathbf{1}_{\mathscr{B}})$ and we set

$$\xi := (\psi * (\mathbf{1}_{\mathscr{A}^o} \times F)) \odot (H_{\mathscr{A}} * (i_{\mathscr{A}^o} \times \eta')) : H_{\mathscr{A}} \xrightarrow{\sim} H_{\mathscr{B}}(F, F).$$

Since ξ is pseudo-natural, for every 1-cell $f : A' \to A$ in $\mathscr{A}$, the coherence constraint τ^ξ of ξ yields invertible 2-cells :

$$Ff \circ \xi_{A'A'}(\mathbf{1}_{A'}) \xrightarrow{\tau^\xi_{(f,\mathbf{1}_A),\mathbf{1}_{A'}}} \xi_{A'A}(f) \xleftarrow{\tau^\xi_{(\mathbf{1}_{A'},f),\mathbf{1}_A}} \xi_{AA}(\mathbf{1}_A) \circ Ff. \tag{2.4.30}$$

Moreover, since $\xi_{AA} : \mathscr{A}(A, A) \xrightarrow{\sim} \mathscr{B}(FA, FA)$ is an equivalence for every $A \in \mathrm{Ob}(\mathscr{A})$, there exists a 1-cell $h : A \to A$ with an isomorphism $\xi_{AA}(h) \xrightarrow{\sim} \mathbf{1}_A$. Taking $A' = A$ and

$f := h$ in (2.4.30), we conclude that the 1-cell $g_A := \xi_{AA}(\mathbf{1}_A)$ is an equivalence in $\mathscr{A}$ for every $A \in \mathrm{Ob}(\mathscr{A})$. On the other hand, the naturality of τ^ξ easily implies that the rule :

$$(f : A' \to A) \mapsto \tau^\xi_{(f,\mathbf{1}_A),\mathbf{1}_A} : g_A \circ Ff \stackrel{\sim}{\to} \xi_{A'A}(f)$$

defines an isomorphism of functors $\mathscr{A}(\mathbf{1}_A, g_A) \stackrel{\sim}{\to} \xi_{A'A}$. But since g_A is an equivalence, $\mathscr{A}(\mathbf{1}_A, g_A)$ is an equivalence of categories, and the same holds for $\xi_{A'A}$. It follows that the rule : $f \mapsto Ff$ yields an equivalence $F_{A'A} : \mathscr{A}(A', A) \stackrel{\sim}{\to} \mathscr{B}(FA', FA)$, *i.e.* (a) holds for F.

Lastly, for a given η' as in (c), there exists a pseudo-natural equivalence $\omega : \mathbf{1}_\mathscr{A} \stackrel{\sim}{\to} \mathbf{1}_\mathscr{A}$ such that $\eta \odot \omega = \eta'$ in $\overline{2\text{-}\mathbf{Cat}}$. We set $\varepsilon' := \varepsilon \odot (F * \omega^{-1} * G) : FG \Rightarrow \mathbf{1}_\mathscr{B}$. We compute in $\overline{2\text{-}\mathbf{Cat}}$:

$$\begin{aligned}
(G * \varepsilon') \odot (\eta' * G) &= (G * \varepsilon) \odot (GF * \omega^{-1} * G) \odot (\eta' * G) \\
&= (G * \varepsilon) \odot (GF * \omega^{-1} * G) \odot (\eta * G) \odot (\omega * G) \\
&= (G * \varepsilon) \odot (\eta * G) \odot (\omega^{-1} * G) \odot (\omega * G) \\
&= (G * \varepsilon) \odot (\eta * G) \\
&= \mathbf{1}_G.
\end{aligned}$$

Likewise we get $(\varepsilon' * F) \odot (F * \eta') = \mathbf{1}_F$ in $\overline{2\text{-}\mathbf{Cat}}$: the details shall be left to the reader. □

Remark 2.4.31. Let $F : \mathscr{A} \to \mathscr{B}$ be a strict and strong 2-equivalence between two 2-categories. Then there exists a strict and strong 2-equivalence $G : \mathscr{B} \to \mathscr{A}$ with strict pseudo-natural isomorphisms $\eta : \mathbf{1}_\mathscr{B} \stackrel{\sim}{\to} F \circ G$ and $\varepsilon : G \circ F \stackrel{\sim}{\to} \mathbf{1}_\mathscr{A}$. Indeed, for every $B \in \mathrm{Ob}(\mathscr{B})$ let us pick $GB \in \mathrm{Ob}(\mathscr{A})$ with an isomorphism $\eta_B : B \stackrel{\sim}{\to} FGB$; for every $B, B' \in \mathrm{Ob}(\mathscr{B})$ we then get an isomorphism of categories

$$\mathscr{A}(GB, GB') \xrightarrow{F_{GB,GB'}} \mathscr{B}(FGB, FGB') \xrightarrow{\mathscr{B}(\eta_B, \eta^{-1}_{B'})} \mathscr{B}(B, B')$$

where $\mathscr{B}(\eta_B, \eta^{-1}_{B'})$ denotes the functor given by the rule : $\varphi \mapsto \eta^{-1}_{B'} \circ \varphi \circ \eta_B$ for every 1-cell $\varphi : FGB \to FGB'$ of $\mathscr{B}$, and $\beta \mapsto \eta^{-1}_{B'} * \beta * \eta_B$ for every 2-cell β between 1-cells $\varphi, \varphi' : FGB \to FGB'$. We let $G_{B,B'} : \mathscr{B}(B, B') \to \mathscr{A}(GB, GB')$ be the inverse of this isomorphism. Then the sought G is given by the rules : $B \mapsto GB$ for every $B \in \mathrm{Ob}(\mathscr{B})$, and $(B, B') \mapsto G_{B,B'}$ for every pair (B, B') of objects of $\mathscr{B}$: the straightforward verification is left to the reader. It is also clear that the rule : $B \mapsto \eta_B$ for every $B \in \mathrm{Ob}(\mathscr{B})$ defines a strict pseudo-natural isomorphism η. Lastly, we define ε by the rule : $A \mapsto F^{-1}_{GFA,A}(\eta^{-1}_{FA})$ for every $A \in \mathrm{Ob}(\mathscr{A})$. The following corollary extends this simple observation to every 2-equivalence.

Corollary 2.4.32. *Let $\mathscr{A}, \mathscr{B}$ be two 2-categories, $F : \mathscr{A} \to \mathscr{B}$ a pseudo-functor. We have :*

(i) *The following conditions are equivalent :*
  (a) *F is a 2-equivalence.*
  (b) *There exists a pseudo-functor $L : \mathscr{B} \to \mathscr{A}$ with two pseudo-natural equivalences*

$$\eta^* : \mathbf{1}_\mathscr{B} \Rightarrow F \circ L \qquad \text{and} \qquad \varepsilon^* : L \circ F \Rightarrow \mathbf{1}_\mathscr{A}.$$

We say that L is a pseudo-inverse *for F.*

(ii) *Suppose that F is a 2-equivalence. Then, for every L and η^* as in* (i) *there exist a pseudo-natural equivalence $\varepsilon^* : L \circ F \Rightarrow \mathbf{1}_\mathscr{A}$ and invertible modifications*

$$\Theta : (F * \varepsilon^*) \odot (\eta^* * F) \rightsquigarrow \mathbf{1}_F \qquad \Theta' : (\varepsilon^* * L) \odot (L * \eta^*) \rightsquigarrow \mathbf{1}_L.$$

Proof. (i.b)⇒(i.a): The coherence constraint of ε^* gives an isomorphism of functors

$$H_\mathscr{A}(\mathbf{1}_{LFA}, \varepsilon^*_{A'}) \circ L_{FA,FA'} \circ F_{AA'} \Rightarrow H_\mathscr{A}(\varepsilon^*_A, \mathbf{1}_{A'}) \qquad \text{for every } A, A' \in \mathrm{Ob}(\mathscr{A}).$$

However, it is easily seen that both $H_\mathscr{A}(\mathbf{1}_{LFA}, \varepsilon_{A'})$ and $H_\mathscr{A}(\varepsilon_A, \mathbf{1}_{A'})$ are equivalences of categories (cp. the proof of lemma 2.1.13), hence the same holds for $L_{FA,FA'} \circ F_{AA'}$. Arguing likewise with the coherence constraint for η^*, we see as well that $F_{LB,LB'} \circ L_{BB'}$ is an equivalence for every $B, B' \in \mathrm{Ob}(\mathscr{B})$. It follows easily that $F_{AA'}$ and $L_{BB'}$ are faithful for every such A, A' and B, B'; especially, $L_{FA,FA'}$ and $F_{LB,LB'}$ are both faithful, and therefore $F_{AA'}$ and $L_{BB'}$ must also be full. Lastly, let $f : FA \to FA'$ be any 1-cell of $\mathscr{B}$; by the foregoing there exists a 1-cell $g : A \to A'$ with an invertible 2-cell $b : LFg \Rightarrow Lf$, and since $L_{FA,FA'}$ is fully faithful, we have $b = Lc$ for an invertible 2-cell $c : Fg \Rightarrow f$, which shows that $F_{AA'}$ is essentially surjective, so it is an equivalence of categories. Furthermore, theorem 2.4.12 says that the 1-cell $\eta^*_B : B \to FLB$ is an equivalence for every $B \in \mathrm{Ob}(\mathscr{B})$, so (i.a) holds.

(i.b)⇒(i.a): Without loss of generality, we may assume that both $\mathscr{A}$ and $\mathscr{B}$ are small 2-categories. Now, consider the category $\mathscr{A}^F$ and the pseudo-functors $F^\flat : \mathscr{A}^F \to \mathscr{B}$, $\pi^F : \mathscr{A}^F \to \mathscr{A}$ and $\sigma^F : \mathscr{A} \to \mathscr{A}^F$ provided by proposition 2.4.6; we remark that π^F is a 2-equivalence, and σ^F is its pseudo-inverse : indeed, we have $\pi^F \circ \sigma^F = \mathbf{1}_{\mathscr{A}}$, and a pseudo-natural equivalence $\mu^F : \sigma^F \circ \pi^F \xrightarrow{\sim} \mathbf{1}_{\mathscr{A}}$, so the assertion follows from the foregoing. It is then clear that F is a 2-equivalence if and only if the same holds for $F \circ \pi^F$, if and only if the same holds for $F^\flat$. Now, suppose we have found a pseudo-functor $L' : \mathscr{B} \to \mathscr{A}^F$, and pseudo-natural equivalences $\eta^{**} : \mathbf{1}_{\mathscr{B}} \xrightarrow{\sim} F^\flat \circ L'$, $\varepsilon^{**} : L' \circ F^\flat \xrightarrow{\sim} \mathbf{1}_{\mathscr{A}^F}$; then set $L := \pi^F \circ L'$, and notice that we get pseudo-natural equivalences

$$(F^\flat * \mu^F * L')^{-1} \odot \eta^{**} : \mathbf{1}_{\mathscr{B}} \xrightarrow{\sim} F \circ L \qquad \pi^F * \varepsilon^{**} * \sigma^F : L \circ F \xrightarrow{\sim} \mathbf{1}_{\mathscr{A}}.$$

We may thus replace F by $F^\flat$, and assume from start that F is strict. Next, consider the pseudo-natural transformation $H_F : H_{\mathscr{A}} \Rightarrow H_{\mathscr{B}}(F, F)$ assigned to $\mathbf{1}_F : F \Rightarrow F$, as in lemma 2.4.18(i). Notice that our assumption on F means that $H_{F,(A,A')}$ is an equivalence in $\mathbf{Cat}$, for every object (A, A') of $\mathscr{A}^o \times \mathscr{A}$, so theorem 2.4.12 implies that H_F is a pseudo-natural equivalence of pseudo-functors. We may therefore find a pseudo-natural equivalence $G : H_{\mathscr{B}}(F, F) \Rightarrow H_{\mathscr{A}}$, with coherence constraint τ^G, and an adjunction :

$$\eta : \mathbf{1}_{H_{\mathscr{A}}} \rightsquigarrow G \odot H_F \qquad \varepsilon : H_F \odot G \rightsquigarrow \mathbf{1}_{H_{\mathscr{B}} \circ (F,F)}$$

for the pair (H_F, G). Explicitly, G attaches to every $A, A' \in \mathrm{Ob}(\mathscr{A})$ a functor

$$G_{AA'} : \mathscr{B}(FA, FA') \to \mathscr{A}(A, A')$$

which is an equivalence of categories, and η and ε are given by natural isomorphisms of functors

$$\eta_{AA'} : \mathbf{1}_{\mathscr{A}(A,A')} \Rightarrow G_{AA'} \circ F_{AA'} \qquad \varepsilon_{AA'} : F_{AA'} \circ G_{AA'} \Rightarrow \mathbf{1}_{\mathscr{B}(FA,FA')}.$$

On the other hand, by assumption, for every $B \in \mathrm{Ob}(\mathscr{B})$ we may find $LB \in \mathrm{Ob}(\mathscr{A})$ with an equivalence $f_B : B \to FLB$; then lemma 2.1.13(i) says that we may also find a 1-cell $g_B : FLB \to B$ and invertible 2-cells

$$\eta'_B : \mathbf{1}_{FLB} \Rightarrow f_B \circ g_B \qquad \varepsilon'_B : g_B \circ f_B \Rightarrow \mathbf{1}_B$$

that form an adjunction for the pair (g_B, f_B). To ease notation, we set $T_{BB'} := \mathscr{B}(g_B, f_{B'})$ for every $B, B' \in \mathrm{Ob}(\mathscr{B})$ and we define

$$L_{BB'} := G_{LB,LB'} \circ T_{BB'} : \mathscr{B}(B, B') \to \mathscr{A}(LB, LB').$$

We shall henceforth drop the subscripts from the notation for F, G, L, T, η and ε. We claim that L is a pseudo-functor $\mathscr{B} \to \mathscr{A}$, with coherence constraint given by the system of 2-cells in $\mathscr{A}$

$$\gamma^L_{h,k} := L(k * \varepsilon'_{B'} * h) \odot G(\varepsilon_{Tk} * \varepsilon_{Th}) \odot \eta_{Lk \circ Lh} : Lk \circ Lh \Rightarrow L(k \circ h)$$
$$\delta^L_B := G(\eta'_B) \odot \eta_{\mathbf{1}_{LB}} : \mathbf{1}_{LB} \Rightarrow L(\mathbf{1}_B)$$

for every $B, B', B'' \in \mathrm{Ob}(\mathscr{B})$ and every pair of 1-cells $h : B \to B'$ and $k : B' \to B''$ in $\mathscr{B}$. Indeed, the naturality of the rule $(h, k) \mapsto \gamma^L_{h,k}$ is clear from the definition; to check the composition axiom, we consider another 1-cell $l : B'' \to B'''$ in $\mathscr{B}$, and we need to show that

$$\gamma^L_{k \circ h, l} \odot (Ll * \gamma^L_{h,k}) = \gamma^L_{h, l \circ k} \odot (\gamma^L_{l,k} * Lh).$$

However, the naturality of η yields the identities :

$$\eta_{Ll\circ L(k\circ h)} \odot (Ll * \gamma^L_{h,k}) = GF(Ll * \gamma^L_{h,k}) \odot \eta_{Ll\circ Lk\circ Lh}$$
$$\eta_{L(l\circ k)\circ Lh} \odot (\gamma^L_{k,l} * Lh) = GF(\gamma^L_{k,l} * Lh) \odot \eta_{Ll\circ Lk\circ Lh}.$$

To ease notation, set

$$X := (\varepsilon_{T(l\circ k)} * \varepsilon_{Th}) \odot F(\gamma^L_{k,l} * Lh) \qquad Y := (\varepsilon_{Tl} * \varepsilon_{T(k\circ h)}) \odot F(Ll * \gamma^L_{h,k}).$$

We may compute :

$$\begin{aligned} X &= (\varepsilon_{T(l\circ k)} * \varepsilon_{Th}) \odot (F(L(l * \varepsilon'_{B''} * k) \odot G(\varepsilon_{Tl} * \varepsilon_{Tk}) \odot \eta_{Ll\circ Lk}) * FLh) \\ &= (\varepsilon_{T(l\circ k)} * \varepsilon_{Th}) \odot (F(L(l * \varepsilon'_{B''} * k) \odot G(\varepsilon_{Tl} * \varepsilon_{Tk})) * FLh) \odot F(\eta_{Ll\circ Lk} * Lh) \\ &= (\varepsilon_{T(l\circ k)} * \varepsilon_{Th}) \odot (FG(T(l * \varepsilon'_{B''} * k) \odot (\varepsilon_{Tl} * \varepsilon_{Tk})) * FLh) \odot F(\eta_{Ll\circ Lk} * Lh) \\ &= ((T(l * \varepsilon'_{B''} * k) \odot (\varepsilon_{Tl} * \varepsilon_{Tk})) * Th) \odot (\varepsilon_{F(Ll\circ Lk)} * \varepsilon_{Th}) \odot F(\eta_{Ll\circ Lk} * Lh) \\ &= ((T(l * \varepsilon'_{B''} * k) \odot (\varepsilon_{Tl} * \varepsilon_{Tk})) * Th) \odot (F(Ll \circ Lk) * \varepsilon_{Th}) \\ &= ((T(l * \varepsilon'_{B''} * k) \odot (\varepsilon_{Tl} * \varepsilon_{Tk})) * Th) \odot (FLl * FLk * \varepsilon_{Th}) \\ &= (T(l * \varepsilon'_{B''} * k) * Th) \odot (\varepsilon_{Tl} * \varepsilon_{Tk} * \varepsilon_{Th}) \end{aligned}$$

where the fourth equality follows from the naturality of ε, the fifth follows from the triangular identities (1.1.13) for the adjunction (η, ε), and the sixth and seventh follow from remark 2.1.1(i). Likewise, a similar calculation gives :

$$Y = (Tl * T(k * \varepsilon'_{B'} * h)) \odot (\varepsilon_{Tl} * \varepsilon_{Tk} * \varepsilon_{Th}).$$

To conclude, it suffices to notice the identity :

$$T(l * k * \varepsilon'_{B'} * h) \odot (T(l * \varepsilon'_{B''} * k) * Th) = T(l * \varepsilon'_{B''} * k * h) \odot (Tl * T(k * \varepsilon'_{B'} * h)).$$

Next, let us show the unit axiom : we come down to checking the identity

$$\gamma^L_{\mathbf{1}_B,f} \odot (Lf * \delta^L_B) = \mathbf{1}_{Lf} \qquad \text{for every 1-cell } f : B \to B' \text{ in } \mathscr{B}.$$

However,

$$\begin{aligned} \gamma^L_{\mathbf{1}_B,f} \odot (Lf * \delta^L_B) &= L(f * \varepsilon'_B) \odot G(\varepsilon_{Tf} * \varepsilon_{T\mathbf{1}_B}) \odot \eta_{Lf\circ L\mathbf{1}_B} \odot (Lf * \delta^L_B) \\ &= L(f * \varepsilon'_B) \odot G(\varepsilon_{Tf} * \varepsilon_{T\mathbf{1}_B}) \odot GF(Lf * \delta^L_B) \odot \eta_{Lf} \end{aligned}$$

by the naturality of η; taking into account the triangularity identities (1.1.13) for the adjunction (η, ε), we are then reduced to showing that

$$Z := T(f * \varepsilon'_B) \odot (\varepsilon_{Tf} * \varepsilon_{T\mathbf{1}_B}) \odot F(Lf * \delta^L_B) = \varepsilon_{Tf}.$$

However, from the triangular identities for the adjunction $(\eta'_B, \varepsilon'_B)$, we easily obtain :

$$T(f * \varepsilon'_B) = Tf * \eta'^{-1}_B$$

and on the other hand, the naturality of ε yields

$$\eta'^{-1}_B \odot \varepsilon_{T\mathbf{1}_B} = \varepsilon_{\mathbf{1}_{FLB}} \odot FG\eta'^{-1}_B$$

whence :

$$\begin{aligned} Z &= (\varepsilon_{Tf} * (\varepsilon_{\mathbf{1}_{FLB}} \odot FG\eta'^{-1}_B)) \odot (FLf * F\delta^L_B) \\ &= (\varepsilon_{Tf} * (\varepsilon_{\mathbf{1}_{FLB}} \odot FG\eta'^{-1}_B)) \odot (FLf * FG\eta'_B) \odot (FLf * F\eta_{\mathbf{1}_{LB}}) \\ &= (\varepsilon_{Tf} * \varepsilon_{\mathbf{1}_{FLB}}) \odot (FLf * F\eta_{\mathbf{1}_{LB}}). \end{aligned}$$

From the triangular identities for (η, ε) we also see that $F(\eta_{\mathbf{1}_{LB}}) = \varepsilon^{-1}_{F\mathbf{1}_{LB}} = \varepsilon^{-1}_{\mathbf{1}_{FLB}}$, so finally :

$$Z = (\varepsilon_{Tf} * \varepsilon_{\mathbf{1}_{FLB}}) \odot (FLf * \varepsilon^{-1}_{\mathbf{1}_{FLB}}) = \varepsilon_{Tf}$$

as required. A similar computation establishes the second identity for the unit axiom of L. This completes the construction of the pseudo-functor L. Next we set, for every $B, B' \in \mathrm{Ob}(\mathscr{B})$ and every 1-cell $h : B \to B'$ in $\mathscr{B}$

$$\eta^*_B := f_B : B \to FLB$$
$$\tau^*_h := (f_{B'} * h * \varepsilon'_B) \odot (\varepsilon_{Th} * f_B) : FL(h) \circ f_B \Rightarrow f_{B'} \circ h.$$

Claim 2.4.33. The system of 1-cells $\eta^*_\bullet$ defines a pseudo-natural equivalence $\eta^* : \mathbf{1}_{\mathscr{B}} \Rightarrow FL$ with coherence constraint given by the system of 2-cells $\tau^*_\bullet$.

Proof of the claim. By remark 2.2.5(i), the naturality of τ^* amounts to the identity

$$(f_{B'} * \beta) \odot (f_{B'} * h_1 * \varepsilon'_B) \odot (\varepsilon_{Th_1} * f_B) = (f_{B'} * h_2 * \varepsilon'_B) \odot (\varepsilon_{Th_2} * f_B) \odot (FL(\beta) * f_B)$$

for every pair of 1-cells $h_1, h_2 : B \to B'$ and every 2-cell $\beta : h_1 \Rightarrow h_2$. However, a simple inspection shows that

$$(f_{B'} * \beta) \odot (f_{B'} * h_1 * \varepsilon'_B) = (f_{B'} * h_2 * \varepsilon'_B) \odot (T(\beta) * f_B)$$

so we are reduced to checking that

$$T(\beta) \odot \varepsilon_{Th_1} = \varepsilon_{Th_2} \odot FL\beta$$

which in turns follows from the naturality of ε. Next, let us check the coherence axioms for τ^*. Denote by $(\delta^{FL}, \gamma^{FL})$ the coherence constraint of FL (see remark 2.1.17(v)); for given 1-cells $h : B \to B'$ and $h' : B' \to B''$ of $\mathscr{B}$ we have to verify the identities

$$(2.4.34) \qquad \tau^*_{\mathbf{1}_B} \odot (\delta^{FL}_B * f_B) = \mathbf{1}_{f_B} \qquad (\tau^*_{h'} * h) \odot (FLh' * \tau^*_h) = \tau^*_{h' \circ h} \odot (\gamma^{FL}_{h,h'} * f_B).$$

However, a direct inspection gives the identity

$$(\tau^*_{h'} * h) \odot (FLh' * \tau^*_h) = (f_{B''} * h' * \varepsilon'_{B'} * h * \varepsilon'_B) \odot (\varepsilon_{Th'} * \varepsilon_{Th} * f_B)$$

so we are reduced to showing that

$$T(h' * \varepsilon'_{B'} * h) \odot (\varepsilon_{Th'} * \varepsilon_{Th}) = \varepsilon_{T(h' \circ h)} \odot \gamma^{FL}_{h,h'}.$$

Now, we compute :

$$\begin{aligned}\varepsilon_{T(h'\circ h)} \odot \gamma^{FL}_{h,h'} &= \varepsilon_{T(h'\circ h)} \odot F(\gamma^{L}_{h,h'}) \\ &= \varepsilon_{T(h'\circ h)} \odot FG(T(h' * \varepsilon'_{B'} * h) \odot (\varepsilon_{Th'} * \varepsilon_{Th})) \odot F(\eta_{Lh'\circ Lh}) \\ &= T(h' * \varepsilon'_{B'} * h) \odot (\varepsilon_{Th'} * \varepsilon_{Th}) \odot \varepsilon_{F(Lh'\circ Lh)} \odot F(\eta_{Lh'\circ Lh}) \\ &= T(h' * \varepsilon'_{B'} * h) \odot (\varepsilon_{Th'} * \varepsilon_{Th})\end{aligned}$$

where the third identity follows from the naturality of ε, and the fourth follows from the triangular identities (1.1.13) for the pair (η, ε). For the first identity (2.4.34), we notice that

$$\tau^*_{\mathbf{1}_B} = (f_B * \varepsilon'_B) \odot (\varepsilon_{T\mathbf{1}_B} * f_B) = (\eta'^{-1}_B * f_B) \odot (\varepsilon_{T\mathbf{1}_B} * f_B)$$

due to the triangular identities for the pair (η', ε'), so we are reduced to checking that

$$\varepsilon_{T\mathbf{1}_B} \odot \delta^{FL}_B = \eta'_B.$$

We compute :

$$\varepsilon_{T\mathbf{1}_B} \odot \delta^{FL}_B = \varepsilon_{T\mathbf{1}_B} \odot F\delta^L_B = \varepsilon_{T\mathbf{1}_B} \odot F(G(\eta'_B) \odot \eta_{\mathbf{1}_{LB}}) = \eta'_B \odot \varepsilon_{\mathbf{1}_{FLB}} \odot F(\eta_{\mathbf{1}_{LB}}) = \eta'_B$$

which concludes the verification of the pseudo-naturality of η^*. Lastly, η^*_B is an equivalence for every $B \in \mathrm{Ob}(\mathscr{B})$, so η^* is a pseudo-natural equivalence, by theorem 2.4.12. ◊

Claim 2.4.35. For every $A \in \mathrm{Ob}(\mathscr{A})$, the 1-cell $G(g_{FA}) : LFA \to A$ is an equivalence in $\mathscr{A}$.

Proof of the claim. We have an invertible 2-cell $\varepsilon_{g_{FA}} : FG(g_{FA}) \Rightarrow g_{FA}$, and since g_{FA} is an equivalence, it follows that the same holds for $FG(g_{FA})$. Then the assertion follows from lemma 2.4.11(iii). ◊

A simple inspection shows that $L_{BB'}$ is equivalence of categories, for every $B, B' \in \mathrm{Ob}(\mathscr{B})$. Taking into account claim 2.4.35, we deduce that L is a 2-equivalence; therefore, the foregoing discussion applies as well with F replaced by L, and yields another 2-equivalence $M : \mathscr{A} \to \mathscr{B}$ with a pseudo-natural equivalence $\eta^{**} : \mathbf{1}_{\mathscr{A}} \Rightarrow LM$. Pick also pseudo-natural equivalences

$$\mu^* : FL \Rightarrow \mathbf{1}_{\mathscr{B}} \qquad \text{and} \qquad \mu^{**} : LM \Rightarrow \mathbf{1}_{\mathscr{A}}$$

representing the inverses of the 2-cells η^*, η^{**} in $\overline{2\text{-}\mathbf{Cat}}$. We get a pseudo-natural equivalence

$$\beta := (\mu^* * M) \odot (F * \eta^{**}) : F \Rightarrow M$$

(corollary 2.4.14), whence the sought pseudo-natural equivalence

$$\varepsilon^* := \mu^{**} \odot (L * \beta) : LF \Rightarrow \mathbf{1}_{\mathscr{A}}.$$

(ii): Assertion (i) says that F is a 2-equivalence if and only if it is an equivalence in the 2-category $\overline{2\text{-}\mathbf{Cat}}$. Then the assertion follows immediately from lemma 2.1.13(i). □

2.5. **2-Limits and 2-colimits.** Consider 2-categories $\mathscr{A}$, $\mathscr{B}$. For every object B of $\mathscr{B}$, one may define the *constant pseudo-functor* with value B : this is the pseudo-functor

$$\mathsf{F}_B : \mathscr{A} \to \mathscr{B} \qquad \text{such that} \qquad \mathsf{F}_B(A) := B \qquad \mathsf{F}_B(f) := \mathbf{1}_B \qquad \mathsf{F}_B(\alpha) := i_B$$

for every $A \in \mathrm{Ob}(\mathscr{A})$, every 1-cell f, and every 2-cell α of $\mathscr{A}$. The coherence constraint for F_B consists of identities. Given a pseudo-functor $F : \mathscr{A} \to \mathscr{B}$, a *pseudo-cone on F with vertex B* is a pseudo-natural transformation $\mathsf{F}_B \Rightarrow F$. Dually, a *pseudo-cocone on F with vertex B* is a pseudo-natural transformation $F \Rightarrow \mathsf{F}_B$. Especially, every 1-cell $f : B \to B'$ induces a pseudo-cone with vertex B :

$$\mathsf{F}_f : \mathsf{F}_B \Rightarrow \mathsf{F}_{B'} \quad : \quad (\mathsf{F}_f)_A := f \qquad \text{for every } A \in \mathrm{Ob}(\mathscr{A})$$

whose coherence constraint consists of identities. Notice that F_f can also be viewed as a pseudo-cocone with vertex B'. Also, every 2-cell $\beta : f \Rightarrow f'$ induces a modification

$$\mathsf{F}_\beta : \mathsf{F}_f \rightsquigarrow \mathsf{F}_{f'} \quad : \quad A \mapsto \beta \qquad \text{for every } A \in \mathrm{Ob}(\mathscr{A}).$$

Definition 2.5.1. Let I and $\mathscr{B}$ be two 2-categories, $F : I \to \mathscr{B}$ a pseudo-functor.

(i) A *2-limit* (resp. a *strong 2-limit*) of F is a pair :

$$2\text{-}\lim_I F := (L, \pi)$$

consisting of an object L of $\mathscr{B}$ and a pseudo-cone $\pi : \mathsf{F}_L \Rightarrow F$, such that the functor :

$$\mathscr{B}(B, L) \to \mathsf{PsNat}(\mathsf{F}_B, F) \quad : \quad f \mapsto \pi \odot \mathsf{F}_f \qquad (\beta : f \Rightarrow f') \mapsto (\pi * \mathsf{F}_\beta : \mathsf{F}_f \rightsquigarrow \mathsf{F}_{f'})$$

is an equivalence (resp. an isomorphism) of categories, for every $B \in \mathrm{Ob}(\mathscr{B})$. In this case, we also say that π is a *universal pseudo-cone* (resp. a *strong universal pseudo-cone*).

(ii) Dually, a *2-colimit* (resp. a *strong 2-colimit*) of F is a pair :

$$2\text{-}\operatorname{colim}_I F := (L, \pi)$$

consisting of an object L of $\mathscr{B}$ and a pseudo-cocone $\pi : F \Rightarrow \mathsf{F}_L$, such that the functor:

$$\mathscr{B}(L, B) \to \mathsf{PsNat}(F, \mathsf{F}_B) \quad : \quad f \mapsto \mathsf{F}_f \odot \pi \qquad (\beta : f \Rightarrow f') \mapsto (\mathsf{F}_\beta * \pi : \mathsf{F}_f \rightsquigarrow \mathsf{F}_{f'})$$

is an equivalence (resp. an isomorphism) of categories, for every $B \in \mathrm{Ob}(\mathscr{B})$. In this case, we also say that π is a *universal pseudo-cocone* (resp. a *strong universal pseudo-cocone*).

(iii) We say that $\mathscr{B}$ is 2-*complete* (resp. 2-*cocomplete*, resp. *strongly* 2-*complete*, resp. *strongly* 2-*cocomplete*) if, for every small 2-category I, every pseudo-functor $I \to \mathscr{B}$ admits a 2-limit (resp. a 2-colimit, resp. a strong 2-limit, resp. a strong 2-colimit).

(iv) Let $\mathscr{C}$ be any other 2-category, $G : \mathscr{B} \to \mathscr{C}$ any pseudo-functor, and (L, π) a 2-limit (resp. a 2-colimit) for F. We say that G *commutes with the* 2-*limit of* F (resp. that G *commutes with the* 2-*colimit of* F) if the pseudo-cone $G * \pi : \mathsf{F}_{GL} \Rightarrow G \circ F$ (resp. the pseudo-cocone $G * \pi : G \circ F \Rightarrow \mathsf{F}_{GL}$) is universal.

Remark 2.5.2. (i) As usual, if the 2-limit of a pseudo-functor exists, it is unique up to (non-unique) equivalence : if (L', π') is another 2-limit, there exist an equivalence $h : L \to L'$ and an isomorphism $\beta : \pi' \odot \mathsf{F}_h \stackrel{\sim}{\to} \pi$; moreover, the pair (h, β) is unique up to unique isomorphism, in a suitable sense, that the reader may spell out, as an exercise. A similar remark holds for 2-colimits.

(ii) Likewise, in the situation of definition 2.5.1(iv), the commutation of G with the 2-limit or 2-colimit of F is an intrinsic property of G, *i.e.* is independent of the choice of a 2-limit or 2-colimit for F.

(iii) To be in keeping with the terminology of [28, Ch.VII], we should write pseudo-bilimit instead of 2-limit (and likewise for 2-colimit). The term “2-limit” denotes in *loc.cit.* a related notion, which is unique up to isomorphism, not just up to equivalence. However, the notion introduced in definition 2.5.1 occurs more frequently in applications.

(iv) Let $A_\bullet := (A_i \mid i \in I)$ be any family of objects of the 2-category $\mathscr{A}$, indexed by a small set I. We may regard $A_\bullet$ as a strict pseudo-functor from the discrete category I to $\mathscr{A}$. Then a 2-limit (resp. a 2-colimit) of this pseudo-functor shall also be called a 2-*product* (resp. a 2-coproduct) of $A_\bullet$.

(v) Let L be the small category with $\mathrm{Ob}(L) := \{0, 1, 2\}$, and whose set of arrows consists of the identity morphisms, and two more arrows $1 \to 0$ and $2 \to 0$. An essentially commutative square (2.1.6) can be regarded as a pseudo-cone π with vertex A, on the functor $F : L \to \mathscr{A}$ such that $F(0) := D$, $F(1) := B$, $F(2) := C$, $F(1 \to 0) := h$ and $F(2 \to 0) := k$. We say that (2.1.6) is 2-*cartesian* if (A, π) is a 2-limit of the functor F. This 2-limit shall be called the 2-*fibre product* of k and h, and shall be denoted

$$B \overset{2}{\underset{(h,k)}{\times}} C$$

or sometimes, just $B \overset{2}{\times}_D C$, if there is no danger of ambiguity.

(vi) In view of (2.2.13), it is easily seen that a pair (L, π) is a 2-colimit for the pseudo-functor $F : I \to \mathscr{B}$ if and only if the pair (L^o, π^o) is a 2-limit for the pseudo-functor $F^o : I^o \to \mathscr{B}^o$, and if and only if $({}^oL, {}^o\pi)$ is a 2-limit for the pseudo-functor ${}^oF : {}^oI \to {}^o\mathscr{B}$. Especially, $\mathscr{B}$ is 2-cocomplete if and only if $\mathscr{B}^o$ is 2-complete, and if and only if ${}^o\mathscr{B}$ is 2-cocomplete.

(vii) Let I and $\mathscr{B}$ be two categories, which we regard as 2-categories with trivial 2-cells. Then every pseudo-functor $F : I \to \mathscr{B}$ is a functor, every pseudo-natural transformation $F \Rightarrow G$ of functors $F, G : I \to \mathscr{B}$ is a natural transformation, and the category $\mathsf{PsNat}(F, G)$ is discrete. It follows easily that a 2-limit (L, π) of any such functor $F : I \to \mathscr{B}$ is a strong 2-limit, it represents also the (usual) limit of F, and the universal pseudo-cone π is also a universal cone. Likewise, any 2-colimit of F represents the colimit of F, and every universal pseudo-cocone is a universal cocone.

The following lemma 2.5.3 indicates that the framework of 2-categories does indeed provide an adequate answer to the issues raised in the introduction of this chapter.

Lemma 2.5.3. *Let I be a* 2*-category, $F, G : I \to \mathscr{B}$ two pseudo-functors, $\omega : F \Rightarrow G$ a pseudo-natural equivalence, and suppose that the* 2*-limit of F exists. Then the same holds for* *the* 2*-limit of* G*, and there is a natural equivalence in* $\mathscr{B}$ *:*

$$2\text{-}\lim_I F \stackrel{\sim}{\to} 2\text{-}\lim_I G.$$

More precisely, if (L,π) *is a pair with* $L \in \mathrm{Ob}(\mathscr{B})$ *and a pseudo-cone* $\pi : \mathsf{F}_L \Rightarrow F$ *representing the* 2*-limit of* F*, then the pair* $(L, \omega \odot \pi)$ *represents the* 2*-limit of* G*.*

A dual assertion holds for 2*-colimits.*

Proof. It is easily seen that the rule $\alpha \mapsto \omega \odot \alpha$ induces an equivalence of categories :

$$\mathsf{PsNat}(\mathsf{F}_B, F) \to \mathsf{PsNat}(\mathsf{F}_B, G) \qquad \text{for every } B \in \mathrm{Ob}(\mathscr{B}).$$

The claim is an immediate consequence. □

2.5.4. Let $\mathscr{C}$ be any 2-category with small Hom-categories. To every object X of $\mathscr{C}$ we attach the strict pseudo-functor

$$h_X : \mathscr{C}^o \to \mathbf{Cat} \qquad Y \mapsto \mathscr{C}(Y,X) \subset Y/\mathscr{C}$$

that assigns to every 1-cell $Y' \xrightarrow{f} Y$ of $\mathscr{C}$ the restriction $\mathscr{C}(Y,X) \to \mathscr{C}(Y',X)$ of the strict pseudo-functor $f^* : Y/\mathscr{C} \to Y'/\mathscr{C}$, and to every 2-cell $\beta : f \Rightarrow f'$ the restriction of the strict pseudo-natural transformation $\beta^* : f^* \Rightarrow f'^*$ (notation of example 2.2.8(i)).

Notice that $h_{X^o} : \mathscr{C} \to \mathbf{Cat}$ is the strict pseudo-functor given by the rule : $Y \mapsto \mathscr{C}(X,Y)$ for every $Y \in \mathrm{Ob}(\mathscr{C})$, and which assigns to the 1-cell f as in the foregoing the restriction of the strict pseudo-functor $f_* : \mathscr{C}/Y \to \mathscr{C}/Y'$, and to the 2-cell β the restriction of $\beta_* : f_* \Rightarrow f'_*$ (again, with the notation of example 2.2.8(i)).

Proposition 2.5.5. *With the notation of* (2.5.4)*, let* I *be another* 2*-category,* $F : I \to \mathscr{C}$ *a pseudo-functor, and* $L \in \mathrm{Ob}(\mathscr{C})$ *any object. We have :*

- (i) *For every pseudo-cone* $\pi : \mathsf{F}_L \Rightarrow F$*, the following conditions are equivalent :*
  - (a) *The pseudo-cone* π *is universal.*
  - (b) *For every* $X \in \mathrm{Ob}(\mathscr{C})$ *the pseudo-cone* $h_{X^o} * \pi : \mathsf{F}_{\mathscr{C}(X,L)} \Rightarrow h_{X^o} \circ F$ *is universal.*
- (ii) *For every pseudo-cocone* $\pi : F \Rightarrow \mathsf{F}_L$*, the following conditions are equivalent :*
  - (a) *The pseudo-cocone* π *is universal.*
  - (b) *For every* $X \in \mathrm{Ob}(\mathscr{C})$ *the pseudo-cone* $h_X * \pi : \mathsf{F}_{\mathscr{C}(L,X)} \Rightarrow h_X \circ F$ *is universal.*

Proof. (i.a)⇒(i.b): We have to check that for every small category $\mathscr{B}$, the functor

$$T : \mathsf{Fun}(\mathscr{B}, \mathscr{C}(X,L)) \to \mathsf{PsNat}(\mathsf{F}_{\mathscr{B}}, h_{X^o} * \pi) \qquad (G : \mathscr{B} \to \mathscr{C}(X,L)) \mapsto (h_{X^o} * \pi) \odot \mathsf{F}_G$$

is an equivalence. To this aim, consider any pseudo-cone $\lambda : \mathsf{F}_{\mathscr{B}} \Rightarrow h_{X^o} * \pi$. Explicitly, λ is the datum of a system of functors $(\lambda_i : \mathscr{B} \to \mathscr{C}(X, Fi) \mid i \in \mathrm{Ob}(I))$ and a system of natural isomorphisms of functors $(\tau^\lambda_\varphi : h_{X^o}(F\varphi) \circ \lambda_i \Rightarrow \lambda_j \mid (i \xrightarrow{\varphi} j) \in \mathrm{Morph}(I))$ fulfilling the usual coherence axioms. To every $B \in \mathrm{Ob}(\mathscr{B})$ we attach the system

$$\lambda_\bullet(B) := (\lambda_i(B) : X \to Fi \mid i \in \mathrm{Ob}((I)).$$

It is easily seen that $\lambda_\bullet(B) : \mathsf{F}_X \Rightarrow F$ is a pseudo-cone with coherence constraint given by the system $(\tau^\lambda_{\varphi,B} : F(\varphi) \circ \lambda_i(B) \Rightarrow \lambda_j(B) \mid (i \xrightarrow{\varphi} j) \in \mathrm{Morph}(I))$. Since (L,π) is universal, we may then find a 1-cell $\lambda^\dagger_B : X \to L$ and an isomorphism of pseudo-cones

$$\Omega^\lambda_B : \pi \odot \mathsf{F}_{\lambda^\dagger_B} \rightsquigarrow \lambda_\bullet(B) \qquad \text{for every } B \in \mathrm{Ob}(\mathscr{B}).$$

Moreover, for every morphism $f : B \to B'$ in $\mathscr{B}$, the system $(\lambda_i(f) : \lambda_i(B) \Rightarrow \lambda_i(B') \mid i \in \mathrm{Ob}(I))$ yields a modification $\lambda_\bullet(f) : \lambda_\bullet(B) \rightsquigarrow \lambda_\bullet(B')$. Then, there exists a unique 2-cell in $\mathscr{C}$

$$\lambda^\dagger_f : \lambda^\dagger_B \Rightarrow \lambda^\dagger_{B'} \qquad \text{such that} \qquad \Omega^\lambda_{B'} \odot (\pi * \mathsf{F}_{\lambda^\dagger_f}) = \lambda_\bullet(f) \odot \Omega^\lambda_B. \tag{2.5.6}$$

Clearly $\lambda_\bullet(g) \odot \lambda_\bullet(f) = \lambda_\bullet(g \circ f)$ for every composable pair of morphisms $B \xrightarrow{f} B' \xrightarrow{g} B''$ of $\mathscr{B}$, whence $\lambda^\dagger_g \odot \lambda^\dagger_f = \lambda^\dagger_{g\circ f}$. We have thus associated with λ a well defined functor $\lambda^\dagger : \mathscr{B} \to \mathscr{C}(X, L)$, together with an isomorphism

$$\Omega^\lambda_\bullet : (h_{X^o} * \pi) \odot \mathsf{F}_{\lambda^\dagger} \xrightarrow{\sim} \lambda \qquad \text{in } \mathsf{PsNat}(\mathsf{F}_{\mathscr{B}}, h_{X^o} \circ F).$$

This proves that T is essentially surjective. Next, let us check that T is faithful : indeed, consider two functors $G, H : \mathscr{B} \to \mathscr{C}(X, L)$ and two natural transformations $\alpha, \beta : G \Rightarrow H$ such that $(h_{X^o} * \pi) \odot \mathsf{F}_\alpha = (h_{X^o} * \pi) \odot \mathsf{F}_\beta$; the latter identity means that

$$\pi_i * \alpha_B = \pi_i * \beta_B : \pi_i \circ GB \Rightarrow \pi_i \circ HB \qquad \text{for every } i \in \mathrm{Ob}(I) \text{ and every } B \in \mathrm{Ob}(\mathscr{B}).$$

In other words, for every $B \in \mathrm{Ob}(\mathscr{B})$, the morphisms of pseudo-cones $\pi * \mathsf{F}_{\alpha_B}, \pi * \mathsf{F}_{\beta_B} : \pi \odot \mathsf{F}_{GB} \Rightarrow \pi \odot \mathsf{F}_{HB}$ coincide; then the universality of π implies that $\alpha_B = \beta_B$ for every $B \in \mathrm{Ob}(\mathscr{B})$, whence the assertion. Lastly, to check that T is full we consider, for G and H as in the foregoing, any modification $\Xi : (h_{X^o} * \pi) \odot \mathsf{F}_G \rightsquigarrow (h_{X^o} * \pi) \odot \mathsf{F}_H$; hence, Ξ amounts to a system of natural transformations $(\Xi_i : \pi_{i,*} \circ G \Rightarrow \pi_{i,*} \circ H \mid i \in \mathrm{Ob}(I))$ related by the compatibility conditions :

$$\Xi_j = (\tau^\pi_{\varphi *} * H) \odot ((F\varphi)_* * \Xi_i) \qquad \text{for every 1-cell } \varphi : i \to j \text{ of } I$$

where $\pi_{i,*} : \mathscr{C}(X, L) \to \mathscr{C}(X, Fi)$ and $(F\varphi)_* : \mathscr{C}(X, Fi) \to \mathscr{C}(X, Fj)$ are the functors induced by $\pi_i : L \to Fi$ and respectively $F\varphi : Fi \to Fj$, and where $\tau^\pi_{\varphi*} : (F\varphi)_* \circ \pi_{i*} \xrightarrow{\sim} \pi_{j*}$ denotes the isomorphism of functors induced by the coherence constraint τ^π of π. Then it is easily seen that for every $B \in \mathrm{Ob}(\mathscr{B})$ the system

$$(\Xi_{i,B} : \pi_i \circ GB \Rightarrow \pi_i \circ HB \mid i \in \mathrm{Ob}(I))$$

yields a morphism of pseudo-cones $\Xi_{\bullet,B} : \pi * \mathsf{F}_{GB} \rightsquigarrow \pi * \mathsf{F}_{HB}$, and the naturality of Ξ_i gives the identities

(2.5.7) $$(\pi * \mathsf{F}_{Hf}) \odot \Xi_{\bullet,B} = \Xi_{\bullet,B'} \odot (\pi * \mathsf{F}_{Gf}) \qquad \text{for every morphism } f : B \to B' \text{ of } \mathscr{B}.$$

By the universality of π, the morphisms $\Xi_{\bullet,B}$ corresponds to a unique 2-cell $\Xi^*_B : GB \Rightarrow HB$, and (2.5.7) implies that $(\Xi^*_B \mid B \in \mathrm{Ob}(\mathscr{B}))$ is a well defined natural transformation $\Xi^* : G \Rightarrow H$. By direct inspection, we see that $(h_{X^o} * \pi) * \mathsf{F}_{\Xi^*} = \Xi$, whence the contention.

(i.b)⇒(i.a): To any $X \in \mathrm{Ob}(\mathscr{C})$ and any pseudo-cone $\varphi : \mathsf{F}_X \Rightarrow F$ we attach the pseudo-cone

$$\varphi_* : \mathsf{F}_{\mathscr{C}(X,X)} \Rightarrow h_{X^o} \circ F \qquad i \mapsto (\varphi_{i*} : \mathscr{C}(X, X) \to \mathscr{C}(X, Fi))$$

with coherence constraint given by the system of isomorphisms of functors

$$(\tau^\varphi_{\psi*} : (F\psi)_* \circ \varphi_{i*} \Rightarrow \varphi_{j*} \mid (i \xrightarrow{\psi} j) \in \mathrm{Morph}(I))$$

where $\tau^\varphi_\bullet$ denotes the coherence constraint of φ. By assumption, there exist a functor $\varphi^\dagger : \mathscr{C}(X, X) \to \mathscr{C}(X, L)$ and an isomorphism of pseudo-cones $\omega : \varphi_* \xrightarrow{\sim} (h_{X^o} * \pi) \odot \mathsf{F}_{\varphi^\dagger}$. Then it is easily seen that the system

$$(\omega_{i,\mathbf{1}_X} : \varphi_i \Rightarrow \pi_i \circ \varphi^\dagger(\mathbf{1}_X) \mid i \in \mathrm{Ob}(I))$$

yields an isomorphism of pseudo-cones $\varphi \xrightarrow{\sim} \pi \odot \mathsf{F}_{\varphi^\dagger(\mathbf{1}_X)}$. This shows that the functor

(2.5.8) $$\mathscr{C}(X, L) \to \mathsf{PsNat}(\mathsf{F}_X, F)$$

as in definition 2.5.1(i) is essentially surjective. Next, in order to check that (2.5.8) is faithful, consider two 1-cells $g, h : X \to L$ in $\mathscr{C}$ and two 2-cells $\alpha, \beta : g \Rightarrow h$ such that $\pi_i * \alpha = \pi_i * \beta$ for every $i \in \mathrm{Ob}(I)$; let also $\mathscr{B}$ be the initial object of $\mathbf{Cat}$, *i.e.* $\mathscr{B}$ is a category with a unique object b and a unique morphism $\mathbf{1}_b$. We let $G, H : \mathscr{B} \to \mathscr{C}(X, L)$ be the functors such that $Gb := g$ and $Hb := h$; then α and β correspond to natural transformations $\alpha', \beta' : G \Rightarrow H$ such that $(h_{X^o} * \pi) \odot \mathsf{F}_{\alpha'} = (h_{X^o} * \pi) \odot \mathsf{F}_{\beta'}$, and the universality of $h_{X^o} * \pi$ implies that $\alpha' = \beta'$, whence

$\alpha = \beta$, as required. Lastly, for g, h, $\mathscr{B}$, G and H as in the foregoing, let $\Xi : \pi \odot \mathsf{F}_g \rightsquigarrow \pi \odot \mathsf{F}_h$ be any modification; we deduce a modification $\Xi' : (h_{X^o} * \pi) \odot \mathsf{F}_G \rightsquigarrow (h_{X^o} * \pi) \odot \mathsf{F}_H$, whence – by the universality of $(h_{X^o} * \pi)$ – a unique natural transformation $\beta' : G \Rightarrow H$ such that $\Xi' = (h_{X^o} * \pi) \odot \mathsf{F}_\beta$. Then β' is the datum of a 2-cell $\beta : g \Rightarrow h$ such that $\Xi = \pi * \mathsf{F}_\beta$, which shows that the functor (2.5.8) is also full.

Assertion (ii) is easily deduced from (i), by considering the pseudo-cone $\pi^\circ : \mathsf{F}_{L^\circ} \Rightarrow F^\circ$ (details left to the reader). □

Proposition 2.5.9. *Let I, $\mathscr{B}$ and $\mathscr{C}$ be three 2-categories, $G : \mathscr{B} \to \mathscr{C}$, and $H : \mathscr{C} \to \mathscr{B}$ two pseudo-functors, and suppose that (H, G) is a 2-adjoint pair. We have :*

(i) *The pseudo-functor G commutes with the 2-limit of every pseudo-functor $F : I \to \mathscr{B}$.*

(ii) *Dually, H commutes with the 2-colimit of every pseudo-functor $F' : I \to \mathscr{C}$.*

Proof. (i): Let $F : I \to \mathscr{B}$ be any pseudo-functor, and (L, π) any 2-limit for F. We fix a unit and a counit (η, ε) for the 2-adjoint pair (H, G), and a pair of invertible modifications

$$\Sigma : (G * \varepsilon) \odot (\eta * G) \rightsquigarrow \mathbf{1}_G \qquad \Sigma' : (\varepsilon * H) \odot (H * \eta) \rightsquigarrow \mathbf{1}_H .$$

For every $Y \in \mathrm{Ob}(\mathscr{C})$ we consider the diagram of functors :

$$\mathscr{D} \quad : \quad \begin{array}{ccc} \mathscr{C}(Y, GL) & \xrightarrow{\ \vartheta\ } & \mathscr{B}(HY, L) \\ \big\downarrow & & \big\downarrow \\ \mathsf{PsNat}(\mathsf{F}_Y, GF) & \xrightarrow{\ \vartheta^\dagger\ } & \mathsf{PsNat}(\mathsf{F}_{HY}, F) \end{array}$$

where ϑ is given by the rules :

$$(f : Y \to GL) \mapsto (\varepsilon_L \circ Hf : HY \to HGL) \qquad \text{and} \qquad (\beta : f \Rightarrow f') \mapsto \varepsilon_L * H\beta$$

for every 1-cell f and every 2-cell β of $\mathscr{C}(Y, GL)$. The functor $\vartheta^\dagger$ assigns to every pseudo-natural transformation $\psi : \mathsf{F}_Y \Rightarrow GF$ the pseudo-natural transformation $(\varepsilon * F) \odot (H * \psi) : \mathsf{F}_{HY} \Rightarrow F$, and to every modification $\Xi : \psi \rightsquigarrow \psi'$ the modification $(\varepsilon * F) * (H \circ \Xi)$ (notation of remark 2.2.10(vi)). The left (resp. right) vertical arrow is the functor associated with the pseudo-cone $G * \pi$ (resp. π) as in definition 2.2.10(i). By inspecting the proof of theorem 2.2.10(ii), it is easily seen that ϑ is an equivalence, and by assumption, the same holds for the right vertical arrow. We notice :

Claim 2.5.10. Diagram $\mathscr{D}$ is essentially commutative.

Proof of the claim. The composition of ϑ with the right vertical arrow assigns to every 1-cell $f : Y \to GL$ the pseudo-natural transformation $\pi \odot \mathsf{F}_{\varepsilon_L \circ Hf} = \pi \odot \mathsf{F}_{\varepsilon_L} \odot \mathsf{F}_{Hf} = \pi \odot (\varepsilon * \mathsf{F}_L) \odot \mathsf{F}_{Hf}$. On the other hand, the composition of $\vartheta^\dagger$ with the left vertical arrow assigns to f the pseudo-natural transformation $(\varepsilon * F) \odot (H * ((G * \pi) \odot \mathsf{F}_f))$. However, we have invertible modifications:

$$\Theta_f : (\varepsilon * F) \odot (H * ((G * \pi) \odot \mathsf{F}_f)) \rightsquigarrow (\varepsilon * F) \odot (HG * \pi) \odot (H * \mathsf{F}_f) \rightsquigarrow \pi \odot (\varepsilon * \mathsf{F}_L) \odot \mathsf{F}_{Hf}$$

for every such f (example 2.2.15(i,ii)). We need to check that the rule : $f \mapsto \Theta_f$ yields a natural transformation; by unwinding the definitions, we come down to showing the commutativity of the diagram :

$$\begin{array}{ccc} \varepsilon_{Fi} \circ H(G\pi_i \circ f) & \xRightarrow{\ \varepsilon_{Fi} * H(G\pi_i * \beta)\ } & \varepsilon_{Fi} \circ H(G\pi_i \circ f') \\ {\scriptstyle \varepsilon_{Fi} * \gamma^H_{f, G\pi_i}} \Big\Downarrow & & \Big\Downarrow {\scriptstyle \varepsilon_{Fi} * \gamma^H_{f', G\pi_i}} \\ \varepsilon_{Fi} \circ HG\pi_i \circ Hf & \xRightarrow{\ \varepsilon_{Fi} * HG\pi_i * H\beta\ } & \varepsilon_{Fi} \circ HG\pi_i \circ Hf' \\ {\scriptstyle (\tau^\varepsilon_{\pi_i})^{-1} * Hf} \Big\Downarrow & & \Big\Downarrow {\scriptstyle (\tau^\varepsilon_{\pi_i})^{-1} * Hf'} \\ \pi_i \circ \varepsilon_L \circ Hf & \xRightarrow{\ (\pi_i \circ \varepsilon_L) * H\beta\ } & \pi_i \circ \varepsilon_L \circ Hf' \end{array}$$

where (δ^H, γ^H) is the coherence constraint of H and τ^ε is the coherence constraint of ε. However, the commutativity of the top square subdiagram follows from the naturality of γ^H (remark 2.1.17(ii)). The commutativity of the bottom square subdiagram is obvious, since the compositions of both pairs of arrows equal $(\tau^\varepsilon_{\pi_i})^{-1} * H\beta$ (remark 2.1.1(i)). $\diamondsuit$

In view of claim 2.5.10, we are reduced to checking that $\vartheta^\dagger$ is an equivalence of categories. To this aim, let us consider the functor

$$\mu : \mathsf{PsNat}(\mathsf{F}_{HY}, F) \to \mathsf{PsNat}(\mathsf{F}_Y, GF) \qquad (\varphi : \mathsf{F}_{HY} \Rightarrow F) \mapsto (G * \varphi) \odot \mathsf{F}_{\eta_Y}$$

that assigns to every modification $\Theta : \varphi \rightsquigarrow \varphi'$ between pseudo-cones $\varphi, \varphi' : \mathsf{F}_{HY} \Rightarrow F$ the modification $(G \circ \Theta) * \mathsf{F}_{\eta_Y}$. We claim that μ is a quasi-inverse for $\vartheta^\dagger$. Indeed, for every pseudo-cone $\psi : \mathsf{F}_Y \Rightarrow GF$ we have $\mu \circ \vartheta^\dagger(\psi) = (G * ((\varepsilon * F) \odot (H * \psi))) \odot \mathsf{F}_{\eta_Y}$, and example 2.2.15(i) and remark 2.2.10(iv) yield an invertible modification

$$\gamma^G_{\varepsilon * F, H * \psi} * \mathsf{F}_{\eta_Y} : \mu \circ \vartheta^\dagger(\psi) \rightsquigarrow (G * \varepsilon * F) \odot (GH * \psi) \odot \mathsf{F}_{\eta_Y}.$$

Moreover, example 2.2.15(ii) and remark 2.2.10(iv) yield an invertible modification

$$(G * \varepsilon * F) * \tau^\eta_\psi : (G * \varepsilon * F) \odot (GH * \psi) \odot \mathsf{F}_{\eta_Y} \rightsquigarrow (G * \varepsilon * F) \odot (\eta * GF) \odot \psi$$

which we may compose with the invertible modification

$$(\Sigma \circ F) * \psi : (G * \varepsilon * F) \odot (\eta * GF) \odot \psi \rightsquigarrow \mathbf{1}_{GF} \odot \psi = \psi.$$

Summing up, we obtain an invertible modification

$$\Lambda_\psi : \mu \circ \vartheta^\dagger(\psi) \rightsquigarrow \psi \qquad i \mapsto (\Sigma_{Fi} * \psi_i) \odot (G(\varepsilon_{Fi}) * \tau^\eta_{\psi_i}) \odot (\gamma^G_{\varepsilon_{Fi}, H\psi_i} * \eta_Y)$$

and we need to check that the rule $\psi \mapsto \Lambda_\psi$ yields a natural transformation

$$\Lambda : \mu \circ \vartheta^\dagger \Rightarrow \mathbf{1}_{\mathsf{PsNat}(\mathsf{F}_Y, GF)}.$$

Thus, let $\Xi : \psi \rightsquigarrow \psi'$ be any modification, and notice that $\mu \circ \vartheta^\dagger(\Xi)_i = G(\varepsilon_{Fi} * H(\Xi_i)) * \eta_Y$ for every $i \in \mathrm{Ob}(I)$; we compute :

$$\begin{aligned}
\Xi_i \odot \Lambda_{\psi,i} &= (\Sigma_{Fi} * \psi'_i) \odot (G(\varepsilon_{Fi}) * \eta_{GFi} * \Xi_i) \odot (G(\varepsilon_{Fi}) * \tau^\eta_{\psi_i}) \odot (\gamma^G_{\varepsilon_{Fi}, H\psi_i} * \eta_Y) \\
&= (\Sigma_{Fi} * \psi'_i) \odot (G(\varepsilon_{Fi}) * \tau^\eta_{\psi'_i}) \odot (G(\varepsilon_{Fi}) * GH(\Xi_i) * \eta_Y) \odot (\gamma^G_{\varepsilon_{Fi}, H\psi_i} * \eta_Y) \\
&= (\Sigma_{Fi} * \psi'_i) \odot (G(\varepsilon_{Fi}) * \tau^\eta_{\psi'_i}) \odot (\gamma^G_{\varepsilon_{Fi}, H\psi'_i} * \eta_Y) \odot (G(\varepsilon_{Fi} * H(\Xi_i)) * \eta_Y) \\
&= \Lambda_{\psi',i} \odot \mu \circ \vartheta^\dagger(\Xi)_i
\end{aligned}$$

where the first identity follows from remark 2.1.1(i), the second one from remark 2.2.5(i), and the third one from remark 2.1.17(ii). The assertion follows.

Lastly, we have $\vartheta^\dagger \circ \mu(\varphi) = (\varepsilon * F) \odot H * ((G * \varphi) \odot \mathsf{F}_{\eta_Y})$ for every pseudo-cone $\varphi : \mathsf{F}_{HY} \Rightarrow F$, and we get an invertible modification

$$\Lambda'_\varphi : \vartheta^\dagger \circ \mu(\varphi) \rightsquigarrow (\varepsilon * F) \odot (HG * \varphi) \odot \mathsf{F}_{H\eta_Y} \rightsquigarrow \varphi \odot \mathsf{F}_{\varepsilon_{HY}} \odot \mathsf{F}_{H\eta_Y} \rightsquigarrow \varphi$$

as the composition of $(\varepsilon * F) * (\gamma^H_{\mathsf{F}_{\eta_Y}, G * \varphi})^{-1}$ from example 2.2.15(i), together with $(\tau^\varepsilon_\varphi)^{-1} * \mathsf{F}_{H\eta_Y}$ from example 2.2.15(ii) and $\varphi * \mathsf{F}_{\Sigma'_Y}$. An explicit calculation as in the foregoing shows that the rule $\varphi \mapsto \Lambda'_\varphi$ yields a natural transformation $\Lambda : \vartheta^\dagger \circ \mu \Rightarrow \mathbf{1}_{\mathsf{PsNat}(\mathsf{F}_{HY}, F)}$ (details left to the reader), and concludes the proof of (i).

Assertion (ii) follows from (i), by considering the opposite 2-categories : details left to the reader. $\square$

Proposition 2.5.11. *Let $\mathscr{A}$, $\mathscr{B}$ be two 2-categories, $F : \mathscr{A} \to \mathscr{B}$ a fully faithful pseudo-functor that admits a left 2-adjoint $G : \mathscr{B} \to \mathscr{A}$. The following holds :*

(i) *For every small 2-category I and every pseudo-functor $\varphi : I \to \mathscr{A}$, if $F \circ \varphi : I \to \mathscr{B}$ admits a 2-limit (L, π^L), then φ admits a 2-limit (M, π^M).*

(ii) *In the situation of* (i)*, we have equivalences* $L \xrightarrow{\sim} FM$ *in* $\mathscr{B}$ *and* $M \xrightarrow{\sim} GL$ *in* $\mathscr{A}$*.*

Proof. Let $\eta : \mathbf{1}_{\mathscr{B}} \Rightarrow FG$ and $\varepsilon : GF \Rightarrow \mathbf{1}_{\mathscr{A}}$ be a unit and a counit for the 2-adjoint pair (G, F), and set $\pi^{GL} := (\varepsilon * \varphi) \odot (G * \pi^L) : \mathsf{F}_{GL} \Rightarrow \varphi$. By the universality of π^L, there exist a 1-cell $f : FGL \to L$ in $\mathscr{B}$ and an isomorphism of pseudo-cones $\pi^L \odot \mathsf{F}_f \xrightarrow{\sim} F * \pi^{GL}$. Invoking example 2.2.15(i,ii) we deduce isomorphisms of pseudo-cones :

$$\pi^L \odot \mathsf{F}_{f \circ \eta_L} = \pi^L \odot \mathsf{F}_f \odot \mathsf{F}_{\eta_L} \xrightarrow{\sim} (F * \pi^{GL}) \odot (\eta * \mathsf{F}_L) \xrightarrow{\sim} (F * \varepsilon * \varphi) \odot (FG * \pi^L) \odot (\eta * \mathsf{F}_L)$$
$$\xrightarrow{\sim} (F * \varepsilon * \varphi) \odot (\eta * F\varphi) \circ \pi^L \xrightarrow{\sim} \pi^L$$

whence an isomorphism $f \circ \eta_L \xrightarrow{\sim} \mathbf{1}_L$ in $\mathscr{B}$, by universality of π^L. On the other hand, since F is fully faithful, there exists a 1-cell $h : GL \to GL$ in $\mathscr{A}$ with an isomorphism $Fh \xrightarrow{\sim} \eta_L \circ f : FGL \to FGL$. Notice the isomorphisms :

(2.5.12) $$Fh \circ \eta_L \xrightarrow{\sim} \eta_L \circ f \circ \eta_L \xrightarrow{\sim} \eta_L \xrightarrow{\sim} F\mathbf{1}_{GL} \circ \eta_L.$$

On the other hand, using the pseudo-naturality of ε and the triangular modifications for the pair (η, ε) we get isomorphisms :

$$\varepsilon_{GL} \circ G(Fh \circ \eta_L) \xrightarrow{\sim} \varepsilon_{GL} \circ GFh \circ G\eta_L \xrightarrow{\sim} h \circ \varepsilon_{GL} \circ G\eta_L \xrightarrow{\sim} h$$

and similarly : $\varepsilon_{GL} \circ G(F\mathbf{1}_{GL} \circ \eta_L) \xrightarrow{\sim} \mathbf{1}_{GL}$. Combining with (2.5.12), we deduce an isomorphism $h \xrightarrow{\sim} \mathbf{1}_{GL}$, whence $\eta_L \circ f \xrightarrow{\sim} F\mathbf{1}_{GL} \xrightarrow{\sim} \mathbf{1}_{FGL}$, which shows that f is an equivalence in $\mathscr{B}$. Especially, (ii) holds with $M := GL$, and it also follows that $(FGL, F * \pi^{GL})$ is a 2-limit of $F \circ \varphi$. It remains to check that (GL, π^{GL}) is a 2-limit of φ. To this aim, we consider for every $A \in \mathrm{Ob}(\mathscr{A})$ the essentially commutative diagram of categories :

$$\begin{array}{ccc}
\mathscr{A}(A, GL) & \longrightarrow & \mathsf{PsNat}(\mathsf{F}_A, \varphi) \\
{\scriptstyle F_{A,GL}} \big\downarrow & & \big\downarrow {\scriptstyle U} \\
\mathscr{B}(FA, FGL) & \longrightarrow & \mathsf{PsNat}(\mathsf{F}_{FA}, F \circ \varphi)
\end{array}$$

whose top (resp. bottom) horizontal arrow is the pseudo-functor induced by the pseudo-cone π^{GL} (resp. $F * \pi^{GL}$) as in definition 2.5.1(i). The pseudo-functor U is given by the rules : $\beta \mapsto F * \beta$ and $\Theta \mapsto F * \Theta$ for every pseudo-cone $\beta : \mathsf{F}_A \Rightarrow \varphi$ and every modification of pseudo-cones $\Theta : \beta \rightsquigarrow \beta'$. Since F is fully faithful, the functor $F_{A,GL}$ is an equivalence, and the same holds for the bottom horizontal arrow. We are then reduced to checking that U is an equivalence. To this aim, recall that ε is a pseudo-natural equivalence (corollary 2.4.29) and fix a 1-cell $\omega_A : A \to GFA$ in $\mathscr{A}$ and an isomorphism

$$\rho_A : \varepsilon_A \circ \omega_A \xrightarrow{\sim} \mathbf{1}_A.$$

There follows an isomorphism $F\varepsilon_A \circ F\omega_A \xrightarrow{\sim} \mathbf{1}_{FA}$; on the other hand, the triangular modifications for (η, ε) give an isomorphism $F\varepsilon_A \circ \eta_{FA} \xrightarrow{\sim} \mathbf{1}_{FA}$, so we get as well an isomorphism

$$\rho'_A : F\omega_A \xrightarrow{\sim} \eta_{FA}.$$

We consider the pseudo-functor

$$V : \mathsf{PsNat}(\mathsf{F}_{FA}, F \circ \varphi) \to \mathsf{PsNat}(\mathsf{F}_A, \varphi) \qquad \beta \mapsto (\varepsilon * \varphi) \odot (G * \beta) \odot \mathsf{F}_{\omega_A}$$

that assigns to every modification $\Theta : \beta \rightsquigarrow \beta'$ of pseudo-cones $\beta, \beta' : \mathsf{F}_{FA} \Rightarrow F \circ \varphi$ the modification $(\varepsilon * \varphi) * (G * \Theta) * \mathsf{F}_{\omega_A}$. We construct as follows isomorphisms of functors

$$\Lambda : VU \xrightarrow{\sim} \mathbf{1}_{\mathsf{PsNat}(\mathsf{F}_A, \varphi)} \qquad \Lambda' : UV \xrightarrow{\sim} \mathbf{1}_{\mathsf{PsNat}(\mathsf{F}_{FA}, F \circ \varphi)}.$$

For every pseudo-cone $\beta : \mathsf{F}_A \Rightarrow \varphi$ we have $VU(\beta) = (\varepsilon * \varphi) \odot (GF * \beta) \odot \mathsf{F}_{\omega_A}$. By example 2.2.15(ii), we have an invertible modification

$$\Xi_\beta : \beta \odot (\varepsilon * \mathsf{F}_A) \rightsquigarrow (\varepsilon * \varphi) \odot (GF * \beta) \qquad i \mapsto \tau^\varepsilon_{\beta_i}$$

where τ^ε denotes the coherence constraint of ε. There follows an isomorphism $\Xi^{-1}_\beta * \mathsf{F}_{\omega_A}$: $VU(\beta) \xrightarrow{\sim} \beta \odot (\varepsilon * \mathsf{F}_A) \odot \mathsf{F}_{\omega_A} = \beta \odot \mathsf{F}_{\varepsilon_A \circ \omega_A}$. We compose the latter with the isomorphism $\beta * \mathsf{F}_{\rho_A} : \beta \odot \mathsf{F}_{\varepsilon_A \circ \omega_A} \xrightarrow{\sim} \beta$ to obtain the isomorphism $\Lambda_\beta : VU(\beta) \xrightarrow{\sim} \beta$. In order to check the naturality of the rule : $\beta \mapsto \Lambda_\beta$, we come down to verifying the identity :

$$((\varepsilon * \varphi) * (GF * \Theta)) \odot \Xi_\beta = \Xi_{\beta'} \odot (\Theta * (\varepsilon * \mathsf{F}_A))$$

for every morphism of pseudo-cones $\Theta : \beta \rightsquigarrow \beta'$. This follows from the naturality of τ^ε, applied to the system of 2-cells $\Theta_i : \beta_i \Rightarrow \beta'_i$: details left to the reader. Lastly, for every pseudo-cone $\beta : \mathsf{F}_{FA} \Rightarrow F \circ \varphi$ we have $UV(\beta) = F * ((\varepsilon * \varphi) \odot (G * \beta) \odot \mathsf{F}_{\omega_A})$. By example 2.2.15(i) we have the invertible modification

$$\Gamma_\beta : (F * \varepsilon * \varphi) \odot (FG * \beta) \odot \mathsf{F}_{F\omega_A} \rightsquigarrow F * ((\varepsilon * \varphi) \odot (G * \beta) \odot \mathsf{F}_{\omega_A}) \qquad i \mapsto \gamma^F_{\omega_A, G\beta_i, \varepsilon_{\varphi(i)}}$$

where $\gamma^F_{\bullet\bullet\bullet}$ is coherence constraint of F (for compositions of three 1-cells). We compose Γ^{-1}_β with the invertible modification $((F * \varepsilon * \varphi) \odot (FG * \beta)) * \mathsf{F}_{\rho'_A}$ to get an isomorphism $UV(\beta) \xrightarrow{\sim} (F * \varepsilon * \varphi) \odot (FG * \beta) \odot (\eta * \mathsf{F}_A)$. Then, example 2.2.15(ii) yields the invertible modification

$$\Xi'_\beta : (FG * \beta) \odot (\eta * \mathsf{F}_A) \rightsquigarrow (\eta * F * \varphi) \odot \beta \qquad i \mapsto \tau^\eta_{\beta_i}$$

where τ^η denotes the coherence constraint of η. On the other hand we have the triangular modification $\Sigma : (F * \varepsilon) \odot (\eta * F) \rightsquigarrow \mathbf{1}_F$, so we can compose further with

$$((\Sigma * \varphi) * \beta) \odot ((F * \varepsilon * \varphi) * \Xi'_\beta) : (F * \varepsilon * \varphi) \odot (FG * \beta) \odot (\eta * \mathsf{F}_A) \xrightarrow{\sim} \beta$$

to get the sought isomorphism $\Lambda'_\beta : UV(\beta) \xrightarrow{\sim} \beta$. Again, the naturality of the rule : $\beta \mapsto \Lambda'_\beta$ follows easily from the naturality of $\gamma^F_{\bullet\bullet\bullet}$ and τ^η. □

2.5.13. Let I, J be two small 2-categories, $\mathscr{C}$ another 2-category, and $\varphi : J \to I$, $F : I \to \mathscr{C}$ two pseudo-functors. Let also (L, π) and (L', π') be 2-limits for F and respectively $F \circ \varphi$. Then there exist a 1-cell $f : L \to L'$ of $\mathscr{C}$ and an invertible modification

$$\Theta : \pi * \varphi \rightsquigarrow \pi' \odot \mathsf{F}_f.$$

Moreover, if $f' : L \to L'$ is another such 1-cell with an invertible modification $\Theta' : \pi * \varphi \rightsquigarrow \pi' \odot \mathsf{F}_{f'}$, there exists a unique invertible 2-cell in $\mathscr{C}$:

$$\beta : f \xrightarrow{\sim} f' \qquad \text{such that} \qquad \mathsf{F}_\beta \odot \Theta = \Theta'.$$

In particular, f shall be an equivalence in $\mathscr{C}$ if and only if the same holds for f' (remark 2.1.4(iii)), so the property that f is an equivalence is an intrinsic feature of the pair (φ, F). Furthermore, Θ induces an essentially commutative diagram of categories :

$$\begin{array}{ccc} \mathscr{C}(X, L) & \xrightarrow{T} & \mathsf{PsNat}(\mathsf{F}_X, F) \\ {\scriptstyle f^X_*} \downarrow & & \downarrow {\scriptstyle S^X} \\ \mathscr{C}(X, L') & \xrightarrow{T'} & \mathsf{PsNat}(\mathsf{F}_X, F \circ \varphi) \end{array} \qquad \text{for every } X \in \mathrm{Ob}(\mathscr{C})$$

whose horizontal arrows are the equivalences of definition 2.5.1(i), where f^X_* is defined as in example 2.1.11(iii), and S^X is given by the rules : $(\beta : \mathsf{F}_X \Rightarrow F) \mapsto \beta * \varphi$ and $(\Xi : \beta \rightsquigarrow \beta') \mapsto \Xi \circ \varphi$. Then the required isomorphism of functors $S^X \circ T \xrightarrow{\sim} T' \circ f^X_*$ is given by the rule : $(g : X \to L) \mapsto \Theta * \mathsf{F}_g$. Hence, f is an equivalence in $\mathscr{C}$ if and only if S^X is an equivalence of categories for every $X \in \mathrm{Ob}(\mathscr{C})$.

Definition 2.5.14. Let I, J be two small 2-categories. We say that a pseudo-functor $\varphi : J \to I$ is 2-*coinitial* if for every 2-complete 2-category $\mathscr{C}$ and every pseudo-functor $F : I \to \mathscr{C}$ the induced 1-cell $f : L \to L'$ as in (2.5.13) is an equivalence. Likewise, we say that φ is 2-*cofinal* if $\varphi^o : J^o \to I^o$ is 2-coinitial.

Remark 2.5.15. Unlike as in the case for usual category, we do not provide a general characterization of 2-coinitial or 2-cofinal pseudo-functors, but we shall point out at least two simple, though useful, classes of such pseudo-functors. Such a characterization – in terms of simple connectivity of comma categories – can be found in [146, Prop.A.2.2], though under slightly different assumptions.

For our first example, let $\mathbb{1}$ be a final object of $\mathbf{Cat}$, *i.e.* a category with a unique object and a unique morphism (then the unique morphism is the identity of the unique object); let likewise $\mathbb{I}$ be a final object of 2-$\mathbf{Cat}$, *i.e.* a 2-category with a unique object, a unique 1-cell and a unique 2-cell. We say that a given object i_0 of I is *pseudo-initial* (resp. *pseudo-final*) if the category $I(i_0, i)$ (resp. $I(i, i_0)$) is equivalent to $\mathbb{1}$, for every $i \in \mathrm{Ob}(I)$.

Proposition 2.5.16. *Let i_0 be a pseudo-initial (resp. pseudo-final) object of I, and $\varphi : \mathbb{I} \to I$ the strict pseudo-functor that maps the object of $\mathbb{I}$ to i_0. Then φ is 2-coinitial (resp. 2-cofinal).*

Proof. Let i_0 be pseudo-initial; we need to check that for every 2-complete 2-category $\mathscr{C}$, every pseudo-functor $F : I \to \mathscr{C}$ and every 2-limit (L, π) of F, the pair $(L, \pi * \varphi)$ is a 2-limit of $F \circ \varphi$, and notice that the latter condition means that $\pi_{i_0} : L \to L' := Fi_0$ is an equivalence. To this aim, it suffices to construct a universal pseudo-cone $\pi' : \mathsf{F}_{L'} \Rightarrow F$ with $\pi'_{i_0} = \mathbf{1}_{L'}$. We may also assume that F is unital, by lemma 2.5.3 and proposition 2.4.3. Thus, let $i \in \mathrm{Ob}(I)$ be any object; by assumption, there exists a 1-cell $f_i : i_0 \to i$ in I, and we set $\pi'_i := Ff_i : L' \to Fi$. Naturally, we take $f_{i_0} := \mathbf{1}_{i_0}$, so that $\pi'_{i_0} = \mathbf{1}_{L'}$, since F is unital. Next, for every one-cell $g : i \to i'$ in I we have by assumption a unique 2-cell $\beta_g : g \circ f_i \Rightarrow f_{i'}$ in I, and it is easily seen that β_g must then be invertible (details left to the reader); we set $\tau_g := (F\beta_g) \odot \gamma^F_{f_i,g} : Ff \circ \pi_i \Rightarrow \pi'_{i'}$, where γ^F denotes the coherence constraint of F.

Claim 2.5.17. The rule : $i \mapsto \pi'_i$ defines a pseudo-cone $\pi' : \mathsf{F}_{L'} \Rightarrow F$ with coherence constraint given by the system of 2-cells $\tau_\bullet$.

Proof of the claim. The uniqueness of β_g implies that $\beta_{\mathbf{1}_i} = \mathbf{1}_{f_i}$ for every $i \in \mathrm{Ob}(I)$ and

$$\beta_h \odot (h * \beta_g) = \beta_{h \circ g} \qquad \text{for every pair of 1-cells } i \xrightarrow{g} i' \xrightarrow{h} i''.$$

Since F is unital, it follows already that $\tau_{\mathbf{1}_i} = \mathbf{1}_{\pi'_i}$ for every $i \in \mathrm{Ob}(I)$. It remains to check that

$$\tau_h \odot (Fh * \tau_g) = \tau_{h \circ g} \odot (\gamma^F_{g,h} * \pi'_i)$$

for every pair of 1-cells (g, h) as in the foregoing. We compute :

$$\begin{aligned} \tau_h \odot (Fh * \tau_g) &= F\beta_h \odot \gamma^F_{f_{i'},h} \odot (Fh * F\beta_g) \odot (Fh * \gamma^F_{f_i,g}) \\ &= F\beta_h \odot F(h * \beta_g) \odot \gamma^F_{g \circ f_i,h} \odot (Fh * \gamma^F_{f_i,g}) \\ &= F\beta_{h \circ g} \odot \gamma^F_{g \circ f_i,h} \odot (Fh * \gamma^F_{f_i,g}) \end{aligned}$$

whereas $\tau_{h \circ g} \odot (\gamma^F_{g,h} * \pi'_i) = F\beta_{g \circ h} \odot \gamma^F_{f_i,g} \odot (\gamma^F_{g,h} * Ff_i)$, so the sought identity follows from the composition axiom for γ^F. ◊

To conclude, we need to check that functor

$$\mathscr{C}(X, L) \to \mathsf{PsNat}(c_X, F) \qquad t \mapsto \pi' \odot \mathsf{F}_t$$

is an equivalence for every $X \in \mathrm{Ob}(\mathscr{C})$. The faithfulness is clear, since $\pi'_{i_0} = \mathbf{1}_{L'}$. To check that the functor is full, consider two 1-cells $t, t' : X \to L$ and a modification $\Xi : \pi' \odot \mathsf{F}_t \rightsquigarrow \pi' \odot \mathsf{F}_{t'}$;

we set $\mu := \Xi_{i_0} : t \Rightarrow t'$. Now, notice that $\beta_{f_i} = \mathbf{1}_{f_i}$ for every $i \in \mathrm{Ob}(I)$, so that $\tau_{f_i} = \mathbf{1}_{Ff_i}$, since F is unital; then the compatibility condition for Ξ, relative to the 1-cell $f_i : i_0 \to i$ yields:

$$\Xi_i = Ff_i * \mu \qquad \text{for every } i \in \mathrm{Ob}(I)$$

which shows that $\Xi = \pi' * \mathsf{F}_\mu$, as required. Lastly, let $\alpha : \mathsf{F}_X \Rightarrow F$ be any pseudo-cone; set $t := \alpha_{i_0} : X \to L'$, and $\Xi_i := \tau^\alpha_{f_i} : Ff_i \circ t \Rightarrow \alpha_i$ for every $i \in \mathrm{Ob}(I)$, where τ^α is the coherence constraint of α. To conclude, it suffices to check that the rule : $i \mapsto \Xi_i$ yields an invertible modification $\Xi : \pi' \odot \mathsf{F}_t \rightsquigarrow \alpha$. This comes down to showing :

$$\tau^\alpha_g \odot (Fg * \tau^\alpha_{f_i}) = \tau^\alpha_{f_{i'}} \odot (\tau_g * \alpha_{i_0}) \qquad \text{for every 1-cell } g : i \to i' \text{ of } I.$$

However, the left-hand side of the latter identity equals $\tau^\alpha_{g \circ f_i} \odot (\gamma^F_{f_i,g} * \alpha_{i_0})$, by the coherence axiom for τ^α, so we are reduced to checking the identity :

$$\tau^\alpha_{g \circ f_i} = \tau^\alpha_{f_{i'}} \odot (F\beta_g * \alpha_{i_0})$$

which holds by naturality of τ^α. $\square$

2.5.18. For our second example, consider a small category I (which we regard as a 2-category with trivial 2-cells, as usual), and a full subcategory J of I, such that the inclusion functor $\varphi : J \to I$ admits a left adjoint $\sigma : I \to J$ with $\sigma \circ \varphi = \mathbf{1}_J$.

Proposition 2.5.19. *In the situation of* (2.5.18)*, the pseudo-functor* $\varphi : J \to I$ *is* 2*-cofinal.*

Proof. For every pseudo-functor $F : I \to \mathscr{C}$, every pseudo-cocone $\pi : F \Rightarrow \mathsf{F}_L$, and every $C \in \mathrm{Ob}(\mathscr{C})$ we have a commutative diagram of categories :

$$\begin{array}{ccc} & \mathscr{C}(L,C) & \\ \swarrow & & \searrow \\ \mathsf{PsNat}(F, \mathsf{F}_C) & \xrightarrow{\ S_C\ } & \mathsf{PsNat}(F \circ \varphi, \mathsf{F}_C) \end{array}$$

whose downward arrows are the functors $f \mapsto \pi \odot \mathsf{F}_f$ and $f \mapsto (\pi * \varphi) \odot \mathsf{F}_f$ of definition 2.5.1(ii), and where S_C is given by the rules : $(\beta : F \Rightarrow \mathsf{F}_C) \mapsto \beta * \varphi$ and $(\Xi : \beta \rightsquigarrow \beta') \mapsto \Xi \circ \varphi$. We come down to checking that S_C is an equivalence, for every such F. To this aim, notice first that the adjoint pair (σ, φ) admits an adjunction whose counit $\sigma \circ \varphi \Rightarrow \mathbf{1}_J$ is the identity natural transformation (proposition 1.1.20(iii)); in view of the triangular identities of (1.1.13), it follows that the unit $\eta : \mathbf{1}_I \Rightarrow \varphi \circ \sigma$ fulfills as well the condition :

$$\eta * \varphi = \mathbf{1}_\varphi.$$

We consider for every $C \in \mathrm{Ob}(\mathscr{C})$ the functor

$$S'_C : \mathsf{PsNat}(F \circ \varphi, \mathsf{F}_C) \to \mathsf{PsNat}(F, \mathsf{F}_C) \qquad (\beta : F \circ \varphi \Rightarrow \mathsf{F}_C) \mapsto (\beta * \sigma) \odot (F * \eta)$$

that assigns to every modification $\Xi : \beta \rightsquigarrow \beta'$ the modification $(\Xi \circ \sigma) * (F * \eta) : S'_C(\beta) \rightsquigarrow S'_C(\beta')$. Notice that for every pseudo-natural transformation $\beta : F \circ \varphi \Rightarrow \mathsf{F}_C$ we have

$$S_C \circ S'_C(\beta) = ((\beta * \sigma) \odot (F * \eta)) * \varphi = (\beta * \sigma * \varphi) \odot (F * \eta * \varphi) = \beta$$

and likewise, if $\beta' : F \circ \varphi \Rightarrow \mathsf{F}_C$ is another such pseudo-natural transformation, and $\Xi : \beta \rightsquigarrow \beta'$ is any modification, we get $S_C \circ S'_C(\Xi) = \Xi$. On the other hand, for every pseudo-natural transformation $\beta : F \Rightarrow \mathsf{F}_C$, example 2.2.15(ii) yields an invertible modification

$$\tau^\beta_\eta : \beta = (\mathsf{F}_C * \eta) \odot (\beta * \mathbf{1}_I) \rightsquigarrow (\beta * \varphi * \sigma) \odot (F * \eta) = S'_C \circ S_C(\beta).$$

To conclude the proof, it suffices to check that the rule $\beta \mapsto \tau_\eta^\beta$ yields a natural transformation $\mathbf{1}_{\mathsf{PsNat}(F,\mathsf{F}_C)} \Rightarrow S'_C \circ S_C$. The latter comes down to the commutativity of the diagram of modifications :

$$\begin{array}{ccc} \beta & \overset{\tau_\eta^\beta}{\rightsquigarrow} & (\beta * \varphi * \sigma) \odot (F * \eta) \\ {\scriptstyle\Xi}\big\wr\downarrow & & \big\wr\downarrow{\scriptstyle (\Xi\circ\varphi\circ\sigma)*(F*\eta)} \\ \beta' & \overset{\tau_\eta^{\beta'}}{\rightsquigarrow} & (\beta' * \varphi * \sigma) \odot (F * \eta) \end{array}$$

for every modification $\Xi : \beta \rightsquigarrow \beta'$. This in turn amounts to the compatibility condition for Ξ, relative to the 1-cell $\eta_i : i \to \varphi \circ \sigma(i)$, for every $i \in \mathrm{Ob}(I)$. □

2.6. **2-Categorical Kan extensions.** Let $I, J, \mathscr{C}$ be three 2-categories, $\varphi : I \to J$ a *unital* pseudo-functor, and suppose that $\mathscr{C}$ is 2-complete, I is small and J has small Hom-categories. We define as follows a pseudo-functor

$$2\text{-}\!\int_\varphi : \mathsf{PsFun}(I, \mathscr{C}) \to \mathsf{PsFun}(J, \mathscr{C}).$$

For every *unital* pseudo-functor $F : I \to \mathscr{C}$ and every $j \in \mathrm{Ob}(J)$, notice that the 2-category $j/\varphi I$ is small under our assumptions, and choose a 2-limit $(L_{F,j}, \pi^{F,j})$ of $F \circ \mathsf{t}_j : j/\varphi I \to \mathscr{C}$ (notation of example 2.2.8(ii)). Then, for every 1-cell $g : j \to j'$ in J there exist a 1-cell $L_{F,g} : L_{F,j} \to L_{F,j'}$ in $\mathscr{C}$ and an invertible modification

$$\Omega^{F,g} : \pi^{F,j} * g^* \rightsquigarrow \pi^{F,j'} \odot \mathsf{F}_{L_{F,g}}$$

with the strict pseudo-functor $g^* : j'/\varphi I \to j/\varphi I$ defined as in example 2.2.8(ii), and we set

$$(2.6.1) \qquad 2\text{-}\!\int_\varphi F(j) := L_{F,j} \qquad 2\text{-}\!\int_\varphi F(g) := L_{F,g}.$$

Obviously, for $g = \mathbf{1}_j$, we take $L_{F,g}$ to be the identity 1-cell of $L_{F,j}$, and $\Omega^{F,g}$ the identity modification of $\pi^{F,j}$. If $h : j' \to j''$ is another 1-cell in J, there exists a unique invertible 2-cell $L_{F,g,h} : L_{F,h} \circ L_{F,g} \Rightarrow L_{F,h\circ g}$ in $\mathscr{C}$ such that

$$(2.6.2) \qquad (\pi^{F,j''} * \mathsf{F}_{L_{F,g,h}}) \odot (\Omega^{F,h} * \mathsf{F}_{L_{F,g}}) \odot (\Omega^{F,g} \circ h^*) = \Omega^{F,h\circ g}.$$

Moreover, if $g, g' : j \to j'$ are two 1-cells of J, and $\beta : g \Rightarrow g'$ is a 2-cell, we set

$$\pi_f^{F,\beta} := \tau_{\beta^*(f)}^{F,j} \qquad \text{for every } (f : j' \to \varphi(i)) \in \mathrm{Ob}(j'/\varphi I)$$

where $\tau^{F,j}$ denotes the coherence constraint of $\pi^{F,j}$, and $\beta^* : g^* \Rightarrow g'^*$ is the strict pseudo-natural transformation associated with β as in example 2.2.8(ii). Since F is unital, $\pi_f^{F,\beta}$ is a 2-cell $\pi_{g^*(f)}^{F,j} \Rightarrow \pi_{g'^*(f)}^{F,j}$ in $\mathscr{C}$, between two 1-cells $L_{F,j} \to Fi$. We claim that the rule : $(f : j' \to \varphi(i)) \mapsto \pi_f^{F,\beta}$ defines an invertible modification $\pi^{F,\beta} : \pi^{F,j} * g \rightsquigarrow \pi^{F,j} * g'$. Indeed, notice first that for every pair of 1-cells $(i_1, g_1) \xrightarrow{(f,\alpha)} (i_2, g_2) \xrightarrow{(f',\alpha')} (i_3, g_3)$ of $j/\varphi(I)$ we have by definition :

$$(2.6.3) \qquad (f', \alpha') \circ (f, \alpha) = \alpha' \odot (\varphi(f') * \alpha) \odot ((\gamma_{f,f'}^\varphi)^{-1} * g_1) \qquad \text{in } j/\varphi(I).$$

Now, again using the unital assumption for F, the assertion comes down to the identity :

$$\tau_{g'^*(h,\alpha)}^{F,j} \odot (Fh * \tau_{\beta^*(f)}^{F,j}) = \tau_{\beta^*(f')}^{F,j} \odot \tau_{g^*(h,\alpha)}^{F,j}$$

for every 1-cell $(h, \alpha) : (f : j' \to \varphi(i)) \to (f' : j' \to \varphi(i'))$ of $j'/\varphi I$. Let us also remark that $\beta^*(f') \circ g^*(\alpha, f) = g'^*(\alpha, f) \circ \beta^*(f)$, and $\beta^*(f) = (\mathbf{1}_i, f * \beta)$ and likewise for $\beta^*(f')$; since also φ is unital, we then have $\gamma_{\mathbf{1}_i,h}^\varphi = \mathbf{1}_h = \gamma_{h,\mathbf{1}_{i'}}^\varphi$, and the sought identity follows from (2.6.3) and the coherence axiom for $\tau^{F,j}$. Consequently, there exists a unique 2-cell $L_{F,\beta} : L_{F,g} \Rightarrow L_{F,g'}$ in $\mathscr{C}$ such that

$$\Omega^{F,g'} \odot \pi^{F,\beta} = (\pi^{F,j'} * \mathsf{F}_{L_{F,\beta}}) \odot \Omega^{F,g} \tag{2.6.4}$$

and we set

$$2\text{-}\!\!\int_\varphi F(\beta) := L_{F,\beta}. \tag{2.6.5}$$

Lemma 2.6.6. *In the situation of* (2.6)*, the system of invertible* 2*-cells* $L_{F,\bullet\bullet}$ *yields a coherence constraint for a unital pseudo-functor* $2\text{-}\!\int_\varphi F : J \to \mathscr{C}$ *defined on objects and* 1*-cells by* (2.6.1)*, and on* 2*-cells by* (2.6.5).

Proof. Let $\beta : g \Rightarrow g'$ and $\beta' : g' \Rightarrow g''$ be two 2-cells of J; a simple inspection shows that

$$\beta'^*(f) \circ \beta^*(f) = (\beta' \odot \beta)^*(f) \qquad \text{for every } (f : j' \to \varphi(i)) \in \mathrm{Ob}(j'/\varphi I)$$

whence $\pi^{F,\beta'} \odot \pi^{F,\beta} = \pi^{F,\beta'\odot\beta}$. On the other hand, we have

$$\begin{aligned}(\pi^{F,j'} * \mathsf{F}_{L_{F,\beta'}}) \odot (\pi^{F,j'} * \mathsf{F}_{L_{F,\beta}}) \odot \Omega^{F,g} &= (\pi^{F,j'} * \mathsf{F}_{L_{F,\beta'}}) \odot \Omega^{F,g'} \odot \pi^{F,\beta} \\ &= \Omega^{F,g''} \odot \pi^{F,\beta'} \odot \pi^{F,\beta}.\end{aligned}$$

Summing up, we already see that $2\text{-}\!\int_\varphi F(\beta') \odot 2\text{-}\!\int_\varphi F(\beta) = 2\text{-}\!\int_\varphi F(\beta' \odot \beta)$. Moreover, notice that $\pi^{F,\mathbf{1}_g}$ is the identity automorphism of $\pi^{F,j} * g$ (remark 2.4.2(ii)), whence

$$2\text{-}\!\!\int_\varphi F(\mathbf{1}_g) = \mathbf{1}_{L_{F,g}} \qquad \text{for every 1-cell } g \text{ of } J.$$

Next, let $g, g' : j \to j'$ and $h, h' : j' \to j''$ be four 1-cells of J and $\beta : g \Rightarrow g'$, $\beta : h \Rightarrow h'$ two 2-cells; we need to check that

$$\Big(2\text{-}\!\!\int_\varphi F(\beta' * \beta)\Big) \odot L_{F,g,h} = L_{F,g',h'} \odot \Big(2\text{-}\!\!\int_\varphi F(\beta') * 2\text{-}\!\!\int_\varphi F(\beta)\Big)$$

and in light of the foregoing, we may assume that either $\beta = \mathbf{1}_g$ or $\beta' = \mathbf{1}_h$. In these cases, the assertion comes down to the identities :

$$\begin{aligned}L_{F,h*\beta} \odot L_{F,g,h} &= L_{F,g',h} \odot (L_{F,h} * L_{F,\beta}) \\ L_{F,\beta'*g} \odot L_{F,g,h} &= L_{F,g,h'} \odot (L_{F,\beta'} * L_{F,g}).\end{aligned}$$

However, set $X := (\Omega^{F,h} * \mathsf{F}_{L_{F,g}}) \odot (\Omega^{F,g} \circ h^*)$. It suffices to show that

$$\begin{aligned}Y := (\pi^{F,j''} * (\mathsf{F}_{L_{F,g',h}} \odot (\mathsf{F}_{L_{F,h}} * \mathsf{F}_{L_{F,\beta}})) \odot X \ &= Z := (\pi^{F,j''} * (\mathsf{F}_{L_{F,h*\beta}} \odot \mathsf{F}_{L_{F,g,h}})) \odot X \\ Y' := (\pi^{F,j''} * (\mathsf{F}_{L_{F,g,h'}} \odot (\mathsf{F}_{L_{F,\beta'}} * \mathsf{F}_{L_{F,g}})) \odot X \ &= Z' := (\pi^{F,j''} * (\mathsf{F}_{L_{F,\beta'*g}} \odot \mathsf{F}_{L_{F,g,h}})) \odot X.\end{aligned}$$

We compute :

$$\begin{aligned}Y &= (\pi^{F,j''} * \mathsf{F}_{L_{F,g',h}}) \odot (\Omega^{F,h} * \mathsf{F}_{L_{F,g'}}) \odot ((\pi^{F,j'} * h^*) * \mathsf{F}_{L_{F,\beta}}) \odot (\Omega^{F,g} \circ h^*) \\ &= (\pi^{F,j''} * \mathsf{F}_{L_{F,g',h}}) \odot (\Omega^{F,h} * \mathsf{F}_{L_{F,g'}}) \odot (\Omega^{F,g'} \circ h^*) \odot (\pi^{F,\beta} \circ h^*) \\ &= \Omega^{F,h\circ g'} \odot (\pi^{F,\beta} \circ h^*).\end{aligned}$$

and on the other hand, $Z = (\pi^{F,j''} * \mathsf{F}_{L_{F,h*\beta}}) \odot \Omega^{F,h\circ g} = \Omega^{F,h\circ g'} \odot \pi^{F,\mathbf{1}_h*\beta}$. So, for the first identity we are reduced to showing that $\pi^{F,\beta} \circ h^* = \pi^{F,\mathbf{1}_h*\beta}$. The latter follows by direct inspection. Next, we compute :

$$\begin{aligned}Y' &= (\pi^{F,j''} * \mathsf{F}_{L_{F,g,h'}}) \odot (\Omega^{F,h'} * \mathsf{F}_{L_{F,g}}) \odot (\pi^{F,\beta'} * \mathsf{F}_{L_{F,g}}) \odot (\Omega^{F,g} \circ h^*) \\ &= (\Omega^{F,h'\circ g}) \odot (\Omega^{F,g} * h'^*)^{-1} \odot (\pi^{F,\beta'} * \mathsf{F}_{L_{F,g}}) \odot (\Omega^{F,g} \circ h^*)\end{aligned}$$

and on the other hand, $Z' = (\pi^{F,j''} * \mathsf{F}_{L_{F,\beta'*\mathbf{1}_g}}) \odot \Omega^{F,h\circ g} = \Omega^{F,h'\circ g} \odot \pi^{F,\beta'*g}$. So, for the second identity we are reduced to checking that

$$(\Omega^{F,g} * h'^*) \odot \pi^{F,\beta'*g} = (\pi^{F,\beta'} * \mathsf{F}_{L_{F,g}}) \odot (\Omega^{F,g} \circ h^*).$$

But this identity follows from the compatibility condition for $\Omega^{F,g}$. Lastly, let us check the composition and unit axioms for $2\text{-}\int_\varphi F$. Hence, let $j \xrightarrow{g} j' \xrightarrow{h} j'' \xrightarrow{k} j'''$ be three 1-cells of J; we need to prove that

$$L_{F,h\circ g,k} \odot (L_{F,k} * L_{F,g,h}) = L_{F,g,k\circ h} \odot (L_{F,h,k} * L_{F,g}) \qquad \text{and} \qquad L_{F,h,\mathbf{1}_{j'}} = \mathbf{1}_{L_{F,h}} = L_{F,\mathbf{1}_j,h}.$$

The two identities that express the unit axiom follow by a simple inspection. For the composition axiom, set $X := (\Omega^{F,k} * \mathsf{F}_{L_{F,h}} * \mathsf{F}_{L_{F,g}}) \odot ((\Omega^{F,h} \circ k^*) * \mathsf{F}_{L_{F,g}}) \odot (\Omega^{F,g} \circ h^* \circ k^*)$. It suffices to show that

$$Y := (\pi^{F,j'''} * (\mathsf{F}_{L_{F,h\circ g,k}} \odot (\mathsf{F}_{L_{F,k}} * \mathsf{F}_{L_{F,g,h}}))) \odot X = Z := (\pi^{F,j'''} * (\mathsf{F}_{L_{F,g,k\circ h}} \odot (\mathsf{F}_{L_{F,h,k}} * \mathsf{F}_{L_{F,g}}))) \odot X.$$

We compute :

$$\begin{aligned} Y &= (\pi^{F,j'''} * \mathsf{F}_{L_{F,h\circ g,k}}) \odot (\Omega^{F,k} * \mathsf{F}_{L_{F,h\circ g}}) \odot ((\pi^{F,j''} * k^*) * \mathsf{F}_{L_{F,g,h}}) \odot ((\Omega^{F,h} \circ k^*) * \mathsf{F}_{L_{F,g}}) \odot (\Omega^{F,g} \circ h^* \circ k^*) \\ &= (\pi^{F,j'''} * \mathsf{F}_{L_{F,h\circ g,k}}) \odot (\Omega^{F,k} * \mathsf{F}_{L_{F,h\circ g}}) \odot (\Omega^{F,h\circ g} \circ k^*) \\ &= \Omega^{F,k\circ h\circ g} \end{aligned}$$

and a similar calculation yields as well $Z = \Omega^{F,k\circ h\circ g}$: details left to the reader. □

Remark 2.6.7. In the situation of (2.6), suppose moreover that $\mathscr{C}$ is strongly 2-complete. In this case we can choose for $(L_{F,j}, \pi^{F,j})$ a strong 2-limit of $F \circ \mathsf{t}_j$, for every $j \in \mathrm{Ob}(J)$. Then, we may also choose $L_{F,g}$ for every 1-cell $g : j \to j'$ in J such that $\pi^{F,j} * g^* = \pi^{F,j'} \odot \mathsf{F}_{L_{F,g}}$, and take $\Omega^{F,g}$ to be the identity modification. With these choices, it follows easily that $L_{F,g,h}$ is an identity 2-cell, for every composable pair of 1-cells g, h of J. We conclude that, in this situation, the pseudo-functor $2\text{-}\int_\varphi F$ of lemma 2.6.6 shall be *strict*.

2.6.8. Keep the notation of (2.6), and let $F, F' : I \to \mathscr{C}$ be two unital pseudo-functors, and $\beta : F \Rightarrow F'$ a pseudo-natural transformation. Then, for every $j \in \mathrm{Ob}(J)$ there exist a 1-cell $L_{\beta,j} : L_{F,j} \to L_{F',j}$ and an invertible modification

$$\Omega^{\beta,j} : (\beta * \mathsf{t}_j) \odot \pi^{F,j} \rightsquigarrow \pi^{F',j} \odot \mathsf{F}_{L_{\beta,j}}.$$

Evidently, we take $L_{\mathbf{1}_F,j} := \mathbf{1}_{L_{F,j}}$ and $\Omega^{\mathbf{1}_F,j} := \mathbf{1}_{\pi^{F,j}}$ for every such F and j. If $g : j \to j'$ is any 1-cell of J, there exists a unique invertible 2-cell $L_{\beta,g} : L_{F',g} \circ L_{\beta,j} \Rightarrow L_{\beta,j'} \circ L_{F,g}$ such that

$$(2.6.9) \quad (\pi^{F',j'} * \mathsf{F}_{L_{\beta,g}}) \odot (\Omega^{F',g} * \mathsf{F}_{L_{\beta,j}}) \odot (\Omega^{\beta,j} \circ g^*) = (\Omega^{\beta,j'} * \mathsf{F}_{L_{F,g}}) \odot ((\beta * \mathsf{t}_{j'}) * \Omega^{F,g}).$$

A direct inspection shows that $L_{\mathbf{1}_F,g} = \mathbf{1}_{L_{F,g}}$ for every such F and g.

If $\beta' : F' \Rightarrow F''$ is another pseudo-natural transformation of unital pseudo-functors, there exists a unique invertible 2-cell $L_{\beta,\beta',j} : L_{\beta',j} \circ L_{\beta,j} \Rightarrow L_{\beta'\odot\beta,j}$ such that

$$(2.6.10) \qquad (\pi^{F'',j} * \mathsf{F}_{L_{\beta,\beta',j}}) \odot (\Omega^{\beta',j} * \mathsf{F}_{L_{\beta,j}}) \odot ((\beta' * \mathsf{t}_j) * \Omega^{\beta,j}) = \Omega^{\beta'\odot\beta,j}.$$

Lemma 2.6.11. *In the situation of* (2.6.8)*, the system of* 2*-cells* $(L_{\beta,g} \mid g : j \to j')$ *yields a coherence constraint for a pseudo-natural transformation*

$$2\text{-}\int_\varphi \beta : 2\text{-}\int_\varphi F \Rightarrow 2\text{-}\int_\varphi F' \qquad j \mapsto L_{\beta,j}.$$

Proof. Clearly $L_{\beta,\mathbf{1}_j} = \mathbf{1}_{L_{\beta,j}}$ for every $j \in \mathrm{Ob}(J)$. Next, let $j \xrightarrow{g} j' \xrightarrow{g'} j''$ be two 1-cells of J; we need to check the identity :

$$(L_{\beta,j''} * L_{F,g,g'}) \odot (L_{\beta,g'} * L_{F,g}) \odot (L_{F',g'} * L_{\beta,g}) = L_{\beta,g'\circ g} \odot (L_{F',g,g'} * L_{\beta,j}).$$

To this aim, set

$$\begin{aligned} X &:= (\Omega^{F',g'} * \mathsf{F}_{L_{F',g}} * \mathsf{F}_{L_{\beta,j}}) \odot ((\Omega^{F',g} \circ g'^*) * \mathsf{F}_{L_{\beta,j}}) \odot (\Omega^{\beta,j} \circ g^* \circ g'^*) \\ Y &:= \pi^{F',j''} * ((\mathsf{F}_{L_{\beta,j''}} * \mathsf{F}_{L_{F,g,g'}}) \odot (\mathsf{F}_{L_{\beta,g'}} * \mathsf{F}_{L_{F,g}})) \\ Z &:= Y \odot (\pi^{F',j''} * \mathsf{F}_{L_{F',g'}} * \mathsf{F}_{L_{\beta,g}}) \odot X \\ U &:= (\pi^{F',j''} * (\mathsf{F}_{L_{\beta,g'\circ g}} \odot (\mathsf{F}_{L_{F',g,g'}} * \mathsf{F}_{L_{\beta,j}}))) \odot X. \end{aligned}$$

It suffices to show that $U = Z$. We compute :

$$\begin{aligned} Z &= Y \odot (\Omega^{F',g'} * \mathsf{F}_{L_{\beta,j'}} * \mathsf{F}_{L_{F,g}}) \odot ((\pi^{F',j'} * g'^*) * \mathsf{F}_{L_{\beta,g}}) \odot ((\Omega^{F',g} \circ g'^*) * \mathsf{F}_{L_{\beta,j}}) \odot (\Omega^{\beta,j} \circ g^* \circ g'^*) \\ &= Y \odot (\Omega^{F',g'} * \mathsf{F}_{L_{\beta,j'}} * \mathsf{F}_{L_{F,g}}) \odot ((\Omega^{\beta,j'} \circ g'^*) * \mathsf{F}_{L_{F,g}}) \odot ((\beta * \mathsf{t}_{j''}) * (\Omega^{F,g} \circ g'^*)) \\ &= (\pi^{F',j''} * \mathsf{F}_{L_{\beta,j''} * L_{F,g,g'}}) \odot (\Omega^{\beta,j''} * \mathsf{F}_{L_{F,g'} * L_{F,g}}) \odot ((\beta * \mathsf{t}_{j''}) * \Omega^{F,g'} * \mathsf{F}_{L_{F,g}}) \odot ((\beta * \mathsf{t}_{j''}) * (\Omega^{F,g} \circ g'^*)) \\ &= (\Omega^{\beta,j''} * \mathsf{F}_{L_{F,g'\circ g}}) \odot ((\beta * \mathsf{t}_{j''}) * \pi^{F,j''} * \mathsf{F}_{L_{F,g,g'}}) \odot ((\beta * \mathsf{t}_{j''}) * \Omega^{F,g'} * \mathsf{F}_{L_{F,g}}) \odot ((\beta * \mathsf{t}_{j''}) * (\Omega^{F,g} \circ g'^*)) \\ &= (\Omega^{\beta,j''} * \mathsf{F}_{L_{F,g'\circ g}}) \odot ((\beta * \mathsf{t}_{j''}) * \Omega^{F,g'\circ g}) \end{aligned}$$

and on the other hand $U = (\pi^{F',j''} * \mathsf{F}_{L_{\beta,g'\circ g}}) \odot (\Omega^{F',g'\circ g} * \mathsf{F}_{L_{\beta,j}}) \odot (\Omega^{\beta,j} \circ g^* \circ g'^*)$, whence the contention. Lastly, we check the naturality condition for $L_{\beta,g}$: let $f, g : j \to j'$ be two 1-cells of J and $\alpha : f \Rightarrow g$ a 2-cell; we need to check the identity :

$$L_{\beta,g} \odot (L_{F',\alpha} * L_{\beta,j}) = (L_{\beta,j'} * L_{F,\alpha}) \odot L_{\beta,f}.$$

To this aim, set $X := (\Omega^{F',f} * \mathsf{F}_{L_{\beta,j}}) \odot (\Omega^{\beta,j} \circ f^*)$; it suffices to show that

$$Y' := (\pi^{F',j'} * (\mathsf{F}_{L_{\beta,g}} \odot (\mathsf{F}_{L_{F',\alpha}} * \mathsf{F}_{L_{\beta,j}}))) \odot X = Z' := (\pi^{F',j'} * ((\mathsf{F}_{L_{\beta,j'}} * \mathsf{F}_{L_{F,\alpha}}) \odot \mathsf{F}_{L_{\beta,f}})) \odot X.$$

Claim 2.6.12. $(\pi^{F',\alpha} * \mathsf{F}_{L_{\beta,j}}) \odot (\Omega^{\beta,j} \circ f^*) = (\Omega^{\beta,j} \circ g^*) \odot ((\beta * \mathsf{t}_{j'}) * \pi^{F,\alpha})$.

Proof of the claim. Recall that α induces a pseudo-natural transformation $\alpha^* : f^* \Rightarrow g^*$, and $\alpha^*(h) = (\mathbf{1}_i, \alpha) : h \circ f \to h \circ g$ for every $(h : j' \to \varphi(i)) \in \mathrm{Ob}(j'/\varphi I)$. The compatibility condition of $\Omega^{\beta,j}$ then yields a commutative diagram

$$\begin{array}{ccc} \beta_i \circ \pi^{F,j}_{f^*(h)} & \xrightarrow{\Omega^{\beta,j}_{f^*(h)}} & \pi^{F',j}_{f^*(h)} \circ L_{\beta,j} \\ {\scriptstyle \beta_i * \pi^{F,\alpha}_h} \Big\Downarrow & & \Big\Downarrow {\scriptstyle \pi^{F',\alpha}_h} \\ \beta_i \circ \pi^{F,j}_{g^*(h)} & \xrightarrow{\Omega^{\beta,j}_{g^*(h)}} & \pi^{F',j}_{g^*(h)} \circ L_{\beta,j}. \end{array}$$

The claim is an immediate consequence. ◊

Using claim 2.6.12, we may compute :

$$\begin{aligned} Y' &= (\pi^{F',j'} * \mathsf{F}_{L_{\beta,g}}) \odot (\Omega^{F',g} * \mathsf{F}_{L_{\beta,j}}) \odot (\pi^{F',\alpha} * \mathsf{F}_{L_{\beta,j}}) \odot (\Omega^{\beta,j} \circ f^*) \\ &= (\pi^{F',j'} * \mathsf{F}_{L_{\beta,g}}) \odot (\Omega^{F',g} * \mathsf{F}_{L_{\beta,j}}) \odot (\Omega^{\beta,j} \circ g^*) \odot ((\beta * \mathsf{t}_{j'}) * \pi^{F,\alpha}) \\ &= (\Omega^{\beta,j'} * \mathsf{F}_{L_{F,g}}) \odot ((\beta * \mathsf{t}_{j'}) * \Omega^{F,g}) \odot ((\beta * \mathsf{t}_{j'}) * \pi^{F,\alpha}) \\ &= (\Omega^{\beta,j'} * \mathsf{F}_{L_{F,g}}) \odot (\beta * \mathsf{t}_{j'} * ((\pi^{F,j'} * \mathsf{F}_{L_{F,\alpha}}) \odot \Omega^{F,f})) \end{aligned}$$

and on the other hand $Z' = (\pi^{F',j'} * (\mathsf{F}_{L_{\beta,j'}} * \mathsf{F}_{L_{F,\alpha}})) \odot (\Omega^{\beta,j'} * \mathsf{F}_{L_{F,f}}) \odot ((\beta * \mathsf{t}_{j'}) * \Omega^{F,f})$. So, we are reduced to proving that

$$(\Omega^{\beta,j'} * \mathsf{F}_{L_{F,g}}) \odot (\beta * \mathsf{t}_{j'} * \pi^{F,j'} * \mathsf{F}_{L_{F,\alpha}}) = (\pi^{F',j'} * (\mathsf{F}_{L_{\beta,j'}} * \mathsf{F}_{L_{F,\alpha}})) \odot (\Omega^{\beta,j'} * \mathsf{F}_{L_{F,f}}).$$

But the latter follows as usual from remark 2.1.1(i). □

Lemma 2.6.13. *In the situation of* (2.6.8)*, the system of* 2*-cells* $(L_{\beta,\beta',j} \mid j \in \mathrm{Ob}(J))$ *yields an invertible modification*

$$L_{\beta,\beta'} : 2\text{-}\!\int_\varphi \beta' \odot 2\text{-}\!\int_\varphi \beta \rightsquigarrow 2\text{-}\!\int_\varphi \beta' \odot \beta.$$

Proof. Let $g : j \to j'$ be any 1-cell of J; we need to show the identity

$$(L_{\beta,\beta',j'} * L_{F,g}) \odot (L_{\beta',j'} * L_{\beta,g}) \odot (L_{\beta',g} * L_{\beta,j}) = L_{\beta'\odot\beta,g} \odot (L_{F'',g} * L_{\beta,\beta',j}).$$

To this aim, let us set $X := ((\beta' * \mathsf{t}_{j'}) * \Omega^{\beta,j'} * \mathsf{F}_{L_{F,g}})^{-1} \odot (\Omega^{\beta',j'} * \mathsf{F}_{L_{\beta,j'}} * \mathsf{F}_{L_{F,g}})^{-1}$ and

$$\begin{aligned} Y_1 &:= X \odot (\pi^{F'',j'} * \mathsf{F}_{L_{\beta',j'}} * \mathsf{F}_{L_{\beta,g}}) \\ Y_2 &:= \pi^{F'',j'} * ((\mathsf{F}_{L_{\beta',g}} * \mathsf{F}_{L_{\beta,j}}) \odot (\mathsf{F}_{L_{F'',g}} * \mathsf{F}_{L_{\beta,\beta',j}})^{-1}) \\ Z &:= X \odot (\pi^{F'',j'} * ((\mathsf{F}_{L_{\beta,\beta',j'}} * \mathsf{F}_{L_{F,g}})^{-1} \odot \mathsf{F}_{L_{\beta'\odot\beta,g}})). \end{aligned}$$

It suffices to check that $Y := Y_1 \odot Y_2 = Z$. Set as well $U := (((\beta' \odot \beta) * \mathsf{t}_{j'}) * \Omega^{F,g})$; we compute :

$$\begin{aligned} Z &= (\Omega^{\beta'\odot\beta,j'} * \mathsf{F}_{L_{F,g}})^{-1} \odot (\pi^{F'',j'} * \mathsf{F}_{L_{\beta'\odot\beta,g}}) \\ &= U \odot (\Omega^{\beta'\odot\beta,j} \circ g^*)^{-1} \odot (\Omega^{F'',g} * \mathsf{F}_{L_{\beta'\odot\beta,j}})^{-1}. \end{aligned}$$

On the other hand, set $U' := (\Omega^{F'',g} * \mathsf{F}_{L_{\beta',j}})^{-1} \odot (\pi^{F'',j'} * \mathsf{F}_{L_{\beta',g}})^{-1}) * \mathsf{F}_{L_{\beta,j}}$; then

$$\begin{aligned} Y &= ((\beta' * \mathsf{t}_{j'}) * \Omega^{\beta,j'} * \mathsf{F}_{L_{F,g}})^{-1} \odot ((\beta' * \mathsf{t}_{j'}) * \pi^{F',j'} * \mathsf{F}_{L_{\beta,g}}) \odot (\Omega^{\beta',j'} * \mathsf{F}_{L_{F',g}} * \mathsf{F}_{L_{\beta,j}})^{-1} \odot Y_2 \\ &= U \odot ((\beta' * \mathsf{t}_{j'}) * ((\Omega^{\beta,j} \circ g^*)^{-1} \odot (\Omega^{F',g} * \mathsf{F}_{L_{\beta,j}})^{-1}) \odot (\Omega^{\beta',j'} * \mathsf{F}_{L_{F',g}} * \mathsf{F}_{L_{\beta,j}})^{-1} \odot Y_2 \\ &= U \odot ((\beta' * \mathsf{t}_{j'}) * (\Omega^{\beta,j} \circ g^*)^{-1}) \odot ((\Omega^{\beta',j} \circ g^*)^{-1} * \mathsf{F}_{L_{\beta,j}}) \odot U' \odot Y_2. \end{aligned}$$

Therefore, we are reduced to checking the identity :

$$((\pi^{F'',j} * g^*) * \mathsf{F}_{\beta,\beta',j})^{-1} \odot (\Omega^{F'',g} * \mathsf{F}_{L_{\beta'\odot\beta,j}})^{-1} = U' \odot Y_2.$$

However, we have :

$$((\pi^{F'',j} * g^*) * \mathsf{F}_{\beta,\beta',j})^{-1} \odot (\Omega^{F'',g} * \mathsf{F}_{L_{\beta'\odot\beta,j}})^{-1} = (\Omega^{F'',g} * \mathsf{F}_{L_{\beta',j} * L_{\beta,j}})^{-1} \odot (\pi^{F'',j'} * \mathsf{F}_{L_{F'',g}} * \mathsf{F}_{\beta,\beta',j})^{-1}$$

so we are further reduced to showing that :

$$(\pi^{F'',j'} * \mathsf{F}_{L_{F'',g}} * \mathsf{F}_{\beta,\beta',j})^{-1} = (\pi^{F'',j'} * \mathsf{F}_{L_{\beta',g}} * \mathsf{F}_{L_{\beta,j}})^{-1} \odot Y_2$$

which is obvious. □

2.6.14. Let $F, F' : I \to \mathscr{C}$ be two unital pseudo-functors, $\beta, \beta' : F \Rightarrow F'$ two pseudo-natural transformations, and $\Xi : \beta \rightsquigarrow \beta'$ a modification. Then for every $j \in \mathrm{Ob}(J)$ there exists a unique modification $L_{\Xi,j} : L_{\beta,j} \rightsquigarrow L_{\beta',j}$ such that

$$\Omega^{\beta',j} \odot ((\Xi \circ \mathsf{t}_j) * \pi^{F,j}) = (\pi^{F',j} * \mathsf{F}_{L_{\Xi,j}}) \odot \Omega^{\beta,j}.$$

Lemma 2.6.15. *In the situation of* (2.6.14)*, the rule* $j \mapsto L_{\Xi,j}$ *defines a modification*

$$2\text{-}\!\int_\varphi \Xi : 2\text{-}\!\int_\varphi \beta \rightsquigarrow 2\text{-}\!\int_\varphi \beta'.$$

Proof. Let $g : j \to j'$ be any 1-cell of J; we need to check the identity :

$$(L_{\Xi,j'} * L_{F,g}) \odot L_{\beta,g} = L_{\beta',g} \odot (L_{F',g} * L_{\Xi,j})$$

and as usual, it suffices to show that

$$Y := \pi^{F',j'} * ((\mathsf{F}_{L_{\Xi,j'}} * \mathsf{F}_{L_{F,g}}) \odot \mathsf{F}_{L_{\beta,g}}) = Z := \pi^{F',j'} * (\mathsf{F}_{L_{\beta',g}} \odot (\mathsf{F}_{L_{F',g}} * \mathsf{F}_{L_{\Xi,j}})).$$

We compute :

$$\begin{aligned}
Y &= ((\Omega^{\beta',j'} \odot ((\Xi \circ \mathsf{t}_{j'}) * \pi^{F,j'}) \odot (\Omega^{\beta,j'})^{-1}) * \mathsf{F}_{L_{F,g}}) \odot (\pi^{F',j'} * \mathsf{F}_{L_{\beta,g}}) \\
&= ((\Omega^{\beta',j'} \odot ((\Xi \circ \mathsf{t}_{j'}) * \pi^{F,j'})) * \mathsf{F}_{L_{F,g}}) \odot ((\beta * \mathsf{t}_{j'}) * \Omega^{F,g}) \odot (\Omega^{\beta,j} \circ g^*)^{-1} \odot (\Omega^{F',g} * \mathsf{F}_{L_{\beta,j}})^{-1} \\
&= (\Omega^{\beta',j'} * \mathsf{F}_{L_{F,g}}) \odot ((\beta' * \mathsf{t}_{j'}) * \Omega^{F,g}) \odot ((\Xi \circ \mathsf{t}_{j'}) * (\pi^{F,j} * g^*)) \odot (\Omega^{\beta,j} \circ g^*)^{-1} \odot (\Omega^{F',g} * \mathsf{F}_{L_{\beta,j}})^{-1} \\
&= (\Omega^{\beta',j'} * \mathsf{F}_{L_{F,g}}) \odot ((\beta' * \mathsf{t}_{j'}) * \Omega^{F,g}) \odot (\Omega^{\beta',j} \circ g^*)^{-1} \odot ((\pi^{F',j} * g^*) * \mathsf{F}_{L_{\Xi,j}}) \odot (\Omega^{F',g} * \mathsf{F}_{L_{\beta,j}})^{-1} \\
&= (\pi^{F',j'} * \mathsf{F}_{L_{\beta',g}}) \odot (\Omega^{F',g} * \mathsf{F}_{L_{\beta',j}}) \odot ((\pi^{F',j} * g^*) * \mathsf{F}_{L_{\Xi,j}}) \odot (\Omega^{F',g} * \mathsf{F}_{L_{\beta,j}})^{-1} \\
&= (\pi^{F',j'} * \mathsf{F}_{L_{\beta',g}}) \odot (\Omega^{F',g} * \mathsf{F}_{L_{\beta',j}}) \odot (\Omega^{F',g} * \mathsf{F}_{\beta',j})^{-1} \odot (\pi^{F',j'} * \mathsf{F}_{L_{F,g}} * \mathsf{F}_{\Xi,j}) \\
&= (\pi^{F',j'} * \mathsf{F}_{L_{\beta',g}}) \odot (\pi^{F',j'} * \mathsf{F}_{L_{F,g}} * \mathsf{F}_{\Xi,j})
\end{aligned}$$

whence the contention. □

Proposition 2.6.16. *Let $I, J, \mathscr{C}$ be as in* (2.6)*, and $\varphi : I \to J$ any pseudo-functor. The rules:*

$$F \mapsto 2\text{-}\int_{\varphi^u} F^u \qquad \beta \mapsto 2\text{-}\int_{\varphi^u} \beta^u \qquad \Xi \mapsto 2\text{-}\int_{\varphi^u} \Xi^u$$

for every pseudo-functor $F : I \to \mathscr{C}$, every pseudo-natural transformation β between pseudo-functors $F, F' : I \to \mathscr{C}$, and every modification Ξ of such pseudo-natural transformations (notation of remark 2.4.4(i)*) define a unital pseudo-functor*

$$2\text{-}\int_{\varphi} : \mathsf{PsFun}(I, \mathscr{C}) \to \mathsf{PsFun}(J, \mathscr{C})$$

with coherence constraint given by the system of invertible modifications $L_{\beta,\beta'}$ of lemma 2.6.13*. We call this pseudo-functor the* right 2-Kan extension along φ.

Proof. The sought functor shall be the composition of the strict 2-equivalence of remark 2.4.4(i) and a similar pseudo-functor $\mathsf{uniPsFun}(I, \mathscr{C}) \to \mathsf{uniPsFun}(J, \mathscr{C})$ given by the foregoing rules; we are therefore reduced to prove the existence of the latter. Now, a direct inspection shows that $L_{\mathbf{1}_F,\beta,j} = \mathbf{1}_{L_{\beta,j}} = L_{\beta,\mathbf{1}_{F'},j}$ for every pair of unital pseudo-functors $F, F' : I \to \mathscr{C}$, every $\beta : F \Rightarrow F'$ and every $j \in \mathrm{Ob}(J)$, whence the unit axiom for the system of modifications $L_{\beta,\beta'}$. In order to check the composition axiom, we need to show the identity :

$$L_{\beta' \odot \beta, \beta'', j} \odot (L_{\beta'',j} * L_{\beta,\beta',j}) = L_{\beta, \beta'' \odot \beta', j} \odot (L_{\beta',\beta'',j} * L_{\beta,j})$$

for every $j \in \mathrm{Ob}(J)$ and every three pseudo-natural transformations of unital pseudo-functors $\beta : F \Rightarrow F'$, $\beta' : F' \Rightarrow F''$, $\beta'' : F'' \Rightarrow F'''$. To this aim, set $X := \Omega^{\beta'' \odot \beta' \odot \beta, j}$, as well as :

$$\begin{aligned}
Y &:= (\pi^{F''',j} * ((\mathsf{F}_{L_{\beta'',j}} * \mathsf{F}_{L_{\beta,\beta',j}})^{-1} \odot \mathsf{F}^{-1}_{L_{\beta' \odot \beta, \beta'', j}})) \odot X \\
Z &:= (\pi^{F''',j} * ((\mathsf{F}_{L_{\beta',\beta'',j}} * \mathsf{F}_{L_{\beta,j}})^{-1} \odot \mathsf{F}^{-1}_{L_{\beta, \beta'' \odot \beta', j}})) \odot X.
\end{aligned}$$

It suffices to prove that $Y = Z$. We compute :

$$\begin{aligned}
Y &= (\pi^{F''',j} * \mathsf{F}_{L_{\beta'',j}} * \mathsf{F}_{L_{\beta,\beta',j}})^{-1} \odot (\Omega^{\beta'',j} * \mathsf{F}_{L_{\beta' \odot \beta, j}}) \odot ((\beta'' * \mathsf{t}_j) * \Omega^{\beta' \odot \beta, j}) \\
&= (\Omega^{\beta'',j} * \mathsf{F}_{L_{\beta',j}} * \mathsf{F}_{L_{\beta,j}}) \odot ((\beta'' * \mathsf{t}_j) * \pi^{F'',j} * \mathsf{F}_{L_{\beta,\beta',j}})^{-1} \odot ((\beta'' * \mathsf{t}_j) * \Omega^{\beta' \odot \beta, j}) \\
&= (\Omega^{\beta'',j} * \mathsf{F}_{L_{\beta',j}} * \mathsf{F}_{L_{\beta,j}}) \odot ((\beta'' * \mathsf{t}_j) * ((\Omega^{\beta',j} * \mathsf{F}_{L_{\beta,j}}) \odot ((\beta' * \mathsf{t}_j) * \Omega^{\beta,j})))
\end{aligned}$$

and likewise :

$$\begin{aligned} Z &= (\pi^{F''',j} * \mathsf{F}_{L_{\beta',\beta'',j}} * \mathsf{F}_{L_{\beta,j}})^{-1} \odot (\Omega^{\beta''\odot\beta',j} * \mathsf{F}_{L_{\beta,j}}) \odot ((\beta'' \odot \beta' * \mathsf{t}_j) * \Omega^{\beta,j}) \\ &= (((\Omega^{\beta'',j} * \mathsf{F}_{L_{\beta',j}}) \odot ((\beta'' * \mathsf{t}_j) * \Omega^{\beta',j})) * \mathsf{F}_{L_{\beta,j}}) \odot ((\beta'' \odot \beta' * \mathsf{t}_j) * \Omega^{\beta,j}) \end{aligned}$$

whence the contention. Next, consider three pseudo-natural transformations $\beta, \beta', \beta'' : F \Rightarrow F'$ and two modifications $\Xi : \beta \rightsquigarrow \beta'$ and $\Xi' : \beta' \rightsquigarrow \beta''$. For every $j \in \mathrm{Ob}(J)$ we have :

$$\Omega^{\beta'',j} \odot (((\Xi' \odot \Xi) \circ \mathsf{t}_j) * \pi^{F,j}) = (\pi^{F',j} * \mathsf{F}_{L_{\Xi',j}}) \odot \Omega^{\beta',j} \odot ((\Xi \circ \mathsf{t}_j) * \pi^{F,j}) = (\pi^{F',j} * \mathsf{F}_{L_{\Xi',j}} * \mathsf{F}_{L_{\Xi,j}}) \odot \Omega^{\beta,j}$$

whence $L_{\Xi'\odot\Xi,j} = L_{\Xi',j} \odot L_{\Xi,j}$. This shows that $2\text{-}\int_\varphi(\Xi' \odot \Xi) = 2\text{-}\int_\varphi \Xi' \odot 2\text{-}\int_\varphi \Xi$. Moreover, a simple inspection shows that $L_{\mathbf{1}_\beta,j} = \mathbf{1}_{L_{\beta,j}}$ for every pseudo-natural transformation β and every $j \in \mathrm{Ob}(J)$, so $2\text{-}\int_\varphi(\mathbf{1}_\beta)$ is the identity modification of $2\text{-}\int_\varphi \beta$.

Lastly, let $F, F', F'' : I \to \mathscr{C}$ be three unital pseudo-functors, $\beta, \beta : F \Rightarrow F'$ and $\alpha, \alpha' : F' \Rightarrow F''$ four pseudo-natural transformations, and $\Xi : \beta \rightsquigarrow \beta'$, $\Theta : \alpha \rightsquigarrow \alpha'$ two modifications; we need to show that

$$L_{\Theta*\Xi,j} \odot L_{\beta,\alpha,j} = L_{\beta',\alpha',j} \odot (L_{\Theta,j} * L_{\Xi,j}) \qquad \text{for every } j \in \mathrm{Ob}(J)$$

and by the foregoing, we may assume that either $\Xi = \mathbf{1}_\beta$ or $\Theta = \mathbf{1}_\alpha$. Suppose first that $\Theta = \mathbf{1}_\alpha$, and let $X := (\Omega^{\alpha,j} * \mathsf{F}_{\beta,j}) \odot ((\alpha * \mathsf{t}_j) * \Omega^{\beta,j})$; it suffices to prove that

$$Y := (\pi^{F'',j} * (\mathsf{F}_{L_{\alpha*\Xi,j}} \odot \mathsf{F}_{L_{\beta,\alpha,j}})) \odot X = Z := (\pi^{F'',j} * (\mathsf{F}_{L_{\beta',\alpha,j}} \odot (\mathsf{F}_{L_{\alpha,j}} * \mathsf{F}_{L_{\Xi,j}}))) \odot X.$$

We compute :

$$\begin{aligned} Z &= (\pi^{F'',j} * \mathsf{F}_{L_{\beta',\alpha,j}}) \odot (\Omega^{\alpha,j} * \mathsf{F}_{\beta',j}) \odot ((\alpha * \mathsf{t}_j) * \pi^{F',j} * \mathsf{F}_{L_{\Xi,j}}) \odot ((\alpha * \mathsf{t}_j) * \Omega^{\beta,j}) \\ &= (\pi^{F'',j} * \mathsf{F}_{L_{\beta',\alpha,j}}) \odot (\Omega^{\alpha,j} * \mathsf{F}_{\beta',j}) \odot ((\alpha * \mathsf{t}_j) * (\Omega^{\beta',j} \odot ((\Xi \circ \mathsf{t}_j) * \pi^{F,j}))) \\ &= \Omega^{\alpha\odot\beta',j} \odot ((\alpha * \mathsf{t}_j) * ((\Xi \circ \mathsf{t}_j) * \pi^{F,j}))) \end{aligned}$$

and on the other hand, $Y = (\pi^{F'',j} * \mathsf{F}_{L_{\alpha*\Xi,j}}) \odot \Omega^{\alpha\odot\beta}$, whence the contention. A similar calculation, left to the reader, settles the case where $\Xi = \mathbf{1}_\beta$, and concludes the proof. □

2.6.17. On the other hand, if $\mathscr{C}$ is 2-cocomplete, then $\mathscr{C}^o$ is 2-complete (remark 2.5.2(vi)), and in view of the strict isomorphisms of (2.2.13), we get a *left 2-Kan extension along* φ :

$$2\text{-}\int^\varphi : \mathsf{PsFun}(I, \mathscr{C}) \to \mathsf{PsFun}(J, \mathscr{C}) \qquad F \mapsto \Big(2\text{-}\int_{\varphi^o} F^o\Big)^o.$$

Explicitly, for every $j \in \mathrm{Ob}(J)$, the object $(2\text{-}\int^\varphi F)(j)$ represents the 2-colimit of the pseudo-functor $F \circ {}^o\mathsf{s}_{{}^oj} : {}^o({}^o\varphi({}^oI)/{}^oj) \to \mathscr{C}$.

Remark 2.6.18. (i) In the situation of (2.6.8), suppose that $\mathscr{C}$ is strongly 2-complete, and choose strong 2-limits $(L_{F,j}, \pi^{F,j})$, identity modifications $\Omega^{F,g}$, and 1-cells $L_{F,g}$ as in remark 2.6.7. Then we may further choose $L_{\beta,j}$ such that $(\beta * \mathsf{t}_j) \odot \pi^{F,j} = \pi^{F',j} \odot \mathsf{F}_{L_{\beta,j}}$, and let $\Omega^{\beta,j}$ as well be the corresponding identity modification. With such choices, it follows easily that $L_{\beta,g}$ shall be an identity 2-cell, for every 1-cell g of J, and the same for $L_{\beta,\beta'}$. Therefore, in this situation the pseudo-natural transformation $2\text{-}\int_\varphi \beta$ of lemma 2.6.11 shall be *strict* as well, and moreover $(2\text{-}\int_\varphi \beta') \odot (2\text{-}\int_\varphi \beta) = 2\text{-}\int_\varphi(\beta' \odot \beta)$.

(ii) We conclude that, with these choices, the pseudo-functor $2\text{-}\int_\varphi$ of proposition 2.6.16 factors through a *strict* pseudo-functor called the *strong right 2-Kan extension along* φ :

$$\mathsf{PsFun}(I, \mathscr{C}) \to \mathsf{stPsFun}(J, \mathscr{C})$$

(notation of definition 2.2.12(iii)). Likewise, if $\mathscr{C}$ is strongly 2-cocomplete, we obtain as in (2.6.17), a *strong left 2-Kan extension along* φ which is another strict pseudo-functors with values in $\mathsf{stPsFun}(J, \mathscr{C})$, namely the opposite of the strong right 2-Kan extension along φ^o.

2.6.19. Keep the notation of (2.6), and suppose now that *all the* 2*-cells of* J *are invertible*. Under this assumption, we easily see that the rule : $(i, f : j \to \varphi(i)) \mapsto f$ yields a pseudo-cone

$$\varphi_j^* : \mathsf{F}_j \Rightarrow \varphi \circ \mathsf{t}_j \qquad \text{for every } j \in \mathrm{Ob}(J)$$

whose coherence constraint is given by the rule : $(h, \alpha) \mapsto \alpha$ (notation of example 2.2.8(ii); notice that this would not be a coherence constraint, if α were not invertible : details left to the reader). Moreover, we have :

$$\varphi_{j'}^* \odot \mathsf{F}_g = \varphi_j^* * g^* \qquad \text{for every 1-cell } g : j \to j' \text{ of } J. \tag{2.6.20}$$

Furthermore, for every pair of 1-cells $g, g' : j \to j'$ and every 2-cell $\alpha : g \Rightarrow g'$ we get a modification

$$\varphi_j^* * \alpha^* : \varphi_j^* * g^* \rightsquigarrow \varphi_j^* * g'^* \qquad (f : j' \to \varphi(i)) \mapsto (\alpha * f : g^*(f) \Rightarrow g'^*(f)).$$

Let $F : I \to \mathscr{C}$ and $G : J \to \mathscr{C}$ be two *unital* pseudo-functors, and $\beta : G \circ \varphi \Rightarrow F$ a pseudo-natural transformation; there exist a 1-cell $\beta_j^\dagger : Gj \to L_{F,j}$ and an invertible modification

$$\Theta^{\beta,j} : (\beta * \mathsf{t}_j) \odot (G * \varphi_j^*) \rightsquigarrow \pi^{F,j} \odot \mathsf{F}_{\beta_j^\dagger} \qquad \text{for every } j \in \mathrm{Ob}(J).$$

Next, let $g : j \to j'$ be any 1-cell of J, and notice that $G * \varphi_j^* * g^* = G * (\varphi_{j'}^* \odot \mathsf{F}_g)$, due to (2.6.20); according to example 2.2.15(i) the coherence constraint γ^G of G induces an invertible modification

$$\Gamma^{G,g} : (G * \varphi_{j'}^*) \odot \mathsf{F}_{Gg} \rightsquigarrow G * \varphi_j^* * g^* \qquad (f : j' \to \varphi(i)) \mapsto \gamma_{g,f}^G.$$

Then there exists a unique invertible 2-cell $\tau_g^{\beta^\dagger} : L_{F,g} \circ \beta_j^\dagger \Rightarrow \beta_{j'}^\dagger \circ Gg$ such that :

$$(\pi^{F,j'} * \mathsf{F}_{\tau_g^{\beta^\dagger}}) \odot (\Omega^{F,g} * \mathsf{F}_{\beta_j^\dagger}) \odot (\Theta^{\beta,j} \circ g^*) \odot ((\beta * \mathsf{t}_{j'}) * \Gamma^{G,g}) = \Theta^{\beta,j'} * \mathsf{F}_{Gg}. \tag{2.6.21}$$

Lemma 2.6.22. *With the notation of* (2.6.19)*, the rule :* $j \mapsto \beta_j^\dagger$ *yields a pseudo-natural transformation*

$$\beta^\dagger : G \Rightarrow 2\text{-}\!\int_\varphi F$$

with coherence constraint given by the system of 2*-cells* $\tau_\bullet^{\beta^\dagger}$.

Proof. A simple inspection shows that $\tau_{\mathbf{1}_j}^{\beta^\dagger} = \mathbf{1}_{\beta_j^\dagger}$ for every $j \in \mathrm{Ob}(J)$. Next, let $g : j \to g'$ and $g' : j' \to j''$ be two 1-cells of J; we have to check the identity :

$$(\beta_{j''}^\dagger * \gamma_{g,g'}^G) \odot (\tau_{g'}^{\beta^\dagger} * Gg) \odot (L_{F,g'} * \tau_g^{\beta^\dagger}) = \tau_{g' \circ g}^{\beta^\dagger} \odot (L_{F,g,g'} * \beta_j^\dagger).$$

To this aim, set

$$\begin{aligned}
X &:= (\Theta^{\beta,j} \circ g^* \circ g'^*) \odot ((\beta * \mathsf{t}_{j''}) * (\Gamma^{G,g} \circ g'^*)) \odot ((\beta * \mathsf{t}_{j''}) * \Gamma^{G,g'} * \mathsf{F}_{Gg}) \\
X' &:= (\Omega^{F,g'} * \mathsf{F}_{L_{F,g}} * \mathsf{F}_{\beta_j^\dagger}) \odot ((\Omega^{F,g} \circ g'^*) * \mathsf{F}_{\beta_j^\dagger}) \odot X \\
Y &:= (\pi^{F,j''} * ((\mathsf{F}_{\beta_{j''}^\dagger} * \mathsf{F}_{\gamma_{g,g'}^G}) \odot (\mathsf{F}_{\tau_{g'}^{\beta^\dagger}} * \mathsf{F}_{Gg}) \odot (\mathsf{F}_{L_{F,g'}} * \mathsf{F}_{\tau_g^{\beta^\dagger}}))) \odot X' \\
Z &:= (\pi^{F,j''} * (\mathsf{F}_{\tau_{g' \circ g}^{\beta^\dagger}} \odot (\mathsf{F}_{L_{F,g,g'}} * \mathsf{F}_{\beta_j^\dagger}))) \odot X'.
\end{aligned}$$

It suffices to show that $Y = Z$. Set as well $Y' := \pi^{F,j''} * ((\mathsf{F}_{\beta^\dagger_{j''}} * \mathsf{F}_{\gamma^G_{g,g'}}) \odot (\mathsf{F}_{\tau^{\beta\dagger}_{g'}} * \mathsf{F}_{Gg}))$; we compute :

$$\begin{aligned} Y &= Y' \odot (\Omega^{F,g'} * \mathsf{F}_{\beta^\dagger_{j'}} * \mathsf{F}_{Gg}) \odot ((\pi^{F,j'} * g'^*) * \mathsf{F}_{\tau^{\beta\dagger}_{g}}) \odot ((\Omega^{F,g} \circ g'^*) * \mathsf{F}_{\beta^\dagger_j}) \odot X' \\ &= Y' \odot (\Omega^{F,g'} * \mathsf{F}_{\beta^\dagger_{j'}} * \mathsf{F}_{Gg}) \odot ((\Theta^{\beta,j'} \circ g'^*) * \mathsf{F}_{Gg}) \odot ((\beta * \mathsf{t}_{j''}) * \Gamma^{G,g'} * \mathsf{F}_{Gg}) \\ &= (\pi^{F,j''} * \mathsf{F}_{\beta^\dagger_{j''}} * \mathsf{F}_{\gamma^G_{g,g'}}) \odot (\Theta^{\beta,j''} * \mathsf{F}_{Gg'} * \mathsf{F}_{Gg}) \\ &= (\Theta^{\beta,j''} * \mathsf{F}_{G(g'\circ g)}) \odot ((\beta * \mathsf{t}_{j''}) * (G * \varphi^*_{j''}) * \mathsf{F}_{\gamma^G_{g,g'}}) \end{aligned}$$

whereas : $Z = (\pi^{F,j''} * \mathsf{F}_{\tau^{\beta\dagger}_{g'\circ g}}) \odot (\Omega^{F,g'\circ g} * \mathsf{F}_{\beta^\dagger_j}) \odot X$. So, we are reduced to checking that

$$(\Gamma^{G,g} \circ g'^*) \odot (\Gamma^{G,g'} * \mathsf{F}_{Gg}) = \Gamma^{G,g'\circ g} \odot ((G * \varphi^*_{j''}) * \mathsf{F}_{\gamma^G_{g,g'}}).$$

The latter follows from the composition axiom for γ^G. Lastly, the naturality of $\tau^{\beta^\dagger}$ amounts to the identity :

$$\tau^{\beta\dagger}_{g'} \odot (L_{F,\alpha} * \beta^\dagger_j) = (\beta^\dagger_{j'} * G\alpha) \odot \tau^{\beta\dagger}_g \qquad \text{for every 2-cell } \alpha : g \Rightarrow g' \text{ of } J.$$

To prove the latter, set $X := (\Omega^{F,g} * \mathsf{F}_{\beta^\dagger_j}) \odot (\Theta^{\beta,j} \circ g^*) \odot ((\beta * \mathsf{t}_{j'}) * \Gamma^{G,g})$; it suffices to show:

$$Y := (\pi^{F,j'} * (\mathsf{F}_{\tau^{\beta\dagger}_{g'}} \odot (\mathsf{F}_{L_{F,\alpha}} * \mathsf{F}_{\beta^\dagger_j}))) \odot X = Z := (\pi^{F,j'} * ((\mathsf{F}_{\beta^\dagger_{j'}} * \mathsf{F}_{G\alpha}) \odot \mathsf{F}_{\tau^{\beta\dagger}_g})) \odot X.$$

To this aim, we remark :

Claim 2.6.23. $(\pi^{F,\alpha} * \mathsf{F}_{\beta^\dagger_j}) \odot (\Theta^{\beta,j} \circ g^*) = (\Theta^{\beta,j} \circ g'^*) \odot ((\beta * \mathsf{t}_{j'}) * (G \circ (\varphi^*_j * \alpha^*))$.

Proof of the claim. We have $\alpha^*(f) = (\mathbf{1}_i, \alpha) : g \circ f \to g' \circ f$ for every $(f : j' \to \varphi(i)) \in \mathrm{Ob}(j'/\varphi I)$. Now, let $\tau^{G*\varphi^*_j}$ be the coherence constraint of $G * \varphi^*_j$; since φ is unital, it is easily seen that $\tau^{G*\varphi^*_j}_{\alpha^*(f)} = G(f * \alpha)$ (details left to the reader). Then the compatibility condition of $\Theta^{\beta,j}$ yields the commutative diagram :

$$\begin{array}{ccc} \beta_i \circ G(f \circ g) & \xrightarrow{\ \Theta^{\beta,j}_{g^*(f)}\ } & \pi^{F,j}_{g^*(f)} \circ \beta^\dagger_j \\ {\scriptstyle \beta_i * G(f*\alpha)} \Downarrow & & \Downarrow {\scriptstyle \pi^{F,\alpha}} \\ \beta_i \circ G(f \circ g') & \xrightarrow{\ \Theta^{\beta,j}_{g'^*(f)}\ } & \pi^{F,j}_{g'^*(f)} \odot \beta^\dagger_j \end{array}$$

whence the claim. ◊

Using claim 2.6.23 we compute :

$$\begin{aligned} Y &= (\pi^{F,j'} * \mathsf{F}_{\tau^{\beta\dagger}_{g'}}) \odot ((\Omega^{F,g'} \odot \pi^{F,\alpha}) * \mathsf{F}_{\beta^\dagger_j}) \odot (\Theta^{\beta,j} \circ g^*) \odot ((\beta * \mathsf{t}_{j'}) * \Gamma^{G,g}) \\ &= (\pi^{F,j'} * \mathsf{F}_{\tau^{\beta\dagger}_{g'}}) \odot (\Omega^{F,g'} * \mathsf{F}_{\beta^\dagger_j}) \odot (\Theta^{\beta,j} \circ g'^*) \odot ((\beta * \mathsf{t}_{j'}) * (G \circ (\varphi^*_j * \alpha^*)) \odot ((\beta * \mathsf{t}_{j'}) * \Gamma^{G,g}) \\ &= (\Theta^{\beta,j'} * \mathsf{F}_{Gg'}) \odot ((\beta * \mathsf{t}_{j'}) * \Gamma^{G,g'})^{-1} \odot ((\beta * \mathsf{t}_{j'}) * (G \circ (\varphi^*_j * \alpha^*)) \odot ((\beta * \mathsf{t}_{j'}) * \Gamma^{G,g}) \end{aligned}$$

and on the other hand

$$Z = (\pi^{F,j'} * \mathsf{F}_{\beta^\dagger_{j'}} * \mathsf{F}_{G\alpha}) \odot (\Theta^{\beta,j'} * \mathsf{F}_{Gg}) = (\Theta^{\beta,j'} * \mathsf{F}_{Gg'}) \odot ((\beta * \mathsf{t}_{j'}) * (G * \varphi^*_{j'}) * \mathsf{F}_{G\alpha})$$

so we are reduced to showing that :

$$(G \circ (\varphi^*_j * \alpha^*)) \odot \Gamma^{G,g} = \Gamma^{G,g'} \odot ((G * \varphi^*_{j'}) * \mathsf{F}_{G\alpha}).$$

But the latter follows from the naturality condition for γ^G. □

Remark 2.6.24. In the situation of (2.6.19), suppose that $\mathscr{C}$ is strongly 2-complete, so that we may take for 2-$\int_\varphi$ the strong 2-right Kan extension of remark 2.6.18(ii). Then we may also choose $\beta^\dagger_j$ such that $(\beta * \mathsf{t}_j) \odot (G * \varphi^*_j) = \pi^{F,j} \odot \mathsf{F}_{\beta^\dagger_j}$, and we may let $\Theta^{\beta,j}$ be the corresponding identity modification. Suppose now additionally that G is a strict pseudo-functor; by inspecting (2.6.21), we deduce that with such choices, $\beta^\dagger$ is then a *strict* pseudo-natural transformation.

2.6.25. Keep the notation of (2.6.19), and let $\beta, \beta' : G \circ \varphi \Rightarrow F$ be two pseudo-natural transformations and $\Xi : \beta \rightsquigarrow \beta'$ a modification. Then there exists for every $j \in \mathrm{Ob}(J)$ a unique 2-cell $\Xi^\dagger_j : \beta^\dagger_j \Rightarrow \beta'^\dagger_j$ such that

$$(2.6.26) \qquad (\pi^{F,j} * \mathsf{F}_{\Xi^\dagger_j}) \odot \Theta^{\beta,j} = \Theta^{\beta',j} \odot ((\Xi \circ \mathsf{t}_j) * (G * \varphi^*_j)).$$

Lemma 2.6.27. *In the situation of* (2.6.25)*, the rule :* $j \mapsto \Xi^\dagger_j$ *defines a modification*

$$\Xi^\dagger : \beta^\dagger \rightsquigarrow \beta'^\dagger.$$

Proof. Let $g : j \to j'$ be any 1-cell of J; we need to show the identity :

$$\tau^{\beta'^\dagger}_g \odot (L_{F,g} * \Xi^\dagger_j) = (\Xi^\dagger_{j'} * Gg) \odot \tau^{\beta^\dagger}_g.$$

To this aim, set $X := (\Omega^{F,g} * \mathsf{F}_{\beta^\dagger_j}) \odot (\Theta^{\beta,j} \circ g^*) \odot ((\beta * \mathsf{t}_{j'}) * \Gamma^{G,g})$; it suffices to show that

$$Y := (\pi^{F,j'} * (\mathsf{F}_{\tau^{\beta'\dagger}_g} \odot (\mathsf{F}_{L_{F,g}} * \mathsf{F}_{\Xi^\dagger_j}))) \odot X = Z := (\pi^{F,j'} * ((\mathsf{F}_{\Xi^\dagger_{j'}} * \mathsf{F}_{Gg}) \odot \mathsf{F}_{\tau^{\beta\dagger}_g})) \odot X.$$

We compute :

$$\begin{aligned}
Y &= (\pi^{F,j'} * \mathsf{F}_{\tau^{\beta'\dagger}_g}) \odot (\Omega^{F,g} * \mathsf{F}_{\beta'^\dagger_j}) \odot ((\pi^{F,j} * g^*) * \mathsf{F}_{\Xi^\dagger_j}) \odot (\Theta^{\beta,j} \circ g^*) \odot ((\beta * \mathsf{t}_{j'}) * \Gamma^{G,g}) \\
&= (\pi^{F,j'} * \mathsf{F}_{\tau^{\beta'\dagger}_g}) \odot (\Omega^{F,g} * \mathsf{F}_{\beta'^\dagger_j}) \odot (\Theta^{\beta',j} \circ g^*) \odot ((\Xi \circ \mathsf{t}_{j'}) * (G * \varphi^*_j * g^*)) \odot ((\beta * \mathsf{t}_{j'}) * \Gamma^{G,g}) \\
&= (\pi^{F,j'} * \mathsf{F}_{\tau^{\beta'\dagger}_g}) \odot (\Omega^{F,g} * \mathsf{F}_{\beta'^\dagger_j}) \odot (\Theta^{\beta',j} \circ g^*) \odot ((\beta' * \mathsf{t}_{j'}) * \Gamma^{G,g}) \odot ((\Xi \circ \mathsf{t}_{j'}) * (G * \varphi^*_{j'}) * \mathsf{F}_{Gg}) \\
&= (\Theta^{\beta',j'} * \mathsf{F}_{Gg}) \odot ((\Xi \circ \mathsf{t}_{j'}) * (G * \varphi^*_{j'}) * \mathsf{F}_{Gg}).
\end{aligned}$$

whereas : $Z = (\pi^{F,j'} * \mathsf{F}_{\Xi^\dagger_{j'}} * \mathsf{F}_{Gg}) \odot (\Theta^{\beta,j'} * \mathsf{F}_{Gg})$, whence the contention. □

Proposition 2.6.28. *With the notation of lemmata* 2.6.22 *and* 2.6.27*, the rules :* $\beta \mapsto \beta^\dagger$ *and* $\Xi \mapsto \Xi^\dagger$ *define a functor*

$$(-)^\dagger_{F,G} : \mathsf{PsNat}(G \circ \varphi, F) \to \mathsf{PsNat}\Big(G, 2\text{-}\int_\varphi F\Big).$$

Proof. A simple inspection shows that $(\mathbf{1}_\beta)^\dagger = \mathbf{1}_{\beta^\dagger}$ for every pseudo-natural transformation $\beta : G \circ \varphi \Rightarrow F$. It remains to check that $(\Xi' \odot \Xi)^\dagger = \Xi'^\dagger \odot \Xi$ for every three pseudo-natural transformations $\beta, \beta', \beta'' : G \circ \varphi \Rightarrow F$ and every pair of modifications $\Xi : \beta \rightsquigarrow \beta'$ and $\Xi' : \beta' \rightsquigarrow \beta''$. However, we have :

$$\begin{aligned}
(\pi^{F,j} * \mathsf{F}_{\Xi'^\dagger_j} * \mathsf{F}_{\Xi^\dagger_j}) \odot \Theta^{\beta,j} &= (\pi^{F,j} * \mathsf{F}_{\Xi'^\dagger_j}) \odot \Theta^{\beta',j} \odot ((\Xi \circ \mathsf{t}_j) * (G * \varphi^*_j)) \\
&= \Theta^{\beta'',j} \odot ((\Xi' \circ \mathsf{t}_j) * (G * \varphi^*_j)) \odot ((\Xi \circ \mathsf{t}_j) * (G * \varphi^*_j)) \\
&= \Theta^{\beta'',j} \odot (((\Xi' \odot \Xi) \circ \mathsf{t}_j) * (G * \varphi^*_j))
\end{aligned}$$

whence the contention. □

2.6.29. Keep the notation of (2.6.19). We set

$$\omega_i^F := \pi^{F,\varphi(i)}_{(i,\mathbf{1}_{\varphi(i)})} : L_{F,\varphi(i)} \to Fi \qquad \text{for every } i \in \mathrm{Ob}(I)$$

$$\tau_f^{\omega^F} := \Omega^{F,\varphi(f)}_{(i',\mathbf{1}_{\varphi(i')})} \odot \tau^{F,\varphi(i)}_{(f,\mathbf{1}_{\varphi(f)})} : Ff \circ \omega_i^F \Rightarrow \omega_{i'}^F \circ L_{F,\varphi(f)} \quad \text{for every 1-cell } f : i \to i' \text{ of } I$$

where $\tau^{F,\varphi(i)}$ denotes the coherence constraint of $\pi^{F,\varphi(i)}$.

Lemma 2.6.30. *The rule : $i \mapsto \omega_i^F$ for every $i \in \mathrm{Ob}(I)$ defines a pseudo-natural transformation*

$$\omega^F : \Big(2\text{-}\!\int_\varphi F\Big) \circ \varphi \Rightarrow F$$

whose coherence constraint is given by the system of 2*-cells* $\tau_\bullet^{\omega^F}$.

Proof. Let $f, f' : i \to i'$ be two 1-cells of I and $\alpha : f \Rightarrow f'$ a 2-cell; for the naturality of τ^{ω^F} we need to check the identity :

$$\tau_{f'}^{\omega^F} \odot (F\alpha * \omega_i^F) = (\omega_{i'}^F * L_{F,\varphi(\alpha)}) \odot \tau_f^{\omega^F}.$$

However, notice that α induces two 1-cells $(f, \varphi(\alpha)), (f', \mathbf{1}_{\varphi(f')}) : (i, \mathbf{1}_{\varphi(i)}) \to (i', \varphi(f'))$ and a 2-cell $\alpha : (f, \varphi(\alpha)) \Rightarrow (f', \mathbf{1}_{\varphi(f')})$ of $\varphi(i)/\varphi I$. By naturality of $\tau^{F,\varphi(i)}$ we deduce the identity :

$$\tau^{F,\varphi(i)}_{(f,\varphi(\alpha))} = \pi^{F,\varphi(i)}_{(f',\mathbf{1}_{\varphi(f')})} \odot (F\alpha * \pi^{F,\varphi(i)}_{(i,\mathbf{1}_{\varphi(i)})})$$

whence :

$$\tau_{f'}^{\omega^F} \odot (F\alpha * \omega_i^F) = \Omega^{F,\varphi(f')}_{(i',\mathbf{1}_{\varphi(i')})} \odot \tau^{F,\varphi(i)}_{(f,\varphi(\alpha))}.$$

On the other hand, from (2.6.4) we get :

$$(\omega_{i'}^F * L_{F,\varphi(\alpha)}) \odot \tau_f^{\omega^F} = \Omega^{F,\varphi(f')}_{(i',\mathbf{1}_{\varphi(i')})} \odot \pi^{F,\varphi(\alpha)}_{(i',\mathbf{1}_{\varphi(i')})} \odot \tau^{F,\varphi(i)}_{(f,\mathbf{1}_{\varphi(f)})}.$$

So we are reduced to checking the identity : $\tau^{F,\varphi(i)}_{(f,\varphi(\alpha))} = \tau^{F,\varphi(i)}_{(\mathbf{1}_{i'},\varphi(\alpha))} \odot \tau^{F,\varphi(i)}_{(f,\mathbf{1}_{\varphi(f)})}$. The latter follows from the coherence axiom for $\tau^{F,\varphi(i)}$. Next, we check the coherence axioms for ω^F. To this aim, notice that for every $i \in \mathrm{Ob}(I)$ and $f := \mathbf{1}_i$, the two-cell $\tau^{F,\varphi(i)}_{(f,\mathbf{1}_{\varphi(f)})}$ is the identity of the 1-cell $\pi^{F,\varphi(i)}_{(i,\mathbf{1}_{\varphi(i)})}$ (remark 2.4.2(ii)); it follows easily that $\tau_{\mathbf{1}_i}^{\omega^F} = \mathbf{1}_{\omega_i^F}$. Lastly, consider two 1-cells $f : i \to i'$, $f' : i' \to i''$ of I, and set $X := \omega_{i''}^F * (L_{F,\gamma^\varphi_{f,f'}} \odot L_{F,\varphi(f),\varphi(f')})$; we need to show that

$$Y := X \odot (\tau_{f'}^{\omega^F} * L_{F,\varphi(f)}) \odot (Ff' * \tau_f^{\omega^F}) = Z := \tau_{f'\circ f}^{\omega^F} \odot (\gamma^F_{f,f'} * \omega_i^F)$$

where γ^φ and γ^F are the coherence constraints of φ and F. We compute :

$$\begin{aligned}
Y &= X \odot ((\Omega^{F,\varphi(f')}_{(i'',\mathbf{1}_{\varphi(i'')})} \odot \tau^{F,\varphi(i')}_{(f',\mathbf{1}_{\varphi(f')})}) * L_{F,\varphi(f)}) \odot (Ff' * (\Omega^{F,\varphi(f)}_{(i',\mathbf{1}_{\varphi(i')})} \odot \tau^{F,\varphi(i)}_{(f,\mathbf{1}_{\varphi(f)})})) \\
&= X \odot (\Omega^{F,\varphi(f')}_{(i'',\mathbf{1}_{\varphi(i'')})} * L_{F,\varphi(f)}) \odot \Omega^{F,\varphi(f)}_{(i',\varphi(f'))} \odot \tau^{F,\varphi(i)}_{(f',\mathbf{1}_{\varphi(f')\circ\varphi(f)})} \odot (Ff' * \tau^{F,\varphi(i)}_{(f,\mathbf{1}_{\varphi(f)})}) \\
&= (\omega_{i''}^F * L_{F,\gamma^\varphi_{f,f'}}) \odot \Omega^{F,\varphi(f')\circ\varphi(f)}_{(i'',\mathbf{1}_{\varphi(i'')})} \odot \tau^{F,\varphi(i)}_{(f',\mathbf{1}_{\varphi(f')\circ\varphi(f)})} \odot (Ff' * \tau^{F,\varphi(i)}_{(f,\mathbf{1}_{\varphi(f)})}) \\
&= \Omega^{F,\varphi(f'\circ f)}_{(i'',\mathbf{1}_{\varphi(i'')})} \odot \pi^{F,\gamma^\varphi_{f,f'}}_{(i'',\mathbf{1}_{\varphi(i'')})} \odot \tau^{F,\varphi(i)}_{(f',\mathbf{1}_{\varphi(f')\circ\varphi(f)})} \odot (Ff' * \tau^{F,\varphi(i)}_{(f,\mathbf{1}_{\varphi(f)})}) \\
&= \Omega^{F,\varphi(f'\circ f)}_{(i'',\mathbf{1}_{\varphi(i'')})} \odot \pi^{F,\gamma^\varphi_{f,f'}}_{(i'',\mathbf{1}_{\varphi(i'')})} \odot \tau^{F,\varphi(i)}_{(f'\circ f,(\gamma^\varphi_{f,f'})^{-1})} \odot (\gamma^F_{f,f'} * \pi^{F,\varphi(i)}_{i,\mathbf{1}_{\varphi(i)}})
\end{aligned}$$

where the second equality follows from the compatibility condition for $\Omega^{F,\varphi(f)}$, the third follows from (2.6.2), the fourth follows from (2.6.4), and the fifth holds by virtue of the coherence axiom for $\tau^{F,\varphi(i)}$, taking into account (2.6.3). So, we are reduced to checking that

$$\pi^{F,\gamma^\varphi_{f,f'}}_{(i'',\mathbf{1}_{\varphi(i'')})} \odot \tau^{F,\varphi(i)}_{(f'\circ f,(\gamma^\varphi_{f,f'})^{-1}))} = \tau^{F,\varphi(i)}_{(f'\circ f,\mathbf{1}_{\varphi(f'\circ f)})}$$

which follows again from (2.6.3) and the coherence axiom for $\tau^{F,\varphi(i)}$. □

2.6.31. In view of lemma 2.6.30 we may define a functor :

$$(-)^{\ddagger}_{F,G} : \mathsf{PsNat}\Big(G, 2\text{-}\int_{\varphi} F\Big) \to \mathsf{PsNat}(G \circ \varphi, F) \qquad \beta \mapsto \beta^{\ddagger} := \omega^F \odot (\beta * \varphi)$$

attaching to every modification $\Xi : \beta \rightsquigarrow \beta'$ of pseudo-natural transformations $\beta, \beta' : G \Rightarrow 2\text{-}\int_{\varphi} F$, the modification $\omega^F * (\Xi \circ \varphi)$. For any pseudo-natural transformation $\beta : G \circ \varphi \Rightarrow F$ we set

$$\Lambda^{\beta}_i := \Theta^{\beta,\varphi(i)}_{(i,\mathbf{1}_{\varphi(i)})} : \beta_i \Rightarrow (\beta^{\dagger})^{\ddagger}_i \qquad \text{for every } i \in \mathrm{Ob}(I).$$

Lemma 2.6.32. *The rule : $i \mapsto \Lambda^{\beta}_i$ defines an invertible modification*

$$\Lambda^{\beta} : \beta \rightsquigarrow (\beta^{\dagger})^{\ddagger}$$

and the rule : $\beta \mapsto \Lambda^{\beta}$ defines an isomorphism of functors :

$$\Lambda : \mathbf{1}_{\mathsf{PsNat}(G\circ\varphi,F)} \xrightarrow{\sim} (-)^{\ddagger}_{F,G} \circ (-)^{\dagger}_{F,G}.$$

Proof. Let $f : i \to i'$ be any 1-cell of I; we need to check the identity :

$$Y := (\pi^{F,\varphi(i')}_{(i',\mathbf{1}_{\varphi(i')})} * \tau^{\beta^{\dagger}}_{\varphi(f)}) \odot (\tau^{\omega^F}_f * \beta^{\dagger}_{\varphi(i)}) \odot (Ff * \Theta^{\beta,\varphi(i)}_{(i,\mathbf{1}_{\varphi(i)})}) = Z := (\Theta^{\beta,\varphi(i')}_{i',\mathbf{1}_{\varphi(i')}} * G\varphi(f)) \odot \tau^{\beta}_f$$

where τ^{β} denotes the coherence constraint of β. Let also $\tau^{(\beta*\mathsf{t}_{\varphi(i)})\odot(G*\varphi^*_{\varphi(i)})}$ be the coherence constraint of $(\beta * \mathsf{t}_{\varphi(i)}) \odot (G * \varphi^*_{\varphi(i)})$, and notice that $\tau^{(\beta*\mathsf{t}_{\varphi(i)})\odot(G*\varphi^*_{\varphi(i)})}_{(f,\mathbf{1}_{\varphi(f)})} = \tau^{\beta}_f$. Combining with the compatibility condition for $\Theta^{\beta,\varphi(i)}$ relative to the 1-cell $(f, \mathbf{1}_{\varphi(f)})$ of $\varphi(i)/\varphi I$, we get :

$$Y = (\pi^{F,\varphi(i')}_{(i',\mathbf{1}_{\varphi(i')})} * \tau^{\beta^{\dagger}}_{\varphi(f)}) \odot (\Omega^{F,\varphi(f)}_{(i',\mathbf{1}_{\varphi(i')})} * \beta^{\dagger}_{\varphi(i)}) \odot (\Theta^{\beta,\varphi(i)}_{(i',\varphi(f))} * G\varphi(i)) \odot \tau^{\beta}_f.$$

Then the sought identity follows from (2.6.21), applied with $g := (f, \mathbf{1}_{\varphi(i)})$, after noticing that $\Gamma^{G,g}_{(i',\mathbf{1}_{\varphi(i')})} = \mathbf{1}_{G\varphi(f)}$, since G is unital. Lastly, let $\beta, \beta' : G \Rightarrow 2\text{-}\int_{\varphi} F$ be two pseudo-natural transformations, and $\Xi : \beta \rightsquigarrow \beta'$ a modification; the naturality of Λ amounts to the identity :

$$(\omega^F_i * \Xi^{\dagger}_{\varphi(i)}) \odot \Lambda^{\beta}_i = \Lambda^{\beta'}_i \odot \Xi_i \qquad \text{for every } i \in \mathrm{Ob}(I)$$

which follows directly from (2.6.26). □

Lemma 2.6.33. *In the situation of* (2.6.19)*, for every $j \in \mathrm{Ob}(J)$ the rule*

$$(i, j \xrightarrow{g} \varphi(i)) \mapsto \Delta^{F,j}_{(i,g)} := \Omega^{F,g}_{(i,\mathbf{1}_{\varphi(i)})} \qquad \textit{for every } (i,g) \in \mathrm{Ob}(j/\varphi I)$$

defines an invertible modification

$$\Delta^{F,j} : \pi^{F,j} \rightsquigarrow (\omega^F * \mathsf{t}_j) \odot \Big(\Big(2\text{-}\int_{\varphi} F\Big) * \varphi^*_j\Big).$$

Proof. Let $(f, \alpha) : (i, g) \to (i', g')$ be any 1-cell of $j/\varphi I$; we need to check the identity :

$$\Omega^{F,g'}_{(i',\mathbf{1}_{\varphi(i')})} \odot \tau^{F,j}_{(f,\alpha)} = Y := (\pi^{F,\varphi(i')}_{(i',\mathbf{1}_{\varphi(i')})} * (L_{F,\alpha} \odot L_{F,g,\varphi(f)})) \odot (\tau^{\omega^F}_f * L_{F,g}) \odot (Ff * \Omega^{F,g}_{(i,\mathbf{1}_{\varphi(i)})}).$$

We compute :

$$\begin{aligned}
Y &= (\pi^{F,\varphi(i')}_{(i',\mathbf{1}_{\varphi(i')})} * (L_{F,\alpha} \odot L_{F,g,\varphi(f)})) \odot ((\Omega^{F,\varphi(f)}_{(i',\mathbf{1}_{\varphi(i')})} \odot \tau^{F,\varphi(i)}_{(f,\mathbf{1}_{\varphi(f)})}) * L_{F,g}) \odot (Ff * \Omega^{F,g}_{(i,\mathbf{1}_{\varphi(i)})}) \\
&= (\pi^{F,\varphi(i')}_{(i',\mathbf{1}_{\varphi(i')})} * (L_{F,\alpha} \odot L_{F,g,\varphi(f)})) \odot (\Omega^{F,\varphi(f)}_{(i',\mathbf{1}_{\varphi(i')})} * L_{F,g}) \odot \Omega^{F,g}_{(i',\varphi(f))} \odot \tau^{F,j}_{(f,g^*(\mathbf{1}_{\varphi(f)}))} \\
&= (\pi^{F,\varphi(i')}_{(i',\mathbf{1}_{\varphi(i')})} * L_{F,\alpha}) \odot \Omega^{F,\varphi(f)\circ g}_{(i',\mathbf{1}_{\varphi(i')})} \odot \tau^{F,j}_{(f,g^*(\mathbf{1}_{\varphi(f)}))} \\
&= \Omega^{F,g'}_{(i',\mathbf{1}_{\varphi(i')})} \odot \pi^{F,\alpha}_{(i',\mathbf{1}_{\varphi(i')})} \odot \tau^{F,j}_{(f,g^*(\mathbf{1}_{\varphi(f)}))}
\end{aligned}$$

where the second equality holds by the compatibility condition of $\Omega^{F,g}$, applied to the 1-cell $(f,\mathbf{1}_{\varphi(f)}):(i,\mathbf{1}_{\varphi(i)})\to(i',\varphi(f))$ of $\varphi(i)/\varphi I$, the third equality follows from (2.6.2), and the fourth equality follows from (2.6.4). So we are reduced to checking that

$$\tau^{F,j}_{(f,\alpha)}=\pi^{F,\alpha}_{(i',\mathbf{1}_{\varphi(i')})}\odot\tau^{F,j}_{(f,g^*(\mathbf{1}_{\varphi(f)}))}$$

which follows from the coherence condition for $\tau^{F,j}$, relative to the composition of 1-cells $(i,g)\xrightarrow{(f,\mathbf{1}_{\varphi(f)\circ g})}(i',\varphi(f)\circ g)\xrightarrow{(\mathbf{1}_{i'},\alpha)}(i',g')$. □

2.6.34. Now, let $\beta:G\Rightarrow 2\text{-}\int_\varphi F$ be a pseudo-natural transformation. For every $j\in\mathrm{Ob}(J)$ we get the 1-cell $(\beta^\ddagger)^\dagger_j:Gj\to L_{F,j}$, as well as the invertible modification

$$\Theta^{\beta^\ddagger,j}:(\omega^F*\mathsf{t}_j)\odot(\beta*(\varphi\circ\mathsf{t}_j))\odot(G*\varphi^*_j)\rightsquigarrow\pi^{F,j}\odot\mathsf{F}_{(\beta^\ddagger)^\dagger_j}.$$

On the other hand, by example 2.2.15(ii) we have the invertible modification

$$\Upsilon^{\beta,j}:\left(\left(2\text{-}\int_\varphi F\right)*\varphi^*_j\right)\odot(\beta*\mathsf{F}_j)\rightsquigarrow(\beta*(\varphi\circ\mathsf{t}_j))\odot(G*\varphi^*_j)\qquad(i,f:j\to\varphi(i))\mapsto\tau^\beta_{\varphi^*_{j,(i,f)}}$$

where τ^β is the coherence constraint of β, and notice that $\beta*\mathsf{F}_j=\mathsf{F}_{\beta_j}$. We set

$$\Delta^{\beta,j}:=\Theta^{\beta^\ddagger,j}\odot((\omega^F*\mathsf{t}_j)*\Upsilon^{\beta,j})\odot(\Delta^{F,j}*\mathsf{F}_{\beta_j}):\pi^{F,j}\odot\mathsf{F}_{\beta_j}\rightsquigarrow\pi^{F,j}\odot\mathsf{F}_{(\beta^\ddagger)^\dagger_j}.$$

There exists then a unique invertible 2-cell

$$\Psi^\beta_j:\beta_j\Rightarrow(\beta^\ddagger)^\dagger_j\qquad\text{such that}\qquad\pi^{F,j}*\mathsf{F}_{\Psi^\beta_j}=\Delta^{\beta,j}.$$

Lemma 2.6.35. *The rule : $j\mapsto\Psi^\beta_j$ for every $j\in\mathrm{Ob}(J)$ yields an invertible modification*

$$\Psi^\beta:\beta\rightsquigarrow(\beta^\ddagger)^\dagger.$$

Proof. Let $g:j\to j'$ be any 1-cell of J; we need to show that

$$\tau^{(\beta^\ddagger)^\dagger}_g\odot(L_{F,g}*\Psi^\beta_j)=(\Psi^\beta_{j'}*Gg)\odot\tau^\beta_g$$

and recall that the coherence constraint $\tau^{(\beta^\ddagger)^\dagger}_g$ of $(\beta^\ddagger)^\dagger$ is characterized by the identity :

$$(\pi^{F,j'}*\mathsf{F}_{\tau^{(\beta^\ddagger)^\dagger}_g})\odot(\Omega^{F,g}*\mathsf{F}_{(\beta^\ddagger)^\dagger_j})\odot(\Theta^{\beta^\ddagger,j}\circ g^*)\odot((\beta^\ddagger*\mathsf{t}_{j'})*\Gamma^{G,g})=\Theta^{\beta^\ddagger,j'}*\mathsf{F}_{Gg}.$$

As usual, we reduce to checking the identity :

$$Y:=\pi^{F,j'}*(\mathsf{F}_{\tau^{(\beta^\ddagger)^\dagger}_g}\odot(\mathsf{F}_{L_{F,g}}*\mathsf{F}_{\Psi^\beta_j}))=Z:=\pi^{F,j'}*((\mathsf{F}_{\Psi^\beta_{j'}}*\mathsf{F}_{Gg})\odot\mathsf{F}_{\tau^\beta_g}).$$

To ease notation, set $H:=2\text{-}\int_\varphi F$; we remark :

Claim 2.6.36. $((\beta*\varphi*\mathsf{t}_{j'})*\Gamma^{G,g})^{-1}\odot(\Upsilon^{\beta,j}\circ g^*)=(\Upsilon^{\beta,j'}*\mathsf{F}_{Gg})\odot((H*\varphi^*_{j'})*\mathsf{F}_{\tau^\beta_g})\odot(\Gamma^{H,g}*\mathsf{F}_{\beta_j})^{-1}$.

Proof of the claim. The left-hand side is the modification

$$(H*\varphi^*_j*g^*)\odot\mathsf{F}_{\beta_j}\rightsquigarrow(\beta*(\varphi\circ\mathsf{t}_{j'}))\odot(G*\varphi^*_{j'})\odot\mathsf{F}_{Gg}\qquad(i,h:j'\to\varphi(i))\mapsto\tau^\beta_{h\circ g}\odot(\beta_{\varphi(i)}*\gamma^G_{g,h})^{-1}$$

and the coherence axiom for τ^β gives :

$$(\beta_{\varphi(i)}*\gamma^G_{g,h})^{-1}\odot\tau^\beta_{h\circ g}=(\tau^\beta_h*Gg)\odot(L_{F,h}*\tau^\beta_g)\odot(L_{F,g,h}*\beta_j)^{-1}$$

where γ^G is the coherence constraint of G. The claim translates the latter identity. ◊

Now, let $X := \Theta^{\beta^\ddagger,j'} * \mathsf{F}_{Gg}$ and $X' := X \odot (\omega^F * \mathsf{t}_{j'}) * \Upsilon^{\beta',j} * \mathsf{F}_{Gg}$. We compute :

$$\begin{aligned} Y &= X \odot ((\beta^\ddagger * \mathsf{t}_{j'}) * \Gamma^{G,g})^{-1} \odot (\Theta^{\beta^\ddagger,j} \circ g^*)^{-1} \odot (\Omega^{F,g} * \mathsf{F}_{(\beta^\ddagger)^\dagger_j})^{-1} \odot (\pi^{F,j'} * \mathsf{F}_{L_{F,g}} * \mathsf{F}_{\Psi^\beta_j}) \\ &= X \odot ((\beta^\ddagger * \mathsf{t}_{j'}) * \Gamma^{G,g})^{-1} \odot (\Theta^{\beta^\ddagger,j} \circ g^*)^{-1} \odot ((\pi^{F,j} * g^*) * \mathsf{F}_{\psi^\beta_j}) \odot (\Omega^{F,g} * \mathsf{F}_{\beta_j})^{-1} \\ &= X \odot ((\beta^\ddagger * \mathsf{t}_{j'}) * \Gamma^{G,g})^{-1} \odot (\Theta^{\beta^\ddagger,j} \circ g^*)^{-1} \odot (\Delta^{\beta,j} \circ g^*) \odot (\Omega^{F,g} * \mathsf{F}_{\beta_j})^{-1} \\ &= X \odot ((\beta^\ddagger * \mathsf{t}_{j'}) * \Gamma^{G,g})^{-1} \odot ((\omega^F * \mathsf{t}_{j'}) * (\Upsilon^{\beta,j} \circ g^*)) \odot ((\Delta^{F,j} \circ g^*) * \mathsf{F}_{\beta_j}) \odot (\Omega^{F,g} * \mathsf{F}_{\beta_j})^{-1} \\ &= X' \odot ((\omega^F * \mathsf{t}_{j'}) * (((H * \varphi^*_{j'}) * \mathsf{F}_{\tau^\beta_g}) \odot (\Gamma^{H,g} * \mathsf{F}_{\beta_j})^{-1})) \odot ((\Delta^{F,j} \circ g^*) * \mathsf{F}_{\beta_j}) \odot (\Omega^{F,g} * \mathsf{F}_{\beta_j})^{-1} \end{aligned}$$

where the last equality follows from claim 2.6.36. On the other hand :

$$Z = X' \odot (\Delta^{F,j'} * \mathsf{F}_{\beta_{j'}} * \mathsf{F}_{Gg}) \odot (\pi^{F,j'} * \mathsf{F}_{\tau^\beta_g}) = X' \odot ((\omega^F * \mathsf{t}_{j'}) * (H * \varphi^*_{j'}) * \mathsf{F}_{\tau^\beta_g}) \odot (\Delta^{F,j'} * \mathsf{F}_{Hg} * \mathsf{F}_{\beta_j})$$

so we are reduced to showing that

$$\Delta^{F,j} \circ g^* = ((\omega^F * \mathsf{t}_{j'}) * \Gamma^{H,g}) \odot (\Delta^{F,j'} * \mathsf{F}_{Hg}) \odot \Omega^{F,g}$$

which follows from (2.6.2). □

Proposition 2.6.37. *For every unital pseudo-functors $F : I \to \mathscr{C}$ and $G : J \to \mathscr{C}$, the functors $(-)^\dagger_{F,G}$ of proposition 2.6.28 and $(-)^\ddagger_{F,G}$ of (2.6.31) are equivalences.*

Proof. In view of lemmata 2.6.32 and 2.6.35, it suffices to prove that the rule : $\beta \mapsto \Psi^\beta$ defines a natural transformation

$$\Psi : \mathbf{1}_{\mathsf{PsNat}(G,2\text{-}\int_\varphi F)} \Rightarrow (-)^\dagger_{F,G} \circ (-)^\ddagger_{F,G}.$$

Thus, let $\alpha, \beta : G \Rightarrow 2\text{-}\int_\varphi F$ be two pseudo-natural transformations, and $\Xi : \alpha \rightsquigarrow \beta$ a modification; we need to check the identity :

$$\Psi^\beta_j \odot \Xi_j = (\omega^F * (\Xi \circ \varphi))^\dagger_j \odot \Psi^\alpha_j \qquad \text{for every } j \in \mathrm{Ob}(J)$$

and as usual, it suffices to show that

$$Y := \pi^{F,j} * (\mathsf{F}_{\Psi^\beta_j} \odot \mathsf{F}_{\Xi_j}) = Z := \pi^{F,j} * (\mathsf{F}_{(\omega^F * (\Xi \circ \varphi))^\dagger_j} \odot \mathsf{F}_{\Psi^\alpha_j}).$$

To ease notation, set $H := 2\text{-}\int_\varphi F$; we remark :

Claim 2.6.38. $((\Xi \circ \varphi \circ \mathsf{t}_j) * (G * \varphi^*_j)) \odot \Upsilon^{\alpha,j} = \Upsilon^{\beta,j} \odot ((H * \varphi^*_j) * (\Xi \circ \mathsf{F}_j))$.

Proof of the claim. The left-hand side is the modification :

$$(H * \varphi^*_j) \odot (\alpha * \mathsf{F}_j) \rightsquigarrow (\beta * (\varphi \circ \mathsf{t}_j)) \odot (G * \varphi^*_j) \qquad (j, g \mapsto \varphi(i)) \mapsto (\Xi_{\varphi(i)} * Gg) \odot \tau^\alpha_g$$

where τ^α is the coherence constraint of α; then the compatibility condition for Ξ gives :

$$(\Xi_{\varphi(i)} * Gg) \odot \tau^\alpha_g = \tau^\beta_g \odot (Hg * \Xi_j)$$

and the claim translates the latter identity. ◊

We have : $Y = \Delta^{\beta,j} \odot (\pi^{F,j} * \mathsf{F}_{\Xi_j}) = \Theta^{\beta^\ddagger,j} \odot ((\omega^F * \mathsf{t}_j) * \Upsilon^{\beta,j}) \odot (\Delta^{F,j} * \mathsf{F}_{\beta_j}) \odot (\pi^{F,j} * \mathsf{F}_{\Xi_j})$, and

$$\begin{aligned} Z &= (\pi^{F,j} * \mathsf{F}_{(\omega^F * (\Xi \circ \varphi))^\dagger_j}) \odot \Delta^\alpha_j \\ &= (\pi^{F,j} * \mathsf{F}_{(\omega^F * (\Xi \circ \varphi))^\dagger_j}) \odot \Theta^{\alpha^\ddagger,j} \odot ((\omega^F * \mathsf{t}_j) * \Upsilon^{\alpha,j}) \odot (\Delta^{F,j} * \mathsf{F}_{\alpha_j}) \\ &= \Theta^{\beta^\ddagger,j} \odot (((\omega^F * (\Xi \circ \varphi)) \circ \mathsf{t}_j) * (G * \varphi^*_j)) \odot ((\omega^F * \mathsf{t}_j) * \Upsilon^{\alpha,j}) \odot (\Delta^{F,j} * \mathsf{F}_{\alpha_j}) \\ &= \Theta^{\beta^\ddagger,j} \odot ((\omega^F * \mathsf{t}_j) * (\Xi \circ \varphi \circ \mathsf{t}_j) * (G * \varphi^*_j)) \odot ((\omega^F * \mathsf{t}_j) * \Upsilon^{\alpha,j}) \odot (\Delta^{F,j} * \mathsf{F}_{\alpha_j}) \\ &= \Theta^{\beta^\ddagger,j} \odot ((\omega^F * \mathsf{t}_j) * (\Upsilon^{\beta,j} \odot ((H * \varphi^*_j) * (\Xi \circ \mathsf{F}_j)))) \odot (\Delta^{F,j} * \mathsf{F}_{\alpha_j}) \end{aligned}$$

where the last equality holds by claim 2.6.38. Thus, we are reduced to checking the identity :

$$\Delta^{F,j} * \mathsf{F}_{\beta_j} \odot (\pi^{F,j} * \mathsf{F}_{\Xi_j}) = ((\omega^F * \mathsf{t}_j) * (H * \varphi_j^*) * (\Xi \circ \mathsf{F}_j)) \odot (\Delta^{F,j} * \mathsf{F}_{\alpha_j})$$

which follows as usual from remark 2.1.1(i). □

Theorem 2.6.39. *Let I be a small 2-category, J a 2-category with small* Hom*-categories, $\varphi : I \to J$ any pseudo-functor, and suppose that all the 2-cells of J are invertible. Then for every 2-complete (resp. 2-cocomplete) 2-category $\mathscr{C}$ the right (resp. left) 2-Kan extension along φ is a right (resp. left) 2-adjoint for the pseudo-functor*

$$\mathsf{PsFun}(\varphi, \mathscr{C}) : \mathsf{PsFun}(J, \mathscr{C}) \to \mathsf{PsFun}(I, \mathscr{C}).$$

Proof. Let us show first that the rule : $(F, G) \mapsto (-)^\ddagger_{F,G}$ yields a pseudo-natural equivalence :

$$(-)^\ddagger : \mathsf{PsNat}\Big(\mathbf{1}_{\mathsf{uniPsFun}(J,\mathscr{C})}, 2\text{-}\!\int_\varphi\Big) \xrightarrow{\sim} \mathsf{PsNat}(\mathsf{uniPsFun}(\varphi^u, \mathscr{C}), \mathbf{1}_{\mathsf{uniPsFun}(I,\mathscr{C})})$$

(notation of remark 2.4.4(i)). To this aim let us remark :

Claim 2.6.40. Let $F, F' : I \to \mathscr{C}$ be two unital pseudo-functors. Every pseudo-natural transformation $\mu : F \Rightarrow F'$ induces an invertible modification :

$$\Sigma^\mu : \mu \odot \omega^F \rightsquigarrow \omega^{F'} \odot \Big(\Big(2\text{-}\!\int_\varphi \mu\Big) * \varphi\Big) \qquad i \mapsto \Omega^{\mu,\varphi(i)}_{(i,\mathbf{1}_{\varphi(i)})}.$$

Proof of the claim. Let $f : i \to i'$ be any 1-cell of I, and τ^μ the coherence constraint of μ; set :

$$Y := (\pi^{F',\varphi(i')}_{(i',\mathbf{1}_{\varphi(i')})} * L_{\mu,\varphi(f)}) \odot (\tau^{\omega^{F'}}_f * L_{\mu,\varphi(i)}) \odot (F'f * \Omega^{\mu,\varphi(i)}_{(i,\mathbf{1}_{\varphi(i)})})$$
$$Z := (\Omega^{\mu,\varphi(i')}_{(i',\mathbf{1}_{\varphi(i')})} * L_{F,\varphi(f)}) \odot (\mu_{i'} * \tau^{\omega^F}_f) \odot (\tau^\mu_f * \pi^{F,\varphi(i)}_{(i,\mathbf{1}_{\varphi(i)})})$$

so that we need to check the identity $Y = Z$. We compute :

$$\begin{aligned} Y &= (\pi^{F',\varphi(i')}_{(i',\mathbf{1}_{\varphi(i')})} * L_{\mu,\varphi(f)}) \odot ((\Omega^{F',\varphi(f)}_{(i',\mathbf{1}_{\varphi(i')})} \odot \tau^{F',\varphi(i)}_{(f,\mathbf{1}_{\varphi(f)})}) * L_{\mu,\varphi(i)}) \odot (F'f * \Omega^{\mu,\varphi(i)}_{(i,\mathbf{1}_{\varphi(i)})}) \\ &= (\pi^{F',\varphi(i')}_{(i',\mathbf{1}_{\varphi(i')})} * L_{\mu,\varphi(f)}) \odot (\Omega^{F',\varphi(f)}_{(i',\mathbf{1}_{\varphi(i')})} * L_{\mu,\varphi(i)}) \odot \Omega^{\mu,\varphi(i)}_{(i',\varphi(f))} \odot (\mu_{i'} * \tau^{F,\varphi(i)}_{(f,\mathbf{1}_{\varphi(f)})}) \odot (\tau^\mu_f * \pi^{F,\varphi(i)}_{(i,\mathbf{1}_{\varphi(i)})}) \end{aligned}$$

where the second equality follows from the compatibility condition of $\Omega^{\mu,\varphi(i)}$, relative to the 1-cell $(f, \mathbf{1}_{\varphi(f)}) : (i, \mathbf{1}_{\varphi(i)}) \to (i', \varphi(f))$ of $\varphi(i)/\varphi I$. We are thus reduced to showing that :

$$(\pi^{F',\varphi(i')}_{(i',\mathbf{1}_{\varphi(i')})} * L_{\mu,\varphi(f)}) \odot (\Omega^{F',\varphi(f)}_{(i',\mathbf{1}_{\varphi(i')})} * L_{\mu,\varphi(i)}) \odot \Omega^{\mu,\varphi(i)}_{(i',\varphi(f))} = (\Omega^{\mu,\varphi(i')}_{(i',\mathbf{1}_{\varphi(i')})} * L_{F,\varphi(f)}) \odot (\mu_{i'} * \Omega^{F,\varphi(f)}_{(i',\mathbf{1}_{\varphi(i')})})$$

which follows from (2.6.9). ♢

Claim 2.6.41. Let $F, F' : I \to \mathscr{C}$ and $G, G' : J \to \mathscr{C}$ be four unital pseudo-functors, $\lambda : G' \Rightarrow G$ and $\mu : F \Rightarrow F'$ two pseudo-natural transformations. The rule

$$\beta \mapsto \tau^\ddagger_{(\lambda,\mu),\beta} := \Sigma^\mu * ((\beta \odot \lambda) * \varphi)$$

yields the orientation for an essentially commutative square diagram of functors :

$$\begin{array}{ccc} \mathsf{PsNat}(G, 2\text{-}\!\int_\varphi F) & \xrightarrow{(-)^\ddagger_{F,G}} & \mathsf{PsNat}(G \circ \varphi, F) \\ {\scriptstyle \mathsf{PsNat}(\lambda, 2\text{-}\!\int_\varphi \mu)} \downarrow & \swarrow \tau^\ddagger_{(\lambda,\mu)} & \downarrow {\scriptstyle \mathsf{PsNat}(\lambda * \varphi, \mu)} \\ \mathsf{PsNat}(G', 2\text{-}\!\int_\varphi F') & \xrightarrow[(-)^\ddagger_{F',G'}]{} & \mathsf{PsNat}(G' \circ \varphi, F'). \end{array}$$

Proof of the claim. Let $\beta, \beta' : G \to 2\text{-}\int_\varphi F$ be two pseudo-natural transformations, and $\Xi : \beta \rightsquigarrow \beta'$ a modification; we need to check the identity :

$$\tau^\ddagger_{(\lambda,\mu),\beta'} \odot (\mu * \omega^F * (\Xi \circ \varphi) * (\lambda * \varphi)) = \Big(\omega^{F'} * \Big(\Big(\Big(2\text{-}\int_\varphi \mu\Big) * \Xi * \lambda\Big) \circ \varphi\Big)\Big) \odot \tau^\ddagger_{(\lambda,\mu),\beta}$$

and we come down to showing that :

$$(\Sigma^\mu * (\beta' * \varphi)) \odot (\mu * \omega^F * (\Xi \circ \varphi)) = \Big(\omega^{F'} * \Big(\Big(2\text{-}\int_\varphi \mu\Big) * \varphi\Big) * (\Xi \circ \varphi)\Big) \odot (\Sigma^\mu * (\beta * \varphi))$$

which follows from remark 2.1.1(i). $\Diamond$

To conclude the proof, it remains to check that the rule $(\lambda, \mu) \mapsto \tau^\ddagger_{(\lambda,\mu)}$ yields a coherence constraint for the sought pseudo-functor $(-)^\ddagger$. For the first coherence axiom, notice that both the source and target of $(-)^\ddagger$ are themselves unital pseudo-functors, hence it suffices to check that $\tau^\ddagger_{(\mathbf{1}_G,\mathbf{1}_F)}$ is the identity automorphism of $(-)^\ddagger_{F,G}$, for every pair of unital pseudo-functors $F : I \to \mathscr{C}$ and $G : J \to \mathscr{C}$ (remark 2.4.2(ii)). We then come down to checking that $\Sigma^{\mathbf{1}_F} = \mathbf{1}_{\omega^F}$ for every such F; the latter identity follows by a direct inspection of the definitions.

Next, consider unital pseudo-functors $F, F', F'' : I \to \mathscr{C}$, $G, G', G'' : J \to \mathscr{C}$ and two pairs of pseudo-natural transformations $(\lambda' : G'' \Rightarrow G', \mu' : F' \Rightarrow F'')$ and $(\lambda : G' \Rightarrow G, \mu : F \Rightarrow F')$; we need to check the identity :

$$((-)^\ddagger_{F'',G''} * \mathsf{PsNat}(G'', L_{\mu,\mu'})) \odot \Big(\tau^\ddagger_{(\lambda',\mu')} * \mathsf{PsNat}\Big(\lambda, 2\text{-}\int_\varphi \mu\Big)\Big) \odot (\mathsf{PsNat}(\lambda' * \varphi, \mu') * \tau^\ddagger_{(\lambda,\mu)}) = \tau^\ddagger_{(\lambda'\odot\lambda,\mu'\odot\mu)}.$$

Thus, let $\beta : G \Rightarrow 2\text{-}\int_\varphi F$ be any pseudo-natural transformation, and set $\rho := (\beta \odot \lambda \odot \lambda') * \varphi$; the assertion comes down to the identity :

$$(\omega^F * (L_{\mu,\mu'} \circ \varphi) * \rho) \odot \Big(\Sigma^{\mu'} * \Big(\Big(\Big(2\text{-}\int_\varphi \mu\Big) * \varphi\Big) \odot \rho\Big)\Big) \odot (\mu' * \Sigma^\mu * \rho) = \Sigma^{\mu'\odot\mu} * \rho$$

so it suffices to show that :

$$(\omega^F * (L_{\mu,\mu'} \circ \varphi)) \odot \Big(\Sigma^{\mu'} * \Big(\Big(2\text{-}\int_\varphi \mu\Big) * \varphi\Big)\Big) \odot (\mu' * \Sigma^\mu) = \Sigma^{\mu'\odot\mu}$$

which holds by (2.6.10). Lastly, the source and target of $(-)^\ddagger$ are pseudo-functors

$$\mathsf{uniPsFun}(J, \mathscr{C})^o \times \mathsf{uniPsFun}(I, \mathscr{C}) \to \mathbf{Cat}.$$

On the other hand, the strict 2-equivalences $(-)^u$ of remark 2.4.4(i) yield a strict 2-equivalence

$$U : \mathsf{PsFun}(J, \mathscr{C})^o \times \mathsf{PsFun}(I, \mathscr{C}) \xrightarrow{\sim} \mathsf{uniPsFun}(J, \mathscr{C})^o \times \mathsf{uniPsFun}(I, \mathscr{C})$$

and the strict isomorphisms of pseudo-functors (2.4.5) yield pseudo-natural isomorphisms

$$\mathsf{PsNat}\Big(\mathbf{1}_{\mathsf{uniPsFun}(J,\mathscr{C})}, 2\text{-}\int_\varphi\Big) \circ U \xrightarrow{\sim} \mathsf{PsNat}\Big(\mathbf{1}_{\mathsf{PsFun}(J,\mathscr{C})}, 2\text{-}\int_\varphi\Big)$$

$$\mathsf{PsNat}(\mathsf{uniPsFun}(\varphi^u, \mathscr{C}), \mathbf{1}_{\mathsf{uniPsFun}(I,\mathscr{C})}) \circ U \xrightarrow{\sim} \mathsf{PsNat}(\mathsf{PsFun}(\varphi, \mathscr{C}), \mathbf{1}_{\mathsf{PsFun}(I,\mathscr{C})}).$$

Summing up, we deduce the sought 2-adjunction between $\mathsf{PsFun}(\varphi, \mathscr{C})$ and $2\text{-}\int_\varphi$.

The assertion concerning the left 2-Kan extension along φ is an immediate consequence, in view of remark 2.4.28(i). $\square$

Corollary 2.6.42. *In the situation of theorem* 2.6.39*, suppose that φ is fully faithful. Then the same holds for the pseudo-functor* $2\text{-}\int_\varphi$ *(resp. for* $2\text{-}\int^\varphi$*).*

Proof. As usual, it suffices to check the assertion concerning right 2-Kan extensions. To this aim, we notice :

Claim 2.6.43. Suppose that $\varphi : I \to J$ is fully faithful and that all the 2-cells of J are invertible. Then for every $i \in \mathrm{Ob}(I)$, the object $(i, \mathbf{1}_{\varphi(i)})$ is pseudo-initial in $\varphi(i)/\varphi I$.

Proof of the claim. Let (i', f) be any othe object of $\varphi(i)/\varphi I$; by assumption, there exist a 1-cell $g : i \to i'$ in I and an invertible 2-cell $\alpha : \varphi(g) \xrightarrow{\sim} f$ in J. Then we get the 1-cell

$$(i, \mathbf{1}_{\varphi(i)}) \xrightarrow{(g,\mathbf{1}_{\varphi(g)})} (i', \varphi(g)) \xrightarrow{(\mathbf{1}_{i'},\alpha)} (i', f).$$

Next, let $(t, \alpha), (s, \beta) : (i, \mathbf{1}_{\varphi(i)}) \to (i', f)$ be two 1-cells in $\varphi(i)/\varphi I$; by assumption β and α are invertible, hence we get the invertible 2-cell $\beta^{-1} \odot \alpha : \varphi(t) \xrightarrow{\sim} \varphi(s)$, and since φ is fully faithful, there exists a unique invertible 2-cell $\lambda : t \xrightarrow{\sim} s$ in I such that $\varphi(\lambda) = \beta^{-1} \odot \alpha$. We deduce an invertible 2-cell $\lambda : (t, \alpha) \xrightarrow{\sim} (s, \beta)$ in $\varphi(i)/\varphi I$, and it is easily seen that this is the unique 2-cell from (t, α) to (s, β). The claim follows. ◊

Let now $F : I \to \mathscr{C}$ be any pseudo-functor, and for every $j \in \mathrm{Ob}(J)$, let $(L_{F,j}, \pi^{F,j})$ be a 2-limit of $F \circ \mathrm{t}_j$, as in (2.6); in view of claim 2.6.43 and proposition 2.5.16, we deduce that the 1-cell $\pi^{F,\varphi(i)}_{(i,\mathbf{1}_{\varphi(i)})} : L_{F,j} \to Fi$ is an equivalence for every $i \in \mathrm{Ob}(I)$, and therefore the pseudo-natural transformation ω^F of lemma 2.6.30 is a pseudo-natural equivalence. To conclude, it suffices now to invoke corollary 2.4.29. □

3. Special categories

3.1. **Fibrations.** Let $\varphi : \mathscr{A} \to \mathscr{B}$ be a functor, $f : A' \to A$ a morphism in $\mathscr{A}$. We say that f is *φ-cartesian*, or – slightly abusively – that f is *$\mathscr{B}$-cartesian*, if the induced commutative diagram of sets (notation of (1.1.25)) :

$$\begin{array}{ccc} \mathrm{Hom}_{\mathscr{A}}(X, A') & \xrightarrow{f_*} & \mathrm{Hom}_{\mathscr{A}}(X, A) \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ \mathrm{Hom}_{\mathscr{B}}(\varphi X, \varphi A') & \xrightarrow{(\varphi f)_*} & \mathrm{Hom}_{\mathscr{B}}(\varphi X, \varphi A) \end{array}$$

is cartesian for every $X \in \mathrm{Ob}(\mathscr{A})$. In this case, one also says that f *is an inverse image of* A over φf, or – slightly abusively – that A' *is an inverse image of* A over φf.

Remark 3.1.1. Keep the notation of (3.1).

(i) It is easily seen that the composition of two $\mathscr{B}$-cartesian morphisms is $\mathscr{B}$-cartesian.

(ii) Let $g : B' \to B$ and $g' : B'' \to B'$ be two morphisms of $\mathscr{B}$, and $f : A' \to A$ and $f'' : A'' \to A$ two $\mathscr{B}$-cartesian morphisms of $\mathscr{A}$ such that $\varphi f = g$ and $\varphi f'' = g \circ g'$. Then there exists a unique morphism $f' : A'' \to A'$ of $\mathscr{A}$ such that $\varphi f' = g'$ and $f'' = f \circ f'$, and this morphism is $\mathscr{B}$-cartesian : the detailed verification shall be left to the reader.

Definition 3.1.2. Let $\varphi : \mathscr{A} \to \mathscr{B}$ be a functor, and B any object of $\mathscr{B}$.

(i) The *fibre of φ over B* is the category whose objects are all the $A \in \mathrm{Ob}(\mathscr{A})$ such that $\varphi A = B$, and whose morphisms $f : A' \to A$ are the elements of $\mathrm{Hom}_{\mathscr{A}}(A', A)$ such that $\varphi f = \mathbf{1}_B$. We denote this category by

$$\varphi^{-1}B \qquad \text{or more simply by} \qquad \mathscr{A}_B$$

in the contexts where the latter notation does not give rise to ambiguities. We denote the natural faithful embedding of $\mathscr{A}_B$ into $\mathscr{A}$ by :

$$\iota_B : \mathscr{A}_B \to \mathscr{A}.$$

(ii) We say that φ is a *fibration* if, for every morphism $g : B' \to B$ in $\mathscr{B}$, and every $A \in \mathrm{Ob}(\mathscr{A}_B)$, there exists an inverse image $f : A' \to A$ of A over g. In this case, we also say that $(\mathscr{A}, \varphi)$ is a *fibred $\mathscr{B}$-category*, and φ is called the *structure functor* of $\mathscr{A}$.

Example 3.1.3. Let $\mathscr{B}, \mathscr{C}$ be any two categories, $F : \mathscr{C} \to \mathscr{B}$ a functor, X any object of $\mathscr{B}$.

(i) The source functor $\mathsf{s}_X : F\mathscr{C}/X \to \mathscr{C}$ of (1.1.27) is a fibration, and all the morphisms in $F\mathscr{C}/X$ are $\mathscr{C}$-cartesian. The easy verification shall be left to the reader.

(ii) The source functor $\mathsf{s} : \mathscr{B}/F\mathscr{C} \to \mathscr{B}$ of (1.1.28) is a fibration. Namely, the s-cartesian morphisms are the commutative diagrams

$$\begin{array}{ccc} B & \xrightarrow{f} & FC \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle Fg'} \\ B' & \xrightarrow{f'} & FC' \end{array}$$

such that g' is an isomorphism in $\mathscr{C}$. The fibre $\mathsf{s}^{-1}X$ is the category $X/F\mathscr{C}$. As a special case, the source functor $\mathsf{s} : \mathsf{Morph}(\mathscr{B}) \to \mathscr{B}$ of (1.1.30) is a fibration.

(iii) Suppose that all fibre products are representable in $\mathscr{B}$. Then also the target functor $\mathsf{t} : \mathsf{Morph}(\mathscr{B}) \to \mathscr{B}$ is a fibration; more precisely, the t-cartesian morphisms are the square diagrams as in (ii) (with $F := \mathbf{1}_{\mathscr{B}}$) which are cartesian (*i.e.* fibred). We have $\mathsf{t}^{-1}X = \mathscr{B}/X$.

Definition 3.1.4. Let $\mathscr{A}$, $\mathscr{A}'$, and $\mathscr{B}$ be three categories, $\varphi : \mathscr{A} \to \mathscr{B}$ and $\varphi' : \mathscr{A}' \to \mathscr{B}$ two functors, and $F : \mathscr{A} \to \mathscr{A}'$ a *$\mathscr{B}$-functor*, *i.e.* F verifies the identity $\varphi' \circ F = \varphi$.

(i) We say that F is *cartesian* if it sends $\mathscr{B}$-cartesian morphisms of $\mathscr{A}$ to $\mathscr{B}$-cartesian morphisms in $\mathscr{A}'$. We denote by :

$$\mathsf{Cart}_{\mathscr{B}}(\mathscr{A}, \mathscr{A}')$$

the category whose objects are the cartesian $\mathscr{B}$-functors $F : \mathscr{A} \to \mathscr{A}'$, and whose morphisms are the *natural $\mathscr{B}$-transformations*, *i.e.* the natural transformations such that :

$$\alpha : F \Rightarrow G \qquad \text{such that} \qquad \varphi' * \alpha = \mathbf{1}_\varphi.$$

The composition law is the usual composition of natural transformations : $(\beta, \alpha) \mapsto \alpha \odot \beta$. Notice that if $\mathscr{A}$ and $\mathscr{A}'$ are small, the same holds for $\mathsf{Cart}_{\mathscr{B}}(\mathscr{A}, \mathscr{A}')$: details left to the reader.

(ii) For any two other $\mathscr{B}$-categories $\mathscr{C} \to \mathscr{B}$ and $\mathscr{C}' \to \mathscr{B}$, every pair of cartesian $\mathscr{B}$-functors $H : \mathscr{C} \to \mathscr{A}$ and $K : \mathscr{A}' \to \mathscr{C}'$ induces a functor :

$$\mathsf{Cart}_{\mathscr{B}}(H, K) : \mathsf{Cart}_{\mathscr{A}}(\mathscr{A}, \mathscr{A}') \to \mathsf{Cart}_{\mathscr{B}}(\mathscr{C}, \mathscr{C}') \qquad G \mapsto K \circ G \circ H.$$

To any morphism $\alpha : G \Rightarrow G'$ in $\mathsf{Cart}_{\mathscr{B}}(\mathscr{A}, \mathscr{A}')$, the functor $\mathsf{Cart}_{\mathscr{B}}(H, K)$ assigns the natural transformation $K * \alpha * H : K \circ G \circ H \Rightarrow K \circ G' \circ H$. In case $H = \mathbf{1}_{\mathscr{A}}$ (resp. $K = \mathbf{1}_{\mathscr{A}'}$) we also denote this functor by $\mathsf{Cart}_{\mathscr{B}}(\mathscr{A}, K)$ (resp. by $\mathsf{Cart}_{\mathscr{B}}(H, \mathscr{A}')$).

Likewise, if $H, H' : \mathscr{C} \to \mathscr{A}$ and $K, K' : \mathscr{A}' \to \mathscr{C}'$ are four $\mathscr{B}$-cartesian functors, every pair of natural $\mathscr{B}$-transformations $\beta : H \Rightarrow H'$ and $\gamma : K \Rightarrow K'$ induces a natural transformation :

$$\mathsf{Cart}_{\mathscr{B}}(\beta, \gamma) : \mathsf{Cart}_{\mathscr{B}}(H, K) \Rightarrow \mathsf{Cart}_{\mathscr{B}}(H', K') \qquad G \mapsto \gamma * G * \beta$$

and again, if $\beta = \mathbf{1}_H$ (resp. $\gamma = \mathbf{1}_K$) we also denote this natural transformation by $\mathsf{Cart}_{\mathscr{B}}(H, \gamma)$ (resp. $\mathsf{Cart}_{\mathscr{B}}(\alpha, K)$).

(iii) Let U, V be two universes, with $\mathsf{U} \subset \mathsf{V}$, and such that $\mathscr{B}$ is V-small; we say that the fibration φ has *essentially* U*-small fibres* if $\varphi^{-1}B$ is an essentially U-small category for every $B \in \mathrm{Ob}(\mathscr{B})$. The V-small fibrations over $\mathscr{B}$ with essentially U-small fibres form a 2-category

$$(\mathsf{U}, \mathsf{V})\text{-}\mathsf{Fib}(\mathscr{B})$$

with Hom-category $\mathsf{Cart}_{\mathscr{B}}(\mathscr{C}, \mathscr{C}')$, for every pair of fibrations $\mathscr{C} \to \mathscr{B} \leftarrow \mathscr{C}'$, and with composition functors given by (ii). When there is no danger of ambiguities, we shall often write U-$\mathsf{Fib}(\mathscr{B})$, or even just $\mathsf{Fib}(\mathscr{B})$ for this 2-category. Notice that if $\mathscr{B}$ is essentially U-small, then (U, V)-$\mathsf{Fib}(\mathscr{B})$ has essentially U-small Hom-categories.

Remark 3.1.5. Let $\mathscr{B}$ and $\mathscr{B}'$ be two categories and $\psi : \mathscr{B}' \to \mathscr{B}$ any functor.

(i) For every category $\mathscr{A}$ and every functor $\varphi : \mathscr{A} \to \mathscr{B}$, the cartesian morphisms of the projection $\varphi' : \mathscr{A} \times_{(\varphi,\psi)} \mathscr{B}' \to \mathscr{B}'$ are the pairs (f, f') where f is a cartesian morphism of $\mathscr{A}$, f' is a morphism of $\mathscr{B}'$, and $\varphi f = \psi f'$ (see example 1.2.25(i)). Moreover, the projection $\mathscr{A} \times_{(\varphi,\psi)} \mathscr{B}' \to \mathscr{A}$ restricts to a natural isomorphism of fibre categories

$$\varphi'^{-1}B' \xrightarrow{\sim} \varphi^{-1}(\psi B') \qquad \text{for every } B' \in \mathrm{Ob}(\mathscr{B}').$$

If φ is a fibration, the same then holds for φ'. Therefore, for every pair of universes $\mathsf{U} \subset \mathsf{V}$ such that $\mathscr{B}$ and $\mathscr{B}'$ are V-small, we get a well defined strict pseudo-functor

$$(\mathsf{U},\mathsf{V})\text{-}\mathsf{Fib}(\psi)^* : (\mathsf{U},\mathsf{V})\text{-}\mathsf{Fib}(\mathscr{B}) \to (\mathsf{U},\mathsf{V})\text{-}\mathsf{Fib}(\mathscr{B}') \qquad (\mathscr{A} \xrightarrow{\varphi} \mathscr{B}) \mapsto (\mathscr{A} \times_{(\varphi,\psi)} \mathscr{B}' \xrightarrow{\varphi'} \mathscr{B}')$$

which assigns to every $\mathscr{B}$-cartesian functor $F : \mathscr{A}_1 \to \mathscr{A}_2$ the $\mathscr{B}'$-cartesian functor $F \times_{\mathscr{B}} \mathscr{B}' : \mathscr{A}_1 \times_{\mathscr{B}} \mathscr{B}' \to \mathscr{A}_2 \times_{\mathscr{B}} \mathscr{B}'$, and to any natural $\mathscr{B}$-transformation $\alpha : F_1 \Rightarrow F_2$ between $\mathscr{B}$-cartesian functors $F_1, F_2 : \mathscr{A}_1 \to \mathscr{A}_2$ the induced $\mathscr{B}'$-transformation $\alpha \times_{\mathscr{B}} \mathscr{B}' : F_1 \times_{\mathscr{B}} \mathscr{B}' \Rightarrow F_2 \times_{\mathscr{B}} \mathscr{B}'$. As usual, we often write $\mathsf{U}\text{-}\mathsf{Fib}(\psi)^*$, or just $\mathsf{Fib}(\psi)^*$ instead of $(\mathsf{U},\mathsf{V})\text{-}\mathsf{Fib}(\psi)^*$.

(ii) Suppose that ψ is a fibration with essentially U-small fibres; then for every fibration $F' : \mathscr{A}' \to \mathscr{B}'$ with essentially U-small fibres, the composition $\psi \circ F' : \mathscr{A}' \to \mathscr{B}$ is also a fibration with essentially U-small fibres, and in this case we get therefore also a well defined strict pseudo-functor

$$(\mathsf{U},\mathsf{V})\text{-}\mathsf{Fib}(\mathscr{B}') \to (\mathsf{U},\mathsf{V})\text{-}\mathsf{Fib}(\mathscr{B}) \qquad (F' : \mathscr{A}' \to \mathscr{B}') \mapsto (\psi \circ F : \mathscr{A}' \to \mathscr{B}).$$

(iii) For $i = 1, 2$, let $\mathscr{A}_i \to \mathscr{B}$ be two fibrations; combining (i) and (ii) we see that the induced functor $\mathscr{A}_1 \times_{\mathscr{B}} \mathscr{A}_2 \to \mathscr{B}$ is also a fibration.

(iv) In the situation of (i), it is easily seen that $\mathsf{Cart}_{\mathscr{B}'}(\mathscr{B}', \mathscr{A} \times_{(\varphi,\psi)} \mathscr{B}')$ is naturally identified with the full subcategory of $\mathsf{Cart}_{\mathscr{B}}(\mathscr{B}', \mathscr{A})$ whose objects are the $\mathscr{B}$-functors $\mathscr{B}' \to \mathscr{A}$ that send *every* morphism of $\mathscr{B}'$ to a $\mathscr{B}$-cartesian morphism of $\mathscr{A}$: the detailed verification shall be left to the reader.

(v) Suppose that $F : \mathscr{A} \to \mathscr{A}'$ is a *$\mathscr{B}$-equivalence*, *i.e.* an equivalence in the 2-category $\mathsf{V}\text{-}\mathbf{Cat}/\mathscr{B}$ (in the sense of definition 2.1.3(iii)). Then it is easily seen that F is $\mathscr{B}$-cartesian.

(vi) In the situation of (i), let $\psi' : \mathscr{B}'' \to \mathscr{B}'$ be another functor between V-small categories; then we have a natural isomorphism of categories :

$$(\mathsf{U},\mathsf{V})\text{-}\mathsf{Fib}(\psi')^* \circ (\mathsf{U},\mathsf{V})\text{-}\mathsf{Fib}(\psi)^*(\mathscr{A}) \xrightarrow{\sim} (\mathsf{U},\mathsf{V})\text{-}\mathsf{Fib}(\psi \circ \psi')^*(\mathscr{A})$$

that assigns to every object (A, B', B'') of $(\mathscr{A} \times_{(\varphi,\psi)} \mathscr{B}') \times_{(\varphi',\psi')} \mathscr{B}''$ the object (A, B'') of $\mathscr{A} \times_{(\varphi,\psi\circ\psi')} \mathscr{B}''$, and it is likewise defined on morphisms. Clearly this isomorphism is strictly pseudo-natural with respect to $\mathscr{A}$, so we get a strict pseudo-natural isomorphism of strict pseudo-functors :

$$\gamma^{(\mathsf{U},\mathsf{V})\text{-}\mathsf{Fib}}_{\psi',\psi} : (\mathsf{U},\mathsf{V})\text{-}\mathsf{Fib}(\psi')^* \circ (\mathsf{U},\mathsf{V})\text{-}\mathsf{Fib}(\psi)^* \xrightarrow{\sim} (\mathsf{U},\mathsf{V})\text{-}\mathsf{Fib}(\psi \circ \psi')^*.$$

3.1.6. *Cleavages.* Let $\varphi : \mathscr{A} \to \mathscr{B}$ be a fibration, and for every object $(A, g : B \to FA)$ of $\mathscr{B}/\varphi\mathscr{A}$ let us choose an inverse image of A over g :

$$g_A : g^*A \to A.$$

Then we claim that the rule $(A, g) \mapsto g_A$ extends uniquely to a functor

$$\boldsymbol{\lambda} : \mathscr{B}/\varphi\mathscr{A} \to \mathsf{Morph}(\mathscr{A})$$

that makes commute the resulting diagram (notation of (1.1.30)) :

(3.1.7)
$$\begin{array}{ccc} \mathscr{B}/\varphi\mathscr{A} & \xrightarrow{\ \boldsymbol{\lambda}\ } & \mathsf{Morph}(\mathscr{A}) \\ & {\scriptstyle \mathsf{S}}\searrow \quad \swarrow{\scriptstyle \mathsf{Morph}(\varphi)} & \\ & \mathsf{Morph}(\mathscr{B}). & \end{array}$$

Indeed, since g_A is φ-cartesian, for every morphism $(h,k) : (A', B' \xrightarrow{g'} \varphi A') \to (A, B \xrightarrow{g} \varphi A)$ of $\mathscr{B}/\varphi\mathscr{A}$ there exists a unique morphism $f : g'^*A' \to g^*A$ that makes commute the diagram

(3.1.8)
$$\begin{array}{ccc} g'^*A' & \xrightarrow{g'_{A'}} & A' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle h} \\ g^*A & \xrightarrow{g_A} & A \end{array} \qquad \text{and such that} \qquad \varphi(f) = k$$

and this square diagram can be regarded as a morphism $(f,h) : g'_{A'} \to g_A$ in $\mathsf{Morph}(\mathscr{A})$ such that $\mathsf{Morph}(\varphi)(f,h) = (k, \varphi(h)) = \mathsf{S}(h,k)$. The uniqueness of f easily implies that the rule $(h,k) \mapsto (f,h)$ yields a well defined functor as sought. We call a *cleavage* for φ any such functor (*"clivage"* in french); hence, the cleavages for φ are characterized as the functors $\boldsymbol{\lambda}$ that make commute (3.1.7) and that map every object of $\mathscr{B}/\varphi\mathscr{A}$ to a φ-cartesian morphism.

Notice that we may always choose a cleavage such that $\boldsymbol{\lambda}(A, \mathbf{1}_{FA}) = \mathbf{1}_A$ for every $A \in \mathrm{Ob}(\mathscr{A})$. We call *unital* a cleavage fulfilling this condition.

Example 3.1.9. Let $\varphi : \mathscr{A} \to \mathscr{B}$ be any fibration. Then $\mathsf{Morph}(\varphi) : \mathsf{Morph}(\mathscr{A}) \to \mathsf{Morph}(\mathscr{B})$ is a fibration as well (notation of (1.1.30)). Indeed, pick a cleavage $\boldsymbol{\lambda}$ for φ; consider any object $(A' \xrightarrow{h} A)$ of $\mathsf{Morph}(\mathscr{A})$ and a morphism $(g', g) : (B' \xrightarrow{k} B) \to (\varphi A' \xrightarrow{\varphi(h)} \varphi A)$ of $\mathsf{Morph}(\mathscr{B})$. By definition, $g : B \to \varphi A$ and $g' : B' \to \varphi A'$ are morphisms of $\mathscr{B}$ such that $\varphi(h) \circ g' = g \circ k$, hence $(h,k) : (A', B' \xrightarrow{g'} \varphi A') \to (A, B \xrightarrow{g} \varphi A)$ is a morphism of $\mathscr{B}/\varphi\mathscr{A}$, and $(f,h) := \boldsymbol{\lambda}(h,k)$ is a commutative square (3.1.8). With this notation, it is easily seen that $(g'^*A' \xrightarrow{f} g^*A)$ is an inverse image of $(A' \xrightarrow{h} A)$ over (g', g) : the details are left to the reader.

3.1.10. *The pseudo-functor associated with a cleavage.* Let now $\boldsymbol{\lambda}$ be any cleavage for a given fibration $\varphi : \mathscr{A} \to \mathscr{B}$; to any morphism $g : B' \to B$ in $\mathscr{B}$ we attach a functor

$$\mathsf{c}_g : \varphi^{-1}B \to \varphi^{-1}B'$$

as follows. First, we consider the functor

$$(-, g) : \varphi^{-1}B \to B'/\varphi\mathscr{A} \qquad A \mapsto (A, g) \qquad (A_1 \xrightarrow{h} A_2) \mapsto ((A_1, g) \xrightarrow{(h, \mathbf{1}_{B'})} (A_2, g)).$$

Let also $\mathsf{s} : \mathsf{Morph}(\mathscr{A}) \to \mathscr{A}$ be the source functor (see (1.1.30)), and notice that the composition $\mathsf{s} \circ \boldsymbol{\lambda} \circ (-, g)$ factors through $\varphi^{-1}B'$, and yields therefore the sought functor c_g. Moreover, the commutativity of the diagram (3.1.8) also means that the rule : $A \mapsto \boldsymbol{\lambda}(A, g)$ defines a natural transformation

(3.1.11)
$$g_\bullet : \iota_{B'} \circ \mathsf{c}_g \Rightarrow \iota_B$$

(notation of definition 3.1.2(i)). Notice especially the natural isomorphism of functors

$$\delta_B : \mathbf{1}_{\varphi^{-1}B} \Rightarrow \mathsf{c}_{\mathbf{1}_B} \qquad A \mapsto ((\mathbf{1}_B)_A^{-1} : A \xrightarrow{\sim} (\mathbf{1}_B)^*A) \qquad \text{for every } B \in \mathrm{Ob}(\mathscr{B}).$$

Furthermore, let $B'' \xrightarrow{h} B' \xrightarrow{g} B$ be two morphisms in $\mathscr{B}$, so we get the functors

$$\mathsf{c}_g : \varphi^{-1}B \to \varphi^{-1}B' \qquad \mathsf{c}_h : \varphi^{-1}B' \to \varphi^{-1}B'' \qquad \mathsf{c}_{g \circ h} : \varphi^{-1}B \to \varphi^{-1}B''$$

as well as the natural transformations :

$$g_\bullet : \iota_{B'} \circ \mathsf{c}_g \Rightarrow \iota_B \qquad h_\bullet : \iota_{B''} \circ \mathsf{c}_h \Rightarrow \iota_{B'} \qquad (g \circ h)_\bullet : \iota_{B''} \circ \mathsf{c}_{g \circ h} \Rightarrow \iota_B.$$

By inspecting the constructions, one easily finds a unique natural isomorphism :

$$\gamma_{(h,g)} : \mathsf{c}_h \circ \mathsf{c}_g \Rightarrow \mathsf{c}_{g \circ h}$$

which fits into a commutative diagram :

$$\begin{array}{ccc} \iota_{B''} \circ \mathsf{c}_h \circ \mathsf{c}_g & \xRightarrow{h_\bullet * \mathsf{c}_g} & \iota_{B'} \circ \mathsf{c}_g \\ {\scriptstyle \iota_{B''} * \gamma_{(h,g)}} \Downarrow & & \Downarrow {\scriptstyle g_\bullet} \\ \iota_{B''} \circ \mathsf{c}_{g \circ h} & \xRightarrow{(g \circ h)_\bullet} & \iota_B. \end{array}$$

Moreover, if $k : B''' \to B''$ is a third morphism of $\mathscr{B}$, we can compute :

$$\begin{aligned} (g \circ h \circ k)_A \circ \gamma_{(k,g \circ h),A} \circ \mathsf{c}_k(\gamma_{(h,g),A}) &= (g \circ h)_A \circ k_{\mathsf{c}_{g \circ h} A} \circ \mathsf{c}_k(\gamma_{(h,g),A}) \\ &= (g \circ h)_A \circ \gamma_{(h,g),A} \circ k_{\mathsf{c}_h \mathsf{c}_g A} \\ &= g_A \circ h_{\mathsf{c}_g A} \circ k_{\mathsf{c}_h \mathsf{c}_g A} \\ &= g_A \circ (h \circ k)_{\mathsf{c}_g A} \circ \gamma_{(k,h),\mathsf{c}_g A} \\ &= (g \circ h \circ k)_A \circ \gamma_{(h \circ k, g),A} \circ \gamma_{(k,h),\mathsf{c}_g A} \end{aligned}$$

for every $A \in \mathrm{Ob}(\mathscr{A})$, and since $(g \circ h \circ k)_A$ is $\mathscr{B}$-cartesian, we deduce :

$$\gamma_{(k,g \circ h),A} \circ \mathsf{c}_k(\gamma_{(h,g),A}) = \gamma_{(h \circ k,g),A} \circ \gamma_{(k,h),\mathsf{c}_g A}. \tag{3.1.12}$$

Likewise, for every morphism $f : B' \to B$ of $\mathscr{B}$ and every $A \in \mathrm{Ob}(\varphi^{-1}B)$ we have the commutative diagram :

$$\begin{array}{ccccc} f^* \mathbf{1}_B^* A & \xrightarrow{\mathsf{c}_f(\mathbf{1}_B)_A} & f^* A & \xleftarrow{(\mathbf{1}_{B'})_{f^* A}} & \mathbf{1}_{B'}^* f^* A \\ {\scriptstyle f_{\mathbf{1}_B^* A}} \downarrow & & \downarrow {\scriptstyle f_A} & & \downarrow {\scriptstyle \gamma_{(\mathbf{1}_{B'},f),A}} \\ \mathbf{1}_B^* A & \xrightarrow{(\mathbf{1}_B)_A} & A & \xleftarrow{f_A} & f^* A \end{array}$$

which implies :

$$\gamma_{(f,\mathbf{1}_B),A} = \mathsf{c}_f((\mathbf{1}_B)_A) \qquad \gamma_{(\mathbf{1}_{B'},f),A} = (\mathbf{1}_{B'})_{\mathsf{c}_f A}. \tag{3.1.13}$$

Suppose now that φ is an object of (U,V)-Fib, for some universe V, and denote by

$$(\mathsf{U},\mathsf{V})\text{-}\mathbf{Cat}$$

the strong sub-2-category of V-$\mathbf{Cat}$ whose objects are the essentially U-small categories (see definition 2.4.9(iii)). Then the identities (3.1.12) and (3.1.13) mean that the rule which assigns to each $B \in \mathrm{Ob}(\mathscr{B})$ the small category $\varphi^{-1}B$ and to each morphism g in $\mathscr{B}$ the functor c_g defines a pseudo-functor

$$\mathsf{c} : \mathscr{B}^o \to (\mathsf{U},\mathsf{V})\text{-}\mathbf{Cat}$$

whose coherence constraint is the system of natural isomorphisms $(\delta_\bullet, \gamma_\bullet)$. Moreover, the system of inclusion functors $(\iota_B \mid B \in \mathrm{Ob}(\mathscr{B}))$ amounts to a pseudo-cocone

$$\iota : \mathsf{c} \Rightarrow \mathsf{F}_{\mathscr{A}}$$

whose coherence constraint is given by the system of natural transformations (3.1.11). Notice that if $\boldsymbol{\lambda}$ is a unital cleavage, c will be a unital pseudo-functor. (Here we view $\mathscr{B}^o$ as a 2-category, as explained in example 2.2.4(i); also, the 2-category structure on V-$\mathbf{Cat}$ is the one of remark 2.1.1(ii)). We call c and ι respectively *the pseudo-functor and the pseudo-cocone associated with the cleavage* $\boldsymbol{\lambda}$.

Example 3.1.14. As an application, we may generalize example 1.5.15 as follows. Let I, J be two small categories, $\varphi : I \to J$ a fibration, fix a cleavage $\boldsymbol{\lambda}$ for φ, and denote by c the associated pseudo-functor.

(i) First, we claim that for every $j \in \mathrm{Ob}(J)$ the functor $(-, \mathbf{1}_j) : \varphi^{-1}j \to j/\varphi I$ is coinitial (notation of (3.1.10)). Indeed, for any $(i_0, f : j \to \varphi i_0) \in \mathrm{Ob}(j/\varphi I)$ we have the cartesian morphism $\boldsymbol{\lambda}(i_0, f) : \mathsf{c}_f(i_0) \to i_0$, and it is easily seen that the category $(-, \mathbf{1}_j)(\varphi^{-1}j)/(i_o, f)$ is isomorphic to $(\varphi^{-1}j)/\mathsf{c}_f(i_0)$, which is obviously connected, whence the claim.

(ii) Notice that the composition $\mathsf{t}_j \circ (-, \mathbf{1}_j) : \varphi^{-1}j \to I$ equals the inclusion functor ι_j (here $\mathsf{t}_j : j/\varphi I \to I$ is the target functor). Moreover, for every morphism $f : j \to j'$ in J, the natural transformation $f_\bullet : \iota_j \circ \mathsf{c}_f \Rightarrow \iota_{j'}$ associated with $\boldsymbol{\lambda}$ induces a morphism

$$\omega_F^f : \lim_{\varphi^{-1}j} F \circ \iota_j \xrightarrow{\lim_{\mathsf{c}_f} \mathbf{1}_{F\circ\iota_j}} \lim_{\varphi^{-1}j'} F \circ \iota_j \circ \mathsf{c}_f \xrightarrow{\lim_{\varphi^{-1}j'} F * f_\bullet} \lim_{\varphi^{-1}j'} F \circ \iota_{j'}$$

for every functor $F : I \to \mathscr{C}$, fitting into a commutative diagram

$$\begin{array}{ccc}
\int_\varphi^\wedge F(j) & \xrightarrow{\int_\varphi^\wedge F(f)} & \int_\varphi^\wedge F(j') \\
\downarrow & & \downarrow \\
\lim\limits_{\varphi^{-1}j} F \circ \iota_j & \xrightarrow{\omega_F^f} & \lim\limits_{\varphi^{-1}j'} F \circ \iota_{j'}
\end{array}$$

whose vertical arrows are the natural isomorphisms provided by (i) and remark 1.5.5(ii). In other words, $\int_\varphi^\wedge$ is naturally isomorphic to the functor that assigns to every such F the functor

$$\int_{\boldsymbol{\lambda}}^\wedge F : J \to \mathscr{C}^\wedge \qquad j \mapsto \lim_{\varphi^{-1}j} F \circ \iota_j \qquad (f : j \to j') \mapsto \omega_F^f.$$

(iii) Furthermore, in this situation, proposition 1.3.2 says that there is a natural isomorphism

$$\lim_I F \xrightarrow{\sim} \operatorname*{Lim}_J \int_{\boldsymbol{\lambda}}^\wedge F \qquad \text{in } \mathscr{C}^\wedge$$

for every functor $F : I \to \mathscr{C}$. If $\mathscr{C}$ is complete, the functor $\int_{\boldsymbol{\lambda}}^\wedge$ is isomorphic, by remark 1.5.16, to the composition of $\mathsf{Fun}(J, h_{\mathscr{C}})$ and a functor well defined up to isomorphism

$$\int_{\boldsymbol{\lambda}} : \mathsf{Fun}(I, \mathscr{C}) \to \mathsf{Fun}(J, \mathscr{C}) \qquad F \mapsto (j \mapsto \operatorname*{Lim}_{\varphi^{-1}j} F \circ \iota_j)$$

and we can restate the foregoing "Fubini" isomorphism in terms of this latter functor.

(iv) Lastly, let $G : I^o \to \mathscr{C}$ be any functor. It follows from (ii) that the functor $\int_\wedge^{\varphi^o} G$ is naturally isomorphic to the functor

$$\int_\wedge^{\boldsymbol{\lambda}} G : J^o \to \mathscr{C}^{o\wedge o} \qquad j^o \mapsto \operatorname*{colim}_{(\varphi^{-1}j)^o} G \circ \iota_j^o \qquad (f^o : j'^o \to j^o) \mapsto \omega_{G^o}^f$$

and we have a natural isomorphism

$$\operatorname*{colim}_{I^o} G \xrightarrow{\sim} \operatorname*{colim}_{J^o} \int_\wedge^{\boldsymbol{\lambda}} G \qquad \text{in } \mathscr{C}^{o\wedge o}.$$

Again, if $\mathscr{C}$ is cocomplete, the functor $\int_\wedge^{\boldsymbol{\lambda}}$ is the composition of $\mathsf{Fun}(J^o, h_{\mathscr{C}^o}^o)$ and a functor

$$\int^{\boldsymbol{\lambda}} : \mathsf{Fun}(I^o, \mathscr{C}) \to \mathsf{Fun}(J^o, \mathscr{C}) \qquad G \mapsto (j^o \mapsto \operatorname*{Colim}_{(\varphi^{-1}j)^o} G \circ \iota_j^o)$$

and we may state the foregoing "Fubini" isomorphism for colimits in terms of this latter functor.

3.1.15. *Fibration associated with a presheaf.* Let $\mathscr{B}$ be a category, F a presheaf on $\mathscr{B}$. As in (1.4.7), we let $\mathscr{F}ib(F)$ be the category of elements of F, and we notice that the source functor

$$\mathsf{s}_F : \mathscr{F}ib(F) \to \mathscr{B}$$

is a fibration. For every $X \in \mathrm{Ob}(\mathscr{B})$, the fibre $\mathsf{s}_F^{-1}(X)$ is (naturally isomorphic to) the discrete category FX (see example 1.1.6(ii)). Notice also that every morphism in $\mathscr{F}ib(F)$ is cartesian, and for every $\mathscr{B}$-category $\mathscr{C}$, the category $\mathsf{Cart}_{\mathscr{B}}(\mathscr{C}, \mathscr{F}ib(F))$ is discrete.

For every pair of presheaves F, G on $\mathscr{B}$, we have a natural bijection:

$$\mathrm{Hom}_{\mathscr{C}^\wedge}(F, G) \stackrel{\sim}{\to} \mathrm{Ob}(\mathsf{Cart}_{\mathscr{B}}(\mathscr{F}ib(F), \mathscr{F}ib(G)))$$

(which we can view as an isomorphism of discrete categories). Namely, to a morphism $\psi : F \to G$ one assigns the functor

$$\mathscr{F}ib(\psi) : \mathscr{F}ib(F) \to \mathscr{F}ib(G) \qquad (X, s) \mapsto (X, \psi_X(s)) \quad \text{for every } (X, s) \in \mathrm{Ob}(\mathscr{F}ib(F)).$$

Example 3.1.16. Let U be a universe, and $\mathscr{C}$ a category with U-small Hom-sets.

(i) We have already remarked in example 1.4.9(ii) the identification of $\mathscr{C}$-fibrations

$$\mathscr{F}ib(h_X) = \mathscr{C}/X \qquad \text{for every } X \in \mathrm{Ob}(\mathscr{C})$$

where h_X is the presheaf represented by X (see (1.2.4)), and $\mathscr{C}/X$ is regarded as a $\mathscr{C}$-fibration as in example 3.1.3(i). Moreover, every morphism $f : X \to Y$ of $\mathscr{C}$ induces a morphism of presheaves $h_f : h_Y \to h_X$, and under the foregoing identification, $\mathscr{F}ib(h_f)$ corresponds to the functor $f_* : \mathscr{C}/X \to \mathscr{C}/Y$ of (1.1.25).

(ii) In the situation of (i), let $u : \mathscr{C}' \to \mathscr{C}$ be any functor; as in example 1.4.9(iii), by direct inspection we get a natural isomorphism of categories

$$\mathsf{Fib}(u)^*(\mathscr{C}/X) \stackrel{\sim}{\to} \mathscr{F}ib(h_X \circ u) = u\mathscr{C}'/X \qquad \text{for every } X \in \mathrm{Ob}(\mathscr{C})$$

(notation of remark 3.1.5(i)) which identifies the fibration $\mathsf{Fib}(u)^*(\mathscr{C}/X) \to \mathscr{C}'$ with the source functor $\mathsf{s}_X : u\mathscr{C}'/X \to \mathscr{C}'$ of (1.1.27). Likewise, for every morphism $f : X \to Y$ in $\mathscr{C}$, the cartesian functor $\mathsf{Fib}(u)^*(f_*) : \mathsf{Fib}(u)^*(\mathscr{C}/X) \to \mathsf{Fib}(u)^*(\mathscr{C}/Y)$ is identified with the functor $u\mathscr{C}'/f : u\mathscr{C}'/X \to u\mathscr{C}'/Y$ as in (1.1.27).

(iii) As a special case, let $h_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}^\wedge_{\mathsf{U}}$ be the Yoneda embedding, and F any U-presheaf on $\mathscr{C}$. As noted in (1.4.7), we have a natural isomorphism of categories $h_{\mathscr{C}}\mathscr{C}/F \stackrel{\sim}{\to} \mathscr{F}ib(F)$, so (i) generalizes to a natural isomorphism of $\mathscr{C}$-fibrations :

$$\mathscr{F}ib(F) \stackrel{\sim}{\to} (\mathsf{V}, \mathsf{V})\text{-}\mathsf{Fib}(h_{\mathscr{C}})^*(\mathscr{C}^\wedge_{\mathsf{U}}/F)$$

for any universe V with $\mathsf{U} \subset \mathsf{V}$, and such that $\mathscr{C}$ is V-small.

3.1.17. Conversely, let $\varphi : \mathscr{A} \to \mathscr{B}$ be a fibration *with discrete fibres*, *i.e.* such that $\varphi^{-1}B$ is a discrete small category for every $B \in \mathrm{Ob}(\mathscr{B})$. Then it is easily seen that for every morphism $g : B' \to B$ of $\mathscr{B}$, every $A \in \varphi^{-1}B$ admits a unique inverse image over g. It follows that φ admits a unique cleavage. Moreover the associated pseudo-functor $\mathsf{c} : \mathscr{B}^o \to \mathbf{Cat}$ is strict, hence it may be regarded as a functor with values in the subcategory $\mathbf{Set}$ of $\mathbf{Cat}$; *i.e.* c is a presheaf on $\mathscr{B}$, and a simple inspection shows that the fibration $\mathscr{F}ib(\mathsf{c})$ is naturally isomorphic to φ. Summing up, we have obtained a fully faithful functor

$$\mathscr{F}ib : \mathscr{B}^\wedge \to \mathsf{Fib}(\mathscr{B})$$

whose essential image is the full subcategory of $\mathsf{Fib}(\mathscr{B})$ whose objects are the fibrations with discrete fibres. We shall show next how to extend this equivalence to the whole of $\mathsf{Fib}(\mathscr{B})$.

3.1.18. *Fibration associated with a pseudo-functor.* Let $\mathscr{B}$ be any V-small category, and $\mathsf{c} : \mathscr{B}^o \to (\mathsf{U},\mathsf{V})\mathbf{Cat}$ any pseudo-functor, with coherence constraint $(\delta^{\mathsf{c}}, \gamma^{\mathsf{c}})$. We attach to c the $\mathscr{B}$-category

$$\pi^{\mathsf{c}} : \mathscr{F}ib(\mathsf{c}) \to \mathscr{B}$$

such that $\mathrm{Ob}(\mathscr{F}ib(\mathsf{c})) := \{(B,X) \mid B \in \mathrm{Ob}(\mathscr{B}),\ X \in \mathrm{Ob}(\mathsf{c}_B)\}$, and where

$$\mathrm{Hom}_{\mathscr{F}ib(\mathsf{c})}((B,X),(B',Y)) := \{(\varphi,f) \mid \varphi \in \mathrm{Hom}_{\mathscr{B}}(B,B'),\ f \in \mathrm{Hom}_{\mathsf{c}_B}(X,\mathsf{c}_\varphi Y)\}$$

for every two objects $(B,X),(B',Y)$. The composition of two morphisms $f : X \to \mathsf{c}_\varphi Y$ and $g : Y \to \mathsf{c}_\psi Z$ is the pair $(\psi\circ\varphi, t)$, where t is the composition

$$X \xrightarrow{\ f\ } \mathsf{c}_\varphi Y \xrightarrow{\mathsf{c}_\varphi(g)} \mathsf{c}_\varphi\mathsf{c}_\psi Z \xrightarrow{\gamma^{\mathsf{c}}_{(\varphi,\psi),Z}} \mathsf{c}_{\psi\circ\varphi} Z.$$

The unit axiom for δ^{c} implies easily that for every object (B,X) the morphism

$$(\mathbf{1}_B, \delta^{\mathsf{c}}_{B,X} : X \to \mathsf{c}_{\mathbf{1}_B} X)$$

is neutral for left and right composition with any other morphism of $\mathscr{F}ib(\mathsf{c})$. Let us show the associativity : if $h : Z \to \mathsf{c}_\lambda W$ is a third morphism, we need to verify the identity

$$\gamma^{\mathsf{c}}_{(\psi\circ\varphi,\lambda),W} \circ \mathsf{c}_{\psi\circ\varphi}(h) \circ \gamma^{\mathsf{c}}_{(\varphi,\psi),Z} \circ \mathsf{c}_\varphi(g) \circ f = \gamma^{\mathsf{c}}_{(\varphi,\lambda\circ\psi),W} \circ \mathsf{c}_\varphi(\gamma^{\mathsf{c}}_{(\psi,\lambda),W}) \circ \mathsf{c}_\varphi\mathsf{c}_\psi(h) \circ \mathsf{c}_\varphi(g) \circ f.$$

But we have $\mathsf{c}_{\psi\circ\varphi}(h) \circ \gamma^{\mathsf{c}}_{(\varphi,\psi),Z} = \gamma^{\mathsf{c}}_{(\varphi,\psi),\mathsf{c}_\lambda W} \circ \mathsf{c}_\varphi\mathsf{c}_\psi(h)$ by the naturality of $\gamma^{\mathsf{c}}_{(\varphi,\psi)}$, hence we are reduced to checking that

$$\gamma^{\mathsf{c}}_{(\psi\circ\varphi,\lambda),W} \circ \gamma^{\mathsf{c}}_{(\varphi,\psi),\mathsf{c}_\lambda W} = \gamma^{\mathsf{c}}_{(\varphi,\lambda\circ\psi),W} \circ \mathsf{c}_\varphi(\gamma^{\mathsf{c}}_{(\psi,\lambda),W})$$

which follows from the composition axioms for γ^{c}. The functor π^{c} is given by the rules : $(B,X) \mapsto B$ and $(\varphi,f) \mapsto \varphi$ for every object (B,X) and every morphism (φ,f) of $\mathscr{F}ib(\mathsf{c})$.

3.1.19. Let $\mathsf{d} : \mathscr{B}^o \to (\mathsf{U},\mathsf{V})\text{-}\mathbf{Cat}$ be another pseudo-functor, with coherence constraint $(\delta^{\mathsf{d}}, \gamma^{\mathsf{d}})$, and

$$\omega : \mathsf{c} \Rightarrow \mathsf{d}$$

any pseudo-natural transformation, with coherence constraint τ^ω. We define a $\mathscr{B}$-functor

$$\mathscr{F}ib(\omega) : \mathscr{F}ib(\mathsf{c}) \to \mathscr{F}ib(\mathsf{d})$$

by assigning to every object (B,X) of $\mathscr{F}ib(\mathsf{c})$ the object $(B,\omega_B X)$ of $\mathscr{F}ib(\mathsf{d})$, and to every morphism $(\varphi,f) : (B,X) \to (B',Y)$ the morphism (φ,t), where t is the composition

$$\omega_B X \xrightarrow{\omega_B f} \omega_B \circ \mathsf{c}_\varphi Y \xrightarrow{(\tau^\omega_{\varphi,Y})^{-1}} \mathsf{d}_\varphi \circ \omega_{B'} Y.$$

To check that these rules do yield a functor, consider any other morphism $(\psi,g) : (B',Y) \to (B'',Z)$ of $\mathscr{F}ib(\mathsf{c})$; we need to show that

$$\tau^{\omega\ -1}_{\psi\circ\varphi,Z} \circ \omega_B(\gamma^{\mathsf{c}}_{(\varphi,\psi),Z} \circ \mathsf{c}_\varphi(g) \circ f) = \gamma^{\mathsf{d}}_{(\varphi,\psi),\omega_{B''}Z} \circ \mathsf{d}_\varphi(\tau^{\omega\ -1}_{\psi,Z} \circ \omega_{B'}(g)) \circ \tau^{\omega\ -1}_{\varphi,Y} \circ \omega_B(f).$$

However, the naturality of τ^ω_φ implies that $\tau^{\omega\ -1}_{\varphi,\mathsf{c}_\psi Z} \circ \omega_B(\mathsf{c}_\varphi g) = \mathsf{d}_\varphi(\omega_{B'} g) \circ \tau^{\omega\ -1}_{\varphi,Y}$, so we are reduced to checking that :

$$\tau^{\omega\ -1}_{\psi\circ\varphi,Z} \circ \omega_B(\gamma^{\mathsf{c}}_{(\varphi,\psi),Z}) = \gamma^{\mathsf{d}}_{(\varphi,\psi),\omega_{B''}Z} \circ \mathsf{d}_\varphi(\tau^{\omega\ -1}_{\psi,Z}) \circ \tau^{\omega\ -1}_{\varphi,\mathsf{c}_\psi Z}.$$

But the latter follows directly from the coherence axioms for τ^ω. Likewise, the assertion that $\mathscr{F}ib(\omega)$ respects identity morphisms comes down to the equalities :

$$\tau^{\omega\ -1}_{\mathbf{1}_B,X} \circ \omega_B(\delta^{\mathsf{c}}_{B,X}) = \delta^{\mathsf{d}}_{B,\omega_B X}$$

which again follows from the coherence axioms for τ^ω. Moreover, if $\mathsf{e} : \mathscr{B}^o \to (\mathsf{U},\mathsf{V})\text{-}\mathbf{Cat}$ is a third pseudo-functor, and

$$\mu : \mathsf{d} \Rightarrow \mathsf{e}$$

a second pseudo-natural transformation, we have

$$\mathscr{F}ib(\mu) \circ \mathscr{F}ib(\omega) = \mathscr{F}ib(\mu \odot \omega).$$

Indeed, if τ^{μ} and $\tau^{\mu\odot\omega}$ are the coherence constraints for μ and $\mu \odot \omega$, the assertion amounts to :

$$\tau^{\mu\ -1}_{\varphi,\omega_{B'}Y} \circ \mu_B(\tau^{\omega}_{\varphi,Y})^{-1} \circ \mu_B(\omega_B f) = \tau^{\mu\odot\omega\ -1}_{B,Y} \circ \mu_B(\omega_B f)$$

for every morphism $(\varphi, f) : (B, X) \to (B', Y)$ in $\mathscr{F}ib(\mathsf{c})$, which is clear from the definition of $\mu \odot \omega$.

Lemma 3.1.20. *With the notation of* (3.1.19)*, the following holds :*

(i) *A morphism* $(\varphi, f) : (B, X) \to (B', Y)$ *of* $\mathscr{F}ib(\mathsf{c})$ *is* π^{c}*-cartesian if and only if* $f : X \to \mathsf{c}_{\varphi}Y$ *is an isomorphism in* c_B*.*

(ii) *The functor* π^{c} *is a fibration, and is endowed with a distinguished cleavage*

$$\boldsymbol{\lambda}^* : \mathscr{B}/\pi^{\mathsf{c}}\mathscr{F}ib(\mathsf{c}) \to \mathsf{Morph}(\mathscr{F}ib(\mathsf{c})) \qquad (X, \varphi) \mapsto (\varphi, \mathbf{1}_{\mathsf{c}_{\varphi}X}).$$

(iii) *The functor* $\mathscr{F}ib(\omega)$ *is* $\mathscr{B}$*-cartesian.*

Proof. (i): In view of proposition 2.4.3 and the discussion of (3.1.19), we may assume that $\boldsymbol{\lambda}$ is unital (details left to the reader). Now, suppose first that (φ, f) is $\mathscr{B}$-cartesian; then there exists a unique morphism $(\mathbf{1}_B, g) : (B, \mathsf{c}_{\varphi}Y) \to (B, X)$ such that $(\varphi, f) \circ (\mathbf{1}_{\mathbb{B}}, g) = (\varphi, \mathbf{1}_{\mathsf{c}_{\varphi}Y})$. Since c is unital, we easily deduce that $f \circ g = \mathbf{1}_{\mathsf{c}_{\varphi}Y}$. Moreover, $(\mathbf{1}_B, g \circ f) : (B, X) \to (B, X)$ is a morphism of $\mathscr{F}ib(\mathsf{c})$ such that $(\varphi, f) \circ (\mathbf{1}_B, g \circ f) = (\varphi, f) = (\varphi, f) \circ (\mathbf{1}_B, \mathbf{1}_X)$, whence $g \circ f = \mathbf{1}_X$. Conversely, suppose that f is an isomorphism, and let $\psi : C \to B$ any morphism in $\mathscr{B}$, $Z \in \mathrm{Ob}(\mathsf{c}_C)$ any object, and $(\varphi \circ \psi, h) : (C, Z) \to (B', Y)$ any morphism of $\mathscr{F}ib(\mathsf{c})$; we let $k : Z \to \mathsf{c}_{\psi}X$ be the composition

$$Z \xrightarrow{h} \mathsf{c}_{\varphi\circ\psi}Y \xrightarrow{\gamma^{\mathsf{c}\ -1}_{(\psi,\varphi),Y}} \mathsf{c}_{\psi}\mathsf{c}_{\varphi}Y \xrightarrow{\mathsf{c}_{\varphi}(f^{-1})} \mathsf{c}_{\psi}X.$$

Then $(\psi, k) : (C, Z) \to (B, X)$ is the unique morphism of $\mathscr{F}ib(\mathsf{c})$ such that $(\varphi, f) \circ (\psi, k) = (\varphi \circ \psi, h)$; this shows that (φ, f) is $\mathscr{B}$-cartesian.

(ii) and (iii) are immediate consequences of (i). □

3.1.21. Lastly, consider pseudo-natural transformations

$$\omega, \omega' : \mathsf{c} \Rightarrow \mathsf{d} \qquad \text{and a modification} \qquad \Xi : \omega \rightsquigarrow \omega'.$$

We attach to Ξ the natural transformation of functors

$$\mathscr{F}ib(\Xi) : \mathscr{F}ib(\omega) \Rightarrow \mathscr{F}ib(\omega')$$

assigning to every object (B, X) of $\mathscr{F}ib(\mathsf{c})$ the morphism $(\mathbf{1}_B, t)$ of $\mathscr{F}ib(\mathsf{d})$, where t is the composition

$$\omega_B X \xrightarrow{\Xi_{B,X}} \omega'_B X \xrightarrow{\delta^{\mathsf{d}}_{B,\omega'_B X}} \mathsf{d}_{\mathbf{1}_B}(\omega'_B X).$$

In order to verify the naturality of $\mathscr{F}ib(\Xi)$, it suffices to check the commutativity of the diagram:

$$\begin{array}{ccccc}
\omega_B X & \xrightarrow{\Xi_{B,X}} & \omega'_B X & \xrightarrow{\delta^{\mathsf{d}}_{B,\omega'_B X}} & \mathsf{d}_{\mathbf{1}_B}(\omega'_B X) \\
{\scriptstyle \omega_B f}\downarrow & & {\scriptstyle \omega'_B f}\downarrow & & \downarrow{\scriptstyle \mathsf{d}_{\mathbf{1}_B}(\omega'_B f)} \\
\omega_B \mathsf{c}_{\varphi} Y & \xrightarrow{\Xi_{B,\mathsf{c}_{\varphi}Y}} & \omega'_B \mathsf{c}_{\varphi} Y & \xrightarrow{\delta^{\mathsf{d}}_{B,\omega'_B \mathsf{c}_{\varphi}Y}} & \mathsf{d}_{\mathbf{1}_B}(\omega'_B \mathsf{c}_{\varphi} Y) \\
{\scriptstyle \tau^{\omega\ -1}_{\varphi,Y}}\downarrow & & {\scriptstyle \tau^{\omega'\ -1}_{\varphi,Y}}\downarrow & & \downarrow{\scriptstyle \gamma^{\mathsf{d}}_{(\mathbf{1}_B,\varphi),\omega'_{B'}Y} \circ \mathsf{d}_{\mathbf{1}_B}(\tau^{\omega'\ -1}_{\varphi,Y})} \\
\mathsf{d}_{\varphi}\omega_{B'} Y & \xrightarrow{\mathsf{d}_{\varphi}(\Xi_{B',Y})} & \mathsf{d}_{\varphi}\omega'_{B'} Y & \xrightarrow{\gamma^{\mathsf{d}}_{(\varphi,\mathbf{1}_B),\omega'_{B'}Y} \circ \mathsf{d}_{\varphi}(\delta^{\mathsf{d}}_{B',\omega'_{B'}Y})} & \mathsf{d}_{\varphi}\omega'_{B'} Y
\end{array}$$

for every morphism $(\varphi, f) : (B, X) \to (B', Y)$ of $\mathscr{F}ib(\mathsf{c})$. However, the commutativity of the two top squares follows from the naturality of Ξ_B and δ^{d}_B, and that of the left bottom square translates the compatibility condition for Ξ. Then, since

$$\gamma^{\mathsf{d}}_{(\mathbf{1}_B,\varphi),\omega'_{B'}Y} = \delta^{\mathsf{d}\ -1}_{B',\mathsf{d}_\varphi\omega'_{B'}Y} \qquad \text{and} \qquad \gamma^{\mathsf{d}}_{(\varphi,\mathbf{1}_B),\omega'_{B'}Y} \circ \mathsf{d}_\varphi(\delta^{\mathsf{d}}_{B',\omega'_{B'}Y}) = \mathbf{1}_{\mathsf{d}_\varphi\omega'_{B'}Y}$$

we are reduced to checking the identity

$$\mathsf{d}_{\mathbf{1}_B}(\tau^{\omega'}_{\varphi,Y}) \circ \delta^{\mathsf{d}}_{B',\mathsf{d}_\varphi\omega'_{B'}Y} = \delta^{\mathsf{d}}_{B,\omega'_B\mathsf{c}_\varphi Y} \circ \tau^{\omega'}_{\varphi,Y}$$

which follows from the naturality of δ^{d}_B. Furthermore, suppose we have a third pseudo-natural transformation

$$\omega'' : \mathsf{c} \Rightarrow \mathsf{d} \qquad \text{and a modification} \qquad \Xi' : \omega' \rightsquigarrow \omega''.$$

Then we have as well

$$\mathscr{F}ib(\Xi' \odot \Xi) = \mathscr{F}ib(\Xi') \odot \mathscr{F}ib(\Xi). \tag{3.1.22}$$

Indeed, the assertion amounts to checking, for every object (B, X) of $\mathscr{F}ib(\mathsf{c})$, the identity

$$\gamma^{\mathsf{d}}_{(\mathbf{1}_B,\mathbf{1}_B),\omega''_BX} \circ \mathsf{d}_{\mathbf{1}_B}(\delta^{\mathsf{d}}_{B,\omega''_BX}) \circ \mathsf{d}_{\mathbf{1}_B}(\Xi'_{B,X}) \circ \delta^{\mathsf{d}}_{B,\omega'_BX} \circ \Xi_{B,X} = \delta^{\mathsf{d}}_{B,\omega''_BX} \circ \Xi'_{B,X} \circ \Xi_{B,X}$$

which is clear, since $\gamma^{\mathsf{d}}_{(\mathbf{1}_B,\mathbf{1}_B),\omega''_BX} \circ \mathsf{d}_{\mathbf{1}_B}(\delta^{\mathsf{d}}_{B,\omega''_BX}) = \mathbf{1}_{\mathsf{d}_{\mathbf{1}_B}\omega''_BX}$ by the unit axiom for δ^{d}, and $\mathsf{d}_{\mathbf{1}_B}(\Xi'_{B,X}) \circ \delta^{\mathsf{d}}_{B,\omega'_BX} = \delta^{\mathsf{d}}_{B,\omega''_BX} \circ \Xi'_{B,X}$, by the naturality of δ^{d}_B. Likewise, suppose we have two pseudo-natural transformations

$$\mu, \mu' : \mathsf{d} \Rightarrow \mathsf{e} \qquad \text{and a modification} \qquad \Theta : \mu \rightsquigarrow \mu'.$$

Then we have as well

$$\mathscr{F}ib(\Theta) * \mathscr{F}ib(\Xi) = \mathscr{F}ib(\Theta * \Xi).$$

For the proof, in light of (3.1.22) we may assume that either $\Theta = \mathbf{1}_\mu$ or $\Xi = \mathbf{1}_\omega$. In the first case, the assertion comes down to the identity

$$(\tau^{\mu}_{\mathbf{1}_B,\omega'_BX})^{-1} \circ \mu_B(\delta^{\mathsf{d}}_{B,\omega'_BX} \circ \Xi_{B,X}) = \delta^{\mathsf{d}}_{B,\mu_B\omega'_BX} \circ \mu_B(\Xi_{B,X})$$

which is equivalent to : $\mu_B(\delta^{\mathsf{d}}_{B,\omega'_BX}) = \tau^{\mu}_{\mathbf{1}_B,\omega'_BX} \circ \delta^{\mathsf{d}}_{B,\mu_B\omega'_BX}$, which in turns follows from the coherence axioms for τ^μ. The case where $\Xi = \mathbf{1}_\omega$ is clear by a simple inspection. Summing up, for every universe V, and every V-small category $\mathscr{B}$, we have obtained a strict pseudo-functor

$$\mathscr{F}ib_{\mathscr{B}} : \mathsf{PsFun}(\mathscr{B}^o, (\mathsf{U},\mathsf{V})\text{-}\mathbf{Cat}) \to (\mathsf{U},\mathsf{V})\text{-}\mathsf{Fib}(\mathscr{B}).$$

Remark 3.1.23. (i) Notice that the discussion of (3.1.19) requires the invertibility of the coherence constraint τ^ω, and therefore it does not apply to general lax-natural transformations $\omega : \mathsf{c} \Rightarrow \mathsf{d}$. However, with the obvious changes, it does apply to lax-natural transformations $\omega : {}^o\mathsf{c} \Rightarrow {}^o\mathsf{d}$ (see example 2.2.6(ii)) : the details shall be left to the reader. From lemma 3.1.20 and its proof, we then easily see that the resulting functor $\mathscr{F}ib(\omega) : \mathscr{F}ib(\mathsf{c}) \to \mathscr{F}ib(\mathsf{d})$ will be cartesian if and only if ω is pseudo-natural.

(ii) Likewise, the discussion of (3.1.21) applies more generally to a modification $\Xi : \omega' \rightsquigarrow \omega$ between any pair of lax-natural transformations $\omega, \omega' : {}^o\mathsf{c} \Rightarrow {}^o\mathsf{d}$; namely, any such modification induces a natural transformation $\mathscr{F}ib(\Xi) : \mathscr{F}ib(\omega) \Rightarrow \mathscr{F}ib(\omega')$.

Theorem 3.1.24. *For every category $\mathscr{B}$, the pseudo-functor $\mathscr{F}ib_{\mathscr{B}}$ is a strong 2-equivalence.*

Proof. Let $\varphi : \mathscr{A} \to \mathscr{B}$ be any fibration with essentially small fibres; pick a unital cleavage $\boldsymbol{\lambda}$ for φ, and let $\mathsf{c} : \mathscr{B}^o \to (\mathsf{U},\mathsf{V})\text{-}\mathbf{Cat}$ be the pseudo-functor associated with $\boldsymbol{\lambda}$.

Claim 3.1.25. There exists an isomorphism of $\mathscr{B}$-categories

$$\psi^{\boldsymbol{\lambda}} : \mathscr{F}ib(\mathsf{c}) \xrightarrow{\sim} \mathscr{A} \qquad (B, X) \mapsto X \qquad ((B, X) \xrightarrow{(g,f)} (B', X')) \mapsto \boldsymbol{\lambda}(X', g) \circ f.$$

Proof of the claim. Indeed, since c is unital, it is clear that $\psi^{\boldsymbol{\lambda}}(\mathbf{1}_{(B,X)}) = \mathbf{1}_{\psi(B,X)}$ for every $(B,X) \in \mathrm{Ob}(\mathscr{F}ib(\mathsf{c}))$. Next, let $(g,f) : (B,X) \to (B',X')$ and $(g',f') : (B',X') \to (B'',X'')$ be two morphisms; then $(g',f') \circ (g,f) = (g' \circ g, \gamma^{\mathsf{c}}_{(g,g'),X''} \circ \mathsf{c}_g(f') \circ f)$, and on the other hand, we may compute

$$\begin{aligned}\psi^{\boldsymbol{\lambda}}(g',f') \circ \psi^{\boldsymbol{\lambda}}(g,f) &= \boldsymbol{\lambda}(X'',g') \circ f' \circ \boldsymbol{\lambda}(X',g) \circ f \\ &= \boldsymbol{\lambda}(X'',g') \circ \boldsymbol{\lambda}(\mathsf{c}_{g'}X'',g) \circ \mathsf{c}_g(f') \circ f \\ &= \boldsymbol{\lambda}(X'',g'g) \circ \gamma^{\mathsf{c}}_{(g,g'),X''} \circ \mathsf{c}_g(f') \circ f \\ &= \psi^{\boldsymbol{\lambda}}((g',f') \circ (g,f))\end{aligned}$$

as required. Clearly $\psi^{\boldsymbol{\lambda}}$ is bijective on objects, and since $\boldsymbol{\lambda}(A,g)$ is a φ-cartesian morphism for every object (A,g) of $\mathscr{B}/\varphi\mathscr{A}$, it is easily seen that $\psi^{\boldsymbol{\lambda}}$ is fully faithful, hence it is an isomorphism of $\mathscr{B}$-categories. ◊

In view of claim 3.1.25, it remains only to check that for every pair of pseudo-functors $\mathsf{c}, \mathsf{d} : \mathscr{B}^o \to (\mathsf{U},\mathsf{V})\text{-}\mathbf{Cat}$ the induced functor

$$\mathsf{PsNat}(\mathsf{c},\mathsf{d}) \to \mathsf{Cart}_{\mathscr{B}}(\mathscr{F}ib(\mathsf{c}), \mathscr{F}ib(\mathsf{d})) \tag{3.1.26}$$

is an isomorphism of categories, and due to proposition 2.4.3 and the discussion of (3.1.19), we may assume that c and d are unital. Thus, let $\omega, \omega' : \mathsf{c} \Rightarrow \mathsf{d}$ be two pseudo-natural transformations, with respective coherence constraints τ^ω and $\tau^{\omega'}$, and suppose that $\mathscr{F}ib(\omega) = \mathscr{F}ib(\omega')$; then clearly $\omega_B X = \omega'_B X$ for every object (B,X) of $\mathscr{F}ib(\mathsf{c})$. Likewise, in light of remark 2.4.2(ii) we see that $\tau^\omega_{\mathbf{1}_B,X} = \tau^{\omega'}_{\mathbf{1}_B,X} = \mathbf{1}_{\omega_B X}$ for every such (B,X), whence $\omega_B f = \omega'_B f$ for every $B \in \mathrm{Ob}(\mathscr{B})$ and every morphism f of c_B. Then it is also clear that $\tau^\omega_{\varphi,Y} = \tau^{\omega'}_{\varphi,Y}$ for every morphism $\varphi : B \to B'$ of $\mathscr{B}$ and every $Y \in \mathrm{Ob}(\mathsf{c}_{B'})$, *i.e.* $\omega = \omega'$.

Next, let $F : \mathscr{F}ib(\mathsf{c}) \to \mathscr{F}ib(\mathsf{d})$ be any $\mathscr{B}$-cartesian functor; for every object (B,X) of $\mathscr{F}ib(\mathsf{c})$ we define $\omega_B X$ as the unique object of d_B such that $F(B,X) = (B, \omega_B X)$. Likewise – and since d is unital – for every morphism of $\mathscr{F}ib(\mathsf{c})$ of the form $(\mathbf{1}_B, f) : (B,X) \to (B,Y)$ we may define $\omega_B f : \omega_B X \to \omega_B Y$ so that $F(\mathbf{1}_B, f) = (\mathbf{1}_B, \omega_B f)$. Since both c and d are unital, it is easily seen that these rules yield a well defined functor $\omega_B : \mathsf{c}_B \to \mathsf{d}_B$. To define a coherence constraint τ^ω for this system $(\omega_B \mid B \in \mathrm{Ob}(\mathscr{B}))$, notice that for every morphism $\varphi : B \to B'$ of $\mathscr{B}$ and every $Y \in \mathrm{Ob}(\mathsf{c}_{B'})$ we have the $\mathscr{B}$-cartesian morphism $(\varphi, \mathbf{1}_{\mathsf{c}_\varphi Y}) : (B, \mathsf{c}_\varphi Y) \to (B', Y)$ of $\mathscr{F}ib(\mathsf{c})$, and let $t : \mathsf{d}_\varphi \circ \omega_{B'} Y \to \omega_B \circ \mathsf{c}_\varphi Y$ be the unique morphism of d_B such that $F(\varphi, \mathbf{1}_{\mathsf{c}_\varphi Y}) = (\varphi, t) : (B, \omega_B \mathsf{c}_\varphi Y) \to (B', \omega_{B'} Y)$; in light of lemma 3.1.20(i) we see that t is an isomorphism of d_B, and we set $\tau^\omega_{\varphi,Y} := t^{-1} : \mathsf{d}_\varphi \circ \omega_{B'} Y \to \omega_B \circ \mathsf{c}_\varphi Y$. To check the naturality of the rule $Y \mapsto \tau^\omega_{\varphi,Y}$, notice that every morphism $f : X \to Y$ in $\mathsf{c}_{B'}$ yields a commutative diagram :

$$\begin{array}{ccc} (B, \mathsf{c}_\varphi X) & \xrightarrow{(\mathbf{1}_B, \mathsf{c}_\varphi f)} & (B, \mathsf{c}_\varphi Y) \\ {\scriptstyle (\varphi, \mathbf{1}_{\mathsf{c}_\varphi X})}\downarrow & & \downarrow{\scriptstyle (\varphi, \mathbf{1}_{\mathsf{c}_\varphi Y})} \\ (B', X) & \xrightarrow{(\mathbf{1}_{B'}, f)} & (B', Y) \end{array}$$

in $\mathscr{F}ib(\mathsf{c})$, whence the identity :

$$(\varphi, \tau^{\omega\,-1}_{\varphi,Y}) \circ (\mathbf{1}_B, \omega_B \mathsf{c}_\varphi f) = (\mathbf{1}_{B'}, \omega_{B'} f) \circ (\varphi, \tau^{\omega\,-1}_{\varphi,X}) \qquad \text{in } \mathscr{F}ib(\mathsf{d})$$

which in turn translates as the identity $\tau^{\omega\,-1}_{\varphi,Y} \circ \omega_B \mathsf{c}_\varphi(f) = \mathsf{d}_\varphi \omega_{B'}(f) \circ \tau^{\omega\,-1}_{\varphi,X}$ in d_B, as required. Lastly, we need to verify the coherence axioms for τ^ω. However, since c and d are unital, the first coherence axiom amounts to the identity

$$\tau^\omega_{\mathbf{1}_B} = \mathbf{1}_{\omega_B} \qquad \text{for every } B \in \mathrm{Ob}(\mathscr{B})$$

which is immediate from the construction of τ^ω. Next, let $\varphi : B \to B'$ and $\psi : B' \to B''$ be two morphisms of $\mathscr{B}$, and X any object of $\mathsf{c}_{B''}$; we notice the commutative diagram :

$$\begin{array}{ccc} (B, \mathsf{c}_\varphi \mathsf{c}_\psi X) & \xrightarrow{(\varphi, \mathbf{1}_{\mathsf{c}_\varphi \mathsf{c}_\psi X})} & (B', \mathsf{c}_\psi X) \\ {\scriptstyle (\mathbf{1}_B, \gamma^{\mathsf{c}}_{(\varphi,\psi),X})}\downarrow & & \downarrow{\scriptstyle (\psi, \mathbf{1}_{\mathsf{c}_\psi X})} \\ (B, \mathsf{c}_{\psi\varphi} X) & \xrightarrow{(\psi\varphi, \mathbf{1}_{\mathsf{c}_{\psi\varphi} X})} & (B'', X). \end{array} \qquad \text{in } \mathscr{F}ib(\mathsf{c})$$

which, after applying the functor F, yields the identity in $\mathscr{F}ib(\mathsf{d})$:

$$(\psi, \tau^{\omega\ -1}_{\psi,X}) \circ (\varphi, \tau^{\omega\ -1}_{\varphi,\mathsf{c}_\psi X}) = (\psi\varphi, \tau^{\omega\ -1}_{\psi\varphi,X}) \circ (\mathbf{1}_B, \omega_B(\gamma^{\mathsf{c}}_{(\varphi,\psi),X})).$$

which in turn translates as the second coherence axiom. A direct inspection now gives :

$$\mathscr{F}ib(\omega) = F.$$

Summing up, we have shown that (3.1.26) is bijective on objects. A simple inspection shows that (3.1.26) is also injective on morphisms. To conclude, consider therefore two pseudo-natural transformations $\omega, \omega' : \mathsf{c} \to \mathsf{d}$ and a natural transformation $\xi : \mathscr{F}ib(\omega) \Rightarrow \mathscr{F}ib(\omega')$ such that $\pi^{\mathsf{d}} * \xi = \mathbf{1}_{\pi^{\mathsf{c}}}$, and for every object (B, X) of $\mathscr{F}ib(\mathsf{c})$, let $\Xi_{B,X} : \omega_B X \to \omega'_B X$ be the unique morphism of d_B such that $\xi_{(B,X)} = (\mathbf{1}_B, \Xi_{B,X})$. We need to check that the rule: $(B, X) \mapsto \Xi_{B,X}$ yields a well defined modification $\omega \rightsquigarrow \omega'$, in which case we shall have $\mathscr{F}ib(\Xi) = \xi$, by inspection. However, for every $B \in \mathrm{Ob}(\mathscr{B})$, the naturality of the rule : $X \mapsto \Xi_{B,X}$ follows from a simple inspection. It remains thus only to verify the compatibility condition for Ξ. To this aim, let $\varphi : B \to B'$ be any morphism of $\mathscr{B}$, and Y any object of $\mathsf{c}_{B'}$; the morphism $(\varphi, \mathbf{1}_{\mathsf{c}_\varphi Y}) : (B, \mathsf{c}_\varphi Y) \to (B', Y)$ of $\mathscr{F}ib(\mathsf{c})$ yields a commutative diagram

$$\begin{array}{ccc} (B, \omega_B \mathsf{c}_\varphi Y) & \xrightarrow{\xi_{(B,\mathsf{c}_\varphi Y)}} & (B, \omega'_B \mathsf{c}_\varphi Y) \\ {\scriptstyle (\varphi, \tau^{\omega\ -1}_{\varphi,Y})}\downarrow & & \downarrow{\scriptstyle (\varphi, \tau^{\omega'\ -1}_{\varphi,Y})} \\ (B', \omega_{B'} Y) & \xrightarrow{\xi_{(B',Y)}} & (B', \omega'_{B'} Y) \end{array} \qquad \text{in } \mathscr{F}ib(\mathsf{d})$$

which in turn is equivalent to the required identity : we leave the details to the reader. □

Remark 3.1.27. (i) Let $\varphi : \mathscr{A} \to \mathscr{B}$ be any fibration, and c the pseudo-functor associated with a given cleavage $\boldsymbol{\lambda}$ for φ. Claim 3.1.25 exhibits a natural isomorphism $\psi^{\boldsymbol{\lambda}} : \mathscr{F}ib(\mathsf{c}) \xrightarrow{\sim} \mathscr{A}$ and the fibration $\pi^{\mathsf{c}} : \mathscr{F}ib(\mathsf{c}) \to \mathscr{B}$ carries the distinguished cleavage $\boldsymbol{\lambda}^*$ provided by lemma 3.1.20(ii). By inspecting the constructions, we find

$$\psi^{\boldsymbol{\lambda}}(\boldsymbol{\lambda}^*(X, g)) = \psi^{\boldsymbol{\lambda}}(g, \mathbf{1}_{\mathsf{c}_\varphi X}) = \boldsymbol{\lambda}(X, g)$$

for every morphism $g : B' \to B$ of $\mathscr{B}$ and every object X of $\varphi^{-1}B$, whence :

$$\psi^{\boldsymbol{\lambda}} \circ \boldsymbol{\lambda}^* = \boldsymbol{\lambda}.$$

Denote by c^* the pseudo-functor associated with $\boldsymbol{\lambda}^*$; it follows that the restrictions of $\psi^{\boldsymbol{\lambda}}$

$$\psi^{\boldsymbol{\lambda}}_{|B} : \mathscr{F}ib(\mathsf{c})_B \xrightarrow{\sim} \mathscr{A}_B \qquad \text{for every } B \in \mathrm{Ob}(\mathscr{B})$$

define a strict pseudo-natural isomorphism of pseudo-functors

$$\psi^{\boldsymbol{\lambda}}_{|\bullet} : \mathsf{c}^* \xrightarrow{\sim} \mathsf{c}.$$

(ii) Let $\mathscr{B}, \mathscr{C}$ be two V-small categories (for some universe V containing U), $\rho : \mathscr{C} \to \mathscr{B}$ a functor, and $\mathsf{c} : \mathscr{B}^o \to (\mathsf{U}, \mathsf{V})\text{-}\mathbf{Cat}$ a pseudo-functor. We have a natural commutative diagram

$$\begin{array}{ccc} \mathscr{F}ib(\mathsf{c} \circ \rho^o) & \xrightarrow{\mathscr{F}ib(\rho)} & \mathscr{F}ib(\mathsf{c}) \\ {\scriptstyle \pi^{\mathsf{c}\circ\rho^o}} \downarrow & & \downarrow {\scriptstyle \pi^{\mathsf{c}}} \\ \mathscr{C} & \xrightarrow{\quad \rho \quad} & \mathscr{B} \end{array}$$

where $\mathscr{F}ib(\rho)$ is the functor given by the rules : $(C, X) \mapsto (\rho(C), X)$ and $(g, f) \mapsto (\rho(g), f)$ for every object (C, X) and morphism (g, f) of $\mathscr{F}ib(\mathsf{c} \circ \rho)$. This diagram induces a natural identification of fibrations over $\mathscr{C}$:

$$\mathsf{T}^\rho_{\mathsf{c}} : \mathscr{F}ib(\mathsf{c} \circ \rho^o) \xrightarrow{\sim} \mathsf{Fib}(\rho)^*(\mathscr{F}ib(\mathsf{c})) \qquad (C, X) \mapsto (C, (\rho(C), X))$$

(notation of remark 3.1.5(i)). Especially, via this identification, $\mathsf{c} \circ \rho^o$ is the pseudo-functor associated with a cleavage for $\mathsf{Fib}(\rho)^*(\mathscr{F}ib(\mathsf{c}))$. By a simple inspection, it is easily seen that the rule : $\mathsf{c} \mapsto \mathsf{T}^\rho_{\mathsf{c}}$ yields a strict pseudo-natural isomorphism of strict pseudo-functors :

$$\begin{array}{ccc} \mathsf{PsFun}(\mathscr{B}^o, (\mathsf{U}, \mathsf{V})\text{-}\mathbf{Cat}) & \xrightarrow{\mathscr{F}ib_{\mathscr{B}}} & (\mathsf{U}, \mathsf{V})\text{-}\mathsf{Fib}(\mathscr{B}) \\ {\scriptstyle \mathsf{PsFun}(\rho^o, \mathbf{Cat})} \downarrow & \mathsf{T}^\rho \nearrow & \downarrow {\scriptstyle \mathsf{Fib}(\rho)^*} \\ \mathsf{PsFun}(\mathscr{C}^o, (\mathsf{U}, \mathsf{V})\text{-}\mathbf{Cat}) & \xrightarrow[\mathscr{F}ib_{\mathscr{C}}]{} & (\mathsf{U}, \mathsf{V})\text{-}\mathsf{Fib}(\mathscr{C}). \end{array}$$

(iii) Let $\mathscr{A} \xrightarrow{\varphi} \mathscr{B} \xleftarrow{\varphi'} \mathscr{A}'$ be two fibrations, and $G : \mathscr{A} \to \mathscr{A}'$ a $\mathscr{B}$-cartesian functor; let us fix cleavages $\boldsymbol{\lambda}$ for $\mathscr{A}$ and $\boldsymbol{\lambda}'$ for $\mathscr{A}'$, and let c and c' be the respective associated pseudo-functors. By theorem 3.1.24, there exists a unique pseudo-natural transformation

$$\mathsf{c}^G : \mathsf{c} \Rightarrow \mathsf{c}'$$

that makes commute the diagram

$$\begin{array}{ccc} \mathscr{F}ib(\mathsf{c}) & \xrightarrow{\psi^{\boldsymbol{\lambda}}} & \mathscr{A} \\ {\scriptstyle \mathscr{F}ib(\mathsf{c}^G)} \downarrow & & \downarrow {\scriptstyle G} \\ \mathscr{F}ib(\mathsf{c}') & \xrightarrow{\psi^{\boldsymbol{\lambda}'}} & \mathscr{A}'. \end{array}$$

By inspecting the constructions, we can extract the following explicit description of c^G :

- for every $B \in \mathrm{Ob}(\mathscr{B})$, the functor $\mathsf{c}^G_B : \mathscr{A}_B \to \mathscr{A}'_B$ is the restriction of G
- for every morphism $g : B \to B'$ in $\mathscr{B}$, the coherence constraint of c^G is the isomorphism of functors $\tau_g : \mathsf{c}'_g \circ \mathsf{c}^G_B \Rightarrow \mathsf{c}^G_{B'} \circ \mathsf{c}_g$ that assigns to every $A \in \mathrm{Ob}(\mathscr{A}_{B'})$ the unique isomorphism of $\mathscr{A}'_B$

$$\tau_{g,A} : \mathsf{c}'_g(GA) \xrightarrow{\sim} G(\mathsf{c}_g A) \qquad \text{such that} \qquad G(\boldsymbol{\lambda}(A, g)) \circ \tau_{g,A} = \boldsymbol{\lambda}'(GA, g).$$

Corollary 3.1.28. *Let $\varphi : \mathscr{A} \to \mathscr{B}$ and $\varphi' : \mathscr{A}' \to \mathscr{B}$ be two fibrations, $F : \mathscr{A} \to \mathscr{A}'$ a cartesian $\mathscr{B}$-functor, and $i \leq 2$ an integer. We have :*

(i) *The following conditions are equivalent :*
  (a) *F is* fibrewise i-faithful, i.e. *the restriction $\mathscr{A}_B \to \mathscr{A}'_B$ of the functor F is i-faithful for every $B \in \mathrm{Ob}(\mathscr{B})$ (see remark* 1.1.5*).*
  (b) *F is i-faithful, and in case $i = 2$, it is even a $\mathscr{B}$-equivalence (see remark* 3.1.5(v)*).*

(ii) *Let $\mathscr{C} \to \mathscr{B}$ be any fibration, and suppose that the conditions of* (i) *hold. Then :*
  (a) *The functor* $\mathsf{Cart}_{\mathscr{B}}(\mathscr{C}, F)$ *is i-faithful.*
  (b) *If $i = 2$, also* $\mathsf{Cart}_{\mathscr{B}}(F, \mathscr{C})$ *is i-faithful.*

Proof. (i): The implication (i.b)⇒(i.a) is trivial. For the converse, we consider first the case where $i = 2$: by theorem 3.1.24 we may assume that $\mathscr{A} = \mathscr{F}ib(\mathsf{c})$ and $\mathscr{A}' = \mathscr{F}ib(\mathsf{c}')$ for (unital) pseudo-functors $\mathsf{c}, \mathsf{c}' : \mathscr{B}^o \to (\mathsf{U}, \mathsf{V})\text{-}\mathbf{Cat}$ (with suitable universes U and V), and $F = \mathscr{F}ib(\omega)$ for a pseudo-natural transformation $\omega : \mathsf{c} \Rightarrow \mathsf{c}'$. Then condition (a) means that $\omega_B : \mathsf{c}_B \to \mathsf{c}'_B$ is an equivalence of categories for every $B \in \mathrm{Ob}(\mathscr{B})$. By theorem 2.4.12, the latter holds if and only if ω is a pseudo-natural equivalence, in which case (i.b) follows.

Next, if $i = 0$, let $A, A' \in \mathrm{Ob}(\mathscr{A})$ and $f_1, f_2 : A \to A'$ be two morphisms of $\mathscr{A}$ such that $Ff_1 = Ff_2$; set $B := \varphi A$ and $t := \varphi(f_1)$, and notice that $\varphi(f_2) = \varphi'(Ff_2) = \varphi'(Ff_1) = t$ as well. Pick any cartesian morphism $h : A'' \to A'$ with $\varphi(h) = t$; then there exist unique morphisms $g_1, g_2 : A \to A''$ in $\varphi^{-1}B$ with $h \circ g_i = f_i$ for $i = 1, 2$. Therefore $Fh \circ Fg_1 = Fh \circ Fg_2$, and since F is cartesian we deduce that $Fg_1 = Fg_2$; by assumption, it follows that $g_1 = g_2$, whence $f_1 = f_2$, as required.

The case where $i = 1$ is similar : let A, A' and B as in the foregoing, and $f' : FA \to FA'$ a morphism of $\mathscr{A}'$; set $t := \varphi'(f')$, and pick a cartesian morphism $h : A'' \to A'$ with $\varphi(h) = t$; then Fh is still cartesian, so there exists a unique morphism $g' : FA \to FA''$ in $\varphi'^{-1}B$ with $Fh \circ g' = f'$. By assumption, $g' = Fg$ for some morphism $g : A \to A''$ in $\varphi^{-1}B$, so that $F(h \circ g) = f'$, as required.

(ii): we consider again first the case where $i = 2$: then, notice that every $\mathscr{B}$-functor $G : \mathscr{A}' \to \mathscr{A}$ that is quasi-inverse to F is also $\mathscr{B}$-cartesian (remark 3.1.5(v)); both assertions (ii.a) and (ii.b) are immediate consequences.

Next, suppose that $i = 0$, and let $G, G' : \mathscr{C} \to \mathscr{A}$ be two functors, $\alpha, \beta : G \Rightarrow G'$ two natural transformations with $F * \alpha = F * \beta$. The latter means that $F(\alpha_C) = F(\beta_C)$ for every $C \in \mathrm{Ob}(\mathscr{C})$; by assumption, we then have $\alpha_C = \beta_C$ for every such C, as required.

Lastly, suppose that $i = 1$, and let $\alpha' : F \circ G \Rightarrow F \circ G'$ be a natural transformation; by assumption, there exists for every $C \in \mathrm{Ob}(\mathscr{C})$ a morphism $\alpha_C : GC \to G'C$ such that $F(\alpha_C) = \alpha'_C$. We claim that the rule : $C \mapsto \alpha_C$ yields a natural transformation $\alpha : G \Rightarrow G'$; indeed, the assertion amounts to the identity $G'f \circ \alpha_C = \alpha_{C'} \circ Gf$ for every morphism $f : C \to C'$ of $\mathscr{C}$. The latter can be checked after composition with F, where it is clear. □

3.1.29. By remark 2.2.17(v), for every universe V with $\mathsf{U} \in \mathsf{V}$, we get a strict pseudo-functor

$$\mathsf{PsFun}((-)^o, \mathbf{Cat}) : {}^o\mathsf{V}\text{-}\mathbf{Cat}^o \to \mathsf{V}\text{-}\overline{2\text{-}\mathbf{Cat}}.$$

We wish now to show how, likewise, the rule $\mathscr{B} \mapsto (\mathsf{U}, \mathsf{V})\text{-}\mathsf{Fib}(\mathscr{B})$ yields a well defined pseudo-functor. Indeed, remark 3.1.5 already says that every functor $\rho : \mathscr{B}' \to \mathscr{B}$ between V-small categories induces a strict pseudo-functor $(\mathsf{U}, \mathsf{V})\text{-}\mathsf{Fib}(\rho)^* : (\mathsf{U}, \mathsf{V})\text{-}\mathsf{Fib}(\mathscr{B}) \to (\mathsf{U}, \mathsf{V})\text{-}\mathsf{Fib}(\mathscr{B}')$, and we have an obvious strict pseudo-natural isomorphism

$$\delta^{(\mathsf{U},\mathsf{V})\text{-}\mathsf{Fib}}_{\mathscr{B}} : \mathbf{1}_{(\mathsf{U},\mathsf{V})\text{-}\mathsf{Fib}(\mathscr{B})} \xrightarrow{\sim} (\mathsf{U}, \mathsf{V})\text{-}\mathsf{Fib}(\mathbf{1}_{\mathscr{B}})^* \qquad \text{for every small category } \mathscr{B}.$$

Hence, let V′ be yet another universe, with $\mathsf{V} \in \mathsf{V}'$; combining with the strict pseudo-natural isomorphisms of remark 3.1.5(vi), we have a datum $(\delta^{(\mathsf{U},\mathsf{V})\text{-}\mathsf{Fib}}_{\bullet}, \gamma^{(\mathsf{U},\mathsf{V})\text{-}\mathsf{Fib}}_{\bullet,\bullet})$ fulfilling the unit and associativity axioms for a pseudo-functor

$$(\mathsf{U}, \mathsf{V})\text{-}\mathsf{Fib} : {}^o\mathsf{V}\text{-}\mathbf{Cat}^o \to \mathsf{V}'\text{-}\overline{2\text{-}\mathbf{Cat}}$$

for which we must however still assign the rule prescribing its action on natural transformations. To this aim, we choose for every $\mathscr{B} \in \mathrm{Ob}(\mathsf{V}\text{-}\mathbf{Cat})$ and every V-small fibration $F : \mathscr{A} \to \mathscr{B}$ with essentially U-small fibres, a cleavage $\boldsymbol{\lambda}^F$, and we denote by $\mathsf{c}^F : \mathscr{B}^o \to (\mathsf{U}, \mathsf{V})\text{-}\mathbf{Cat}$ the corresponding pseudo-functor. With the notation of remark 3.1.27, we deduce for every functor $\rho : \mathscr{B}' \to \mathscr{B}$ between V-small categories, a cartesian isomorphism of $\mathscr{B}'$-fibrations

$$\mathsf{T}^\rho_F := (\mathsf{U}, \mathsf{V})\text{-}\mathsf{Fib}(\rho)^*(\psi^{\boldsymbol{\lambda}^F}) \circ \mathsf{T}^\rho_{\mathsf{c}^F} : \mathscr{F}ib(\mathsf{c}^F \circ \rho^o) \xrightarrow{\sim} (\mathsf{U}, \mathsf{V})\text{-}\mathsf{Fib}(\rho)^*(\mathscr{A}).$$

Let now $\rho, \rho' : \mathscr{B}' \to \mathscr{B}$ be two functors, and $\alpha : \rho \Rightarrow \rho'$ a natural transformation. There exists a unique $\mathscr{B}'$-cartesian functor $(\mathsf{U},\mathsf{V})\text{-}\mathsf{Fib}(\alpha)^*_{\mathscr{A}}$ making commute the diagram :

$$\begin{array}{ccc} \mathscr{F}ib(\mathsf{c}^F \circ \rho'^o) & \xrightarrow{\mathscr{F}ib(\mathsf{c}^F * \alpha^o)} & \mathscr{F}ib(\mathsf{c}^F \circ \rho^o) \\ {\scriptstyle \mathsf{T}^{\rho'}_F}\downarrow & & \downarrow{\scriptstyle \mathsf{T}^{\rho}_F} \\ (\mathsf{U},\mathsf{V})\text{-}\mathsf{Fib}(\rho')^*\mathscr{A} & \xrightarrow{(\mathsf{U},\mathsf{V})\text{-}\mathsf{Fib}(\alpha)^*_{\mathscr{A}}} & (\mathsf{U},\mathsf{V})\text{-}\mathsf{Fib}(\rho)^*\mathscr{A}. \end{array}$$

To ease notation, throughout the rest of this section we shall write Fib *instead of* $(\mathsf{U},\mathsf{V})\text{-}\mathsf{Fib}$. Next, let $F' : \mathscr{A}' \to \mathscr{B}$ be another fibration with essentially U-small fibres, $G : \mathscr{A} \to \mathscr{A}'$ a cartesian functor between $\mathscr{B}$-fibrations, and define $\mathsf{c}^G : \mathsf{c}^F \Rightarrow \mathsf{c}^{F'}$ as in remark 3.1.27(iii); by example 2.2.15(ii), the coherence constraint τ^G of c^G defines an invertible modification :

$$\tau^G_\alpha : (\mathsf{c}^{F'} * \alpha^o) \odot (\mathsf{c}^G * \rho'^o) \rightsquigarrow (\mathsf{c}^G * \rho^o) \odot (\mathsf{c}^F * \alpha^o).$$

We consider the induced diagram of $\mathscr{B}'$-cartesian functors :

$$\begin{array}{ccccccc} \mathscr{F}ib(\mathsf{c}^{F'} \circ \rho'^o) & & & \xrightarrow{\mathscr{F}ib(\mathsf{c}^{F'} * \alpha^o)} & & & \mathscr{F}ib(\mathsf{c}^{F'} \circ \rho^o) \\ & \nwarrow^{\mathscr{F}ib(\mathsf{c}^G * \rho'^o)} & & & & \nearrow^{\mathscr{F}ib(\mathsf{c}^G * \rho^o)} & \\ & & \mathscr{F}ib(\mathsf{c}^F \circ \rho'^o) & \xrightarrow{\mathscr{F}ib(\mathsf{c}^F * \alpha^o)} & \mathscr{F}ib(\mathsf{c}^F \circ \rho^o) & & \\ {\scriptstyle \mathsf{T}^{\rho'}_{F'}}\downarrow & & {\scriptstyle \mathsf{T}^{\rho'}_F}\downarrow & & \downarrow{\scriptstyle \mathsf{T}^{\rho}_F} & & \downarrow{\scriptstyle \mathsf{T}^{\rho}_{F'}} \\ & & \mathsf{Fib}(\rho')^*\mathscr{A} & \xrightarrow{\mathsf{Fib}(\alpha)^*_{\mathscr{A}}} & \mathsf{Fib}(\rho)^*\mathscr{A} & & \\ & \swarrow_{\mathsf{Fib}(\rho')^*(G)} & & & & \searrow_{\mathsf{Fib}(\rho)^*(G)} & \\ \mathsf{Fib}(\rho')^*\mathscr{A}' & & & \xrightarrow{\mathsf{Fib}(\alpha)^*_{\mathscr{A}'}} & & & \mathsf{Fib}(\rho)^*\mathscr{A}' \end{array}$$

whose inner and outer rectangular subdiagrams commute by definition, and whose left and right trapezoidal subdiagrams commute by the strict pseudo-naturality of T^ρ and $\mathsf{T}^{\rho'}$. If we orient the upper trapezoidal subdiagram with the natural transformation $\mathscr{F}ib(\tau^G_\alpha)^{-1}$, there exists therefore a unique natural transformation

$$\tau^\alpha_G : \mathsf{Fib}(\alpha)^*_{\mathscr{A}'} \circ \mathsf{Fib}(\rho')^*(G) \Rightarrow \mathsf{Fib}(\rho)^*(G) \circ \mathsf{Fib}(\alpha)^*_{\mathscr{A}}$$

orienting the lower trapezoidal subdiagram, such that the resulting oriented cubical diagram (whose other four faces are oriented by identities) commutes on 2-cells, in the sense of (2.3.21); the latter means that :

$$\mathsf{T}^\rho_{F'} * \mathscr{F}ib(\tau^G_\alpha)^{-1} = \tau^\alpha_G * \mathsf{T}^{\rho'}_F.$$

Lemma 3.1.30. *With the notation of* (3.1.29)*, the rule :* $\mathscr{A} \mapsto \mathsf{Fib}(\alpha)^*_{\mathscr{A}}$ *defines a pseudo-natural transformation* $\mathsf{Fib}(\alpha)^* : \mathsf{Fib}(\rho')^* \Rightarrow \mathsf{Fib}(\rho)^*$ *whose coherence constraint is given by the system of natural transformations* $\tau^\alpha_\bullet$.

Proof. By inspecting the construction, we are easily reduced to checking that $\tau^{G'}_\alpha \boxminus \tau^G_\alpha = \tau^{G' \circ G}_\alpha$ for every pair of cartesian functors of $\mathscr{B}$-fibrations $\mathscr{A} \xrightarrow{G} \mathscr{A}' \xrightarrow{G'} \mathscr{A}''$. But this is clear, since $\mathsf{c}^{G' \circ G} = \mathsf{c}^{G'} \odot \mathsf{c}^G$. □

3.1.31. Next, let $\rho'' : \mathscr{B}' \to \mathscr{B}$ be another functor, and $\alpha' : \rho' \Rightarrow \rho''$ a second natural transformation. According to example 2.2.15(i), we have an invertible modification

$$\gamma^F_{\alpha,\alpha'} : (\mathsf{c}^F * \alpha^o) \odot (\mathsf{c}^F * \alpha'^o) \rightsquigarrow \mathsf{c}^F * (\alpha' \odot \alpha)^o$$

induced by the coherence constraint $\gamma^F_{\bullet\bullet}$ of c^F. Whence, a unique natural transformation

$$\Xi^{\alpha,\alpha'}_{\mathscr{A}} : \mathsf{Fib}(\alpha)^*_{\mathscr{A}} \circ \mathsf{Fib}(\alpha')^*_{\mathscr{A}} \Rightarrow \mathsf{Fib}(\alpha' \odot \alpha)^*_{\mathscr{A}} \quad \text{such that} \quad \mathsf{T}^\rho_F * \mathscr{F}ib(\gamma^F_{\alpha,\alpha'}) = \Xi^{\alpha,\alpha'}_{\mathscr{A}} * \mathsf{T}^{\rho''}_F.$$

Lemma 3.1.32. *With the notation of* (3.1.31)*, the following holds :*

(i) *The rule :* $\mathscr{A} \mapsto \Xi^{\alpha,\alpha'}_{\mathscr{A}}$ *is a modification* $\Xi^{\alpha,\alpha'} : \mathsf{Fib}(\alpha)^* \odot \mathsf{Fib}(\alpha')^* \rightsquigarrow \mathsf{Fib}(\alpha' \odot \alpha)^*$.
(ii) *Especially, we have* $\mathsf{Fib}(\alpha)^* \odot \mathsf{Fib}(\alpha')^* = \mathsf{Fib}(\alpha' \odot \alpha)^*$ *in* $\mathsf{V}'\text{-}\overline{2\text{-}\mathbf{Cat}}$.

Proof. Recall that the coherence constraint of $\mathsf{Fib}(\alpha)^* \odot \mathsf{Fib}(\alpha')^*$ assigns to every cartesian functor $G : \mathscr{A} \to \mathscr{A}'$ of $\mathscr{B}$-fibrations the natural transformation

$$\beta := (\mathsf{Fib}(\alpha)^*_{\mathscr{A}'} * \tau^{\alpha'}_G) \odot (\tau^{\alpha}_G * \mathsf{Fib}(\alpha')^*_{\mathscr{A}})$$

and we need to check the identity :

$$X := (\Xi^{\alpha,\alpha'}_{\mathscr{A}'} * \mathsf{Fib}(\rho'')^*(G)) \odot \beta = Y := \tau^{\alpha'\odot\alpha}_G \odot (\mathsf{Fib}(\rho)^*(G) * \Xi^{\alpha,\alpha'}_{\mathscr{A}}).$$

To this aim, it suffices to check that $X * \mathsf{T}^{\rho''}_F = Y * \mathsf{T}^{\rho''}_F$. We compute :

$$\begin{aligned} X*\mathsf{T}^{\rho''}_F &= (\Xi^{\alpha,\alpha'}_{\mathscr{A}'} * \mathsf{Fib}(\rho'')^*(G) * \mathsf{T}^{\rho''}_F) \odot (\beta * \mathsf{T}^{\rho''}_F) \\ &= (\Xi^{\alpha,\alpha'}_{\mathscr{A}'} * \mathsf{T}^{\rho''}_{F'} * \mathscr{F}ib(\mathsf{c}^G * \rho''^o)) \odot (\mathsf{Fib}(\alpha)^*_{\mathscr{A}'} * \mathsf{T}^{\rho'}_{F'} * \mathscr{F}ib(\tau^G_{\alpha'})^{-1}) \odot (\tau^\alpha_G * \mathsf{T}^{\rho'}_F * \mathscr{F}ib(\mathsf{c}^F * \alpha'^o)) \\ &= \mathsf{T}^\rho_{F'} * (\mathscr{F}ib(\gamma^{F'}_{\alpha,\alpha'} * (\mathsf{c}^G * \rho''^o)) \odot \mathscr{F}ib((\mathsf{c}^{F'} * \alpha^o) * \tau^G_{\alpha'})^{-1} \odot \mathscr{F}ib(\tau^G_\alpha * (\mathsf{c}^F * \alpha'^o))^{-1}) \end{aligned}$$

and :

$$\begin{aligned} Y * \mathsf{T}^{\rho''}_F &= (\mathsf{T}^\rho_{F'} * \mathscr{F}ib(\tau^G_{\alpha'\odot\alpha})^{-1}) \odot (\mathsf{Fib}(\rho)^*(G) * \mathsf{T}^\rho_F * \mathscr{F}ib(\gamma^F_{\alpha,\alpha'})) \\ &= (\mathsf{T}^\rho_{F'} * \mathscr{F}ib(\tau^G_{\alpha'\odot\alpha})^{-1}) \odot (\mathsf{T}^\rho_{F'} * \mathscr{F}ib((\mathsf{c}^G * \rho^o) * \gamma^F_{\alpha,\alpha'}). \end{aligned}$$

Thus, we are reduced to showing the identity :

$$\tau^G_{\alpha'\odot\alpha} \odot (\gamma^{F'}_{\alpha,\alpha'} * (\mathsf{c}^G * \rho''^o)) = ((\mathsf{c}^G * \rho^o) * \gamma^F_{\alpha,\alpha'}) \odot (\tau^G_\alpha * (\mathsf{c}^F * \alpha'^o)) \odot ((\mathsf{c}^{F'} * \alpha^o) * \tau^G_{\alpha'}).$$

which follows directly from the coherence axiom for τ^G. ◻

In light of lemma 3.1.32(ii), the construction of the pseudo-functor (U,V)-Fib of (3.1.29) shall be complete, once we have shown :

Lemma 3.1.33. *Let* $\mu, \mu' : \mathscr{B}'' \to \mathscr{B}'$ *and* $\rho, \rho' : \mathscr{B}' \to \mathscr{B}$ *be four functors, and* $\alpha : \rho \Rightarrow \rho'$, $\beta : \mu \Rightarrow \mu'$ *two natural transformations. Then the following diagram commutes in* $\mathsf{V}'\text{-}\overline{2\text{-}\mathbf{Cat}}$ *:*

$$\begin{array}{ccc} \mathsf{Fib}(\mu')^* \circ \mathsf{Fib}(\rho')^* & \xRightarrow{\mathsf{Fib}(\beta)^* * \mathsf{Fib}(\alpha)^*} & \mathsf{Fib}(\mu)^* \circ \mathsf{Fib}(\rho)^* \\ {\scriptstyle \gamma^{\mathsf{Fib}}_{\mu',\rho'}} \Downarrow & & \Downarrow {\scriptstyle \gamma^{\mathsf{Fib}}_{\mu,\rho}} \\ \mathsf{Fib}(\rho' \circ \mu')^* & \xRightarrow{\mathsf{Fib}(\alpha * \beta)^*} & \mathsf{Fib}(\rho \circ \mu)^*. \end{array}$$

Proof. In view of lemma 3.1.32(ii), it suffices to consider separately the cases where $\mu = \mu'$ and $\beta = \mathbf{1}_\mu$, and where $\rho = \rho'$ and $\alpha = \mathbf{1}_\rho$. In the first case, we have to check the identities :

$$\begin{aligned} \gamma^{\mathsf{Fib}}_{(\mu,\rho),\mathscr{A}} \circ \mathsf{Fib}(\mu)^*(\mathsf{Fib}(\alpha)^*_{\mathscr{A}}) &= \mathsf{Fib}(\alpha * \mu)^*_{\mathscr{A}} \circ \gamma^{\mathsf{Fib}}_{(\mu,\rho'),\mathscr{A}} \\ X := \tau^{\alpha*\mu}_G * \gamma^{\mathsf{Fib}}_{(\mu,\rho'),\mathscr{A}} &= Y := \gamma^{\mathsf{Fib}}_{(\mu,\rho),\mathscr{A}'} * \mathsf{Fib}(\mu)^*(\tau^\alpha_G) \end{aligned}$$

for every fibration $F : \mathscr{A} \to \mathscr{B}$ and every cartesian functor of $\mathscr{B}$-fibrations $G : \mathscr{A} \to \mathscr{A}'$. For the first identity, we consider the diagram :

$$\begin{array}{ccc}
\mathscr{F}ib(\mathsf{c}^F \circ \rho'^o \circ \mu^o) & \xrightarrow{\mathscr{F}ib(\mathsf{c}^F * \alpha^o * \mu^o)} & \mathscr{F}ib(\mathsf{c}^F \circ \rho^o \circ \mu^o) \\
{\scriptstyle \mathsf{T}^\mu_{\mathsf{c}^F \circ \rho'^o}} \downarrow & & \downarrow {\scriptstyle \mathsf{T}^\mu_{\mathsf{c}^F \circ \rho^o}} \\
\mathsf{Fib}(\mu)^* \mathscr{F}ib(\mathsf{c}^F \circ \rho'^o) & \xrightarrow{\mathsf{Fib}(\mu)^* \mathscr{F}ib(\mathsf{c}^F * \alpha^o)} & \mathsf{Fib}(\mu)^* \mathscr{F}ib(\mathsf{c}^F \circ \rho^o) \\
{\scriptstyle \mathsf{Fib}(\mu)^*(\mathsf{T}^{\rho'}_F)} \downarrow & & \downarrow {\scriptstyle \mathsf{Fib}(\mu)^*(\mathsf{T}^{\rho}_F)} \\
\mathsf{Fib}(\mu)^* \mathsf{Fib}(\rho')^* \mathscr{A} & \xrightarrow{\mathsf{Fib}(\mu)^* \mathsf{Fib}(\alpha)^*_{\mathscr{A}}} & \mathsf{Fib}(\mu)^* \mathsf{Fib}(\rho)^* \mathscr{A} \\
{\scriptstyle \gamma^{\mathsf{Fib}}_{(\mu,\rho'),\mathscr{A}}} \downarrow & & \downarrow {\scriptstyle \gamma^{\mathsf{Fib}}_{(\mu,\rho),\mathscr{A}}} \\
\mathsf{Fib}(\rho' \circ \mu)^* \mathscr{A} & \xrightarrow{\mathsf{Fib}(\alpha * \mu)^*_{\mathscr{A}}} & \mathsf{Fib}(\rho \circ \mu)^* \mathscr{A}
\end{array}$$

whose central square subdiagrams commutes by construction, and whose upper square subdiagram commutes by strict naturality of T^μ. Notice that the composition of the three left (resp. right) vertical arrows equals $\mathsf{T}^{\rho'\circ\mu}_F$ (resp. $\mathsf{T}^{\rho\circ\mu}_F$); it then follows that also the external square diagram commutes. We conclude that the bottom square subdiagram commutes as well, whence the sought identity. For the second identity, set $Z := \mathsf{Fib}(\mu)^*(\mathsf{T}^{\rho'}_F) \circ \mathsf{T}^\mu_{\mathsf{c}^F \circ \rho'^o}$; it suffices to check that $X * Z = Y * Z$. We compute :

$$\begin{aligned}
X * Z &= \tau^{\alpha*\mu}_G * \mathsf{T}^{\rho'\circ\mu}_F = \mathsf{T}^{\rho\circ\mu}_{F'} * \mathscr{F}ib(\tau^G_{\alpha*\mu})^{-1} \\
Y * Z &= \gamma^{\mathsf{Fib}}_{(\mu,\rho),\mathscr{A}'} * \mathsf{Fib}(\mu)^*(\tau^\alpha_G * \mathsf{T}^{\rho'}_F) * \mathsf{T}^\mu_{\mathsf{c}^F \circ \rho'^o} \\
&= \gamma^{\mathsf{Fib}}_{(\mu,\rho),\mathscr{A}'} * \mathsf{Fib}(\mu)^*(\mathsf{T}^\rho_{F'} * \mathscr{F}ib(\tau^G_\alpha)^{-1}) * \mathsf{T}^\mu_{\mathsf{c}^F \circ \rho'^o} \\
&= \mathsf{T}^{\rho\circ\mu}_{F'} * \mathsf{T}^{\mu\,-1}_{\mathsf{c}^{F'} \circ \rho^o} * \mathsf{Fib}(\mu)^* \mathscr{F}ib(\tau^G_\alpha)^{-1} * \mathsf{T}^\mu_{\mathsf{c}^F \circ \rho'^o} \\
&= \mathsf{T}^{\rho\circ\mu}_{F'} * \mathscr{F}ib(\tau^G_\alpha * \mu^o)
\end{aligned}$$

where the last equality holds by the strict pseudo-naturality of T^μ. So we are reduced to checking that $\tau^G_{\alpha*\mu} = \tau^G_\alpha * \mu^o$. The latter follows by direct inspection.

For the case where $\alpha = \mathbf{1}_\rho$, let $F : \mathscr{A} \to \mathscr{B}$ be any fibration, set $\rho^*\mathscr{A} := \mathsf{Fib}(\rho)^*\mathscr{A}$, and let $\rho^*F : \rho^*\mathscr{A} \to \mathscr{B}'$ be the induced functor. There exists a unique pseudo-natural isomorphism

$$\omega_{F,\rho} : \mathsf{c}^F \circ \rho^o \xrightarrow{\sim} \mathsf{c}^{\rho^* F}$$

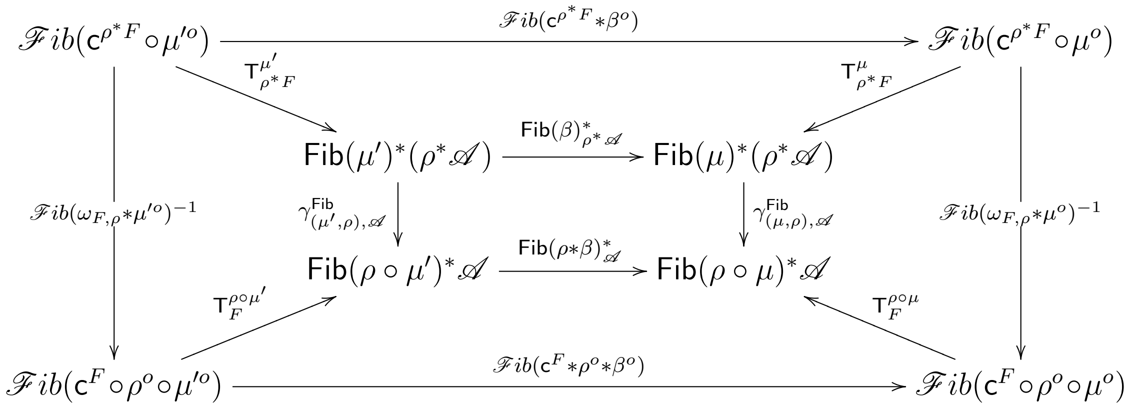
that makes commute the diagram :

$$\begin{array}{ccc}
\mathscr{F}ib(\mathsf{c}^F \circ \rho^o) & \xrightarrow{\mathscr{F}ib(\omega_{F,\rho})} & \mathscr{F}ib(\mathsf{c}^{\rho^* F}) \\
& {\scriptstyle \mathsf{T}^\rho_F} \searrow \quad \swarrow {\scriptstyle \psi\lambda^{\rho^* F}} & \\
& \rho^* \mathscr{A}. &
\end{array}$$

It follows easily that for every cartesian functor $G : \mathscr{A} \to \mathscr{A}'$ we have :

$$\mathsf{c}^{\mathsf{Fib}(\rho)^* G} = \omega_{F',\rho} \odot (\mathsf{c}^G * \rho) \odot \omega^{-1}_{F,\rho} \qquad (3.1.34)$$

(details left to the reader). We consider the diagram :



whose upper and lower trapezoidal subdiagrams commute by construction; it is easily seen that the same holds for the right and left trapezoidal subdiagrams (details left to the reader). Moreover, by example 2.2.15(ii) there exists an invertible modification

$$\tau^{\omega^{-1}_{F,\rho}}_{\beta^o} : (\omega^{-1}_{F,\rho} * \mu^o) \odot (\mathsf{c}^{\rho^*F} * \beta^o) \rightsquigarrow ((\mathsf{c}^F \circ \rho^o) * \beta^o) \odot (\omega^{-1}_{F,\rho} * \mu'^o).$$

Hence, there exists a unique orientation

$$\Xi_{\mathscr{A}} : \gamma^{\mathsf{Fib}}_{(\mu,\rho),\mathscr{A}} \circ \mathsf{Fib}(\beta)^*_{\rho^*\mathscr{A}} \Rightarrow \mathsf{Fib}(\beta*\rho)^*_{\mathscr{A}} \circ \gamma^{\mathsf{Fib}}_{(\mu',\rho),\mathscr{A}}$$

for the inner square subdiagram, such that :

$$\Xi_{\mathscr{A}} * \mathsf{T}^{\mu'}_{\rho^*F} = \mathsf{T}^{\rho\circ\mu}_{F} * \mathscr{F}ib(\tau^{\omega^{-1}_{F,\rho}}_{\beta^o}).$$

To conclude the proof, it suffices then to show :

Claim 3.1.35. The rule : $\mathscr{A} \mapsto \Xi_{\mathscr{A}}$ defines an invertible modification

$$\Xi : \gamma^{\mathsf{Fib}}_{\mu,\rho} \odot (\mathsf{Fib}(\beta)^* * \mathsf{Fib}(\rho)^*) \rightsquigarrow \mathsf{Fib}(\beta*\rho)^* \odot \gamma^{\mathsf{Fib}}_{\mu',\rho}.$$

Proof of the claim. Notice that the coherence constraint of $\gamma^{\mathsf{Fib}}_{\mu,\rho} \odot (\mathsf{Fib}(\beta)^* * \mathsf{Fib}(\rho)^*)$ assigns to every cartesian functor $G : \mathscr{A} \to \mathscr{A}'$ of $\mathscr{B}$-fibrations the natural transformation $X := \gamma^{\mathsf{Fib}}_{(\mu,\rho),\mathscr{A}'} * \tau^{\beta}_{\mathsf{Fib}(\rho)^*(G)}$, and the coherence constraint of $\mathsf{Fib}(\beta*\rho)^* \odot \gamma^{\mathsf{Fib}}_{\mu',\rho}$ assigns to G the natural transformation $Y := \tau^{\rho*\beta}_{G} * \gamma^{\mathsf{Fib}}_{(\mu',\rho),\mathscr{A}}$. Then the assertion comes down to the identity:

$$X' := (\Xi_{\mathscr{A}'} * (\mathsf{Fib}(\mu')^*\mathsf{Fib}(\rho)^*G)) \odot X = Y' := Y \odot ((\mathsf{Fib}(\rho\circ\mu)^*G) * \Xi_{\mathscr{A}})$$

and it suffices to check that $X' * \mathsf{T}^{\mu'}_{\rho^*F} = Y' * \mathsf{T}^{\mu'}_{\rho^*(F)}$. We compute :

$$\begin{aligned}
Y' * \mathsf{T}^{\mu'}_{\rho^*F} &= (\tau^{\rho*\beta}_{G} * \mathsf{T}^{\rho\circ\mu'}_{F} * \mathscr{F}ib(\omega_{F,\rho} * \mu'^o)^{-1}) \odot ((\mathsf{Fib}(\rho\circ\mu)^*G) * \mathsf{T}^{\rho\circ\mu}_{F} * \mathscr{F}ib(\tau^{\omega^{-1}_{F,\rho}}_{\beta^o})) \\
&= (\mathsf{T}^{\rho\circ\mu}_{F'} * \mathscr{F}ib(\tau^{G\,-1}_{\rho*\beta} * (\omega_{F,\rho} * \mu'^o)^{-1})) \odot ((\mathsf{Fib}(\rho\circ\mu)^*G) * \mathsf{T}^{\rho\circ\mu}_{F} * \mathscr{F}ib(\tau^{\omega^{-1}_{F,\rho}}_{\beta^o})) \\
&= (\mathsf{T}^{\rho\circ\mu}_{F'} * \mathscr{F}ib(\tau^{G\,-1}_{\rho*\beta} * (\omega_{F,\rho} * \mu'^o)^{-1})) \odot (\mathsf{T}^{\rho\circ\mu}_{F'} * \mathscr{F}ib(\mathsf{c}^G * \rho^o * \mu^o) * \mathscr{F}ib(\tau^{\omega^{-1}_{F,\rho}}_{\beta^o})) \\
&= \mathsf{T}^{\rho\circ\mu}_{F'} * \mathscr{F}ib((\tau^{G\,-1}_{\rho*\beta} * (\omega_{F,\rho} * \mu'^o)^{-1}) \odot ((\mathsf{c}^G * \rho^o * \mu^o) * \tau^{\omega^{-1}_{F,\rho}}_{\beta^o})) \\
X * \mathsf{T}^{\mu'}_{\rho^*F} &= \gamma^{\mathsf{Fib}}_{(\mu,\rho),\mathscr{A}'} * \mathsf{T}^{\mu}_{\rho^*F'} * \mathscr{F}ib(\tau^{\mathsf{Fib}(\rho)^*G}_{\beta})^{-1} \\
&= \mathsf{T}^{\rho\circ\mu}_{F'} * \mathscr{F}ib((\omega_{F',\rho} * \mu^o)^{-1} * (\tau^{\mathsf{Fib}(\rho)^*G}_{\beta})^{-1})
\end{aligned}$$

and :

$$\begin{aligned}
\Xi_{\mathscr{A}'} * (\mathsf{Fib}(\mu')^*\mathsf{Fib}(\rho)^*G) * \mathsf{T}^{\mu'}_{\rho^*F} &= \Xi_{\mathscr{A}'} * \mathsf{T}^{\mu'}_{\rho^*F'} * \mathscr{F}ib(\mathsf{c}^{\mathsf{Fib}(\rho)^*G} * \mu'^o) \\
&= \mathsf{T}^{\rho\circ\mu}_{F'} * \mathscr{F}ib(\tau^{\omega^{-1}_{F',\rho}}_{\beta^o} * (\mathsf{c}^{\mathsf{Fib}(\rho)^*G} * \mu'^o))
\end{aligned}$$

so that $X' * Z = \mathsf{T}^{\rho\circ\mu}_{F'} * \mathscr{F}ib((\tau_{\beta^o}^{\omega^{-1}_{F',\rho}} * (\mathsf{c}^{\mathsf{Fib}(\rho)^*G} * \mu'^o)) \odot ((\omega_{F',\rho} * \mu^o)^{-1} * (\tau_\beta^{\mathsf{Fib}(\rho)^*G})^{-1}))$. We are thus reduced to checking that :

$$((\mathsf{c}^G * \rho^o * \mu^o) * \tau_{\beta^o}^{\omega^{-1}_{F,\rho}}) \odot ((\omega^{-1}_{F',\rho} * \mu^o) * \tau_\beta^{\mathsf{Fib}(\rho)^*G}) = (\tau^G_{\rho*\beta} * (\omega^{-1}_{F,\rho} * \mu'^o)) \odot (\tau_{\beta^o}^{\omega^{-1}_{F',\rho}} * (\mathsf{c}^{\mathsf{Fib}(\rho)^*G} * \mu'^o)).$$

The latter follows easily from (3.1.34) : details left to the reader. $\square$

Corollary 3.1.36. *We have a pseudo-natural equivalence*

$$\mathscr{F}ib_\bullet : \mathsf{PsFun}((-)^o, \mathbf{Cat}) \Rightarrow (\mathsf{U}, \mathsf{V})\text{-}\mathsf{Fib} \qquad \mathscr{C} \mapsto \mathscr{F}ib_{\mathscr{C}}$$

whose coherence constraint attaches to every functor ρ of small categories the strict pseudo-natural isomorphism $(\mathsf{T}^\rho)^{-1}$.

Proof. The required coherence axioms for the rule $\rho \mapsto (\mathsf{T}^\rho)^{-1}$ hold by direct inspection (details left to the reader). There remains therefore only to check the naturality of this rule, relative to natural transformations $\beta : \rho \Rightarrow \rho'$ between functors $\rho, \rho' : \mathscr{B}' \to \mathscr{B}$ of V-small categories. The assertion amounts to checking the following identity in the 2-category $\mathsf{V}\text{-}\overline{2\text{-}\mathbf{Cat}}$:

$$(\mathsf{T}^\rho)^{-1} \odot (\mathsf{Fib}(\beta)^* * \mathscr{F}ib_{\mathscr{B}}) = (\mathscr{F}ib_{\mathscr{B}'} * \mathsf{PsFun}(\beta^o, \mathbf{Cat})) \odot (\mathsf{T}^{\rho'})^{-1}. \tag{3.1.37}$$

Thus, let $\mathsf{d} : \mathscr{B}^o \to \mathbf{Cat}$ be any pseudo-functor, and denote by $F_\mathsf{d} : \mathscr{F}ib(\mathsf{d}) \to \mathscr{B}$ the induced fibration. With the notation of (3.1.29), there exists a unique pseudo-natural isomorphism

$$\omega^\mathsf{d} : \mathsf{c}^{F_\mathsf{d}} \xrightarrow{\sim} \mathsf{d} \qquad \text{such that} \qquad \mathscr{F}ib(\omega^\mathsf{d}) = \psi^{\lambda^{F_\mathsf{d}}}.$$

We consider the diagram :

$$\begin{array}{ccccc}
\mathscr{F}ib(\mathsf{c}^{F_\mathsf{d}} \circ \rho'^o) & \xrightarrow{\mathsf{T}^{\rho'}_{F_\mathsf{d}}} & \mathsf{Fib}(\rho')^*\mathscr{F}ib(\mathsf{d}) & \xrightarrow{(\mathsf{T}^{\rho'}_\mathsf{d})^{-1}} & \mathscr{F}ib(\mathsf{d} \circ \rho'^o) \\
\Big\downarrow{\scriptstyle \mathscr{F}ib(\mathsf{c}^{F_\mathsf{d}} * \beta^o)} & & \Big\downarrow{\scriptstyle \mathsf{Fib}(\beta)^*\mathscr{F}ib(\mathsf{d})} & & \Big\downarrow{\scriptstyle \mathscr{F}ib(\mathsf{d} * \beta^o)} \\
\mathscr{F}ib(\mathsf{c}^{F_\mathsf{d}} \circ \rho^o) & \xrightarrow[\mathsf{T}^{\rho}_{F_\mathsf{d}}]{} & \mathsf{Fib}(\rho)^*\mathscr{F}ib(\mathsf{d}) & \xrightarrow[(\mathsf{T}^{\rho}_\mathsf{d})^{-1}]{} & \mathscr{F}ib(\mathsf{d} \circ \rho'^o)
\end{array}$$

whose left square subdiagram commutes by construction. Invoking the strict pseudo-naturality of T^ρ, it is easily seen that the composition of the two bottom horizontal arrows agrees with $\mathscr{F}ib(\omega^\mathsf{d} * \rho^o)$; likewise, the composition of the two top horizontal arrows yields $\mathscr{F}ib(\omega^\mathsf{d} * \rho'^o)$. Now, according to example 2.2.15(ii), there exists an invertible modification :

$$\tau^{\omega^\mathsf{d}}_\beta : (\mathsf{d} * \beta^o) \odot (\omega^\mathsf{d} * \rho'^o) \rightsquigarrow (\omega^\mathsf{d} * \rho^o) \odot (\mathsf{c}^{F_\mathsf{d}} * \beta^o).$$

Hence, there exists a unique isomorphism of functors

$$\vartheta^\beta_\mathsf{d} : \mathscr{F}ib(\mathsf{d} * \beta^o) \circ (\mathsf{T}^{\rho'}_\mathsf{d})^{-1} \xrightarrow{\sim} (\mathsf{T}^\rho_\mathsf{d})^{-1} \circ \mathsf{Fib}(\rho)^*\mathscr{F}ib(\mathsf{d}) \quad \text{such that} \quad \vartheta^\beta_\mathsf{d} * \mathsf{T}^{\rho'}_{F_\mathsf{d}} = \mathscr{F}ib(\tau^{\omega^\mathsf{d}}_\beta)$$

and we are reduced to checking that the rule : $\mathsf{d} \mapsto \vartheta^\beta_\mathsf{d}$ defines an invertible modification from the right to the left side of (3.1.37). Hence, let $\alpha : \mathsf{d} \Rightarrow \mathsf{d}'$ be a pseudo-natural transformation, and define $\mathsf{c}^{\mathscr{F}ib(\alpha)} : \mathsf{c}^{F_\mathsf{d}} \Rightarrow \mathsf{c}^{F_{\mathsf{d}'}}$ as in remark 3.1.27(iii); it is easily seen that

$$\mathsf{c}^{\mathscr{F}ib(\alpha)} = \omega^{\mathsf{d}'\,-1} \odot \alpha \odot \omega^\mathsf{d}.$$

Now, the coherence constraint of the left-hand side of (3.1.37) assigns to α the natural transformation $X := (\mathsf{T}^\rho_{\mathsf{d}'})^{-1} * \tau^\beta_{\mathscr{F}ib(\alpha)}$, where $\tau^\beta_{\mathscr{F}ib(\alpha)}$ is characterized by the identity :

$$\tau^\beta_{\mathscr{F}ib(\alpha)} * \mathsf{T}^{\rho'}_{F_\mathsf{d}} = \mathsf{T}^\rho_{F_{\mathsf{d}'}} * \mathscr{F}ib(\tau_\beta^{\mathscr{F}ib(\alpha)})^{-1}$$

and $\tau_\beta^{\mathscr{F}ib(\alpha)} : (\mathsf{c}^{F_{\mathsf{d}'}} * \beta^o) \odot (\mathsf{c}^{\mathscr{F}ib(\alpha)} * \rho'^o) \rightsquigarrow (\mathsf{c}^{\mathscr{F}ib(\alpha)} * \rho^o) \odot (\mathsf{c}^{F_\mathsf{d}} * \beta^o)$ is given as well by example 2.2.15(ii). According to remark 2.2.17(ii), the right-hand side of (3.1.37) assigns to α

the natural transformation $Y := \mathscr{F}ib(\tau^{\alpha}_{\beta})^{-1} * (\mathsf{T}^{\rho'}_{\mathsf{d}})^{-1}$, where again $\tau^{\alpha}_{\beta} : (\mathsf{d}' * \beta^o) \odot (\alpha * \rho'^o) \Rightarrow (\alpha * \rho^o) \odot (\mathsf{d} * \beta^o)$ is given by example 2.2.15(ii). We need then to check that :

$$Y' := (\vartheta^{\beta}_{\mathsf{d}'} * \mathsf{Fib}(\rho')^* \mathscr{F}ib(\alpha)) \odot Y = X' := X \odot (\mathscr{F}ib(\alpha * \rho^o) * \vartheta^{\beta}_{\mathsf{d}})$$

and it suffices to check that $Y' * \mathsf{T}^{\rho'}_{F_{\mathsf{d}}} = X' * \mathsf{T}^{\rho'}_{F_{\mathsf{d}}}$. We compute :

$$\begin{aligned} Y' * \mathsf{T}^{\rho'}_{F_{\mathsf{d}}} &= (\vartheta^{\beta}_{\mathsf{d}'} * \mathsf{T}^{\rho'}_{F'_{\mathsf{d}}} * \mathscr{F}ib(\mathsf{c}^{\mathscr{F}ib(\alpha)} * \rho'^o)) \odot \mathscr{F}ib(\tau^{\alpha}_{\beta} * (\omega^{\mathsf{d}} * \rho'^o))^{-1} \\ &= (\mathscr{F}ib(\tau^{\omega^{\mathsf{d}'}}_{\beta}) * \mathscr{F}ib(\mathsf{c}^{\mathscr{F}ib(\alpha)} * \rho'^o)) \odot \mathscr{F}ib(\tau^{\alpha}_{\beta} * (\omega^{\mathsf{d}} * \rho'^o))^{-1} \\ X' * \mathsf{T}^{\rho'}_{F_{\mathsf{d}}} &= ((\mathsf{T}^{\rho}_{\mathsf{d}'})^{-1} * \mathsf{T}^{\rho}_{F_{\mathsf{d}'}} * \mathscr{F}ib(\tau^{\mathscr{F}ib(\alpha)}_{\beta})^{-1}) \odot (\mathscr{F}ib(\alpha * \rho^o) * \mathscr{F}ib(\tau^{\omega^{\mathsf{d}}}_{\beta})) \\ &= (\mathscr{F}ib(\omega^{\mathsf{d}'} * \rho^o) * \mathscr{F}ib(\tau^{\mathscr{F}ib(\alpha)}_{\beta})^{-1}) \odot (\mathscr{F}ib(\alpha * \rho^o) * \mathscr{F}ib(\tau^{\omega^{\mathsf{d}}}_{\beta})) \end{aligned}$$

hence we are further reduced to showing that :

$$((\omega^{\mathsf{d}'} * \rho^o) * \tau^{\mathscr{F}ib(\alpha)}_{\beta}) \odot (\tau^{\omega^{\mathsf{d}'}}_{\beta} * (\mathsf{c}^{\mathscr{F}ib(\alpha)} * \rho'^o)) = ((\alpha * \rho^o) * \tau^{\omega^{\mathsf{d}}}_{\beta}) \odot (\tau^{\alpha}_{\beta} * (\omega^{\mathsf{d}} * \rho'^o))$$

but this is clear, since both sides agree with $\tau^{\alpha\odot\omega^{\mathsf{d}}}_{\beta}$, where $\tau^{\alpha\odot\omega^{\mathsf{d}}}$ denotes as usual the coherence constraint of $\alpha \odot \omega^{\mathsf{d}}$. ☐

3.2. **Split fibrations and cartesian sections.** As usual, we fix a universe U, but some of the constructions in this section shall produce fibrations whose fibre categories will be *equivalent, though not necessarily isomorphic* to a U-small category.

Definition 3.2.1. Let $\mathscr{B}$ be any category.

(i) A *split fibration* over $\mathscr{B}$ is a pair $(\varphi, \boldsymbol{\lambda})$ where $\varphi : \mathscr{A} \to \mathscr{B}$ is a fibration and $\boldsymbol{\lambda}$ a cleavage for φ whose associated pseudo-functor is strict, in which case we also say that $\boldsymbol{\lambda}$ is *split*.

(ii) For $i = 1, 2$, let $(\varphi_i : \mathscr{A}_i \to \mathscr{B}, \boldsymbol{\lambda}_i)$ be two split fibrations, and $F : \mathscr{A}_1 \to \mathscr{A}_2$ a cartesian functor. Let also c_i be the pseudo-functor associated with $\boldsymbol{\lambda}_i$, for $i = 1, 2$. For every $B \in \mathrm{Ob}(\mathscr{B})$, let $F_B : \varphi_1^{-1}B \to \varphi_2^{-1}B$ denote the restriction of F. We say that F is *a split cartesian functor* $(\varphi_1, \mathsf{c}_1) \to (\varphi_2, \mathsf{c}_2)$ if the rule : $B \mapsto F_B$ defines a strict pseudo-natural transformation $\mathsf{c}_1 \Rightarrow \mathsf{c}_2$.

Remark 3.2.2. (i) With the notation of definition 3.2.1(i), it is easily seen that $\boldsymbol{\lambda}$ is split if and only if for every $B \in \mathrm{Ob}(\mathscr{B})$ we have $\mathsf{c}_{\mathbf{1}_B} = \mathbf{1}_{\mathsf{c}_B}$, and for every pair of morphisms $B'' \xrightarrow{h} B' \xrightarrow{g} B$ of $\mathscr{B}$ we have $\mathsf{c}_{g\circ h} = \mathsf{c}_g \mathsf{c}_h$ and the following diagram commutes :

$$\begin{array}{ccc} \mathsf{c}_{g\circ h}A & \xrightarrow{\boldsymbol{\lambda}(\mathsf{c}_g A, h)} & \mathsf{c}_g A \\ & {\scriptstyle \boldsymbol{\lambda}(A, g\circ h)} \searrow \quad \swarrow {\scriptstyle \boldsymbol{\lambda}(A,g)} & \\ & A & \end{array} \qquad \text{for every } A \in \mathrm{Ob}(\varphi^{-1}B).$$

(ii) Likewise, with the notation of definition 3.2.1(ii), and in light of remark 3.1.27(iii), we see that the $\mathscr{B}$-cartesian functor $F : \mathscr{A}_1 \to \mathscr{A}_2$ is split if and only if for every morphism $g : B' \to B$ of $\mathscr{B}$ and every $A \in \mathrm{Ob}(\varphi_1^{-1}B)$ we have

$$\mathsf{c}'_g(FA) = F(\mathsf{c}_g A) \qquad \text{and} \qquad F(\boldsymbol{\lambda}(A, g)) = \boldsymbol{\lambda}'(FA, g).$$

(details left to the reader).

Example 3.2.3. The fibration $\mathsf{s} : \mathscr{B}/F\mathscr{C} \to \mathscr{B}$ associated with any functor $F : \mathscr{C} \to \mathscr{B}$ (see example 3.1.3(ii)) is split. Indeed, we have a natural cleavage, defined as follows. An object of

$\mathscr{B}/\mathsf{s}(\mathscr{B}/F\mathscr{C})$ is the datum of objects $B \in \mathrm{Ob}(\mathscr{B})$ and $(f : B' \to FC') \in \mathrm{Ob}(\mathscr{B}/F\mathscr{C})$, and a morphism $g : B \to B'$ in $\mathscr{B}$; to such a datum we assign the commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{f\circ g} & FC' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle F\mathbf{1}_{C'}} \\ B' & \xrightarrow{f} & FC' \end{array}$$

which is an object $\boldsymbol{\lambda}(B, f, g)$ of $\mathsf{Morph}(\mathscr{B}/F\mathscr{C})$. Since this object is a $\mathscr{B}$-cartesian morphism of $\mathscr{B}/F\mathscr{C}$, the rule $(B, f, g) \mapsto \boldsymbol{\lambda}(B, f, g)$ extends uniquely to a cleavage for s. By a simple inspection, we find that the corresponding pseudo-functor c associates with every morphism $g : B \to B'$ of $\mathscr{B}$ the functor $g/F\mathscr{C} : B'/F\mathscr{C} \to B/F\mathscr{C}$ (notation of (1.1.27)). Then it is clear that c is strict.

3.2.4. *Cartesian sections of a fibration.* Let $\mathscr{B}$ be a category, V a universe such that $\mathsf{U} \subset \mathsf{V}$ and $\mathscr{B}$ is V-small. The V-small split fibrations over $\mathscr{B}$ with essentially U-small fibres are the objects of a 2-category

$$(\mathsf{U}, \mathsf{V})\text{-}\mathsf{Split}(\mathscr{B})$$

whose 1-cells are given by the split cartesian functors, and whose 2-cells are the natural $\mathscr{B}$-transformations. As usual, we also often write $\mathsf{U}\text{-}\mathsf{Split}(\mathscr{B})$ or just $\mathsf{Split}(\mathscr{B})$ for this 2-category. Furthermore, we have an obvious forgetful strict pseudo-functor :

$$\mathsf{F} : (\mathsf{U}, \mathsf{V})\text{-}\mathsf{Split}(\mathscr{B}) \to (\mathsf{U}, \mathsf{V})\text{-}\mathsf{Fib}(\mathscr{B}) \qquad (\varphi, \boldsymbol{\lambda}) \mapsto \varphi.$$

On the other hand, *if $\mathscr{B}$ is U-small*, we have as well a natural strict pseudo-functor

$$\mathsf{C} : (\mathsf{U}, \mathsf{V})\text{-}\mathsf{Fib}(\mathscr{B}) \to (\mathsf{U}, \mathsf{V})\text{-}\mathsf{Split}(\mathscr{B}).$$

Namely, for a fibration $\varphi : \mathscr{A} \to \mathscr{B}$ with essentially U-small fibres, we consider the strict pseudo-functor

$$\mathscr{A}(-) : \mathscr{B}^o \to (\mathsf{U}, \mathsf{V})\text{-}\mathbf{Cat} \qquad B \mapsto \mathscr{A}(B) := \mathsf{Cart}_{\mathscr{B}}(\mathscr{B}/B, \mathscr{A})$$

that assigns to every morphism $f : B \to B'$ of $\mathscr{B}$ the functor $\mathsf{Cart}(f_*, \mathscr{A})$ for (recall that $\mathscr{B}/B$ is fibred over $\mathscr{B}$, by example 3.1.3(i); also $f_* : \mathscr{B}/B \to \mathscr{B}/B'$ is the $\mathscr{B}$-cartesian functor as in (1.1.25)). The objects of $\mathscr{A}(B)$ are called the *cartesian sections* of $\mathscr{A}$ over B, and the morphisms of $\mathscr{A}(B)$ are also called *morphisms of cartesian sections*. Then we set

$$\mathsf{C}(\mathscr{A}) := \mathscr{F}ib(\mathscr{A}(-))$$

and we let $\mathsf{C}(\varphi) : \mathsf{C}(\mathscr{A}) \to \mathscr{B}$ be the resulting fibration; recall that the fibre category $\mathsf{C}(\varphi)^{-1}B$ is naturally isomorphic to $\mathscr{A}(B)$ for every $B \in \mathrm{Ob}(\mathscr{B})$, and $\mathsf{C}(\mathscr{A})$ carries a distinguished cleavage whose associated pseudo-functor corresponds – under this identification – to the strict pseudo-functor $\mathscr{A}(-)$ (remark 3.1.27(i)).

Every $\mathscr{B}$-cartesian functor $F : \mathscr{A}_1 \to \mathscr{A}_2$ induces a strict pseudo-natural transformation

$$\mathsf{Cart}_{\mathscr{B}}(\mathscr{B}/-, F) : \mathscr{A}_1(-) \Rightarrow \mathscr{A}_2(-) \qquad B \mapsto \mathsf{Cart}_{\mathscr{B}}(\mathscr{B}/B, F)$$

whence a $\mathscr{B}$-cartesian functor

$$\mathsf{C}(F) := \mathscr{F}ib(\mathsf{Cart}_{\mathscr{B}}(\mathscr{B}/-, F)) : \mathsf{C}(\mathscr{A}_1) \to \mathsf{C}(\mathscr{A}_2)$$

and every natural $\mathscr{B}$-transformation $\alpha : F_1 \Rightarrow F_2$ of $\mathscr{B}$-cartesian functors $F_1, F_2 : \mathscr{A}_1 \to \mathscr{A}_2$ induces a modification

$$\mathsf{Cart}_{\mathscr{B}}(\mathscr{B}/-, \alpha) : \mathsf{Cart}_{\mathscr{B}}(\mathscr{B}/-, F_1) \rightsquigarrow \mathsf{Cart}_{\mathscr{B}}(\mathscr{B}/-, F_2) \qquad B \mapsto \mathsf{Cart}_{\mathscr{B}}(\mathbf{1}_{\mathscr{B}/B}, \alpha)$$

whence a natural $\mathscr{B}$-transformation

$$\mathsf{C}(\alpha) := \mathscr{F}ib(\mathsf{Cart}_{\mathscr{B}}(\mathscr{B}/-, \alpha)) : \mathsf{C}(F_1) \Rightarrow \mathsf{C}(F_2).$$

Taking into account the discussion of (3.1.18)-(3.1.21) we easily conclude that the foregoing rules yield a well defined strict pseudo-functor C as sought.

3.2.5. *The evaluation functor of a fibration.* Let $\mathscr{B}$ be a category, and $\varphi : \mathscr{A} \to \mathscr{B}$ a fibration; we claim that there exists a $\mathscr{B}$-cartesian functor

$$\mathsf{ev}^{\mathscr{A}} : \mathsf{C}(\mathscr{A}) \to \mathscr{A}$$

that assigns to every object $(B, G : \mathscr{B}/B \to \mathscr{A})$ of $\mathsf{C}(\mathscr{A})$ its evaluation $G\mathbf{1}_B \in \mathrm{Ob}(\mathscr{A}_B)$, and to any morphism $(f, \alpha : G \Rightarrow G' \circ f_*) : (B, G) \to (B', G')$, the composition

$$G\mathbf{1}_B \xrightarrow{\alpha_{\mathbf{1}_B}} G'f \xrightarrow{G'(f/B')} G'\mathbf{1}_{B'}$$

where $f/B' : f \to \mathbf{1}_{B'}$ is the morphism of $\mathscr{B}/B'$ determined by f. In order to check that these rules define a functor, let $(g, \beta : G' \Rightarrow G'' \circ g_*) : (B', G') \to (B'', G'')$ be any other morphism; then $(g, \beta) \circ (f, \alpha) = (g \circ f, (\beta * f_*) \odot \alpha)$, and the assertion comes down to the identity

$$G''(g/B'') \circ \beta_{\mathbf{1}_{B'}} \circ G'(f/B') \circ \alpha_{\mathbf{1}_B} = G''(g \circ f/B'') \circ ((\beta * f_*) \odot \alpha)_{\mathbf{1}_B}.$$

But we have

$$\beta_{\mathbf{1}_{B'}} \circ G'(f/B') = G''(g_*(f/B')) \circ \beta_f \quad \text{and} \quad G''(g/B'') \circ G''(g_*(f/B')) = G''(g \circ f/B'')$$

whence the contention. Moreover, a simple inspection shows that every cartesian functor $F : \mathscr{A}_1 \to \mathscr{A}_2$ of fibred $\mathscr{B}$-categories yields a commutative diagram of $\mathscr{B}$-categories :

$$\begin{array}{ccc} \mathsf{C}(\mathscr{A}_1) & \xrightarrow{\mathsf{C}(F)} & \mathsf{C}(\mathscr{A}_2) \\ {\scriptstyle \mathsf{ev}^{\mathscr{A}_1}}\downarrow & & \downarrow{\scriptstyle \mathsf{ev}^{\mathscr{A}_2}} \\ \mathscr{A}_1 & \xrightarrow{F} & \mathscr{A}_2 \end{array}$$

and for every natural $\mathscr{B}$-transformation $\alpha : F_1 \Rightarrow F_2$ of $\mathscr{C}$-cartesian functors $F_1, F_2 : \mathscr{A}_1 \to \mathscr{A}_2$ we have the identity :

$$\mathsf{ev}^{\mathscr{A}_2} * \mathsf{C}(\alpha) = \alpha * \mathsf{ev}^{\mathscr{A}_1}.$$

Hence, *if* B *is* U*-small*, the rule $\mathscr{A} \mapsto \mathsf{ev}^{\mathscr{A}}$ defines a strict pseudo-natural transformation

$$\mathsf{ev}^\bullet : \mathsf{F} \circ \mathsf{C} \Rightarrow \mathbf{1}_{\mathsf{Fib}(\mathscr{B})}.$$

Remark 3.2.6. With the notation of (3.2.5), let $G, G' \in \mathscr{A}(B)$ be two cartesian sections over an object $B \in \mathrm{Ob}(\mathscr{B})$, and $\alpha : G \Rightarrow G'$ a natural $\mathscr{B}$-transformation. Then it is easily seen that α is uniquely determined by its evaluation $\mathsf{ev}^{\mathscr{A}}(\mathbf{1}_B, \alpha) = \alpha_{\mathbf{1}_B} : G\mathbf{1}_B \to G'\mathbf{1}_B$. Indeed, every object $(f : B' \to B) \in \mathrm{Ob}(\mathscr{B}/B)$ yields a morphism $f/B : f \to \mathbf{1}_B$ in $\mathscr{B}/B$, whence a commutative diagram

$$\begin{array}{ccc} Gf & \xrightarrow{\alpha_f} & G'f \\ {\scriptstyle G(f/B)}\downarrow & & \downarrow{\scriptstyle G'(f/B)} \\ G\mathbf{1}_B & \xrightarrow{\alpha_{\mathbf{1}_B}} & G'\mathbf{1}_B \end{array}$$

which determines α_f uniquely, since $G'(f/B)$ is cartesian. By the same token, given any morphism $g : G\mathbf{1}_B \to G'\mathbf{1}_B$ in $\varphi^{-1}B$, let us define $\alpha^g_f : Gf \to G'f$ as the unique morphism in $\varphi^{-1}B'$ such that $G'(f/B) \circ \alpha^g_f = g \circ G(f/B)$. Then it is easily seen that the rule $f \mapsto \alpha^g_f$ defines a natural $\mathscr{B}$-transformation $\alpha^g : G \Rightarrow G'$ such that $\alpha^g_{\mathbf{1}_B} = g$. If $G'' : \mathscr{B}/B \to \mathscr{A}$ is another cartesian functor, and $g' : G'\mathbf{1}_B \to G''\mathbf{1}_B$ is any morphism in $\varphi^{-1}B$, clearly we have :

$$\alpha^{g'} \odot \alpha^g = \alpha^{g' \circ g}.$$

Theorem 3.2.7. *The pseudo-functor* C *is a strong right* 2*-adjoint for* F*, and is fully faithful.*

Proof. According to proposition 1.1.15(i) and theorem 2.4.24(i), in order to prove the first assertion it suffices to exhibit a unit and a counit fulfilling the triangular identities (1.1.13). Our candidate counit shall be the pseudo-natural transformation $\mathsf{ev}^\bullet$. Taking into account remark 3.1.5(v) and corollary 2.4.29, in order to see that $\mathsf{ev}^{\mathscr{A}}$ is $\mathscr{B}$-cartesian for every fibration $\varphi : \mathscr{A} \to \mathscr{B}$, and to prove the second assertion of the theorem, it then suffices to notice :

Claim 3.2.8. The functor $\mathsf{ev}^{\mathscr{A}}$ is a $\mathscr{B}$-equivalence.

Proof of the claim. It suffices to show that $\mathsf{ev}^{\mathscr{A}}$ is a fibrewise equivalence (corollary 3.1.28(i)). However, from remark 3.2.6 we see already that for every $B \in \mathrm{Ob}(\mathscr{B})$ the restriction

$$\mathsf{ev}^{\mathscr{A}}_{|B} : \mathscr{A}(B) \to \mathscr{A}_B$$

of $\mathsf{ev}^{\mathscr{A}}$ is fully faithful. To show that $\mathsf{ev}^{\mathscr{A}}_{|B}$ is essentially surjective, let $\boldsymbol{\lambda}$ be a unital cleavage for φ, and c its associated unital pseudo-functor. For every $A \in \mathrm{Ob}(\mathscr{A}_B)$ consider the functor

$$i_A : \mathscr{B}/B \to \mathscr{B}/\varphi\mathscr{A} \qquad (g : B' \to B) \mapsto (A, B, g)$$

that assigns to every morphism $(h/B) : g \to g'$ of $\mathscr{B}/B$ the morphism $(A, B, (h/A)) : (A, B, g) \to (A, B, g')$ of $\mathscr{B}/\varphi\mathscr{A}$. Let also $\mathsf{s} : \mathsf{Morph}(\mathscr{A}) \to \mathscr{A}$ be the source functor (notation of (1.1.30)); it suffices to notice that

$$\beta^{\boldsymbol{\lambda}}_{B,A} := \mathsf{s} \circ \boldsymbol{\lambda} \circ i_A : \mathscr{B}/B \to \mathscr{A}.$$

is a cartesian section with $\beta^{\boldsymbol{\lambda}}_{B,A}(\mathbf{1}_B) = A$. ◊

Next, let $(\varphi : \mathscr{A} \to \mathscr{B}, \boldsymbol{\lambda})$ be any split fibration with small fibres, and c the strict pseudo-functor associated with $\boldsymbol{\lambda}$; denote also by $\boldsymbol{\lambda}^*$ the distinguished cleavage of $\mathsf{C}(\mathscr{A})$, whose associated pseudo-functor is $\mathscr{A}(-)$ (see (3.2.4)). To define a unit for our adjunction, we need to exhibit a natural split cartesian functor :

$$(\varphi, \boldsymbol{\lambda}) \to (\mathsf{C}(\mathscr{A}), \boldsymbol{\lambda}^*).$$

Now, for every $B \in \mathrm{Ob}(\mathscr{B})$ any $A \in \mathrm{Ob}(\mathscr{A}_B)$ define $\beta^{\boldsymbol{\lambda}}_{B,A} \in \mathscr{A}(B)$ as in the proof of claim 3.2.8; by virtue of remark 3.2.6, for every morphism $f : A' \to A$ in $\mathscr{A}_B$ there exists a unique natural $\mathscr{C}$-transformation $\beta^{\boldsymbol{\lambda}}_{B,f} : \beta^{\boldsymbol{\lambda}}_{B,A'} \Rightarrow \beta^{\boldsymbol{\lambda}}_{B,A}$ such that $(\beta^{\boldsymbol{\lambda}}_{B,f})_{\mathbf{1}_B} = f$, and the rules $A \mapsto \beta^{\boldsymbol{\lambda}}_{B,A}$, $f \mapsto \beta^{\boldsymbol{\lambda}}_{B,f}$ yield a well defined functor

$$\beta^{\boldsymbol{\lambda}}_B : \mathscr{A}_B \to \mathscr{A}(B).$$

According to theorem 3.1.24, it then suffices to show that the rule : $B \mapsto \beta^{\boldsymbol{\lambda}}_B$ defines a strict pseudo-natural transformation

$$\beta^{\boldsymbol{\lambda}} : \mathsf{c} \Rightarrow \mathscr{A}(-).$$

The assertion amounts to checking the identities

$$\beta^{\boldsymbol{\lambda}}_{B,A} \circ g_* = \beta^{\boldsymbol{\lambda}}_{B',\mathsf{c}_g A} \qquad \text{for every } (B' \xrightarrow{g} B) \in \mathrm{Ob}(\mathscr{B}/B) \text{ and every } A \in \mathrm{Ob}(\mathscr{A}_B).$$

However, we have :

$$\beta^{\boldsymbol{\lambda}}_{B',\mathsf{c}_g A}(g') = \mathsf{c}_{g'}(\mathsf{c}_g A) = \mathsf{c}_{g\circ g'} A = \beta^{\boldsymbol{\lambda}}_{B,A}(g \circ g') \qquad \text{for every object } g' : B'' \to B' \text{ of } \mathscr{B}/B'$$

which shows that $\beta^{\boldsymbol{\lambda}}_{B,A} \circ g_*$ and $\beta^{\boldsymbol{\lambda}}_{B',\mathsf{c}_g A}$ agree on objects; to see that they also agree on morphisms, we consider two objects $f : C \to B'$ and $f' : C' \to B'$ of $\mathscr{B}/B'$ and a morphism $h/B' : f \to f'$ of $\mathscr{B}/B'$ (notation as in the proof of claim 3.2.8), and we notice that, due to remark 3.2.2(i), the three subdiagrams of the following diagram commute

$$\begin{array}{ccccc} \mathsf{c}_{gf}A & & \xrightarrow{\quad\beta^{\boldsymbol{\lambda}}_{B',\mathsf{c}_gA}(h/B')\quad} & & \mathsf{c}_{gf'}A \\ & \searrow^{\boldsymbol{\lambda}(\mathsf{c}_gA,f)} & & {}^{\boldsymbol{\lambda}(\mathsf{c}_gA,f')}\swarrow & \\ & & \mathsf{c}_gA & & \\ & & \downarrow{\scriptstyle \boldsymbol{\lambda}(A,g)} & & \\ & {\scriptstyle\boldsymbol{\lambda}(A,gf)}\searrow & A & \swarrow{\scriptstyle\boldsymbol{\lambda}(A,gf')} & \end{array}$$

The assertion is an immediate consequence. We need to verify that the rule $(\varphi,\boldsymbol{\lambda}) \mapsto \beta^{\boldsymbol{\lambda}}$ yields a strict pseudo-natural transformation

$$\beta^{\bullet} : \mathbf{1}_{\mathsf{Split}(\mathscr{B})} \Rightarrow \mathsf{C} \circ \mathsf{F}.$$

To this aim, let $G : (\varphi : \mathscr{A} \to \mathscr{B}, \boldsymbol{\lambda}) \to (\varphi' : \mathscr{A}' \to \mathscr{B}, \boldsymbol{\lambda}')$ be any split $\mathscr{B}$-cartesian functor, and let c and c' be the pseudo-functors associated with $\boldsymbol{\lambda}$ and respectively $\boldsymbol{\lambda}'$; we come down to showing the commutativity of the diagram

$$\begin{array}{ccc} \mathscr{A}_B & \xrightarrow{\beta^{\boldsymbol{\lambda}}_B} & \mathscr{A}(B) \\ {\scriptstyle G_B}\downarrow & & \downarrow{\scriptstyle \mathsf{Cart}_{\mathscr{B}}(\mathscr{B}/B,G)} \\ \mathscr{A}'_B & \xrightarrow{\beta^{\boldsymbol{\lambda}'}_B} & \mathscr{A}'(B) \end{array} \qquad \text{for every } B \in \mathrm{Ob}(\mathscr{B}).$$

However, the composition $\mathsf{Cart}_{\mathscr{B}}(\mathscr{B}/B, G) \circ \beta^{\boldsymbol{\lambda}}_B$ assigns to every $A \in \mathrm{Ob}(\mathscr{A}_B)$ the cartesian section $\mathscr{B}/B \to \mathscr{A}'$ given by the rule : $(B' \xrightarrow{g} B) \mapsto G(\mathsf{c}_gA)$, whereas $\beta^{\boldsymbol{\lambda}'}_B \circ G$ assigns to the same object the cartesian section given by the rule : $(B' \xrightarrow{g} B) \mapsto \mathsf{c}'_g(GA)$; moreover, if $B'' \xrightarrow{g'} B$ is any other object of $\mathscr{B}/B$ and $h/B : g \to g'$ is any morphism, we get two commutative diagrams :

$$\begin{array}{ccc} G(\mathsf{c}_gA) & \xrightarrow{G(\beta^{\boldsymbol{\lambda}}_{B,A}(h/B))} & G(\mathsf{c}_{g'}A) \\ & {\scriptstyle G(\boldsymbol{\lambda}(A,g))}\searrow \quad \swarrow{\scriptstyle G(\boldsymbol{\lambda}(A,g'))} & \\ & GA & \end{array} \qquad \begin{array}{ccc} \mathsf{c}'_g(GA) & \xrightarrow{\beta^{\boldsymbol{\lambda}'}_{B,GA}(h/B)} & \mathsf{c}'_{g'}(GA) \\ & {\scriptstyle \boldsymbol{\lambda}'(GA,g)}\searrow \quad \swarrow{\scriptstyle \boldsymbol{\lambda}'(GA,g')} & \\ & GA. & \end{array}$$

Since G is split, remark 3.2.2(ii) says that these two diagrams coincide, i.e. the functors $\mathsf{Cart}_{\mathscr{B}}(\mathscr{B}/B, G) \circ \beta^{\boldsymbol{\lambda}}_B$ and $\beta^{\boldsymbol{\lambda}'}_B \circ G$ agree on all objects of $\varphi^{-1}B$. To see that they agree also on morphisms, let $f : A \to A'$ be any morphism of $\varphi^{-1}B$; we have to show that $G * \beta^{\boldsymbol{\lambda}}_{B,f} = \beta^{\boldsymbol{\lambda}'}_{B,Gf}$. By remark 3.2.6, it suffices then to compare the evaluations at $\mathbf{1}_B$ of both of these natural $\mathscr{B}$-transformation; but by definition we have $(G * \beta^{\boldsymbol{\lambda}}_{B,f})_{\mathbf{1}_B} = Gf = (\beta^{\boldsymbol{\lambda}'}_{B,Gf})_{\mathbf{1}_B}$, as required.

Lastly, let $G, G' : (\varphi : \mathscr{A} \to \mathscr{B}, \boldsymbol{\lambda}) \to (\varphi' : \mathscr{A}' \to \mathscr{B}, \boldsymbol{\lambda}')$ be two split $\mathscr{B}$-cartesian functors, and $\gamma : G \Rightarrow G'$ a natural $\mathscr{B}$-transformation; we need to show :

$$\mathsf{Cart}_{\mathscr{B}}(\mathscr{B}/B, \gamma) * \beta^{\boldsymbol{\lambda}}_B = \beta^{\boldsymbol{\lambda}'}_B * (\gamma_{|\mathscr{A}_B}) \qquad \text{for every } B \in \mathrm{Ob}(\mathscr{B})$$

which translates as the identity : $\gamma_{\mathsf{c}_gA} = \mathsf{c}'_g(\gamma_A)$ for every $g \in \mathrm{Ob}(\mathscr{B}/B)$ and $A \in \mathrm{Ob}(\mathscr{A}_B)$. However, again by virtue of remark 3.2.2(ii), the following commutative diagrams coincide :

$$\begin{array}{ccc} G(\mathsf{c}_gA) & \xrightarrow{\gamma_{\mathsf{c}_gA}} & G'(\mathsf{c}_gA) \\ {\scriptstyle G(\boldsymbol{\lambda}(A,g))}\downarrow & & \downarrow{\scriptstyle G'(\boldsymbol{\lambda}(A,g))} \\ GA & \xrightarrow{\gamma_A} & G'A \end{array} \qquad \begin{array}{ccc} \mathsf{c}'_g(GA) & \xrightarrow{\mathsf{c}'_g(\gamma_A)} & \mathsf{c}'_g(G'A) \\ {\scriptstyle \boldsymbol{\lambda}'(GA,g)}\downarrow & & \downarrow{\scriptstyle \boldsymbol{\lambda}'(G'A,g)} \\ GA & \xrightarrow{\gamma_A} & G'A \end{array}$$

whence the contention.

It remains to show that the unit and counit thus defined fulfill the triangular identities. Now, let $\varphi : \mathscr{A} \to \mathscr{B}$ be any fibration, $(\varphi' : \mathscr{A}' \to \mathscr{B}, \boldsymbol{\lambda}')$ any split fibration, and c' the pseudo-functor associated with $\boldsymbol{\lambda}'$. Denote by $\boldsymbol{\lambda}^*$ the distinguished cleavage of $\mathsf{C}(\mathscr{A})$ as in (3.2.4), whose associated pseudo-functor is (naturally identified with) $\mathscr{A}(-)$; then, in view of theorem 3.1.24, the triangular identities come down to the commutativity of the diagrams

$$\begin{array}{ccc} \mathscr{A}(B) & \xrightarrow{\beta_B^{\boldsymbol{\lambda}^*}} & \mathsf{C}(\mathscr{A})(B) \\ & \diagdown\!\!\diagdown & \downarrow{\scriptstyle \mathsf{Cart}_{\mathscr{B}}(\mathscr{B}/B,\mathsf{ev}^{\mathscr{A}})} \\ & & \mathscr{A}(B) \end{array} \qquad\qquad \begin{array}{ccc} \mathsf{c}'_B & \xrightarrow{\beta_B^{\boldsymbol{\lambda}'}} & \mathscr{A}'(B) \\ & \diagdown\!\!\diagdown & \downarrow{\scriptstyle \mathsf{ev}^{\mathscr{A}'}_{|B}} \\ & & \mathsf{c}'_B \end{array}$$

for every $B \in \mathrm{Ob}(\mathscr{B})$. The latter follows by a direct inspection. $\square$

Remark 3.2.9. (i) Let $\varphi : \mathscr{A} \to \mathscr{B}$ be a fibration, and c the pseudo-functor associated with a given cleavage $\boldsymbol{\lambda}$ for φ. The proof of theorem 3.2.7 furnishes a pseudo-natural equivalence

$$\mathsf{ev}^{\mathscr{A}}_{|\bullet} : \mathscr{A}(-) \Rightarrow \mathsf{c}.$$

In case $\boldsymbol{\lambda}$ is split, it also furnishes a quasi-inverse for $\mathsf{ev}^{\mathscr{A}}_{|\bullet}$ which is a *strict* pseudo-natural equivalence $\beta^{\boldsymbol{\lambda}} : \mathsf{c} \Rightarrow \mathscr{A}(-)$. However, we cannot deduce from this that the unit of the 2-adjunction of theorem 3.2.7 is an equivalence, since we cannot ensure the existence of a *strict* quasi-inverse for $\beta^{\boldsymbol{\lambda}}$. (Especially, $\mathsf{ev}^{\mathscr{A}}_{|\bullet}$ is not strict, in general.)

(ii) On the other hand, denote by $\mathsf{spFib}(\mathscr{B})$ the full 2-subcategory of $\mathsf{Fib}(\mathscr{B})$ whose objects are the fibrations that admit a split cleavage. Then claim 3.2.8 implies that the inclusion functor

$$\mathsf{spFib}(\mathscr{B}) \to \mathsf{Fib}(\mathscr{B})$$

is a 2-equivalence which admits a pseudo-inverse given by the rule : $(\varphi : \mathscr{A} \to \mathscr{B}) \mapsto \mathsf{C}(\mathscr{A})$ for every fibration φ.

3.2.10. For every pair of universes U, V with $\mathsf{U} \subset \mathsf{V}$, and every $\mathscr{C} \in \mathrm{Ob}((\mathsf{U},\mathsf{V})\text{-}\mathbf{Cat})$, the set of connected components $\pi_0(\mathscr{C})$ is essentially U-small (see remark 1.2.21(ii)); hence, after replacing $\pi_0(\mathscr{C})$ by a U-small set of the same cardinality, we get a well defined functor

$$\pi_0 : (\mathsf{U},\mathsf{V})\text{-}\mathbf{Cat} \to \mathbf{Set}$$

that is left adjoint to the fully faithful imbedding $\mathbf{Set} \to (\mathsf{U},\mathsf{V})\text{-}\mathbf{Cat}$ which assigns to every U-small set the associated discrete category (see example 1.1.6(ii)). Moreover, π_0 is also a well defined strict pseudo-functor, for the natural 2-category structure on $(\mathsf{U},\mathsf{V})\text{-}\mathbf{Cat}$ (and here we regard $\mathbf{Set}$ as a 2-category whose only 2-cells are identities : see example 2.2.4(i)).

Now, if $\mathscr{B}$ is any V-small category, and $\mathsf{c} : \mathscr{B}^o \to (\mathsf{U},\mathsf{V})\text{-}\mathbf{Cat}$ any pseudo-functor, it follows that the composition $\pi_0 \circ \mathsf{c} : \mathscr{B}^o \to \mathbf{Set}$ is a strict pseudo-functor, *i.e.* it is a presheaf on $\mathscr{B}$. Also, every pseudo-natural transformation $\omega : \mathsf{c} \Rightarrow \mathsf{d}$ of pseudo-functors $\mathsf{c}, \mathsf{d} : \mathscr{B}^o \to (\mathsf{U},\mathsf{V})\text{-}\mathbf{Cat}$ induces a morphism of presheaves $\pi_0 * \omega : \pi_0 \circ \mathsf{c} \to \pi_0 \circ \mathsf{d}$, and notice that if $\Xi : \omega \rightsquigarrow \omega'$ is any modification of such pseudo-natural transformations, then $\pi_0 * \omega = \pi_0 * \omega'$ and the induced modification $\pi_0 \circ \Xi : \pi_0 * \omega \rightsquigarrow \pi_0 * \omega$ is just the identity. In view of theorem 3.1.24, it then follows that the rule $\mathsf{c} \mapsto \pi_0 \circ \mathsf{c}$ yields a well defined strict pseudo-functor

$$\pi_0^{\mathscr{B}} : (\mathsf{U},\mathsf{V})\text{-}\mathsf{Fib}(\mathscr{B}) \to \mathscr{B}^\wedge_{\mathsf{U}}$$

Explicitly, for every fibration $\varphi : \mathscr{A} \to \mathscr{B}$, the presheaf $\pi_0^{\mathscr{B}}(\mathscr{A})$ is given by the rule : $X \mapsto \pi_0(\varphi^{-1}X)$ for every $X \in \mathrm{Ob}(\mathscr{B})$. It is then easily seen that $\pi_0^{\mathscr{B}}$ is a strong left 2-adjoint to the strict pseudo-functor $\mathscr{F}ib$ of (3.1.17).

Remark 3.2.11. The functor $\mathscr{F}ib$ of (3.1.17) also admits a right adjoint. Indeed, notice that $\mathscr{F}ib$ factors uniquely as the composition of a strict pseudo-functor

$$sp\mathscr{F}ib : \mathscr{B}^\wedge \to \mathsf{Split}(\mathscr{B})$$

and the forgetful strict pseudo-functor F (notation of (3.2.4)). In light of theorem 3.2.7, it then suffices to find a right adjoint for $sp\mathscr{F}ib$. Now, if $(\varphi : \mathscr{A} \to \mathscr{B}, \boldsymbol{\lambda})$ is any split fibration, and $\mathsf{c} : \mathscr{B}^o \to \mathbf{Cat}$ the associated pseudo-functor, notice that the composition

$$\mathrm{Ob} \circ \mathsf{c} : \mathscr{B}^o \to \mathbf{Cat} \to \mathbf{Set} \qquad B \mapsto \mathrm{Ob}(\mathscr{A}_B)$$

is a presheaf. It is easily seen that the resulting functor

$$\mathrm{Ob}^{\mathscr{B}} : \mathsf{Split}(\mathscr{B}) \to \mathscr{B}^o \qquad (\varphi : \mathscr{A} \to \mathscr{B}, \boldsymbol{\lambda}) \mapsto \mathrm{Ob} \circ \mathsf{c}$$

is a right adjoint for $sp\mathscr{F}ib$. However, we do not get a right 2-adjoint by this rule.

3.3. 2**-Fibrations.** We wish now to consider the generalization of (3.1.18) where $\mathscr{B}$ is an arbitrary V-small 2-category. In this case, for every pseudo-functor $\mathsf{c} : \mathscr{B}^o \to (\mathsf{U}, \mathsf{V})\text{-}\mathbf{Cat}$ we get a natural 2-category structure on $\mathscr{F}ib(\mathsf{c})$, such that the resulting functor $\pi^c : \mathscr{F}ib(\mathsf{c}) \to \mathscr{B}$ is a strict pseudo-functor. Indeed, for every two objects $(B, X), (B', Y)$ of $\mathscr{F}ib(\mathsf{c})$, and any two morphisms $(\varphi_1, f_1), (\varphi_2, f_2) : (B, X) \to (B', Y)$ of $\mathscr{F}ib(\mathsf{c})$, we declare that the 2-cells $(\varphi_1, f_1) \Rightarrow (\varphi_2, f_2)$ are the 2-cells $\alpha : \varphi_1 \Rightarrow \varphi_2$ that make commute the diagram

$$\xymatrix{ & X \ar[ld]_{f_1} \ar[rd]^{f_2} & \\ \mathsf{c}_{\varphi_1}Y \ar[rr]^{\mathsf{c}_{\alpha,Y}} & & \mathsf{c}_{\varphi_2}Y. }$$

If $(\varphi_3, f_3) : (B, X) \to (B', Y)$ is another morphism and $\beta : (\varphi_2, f_2) \Rightarrow (\varphi_3, f_3)$ another 2-cell, the composition $\beta \odot \alpha : (\varphi_1, f_1) \Rightarrow (\varphi_3, f_3)$ is given by the composition law for 2-cells of $\mathscr{B}(B, B')$. Since $\mathsf{c}_{\beta\odot\alpha,Y} = \mathsf{c}_{\beta,Y} \circ \mathsf{c}_{\alpha,Y}$, it is easily seen that this composition rule is well defined. Likewise, if (B'', Z) is any other object of $\mathscr{F}ib(\mathsf{c})$, $(\psi_1, g_1), (\psi_2, g_2) : (B', Y) \to (B'', Z)$ any two morphisms and $\alpha' : (\psi_1, g_1) \Rightarrow (\psi_2, g_2)$ any 2-cell, we let $\alpha' * \alpha : (\psi_1, g_1) \circ (\varphi_1, f_1) \Rightarrow (\psi_2, g_2) \circ (\varphi_2, f_2)$ be defined by the corresponding composition law for 2-cells in $\mathscr{B}$. In order to check that this rule is well defined, it suffices to show the commutativity of the diagram :

$$\xymatrix{ X \ar[r]^{f_1} \ar[rd]_{f_2} & \mathsf{c}_{\varphi_1}Y \ar[r]^{\mathsf{c}_{\varphi_1}(g_1)} \ar[d]^{\mathsf{c}_{\alpha,Y}} & \mathsf{c}_{\varphi_1}\mathsf{c}_{\psi_1}Z \ar[r]^{\gamma^{\mathsf{c}}_{(\varphi_1,\psi_1),Z}} \ar[d]^{\mathsf{c}_{\alpha,\mathsf{c}_{\psi_1}Z}} & \mathsf{c}_{\psi_1\circ\varphi_1}Z \ar[dd]^{\mathsf{c}_{\alpha'*\alpha,Z}} \\ & \mathsf{c}_{\varphi_2}Y \ar[r]^{\mathsf{c}_{\varphi_2}(g_1)} \ar[rd]_{\mathsf{c}_{\varphi_2}(g_2)} & \mathsf{c}_{\varphi_2}\mathsf{c}_{\psi_1}Z \ar[d]^{\mathsf{c}_{\varphi_2}(\mathsf{c}_{\alpha',Z})} & \\ & & \mathsf{c}_{\varphi_2}\mathsf{c}_{\psi_2}Z \ar[r]_{\gamma^{\mathsf{c}}_{(\varphi_2,\psi_2),Z}} & \mathsf{c}_{\psi_2\circ\varphi_2}Z }$$

(where γ^{c} is the coherence constraint of c); the latter follows directly from remark 2.1.17(ii) and the identity : $\mathsf{c}_{\varphi_2}(\mathsf{c}_{\alpha',Z}) \circ \mathsf{c}_{\alpha,\mathsf{c}_{\psi_1}Z} = (\mathsf{c}_{\alpha'} * \mathsf{c}_\alpha)_Z$. It is then obvious that $\mathscr{F}ib(\mathsf{c})$ is a 2-category with these composition laws (and with the composition law for 1-cells given by (3.1.18)), since the required associativity and unit axioms hold already in $\mathscr{B}$.

In analogy with the case of usual categories, it is natural to make the following :

Definition 3.3.1. Let $\mathscr{A}$, $\mathscr{B}$ be two V-small 2-categories, $\pi : \mathscr{A} \to \mathscr{B}$ a strict pseudo-functor.

(i) We say that a 1-cell $f : A' \to A$ of $\mathscr{A}$ is *$\mathscr{B}$-cartesian* if for every $X \in \mathrm{Ob}(\mathscr{A})$ the following diagram is cartesian in the category $\mathsf{V}\text{-}\mathbf{Cat}$:

$$\begin{array}{ccc} \mathscr{A}(X, A') & \xrightarrow{f_*} & \mathscr{A}(X, A) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \mathscr{B}(\pi X, \pi A') & \xrightarrow{(\pi f)_*} & \mathscr{B}(\pi X, \pi A). \end{array}$$

(ii) We say that π (or $\mathscr{A}$) is a 2-fibration over $\mathscr{B}$, if for every 1-cell $\varphi : B' \to B$ in $\mathscr{B}$ and every $A \in \pi^{-1}B$ there exists a $\mathscr{B}$-cartesian 1-cell $f : A' \to A$ in $\mathscr{A}$ such that $\pi(f) = \varphi$.

(iii) For every $B \in \mathrm{Ob}(\mathscr{B})$, the *fibre* $\pi^{-1}B$ is the 2-subcategory of $\mathscr{A}$ whose underlying category is the fibre over B of the functor underlying π, and whose 2-cells are the 2-cells α of $\mathscr{A}$ such that $\pi(\alpha) = i_B$. We say that π *has essentially* U*-small fibres* if $\pi^{-1}B$ is an essentially U-small 2-category for every $B \in \mathrm{Ob}(\mathscr{B})$.

Remark 3.3.2. (i) With the notation of definition 3.3.1, notice that a $\mathscr{B}$-cartesian 1-cell of $\mathscr{A}$ is also a cartesian morphism of the underlying category, since the forgetful functor $\mathsf{V}\text{-}\mathbf{Cat} \to \mathsf{V}\text{-}\mathbf{Set}$ commutes with fibre products.

(ii) It is easily seen that a composition of cartesian 1-cells is cartesian, and every invertible 1-cell is cartesian (details left to the reader).

(iii) It follows from (i) that if π is a 2-fibration, then it is also a (usual) fibration on the underlying categories. Moreover, in this case a 1-cell of $\mathscr{A}$ is cartesian if and only if it is a cartesian morphism of the underlying category : indeed, the necessity has already been remarked in (i). Conversely, let $f : A' \to A$ be a 1-cell which is a cartesian morphism in the underlying category, and pick any 1-cell $g : A'' \to A$ of $\mathscr{A}$ which is cartesian in the sense of definition 3.3.1(i), and with $\pi(g) = \pi(f)$; then there exists an isomorphism $h : A' \xrightarrow{\sim} A''$ in the underlying category of $\mathscr{A}$ such that $f = g \circ h$. By (ii), we deduce that f is a cartesian 1-cell.

(iv) Notice that our definition of 2-fibrations (which follows [81, pp.35–36]) does not include any condition on the existence of cartesian liftings of 2-cells of $\mathscr{B}$. This is somewhat inadequate, and thus our definition should be regarded as provisional. A much more comprehensive treatment of 2-fibrations (including the relevant condition of cartesian liftings of 2-cells) can be found in the paper [44].

Lemma 3.3.3. *Let $\mathscr{B}$ be a 2-category, and $\mathsf{c} : \mathscr{B}^o \to (\mathsf{U}, \mathsf{V})\text{-}\mathbf{Cat}$ a pseudo-functor. We have :*

(i) *The 2-category $\mathscr{F}ib(\mathsf{c})$ is a 2-fibration over $\mathscr{B}$.*

(ii) *A 1-cell $(\varphi, f) : (B', X) \to (B, Y)$ of $\mathscr{F}ib(\mathsf{c})$ is cartesian if and only if $f : X \to \mathsf{c}_\varphi Y$ is an isomorphism in $\mathsf{c}_{B'}$.*

Proof. It suffices to prove (ii). Now, notice that the condition of (ii) is necessary, due to lemma 3.1.20(i). Conversely, let (φ, f) be such a 1-cell with f an isomorphism; consider any object (B'', Z) of $\mathscr{F}ib(\mathsf{c})$, any two 1-cells $\psi_1, \psi_2 : B'' \to B'$ in $\mathscr{B}$, a 2-cell $\alpha : \psi_1 \Rightarrow \psi_2$ in $\mathscr{B}$, and any 2-cell $\beta : (\varphi \circ \psi_1, g_1) \Rightarrow (\varphi \circ \psi_2, g_2)$ in $\mathscr{F}ib(\mathsf{c})$ such that $\beta = \varphi_* * \alpha$. We know already by lemma 3.1.20(i) that for $i = 1, 2$ there exists a unique 1-cell $(\psi_i, g'_i) : (B'', Z) \to (B', X)$ of $\mathscr{F}ib(\mathsf{c})$ such that $(\varphi, f) \circ (\psi_i, g'_i) = (\varphi \circ \psi_i, g_i)$, and it remains to check that α yields a 2-cell $(\psi_1, g'_1) \Rightarrow (\psi_2, g'_2)$. To this aim, we consider the diagram :

$$\begin{array}{ccccccc} Z & \xrightarrow{g'_1} & \mathsf{c}_{\psi_1} X & \xrightarrow{\mathsf{c}_{\psi_1}(f)} & \mathsf{c}_{\psi_1}\mathsf{c}_\varphi Y & \xrightarrow{\gamma^{\mathsf{c}}_{(\psi_1,\varphi),Y}} & \mathsf{c}_{\varphi\circ\psi_1} Y \\ & \searrow^{g'_2} & \downarrow{\scriptstyle \mathsf{c}_{\alpha,X}} & & \downarrow{\scriptstyle \mathsf{c}_{\alpha,\mathsf{c}_\varphi Y}} & & \downarrow{\scriptstyle \mathsf{c}_{\varphi*\alpha,Y}} \\ & & \mathsf{c}_{\psi_2} X & \xrightarrow{\mathsf{c}_{\psi_2}(f)} & \mathsf{c}_{\psi_2}\mathsf{c}_\varphi Y & \xrightarrow{\gamma^{\mathsf{c}}_{(\psi_2,\varphi),Y}} & \mathsf{c}_{\varphi\circ\psi_2} Y \end{array}$$

(where γ^{c} is the coherence constraint of c). By assumption, the subdiagram obtained after removing the two middle vertical arrows commutes, and we need to check that the triangular subdiagram on the left commutes as well. However, the square subdiagram on the right commutes by remark 2.1.17(ii), and the central square subdiagram commutes by naturality oc c_α. Moreover, since f is an isomorphism, the two bottom horizontal arrows are both isomorphisms. The contention follows straightforwardly. □

Example 3.3.4. Let $\mathscr{B}$ be any 2-category.

(i) Denote by $\mathbb{1}$ a chosen final object of $\mathbf{Cat}$, *i.e.* a category with only one object and one morphism, and consider the constant pseudo-functor $\mathsf{F}_{\mathbb{1}} : \mathscr{B}^o \to \mathbf{Cat}$ with value $\mathbb{1}$ (notation of (2.5)). It is easily seen that the induced projection :

$$\mathscr{F}ib(\mathsf{F}_{\mathbb{1}}) \to \mathscr{B}$$

is an isomorphism of 2-categories.

(ii) If $\mathscr{B}$ has small Hom-categories, fix $B \in \mathscr{B}$ and consider the strict pseudo-functor

$$h_B : \mathscr{B}^o \to \mathbf{Cat} \qquad B' \mapsto \mathscr{B}(B', B)$$

that assigns to every 1-cell $f : B' \to B''$ the functor $f^* : \mathscr{B}(B'', B) \to \mathscr{B}(B', B)$, and to every 2-cell $\beta : f \Rightarrow f'$ between 1-cells $f, f' : B' \to B''$ the natural transformation $\beta^* : f^* \Rightarrow f'^*$ (see example 2.2.8(i)). Then we have a natural isomorphism of 2-categories

$$\mathscr{F}ib(h_B) \xrightarrow{\sim} \mathscr{B}/B$$

(notation of example 2.1.11(ii)) which identifies the natural projection $\mathscr{F}ib(h_B) \to \mathscr{B}$ with the source strict pseudo-functor $\mathsf{s}_B : \mathscr{B}/B \to \mathscr{B}$.

3.3.5. For any pair of 2-fibrations $\mathscr{A}, \mathscr{A}'$ over $\mathscr{B}$ we denote by

$$2\text{-}\mathsf{Cart}_{\mathscr{B}}(\mathscr{A}, \mathscr{A}')$$

the category whose objects are the *$\mathscr{B}$-cartesian strict pseudo-functors*, *i.e.* the strict pseudo-functors $\mathscr{A} \to \mathscr{A}'$ that are $\mathscr{B}$-cartesian functors on the underlying categories; the morphisms are the strict pseudo-natural transformations that are natural $\mathscr{B}$-transformations on the underlying categories. Notice that $2\text{-}\mathsf{Cart}_{\mathscr{B}}(\mathscr{A}, \mathscr{F}ib(\mathsf{c}))$ is a full subcategory of $\mathsf{Cart}_{\mathscr{B}}(\mathscr{A}, \mathscr{F}ib(\mathsf{c}))$, for every pseudo-functor $\mathsf{c} : \mathscr{B}^o \to \mathbf{Cat}$ and every 2-fibration $\mathscr{A}$.

Just as in definition 3.1.4(ii), every pair of $\mathscr{B}$-cartesian strict pseudo-functors $H : \mathscr{C} \to \mathscr{A}$ and $K : \mathscr{A}' \to \mathscr{C}'$ induces a functor

$$2\text{-}\mathsf{Cart}_{\mathscr{B}}(H, K) : 2\text{-}\mathsf{Cart}_{\mathscr{B}}(\mathscr{A}, \mathscr{A}') \to 2\text{-}\mathsf{Cart}_{\mathscr{B}}(\mathscr{C}, \mathscr{C}') \qquad G \mapsto K \circ G \circ H$$

which assigns to any morphism $\alpha : G \Rightarrow G'$ in $2\text{-}\mathsf{Cart}_{\mathscr{B}}(\mathscr{A}, \mathscr{A}')$, the natural transformation $K * \alpha * H : K \circ G \circ H \Rightarrow K \circ G' \circ H$. As usual, in case $H = \mathbf{1}_{\mathscr{A}}$ (resp. $K = \mathbf{1}_{\mathscr{A}'}$) we also denote this functor by $2\text{-}\mathsf{Cart}_{\mathscr{B}}(\mathscr{A}, K)$ (resp. by $2\text{-}\mathsf{Cart}_{\mathscr{B}}(H, \mathscr{A}')$).

Likewise, if $H, H' : \mathscr{C} \to \mathscr{A}$ and $K, K' : \mathscr{A}' \to \mathscr{C}'$ are four $\mathscr{B}$-cartesian strict pseudo-functors, every pair of strict pseudo-natural $\mathscr{B}$-transformations $\beta : H \Rightarrow H'$ and $\gamma : K \Rightarrow K'$ induces a natural transformation :

$$2\text{-}\mathsf{Cart}_{\mathscr{B}}(\beta, \gamma) : 2\text{-}\mathsf{Cart}_{\mathscr{B}}(H, K) \Rightarrow 2\text{-}\mathsf{Cart}_{\mathscr{B}}(H', K') \qquad G \mapsto \gamma * G * \beta$$

and again, if $\beta = \mathbf{1}_H$ (resp. $\gamma = \mathbf{1}_K$) we denote this natural transformation by $2\text{-}\mathsf{Cart}_{\mathscr{B}}(H, \gamma)$ (resp. $2\text{-}\mathsf{Cart}_{\mathscr{B}}(\alpha, K)$).

3.3.6. In the situation of (3.3), let $\mathsf{d} : \mathscr{B}^o \to (\mathsf{U},\mathsf{V})\text{-}\mathbf{Cat}$ be another pseudo-functor, and $\omega : \mathsf{c} \Rightarrow \mathsf{d}$ any pseudo-natural transformation. We claim that the functor $\mathscr{F}ib(\omega) : \mathscr{F}ib(\mathsf{c}) \to \mathscr{F}ib(\mathsf{d})$ defined as in (3.1.19) is a $\mathscr{B}$-cartesian strict pseudo-functor. The assertion amounts to saying that any 2-cell $\alpha : (\varphi_1, f_1) \Rightarrow (\varphi_2, f_2)$ as in (3.3) induces a 2-cell

$$\alpha : \mathscr{F}ib(\omega)(\varphi_1, f_1) \Rightarrow \mathscr{F}ib(\omega)(\varphi_2, f_2).$$

In turn, the latter boils down to the commutativity of the diagram :

$$\begin{array}{ccccc}
\omega_B \circ \mathsf{c}_{\varphi_1} Y & \xleftarrow{\omega_B f_1} & \omega_B X & \xrightarrow{\omega_B f_2} & \omega_B \circ \mathsf{c}_{\varphi_2} Y \\
\downarrow{\scriptstyle (\tau^\omega_{\varphi_1,Y})^{-1}} & & & & \downarrow{\scriptstyle (\tau^\omega_{\varphi_2,Y})^{-1}} \\
\mathsf{d}_{\varphi_1} \circ \omega_{B'} Y & & \xrightarrow{\quad \mathsf{d}_{\alpha,\omega_{B'}Y} \quad} & & \mathsf{d}_{\varphi_2} \circ \omega_{B'} Y.
\end{array}$$

However, since α is a 2-cell in $\mathscr{F}ib(\mathsf{c})$, we have the identity :

$$\omega_B f_2 = \omega_B(\mathsf{c}_{\alpha,Y}) \circ \omega_B f_1$$

and on the other hand, by remark 2.2.5(i), the naturality of τ^ω implies the identity

$$(\tau^\omega_{\varphi_2,Y})^{-1} \circ \omega_B(\mathsf{c}_{\alpha,Y}) = \mathsf{d}_{\alpha,\omega_{B'}Y} \circ (\tau^\omega_{\varphi_1,Y})^{-1}$$

whence the assertion. Lastly, for any two pseudo-natural transformations $\omega, \omega' : \mathsf{c} \Rightarrow \mathsf{d}$ and every modification $\Xi : \omega \rightsquigarrow \omega'$, a simple inspection shows that the natural transformation $\mathscr{F}ib(\Xi) : \mathscr{F}ib(\omega) \Rightarrow \mathscr{F}ib(\omega')$ of (3.1.21) is also a strict pseudo-natural transformation.

Lemma 3.3.7. (i) *For every two pseudo-functors* $\mathsf{c}, \mathsf{d} : \mathscr{B}^o \to (\mathsf{U},\mathsf{V})\text{-}\mathbf{Cat}$*, the induced functor:*

$$\mathscr{F}ib : \mathsf{PsNat}(\mathsf{c}, \mathsf{d}) \to 2\text{-}\mathsf{Cart}_{\mathscr{B}^o}(\mathscr{F}ib(\mathsf{c}), \mathscr{F}ib(\mathsf{d}))$$

is an isomorphism of categories.

(ii) (2-Yoneda's Lemma) *For every pseudo-functor* $\mathsf{c} : \mathscr{B}^o \to (\mathsf{U},\mathsf{V})\mathbf{Cat}$ *and every* $X \in \mathrm{Ob}(\mathscr{B})$*, we have an equivalence of categories pseudo-natural in both arguments :*

$$\mathsf{PsNat}(h_X, \mathsf{c}) \xrightarrow{\sim} \mathsf{c}_X.$$

Proof. (i): Notice that a $\mathscr{B}$-cartesian strict pseudo-functor $\mathscr{F}ib(\mathsf{c}) \to \mathscr{F}ib(\mathsf{d})$ is completely determined by its underlying cartesian functor of (usual) $\mathscr{B}$-fibrations. Then the assertion is already known from the proof of theorem 3.1.24.

(ii) follows from (i), example 3.3.4(ii) and claim 3.2.8. □

3.3.8. Clearly the 2-fibrations with V-small and essentially U-small fibres over a given V-small 2-category $\mathscr{B}$ form a 2-category

$$(\mathsf{U},\mathsf{V})\text{-}2\text{-}\mathsf{Fib}(\mathscr{B})$$

whose Hom-categories are given by the categories $2\text{-}\mathsf{Cart}_{\mathscr{B}}(-,-)$ as in (3.3.5). As usual, this 2-category shall be often just denoted $2\text{-}\mathsf{Fib}(\mathscr{B})$. We have a strict and strongly faithful pseudo-functor

$$\mathscr{F}ib_{\mathscr{B}} : \mathsf{PsFun}(\mathscr{B}^o, \mathbf{Cat}) \to 2\text{-}\mathsf{Fib}(\mathscr{B}) \qquad F \mapsto \mathscr{F}ib(F)$$

but unlike for usual fibrations, $\mathscr{F}ib_{\mathscr{B}}$ is not a 2-equivalence. As an application of these constructions, we deduce :

Theorem 3.3.9. *The* 2*-category* $(\mathsf{U},\mathsf{V})\text{-}\mathbf{Cat}$ *is strongly* 2*-complete and strongly* 2*-cocomplete.*

Proof. Let $F : \mathscr{B} \to (\mathsf{U},\mathsf{V})\mathbf{Cat}$ be any pseudo-functor from a U-small 2-category $\mathscr{B}$; we set :

$$\mathscr{L}_F := 2\text{-}\mathsf{Cart}_{\mathscr{B}^o}(\mathscr{B}^o, \mathscr{F}ib(F))$$

and we define a pseudo-cone $\pi : \mathsf{F}_{\mathscr{L}_F} \Rightarrow F$ as follows. First, in light of lemma 3.3.7(i) and example 3.3.4, we have a natural isomorphism of categories

$$\omega : \mathscr{L}_F \xrightarrow{\sim} \mathsf{PsNat}(\mathsf{F}_{\mathbf{1}}, F)$$

where $\mathsf{F}_{\mathbf{1}} : \mathscr{B} \to \mathbf{Cat}$ is the constant pseudo-functor with value $\mathbf{1}$. Now, let $\beta : \mathsf{F}_{\mathbf{1}} \Rightarrow F$ be any pseudo-cone; for every $B \in \mathrm{Ob}(\mathscr{B})$ we have then the functor $\beta_B : \mathbf{1} \to FB$, which is identified with the object $\beta_B(\varnothing) \in \mathrm{Ob}(FB)$, under the isomorphism of categories :

$$\mathsf{Fun}(\mathbf{1}, FB) \xrightarrow{\sim} FB \qquad G \mapsto G(\varnothing) \qquad (\alpha : G \Rightarrow G') \mapsto \alpha_\varnothing : G(\varnothing) \to G'(\varnothing)$$

(here $\mathrm{Ob}(\mathbf{1}) = \{\varnothing\}$). Clearly, the rule $\beta \mapsto \beta_B$ yields a functor

$$\pi_B : \mathsf{PsNat}(\mathsf{F}_{\mathbf{1}}, F) \to FB$$

that assigns to every modification $\Xi : \beta \rightsquigarrow \beta'$ the natural transformation $\Xi_B : \beta_B \Rightarrow \beta'_B$, identified with the morphism $\Xi_{B,\varnothing} : \beta_B(\varnothing) \to \beta'_B(\varnothing)$ in FB. Lastly, it is easily seen that the rule $B \mapsto \pi_B \circ \omega$ yields a pseudo-cone as sought, with coherence constraint

$$\tau^\pi_f : Ff \circ \pi_B \Rightarrow \pi_{B'} \qquad \text{such that} \qquad \tau^\pi_{f,\beta} := \tau^\beta_{f,\varnothing}$$

for every pseudo-cone $\beta : \mathsf{F}_{\mathbf{1}} \to F$ with coherence constraint τ^β.

Explicitly, for every $B \in \mathrm{Ob}(\mathscr{B})$ the functor $\pi_B : \mathscr{L}_F \to FB$ is determined by the identity :

$$\varphi(B) = (B, \pi_B(\varphi)) \qquad \text{for every cartesian strict pseudo-functor } \varphi : \mathscr{B}^o \to \mathscr{F}ib(F)$$

and for every pair of cartesian strict pseudo-functors $\varphi, \varphi' : \mathscr{B}^o \to \mathscr{F}ib(F)$ and every strict pseudo-natural $\mathscr{B}^o$-transformation $\beta : \varphi \Rightarrow \varphi'$, we have $\beta_B = (\mathbf{1}_B, \pi_B(\beta))$.

We claim that $(\mathscr{L}_F, \pi)$ is a strong 2-limit of F, *i.e.* for every small category $\mathscr{C}$, the functor

$$\text{(3.3.10)} \qquad \mathsf{Fun}(\mathscr{C}, \mathscr{L}_F) \to \mathsf{PsNat}(\mathsf{F}_\mathscr{C}, F) \qquad (\varphi : \mathscr{C} \to \mathscr{L}_F) \mapsto \pi \odot \mathsf{F}_\varphi$$

is an isomorphism of categories (see definition 2.5.1(i)). Indeed, let $\alpha : \mathsf{F}_\mathscr{C} \Rightarrow F$ be any pseudo-cone with vertex $\mathscr{C}$ and basis F; there follows a $\mathscr{B}^o$-cartesian strict pseudo-functor $\mathscr{F}ib(\alpha) : \mathscr{F}ib(\mathsf{F}_\mathscr{C}) \to \mathscr{F}ib(F)$, whence a functor

$$\alpha^\dagger := 2\text{-}\mathsf{Cart}_{\mathscr{B}^o}(\mathscr{B}^o, \mathscr{F}ib(\alpha)) : 2\text{-}\mathsf{Cart}_{\mathscr{B}^o}(\mathscr{B}^o, \mathscr{F}ib(\mathsf{F}_\mathscr{C})) \to \mathscr{L}_F.$$

If $\Theta : \alpha \rightsquigarrow \beta$ is any modification, we set as well $\Theta^\dagger := 2\text{-}\mathsf{Cart}_{\mathscr{B}^o}(\mathscr{B}^o, \mathscr{F}ib(\Theta)) : \alpha^\dagger \to \beta^\dagger$. However, a simple inspection shows that $\mathscr{F}ib(\mathsf{F}_\mathscr{C}) = \mathscr{B}^o \times \mathscr{C}$, the product in the category of 2-categories (where $\mathscr{C}$ is regarded as a 2-category with trivial 2-cells), fibred over $\mathscr{B}^o$ via the natural projection $p : \mathscr{B}^o \times \mathscr{C} \to \mathscr{B}^o$. There is an obvious functor

$$c^\mathscr{C} : \mathscr{C} \to 2\text{-}\mathsf{Cart}_{\mathscr{B}^o}(\mathscr{B}^o, \mathscr{F}ib(\mathsf{F}_\mathscr{C})) \qquad X \mapsto c_X$$

that assigns to every object X of $\mathscr{C}$ the constant functor $c_X : \mathscr{B}^o \to \mathscr{C}$ with value X, naturally identified with a section $\mathscr{B}^o \to \mathscr{B}^o \times \mathscr{C}$ of the projection p. Then, a simple inspection shows that the functor

$$\mathsf{PsNat}(\mathsf{F}_\mathscr{C}, F) \to \mathsf{Fun}(\mathscr{C}, \mathscr{L}_F) \qquad \alpha \mapsto \alpha^\dagger \circ c^\mathscr{C} \qquad \Theta \mapsto \Theta^\dagger * c^\mathscr{C}$$

is inverse to the functor (3.3.10), whence the contention.

Next we consider the category $\overline{\mathscr{F}ib}(F)$ such that $\mathrm{Ob}(\overline{\mathscr{F}ib}(F)) := \mathrm{Ob}(\mathscr{F}ib(F))$ and

$$\mathrm{Hom}_{\overline{\mathscr{F}ib}(F)}(X, X') := \pi_0(\mathscr{F}ib(F)(X, X')) \qquad \text{for every } X, X' \in \mathrm{Ob}(\mathscr{F}ib(F))$$

(notation of (3.2.10)). The composition law for morphisms in $\overline{\mathscr{F}ib}(F)$ is induced in the obvious way by that of 1-cells of $\mathscr{F}ib(F)$. For every 1-cell f of $\mathscr{F}ib(F)$ we denote by $[f]$ the class of f in $\mathrm{Morph}(\overline{\mathscr{F}ib}(F))$ and we let $\Sigma := \{[f] \in \mathrm{Morph}(\overline{\mathscr{F}ib}(F)) \mid f \text{ is } \mathscr{B}\text{-cartesian}\}$. We set

$$\mathscr{L}'_F := \overline{\mathscr{F}ib}(F))[\Sigma^{-1}]$$

(notation of theorem 1.6.9). We have a natural functor

$$\mathscr{F}ib(F) \to \mathscr{L}'_F$$

namely, the composition of the obvious projection $\mathscr{F}ib(F) \to \overline{\mathscr{F}ib}(F)$ with the localization $\overline{\mathscr{F}ib}(F) \to \mathscr{L}'_F$. Recall that $\mathscr{F}ib(F)$ carries a distinguished cleavage $\boldsymbol{\lambda}^*$, whose associated pseudo-functor is naturally identified with F (remark 3.1.27(i)). Then, the pseudo-cocone $\iota : F \to \mathsf{F}_{\mathscr{F}ib(F)}$ associated with $\boldsymbol{\lambda}^*$ induces a a pseudo-cocone $\iota' : F \Rightarrow \mathsf{F}_{\mathscr{L}'_F}$. We claim that $(\mathscr{L}'_F, \iota')$ is a strong 2-colimit of F, *i.e.* for every small category $\mathscr{C}$, the induced functor

$$\mathsf{Fun}(\mathscr{L}'_F, \mathscr{C}) \to \mathsf{PsNat}(F, \mathsf{F}_{\mathscr{C}}) \qquad (\varphi : \mathscr{L}'_F \to \mathscr{C}) \mapsto \mathsf{F}_\varphi \odot \iota' \tag{3.3.11}$$

is an isomorphism of categories. Indeed, let $\alpha : F \Rightarrow \mathsf{F}_{\mathscr{C}}$ be any pseudo-cocone; there follows a $\mathscr{B}^o$-cartesian strict pseudo-functor $\mathscr{F}ib(\alpha) : \mathscr{F}ib(F) \to \mathscr{F}ib(\mathsf{F}_{\mathscr{C}}) \stackrel{\sim}{\to} \mathscr{B}^o \times \mathscr{C}$. After composing with the projection $p : \mathscr{B}^o \times \mathscr{C} \to \mathscr{C}$ we get a functor

$$\alpha^* : \mathscr{F}ib(F) \to \mathscr{C}.$$

Since $\mathscr{C}$ has trivial 2-cells, α^* factors through a unique functor $\overline{\alpha}^* : \overline{\mathscr{F}ib}(F) \to \mathscr{C}$. By lemma 3.3.3(ii), a 1-cell $(f, g) : (B, X) \to (B', X')$ of $\mathscr{F}ib(\mathsf{F}_{\mathscr{C}})$ is cartesian if and only if $g : X \to X'$ is an isomorphism of $\mathscr{C}$. It follows that $\overline{\alpha}^*$ in turn factors uniquely through a functor

$$\alpha^\ddagger : \mathscr{L}'_F \to \mathscr{C}.$$

On the other hand, notice that a functor $\mathscr{F}ib(F) \to \mathscr{C}$ factors through $\overline{\mathscr{F}ib}(F)$ if and only if it induces a strict pseudo-functor $\mathscr{F}ib(F) \to \mathscr{B}^o \times \mathscr{C}$, and the latter is $\mathscr{B}^o$-cartesian if and only if the resulting functor $\overline{\mathscr{F}ib}(F) \to \mathscr{C}$ factors through $\mathscr{L}'_F$.

If $\Theta : \alpha \rightsquigarrow \beta$ is any modification, we get a natural $\mathscr{B}$-transformation $\Theta^* := p * \mathscr{F}ib(\Theta) : \alpha^* \Rightarrow \beta^*$, which in turn induces a natural transformation

$$\Theta^\ddagger : \alpha^\ddagger \Rightarrow \beta^\ddagger.$$

By a direct inspection, we see that the functor

$$\mathsf{PsNat}(F, \mathsf{F}_{\mathscr{C}}) \to \mathsf{Fun}(\mathscr{L}'_F, \mathscr{C}) \qquad \alpha \mapsto \alpha^\ddagger \qquad \Theta \mapsto \Theta^\ddagger$$

is inverse to (3.3.11), whence the contention. □

Example 3.3.12. (i) For instance, let $\mathscr{C}_\bullet := (\mathscr{C}_i \mid i \in I)$ be any small family of small categories. By inspecting the proof of theorem 3.3.9, we see that the 2-product of $\mathscr{C}_\bullet$ is represented by the product $\mathscr{C} := \prod_{i \in I} \mathscr{C}_i$, as constructed in example 1.2.25(i), and the universal cone for $\mathscr{C}$ is also a universal pseudo-cone for the 2-product of $\mathscr{C}_\bullet$.

(ii) Let $F : \mathscr{C} \to \mathscr{B}$ and $F' : \mathscr{C}' \to \mathscr{B}$ be two functors between small categories; by inspecting the proof of theorem 3.3.9, we see that the 2-fibre product of F and F' (remark 2.5.2(v)) is represented by the category whose objects are all data of the form $X := (c, c', \xi)$, where $c \in \mathrm{Ob}(\mathscr{C})$, $c' \in \mathrm{Ob}(\mathscr{C}')$, and $\xi : F(c) \stackrel{\sim}{\to} F'(c')$ is an isomorphism in $\mathscr{B}$. If $X' := (d, d', \zeta)$ is another such datum, the morphisms $X \to X'$ are the pairs (φ, φ'), where $\varphi : c \to d$ (resp. $\varphi' : c' \to d'$) is a morphism in $\mathscr{C}$ (resp. in $\mathscr{C}'$), and $\zeta \circ F(\varphi) = F'(\varphi') \circ \xi$.

Example 3.3.13. (i) Consider a small category $\mathscr{B}$ and a pseudo-functor $F : \mathscr{B} \to (\mathsf{U}, \mathsf{V})$-$\mathbf{Cat}$. In this case, obviously $\overline{\mathscr{F}ib}(F) = \mathscr{F}ib(F)$, so the strong 2-colimit of F is represented by $\mathscr{F}ib(F)[\Sigma^{-1}]$, where $\Sigma \subset \mathrm{Morph}(\mathscr{F}ib(F))$ is the set of cartesian morphisms.

(ii) Suppose next that $\mathscr{B}$ is *filtered*. Then we claim that Σ admits a right calculus of fractions. Indeed, it is clear that Σ satisfies conditions (CF1) and (CF2) of definition 1.6.14(i). Next, let $(\varphi, f) : (B, X) \to (B', Y)$ and $(\psi, g) : (B'', Z) \to (B', Y)$ be two morphisms of $\mathscr{F}ib(F)$, with $(\psi, g) \in \Sigma$; hence $\varphi : B' \to B$ and $\psi : B' \to B''$ are morphisms of $\mathscr{B}$, so there exist morphisms $\varphi' : B \to B'''$ and $\psi' : B'' \to B'''$ in $\mathscr{B}$ with $\varphi' \circ \varphi = \psi' \circ \psi$. It follows that $(\varphi', \mathbf{1}_{F_{\varphi'}X}) : (B''', F_{\varphi'}X) \to (B, X)$ lies in Σ, and since (ψ, g) is cartesian there exists a unique morphism $(\psi',h) : (B''',F_{\varphi'}X) \to (B'',Y)$ such that $(\psi,g)\circ(\psi',h) = (\varphi,f)\circ(\varphi',\mathbf{1}_{F_{\varphi'}X})$. This proves that condition (CF3) holds as well for Σ. Lastly, let $(\varphi,f),(\varphi',f') : (B,X) \to (B',Y)$ and $(\psi,g) : (B',Y) \to (B'',Z)$ be three morphisms of $\mathscr{F}ib(F)$, with (ψ,g) cartesian, and suppose that $(\psi,g)\circ(\varphi,f) = (\psi,g)\circ(\varphi',f')$; hence $\varphi,\varphi' : B' \to B$ are two morphisms of $\mathscr{B}$, and by assumption there exists a morphism $\mu : B \to B'''$ such that $\mu\circ\varphi = \mu\circ\varphi'$. Then we have the cartesian morphism $(\mu,\mathbf{1}_{F_\mu X}) : (B''',F_\mu X) \to (B,X)$, and obviously $(\psi,g)\circ(\varphi,f)\circ(\mu,\mathbf{1}_{F_\mu X}) = (\psi,g)\circ(\varphi',f')\circ(\mu,\mathbf{1}_{F_\mu X})$. Since (ψ,g) is cartesian, it follows that $(\varphi,f)\circ(\mu,\mathbf{1}_{F_\mu X}) = (\varphi',f')\circ(\mu,\mathbf{1}_{F_\mu X})$. This shows that condition (CF4) holds for Σ.

(iii) In particular, in the situation of (ii), the morphisms in $\mathscr{F}ib(F)[\Sigma^{-1}]$ can be expressed as fractions with denominators in Σ, as detailed by proposition 1.6.16 (and remark 1.6.19).

(iv) Let I be a small filtered category, and $F : I \to (\mathsf{U},\mathsf{V})\text{-}\mathbf{Cat}$ any functor; let $\mathscr{L}$ be the colimit of F, as explicitly given by example 1.5.10. In particular, the objects of $\mathscr{L}$ are equivalence classes $[i,X]$ of pairs $(i,X) \in \mathrm{Ob}(\mathscr{F}ib(F))$, and the morphisms $[f] : [i,X] \to [i',X']$ are equivalence classes of morphisms $f : F_\varphi X \to F_{\varphi'}X'$, for all pairs of morphism $(i \xrightarrow{\varphi} j \xleftarrow{\varphi_2} i')$ of I. By direct inspection of the construction of $\mathscr{L}$, we get a well defined functor

$$\mathscr{F}ib(F) \to \mathscr{L} \qquad (i,X) \mapsto [i,X] \qquad ((i' \to i, f) : (i,X) \to (i',X')) \mapsto [f].$$

By the universal property of the localization, this functor extends uniquely to a well defined functor $\mathscr{F}ib(F)[\Sigma^{-1}] \to \mathscr{L}$, where Σ is as in (ii), and in view of (iii) and lemma 3.1.20(i), it is easily seen that the latter functor is an equivalence. In other words, the colimit of F represents as well the 2-colimit of the same functor, and more precisely, every universal cocone $F \Rightarrow c_{\mathscr{L}}$ is also a (strict) universal pseudo-cocone.

3.3.14. Let $\mathscr{C}$ a U-small category, $\mathscr{B}$ a V-small category with U-small Hom-sets, and $\rho : \mathscr{C} \to \mathscr{B}$ a functor; theorem 3.3.9 enables us to construct both right and left 2-adjoints for the strict pseudo-functor $\mathsf{V}\text{-}\mathsf{Fib}(\rho)^*$ of remark 3.1.5(i). Indeed, combining with theorem 2.6.39 and remark 2.6.18(ii) we see first that the strong left 2-Kan extension along ρ^o is a strict left 2-adjoint L for $\mathsf{P} := \mathsf{PsFun}(\rho^o,(\mathsf{U},\mathsf{V})\text{-}\mathbf{Cat})$. On the other hand, by remark 2.4.31, the strict and strong 2-equivalence $\mathscr{F}ib_{\mathscr{B}}$ of theorem 3.1.24 admits a strict and strong pseudo-inverse

$$\mathsf{Q}_{\mathscr{B}} : (\mathsf{U},\mathsf{V})\text{-}\mathsf{Fib}(\mathscr{B}) \xrightarrow{\sim} \mathsf{PsFun}(\mathscr{B}^o,(\mathsf{U},\mathsf{V})\text{-}\mathbf{Cat}).$$

It follows that the strict pseudo-functor

$$(\mathsf{U},\mathsf{V})\text{-}\mathsf{Fib}(\rho)_! := \mathscr{F}ib_{\mathscr{B}}\circ\mathsf{L}\circ\mathsf{Q}_{\mathscr{C}} : (\mathsf{U},\mathsf{V})\text{-}\mathsf{Fib}(\mathscr{C}) \to (\mathsf{U},\mathsf{V})\text{-}\mathsf{Fib}(\mathscr{B})$$

is a left 2-adjoint for $\mathscr{F}ib_{\mathscr{C}}\circ\mathsf{P}\circ\mathsf{Q}_{\mathscr{B}}$. On the other hand, remark 3.1.27(ii) yields a pseudo-natural isomorphism $\mathscr{F}ib_{\mathscr{C}}\circ\mathsf{P} \xrightarrow{\sim} (\mathsf{U},\mathsf{V})\text{-}\mathsf{Fib}(\rho)^*\circ\mathscr{F}ib_{\mathscr{B}}$, so $(\mathsf{U},\mathsf{V})\text{-}\mathsf{Fib}(\rho)_!$ is also left 2-adjoint to $(\mathsf{U},\mathsf{V})\text{-}\mathsf{Fib}(\rho)^*$. Likewise, from the strong right 2-Kan extension R along ρ^o we get a strict right 2-adjoint for $(\mathsf{U},\mathsf{V})\text{-}\mathsf{Fib}(\rho)^*$, namely the pseudo-functor

$$(\mathsf{U},\mathsf{V})\text{-}\mathsf{Fib}(\rho)_* := \mathscr{F}ib_{\mathscr{B}}\circ\mathsf{R}\circ\mathsf{Q}_{\mathscr{C}} : (\mathsf{U},\mathsf{V})\text{-}\mathsf{Fib}(\mathscr{C}) \to (\mathsf{U},\mathsf{V})\text{-}\mathsf{Fib}(\mathscr{B}).$$

As usual, we will omit mentioning V, when no ambiguities are likely to arise from the omission.

Remark 3.3.15. (i) Like all 2-adjoints, the pseudo-functors $\mathsf{Fib}(\rho)_!$ and $\mathsf{Fib}(\rho)_*$ are well defined up to pseudo-natural equivalence. However, our construction yields more canonical representatives, that are well defined up to pseudo-natural *isomorphisms*, since it relies on the strong 2-Kan extensions along ρ^o. Moreover, both these 2-adjoints factor naturally through pseudo-functors

$$\mathsf{Fib}(\mathscr{C}) \to \mathsf{Split}(\mathscr{B})$$

again because this is a feature of strong 2-Kan extensions : see remark 2.6.18(ii).

(ii) In the situation of (3.3.14), we can describe more explicitly the pseudo-functor $\mathsf{Fib}(\rho)_*$ as follows. Let $\mathscr{A} \to \mathscr{C}$ be any fibration with essentially small fibres; then $\mathsf{C}(\mathsf{Fib}(\rho)_*(\mathscr{A}))$ is the fibration associated with the strict pseudo-functor $\mathsf{Fib}(\rho)_*(\mathscr{A})(-) : \mathscr{B}^o \to (\mathsf{U},\mathsf{V})\text{-}\mathbf{Cat}$, and by example 3.1.16(ii), for every $B \in \mathrm{Ob}(\mathscr{B})$ we have an equivalence of categories :

$$\mathsf{Fib}(\rho)_*(\mathscr{A})(B) \stackrel{\sim}{\to} \mathsf{Cart}_{\mathscr{C}}(\rho\mathscr{C}/B,\mathscr{A})$$

that is pseudo-natural with respect to morphisms $B' \to B$ in $\mathscr{B}$. Thus, $\mathsf{Fib}(\rho)_*(\mathscr{A})$ is naturally equivalent to the fibration associated with the strict pseudo-functor

$$\mathsf{Cart}_{\mathscr{C}}(\rho\mathscr{C}/-,\mathscr{A}) : \mathscr{B}^o \to (\mathsf{U},\mathsf{V})\text{-}\mathbf{Cat} \qquad B \mapsto \mathsf{Cart}_{\mathscr{C}}(\rho\mathscr{C}/B,\mathscr{A})$$

that assigns to every morphism $f : B' \to B$ of $\mathscr{B}$ the functor $\mathsf{Cart}_{\mathscr{C}}(\rho\mathscr{C}/f,\mathscr{A})$.

(iii) Moreover, we have a pseudo-commutative diagram of 2-categories :

$$\begin{array}{ccc} \mathscr{C}^\wedge & \xrightarrow{\mathscr{F}ib_{\mathscr{C}}} & \mathsf{Fib}(\mathscr{C}) \\ {\scriptstyle\rho_*}\downarrow & & \downarrow{\scriptstyle\mathsf{Fib}(\rho)_*} \\ \mathscr{B}^\wedge & \xrightarrow{\mathscr{F}ib_{\mathscr{B}}} & \mathsf{Fib}(\mathscr{B}) \end{array}$$

(see remark 2.4.10). Indeed, let $j : \mathbf{Set} \to \mathbf{Cat}$ be the inclusion functor (that assigns to every set the associated discrete category); by inspecting the constructions of $\mathsf{Fib}(\rho)_*$ and ρ_* (see remark 1.3.6(i)), we come down to exhibiting an equivalence :

$$2\text{-}\!\!\int_{\rho^o} j \circ F \stackrel{\sim}{\to} j \circ \int_{\rho^o} F \qquad\qquad \text{for every presheaf } F : \mathscr{C}^o \to \mathbf{Set}$$

pseudo-natural with respect to F. Now, for every $B \in \mathrm{Ob}(\mathscr{B})$ the category $2\text{-}\!\int_{\rho^o} j \circ F(B)$ is the 2-limit of the functor $j \circ F \circ \mathsf{t}_{B^o} : B^o/\varphi^o\mathscr{C}^o \to \mathbf{Cat}$. But since j is fully faithful and admits a left 2-adjoint (see (3.2.10)), proposition 2.5.11 says that such 2-limit is the discrete category associated with the set representing the 2-limit of $F \circ \mathsf{t}_{B^o} : B^o/\varphi^o\mathscr{C}^o \to \mathbf{Set}$. By remark 2.5.2(vii), the latter is represented by the (usual) limit of the same functor. But this in turn is none else than the definition of $\int_\varphi F(B^o)$, whence the contention.

(iv) In view of corollary 2.6.42, we also see that if ρ is a fully faithful functor, then $\mathsf{Fib}(\rho)_*$ and $\mathsf{Fib}(\rho)_!$ are both fully faithful pseudo-functors.

(v) For every other universe V' with $\mathsf{V} \subset \mathsf{V}'$, we get a diagram of pseudo-functors :

$$\begin{array}{ccc} (\mathsf{U},\mathsf{V})\text{-}\mathsf{Fib}(\mathscr{C}) & \xrightarrow{(\mathsf{U},\mathsf{V})\text{-}\mathsf{Fib}(\rho)_!} & (\mathsf{U},\mathsf{V})\text{-}\mathsf{Fib}(\mathscr{B}) \\ \downarrow & & \downarrow \\ (\mathsf{U},\mathsf{V}')\text{-}\mathsf{Fib}(\mathscr{C}) & \xrightarrow{(\mathsf{U},\mathsf{V}')\text{-}\mathsf{Fib}(\rho)_!} & (\mathsf{U},\mathsf{V}')\text{-}\mathsf{Fib}(\mathscr{B}) \end{array} \qquad \begin{array}{ccc} (\mathsf{U},\mathsf{V})\text{-}\mathsf{Fib}(\mathscr{C}) & \xrightarrow{(\mathsf{U},\mathsf{V})\text{-}\mathsf{Fib}(\rho)_*} & (\mathsf{U},\mathsf{V})\text{-}\mathsf{Fib}(\mathscr{B}) \\ \downarrow & & \downarrow \\ (\mathsf{U},\mathsf{V}')\text{-}\mathsf{Fib}(\mathscr{C}) & \xrightarrow{(\mathsf{U},\mathsf{V}')\text{-}\mathsf{Fib}(\rho)_*} & (\mathsf{U},\mathsf{V}')\text{-}\mathsf{Fib}(\mathscr{B}) \end{array}$$

whose vertical arrows are the inclusions. A direct inspection of the constructions easily shows that both diagrams are commutative : the details shall be left to the reader.

(vi) Let $\mathscr{C}$ and $\mathscr{C}'$ be two small categories, $u : \mathscr{C} \to \mathscr{C}'$ and $v : \mathscr{C}' \to \mathscr{C}$ two functors, such that v is left adjoint to u. In light of (1.1.32), we get for every fibration $\mathscr{A} \to \mathscr{C}'$ and every $X \in \mathrm{Ob}(\mathscr{C})$ a natural isomorphism of categories :

$$\omega_X : \mathsf{Cart}_{\mathscr{C}'}(v\mathscr{C}'/X,\mathscr{A}) \stackrel{\sim}{\to} \mathscr{A}(uX)$$

and for every morphism $f : Y \to X$ in $\mathscr{C}$, a commutative diagram of categories :

$$\begin{array}{ccc} \mathsf{Cart}_{\mathscr{C}'}(v\mathscr{C}'/X,\mathscr{A}) & \xrightarrow{\omega_X} & \mathscr{A}(uX) \\ {\scriptstyle\mathsf{Cart}_{\mathscr{C}'}(v\mathscr{C}'/f,\mathscr{A})}\downarrow & & \downarrow{\scriptstyle\mathscr{A}(uf)} \\ \mathsf{Cart}_{\mathscr{C}'}(v\mathscr{C}'/Y,\mathscr{A}) & \xrightarrow{\omega_Y} & \mathscr{A}(uY). \end{array}$$

Combining with (i) and claim 3.2.8, we deduce a natural equivalence of fibrations over $\mathscr{C}$:

$$\mathsf{Fib}(v)_*(\mathscr{A}) \stackrel{\sim}{\to} \mathsf{Fib}(u)^*(\mathscr{A})$$

and it is easily seen that the system of such equivalences amounts to a pseudo-natural equivalence of pseudo-functors :

$$\mathsf{Fib}(v)_* \stackrel{\sim}{\to} \mathsf{Fib}(u)^*.$$

3.3.16. For every small category $\mathscr{C}$ we have a natural functor

$$\mathscr{C}/- : \mathscr{C} \to \mathsf{Fib}(\mathscr{C}) \qquad X \mapsto (\mathsf{s}_X : \mathscr{C}/X \to \mathscr{C})$$

that assigns to every morphism $f : X \to Y$ of $\mathscr{C}$ the (cartesian) functor $f_* : \mathscr{C}/X \to \mathscr{C}/Y$ as in (1.1.25). The following proposition upgrades remark 1.3.6(ii) from presheaves to fibrations :

Proposition 3.3.17. *Let $u : \mathscr{C} \to \mathscr{B}$ be any functor between small categories. With the notation of* (3.3.16)*, we have a pseudo-commutative diagram of 2-categories :*

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{\ u\ } & \mathscr{B} \\ {\scriptstyle \mathscr{C}/-}\downarrow & & \downarrow{\scriptstyle \mathscr{B}/-} \\ \mathsf{Fib}(\mathscr{C}) & \xrightarrow{\mathsf{Fib}(u)_!} & \mathsf{Fib}(\mathscr{B}). \end{array}$$

Proof. Set $\mathsf{Fib}(u)_!(\mathscr{C}/-) := \mathsf{Fib}(u)_! \circ (\mathscr{C}/-) : \mathscr{C} \to \mathsf{Fib}(\mathscr{B})$ and let $\mathscr{B}/u- : \mathscr{C} \to \mathsf{Fib}(\mathscr{B})$ be the composition of u with the pseudo-funtor $\mathscr{B}/-$. To ease notation, we set :

$$F := \mathsf{Cart}_{\mathscr{B}}(\mathsf{Fib}(u)_!(\mathscr{C}/-), -) : \mathscr{C}^o \times \mathsf{Fib}(\mathscr{B}) \to (\mathsf{U},\mathsf{V})\text{-}\mathbf{Cat}$$
$$G := \mathsf{Cart}_{\mathscr{B}}(\mathscr{B}/u-, -) : \mathscr{C}^o \times \mathsf{Fib}(\mathscr{B}) \to (\mathsf{U},\mathsf{V})\text{-}\mathbf{Cat}$$

(notation of example 2.2.7(ii)). Let ϑ be the 2-adjunction for the pair $(\mathsf{Fib}(u)_!, \mathsf{Fib}(u)^*)$; we deduce a pseudo-natural equivalence

$$\omega := \vartheta * ((\mathscr{C}/-)^o \times \mathbf{1}_{\mathsf{Fib}(\mathscr{B})}) : F \Rightarrow \mathsf{Cart}_{\mathscr{C}}(\mathscr{C}/-, \mathsf{Fib}(u)^*).$$

On the other hand, we have a strict pseudo-natural equivalence :

$$\lambda : G \Rightarrow \mathsf{Cart}_{\mathscr{C}}(\mathscr{C}/-, \mathsf{Fib}(u)^*).$$

Namely, for every $X \in \mathrm{Ob}(\mathscr{C})$ and every $\mathscr{B}$-fibration $\mathscr{A}$, the functor

$$\lambda_{X,\mathscr{A}} : \mathscr{A}(uX) \to \mathsf{Fib}(u)^*\mathscr{A}(X)$$

assigns to every cartesian section $\varphi \in \mathscr{A}(uX)$ the unique cartesian section $\varphi^* \in \mathscr{A} \times_{\mathscr{B}} \mathscr{C}(X)$ whose composition with the projection $\pi : \mathscr{A} \times_{\mathscr{B}} \mathscr{C} \to \mathscr{A}$ equals $\varphi \circ u_{|X}$. To every natural $\mathscr{B}$-transformation $\alpha : \varphi \Rightarrow \varphi'$ between such functors, $\lambda_{X,\mathscr{A}}$ assigns the unique natural $\mathscr{C}$-transformation α^* such that $\pi * \alpha^* = \alpha * u_{|X}$. We may then pick a quasi-inverse μ for λ, and consider the pseudo-natural equivalence

$$\gamma := \mu \odot \omega : F \stackrel{\sim}{\to} G.$$

For every $X \in \mathrm{Ob}(\mathscr{C})$, let $i_X : \mathsf{Fib}(\mathscr{B}) \to \mathscr{C}^o \times \mathsf{Fib}(\mathscr{B})$ be the unique strict pseudo-functor whose composition with the projection $\mathscr{C}^o \times \mathsf{Fib}(\mathscr{B}) \to \mathsf{Fib}(\mathscr{B})$ equals $\mathbf{1}_{\mathsf{Fib}(\mathscr{B})}$, and whose composition with the projection $\mathscr{C}^o \times \mathsf{Fib}(\mathscr{B}) \to \mathscr{C}^o$ is the constant pseudo-functor F_{X^o} (notation of (2.5)); we deduce the pseudo-natural equivalence

$$\gamma * i_X : \mathsf{Cart}_{\mathscr{B}}(\mathsf{Fib}(u)_!(\mathscr{C}/X), -) \stackrel{\sim}{\to} \mathsf{Cart}_{\mathscr{B}}(\mathscr{B}/uX, -).$$

We show more generally :

Claim 3.3.18. Let $\mathscr{F}$ and $\mathscr{G}$ be two $\mathscr{B}$-fibrations, and $\beta_\bullet : \mathsf{Cart}_{\mathscr{B}}(\mathscr{F}, -) \stackrel{\sim}{\to} \mathsf{Cart}_{\mathscr{B}}(\mathscr{G}, -)$ a pseudo-natural equivalence. Then $\beta^* := \beta_{\mathscr{F}}(\mathbf{1}_{\mathscr{F}})$ is a $\mathscr{B}$-equivalence of categories $\mathscr{G} \stackrel{\sim}{\to} \mathscr{F}$.

Proof of the claim. Let $\alpha_\bullet : \mathsf{Cart}_{\mathscr{B}}(\mathscr{G},-) \to \mathsf{Cart}_{\mathscr{B}}(\mathscr{F},-)$ be a quasi-inverse of $\beta_\bullet$, and set $\alpha^* := \alpha_{\mathscr{G}}(\mathbf{1}_{\mathscr{G}}) : \mathscr{F} \to \mathscr{G}$. There follow essentially commutative diagrams :

$$\begin{array}{ccc} \mathsf{Cart}_{\mathscr{B}}(\mathscr{F},\mathscr{F}) & \xrightarrow{\mathsf{Cart}_{\mathscr{B}}(\mathscr{F},\alpha^*)} & \mathsf{Cart}_{\mathscr{B}}(\mathscr{F},\mathscr{G}) \\ {\scriptstyle\beta_{\mathscr{F}}}\downarrow & & \downarrow{\scriptstyle\beta_{\mathscr{G}}} \\ \mathsf{Cart}_{\mathscr{B}}(\mathscr{G},\mathscr{F}) & \xrightarrow{\mathsf{Cart}_{\mathscr{B}}(\mathscr{G},\alpha^*)} & \mathsf{Cart}_{\mathscr{B}}(\mathscr{G},\mathscr{G}) \end{array} \qquad \begin{array}{ccc} \mathsf{Cart}_{\mathscr{B}}(\mathscr{G},\mathscr{G}) & \xrightarrow{\mathsf{Cart}_{\mathscr{B}}(\mathscr{G},\beta^*)} & \mathsf{Cart}_{\mathscr{B}}(\mathscr{G},\mathscr{F}) \\ {\scriptstyle\alpha_{\mathscr{G}}}\downarrow & & \downarrow{\scriptstyle\alpha_{\mathscr{F}}} \\ \mathsf{Cart}_{\mathscr{B}}(\mathscr{F},\mathscr{G}) & \xrightarrow{\mathsf{Cart}_{\mathscr{B}}(\mathscr{F},\beta^*)} & \mathsf{Cart}_{\mathscr{B}}(\mathscr{F},\mathscr{F}). \end{array}$$

From the left diagram we get isomorphisms : $\alpha^* \circ \beta^* \xrightarrow{\sim} \beta_{\mathscr{G}}(\alpha^*) = \beta_{\mathscr{G}} \circ \alpha(\mathbf{1}_{\mathscr{G}}) \xrightarrow{\sim} \mathbf{1}_{\mathscr{G}}$. Likewise, from the right diagram we get an isomorphism $\beta^* \circ \alpha^* \xrightarrow{\sim} \mathbf{1}_{\mathscr{F}}$, whence the contention. ◊

To ease notation, for every $X \in \mathrm{Ob}(\mathscr{C})$ and every morphism $f : X \to Y$ of $\mathscr{C}$ set

$$[X] := \mathsf{Fib}(u)_!(\mathscr{C}/X) \qquad \text{and} \qquad [f] := \mathsf{Fib}(u)_!(f_*) : [X] \to [Y]$$

According to claim 3.3.18, for every $X \in \mathrm{Ob}(\mathscr{C})$ we get an equivalence of $\mathscr{B}$-categories :

$$\Gamma_X := \gamma_{X,[X]}(\mathbf{1}_{[X]}) : \mathscr{B}/uX \xrightarrow{\sim} [X].$$

To conclude, it remains to show that the rule : $X \mapsto \Gamma_X$ yields a pseudo-natural equivalence as sought. To this aim, we need to exhibit a coherence constraint for Γ. Now, notice that, after possibly replacing it by a pseudo-naturally isomorphic pseudo-functor, we may assume that $\mathsf{Fib}(u)_!$ is unital (proposition 2.4.3); then, for every morphism $f : X \to Y$ in $\mathscr{C}$ consider the diagram of oriented squares :

$$\begin{array}{ccc} F(Y,[Y]) & \xrightarrow{\gamma_{Y,[Y]}} & G(Y,[Y]) \\ {\scriptstyle F(f,\mathbf{1}_{[Y]})}\downarrow & \Downarrow \tau^{\gamma}_{f,\mathbf{1}_{[Y]}} & \downarrow{\scriptstyle G(f,\mathbf{1}_{[Y]})} \\ F(X,[Y]) & \xrightarrow{\gamma_{X,[Y]}} & G(X,[Y]) \\ {\scriptstyle F(\mathbf{1}_X,[f])}\uparrow & \Uparrow \tau^{\gamma}_{\mathbf{1}_X,[f]} & \uparrow{\scriptstyle G(\mathbf{1}_X,[f])} \\ F(X,[X]) & \xrightarrow[\gamma_{X,[X]}]{} & G(X,[X]). \end{array}$$

where $\tau^{\gamma}_{\bullet,\bullet}$ is the coherence constraint of γ. We deduce isomorphisms of $\mathscr{B}$-cartesian functors :

$$[f] \circ \Gamma_X \xrightarrow{(\tau^{\gamma}_{\mathbf{1}_X,[f]})_{\mathbf{1}_{[X]}}} \gamma_{X,[Y]}([f]) \xleftarrow{(\tau^{\gamma}_{f,\mathbf{1}_{[Y]}})_{\mathbf{1}_{[Y]}}} \Gamma_Y \circ (uf)_*$$

so our candidate coherence constraint is :

$$\tau^{\Gamma}_f := (\tau^{\gamma}_{f,\mathbf{1}_{[Y]}})^{-1}_{\mathbf{1}_{[Y]}} \odot (\tau^{\gamma}_{\mathbf{1}_X,[f]})_{\mathbf{1}_{[X]}} : [f] \circ \Gamma_X \xrightarrow{\sim} \Gamma_Y \circ (uf)_*.$$

With this definition, it is already clear that $\tau^{\Gamma}_{\mathbf{1}_X} = \mathbf{1}_{\Gamma_X}$ for every $X \in \mathrm{Ob}(\mathscr{C})$. Next, let $f : X \to Y$ and $g : Y \to Z$ be two morphisms of $\mathscr{C}$; we need to check that :

$$(\tau^{\Gamma}_g * (uf)_*) \odot ([g] * \tau^{\Gamma}_f) = \tau^{\Gamma}_{g\circ f} \odot (\gamma^{\mathsf{Fib}(u)_!}_{f_*,g_*} * \Gamma_X)$$

where $\gamma^{\mathsf{Fib}(u)_!}$ denotes the coherence constraint of $\mathsf{Fib}(u)_!$ But the coherence condition for τ^{γ}, relative to the composition of 1-cells $(\mathbf{1}_{X^o},[g]) \circ (f^o,\mathbf{1}_{[Y]}) = (f^o,[g]) = (f^o,\mathbf{1}_{[Z]}) \circ (\mathbf{1}_{Y^o},[g])$ yields the identities :

$$(\tau^{\gamma}_{\mathbf{1}_X,[g]})_{[f]} \odot ([g] * (\tau^{\gamma}_{f,\mathbf{1}_{[Y]}})) = (\tau^{\gamma}_{f,[g]})_{\mathbf{1}_{[Y]}} = (\tau^{\gamma}_{f,\mathbf{1}_{[Z]}})_{[g]} \odot (\tau^{\gamma}_{\mathbf{1}_Y,[g]} * (uf)_*)$$

By the same token, from the identity : $(f^o,\mathbf{1}_{[Z]}) \circ (g^o,\mathbf{1}_{[Z]}) = ((g\circ f)^o,\mathbf{1}_{[Z]})$ we get :

$$\gamma_{X,[Z]}(\gamma^{\mathsf{Fib}(u)_!}_{f_*,g_*}) \odot (\tau^{\gamma}_{f,\mathbf{1}_{[Z]}})_{[g]} \odot ((\tau^{\gamma}_{g,\mathbf{1}_{[Z]}})_{\mathbf{1}_{[Z]}} * (uf)_*) = (\tau^{\gamma}_{g\circ f,\mathbf{1}_{[Z]}})_{\mathbf{1}_{[Z]}}$$

whence :

$$(\tau^\Gamma_g * (uf)_*)\odot([g] * \tau^\Gamma_f) = ((\tau^\gamma_{g,\mathbf{1}_{[Z]}})^{-1}_{\mathbf{1}_{[Z]}} * (uf)_*)\odot(\tau^\gamma_{f,\mathbf{1}_{[Z]}})^{-1}_{[g]}\odot(\tau^\gamma_{\mathbf{1}_X,[g]})_{[f]}\odot([g]*(\tau^\gamma_{\mathbf{1}_X,[f]})_{\mathbf{1}_{[X]}})$$
$$= (\tau^\gamma_{g\circ f,\mathbf{1}_{[Z]}})^{-1}_{\mathbf{1}_{[Z]}} \odot \gamma_{X,[Z]}(\gamma^{\mathsf{Fib}(u)_!}_{f_*,g_*})\odot(\tau^\gamma_{\mathbf{1}_X,[g]})_{[f]}\odot([g]*(\tau^\gamma_{\mathbf{1}_X,[f]})_{\mathbf{1}_{[X]}}).$$

Thus, we are reduced to checking that :

$$(\tau^\gamma_{\mathbf{1}_X,[g\circ f]})_{\mathbf{1}_{[X]}} \odot (\gamma^{\mathsf{Fib}(u)_!}_{f_*,g_*} * \Gamma_X) = \gamma_{X,[Z]}(\gamma^{\mathsf{Fib}(u)_!}_{f_*,g_*}) \odot (\tau^\gamma_{\mathbf{1}_X,[g]})_{[f]} \odot ([g] * (\tau^\gamma_{\mathbf{1}_X,[f]})_{\mathbf{1}_{[X]}}).$$

But the latter follows once again from the coherence condition for τ^γ, applied to the identity : $(\mathbf{1}_X, [g]) \circ (\mathbf{1}_X, [f]) = (\mathbf{1}_X, [g \circ f])$. □

In the same vein, we point out the following :

Proposition 3.3.19. *Let $\mathscr{C}$, $\mathscr{C}'$ be two small categories, $u : \mathscr{C} \to \mathscr{C}'$ a functor, and V a universe such that $\mathscr{C}^\wedge$ and $\mathscr{C}'^\wedge$ are V-small. We have a pseudo-commutative diagram of 2-categories :*

$$\begin{array}{ccc} (\mathsf{U},\mathsf{V})\text{-}\mathsf{Fib}(\mathscr{C}'^\wedge_\mathsf{U}) & \xrightarrow{(\mathsf{U},\mathsf{V})\text{-}\mathsf{Fib}(u^\wedge_\mathsf{U})_!} & (\mathsf{U},\mathsf{V})\text{-}\mathsf{Fib}(\mathscr{C}^\wedge_\mathsf{U}) \\ \downarrow{\scriptstyle (\mathsf{U},\mathsf{V})\text{-}\mathsf{Fib}(h_{\mathscr{C}'})^*} & & \downarrow{\scriptstyle (\mathsf{U},\mathsf{V})\text{-}\mathsf{Fib}(h_{\mathscr{C}})^*} \\ (\mathsf{U},\mathsf{V})\text{-}\mathsf{Fib}(\mathscr{C}') & \xrightarrow{(\mathsf{U},\mathsf{V})\text{-}\mathsf{Fib}(u)^*} & (\mathsf{U},\mathsf{V})\text{-}\mathsf{Fib}(\mathscr{C}). \end{array}$$

Proof. Let $\mathscr{E}'$ be any fibration with V-small fibres over $\mathscr{C}'^\wedge$; pick a unital cleavage for $\mathscr{E}'$ and let c' be its associated pseudo-functor. Set $\mathscr{E} := (\mathsf{U},\mathsf{V})\text{-}\mathsf{Fib}(u^\wedge_\mathsf{U})_!\mathscr{E}'$; by inspecting (3.3.14) we see that for every $F \in \mathrm{Ob}(\mathscr{C}^\wedge)$ the fibre category $\mathscr{E}_F$ represents the strong 2-colimit of

$$\mathsf{c}' \circ \mathsf{t}^o_F : (F/u^\wedge\mathscr{C}'^\wedge) \to (\mathsf{U},\mathsf{V})\text{-}\mathbf{Cat}$$

where $\mathsf{t}_F : (F/u^\wedge\mathscr{C}'^\wedge) \to \mathscr{C}'^\wedge$ denotes the target functor. Fix a universal pseudo-cocone

$$e^F_\bullet : \mathsf{c}' \circ \mathsf{t}^o_F \Rightarrow \mathsf{F}_{\mathscr{E}_F} \qquad (F \xrightarrow{\varphi} u^\wedge G) \mapsto (\mathscr{E}'_G \xrightarrow{e^F_\varphi} \mathscr{E}_F)$$

and let $\tau^F_\bullet$ be the coherence constraint of $e^F_\bullet$. Then $\mathscr{E}$ admits a split cleavage d, that assigns to every morphism $\psi : F' \to F$ of $\mathscr{C}^\wedge$ the unique functor $\mathscr{E}_\psi : \mathscr{E}_F \to \mathscr{E}_{F'}$ such that

$$\mathsf{F}_{\mathscr{E}_\psi} \odot e^F = e^{F'} * \psi^{*o}$$

where $\psi^* : F/u^\wedge\mathscr{C}'^\wedge \to F'/u^\wedge\mathscr{C}'^\wedge$ is defined as in (1.1.25). Hence, $\mathsf{V}\text{-}\mathsf{Fib}(h_\mathscr{C})^*$ admits the split cleavage $\mathsf{d} \circ h^o_\mathscr{C}$. For every $X \in \mathrm{Ob}(\mathscr{C})$, let $\eta_X : h_X \to u^\wedge h_{uX}$ be the morphism given by the rule : $\varphi \mapsto u(\varphi) \in h_{uX}(uY)$ for every $Y \in \mathrm{Ob}(\mathscr{C})$ and every $\varphi \in h_X(Y)$. We notice :

Claim 3.3.20. For every $X \in \mathrm{Ob}(\mathscr{C})$, the morphism $\eta_\mathscr{E}$ is an initial object of $h_X/u^\wedge\mathscr{C}'^\wedge$.

Proof of the claim. Indeed, In light of remark 1.3.6(ii,iii), we may regard η_X as the unit of the natural adjunction for the adjoint pair $(u_!, u^\wedge)$. Hence, for every $G \in \mathrm{Ob}(\mathscr{C}'^\wedge)$ and every morphism of presheaves $\beta : h_X \to u^\wedge G$ there exists a unique morphism of presheaves $\beta' : h_{uX} \to G$ on $\mathscr{C}'$ such that $\beta = u^\wedge(\beta') \circ \eta_X$ (explicitly, under the canonical bijection provided by Yoneda's lemma, β corresponds to a unique section $s_\beta \in (u^\wedge G)X$, and β' corresponds then to the same s_β, regarded as an element of $G(uX)$). The claim is an immediate consequence. ◊

From claim 3.3.20 and proposition 2.5.16, we get an equivalence of categories

$$\varepsilon_X := e^{h_X}_{\eta_X} : \mathscr{E}'_{h_{uX}} \to \mathscr{E}_{h_X} \qquad \text{for every } X \in \mathrm{Ob}(\mathscr{C}).$$

Claim 3.3.21. The rule : $X \mapsto \varepsilon_X$ yields a pseudo-natural equivalence $\varepsilon : \mathsf{c}'\circ(h_{\mathscr{C}'}\circ u)^o \xrightarrow{\sim} \mathsf{d}\circ h^o_\mathscr{C}$.

Proof of the claim. For every morphism $f : Y \to X$ in $\mathscr{C}$ we have a natural isomorphism :

$$\tau^{\varepsilon}_f : \mathsf{d}_{h_f} \circ \varepsilon_X = e^{h_Y}_{\eta_X \circ h_f} = e^{h_Y}_{u^\wedge(h_{uf}) \circ \eta_Y} \xrightarrow{\tau^{h_Y}_{h_{uf}}} \varepsilon_Y \circ \mathsf{c}'_{h_{uf}}$$

and we check that the system $(\tau^{\varepsilon}_f \mid f \in \mathrm{Morph}(\mathscr{C}))$ yields a coherence constraint for ε. First, we get $\tau^{\varepsilon}_{\mathbf{1}_X} = \mathbf{1}_{\varepsilon_X}$ for every $X \in \mathrm{Ob}(\mathscr{C})$, by remark 2.4.2(ii). Next, notice that for every morphism $\psi : F' \to F$ in $\mathscr{C}^\wedge$ we have :

$\mathscr{E}_\psi * \tau^F_\mu = \tau^{F'}_\mu$ for every morphism $F'/\mu : (F' \to u^\wedge G_1) \to (F' \to u^\wedge G_2)$ in $F'/u^\wedge \mathscr{C}'^\wedge$.

Thus, let $f : X \to Y$ and $g : Y \to Z$ be two morphisms in $\mathscr{C}$; we deduce :

$$\begin{aligned}(\varepsilon_Z * \gamma^{\mathsf{c}'}_{h_{uf},h_{ug}}) \odot (\tau^{\varepsilon}_g * \mathsf{c}'_{uf}) \odot (\mathsf{d}_{h_g} * \tau^{\varepsilon}_f) &= (\varepsilon_Z * \gamma^{\mathsf{c}'}_{h_{uf},h_{ug}}) \odot (\tau^{h_Z}_{h_{ug}} * \mathsf{c}'_{uf}) \odot \tau^{h_Z}_{h_{uf}} \\ &= (\varepsilon_Z * \gamma^{\mathsf{c}'}_{h_{uf},h_{ug}}) \odot \tau^{h_Z}_{h_{u(g\circ f)}} \\ &= \tau^{\varepsilon}_{g\circ f}\end{aligned}$$

whence the contention. ◊

From claim 3.3.21 we deduce an equivalence of fibrations over $\mathscr{C}$:

$$\Omega_{\mathscr{E}'} : \mathsf{Fib}(h_{\mathscr{C}'} \circ u)^* \mathscr{E}' \xrightarrow{\sim} \mathsf{Fib}(h_{\mathscr{C}})^* \circ \mathsf{Fib}(u^\wedge_{\mathsf{U}})_! \mathscr{E}'.$$

Next, let $\varphi' : \mathscr{E}'_1 \to \mathscr{E}'_2$ be a cartesian functor of fibrations over $\mathscr{C}'^\wedge$; let $\mathscr{E}_i := \mathsf{Fib}(u^\wedge)_! \mathscr{E}'_i$ for $i = 1, 2$, and set $\varphi := \mathsf{Fib}(u^\wedge_{\mathsf{U}})_! \varphi' : \mathscr{E}_1 \to \mathscr{E}_2$. For $i = 1, 2$, pick also unital a cleavage c'_i for $\mathscr{E}'$, so that φ' corresponds to a pseudo-natural transformation $\omega' : \mathsf{c}'_1 \Rightarrow \mathsf{c}'_2$. For every $F \in \mathrm{Ob}(\mathscr{C}^\wedge)$ we pick universal cocones $e^F_{1,\bullet} : \mathsf{c}'_i \circ \mathsf{t}^o_F \Rightarrow \mathsf{F}_{\mathscr{E}_{i,F}}$, and denote by d_1 and d_2 the natural split cleavages for $\mathscr{E}_1$ and $\mathscr{E}_2$ described in the foregoing. Then φ corresponds to the strict pseudo-natural transformation $\omega : \mathsf{d}_1 \Rightarrow \mathsf{d}_2$ that assigns to every $F \in \mathrm{Ob}(\mathscr{C}^\wedge)$ the unique functor $\omega_F : \mathscr{E}_{1,F} \to \mathscr{E}_{2,F}$ such that

$$e^F_{2,\bullet} \odot (\omega' * \mathsf{t}^o_F) = \mathsf{F}_{\omega_F} \odot e^F_{1,\bullet}. \tag{3.3.22}$$

Let $\varepsilon_i : \mathsf{c}'_i \circ (h_{\mathscr{C}'} \circ u)^o \xrightarrow{\sim} \mathsf{d}_i \circ h^o_{\mathscr{C}}$ be the pseudo-natural equivalence of claim 3.3.21, for $i = 1, 2$.

Claim 3.3.23. $(\omega * h^o_{\mathscr{C}}) \odot \varepsilon_1 = \varepsilon_2 \odot (\omega' * (h_{\mathscr{C}'} \circ u)^o)$.

Proof of the claim. Directly from (3.3.22) we see that the two sides of the identity of the claim agree on every $X \in \mathrm{Ob}(\mathscr{C})$. It remains to check that the respective coherence constraints agree as well. However, for every $F \in \mathrm{Ob}(\mathscr{C}^\wedge)$ denote by $\tau^F_{i,\bullet}$ the coherence constraint of $e^F_{i,\bullet}$, for $i = 1, 2$; the coherence constraint of $(\omega * h^o_{\mathscr{C}}) \odot \varepsilon_1$ assigns to every morphism $f : X \to Y$ of $\mathscr{C}$ the natural isomorphism of functors $\omega_{h_X} * \tau^{h_Y}_{1,h_{uf}}$, whereas the coherence constraint of $\varepsilon_2 \odot (\omega' * (h_{\mathscr{C}'} \circ u)^o)$ assigns to f the natural isomorphism $\tau^{h_Y}_{2,h_{uf}} * \omega'_{h_{uY}}$. Then again the required identity follows directly from (3.3.22). ◊

From claim 3.3.23 we deduce a commutative diagram of cartesian functors :

$$\begin{array}{ccc} \mathsf{Fib}(h_{\mathscr{C}'} \circ u)^* \mathscr{E}'_1 & \xrightarrow{\Omega_{\mathscr{E}'_1}} & \mathsf{Fib}(h_{\mathscr{C}})^* \mathscr{E}_1 \\ {\scriptstyle \mathsf{Fib}(h_{\mathscr{C}'} \circ u)^* \varphi'} \downarrow & & \downarrow {\scriptstyle \mathsf{Fib}(h_{\mathscr{C}})^* \varphi} \\ \mathsf{Fib}(h_{\mathscr{C}'} \circ u)^* \mathscr{E}'_2 & \xrightarrow{\Omega_{\mathscr{E}'_2}} & \mathsf{Fib}(h_{\mathscr{C}})^* \mathscr{E}_2. \end{array}$$

Lastly, for $\mathscr{E}'_1, \mathscr{E}'_2$ as in the foregoing, let $\varphi'_1, \varphi'_2 : \mathscr{E}' \to \mathscr{E}'_2$ be two cartesian functors, $\beta' : \varphi'_1 \Rightarrow \varphi'_2$ a natural $\mathscr{C}'^\wedge$-transformation, and set $\varphi_i := \mathsf{Fib}(u^\wedge_{\mathsf{U}})_! \varphi'_i$ for $i = 1, 2$ and $\beta := \mathsf{Fib}(u^\wedge_{\mathsf{U}})_! \beta' : \varphi_1 \Rightarrow \varphi_2$. Say that φ'_1 and φ'_2 correspond to pseudo-natural transformations $\omega'_1, \omega'_2 : \mathsf{c}'_1 \Rightarrow \mathsf{c}'_2$, and β' corresponds to a modification $\Xi' : \omega'_1 \rightsquigarrow \omega'_2$; then φ_1 and φ_2 correspond to strict pseudo-natural transformations $\omega_1, \omega_2 : \mathsf{d}_1 \Rightarrow \mathsf{d}_2$ described as in the foregoing, and β corresponds to the modification $\Xi : \omega_1 \rightsquigarrow \omega_2$ assigning to every $F \in \mathrm{Ob}(\mathscr{C}^\wedge)$ the unique natural transformation $\Xi_F : \omega_{1,F} \Rightarrow \omega_{2,F}$ with :

$$e^F_{2,\bullet} * (\Xi' \circ \mathsf{t}^o_F) = \mathsf{F}_{\Xi_F} * e^F_{1,\bullet}$$

which specializes to the identity : $\Xi_{h_X} \odot e^{h_X}_{1,\eta_X} = e^{h_X}_{2,\eta_X} \odot \Xi'_{h_{uX}}$ for every $X \in \mathrm{Ob}(\mathscr{C})$. The latter in turns yields the identity :

$$(\Xi \circ h^o_{\mathscr{C}}) * \varepsilon_1 = \varepsilon_2 * (\Xi' \circ (h_{\mathscr{C}'} \circ u)^o)$$

which finally shows that :

$$\mathsf{Fib}(h_{\mathscr{C}})^*(\beta) * \Omega_{\mathscr{E}'_1} = \Omega_{\mathscr{E}'_2} * \mathsf{Fib}(h_{\mathscr{C}'} \circ u)^*(\beta').$$

Summing up, this shows that the system of equivalences $\Omega_\bullet$ yields a strict pseudo-natural equivalence of strict pseudo-functors $\mathsf{Fib}(h_{\mathscr{C}'} \circ u)^* \stackrel{\sim}{\to} \mathsf{Fib}(h_{\mathscr{C}})^* \circ \mathsf{Fib}(u^\wedge_{\mathsf{U}})_!$, as stated. □

3.3.24. We conclude this section by showing that the 2-limits and 2-colimits are computed fibrewise in the 2-category (U,V)-$\mathsf{Fib}(\mathscr{B})$, a result that extends the corresponding assertion for the category of presheaves (corollary 1.4.3(ii)). To begin with, let $\mathscr{B}$ be any V-small category; with every $B \in \mathrm{Ob}(\mathscr{B})$ we associate a strict pseudo-functor

$$\mathsf{fib}_B : (\mathsf{U},\mathsf{V})\text{-}\mathsf{Fib}(\mathscr{B}) \to (\mathsf{U},\mathsf{V})\text{-}\mathbf{Cat} \qquad \mathscr{A} \mapsto \mathscr{A}_B.$$

If $\mathscr{A}$ and $\mathscr{A}'$ are two $\mathscr{B}$-fibrations, $\psi, \psi' : \mathscr{A} \to \mathscr{A}'$ two $\mathscr{B}$-cartesian functors, and $\beta : \psi \Rightarrow \psi'$ a natural $\mathscr{B}$-transformation, then $\mathsf{fib}_B(\psi) : \mathscr{A}_B \to \mathscr{A}'_B$ is the restriction of ψ, and $\mathsf{fib}_B(\beta) : \mathsf{fib}_B(\psi) \Rightarrow \mathsf{fib}_B(\psi')$ is the restriction of β. We may then state :

Theorem 3.3.25. *With the notation of* (3.3.24)*, we have :*

(i) *The* 2*-category* (U,V)-$\mathsf{Fib}(\mathscr{B})$ *is strongly* 2*-complete and strongly* 2*-cocomplete.*
(ii) *For every* $B \in \mathrm{Ob}(\mathscr{B})$ *and any small* 2*-category* I*, the pseudo-functor* fib_B *commutes with the strong* 2*-limit and strong* 2*-colimit of any pseudo-functor* $I \to (\mathsf{U},\mathsf{V})$-$\mathsf{Fib}(\mathscr{B})$.

Proof. We prove the 2-completeness of $\mathsf{Fib}(\mathscr{B})$; the 2-cocompleteness follows by the same argument, considering the opposite 2-categories : the details shall be left to the reader. In view of theorem 3.1.24, it suffices to show the corresponding assertions for the 2-category $\mathsf{PsFun}(\mathscr{B}^o, \mathbf{Cat})$ and the similar strict pseudo-functors that we denote as well by

$$\mathsf{fib}_B : \mathsf{PsFun}(\mathscr{B}^o, \mathbf{Cat}) \to \mathbf{Cat} \qquad \mathsf{c} \mapsto \mathsf{c}_B.$$

Notice that every morphism $g : B' \to B$ of $\mathscr{B}$ induces a pseudo-natural transformation

$$\mathsf{fib}_g : \mathsf{fib}_B \Rightarrow \mathsf{fib}_{B'} \qquad \mathsf{c} \mapsto (\mathsf{c}_g : \mathsf{c}_B \to \mathsf{c}_{B'})$$

with coherence constraint given by the rule :

$$\tau^{\mathsf{fib}_g}_\beta := (\tau^\beta_g)^{-1}$$

for every pair of pseudo-natural functors $\mathsf{c}, \mathsf{d} : \mathscr{B}^o \to \mathbf{Cat}$ and every pseudo-natural transformation $\beta : \mathsf{c} \Rightarrow \mathsf{d}$ with coherence constraint τ^β. Moreover, if $h : B'' \to B'$ is any other morphism of $\mathscr{B}$, we get the invertible modification

$$\Gamma_{h,g} : \mathsf{fib}_h \odot \mathsf{fib}_g \rightsquigarrow \mathsf{fib}_{g \circ h} \qquad \mathsf{c} \mapsto \gamma^{\mathsf{c}}_{h,g}$$

where $(\delta^{\mathsf{c}}, \gamma^{\mathsf{c}})$ denotes the coherence constraint of c. Indeed, it is easily seen that the compatibility condition for $(\Gamma_{h,g})_\beta$ corresponding to a pseudo-natural transformation $\beta : \mathsf{c} \Rightarrow \mathsf{d}$ is equivalent to the coherence axiom for τ^β : details left to the reader. Likewise, for every third morphism $k : B''' \to B''$ of $\mathscr{B}$, the composition axiom for pseudo-functors translates as :

$$\Gamma_{k,g\circ h} \odot (\mathsf{fib}_k * \Gamma_{h,g}) = \Gamma_{h\circ k,g} \odot (\Gamma_{k,h} * \mathsf{fib}_g). \tag{3.3.26}$$

Let now $F : I \to \mathsf{PsFun}(\mathscr{B}^o, \mathbf{Cat})$ be any pseudo-functor; we need to exhibit a 2-limit for F, and by virtue of proposition 2.4.3, we may assume that $Fi : \mathscr{B}^o \to \mathbf{Cat}$ is unital for every

$i \in \mathrm{Ob}(I)$. Then, for every $B \in \mathrm{Ob}(\mathscr{B})$ choose a strong 2-limit $(L(B), \pi^B)$ of the pseudo-functor $\mathsf{fib}_B \circ F$ (theorem 3.3.9), and let τ^B be the coherence constraint of the pseudo-cone $\pi^B : \mathsf{F}_{L(B)} \Rightarrow \mathsf{fib}_B \circ F$. Let $g : B' \to B$ be any morphism of $\mathscr{B}$; by the (strong) universality of $\pi^{B'}$, we find a unique functor $L(g) : L(B) \to L(B')$ such that

$$\pi^{B'} \odot \mathsf{F}_{L(g)} = (\mathsf{fib}_g * F) \odot \pi^B. \tag{3.3.27}$$

Notice that $\mathsf{fib}_{\mathbf{1}_B} * F = \mathbf{1}_{\mathsf{fib}_B \circ F}$, since Fi is unital for every $i \in \mathrm{Ob}(I)$; therefore :

$$L(\mathbf{1}_B) = \mathbf{1}_{L(B)} \qquad \text{for every } B \in \mathrm{Ob}(\mathscr{B}).$$

Moreover, if $h : B'' \to B'$ is another morphism of $\mathscr{B}$, we have identities :

$$\begin{aligned}
\pi^{B''} \odot \mathsf{F}_{L(h)\circ L(g)} &= (\mathsf{fib}_h * F) \odot (\mathsf{fib}_g * F) \odot \pi^B = ((\mathsf{fib}_h \odot \mathsf{fib}_g) * F) \odot \pi^B \\
\pi^{B''} \odot \mathsf{F}_{L(h\circ g)} &= (\mathsf{fib}_{h\circ g}) \odot \pi^B
\end{aligned}$$

so there exists a unique natural transformation $\gamma^L_{h,g} : L(h) \circ L(g) \Rightarrow L(g \circ h)$ such that

$$\pi^{B''} * \mathsf{F}_{\gamma^L_{h,g}} = (\Gamma_{h,g} \circ F) * \pi^B.$$

Claim 3.3.28. The rules $B \mapsto L(B)$ and $(g : B' \to B) \mapsto L(g)$ for every $B \in \mathrm{Ob}(\mathscr{B})$ and every morphism g of $\mathscr{B}$ define a unital pseudo-functor $L : \mathscr{B}^o \to \mathbf{Cat}$ with coherence constraint given by the system of natural transformations $\gamma^L_{\bullet,\bullet}$.

Proof of the claim. The unit axiom for γ^L is clear from the construction. To check the composition axiom, let g and h be as in the foregoing, and consider as well a third morphism $k : B''' \to B''$ of $\mathscr{B}$. Since $\pi^{B'''}$ is a universal pseudo-cone, it suffices to show that

$$X := \pi^{B'''} * (\mathsf{F}_{\gamma^L_{k,g\circ h}} \odot \mathsf{F}_{L(k)*\gamma^L_{h,g}}) = Y := \pi^{B'''} * (\mathsf{F}_{\gamma^L_{h\circ k,g}} \odot \mathsf{F}_{\gamma^L_{k,h}*L(g)}).$$

However, by unwinding the definitions we find :

$$\begin{aligned}
X &= ((\Gamma_{k,g\circ h} \circ F) * \pi^B) \odot (\pi^{B'''} * \mathsf{F}_{L(k)} * \mathsf{F}_{\gamma^L_{h,g}}) \\
&= ((\Gamma_{k,g\circ h} \circ F) * \pi^B) \odot ((\mathsf{fib}_k * F) * \pi^{B''} * \mathsf{F}_{\gamma^L_{h,g}}) \\
&= ((\Gamma_{k,g\circ h} \circ F) * \pi^B) \odot ((\mathsf{fib}_k * F) * (\Gamma_{h,g} \circ F) * \pi^B) \\
&= ((\Gamma_{k,g\circ h} \circ F) * \pi^B) \odot (((\mathsf{fib}_k * \Gamma_{h,g}) \circ F) * \pi^B) \\
Y &= ((\Gamma_{h\circ k,g} \circ F) * \pi^B) \odot (\pi^{B'''} * \mathsf{F}_{\gamma^L_{k,h}} * \mathsf{F}_{L(g)}) \\
&= ((\Gamma_{h\circ k,g} \circ F) * \pi^B) \odot ((\Gamma_{k,h} \circ F) * \pi^{B'} * \mathsf{F}_{L(g)}) \\
&= ((\Gamma_{h\circ k,g} \circ F) * \pi^B) \odot ((\Gamma_{k,h} \circ F) * (\mathsf{fib}_g * F) * \pi^B) \\
&= ((\Gamma_{h\circ k,g} \circ F) * \pi^B) \odot ((\Gamma_{k,h} * \mathsf{fib}_g) \circ F) \odot \pi^B
\end{aligned}$$

so it suffices to show:

$$(\Gamma_{h\circ k,g} \circ F) \odot ((\Gamma_{k,h} * \mathsf{fib}_g) \circ F) = (\Gamma_{k,g\circ h} \circ F) \odot ((\mathsf{fib}_k * \Gamma_{h,g}) \circ F)$$

which follows straightforwardly from (3.3.26). ◊

Next, we notice that for every $i \in \mathrm{Ob}(I)$ the rule : $B \mapsto (\pi^B_i : L(B) \to Fi(B))$ defines a strict pseudo-natural transformation

$$\pi_i : L \Rightarrow Fi.$$

Indeed, the first coherence axiom for π_i is easily checked, recalling that both L and Fi are unital, and the second one follows directly from the definition of γ^L : details left to the reader.

Claim 3.3.29. (i) For every 1-cell $\varphi : i \to j$ in I, the rule :

$$B \mapsto \tau_\varphi^B$$

defines an invertible modification $\Xi^\varphi : F(\varphi) \odot \pi_i \rightsquigarrow \pi_j$.

(ii) The system $(\pi_i \mid i \in \mathrm{Ob}(I))$ defines a pseudo-cone

$$\pi : \mathsf{F}_L \Rightarrow F$$

with coherence constraint given by the invertible modifications $\Xi^\bullet$.

Proof of the claim. (i): Let $\tau^{F\varphi}$ and $\tau^{\mathsf{fib}_g * F}$ be the coherence constraints of $F\varphi$ and respectively $\mathsf{fib}_g * F$. We need to verify the compatibility condition :

$$Fj(g) * \tau_\varphi^B = (\tau_j^{B'} * L(g)) \odot (\tau_g^{F\varphi} * \pi_i^B)$$

for every morphism $g : B' \to B$ of $\mathscr{B}$. However, comparing the coherence constraints of the two sides of (3.3.27) we get :

$$(Fj(g) * \tau_\varphi^B) \odot (\tau_\varphi^{\mathsf{fib}_g * F} * \pi_i^B) = \tau_\varphi^{B'} * L(g).$$

Since $\tau_\varphi^{\mathsf{fib}_g * F} = (\tau_g^{F\varphi})^{-1}$, the assertion follows.

(ii): Denote by (δ^F, γ^F) the coherence constraint of F; we need to check the identities

$$\Xi^{\mathbf{1}_i} \odot (\delta_i^F * \pi_i) = \mathbf{1}_{\pi_i} \qquad \Xi^\psi \odot (F\psi * \Xi^\varphi) = \Xi^{\psi\circ\varphi} \odot (\gamma_{\varphi,\psi}^F * \pi_i)$$

for every $i, j, k \in \mathrm{Ob}(I)$ and every pair of 1-cells $\varphi : i \to j$ and $\psi : j \to k$ of I. However, the latter are none else than a rephrasing of the coherence axioms for the pseudo-cones π^B, for B ranging over all the objects of $\mathscr{B}$. ◊

It remains to check that the pseudo-cone $\pi : \mathsf{F}_L \Rightarrow F$ of claim 3.3.29(ii) is universal. To this aim, let $X : \mathscr{B}^o \to \mathbf{Cat}$ be any pseudo-functor, and $\beta : \mathsf{F}_X \Rightarrow F$ any pseudo-cone; we need to exhibit a pseudo-natural trasformation $t : X \Rightarrow L$ such that $\pi \odot \mathsf{F}_t = \beta$. Let $\omega : X \stackrel{\sim}{\to} X^u$ be the isomorphism furnished by proposition 2.4.3, with X^u a unital pseudo-functor; we get a commutative diagram of categories :

$$\begin{array}{ccc} \mathsf{PsNat}(X^u, L) & \longrightarrow & \mathsf{PsNat}(\mathsf{F}_{X^u}, F) \\ {\scriptstyle \mathsf{PsNat}(\omega, L)}\downarrow & & \downarrow{\scriptstyle \mathsf{PsNat}(\mathsf{F}_\omega, F)} \\ \mathsf{PsNat}(X, L) & \longrightarrow & \mathsf{PsNat}(\mathsf{F}_X, F) \end{array}$$

whose vertical arrows are isomorphisms. We need to check that the bottom horizontal arrow is an isomorphism, so it suffices to show that the same holds for the top horizontal arrow; we may thus assume that X is unital. Now, $\mathsf{fib}_B * \beta$ is a pseudo-cone with vertex $X(B)$ and basis $\mathsf{fib}_B \circ F : I \to \mathbf{Cat}$; then there exists a unique functor $t_B : X(B) \to L(B)$ such that

$$\pi^B \odot \mathsf{F}_{t_B} = \mathsf{fib}_B * \beta \qquad \text{for every } B \in \mathrm{Ob}(\mathscr{B}).$$

Denote by τ^β the coherence constraint of β, and by τ^{β_i} the coherence constraint of the pseudo-natural transformation $\beta_i : X \Rightarrow Fi$, for every $i \in \mathrm{Ob}(I)$; we remark :

Claim 3.3.30. For every morphism $g : B' \to B$ of $\mathscr{B}$, the rule :

$$i \mapsto \tau_g^{\beta_i} \qquad \text{for every } i \in \mathrm{Ob}(I)$$

defines an invertible modification $\Lambda^g : (\mathsf{fib}_g * F) \odot (\mathsf{fib}_B * \beta) \rightsquigarrow (\mathsf{fib}_{B'} * \beta) \odot \mathsf{F}_{X(g)}$.

Proof of the claim. The coherence constraint of $(\mathsf{fib}_g * F) \odot (\mathsf{fib}_B * \beta)$ assigns to every 1-cell $\varphi : i \to j$ of I the composition of oriented squares :

$$\begin{array}{ccccc} X(B) & \xrightarrow{\beta_{i,B}} & Fi(B) & \xrightarrow{Fi(g)} & Fi(B') \\ \| & \Downarrow \tau^{\beta}_{\varphi,B} & \downarrow F\varphi(B) & \Downarrow (\tau^{F\varphi}_g)^{-1} & \downarrow F\varphi(B') \\ X(B) & \xrightarrow[\beta_{j,B}]{} & Fj(B) & \xrightarrow[Fj(g)]{} & Fj(B') \end{array}$$

whereas the coherence constraint of $(\mathsf{fib}_{B'} * \beta) \odot \mathsf{F}_{X(g)}$ assigns to φ the natural transformation

$$\tau^{\beta}_{\varphi,B'} * X(g) : F\varphi(B') \circ \beta_{i,B'} \circ X(g) \Rightarrow \beta_{j,B'} \circ X(g).$$

Thus, for every such φ we need to show the identity :

$$\tau^{\beta_j}_g \odot (Fj(g) * \tau^{\beta}_{\varphi,B}) \odot ((\tau^{F\varphi}_g)^{-1} * \beta_{i,B}) = \tau^{\beta}_{\varphi,B'} \odot (F\varphi(B') * \tau^{\beta_i}_g).$$

But the latter is equivalent to the compatibility condition for the invertible modification $\tau^{\beta}_{\varphi} : F\varphi \odot \beta_i \rightsquigarrow \beta_j$, corresponding to the morphism g. $\Diamond$

Notice now that $(\mathsf{fib}_g * F) \odot (\mathsf{fib}_B * \beta) = \pi^{B'} \odot \mathsf{F}_{L(g)\circ t_B}$ and $(\mathsf{fib}_{B'} * \beta) \odot \mathsf{F}_{X(g)} = \pi^{B'} \odot \mathsf{F}_{t_{B'}\circ X(g)}$ for every morphism $g : B' \to B$ of $\mathscr{B}$. In view of claim 3.3.30, and since $\pi^{B'}$ is universal, there exists therefore a unique natural isomorphism of functors

$$\tau^t_g : L(g) \circ t_B \Rightarrow t_{B'} \circ X(g) \qquad \text{such that} \qquad \pi^{B'} * \mathsf{F}_{\tau^t_g} = \Lambda^g.$$

Notice that, by virtue of remark 2.4.2(ii), we have $\Lambda^{\mathbf{1}_B} = \mathbf{1}_{\mathsf{fib}_B * \beta}$ for every $B \in \mathrm{Ob}(\mathscr{B})$, and therefore $\tau^t_{\mathbf{1}_B} = \mathbf{1}_{t_B}$ for every such B. Let us check that the system $(t_B \mid B \in \mathrm{Ob}(\mathscr{B}))$ yields the sought pseudo-natural transformation, with coherence constraint given by the system of isomorphisms of functors $\tau^t_\bullet$. Indeed, the first coherence axiom follows easily from the foregoing; it remains therefore only to show that for every pair of morphisms $g : B' \to B$ and $h : B'' \to B'$ of $\mathscr{B}$ we have :

$$(t_{B''} * \gamma^X_{h,g}) \odot (\tau^t_h * X(g)) \odot (L(h) * \tau^t_g) = \tau^t_{g\circ h} \odot (\gamma^L_{g,h} * t_B)$$

where γ^X denotes the coherence constraint of X. Using the universality of $\pi^{B''}$, we are then reduced to showing the identity :

$$U := (\pi^{B''} * \mathsf{F}_{t_{B''}} * \mathsf{F}_{\gamma^X_{h,g}}) \odot (\Lambda^h * \mathsf{F}_{X(g)}) \odot (\pi^{B''} * \mathsf{F}_{L(h)} * \mathsf{F}_{\tau^t_g}) = V := \Lambda^{g\circ h} \odot (\pi^{B''} * \mathsf{F}_{\gamma^L_{g,h}} * \mathsf{F}_{t_B}).$$

However, we have :

$$\begin{aligned} U &= (\pi^{B''} * \mathsf{F}_{t_{B''}} * \mathsf{F}_{\gamma^X_{h,g}}) \odot (\Lambda^h * \mathsf{F}_{X(g)}) \odot ((\mathsf{fib}_h * F) * \pi^{B'} * \mathsf{F}_{\tau^t_g}) \\ &= ((\mathsf{fib}_{B''} * \beta) * \mathsf{F}_{\gamma^X_{h,g}}) \odot (\Lambda^h * \mathsf{F}_{X(g)}) \odot ((\mathsf{fib}_h * F) * \pi^{B'} * \mathsf{F}_{\tau^t_g}) \\ &= ((\mathsf{fib}_{B''} * \beta) * \mathsf{F}_{\gamma^X_{h,g}}) \odot (\Lambda^h * \mathsf{F}_{X(g)}) \odot ((\mathsf{fib}_h * F) * \Lambda^g) \\ V &= \Lambda^{g\circ h} \odot ((\Gamma_{h,g} \circ F) * \pi^B * \mathsf{F}_{t_B}). \end{aligned}$$

Thus, it suffices to show :

$$((\mathsf{fib}_{B''} * \beta) * \mathsf{F}_{\gamma^X_{h,g}}) \odot (\Lambda^h * \mathsf{F}_{X(g)}) \odot ((\mathsf{fib}_h * F) * \Lambda^g) = \Lambda^{g\circ h} \odot ((\Gamma_{h,g} \circ F) * (\mathsf{fib}_B * \beta)).$$

But a simple inspection shows that the latter identity translates the coherence axiom for β_i, relative to the pair of 1-cells (h, g), and for i ranging over all the objects of I.

Claim 3.3.31. (i) $\pi_i \odot t = \beta_i$ for every $i \in \mathrm{Ob}(I)$.
(ii) $\pi \odot \mathsf{F}_t = \beta$.

Proof of the claim. We need to check that the coherence constraint $\tau^{\pi_i \odot t}$ of $\pi_i \odot t$ equals τ^{β_i}. But for every morphism $g : B' \to B$ of $\mathscr{B}$ we have $\tau_g^{\pi_i \odot t} = \pi_i^{B'} * \tau_g^t = \Lambda_i^g$, whence the contention.

(ii): In view of (i), it suffices to check that β and $\pi \odot \mathsf{F}_t$ have the same coherence constraints. But the coherence constraint of $\pi \odot \mathsf{F}_t$ assigns to every 1-cell $\varphi : i \to j$ in I the invertible modification $\Xi^\varphi * t$, so the assertion follows by a direct inspection of the definitions. ◊

Claim 3.3.31(ii) shows that the functor

$$\mathsf{PsNat}(X, L) \to \mathsf{PsNat}(\mathsf{F}_X, F) \qquad t \mapsto \pi \odot \mathsf{F}_t \tag{3.3.32}$$

is surjective on objects. To check injectivity on objects, consider two pseudo-natural transformations $t, t' : X \Rightarrow L$ such that $\pi \odot \mathsf{F}_t = \pi \odot \mathsf{F}_{t'}$. Then we have

$$\pi^B \odot \mathsf{F}_{t_B} = \mathsf{fib}_B * (\pi \odot \mathsf{F}_t) = \mathsf{fib}_B * (\pi \odot \mathsf{F}_{t'}) = \pi^B \odot \mathsf{F}_{t'_B} \qquad \text{for every } B \in \mathrm{Ob}(\mathscr{B}).$$

By strong universality of π^B, we deduce that $t_B = t'_B$ for every such B. Moreover, recall that we have attached to $\pi \odot \mathsf{F}_t$ and every morphism $g : B' \to B$ an invertible modification $\Lambda^g : \pi^{B'} \odot \mathsf{F}_{L(g) \circ t_B} \rightsquigarrow \pi^{B'} \odot \mathsf{F}_{t_{B'} \circ X(g)}$. Let τ^t and $\tau^{t'}$ be the coherence constraint of t and t'; by inspecting the definition, we find that

$$\Lambda_i^g = (\tau_g^B * t_B) \odot (\pi_i^{B'} * \tau_g^t) \qquad \text{for every } i \in \mathrm{Ob}(I).$$

Since $\pi \odot \mathsf{F}_t = \pi \odot \mathsf{F}_{t'}$ and $t_B = t'_B$, we get $\pi_i^{B'} * \tau_g^t = \pi_i^{B'} * \tau_g^{t'}$ for every $i \in \mathrm{Ob}(I)$, *i.e.* $\pi^{B'} * \mathsf{F}_{\tau_g^t} = \pi^{B'} * \mathsf{F}_{\tau_g^{t'}}$, whence $\tau_g^t = \tau_g^{t'}$, by the universality of $\pi^{B'}$. Thus, $t = t'$ as required.

To check that (3.3.32) is faithful, consider pseudo-natural transformations $t, t' : X \Rightarrow L$ and modifications $\Delta, \Delta' : t \rightsquigarrow t'$ with $\pi * \mathsf{F}_\Delta = \pi * \mathsf{F}_{\Delta'}$; we deduce:

$$\pi^B * (\mathsf{fib}_B \circ \mathsf{F}_\Delta) = \mathsf{fib}_B \circ (\pi * \mathsf{F}_\Delta) = \mathsf{fib}_B \circ (\pi * \mathsf{F}_{\Delta'}) = \pi^B * (\mathsf{fib}_B \circ \mathsf{F}_{\Delta'}) \qquad \text{for every } B \in \mathrm{Ob}(\mathscr{B})$$

and clearly $\mathsf{fib}_B \circ \mathsf{F}_\Delta$ and $\mathsf{fib}_B \circ \mathsf{F}_{\Delta'}$ are the constant modifications $\mathsf{F}_{t_B} \rightsquigarrow \mathsf{F}_{t'_B}$ associated with Δ and Δ'. By the universality of π^B, it follows that $\mathsf{fib}_B \circ \mathsf{F}_\Delta = \mathsf{fib}_B \circ \mathsf{F}_{\Delta'}$ for every $B \in \mathrm{Ob}(\mathscr{B})$, whence $\Delta = \Delta'$, as required.

Lastly, let $\Delta : \pi \odot \mathsf{F}_t \rightsquigarrow \pi \odot \mathsf{F}_{t'}$ be any modification; for every $B \in \mathrm{Ob}(\mathscr{B})$ we obtain the modification $\mathsf{fib}_B \circ \Delta : \pi^B \odot \mathsf{F}_{t_B} \rightsquigarrow \pi^B \odot \mathsf{F}_{t'_B}$, and since π^B is universal, there exists a unique natural transformation $\delta^B : t_B \Rightarrow t'_B$ such that $\mathsf{fib}_B \circ \Delta = \pi^B * \mathsf{F}_{\delta^B}$. It remains to check that the rule : $B \mapsto \delta^B$ yields a modification $\delta : t \rightsquigarrow t'$, since in this case we get $\Delta = \pi * \mathsf{F}_\delta$, which will prove that (3.3.32) is also full, thus concluding the proof of the theorem. However, Δ is the datum of a modification $\Delta_i : \pi_i \odot t \rightsquigarrow \pi_i \odot t'$ for every $i \in \mathrm{Ob}(I)$; the compatibility condition for Δ_i asserts that the following two compositions of oriented squares coincide for every morphism $g : B' \to B$ of $\mathscr{B}$:

$$\begin{array}{ccccc} X(B) & \xrightarrow{t_B} & L(B) & \xrightarrow{\pi_i^B} & Fi(B) \\ \| & \Swarrow_{\delta^B} & \| & \Swarrow_{\mathbf{1}_{\pi_i^B}} & \| \\ X(B) & \xrightarrow{t'_B} & L(B) & \xrightarrow{\pi_i^B} & Fi(B) \\ {\scriptstyle X(g)}\downarrow & \Swarrow_{\tau_g^{t'}} & \downarrow{\scriptstyle L(g)} & \Swarrow_{\tau_g^{\pi_i}} & \downarrow{\scriptstyle Fi(g)} \\ X(B') & \xrightarrow[t'_{B'}]{} & L(B') & \xrightarrow[\pi_i^{B'}]{} & Fi(B') \end{array} \qquad \begin{array}{ccccc} X(B) & \xrightarrow{t_B} & L(B) & \xrightarrow{\pi_i^B} & Fi(B) \\ {\scriptstyle X(g)}\downarrow & \Swarrow_{\tau_g^{t}} & \downarrow{\scriptstyle L(g)} & \Swarrow_{\tau_g^{\pi_i}} & \downarrow{\scriptstyle Fi(g)} \\ X(B') & \xrightarrow{t_{B'}} & L(B') & \xrightarrow{\pi_i^{B'}} & Fi(B') \\ \| & \Swarrow_{\delta^{B'}} & \| & \Swarrow_{\mathbf{1}_{\pi_i^{B'}}} & \| \\ X(B') & \xrightarrow[t'_{B'}]{} & L(B') & \xrightarrow[\pi_i^{B'}]{} & Fi(B') \end{array}$$

where τ_g^t, $\tau_g^{t'}$ and $\tau_g^{\pi_i}$ are the coherent constraints of respectively t, t' and π_i. We deduce that

$$\pi_i^{B'} * (\tau_g^{t'} \odot (L(g) * \delta^B)) = \pi_i^{B'} * ((\delta^{B'} * X(g)) \odot \tau_g^t) \qquad \text{for every } i \in \mathrm{Ob}(I)$$

and invoking again the universality of $\pi^{B'}$ we conclude that $\tau_g^{t'} \odot (L(g) * \delta^B) = (\delta^{B'} * X(g)) \odot \tau_g^t$, which is the required compatibility condition for δ. □

3.4. **Fibrations in groupoids.** Recall that a *groupoid* is a category all whose morphisms are invertible. To every category $\mathscr{B}$, we may attach the groupoid $\mathscr{B}^\times$ such that $\mathrm{Ob}(\mathscr{B}^\times) = \mathrm{Ob}(\mathscr{B})$, and whose morphisms are the isomorphisms of $\mathscr{B}$; the composition law for morphisms in $\mathscr{B}^\times$ is the same as that of $\mathscr{B}$, so $\mathscr{B}^\times$ is a subcategory of $\mathscr{B}$. Then clearly every functor $G : \mathscr{A} \to \mathscr{B}$ restricts to a functor $G^\times : \mathscr{A}^\times \to \mathscr{B}^\times$, and for every universe U, the rules : $\mathscr{C} \mapsto \mathscr{C}^\times$ and $(F : \mathscr{A} \to \mathscr{B}) \mapsto F^\times$ yield a right adjoint $(-)^\times : \mathsf{U}\text{-}\mathbf{Cat} \to \mathsf{U}\text{-}\mathsf{Gpd}$ for the inclusion functor

$$\mathsf{U}\text{-}\mathsf{Gpd} \to \mathsf{U}\text{-}\mathbf{Cat}$$

of the full subcategory $\mathsf{U}\text{-}\mathsf{Gpd}$ of $\mathsf{U}\text{-}\mathbf{Cat}$ whose objects are the U-small groupoids.

Remark 3.4.1. (i) Notice that every isomorphism of functors $\alpha : G \xrightarrow{\sim} H$ restricts to an isomorphism of functors $\alpha^\times : G^\times \xrightarrow{\sim} H^\times$.

(ii) Let $\mathscr{A}$ and $\mathscr{B}$ be any two categories, and $G : \mathscr{A} \to \mathscr{B}$ any functor. Then it is easily seen that G is essentially surjective if and only if the same holds for $G^\times$. Moreover, if G is i-faithful for some $i \in \{0, 1, 2\}$, then the same holds for $G^\times$. The verification in case $i = 0$ shall be left to the reader. Next, let $f : GX \to GY$ be a morphism in $\mathscr{B}^\times$; if G is fully faithful, there exists $g : X \to Y$ in $\mathscr{A}$ such that $Gg = f$. But f is invertible, so by the same token there exists $h : Y \to X$ in $\mathscr{A}$ such that $Gh = f^{-1}$; since $G(f \circ g) = G\mathbf{1}_Y$ and $G(g \circ f) = G\mathbf{1}_X$, we deduce $f \circ g = \mathbf{1}_Y$ and $g \circ f = \mathbf{1}_X$, since G is faithful. This shows that g is a morphism in $\mathscr{A}^\times$, and proves the assertion for $i = 1$. For $i = 2$, the functor G is an equivalence, so we know already that $G^\times$ is fully faithful; but we have also already noticed that $G^\times$ is essentially surjective, whence the assertion for $i = 2$.

(iii) Since $(-)^\times$ is a right adjoint, it commutes with all limits of $\mathsf{V}\text{-}\mathbf{Cat}$ (see proposition 1.3.25(iii)). Moreover, by inspecting the construction of example 1.5.10, it is easily seen that $(-)^\times$ also commutes with all filtered (small) colimits : details left to the reader.

3.4.2. We wish now to upgrade the associated groupoid construction to the 2-category of $\mathscr{C}$-fibrations, for any given base category $\mathscr{C}$. Indeed, let $F : \mathscr{A} \to \mathscr{C}$ be any fibration; we associate with $\mathscr{A}$ the category

$$\mathscr{A}^\times$$

with $\mathrm{Ob}(\mathscr{A}^\times) = \mathscr{A}$, and whose morphisms are the cartesian morphisms of $\mathscr{A}$. The composition law for morphisms in $\mathscr{A}^\times$ is the restriction of that of $\mathscr{A}$, hence $\mathscr{A}^\times$ is a subcategory of $\mathscr{A}$, and we denote by $F^\times : \mathscr{A}^\times \to \mathscr{C}$ the restriction of F. Then it is easily seen that $F^\times$ is also a fibration, and all morphisms of $\mathscr{A}^\times$ are cartesian for this fibration; moreover, for every $X \in \mathrm{Ob}(\mathscr{C})$ the fibre category $(F^\times)^{-1}X$ is a groupoid. The latter assertion can be easily checked directly, and it follows also straightforwardly from lemma 3.1.20(i). We call $F^\times$ the *fibration in groupoids associated with* F. We say that F is a *fibration in groupoids* if $\mathscr{A} = \mathscr{A}^\times$. Clearly every cartesian functor $G : \mathscr{A} \to \mathscr{B}$ of $\mathscr{C}$-fibrations restricts to a (cartesian) functor

$$G^\times : \mathscr{A}^\times \to \mathscr{B}^\times.$$

On the other hand, a given natural $\mathscr{C}$-transformation $\beta : G \Rightarrow H$ between $\mathscr{C}$-cartesian functors $G, H : \mathscr{A} \to \mathscr{B}$ induces a natural $\mathscr{C}$-transformation $\beta^\times : G^\times \Rightarrow H^\times$ if and only if β is an isomorphism of functors, *i.e.* if and only if β is a morphism of the groupoid $\mathsf{Cart}_\mathscr{C}(\mathscr{A}, \mathscr{B})^\times$ associated with $\mathsf{Cart}_\mathscr{C}(\mathscr{A}, \mathscr{B})$ as in (3.4). Thus, we get a natural functor

$$\mathsf{Cart}_\mathscr{C}(\mathscr{A}, \mathscr{B})^\times \to \mathsf{Cart}_\mathscr{C}(\mathscr{A}^\times, \mathscr{B}^\times) \qquad G \mapsto G^\times \qquad (\beta : G \Rightarrow H) \mapsto \beta^\times \tag{3.4.3}$$

3.4.4. Let U, V be universes such that $\mathsf{U} \subset \mathsf{V}$, and $\mathscr{C}$ a V-small category; we denote by

$$(\mathsf{U}, \mathsf{V})\text{-}\mathsf{Fib}^\times(\mathscr{C}) \qquad \text{and} \qquad (\mathsf{U}, \mathsf{V})\text{-}\mathsf{Gpd}(\mathscr{C})$$

respectively : the sub-2-category of $(\mathsf{U}, \mathsf{V})\text{-}\mathsf{Fib}(\mathscr{C})$ whose objects are the same as those of $(\mathsf{U}, \mathsf{V})\text{-}\mathsf{Fib}(\mathscr{C})$, and whose Hom-category is $\mathsf{Cart}_\mathscr{C}(\mathscr{A}, \mathscr{B})^\times$ for every pair $(\mathscr{A}, \mathscr{B})$ of such $\mathscr{C}$-fibrations, and : the strong sub-2-category of $(\mathsf{U}, \mathsf{V})\text{-}\mathsf{Fib}(\mathscr{C})$ whose objects are the fibrations in groupoids. With this notation, clearly we get as well a strict pseudo-functor

$$(-)^{\times}_{\mathscr{C}} : (\mathsf{U},\mathsf{V})\text{-}\mathsf{Fib}^{\times}(\mathscr{C}) \to (\mathsf{U},\mathsf{V})\text{-}\mathsf{Gpd}(\mathscr{C}) \qquad \mathscr{A} \mapsto \mathscr{A}^{\times}$$

which is given on Hom-categories by the foregoing system of functors (3.4.3). As usual, we may omit mentioning the universes in the above notation, if no ambiguities are likely to arise.

3.4.5. Moreover, every functor $u : \mathscr{C} \to \mathscr{C}'$ induces strict pseudo-functors

$$(\mathsf{U},\mathsf{V})\text{-}\mathsf{Fib}^{\times}(u)^* : \mathsf{Fib}^{\times}(\mathscr{C}') \to \mathsf{Fib}^{\times}(\mathscr{C}) \qquad (\mathsf{U},\mathsf{V})\text{-}\mathsf{Gpd}(u)^* : \mathsf{Gpd}(\mathscr{C}') \to \mathsf{Gpd}(\mathscr{C})$$

defined as the restrictions of $\mathsf{Fib}(u)^* : \mathsf{Fib}(\mathscr{C}') \to \mathsf{Fib}(\mathscr{C})$. As usual, these functors will be sometimes just denoted $\mathsf{Fib}^{\times}(u)^*$ and respectively $\mathsf{Gpd}(u)^*$. By simple inspection we get a commutative diagram of 2-categories :

$$\begin{array}{ccc} \mathsf{Fib}^{\times}(\mathscr{C}') & \xrightarrow{\mathsf{Fib}^{\times}(u)^*} & \mathsf{Fib}^{\times}(\mathscr{C}) \\ {\scriptstyle (-)^{\times}_{\mathscr{C}'}}\downarrow & & \downarrow{\scriptstyle (-)^{\times}_{\mathscr{C}}} \\ \mathsf{Gpd}(\mathscr{C}') & \xrightarrow{\mathsf{Gpd}(u)^*} & \mathsf{Gpd}(\mathscr{C}). \end{array}$$

3.4.6. Let $u : \mathscr{C} \to \mathscr{C}'$ be a functor from a U-small category $\mathscr{C}$ to a V-small category $\mathscr{C}'$ with U-small Hom-sets. Recall that the source functor $\mathsf{s}_X : u\mathscr{C}/X \to \mathscr{C}$ is a fibration in groupoids for every $X \in \mathrm{Ob}(\mathscr{C}')$ (example 3.1.3(i)); taking into account remark 3.3.15(ii) we deduce for every $\mathscr{C}$-fibration $\mathscr{A}$ with V-small fibres and every such X, natural equivalences of categories :

$$(\mathsf{Fib}(u)_*\mathscr{A}(X))^{\times} \xrightarrow{\sim} \mathsf{Cart}_{\mathscr{C}}(u\mathscr{C}/X, \mathscr{A})^{\times} \xrightarrow{\sim} \mathsf{Cart}_{\mathscr{C}}(u\mathscr{C}/X, \mathscr{A}^{\times}) \xrightarrow{\sim} (\mathsf{Fib}(u)_*(\mathscr{A}^{\times}))(X).$$

There follows a pseudo-commutative diagram of 2-categories :

$$\begin{array}{ccc} \mathsf{Fib}^{\times}(\mathscr{C}) & \xrightarrow{(\mathsf{U},\mathsf{V})\text{-}\mathsf{Fib}^{\times}(u)_*} & \mathsf{Fib}^{\times}(\mathscr{C}') \\ {\scriptstyle (-)^{\times}_{\mathscr{C}}}\downarrow & & \downarrow{\scriptstyle (-)^{\times}_{\mathscr{C}'}} \\ \mathsf{Gpd}(\mathscr{C}) & \xrightarrow{(\mathsf{U},\mathsf{V})\text{-}\mathsf{Gpd}(u)_*} & \mathsf{Gpd}(\mathscr{C}') \end{array}$$

where $(\mathsf{U},\mathsf{V})\text{-}\mathsf{Fib}^{\times}(u)_*$ and $(\mathsf{U},\mathsf{V})\text{-}\mathsf{Gpd}(u)_*$ are the restrictions of $(\mathsf{U},\mathsf{V})\text{-}\mathsf{Fib}(u)_*$. *Suppose moreover that the category $X/u\mathscr{C}$ is cofiltered for every $X \in \mathrm{Ob}(\mathscr{C}')$*; then, in view of remarks 3.4.1(iii) and 5.5.14(iii), for every such X we have as well a natural equivalence of categories :

$$(\mathsf{Fib}(u)_!\mathscr{A}(X))^{\times} \xrightarrow{\sim} \operatorname*{colim}_{X \to uY} \mathscr{A}(Y)^{\times} \xrightarrow{\sim} \mathsf{Fib}(u)_!(\mathscr{A}^{\times})(X)$$

where the colimit ranges over the small filtered category $(X/u\mathscr{C})^o$. We get therefore a pseudo-commutative diagram of 2-categories :

$$\begin{array}{ccc} \mathsf{Fib}^{\times}(\mathscr{C}) & \xrightarrow{(\mathsf{U},\mathsf{V})\text{-}\mathsf{Fib}^{\times}(u)_!} & \mathsf{Fib}^{\times}(\mathscr{C}') \\ {\scriptstyle (-)^{\times}_{\mathscr{C}}}\downarrow & & \downarrow{\scriptstyle (-)^{\times}_{\mathscr{C}'}} \\ \mathsf{Gpd}(\mathscr{C}) & \xrightarrow{(\mathsf{U},\mathsf{V})\text{-}\mathsf{Gpd}(u)_!} & \mathsf{Gpd}(\mathscr{C}) \end{array} \tag{3.4.7}$$

where $(\mathsf{U},\mathsf{V})\text{-}\mathsf{Fib}^{\times}(u)_!$ and $(\mathsf{U},\mathsf{V})\text{-}\mathsf{Gpd}(u)_!$ denote the restrictions of $(\mathsf{U},\mathsf{V})\text{-}\mathsf{Fib}(u)_!$.

Remark 3.4.8. Let $\mathscr{C}$, U and V be as in (3.4.4).

(i) It is easily seen that the pseudo-functor $(-)^{\times}_{\mathscr{C}}$ is a strong right 2-adjoint for the inclusion pseudo-functor $\mathsf{Gpd}(\mathscr{C}) \to \mathsf{Fib}^{\times}(\mathscr{C})$, and the counit of the resulting 2-adjunction assigns to every fibration $\mathscr{A} \to \mathscr{C}$ the inclusion functor $\mathscr{A}^{\times} \to \mathscr{A}$: details left to the reader.

(ii) Notice that a fibration $\pi : \mathscr{A} \to \mathscr{C}$ is a fibration in groupoids if and only if $\pi^{-1}X$ is a groupoid for every $X \in \mathrm{Ob}(\mathscr{C})$. Indeed, the condition is obviously necessary. Conversely, recall that every morphism f of $\mathscr{A}$ is the composition of a cartesian morphism and a morphism g such that $\pi(g) = \mathbf{1}_X$ for some $X \in \mathrm{Ob}(\mathscr{C})$; but if $\pi^{-1}X$ is a groupoid, then g is an isomorphism in $\mathscr{A}$, in particular it is cartesian, so the same holds for f, which shows that $\mathscr{A} = \mathscr{A}^\times$.

(iii) By the same token, to every pseudo-functor $F : \mathscr{C}^o \to \mathbf{Cat}$ we attach the pseudo-functor

$$F^\times : \mathscr{C}^o \to \mathbf{Cat} \qquad X \mapsto (FX)^\times$$

(whose coherence constraints are the same as those of F). Then we easily see that :

$$\mathscr{F}ib(F)^\times = \mathscr{F}ib(F^\times).$$

3.4.9. With the notation of (3.4.4), notice that for every presheaf $F \in \mathrm{Ob}(\mathscr{C}^\wedge_{\mathsf{U}})$, the fibration $\mathscr{F}ib_{\mathscr{C}}(F)$ is a fibration in groupoid, so we get a well defined strict pseudo-functor

$$\mathscr{F}ib_{\mathscr{C}} : \mathscr{C}^\wedge_{\mathsf{U}} \to (\mathsf{U},\mathsf{V})\text{-}\mathsf{Gpd}(\mathscr{C})$$

and by restriction, the pseudo-functor $\pi_0^{\mathscr{C}}$ (see (3.2.10)) yields a strong left 2-adjoint

$$\pi_0^{\mathscr{C}} : (\mathsf{U},\mathsf{V})\text{-}\mathsf{Gpd}(\mathscr{C}) \to \mathscr{C}^\wedge_{\mathsf{U}}.$$

Moreover, for every functor $u : \mathscr{C} \to \mathscr{C}'$, a simple inspection yields an essentially commutative diagram

$$\begin{array}{ccc}
\mathsf{Gpd}(\mathscr{C}') & \xrightarrow{\mathsf{Gpd}(u)^*} & \mathsf{Gpd}(\mathscr{C}) \\
{\scriptstyle \pi_0^{\mathscr{C}'}}\downarrow & & \downarrow{\scriptstyle \pi_0^{\mathscr{C}}} \\
\mathscr{C}'^\wedge_{\mathsf{U}} & \xrightarrow{u^\wedge_{\mathsf{U}}} & \mathscr{C}^\wedge_{\mathsf{U}}.
\end{array}$$

Furthermore, if $\mathscr{C}$ is U-small and $\mathscr{C}'$ is V-small with U-small Hom-sets, we also get the essentially commutative diagram :

$$(3.4.10) \qquad \begin{array}{ccc}
\mathsf{Gpd}(\mathscr{C}) & \xrightarrow{\mathsf{Gpd}(u)_!} & \mathsf{Gpd}(\mathscr{C}') \\
{\scriptstyle \pi_0^{\mathscr{C}}}\downarrow & & \downarrow{\scriptstyle \pi_0^{\mathscr{C}'}} \\
\mathscr{C}^\wedge_{\mathsf{U}} & \xrightarrow{u_{\mathsf{U}!}} & \mathscr{C}'^\wedge_{\mathsf{U}}.
\end{array}$$

Indeed, for every $X \in \mathrm{Ob}(\mathscr{C}')$ and every fibration $\mathscr{A}$ with essentially U-small fibres we have natural equivalences of categories :

$$(\pi_0^{\mathscr{C}'}\mathsf{Gpd}(u)_!\mathscr{A})(X) \xrightarrow{\sim} \pi_0((\mathsf{Fib}(u)_!\mathscr{A})(X)) \xrightarrow{\sim} \pi_0(\underset{X\to uY}{2\text{-colim}}\,\mathscr{A}(Y)) \xrightarrow{\sim} \underset{X\to uY}{2\text{-colim}}\,\pi_0(\mathscr{A}(Y))$$

since π_0 is a 2-left adjoint pseudo-functor (proposition 2.5.9(ii)). However, remark 2.5.2(vii) says that the 2-colimit of the (strict) pseudo-functor given by the rule $(X \to uY) \mapsto \pi_0(\mathscr{A}(Y))$ also represents the colimit of the same functor, in the category of U-small sets; on the other hand, the latter colimit is represented by $u_{\mathsf{U}!}(\pi_0^{\mathscr{C}}\mathscr{A})(X)$, whence the contention.

3.5. **Sieves and descent theory.** This section develops the basics of descent theory, in the general framework of fibred categories.

Definition 3.5.1. Let $\mathscr{B}$ and $\mathscr{C}$ be two categories, $F : \mathscr{B} \to \mathscr{C}$ a functor.

(i) A *sieve* of $\mathscr{C}$ is a full subcategory $\mathscr{S}$ of $\mathscr{C}$ such that the following holds. If $A \in \mathrm{Ob}(\mathscr{S})$, and $B \to A$ is any morphism in $\mathscr{C}$, then $B \in \mathrm{Ob}(\mathscr{S})$.

(ii) If $S \subset \mathrm{Ob}(\mathscr{C})$ is any subset, there is a smallest sieve $\mathscr{S}_S$ of $\mathscr{C}$ such that $S \subset \mathrm{Ob}(\mathscr{S}_S)$; we call $\mathscr{S}_S$ the *sieve generated by* S. If $S' \subset \mathrm{Ob}(\mathscr{C})$ is another subset and $\mathscr{S}_{S'} \subset \mathscr{S}_S$, we say that S' is a *refinement* of S.

(iii) If $\mathscr{S}$ is a sieve of $\mathscr{C}$, the *inverse image of* $\mathscr{S}$ *under* F is the full subcategory $F^{-1}\mathscr{S}$ of $\mathscr{B}$ with $\mathrm{Ob}(F^{-1}\mathscr{S}) = \{B \in \mathrm{Ob}(\mathscr{B}) \mid FB \in \mathrm{Ob}(\mathscr{S})\}$ (notice that $F^{-1}\mathscr{S}$ is a sieve).

(iv) If $f : X \to Y$ is any morphism in $\mathscr{C}$, and $\mathscr{S}$ is any sieve of $\mathscr{C}/Y$, we shall write $\mathscr{S} \times_Y f$ for the inverse image of $\mathscr{S}$ under the functor f_* (notation of (1.1.25)).

Remark 3.5.2. Let $\mathscr{C}$ be a category with small Hom-sets and X an object of $\mathscr{C}$.

(i) Every sieve $\mathscr{T}$ of $\mathscr{C}$ yields a subobject $F_{\mathscr{T}}$ of the final object $\mathbb{1}_{\mathscr{C}}$ of $\mathscr{C}^\wedge$, by declaring that $F_{\mathscr{T}}(Y) \neq \varnothing$ if and only if $Y \in \mathrm{Ob}(\mathscr{T})$; hence $F_{\mathscr{T}}(Y)$ is a set with one element for every $Y \in \mathrm{Ob}(\mathscr{T})$, and is empty for $Y \in \mathrm{Ob}(\mathscr{C}) \setminus \mathrm{Ob}(\mathscr{T})$. It is easily seen that the rule $\mathscr{T} \mapsto F_{\mathscr{T}}$ establishes a bijection between the set of sieves of $\mathscr{C}$ and the set of subobjects of $\mathbb{1}_{\mathscr{C}}$. The inverse mapping is given by the rule :

$$(F \subset \mathbb{1}_{\mathscr{C}}) \mapsto (\mathscr{F}ib(F) \subset \mathscr{F}ib(\mathbb{1}_{\mathscr{C}}) = \mathscr{C}).$$

(ii) For every sieve $\mathscr{S}$ of the category $\mathscr{C}/X$ we may then also consider the presheaf on $\mathscr{C}$

$$h_{\mathscr{S}} := \mathsf{s}_{X!}(F_{\mathscr{S}})$$

where $F_{\mathscr{S}}$ is the presheaf on $\mathscr{C}/X$ defined as in (i), and $\mathsf{s}_{X!} : (\mathscr{C}/X)^\wedge \to \mathscr{C}^\wedge$ is the left adjoint to the functor $\mathsf{s}_X^\wedge$ induced by the source functor $\mathsf{s}_X : \mathscr{C}/X \to \mathscr{C}$ (proposition 1.4.13(vi.a)). By proposition 1.4.13(vi.b,c), the presheaf $h_{\mathscr{S}}$ is a subobject of h_X (notation of (1.2.4)), and by inspecting (1.4.11) we see that

$$h_{\mathscr{S}}(Y) = \{f \in \mathrm{Hom}_{\mathscr{C}}(Y, X) \mid (Y, f) \in \mathrm{Ob}(\mathscr{S})\} \qquad \text{for every } Y \in \mathrm{Ob}(\mathscr{C}).$$

For a given morphism $f : Y' \to Y$ in $\mathscr{C}$, the map $h_{\mathscr{S}}(f)$ is just the restriction of $\mathrm{Hom}_{\mathscr{C}}(f, X)$. By the same token, it also follows that the rule $\mathscr{S} \mapsto h_{\mathscr{S}}$ sets up a natural bijection between the subobjects of h_X in $\mathscr{C}^\wedge$ and the sieves of $\mathscr{C}/X$. The inverse mapping is given by the rule :

$$(F \subset h_X) \mapsto (\mathscr{F}ib(F) \subset \mathscr{F}ib(h_X) = \mathscr{C}/X)$$

(see example 3.1.16(i)). Especially, the restriction $\mathscr{S} \to \mathscr{C}$ of the source functor $\mathsf{s}_X : \mathscr{C}/X \to \mathscr{C}$ is a fibration, and example 3.1.16(i) generalizes to a natural isomorphism of $\mathscr{C}$-fibrations :

$$\mathscr{F}ib(h_{\mathscr{S}}) \xrightarrow{\sim} \mathscr{S}.$$

Combining with lemma 1.4.8, we also deduce a natural isomorphism in $\mathscr{C}^\wedge$:

$$\operatorname*{colim}_{\mathscr{S}} h_{\mathscr{C}} \circ \mathsf{s}_{\mathscr{S}} \xrightarrow{\sim} h_{\mathscr{S}} \tag{3.5.3}$$

where $\mathsf{s}_{\mathscr{S}} : \mathscr{S} \to \mathscr{C}$ is the restriction of the source functor s_X of (1.1.24).

(iii) Let $S := \{X_i \to X \mid i \in I\}$ be any family of morphisms of $\mathscr{C}$. Then S generates a given sieve $\mathscr{S}$ of $\mathscr{C}/X$ if and only if :

$$h_{\mathscr{S}} = \bigcup_{i \in I} \mathrm{Im}(h_{X_i} \to h_X).$$

(Notice that the above union is well defined even in case I is not small.)

(iv) Moreover, if S as in (iii) generates $\mathscr{S}$ and $f : Y \to X$ is any morphism such that the fibre product $Y_i := Y \times_X X_i$ is representable in $\mathscr{C}$ for every $i \in I$, then $\mathscr{S} \times_X f$ is the sieve generated by the family of induced projections $\{Y_i \to Y \mid i \in I\} \subset \mathrm{Ob}(\mathscr{C}/Y)$.

(v) Furthermore, let $f : Y \to X$ be any morphism of $\mathscr{C}$; it is easily seen that the correspondence of (ii) induces a natural identification of subobjects of h_Y:

$$h_{\mathscr{S} \times_X f} = h_{\mathscr{S}} \times_{h_X} h_Y \qquad \text{for every sieve } \mathscr{S} \text{ of } \mathscr{C}/X.$$

Since the Yoneda embedding is fully faithful, we may sometimes abuse notation, to identify X with the corresponding representable presheaf h_X; then we may write $h_{\mathscr{S} \times_X f} = h_{\mathscr{S}} \times_X Y$. In view of (ii) we deduce an isomorphism of fibred $\mathscr{C}$-categories $\mathscr{F}ib(h_{\mathscr{S}} \times_X Y) \xrightarrow{\sim} \mathscr{S} \times_X f$. On the other hand, let $\pi : h_{\mathscr{S}} \times_X Y \to h_{\mathscr{S}}$ be the natural projection; a direct inspection shows that the foregoing isomorphisms fit into the commutative diagram :

$$\begin{array}{ccc} \mathscr{F}ib(h_{\mathscr{S}} \times_X Y) & \xrightarrow{\sim} & \mathscr{S} \times_X f \\ {\scriptstyle \mathscr{F}ib(\pi)} \downarrow & & \downarrow {\scriptstyle f_{*|\mathscr{S}}} \\ \mathscr{F}ib(h_{\mathscr{S}}) & \xrightarrow{\sim} & \mathscr{S} \end{array}$$

where the right vertical arrow is the restriction of the functor $f_* : \mathscr{C}/Y \to \mathscr{C}/X$ (details left to the reader). (Especially, $f_{*|\mathscr{S}}$ is a $\mathscr{C}$-cartesian functor, but this assertion is trivial, since every morphism of $\mathscr{C}/X$ is $\mathscr{C}$-cartesian).

3.5.4. Let $\mathscr{C}$ be a small category, and $\mathscr{S}$ the sieve of $\mathscr{C}$ generated by a subset $S \subset \mathrm{Ob}(\mathscr{C})$. Say that $S = \{S_i \mid i \in I\}$ for a small set I; for every $i \in I$ there is a faithful embedding $\varepsilon_i : \mathscr{C}/S_i \to \mathscr{S}$, and for every pair $(i, j) \in I \times I$, we define

$$\mathscr{C}/S_{ij} := \mathscr{C}/S_i \times_{(\varepsilon_i,\varepsilon_j)} \mathscr{C}/S_j$$

(notation of example 1.2.25(i)). Hence, the objects of $\mathscr{C}/S_{ij}$ are all the triples (X, g_i, g_j), where $X \in \mathrm{Ob}(\mathscr{C})$ and $g_l \in \mathrm{Hom}_{\mathscr{C}}(X, S_l)$ for $l = i, j$. The natural projections :

$$\pi^1_{ij*} : \mathscr{C}/S_{ij} \to \mathscr{C}/S_i \qquad \pi^0_{ij*} : \mathscr{C}/S_{ij} \to \mathscr{C}/S_j$$

are faithful embeddings. We deduce a natural diagram of categories :

$$\coprod_{(i,j)\in I\times I} \mathscr{C}/S_{ij} \overset{\partial_0}{\underset{\partial_1}{\rightrightarrows}} \coprod_{i\in I} \mathscr{C}/S_i \xrightarrow{\varepsilon} \mathscr{S}$$

where :

$$\partial_0 := \coprod_{(i,j)\in I\times I} \pi^0_{ij*} \qquad \partial_1 := \coprod_{(i,j)\in I\times I} \pi^1_{ij*} \qquad \varepsilon := \coprod_{i\in I} \varepsilon_i.$$

Remark 3.5.5. Notice that, under the current assumptions, the product $S_{ij} := S_i \times S_j$ is not necessarily representable in $\mathscr{C}$. In case it is, we may consider another category, also denoted $\mathscr{C}/S_{ij}$, namely the category of S_{ij}-objects of $\mathscr{C}$ (as in (1.1.24)). The latter is naturally isomorphic to the category with the same name introduced in (3.5.4). Moreover, under this natural isomorphism, the projections π^0_{ij*} and π^1_{ij*} are identified with the functors induced by the natural morphisms $\pi^0_{ij} : S_{ij} \to S_j$ and respectively $\pi^1_{ij} : S_{ij} \to S_i$. Hence, in this case, the notation of (3.5.4) is compatible with (1.1.25).

Lemma 3.5.6. *With the notation of* (3.5.4)*, the functor* ε *induces an isomorphism between* $\mathscr{S}$ *and the coequalizer (in the category* $\mathbf{Cat}$*) of the pair of functors* (∂_0, ∂_1)*.*

Proof. Let $\mathscr{A}$ be any other object of $\mathbf{Cat}$, and $F : \coprod_{i\in I} \mathscr{C}/S_i \to \mathscr{A}$ a functor such that $F \circ \partial_0 = F \circ \partial_1$. We have to show that F factors uniquely through ε. To this aim, we construct explicitly a functor $G : \mathscr{S} \to \mathscr{A}$ such that $G \circ \varepsilon = F$. First of all, by the universal property of the coproduct, F is the same as a family of functors $(F_i : \mathscr{C}/S_i \to \mathscr{A} \mid i \in I)$, and the assumption on F amounts to the system of identities :

$$F_i \circ \pi^1_{ij*} = F_j \circ \pi^0_{ij*} \qquad \text{for every } i, j \in I. \tag{3.5.7}$$

Hence, let $X \in \mathrm{Ob}(\mathscr{S})$; by assumption there exist $i \in I$ and a morphism $f : X \to S_i$ in $\mathscr{C}$, so we may set $GX := F_i f$. In case $g : X \to S_j$ is another morphism in $\mathscr{C}$, we deduce an object $h := (X, f, g) \in \mathrm{Ob}(\mathscr{C}/S_{ij})$, so $f = \pi^1_{ij*}h$ and $g = \pi^0_{ij*}h$; then (3.5.7) shows that $F_i f = F_j g$, *i.e.* GX is well-defined.

Next, let $\varphi : X \to Y$ be any morphism in $\mathscr{S}$; choose $i \in I$ and a morphism $f_Y : Y \to S_i$, and set $f_X := f_Y \circ \varphi$. We let $G\varphi := F_i(\varphi : f_X \to f_Y)$. Arguing as in the foregoing, one verifies easily that $G\varphi$ is independent of all the choices, and then it follows easily that $G(\psi \circ \varphi) = G\psi \circ G\varphi$ for every other morphism $\psi : Y \to Z$ in $\mathscr{S}$. It is also clear that $G\mathbf{1}_X = \mathbf{1}_{GX}$, whence the contention. □

3.5.8. In the situation of (3.5.4), suppose that the set of generators S is the whole of $\mathrm{Ob}(\mathscr{S})$; in this case, the augmentation ε can also be used to produce the following 2-categorical presentation of $\mathscr{S}$, which upgrades the isomorphism (3.5.3). Consider the strict pseudo-functor

$$G_{\mathscr{S}} : \mathscr{S} \to \mathsf{Fib}(\mathscr{C}) \quad : \quad Y \mapsto \mathscr{C}/Y \qquad (Z \xrightarrow{f} Y) \mapsto (\mathscr{C}/Z \xrightarrow{f_*} \mathscr{C}/Y).$$

We have a natural strict pseudo-cocone

$$\widehat{\varepsilon} : G_{\mathscr{S}} \Rightarrow \mathsf{F}_{\mathscr{S}}$$

where $\mathsf{F}_{\mathscr{S}}$ is the constant pseudo-functor $\mathscr{S} \to \mathsf{Fib}(\mathscr{C})$ with value $\mathscr{S}$, and $\widehat{\varepsilon}_X : \mathscr{C}/X \to \mathscr{S}$ is the faithful embedding as in (3.5.4), for every $X \in \mathrm{Ob}(\mathscr{S})$. We may then state :

Lemma 3.5.9. *The pseudo-cocone $\widehat{\varepsilon}$ induces an equivalence of fibrations over $\mathscr{C}$:*

$$2\text{-}\operatorname*{colim}_{\mathscr{S}} G_{\mathscr{S}} \xrightarrow{\sim} \mathscr{S}.$$

Proof. By theorem 3.3.25(ii), the 2-colimits are computed fibrewise in the 2-category $\mathsf{Fib}(\mathscr{C})$, so we are reduced to checking that $\widehat{\varepsilon}$ induces an equivalence of categories

$$2\text{-}\operatorname*{colim}_{\mathscr{S}} G_{\mathscr{S},X} \xrightarrow{\sim} \mathscr{S}_X \qquad \text{for every } X \in \mathrm{Ob}(\mathscr{C})$$

where $\mathscr{S}_X$ is the fibre category over X of the fibration $\mathscr{S} \to \mathscr{C}$, and $G_{\mathscr{S},X} : \mathscr{S} \to \mathbf{Cat}$ is the strict pseudo-functor that assigns to every $Y \in \mathrm{Ob}(\mathscr{S})$ the fibre category $(\mathscr{C}/Y)_X$ of the fibration $\mathscr{C}/Y \to \mathscr{C}$ (for every morphism $f : Z \to Y$ in $\mathscr{S}$, the corresponding functor $G_{\mathscr{S},X}(f)$ is the restriction of $G_{\mathscr{S}}(f) = f_*$). However, if $X \notin \mathrm{Ob}(\mathscr{S})$, the category $\mathscr{S}_X$ is empty, and $G_{\mathscr{S},X}$ is the constant pseudo-functor with value equal to the empty category, so the assertion is clear in this case. Suppose then that $X \in \mathrm{Ob}(\mathscr{S})$, in which case $\mathscr{S}_X$ is the category whose unique object is X and whose unique morphism is $\mathbf{1}_X$. Also, for every $Y \in \mathrm{Ob}(\mathscr{S})$, the category $(\mathscr{C}/Y)_X$ is discrete, with set of objects given by $\mathrm{Hom}_{\mathscr{C}}(X, Y)$. According to example 3.3.13(i), the strong 2-colimit of $G_{\mathscr{S},X}$ is represented by the category $\mathscr{F}ib(G_{\mathscr{S},X})[\Sigma^{-1}]$, where Σ is the set of cartesian morphisms of the fibration $\mathscr{F}ib(G_{\mathscr{S},X}) \to \mathscr{S}^o$. Now, it is easily seen that $\mathscr{F}ib(G_{\mathscr{S},X}) = (X/\mathscr{S})^o = \mathscr{S}^o/X^o$, with structure functor given by the usual source functor $\mathscr{S}^o/X^o \to \mathscr{S}^o$. Hence, Σ is the set of all morphisms of $\mathscr{C}^o/X^o$, and since $\mathscr{C}^o/X^o$ has the final object $\mathbf{1}_{X^o}$, the assertion follows easily from example 1.6.12. □

Definition 3.5.10. Let $\varphi : \mathscr{A} \to \mathscr{B}$ be a fibration, $B \in \mathrm{Ob}(\mathscr{B})$ an object, $\mathscr{S} \subset \mathscr{B}/B$ a sieve, $i \in \{0, 1, 2\}$, and denote by $\iota_{\mathscr{S}} : \mathscr{S} \to \mathscr{B}/B$ the fully faithful embedding.

(i) We say that $\mathscr{S}$ is a *sieve of φ-i-descent*, if the restriction functor :

$$\mathsf{Cart}_{\mathscr{B}}(\iota_{\mathscr{S}}, \mathscr{A}) : \mathscr{A}(B) \to \mathsf{Cart}_{\mathscr{B}}(\mathscr{S}, \mathscr{A})$$

is i-faithful (see remark 1.1.5).

(ii) We say that $\mathscr{S}$ is a *sieve of universal φ-i-descent* if the sieve $\mathscr{S} \times_B f$ is of φ-i-descent for every morphism $f : B' \to B$ of $\mathscr{B}$ (notation of definition 3.5.1(iv)).

(iii) Let $f : B' \to B$ be a morphism in $\mathscr{B}$. We say that f is a *morphism of φ-i-descent* (resp. a *morphism of universal φ-i-descent*), if the sieve generated by $\{f\}$ is of φ-i-descent (resp. of universal φ-i-descent).

Remark 3.5.11. In the situation of definition 3.5.10, suppose that $\mathscr{S}$ is the sieve generated by a set of objects $\{S_i \to B \mid i \in I\} \subset \mathrm{Ob}(\mathscr{B}/B)$. There follows a natural diagram of categories (notation of (3.5.4)) :

$$\mathsf{Cart}_{\mathscr{B}}(\mathscr{S},\mathscr{A}) \xrightarrow{\varepsilon^*} \prod_{i\in I} \mathscr{A}(S_i) \underset{\partial_1^*}{\overset{\partial_0^*}{\rightrightarrows}} \prod_{(i,j)\in I\times I} \mathsf{Cart}_{\mathscr{B}}(\mathscr{B}/S_{ij},\mathscr{A})$$

where $\varepsilon^* := \mathsf{Cart}_{\mathscr{B}}(\varepsilon,\mathscr{A})$ and $\partial_i^* := \mathsf{Cart}_{\mathscr{B}}(\partial_i,\mathscr{A})$, for $i = 0, 1$. With this notation, lemma 3.5.6 easily implies that ε^* induces an isomorphism between $\mathsf{Cart}_{\mathscr{B}}(\mathscr{S},\mathscr{A})$ and the equalizer (in the category $\mathbf{Cat}$) of the pair of functors $(\partial_0^*, \partial_1^*)$.

3.5.12. We would like to exploit the presentation of $\mathsf{Cart}_{\mathscr{B}}(\mathscr{S},\mathscr{A})$ in remark 3.5.11, in order to translate definition 3.5.10 in terms of the fibre categories $\mathscr{A}_{S_i}$ and $\mathscr{A}_{S_{ij}}$. The problem is that such a translation must be carried out via a pseudo-natural equivalence (namely ev), and such equivalences do not respect a presentation as above in terms of equalizers in the category $\mathbf{Cat}$. What we need is to upgrade our presentation of $\mathscr{S}$ to a new one, which is preserved by pseudo-natural transformations. This is achieved as follows. Resume the general situation of (3.5.4). For every $i, j, k \in I$, set $\mathscr{C}/S_{ijk} := \mathscr{C}/S_{ij} \times_{\mathscr{C}} \mathscr{C}/S_k$. We have a natural diagram of categories :

$$\coprod_{(i,j,k)\in I^3} \mathscr{C}/S_{ijk} \overset{\partial_0'}{\underset{\partial_2'}{\overset{\partial_1'}{\rightrightarrows\!\!\!\to}}} \coprod_{(i,j)\in I^2} \mathscr{C}/S_{ij} \underset{\partial_1}{\overset{\partial_0}{\rightrightarrows}} \coprod_{i\in I} \mathscr{C}/S_i \xrightarrow{\varepsilon} \mathscr{S} \qquad (3.5.13)$$

where ∂_0' is the coproduct of the natural projections $\pi^0_{ijk*} : \mathscr{C}/S_{ijk} \to \mathscr{C}/S_{jk}$ for every $i, j, k \in I$, and likewise ∂_1' (resp. ∂_2') is the coproduct of the projections $\pi^1_{ijk*} : \mathscr{C}/S_{ijk} \to \mathscr{C}/S_{ik}$ (resp. $\pi^2_{ijk*} : \mathscr{C}/S_{ijk} \to \mathscr{C}/S_{ij}$). We can view (3.5.13) as an *augmented* 2*-truncated semi-simplicial object* in $\mathsf{Fib}(\mathscr{C})$, *i.e.* a functor :

$$F_{\mathscr{S}} : \Sigma_2^{+o} \to \mathsf{Fib}(\mathscr{C})$$

from the opposite of the category Σ_2^+ whose objects are the ordered sets $[-1]$, $[0]$, $[1]$ and $[2]$, and whose morphisms are the non-decreasing injective maps (this is a subcategory of the category $\Delta_2^\wedge$ of definition 7.4.1(iii)).

Remark 3.5.14. Suppose that finite products are representable in $\mathscr{B}$, and for every $i, j, k \in I$, set $S_{ij} := S_i \times S_j$, and $S_{ijk} := S_{ij} \times S_k$. Just as in remark 3.5.5, the category $\mathscr{C}/S_{ijk}$ of S_{ijk}-objects of $\mathscr{C}$ is naturally isomorphic to the category with the same name introduced in (3.5.12), and under this isomorphism, the functors π^0_{ijk*} are identified with the functors arising from the natural projections $\pi^0_{ijk} : S_{ijk} \to S_{jk}$ (and likewise for π^1_{ijk*} and π^2_{ijk*}).

With this notation, denote by Σ_2 the full subcategory of Σ_2^+ whose objects are the non-empty sets; we have the following 2-category analogue of lemma 3.5.6 :

Proposition 3.5.15. *The augmentation ε induces an equivalence of categories :*

$$2\text{-}\operatorname*{colim}_{\Sigma_2^o} F_{\mathscr{S}} \xrightarrow{\sim} \mathscr{S}.$$

Proof. By theorem 3.3.25(ii), the 2-colimits are computed fibrewise in the 2-category $\mathsf{Fib}(\mathscr{C})$, so we are reduced to checking that ε induces an equivalence of categories

$$2\text{-}\operatorname*{colim}_{\Sigma_2^o} F_{\mathscr{S},X} \xrightarrow{\sim} \mathscr{S}_X \qquad \text{for every } X \in \mathrm{Ob}(\mathscr{C})$$

where $\mathscr{S}_X$ is the fibre category over X of the fibration $\mathscr{S} \to \mathscr{C}$, and $F_{\mathscr{S},X} := \mathsf{fib}_X \circ F_{\mathscr{S}}$, where $\mathsf{fib}_X : \mathsf{Fib}(\mathscr{C}) \to \mathbf{Cat}$ is defined as in (3.3.24). Now, if $X \notin \mathrm{Ob}(\mathscr{S})$, the category $\mathscr{S}_X$ is empty, and $\mathsf{fib}_X \circ F_{\mathscr{S}}$ is the constant pseudo-functor with value equal to the empty category, so the assertion trivially holds in this case. Suppose then that $X \in \mathrm{Ob}(\mathscr{S})$, in which case $\mathscr{S}_X$ is the category whose unique object is X and whose unique morphism is $\mathbf{1}_X$. Set

$$T := \coprod_{i\in I} \mathrm{Hom}_{\mathscr{C}}(X, S_i)$$

and notice that $F_{\mathscr{S},J}[0], F_{\mathscr{S},J}[1]$ and $F_{\mathscr{S},J}[2]$ are the discrete categories whose sets of objects are respectively T, $T\times T$ and $T\times T\times T$. Moreover, the strict pseudo-functor $F_{\mathscr{S},X}$ corresponds to the 2-truncated semi-simplicial diagram of sets :

$$T\times T\times T \underset{\partial'_2}{\overset{\partial'_0}{\rightrightarrows}}\!\!\xrightarrow{\partial'_1} T\times T \underset{\partial_1}{\overset{\partial_0}{\rightrightarrows}} T \tag{3.5.16}$$

whose maps ∂_i and ∂'_i are the natural projections. To conclude, it therefore suffices to show :

Claim 3.5.17. For every set $T\neq\varnothing$, the 2-colimit in the 2-category $\mathbf{Cat}$ of the system of discrete categories (3.5.16) is represented by the discrete category with one object.

Proof of the claim. The diagram (3.5.16) can be regarded as a presheaf $T_\bullet$ on the category Σ_2, and according to example 3.3.13(i), the strong 2-colimit of $T_\bullet$ is represented by the category $\mathscr{T} := \mathscr{F}ib(T_\bullet)[\Lambda^{-1}]$, where Λ is the set of cartesian morphisms of the fibration $\mathscr{F}ib(T_\bullet)\to\Sigma_2$; but every morphism of $\mathscr{F}ib(T_\bullet)$ is cartesian (see (3.1.15)). Denote by $\mathscr{T}_0$ the category with $\mathrm{Ob}(\mathscr{T}_0) = T$, and such that $\mathrm{Hom}_{\mathscr{T}_0}(t,t')$ contains a unique element $\tau_{t,t'}$, for every $t,t'\in T$. We define a functor $q:\mathscr{T}_0\to\mathscr{T}$ as follows. Recall that the objects of $\mathscr{F}ib(T_\bullet)$ are the pairs $(j,\underline{t})$, where $j\in\{0,1,2\}$ and $\underline{t}\in T^{j+1}$. For every $\underline{t} := (t_0,t_1)\in T\times T$ we have morphisms $(0,t_1)\xleftarrow{\delta_{0,\underline{t}}}(1,\underline{t})\xrightarrow{\delta_{1,\underline{t}}}(0,t_0)$ in $\mathscr{F}ib(T_\bullet)$, and we set

$$q(t_0) := (0,t_0) \quad\text{and}\quad q(\tau_{t_0,t_1}) := [\delta_{0,(t_0,t_1)}]\circ[\delta_{1,(t_0,t_1)}]^{-1} \qquad \text{for every } t_0,t_1\in T$$

where we denote by $[f]$ the image in $\mathscr{T}$ of every morphism f of $\mathscr{F}ib(T_\bullet)$. Indeed, for every $\underline{t} := (t_0,t_1,t_2)\in T^3$ and $j = 0,1,2$, we have as well morphisms $\delta'_{j,\underline{t}} : (2,\underline{t})\to(1,\partial'_j(\underline{t}))$ in $\mathscr{F}ib(T_\bullet)$, and we may compute :

$$\begin{aligned} q(\tau_{t_1,t_2})\circ q(\tau_{t_0,t_1}) &= [\delta_{0,\partial'_0\underline{t}}]\circ[\delta_{1,\partial'_0\underline{t}}]^{-1}\circ[\delta_{0,\partial'_2\underline{t}}]\circ[\delta_{1,\partial'_2\underline{t}}]^{-1} \\ &= [\delta_{0,\partial'_0\underline{t}}]\circ[\delta'_{0,\underline{t}}]\circ[\delta'_{0,\underline{t}}]^{-1}\circ[\delta_{1,\partial'_0\underline{t}}]^{-1}\circ[\delta_{0,\partial'_2\underline{t}}]\circ[\delta'_{2,\underline{t}}]\circ[\delta'_{2,\underline{t}}]^{-1}\circ[\delta_{1,\partial'_2\underline{t}}]^{-1} \\ &= [\delta_{0,\partial'_0\underline{t}}]\circ[\delta'_{0,\underline{t}}]\circ[\delta'_{0,\underline{t}}]^{-1}\circ[\delta_{1,\partial'_0\underline{t}}]^{-1}\circ[\delta_{1,\partial'_0\underline{t}}]\circ[\delta'_{0,\underline{t}}]\circ[\delta'_{1,\underline{t}}]^{-1}\circ[\delta_{1,\partial'_1\underline{t}}]^{-1} \\ &= [\delta_{0,\partial'_0\underline{t}}]\circ[\delta'_{0,\underline{t}}]\circ[\delta'_{1,\underline{t}}]^{-1}\circ[\delta_{1,\partial'_1\underline{t}}]^{-1} \\ &= [\delta_{0,\partial'_1\underline{t}}]\circ[\delta'_{1,\underline{t}}]\circ[\delta'_{1,\underline{t}}]^{-1}\circ[\delta_{1,\partial'_1\underline{t}}]^{-1} \\ &= q(\tau_{t_0,t_2}) \end{aligned}$$

as required. Next, let also $q':\mathscr{T}\to\mathscr{T}_0$ be the unique functor given by the rules :

$$(0,t_0)\mapsto t_0 \qquad (1,(t_0,t_1))\mapsto t_0 \qquad (2,(t_0,t_1,t_2))\mapsto t_0 \qquad \text{for every } t_0,t_1,t_2\in T.$$

Obviously $q'\circ q = \mathbf{1}_{\mathscr{T}_0}$. Let $L:\mathscr{F}ib(T_\bullet)\to\mathscr{T}$ be the localization; to conclude, it suffices to exhibit an isomorphism of functors $\mathbf{1}_{\mathscr{T}}\xrightarrow{\sim} q\circ q'$, and corollary 1.6.11 further reduces to exhibiting an isomorphism $\omega : L\xrightarrow{\sim} q\circ q'\circ L$. We define ω by the rules :

$$(0,t_0)\mapsto\mathbf{1}_{(0,t_0)} \qquad (1,(t_0,t_1))\mapsto[\delta_{1,(t_0,t_1)}] \qquad (2,(t_0,t_1,t_2))\mapsto[\delta_{1,(t_0,t_2)}\circ\delta'_{2,(t_0,t_1,t_2)}]$$

for every $t_0,t_1,t_2\in T$. The naturality of ω follows by a straightforward verification. □

3.5.18. Resume the situation of remark 3.5.11, and notice that all the categories appearing in (3.5.13) are fibred over $\mathscr{B}$: indeed, every morphism in each of these categories is cartesian, hence all the functors appearing in (3.5.13) are cartesian. Let us consider now the functor :

$$\mathsf{Cart}_{\mathscr{B}}(-,\mathscr{A}) : (\mathbf{Cat}/\mathscr{B})^o \to \mathbf{Cat}$$

that assigns to every $\mathscr{B}$-category $\mathscr{C}$ the category $\mathsf{Cart}_{\mathscr{B}}(\mathscr{C},\mathscr{A})$. With the notation of remark 3.5.11, we deduce a functor :

$$\mathsf{Cart}_{\mathscr{B}}(F_{\mathscr{S}},\mathscr{A}) : \Sigma_2^+ \to \mathbf{Cat}$$

and in light of the foregoing observations, proposition 3.5.15 easily implies that $\mathsf{Cart}_{\mathscr{B}}(\varepsilon,\mathscr{A})$ induces an equivalence of categories :

$$2\text{-}\lim_{\Sigma_2} \mathsf{Cart}_{\mathscr{B}}(F_{\mathscr{S}},\mathscr{A}) \xrightarrow{\sim} \mathsf{Cart}_{\mathscr{B}}(\mathscr{S},\mathscr{A}). \qquad (3.5.19)$$

Next, suppose that the fibre products $S_{ij} := S_i \times S_j$ and $S_{ijk} := S_{ij} \times S_k$ are representable in $\mathscr{B}$ (see remark 3.5.14); in this case, we may compose with the pseudo-equivalence $\mathsf{ev}^{\mathscr{A}}_{|\bullet}$ of remark 3.2.9(i) : combining with lemma 2.5.3 we finally obtain an equivalence between the category $\mathsf{Cart}_{\mathscr{B}}(\mathscr{S},\mathscr{A})$, and the 2-limit of the pseudo-functor $\mathsf{d} := \mathsf{ev} \circ \mathsf{Cart}_{\mathscr{B}}(F_{\mathscr{S}},\mathscr{A}) : \Sigma_2 \to \mathbf{Cat}$:

$$\prod_{i\in I} \mathscr{A}_{S_i} \overset{\partial^0}{\underset{\partial^1}{\rightrightarrows}} \prod_{(i,j)\in I^2} \mathscr{A}_{S_{ij}} \overset{\partial^0,\ \partial^1}{\underset{\partial^2}{\rightrightarrows}} \prod_{(i,j,k)\in I^3} \mathscr{A}_{S_{ijk}}.$$

Of course, the coface operators ∂^s on $\prod_{i\in I} \mathscr{A}_{S_i}$ decompose as products of pull-back functors:

$$\pi^{0*}_{ij} : \mathscr{A}_{S_j} \to \mathscr{A}_{S_{ij}} \qquad \pi^{1*}_{ij} : \mathscr{A}_{S_i} \to \mathscr{A}_{S_{ij}}$$

attached – via the chosen cleavage c of φ – to the projections $\pi^0_{ij} : S_{ij} \to S_j$ and $\pi^1_{ij} : S_{ij} \to S_i$ (and likewise for the components π^{t*}_{ijk} of the other coface operators).

3.5.20. By inspecting the proof of theorem 3.3.9, we may give the following explicit description of this 2-limit. Namely, it is the category whose objects are the data

$$\underline{X} := (X_i, X_{ij}, X_{ijk}, \xi^u_i, \xi^s_{ij}, \xi^t_{ijk} \mid i,j,k \in I; s \in \{0,1\}; u,t \in \{0,1,2\})$$

where :

$$X_i \in \mathrm{Ob}(\mathscr{A}_{S_i}) \quad X_{ij} \in \mathrm{Ob}(\mathscr{A}_{S_{ij}}) \quad X_{ijk} \in \mathrm{Ob}(\mathscr{A}_{S_{ijk}}) \qquad \text{for every } i,j,k \in I$$

and for every $i,j,k \in I$:

$$\begin{array}{lll} \xi^0_i : (\pi^1_{ik}\pi^1_{ijk})^* X_i \xrightarrow{\sim} X_{ijk} & \xi^0_{ij} : \pi^{0*}_{ij} X_j \xrightarrow{\sim} X_{ij} & \xi^0_{ijk} : \pi^{0*}_{ijk} X_{jk} \xrightarrow{\sim} X_{ijk} \\ \xi^1_j : (\pi^1_{jk}\pi^0_{ijk})^* X_j \xrightarrow{\sim} X_{ijk} & \xi^1_{ij} : \pi^{1*}_{ij} X_i \xrightarrow{\sim} X_{ij} & \xi^1_{ijk} : \pi^{1*}_{ijk} X_{ik} \xrightarrow{\sim} X_{ijk} \\ \xi^2_k : (\pi^0_{jk}\pi^0_{ijk})^* X_k \xrightarrow{\sim} X_{ijk} & & \xi^2_{ijk} : \pi^{2*}_{ijk} X_{ij} \xrightarrow{\sim} X_{ijk} \end{array}$$

are isomorphisms related by the cosimplicial identities :

$$\begin{array}{ll} \xi^0_{ijk} \circ \pi^{0*}_{ijk}\xi^0_{jk} = \xi^2_k \circ \gamma^{00}_X & \xi^1_{ijk} \circ \pi^{1*}_{ijk}\xi^0_{ik} = \xi^2_k \circ \gamma^{01}_X \\ \xi^2_{ijk} \circ \pi^{2*}_{ijk}\xi^0_{ij} = \xi^1_j \circ \gamma^{02}_X & \xi^0_{ijk} \circ \pi^{0*}_{ijk}\xi^1_{jk} = \xi^1_j \circ \gamma^{10}_X \\ \xi^2_{ijk} \circ \pi^{2*}_{ijk}\xi^1_{ij} = \xi^0_i \circ \gamma^{12}_X & \xi^1_{ijk} \circ \pi^{1*}_{ijk}\xi^1_{ik} = \xi^0_i \circ \gamma^{11}_X \end{array}$$

where $\gamma^{st} := \gamma_{\mathsf{d}(\partial^s),\mathsf{d}(\partial^t)} : \mathsf{d}(\partial^s) \circ \mathsf{d}(\partial^t) \Rightarrow \mathsf{d}(\partial^s \circ \partial^t)$ denotes the coherence constraint of the cleavage c, for any pair of arrows (∂^s, ∂^t) in the category Σ_2. The morphisms $\underline{X} \to \underline{Y}$ in this category are the systems of morphisms :

$$(X_i \to Y_i, X_{ij} \to Y_{ij}, X_{ijk} \to Y_{ijk} \mid i,j,k \in I)$$

that are compatible in the obvious way with the various isomorphisms. However, one may argue as in the proof of proposition 3.5.15, to replace this category by an equivalent one which admits a handier description : given a datum $\underline{X}$, one can make up an isomorphic datum $\underline{X}^* := (X_i, X^*_{ij}, X^*_{ijk}, \eta_i, \eta_{ij}, \eta_{ijk})$, by the rule :

$$X^*_{ij} := \pi^{1*}_{ij} X_i \qquad X^*_{ijk} := (\pi^1_{ik} \circ \pi^1_{ijk})^* X_i$$

$$\begin{array}{lll} \eta^0_i := \mathbf{1} & \eta^1_{ij} := \mathbf{1} & \eta^0_{ijk} := \eta^1_j \circ \gamma^{10}_X \\ \eta^1_j := (\xi^0_i)^{-1} \circ \xi^1_j & \eta^0_{ij} := (\xi^1_{ij})^{-1} \circ \xi^0_{ij} & \eta^1_{ijk} := \gamma^{11}_X \\ \eta^2_k := (\xi^0_i)^{-1} \circ \xi^2_k & & \eta^2_{ijk} := \gamma^{12}_X. \end{array}$$

The cosimplicial identities for this new object are subsumed into a single cocycle identity for $\omega_{ij} := \eta^0_{ij}$. Summing up, we arrive at the following description of our 2-limit :

- The objects are all the systems $\underline{X} := (X_i, \omega^X_{ij} \mid i, j \in I)$ where $X_i \in \mathrm{Ob}(\mathscr{A}_{S_i})$ for every $i \in I$, and
$$\omega^X_{ij} : \pi^{0*}_{ij} X_j \xrightarrow{\sim} \pi^{1*}_{ij} X_i$$
is an isomorphism in $\mathscr{A}_{S_{ij}}$, for every $i, j \in I$, fulfilling the cocycle identity :
$$p^{2*}_{ijk}\omega^X_{ij} \circ p^{0*}_{ijk}\omega^X_{jk} = p^{1*}_{ijk}\omega^X_{ik} \qquad \text{for every } i, j, k \in I$$
where, for every $i, j, k \in I$, and $t = 0, 1, 2$ we have set :
$$p^{t*}_{ijk}\omega^X_{\bullet\bullet} := \gamma^{1t}_X \circ \pi^{t*}_{ijk}\omega^X_{\bullet\bullet} \circ (\gamma^{0t}_X)^{-1}.$$
- The morphisms $\underline{X} \to \underline{Y}$ are the systems of morphisms $(f_i : X_i \to Y_i \mid i \in I)$ with :

(3.5.21) $$\omega^Y_{ij} \circ \pi^{0*}_{ij} f_j = \pi^{1*}_{ij} f_i \circ \omega^X_{ij} \qquad \text{for every } i, j \in I.$$

3.5.22. We shall call any pair $(X_\bullet, \omega_\bullet)$ of the above form, a *descent datum for the fibration* φ, *relative to the family* $\underline{S} := (\pi_i : S_i \to B \mid i \in I)$ *and the cleavage* c. The category of such descent data shall be denoted:

$$\mathsf{Desc}(\varphi, \underline{S}, \mathsf{c}).$$

Sometimes we may also denote it by $\mathsf{Desc}(\mathscr{A}, \underline{S}, \mathsf{c})$, if the notation is not ambiguous. Of course, two different choices of cleavage lead to equivalent categories of descent data, so usually we omit mentioning explicitly c, and write simply $\mathsf{Desc}(\varphi, \underline{S})$ or $\mathsf{Desc}(\mathscr{A}, \underline{S})$. The foregoing discussion can be summarized, by saying that there is a commutative diagram of categories :

(3.5.23) $$\begin{array}{ccc} \mathscr{A}(B) & \xrightarrow{\mathsf{Cart}_{\mathscr{B}}(\iota_{\mathscr{S}}, \mathscr{A})} & \mathsf{Cart}_{\mathscr{B}}(\mathscr{S}, \mathscr{A}) \\ {\scriptstyle \mathsf{ev}_B}\downarrow & & \downarrow{\scriptstyle \delta_{\underline{S}}} \\ \mathscr{A}_B & \xrightarrow{\rho_{\underline{S}}} & \mathsf{Desc}(\varphi, \underline{S}, \mathsf{c}) \end{array}$$

whose vertical arrows are equivalences, and where $\rho_{\underline{S}}$ is determined by c. Explicitly, $\rho_{\underline{S}}$ assigns to every $C \in \mathrm{Ob}(\mathscr{A}_B)$ the pair $(C_\bullet, \omega^C_\bullet)$ where $C_i := \pi^*_i C$, and ω^C_{ij} is the composition :

$$\pi^{0*}_{ij} \circ \pi^*_j C \xrightarrow[\sim]{\gamma_{(\pi^0_{ij}, \pi_j)}} (\pi_j \circ \pi^0_{ij})^* C = (\pi_i \circ \pi^1_{ij})^* C \xrightarrow[\sim]{\gamma^{-1}_{(\pi^1_{ij}, \pi_i)}} \pi^{1*}_{ij} \circ \pi^*_i C$$

where $\gamma_{(\pi^1_{ij}, \pi_i)}$ and $\gamma_{(\pi^0_{ij}, \pi_j)}$ are the coherence constraints for the cleavage c (see (3.1.6)). The descent datum $(X_\bullet, \omega_\bullet)$ is said to be *effective*, if it lies in the essential image of $\rho_{\underline{S}}$.

We also have an obvious functor :

$$\mathsf{p}_{\underline{S}} : \mathsf{Desc}(\varphi, \underline{S}) \to \prod_{i \in I} \mathscr{A}_{S_i} \qquad (X_i, \omega_{ij} \mid i, j \in I) \mapsto (X_i \mid i \in I)$$

such that :

$$\mathsf{p}_{\underline{S}} \circ \rho_{\underline{S}} = \prod_{i \in I} \pi^*_i : \mathscr{A}_B \to \prod_{i \in I} \mathscr{A}_{S_i}.$$

3.5.24. Furthermore, for every morphism $f : B' \to B$ in $\mathscr{B}$, set

$$\underline{S} \times_B f := (\pi_i \times_B B' : S_i \times_B B' \to B' \mid i \in I)$$

which is a generating family for the sieve $\mathscr{S} \times_B f$ (notation of (1.1.24)); then we deduce a pseudo-natural transformation of pseudo-functors $(\mathscr{B}/B)^o \to \mathbf{Cat}$:

$$\rho : \mathsf{c} \circ i_B^o \Rightarrow \mathsf{Desc}(\varphi, \underline{S} \times_B -, \mathsf{c}) \qquad (f : B' \to B) \mapsto \rho_{\underline{S}\times_B f}$$

(where $\mathsf{s}_B : \mathscr{B}/B \to \mathscr{B}$ is the source functor of (1.1.24)) fitting into a commutative diagram :

$$\begin{array}{ccc} \mathscr{A}(-) \circ \mathsf{s}_B^o & \stackrel{\mathsf{Cart}_{\mathscr{B}}(\iota_{\mathscr{S}\times_B -}, \mathscr{A})}{\Longrightarrow} & \mathsf{Cart}_{\mathscr{B}}(\mathscr{S} \times_B -, \mathscr{A}) \\ {\scriptstyle \mathsf{ev}*\mathsf{s}_B^o}\Downarrow & & \Downarrow{\scriptstyle \delta_{\underline{S}\times_B -}} \\ \mathsf{c} \circ \mathsf{s}_B^o & \stackrel{\rho}{\Longrightarrow} & \mathsf{Desc}(\varphi, \underline{S} \times_B -, \mathsf{c}) \end{array}$$

using which, one can figure out the pseudo-functoriality of the rule : $f \mapsto \mathsf{Desc}(\varphi, \underline{S} \times_B f, \mathsf{c})$. Namely, every pair of objects $f : C \to B$ and $f' : C' \to B$, and any morphism $h : C' \to C$ in $\mathscr{B}/B$, yield a commutative diagram :

$$\begin{array}{ccccccc} S_j \times_B C' & \xleftarrow{\widetilde{\pi}^0_{ij}} & S_{ij} \times_B C' & \xrightarrow{\widetilde{\pi}^1_{ij}} & S_i \times_B C' & \xrightarrow{\widetilde{\pi}_i} & C' \\ {\scriptstyle h_j := S_j \times_B h}\downarrow & & {\scriptstyle h_{ij} := S_{ij} \times_B h}\downarrow & & \downarrow{\scriptstyle h_i := S_i \times_B h} & & \downarrow{\scriptstyle h} \\ S_j \times_B C & \xleftarrow[\pi^0_{ij}]{} & S_{ij} \times_B C & \xrightarrow[\pi^1_{ij}]{} & S_i \times_B C & \xrightarrow[\pi_i]{} & C. \end{array}$$

Hence one obtains a functor :

$$\mathsf{Desc}(\varphi, h, \mathsf{c}) : \mathsf{Desc}(\varphi, \underline{S} \times_B f, \mathsf{c}) \to \mathsf{Desc}(\varphi, \underline{S} \times_B f', \mathsf{c})$$

by the rule :

$$(X_i, \omega^X_{ij} \mid i, j \in I) \mapsto (h_i^* X_i, \widetilde{\omega}^X_{ij} \mid i, j \in I)$$

where $\widetilde{\omega}^X_{ij}$ is the isomorphism that makes commute the diagram :

$$\begin{array}{ccccc} h_{ij}^* \pi^{0*}_{ij} X_j & \xrightarrow{\gamma_{(h_{ij}, \pi^0_{ij})}} & (\pi^0_{ij} \circ h_{ij})^* X_j & \xleftarrow{\gamma_{(\widetilde{\pi}^0_{ij}, h_j)}} & \widetilde{\pi}^{0*}_{ij} \circ h_j^* X_j \\ {\scriptstyle h_{ij}^* \omega_{ij}}\downarrow & & & & \downarrow{\scriptstyle \widetilde{\omega}^X_{ij}} \\ h_{ij}^* \pi^{1*}_{ij} X_i & \xrightarrow{\gamma_{(h_{ij}, \pi^1_{ij})}} & (\pi^1_{ij} \circ h_{ij})^* X_i & \xleftarrow{\gamma_{(\widetilde{\pi}^1_{ij}, h_i)}} & \widetilde{\pi}^{1*}_{ij} \circ h_i^* X_i \end{array}$$

and if $f'' : C'' \to B$ is a third object, with a morphism $g : C'' \to C'$, we have a natural isomorphism of functors :

$$\mathsf{Desc}(\varphi, g, \mathsf{c}) \circ \mathsf{Desc}(\varphi, h, \mathsf{c}) \Rightarrow \mathsf{Desc}(\varphi, h \circ g, \mathsf{c})$$

which is induced by the cleavage c, in the obvious fashion.

Theorem 3.5.25. *For $i = 1, 2$, let $\varphi_i : \mathscr{A}_i \to \mathscr{B}$ be two fibrations, $F : \mathscr{A}_1 \to \mathscr{A}_2$ a cartesian functor of $\mathscr{B}$-categories, B an object of $\mathscr{B}$ and $\mathscr{S}$ a sieve of $\mathscr{B}/B$ generated by the family $(S_i \to B \mid i \in I)$. We assume that S_{ij} and S_{ijk} are representable in $\mathscr{B}$, for every $i, j, k \in I$ (see remark* 3.5.14*); then we have :*

(i) *For $n \in \{0, 1, 2\}$ and every $i, j, k \in I$, suppose that*
  (a) *$\mathscr{S}$ is a sieve both of φ_1-n-descent and of φ_2-$(n-1)$-descent.*
  (b) *The restriction $F_i : \varphi_1^{-1} S_i \to \varphi_2^{-1} S_i$ of F is n-faithful.*
  (c) *The restriction $F_{ij} : \varphi_1^{-1} S_{ij} \to \varphi_2^{-1} S_{ij}$ of F is $(n-1)$-faithful.*
  (d) *The restriction $F_{ijk} : \varphi_1^{-1} S_{ijk} \to \varphi_2^{-1} S_{ijk}$ of F is $(n-2)$-faithful.*

Then the restriction $F_B : \varphi_1^{-1} B \to \varphi_2^{-1} B$ of F is n-faithful.

(ii) *Suppose that the functors F_{ij} are fully faithful, and the functors F_{ijk} are faithful, for every $i, j, k \in I$. Then the natural commutative diagram*

$$\begin{array}{ccc} \mathsf{Cart}_{\mathscr{B}}(\mathscr{S}, \mathscr{A}_1) & \xrightarrow{\mathsf{Cart}_{\mathscr{B}}(\mathscr{S}, F)} & \mathsf{Cart}_{\mathscr{B}}(\mathscr{S}, \mathscr{A}_2) \\ \downarrow & & \downarrow \\ \prod_{i \in I} \varphi_1^{-1} S_i & \xrightarrow{\prod_{i \in I} F_i} & \prod_{i \in I} \varphi_2^{-1} S_i \end{array}$$

is 2*-cartesian.*

Proof. (i): In view of theorem 3.2.7, we may assume that both $\mathscr{A}_1$ and $\mathscr{A}_2$ are split fibrations (with a suitable choice of cleavages), and F is a split cartesian functor. Recall that the latter condition means the following. For every morphism $g : X \to Y$ in $\mathscr{B}$, the induced diagram

$$\begin{array}{ccc} \varphi_1^{-1} Y & \xrightarrow{g^*} & \varphi_1^{-1} X \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F} \\ \varphi_2^{-1} Y & \xrightarrow{g^*} & \varphi_2^{-1} X \end{array}$$

commutes (where the horizontal arrows are the pull-back functor given by the chosen cleavages). In this situation, we have a commutative diagram

$$(3.5.26) \qquad \begin{array}{ccc} \varphi_1^{-1} B & \xrightarrow{\rho_{\underline{S}}} & \mathsf{Desc}(\varphi_1, \underline{S}) \\ {\scriptstyle F_B}\downarrow & & \downarrow{\scriptstyle \underline{F}} \\ \varphi_2^{-1} B & \xrightarrow{\rho_{\underline{S}}} & \mathsf{Desc}(\varphi_2, \underline{S}) \end{array}$$

whose right vertical arrow is the functor given by the rule :

$$\underline{X} := (X_i, \omega_{ij} \mid i, j \in I) \mapsto \underline{F}(\underline{X}) := (F_i X_i, F_{ij} \omega_{ij} \mid i, j \in I).$$

for every object $\underline{X}$ of $\mathsf{Desc}(\varphi_1, \underline{S})$. By assumption, the top horizontal arrow of (3.5.26) is n-faithful, and the bottom horizontal arrow is $(n-1)$-faithful. We need to prove that the left vertical arrow is n-faithful, and it is easily seen that this will follow, once we have shown that the same holds for the right vertical arrow.

Suppose first that $n = 0$; we have to show that $\underline{F}$ is faithful. However, let $\underline{X}$ and $\underline{Y}$ be two objects of $\mathsf{Desc}(\varphi_1, \underline{S})$, and $\underline{h}_1, \underline{h}_2 : \underline{X} \to \underline{Y}$ two morphisms. By definition, $\underline{h}_t$ (for $t = 1, 2$) is a compatible system $(h_{t,i} : X_i \to Y_i \mid i \in I)$, where each $h_{t,i}$ is a morphism in $\varphi_1^{-1} S_i$. Then, $\underline{F}(\underline{h}_t)$ is the compatible system $(F_i h_{t,i} \mid i \in I)$. Thus, the condition $\underline{F}(\underline{h}_1) = \underline{F}(\underline{h}_2)$ translates the system of identities $F_i h_{1,i} = F_i h_{2,i}$ for every $i \in I$. By assumption, each F_i is faithful, therefore $\underline{h}_1 = \underline{h}_2$, as stated.

For $n = 1$, assumption (d) is empty, (b) means that F_i is fully faithful, and (c) means that F_{ij} is faithful for every $i, j \in I$. In light of the previous case, we have only to show that $\underline{F}$ is full. Hence, let $\underline{X}, \underline{Y}$ be as in the foregoing, and $(h_i : F_i X_i \to F_i Y_i \mid i \in I)$ a morphism $\underline{F}(\underline{X}) \to \underline{F}(\underline{Y})$ in $\mathsf{Desc}(\varphi_2, \underline{S})$. By assumption, for every $i \in I$ we may find a unique morphism $f_i : X_i \to Y_i$ such that $F_i f_i = h_i$. It remains only to check that the system $(f_i \mid i \in I)$ fulfills condition (3.5.21), and since the functors F_{ij} are faithful, it suffices to verify that F_{ij}(3.5.21) holds. However, since F is split cartesian, we have :

$$F_{ij} \circ \pi_{ij}^{0*} f_j = \pi_{ij}^{0*} \circ F_j f_j \qquad F_{ij} \circ \pi_{ij}^{1*} f_i = \pi_{ij}^{1*} \circ F_i f_i$$

hence we reduce to showing that $(F_{ij} \omega_{ij}^Y) \circ \pi_{ij}^{0*} h_j = \pi_{ij}^{1*} h_i \circ (F_{ij} \omega_{ij}^X)$, which holds by assumption.

Next, we consider assertion (ii) : the contention is that the functors $\underline{F}$ and :

$$\mathsf{p}_{1,\underline{S}} : \mathsf{Desc}(\varphi_1, \underline{S}) \to \prod_{i\in I} \varphi_1^{-1} S_i$$

as in (3.5.22), induce an equivalence $(\mathsf{p}_{1,\underline{S}}, \underline{F})$ between $\mathsf{Desc}(\varphi_1, \underline{S})$ and the category $\mathscr{C}$ consisting of all data of the form $\underline{G} := (G_i, H_i, \alpha_i, \omega_{ij}^H \mid i, j \in I)$, where $G_i \in \mathrm{Ob}(\varphi_1^{-1} S_i)$, $H_i \in \mathrm{Ob}(\varphi_2^{-1} S_i)$, $\alpha_i : F_i G_i \xrightarrow{\sim} H_i$ are isomorphisms in $\varphi_2^{-1} S_i$, and $\underline{H} := (H_i, \omega_{ij}^H \mid i, j \in I)$ is an object of $\mathsf{Desc}(\varphi_2, \underline{S})$. However, given an object as above, set :

$$\omega_{ij}^{H'} := (\pi_{ij}^{1*} \alpha_i^{-1}) \circ \omega_{ij}^H \circ (\pi_{ij}^{0*} \alpha_j).$$

Since φ_2 is a split fibration, one verifies easily that the datum $\underline{H}' := (H'_i := F_i G_i, \omega_{ij}^{H'} \mid i, j \in I)$ is an object of $\mathsf{Desc}(\varphi_2, \underline{S})$ isomorphic to $\underline{H}$, and the new datum $(F_i, H'_i, \mathbf{1}_{H'_i}, \omega_{ij}^{H'} \mid i, j \in I)$ is isomorphic to $\underline{G}$; hence $\mathscr{C}$ is equivalent to the category $\mathscr{C}'$ whose objects are all data of the form $(G_i, \omega_{ij} \mid i, j \in I)$ where $G_i \in \mathrm{Ob}(\varphi_1^{-1} S_i)$, and $(F_i G_i, \omega_{ij} \mid i, j \in I)$ is an object of $\mathsf{Desc}(\varphi_2, \underline{S})$. By assumption F_{ij} is fully faithful, and F is a split cartesian functor; hence we may find unique isomorphisms $\widetilde{\omega}_{ij}^G : \pi_{ij}^{0*} G_j \xrightarrow{\sim} \pi_{ij}^{1*} G_i$ such that $\widetilde{\omega}_{ij} = F_{ij} \widetilde{\omega}_{ij}^G$. We claim that the datum $(G_i, \omega_{ij}^G \mid i, j \in I)$ is an object of $\mathsf{Desc}(\varphi_1, \underline{S})$, *i.e.* the isomorphisms ω_{ij}^G satisfy the cocycle condition

$$\pi_{ijk}^{2*} \omega_{ij}^G \circ \pi_{ijk}^{0*} \omega_{jk}^G = \pi_{ijk}^{1*} \omega_{ik}^G \qquad \text{for every } i, j, k \in I. \tag{3.5.27}$$

To check this identity, since by assumption the functors F_{ijk} are faithful, it suffices to see that F_{ijk}(3.5.27) holds, which is clear, since the cocycle condition holds for the isomorphisms ω_{ij} (and since F is split cartesian). This shows that $(\mathsf{p}_{1,\underline{S}}, \underline{F})$ is essentially surjective. Next, since the functor $\mathsf{p}_{1,\underline{S}}$ is faithful, the same holds for $(\mathsf{p}_{1,\underline{S}}, \underline{F})$. Finally, let

$$\underline{G} := (G_i, \omega_{ij} \mid i, j \in I) \qquad \underline{G}' := (G'_i, \omega'_{ij} \mid i, j \in I)$$

be two objects of $\mathscr{C}'$. A morphism $\underline{G} \to \underline{G}'$ consists of a system $(\alpha_i : G_i \to G'_i \mid i \in I)$ of morphisms such that $(F_i \alpha_i \mid i \in I)$ is a morphism

$$(F_i G_i, \omega_{ij} \mid i, j \in I) \to (F_i G'_i, \omega'_{ij} \mid i, j \in I)$$

in $\mathsf{Desc}(\varphi_2, \underline{S})$. To show that $(\mathsf{p}_{1\underline{S}}, \underline{F})$ is full, and since we know already that this functor is essentially surjective, we may assume that there exist $(G_i, \omega_{ij}^G \mid i, j \in I)$, $(G'_i, \omega_{ij}^{G'} \mid i, j \in I)$ in $\mathsf{Desc}(\varphi_1, \underline{S})$ such that $\omega_{ij} = F_{ij} \omega_{ij}^G$ and $\omega'_{ij} = F_{ij} \omega_{ij}^{G'}$ for every $i, j \in I$; in this case, it suffices to verify the identity

$$\omega_{ij}^{G'} \circ \pi_{ij}^{0*} \alpha_j = \pi_{ij}^{1*} \alpha_i \circ \omega_{ij}^G \qquad \text{for every } i, j \in I. \tag{3.5.28}$$

Again, the faithfulness of F_{ij} reduces to checking that F_{ij}(3.5.28) holds, which is clear, since F is split cartesian.

Lastly, notice that the case $n = 2$ of assertion (i) is a formal consequence of (ii). □

3.5.29. In the situation of (3.5.22), let $\mathscr{S}$ be the sieve generated by the family $\underline{S}$, and $g : B' \to B$ any morphism in $\mathscr{B}$. We let :

$$B'_i := B' \times_B S_i \quad B'_{ij} := B' \times_B S_{ij} \quad B'_{ijk} := B' \times_B S_{ijk} \qquad \text{for every } i, j, k \in I$$

and denote $g_i : B'_i \to S_i$, $g_{ij} : B'_{ij} \to S_{ij}$ and $g_{ijk} : B'_{ijk} \to S_{ijk}$ the induced projections.

Corollary 3.5.30. *With the notation of* (3.5.29)*, let* $n \in \{0, 1, 2\}$. *The following holds :*

(i) *$\mathscr{S}$ is a sieve of φ-n-descent, if and only if $\rho_{\underline{S}}$ is n-faithful (see* (3.5.23)*).*

(ii) *Suppose that :*

  (a) *$\mathscr{S}$ is a sieve of universal φ-n-descent.*

  (b) *The pull-back functors $g_i^* : \varphi^{-1} S_i \to \varphi^{-1} B'_i$ are n-faithful.*

(c) *The pull-back functors* $g^*_{ij} : \varphi^{-1}S_{ij} \to \varphi^{-1}B'_{ij}$ *are* $(n-1)$*-faithful.*
(d) *The pull-back functors* $g^*_{ijk} : \varphi^{-1}S_{ijk} \to \varphi^{-1}B'_{ijk}$ *are* $(n-2)$*-faithful.*
Then the pull-back functor g^* *is* n*-faithful.*

(iii) *Suppose that the functors* g^*_{ij} *are fully faithful, and the functors* g^*_{ijk} *are faithful, for every* $i, j, k \in I$*. Then the natural essentially commutative diagram :*

$$\begin{array}{ccc} \mathsf{Desc}(\varphi, \underline{S}) & \xrightarrow{\mathsf{Desc}(\varphi,g)} & \mathsf{Desc}(\varphi, \underline{S} \times_B B') \\ {\scriptstyle \mathsf{p}_{\underline{S}}}\downarrow & & \downarrow{\scriptstyle \mathsf{p}_{\underline{S}\times_B B'}} \\ \prod_{i\in I} \varphi^{-1}S_i & \xrightarrow{\prod_{i\in I} g^*_i} & \prod_{i\in I} \varphi^{-1}B'_i \end{array}$$

is 2*-cartesian (see remark* 2.5.2(v) *and example* 3.3.12(ii)*).*

Proof. (i) follows by inspecting (3.5.23).

(ii): Thanks to theorem 3.2.7, we may assume that φ is a split fibration. Now, set

$$\mathscr{C} := \mathsf{Morph}(\mathscr{B}) \qquad \mathscr{A}_1 := \mathscr{C} \times_{(\mathsf{t},\varphi)} \mathscr{A} \qquad \mathscr{A}_2 := \mathscr{C} \times_{(\mathsf{s},\varphi)} \mathscr{A}$$

where $\mathsf{s}, \mathsf{t} : \mathscr{C} \to \mathscr{B}$ are the source and target functors (see (1.1.30)). The natural projections $\varphi_i : \mathscr{A}_i \to \mathscr{C}$ (for $i = 1, 2$) are two fibrations (see remark 3.1.5(i)). Moreover, t induces a functor $\mathsf{t}_{|g} : \mathscr{C}/g \to \mathscr{B}/B$ (notation of (1.1.26)) and we let $\mathscr{S}/g := \mathsf{t}^{-1}_{|g}\mathscr{S}$, which is the sieve of $\mathscr{C}/g$ generated by the cartesian diagrams

$$D_i \quad : \quad \begin{array}{ccc} B'_i & \xrightarrow{g_i} & S_i \\ \downarrow & & \downarrow \\ B' & \xrightarrow{g} & B \end{array} \qquad \text{for every } i \in I.$$

Notice that the products $D_{ij} := D_i \times D_j$ are represented in $\mathscr{C}/g$ by the diagrams

$$\begin{array}{ccc} B'_{ij} & \xrightarrow{g_{ij}} & S_{ij} \\ \downarrow & & \downarrow \\ B' & \xrightarrow{g} & B \end{array} \qquad \text{for every } i, j \in I$$

and likewise one may represent the triple products $D_{ijk} := D_{ij} \times D_k$.

By definition, the objects of $\mathscr{A}_1$ (resp. $\mathscr{A}_2$) are the pairs $(h : X \to Y, a)$, where h is a morphism in $\mathscr{B}$ and $a \in \mathrm{Ob}(\varphi^{-1}Y)$ (resp. $a \in \mathrm{Ob}(\varphi^{-1}X)$). A morphism of $\mathscr{A}_1$ (resp. of $\mathscr{A}_2$)

$$(h : X \to Y, a) \to (h' : X' \to Y', a')$$

is a datum (f_1, f_2, t), where $f_1 : X \to X'$ and $f_2 : Y \to Y'$ are morphisms in $\mathscr{B}$ with $f_2 \circ h = h' \circ f_1$, and $t : a \to f^*_2 a'$ (resp. $t : a \to f^*_1 a'$) is a morphism in $\varphi^{-1}Y$ (resp. in $\varphi^{-1}X$). Now, we define a functor $F : \mathscr{A}_1 \to \mathscr{A}_2$ of $\mathscr{C}$-categories, by the rule :

- $(h, a) \mapsto (h, h^*a)$ for every $(h, a) \in \mathrm{Ob}(\mathscr{A}_1)$.
- $(f_1, f_2, t) \mapsto (f_1, f_2, h^*t)$ for every morphism (f_1, f_2, t) of $\mathscr{A}_1$ as above. Notice that if $t : a \to f^*_2 a'$ is a morphism in $\varphi^{-1}Y$, then $h^*t : h^*a \to h^* f^*_2 a' = f^*_1 h'^* a'$ is a morphism of $\varphi^{-1}Y'$, since φ is a split fibration.

Notice that a morphism (f_1, f_2, t) of either $\mathscr{A}_1$ or $\mathscr{A}_2$ is cartesian if and only if t is an isomorphism; especially, it is clear that F is a cartesian functor. Moreover, for every object $h : X \to Y$ of $\mathscr{C}$, the restriction $\varphi_1^{-1}h \to \varphi_2^{-1}h$ of F is isomorphic to the pull-back functor $h^* : \varphi^{-1}Y \to \varphi^{-1}h$. Especially, conditions (b)–(d) say that the restriction $F_i : \varphi_1^{-1}g_i \to \varphi_2^{-1}g_i$ (resp. $F_{ij} : \varphi_1^{-1}g_{ij} \to \varphi_2^{-1}g_{ij}$, resp. $F_{ijk} : \varphi_1^{-1}g_{ijk} \to \varphi_2^{-1}g_{ijk}$) are n-faithful (resp. $(n-1)$-faithful, resp. $(n-2)$-faithful). In light of theorem 3.5.25(i), we are then reduced to showing

Claim 3.5.31. $\mathscr{S}/g$ is a sieve both of φ_1-n-descent and of φ_2-n-descent.

Proof of the claim. Let $\mathscr{D}$ be any (small) category; we remark first that a functor $\mathscr{D} \to \mathscr{A}_1$ is the same as a pair of functors $(H : \mathscr{D} \to \mathscr{A}, K : \mathscr{D} \to \mathscr{C})$ such that $\varphi \circ H = \mathsf{t} \circ K$, and likewise one can describe the functors $\mathscr{D} \to \mathscr{A}_2$. Then, it is easily seen that the functors

$$\mathscr{A}(B) \to \mathsf{Cart}_{\mathscr{C}}(\mathscr{C}/g, \mathscr{A}_1) \qquad G \mapsto (G \circ \mathsf{t}, \mathsf{t})$$
$$\mathscr{A}(B') \to \mathsf{Cart}_{\mathscr{C}}(\mathscr{C}/g, \mathscr{A}_2) \qquad G \mapsto (G \circ \mathsf{s}, \mathsf{s})$$

are equivalences, and induce equivalences

$$\mathsf{Cart}_{\mathscr{B}}(\mathscr{S}, \mathscr{A}) \to \mathsf{Cart}_{\mathscr{C}}(\mathscr{S}/g, \mathscr{A}_1)$$
$$\mathsf{Cart}_{\mathscr{B}}(\mathscr{S} \times_B g, \mathscr{A}) \to \mathsf{Cart}_{\mathscr{C}}(\mathscr{S}/g, \mathscr{A}_2)$$

(details left to the reader). The claim follows immediately. □

3.5.32. Quite generally, if $\varphi : \mathscr{A} \to \mathscr{B}$ is a fibration over a category $\mathscr{B}$ that admits fibre products, the descent data for φ (relative to a fixed cleavage c) also form a fibration :

$$\mathsf{D}\varphi : \varphi\text{-}\mathsf{Desc} \to \mathsf{Morph}(\mathscr{B}).$$

Namely, for every morphism $f : T' \to T$ of $\mathscr{B}$, the fibre over f is the category $\mathsf{Desc}(\varphi, f)$ of all descent data (f, A, ξ) relative to the family $\{f\}$, and the cleavage c, so A is an object of $\varphi^{-1}T'$ and $\xi : p_1^*A \xrightarrow{\sim} p_2^*A$ is an isomorphism in the category $\varphi^{-1}(T' \times_T T')$ satisfying the usual cocycle condition (here $p_1, p_2 : T' \times_T T' \to T'$ denote the two natural morphisms). Given two objects $\underline{A} := (f : T' \to T, A, \xi)$, $\underline{A}' := (g : W' \to W, A', \zeta)$ of φ-Desc, the morphisms $\underline{A} \to \underline{A}'$ are the data (h, α) consisting of a commutative diagram :

$$\begin{array}{ccc} W' & \xrightarrow{h} & T' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ W & \longrightarrow & T \end{array}$$

and a morphism $\alpha : A \to A'$ such that $\varphi(\alpha) = h$ and $p_1^*(\alpha) \circ \xi = \zeta \circ p_2^*(\alpha)$.

We have a natural cartesian functor of fibrations :

$$\begin{array}{ccc} \mathscr{A} \times_{(\varphi,\mathsf{t})} \mathsf{Morph}(\mathscr{B}) & \xrightarrow{\quad \mathsf{d} \quad} & \varphi\text{-}\mathsf{Desc} \\ & {\scriptstyle \mathsf{p}} \searrow \quad \swarrow {\scriptstyle \mathsf{D}\varphi} & \\ & \mathsf{Morph}(\mathscr{B}) & \end{array}$$

where $\mathsf{t} : \mathsf{Morph}(\mathscr{B}) \to \mathscr{B}$ is the target functor (see (1.1.30) and example 1.2.25(i)). Namely, to any pair $(T, f : S \to \varphi T)$ with $T \in \mathrm{Ob}(\mathscr{A})$ and $f \in \mathrm{Ob}(\mathsf{Morph}(\mathscr{B}))$, one assigns the canonical descent datum $\mathsf{d}(T, f) := \rho_{\{f\}}(T)$ in $\mathsf{Desc}(\varphi, f)$ associated with the pair as in (3.5.23).

Corollary 3.5.33. *In the situation of* (3.5.32)*, let $f : B' \to B$ be a morphism of $\mathscr{B}$, and $\mathscr{S}$ a sieve of $\mathscr{B}/B$, generated by a family $(S_i \to B \mid i \in I)$. Let $n \in \{0, 1, 2\}$, and suppose that :*

(a) *$\mathscr{S}$ is a sieve of universal φ-n-descent.*
(b) *For every $i \in I$, the morphism $S_i \times_B f$ is of φ-n-descent.*
(c) *For every $i, j \in I$, the morphism $S_{ij} \times_B f$ is of φ-$(n-1)$-descent.*
(d) *For every $i, j, k \in I$, the morphism $S_{ijk} \times_B f$ is of φ-$(n-2)$-descent.*

Then f is a morphism of φ-n-descent.

Proof. In view of corollary 3.5.30(i), it is easily seen that a morphism $g : T' \to T$ in $\mathscr{B}$ is of φ-n-descent if and only if the restriction $\varphi^{-1}T \to \mathsf{D}\varphi^{-1}g$ of d is n-faithful. Set $\mathscr{C} := \mathsf{Morph}(\mathscr{B})$; as in the proof of corollary 3.5.30(ii), the functor t induces a functor $\mathsf{t}_{|f} : \mathscr{C}/f \to \mathscr{B}/B$, and we let $\mathscr{S}/f := \mathsf{t}_{|f}^{-1}\mathscr{S}$. With this notation, theorem 3.5.25(i) reduces to showing :

Claim 3.5.34. The sieve $\mathscr{S}/f$ is both of p-n-descent and of Dφ-n-descent.

Proof of the claim. By claim 3.5.31, it is already known that $\mathscr{S}/f$ is of p-n-descent. To show that $\mathscr{S}/f$ is of Dφ-n-descent, we consider the commutative diagram

$$\begin{array}{ccc} \mathscr{A}(B') & \longrightarrow & \mathsf{Cart}_{\mathscr{C}}(\mathscr{C}/f, \varphi\text{-}\mathsf{Desc}) \\ \downarrow & & \downarrow \\ \mathsf{Cart}_{\mathscr{B}}(\mathscr{S} \times_B f, \mathscr{A}) & \longrightarrow & \mathsf{Cart}_{\mathscr{C}}(\mathscr{S}/f, \varphi\text{-}\mathsf{Desc}) \end{array}$$

whose left (resp. right) vertical arrow is induced by the inclusion $\mathscr{S} \times_B f \to \mathscr{B}/B'$ (resp. $\mathscr{S}/f \to \mathscr{S} \times_B f$) and whose top horizontal arrow is defined as follows. Given a cartesian functor $G : \mathscr{B}/B' \to \mathscr{A}$, we let D$G : \mathscr{C}/f \to \varphi$-Desc be the unique cartesian functor determined on the objects of $\mathscr{C}/f$ by the rule :

$$\left(\begin{array}{ccc} T' & \xrightarrow{g} & T \\ {\scriptstyle h}\downarrow & & \downarrow \\ B' & \xrightarrow{f} & B \end{array}\right) \mapsto \mathsf{d}(G(h), g).$$

We leave to the reader the verification the rule $G \mapsto \mathrm{D}G$ extends to a well defined functor, and then there exists a unique (similarly defined) bottom horizontal arrow that makes commute the foregoing diagram. Moreover, both horizontal arrow thus obtained are equivalences of categories. The claim follows. ◻

Lemma 3.5.35. *Let $i \leq 2$ be an integer, $F : \mathscr{E} \to \mathscr{C}$ a fibration, $X \in \mathrm{Ob}(\mathscr{C})$, and $\mathscr{T} \subset \mathscr{S}$ two sieves of $\mathscr{C}/X$. The following holds :*

(i) *If $\mathscr{T} \times_X f$ is of F-i-descent for every $(Y \xrightarrow{f} X) \in \mathscr{S}$, then the induced functor*

$$j : \mathsf{Cart}_{\mathscr{C}}(\mathscr{S}, \mathscr{E}) \to \mathsf{Cart}_{\mathscr{C}}(\mathscr{T}, \mathscr{E})$$

is i-faithful.

(ii) *If $\mathscr{T}$ is of universal F-i-descent, the same holds for $\mathscr{S}$.*

Proof. (i): The assertion is trivial for $i < 0$. We consider first the case where $i = 0$; thus, let $\varphi, \psi : \mathscr{S} \to \mathscr{E}$ be two $\mathscr{C}$-cartesian functors, and $\alpha, \beta : \varphi \Rightarrow \psi$ two natural $\mathscr{C}$-transformations such that $\alpha_{|\mathscr{T}} = \beta_{|\mathscr{T}}$. Let also $(Y \xrightarrow{f} X) \in \mathrm{Ob}(\mathscr{S})$ be any object and $f_* : \mathscr{C}/Y \to \mathscr{S}$ the induced functor; then $\mathscr{T}' := \mathscr{T} \times_X f \subset \mathscr{C}/Y$ is a sieve of F-0-descent and

$$(\alpha * f_*)_{|\mathscr{T}'} = (\beta * f_*)_{|\mathscr{T}'}.$$

We deduce that $\alpha * f_* = \beta * f_*$; especially $\alpha_f = \beta_f$, which shows that j is faithful.

Next, suppose that $i = 1$, and for φ and ψ as in the foregoing, let $\alpha : \varphi_{|\mathscr{T}} \Rightarrow \psi_{|\mathscr{T}}$ be a given natural $\mathscr{C}$-transformation, and $(Y \xrightarrow{f} X) \in \mathrm{Ob}(\mathscr{S})$; then the sieve $\mathscr{T}' := \mathscr{T} \times_X f \subset \mathscr{C}/Y$ is of F-1-descent, and f_* restricts to a functor $f_{*|\mathscr{T}'} : \mathscr{T}' \to \mathscr{T}$, so the natural $\mathscr{C}$-transformation

$$\alpha * (f_{*|\mathscr{T}'}) : (\varphi \circ f_*)_{|\mathscr{T}'} \Rightarrow (\psi \circ f_*)_{|\mathscr{T}'}$$

extends to a unique natural $\mathscr{C}$-transformation $\alpha^{(f)} : \varphi \circ f_* \Rightarrow \psi \circ f_*$. Now, let us define

$$\widetilde{\alpha}_f := \alpha^{(f)}_{\mathbf{1}_Y} \qquad \text{for every } (Y \xrightarrow{f} X) \in \mathrm{Ob}(\mathscr{S}).$$

Clearly, $\widetilde{\alpha}_f = \alpha_f$ whenever $f \in \mathrm{Ob}(\mathscr{T})$, and it remains to check that $\widetilde{\alpha}$ is a natural $\mathscr{C}$-transformation $\varphi \Rightarrow \psi$. Thus, let $h/X : (Y' \xrightarrow{f'} X) \to (Y \xrightarrow{f} X)$ be a morphism of $\mathscr{S}$; we need to show that

$$\psi(h/X) \circ \widetilde{\alpha}_f = \widetilde{\alpha}_{f'} \circ \varphi(h/X).$$

Considering the morphism $h/Y : (Y' \xrightarrow{h} Y) \to \mathbf{1}_Y$ of $\mathscr{C}/Y$, and noticing that $\varphi(h/X) = (\varphi \circ f_*)(h/Y)$ and $\psi(h/X) = (\psi \circ f_*)(h/Y)$, we come down to checking that $\alpha^{(f)}_h = \alpha^{(f')}_{\mathbf{1}_{Y'}}$. We show more precisely that $\alpha^{(f)} * h_* = \alpha^{(f')}$. Indeed, notice that

$$(\alpha^{(f)} * h_*)_{|\mathscr{T}\times_X f'} = \alpha * (f'_{*|\mathscr{T}\times_X f'}) = \alpha^{(f')}_{|\mathscr{T}\times_X f'}$$

and since $\mathscr{T} \times_X f'$ is of F-0-descent, the assertion follows.

Lastly, for $i = 2$, we know already that j is fully faithful, so it remains only to show that every cartesian functor $\varphi : \mathscr{T} \to \mathscr{E}$ is isomorphic to the restriction of a cartesian functor $\mathscr{S} \to \mathscr{E}$. Now, for every $(f : Y \to X) \in \mathrm{Ob}(\mathscr{S})$ set $\mathscr{T}^{(f)} := \mathscr{T} \times_X f$; by assumption there exists a $\mathscr{C}$-cartesian functor $\varphi^{(f)} : \mathscr{C}/Y \to \mathscr{E}$ with an isomorphism of functors :

$$\omega_f : \varphi^{(f)}_{|\mathscr{T}^{(f)}} \xrightarrow{\sim} \varphi \circ (f_{*|\mathscr{T}^{(f)}}).$$

Notice that if $f \in \mathrm{Ob}(\mathscr{T})$, we have $\mathscr{T}^{(f)} = \mathscr{C}/Y$, and in this case we take $\varphi^{(f)} := \varphi \circ f_*$ and $\omega_f := \mathbf{1}_{\varphi^{(f)}}$. Then, for every morphism $h/X : (Y' \xrightarrow{f'} X) \to (Y \xrightarrow{f'} X)$ in $\mathscr{S}$ we get the isomorphism :

$$\varphi^{(f')}_{|\mathscr{T}^{(f')}} \xrightarrow{\omega_{f'}} \varphi \circ f'_{*|\mathscr{T}^{(f')}} \xrightarrow{(\omega_f * h_*)^{-1}_{|\mathscr{T}^{(f')}}} (\varphi^{(f)} \circ h_*)_{*|\mathscr{T}^{(f')}}$$

which, as $\mathscr{T}^{(f')}$ is of F-1-descent, extends uniquely to an isomorphism of functors $\mathscr{C}/Y' \to \mathscr{E}$

$$\tau_{h/X} : \varphi^{(f')} \xrightarrow{\sim} \varphi^{(f)} \circ h_*.$$

Claim 3.5.36. The rule : $(f : Y \to X) \mapsto (\varphi^{(f)} : \mathscr{C}/Y \to \mathscr{E})$ for every $f \in \mathrm{Ob}(\mathscr{S})$ defines a pseudo-cocone

$$\varphi^{(\bullet)} : G_{\mathscr{S}} \Rightarrow \mathsf{F}_{\mathscr{E}}$$

whose coherence constraint is given by the system of isomorphisms $\tau_{\bullet/X}$, and where $G_{\mathscr{S}} : \mathscr{S} \to \mathbf{Cat}$ is the functor associated with $\mathscr{S}$ as in (3.5.8).

Proof of the claim. Clearly $(\tau_{\mathbf{1}_Y/Y})_{|\mathscr{T}^{(f)}} = \mathbf{1}_{\varphi^{(f)}_{|\mathscr{T}^{(f)}}}$ for every $(f : Y \to X) \in \mathrm{Ob}(\mathscr{S})$, whence $\tau_{\mathbf{1}_Y/Y} = \mathbf{1}_{\varphi^{(f)}}$ by the uniqueness of $\tau_{\mathbf{1}_Y/Y}$. Next, let $(Y'' \xrightarrow{f''} X) \xrightarrow{h'/X} (Y' \xrightarrow{f'} X) \xrightarrow{h/X} (Y \xrightarrow{f} X)$ be two morphism of $\mathscr{S}$; the coherence axiom for $\tau_{\bullet/X}$ comes down to the identity :

$$(\tau_{h/X} * h'_*) \circ \tau_{h'/X} = \tau_{(h\circ h')/X}.$$

But the latter can be checked after restriction to the sieve $\mathscr{T}^{(f'')}$, where it follows by a simple inspection : the details are left to the reader. ◊

From claim 3.5.36 it follows that there exists a functor

$$\widetilde{\varphi} : \mathscr{S} \to \mathscr{E} \qquad \text{and an invertible modification} \qquad \Xi : \mathsf{F}_{\widetilde{\varphi}} \odot \widehat{\varepsilon} \xrightarrow{\sim} \varphi^{(\bullet)}$$

where $\widehat{\varepsilon} : G_{\mathscr{S}} \Rightarrow \mathsf{F}_{\mathscr{E}}$ is the universal pseudo-cocone provided by lemma 3.5.9. Explicitly, Ξ is a system of isomorphisms of functors

$$\Xi_f : \widetilde{\varphi} \circ f_* \xrightarrow{\sim} \varphi^{(f)} \qquad \text{for every } (f : Y \to X) \in \mathrm{Ob}(\mathscr{S})$$

from which it follows easily that $\widetilde{\varphi}$ is $\mathscr{C}$-cartesian. The compatibility conditions for Ξ amount to the identities :

(3.5.37) $$\Xi_f * h_* = \tau_h \odot \Xi_{f'} \qquad \text{for every morphism } h/X : f \to f' \text{ in } \mathscr{S}.$$

To conclude, it suffices to exhibit an isomorphism of functors $\xi : \widetilde{\varphi}_{|\mathscr{T}} \xrightarrow{\sim} \varphi$. To this aim, we set

$$\xi_g := (\Xi_g)_{\mathbf{1}_Y} : \widetilde{\varphi}(g) \xrightarrow{\sim} \varphi^{(g)}(\mathbf{1}_Y) = \varphi(g) \qquad \text{for every } (g : Y \to X) \in \mathrm{Ob}(\mathscr{T}).$$

To check the naturality of ξ, let $h/X : (Y' \xrightarrow{g'} X) \to (Y \xrightarrow{g} X)$ be any morphism of $\mathscr{T}$; notice that $\tau_{h/X} = \mathbf{1}_{\varphi(g')}$, whence $\xi_{g'} = (\Xi_g)_h$, due to (3.5.37). On the other hand, the naturality of Ξ_g yields the commutative diagram

$$\begin{array}{ccc}
\widetilde{\varphi}(g') = \widetilde{\varphi} \circ g_*(h) & \xrightarrow{(\Xi_g)_h} & \varphi^{(g)}(h) = \varphi(g') \\
{\scriptstyle \widetilde{\varphi}\circ g_*(h/Y)}\downarrow & & \downarrow{\scriptstyle \varphi^{(g)}(h/Y)} \\
\widetilde{\varphi}(g) = \widetilde{\varphi} \circ g_*(\mathbf{1}_Y) & \xrightarrow{\xi_g} & \varphi^{(g)}(\mathbf{1}_Y) = \varphi(g)
\end{array}$$

where $\widetilde{\varphi} \circ g_*(h/Y) = \widetilde{\varphi}(h/x)$ and $\varphi^{(g)}(h/Y) = \varphi(h/X)$, whence the contention.

(ii): We consider the induced functors

$$\mathscr{E}(X) \xrightarrow{j'} \mathsf{Cart}_{\mathscr{C}}(\mathscr{S}, \mathscr{E}) \xrightarrow{j''} \mathsf{Cart}_{\mathscr{C}}(\mathscr{T}, \mathscr{E})$$

and notice that j'' is i-faithful, by virtue of (i); the same holds for $j'' \circ j'$, by assumption. Then it follows easily that j' is i-faithful as well, whence the assertion. □

Proposition 3.5.38. *Let $i \leq 2$ be an integer, $F : \mathscr{E} \to \mathscr{C}$ a fibration, $X \in \mathrm{Ob}(\mathscr{C})$, and $\mathscr{T}, \mathscr{S}$ two sieves of $\mathscr{C}/X$, such that :*

(a) *$\mathscr{S}$ is of universal F-i-descent.*

(b) *$\mathscr{T} \times_X f$ is of universal F-i-descent for every $(Y \xrightarrow{f} X) \in \mathscr{S}$.*

Then $\mathscr{T}$ is of universal F-i-descent.

Proof. According to lemma 3.5.35(ii), the sieve $\mathscr{S}' := \mathscr{S} \bigcup \mathscr{T}$ is also of universal F-i-descent. It is also clear that $\mathscr{T} \times_X f$ is of F-i-descent for every $(Y \xrightarrow{f} X) \in \mathscr{S}'$. Thus, we may replace $\mathscr{S}$ by $\mathscr{S}'$ and assume that $\mathscr{T} \subset \mathscr{S}$. Next, let $f : Y \to X$ be any morphism of $\mathscr{C}$; we need to check that $\mathscr{T} \times_X f$ is a sieve of F-i-descent, and by assumption the same holds for $\mathscr{S} \times_X f$, so it suffices to show that the induced functor $\mathsf{Cart}_{\mathscr{C}}(\mathscr{S} \times_X f, \mathscr{E}) \to \mathsf{Cart}_{\mathscr{C}}(\mathscr{T} \times_X f, \mathscr{E})$ is i-faithful. In view of lemma 3.5.35(i), we are then reduced to checking that $(\mathscr{T} \times_X f) \times_Y g$ is a sieve of F-i-descent for every $g \in \mathrm{Ob}(\mathscr{S} \times_X f)$. But we have $(\mathscr{T} \times_X f) \times_Y g = \mathscr{T} \times_X (f \circ g)$, and $f \circ g \in \mathrm{Ob}(\mathscr{S})$, so the assertion follows from (b). □

3.6. **Profinite groups and Galois categories.** Quite generally, for any profinite group P, let P-$\mathbf{Set}$ denote the category of discrete finite sets, endowed with a continuous left action of P (the morphisms in P-$\mathbf{Set}$ are the P-equivariant maps). Any continuous group homomorphism $\omega : P \to Q$ of profinite groups induces a *restriction functor*

$$\mathrm{Res}(\omega) : Q\text{-}\mathbf{Set} \to P\text{-}\mathbf{Set}$$

in the obvious way. In case the notation is not ambiguous, one writes also Res^P_Q for this functor.

For any two profinite groups P and Q, we denote by

$$\mathrm{Hom}_{\mathrm{cont}}(P, Q)$$

the set of all continuous group homomorphisms $P \to Q$. If φ_1, φ_2 are two such group homomorphisms, we say that φ_1 is *conjugate* to φ_2, and we write $\varphi_1 \sim \varphi_2$, if there exists an inner automorphism ω of G, such that $\varphi_2 = \omega \circ \varphi_1$. Clearly the trivial map $\pi \to G$ (whose image is the neutral element of G), is the unique element of a distinguished conjugacy class.

3.6.1. Let P be any profinite group; for any (discrete) finite group G, consider the pointed set $\mathrm{Hom}_{\mathrm{cont}}(P, G)/\!\sim$ of conjugacy classes of continuous group homomorphisms $P \to G$. This is also denoted

$$H^1_{\mathrm{cont}}(P, G)$$

and called the first *non-abelian continuous cohomology group* of P with coefficients in G (so G is regarded as a P-module with trivial P-action). Clearly the formation of $H^1(P,G)$ is covariant on the argument G, and controvariant for continuous homomorphisms of profinite groups.

Lemma 3.6.2. *Let $\varphi : P \to P'$ be a continuous homomorphism of profinite groups, and suppose that the induced map of pointed sets :*

$$H^1_{\mathrm{cont}}(P',G) \to H^1_{\mathrm{cont}}(P,G) \quad : \quad f \mapsto f \circ \varphi$$

is bijective, for every finite group G. Then φ is an isomorphism of topological groups.

Proof. First we show that φ is injective. Indeed, let $x \in P$ be any element; we may find an open normal subgroup $H \subset P$ such that $x \notin H$; taking $G := P/H$, we deduce that the projection $P \to P/H$ factors through φ and a group homomorphism $f : P' \to P/H$, hence $x \notin \mathrm{Ker}\,\varphi$, as claimed. Moreover, let $H' := \mathrm{Ker}\, f$; clearly $H' \cap P = H$, so the topology of P is induced from that of P'. It remains only to show that φ is surjective; to this aim, we consider any continuous surjection $f' : P' \to G'$ onto a finite group, and it suffices to show that the restriction of f' to φP is still surjective. Indeed, let G be the image of φP in G', denote by $i : G \to G'$ the inclusion map, and let $f : P \to G$ be the unique continuous map such that $i \circ f = f' \circ \varphi$; by assumption, there exists a continuous group homomorphism $g : P' \to G$ such that $f = g \circ \varphi$. On the other hand, $(i \circ g) \circ \varphi = f' \circ \varphi$, hence the conjugacy class of $i \circ g$ equals the conjugacy class of f', especially $i \circ g$ is surjective, hence the same holds for i, as required. $\square$

3.6.3. Let G be a profinite group, and $H \subset G$ an open subgroup. It is easily seen that H is also a profinite group, with the topology induced from G. Moreover, the restriction functor

$$\mathrm{Res}^H_G : G\text{-}\mathbf{Set} \to H\text{-}\mathbf{Set}$$

admits a left adjoint

$$\mathrm{Ind}^G_H : H\text{-}\mathbf{Set} \to G\text{-}\mathbf{Set}.$$

Namely, to any finite set Σ with a continuous left action of H, one assigns the set $\mathrm{Ind}^G_H\Sigma := G \times \Sigma/\!\sim$, where $\sim$ is the equivalence relation such that

$$(gh, \sigma) \sim (g, h\sigma) \qquad \text{for every } g \in G,\ h \in H \text{ and } \sigma \in \Sigma.$$

The left G-action on $\mathrm{Ind}^G_H\Sigma$ is given by the rule : $(g', (g, \sigma)) \mapsto (g'g, \sigma)$ for every $g, g' \in G$ and $\sigma \in \Sigma$. It is easily seen that this action is continuous, and the reader may check that the functor Ind^G_H is indeed left adjoint to Res^H_G.

3.6.4. Moreover, let 1 denote the final object of H-$\mathbf{Set}$; notice that $\mathrm{Ind}^G_H 1 = G/H$, the set of orbits of G under its right translation action by H. Hence, for any finite set Σ with continuous H-action, the unique map $t_\Sigma : \Sigma \to 1$ yields a G-equivariant map

$$\mathrm{Ind}^G_H t_\Sigma : \mathrm{Ind}^G_H \to G/H$$

and therefore Ind^G_H factors through a functor

$$H\text{-}\mathbf{Set} \to G\text{-}\mathbf{Set}/(G/H). \tag{3.6.5}$$

It is easily seen that (3.6.5) is an equivalence. Indeed, one obtains a natural quasi-inverse, by the rule : $(f : \Sigma \to G/H) \mapsto f^{-1}(H)$. The detailed verification shall be left to the reader.

Definition 3.6.6. ([82, Exp.V, Def.5.1]) Let $\mathscr{C}$ be a category, and $F : \mathscr{C} \to \mathbf{Set}$ a functor.

(i) We say that $\mathscr{C}$ is a *Galois category*, if $\mathscr{C}$ is equivalent to P-$\mathbf{Set}$, for some profinite group P. We denote $\mathbf{Galois}$ the category whose objects are all the Galois categories, and whose morphisms are the exact functors between Galois categories.

(ii) We say that F is a *fibre functor*, if F is exact and conservative, and $F(X)$ is a finite set for every $X \in \mathrm{Ob}(\mathscr{C})$.

(iii) We denote $\mathbf{fibre.Fun}$ the 2-*category of fibre functors*, defined as follows :
  (a) The objects are all the pairs $(\mathscr{C}, F)$ consisting of a Galois category $\mathscr{C}$ and a fibre functor F for $\mathscr{C}$.
  (b) The 1-cells $(\mathscr{C}_1, F_1) \to (\mathscr{C}_2, F_2)$ are all the pairs (G, β) consisting of an exact functor $G : \mathscr{C}_1 \to \mathscr{C}_2$ and an isomorphism of functors $\beta : F_1 \stackrel{\sim}{\to} F_2 \circ G$.
  (c) And for every pair of 1-cells $(G', \beta'), (G, \beta) : (\mathscr{C}_1, F_1) \to (\mathscr{C}_2, F_2)$, the 2-cells $(G', \beta') \to (G, \beta)$ are the isomorphisms $\gamma : G' \stackrel{\sim}{\to} G$ such that $(F_2 * \gamma) \odot \beta' = \beta$.

Composition of 1-cells and 2-cells is defined in the obvious way. We shall also denote simply by G a 1-cell (G, β) as in (b), such that $F_1 = F_2 \circ G$ and β is the identity automorphism of F_1.

3.6.7. Notice that any Galois category $\mathscr{C}$ admits a fibre functor : indeed, if P is any profinite group, the forgetful functor

$$\mathsf{f}_P : P\text{-}\mathbf{Set} \to \mathbf{Set}$$

fulfills the conditions of definition 3.6.6(ii), therefore the same holds for the functor $\mathsf{f}_P \circ \beta$, if $\beta : \mathscr{C} \to P\text{-}\mathbf{Set}$ is any equivalence. For any Galois category $\mathscr{C}$ and any fibre functor $F : \mathscr{C} \to \mathbf{Set}$, we denote

$$\pi_1(\mathscr{C}, F)$$

the group of automorphisms of F, and we call it the *fundamental group of $\mathscr{C}$ pointed at F*. By definition, for every $X \in \mathrm{Ob}(\mathscr{C})$, the finite set $F(X)$ is endowed with a natural left action of $\pi_1(\mathscr{C}, F)$. For every $X \in \mathrm{Ob}(\mathscr{C})$ and every $\xi \in F(X)$, the stabilizer $H_{X,\xi} \subset \pi_1(\mathscr{C}, F)$ is a subgroup of finite index, and we endow $\pi_1(\mathscr{C}, F)$ with the coarsest group topology for which all such $H_{X,\xi}$ are open subgroups. The resulting topological group $\pi_1(\mathscr{C}, F)$ is profinite, and its natural left action on every $F(X)$ is continuous. Thus, F upgrades to a functor denoted

$$F^\dagger : \mathscr{C} \stackrel{\sim}{\to} \pi_1(\mathscr{C}, F)\text{-}\mathbf{Set}.$$

A basic result states that $F^\dagger$ is an equivalence ([82, Exp.V, Th.4.1]).

Example 3.6.8. Let P be any profinite group. Then there is an obvious injective map

$$P \to \pi_1(P\text{-}\mathbf{Set}, \mathsf{f}_P)$$

and [82, Exp.V, Th.4.1] implies that this map is an isomorphism of profinite groups. In other words, the group P can be recovered, up to unique isomorphism, from the category P-$\mathbf{Set}$ together with its forgetful functor f_P.

Remark 3.6.9. Let $\mathscr{C}, \mathscr{C}'$ be two Galois categories, and $F : \mathscr{C} \to \mathbf{Set}$ a fibre functor.
(i) Any exact functor $G : \mathscr{C}' \to \mathscr{C}$ induces a continuous group homomorphism :

$$\pi_1(G) : \pi_1(\mathscr{C}, F) \to \pi_1(\mathscr{C}', F \circ G) \qquad \omega \mapsto \omega * G.$$

(ii) Furthermore, any isomorphism $\beta : F' \stackrel{\sim}{\to} F$ of fibre functors of $\mathscr{C}$ induces an isomorphism of profinite groups :

$$\pi_1(\beta) : \pi_1(\mathscr{C}', F) \stackrel{\sim}{\to} \pi_1(\mathscr{C}', F) \qquad \omega \mapsto \beta^{-1} \odot \omega \odot \beta$$

(see [82, Exp.V, §4] for all these generalities).
(iii) Let now $(G, \beta) : (\mathscr{C}_1, F_1) \to (\mathscr{C}_2, F_2)$ be a 1-cell of $\mathbf{fibre.Fun}$. Combining (i) and (ii), we deduce a natural continuous group homomorphism

$$\pi_1(G, \beta) : \pi_1(\mathscr{C}_2, F_2) \xrightarrow{\pi_1(G)} \pi_1(\mathscr{C}_1, F_2 \circ G) \xrightarrow{\pi_1(\beta)} \pi_1(\mathscr{C}_1, F_1).$$

Proposition 3.6.10. *With the notation of remark* 3.6.9*, the rule that assigns :*
- *To any object $(\mathscr{C}, F)$ of* $\mathbf{fibre.Fun}$*, the profinite group $\pi_1(\mathscr{C}, F)$*
- *To any* 1*-cell (G, β) of* $\mathbf{fibre.Fun}$*, the continuous map $\pi_1(G, \beta)$*

defines a pseudo-functor

$$\pi_1 : \mathbf{fibre.Fun} \to \mathbf{pf.Grp}^o$$

from the 2*-category of fibre functors, to the opposite of the category of profinite groups (and continuous group homomorphisms).*

Proof. (Here we regard $\mathbf{pf.Grp}$ as a 2-category with trivial 2-cells : see example 2.2.4(i)). Let

$$(\mathscr{C}_1, F_1) \xrightarrow{(G,\beta_G)} (\mathscr{C}_2, F_2) \xrightarrow{(H,\beta_H)} (\mathscr{C}_3, F_3)$$

be any pair of (composable) 1-cells; functoriality on 1-cells amounts to the identity :

$$\pi_1(G, \beta_G) \circ \pi_1(H, \beta_H) = \pi_1(H \circ G, (\beta_H * G) \circ \beta_G)$$

whose detailed verification we leave to the reader. Next, let $\gamma : (G', \beta') \to (G, \beta)$ be a 2-cell between 1-cells $(G', \beta'), (G, \beta) : (\mathscr{C}_1, F_1) \to (\mathscr{C}_2, F_2)$; we have to check that $\pi_1(G, \beta) = \pi_1(G', \beta')$. This identity boils down to the commutativity of the diagram :

$$\begin{array}{ccc}
\pi_1(\mathscr{C}_2, F_2) & \xrightarrow{\pi_1(G')} & \pi_1(\mathscr{C}_1, F_2 \circ G') \\
{\scriptstyle \pi_1(G)}\downarrow & \nearrow{\scriptstyle \pi_1(F_2 * \gamma)} & \downarrow{\scriptstyle \pi_1(\beta')} \\
\pi_1(\mathscr{C}_2, F_2 \circ G) & \xrightarrow{\pi_1(\beta)} & \pi_1(\mathscr{C}_1, F_1).
\end{array}$$

However, the commutativity of the lower triangular subdiagram is clear, hence we are reduced to checking the commutativity of the upper triangular subdiagram; the latter is a special case of the following more general :

Claim 3.6.11. Let $\mathscr{C}$ and $\mathscr{C}'$ be two Galois categories, $G, G' : \mathscr{C}' \to \mathscr{C}$ two exact functors, $\beta : G' \stackrel{\sim}{\to} G$ an isomorphism, and $F : \mathscr{C} \to \mathbf{Set}$ a fibre functor. Then the induced diagram of profinite groups

$$\begin{array}{ccc}
 & \pi_1(\mathscr{C}, F) & \\
{\scriptstyle \pi_1(G)}\swarrow & & \searrow{\scriptstyle \pi_1(G')} \\
\pi_1(\mathscr{C}', F \circ G) & \xrightarrow{\pi_1(F * \beta)} & \pi_1(\mathscr{C}', F \circ G')
\end{array}$$

commutes.

Proof of the claim. Left to the reader. □

Example 3.6.12. Let $\omega : P \to Q$ be a continuous group homomorphism between profinite groups. Clearly $\mathsf{f}_P \circ \mathrm{Res}(\omega) = \mathsf{f}_Q$, and it is easily seen that the resulting diagram

$$\begin{array}{ccc}
P & \xrightarrow{\omega} & Q \\
\downarrow & & \downarrow \\
\pi_1(P\text{-}\mathbf{Set}, \mathsf{f}_P) & \xrightarrow{\pi_1(\mathrm{Res}(\omega))} & \pi_1(Q\text{-}\mathbf{Set}, \mathsf{f}_Q)
\end{array}$$

commutes, where the vertical arrows are the natural identifications given by example 3.6.8 : the verification is left as an exercise to the reader.

3.6.13. Let $\underline{P} := (P_i \mid i \in I)$ be a cofiltered system of profinite groups, with continuous transition maps, and denote by P the limit of this system, in the category of groups. Then P is naturally a closed subgroup of $Q := \prod_{i \in I} P_i$, and the topology $\mathscr{T}$ induced by the inclusion map $P \to Q$ makes it into a compact and complete topological group. Moreover, since the topology of Q is profinite, the same holds for the topology $\mathscr{T}$ (details left to the reader). It is then easily seen that the resulting topological group $(P, \mathscr{T})$ is the limit of the system $\underline{P}$ in the category of profinite groups.

Proposition 3.6.14. *In the situation of* (3.6.13)*, the natural functor*

$$\text{2-}\operatorname*{colim}_{i\in I} P_i\text{-}\mathbf{Set} \to P\text{-}\mathbf{Set}$$

is an equivalence.

Proof. The functor is obviously faithful; let us show that it is also full. Indeed, let $j \in I$ be any index, Σ, Σ' two objects of P_j-$\mathbf{Set}$, and $\varphi : \Sigma \to \Sigma'$ a P-equivariant map; we need to show that φ is already P_i-equivariant, for some index $i \in I$. To this aim, we may as usual replace I by I/j, and assume that j is the final element of I. We may also find an open normal subgroup $H_j \subset P_j$ that acts trivially on both Σ and Σ'. Then, for every $i \in I$, we let $H_i \subset P_i$ be the preimage of P_j, and we set $\overline{P}_i := P_i/H_i$. Let also $\overline{P} := P/H$, where $H \subset P$ is the preimage of H_j; by construction, φ is $\overline{P}$-equivariant. Clearly, we may find $i \in I$ such that the image of $\overline{P}$ in the finite group $\overline{P}_j$ equals the image of $\overline{P}_i$, and for such index i, the induced map $\overline{P} \to \overline{P}_i$ is an isomorphism. Especially, φ is P_i-equivariant, as sought.

Lastly, we show essential surjectivity. Indeed, let Σ be any object of P-$\mathbf{Set}$; we have to show that the P-action on Σ is the restriction of a continuous P_i-action, for a suitable $i \in I$. However, we may find a normal open subgroup $H \subset P$ that acts trivially on Σ. Then there exists a normal open subgroup $L \subset Q$ (notation of (3.6.13)) such that $P \cap L \subset H$. We may also assume that there exist a finite subset $J \subset I$ and for every $i \in J$ an open normal subgroup $L_i \subset P_i$ such that $L = \prod_{i\in J} L_i \times \prod_{i\in I\setminus J} P_i$. Pick an index $j \in I$ that admits morphisms $f_i : j \to i$ in I, for every $i \in J$, and let $L'_i \subset P_j$ denote the preimage of L_i under the corresponding map $P_j \to P_i$. Finally, set $H_j := \bigcap_{i\in J} L'_i$. By construction, H contains the preimage of H_j in P, and we may therefore assume that H is this preimage. We may replace as usual I by I/j, and assume that j is the final element of I. Then, for every $i \in I$, we let H_i denote the preimage of H_j in P_i, and we set $\overline{P}_i := P_i/H_i$. Set as well $\overline{P} := P/H$. Clearly, there exists $i \in I$ such that the image of the induced map $\overline{P} \to \overline{P}_j$ equals the image of $\overline{P}_i$; for such index i, the induced map $\overline{P} \to \overline{P}_i$ is an isomorphism. Thus, Σ the restriction of an object of P_i-$\mathbf{Set}$, as wished. □

3.6.15. We consider now a situation that generalizes slightly that of (3.6.13). Namely, let I be a small filtered category, and

$$(\mathscr{C}_\bullet, F_\bullet) : I \to \mathbf{fibre.Fun} \qquad i \mapsto (\mathscr{C}_i, F_i)$$

a pseudo-functor. By proposition 3.6.10, the composition of π_1 and $(\mathscr{C}_\bullet, F_\bullet)$ is a functor

$$\pi_1(\mathscr{C}_\bullet, F_\bullet) : I^o \to \mathbf{pf.Grp} \qquad i \mapsto P_i := \pi_1(\mathscr{C}_i, F_i).$$

Let P denote the limit (in $\mathbf{pf.Grp}$) of the cofiltered system $P_\bullet$, and set

$$\mathscr{C} := \text{2-}\operatorname*{colim}_I \mathscr{C}_\bullet$$

where the 2-colimit is formed in the 2-category of small categories. We may then state :

Corollary 3.6.16. *In the situation of* (3.6.15)*, there exists a natural equivalence :*

$$\mathscr{C} \xrightarrow{\sim} P\text{-}\mathbf{Set}.$$

Proof. Recall that $(\mathscr{C}_\bullet, F_\bullet)$ is the datum of isomorphisms

$$\beta_\varphi : F_j \xrightarrow{\sim} F_i \circ \mathscr{C}_\varphi \qquad \text{for every morphism } \varphi : j \to i \text{ in } I$$

and 2-cells :

$$\tau_{\psi,\varphi} : (\mathscr{C}_{\psi\circ\varphi}, \beta_{\psi\circ\varphi}) \xrightarrow{\sim} (\mathscr{C}_\psi, \beta_\psi) \circ (\mathscr{C}_\varphi, \beta_\varphi) \qquad \text{for every composition } j \xrightarrow{\varphi} i \xrightarrow{\psi} k$$

that – by definition – satisfy the identities :

$$(3.6.17) \qquad (\beta_\psi * \mathscr{C}_\varphi) \odot \beta_\varphi = (F_k * \tau_{\psi,\varphi}) \odot \beta_{\psi\circ\varphi} \qquad \text{for every composition } j \xrightarrow{\varphi} i \xrightarrow{\psi} k$$

as well as the composition identities :

$$((\mathscr{C}_\mu, \beta_\mu) * \tau_{\psi,\varphi}) \odot \tau_{\mu,\psi\circ\varphi} = (\tau_{\mu,\psi} * (\mathscr{C}_\varphi, \beta_\varphi)) \odot \tau_{\mu\circ\psi,\varphi} \qquad \text{for compositions } j \xrightarrow{\varphi} i \xrightarrow{\psi} k \xrightarrow{\mu} l.$$

Let $P_\bullet\text{-}\mathbf{Set} : I \to \mathbf{Cat}$ denote the functor given by the rule : $i \mapsto P_i\text{-}\mathbf{Set}$ for every $i \in \mathrm{Ob}(I)$, and $\varphi \mapsto \mathrm{Res}(P_\varphi)$, where $P_\varphi := \pi_1(\mathscr{C}_\varphi, \beta_\varphi)$ for every morphism φ of I. In view of proposition 3.6.14 and lemma 2.5.3, it suffices to show that the rule : $i \mapsto F_i^\dagger$ for every $i \in \mathrm{Ob}(I)$ (notation of (3.6.7)), extends to a pseudo-natural isomorphism

$$F_\bullet^\dagger : \mathscr{C}_\bullet \xrightarrow{\sim} P_\bullet\text{-}\mathbf{Set}.$$

Indeed, let $\varphi : j \to i$ be any morphism of I, and X any object of $\mathscr{C}_j$; we remark that the bijection

$$\beta_\varphi(X) : \mathrm{Res}(P_\varphi)(F_j^\dagger X) \xrightarrow{\sim} F_i^\dagger \circ \mathscr{C}_\varphi(X)$$

is P_i-equivariant; the proof amounts to unwinding the definitions, and shall be left to the reader. Hence we get an isomorphism of functors $\beta_\varphi^\dagger : \mathrm{Res}(P_\varphi) \circ F_j^\dagger \xrightarrow{\sim} F_i^\dagger \circ \mathscr{C}_\varphi$, and from (3.6.17) we deduce the identities :

$$(\beta_\psi^\dagger * \mathscr{C}_\varphi) \odot (\mathrm{Res}(P_\varphi) * \beta_\varphi^\dagger) = (F_k^\dagger * \tau_{\psi,\varphi}) \odot \beta_{\psi\circ\varphi}^\dagger \qquad \text{for every composition } j \xrightarrow{\varphi} i \xrightarrow{\psi} k.$$

The latter show that the system $\beta_\bullet^\dagger$ fulfills the coherence axiom for a pseudo-natural transformation, as required. □

Remark 3.6.18. (i) Keep the situation of (3.6.15), and let $a_\bullet : \mathscr{C}_\bullet \Rightarrow \mathscr{C}$ be the universal pseudo-cocone induced by the pseudo-functor $\mathscr{C}_\bullet$. We may regard the pseudo-functor $(\mathscr{C}_\bullet, F_\bullet)$ as a pseudo-cocone $F_\bullet : \mathscr{C}_\bullet \Rightarrow \mathbf{Set}$ whose vertex is the category $\mathbf{Set}$. Then, by the universal property of colimits, we get a functor $F : \mathscr{C} \to \mathbf{Set}$ and an isomorphism

$$\sigma_\bullet : F * a_\bullet \xrightarrow{\sim} F_\bullet.$$

On the other hand, let $r_\bullet : P_\bullet\text{-}\mathbf{Set} \Rightarrow P\text{-}\mathbf{Set}$ be the natural cocone (so r_i is the restriction functor corresponding to the natural map $P \to P_i$, for every $i \in \mathrm{Ob}(I)$); the equivalence G of corollary 3.6.16 is deduced from the pseudo-cocone $r_\bullet \circ F_\bullet^\dagger : \mathscr{C}_\bullet \to P\text{-}\mathbf{Set}$ (where $F_\bullet^\dagger$ is as in the proof of corollary 3.6.16), so we have an isomorphism of pseudo-functors

$$t_\bullet : G * a_\bullet \xrightarrow{\sim} r_\bullet \circ F_\bullet^\dagger$$

whence an isomorphism

$$\mathsf{f}_P * t_\bullet : (\mathsf{f}_P \circ G) * a_\bullet \xrightarrow{\sim} \mathsf{f}_P * (r_\bullet \circ F_\bullet^\dagger) = F_\bullet$$

(notation of (3.6.7)). There follows an isomorphism $F * a_\bullet \xrightarrow{\sim} (\mathsf{f}_P \circ G) * a_\bullet$; by the universal property of the 2-colimit, the latter must come from a unique isomorphism $\vartheta : F \xrightarrow{\sim} \mathsf{f}_P \circ G$. Especially, we see that F is also a fibre functor, and we get a pseudo-cocone

$$(a_\bullet, \sigma_\bullet) : (\mathscr{C}_\bullet, F_\bullet) \Rightarrow (\mathscr{C}, F).$$

It is now immediate that $(\mathscr{C}, F)$ is the 2-colimit (in the 2-category $\mathbf{fibre.Fun}$) of the pseudo-functor $(\mathscr{C}_\bullet, F_\bullet)$, and $(a_\bullet, \sigma_\bullet)$ is the corresponding universal pseudo-cocone.

(ii) Likewise, $r_\bullet$ may be regarded as a universal pseudo-cocone

$$r_\bullet : (P_\bullet\text{-}\mathbf{Set}, \mathsf{f}_{P_\bullet}) \Rightarrow (P\text{-}\mathbf{Set}, \mathsf{f}_P)$$

(with trivial coherence constraint), and the coherence constraint $\beta_\bullet^\dagger$ as in the proof of corollary 3.6.16 yields a pseudo-natural equivalence

$$F_\bullet^\dagger : (\mathscr{C}_\bullet, F_\bullet) \xrightarrow{\sim} (P_\bullet\text{-}\mathbf{Set}, \mathsf{f}_{P_\bullet})$$

as well as an isomorphism

$$(G, \vartheta) * (a_\bullet, \sigma_\bullet) \xrightarrow{\sim} r_\bullet \odot F_\bullet^\dagger.$$

Thus, for every $i \in \mathrm{Ob}(I)$ we get a 2-cell of $\mathbf{fibre.Fun}$:

$$(G, \vartheta) \circ (a_i, \sigma_i) \to r_i \circ F_i^\dagger$$

whence – by proposition 3.6.10 – a commutative diagram of profinite groups :

$$\begin{array}{ccc} P & \longrightarrow & P_i \\ {\scriptstyle \pi_1(G,\vartheta)} \downarrow & & \downarrow {\scriptstyle \pi_1(F_i^\dagger)} \\ \pi_1(\mathscr{C}, F) & \xrightarrow{\pi_1(a_i,\sigma_i)} & \pi_1(\mathscr{C}_i, F_i). \end{array}$$

3.6.19. Let $(\mathscr{C}, F)$ be a fibre functor, and X a connected object of $\mathscr{C}$ (example 1.2.16(iv)); pick any $\xi \in F(X)$, and let $H_\xi \subset \pi_1(\mathscr{C}, F)$ be the stabilizer of ξ for the natural left action of $\pi_1(\mathscr{C}, F)$ on $F(X)$. For every object $f : Y \to X$ of $\mathscr{C}/X$, we set

$$F_\xi(f) := F(f)^{-1}(\xi) \subset F(Y).$$

It is clear that the rule $f \mapsto F_\xi(f)$ yields a functor $F_\xi^\dagger : \mathscr{C}/X \to H_\xi\text{-}\mathbf{Set}$, which we call the *subfunctor of $F_{|X}$ selected by ξ*.

Proposition 3.6.20. *In the situation of* (3.6.19)*, we have :*

(i) *$\mathscr{C}/X$ is also a Galois category, and $F_\xi := \mathsf{f}_{H_\xi} \circ F_\xi^\dagger$ is a fibre functor for $\mathscr{C}/X$.*

(ii) *The functor $F_\xi^\dagger$ induces a natural isomorphism of profinite groups :*

$$\pi_1(F_\xi^\dagger) : H_\xi \xrightarrow{\sim} \pi_1(\mathscr{C}/X, F_\xi).$$

Proof. The fibre functor F induces an equivalence of categories

$$\mathscr{C}/X \xrightarrow{\sim} \pi_1(\mathscr{C}, F)\text{-}\mathbf{Set}/F(X).$$

On the other hand, since X is connected, there exists a unique isomorphism $\omega : F(X) \xrightarrow{\sim} G/H_\xi$ of G-sets such that $\omega(\xi) = H_\xi$, and then the discussion of (3.6.4) yields an equivalence

$$\pi_1(\mathscr{C}, F)\text{-}\mathbf{Set}/F(X) \xrightarrow{\sim} H_\xi\text{-}\mathbf{Set}.$$

A simple inspection shows that the resulting equivalence $\mathscr{C}/X \xrightarrow{\sim} H_\xi\text{-}\mathbf{Set}$ is none else than the functor $F_\xi^\dagger$, so the assertion follows from remark 3.6.9(i) and example 3.6.8. □

3.6.21. Let $(\mathscr{C}, F)$ be a fibre functor, and let us now fix a cleavage $\mathsf{c} : \mathscr{C}^o \to \mathbf{Cat}$ for the fibred category $\mathsf{t} : \mathsf{Morph}(\mathscr{C}) \to \mathscr{C}$ (see example 3.1.3(iii)). Also, let I be a small cofiltered category, and $X_\bullet : I \to \mathscr{C}$ a functor such that X_i is a connected object of $\mathscr{C}$, for every $i \in \mathrm{Ob}(I)$; we pick an element

$$\xi_\bullet \in \lim_I F \circ X_\bullet.$$

In other words, $\xi_\bullet := (\xi_i \in F(X_i) \mid i \in I)$ is a compatible system of elements such that

$$F(\varphi)(\xi_j) = \xi_i \qquad \text{for every morphism } \varphi : j \to i \text{ in } I.$$

For every $i \in I$, we denote by $H_i \subset \pi_1(\mathscr{C}, F)$ the stabilizer of ξ_i for the left action of $\pi_1(\mathscr{C}, F)$ on $F(X_i)$. Clearly, any morphism $j \to i$ induces an inclusion $H_j \subset H_i$. Furthermore, let

$$F_i^\dagger : \mathscr{C}/X_i \to H_i\text{-}\mathbf{Set} \qquad \text{for every } i \in I$$

be the subfunctor selected by ξ_i, and set $F_i := \mathsf{f}_{H_i} \circ F_i^\dagger$. Let $\varphi : j \to i$ be a morphism of I; to the corresponding morphism $X_\varphi : X_j \to X_i$, the cleavage c attaches a pull-back functor

$$X_\varphi^* : \mathscr{C}/X_i \to \mathscr{C}/X_j.$$

Especially, for any object $Y \in \mathrm{Ob}(\mathscr{C}/X_i)$ we have the cartesian diagram in $\mathscr{C}$:

$$\begin{array}{ccc} X_\varphi^*(Y) & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X_j & \xrightarrow{X_\varphi} & X_i \end{array}$$

whence, since F is exact, a natural bijection :

$$F(X_\varphi^*(Y)) \stackrel{\sim}{\to} F(Y) \times_{F(X_i)} F(X_j)$$

which in turns yields a bijection :

$$F_j^\dagger(X_\varphi^*(Y)) \stackrel{\sim}{\to} F(Y) \times_{F(X_i)} \{\xi_j\} = F_i^\dagger(Y) \times \{\xi_j\}.$$

That is, we have a natural isomorphism of functors :

$$\alpha_\varphi^\dagger : F_j^\dagger \circ X_\varphi^* \stackrel{\sim}{\to} \mathrm{Res}_{H_i}^{H_j} \circ F_i^\dagger \tag{3.6.22}$$

and since the X_φ^* are exact functors, it is easily seen that the isomorphisms $\alpha_\varphi := \mathsf{f}_j * \alpha_\varphi^\dagger$ yield a pseudo-functor

$$I \to \mathbf{fibre.Fun} \qquad i \mapsto (\mathscr{C}/X_i, F_i) \qquad \varphi \mapsto (X_\varphi^*, \alpha_\varphi). \tag{3.6.23}$$

Moreover, $\alpha_\varphi^\dagger$ can be seen as a 2-cell of **fibre.Fun** : $F_j^\dagger \circ (X_\varphi^*, \alpha_\varphi) \stackrel{\sim}{\to} \mathrm{Res}_{H_j}^{H_i} \circ F_i^\dagger$, whence a commutative diagram of profinite groups :

$$\begin{array}{ccc} H_j & \longrightarrow & H_i \\ {\scriptstyle \pi_1(F_j^\dagger)}\downarrow & & \downarrow{\scriptstyle \pi_1(F_i^\dagger)} \\ \pi_1(\mathscr{C}/X_j, F_j) & \xrightarrow{\pi_1(X_\varphi^*, \alpha_\varphi)} & \pi_1(\mathscr{C}/X_i, F_i) \end{array}$$

whose top horizontal arrow is the inclusion map. Especially, notice that the map $\pi_1(X_\varphi^*, \alpha_\varphi)$ does not depend on the chosen cleavage; this can also be seen by remarking that any two cleavages c, c' are related by a pseudo-natural isomorphism $\mathsf{c} \stackrel{\sim}{\to} \mathsf{c}'$ (details left to the reader).

3.6.24. Let $(\mathscr{C}/X, F_\xi)$ be the 2-colimit of the pseudo-functor (3.6.23), as in remark 3.6.18(i), and fix a corresponding universal pseudo-cocone $(a_\bullet, \sigma_\bullet) : (\mathscr{C}/X_\bullet, F_\bullet) \Rightarrow (\mathscr{C}/X, F_\xi)$. We may then state :

Corollary 3.6.25. *In the situation of* (3.6.24)*, there exists a natural isomorphism of profinite groups :*

$$H := \bigcap_{i \in \mathrm{Ob}(I)} H_i \stackrel{\sim}{\to} \pi_1(\mathscr{C}/X, F_\xi)$$

which fits into a commutative diagram :

$$\begin{array}{ccc} H & \longrightarrow & H_i \\ \downarrow & & \downarrow{\scriptstyle \pi_1(F_i^\dagger)} \\ \pi_1(\mathscr{C}/X, F_\xi) & \xrightarrow{\pi_1(a_i, \sigma_i)} & \pi_1(\mathscr{C}/X_i, F_i) \end{array} \qquad \textit{for every } i \in \mathrm{Ob}(I) \tag{3.6.26}$$

whose top horizontal arrow is the inclusion map.

Proof. By corollary 3.6.16, we have an isomorphism of $\pi_1(\mathscr{C}/X, F_\xi)$ with the limit of the cofiltered system $(\pi_1(\mathscr{C}/X, F_i) \mid i \in \mathrm{Ob}(I))$; on the other hand, the discussion of (3.6.21) shows that the latter system is naturally isomorphic to the system $(H_i \mid i \in \mathrm{Ob}(I))$. Lastly, the commutativity of (3.6.26) follows from remark 3.6.18(ii). □

3.7. **Tensor categories and abelian categories.** In this section we assemble some basic definitions and results that pertain to abelian categories and other related classes of categories with extra structure.

Definition 3.7.1. A *tensor category* is a datum $\underline{\mathscr{C}} := (\mathscr{C}, \otimes, \Phi, \Psi)$ consisting of a category $\mathscr{C}$, a functor

$$\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C} \quad : \quad (X, Y) \mapsto X \otimes Y$$

and natural isomorphisms :

$$\Phi_{X,Y,Z} : X \otimes (Y \otimes Z) \xrightarrow{\sim} (X \otimes Y) \otimes Z \qquad \Psi_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$$

for every $X, Y, Z \in \mathrm{Ob}(\mathscr{C})$, called respectively the *associativity* and *commutativity constraints* of $\underline{\mathscr{C}}$, that satisfy the following axioms.

(a) *Coherence axiom* : the diagram

$$\begin{array}{ccccc}
X \otimes (Y \otimes (Z \otimes T)) & \xrightarrow{\Phi_{X,Y,Z\otimes T}} & (X \otimes Y) \otimes (Z \otimes T) & \xrightarrow{\Phi_{X\otimes Y,Z,T}} & ((X \otimes Y) \otimes Z) \otimes T \\
{\scriptstyle X\otimes\Phi_{Y,Z,T}}\downarrow & & & & \uparrow{\scriptstyle \Phi_{X,Y,Z}\otimes T} \\
X \otimes ((Y \otimes Z) \otimes T) & & \xrightarrow{\qquad\qquad\Phi_{X,Y\otimes Z,T}\qquad\qquad} & & (X \otimes (Y \otimes Z)) \otimes T
\end{array}$$

commutes, for every $X, Y, Z, T \in \mathrm{Ob}(\mathscr{C})$.

(b) *Compatibility axiom* : the diagram

$$\begin{array}{ccccc}
X \otimes (Y \otimes Z) & \xrightarrow{\Phi_{X,Y,Z}} & (X \otimes Y) \otimes Z & \xrightarrow{\Psi_{X\otimes Y,Z}} & Z \otimes (X \otimes Y) \\
{\scriptstyle X\otimes\Psi_{Y,Z}}\downarrow & & & & \downarrow{\scriptstyle \Phi_{Z,X,Y}} \\
X \otimes (Z \otimes Y) & \xrightarrow{\Phi_{X,Z,Y}} & (X \otimes Z) \otimes Y & \xrightarrow{\Psi_{X,Z}\otimes Y} & (Z \otimes X) \otimes Y
\end{array}$$

commutes, for every $X, Y, Z \in \mathrm{Ob}(\mathscr{C})$.

(c) *Commutation axiom* : we have $\Psi_{Y,X} \circ \Psi_{X,Y} = \mathbf{1}_{X\otimes Y}$ for every $X, Y \in \mathrm{Ob}(\mathscr{C})$

(d) *Unit axiom* : there exist an object $U \in \mathrm{Ob}(\mathscr{C})$ and an isomorphism $u : U \xrightarrow{\sim} U \otimes U$ such that the functor

$$\mathscr{C} \to \mathscr{C} \quad : \quad X \mapsto U \otimes X \tag{3.7.2}$$

is an equivalence. One says that (U, u) is a *unit object* of $\underline{\mathscr{C}}$.

Lemma 3.7.3. *Let* $\underline{\mathscr{C}} := (\mathscr{C}, \otimes, \Phi, \Psi)$ *be any tensor category. The diagram*

$$\begin{array}{ccccc}
X \otimes (Y \otimes Z) & \xrightarrow{\Phi_{X,Y,z}} & (X \otimes Y) \otimes Z & \xrightarrow{\Psi_{X,Y}\otimes Z} & (Y \otimes X) \otimes Z \\
{\scriptstyle X\otimes\Psi_{Y,Z}}\downarrow & & & & \uparrow{\scriptstyle \Phi_{Y,X,Z}} \\
X \otimes (Z \otimes Y) & \xrightarrow{\Phi_{X,Z,Y}} & (X \otimes Z) \otimes Y & \xrightarrow{\Psi_{X\otimes Z,Y}} & Y \otimes (X \otimes Z)
\end{array}$$

commutes, for every $X, Y, Z \in \mathrm{Ob}(\mathscr{C})$.

Proof. To ease notation, we shall omit the tensor symbol $\otimes$ between objects, and we shall drop the subscript from Φ and Ψ when we display a diagram. It suffices to consider the diagram

$$\begin{tikzcd} & Y(XZ) \arrow[rrr, "\Phi"] \arrow[d, "Y\otimes\Psi"] & & & (YX)Z \arrow[d, "\Psi"] & \\ & Y(ZX) \arrow[r, "\Phi"] & (YZ)X \arrow[r, "\Psi\otimes X"] & (ZY)X & Z(YX) \arrow[l, "\Phi"'] & \\ (XZ)Y \arrow[ruu, "\Psi"] \arrow[r, "\Psi\otimes Y"] & (ZX)Y \arrow[u, "\Psi"] & & & Z(XY) \arrow[lll, "\Phi"'] \arrow[u, "Z\otimes\Psi"'] & (XY)Z \arrow[l, "\Psi"'] \arrow[luu, "\Psi\otimes Z"'] \\ X(ZY) \arrow[u, "\Phi"] & & & & & X(YZ) \arrow[lllll, "X\otimes\Psi"'] \arrow[u, "\Phi"'] \end{tikzcd}$$

whose two triangular subdiagrams commute by naturality of Ψ, and whose three rectangular subdiagrams commute by compatibility. $\square$

Definition 3.7.4. Let $\underline{\mathscr{C}} := (\mathscr{C}, \otimes, \Phi, \Psi)$ and $\underline{\mathscr{C}}' := (\mathscr{C}', \otimes', \Phi', \Psi')$ be two tensor categories. A *tensor functor* $\underline{\mathscr{C}} \to \underline{\mathscr{C}}'$ is a pair (F, c) consisting of a functor $F : \mathscr{C} \to \mathscr{C}'$ and a natural isomorphism

$$c_{X,Y} : FX \otimes FY \xrightarrow{\sim} F(X \otimes Y) \qquad \text{for all } X, Y \in \mathrm{Ob}(\mathscr{C})$$

such that the following holds.

(a) For every objects X, Y, Z of $\mathscr{C}$, the diagram

$$\begin{tikzcd} FX \otimes (FY \otimes FZ) \arrow[r, "FX\otimes c_{YZ}"] \arrow[d, "\Phi'_{FX,FY,FZ}"'] & FX \otimes F(Y \otimes Z) \arrow[r, "c_{X,Y\otimes Z}"] & F(X \otimes (Y \otimes Z)) \arrow[d, "F(\Phi_{X,Y,Z})"] \\ (FX \otimes FY) \otimes FZ \arrow[r, "c_{X,Y}\otimes FZ"] & F(X \otimes Y) \otimes FZ \arrow[r, "c_{X\otimes Y,Z}"] & F((X \otimes Y) \otimes Z) \end{tikzcd}$$

commutes.

(b) For all objects X, Y of $\mathscr{C}$, the diagram

$$\begin{tikzcd} FX \otimes FY \arrow[r, "c_{X,Y}"] \arrow[d, "\Psi_{FX,FY}"'] & F(X \otimes Y) \arrow[d, "F(\Psi_{X,Y})"] \\ FY \otimes FX \arrow[r, "c_{Y,X}"] & F(Y \otimes X) \end{tikzcd}$$

commutes.

(c) If (U, u) is a unit object of $\underline{\mathscr{C}}$, then $(FU, c^{-1}_{U,U} \circ Fu)$ is a unit object of $\underline{\mathscr{C}}'$.

Remark 3.7.5. (i) Lemma 3.7.3 illustrates a general principle valid in every tensor categor $\underline{\mathscr{C}}$: namely, say that $X_1, \dots, X_n$ is a sequence of *distinct* objects of $\mathscr{C}$, and X' and X'' are obtained from these two sequences by taking tensor products several times, and in any order, in which case we say that X' and X'' *have no repetitions*. Now, there will be usually various ways to combine the associativity and commutativity constraints, in order to exhibit some isomorphism $X' \xrightarrow{\sim} X''$. However, the resulting isomorphism shall be independent of the way in which it is expressed as such a combination. This follows from a theorem of Mac Lane. To formalize this result, one could observe that, for any set Σ, there exists a universal tensor category T_Σ "generated by Σ", *i.e.* such that – for any other tensor category $\underline{\mathscr{C}}$ – any mapping $\Sigma \to \mathrm{Ob}(\mathscr{C})$ extends uniquely, up to isomorphism, to a tensor functor $T_\Sigma \to \underline{\mathscr{C}}$. Then Mac Lane's theorem says that, for every set Σ, and every object $X \in \mathrm{Ob}(T_\Sigma)$ that has no repetitions, the group $\mathrm{Aut}_{T_\Sigma}(X)$ is trivial. Instead of relying on such a general result, we shall make *ad hoc* verifications, as in the proof of lemma 3.7.3 and of the forthcoming proposition 3.7.6. However, in view of this principle, in the following we shall often omit a detailed description of the isomorphism that we choose to connect two given objects that are thus related : the reader will be able in any case to produce one isomorphism, and the principle says that these choices cannot be source of ambiguities.

(ii) Let $\mathscr{D}$ be any category, and $\underline{\mathscr{C}}$ a tensor category. Then notice that $\mathsf{Fun}(\mathscr{D}, \mathscr{C})$ inherits from $\underline{\mathscr{C}}$ a natural tensor category structure : we leave to the reader the task of spelling out the details. Moreover notice that if $(F, c) : \underline{\mathscr{C}}_1 \to \underline{\mathscr{C}}_2$ and $(F', c') : \underline{\mathscr{C}}_2 \to \underline{\mathscr{C}}_3$ are any two tensor functors between tensor categories, then the composition

$$(F' \circ F, (F' * c) \odot (c' * (F \times F))) : \underline{\mathscr{C}}_1 \to \underline{\mathscr{C}}_3$$

is again a tensor functor.

Proposition 3.7.6. *Let (U, u) be a unit object of a tensor category $\underline{\mathscr{C}}$. Then there exists a unique natural isomorphism*

$$u_X : X \xrightarrow{\sim} U \otimes X \qquad \text{for every } X \in \mathrm{Ob}(\mathscr{C})$$

such that $u_U = u$, and such that the diagrams

$$\begin{array}{ccc} X \otimes Y & \xrightarrow{u_{X\otimes Y}} & U \otimes (X \otimes Y) \\ & \searrow^{u_X \otimes Y} & \downarrow{\scriptstyle \Phi_{U,X,Y}} \\ & & (U \otimes X) \otimes Y \end{array} \qquad \begin{array}{ccc} X \otimes Y & \xrightarrow{u_X \otimes Y} & (U \otimes X) \otimes Y \\ {\scriptstyle X \otimes u_Y}\downarrow & & \downarrow{\scriptstyle \Psi_{U,X} \otimes Y} \\ X \otimes (U \otimes Y) & \xrightarrow{\Phi_{X,U,Y}} & (X \otimes U) \otimes Y \end{array}$$

commute for every $X, Y \in \mathrm{Ob}(\mathscr{C})$.

Proof. Since (3.7.2) is an equivalence, there exists a unique isomorphism u_X fitting into the commutative diagram

$$\begin{array}{ccc} UX & \xrightarrow{u \otimes X} & (UU)X \\ & \searrow^{U \otimes u_X} & \uparrow{\scriptstyle \Phi} \\ & & U(UX). \end{array}$$

With this definition, the naturality of the rule : $X \mapsto u_X$ is clear. In order to check the commutativity of the first diagram, it suffices to show that

$$(U \otimes \Phi_{U,X,Y}) \circ (U \otimes u_{XY}) = U \otimes (u_X \otimes Y). \tag{3.7.7}$$

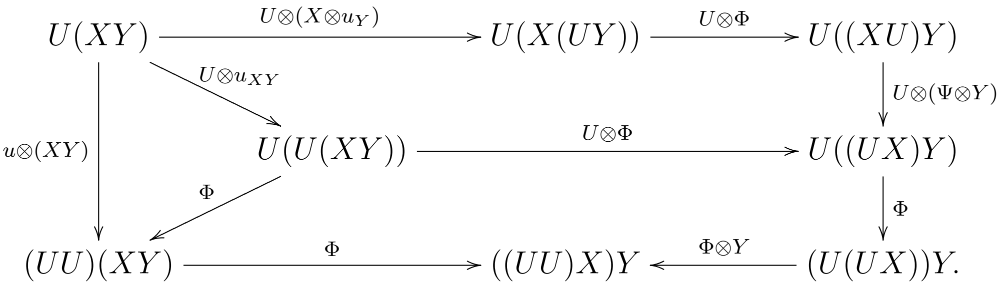
However, set $\Theta := (\Phi_{U,U,X} \otimes Y) \circ \Phi_{U,UX,Y}$; we have :

$$\begin{aligned} \Theta \circ (U \otimes (u_X \otimes Y)) &= (\Phi_{U,U,X} \otimes Y) \circ ((U \otimes u_X) \otimes Y) \circ \Phi_{U,X,Y} \\ &= ((u \otimes X) \otimes Y) \circ \Phi_{U,X,Y} \\ &= \Phi_{UU,X,Y} \circ (u \otimes (XY)) \\ &= \Phi_{UU,X,Y} \circ \Phi_{U,U,XY} \circ (U \otimes u_{XY}) \\ &= \Theta \circ (U \otimes \Phi_{U,X,Y}) \circ (U \otimes u_{XY}) \end{aligned}$$

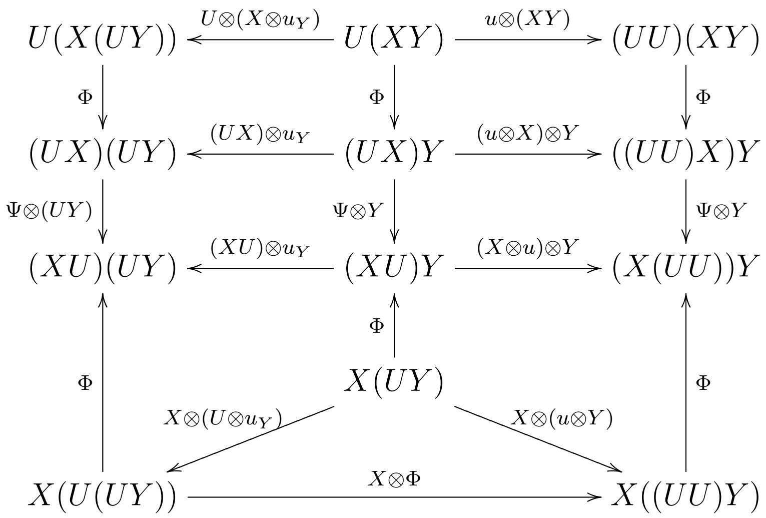
where the first and third identities hold by naturality of Φ, the second and fourth by the definition of u_X and respectively $u_{X \otimes Y}$, and the fifth by coherence. Since Θ is an isomorphism, we get (3.7.7). Next, in light of the foregoing, the second diagram commutes if and only if the diagram

$$\begin{array}{ccccc} XY & \xrightarrow{u_{XY}} & U(XY) & \xrightarrow{\Phi} & (UX)Y \\ & \searrow^{X \otimes u_Y} & & & \downarrow{\scriptstyle \Psi \otimes Y} \\ & & X(UY) & \xrightarrow{\Phi} & (XU)Y \end{array} \tag{3.7.8}$$

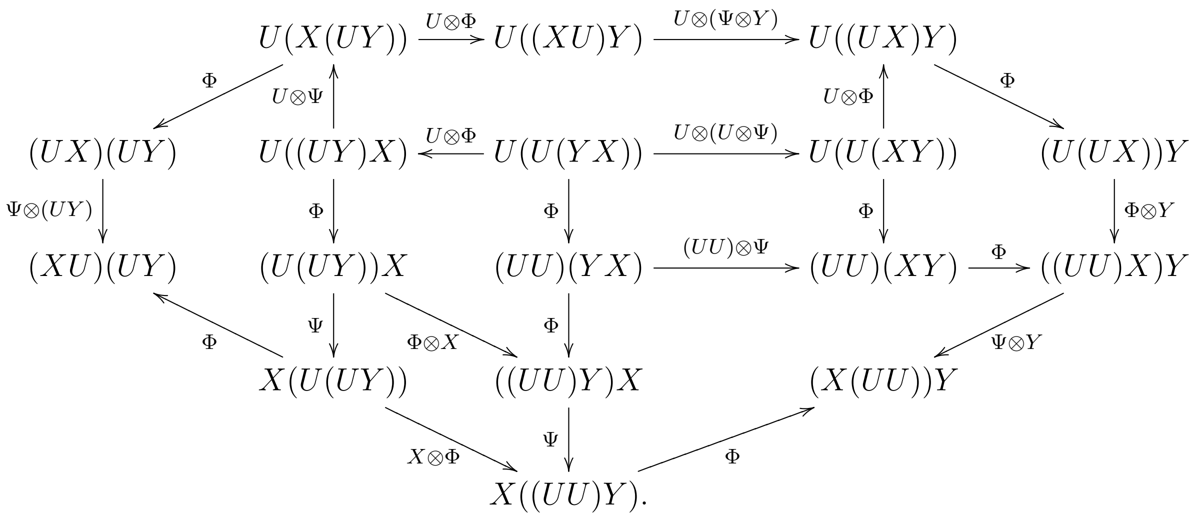
commutes, and it suffices to check that $U \otimes$ (3.7.8) commutes. To this aim, we consider the diagram



whose lower subdiagram commutes by the coherence axiom for (U, U, X, Y), whose upper subdiagram is equivalent to (3.7.8), since $\Psi_{U,X}^{-1} = \Psi_{X,U}$, and whose triangular subdiagram commutes by definition of u_{XY}. Hence, we are reduced to showing that the outer rectangular subdiagram of the above diagram commutes. However, we have a commutative diagram :



so we are further reduced to checking the commutativity of the diagram :



However, the leftmost and the lower triangular subdiagrams commute by compatibility, and the upper rightmost subdiagram commutes by coherence. The lower leftmost subdiagram commutes by naturality of Ψ, and the central square sudiagram commutes by naturality of Φ. The remaining central subdiagram commutes by coherence, and the top rectangular subdiagram is of the form $U \otimes D$, where D is a diagram that commutes by virtue of lemma 3.7.3.

The uniqueness of u_X is clear by inspecting the second diagram, with $Y = U$. □

Remark 3.7.9. (i) Keep the notation of proposition 3.7.6. As a consequence of the naturality of u_X, we get a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{u_X} & U \otimes X \\ {\scriptstyle u_X}\downarrow & & \downarrow{\scriptstyle u\otimes u_X} \\ U \otimes X & \xrightarrow{u_{U\otimes X}} & U \otimes (U \otimes X) \end{array}$$

for every object X of $\mathscr{C}$. In other words :

$$u_{U\otimes X} = U \otimes u_X \qquad \text{for every } X \in \mathrm{Ob}(\mathscr{C}).$$

(ii) Let $X = Y = U$ in the second diagram of proposition 3.7.6; we obtain a commutative diagram

$$\begin{array}{ccc} U \otimes U & \xrightarrow{u\otimes U} & (U \otimes U) \otimes U \\ {\scriptstyle U\otimes u}\downarrow & & \downarrow{\scriptstyle \Psi_{U,U}\otimes U} \\ U \otimes (U \otimes U) & \xrightarrow{\Phi_{U,U,U}} & (U \otimes U) \otimes U \end{array}$$

Since $u = u_U$, we may combine with (i), to deduce that $\Psi_{U,U} \otimes U = \mathbf{1}_{(U\otimes U)\otimes U} = \mathbf{1}_{U\otimes U} \otimes U$. By naturality of Ψ, it follows that $U \otimes \Psi_{U,U} = U \otimes \mathbf{1}_{U\otimes U}$, and since (3.7.2) is an equivalence, we conclude that

$$\Psi_{U,U} = \mathbf{1}_{U\otimes U}.$$

Example 3.7.10. Let $\mathscr{C}$ be any category with small Hom-sets, in which finite products are representable. For every pair (X, Y) of objects of $\mathscr{C}$, pick an object $X \otimes Y$ representing their product, and fix also two projections $p_{X,Y} : X \otimes Y \to X$, $q_{X,Y} : X \otimes Y \to Y$ inducing an isomorphism of functors

$$h_{X\otimes Y} \xrightarrow{\sim} h_X \times h_Y \quad : \quad \varphi \mapsto (p_{X,Y} \circ \varphi, q_{X,Y} \circ \varphi) \qquad \text{for every } Z \in \mathrm{Ob}(\mathscr{C}) \text{ and } \varphi \in h_{X\times Y}(Z)$$

(notation of (1.2.4)). If (X', Y') is another such pair, and $(g, h) : (X, Y) \to (X', Y')$ any morphism in $\mathscr{C} \times \mathscr{C}$, then there exists a unique morphism $f : X \otimes Y \to X' \otimes Y'$ such that

$$(p_{X',Y'} \circ f, q_{X',Y'} \circ f) = (g \circ p_{X,Y}, h \circ p_{X,Y})$$

and we set $g \otimes h := f$. These rules define a functor $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$. For every three objects X, Y, Z there is a natural isomorphism $X \otimes (Y \otimes Z) \xrightarrow{\sim} (X \otimes Y) \otimes Z$ that yields an associativity constraint for $\otimes$; namely, we let

$$\Phi_{X,Y,Z} := (p_{X,Y\otimes Z} \otimes (p_{Y,Z} \otimes q_{X,Y\otimes Z})) \otimes (q_{Y,Z} \circ q_{X,Y\otimes Z}).$$

Likewise, we get a commutativity constraint by setting

$$\Psi_{X,Y} := q_{X,Y} \otimes p_{X,Y} \qquad \text{for every } X, Y \in \mathrm{Ob}(\mathscr{C}).$$

The verifications of the axioms of definition 3.7.1 are lengthy but straightforward, and shall be left to the reader. If U is any final object of $\mathscr{C}$ (example 1.2.16(iv)), then there exists a unique morphism $u : U \to U \otimes U$ which is easily seen to be an isomorphism, and the pair (U, u) yields a unit for $\otimes$. In this way, any category with finite products and small Hom-sets is naturally endowed with a structure of tensor category. Notice that, for this tensor structure on $\mathscr{C}$ (and indeed, for most of the tensor categories that are found in applications), the existence of functorial isomorphisms u_X fulfilling the conditions of proposition 3.7.6, is self-evident.

Definition 3.7.11. Let $(\mathscr{C}, \otimes, \Phi, \Psi)$ be a tensor category, $X \in \mathrm{Ob}(\mathscr{C})$ any object, and suppose that the functor

$$- \otimes X : \mathscr{C} \to \mathscr{C} \qquad Y \mapsto Y \otimes X$$

admits a right adjoint :

$$\mathscr{H}om(X, -) : \mathscr{C} \to \mathscr{C} \qquad Y \mapsto \mathscr{H}om(X, Y).$$

Then, we call $\mathscr{H}om(X, -)$ the *internal* Hom *functor* for the object X.

Remark 3.7.12. (i) As usual, the internal Hom functor is determined up to unique isomorphism, if it exists. The counit of adjunction is a morphism of $\mathscr{C}$

$$\mathrm{ev}_{X,Y} : \mathscr{H}om(X, Y) \otimes X \to Y$$

called the *evaluation morphism*.

(ii) Suppose that every object of $\mathscr{C}$ admits an internal Hom functor; then we say briefly that $\mathscr{C}$ *admits an internal* Hom *functor*, and clearly we get a functor

$$\mathscr{C}^o \times \mathscr{C} \to \mathscr{C} \quad : \quad (X, Y) \mapsto \mathscr{H}om(X, Y) \qquad \text{for every } X, Y \in \mathrm{Ob}(\mathscr{C}).$$

Moreover, for every $X, Y, Z \in \mathrm{Ob}(\mathscr{C})$ the composition

$$\begin{array}{ccc} (\mathscr{H}om(X,Y) \otimes \mathscr{H}om(Y,Z)) \otimes X & \xrightarrow{\sim} & (\mathscr{H}om(X,Y) \otimes X) \otimes \mathscr{H}om(Y,Z) \\ & & \downarrow{\scriptstyle \mathrm{ev}_{X,Y} \otimes \mathscr{H}om(Y,Z)} \\ Z \xleftarrow{\mathrm{ev}_{Y,Z}} \mathscr{H}om(Y,Z) \otimes Y & \xleftarrow{\sim} & Y \otimes \mathscr{H}om(Y,Z) \end{array}$$

corresponds, by adjunction, to a unique *composition morphism*

$$\mathscr{H}om(X, Y) \otimes \mathscr{H}om(Y, Z) \to \mathscr{H}om(X, Z).$$

(iii) In the situation of (ii), notice that the functor $\mathscr{C} \to \mathscr{C}$ given by the rule : $Z \mapsto \mathscr{H}om(X, \mathscr{H}om(Y, Z))$, for every $Z \in \mathrm{Ob}(\mathscr{C})$, is right adjoint to the functor given by the rule : $Z \mapsto (Z \otimes X) \otimes Y \xrightarrow{\sim} Z \otimes (X \otimes Y)$. There follows a natural isomorphism

$$\mathscr{H}om(X, \mathscr{H}om(Y, Z)) \xrightarrow{\sim} \mathscr{H}om(X \otimes Y, Z) \qquad \text{for every } X, Y, Z \in \mathrm{Ob}(\mathscr{C}).$$

(iv) Moreover, for any unit (U, u) of $\underline{\mathscr{C}}$, we get natural bijections :

$$\mathrm{Hom}_{\mathscr{C}}(U, \mathscr{H}om(X, Y)) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{C}}(U \otimes X, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{C}}(X, Y) \qquad \text{for every } X, Y \in \mathrm{Ob}(\mathscr{C}).$$

Also, for every object Y of $\mathscr{C}$, denote by $u_Y : Y \xrightarrow{\sim} U \otimes Y$ the isomorphism given by proposition 3.7.6; for every $X \in \mathrm{Ob}(X)$, it induces natural bijections

$$\mathrm{Hom}_{\mathscr{C}}(Y, \mathscr{H}om(U, X)) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{C}}(Y \otimes U, X) \xrightarrow{\mathrm{Hom}_{\mathscr{C}}(u_Y \circ \Psi_{U,Y}, X)} \mathrm{Hom}_{\mathscr{C}}(Y, X)$$

which correspond, via the Yoneda embedding, to a natural isomorphism

$$\mathscr{H}om(U, X) \xrightarrow{\sim} X \qquad \text{for every } X \in \mathrm{Ob}(\mathscr{C}).$$

(v) Let X, Y, Z be any three objects of $\mathscr{C}$; the natural transformation

$$\mathrm{Hom}_{\mathscr{C}}(W \otimes X, Y) \to \mathrm{Hom}_{\mathscr{C}}(W \otimes (X \otimes Z), Y \otimes Z) \qquad \varphi \mapsto (\varphi \otimes Z) \circ \Phi_{W,X,Z}$$

corresponds, via the Yoneda embedding, to a unique morphism

$$t_{X,Y,Z} : \mathscr{H}om(X, Y) \to \mathscr{H}om(X \otimes Z, Y \otimes Z).$$

The reader can check that $t_{X,Y,Z}$ also corresponds, by adjunction, to the morphism

$$(\mathrm{ev}_{X,Y} \otimes Z) \circ \Phi_{\mathscr{H}om(X,Y),X,Z} : \mathscr{H}om(X, Y) \otimes (X \otimes Z) \to Y \otimes Z.$$

(vi) In the situation of remark 3.7.5(ii), suppose that $\mathscr{C}$ admits an internal Hom functor; then it is easily seen that the resulting tensor category $\mathsf{Fun}(\mathscr{D},\mathscr{C})$ inherits as well an internal Hom functor, in the obvious way.

The formalism of tensor categories provides the language to deal uniformly with the notions of algebras and their modules that occur in various concrete settings.

Definition 3.7.13. Let $(\mathscr{C},\otimes,\Phi,\Psi)$ be a tensor category, A and B any two objects of $\mathscr{C}$.

(i) A *left A-module* (resp. a *right B-module*) is a datum (X,μ_X), consisting of an object X of $\mathscr{C}$, and a morphism in $\mathscr{C}$:

$$\mu_X : A\otimes X\to X \qquad \text{(resp. } \mu_X : X\otimes B\to X\text{)}$$

called the *scalar multiplication* of X.

(ii) A *morphism of left A-modules* $(X,\mu_X)\to(X',\mu_{X'})$ is a morphism $f:X\to X'$ in $\mathscr{C}$ which makes commute the diagram :

$$\begin{array}{ccc} A\otimes X & \xrightarrow{\mu_X} & X \\ {\scriptstyle \mathbf{1}_A\otimes f}\downarrow & & \downarrow{\scriptstyle f} \\ A\otimes X' & \xrightarrow{\mu_{X'}} & X'. \end{array}$$

One defines likewise morphisms of right B-modules.

(iii) An *(A,B)-bimodule* is a datum (X,μ^l_X,μ^r_X) such that (X,μ^l_X) is a left A-module, (X,μ^r_X) is a right B-module, and the scalar multiplications commute, *i.e.* the diagram

$$\begin{array}{ccccc} A\otimes(X\otimes B) & & \xrightarrow{\mathbf{1}_A\otimes\mu^r_X} & & A\otimes X \\ {\scriptstyle \Phi_{A,X,B}}\downarrow & & & & \downarrow{\scriptstyle \mu^l_X} \\ (A\otimes X)\otimes B & \xrightarrow{\mu^l_X\otimes\mathbf{1}_B} & X\otimes B & \xrightarrow{\mu^r_X} & X \end{array}$$

commutes. Of course, a morphism of (A,B)-bimodules must be compatible with both left and right multiplication.

We denote by $A\text{-}\mathbf{Mod}_l$ (resp. $B\text{-}\mathbf{Mod}_r$, resp. $(A,B)\text{-}\mathbf{Mod}$) the category of left A-modules (resp. right B-modules, resp. (A,B)-bimodules). For any two left A-modules (resp. right B-modules, resp. (A,B)-bimodules) X and X', we shall write

$$\mathrm{Hom}_{A_l}(X,X') \qquad (\text{ resp. } \mathrm{Hom}_{B_r}(X,X')\text{ }) \qquad (\text{ resp. } \mathrm{Hom}_{(A,B)}(X,X')\text{ })$$

for the set of morphisms of left A-modules (resp. of right B-modules, resp. of (A,B)-bimodules) $X\to X'$.

3.7.14. In the situation of definition 3.7.13, notice that

$$\mathrm{Hom}_{B_r}(X,X') = \mathrm{Equal}(\ \mathrm{Hom}_{\mathscr{C}}(X,X') \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \mathrm{Hom}_{\mathscr{C}}(X\otimes B,X')\ )$$

where α (resp. β) is given by the rule :

$$f\mapsto f\circ\mu_X \qquad (\text{ resp. } f\mapsto\mu_{X'}\circ(f\otimes B)\text{ }) \qquad \text{for every } f\in\mathrm{Hom}_{\mathscr{C}}(X,X')$$

and similarly for left A-modules. Now, suppose that all equalizers in $\mathscr{C}$ are representable, and that $\mathscr{C}$ admits an internal Hom functor; then we may define

$$\mathscr{H}\!om_{B_r}(X,X') := \mathrm{Equal}(\ \mathscr{H}\!om(X,X') \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \mathscr{H}\!om(X\otimes B,X')\ )$$

where $\alpha := \mathscr{H}om(\mu_X, X')$ and $\beta := \mathscr{H}om(X \otimes B, \mu_{X'}) \circ t_{X,X',B}$ (notation of remark 3.7.12(v)). Then, it is easily seen that the bijections of remark 3.7.12(iv) induce natural identifications

$$\mathrm{Hom}_{\mathscr{C}}(U, \mathscr{H}om_{B_r}(X, X')) \xrightarrow{\sim} \mathrm{Hom}_{B_r}(X, X') \qquad \text{for every } X, X' \in \mathrm{Ob}(B_r\text{-}\mathbf{Mod}).$$

Likewise we may represent in $\mathscr{C}$ the set of morphisms between two left A-modules, and two (A, B)-modules (details left to the reader).

3.7.15. Let $\underline{\mathscr{C}}$ be a tensor category as in (3.7.14) and A, B, C any three objects of $\mathscr{C}$; suppose that (X, μ^l_X, μ^r_X) is an (A, B)-bimodule, and $(X', \mu^l_{X'}, \mu^r_{X'})$ a (C, B)-bimodule. Then we claim that $\mathscr{H} := \mathscr{H}om_{B_r}((X, \mu^r_X), (X', \mu^r_{X'}))$ is naturally a (C, A)-bimodule. For this, we have to exhibit natural morphisms

$$C \otimes \mathscr{H} \xrightarrow{\mu_l} \mathscr{H} \xleftarrow{\mu_r} \mathscr{H} \otimes A$$

fulfilling the condition of definition 3.7.13(iii). However, by adjunction, the datum of μ_l is the same as that of a morphism $C \to \mathscr{H}om(\mathscr{H}, \mathscr{H})$, and since the functor $\mathscr{H}om(X, -)$ is left exact, the latter is the same as a morphism

$$C \to \mathrm{Equal}(\ \mathscr{H}om(\mathscr{H}, \mathscr{H}om(X, X')) \underset{\mathscr{H}om(\mathscr{H},\beta)}{\overset{\mathscr{H}om(\mathscr{H},\alpha)}{\rightrightarrows}} \mathscr{H}om(\mathscr{H}, \mathscr{H}om(X \otimes B, X'))$$

which in turn – by remark 3.7.12(iii) – corresponds to a morphism

$$C \to \mathrm{Equal}(\ \mathscr{H}om(\mathscr{H} \otimes X, X') \underset{\mathscr{H}om(\mathscr{H}\otimes\beta,X')}{\overset{\mathscr{H}om(\mathscr{H}\otimes\alpha,X')}{\rightrightarrows}} \mathscr{H}om(\mathscr{H} \otimes (X \otimes B), X')$$

and again, the latter is the same as an element of

$$\mathrm{Equal}(\ \mathrm{Hom}_{\mathscr{C}}(C \otimes (\mathscr{H} \otimes X), X') \underset{\mathrm{Hom}_{\mathscr{C}}(C\otimes(\mathscr{H}\otimes\beta),X')}{\overset{\mathrm{Hom}_{\mathscr{C}}(C\otimes(\mathscr{H}\otimes\alpha),X')}{\rightrightarrows}} \mathrm{Hom}_{\mathscr{C}}(C \otimes (\mathscr{H} \otimes (X \otimes B)), X').$$

By unwinding the definition, it is easily seen that the composition

$$\overline{\mu}_l : C \otimes (\mathscr{H} \otimes X) \xrightarrow{\mathrm{ev}_{X,X'}} C \otimes X' \xrightarrow{\mu^l_{X'}} X'$$

lies in the above equalizer, and it provides a left C-module structure for $\mathscr{H}$. Likewise, μ_r shall be the morphism corresponding to the composition

$$\overline{\mu}_r : (\mathscr{H} \otimes X) \otimes A \xrightarrow{\sim} \mathscr{H} \otimes (A \otimes X) \xrightarrow{\mathscr{H}\otimes\mu^l_X} \mathscr{H} \otimes X \xrightarrow{\mathrm{ev}_{X,X'}} X'.$$

Then, the condition that $(\mathscr{H}, \mu_l, \mu_r)$ is a bimodule, comes down to the commutativity of the diagram

$$\begin{array}{ccccc}
C \otimes ((\mathscr{H} \otimes X) \otimes A) & & \xrightarrow{\quad \mathbf{1}_C\otimes\overline{\mu}_r \quad} & & C \otimes X' \\
\Big\downarrow {\scriptstyle C\otimes\Phi_{\mathscr{H},X,A}} & & & & \Big\downarrow {\scriptstyle \mu^l_{X'}} \\
C \otimes (\mathscr{H} \otimes (X \otimes A)) & \xrightarrow{C\otimes(\mathscr{H}\otimes(\mu^l_X\circ\Psi_{X,A}))} & C \otimes (\mathscr{H} \otimes X) & \xrightarrow{\overline{\mu}_l} & X'
\end{array}$$

which is immediate (details left to the reader). We have thus obtained a bifunctor :

(3.7.16) $$\mathscr{H}om_{B_r}(-,-) : (A, B)\text{-}\mathbf{Mod}^o \times (C, B)\text{-}\mathbf{Mod} \to (C, A)\text{-}\mathbf{Mod}.$$

Likewise, we may define a bifunctor :

$$\mathscr{H}om_{A_l}(-,-) : (A, B)\text{-}\mathbf{Mod}^o \times (A, C)\text{-}\mathbf{Mod} \to (B, C)\text{-}\mathbf{Mod}.$$

3.7.17. Keep the situation of (3.7.15), and suppose moreover that all coequalizers in $\mathscr{C}$ are representable. Fix an (A, B)-bimodule (X, μ^l_X, μ^r_X); the functor

$$(C, B)\text{-}\mathbf{Mod} \to (C, A)\text{-}\mathbf{Mod} \quad : \quad X' \mapsto \mathscr{H}om_{B_r}(X, X')$$

admits a left adjoint, the *tensor product*

$$(C, A)\text{-}\mathbf{Mod} \to (C, B)\text{-}\mathbf{Mod} \quad : \quad (X', \mu^l_{X'}, \mu^l_{X'}) \mapsto (X', \mu^l_{X'}, \mu^r_{X'}) \otimes_A (X, \mu^l_X, \mu^r_X)$$

given by the coequalizer (in $\mathscr{C}$) of the morphisms :

$$X' \otimes (A \otimes X) \underset{(\mu^r_{X'} \otimes \mathbf{1}_X) \circ \Phi_{X', A, X}}{\overset{\mathbf{1}_{X'} \otimes \mu^l_X}{\rightrightarrows}} X' \otimes X$$

with scalar multiplications induced by μ^r_X and $\mu^l_{X'}$. Likewise, we have a functor :

$$(A, C)\text{-}\mathbf{Mod} \to (B, C)\text{-}\mathbf{Mod} \quad : \quad (X', \mu^l_{X'}, \mu^l_{X'}) \mapsto (X, \mu^l_X, \mu^r_X) \otimes_A (X', \mu^l_{X'}, \mu^l_{X'})$$

which admits a similar description, and is left adjoint to the functor $X' \mapsto \mathscr{H}om_{A_l}(X, X')$ (verifications left to the reader).

3.7.18. Let (U, u) be unit object for $\underline{\mathscr{C}}$. Notice that, for any object A of $\mathscr{C}$, the rule $(Y, \mu_Y) \mapsto (Y, u_Y^{-1}, \mu_Y)$ (where $u_Y : Y \to U \otimes Y$ is the natural isomorphism supplied by proposition 3.7.6), induces a faithful functor $A\text{-}\mathbf{Mod}_r \to (U, A)\text{-}\mathbf{Mod}$. Letting $C := U$ in (3.7.17), we see that any (A, B)-bimodule X also determines a functor :

$$A\text{-}\mathbf{Mod}_r \to B\text{-}\mathbf{Mod}_r \quad : \quad Y \mapsto Y \otimes_A X$$

and likewise for left A-modules.

Example 3.7.19. If $\mathscr{C} = \mathbf{Set}$ is the category of sets (regarded as a tensor category as in example 3.7.21), then a left A-module is just a set X' with a *left action* of A, *i.e.* a map of sets

$$A \times X' \to X' \quad : \quad (a, x) \mapsto a \cdot x.$$

An (A, A)-bimodule X is a set with both left and right actions of A, such that $(a \cdot x) \cdot a' = a \cdot (x \cdot a')$ for every $a, a' \in A$ and every $x \in X$. With this notation, the tensor product $X' \otimes_A X$ is the quotient $(X' \times X)/\!\sim$, where $\sim$ is the smallest equivalence relation such that $(x'a, x) \sim (x', ax)$ for every $x \in X$, $x' \in X'$ and $a \in A$.

Definition 3.7.20. Let $\underline{\mathscr{C}} := (\mathscr{C}, \otimes, \Phi, \Psi)$ be a tensor category, and (U, u) a unit for $\underline{\mathscr{C}}$.

(i) A $\underline{\mathscr{C}}$*-semigroup* is a datum (M, μ_M) consisting of an object M of $\mathscr{C}$ and a morphism $\mu_M : M \otimes M \to M$, the *multiplication law* of $\underline{M}$, such that (M, μ_M, μ_M) is a (M, M)-bimodule. A morphism of $\underline{\mathscr{C}}$-semigroups is a morphism $\varphi : M \to M'$ in $\mathscr{C}$, such that

$$\mu_{M'} \circ (\varphi \otimes \varphi) = \varphi \circ \mu_M.$$

(ii) A $\underline{\mathscr{C}}$*-monoid* is a datum $\underline{M} := (M, \mu_M, 1_M)$, where (M, μ_M) is a semigroup, and $1_M : U \to M$ is a morphism in $\mathscr{C}$, called the *unit* of $\underline{M}$, such that

$$\mu_M \circ (1_M \otimes \mathbf{1}_M) \circ u_M = \mathbf{1}_M = \mu_M \circ (\mathbf{1}_M \otimes 1_M) \circ u_M$$

where $u_M : M \xrightarrow{\sim} U \otimes M$ is the natural isomorphism provided by proposition 3.7.6. We say that $\underline{M}$ is *commutative*, if

$$\mu_M = \mu_M \circ \Psi_{M,M}.$$

A morphism of monoids $\underline{M} \to \underline{M}'$ is a morphism of semigroups $\varphi : M \to M'$ such that $\varphi \circ 1_M = 1_{M'}$.

Example 3.7.21. (i) Let $\mathscr{C}$ be any category with small Hom-sets, in which finite products are representable, endow $\mathscr{C}$ with the tensor category structure described in example 3.7.10, and pick a final object $1_\mathscr{C}$ of $\mathscr{C}$. Then, a $\underline{\mathscr{C}}$-monoid is a datum $\underline{M} := (M, \mu_M, 1_M)$, where $\mu_M : M \times M \to M$ and $1_M : 1_\mathscr{C} \to M$ are morphisms of $\mathscr{C}$, and the axioms for μ_M and 1_M can be rephrased as requiring that, for every object X of $\mathscr{C}$, the set $M(X) := \mathrm{Hom}_\mathscr{C}(X, M)$, endowed with the composition law :

$$M(X) \times M(X) \xrightarrow{\sim} \mathrm{Hom}_\mathscr{C}(X, M \times M) \xrightarrow{\mathrm{Hom}_\mathscr{C}(X,\mu_M)} M(X) \qquad (m, m') \mapsto m \cdot m'$$

is a (usual) monoid, with unit $\mathrm{Im}\, 1_M(X) \in M(X)$. Of course, $\underline{M}$ is commutative, if and only if $m \cdot m' = m' \cdot m$ for all objects X of $\mathscr{C}$ and every $m, m' \in M(X)$.

(ii) In the situation of (i), a $\underline{\mathscr{C}}$-monoid shall also be called simply a $\mathscr{C}$-monoid. The category of $\mathscr{C}$-monoids admits an initial object which is also a final object, namely $\underline{1}_\mathscr{C} := (1_\mathscr{C}, \mu_1, \mathbf{1}_1)$, where μ_1 is the (unique) morphism $1_\mathscr{C} \times 1_\mathscr{C} \to 1_\mathscr{C}$. (Most of the above can be repeated with the theory of semigroups replaced by any "algebraic theory" in the sense of [29, Def.3.3.1] : *e.g.* in this way one can define $\mathscr{C}$-groups, $\mathscr{C}$-rings, and so on.)

3.7.22. Let $\underline{\mathscr{C}}$ and U be as in definition 3.7.20, and $\underline{M} := (M, \mu_M, 1_M)$ a $\underline{\mathscr{C}}$-monoid; of course, we are especially interested in the M-modules which are compatible with the unit and multiplication law of M. Hence we define a *left $\underline{M}$-module* as a left M-module (S, μ_S) such that the following diagrams commute :

$$\begin{array}{ccc} U \otimes S & = & U \otimes S \\ {\scriptstyle 1_M \otimes \mathbf{1}_S}\downarrow & & \uparrow{\scriptstyle u_S} \\ M \otimes S & \xrightarrow{\mu_S} & S \end{array} \qquad \begin{array}{ccccc} M \otimes (M \otimes S) & & \xrightarrow{\mathbf{1}_M \otimes \mu_S} & & M \otimes S \\ {\scriptstyle \Phi_{M,M,S}}\downarrow & & & & \downarrow{\scriptstyle \mu_S} \\ (M \otimes M) \otimes S & \xrightarrow{\mu_M \otimes \mathbf{1}_S} & M \otimes S & \xrightarrow{\mu_S} & S \end{array}$$

where u_S is the isomorphism given by proposition 3.7.6. Likewise we define right $\underline{M}$-modules, and $(\underline{M}, \underline{N})$-bimodules, if $\underline{N}$ is a second $\underline{\mathscr{C}}$-monoid; especially, $(\underline{M}, \underline{M})$-bimodules shall also be called simply $\underline{M}$-bimodules.

A morphism of left $\underline{M}$-modules $(S, \mu_S) \to (S', \mu_{S'})$ is just a morphism of left M-modules, and likewise for right modules and bimodules. For instance, $\underline{M}$ is a $\underline{M}$-bimodule in a natural way, and an *ideal* of M is a sub-$\underline{M}$-bimodule I of $\underline{M}$. We denote by $\underline{M}$-$\mathbf{Mod}_l$ (resp. $\underline{M}$-$\mathbf{Mod}_r$, resp. $\underline{M}$-$\mathbf{Mod}$) the category of left (resp. right, resp. bi-) $\underline{M}$-modules; more generally, if $\underline{N}$ is a second $\underline{\mathscr{C}}$-monoid, we have the category $(\underline{M}, \underline{N})$-$\mathbf{Mod}$ of the corresponding bimodules.

Example 3.7.23. Take $\mathscr{C} := \mathbf{Set}$, regarded as a tensor category, as in example 3.7.10. Then a $\mathscr{C}$-monoid is just a usual monoid M, and a left M-module is a datum (S, μ_S) consisting of a set S and a *scalar multiplication* $M \times S \to S : (m, s) \mapsto m \cdot s$ such that

$$1 \cdot s = s \quad \text{and} \quad x \cdot (y \cdot s) = (x \cdot y) \cdot s \qquad \text{for every } x, y \in M \text{ and every } s \in S.$$

A morphism $\varphi : (S, \mu_S) \to (T, \mu_T)$ of M-modules is then a map of sets $S \to T$ such that

$$x \cdot \varphi(s) = \varphi(x \cdot s) \qquad \text{for every } x \in M \text{ and every } s \in S$$

Likewise, an ideal of M is a subset $I \subset M$ such that $a \cdot x, x \cdot a \in I$ whenever $a \in I$ and $x \in M$.

Remark 3.7.24. Let $\mathscr{C}$ be a complete and cocomplete category, whose colimits are universal (see example 1.4.17), and $\underline{M}$ a $\mathscr{C}$-monoid (see example 3.7.21(ii)).

(i) The categories $\underline{M}$-$\mathbf{Mod}_l$, $\underline{M}$-$\mathbf{Mod}_r$ and $(\underline{M}, \underline{N})$-$\mathbf{Mod}$ are complete and cocomplete, and the forgetful functor $\underline{M}$-$\mathbf{Mod}_l \to \mathscr{C}$ (resp. the same for right modules and bimodules) commutes with all limits and colimits.

(ii) Notice also that the forgetful functor $\underline{M}$-$\mathbf{Mod}_l \to \mathscr{C}$ is conservative. Together with (i) and propositions 1.3.18 and 1.3.21(i), this implies that a morphism of left $\underline{M}$-modules is a monomorphism (resp. an epimorphism) if and only if the same holds for the underlying morphism in $\mathscr{C}$ (and likewise for right modules and bimodules).

(iii) For each of these categories, the initial object is just the initial object $\varnothing_{\mathscr{C}}$ of $\mathscr{C}$, endowed with the trivial scalar multiplication. Likewise, the final object is the final object $1_{\mathscr{C}}$ of $\mathscr{C}$, with scalar multiplication given by the unique morphism $M \times 1_{\mathscr{C}} \to 1_{\mathscr{C}}$. Moreover, the forgetful functor $\underline{M}\text{-}\mathbf{Mod}_l \to \mathscr{C}$ admits a left adjoint, that assigns to any $\Sigma \in \mathrm{Ob}(\mathscr{C})$ the *free $\underline{M}$-module* $\underline{M}^{(\Sigma)}$ generated by Σ; as an object of $\mathscr{C}$, the latter is just $M \times \Sigma$, and the scalar multiplication is derived from the composition law of $\underline{M}$, in the obvious way.

For instance, for any $n \in \mathbb{N}$, and any left (or right or bi-) $\underline{M}$-module S, we denote as usual by $S^{\oplus n}$ the coproduct of n copies of S.

Remark 3.7.25. Let $\underline{\mathscr{C}}$ be a tensor category, $\underline{M}$, $\underline{N}$, $\underline{P}$, $\underline{Q}$ four $\underline{\mathscr{C}}$-monoids.

(i) Let S be a $(\underline{M},\underline{N})$-bimodule, S' a $(\underline{P},\underline{N})$-bimodule and S'' a $(\underline{P},\underline{M})$-bimodule. Then it is easily seen that the (P,M)-bimodule (resp. the (P,N)-bimodule) $\mathscr{H}om_{N_r}(S,S')$ (resp. $S'' \otimes_M S$) is actually a $(\underline{P},\underline{M})$-bimodule (resp. a $(\underline{P},\underline{N})$-bimodule) and the adjunction of (3.7.17) restricts to an adjunction between the corresponding categories of bimodules : the details shall be left to the reader.

(ii) We have as well the analogue of the usual associativity constraints. Namely, for every $(\underline{M},\underline{N})$-bimodule S, every $(\underline{N},\underline{P})$-bimodule S' and every $(\underline{P},\underline{Q})$-bimodule S'', there is a natural isomorphism

$$(S \otimes_N S') \otimes_P S'' \xrightarrow{\sim} S \otimes_N (S' \otimes_P S'') \qquad \text{in } (\underline{M},\underline{Q})\text{-}\mathbf{Mod}$$

and natural isomorphisms $M \otimes_M S \xrightarrow{\sim} S \xrightarrow{\sim} S \otimes_N N$ in $(\underline{M},\underline{N})\text{-}\mathbf{Mod}$.

(iii) Also, if $\underline{M}$ is commutative, every left (or right) $\underline{M}$-module is naturally a $(\underline{M},\underline{M})$-bimodule, and we have a commutative constraint

$$S \otimes_M S' \xrightarrow{\sim} S' \otimes_M S \qquad \text{for all left (or right) } \underline{M}\text{-modules.}$$

And taking into account (ii), it is easily seen that $(\underline{M}\text{-}\mathbf{Mod}_l, \otimes_M)$ is a tensor category.

3.7.26. Let $\varphi : \underline{M}_1 \to \underline{M}_2$ be a morphism of $\underline{\mathscr{C}}$-monoids; we have the $(\underline{M}_1,\underline{M}_2)$-bimodule :

$$M_{1,2} := (M_2, \mu_{M_2} \circ (\varphi \otimes \mathbf{1}_{M_2}), \mu_{M_2}).$$

Letting $X := M_{1,2}$ in (3.7.18), we obtain a *base change functor* for right modules :

$$\underline{M}_1\text{-}\mathbf{Mod}_r \to \underline{M}_2\text{-}\mathbf{Mod}_r \quad : \quad X \mapsto X \otimes_{M_1} M_2 := X \otimes_{M_1} M_{1,2}.$$

The base change is left adjoint to the *restrictions of scalars* associated with φ, *i.e.* the functor :

$$\underline{M}_2\text{-}\mathbf{Mod}_r \to \underline{M}_1\text{-}\mathbf{Mod}_l \quad : \quad (X,\mu_X) \mapsto (X,\mu_X)_{(\varphi)} := (X, \mu_X \circ (\mathbf{1}_X \otimes \varphi))$$

(verifications left to the reader). The same can be repeated, as usual, for left modules; for bimodules, one must take the tensor product on both sides : $X \mapsto M_2 \otimes_{M_1} X \otimes_{M_1} M_2$.

Example 3.7.27. Take $\mathscr{C} = \mathbf{Set}$, and let M be any monoid, Σ any set, and $M^{(\Sigma)}$ the free M-module generated by Σ. From the isomorphism

$$M^{(\Sigma)} \otimes_M \{1\} \xrightarrow{\sim} \{1\}^{(\Sigma)} = \Sigma$$

we see that the cardinality of Σ is an invariant, called the *rank* of the free M-module $M^{(\Sigma)}$, and which we denote $\mathrm{rk}_M M^{(\Sigma)}$.

Definition 3.7.28. (i) A *pre-additive category* is the datum of a category $\mathscr{A}$ and of an abelian group structure on $\mathrm{Hom}_{\mathscr{A}}(A,B)$ for every $A, B \in \mathrm{Ob}(\mathscr{A})$ (especially, the Hom-sets of $\mathscr{A}$ are not empty), such that the following holds. For every $A, B, C \in \mathrm{Ob}(\mathscr{A})$, the composition law

$$\mathrm{Hom}_{\mathscr{A}}(A,B) \times \mathrm{Hom}_{\mathscr{A}}(B,C) \to \mathrm{Hom}_{\mathscr{A}}(A,C)$$

is a bilinear pairing.

(ii) A functor $F : \mathscr{A} \to \mathscr{B}$ between pre-additive categories is *additive* if it induces group homomorphisms

$$\mathrm{Hom}_{\mathscr{A}}(X,Y) \to \mathrm{Hom}_{\mathscr{B}}(FX,FY) \quad : \quad \varphi \mapsto F\varphi \qquad \text{for every } X,Y \in \mathrm{Ob}(\mathscr{A}).$$

Remark 3.7.29. Let $\mathscr{A}$ be any pre-additive category, and choose a universe U such that $\mathscr{A}$ is U-small, so that all the finite limits and finite colimits of $\mathscr{A}$ are well defined as U-presheaves.

(i) If $A \in \mathrm{Ob}(\mathscr{A})$ is any object, denote by $\mathbf{0}_A$ the neutral element of the abelian group $\mathrm{End}_{\mathscr{A}}(A)$. Suppose that the equalizer of the pair of morphisms $\mathbf{1}_A, \mathbf{0}_A : A \to A$ is representable by an object 0 of $\mathscr{A}$ (see example 1.2.16(ii)). Then, the datum of a morphism $B \to 0$ is the same as that of a morphism $\varphi : B \to A$ such that $\varphi = \mathbf{0}_A \circ \varphi$. By the bilinearity of the Hom-pairing, the latter condition holds if and only if φ is the neutral element of $\mathrm{Hom}_{\mathscr{A}}(B,A)$. We conclude that 0 is a final object in $\mathscr{A}$. Dually, if the coequalizer of the pair $(\mathbf{1}_A, \mathbf{0}_A)$ is representable by some object $0'$ of $\mathscr{A}$, then $0'$ is initial in $\mathscr{A}$. Moreover, if $\mathscr{A}$ admits a final object 0, then it is easily seen that the unique morphism $A \to 0$ is also the coequalizer of the pair $(\mathbf{1}_A, \mathbf{0}_A)$, so 0 is also an initial object. Conversely, if $\mathscr{A}$ admits an initial object, then this object is also final in $\mathscr{A}$, and for any two objects A, B of $\mathscr{A}$, the neutral element $\mathbf{0}_{A,B}$ of $\mathrm{Hom}_{\mathscr{A}}(A,B)$ is the unique morphism that factors through 0. We say that 0 is a *zero object* for $\mathscr{A}$.

(ii) Suppose that the product $A_1 \times A_2$ is representable in $\mathscr{A}$ for given $A_1, A_2 \in \mathrm{Ob}(\mathscr{A})$. Denote by $p_i : A_1 \times A_2 \to A_i$ $(i = 1,2)$ the projections; then, there are unique morphisms $e_i : A_i \to A_1 \times A_2$ $(i = 1,2)$ such that

$$p_i \circ e_i = \mathbf{1}_{A_i} \qquad \text{for } i = 1,2 \qquad \text{and} \qquad p_i \circ e_j = \mathbf{0}_{A_j,A_i} \qquad \text{for } i \neq j. \tag{3.7.30}$$

Notice that

$$e_1 \circ p_1 + e_2 \circ p_2 = \mathbf{1}_{A_1 \times A_1}. \tag{3.7.31}$$

Indeed, we have

$$p_i \circ (e_1 \circ p_1 + e_2 \circ p_2) = (p_i \circ e_1 \circ p_1) + (p_i \circ e_2 \circ p_2) = p_i \qquad i = 1,2$$

by bilinearity of the Hom pairing, and $\mathbf{1}_{A_1 \times A_2}$ is the unique endomorphism φ of $A_1 \times A_2$ such that $p_i \circ \varphi = p_i$ for $i = 1,2$. It follows that $A_1 \times A_2$ also represents the coproduct $A_1 \amalg A_2$. Indeed, say that $f_i : A_i \to B$, for $i = 1,2$, are two morphisms to another object B of $\mathscr{A}$, and set $f := f_1 \circ p_1 + f_2 \circ p_2 : A_1 \times A_2 \to B$; it is easily seen that $f \circ e_i = f_i$ for $i = 1,2$, and by virtue of (3.7.31), the morphism f is the unique one that satisfies these identities. Conversely, if the coproduct of A_1 and A_2 is representable, a similar argument shows that also $A_1 \times A_2$ is representable. We say that $A_1 \times A_2$ is a *biproduct* of A_1 and A_2, and denote it by $A_1 \oplus A_2$.

(iii) Notice that the morphisms $(p_i, e_i \mid i = 1,2)$ with the identities (3.7.30) and (3.7.31) characterize $A_1 \oplus A_2$ up to unique isomorphism. Namely, say that B is another object of $\mathscr{A}$, for which exist morphisms $p'_i : B \to A_i$ and $e'_i : A_i \to B$ $(i = 1,2)$ such that $p'_i \circ e'_i = \mathbf{1}_{A_i}$ for $i = 1,2$, and $p'_i \circ e'_j = \mathbf{0}_{A_j A_i}$ for $i \neq j$, and moreover $e'_1 \circ p'_1 + e'_2 \circ p'_2 = \mathbf{1}_B$. Then the pair (e'_1, e'_2) (resp. (p'_1, p'_2)) induces a morphism $e' : A_1 \oplus A_2 \to B$ (resp. $p' : B \to A_1 \oplus A_2$) with

$$p_j \circ p' \circ e' \circ e_i = p'_j \circ e'_i = p_j \circ e_i \qquad \text{for } i,j = 1,2$$

which – by virtue of the universal properties of the biproduct – implies that $p' \circ e' = \mathbf{1}_{A_1 \oplus A_2}$. Likewise, we may compute

$$\begin{aligned} e' \circ p' &= (e' \circ e_1 \circ p_1 + e' \circ e_2 \circ p_2) \circ (e_1 \circ p_1 \circ p' + e_2 \circ p_2 \circ p') \\ &= e' \circ e_1 \circ p_1 \circ p' + e' \circ e_2 \circ p_2 \circ p' \\ &= e'_1 \circ p'_1 + e'_2 \circ p'_2 = \mathbf{1}_B \end{aligned}$$

whence the contention.

(iv) Suppose that B_1 and B_2 are any other two objects of $\mathscr{A}$ such that $B_1 \oplus B_2$ is also representable; given two morphisms $f_1 : A_1 \to B_1$ and $f_2 : A_2 \to B_2$, we denote by $f_1 \oplus f_2 : A_1 \oplus A_2 \to B_1 \oplus B_2$ the unique morphism such that

$$p_{B,i} \circ (f_1 \oplus f_2) \circ e_{A,i} = f_i \quad \text{for } i = 1,2, \text{ and} \quad p_{B,i} \circ (f_1 \oplus f_2) \circ e_{A,j} = \mathbf{0}_{A_j,B_i} \quad \text{for } i \neq j.$$

Notice that

$$f_1 \oplus f_2 = (f_1 \oplus \mathbf{0}_{A_2,B_2}) + (\mathbf{0}_{A_1,B_1} \oplus f_2) \tag{3.7.32}$$

(where the sum is taken in the abelian group $\mathrm{Hom}_{\mathscr{A}}(A_1 \oplus A_2, B_1 \oplus B_2)$); indeed, by bilinearity of the Hom pairing, it is easily seen that the right-hand side of (3.7.32) also satisfies the same identities above that define $f_1 \oplus f_2$.

(v) If $f : A \to B$ is any morphism of $\mathscr{A}$, then we define the *kernel* (resp. *cokernel*) of f as the equalizer (resp. coequalizer)

$$\mathrm{Ker}\, f := \mathrm{Equal}(f, \mathbf{0}_{A,B}) \qquad \mathrm{Coker}\, f := \mathrm{Coequal}(f, \mathbf{0}_{A,B}).$$

Suppose that all kernels and cokernels of $\mathscr{A}$ are representable in $\mathscr{A}$, and denote by

$$\iota_f : \mathrm{Ker}\, f \to A \qquad \pi_f : B \to \mathrm{Coker}\, f$$

the natural morphisms in $\mathscr{A}$. Notice that ι_f is a monomorphism, and π_f an epimorphism. Notice also that f factors uniquely in $\mathscr{A}$ as a composition

$$A \xrightarrow{\pi_{\iota_f}} \mathrm{Coker}\, \iota_f \xrightarrow{\beta_f} \mathrm{Ker}\, \pi_f \xrightarrow{\iota_{\pi_f}} B. \tag{3.7.33}$$

(vi) Suppose that $\mathscr{A}$ admits a zero object 0, and let $f : A \to B$ be any morphism; by definition $\mathrm{Ker}\, f$ is the presheaf such that

$$\mathrm{Ker}\, f(C) = \{g : B \to C \mid g \circ f = g \circ \mathbf{0}_{A,B} = \mathbf{0}_{A,C}\}.$$

If f is a monomorphism, the identity $g \circ f = \mathbf{0}_{A,C} = \mathbf{0}_{B,C} \circ f$ implies that $g = \mathbf{0}_{B,C}$, so $\mathrm{Ker}\, f$ is represented by 0. Dually, if f is an epimorphism, then $\mathrm{Coker}\, f$ is represented by 0.

Remark 3.7.34. Let $\mathscr{A}$, $\mathscr{B}$ be any two pre-additive categories that admit a zero object, and $F : \mathscr{A} \to \mathscr{B}$ a functor.

(i) If F is additive, remark 3.7.29(iii) immediately implies that F transforms representable biproducts into representable biproducts. The latter assertion still holds in case F is not necessarily additive, but is either left or right exact. Indeed, suppose that F is left exact, let $A_1 \oplus A_2$ be any biproduct, and let p_i, e_i be the morphisms described in remark 3.7.29(ii); by left exactness, $F(A_1 \oplus A_2)$ represents the product of FA_1 and FA_2, whence an isomorphism $FA_1 \oplus FA_2 \xrightarrow{\sim} F(A_1 \oplus A_2)$ which identifies Fp_1 and Fp_2 with the natural projections. Moreover, F transforms the final object of $\mathscr{A}$ into the final object of $\mathscr{B}$ (see example 1.2.16(iv)); then, by inspecting the argument in remark 3.7.29(ii), it is easily seen that F identifies as well Fe_i with the natural injections $FA_i \to FA_1 \oplus FA_2$, for $i = 1, 2$, so the assertion follows from remark 3.7.29(iii). A similar argument works in case F is right exact.

(ii) Suppose moreover, that all biproducts of $\mathscr{A}$ are representable. Then we claim that the abelian group structure on $\mathrm{Hom}_{\mathscr{A}}(A, B)$ is determined by the category $\mathscr{A}$, *i.e.* if $\mathscr{B}$ is any other pre-additive category, and $F : \mathscr{A} \to \mathscr{B}$ is any equivalence of categories, then F induces group isomorphisms (and not just bijections) on Hom sets. Indeed, let A and B be any two objects of $\mathscr{A}$, and denote by $\Delta_A : A \to A \oplus A$ (resp. $\mu_B : B \oplus B \to B$) the unique morphism such that $p_i \circ \Delta_A = \mathbf{1}_A$ (resp. $\mu_B \circ e_i = \mathbf{1}_B$) for $i = 1, 2$. Then we have

$$f_1 + f_2 = \mu_B \circ (f_1 \oplus f_2) \circ \Delta_A \qquad \text{for every } f_1, f_2 : A \to B$$

where $f_1 + f_2$ denotes the sum in the abelian group $\mathrm{Hom}_{\mathscr{A}}(A, B)$. Indeed, since clearly $\mathbf{0}_{A,B} \oplus \mathbf{0}_{A,B} = \mathbf{0}_{A \oplus A, B \oplus B}$, identity (3.7.32) reduces to checking that $f_1 = \mu_B \circ (f_1 \oplus \mathbf{0}_{A,B}) \circ \Delta_A$ (and likewise for f_2), which follows easily from (3.7.31) : details left to the reader.

(iii) Combining (i) and (ii) we see that if all biproducts of $\mathscr{A}$ are representable, and F is either left or right exact, then F is additive. More generally, we see that for F to be additive, it suffices that F sends the zero object of $\mathscr{A}$ to the zero object of $\mathscr{B}$, and that F commutes with the biproducts of the form $A \oplus A$, for every $A \in \mathrm{Ob}(\mathscr{A})$.

Definition 3.7.35. (i) An *additive category* is a pre-additive category which admits a zero object, and whose biproducts are representable.

(ii) An *abelian category* is an additive category $\mathscr{A}$ such that the following holds:

(a) All the kernels and cokernels of $\mathscr{A}$ are representable.

(b) For every morphism f of $\mathscr{A}$, the morphism β_f of (3.7.33) is an isomorphism.

Example 3.7.36. Let $\mathscr{A}$ be an additive category, A an object of $\mathscr{A}$, and $e : A \to A$ an endomorphism of A such that :

(a) $e = e \circ e$, *i.e.* e is an *idempotent* element of the ring $\mathrm{End}_{\mathscr{A}}(A)$.

(b) $\mathrm{Ker}\,(e)$ and $\mathrm{Ker}\,(\mathbf{1}_A - e)$ are representable in $\mathscr{A}$.

Then the morphisms $\iota_e : \mathrm{Ker}\,(e) \to A$ and $\iota_{\mathbf{1}_A - e} : \mathrm{Ker}\,(\mathbf{1}_A - e) \to A$ induce an isomorphism

$$\omega : \mathrm{Ker}\,(e) \oplus \mathrm{Ker}\,(\mathbf{1}_A - e) \stackrel{\sim}{\to} A.$$

Indeed, it is easily seen that the morphism $\varepsilon : A \to A \oplus A$ defined by the pair $(\mathbf{1}_A - e, e)$ factors as a composition

$$A \xrightarrow{\overline{\varepsilon}} \mathrm{Ker}\,(e) \oplus \mathrm{Ker}\,(\mathbf{1}_A - e) \xrightarrow{\iota_e \oplus \iota_{\mathbf{1}_A - e}} A \oplus A$$

for a unique morphism $\overline{\varepsilon}$ of $\mathscr{A}$. Then a direct computation shows that $\omega \circ \overline{\varepsilon} = \mathbf{1}_A$ and $\overline{\varepsilon} \circ \omega = \mathbf{1}_{\mathrm{Ker}\,(e) \oplus \mathrm{Ker}\,(\mathbf{1}_A - e)}$, whence the assertion (details left to the reader).

Remark 3.7.37. Let $\mathscr{A}$ be any pre-additive category.

(i) The category $\mathscr{A}^o$ is naturally pre-additive : namely, if $A, B \in \mathrm{Ob}(\mathscr{A}^o)$ are any two objects, one endows $\mathrm{Hom}_{\mathscr{A}^o}(A, B)$ with the group structure of $\mathrm{Hom}_{\mathscr{A}}(B, A)$. Likewise, if $F : \mathscr{A} \to \mathscr{B}$ is any additive functor between the pre-additive categories $\mathscr{A}$, $\mathscr{B}$, then also $F^o : \mathscr{A}^o \to \mathscr{B}^o$ is additive. Moreover, if $\mathscr{A}$ is additive the same holds for $\mathscr{A}^o$. Indeed, the zero object of $\mathscr{A}$ is obviously a zero object also in $\mathscr{A}^o$, and biproducts are representable in $\mathscr{A}^o$, since one can take

$$A^o \oplus B^o := (A \oplus B)^o \qquad \text{for every } A, B \in \mathrm{Ob}(\mathscr{A}).$$

Furthermore, if $\mathscr{A}$ is abelian, the same holds for $\mathscr{A}^o$. Indeed, we can take

$$\mathrm{Ker}\, f^o := (\mathrm{Coker}\, f)^o \qquad \mathrm{Coker}\, f^o := (\mathrm{Ker}\, f)^o \qquad \text{for every morphism } f \text{ in } \mathscr{A}$$

and with these choices we have :

$$\iota_{f^o} = (\pi_f)^o \qquad \pi_{f^o} = (\iota_f)^o \qquad \beta_{f^o} = (\beta_f)^o$$

so β_{f^o} is an isomorphism whenever the same holds for β_f.

(ii) Notice that for every category $\mathscr{C}$, the category $\mathsf{Fun}(\mathscr{C}, \mathscr{A})$ is pre-additive; indeed, if $\tau, \sigma : F \Rightarrow G$ are two natural transformations between functors $F, G : \mathscr{C} \to \mathscr{A}$, then we obtain a natural transformation $\tau + \sigma$ from F to G, by the rule : $(\tau + \sigma)_X := \tau_X + \sigma_X$ for every $X \in \mathrm{Ob}(\mathscr{C})$ (where the sum denotes the addition law of $\mathrm{Hom}_{\mathscr{A}}(FX, GX)$). Clearly, this rule yields an abelian group structure, and the composition of natural transformation defines a bilinear pairing $(\tau, \tau') \mapsto \tau \odot \tau'$ on the resulting groups of natural transformations (verification left to the reader).

Moreover, if $\mathscr{A}$ is an additive (resp. abelian) category, then the same holds for $\mathsf{Fun}(\mathscr{C}, \mathscr{A})$, for every category $\mathscr{C}$. Indeed, if 0 denotes a zero object of $\mathscr{A}$, it follows easily from remark 1.4.2(iii) that the constant functor $0_{\mathscr{C}} : \mathscr{C} \to \mathscr{A}$ with $0_{\mathscr{C}}(C) := 0$ for every $C \in \mathrm{Ob}(\mathscr{C})$ is a zero object for $\mathsf{Fun}(\mathscr{C}, \mathscr{A})$. By the same token, if $F, G : \mathscr{C} \to \mathscr{A}$ are any two functors, then the biproduct $F \oplus G$ is represented by the functor given by the rule : $C \mapsto FC \oplus GC$ for every $C \in \mathrm{Ob}(\mathscr{C})$ (where $FC \oplus GC$ denotes any fixed choice of an object of $\mathscr{A}$ representing the biproduct of FC and GC); then $(F \oplus G)(\varphi)$ is the induced morphism $F\varphi \oplus G\varphi : FC \oplus GC \to FC' \oplus GC'$ for every morphism $\varphi : C \to C'$ in $\mathscr{C}$ (details left to the reader). Likewise, for every natural transformation $\tau : F \Rightarrow G$, the kernel and cokernel of τ are computed argumentwise, and the same holds for the natural morphism β_τ of (3.7.33), so if all kernels and cokernels are representable in $\mathscr{A}$, the same holds for the kernels and cokernels in $\mathsf{Fun}(\mathscr{C}, \mathscr{A})$, and if β_{τ_C} is an isomorphism for every $C \in \mathrm{Ob}(\mathscr{C})$, then the same holds for β_τ, whence the contention.

(iii) For any other pre-additive category $\mathscr{B}$, let us denote by $\mathbf{Add}(\mathscr{B}, \mathscr{A})$ the full subcategory of $\mathsf{Fun}(\mathscr{B}, \mathscr{A})$ whose objects are the additive functors. It follows from (ii) that $\mathbf{Add}(\mathscr{B}, \mathscr{A})$ is a pre-additive category.

(iv) Suppose that $\mathscr{A}$ has small Hom-sets. By definition, for every $A, B \in \mathrm{Ob}(\mathscr{A})$, the set $h_A(B) := \mathrm{Hom}_{\mathscr{A}}(B, A)$ carries an abelian group structure, such that the presheaf h_A factors through an additive functor $h_A^\dagger : \mathscr{A}^o \to \mathbb{Z}\text{-}\mathbf{Mod}$ from $\mathscr{A}^o$ to the category of (small) abelian groups, and the forgetful functor $\mathbb{Z}\text{-}\mathbf{Mod} \to \mathbf{Set}$. Hence, the Yoneda embedding factors through a fully faithful *group-valued Yoneda embedding*

$$h^\dagger : \mathscr{A} \to \mathbf{Add}(\mathscr{A}^o, \mathbb{Z}\text{-}\mathbf{Mod}).$$

In view of (iii), we conclude that every pre-additive category is a full subcategory of an abelian category. Moreover, Yoneda's lemma extends *verbatim* to the group-valued case : namely, by inspecting the proof of proposition 1.2.6, we see that, for every $A \in \mathrm{Ob}(\mathscr{A})$ and every *additive* functor $F : \mathscr{A}^o \to \mathbb{Z}\text{-}\mathbf{Mod}$ there are natural isomorphisms of abelian groups

$$FA \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Add}(\mathscr{A}^o, \mathbb{Z}\text{-}\mathbf{Mod})}(h_A^\dagger, F). \tag{3.7.38}$$

(v) Let $f : \mathscr{A} \to \mathscr{B}$ be any functor between pre-additive categories; then the arguments of remark 1.3.6(i) extend *verbatim* to the present situation : after choosing a universe U large enough so that $\mathscr{A}$ and $\mathscr{B}$ are U-small, f induces functor

$$f^* : \mathsf{Fun}(\mathscr{B}^o, \mathbb{Z}\text{-}\mathbf{Mod}) \to \mathsf{Fun}(\mathscr{A}^o, \mathbb{Z}\text{-}\mathbf{Mod})$$

that admits both left and right adjoints, denoted respectively $f_!$ and f_*, and we have :

Proposition 3.7.39. *In the situation of remark* 3.7.37(v)*, suppose that f is additive and both $\mathscr{A}$ and $\mathscr{B}$ are additive categories. Then :*

(i) *Both f^*, $f_!$ and f_* are additive functors, and restrict to functors*

$$\mathbf{Add}(\mathscr{B}^o, \mathbb{Z}\text{-}\mathbf{Mod}) \underset{f_! \quad f_*}{\overset{f^*}{\rightleftarrows}} \mathbf{Add}(\mathscr{A}^o, \mathbb{Z}\text{-}\mathbf{Mod}).$$

(ii) *The resulting diagram of functors*

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{h^\dagger} & \mathbf{Add}(\mathscr{A}^o, \mathbb{Z}\text{-}\mathbf{Mod}) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_!} \\ \mathscr{B} & \xrightarrow{h^\dagger} & \mathbf{Add}(\mathscr{B}^o, \mathbb{Z}\text{-}\mathbf{Mod}) \end{array}$$

is essentially commutative.

Proof. (i): Since every left (resp. right) adjoint functor is right (resp. left) exact, remark 3.7.34(iii) says that f^*, f_* and $f_!$ are additive. Next, a simple inspection shows that f^* transforms additive functors into additive functors. Let now $F : \mathscr{A}^o \to \mathbb{Z}\text{-}\mathbf{Mod}$ be an additive functor, $B \in \mathrm{Ob}(\mathscr{B})$ any object, and $G := f_! F$; from the proof of theorem 1.3.4, we see that

$$GB = \operatorname*{colim}_{\psi : B \to fA} FA$$

where the colimit ranges over the small category $f\mathscr{A}^o/B$ of all pairs (A, ψ) consisting of an object A of $\mathscr{A}$, and a morphism $\psi : B \to fA$ in $\mathscr{B}$. Denote by $0_\mathscr{A}$ and $0_\mathscr{B}$ the zero objects of $\mathscr{A}$ and $\mathscr{B}$; we wish to show that G is additive, and according to remark 3.7.34(iii), it suffices to check that $G(0_\mathscr{B}) = 0$, and that the natural morphism $G(B \oplus B) \to GB \oplus GB$ (deduced from the projections $p_i : B \oplus B \to B$) is an isomorphism, for every $B \in \mathrm{Ob}(\mathscr{B})$.

However, notice that the functor $\mathsf{s}_{0_\mathscr{B}} : f\mathscr{A}^o/0_\mathscr{B} \to \mathscr{A}^o$ is an isomorphism of categories (notation of (1.1.27)); whence a natural isomorphism

$$G(0_\mathscr{B}) \xrightarrow{\sim} \operatorname*{colim}_{\mathscr{A}^o} F \xrightarrow{\sim} F(0_\mathscr{A}) = 0$$

where the last identity holds, since F is additive. Next, for any $B_1, B_2 \in \mathrm{Ob}(\mathscr{B})$ consider the functor

$$\Phi : (f\mathscr{A}^o/B_1) \times (f\mathscr{A}^o/B_2) \to f\mathscr{A}^o/B_1 \oplus B_2 \qquad ((A_1, \psi_1), (A_2, \psi_2)) \mapsto (A_1 \oplus A_2, \psi_1 \oplus \psi_2).$$

Claim 3.7.40. The functor Φ is cofinal.

Proof of the claim. We apply the criterion of proposition 1.5.2. Indeed, let

$$i := (A, \psi : B_1 \oplus B_2 \to fA)$$

be any object of $f\mathscr{A}^o/B_1 \oplus B_2$ and

$$j := ((A_1, \varphi_1 : B_1 \to fA_1), (A_2, \varphi_2 : B_2 \to fA_2))$$
$$j' := ((A'_1, \varphi'_1 : B_1 \to fA'_1), (A'_2, \varphi'_2 : B_2 \to fA'_2))$$

any two objects of $(f\mathscr{A}^o/B_1) \times (f\mathscr{A}^o/B_2)$, and suppose that $\beta : i \to \Phi j$ and $\beta' : i \to \Phi j'$ are morphisms in $f\mathscr{A}^o/B_1 \oplus B_2$, given by morphisms $\beta : A_1 \oplus A_2 \to A$ and $\beta' : A'_1 \oplus A'_2 \to A$ in $\mathscr{A}$ such that $f\beta \circ (\varphi_1 \oplus \varphi_2) = \psi = f\beta' \circ (\varphi'_1 \oplus \varphi'_2)$. We let

$$j'' := ((A_1 \oplus A'_1, \delta_1 : B_1 \to f(A_1 \oplus A'_1)), (A_2 \oplus A'_2, \delta_2 : B_2 \to f(A_2 \oplus A'_2))$$

where $\delta_1 := (\varphi_1 \oplus \varphi'_1) \circ \Delta_{B_1}$, and likewise for δ_2. Let also

$$A_1 \oplus A_2 \xleftarrow{p} A_1 \oplus A'_1 \oplus A_2 \oplus A'_2 \xrightarrow{p'} A'_1 \oplus A'_2$$

be the natural projections, which define morphisms

$$\Phi p : \Phi j \to \Phi j'' \qquad \Phi p' : \Phi j' \to \Phi j'' \qquad \text{in } f\mathscr{A}^o/B_1 \oplus B_2$$

A simple inspection shows that $\Phi(p) \circ \beta = \Phi(p') \circ \beta'$, and therefore $i/\Phi(f\mathscr{A}^o/B_1 \oplus B_2)$ is connected, whence the claim. ♢

In light of claim 3.7.40, we are reduced to checking that the natural morphism

$$\operatorname*{colim}_{(f\mathscr{A}^o/B_1)\times(f\mathscr{A}^o/B_2)} F \circ \mathsf{s}_{B_1 \oplus B_2} \circ \Phi \to GB_1 \oplus GB_2$$

is an isomorphism, for any $B_1, B_2 \in \mathrm{Ob}(\mathscr{B})$. The latter assertion follows easily by inspecting the definitions, since F is additive. Lastly, a similar argument shows that f_*F is additive, whenever the same holds for F : the reader can spell out the proof as an exercise.

(ii): One may argue as in remark 1.3.6(ii) : in view of (3.7.38), we see that, for every object A of $\mathscr{A}$, both $h^\dagger_{fA}$ and $f_! h^\dagger_A$ represent the same functor : details left to the reader. □

Lemma 3.7.41. *For any abelian category $\mathscr{A}$, the following holds :*

(i) *$\mathscr{A}$ is finitely complete and finitely cocomplete.*

(ii) *The image of any morphism $f : A \to B$ of $\mathscr{A}$ exists in $\mathscr{A}$, and we have*

$$\mathrm{Coker}\, \pi_f = \mathrm{Im}\, f \qquad \textit{in } \mathrm{Sub}(B)$$

(notation of (1.1.2)*, example* 1.2.26 *and remark* 3.7.29(v)*).*

Proof. (i): By proposition 1.2.22(i,ii), it suffices to check that all equalizers and coequalizers are representable in $\mathscr{A}$. However, let $g, g' : X \to Y$ be any pair of morphisms in $\mathscr{A}$ with same soruces and targets; it easily seen that $\mathrm{Ker}\,(g-g')$ (resp. $\mathrm{Coker}\,(g-g')$) represents the equalizer (resp. the coequalizer) of g and g', whence the assertion.

(ii): Suppose that f factors through a subobject $g : C \to B$ of B; there follows a natural morphism

$$\mathrm{Coker}\, f \to \mathrm{Coker}\, g$$

which in turns induces a morphism $\mathrm{Ker}\,\pi_f \to \mathrm{Ker}\,\pi_g$ of subobjects of B. But since $\mathscr{A}$ is abelian, $\mathrm{Ker}\,\pi_g$ is also the cokernel of $\iota_g : \mathrm{Ker}\, g \to C$ (notation of remark 3.7.29(v)). However, $\mathrm{Ker}\, g = 0$ by remark 3.7.29(vi), hence $\mathrm{Coker}\,\iota_g = C$, whence the contention. □

Definition 3.7.42. Let $\mathscr{A}$ be any abelian category with zero object $0 \in \mathrm{Ob}(\mathscr{A})$, and

$$0 \xrightarrow{0} A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{0} 0$$

a sequence of morphisms in $\mathscr{A}$.

(i) We say that the sequence (f, g) is *exact* if $\mathrm{Ker}\, g = \mathrm{Im}\, f$.
(ii) We say that the sequence $(0, f, g)$ is *left exact* if both $(0, f)$ and (f, g) are exact.
(iii) We say that the sequence $(f, g, 0)$ is *right exact* if both (f, g) and $(g, 0)$ are exact.
(iv) We say that the sequence $(0, f, g, 0)$ is *short exact* if $(0, f, g)$ is left exact and $(f, g, 0)$ is right exact.

In other words, the sequence $(0, f, g)$ is left exact if and only if f is a monomorphism, and A represents $\mathrm{Ker}\, g$. Likewise, $(f, g, 0)$ is right exact if and only if g is an epimorphism and the image of f represents $\mathrm{Ker}\, g$. The terminology is explained by the following :

Lemma 3.7.43. *Let $F : \mathscr{A} \to \mathscr{B}$ be any additive functor between abelian categories. Then :*

(i) *F is left exact if and only if it transforms left exact sequences of $\mathscr{A}$ into left exact sequences of $\mathscr{B}$.*
(ii) *F is right exact if and only if it transforms right exact sequences of $\mathscr{A}$ into right exact sequences of $\mathscr{B}$.*
(iii) *F is exact if and only if it transforms short exact sequences of $\mathscr{A}$ into short exact sequences of $\mathscr{B}$.*

Proof. (i): Any left exact functor transforms kernels into kernels and monomorphisms into monomorphisms (proposition 1.3.18(i)), so the condition is necessary. Conversely, suppose that F fulfills this condition; since F is additive, it commutes with finite products, so it suffices to check that it also commutes with equalizers (proposition 1.3.22(i)). Arguing as in the proof of lemma 3.7.41(i), we reduce to checking that F commutes with kernels. But this follows by considering the left exact sequence

$$0 \to \mathrm{Ker}\, f \to A \xrightarrow{f} B \qquad \text{for any morphism } f \text{ of } \mathscr{A}$$

and its image under F. Assertion (ii) admits the dual proof, and (iii) follows by considering the similar short exact sequences

$$0 \to \mathrm{Ker}\, f \to A \to \mathrm{Im}\, f \to 0 \qquad 0 \to \mathrm{Im}\, f \to B \to \mathrm{Coker}\, f \to 0$$

for every f as in the foregoing, and arguing as in the proof of (i). □

Definition 3.7.44. An *abelian tensor category* is a tensor category $(\mathscr{C}, \otimes, \Phi, \Psi)$ such that $\mathscr{C}$ is an abelian category, and the functor $\otimes$ induces bilinear pairings

$$\mathrm{Hom}_{\mathscr{C}}(A, B) \times \mathrm{Hom}_{\mathscr{C}}(A', B') \to \mathrm{Hom}_{\mathscr{C}}(A \otimes A', B \otimes B') \quad : \quad (f, g) \mapsto f \otimes g$$

for every $A, A', B, B' \in \mathrm{Ob}(\mathscr{C})$.

Remark 3.7.45. Let $(\mathscr{A}, \otimes, \Phi, \Psi)$ be a tensor category, such that $\mathscr{A}$ is abelian. If $\mathscr{A}$ admits an internal Hom functor $\mathscr{H}om$, the functor $-\otimes A$ is right exact, and the functor $\mathscr{H}om(A, -)$ is left exact for every $A \in \mathrm{Ob}(\mathscr{A})$, so both are additive, by virtue of remark 3.7.34(iii). Especially, $\mathscr{A}$ is an abelian tensor category, in this case.

Lemma 3.7.46. *Let $\mathscr{A}$ be any additive category with small* Hom*-sets, and $\Sigma \subset \mathrm{Ob}(\mathscr{A})$ a small subset. We have :*

(i) *If $\mathscr{A}$ is abelian, there exists a small full abelian subcategory $\mathscr{B}$ of $\mathscr{A}$ such that $\Sigma \subset \mathrm{Ob}(\mathscr{B})$.*
(ii) *If $\mathscr{A}$ is small, there exist a complete and cocomplete abelian tensor category $(\mathscr{C}, \otimes)$ with internal* Hom *functor, and a fully faithful additive functor $\mathscr{A} \to \mathscr{C}$.*

Proof. (i): Let $\mathscr{B}_0$ be the full subcategory of $\mathscr{A}$ such that $\mathrm{Ob}(\mathscr{B}_0) = \Sigma$; clearly $\mathscr{B}_0$ is small. Next, for any subcategory $\mathscr{D}$ of $\mathscr{A}$, denote by $\mathscr{D}'$ a subcategory of $\mathscr{A}$ obtained as follows. For every morphism φ of $\mathscr{D}$, we pick objects in $\mathscr{A}$ representing the kernel and cokernel of φ, and for any two objects of $\mathscr{D}$, we pick an object in $\mathscr{A}$ representing their product; let $\Sigma' \subset \mathrm{Ob}(\mathscr{A})$ be the resulting subset. Then $\mathscr{D}'$ is the full subcategory of $\mathscr{A}$ such that $\mathrm{Ob}(\mathscr{D}') = \mathrm{Ob}(\mathscr{D}) \cup \Sigma'$. It is easily seen that $\mathscr{D}'$ is small, whenever the same holds for $\mathscr{D}$. Then we set inductively $\mathscr{B}_{i+1} := \mathscr{B}'_i$ for every $i \in \mathbb{N}$. The full subcategory $\mathscr{B}$ of $\mathscr{A}$ with $\mathrm{Ob}(\mathscr{B}) = \bigcup_{i\in\mathbb{N}} \mathrm{Ob}(\mathscr{B}_i)$ is still small, and it is abelian, by construction.

(ii): We let $\mathscr{C} := \mathsf{Fun}(\mathscr{A}, \mathbb{Z}\text{-}\mathbf{Mod})$. Then $\mathscr{C}$ is an abelian category, by virtue of remark 3.7.37(ii), and since $\mathbb{Z}\text{-}\mathbf{Mod}$ is complete and cocomplete, the same holds for $\mathscr{C}$; moreover, the standard tensor product of abelian groups defines a tensor category structure with internal Hom functor on $\mathbb{Z}\text{-}\mathbf{Mod}$, and the latter is inherited by $\mathscr{C}$ (remarks 3.7.5(ii) and 3.7.12(vi)). It is clear that these two structures amount to an abelian tensor category, and the group-valued Yoneda embedding is the sought fully faithful functor. □

3.7.47. Let $\mathscr{A}$ be an additive category with small Hom-sets, and

$$h^\dagger : \mathscr{A}^o \to \mathscr{A}^\dagger := \mathsf{Fun}(\mathscr{A}, \mathbb{Z}\text{-}\mathbf{Mod})$$

the fully faithful group-valued Yoneda embedding. For every abelian group G, denote by $G_\mathscr{A} : \mathscr{A} \to \mathbb{Z}\text{-}\mathbf{Mod}$ the constant functor with value G : so, $G_\mathscr{A}(A) := G$ for every $A \in \mathrm{Ob}(\mathscr{A})$, and $G_\mathscr{A}(\varphi) := \mathbf{1}_G$ for every morphism φ in $\mathscr{A}$. Since $\mathscr{A}^\dagger$ is an abelian tensor category with an internal Hom functor $\mathscr{H}om$ (see the proof of lemma 3.7.46(ii)), we may define

$$G \otimes_\mathbb{Z} A := \mathscr{H}om(G_\mathscr{A}, h^\dagger_{A^o}) \qquad \text{for every } A \in \mathrm{Ob}(\mathscr{A}).$$

If G is free of finite rank, $G \otimes_\mathbb{Z} A$ is a finite direct sum of copies of A, and therefore lies in the essential image of $h^\dagger$; the same holds for a finitely generated G, provided $\mathscr{A}$ is an abelian category : indeed, we may write G as a cokernel of a map $L_1 \to L_2$ of free abelian groups of finite rank, and since the functor $\mathscr{H}om(-, h^\dagger_A)$ is right exact, we see that $G \otimes_\mathbb{Z} A$ is the kernel of a morphism of $\mathscr{A}^o$ (*i.e.* the cokernel of a morphism of $\mathscr{A}$), so it is represented by an object of $\mathscr{A}$. Moreover, if $\varphi : G \to H$ is any morphism of abelian groups, we have an obvious induced morphism $\varphi_\mathscr{A} : G_\mathscr{A} \to H_\mathscr{A}$, whence a morphism $\varphi \otimes_\mathbb{Z} A := \mathscr{H}om(\varphi_\mathscr{A}, h^\dagger_A)$.

Thus, if $\mathscr{A}$ is abelian, after replacing $G \otimes_\mathbb{Z} A$ by an isomorphic object, we obtain a well defined functor

$$\mathbb{Z}\text{-}\mathbf{Mod}_{\mathrm{fg}} \times \mathscr{A} \to \mathscr{A} \qquad (G, A) \mapsto G \otimes_\mathbb{Z} A \tag{3.7.48}$$

where $\mathbb{Z}\text{-}\mathbf{Mod}_{\mathrm{fg}}$ is the full subcategory of $\mathbb{Z}\text{-}\mathbf{Mod}$ whose objects are the finitely generated abelian groups. This functor is not unique, but any two such functors are naturally isomorphic. If $\mathscr{A}$ is only additive, we can still define such a tensor product functor on the category $\mathbb{Z}\text{-}\mathbf{Mod}_{\mathrm{fft}} \times \mathscr{A} \to \mathscr{A}$, where $\mathbb{Z}\text{-}\mathbf{Mod}_{\mathrm{fft}}$ is the full subcategory of $\mathbb{Z}\text{-}\mathbf{Mod}_{\mathrm{fg}}$ whose objects are the free abelian groups of finite rank.

Remark 3.7.49. Keep the notation of (3.7.47); we have :

(i) From the construction, it is clear that (3.7.48) is a *biadditive* functor, *i.e.*, for every abelian group G, and every $A \in \mathrm{Ob}(\mathscr{A})$, the restrictions $G \otimes -$ and $- \otimes A$ of (3.7.48) are additive.

(ii) Suppose that $\mathscr{A}$ is cocomplete; since the tensor product is right exact, it follows easily that (3.7.48) extends to the whole of $\mathbb{Z}$-$\mathbf{Mod}$: details left to the reader.

4. SITES AND TOPOI

In this chapter, we assemble some generalities concerning sites and topoi. The main reference for this material is [8].

4.1. **Topologies and sites.** As in the previous chapter, we fix a universe U such that $\mathbb{N} \in \mathsf{U}$, and small means U-small throughout. Especially, a presheaf on any category takes its values in U, unless explicitly stated otherwise.

Definition 4.1.1. Let $\mathscr{C}$ be a category.

(i) A *topology* on $\mathscr{C}$ is the datum, for every $X \in \mathrm{Ob}(\mathscr{C})$, of a set $J(X)$ of sieves of $\mathscr{C}/X$, fulfilling the following conditions :
   (a) (Stability under base change) For every morphism $f : Y \to X$ of $\mathscr{C}$, and every $\mathscr{S} \in J(X)$, the sieve $\mathscr{S} \times_X f$ lies in $J(Y)$.
   (b) (Local character) Let X be any object of $\mathscr{C}$, and $\mathscr{S}$, $\mathscr{S}'$ two sieves of $\mathscr{C}/X$, with $\mathscr{S} \in J(X)$. Suppose that, for every object $f : Y \to X$ of $\mathscr{S}$, the sieve $\mathscr{S}' \times_X f$ lies in $J(Y)$. Then $\mathscr{S}' \in J(X)$.
   (c) For every $X \in \mathrm{Ob}(\mathscr{C})$, we have $\mathscr{C}/X \in J(X)$.

(ii) In the situation of (i), the elements of $J(X)$ shall be called the *sieves covering* X. Moreover, say that $\mathscr{S}$ is the sieve of $\mathscr{C}/X$ generated by a family $(f_i : X_i \to X \mid i \in I)$ of morphisms. If $\mathscr{S}$ is a sieve covering X, we say that the family $(f_i \mid i \in I)$ *covers* X, or that it is a *covering family of* X.

(iii) The datum $(\mathscr{C}, J)$ of a category $\mathscr{C}$ and a topology $J := (J(X) \mid X \in \mathrm{Ob}(\mathscr{C}))$ on $\mathscr{C}$ is called a *site*, and then $\mathscr{C}$ is also called the *category underlying* the site $(\mathscr{C}, J)$. We say that $(\mathscr{C}, J)$ is a *small site*, if $\mathscr{C}$ is a small category.

(iv) The set of all topologies on $\mathscr{C}$ is partially ordered by inclusion: given two topologies J_1 and J_2 on $\mathscr{C}$, we say that J_1 is *finer* than J_2, if $J_2(X) \subset J_1(X)$ for every $X \in \mathrm{Ob}(\mathscr{C})$.

Example 4.1.2. Let T be any topological space, and $\mathscr{T}$ the set of open subsets of T, which is partially ordered by inclusion, and can thus be regarded naturally as a category, as in example 1.1.6(iii). Then the category $\mathscr{T}$ carries a natural topology J_T in the sense of definition 4.1.1 : namely, for every $U \in \mathscr{T}$ we declare that the elements of $J_T(U)$ are the sieves $\mathscr{S} \subset \mathscr{T}/U$ such that $\bigcup_{(f:U'\to U)\in\mathscr{S}} U' = U$. In other words, a family $(U'_i \to U \mid i \in I)$ of morphisms of $\mathscr{T}$ covers U for the topology J_T if and only if $\bigcup_{i\in I} U_i = U$.

Remark 4.1.3. Let $(\mathscr{C}, J)$ be any site.

(i) Any finite intersection of sieves covering an object X, again covers X. Indeed, say that $\mathscr{S}_1$ and $\mathscr{S}_2$ are two sieves covering X; set $\mathscr{S} := \mathscr{S}_1 \cap \mathscr{S}_2$ and let $f : Y \to X$ be any object of $\mathscr{S}_1$. Then $\mathscr{S} \times_X f = \mathscr{S}_2 \times_X f \in J(Y)$, so the assertion follows from the local character of J.

(ii) Also, any sieve of $\mathscr{C}/X$ containing a covering sieve is again a covering sieve. Indeed, if $\mathscr{S} \subset \mathscr{S}'$, then $\mathscr{S}' \times_X f = \mathscr{C}/Y$ for every object $f : Y \to X$ of $\mathscr{S}$.

(iii) Let $f_\bullet := (f_i : Y_i \to X \mid i \in I)$ be a family of objects of $\mathscr{C}/X$ that generates a sieve $\mathscr{S}$ covering X, and for every $i \in I$, let $(g_{ij} : Z_{ij} \to Y_i \mid i \in J_i)$ be a family of objects of $\mathscr{C}/Y_i$ that generates a sieve $\mathscr{S}_i$ covering Y_i. Then the family $(f_i \circ g_{ij} : Z_{ij} \to X \mid i \in I,\ j \in J_i)$ is a refinement of $f_\bullet$ (see definition 3.5.1(ii)) and generates a sieve $\mathscr{S}'$ covering X. Indeed, say that $f : Y \to X$ lies in $\mathscr{S}$, and pick $i \in I$ such that f factors through f_i and a morphism $g : Y \to Y_i$; then it is easily seen that $\mathscr{S}_i \times_{Y_i} g \subset \mathscr{S}' \times_X f$.

4.1.4. Suppose that $\mathscr{C}$ has small Hom-sets. Then, in view of remark 3.5.2(ii), a topology can also be defined by assigning, to any object X of $\mathscr{C}$, a family $J'(X)$ of subobjects of h_X, called the *subobjects covering* X, such that :

(a) For every $X \in \mathrm{Ob}(\mathscr{C})$, every $R \in J'(X)$, and every morphism $Y \to X$ in $\mathscr{C}$, the fibre product $R \times_X Y$ lies in $J'(Y)$.
(b) Say that $X \in \mathrm{Ob}(\mathscr{C})$, and let R, R' be two subobjects of h_X, such that $R \in J'(X)$. Suppose that, for every $Y \in \mathrm{Ob}(\mathscr{C})$, and every morphism $f : h_Y \to R$, we have $R' \times_X Y \in J'(Y)$. Then $R' \in J'(X)$.
(c) $h_X \in J'(X)$ for every $X \in \mathrm{Ob}(\mathscr{C})$.

This viewpoint is adopted in the following :

Definition 4.1.5. Let V be a universe, $C := (\mathscr{C}, J)$ a site, and $F \in \mathrm{Ob}(\mathscr{C}^\wedge_{\mathsf{V}})$.

(i) We say that F is a *separated* V*-presheaf* (resp. a V*-sheaf*) on C, if for every $X \in \mathrm{Ob}(\mathscr{C})$, every subobject R covering X, and every universe V' containing V and such that the category $\mathscr{C}$ has V'-small Hom-sets, the induced morphism :

$$F(X) = \mathrm{Hom}_{\mathscr{C}^\wedge_{\mathsf{V}'}}(h_X, F) \to \mathrm{Hom}_{\mathscr{C}^\wedge_{\mathsf{V}'}}(R, F)$$

is injective (resp. is bijective). For $\mathsf{V} = \mathsf{U}$, we shall just say separated presheaf instead of separated U-presheaf, and likewise for sheaves.

(ii) We let $C^\sim_{\mathsf{V}}$ (resp. $C^{\mathrm{sep}}_{\mathsf{V}}$) be the full subcategory of $\mathscr{C}^\wedge_{\mathsf{V}}$ whose objects are the V-sheaves (resp. the separated V-presheaves) on C. For $\mathsf{V} = \mathsf{U}$, this category will be usually denoted simply $C^\sim$ (resp. C^{sep}).

4.1.6. In the situation of definition 4.1.5(i), say that $R = h_{\mathscr{S}}$ for some sieve $\mathscr{S}$ covering X. In light of (3.5.3), (1.4.5) and proposition 1.2.6(ii), we get a natural identification :

$$\mathrm{Hom}_{\mathscr{C}^\wedge_{\mathsf{V}'}}(R, F) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{C}^\wedge_{\mathsf{V}'}}(\operatorname*{colim}_{\mathscr{S}} h_{\mathscr{C}} \circ \mathsf{s}_{\mathscr{S}}, F) \xrightarrow{\sim} \lim_{\mathscr{S}^o} F \circ \mathsf{s}^o_{\mathscr{S}}.$$

Let also $(S_i \to X \mid i \in I)$ be a generating family for $\mathscr{S}$. Combining lemma 3.5.6, example 3.1.16(i) and (3.1.15), we get a natural isomorphism :

$$\mathrm{Hom}_{\mathscr{C}^\wedge_{\mathsf{V}'}}(R, F) \xrightarrow{\sim} \mathrm{Equal}\Big(\prod_{i\in I} \mathsf{Cart}_{\mathscr{C}}(\mathscr{C}/S_i, \mathscr{F}ib(F)) \rightrightarrows \prod_{(i,j)\in I\times I} \mathsf{Cart}_{\mathscr{C}}(\mathscr{C}/S_{ij}, \mathscr{F}ib(F))\Big)$$

which – again by (3.1.15) – we may rewrite more simply as :

$$\mathrm{Hom}_{\mathscr{C}^\wedge_{\mathsf{V}'}}(R, F) \xrightarrow{\sim} \mathrm{Equal}\Big(\prod_{i\in I} F(S_i) \rightrightarrows \prod_{(i,j)\in I\times I} \mathrm{Hom}_{\mathscr{C}^\wedge_{\mathsf{V}'}}(h_{S_i} \times_{h_X} h_{S_j}, F)\Big).$$

The above equalizer can be described explicitly as follows. It consists of all the systems

$$(a_i \mid i \in I) \qquad \text{with } a_i \in F(S_i) \text{ for every } i \in I$$

such that, for every $i, j \in I$ and every object $Y \to X$ of $\mathscr{C}/X$, we have :

$$(4.1.7) \qquad F(g_i)(a_i) = F(g_j)(a_j) \qquad \text{for every pair } (S_i \xleftarrow{g_i} Y \xrightarrow{g_j} S_j) \text{ of morphisms in } \mathscr{C}/X.$$

If all $S_{ij} := S_i \times_X S_j$ are representable in $\mathscr{C}$, the latter expression takes the more familiar form:

$$\mathrm{Hom}_{\mathscr{C}^\wedge_{\mathsf{V}'}}(R, F) \xrightarrow{\sim} \mathrm{Equal}\Big(\prod_{i\in I} F(S_i) \rightrightarrows \prod_{(i,j)\in I\times I} F(S_{ij})\Big).$$

Remark 4.1.8. Let $C := (\mathscr{C}, J)$ be a site such that $\mathscr{C}$ is a category with small Hom-sets.

(i) The arguments in (4.1.6) yield also the following. A presheaf F on $\mathscr{C}$ is separated (resp. is a sheaf) on C, if and only if every covering sieve of C is a sieve of 1-descent (resp. of 2-descent) for the fibration $\mathsf{s}_F : \mathscr{F}ib(F) \to \mathscr{C}$ of (3.1.15).

(ii) Let $F : \mathscr{A} \to \mathscr{B}$ be a fibration between two categories, and $i \leq 2$ an integer. For every $X \in \mathrm{Ob}(\mathscr{B})$, let $J^i_F(X)$ denote the set of all sieves $\mathscr{S} \subset \mathscr{B}/X$ of universal F-i-descent. Then we claim that J^i_F is a topology on $\mathscr{B}$. Indeed, it is clear that J^i_F fulfills conditions (a) and (c) of definition 4.1.1(i), and condition (b) follows from proposition 3.5.38.

(iii) Let F be a presheaf on $\mathscr{C}$. We deduce from (i) and (ii) that the topology $J^F := J^2_{\mathsf{s}_F}$ is the finest on $\mathscr{C}$ for which F is a sheaf. A subobject $R \subset h_X$ (for any $X \in \mathrm{Ob}(\mathscr{C})$) lies in $J^F(X)$ if and only if the natural map $F(X') \to \mathrm{Hom}_{\mathscr{C}^\wedge}(R \times_X X', F)$ is bijective for every morphism $X' \to X$ in $\mathscr{C}$.

(iv) More generally, if $(F_i \mid i \in I)$ is any family of presheaves on $\mathscr{C}$, there exists a finest topology for which each F_i is a sheaf : namely, the intersection of the topologies J^{F_i} as in (iii).

(v) As an important special case, we deduce the existence of a finest topology J on $\mathscr{C}$ such that all representable presheaves are sheaves on $(\mathscr{C}, J)$. This topology $\mathrm{Can}_{\mathscr{C}}$ is called the *canonical topology* on $\mathscr{C}$. We thus associate to every category $\mathscr{C}$ a *canonical site*

$$\mathrm{Can}(\mathscr{C}) := (\mathscr{C}, \mathrm{Can}_{\mathscr{C}}).$$

(vi) Another interesting case of (iv) is obtained by taking the family of all presheaves on $\mathscr{C}$. The corresponding topology J on $\mathscr{C}$ can be easily described explicitly : namely, one takes $J(X) := \{\mathscr{C}/X\}$ for every $X \in \mathrm{Ob}(\mathscr{C})$.

Example 4.1.9. Let $\mathscr{C}$ be a category, $X \in \mathrm{Ob}(\mathscr{C})$ any object, and $\mathscr{S} \subset \mathscr{C}/X$ a sieve.

(i) We say that $\mathscr{S}$ is an *epimorphic sieve* (resp. a *strict epimorphic sieve*) if for every $Y \in \mathrm{Ob}(\mathscr{C})$ the natural map

$$h_Y(X) \to \mathrm{Hom}_{\mathscr{C}^\wedge}(h_{\mathscr{S}}, h_Y)$$

is injective (resp. bijective). We say that $\mathscr{S}$ is a *universal epimorphic sieve* (resp. a *universal strict epimorphic sieve*) if $\mathscr{S} \times_X h$ is epimorphic (resp. strict epimorphic) for every morphism $h : Y \to X$ of $\mathscr{C}$. From remark 4.1.8(iii,v) we see that $\mathscr{S}$ is universal strict epimorphic if and only if it covers X in the canonical topology of $\mathscr{C}$.

(ii) We say that a family of morphisms $f_\bullet := (f_i : X_i \to X \mid i \in I)$ of $\mathscr{C}$ is *epimorphic* (resp. *strict epimorphic*) if it generates an epimorphic (resp. strictly epimorphic) sieve. This is the same as saying for every $Y \in \mathrm{Ob}(\mathscr{C})$, the natural map

$$\mathrm{Hom}_{\mathscr{C}}(X, Y) \to \prod_{i \in I} \mathrm{Hom}_{\mathscr{C}}(X_i, Y) \qquad g \mapsto (g \circ f_i \mid i \in I) \tag{4.1.10}$$

is injective (resp. and its image consists of all the systems of morphisms $(g_i : X_i \to Y \mid i \in I)$ such that, for every $Z \in \mathrm{Ob}(\mathscr{C})$, every $i, j \in I$, and every pair of morphisms $X_j \xleftarrow{h_j} Z \xrightarrow{h_i} X_i$ with $f_i \circ h_i = f_j \circ h_j$, we have $g_i \circ h_i = g_j \circ h_j$).

(iii) We say that a family $f_\bullet$ as in (ii) is *effective epimorphic*, if it is strict epimorphic, and moreover the fibre products $X_i \times_X X_j$ are representable in $\mathscr{C}$ for every $i, j \in I$. This is the same as saying that for every $Y \in \mathrm{Ob}(\mathscr{C})$ the map (4.1.10) identifies $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ with the equalizer of the two natural maps

$$\prod_{i \in I} \mathrm{Hom}_{\mathscr{C}}(X_i, Y) \rightrightarrows \prod_{(i,j) \in I \times I} \mathrm{Hom}_{\mathscr{C}}(X_i \times_X X_j, Y).$$

(iv) A family $f_\bullet$ as in (ii) is called *universal epimorphic* (resp. *universal effective epimorphic*) if (a) the fibre products $Y_i := X_i \times_X Y$ are representable in $\mathscr{C}$, for every $i \in I$ and every morphism $Y \to X$ in $\mathscr{C}$, and (b) all the resulting families $(Y_i \to Y \mid i \in I)$ are still epimorphic (resp. effective epimorphic). Then clearly such a family generates a universal epimorphic sieve (resp. a universal strict epimorphic sieve).

(v) We say that a morphism $f : X' \to X$ of $\mathscr{C}$ is an *effective epimorphism* (resp. a *universal epimorphism*, resp. a *universal effective epimorphism*) if the family $\{f\}$ has the corresponding property. Hence, f is an effective epimorphism if the fibre product $X' \times_X X'$ is representable in $\mathscr{C}$, and the induced diagram

$$X' \times_X X' \overset{p_1}{\underset{p_2}{\rightrightarrows}} X' \xrightarrow{f} X$$

identifies X with the coequalizer of the two projections p_1 and p_2. (Notice that this condition implies that f is an epimorphism). Likewise, f is a universal epimorphism (resp. a universal effective epimorphism) if for every morphism $Y \to X$, the fibre product $X' \times_X Y$ is representable in $\mathscr{C}$ and $f \times_X Y : X' \times_X Y \to Y$ is an epimorphism (resp. an effective epimorphism).

(vi) Suppose that all fibre products are representable in $\mathscr{C}$; in this case, proposition 1.2.18(ii) implies that a morphism $X' \to X$ of $\mathscr{C}$ is a universal epimorphism if and only if it is an epimorphism and the coproduct $X \amalg_{X'} X$ is a universal colimit, in the sense of example 1.4.17(ii).

(vii) Combining (vi) with example 1.4.17(iv), we conclude that all epimorphisms of $\mathbf{Set}$ are universal. Also, it is easily seen that all epimorphisms of $\mathbf{Set}$ are effective, and hence universal effective. More generally, If $\mathscr{C}$ is a small category, example 1.2.24(i) and corollary 1.4.3(ii) imply that all epimorphisms in $\mathscr{C}^\wedge$ are universal effective.

4.1.11. Let $C := (\mathscr{C}, J)$ be a small site; directly from definition 4.1.5 (and from (1.4.6)), we see that the category $C^\sim$ is complete, and the fully faithful inclusion $C^\sim \to \mathscr{C}^\wedge$ commutes with all limits. Moreover, given a presheaf F on $\mathscr{C}$, it is possible to construct a solution set for F relative to this functor, and therefore one may apply theorem 1.5.13 to produce a left adjoint. However, a more direct and explicit construction of the left adjoint can be given; the latter also provides some additional information which is hard to extract from the former method. Namely, for every $X \in \mathrm{Ob}(\mathscr{C})$, endow $J(X)$ with the partial ordering induced by inclusion of sieves. Notice that $J(X)$ is small and cofiltered, for every such X, hence the opposite ordered set $J(X)^o$ is filtered. For a given presheaf F on $\mathscr{C}$, set

$$F^+(X) := \operatorname*{colim}_{\mathscr{S} \in J(X)^o} \mathrm{Hom}_{\mathscr{C}^\wedge}(h_{\mathscr{S}}, F)$$

where the transition map $\mathrm{Hom}_{\mathscr{C}^\wedge}(h_{\mathscr{S}}, F) \to \mathrm{Hom}_{\mathscr{C}^\wedge}(h_{\mathscr{S}'}, F)$ is induced by the monomorphism $h_{\mathscr{S}'} \to h_{\mathscr{S}}$ of subobjects of h_X, for every inclusion $\mathscr{S}' \subset \mathscr{S}$ of sieves covering X. Let $(\tau_{\mathscr{S}} : \mathrm{Hom}_{\mathscr{C}^\wedge}(h_{\mathscr{S}}, F) \to F^+(X) \mid \mathscr{S} \in J(X))$ be the universal cocone. For a morphism $f : Y \to X$ in $\mathscr{C}$ and a sieve $\mathscr{S} \in J(X)$, the natural projection $h_{\mathscr{S}} \times_X Y \to h_{\mathscr{S}}$ induces a map

$$\mathrm{Hom}_{\mathscr{C}^\wedge}(h_{\mathscr{S}}, F) \to \mathrm{Hom}_{\mathscr{C}^\wedge}(h_{\mathscr{S}} \times_X Y, F) \xrightarrow{\tau_{\mathscr{S} \times_X f}} F^+(Y)$$

whence a map $F^+f : F^+(X) \to F^+(Y)$, after taking colimits. Likewise, every morphism $F \to F'$ of presheaves on $\mathscr{C}$ induces a map $\mathrm{Hom}_{\mathscr{C}^\wedge}(h_{\mathscr{S}}, F) \to \mathrm{Hom}_{\mathscr{C}^\wedge}(h_{\mathscr{S}}, F')$, for every $X \in \mathrm{Ob}(\mathscr{C})$ and every $\mathscr{S} \in J(X)$, whence a map $F^+(X) \to F'^+(X)$, after taking colimits. Thus, we have a functor :

$$\mathscr{C}^\wedge \to \mathscr{C}^\wedge \qquad F \mapsto F^+. \tag{4.1.12}$$

with a natural transformation $F(X) \to F^+(X)$, since $\mathscr{C}/X \in J(X)$ for every $X \in \mathrm{Ob}(\mathscr{C})$.

Theorem 4.1.13. *In the situation of* (4.1.11)*, the following holds :*

(i) *The inclusion functor* $i : C^\sim \to \mathscr{C}^\wedge$ *admits the left adjoint*

$$\mathscr{C}^\wedge \to C^\sim \qquad F \mapsto F^a := (F^+)^+. \tag{4.1.14}$$

For every presheaf F*, we call* F^a *the* sheaf associated with F.

(ii) *Moreover, the functor* (4.1.14) *is exact.*

Proof. We begin with the following :

Claim 4.1.15. Let F be a separated presheaf on $\mathscr{C}$, and $\mathscr{S}_1 \subset \mathscr{S}_2$ two sieves covering some $X \in \mathrm{Ob}(\mathscr{C})$. Then the natural map $\mathrm{Hom}_{\mathscr{C}^\wedge}(h_{\mathscr{S}_2}, F) \to \mathrm{Hom}_{\mathscr{C}^\wedge}(h_{\mathscr{S}_1}, F)$ is injective.

Proof of the claim. We may find a generating family $(f_i : S_i \to X \mid i \in I_2)$ for $\mathscr{S}_2$, and a subset $I_1 \subset I_2$, such that $(f_i \mid i \in I_1)$ generates $\mathscr{S}_1$. Let $s, s' : h_{\mathscr{S}_2} \to F$, whose images agree in $\mathrm{Hom}_{\mathscr{C}^\wedge}(h_{\mathscr{S}_1}, F)$. By (4.1.6), s and s' correspond to families $(s_i \mid i \in I_2)$, $(s'_i \mid i \in I_2)$ with $s_i, s'_i \in F(S_i)$ for every $i \in I_2$, fulfilling the system of identities (4.1.7), and the foregoing condition means that $s_i = s'_i$ for every $i \in I_1$. We need to show that $s_i = s'_i$ for every $i \in I_2$. Hence, let $i \in I_2$ be any element; by assumption, the natural map

$$F(S_i) \to \mathrm{Hom}_{\mathscr{C}^\wedge}(h_{\mathscr{S}_1} \times_X S_i, F)$$

is injective. However, the objects of $\mathscr{S}_1 \times_X f_i$ are all the morphisms $g_i : Y \to S_i$ in $\mathscr{C}$ such that $f_i \circ g_i = f_j \circ g_j$ for some $j \in I_1$ and some $g_j : Y \to S_j$ in $\mathscr{C}$. If we apply the identities (4.1.7) to these maps g_i, g_j, we deduce that :

$$F(g_i)(s_i) = F(g_j)(s_j) = F(g_j)(s'_j) = F(g_i)(s'_i).$$

In other words, s_i and s'_i have the same image in $\mathrm{Hom}_{\mathscr{C}^\wedge}(h_{\mathscr{S}_1} \times_X S_i, F)$, hence they agree, as claimed. ◊

Claim 4.1.16. (i) The functor (4.1.12) is left exact.

(ii) For every $F \in \mathrm{Ob}(\mathscr{C}^\wedge)$, the presheaf F^+ is separated.

(iii) If F is a separated presheaf on $\mathscr{C}$, then F^+ is a sheaf on C.

Proof of the claim. (i) is clear, since $J(X)^o$ is filtered for every $X \in \mathrm{Ob}(\mathscr{C})$.

(ii): Let $s, s' \in F^+(X)$, and suppose that the images of s and s' agree in $\mathrm{Hom}_{\mathscr{C}^\wedge}(h_{\mathscr{S}}, F^+)$, for some $\mathscr{S} \in J(X)$. We may find a sieve $\mathscr{T} \in J(X)$ such that s and s' come from elements $\overline{s}, \overline{s}' \in \mathrm{Hom}_{\mathscr{C}^\wedge}(h_{\mathscr{T}}, F)$. Let $(g_i : S_i \to X \mid i \in I)$ be a family of generators for $\mathscr{S}$; in view of (4.1.6), the images of s and s' agree in $F^+(S_i)$ for every $i \in I$. The latter means that, for every $i \in I$, there exists $\mathscr{S}_i \in J(S_i)$, refining $\mathscr{T} \times_X g_i$, such that the images of $\overline{s}$ and $\overline{s}'$ agree in $\mathrm{Hom}_{\mathscr{C}^\wedge}(h_{\mathscr{S}_i}, F)$. For every $i \in I$, let $(g_{i\lambda} : T_{i\lambda} \to S_i \mid \lambda \in \Lambda_i)$ be a family of generators for $\mathscr{S}_i$, and consider the sieve $\mathscr{T}'$ of $\mathscr{C}/X$ generated by $(g_i \circ g_{i\lambda} : T_{i\lambda} \to X \mid i \in I, \lambda \in \Lambda_i)$. Then $\mathscr{T}'$ covers X (remark 4.1.3(iii)) and refines $\mathscr{T}$, and the images of $\overline{s}$ and $\overline{s}'$ agree in $\mathrm{Hom}_{\mathscr{C}^\wedge}(h_{\mathscr{T}'}, F)$ (as one sees easily, again by virtue of (4.1.6)). This shows that $s = s'$, whence the contention.

(iii): In view of (ii), it suffices to show that the natural map $F^+(X) \to \mathrm{Hom}_{\mathscr{C}^\wedge}(h_{\mathscr{S}}, F^+)$ is surjective for every $\mathscr{S} \in J(X)$. Hence, say that $s \in \mathrm{Hom}_{\mathscr{C}^\wedge}(h_{\mathscr{S}}, F^+)$, and let $(S_i \mid i \in I)$ be a generating family for $\mathscr{S}$. By (4.1.6), s corresponds to a system $(s_i \in F^+(S_i) \mid i \in I)$ such that the following holds. For every $i, j \in I$, and every pair of morphisms $u_i : Y \to S_i$ and $u_j : Y \to S_j$ in $\mathscr{C}/X$, we have

$$F^+(u_i)(s_i) = F^+(u_j)(s_j). \tag{4.1.17}$$

For every $i \in I$, let $\mathscr{S}_i \in J(S_i)$ such that s_i is the image of some $\overline{s}_i \in \mathrm{Hom}_{\mathscr{C}^\wedge}(h_{\mathscr{S}_i}, F)$. For every u_i, u_j as above, set $\mathscr{S}_{ij} := (\mathscr{S}_i \times_{S_i} u_i) \cap (\mathscr{S}_j \times_{S_j} u_j)$; since F is separated, (4.1.17) and claim 4.1.15 imply that the images of $\overline{s}_i$ and $\overline{s}_j$ agree in $\mathrm{Hom}_{\mathscr{C}^\wedge}(h_{\mathscr{S}_{ij}}, F)$, for every $i, j \in I$.

However, for every $i \in I$, let $(g_{i\lambda} : T_{i\lambda} \to S_i \mid \lambda \in \Lambda_i)$ be a generating family for $\mathscr{S}_i$; then $\overline{s}_i$ corresponds to a compatible system of sections $\overline{s}_{i\lambda} \in F(T_{i\lambda})$, and $\mathscr{S}_{ij}$ is the sieve of all morphisms $f : Z \to Y$ such that

$$u_i \circ f = g_{i\lambda} \circ f'_i \qquad \text{and} \qquad u_j \circ f = g_{j\mu} \circ f'_j$$

for some $\lambda \in \Lambda_i$, $\mu \in \Lambda_j$ and some $f'_i : Z \to T_{i\lambda}$, $f'_j : Z \to T_{j\mu}$, so by construction we have

$$F(f'_i)(\overline{s}_{i\lambda}) = F(f'_j)(\overline{s}_{j\mu}) \qquad \text{for every } i, j \in I \text{ and } \lambda \in \Lambda_i,\ \mu \in \Lambda_\mu. \tag{4.1.18}$$

Lastly, let $\mathscr{T}$ be the sieve of $\mathscr{C}/X$ generated by $(g_i \circ g_{i\lambda} : T_{i\lambda} \to X \mid i \in I, \lambda \in \Lambda_i)$; then $\mathscr{T}$ covers X (remark 4.1.3(iii)), and (4.1.18) shows that the system $(F(g_{i\lambda})(\overline{s}_{i\lambda}) \mid i \in I, \lambda \in \Lambda_i)$ defines an element of $\mathrm{Hom}_{\mathscr{C}^\wedge}(h_{\mathscr{T}}, F)$, whose image in $F^+(X)$ agrees with s. ◊

From claim 4.1.16 we see that the rule : $F \mapsto F^a := (F^+)^+$ defines a left exact functor $\mathscr{C}^\wedge \to C^\sim$, with natural transformations $\eta_F : F \Rightarrow i(F^a)$ for every $F \in \mathrm{Ob}(\mathscr{C}^\wedge)$ and $\varepsilon_G : (iG)^a \Rightarrow G$ for every $G \in \mathrm{Ob}(C^\sim)$ fulfilling the triangular identities of (1.1.13). The theorem follows. □

Remark 4.1.19. Let $C := (\mathscr{C}, J)$ be a small site.

(i) It has already been remarked that $C^\sim$ is complete, and from theorem 4.1.13 we also deduce that $C^\sim$ is cocomplete, and the functor (4.1.14) (resp. the inclusion functor $i : C^\sim \to \mathscr{C}^\wedge$) commutes with all colimits (resp. with all limits); more precisely, if $F : I \to C^\sim$ is any functor from a small category I, we have a natural isomorphism in $C^\sim$ (resp. in $\mathscr{C}^\wedge$) :

$$\operatorname*{colim}_I F \xrightarrow{\sim} (\operatorname*{colim}_I i \circ F)^a \qquad (\text{resp. } \ i(\lim_I F) \xrightarrow{\sim} \lim_I i \circ F)$$

(proposition 1.3.25(iii,iv)); especially, limits in $C^\sim$ are computed argumentwise. Also, all colimits and all epimorphisms are universal in $C^\sim$ (see examples 1.4.17(ii,iv) and 4.1.9(v,vii)), and filtered colimits in $C^\sim$ commute with finite limits, since the same holds in $\mathscr{C}^\wedge$.

(ii) Furthermore, $C^\sim$ is well-powered and co-well-powered, since the same holds for $\mathscr{C}^\wedge$. Especially, every morphism $f : F \to G$ in $C^\sim$ admits a well defined image (see example 1.2.26(i)) that can be constructed explicitly as the subobject $(\mathrm{Im}\, i(f))^a$ (details left to the reader).

(iii) By composing with the Yoneda embedding, we obtain a functor

$$h^a_C : \mathscr{C} \to C^\sim \quad : \quad X \mapsto (h_X)^a \qquad \text{for every } X \in \mathrm{Ob}(\mathscr{C})$$

and since the functor $(-)^a : \mathscr{C}^\wedge \to C^\sim$ commutes with small colimits, lemma 1.4.8 yields a natural isomorphism :

$$\operatorname*{colim}_{\mathscr{F}ib(F)} h^a_C \circ \mathsf{s}_F \xrightarrow{\sim} F \qquad \text{for every sheaf } F \text{ on } C.$$

(iv) From the proof of claim 4.1.16 it is clear that the functor

$$\mathscr{C}^\wedge \to C^{\mathrm{sep}} \qquad F \mapsto F^{\mathrm{sep}} := \mathrm{Im}(F \to F^+)$$

is left adjoint to the inclusion $C^{\mathrm{sep}} \to \mathscr{C}^\wedge$. Moreover, we have a natural identification :

$$F^a \xrightarrow{\sim} (F^{\mathrm{sep}})^+ \qquad \text{for every } F \in \mathrm{Ob}(\mathscr{C}^\wedge).$$

(v) Let V be a universe such that $\mathsf{U} \subset \mathsf{V}$; from the definitions, it is clear that the fully faithful inclusion $\mathscr{C}^\wedge_\mathsf{U} \subset \mathscr{C}^\wedge_\mathsf{V}$ restricts to a fully faithful inclusion

$$C^\sim_\mathsf{U} \subset C^\sim_\mathsf{V}.$$

Moreover, by inspecting the proof of theorem 4.1.13, we deduce an essentially commutative diagram of categories :

$$\begin{array}{ccc} \mathscr{C}^\wedge_\mathsf{U} & \longrightarrow & C^\sim_\mathsf{U} \\ \downarrow & & \downarrow \\ \mathscr{C}^\wedge_\mathsf{V} & \longrightarrow & C^\sim_\mathsf{V} \end{array}$$

whose vertical arrows are the inclusions, and whose horizontal arrows are the functors $F \mapsto F^a$.

In practice, one often encounters sites that are not small, but which share many of the properties of small sites. These more general situations are encompassed by the following :

Definition 4.1.20. Let $C := (\mathscr{C}, J)$ be a site.

(i) A *topologically generating family* for C is a subset $G \subset \mathrm{Ob}(\mathscr{C})$, such that, for every $X \in \mathrm{Ob}(\mathscr{C})$, the family

$$G/X := \bigcup_{Y \in G} \{Y\} \times \mathrm{Hom}_{\mathscr{C}}(Y, X) \subset \mathrm{Ob}(\mathscr{C}/X)$$

generates a sieve covering X.

(ii) We say that C is a U-*site*, if $\mathscr{C}$ has small Hom-sets, and C admits an essentially small topologically generating family.

4.1.21. Let $C := (\mathscr{C}, J)$ be a U-site, and G an essentially small topologically generating family for C. For every $X \in \mathrm{Ob}(\mathscr{C})$, denote by $J_G(X) \subset J(X)$ the set of all sieves covering X which are generated by a subset of G/X (notation of definition 4.1.20(i)).

Lemma 4.1.22. *With the notation of* (4.1.21)*, for every $X \in \mathrm{Ob}(\mathscr{C})$ the following holds :*

(i) *$J_G(X)$ is an essentially small set.*

(ii) *$J_G(X)$ is a coinitial subset of $J(X)$ (for the partial order given by inclusion of sieves).*

Proof. (i) is left to the reader.

(ii): Let $\mathscr{S} \in J(X)$, and say that $\mathscr{S}$ is generated by a family $(f_i : S_i \to X \mid i \in I)$ of objects of $\mathscr{C}/X$ (indexed by some not necessarily small set I). Let $\mathscr{S}'$ be the sieve generated by

$$\bigcup_{i \in I} \{f_i \circ g \mid g \in G/S_i\}.$$

It is easily seen that $\mathscr{S}' \subset \mathscr{S}$ and $\mathscr{S}' \in J_G(X)$. □

Remark 4.1.23. (i) In the situation of (4.1.21), let V be a universe with $\mathsf{U} \subset \mathsf{V}$, and such that C is a V-small site, so that we have a well defined functor $(-)^+ : \mathscr{C}^\wedge_\mathsf{V} \to \mathscr{C}^\wedge_\mathsf{V}$, as in (4.1.11). Lemma 4.1.22 implies that the natural map :

$$\mathop{\mathrm{colim}}_{\mathscr{S} \in J_G(X)^o} \mathrm{Hom}_{\mathscr{C}^\wedge}(h_\mathscr{S}, F) \to F^+(X)$$

is bijective. Therefore, if F is a U-presheaf, $F^+(X)$ is essentially U-small, and then the same holds for $F^a(X)$. In other words, the restriction to $\mathscr{C}^\wedge_\mathsf{U}$ of the functor $\mathscr{C}^\wedge_\mathsf{V} \to C^\sim_\mathsf{V} : F \mapsto F^a$ is isomorphic to a functor that factors through $C^\sim$.

(ii) We deduce that theorem 4.1.13 holds, more generally, when C is an arbitrary U-site. Likewise, a simple inspection shows that remark 4.1.19(i,iv,v) holds when C is only assumed to be a U-site.

Proposition 4.1.24. *Let $C := (\mathscr{C}, J)$ be a U-site. The following holds :*

(i) *A morphism in $C^\sim$ is an isomorphism if and only if it is both a monomorphism and an epimorphism.*

(ii) *All epimorphisms in $C^\sim$ are universal effective.*

Proof. (i): Let $\varphi : F \to G$ be a monomorphism in $C^\sim$; then the morphism of presheaves $i(\varphi) : iF \to iG$ is also a monomorphism, and it is easily seen that the cocartesian diagram

$$\mathscr{D} \quad : \quad \begin{array}{ccc} iF & \xrightarrow{i(\varphi)} & iG \\ {\scriptstyle i(\varphi)}\downarrow & & \downarrow{\scriptstyle \alpha} \\ iG & \longrightarrow & iG \amalg_{iF} iG \end{array}$$

is also cartesian, hence the same holds for the induced diagram of sheaves $\mathscr{D}^a$. If moreover, φ is an epimorphism, then α^a is an isomorphism, hence the same holds for $\varphi = (i(\varphi))^a$.

(ii): Let $f : F \to G$ be an epimorphism in $C^\sim$; in view of remarks 4.1.19(i) and 4.1.23(ii), it suffices to show that f is effective. However, set $G' := \mathrm{Im}(i(f))$, and let $p_i : F \times_G F \to F$ (for $i = 1, 2$) be the two projections. Suppose $\varphi : F \to X$ is a morphism in $C^\sim$ such that $\varphi \circ p_1 = \varphi \circ p_2$; since $i(F \times_G F) = iF \times_{iG} iF = iF \times_{G'} iF$, the morphism $i(\varphi)$ factors through a (unique) morphism $\psi : G' \to X$. On the other hand, it is easily seen that $(G')^a = G$, hence φ factors through the morphism $\psi^a : G \to X$. □

Remark 4.1.25. Proposition 4.1.24(i) implies that every morphism $\varphi : X \to Y$ in $C^{\sim}$ factors uniquely (up to unique isomorphism) as the composition of an epimorphism followed by a monomorphism. Indeed; such a factorization is provided by the natural morphisms $X \to \mathrm{Im}(\varphi)$ and $\mathrm{Im}(\varphi) \to Y$ (see example 1.2.26(i)). If $X \xrightarrow{\varphi'} Z \to Y$ is another such factorization, then by definition φ' factors through a unique monomorphism $\psi : \mathrm{Im}(\varphi) \to Z$. However, ψ is an epimorphism, since the same holds for φ'. Hence ψ is an isomorphism.

Proposition 4.1.26. *Let $(\mathscr{C}, J)$ be a* U*-site, $X \in \mathrm{Ob}(\mathscr{C})$, and R any subobject of h_X. The following conditions are equivalent :*

(a) *The inclusion map $i : R \to h_X$ induces an isomorphism on associated sheaves*

$$i^a : R^a \xrightarrow{\sim} h^a_X.$$

(b) *R covers X.*

Proof. (b)⇒(a) : By definition, the natural map $R^a(X) \to \mathrm{Hom}_{\mathscr{C}^\wedge}(R, R^a)$ is bijective, hence there exists a morphism $f : h_X \to R^a$ in $\mathscr{C}^\wedge$ whose composition with i is the unit of adjunction $R \to R^a$. Therefore, $f^a : h^a_X \to R^a$ is a left inverse for i^a. On the other hand, we have a commutative diagram :

$$\begin{array}{ccc} \mathrm{Hom}_{\mathscr{C}^\wedge}(h^a_X, h^a_X) & \longrightarrow & \mathrm{Hom}_{\mathscr{C}^\wedge}(R^a, h^a_X) \\ \downarrow & & \downarrow \\ h^a_X(X) & \longrightarrow & \mathrm{Hom}_{\mathscr{C}^\wedge}(R, h^a_X) \end{array}$$

whose bottom and vertical arrows are bijective, so that the same holds also for the top arrow. Set $g := i^a \circ f^a$, and notice that $g \circ i^a = i^a$, therefore g must be the identity of h^a_X, whence the contention.

(a)⇒(b): Let $\eta_X : h_X \to h^a_X$ be the unit of adjunction, and set $j := (i^a)^{-1} \circ \eta_X : h_X \to R^a$. By remarks 4.1.19(iv) and 4.1.23(ii), we may find a covering subobject $i_1 : R_1 \to h_X$, and a morphism $j_1 : R_1 \to R^{\mathrm{sep}}$ whose image in $\mathrm{Hom}_{\mathscr{C}^\wedge}(R_1, R^a)$ equals $j \circ i_1$. Denote by $\eta'_X : h_X \to h^{\mathrm{sep}}_X$ the unit of adjunction; by construction, the two morphisms

$$i^{\mathrm{sep}} \circ j_1, \eta'_X \circ i_1 : R_1 \to h^{\mathrm{sep}}_X$$

have the same image in $\mathrm{Hom}_{\mathscr{C}^\wedge}(R_1, h^a_X)$. This means that there exists a subobject $i_2 : R_2 \to R_1$ covering X, such that $i^{\mathrm{sep}} \circ j_1 \circ i_2 = \eta'_X \circ i_1 \circ i_2$.

Next, let $Y_\bullet := (Y_\lambda \to X \mid \lambda \in \Lambda)$ be a generating family for the sieve of $\mathscr{C}/X$ corresponding to R_2. There follows, for every $\lambda \in \Lambda$, a commutative diagram :

$$\begin{array}{ccc} h_{Y_\lambda} & \xrightarrow{j_\lambda} & R^{\mathrm{sep}} \\ {\scriptstyle i_\lambda}\downarrow & & \downarrow{\scriptstyle i^{\mathrm{sep}}} \\ h_X & \xrightarrow{\eta'_X} & h^{\mathrm{sep}}_X. \end{array} \tag{4.1.27}$$

Then, for every $\lambda \in \Lambda$ there exists a covering subobject $s_\lambda : R_\lambda \to h_{Y_\lambda}$ such that j_λ lifts to some $t_\lambda : R_\lambda \to R$, and we pick a generating family $(Z_{\lambda\mu} \to Y_\lambda \mid \mu \in \Lambda_\lambda)$ for the sieve of $\mathscr{C}/Y_\lambda$ corresponding to R_λ; after replacing $Y_\bullet$ by the resulting family $(Z_{\lambda\mu} \to X \mid \lambda \in \Lambda,\ \mu \in \Lambda_\lambda)$ (which still covers X, by virtue of remark 4.1.3(iii)), we may assume that (4.1.27) lifts to a commutative diagram

$$\begin{array}{ccc} h_{Y_\lambda} & \xrightarrow{t_\lambda} & R \\ {\scriptstyle i_\lambda}\downarrow & & \downarrow \\ h_X & \xrightarrow{\eta'_X} & h^{\mathrm{sep}}_X. \end{array}$$

for every $\lambda \in \Lambda$. Then there exists a covering subobject $s'_\lambda : R'_\lambda \to h_{Y_\lambda}$ such that $i \circ t_\lambda \circ s'_\lambda = i_\lambda \circ s'_\lambda$ in $\mathrm{Hom}_{\mathscr{C}^\wedge}(R'_\lambda, h_X)$. Lastly, set

$$R' := \bigcup_{\lambda \in \Lambda} \mathrm{Im}(i_\lambda \circ s'_\lambda : R'_\lambda \to h_X)$$

(notice that $R' \in \mathrm{Ob}(\mathscr{C}^\wedge_{\mathsf{U}})$ even in case Λ is not a small set). It is easily seen that R' is a covering subobject of X, and the inclusion map $R' \to h_X$ factors through R, so R covers X as well. ☐

Definition 4.1.28. Let $(\mathscr{C}, J)$ be a site such that $\mathscr{C}$ has small Hom-sets, and let $\varphi : F \to G$ be a morphism in $\mathscr{C}^\wedge$.

(i) We say that φ is a *covering morphism* if, for every $X \in \mathrm{Ob}(\mathscr{C})$ and every morphism $h_X \to G$ in $\mathscr{C}^\wedge$, the image of the induced morphism $F \times_G h_X \to h_X$ is a covering subobject of h_X.

(ii) We say that φ a *bicovering morphism* if both φ and the diagonal morphism $F \to F \times_G F$ induced by φ, are covering morphisms.

Remark 4.1.29. (i) In the situation of definition 4.1.28, let $S := \{X_i \to X \mid i \in I\}$ be a family of morphisms in $\mathscr{C}$, and pick a universe V containing U, such that I is V-small. Using remark 3.5.2(iii), it is easily seen that S covers X if and only if the induced morphism in $\mathscr{C}^\wedge_{\mathsf{V}}$

$$\coprod_{i \in I} h_{X_i} \to h_X$$

is a covering morphism.

(ii) With the notation of definition 4.1.28, recall that, by Yoneda's lemma, the set of morphisms $h_X \to G$ in $\mathscr{C}^\wedge$ is naturally identified with GX. Fix $s \in GX$, and let $\psi_s : h_X \to G$ be the corresponding morphism; under this identification, for every $Y \in \mathrm{Ob}(\mathscr{C})$ the image of the induced map $(F \times_{(\varphi,\psi_s)} h_X)(Y) \to h_X(Y)$ is then the set of all morphisms $f : Y \to X$ such that $(Gf)(s)$ lies in the image of the map $\varphi(f) : FY \to GY$. Thus φ is a covering morphism if and only if for every $X \in \mathrm{Ob}(\mathscr{C})$ and every $s \in GX$, the sieve of $\mathscr{C}/X$ generated by all morphisms $f : Y \to X$ in $\mathscr{C}$ such that $(Gf)(s) \in \mathrm{Im}(\varphi_Y : FY \to GY)$ covers X. In other words, φ is a covering morphism if and only if for every $X \in \mathrm{Ob}(\mathscr{C})$ and every $s \in GX$ there exists a covering family $(f_i : Y_i \to X \mid i \in I)$ such that $(Gf_i)(s) \in \mathrm{Im}(\varphi_{Y_i})$ for every $i \in I$.

(iii) Likewise, the diagonal morphism $F \to F \times_G F$ induced by φ is a covering morphism if and only if for every $X \in \mathrm{Ob}(\mathscr{C})$ and every two sections $s, s' \in FX$ such that $\varphi_X(s) = \varphi_X(s')$, the set of all morphisms $f : Y \to X$ in $\mathscr{C}$ such that $(Ff)(s) = (Ff)(s')$ generates a sieve covering X. This is the same as saying that for every such X and every such pair (s, s') there exists a covering family $(f_i : Y_i \to X \mid i \in I)$ such that $(Ff_i)(s) = (Ff_i)(s')$ for every $i \in I$.

Corollary 4.1.30. *Let $C := (\mathscr{C}, J)$ be a U-site, and $\varphi : F \to G$ a morphism in $\mathscr{C}^\wedge$. We have :*

(i) *φ is a covering morphism if and only if $\varphi^a : F^a \to G^a$ is an epimorphism in $C^\sim$.*

(ii) *The diagonal morphism $F \to F \times_G F$ induced by φ is a covering morphism if and only if φ^a is a monomorphism in $C^\sim$.*

(iii) *φ is a bicovering morphism if and only if $\varphi^a : F^a \to G^a$ is an isomorphism in $C^\sim$.*

Proof. Let V be a universe with $\mathsf{U} \subset \mathsf{V}$, and such that $\mathscr{C}$ is V-small. Clearly φ is a covering (resp. bicovering) morphism in $\mathscr{C}^\wedge_{\mathsf{U}}$ if and only if the same holds for the image of φ under the fully faithful inclusion $\mathscr{C}^\wedge_{\mathsf{U}} \subset \mathscr{C}^\wedge_{\mathsf{V}}$. So we may replace U by V, and assume that C is a small site.

(i): Suppose that φ^a is an epimorphism; let X be any object of $\mathscr{C}$, and $h_X \to G$ a morphism. Since the epimorphisms of $C^\sim$ are universal (remark 4.1.19(i)), the induced morphism

$$(\varphi \times_G X)^a : (F \times_G h_X)^a \to h_X^a$$

is an epimorphism. Let $R \subset h_X$ be the image of $\varphi \times_G h_X$; then the induced morphism $R^a \to h^a_X$ is both a monomorphism and an epimorphism, so it is an isomorphism, by proposition 4.1.24(i). Hence R is a covering subobject, according to proposition 4.1.26.

Conversely, suppose that φ is a covering morphism. By remark 4.1.19(iii), G is the colimit of a family $(h^a_{X_i} \mid i \in I)$ for certain $X_i \in \mathrm{Ob}(\mathscr{C})$. By definition, the image R_i of the induced morphism $\varphi \times_G X_i : F \times_G h_{X_i} \to h_{X_i}$ covers X_i, for every $i \in I$. Now, the induced morphism $F \times_G h_{X_i} \to R_i$ is an epimorphism, and the morphism $R^a_i \to h^a_{X_i}$ is an isomorphism (proposition 4.1.26), hence $(\varphi \times_G X_i)^a$ is an epimorphism, and then the same holds for

$$\operatorname*{colim}_{i\in I}(\varphi \times_G X_i)^a : \operatorname*{colim}_{i\in I} F^a \times_{G^a} h^a_{X_i} \to \operatorname*{colim}_{i\in I} h^a_{X_i} = G^a$$

which is isomorphic to φ^a, since the colimits of $C^\sim$ are universal (remark 4.1.19(i)); so φ^a is an epimorphism.

(ii): If $\delta : F \to F \times_G F$ is a covering morphism, the foregoing shows that the induced morphism $\delta^a : F^a \to F^a \times_{G^a} F^a$ is both an epimorphism and a monomorphism, hence it is an isomorphism (proposition 4.1.24(i)), so φ^a is a monomorphism (proposition 1.2.18(i)). Conversely, if φ^a is a monomorphism, δ^a is an isomorphism, hence δ is a covering morphism, by (i).

(iii) follows from (i), (ii) and proposition 4.1.24(i). ☐

Remark 4.1.31. (i) Let $C := (\mathscr{C}, J)$ be a site such that $\mathscr{C}$ has small Hom-sets, and V a universe containing U, such that $\mathscr{C}^\wedge$ is V-small. For every presheaf F on $\mathscr{C}$, let $J^\wedge(F)$ be the set of all sieves $\mathscr{S} \subset \mathscr{C}^\wedge/F$ such that the natural morphism

$$\coprod_{(f:G\to F)\in\mathrm{Ob}(\mathscr{S})} G \to F$$

is a covering morphism in $\mathscr{C}^\wedge_{\mathsf{V}}$. Notice that $\mathscr{S} \in J^\wedge(F)$ if and only if there exists a generating family $(f_i : F_i \to F \mid i \in I)$ for $\mathscr{S}$, indexed by a V-small set I, such that the natural morphism $\amalg_{i\in I} F_i \to F$ is a covering morphism. Since the functor $(-)^a : \mathscr{C}^\wedge_{\mathsf{V}} \to C^\sim_{\mathsf{V}}$ commutes with arbitrary V-small colimits, corollary 4.1.30(i) says that the latter condition holds if and only if the induced morphism $\amalg_{i\in I} F^a_i \to F^a$ is an epimorphism in $C^\sim_{\mathsf{V}}$.

(ii) We claim that the system $(J^\wedge(F) \mid F \in \mathrm{Ob}(\mathscr{C}^\wedge))$ is a topology on $\mathscr{C}^\wedge$. Indeed, it is clear from the definition that if $\mathscr{S} \in J^\wedge(F)$, then $\mathscr{S} \times_F f$ lies in $J^\wedge(F')$ for every morphism $f : F' \to F$ in $\mathscr{C}^\wedge$. Next, let $\mathscr{S}, \mathscr{S}' \subset \mathscr{C}^\wedge/F$ be two sieves, with $\mathscr{S} \in J^\wedge(F)$ and such that $\mathscr{S}' \times_F f \in J^\wedge(F')$ for every $(f : F' \to F) \in \mathrm{Ob}(\mathscr{S})$. By the foregoing, this means that the induced morphism $\amalg_{(g:G\to F)\in\mathrm{Ob}(\mathscr{S}')} G^a \times_{F^a} F'^a \to F'^a$ is an epimorphism in $C^\sim_{\mathsf{V}}$ for every such f. Since (in every category) an arbitrary coproduct of epimorphisms is an epimorphism, it follows that both of the following morphisms are epimorphisms :

$$\coprod_{(f:F'\to F)\in\mathrm{Ob}(\mathscr{S})} \; \coprod_{(g:G\to F)\in\mathrm{Ob}(\mathscr{S}')} G^a \times_{F^a} F'^a \to \coprod_{(f:F'\to F)\in\mathrm{Ob}(\mathscr{S})} F'^a \to F^a$$

and then so is their composition. But $f \circ (g \times_F F') : G \times_F F' \to F$ lies in $\mathscr{S}'$ for every $f \in \mathrm{Ob}(\mathscr{S})$ and every $g \in \mathrm{Ob}(\mathscr{S}')$, so $\mathscr{S}' \in J^\wedge(F)$, as required. We denote the resulting site

$$C^\wedge := (\mathscr{C}^\wedge, J^\wedge).$$

In light of remark 4.1.29(ii) it is easily seen that $C^\wedge$ is independent of the choice of V.

4.2. **Continuous and cocontinuous functors.** We now begin the study of those functors that are compatible with given topologies on their domain and codomain of definition; as explained hereafter, such compatibility can manifest itself in two distinct fashions :

Definition 4.2.1. Let $C = (\mathscr{C}, J)$ and $C' = (\mathscr{C}', J')$ be two sites, and $g : \mathscr{C} \to \mathscr{C}'$ a functor.

(i) We say that g is *continuous* for the topologies J and J', if the following holds. For every universe V and every V-sheaf F on C', the V-presheaf $g^\wedge_{\mathsf{V}} F$ is a V-sheaf on C (notation of (1.2.2)). In this case, $g^\wedge_{\mathsf{V}}$ clearly induces by restriction a functor

$$\widetilde{g}_{\mathsf{V}*} : C'^{\sim}_{\mathsf{V}} \to C^{\sim}_{\mathsf{V}}.$$

(ii) We say that g is *cocontinuous* for the topologies J and J' if the following holds. For every $X \in \mathrm{Ob}(\mathscr{C})$, and every covering sieve $\mathscr{S}' \in J'(gX)$, the sieve $g^{-1}_{|X}\mathscr{S}'$ covers X in C (notation of definition 3.5.1(iii) and (1.1.26)).

Example 4.2.2. Let T, T' be two topological spaces, and $f : T \to T'$ a continuous map; denote by $\mathscr{T}$ (resp. $\mathscr{T}'$) the category of open subsets of T (resp. of $\mathscr{T}'$), endowed with its natural topology as in example 4.1.2. Then f induces a functor $f^{-1} : \mathscr{T}' \to \mathscr{T} : U \mapsto f^{-1}U$, and it is easily seen that f^{-1} is a continuous functor, according to definition 4.2.1(i).

Lemma 4.2.3. *Let $C = (\mathscr{C}, J)$ and $C' = (\mathscr{C}', J')$ be two sites, $g : \mathscr{C} \to \mathscr{C}'$ a functor, and V a universe such that $\mathscr{C}$ is V-small and $\mathscr{C}'$ has small* Hom*-sets. The following conditions are equivalent :*

(a) *For every V-sheaf F on C', the presheaf $g^\wedge_{\mathsf{V}} F$ is a V-sheaf on C.*
(b) *For every morphism $\varphi : F \to G$ of V-presheaves on $\mathscr{C}$ that is bicovering for the topology J, the morphism $g_{\mathsf{V}!}(\varphi) : g_{\mathsf{V}!}F \to g_{\mathsf{V}!}G$ is bicovering for the topology J'.*
(c) *For every $X \in \mathrm{Ob}(\mathscr{C})$ and every subobject $R \subset h_X$ covering X for the topology J, the induced morphism of presheaves $g_{\mathsf{V}!}R \to g_{\mathsf{V}!}h_X$ is bicovering for the topology J'.*
(d) *g is continuous for the topologies J and J'.*

Proof. (a)⇒(b): By virtue of remark 1.3.6(iv) we may replace V by a larger universe, and assume that $\mathscr{C}$ and $\mathscr{C}'$ are V-small. Then, by corollary 4.1.30(iii), the morphism $\varphi^a : F^a \to G^a$ is an isomorphism, and we need to check that $g_{\mathsf{V}!}(\varphi)^a$ is an isomorphism. To this aim, let H be any V-sheaf on C'; we get a commutative diagram :

$$\begin{array}{ccccccc}
\mathrm{Hom}_{\mathscr{C}'^\wedge}(g_{\mathsf{V}!}(G)^a, H) & \longrightarrow & \mathrm{Hom}_{\mathscr{C}'^\wedge}(g_{\mathsf{V}!}(G), H) & \longrightarrow & \mathrm{Hom}_{\mathscr{C}^\wedge}(G, g^\wedge_{\mathsf{V}} H) & \longleftarrow & \mathrm{Hom}_{\mathscr{C}^\wedge}(G^a, g^\wedge_{\mathsf{V}} H) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{Hom}_{\mathscr{C}'^\wedge}(g_{\mathsf{V}!}(F)^a, H) & \longrightarrow & \mathrm{Hom}_{\mathscr{C}'^\wedge}(g_{\mathsf{V}!}(F), H) & \longrightarrow & \mathrm{Hom}_{\mathscr{C}^\wedge}(F, g^\wedge_{\mathsf{V}} H) & \longleftarrow & \mathrm{Hom}_{\mathscr{C}^\wedge}(F^a, g^\wedge_{\mathsf{V}} H)
\end{array}$$

whose horizontal arrows are bijections, due to (a). Since φ^a is bijective, the right-most vertical arrow is bijective, hence the same holds for the left-most vertical arrow, whence the contention.

(b)⇒(c) and (d)⇒(a) are trivial.

(c)⇒(d): Let V' be a universe, and F a V'-sheaf on C'; we need to check that $g^\wedge_{\mathsf{V}'} F$ is a sheaf on C. To this aim, we may replace V' by a larger universe, and assume that $\mathsf{V} \subset \mathsf{V}'$, and that $\mathscr{C}$ and $\mathscr{C}'$ are V'-small; then by remark 1.3.6(iv) and corollary 4.1.30(iii) we see that for every $X \in \mathrm{Ob}(\mathscr{C})$ and every covering subobject $R \subset h_X$ the induced morphism $g_{\mathsf{V}'!}(R)^a \to g_{\mathsf{V}'!}(h_X)^a$ is an isomorphism. Now, we have a commutative diagram :

$$\begin{array}{ccccc}
\mathrm{Hom}_{\mathscr{C}^\wedge}(h_X, g^\wedge_{\mathsf{V}'} F) & \longrightarrow & \mathrm{Hom}_{\mathscr{C}'^\wedge}(g_{\mathsf{V}'!}h_X, F) & \longleftarrow & \mathrm{Hom}_{\mathscr{C}'^\wedge}(g_{\mathsf{V}'!}(h_X)^a, F) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Hom}_{\mathscr{C}^\wedge}(R, g^\wedge_{\mathsf{V}'} F) & \longrightarrow & \mathrm{Hom}_{\mathscr{C}'^\wedge}(g_{\mathsf{V}'!}R, F) & \longleftarrow & \mathrm{Hom}_{\mathscr{C}'^\wedge}(g_{\mathsf{V}'!}(R)^a, F)
\end{array}$$

whose horizontal arrows are bijections; moreover, the right-most vertical arrow is bijective as well, hence the same holds for the left-most vertical arrow, which is the contention. □

Lemma 4.2.4. *In the situation of definition* 4.2.1*, consider the following conditions :*

(a) *g is continuous.*

(b) *For every covering family* $(X_i \to X \mid i \in I)$ *in* C*, the family* $(gX_i \to gX \mid i \in I)$ *covers* gX *in* C'*.*
(c) *For every small covering family* $(X_i \to X \mid i \in I)$ *in* C*, the family* $(gX_i \to gX \mid i \in I)$ *covers* gX *in* C'*.*
(d) *For every universe* V*, and every* V*-presheaf* F *on* $\mathscr{C}'$ *that is separated for the topology* J'*, the presheaf* $g^{\wedge}_{\mathsf{V}}F$ *is separated for the topology* J*, so that* $g^{\wedge}_{\mathsf{V}}$ *restricts to a functor*

$$g^{\mathrm{sep}}_{\mathsf{V}*} : C'^{\mathrm{sep}}_{\mathsf{V}} \to C^{\mathrm{sep}}_{\mathsf{V}}.$$

Then (a)⇒(b)⇔(d)⇒(c)*. Moreover, if all fibre products are representable in* $\mathscr{C}$*, and* g *commutes with fibre products, then* (b)⇒(a)*. Furthermore, if* C *is a* U*-site, then* (c)⇒(b)*.*

Proof. Obviously (b)⇒(c).

(a)⇒(b): After replacing U by a larger universe, we may assume that I is a small set and both $\mathscr{C}$ and $\mathscr{C}'$ are small. Let F be any sheaf on C'; by assumption, $g^{\wedge}F$ is a sheaf on C, hence the natural map :

$$F(gX) = g^{\wedge}F(X) \to \prod_{i\in I} g^{\wedge}F(X_i) = \prod_{i\in I} F(gX_i)$$

is injective (by (4.1.6)). This means that the induced morphism

$$\coprod_{i\in I} h^a_{gX_i} \to h^a_{gX} \qquad (4.2.5)$$

is an epimorphism in $C'^{\sim}$. Then the assertion follows from corollary 4.1.30 and remark 4.1.29(i).

(d)⇒(b) is similar : we may assume that I, $\mathscr{C}$ and $\mathscr{C}'$ are small. If F is any separated presheaf on C', then by assumption $g^{\wedge}F$ is separated on C, and arguing as in the foregoing, we deduce that the induced morphism $\amalg_{i\in I} h^{\mathrm{sep}}_{gX_i} \to h^{\mathrm{sep}}_{gX}$ is an epimorphism in C'^{sep}, and then (4.2.5) is an epimorphism in $C'^{\sim}$, so we conclude, again by corollary 4.1.30 and remark 4.1.29(i).

(b)⇒(d): let F be a separated V-presheaf on C', and $(X_i \to X \mid i \in I)$ any covering family in C; in view of (4.1.6), it suffices to show that the induced map

$$F(gX) = g^{\wedge}_{\mathsf{V}}F(X) \to \prod_{i\in I} g^{\wedge}_{\mathsf{V}}F(X_i) = \prod_{i\in I} F(gX_i)$$

is injective. But this is clear, since $(gX_i \to gX \mid i \in I)$ is a covering family in C'.

Next, suppose that (b) holds, the fibre products in $\mathscr{C}$ are representable, and g commutes with all fibre products. For every $i, j \in I$, set $X_{ij} := X_i \times_X X_j$. To show that (a) holds, it suffices – in view of (4.1.6) – to prove :

Claim 4.2.6. The natural map

$$g^{\wedge}F(X) \to \mathrm{Hom}_{\mathscr{C}^{\wedge}}\Big(\mathrm{Coequal}\Big(\coprod_{i,j\in I} h_{X_{ij}} \rightrightarrows \coprod_{i\in I} h_{X_i}\Big), g^{\wedge}F\Big)$$

is bijective.

Proof of the claim. Since $g_!$ is right exact, and due to (1.3.7), this is the same as the natural map

$$F(gX) \to \mathrm{Hom}_{\mathscr{C}'^{\wedge}}\Big(\mathrm{Coequal}\Big(\coprod_{i,j\in I} h_{gX_{ij}} \rightrightarrows \coprod_{i\in I} h_{gX_i}\Big), F\Big).$$

However, by assumption $gX_{ij} = gX_i \times_{gX} gX_j$, and then the claim follows by applying (4.1.6) to the covering family $(gX_i \to gX \mid i \in I)$. ◊

Lastly, suppose that C is a U-site; in order to show that (c)⇒(b), we remark more precisely :

Claim 4.2.7. Let C be a U-site, $\mathscr{F} := (\varphi_i : X_i \to X \mid i \in I)$ any covering family. Then there exists a small set $J \subset I$ such that the subfamily $(\varphi_i \mid i \in J)$ covers X.

Proof of the claim. Let $\mathscr{S} \subset \mathscr{C}/X$ be the sieve generated by $\mathscr{S}$. By lemma 4.1.22, we may find a small covering family $\mathscr{F}' := (\psi_i : X'_i \to X \mid i \in I')$ (*i.e.* such that I' is small), that generates a sieve $\mathscr{S}' \subset \mathscr{S}$. Then, for every $i \in I'$ we may find $\gamma(i) \in I$ such that ψ_i factors through $\varphi_{\gamma(i)}$. The subset $J := \gamma I'$ will do. ◊

Let $\mathscr{F}$ and J be as in claim 4.2.7; then the sieve $g\mathscr{S}$ generated by $(g(\varphi_i) \mid i \in I)$ contains the sieve $g\mathscr{S}'$ generated by $(g(\varphi_i) \mid i \in J)$. Especially, if $g\mathscr{S}'$ is a covering sieve, the same holds for $g\mathscr{S}$, whence the contention. □

4.2.8. In the situation of definition 4.2.1, let V be a universe with $\mathsf{U} \subset \mathsf{V}$, such that C is a V-site, and $\mathscr{C}'$ has V-small Hom-sets. Then we may define a functor

$$\breve{g}^*_{\mathsf{V}} : C'^{\sim}_{\mathsf{V}} \to C^{\sim}_{\mathsf{V}} \qquad F \mapsto (g^{\wedge}_{\mathsf{V}} \circ i_{C'} F)^a$$

(where $i_{C'} : C'^{\sim}_{\mathsf{V}} \to \mathscr{C}'^{\wedge}_{\mathsf{V}}$ is the forgetful functor). On the other hand, if C is a V-small site and C' is a V-site, we can define the functors

$$g^a_{\mathsf{V}!} : \mathscr{C}^{\wedge}_{\mathsf{V}} \to C'^{\sim}_{\mathsf{V}} \qquad F \mapsto (g_{\mathsf{V}!}F)^a \qquad \text{and} \qquad \widetilde{g}^*_{\mathsf{V}} := g^a_{\mathsf{V}!} \circ i_C : C^{\sim}_{\mathsf{V}} \to C'^{\sim}_{\mathsf{V}}.$$

As usual, if $\mathsf{V} = \mathsf{U}$ we often omit the subscript U. Later we shall generalize these constructions to the case where both C and C' are V-sites : see corollary 4.3.19. For now we remark :

Lemma 4.2.9. *In the situation of definition* 4.2.1*, let* V *be a universe with* $\mathsf{U} \subset \mathsf{V}$*, such that* $\mathscr{C}$ *is* V*-small, and* $\mathscr{C}'$ *has* V*-small* Hom*-sets. Then the following conditions are equivalent :*

(a) *g is a cocontinuous functor.*
(b) *For every covering morphism $\varphi : F \to G$ in $\mathscr{C}'^{\wedge}$, the morphism $g^{\wedge}(\varphi) : g^{\wedge}F \to g^{\wedge}G$ is covering.*
(c) *For every bicovering morphism $\varphi : F \to G$ in $\mathscr{C}'^{\wedge}$, the morphism $g^{\wedge}(\varphi) : g^{\wedge}F \to g^{\wedge}G$ is bicovering.*
(d) *For every $X \in \mathrm{Ob}(\mathscr{C}')$ and every covering subobject R of h_X, the induced morphism $g^{\wedge}R \to g^{\wedge}h_X$ is bicovering.*
(e) *For every $F \in \mathrm{Ob}(C^{\sim}_{\mathsf{V}})$, the V-presheaf $g_{\mathsf{V}*}F$ is a V-sheaf on C' (see remark* 1.3.6(i)*).*

When these conditions hold, the restriction of $g_{\mathsf{V}*}$ is a right adjoint to $\breve{g}^*_{\mathsf{V}}$ denoted

$$\breve{g}_{\mathsf{V}*} : C^{\sim}_{\mathsf{V}} \to C'^{\sim}_{\mathsf{V}}$$

Proof. After replacing V by U, we may assume that $\mathscr{C}$ is small, and $\mathscr{C}'$ has small Hom-sets. Recall that the unit of the natural adjunction for the pair $(g_!, g^{\wedge})$ assigns to every $X \in \mathrm{Ob}(\mathscr{B})$ the morphism $\eta_X : h_X \to g^{\wedge}(g_! h_X) \xrightarrow{\sim} g^{\wedge}(h_{gX})$ such that $\eta_X(s) := Fs$ for every $X' \in \mathrm{Ob}(\mathscr{C})$ and every $(s : X' \to X) \in h_X(X')$: see remark 1.3.6(v). We notice :

Claim 4.2.10. Let $X \in \mathrm{Ob}(\mathscr{C})$, and $\mathscr{T}' \subset \mathscr{C}'/gX$ a covering sieve; set $\mathscr{T} := g^{-1}_{|X}\mathscr{T}'$. Then :

$$h_{\mathscr{T}} = g^{\wedge}(h_{\mathscr{T}'}) \times_{g^{\wedge}(h_{gX})} h_X.$$

Proof of the claim. For every $X' \in \mathrm{Ob}(\mathscr{C})$, the set $(g^{\wedge}(h_{\mathscr{T}'}) \times_{g^{\wedge}(h_{gX})} h_X)(X')$ consists of the pairs (t, s) where $t : gX' \to gX$ is an object of $\mathscr{T}'$ and $s : X' \to X$ is a morphism in $\mathscr{C}$, such that $g(s) = t$. In other words, this is the set of all $s \in h_X(X')$ such that $g(s) \in \mathrm{Ob}(\mathscr{T}')$. This is precisely the definition of $h_{\mathscr{T}}(X')$. ◊

(a)⇒(b): Let $X \in \mathrm{Ob}(\mathscr{C})$ and $\psi : h_X \to g^{\wedge}G$ any morphism in $\mathscr{C}^{\wedge}$. By adjunction, ψ corresponds to a morphism $\psi' : h_{gX} \to G$ in $\mathscr{C}'^{\wedge}$, such that $\psi = g^{\wedge}(\psi') \circ \eta_X$. By assumption, ψ' induces a covering morphism $F \times_G h_{gX} \to h_{gX}$, whose image is therefore of the form $h_{\mathscr{S}'}$ for some covering sieve $\mathscr{S}' \in J'(gX)$. Since $g^{\wedge}$ is exact, the epimorphism $\pi : F \times_G h_{gX} \to h_{\mathscr{S}'}$

induces an epimorphism $g^\wedge(\pi) : g^\wedge(F \times_G h_{gX} \to h_{gX}) \to g^\wedge(h_{\mathscr{S}'})$, and notice the following commutative diagram whose three square subdiagrams are cartesian :

$$\begin{array}{ccccc}
g^\wedge(F) \times_{g^\wedge(G)} h_X & \longrightarrow & g^\wedge(h_{\mathscr{S}'}) \times_{g^\wedge(h_{gX})} h_X & \longrightarrow & h_X \\
\downarrow & & \downarrow & & \downarrow {\scriptstyle \eta_X} \\
g^\wedge(F \times_G h_{gX}) & \xrightarrow{g^\wedge(\pi)} & g^\wedge(h_{\mathscr{S}'}) & \longrightarrow & g^\wedge(h_{gX}) \\
\downarrow & & & & \downarrow {\scriptstyle g^\wedge(\psi')} \\
g^\wedge(F) & & \longrightarrow & & g^\wedge(G)
\end{array}$$

In view of claim 4.2.10, we deduce that the image of the morphism $g^\wedge(F) \times_{g^\wedge(G)} h_X \to h_X$ induced by $g^\wedge(\varphi)$ is $h_{\mathscr{S}}$, where $\mathscr{S} := g^{-1}_{|X}\mathscr{S}'$; but $\mathscr{S}$ is a covering sieve of X, since g is cocontinuous. This shows that (b) holds.

(b)⇒(c): If φ is a bicovering morphism, (b) implies that both $g^\wedge(\varphi)$ and the morphism $\delta : g^\wedge(F) \to g^\wedge(F \times_G F)$ induced by φ are covering morphisms. However, $g^\wedge$ is exact, so δ is naturally identified with the diagonal morphism $g^\wedge(F) \to g^\wedge(F) \times_{g^\wedge(G)} g^\wedge(F)$ induced by $g^\wedge(\varphi)$; thus, $g^\wedge(\varphi)$ is bicovering.

(c)⇒(d) is obvious.

(d)⇒(e): Let F be any sheaf on C; we have to show that the natural map

$$g_*F(Y) \to \mathrm{Hom}_{\mathscr{C}'^\wedge}(R, g_*F)$$

is bijective, for every $Y \in \mathrm{Ob}(\mathscr{C}')$, and every covering subobject of h_Y. By adjunction (and by corollary 4.1.30), this is the same as saying that the monomorphism $g^\wedge(R) \to g^\wedge(h_Y)$ is a covering morphism, which holds by assumption (d).

(e)⇒(a): The assumption implies that, for every sheaf F on C, every $X \in \mathrm{Ob}(\mathscr{C})$, and every covering sieve $\mathscr{S}' \subset \mathscr{C}'/gX$, the natural map

$$g_*F(gX) \to \mathrm{Hom}_{\mathscr{C}'^\wedge}(h_{\mathscr{S}'}, g_*F)$$

is bijective. By adjunction, the same then holds for the natural map

$$\mathrm{Hom}_{\mathscr{C}^\wedge}(g^\wedge(h_{gX}), F) \to \mathrm{Hom}_{\mathscr{C}^\wedge}(g^\wedge(h_{\mathscr{S}'}), F)$$

so the induced morphism $g^\wedge(h_{\mathscr{S}'}) \to g^\wedge(h_{gX})$ is bicovering (proposition 4.1.26). Also, this morphism is a monomorphism (since $g^\wedge$ commutes with all limits); therefore, after base change along the unit of adjunction $\eta_X : h_X \to g^\wedge(h_{gX})$, we deduce a covering monomorphism

$$g^\wedge(h_{\mathscr{S}'}) \times_{g^\wedge(h_{gX})} h_X \to h_X.$$

Then the contention follows from claim 4.2.10.

Lastly, the assertion concerning the left adjoint $\breve{g}^*_{\mathsf{U}}$ is immediate from the definitions. □

Lemma 4.2.11. *Let $C' := (\mathscr{C}', J')$ be a U-site, $C := (\mathscr{C}, J)$ a small site, and $g : \mathscr{C} \to \mathscr{C}'$ a continuous functor. Then the following holds :*

(i) *The functor $\widetilde{g}^*_{\mathsf{U}} : C^\sim_{\mathsf{U}} \to C'^\sim_{\mathsf{U}}$ of* (4.2.8) *is left adjoint to $\widetilde{g}_{\mathsf{U}*}$.*

(ii) *The natural diagrams of functors :*

$$\begin{array}{ccc}
\mathscr{C} & \xrightarrow{g} & \mathscr{C}' \\
{\scriptstyle h^a_{\mathscr{C}}} \downarrow & & \downarrow {\scriptstyle h^a_{\mathscr{C}'}} \\
C^\sim_{\mathsf{U}} & \xrightarrow{\widetilde{g}^*_{\mathsf{U}}} & C'^\sim_{\mathsf{U}}
\end{array}
\qquad
\begin{array}{ccc}
\mathscr{C}^\wedge_{\mathsf{U}} & \xrightarrow{(-)^a} & C^\sim_{\mathsf{U}} \\
& {\scriptstyle g^a_{\mathsf{U}!}} \searrow \quad \swarrow {\scriptstyle \widetilde{g}^*_{\mathsf{U}}} & \\
& C'^\sim_{\mathsf{U}} &
\end{array}$$

are essentially commutative. (Notation of theorem 4.1.13.*)*

Proof. The first assertion is straightforward, and the commutativity of the square diagram in (ii) is reduced to the corresponding assertion for (1.3.7), which has already been remarked. To show the commutativity for the triangular diagram in (ii), it suffices to check that for every presheaf F on $\mathscr{C}$ and every sheaf G of C' the natural map $\eta_F : F \to i_C(F^a)$ induces a bijection

$$\mathrm{Hom}_{C'^\sim}(\widetilde{g}^*_{\mathsf{U}}(F^a), G) \stackrel{\sim}{\to} \mathrm{Hom}_{C'^\sim}(g^a_{\mathsf{U}!}(F), G).$$

However, by adjunction, the latter is naturally identified with the map

$$\mathrm{Hom}_{\mathscr{C}^\wedge}(F^a, g^\wedge_{\mathsf{U}} G) \to \mathrm{Hom}_{\mathscr{C}^\wedge}(F, g^\wedge_{\mathsf{U}} G) \tag{4.2.12}$$

induced by η_F; but $g^\wedge_{\mathsf{U}} G$ is a sheaf, since g is continuous, so (4.2.12) is indeed bijective. □

4.2.13. Let $(\mathscr{C}, J)$ be a small site, $(\mathscr{C}', J')$ a U-site, $u : \mathscr{C} \to \mathscr{C}'$ a continuous (resp. cocontinuous) functor. Let also V be a universe such that $\mathsf{U} \subset \mathsf{V}$. Then it follows easily from remark (1.3.6)(iv) and lemma 4.2.11(i) (resp. from lemma 4.2.9), and from remark 4.1.23(ii) that we have essentially commutative diagrams of categories :

$$\begin{array}{ccc} C^\sim_{\mathsf{U}} & \underset{\widetilde{u}_{\mathsf{U}*}}{\overset{\widetilde{u}^*_{\mathsf{U}}}{\rightleftarrows}} & C'^\sim_{\mathsf{U}} \\ \downarrow & & \downarrow \\ C^\sim_{\mathsf{V}} & \underset{\widetilde{u}_{\mathsf{V}*}}{\overset{\widetilde{u}^*_{\mathsf{V}}}{\rightleftarrows}} & C'^\sim_{\mathsf{V}} \end{array} \qquad \text{(resp.} \qquad \begin{array}{ccc} C^\sim_{\mathsf{U}} & \underset{\breve{u}^*_{\mathsf{U}}}{\overset{\breve{u}_{\mathsf{U}*}}{\rightleftarrows}} & C'^\sim_{\mathsf{U}} \\ \downarrow & & \downarrow \\ C^\sim_{\mathsf{V}} & \underset{\breve{u}^*_{\mathsf{V}}}{\overset{\breve{u}_{\mathsf{V}*}}{\rightleftarrows}} & C'^\sim_{\mathsf{V}} \end{array} \text{)}$$

whose vertical arrows are the inclusion functors. More generally, the diagram for $\breve{u}_{\mathsf{U}*}$ is well defined and essentially commutative, whenever C is a U-site, and $\mathscr{C}'$ has small Hom-sets.

Lemma 4.2.14. (i) *Let $C := (\mathscr{C}, J)$ and $C' := (\mathscr{C}', J')$ be two sites, and $v : \mathscr{C} \to \mathscr{C}'$, $u : \mathscr{C}' \to \mathscr{C}$ two functors, such that v is left adjoint to u. The following conditions are equivalent:*

(a) *u is continuous.*
(b) *v is cocontinuous.*

(ii) *Moreover, when these conditions hold, then for every universe V such that $\mathscr{C}$ and $\mathscr{C}'$ are V-small, we have natural isomorphisms of functors :*

$$\widetilde{u}_{\mathsf{V}*} \stackrel{\sim}{\to} \breve{v}_{\mathsf{V}*} \qquad \widetilde{u}^*_{\mathsf{V}} \stackrel{\sim}{\to} \breve{v}^*_{\mathsf{V}}.$$

Proof. In view of lemma 4.2.3, we may replace U by a larger universe, after which we may assume that C and C' are small sites. In this case, the lemma follows from lemma 4.2.9 and proposition 1.3.25(i). □

Lemma 4.2.15. *Let $(\mathscr{C}, J)$ be a small site, $(\mathscr{C}', J')$ a U-site, $u : \mathscr{C} \to \mathscr{C}'$ a continuous and cocontinuous functor. Then we have :*

(i) *$\widetilde{u}_* = \breve{u}^*$ and this functor admits the left adjoint $\widetilde{u}^*$ and the right adjoint $\breve{u}_*$.*
(ii) *We have a natural isomorphism of functors*

$$(-)^a \circ u^\wedge \stackrel{\sim}{\to} \widetilde{u}_* \circ (-)^a \ : \ \mathscr{C}'^\wedge \to C^\sim.$$

(iii) *$\widetilde{u}^*$ is fully faithful if and only if the same holds for $\breve{u}_*$.*
(iv) *If u is fully faithful, then the same holds for $\widetilde{u}^*$. The converse holds, provided the topologies J and J' are coarser than the canonical topologies.*

Proof. (i) is clear by inspecting the definitions.

(ii): For every presheaf G on $\mathscr{C}'$, let $i_G : G \to G^a$ be the natural morphism; due to lemma 4.2.9 the induced morphism $u^\wedge(i_G) : u^\wedge G \to u^\wedge(G^a)$ is bicovering, and notice that $u^\wedge(G^a)$ is a sheaf, since u is continuous. Thus, $u^\wedge(i_G)$ factors through the natural morphism $i_{u^\wedge G} : u^\wedge G \to (u^\wedge G)^a$ and a unique morphism in $C^\sim$

$$\omega_G : (u^\wedge G)^a \to u^\wedge(G^a).$$

Since $i_{u^\wedge G}$ is bicovering, the same holds for ω_G, *i.e.* the latter is an isomorphism of on $C^\sim$. It is then easily seen that the rule $G \mapsto \omega_G$ yields the sought isomorphism.

Assertion (iii) follows from (i) and proposition 1.1.20(iv). Next, suppose that u is fully faithful; then the same holds for $\breve{u}_*$ (corollary 1.5.19(ii)), so the claim follows from (iii). Lastly, suppose that $\widetilde{u}^*$ is fully faithful, and both J and J' are coarser than the canonical topologies on $\mathscr{C}$ and $\mathscr{C}'$. In such case, the Yoneda embedding for $\mathscr{C}$ (resp. $\mathscr{C}'$) realizes $\mathscr{C}$ (resp. $\mathscr{C}'$) as a full subcategory of $C^\sim$ (resp. of $C'^\sim$). Then, from (1.3.7) and the explicit expression of $\widetilde{u}^*$ provided by lemma 4.2.11(i), we deduce that u is fully faithful. □

Definition 4.2.16. Let $C := (\mathscr{C}, J)$ be a site, $\mathscr{B}$ any category, and $g : \mathscr{B} \to \mathscr{C}$ a functor. Pick a universe V such that $\mathscr{B}$ is V-small and C is a V-site. According to remark 4.1.8(iv), there is a finest topology J_g on $\mathscr{B}$ such that, for every V-sheaf F on C, the V-presheaf $g^\wedge F$ is a V-sheaf on $(\mathscr{B}, J_g)$. By lemma 4.2.3, the topology J_g is independent of the chosen universe V. We call J_g the *topology induced via g by J on $\mathscr{B}$*. Clearly g is continuous for the sites $(\mathscr{B}, J_g)$ and C.

Lemma 4.2.17. *In the situation of definition* 4.2.16*, let X be any object of $\mathscr{B}$, and $R \subset h_X$ any subobject in $\mathscr{B}^\wedge$. We have :*

(i) *$R \in J_g(X)$ if and only if for every morphism $Y \to X$ in $\mathscr{B}$, the induced morphism $g_{\mathsf{V}!}(R \times_X Y) \to h_{g(Y)}$ is a bicovering morphism in $\mathscr{C}^\wedge_\mathsf{V}$ (for the topology J on $\mathscr{C}$).*

(ii) *Suppose moreover, that either one of the following condition holds:*
 (a) *The functor $g^a_{\mathsf{V}!} : \mathscr{B}^\wedge_\mathsf{V} \to C^\sim_\mathsf{V}$ commutes with all fibre products.*
 (b) *All fibre products are representable in $\mathscr{B}$, and g commutes with fibre products.*

Then a family $(f_i : B_i \to B \mid i \in I)$ of morphisms of $\mathscr{B}$ generates a covering sieve of J_g if and only if $(g(f_i) : gB_i \to gB \mid i \in I)$ generates a covering sieve of J.

Proof. (i): According to remark 4.1.8(iii), we have $R \in J_g(X)$ if and only if for every morphism $Y \to X$ in $\mathscr{C}$ the natural morphism $\varphi_Y : R \times_X Y \to h_Y$ induces a bijection

$$(g^\wedge_\mathsf{V} G)(Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{B}^\wedge}(R \times_X Y, g^\wedge_\mathsf{V} G) \qquad \text{for every } G \in \mathrm{Ob}(C^\sim_\mathsf{V}).$$

By adjunction, this is equivalent to saying that φ_Y induces a bijection

$$G(gY) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{C}^\wedge}(g_{\mathsf{V}!}(R \times_X Y), G) \qquad \text{for every } G \in \mathrm{Ob}(C^\sim_\mathsf{V}).$$

The latter is in turn naturally identified with the map

$$\mathrm{Hom}_{C^\sim_\mathsf{V}}(g_{\mathsf{V}!}(\varphi_Y)^a, G) : \mathrm{Hom}_{C^\sim_\mathsf{V}}(h^a_{gY}, G) \to \mathrm{Hom}_{C^\sim_\mathsf{V}}(g_{\mathsf{V}!}(R \times_X Y)^a, G).$$

By Yoneda's lemma, it follows that $R \in J_g(R)$ if and only if $g_{\mathsf{V}!}(\varphi_Y)^a$ is an isomorphism, and this is equivalent to the stated condition, by corollary 4.1.30.

(ii): Suppose first that condition (a) holds. Notice that since $\mathscr{B}$ is V-small, we may assume that the indexing set I is V-small as well. Let $R \subset h_B$ (resp. $R' \subset h_{gB}$) be the subobject generated by the family $(f_i \mid i \in I)$ (resp. $(g(f_i) \mid i \in I)$); if R is a covering subobject of h_B for J_g, then R' is a covering subobject of h_{gB} for J, by lemma 4.2.4. Conversely, suppose that R' is a covering subobject of h_{gB} for J; according to (i) and corollary 4.1.30, we need to check that the inclusion $j : R \to h_B$ induces an isomorphism $g^a_{\mathsf{V}!}(R \times_B B') \to h^a_{gB'}$ for every morphism $B' \to B$ in $\mathscr{B}$. Since $g^a_{\mathsf{V}!}$ commutes with fibre products, we are then easily reduced to checking that j induces an isomorphism $g^a_{\mathsf{V}!}(j) : g^a_{\mathsf{V}!}R \to h^a_B$. However, $g^a_{\mathsf{V}!}(j)$ is a monomorphism, since $g^a_{\mathsf{V}!}$ commutes with fibre products (proposition 1.3.18(i)). To show that $g^a_{\mathsf{V}!}(j)$ is an epimorphism, let $S := \coprod_{i \in I} h_{B_i}$ (which is an object of $\mathscr{B}^\wedge_\mathsf{V}$, since I is V-small) and denote by $j' : S \to h_B$ the morphism induced by the morphisms f_i. The image of $g^a_{\mathsf{V}!}(j)$ contains the image of $g^a_{\mathsf{V}!}(j')$; on the other hand, $g^a_{\mathsf{V}!}$ commutes with coproducts, since it is a left adjoint, and we have natural identifications : $g^a_{\mathsf{V}!}(h_{B_i}) = h^a_{gB_i}$ for every $i \in I$. Thus, the image of $g^a_{\mathsf{V}!}(j')$ equals that of the morphism $\varphi : \coprod_{i \in I} h^a_{gB_i} \to h^a_B$ induced by the morphisms $g(f_i)$. But since R' is a covering subobject, φ is an epimorphism, so the same holds for $g^a_{\mathsf{V}!}(j)$.

Next, suppose that condition (b) holds, and let $f_\bullet := (f_i : B_i \to B \mid i \in I)$ be a family of morphisms of $\mathscr{B}$ such that $(g(f_i) \mid i \in I)$ covers gB in the topology J. Set $B_{ij} := B_i \times_B B_j$ for every $i, j \in I$; the subobject $R \subset h_B$ generated by $f_\bullet$ is the coequalizer in $\mathscr{B}^\wedge$ of the natural projections

$$\textstyle\coprod_{i,j\in I} h_{B_{ij}} \rightrightarrows \coprod_{i\in I} h_{B_i}$$

and according to (i) we need to check that for every morphism $B' \to B$ the induced morphism $g_{\mathsf{V}!}(R \times_B B') \to h_{gB'}$ is a bicovering morphism. However, set as well $B'_i := B_i \times_B B'$ and $B'_{ij} := B_{ij} \times_B B'$ for every $i, j \in I$; then $R \times_B B'$ is the coequalizer of the induced projections

$$\textstyle\coprod_{i,j\in I} h_{B'_{ij}} \rightrightarrows \coprod_{i\in I} h_{B'_i}.$$

On the other hand, $g_{\mathsf{V}!}$ commutes with coequalizers and coproducts, since it is a left adjoint; in view of remark 1.3.6(ii) we deduce that $g_{\mathsf{V}!}(R \times_B B')$ is the coequalizer of the projections

$$\textstyle\coprod_{i,j\in I} h_{gB'_{ij}} \rightrightarrows \coprod_{i\in I} h_{gB'_i}.$$

But by assumption we have as well $gB'_i = gB_i \times_{gB} gB'$, whence $h_{gB_i} = h_{gB'_i} \times_{h_{gB}} h_{B'}$, and likewise for $h_{gB'_{ij}}$, for every $i, j \in I$ (corollary 1.4.3(vi)). We conclude that $g_{\mathsf{V}!}(R \times_B B')$ is the subobject of $h_{B'}$ generated by the family $(g(f_i) \times_{gB} gB' \to gB' \mid i \in I)$, and it is therefore a covering subobject, as required. □

Remark 4.2.18. (i) In the situation of lemma 4.2.17, let V' be another universe with $\mathsf{V} \subset \mathsf{V}'$, and F a V'-presheaf on $\mathscr{B}$. According to lemma 1.4.8, F is naturally isomorphic to the colimit of the functor $h_{\mathscr{B}} \circ \mathsf{s}_F : \mathscr{F}ib(F) \to \mathscr{B}^\wedge_{\mathsf{V}} \subset \mathscr{B}^\wedge_{\mathsf{V}'}$, where $\mathsf{s}_F : \mathscr{F}ib(F) \to \mathscr{B}$ is the source functor. For every finite subset $S \subset \mathrm{Ob}(\mathscr{F}ib(F))$, let $F_S \subset F$ be the subobject generated by the union of the images of all the morphisms $h_{(X,s)} : h_B \to F$ with $(B, s) \in S$ (notation of (1.4.7)), and notice that F_S is isomorphic to an object of $\mathscr{B}^\wedge_{\mathsf{V}}$. After replacing each F_S by an isomorphic object of $\mathscr{B}^\wedge_{\mathsf{V}}$, we obtain therefore F as the filtered colimit of a system $(F_S \mid S \in \Sigma_F)$ of objects of $\mathscr{B}^\wedge_{\mathsf{V}}$, indexed by the set Σ_F of all finite subsets of $\mathrm{Ob}(\mathscr{F}ib(F))$.

(ii) Now, let $F_1 \xrightarrow{\varphi_1} F_0 \xleftarrow{\varphi_2} F_2$ be two morphisms in $\mathscr{B}^\wedge_{\mathsf{V}}$. Consider the set $\Sigma(\varphi_1, \varphi_2)$ of all triples (S_0, S_1, S_2) where S_i is a finite subset of $\mathrm{Ob}(\mathscr{F}ib(F_i))$ for $i = 0, 1, 2$, and $(B, \varphi_j \circ s) \in S_0$ for $j = 1, 2$ and every $(B, s) \in S_j$. We endow $\Sigma(\varphi_1, \varphi_2)$ with the partial order such that $(S_0, S_1, S_2) \geq (S'_0, S'_1, S'_2)$ if and only if $S'_i \subset S_i$ for $i = 0, 1, 2$. We have a natural functor

$$\tau : \Sigma(\varphi_1, \varphi_2) \to \mathscr{B}^\wedge_{\mathsf{V}} \qquad (S_0, S_1, S_2) \mapsto F_{S_1} \times_{F_{S_0}} F_{S_2}$$

and by a simple inspection we see that the colimit of τ is naturally isomorphic to $F_1 \times_{F_0} F_2$.

(iii) We deduce that $G := g^a_{\mathsf{V}'!}(F_1 \times_{F_0} F_2)$ is naturally isomorphic to the colimit of $g^a_{\mathsf{V}'!} \circ \tau$, since $g^a_{\mathsf{V}'!}$ is a left adjoint; but $g^a_{\mathsf{V}'!} \circ \tau$ is in turn naturally isomorphic to the functor $g^a_{\mathsf{V}!} \circ \tau$, by lemma 4.2.11(ii). Suppose now that $g^a_{\mathsf{V}!}$ commutes with all fibre products of $\mathscr{B}^\wedge_{\mathsf{V}}$. Then G is also naturally isomorphic to the filtered colimit of the system

$$(g^a_{\mathsf{V}!} F_{1,S_1} \times_{g^a_{\mathsf{V}!} F_{0,S_0}} g^a_{\mathsf{V}!} F_{2,S_2} \mid (S_0, S_1, S_2) \in \Sigma(\varphi_1, \varphi_2)).$$

But since filtered colimits in $\mathscr{B}^\wedge_{\mathsf{V}'}$ commute with fibre products, the latter is naturally isomorphic to the fibre product $G_1 \times_{G_0} G_2$, where G_i is the filtered colimit of the system $(g^a_{\mathsf{V}'!} F_{S_i} \mid S_i \in \Sigma_{F_i})$ (notice that the projections $\Sigma(\varphi_1, \varphi_2) \to \Sigma_{F_i} : (S_0, S_1, S_2) \to S_i$ are obviously coinitial functors). But again, since $g^a_{\mathsf{V}'!}$ commutes with filtered colimits, G_i is naturally isomorphic to $g^a_{\mathsf{V}'!}(F_i)$. We conclude that $g^a_{\mathsf{V}'!}$ commutes with fibre products as well. In particular, condition (ii.a) of lemma 4.2.17 is independent of the choice of universe V.

4.3. **Morphisms of sites.** We observed in example 4.2.2 that every continuous map of topological spaces $f : T \to T'$ induces a continuous functor $u := f^{-1} : \mathscr{T}' \to \mathscr{T}$ between the corresponding categories of open subsets. It is well known that the induced functor on categories of sheaves $\widetilde{u}^* : \mathscr{T}^\sim \to \mathscr{T}'^\sim$ is moreover *exact*, whereas for a general continuous functor g between arbitrary sites, only the right exactness of $\widetilde{g}^*$ is always assured. The exactness of $\widetilde{g}^*$ therefore singles out an interesting class of continuous functors, which will play an important role in our discussion of topoi, and – even more crucially – for the study of the functorial properties of the categories of stacks, in section 5.4. In this section we carry out a preliminary investigation of this class of functors, and prove a useful characterization. Let us begin with :

Definition 4.3.1. (i) Let $C = (\mathscr{C}, J)$ and $C' = (\mathscr{C}', J')$ be two sites, and V a universe such that C and C' are V-small. A *morphism of sites* $C' \to C$ is the datum of a continuous functor $g : \mathscr{C} \to \mathscr{C}'$, such that the left adjoint $\widetilde{g}^*_{\mathsf{V}}$ of $\widetilde{g}_{\mathsf{V}*}$ is exact (notation of definition 4.2.1(i)).

(ii) Let U' be a universe with $\mathsf{U} \in \mathsf{U}'$; the U'-small U-sites are the objects of a 2-category

$$(\mathsf{U}, \mathsf{U}')\text{-}\mathbf{Site}$$

whose 1-cells are the morphisms of sites, and whose 2-cells $\beta : g \Rightarrow g'$ are the natural transformations $g' \Rightarrow g$ between the underlying functors of such morphisms, with the obvious composition laws for 1-cells and 2-cells.

(iii) We shall also be interested in the 2-category of U-*lex-sites*

$$(\mathsf{U}, \mathsf{U}')\text{-lex.}\mathbf{Site}$$

defined as the 2-subcategory of $(\mathsf{U}, \mathsf{U}')$-$\mathbf{Site}$ whose objects are the finitely complete U'-small U-sites and whose 1-cells are the morphisms of sites g as in (i) such that the underlying functor $g : \mathscr{C} \to \mathscr{C}'$ is left exact. For any two such morphisms g, g', the 2-cells $g \Rightarrow g'$ are the same as in $(\mathsf{U}, \mathsf{U}')$-$\mathbf{Site}$.

Remark 4.3.2. (i) With the notation of definition 4.3.1(ii), two different choices of the auxiliary universe U' *do not* necessarily yield 2-equivalent 2-categories; nevertheless, such choices are usually ininfluent in the proofs of our results, so we mostly omit mentioning them explicitly, and write U-$\mathbf{Site}$ and U-lex.$\mathbf{Site}$ for these 2-categories. When the choice of U is clear from the context, we shall likewise drop the mention of U, and write just $\mathbf{Site}$ and lex.$\mathbf{Site}$.

(ii) On the other hand, the following proposition 4.3.9 will show that the definition of morphism of sites $g : (\mathscr{C}', J') \to (\mathscr{C}, J)$ depends only on the underlying functor $g : \mathscr{C} \to \mathscr{C}'$, and not on the choice of a universe V such that C and C' are V-small.

(iii) Let $C := (\mathscr{C}, J)$ and $C' := (\mathscr{C}', J')$ be two sites, $u : C' \to C$ a morphism of sites, and S a set. The *constant presheaf on* $\mathscr{C}$ *with value* S is the constant functor $c_S : \mathscr{C}^o \to \mathbf{Set}$ associated with S (see (1.2.9)), and the *constant sheaf on* C *with value* S is the associated sheaf c_S^a. Clearly we have $c_S = \amalg_{s \in S} c_{\{s\}}$, and each presheaf $c_{\{s\}}$ is a final object of the category $\mathscr{C}^\wedge$. It follows that $c_S^a = \amalg_{s \in S} c_{\{s\}}^a$, and $c_{\{s\}}^a$ is a final object of $C^\sim$ (example 1.2.16(iv)). Since $\widetilde{u}^*$ commutes with all colimits and is left exact by assumption, we deduce that $\widetilde{u}^*(c_S^a)$ is naturally isomorphic to the constant sheaf with value S on C'. This assertion may fail for general continuous functors.

For our characterization of morphisms of sites, we shall need the following :

Definition 4.3.3. Let $(\mathscr{B}, J)$ be a site, $F : \mathscr{C} \to \mathscr{B}$ a fibration, and fix a cleavage $\boldsymbol{\lambda}$ for F, so that for every $X \in \mathrm{Ob}(\mathscr{C})$ and every morphism $f : B \to FX$ we have the cartesian morphism $\boldsymbol{\lambda}(X, f) : f^*X \to X$. We say that F is *locally cofiltered* (relative to the topology J) if the following holds for every $B \in \mathrm{Ob}(\mathscr{B})$ and every $X, X' \in \mathrm{Ob}(\mathscr{C}_B)$:

(a) There exists a covering family $(B_i \to B \mid i \in I)$ in the topology J with $\mathrm{Ob}(\mathscr{C}_{B_i}) \neq \varnothing$ for every $i \in I$.

(b) There exist a covering family $(f_i : B_i \to B \mid i \in I)$ in the topology J, and for every $i \in I$ an object $Y_i \in \mathrm{Ob}(\mathscr{C}_{B_i})$ with two morphisms $f_i^* X \leftarrow Y_i \to f_i^* X'$ in $\mathscr{C}_{B_i}$.
(c) For every pair of morphisms $g, g' : X' \to X$ in $\mathscr{C}_B$ there exist a covering family $(f_i : B_i \to B \mid i \in I)$ in the topology J, and for every $i \in I$ a morphism $h_i : Y_i \to f_i^* X'$ in $\mathscr{C}_{B_i}$ such that $f_i^*(g) \circ h_i = f_i^*(g') \circ h_i$.

4.3.4. In the situation of definition 4.3.3, suppose that $(\mathscr{B}, J)$ and $\mathscr{C}$ are small; we consider the functor

$$\int_a^{\boldsymbol{\lambda}} : \mathscr{C}^\wedge \to (\mathscr{B}, J)^\sim \qquad G \mapsto \Big(\int^{\boldsymbol{\lambda}} G\Big)^a$$

where $\int^{\boldsymbol{\lambda}} : \mathscr{C}^\wedge \to \mathscr{B}^\wedge$ is defined as in example 3.1.14(iv). Recall that the sections of $\int^{\boldsymbol{\lambda}} G(B)$ are the equivalence classes $[X, t]$ of pairs (X, t) consisting of an object X of $F^{-1}B$ and a section $t \in GX$, for the equivalence relations on the set of such pairs explicited in example 1.2.23(i).

Lemma 4.3.5. *With the notation of* (4.3.4)*, let L be any presheaf on $\mathscr{C}$, and $M := \int^{\boldsymbol{\lambda}} L$. Let also $B \in \mathrm{Ob}(\mathscr{B})$ and $(X, t), (X', t')$ two pairs consisting of objects $X, X' \in F^{-1}B$ and sections $t \in LX$, $t' \in LX'$. If $[X, t] = [X', t']$ in $M(B)$, there exist a covering family $(f_j : B_j \to B \mid j \in I)$ for the topology J, and for each $j \in I$ morphisms $g_j : X_i \to X$, $g'_j : X_i \to X'$ in $\mathscr{C}$ such that $Fg_j = Fg'_j = f_j$ and $(Lg_j)(t) = (Lg'_j)(t')$.*

Proof. As in the proof of proposition 1.5.21(ii) we shall say that a diagram of $F^{-1}B$ is a pair of finite sets (A, A') with $A \subset \mathrm{Ob}(F^{-1}B)$, $A' \subset \mathrm{Morph}(F^{-1}B)$, such that the source and target of every element of A' lie in A. We notice :

Claim 4.3.6. Let (A, A') be any diagram of $F^{-1}B$. There exist a covering family $(f_j : B_j \to B \mid j \in I)$ and for every $j \in I$ an object Y_j of $F^{-1}B_j$ with a system of morphisms $(g_{jX} : Y_j \to X \mid X \in A)$ such that the following holds. For every $j \in I$, every $X, X', X'' \in A$ and every $h : X \to X''$, $h' : X' \to X''$ in A' we have $h \circ g_{jX} = h' \circ g_{jX'}$.

Proof of the claim. Let $X_1, \dots, X_n$ be the elements of A. Since F is locally cofiltered, by a simple induction on n we find a covering family $f'_\bullet := (f'_{j'} : B_{j'} \to B \mid j' \in I')$ and morphisms $g'_{j'i} : Y'_{j'} \to X_i$ such that $Fg'_{j'i} = f'_{j'}$ for every $j' \in I'$ and $i = 1, \dots, n$. Next, let $T \subset A' \times A'$ be the subset of all pairs (h, h') such that h and h' have the same target; we show that for every $T' \subset T$, and every $j' \in I'$ there exist a covering family $f''_{\bullet\bullet} := (f''_{j'\lambda} : B_{j'\lambda} \to B_{j'} \mid \lambda \in \Lambda_{j'})$ and morphisms $g''_{j'\lambda} : Y_{j'\lambda} \to Y'_{j'}$ with $Fg''_{j'\lambda} = f''_{j'\lambda}$ and

$$h \circ g'_{j'i} \circ g''_{j'\lambda} = h' \circ g'_{j'i'} \circ g''_{j'\lambda}$$

for every $\lambda \in \Lambda_{j'}$, every $i, i', k = 1, \dots, n$ and every $(h : X_i \to X_k, h' : X_{i'} \to X_k)$ in T'. The claim will follow for $T' = T$, by letting $f_\bullet$ be the covering family $(f'_j \circ f''_{j'\lambda} : B_{j'\lambda} \to B \mid j' \in I, \ \lambda \in \Lambda_{j'})$, and letting $g_{\bullet\bullet}$ be the system of morphisms $g'_{j'i} \circ g''_{j'\lambda}$.

Now, if $T' = \varnothing$, there is nothing to show. Thus, let $(h_0, h_1) \in T'$; by induction on the cardinality of T', we may assume that the assertion is known for the subset $T'' := T' \setminus \{(h_0, h_1)\}$. After replacing $f'_\bullet$ by the covering family $(f'_j \circ f''_{j'\lambda} \mid j' \in I, \ \lambda \in \Lambda_{j'})$ and the system $g'_{\bullet\bullet}$ with the system of morphisms $g'_{j'i} \circ g''_{j'\lambda}$, we may then assume that $h \circ g'_{j'i} = h' \circ g'_{j'i'}$ for every $i, i' = 1, \dots, n$, every $j \in I'$ and every $(h, h') \in T''$ such that the sources of h and h' are i and i' respectively. We may assume that the common target of h_0 and h_1 is X_1, and say that their sources are X_{i_0} and X_{i_1} respectively. Since F is locally cofiltered, there exist a covering family $f''_{\bullet\bullet}$ and a system of morphisms $g''_{\bullet\bullet}$ as in the foregoing, such that $h_0 \circ g'_{j'i_0} \circ g''_{j'\lambda} = h_1 \circ g'_{j'i_1} \circ g''_{j'\lambda}$ for every $j' \in I$ and every $\lambda \in \Lambda_{j'}$. Clearly with these choices for $f''_{\bullet\bullet}$ and $g''_{\bullet\bullet}$ the sought identities hold whenever $(h, h') \in T'$, as required. ◊

Now, the condition $[X, t] = [X', t']$ means that for some $k \in \mathbb{N}$ there exist objects of $F^{-1}B$

$$X_0, Y_0, X_1, Y_1, \dots, Y_k, X_{k+1}$$

and sections $t_i \in LX_i$ for $i = 0, \dots, k+1$, with $X_0 = X$, $X_{k+1} = X'$, $t_0 = t$ and $t_{k+1} = t'$, as well as morphisms

$$X_i \xleftarrow{q_i} Y_i \xrightarrow{q'_i} X_{i+1} \qquad \text{such that } (Lq_i)(t_i) = (Lq'_i)(t_{i+1}) \text{ for } i = 0, \dots, k.$$

We apply claim 4.3.6 to the diagram formed by all the objects X_i, Y_i and all the morphisms q_i, q'_i, to find a covering family $(f_j : B_j \to B \mid j \in I)$ and morphisms $h_j : Z_j \to Y_i$ with $Fh_j = f_j$, such that $g_{ij} := q_i \circ h_j = q'_i \circ h_j$ for every $j \in I$ and $i = 0, \dots, k$. It follows that $(Lg_{0j})(t_0) = (Lg_{1j})(t_1) = \cdots = (Lg_{k+1,j})(t_{k+1})$ for every $j \in I$. Then the assertion holds with $g_j := g_{0j}$ and $g'_j := g_{k+1,j}$ for every $j \in I$. □

Proposition 4.3.7. *With the notation of* (4.3.4)*, we have :*

(i) *The functor* $\int^{\boldsymbol{\lambda}}$ *commutes with all colimits.*

(ii) *If* F *is a locally cofiltered fibration,* $\int_a^{\boldsymbol{\lambda}}$ *is exact.*

Proof. (i): Since the colimits in $\mathscr{B}^\wedge$ are computed argumentwise (corollary 1.4.3(ii)), we are reduced to checking that for every $B \in \mathrm{Ob}(\mathscr{B})$ the functor

$$\mathscr{C}^\wedge \to \mathbf{Set} \qquad G \mapsto \operatorname*{colim}_{(F^{-1}B)^o} G \circ \iota_B^o$$

commutes with colimits. However, the functor $\iota_B^\wedge : \mathscr{C}^\wedge \to (F^{-1}B)^\wedge$ induced by the inclusion functor $\iota_B : F^{-1}B \to \mathscr{C}$ commutes with colimits (again, by corollary 1.4.3(ii)), so we come down to checking that the functor $\mathrm{Colim}_{(F^{-1}B)^o} : (F^{-1}B)^\wedge \to \mathbf{Set}$ of remark 1.3.3(ii) commutes with colimits, which is clear, as the latter is a left adjoint.

(ii): In light of (i), and since the functor $G \mapsto G^a$ commutes with colimits, it suffices to check that $\int_a^{\boldsymbol{\lambda}}$ is left exact, and by proposition 1.3.22(i) we are reduced to showing that $\int_a^{\boldsymbol{\lambda}}$ commutes with equalizers and finite non-empty products, and that it preserves final objects.

Thus, let E be the presheaf on $\mathscr{C}$ such that $E(X) = \{\varnothing\}$ for every $X \in \mathrm{Ob}(\mathscr{C})$, and set $E' := \int^{\boldsymbol{\lambda}} E$. For every $B \in \mathrm{Ob}(\mathscr{B})$ and every $s \in E'^a(B)$ there is a covering family $B_\bullet := (B_i \to B \mid i \in I)$ in the topology J, and for every $i \in I$ an object X_i of $F^{-1}B_i$ such that s is represented by the system of section $([X_i, \varnothing]) \in E'(B_i)$. Conversely, every such system defines a section of $E'^a(B)$. Now, since F is locally cofiltered there exists such a covering family $B_\bullet$ of B with the property that $\mathrm{Ob}(F^{-1}B_i)$ is not empty for every $i \in I$; by picking arbitrary $X_i \in \mathrm{Ob}(F^{-1}B_i)$ for each i we obtain therefore a section of $E'^a(B)$; this shows that $E'^a(B) \neq \varnothing$ for every $B \in \mathrm{Ob}(\mathscr{B})$. Next, let $s := ([X_i, \varnothing] \mid i \in I)$ and $s' := ([X'_i, \varnothing] \mid i \in I)$ be two sections of $E'^a(B)$, with $FX_i = FX'_i = B_i$ for every $i \in I$. Since F is locally cofiltered, we may find for every $i \in I$ a covering family $(f_{i\lambda} : B_{i\lambda} \to B_i \mid \lambda \in \Lambda_i)$ and morphisms $g_{i\lambda} : Y_{i\lambda} \to X_i$, $g'_{i\lambda} : Y_{i\lambda} \to X'_i$ in $\mathscr{C}$ with $Fg_{i\lambda} = Fg'_{i\lambda} = f_{i\lambda}$ for every $\lambda \in \Lambda_i$. Then both s and s' are represented by the system $([Y_{i\lambda}, \varnothing] \mid i \in I,\ \lambda \in \Lambda_i)$. This proves that $E'^a(B)$ contains a unique section for every $B \in \mathrm{Ob}(\mathscr{B})$, *i.e.* E'^a is the final object of $(\mathscr{B}, J)^\sim$, as required.

To check that $\int_a^{\boldsymbol{\lambda}}$ commutes with finite products, let G_1, G_2 be two presheaves on $\mathscr{C}$, and set

$$G := G_1 \times G_2 \qquad H := \int^{\boldsymbol{\lambda}} G \qquad H_i := \int^{\boldsymbol{\lambda}} G_i \qquad \text{for } i = 1, 2.$$

We need to check that the projections $G \to G_i$ induce an isomorphism $\omega : H^a \xrightarrow{\sim} H_1^a \times H_2^a$. To this aim, let as well $B \in \mathrm{Ob}(\mathscr{B})$ and $s \in (H_1^a \times H_2^a)(B)$. Hence s is represented by the datum of a covering family $(f_j : B_j \to B \mid j \in I)$ and sections $s_{ij} \in H_i(B_j)$ for $i = 1, 2$ and every $j \in I$, fulfilling the following compatibility condition. For every $B' \in \mathrm{Ob}(\mathscr{B})$, every $j, j' \in I$ and every pair of morphisms $g : B' \to B_j$, $g' : B' \to B_{j'}$ such that $f_j \circ g_j = f_{j'} \circ g'$, we have $H_i(g)(s_{ij}) = H_i(g')(s_{ij'})$ for $i = 1, 2$. By example 1.2.23(i), each s_{ij} is the class $[X_{ij}, t_{ij}]$ of a pair consisting of an object X_{ij} of $F^{-1}B_j$ and a section $t_{ij} \in G_iX_{ij}$. Since F is locally cofiltered, we may then find for every $j \in I$ a covering family $(f_{j\lambda} : B_{j\lambda} \to B_j \mid \lambda \in \Lambda_j)$ and for every $j \in I$ and every $\lambda \in \Lambda_j$ an object $Y_{j\lambda}$ of $F^{-1}B_{j\lambda}$ and morphisms $g_{ij\lambda} : Y_{j\lambda} \to X_{ij}$ in $\mathscr{C}$ with $Fg_{ij\lambda} = f_{j\lambda}$ for $i = 1, 2$. With this notation, set $t_{ij\lambda} := (G_i g_{ij\lambda})(t_{ij})$; we have

$$[Y_{j\lambda}, t_{ij\lambda}] = H_i(f_{j\lambda})(s_{ij}) \qquad \text{for } i = 1, 2, \text{ every } j \in I \text{ and every } \lambda \in \Lambda_j.$$

Thus, the class $\tau_{j\lambda} := [Y_{j\lambda}, (t_{1j\lambda}, t_{2j\lambda})]$ is a section of $H(Y_{j\lambda})$ for every $j \in I$ and $\lambda \in \Lambda_j$. Lastly, the family $(f_j \circ f_{j\lambda} : B_{j\lambda} \to B \mid j \in I,\ \lambda \in \Lambda_j)$ covers B, and the system of sections $\tau_{\bullet\bullet}$ defines a section of $H^a(B)$ whose image under ω_B agrees with s. This shows that ω is an epimorphism. In order to check that ω is a monomorphism, consider $t, t' \in H^a(B)$ whose images agree in $(H_1^a \times H_2^a)(B)$; we may find a covering family $f_\bullet := (f_j : B_j \to B \mid j \in I)$ such that t and t' can be represented by compatible systems of sections $t_\bullet := ([X_j, (t_{1j}, t_{2j})] \mid j \in I)$, $t'_\bullet := ([X'_j, (t'_{1j}, t'_{2j})] \mid j \in I)$ with $X_j, X'_j \in F^{-1}B_j$ and $t_{ij}, t'_{ij} \in G_i X_j$ for $i = 1, 2$ and every $j \in I$. By assumption, the resulting systems $([X_j, t_{ij}] \mid j \in I)$ and $([X'_j, t'_{ij}] \mid j \in I)$ represent the same sections of $H_i^a(B)$ for $i = 1, 2$. This means that there exists for every $j \in I$ a covering family $(f_{j\lambda} : B_{j\lambda} \to B_j \mid \lambda \in \Lambda_j)$ such that if we set $t_{ij\lambda} := G_i(\boldsymbol{\lambda}(f_{j\lambda}, X_j))(t_{ij})$ and $t'_{ij\lambda} := G_i(\boldsymbol{\lambda}(f_{j\lambda}, X'_j))(t'_{ij})$, we have

$$[f^*_{j\lambda} X_j, t_{ij\lambda}] = [f^*_{j\lambda} X'_j, t'_{ij\lambda}] \qquad \text{in } H_i(B_{j\lambda}) \text{ for } i = 1, 2, \text{ every } j \in I \text{ and every } \lambda \in \Lambda_j.$$

After replacing $f_\bullet$ by the covering family $(f_j \circ f_{j\lambda} : B_{j\lambda} \to B \mid j \in I,\ \lambda \in \Lambda_j)$ and the systems $t_\bullet$, $t'_\bullet$ by $([f^*_{j\lambda} X_j, (t_{1j\lambda}, t_{2j\lambda})] \mid j \in I,\ \lambda \in \Lambda)$ and $([f^*_{j\lambda} X'_j, (t'_{1j\lambda}, t'_{2j\lambda})] \mid j \in I,\ \lambda \in \Lambda)$, we may therefore assume from start that

$$[X_j, t_{ij}] = [X'_j, t'_{ij}] \qquad \text{in } H_i(B_j) \text{ for } i = 1, 2 \text{ and every } j \in I.$$

By lemma 4.3.5, we may then find for $i = 1, 2$ and every $j \in I$ a covering family $(f_{ij\lambda} : B_{ij\lambda} \to B_j \mid \lambda \in \Lambda_{ij})$ and morphisms $g_{ij\lambda} : Y_{ij\lambda} \to X_j$, $g'_{ij\lambda} : Y_{ij\lambda} \to X'_j$ in $\mathscr{C}$ such that $Fg_{ij\lambda} = Fg'_{ij\lambda} = f_{ij\lambda}$ and $t_{ij\lambda} := (G_i g_{ij\lambda})(t_{ij}) = (G_i g'_{ij\lambda})(t'_{ij})$ for every $\lambda \in \Lambda_{ij}$.

For $i = 1, 2$ and every $j \in I$, let $\mathscr{S}_{ij} \subset \mathscr{B}/B_j$ be the sieve generated by the family $f_{ij\bullet}$, and pick a generating family $(f'_\sigma : B_\sigma \to B_j \mid \sigma \in \Sigma_j)$ for the sieve $\mathscr{S}_{1j} \cap \mathscr{S}_{2j}$, which covers B_j as well in the topology J. Set $\Sigma := \bigcup_{j \in I} \{j\} \times \Sigma_j$. By definition, this means that for $i = 1, 2$ and every $(j, \sigma) \in \Sigma$ there exist $\lambda \in \Lambda_{1j}$, $\lambda' \in \Lambda_{2j}$ and a commutative diagram in $\mathscr{B}$:

$$\begin{array}{ccc} B_\sigma & \xrightarrow{h_\sigma} & B_{1j\lambda} \\ {\scriptstyle h'_\sigma}\downarrow & \searrow^{f'_\sigma} & \downarrow{\scriptstyle f_{1j\lambda}} \\ B_{2j\lambda'} & \xrightarrow[f_{2j\lambda'}]{} & B_j. \end{array}$$

With this notation, we set $Z_{1j\sigma} := h^*_\sigma Y_{1j\lambda}$, $Z_{2j\sigma} := h'^*_\sigma Y_{2j\lambda'}$ and :

$$s_{1j\sigma} := G_1(\boldsymbol{\lambda}(h_\sigma, Y_{1j\lambda}))(t_{1j\lambda}) \quad s_{2j\sigma} := G_2(\boldsymbol{\lambda}(h'_\sigma, Y_{2j\lambda}))(t_{2j\lambda}) \qquad \text{for every } (j, \sigma) \in \Sigma.$$

Then, both compatible systems $([X_j, t_{1j}] \mid j \in I)$ and $([X'_j, t'_{1j}] \mid j \in I)$ agree with the compatible system $([Z_{1\sigma}, s_{1\sigma}] \mid \sigma \in \Sigma)$ in $H_1^a(B)$, and both compatible systems $([X_j, t_{2j}] \mid j \in I)$ and $([X'_j, t'_{2j}] \mid j \in I)$ agree with the compatible system $([Z_{2\sigma}, s_{2\sigma}] \mid \sigma \in \Sigma)$ in $H_2^a(B)$.

Since F is locally cofiltered, we may find for every $\sigma \in \Sigma$ a covering family $(f_{\sigma\lambda} : B_{\sigma\lambda} \to B_\sigma \mid \lambda \in \Lambda_\sigma)$ and morphisms $g_{i\sigma\lambda} : Z_{\sigma\lambda} \to Z_{i\sigma}$, with $Fg_{i\sigma\lambda} = f_{\sigma\lambda}$ for $i = 1, 2$. Set $s_{i\sigma\lambda} := (G_i g_{i\sigma\lambda})(s_{i\sigma})$ for $i = 1, 2$, every $\sigma \in \Sigma$ and every $\lambda \in \Lambda_\sigma$. Hence both t and t' are represented by the compatible system of sections $([Z_{\sigma\lambda}, (s_{1\sigma\lambda}, s_{2\sigma\lambda})] \mid \sigma \in \Sigma,\ \lambda \in \Lambda_\sigma)$, so $t = t'$.

Lastly, let $\varphi_1, \varphi_2 : G_1 \to G_2$ be two morphisms of presheaves on $\mathscr{C}$, and $E \subset G_1$ the equalizer of φ_1 and φ_2; set also $H_i := \int^{\boldsymbol{\lambda}} G_i$, $\psi_i := \int^{\boldsymbol{\lambda}} \varphi_i$ for $i = 1, 2$, and denote by E' the equalizer of $\psi_1^a, \psi_2^a : H_1^a \to H_a^2$. We need to check that the induced morphism $E^a \to E'$ is an isomorphism. Thus, let $B \in \mathrm{Ob}(\mathscr{B})$ and $s \in E'(B)$; by remark 4.1.19(i), the set $E'(B)$

is the equalizer of the induced maps $\psi_1^a(B), \psi_2^a(B) : H_1^a(B) \to H_2^a(B)$, so we may find a covering family $(f_j : B_j \to B \mid j \in I)$ such that s is represented by a compatible system $t_\bullet := ([X_j, t_j] \mid j \in I)$ with $X_j \in \mathrm{Ob}(F^{-1}B_j)$ and $t_j \in G_1(B_j)$ for every $j \in I$, such that the systems $([X_j, (\varphi_{1X_j})(t_j)] \mid j \in I)$ and $([X_j, (\varphi_{2X_j})(t_j)] \mid j \in I)$ represent the same section of $H_2^a(B)$. This means that for every $j \in I$ there exists a covering family $f_\bullet := (f_{j\lambda} : B_{j\lambda} \to B_j \mid \lambda \in \Lambda_j)$ such that if we set $t_{j\lambda} := G_1(\boldsymbol{\lambda}(f_{j\lambda}, X_j))(t_j)$ we get the following identity in $H_2(B_{j\lambda})$:

$$[f_{j\lambda}^* X_j, \varphi_{1X_j}(t_{j\lambda})] = [f_{j\lambda}^* X_j, \varphi_{2X_j}(t_{j\lambda})] \qquad \text{for } i = 1, 2, \text{ every } j \in I \text{ and every } \lambda \in \Lambda_j.$$

After replacing $f_\bullet$ by the covering family $(f_{j\lambda} \circ f_j : B_{j\lambda} \to B \mid j \in I, \ \lambda \in \Lambda_j)$ and $t_\bullet$ by the compatible system $([f_{j\lambda}^* X_j, t_{j\lambda}] \mid j \in I, \ \lambda \in \Lambda_j)$ we may then assume that

$$[X_j, (\varphi_{1X_j})(t_j)] = [X_j, (\varphi_{2X_j})(t_j)] \qquad \text{in } H_2(B_j) \text{ for every } j \in I.$$

By lemma 4.3.5 we may then find for every $j \in I$ a covering family $(f_{j\lambda} : B_{j\lambda} \to B_j \mid \lambda \in \Lambda_j)$ and morphisms $g_{j\lambda}, g'_{j\lambda} : Y_{j\lambda} \to X_j$ in $\mathscr{C}$ such that $Fg_{j\lambda} = Fg'_{j\lambda} = f_{j\lambda}$ and

$$G_2(g_{j\lambda})((\varphi_{1X_j})(t_j)) = G_2(g'_{j\lambda})((\varphi_{2X_j})(t_j)) \qquad \text{for every } \lambda \in \Lambda_j.$$

Set $\Sigma := \bigcup_{j\in I}\{j\} \times \Lambda_j$; since F is locally cofiltered, for every $\sigma \in \Sigma$ we may then find a covering family $(f'_{\sigma\lambda'} : B'_{\sigma\lambda'} \to B_\sigma \mid \lambda' \in \Lambda'_\sigma)$ and a morphism $h_{\sigma\lambda'} : Y_{\sigma\lambda'} \to Y_\sigma$ in $\mathscr{C}$ such that $Fh_{\sigma\lambda'} = f'_{\sigma\lambda'}$ and $g_{\sigma\lambda'} := g_\sigma \circ h_{\sigma\lambda'} = g'_\sigma \circ h_{\sigma\lambda'}$ for every $\lambda' \in \Lambda'_\sigma$. Set $t_{j\lambda\lambda'} := G_1(g_{j\lambda\lambda'})(t_j)$ for every $(j, \lambda) \in \Sigma$ and every $\lambda' \in \Lambda'_{(j,\lambda)}$. The compatible system $([Y_{\sigma\lambda'}, t_{\sigma\lambda'}] \mid \sigma \in \Sigma, \ \lambda' \in \Lambda'_\sigma)$ represents again the section s, and by construction we have $\varphi_{1Y_{\sigma\lambda'}}(t_{\sigma\lambda'}) = \varphi_{2Y_{\sigma\lambda'}}(t_{\sigma\lambda'})$ for every $\sigma \in \Sigma$ and every $\lambda' \in \Lambda'_\sigma$. This proves that s lies in the image of the map $E^a(B) \to E'(B)$, so the natural morphism $E^a \to E'$ is an epimorphism. To see that it is also a monomorphism, consider two sections s, s' of $E^a(B)$ whose images agree in $E'(B)$; we may find a covering family $(f_j : B_j \to B \mid i \in I)$ and compatible systems $([X_j, t_j] \mid j \in I)$, $([X'_j, t'_j] \mid j \in I)$ with $FX_j = FX'_j = B_j$ and $t_j \in G_1X_j$, $t'_j \in G_1X'_j$ for every $j \in I$. Arguing as in the foregoing, we may then assume that $[X_j, t_j] = [X'_j, t'_j]$ in $H_1(B_j)$ for every $j \in I$. Then by lemma 4.3.5 there exist for every $j \in I$ a covering family $(f_{j\lambda} : b_{j\lambda} \to B_j \mid j \in \Lambda_j)$ and morphims $g_{j\lambda} : Y_{j\lambda} \to X_j$, $g'_{j\lambda} : Y_{j\lambda} \to X'_j$ in $\mathscr{C}$ such that $Fg_{j\lambda} = Fg'_{j\lambda} = f_{j\lambda}$ and $t_{j\lambda} := (G_1 g_{j\lambda})(t_j) = (G_1 g'_{j\lambda})(t'_j)$ for every $\lambda \in \Lambda_j$. Clearly $\varphi_{1Y_{j\lambda}}(t_{j\lambda}) = \varphi_{2Y_{j\lambda}}(t_{j\lambda})$ for every $j \in I$ and every $\lambda \in \Lambda_j$; thus, both s and s' are represented by the compatible system $([Y_{j\lambda}, t_{j\lambda}] \mid j \in I, \ \lambda \in \Lambda_j)$, and the proof is concluded. ☐

4.3.8. Let $C := (\mathscr{C}, J)$ and $C' := (\mathscr{C}', J')$ be two sites, $g : \mathscr{C} \to \mathscr{C}'$ a given functor, and V a universe such that C and C' are V-small. We shall apply proposition 4.3.7 to the split fibration $\mathsf{s} : \mathscr{C}'/g\mathscr{C} \to \mathscr{C}'$ associated with g, with its canonical cleavage $\boldsymbol{\lambda}$ (see example 3.2.3). Let also $\mathsf{t} : \mathscr{C}'/g\mathscr{C} \to \mathscr{C}$ be the target functor (see (1.1.28)); with this notation, notice that

$$g_{\mathsf{V}!} = \int^{\boldsymbol{\lambda}} \circ \ \mathsf{t}^\wedge.$$

In light of proposition 4.3.7, it follows already that if s is locally cofiltered, then the functors $g^a_{\mathsf{V}!}$ and $\widetilde{g}^*_{\mathsf{V}} : C^\sim_{\mathsf{V}} \to C'^\sim_{\mathsf{V}}$ are exact. In fact, we have :

Proposition 4.3.9. *With the notation of* (4.3.8)*, the following conditions are equivalent :*

(a) *g is a morphism of sites $C' \to C$.*
(b) *The fibration s is locally cofiltered, and for every covering family $(f_i : X_i \to X \mid i \in I)$ for the topology J, the family $(g(f_i) : gX_i \to gX \mid i \in I)$ covers gX relative to J'.*

Proof. (a)⇒(b) : Since g is continuous, we know already that it transforms covering families for J into covering families for J' (lemma 4.2.4). Notice as well that $g^a_{\mathsf{V}!} = \widetilde{g}^*_{\mathsf{V}} \circ (-)^a$ (lemma 4.2.11(ii)); since $\widetilde{g}^*_{\mathsf{V}}$ is exact, the same holds then for $g^a_{\mathsf{V}!}$.

Let us check next that condition (a) of definition 4.3.3 holds for s. To this aim, consider the final object E of $C^\sim$ such that $E(X) = \{\varnothing\}$ for every $X \in \mathrm{Ob}(\mathscr{C})$. By assumption, $\widetilde{g}^*_{\mathsf{V}}(E)$ is the final object of $C'^\sim_{\mathsf{V}}$, *i.e.* $(\widetilde{g}^*_{\mathsf{V}}E)(X')$ is a set with one element for every $X' \in \mathrm{Ob}(\mathscr{C}')$. However, every section of $(\widetilde{g}^*_{\mathsf{V}}E)(X')$ is represented by the datum of a covering family $(X'_j \to X' \mid j \in I)$ in the topology J', and a system of sections $([X'_j \to gX_j, \varnothing] \mid j \in I)$; especially, $\mathrm{Ob}(\mathsf{s}^{-1}X'_j) \neq \varnothing$ for every $j \in I$, as required.

Next, let $l_i : X' \to gX_i$ for $i = 1, 2$ be two objects of $\mathsf{s}^{-1}X'$; in order to check condition (b) of definition 4.3.3 we need to exhibit a covering family $(f_j : X'_j \to X' \mid j \in I)$ and for every $j \in I$ an object $h_j : X'_j \to gY_j$ of $\mathsf{s}^{-1}X'_j$ and morphisms $k_{ij} : Y_j \to X_i$ in $\mathscr{C}$ that make commute the diagram

(4.3.10)
$$\begin{array}{ccccc} X' & \xleftarrow{f_j} & X'_j & \xrightarrow{h_j} & gY_j \\ & {\scriptstyle l_i}\searrow & & \swarrow{\scriptstyle g(k_{ij})} & \\ & & gX_i & & \end{array} \qquad \text{for } i = 1, 2.$$

To this aim, set $H_i := g_{\mathsf{V}!}(h_{X_i}) = h_{gX_i}$ for $i = 1, 2$, and $H := g_{\mathsf{V}!}(h_{X_1} \times h_{X_2})$. The pair (l_1, l_2) yields a section $l_\bullet$ of $(H_1 \times H_2)(X')$; on the other hand, since $g^a_{\mathsf{V}!}$ is exact, the natural morphism $\omega : H \to H_1 \times H_2$ induces an isomorphism on associated sheaves in $C'^\sim$. Especially, there exists a covering family $(f_j \mid j \in I)$ as sought, such that for every $j \in I$ the section $(H_1 \times H_2)(f_j)(l_\bullet) = (l_1 \circ f_j, l_2 \circ f_j)$ agrees with the image under ω of a section $[h_j : X'_j \to gY_j, (k_{1j}, k_{2j})]$ of $H(X'_j)$. Unwinding the definition, we obtain precisely a diagram (4.3.10).

Lastly, let $l_i : X' \to gX_i$ for $i = 1, 2$ be two objects of $\mathsf{s}^{-1}X'$, and $t_1, t_2 : X_1 \to X_2$ two morphisms in $\mathscr{C}$ such that $g(t_1) \circ l_1 = l_2 = g(t_2) \circ l_1$; in order to verify condition (c) of definition 4.3.3 we need to exhibit a covering family $(f_j : X'_j \to X' \mid j \in I)$ and for every $j \in I$ an object $h_j : X'_j \to gY_j$ of $\mathsf{s}^{-1}X'_j$ with a morphism $k_j : Y_j \to X_1$ such that

(4.3.11)
$$l_1 \circ f_j = g(k_j) \circ h_j \qquad \text{and} \qquad t_1 \circ k_j = t_2 \circ k_j.$$

To this aim, let $t_{i*} : h_{X_1} \to h_{X_2}$ be the morphism of presheaves on $\mathscr{C}$ induced by t_i, for $i = 1, 2$; we set $H_i := g_{\mathsf{V}!}(h_{X_i}) = h_{gX_i}$ for $i = 1, 2$, and denote by E the equalizer of t_{1*} and t_{2*}, and by E' the equalizer of $g_{\mathsf{V}!}(t_{1*}), g_{\mathsf{V}!}(t_{2*}) : H_1 \to H_2$. Since $g^a_{\mathsf{V}!}$ is exact, the natural morphism $\omega : g_{\mathsf{V}!}E \to E'$ induces an isomorphism on associated sheaves in $C^\sim$. On the other hand, l_1 defines a section of $E'(X')$; it follows that there exists a covering family $(f_j : X'_j \to X' \mid j \in I)$ such that for every $j \in I$ the section $(E'f_j)(l_1)$ agrees with the image under ω of a section $[h_j : X'_j \to gY_j, k_j]$ of $(g_{\mathsf{V}!}E')(X'_j)$. Unwinding the definitions, we obtain the identities (4.3.11).

(b)⇒(a): We have already noticed that if s is locally cofiltered, the functor $\widetilde{g}^*_{\mathsf{V}}$ is exact. In order to check that g is continuous, let us consider any sheaf F on $\mathscr{C}'$, any $X \in \mathrm{Ob}(\mathscr{C})$, and any covering subobject $R \subset h_X$ for the topology J. We need to show that the natural map

$$F(gX) \simeq \mathrm{Hom}_{\mathscr{C}^\wedge_{\mathsf{V}}}(h_X, g^*_{\mathsf{V}}F) \to \mathrm{Hom}_{\mathscr{C}^\wedge_{\mathsf{V}}}(R, g^*_{\mathsf{V}}F)$$

is bijective. By adjunction, the latter is naturally identified with the induced map

$$\mathrm{Hom}_{\mathscr{C}'^\wedge_{\mathsf{V}}}(g_{\mathsf{V}!}(h_X), F) \to \mathrm{Hom}_{\mathscr{C}'^\wedge_{\mathsf{V}}}(g_{\mathsf{V}!}(R), F)$$

so we come down to checking that the morphism $\varphi : (g_{\mathsf{V}!}R)^a \to (g_{\mathsf{V}!}h_X)^a = h^a_{gX}$ induced by the inclusion $R \to h_X$ is an isomorphism. However, we know already that the functor $g^a_{\mathsf{V}!}$ is exact, so φ is a monomorphism. Next, let $(f_j : X_j \to X \mid j \in I)$ be a V-small family of generators for the covering sieve of X corresponding to R; there follows a covering morphism

$$S := \coprod_{j \in I} h_{X_j} \to R \to h_X \qquad \text{in } \mathscr{C}^\wedge$$

whose image under $g_{\mathsf{V}!}$ is still a covering morphism : indeed, $g_{\mathsf{V}!}$ commutes with coproducts since it is a left adjoint, so $g_{\mathsf{V}!}(S)$ is the coproduct of the family $(h_{gX_j} \mid j \in I)$, and by assumption g transforms covering families for the topology J into covering families for the topology J'. Thus, the morphism $g_{\mathsf{V}!}R \to g_{\mathsf{V}!}h_X$ induces an epimorphism as well on associated sheaves, whence the contention. □

Example 4.3.12. In the situation of (4.3.8), suppose that C is a lex-site and g is left exact. Then g is a morphism of sites if and only if for every covering family $(f_i : X_i \to X \mid i \in I)$ for the topology J, the family $(g(f_i) : gX_i \to gX \mid i \in I)$ covers gX relative to J'. Indeed, under this assumption, the category $X/g\mathscr{C}$ is cofiltered for every $X \in \mathrm{Ob}(\mathscr{C}')$ (example 1.3.16(i)), so the source fibration $\mathsf{s} : \mathscr{C}'/g\mathscr{C} \to \mathscr{C}'$ is trivially locally cofiltered, and the assertion follows from proposition 4.3.9.

Theorem 4.3.13. *Let $(\mathscr{C}, J)$ be a U-site, $\mathscr{C}'$ a category, $g : \mathscr{C}' \to \mathscr{C}$ a functor, and endow $\mathscr{C}'$ with the topology J' induced by g. Suppose that the following conditions hold :*

- (a) *For every $X \in \mathrm{Ob}(\mathscr{C})$ there exists a family $(gY_i \to X \mid i \in I)$ of objects of $g\mathscr{C}'/X$ that covers X in the topology J.*
- (b) *For every $Y, Y' \in \mathrm{Ob}(\mathscr{C}')$ and every morphism $\varphi : gY \to gY'$ in $\mathscr{C}$ there exist :*
  - (b.i) *a family $(\psi_i : Z_i \to Y \mid i \in I)$ of objects of $\mathscr{C}'/Y$ such that the family $(g(\psi_i) : gZ_i \to gY \mid i \in I)$ covers gY in the topology J*
  - (b.ii) *for each $i \in I$ a morphism $\nu_i : Z_i \to Y'$ such that $\varphi \circ g(\psi_i) = g(\nu_i)$.*
- (c) *For every pair of morphisms $\varphi, \varphi' : Y \to Y'$ in $\mathscr{C}'$ such that $g(\varphi) = g(\varphi')$ there exists a family $(\psi_i : Z_i \to Y \mid i \in I)$ of objects of $\mathscr{C}'/Y$ such that the family $(g(\psi_i) : gZ_i \to gY \mid i \in I)$ covers gY in the topology J, and $\varphi \circ \psi_i = \varphi' \circ \psi_i$ for every $i \in I$.*

Then we have :

- (i) *g is cocontinuous for the topologies J, J' and is a morphism of sites $(\mathscr{C}, J) \to (\mathscr{C}', J')$.*
- (ii) *g induces an equivalence $\widetilde{g}_* : (\mathscr{C}, J)^\sim \xrightarrow{\sim} (\mathscr{C}', J')^\sim$.*
- (iii) *A family $(\psi_i : Y_i \to Y \mid i \in I)$ of morphisms of $\mathscr{C}'$ generates a covering sieve of J' if and only if $(g(\psi_i) : gY_i \to gY \mid i \in I)$ generates a covering sieve of J.*

Proof. Suppose first that both $\mathscr{C}$ and $\mathscr{C}'$ are small. Let $\mathsf{s} : \mathscr{C}/g\mathscr{C}' \to \mathscr{C}$ be the fibration associated with g, as in (4.3.8); we remark :

Claim 4.3.14. The fibration s is locally cofiltered.

Proof of the claim. To check condition (a) of definition 4.3.3, let $X \in \mathrm{Ob}(\mathscr{C})$; we need to find a covering family $(X_j \to X \mid j \in I)$ for the topology J, such that $\mathrm{Ob}(\mathsf{s}^{-1}X_j) \neq \varnothing$ for every $j \in I$. However, by condition (a) of the theorem we have a covering family $(gY_j \to X \mid j \in I)$ for X; we may then choose $X_j := gY_j$, since then $\mathbf{1}_{gY_j} \in \mathrm{Ob}(\mathsf{s}^{-1}X_j)$ for every $j \in I$.

Next, we check condition (b) of definition 4.3.3 : consider any $X \in \mathrm{Ob}(\mathscr{C})$ and morphisms $\varphi_i : X \to gY_i$ in $\mathscr{C}'$ for $i = 1, 2$; we need to find a covering family $(f_j : X_j \to X \mid j \in I)$ such that for every $j \in I$ there exist a morphism $\psi_j : X_j \to gZ_j$ in $\mathscr{C}$ and morphisms $\nu_{ij} : Z_j \to Y_i$ with $g(\nu_{ij}) \circ \psi_j = \varphi_i \circ f_j$ for $i = 1, 2$. To this aim, we use first condition (a) of the theorem to find a covering family $f'_\bullet := (f'_j : gY'_j \to X \mid j \in I')$; then, by condition (b) of the theorem we find for every $j \in I'$ a family $(f'_{j\lambda} : Y'_{j\lambda} \to Y'_j \mid \lambda \in \Lambda_j)$ such that $(g(f'_{j\lambda}) \mid \lambda \in \Lambda_j)$ covers gY'_j, and such that for every $\lambda \in \Lambda_j$ there exists a morphism $h_{j\lambda} : Y'_{j\lambda} \to Y_1$ with $g(h_{j\lambda}) = \varphi_1 \circ f'_j \circ g(f'_{j\lambda})$. We may then replace $f'_\bullet$ by the family $(f'_j \circ g(f'_{j\lambda}) : gY'_{j\lambda} \to X \mid j \in I', \lambda \in \Lambda_j)$, and assume from start that for every $j \in I'$ there exists a morphism $h_j : Y'_j \to Y_1$ such that $g(h_j) = \varphi_1 \circ f'_j$. Next, we apply again condition (b) of the theorem to find for every $j \in I'$ a family $(f''_{j\lambda'} : Y''_{j\lambda'} \to Y'_j \mid \lambda' \in \Lambda'_j)$ such that $(g(f''_{j\lambda'}) \mid \lambda' \in \Lambda'_j)$ covers gY'_j, and such that for every $\lambda' \in \Lambda'_j$ there exists a morphism $h'_{j\lambda'} : Y''_{j\lambda'} \to Y_2$ with $g(h'_{j\lambda'}) = \varphi_2 \circ f'_j \circ g(f''_{j\lambda'})$. We may then further replace $f'_\bullet$ by the family $(f'_j \circ g(f''_{j\lambda'}) : gY''_{j\lambda'} \to X \mid j \in I', \lambda' \in \Lambda'_j)$, and the system $(h_j \mid j \in I')$ by the system $(h_j \circ f''_{j\lambda'} : Y''_{j\lambda'} \to Y_1 \mid j \in I',\ \lambda' \in \Lambda'_j)$, and assume from start that there exists as well for every $j \in I'$ a morphism $h'_j : Y'_j \to Y_2$ such that $g(h'_j) = \varphi_2 \circ f'_j$. Then we set $X_j := gY'_j$, $Z_j := Y'_j$, $\psi_j := \mathbf{1}_{X_j}$, $\nu_{1j} := h_j$ and $\nu_{2j} := h'_j$ for every $j \in I'$; clearly the resulting family $f_\bullet$ and systems of morphisms $\psi_\bullet$, $\nu_{\bullet\bullet}$ will do.

Lastly, we check condition (c) of definition 4.3.3 : let $X \in \mathrm{Ob}(\mathscr{C})$ and consider morphisms $\varphi : X \to gY_1$ in $\mathscr{C}$ and $\psi, \psi' : Y_1 \to Y_2$ in $\mathscr{C}'$ such that $g(\psi) \circ \varphi = g(\psi') \circ \varphi$; we need to find a covering family $(f_j : X_j \to X \mid j \in I)$ and for every $j \in I$ morphisms $h_j : X_j \to gZ_j$ in $\mathscr{C}$ and $\nu_j : Z_j \to Y_1$ in $\mathscr{C}'$ such that $\varphi \circ f_j = g(\nu_j) \circ h_j$ and $\psi \circ \nu_j = \psi' \circ \nu_j$. To this aim, we use conditions (a) and (b) of the theorem to get first a covering family $f'_\bullet := (f'_j : gY_j \to X \mid j \in I)$, and then for every $j \in I$ a family $(f''_{j\lambda} : Y_{j\lambda} \to Y_j \mid \lambda \in \Lambda_j)$ such that $(g(f''_{j\lambda}) \mid \lambda \in \Lambda_j)$ covers gY_j, and such that for every $\lambda \in \Lambda_j$ there exists a morphism $\nu'_{j\lambda} : Y_{j\lambda} \to Y_1$ with $g(\nu'_{j\lambda}) = \varphi \circ f'_j \circ g(f''_{j\lambda})$. After replacing $f'_\bullet$ by the family $(f'_j \circ g(f''_{j\lambda}) \mid j \in I,\ \lambda \in \Lambda_j)$, we may assume that for every $j \in I$ there exists a morphism $\nu'_j : Y_j \to Y_1$ such that $g(\nu'_j) = \varphi \circ f'_j$. Especially, notice that $g(\psi \circ \nu'_j) = g(\psi' \circ \nu'_j)$ for every $j \in I$. By condition (c) of the theorem, there exists therefore for every $j \in I$ a family $(\nu'_{j\lambda'} : Y'_{j\lambda'} \to Y_j \mid \lambda' \in \Lambda'_j)$ such that $(g(\nu'_{j\lambda'}) \mid \lambda' \in \Lambda'_j)$ covers gY_j and $\psi \circ \nu'_j \circ \nu'_{j\lambda'} = \psi' \circ \nu'_j \circ \nu_{j\lambda}$ for every $\lambda' \in \Lambda_j$. Set $I := \bigcup_{j \in I} \{j\} \times \Lambda_j$; for every $(j, \lambda') \in I$ we let $X_{j\lambda'} := gY_{j\lambda'}$, $f_{j\lambda'} := f'_j \circ g(\nu'_{j\lambda'})$, $h_{j\lambda'} := \mathbf{1}_{X_{j\lambda'}}$ and $\nu_{j\lambda'} := \nu'_j \circ \nu'_{j\lambda'}$. Clearly the resulting covering family $f_{\bullet\bullet}$ and systems of morphisms $h_{\bullet\bullet}$, $\nu_{\bullet\bullet}$ will do. ◊

(iii) follows from claim 4.3.14, lemmata 4.2.17(ii) and 4.2.4, and proposition 4.3.9.

(i): The continuity of g holds by definition of J'; then g is a morphism of sites, by claim 4.3.14 and proposition 4.3.9. To check the cocontinuity, let $Y \in \mathrm{Ob}(\mathscr{C}')$ and $f_\bullet := (f_j : X_j \to gY \mid j \in I)$ a family of morphisms in $\mathscr{C}$ generating a covering sieve $\mathscr{S}$ of gY; we need to show that $g^{-1}_{|Y}\mathscr{S}$ is a covering sieve of Y for the topology J'. However, by (a) we may find for every $j \in I$ a family $(f_{j\lambda} : gY_{j\lambda} \to X_j \mid \lambda \in \Lambda_j)$ covering X_j in the topology J; let $\mathscr{S}' \subset \mathscr{S}$ be the sieve generated by the system $(f'_{j\lambda} := f_j \circ f_{j\lambda} : gY_{j\lambda} \to gY \mid j \in I,\ \lambda \in \Lambda_j)$. Clearly $\mathscr{S}'$ covers gY in the topology J, and by remark 4.1.3(ii) it suffices to check that $g^{-1}_{|Y}\mathscr{S}'$ is a covering sieve for Y, so we may replace $f_\bullet$ by the family $f'_{\bullet\bullet}$, and assume from start that for every $j \in I$ we have $X_j = gY_j$ for some $Y_j \in \mathrm{Ob}(\mathscr{C}')$. By condition (b), we may then find for every $j \in I$ a family $(h_{j\lambda} : Y_{j\lambda} \to Y_j \mid \lambda \in \Lambda_j)$ of morphisms in $\mathscr{C}'$ such that the family $(g(h_{j\lambda}) \mid \lambda \in \Lambda_j)$ covers gY_j in the topology J, and for every $j \in I$ and $\lambda \in \Lambda_j$ a morphism $k_{j\lambda} : Y_{j\lambda} \to Y$ in $\mathscr{C}'$ such that $f''_{j\lambda} := f_j \circ g(h_{j\lambda}) = g(k_{j\lambda})$. Let $\mathscr{S}'' \subset \mathscr{S}$ be the sieve generated by the system $(f''_{j\lambda} : gY_{j\lambda} \to gY \mid j \in I,\ \lambda \in \Lambda_j)$. Then $\mathscr{S}''$ covers gY in the topology J, and it suffices to check that $\mathscr{T} := g^{-1}_{|Y}\mathscr{S}''$ covers Y in the topology J'. However, $\mathscr{T}$ contains the family $(k_{j\lambda} \mid j \in I,\ \lambda \in \Lambda_j)$, which covers Y, by virtue of (iii). The assertion follows.

Claim 4.3.15. (i) Let $\varphi : G \to G'$ be a morphism of presheaves on $\mathscr{C}$ such that $g^\wedge(\varphi) : g^\wedge G \to g^\wedge G'$ is an isomorphism in $\mathscr{C}'^\wedge$. Then φ is a bicovering morphism.

(ii) Let (η', ε') be a unit and counit for the adjoint pair $(\widetilde{g}^*, \widetilde{g}_*)$. In order to prove assertion (ii) of the theorem, it suffices to check that $\eta' : \mathbf{1}_{(\mathscr{C}', J')^\sim} \Rightarrow \widetilde{g}_*\widetilde{g}^*$ is an isomorphism of functors.

Proof of the claim. (i): Indeed, let $X \in \mathrm{Ob}(\mathscr{C})$ and $s \in G'X$; by condition (a) there exists a covering family $(f_i : gY_i \to X \mid i \in I)$, and by assumption $(G'f_i)(s) \in \mathrm{Im}(\varphi_{gY_i})$ for every $i \in I$, so φ is a covering morphism, by remark 4.1.29(ii). Likewise, let $s, s' \in GX$ such that $t := \varphi_X(s) = \varphi_X(s')$; it follows that $\varphi_{gY_i}((Gf_i)(s)) = \varphi_{gY_i}((Gf_i)(s')) = (G'f_i)(t)$, whence $(Gf_i)(s) = (Gf_i)(s')$ for every $i \in I$. In light of remark 4.1.29(iii), the assertion follows.

(ii): Indeed, if η' is an isomorphism, the triangular identities of (1.1.13) show that $g^\wedge(\varepsilon'_G)$ will also be an isomorphism for every sheaf G on $(\mathscr{C}, J)$, hence ε'_G will be an isomorphism, by (i), and to conclude it will suffice to invoke proposition 1.1.20(i,iii). ◊

(ii): From (i) and lemma 4.2.15(ii) we get an isomorphism of functors $\mathscr{C}^\wedge \to (\mathscr{C}', J')^\sim$

$$\omega : (-)^a \circ g^\wedge \stackrel{\sim}{\to} \widetilde{g}_* \circ (-)^a.$$

Let now $\eta : \mathbf{1}_{\mathscr{C}'^\wedge} \Rightarrow g^\wedge \circ g_!$ and $\varepsilon : g_! \circ g^\wedge \Rightarrow \mathbf{1}_{\mathscr{C}^\wedge}$ be the unit and counit of the natural adjunction for the pair of functors $(g_!, g^\wedge)$, as in remark 1.3.6(iii). We consider the natural transformations of functors on $(\mathscr{C}', J')^\sim$ and respectively on $(\mathscr{C}, J)^\sim$

$$\eta'_F : F \xrightarrow{(\eta_F)^a} (g^\wedge g_! F)^a \xrightarrow{\omega_{g_! F}} (g^\wedge \circ g^a_!)(F) \qquad \varepsilon'_G : (g^a_! \circ g^\wedge)(G) \xrightarrow{(\varepsilon_F)^a} G.$$

A little diagram chase shows that the pair (η', ε') fulfills the triangular identities of (1.1.13), so these are the unit and counit of an adjunction for the pair $(\widetilde{g}^*, \widetilde{g}_*)$. By claim 4.3.15(ii), we are thus reduced to showing that η_F is a bicovering morphisms, for every presheaf F on $\mathscr{C}'$.

We check first that η_F is a covering morphism : thus, let $X \in \mathrm{Ob}(\mathscr{C}'^\wedge)$, and $s \in (g^\wedge g_! F)(X)$; then s is a class $[\varphi : gX \to gY, \sigma]$, where φ is a morphism in $\mathscr{C}$, and $\sigma \in FY$. By condition (b) there exists a family $(f_j : X_j \to X \mid j \in I)$ of morphisms of $\mathscr{C}'$ such that $(\varphi(f_j) \mid j \in I)$ is a covering family for the topology J, and such that for every $j \in I$ there exists a morphism $h_j : X_i \to Y$ with $g(h_j) = \varphi \circ g(f_j)$. Set $\sigma_j := (Fh_j)(\sigma) \in FX_i$ for every $j \in I$; then $\eta_{F,X_j}(\sigma_j)$ is the class $[\mathbf{1}_{gX_j}, \sigma_j]$ in $(g^\wedge g_! F)(X_j)$ for every $j \in I$. On the other hand, $(g^\wedge g_! F)(f_j)(s) = [\varphi \circ g(f_j) : gX_j \to gY, \sigma]$ for every $j \in I$. According to remark 4.1.29(ii), it suffices then to show that $[\mathbf{1}_{gX_j}, \sigma_j] = [\varphi \circ g(f_j), \sigma]$ in $(g^\wedge g_! F)(X_j)$, for every $j \in I$. But h_j yields a morphism $gX_j/h_j : (X_j, \mathbf{1}_{gX_j}) \to (Y, \varphi \circ g(f_j))$ in $gX_j/g\mathscr{C}'$ whence the assertion.

Next, let $t, t' \in FX$ be two sections such that $\eta_{F,X}(t) = \eta_{F,X}(t')$, *i.e.* $[\mathbf{1}_{gX}, t] = [\mathbf{1}_{gX}, t']$ in $(g_! F)(gX)$. According to lemma 4.3.5, there exist a covering family $(f_j : gX_j \to gX \mid j \in I)$ for the topology J, and morphisms $\varphi_j : gX_j \to gY_j$ in $\mathscr{C}$ and $h_j h'_j : Y_j \to X$ in $\mathscr{C}'$ such that

$$g(h_j) \circ \varphi_j = g(h'_j) \circ \varphi_j = f_j \qquad \text{and} \qquad (Fh_j)(t) = (Fh'_j)(t') \qquad \text{for each } j \in I.$$

Then, by condition (b), we may find for every $j \in I$ a family $(f'_{j\lambda} : X_{j\lambda} \to X_j \mid \lambda \in \Lambda_j)$ such that $(g(f'_{j\lambda}) \mid \lambda \in \Lambda_j)$ is a covering family for the topology J, and such that for every $\lambda \in \Lambda_j$ there exists a morphism $\nu_{j\lambda} : X_{j\lambda} \to Y_j$ with $g(\nu_{j\lambda}) = \varphi_j \circ g(f'_{j\lambda})$. After replacing $(f_j \mid j \in \Lambda)$ by the covering family $(f_j \circ g(f'_{j\lambda}) : X_{j\lambda} \to X \mid j \in I,\ \lambda \in \Lambda)$, and $(\varphi_j \mid j \in J)$ by the system of morphisms $(\varphi_j \circ g(f'_{j\lambda}) \mid j \in I,\ \lambda \in \Lambda_j)$, we may then assume that for every $j \in I$ there exists a morphism $\nu_j : X_j \to Y_j$ such that $\varphi_j = g(\nu_j)$, in which case we have $g(h_j \circ \nu_j) = g(h'_j \circ \nu_j) = f_j$ for every $j \in I$. Then, by condition (c) there exists for every $j \in I$ a family $(f''_{j\lambda'} : X'_{j\lambda'} \to X_j \mid \lambda' \in \Lambda'_j)$ of morphisms in $\mathscr{C}'$ such that $(g(f''_{j\lambda'}) \mid \lambda' \in \Lambda'_j)$ is a covering family for the topology J, and such that $h_j \circ \nu_j \circ f''_{j\lambda'} = h'_j \circ \nu_j \circ f''_j$. After replacing $(f_j \mid j \in \Lambda)$ by the covering family $(f_j \circ g(f''_{j\lambda'}) : X_{j\lambda'} \to X \mid j \in I,\ \lambda' \in \Lambda'_j)$ and $(\nu_j \mid j \in I)$ by the system $(\nu_j \circ f''_{j\lambda'} \mid j \in I,\ \lambda' \in \Lambda'_j)$, we may then assume that $\mu_j := h_j \circ \nu_j = h'_j \circ \nu_j$ for every $j \in I$. It follows that $(F\mu_j)(t) = (F\mu_j)(t')$ for every $j \in I$; lastly, the family $(\mu_j : X_j \to X \mid j \in I)$ covers X in the topology J', by virtue of (iii). In view of remark 4.1.29(iii), we conclude that η_F is a bicovering morphism, as stated.

This completes the proof of the theorem in case $\mathscr{C}$ and $\mathscr{C}'$ are small. Lastly, consider the case where $(\mathscr{C}, J)$ is a U-site and $\mathscr{C}'$ is a general category, and pick a universe V such that $\mathscr{C}$ and $\mathscr{C}'$ are V-small. The theorem then applies to the functor $\widetilde{g}_{\mathsf{V}*}$, so the latter is an equivalence, and we also get assertions (i) and (iii), which are independent of the universe U. It remains therefore only to check that $\widetilde{g}_{\mathsf{U}*}$ is an equivalence, and taking into account the commutative diagram

$$\begin{array}{ccc} (\mathscr{C}, J)^\sim_{\mathsf{U}} & \xrightarrow{\widetilde{g}_{\mathsf{U}*}} & (\mathscr{C}', J')^\sim_{\mathsf{U}} \\ \downarrow & & \downarrow \\ (\mathscr{C}, J)^\sim_{\mathsf{V}} & \xrightarrow{\widetilde{g}_{\mathsf{V}*}} & (\mathscr{C}', J')^\sim_{\mathsf{V}} \end{array}$$

(whose vertical arrows are the fully faithful inclusion functors), we already see that $\widetilde{g}_{\mathsf{U}*}$ is fully faithful. We notice :

Claim 4.3.16. For every covering family $(f_i : X_i \to X \mid i \in I)$ in the topology J there exists an essentially small subset $I' \subset I$ such that $(f_i : X_i \to X \mid i \in I')$ is still a covering family.

Proof of the claim. Pick an essentially small topologically generating family $G \subset \mathrm{Ob}(\mathscr{C})$ for the site $(\mathscr{C}, J)$; then for every $i \in I$ there exists a covering family $(f_{i\lambda} : X_{i\lambda} \to X_i \mid \lambda \in \Lambda_i)$ with $X_{i\lambda} \in G$ for every $\lambda \in \Lambda_i$. Set $\Lambda := \bigcup_{i \in I} \{i\} \times \Lambda_i$; the family $(f'_{i\lambda} := f_i \circ f_{i\lambda} : X_{i\lambda} \to X \mid (i, \lambda) \in \Lambda)$ covers X in the topology J. Since $\mathrm{Hom}_{\mathscr{C}}(X_{i\lambda}, X)$ is a small set for every $(i, \lambda) \in \Lambda$, we may then find an essentially small subset $\Lambda' \subset \Lambda$ such that $(f'_{i\lambda} \mid (i, \lambda) \in \Lambda')$ is still covering; then we can take for $I' \subset I$ the image of Λ' under the projection $\Lambda \to I$. ◊

Now, let F be any U-sheaf on $(\mathscr{C}', J')$; we know already that there exists a V-sheaf G on $(\mathscr{C}, J)$ such that $\widetilde{g}_{*\mathsf{V}} G$ is isomorphic to F, and we need to check that GX is essentially small for every $X \in \mathrm{Ob}(\mathscr{C})$. But by condition (a) we may find a covering family $(gY_i \to X \mid i \in I)$, and by claim 4.3.16 we may assume that I is small. Then the induced map $GX \to \prod_{i \in I} G(gY_i)$ is injective, and $G(gY_i) \overset{\sim}{\to} FY_i$ is essentially small for every $i \in I$, whence the contention. □

Remark 4.3.17. (i) Notice that if g is full (resp. faithful), then condition (b) (resp. (c)) of theorem 4.3.13 is trivially satisfied.

(ii) It follows easily that theorem 4.3.13 generalizes [17, §2.1, Prop.], which in turn generalizes the implication (i)⇒(ii) in [8, Exp.III, Th.4.1]. One can also show that conditions (a), (b) and (c) of the theorem are implied by conditions $(\mathscr{L}0)$, $(\mathscr{L}1)$ and $(\mathscr{L}2)$ of [9, Exp.V, Déf.8.1.1], hence theorem 4.3.13 generalizes as well [9, Exp.V, Prop.8.1.12].

(iii) A necessary and sufficient condition for a morphism of sites to induce an equivalence of topoi is given in [46, Th.4.7].

(iv) In the situation of theorem 4.3.13, one can show that $(\mathscr{C}', J')$ does not necessarily have an essentially small topological generating family, and does not necessarily have essentially small Hom-sets, but every essentially small subcategory $\mathscr{B} \subset \mathscr{C}'$ is contained in an essentially small subcategory $\mathscr{B}' \subset \mathscr{C}'$ such that the restriction $\mathscr{B}' \to \mathscr{C}$ of g still satisfies conditions (a)–(c) of theorem 4.3.13.

We may now generalize lemma 4.2.11 to any continuous functor between U-sites. Indeed, we notice the following further application of theorem 4.3.13, which appears in [8, Exp.III, Th.4.1].

Proposition 4.3.18. *Let $C := (\mathscr{C}, J)$ be a U-site, G a topologically generating family for C. Denote by $\mathscr{G}$ the full subcategory of $\mathscr{C}$ with $\mathrm{Ob}(\mathscr{G}) = G$, and endow $\mathscr{G}$ with the topology $J_{\mathscr{G}}$ induced by J via the inclusion functor $u : \mathscr{G} \to \mathscr{C}$. Then :*

(i) *u is cocontinuous for the topologies $J, J_{\mathscr{G}}$ and is a morphism of sites $C \to (\mathscr{G}, J_{\mathscr{J}})$.*
(ii) *The induced functor $\widetilde{u}_* : C^\sim \to (\mathscr{G}, J_{\mathscr{G}})^\sim$ is an equivalence.*
(iii) *A family $(\psi_i : Y_i \to Y \mid i \in I)$ of morphisms of $\mathscr{G}$ generates a covering sieve of $J_{\mathscr{G}}$ if and only if it generates a covering sieve of J.*
(iv) *$(\mathscr{G}, J_{\mathscr{G}})$ is a U-site.*

Proof. (i),(ii),(iii): We apply the criterion of theorem 4.3.13, and taking into account remark 4.3.17(i) we are reduced to checking condition (a) of the theorem. The latter holds by definition of topologically generating family.

(iv): Pick an essentially small topologically generating family $H \subset \mathrm{Ob}(\mathscr{C})$ for the site C. With claim 4.3.16, we get an essentially small subset $G' \subset G$ such that every $X \in H$ has a covering family $(X_i \to X \mid i \in I)$ in the topology J, with $X_i \in G'$ for every $i \in I$. By (iii) and remark 4.1.3(iii), G' is then a topologically generating family for both C and $(\mathscr{G}, J_{\mathscr{G}})$. □

Corollary 4.3.19. *Let $C := (\mathscr{C}, J)$ and $C' := (\mathscr{C}', J')$ be two U-sites, $g : \mathscr{C} \to \mathscr{C}'$ a functor, and V, V' a pair of universes with $\mathsf{U} \subset \mathsf{V} \subset \mathsf{V}'$. We have :*

(i) *If g is continuous, the following holds :*

(a) *The functor $\widetilde{g}_{\mathsf{V}*} : C'^{\sim}_{\mathsf{V}} \to C^{\sim}_{\mathsf{V}}$ admits a left adjoint $\widetilde{g}^*_{\mathsf{V}} : C^{\sim}_{\mathsf{V}} \to C'^{\sim}_{\mathsf{V}}$.*

(b) *There are essentially commutative diagrams of categories :*

$$\begin{array}{ccc} C^{\sim}_{\mathsf{V}} & \xrightarrow{\widetilde{g}^*_{\mathsf{V}}} & C'^{\sim}_{\mathsf{V}} \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i'} \\ C^{\sim}_{\mathsf{V}'} & \xrightarrow{\widetilde{g}^*_{\mathsf{V}'}} & C'^{\sim}_{\mathsf{V}'} \end{array} \qquad \begin{array}{ccc} \mathscr{C} & \xrightarrow{g} & \mathscr{C}' \\ {\scriptstyle h^a_{\mathscr{C}}}\downarrow & & \downarrow{\scriptstyle h^a_{\mathscr{C}'}} \\ C^{\sim}_{\mathsf{U}} & \xrightarrow{\widetilde{g}^*_{\mathsf{U}}} & C'^{\sim}_{\mathsf{U}} \end{array}$$

where i and i' are the inclusion functors.

(c) *Suppose moreover that $\mathscr{C}$ is finitely complete and g is left exact. Then $\widetilde{g}^*_{\mathsf{V}}$ is exact.*

(ii) *If g is cocontinuous, the following holds :*

(a) *The functor $\breve{g}^*_{\mathsf{V}} : C'^{\sim}_{\mathsf{V}} \to C^{\sim}_{\mathsf{V}}$ admits a right adjoint $\breve{g}_{\mathsf{V}*} : C^{\sim}_{\mathsf{V}} \to C'^{\sim}_{\mathsf{V}}$.*

(b) *There is an essentially commutative diagram of categories :*

$$\begin{array}{ccc} C^{\sim}_{\mathsf{V}} & \xrightarrow{\breve{g}_{\mathsf{V}*}} & C'^{\sim}_{\mathsf{V}} \\ \downarrow & & \downarrow \\ C^{\sim}_{\mathsf{V}'} & \xrightarrow{\breve{g}_{\mathsf{V}'*}} & C'^{\sim}_{\mathsf{V}'} \end{array}$$

whose vertical arrows are the inclusion functors.

(iii) *If g is continuous and cocontinuous, we have natural isomorphisms of functors :*

$$\widetilde{g}_{\mathsf{V}*} \xrightarrow{\sim} \breve{g}^*_{\mathsf{V}} \;:\; C'^{\sim}_{\mathsf{V}} \to C^{\sim}_{\mathsf{V}} \qquad\qquad (-)^a \circ g^{\wedge}_{\mathsf{V}} \xrightarrow{\sim} \widetilde{g}_{\mathsf{V}*} \circ (-)^a \;:\; \mathscr{C}'^{\wedge}_{\mathsf{V}} \to C^{\sim}_{\mathsf{V}}.$$

Proof. (i.a): We choose an essentially small topologically generating family G for C, and define the site $(\mathscr{G}, J_{\mathscr{G}})$ and the continuous functor $u : (\mathscr{G}, J_{\mathscr{G}}) \to C$ as in proposition 4.3.18. By applying lemma 4.2.11(i) to the continuous functor $v := g \circ u$, we deduce that $\widetilde{v}_{\mathsf{V}*} = \widetilde{u}_{\mathsf{V}*} \circ \widetilde{g}_{\mathsf{V}*}$ admits a left adjoint. Then the assertion follows from proposition 4.3.18(ii).

(i.b): More precisely, $\widetilde{g}^*_{\mathsf{V}} = \widetilde{v}^*_{\mathsf{V}} \circ \widetilde{u}_{\mathsf{V}*}$. Thus, the essential commutativity of the left diagram follows from (4.2.13). Likewise, the essential commutativity of the right diagram follows from that of the left diagram (with $\mathsf{V} := \mathsf{U}$ and V' large enough so that $\mathscr{C}$ is V'-small) together with lemma 4.2.11(ii).

(i.c) holds by example 4.3.12.

(ii.a): Let u and h be as in the foregoing; from proposition 4.3.18(i) we deduce that h is cocontinuous, hence $\breve{h}^*_{\mathsf{V}} = \breve{u}^*_{\mathsf{V}} \circ \breve{g}^*_{\mathsf{V}}$ admits a right adjoint; however $\breve{u}^*_{\mathsf{V}} = \widetilde{u}_{\mathsf{V}*}$ (lemma 4.2.15(i)), and the latter is an equivalence (proposition 4.3.18(ii)), whence the contention.

(ii.b): More precisely, $\breve{g}_{\mathsf{V}*} = \breve{h}_{\mathsf{V}*} \circ \widetilde{u}^*_{\mathsf{V}}$, hence the assertion follows from (4.2.13).

(iii): In view of (i.b) and (ii.b), in order to prove the assertion, we may assume that $\mathscr{C}$ and $\mathscr{C}'$ are V-small, and this case is already covered by lemma 4.2.15(i,ii). □

The last result of this section will show the representability of the 2-limit of any (small) cofiltered system of lex-sites. The proof shall use the following :

Lemma 4.3.20. *Let I be a small filtered category, U' a universe, and*

$$\mathscr{C}_\bullet : I \to \mathsf{U}'\text{-}\mathbf{Cat} \qquad i \mapsto \mathscr{C}_i \qquad (\varphi : i \to j) \mapsto (\mathscr{C}_\varphi : \mathscr{C}_i \to \mathscr{C}_j)$$

any pseudo-functor such that $\mathscr{C}_i$ is finitely complete for every $i \in \mathrm{Ob}(I)$ and $\mathscr{C}_\varphi$ is a left exact functor for every morphism φ of I. Let also $(\mathscr{C}, \pi_\bullet)$ be a 2-colimit of $\mathscr{C}_\bullet$. Then $\mathscr{C}$ is finitely complete and $\pi_i : \mathscr{C}_i \to \mathscr{C}$ is a left exact functor for every $i \in \mathrm{Ob}(I)$.

Proof. Notice that the assertions depend only on the equivalence class of the category $\mathscr{C}$; hence, we may suppose that $(\mathscr{C}, \pi_\bullet)$ is the strong 2-colimit of $\mathscr{C}_\bullet$ described explicitly by example 3.3.13(i), *i.e.* $\mathscr{C} = \mathscr{F}ib(\mathscr{C}_\bullet)[\Sigma^{-1}]$, where Σ is the set of cartesian morphisms of $\mathscr{F}ib(\mathscr{C}_\bullet)$; then

π_i is induced by the inclusion functor of the fibre category $F^{-1}(i) = \mathscr{C}_i$ into $\mathscr{F}ib(\mathscr{C}_\bullet)$, where $F : \mathscr{F}ib(\mathscr{C}_\bullet) \to I^o$ is the natural projection. By proposition 1.2.22(i)), in order to check that $\mathscr{C}$ is finitely complete it suffices to show the following two claims :

Claim 4.3.21. All finite products are representable in $\mathscr{C}_\bullet$.

Proof of the claim. By a simple induction we are easily reduced to showing that the product of two objects $(i_1^o, X_1), (i_2^o, X_2)$ is representable (here we have $i_1^o, i_2^o \in \mathrm{Ob}(I^o)$ and $X_t \in \mathrm{Ob}(\mathscr{C}_{i_t})$ for $t = 1, 2$: see (3.1.18)). Since I^o is cofiltered, we may find $i \in \mathrm{Ob}(I)$ and morphisms $\varphi_t^o : i^o \to i_t^o$ for $t = 1, 2$; then (i_t^o, X_t) is isomorphic to $(i, \mathscr{C}_{\varphi_t} X_t)$ in $\mathscr{C}$ for $t = 1, 2$, so we may assume that $i = i_1 = i_2$. Since $\mathscr{C}_i$ is finitely complete, the product $X_1 \times X_2$ is representable in $\mathscr{C}_i$, say by an object P, and let $(p_t : P \to X_t \mid t = 1, 2)$ be a universal cone. Let us check that

$$(q_t := (\mathbf{1}_{i^o}, p_t) : (i^o, P) \to (i^o, X_t) \mid t = 1, 2)$$

is a universal cone in $\mathscr{C}$. Indeed, let (j^o, Y) be any object of $\mathscr{C}$ and $(r_t : (j^o, Y) \to (i^o, X_t) \mid t = 1, 2)$ a pair of morphisms of $\mathscr{C}$; we need to show that there exists a unique morphism

$$(4.3.22) \qquad r : (j^o, Y) \to (i^o, P) \qquad \text{in } \mathscr{C} \text{ such that} \qquad q_t \circ r = r_t \qquad \text{for } t = 1, 2.$$

By example 3.3.13(ii) we can write $r_t = g_t \circ s_t^{-1}$, where $s_t : (j_t^o, Y_t) \to (j^o, Y)$ and $g_t : (j_t^o, Y_t) \to (i^o, X_t)$ are morphisms of $\mathscr{F}ib(\mathscr{C}_\bullet)$, and s_t is cartesian, for $t = 1, 2$. By (CF3) of definition 1.6.14(i) we may then find an object (j'^o, Y') of $\mathscr{C}$ and morphisms $s'_t : (j'^o, Y') \to (j_t^o, Y_t)$ of $\mathscr{F}ib(\mathscr{C}_\bullet)$ such that $s'' := s_1 \circ s'_1 = s_2 \circ s'_2$ and such that s'_1 is cartesian; therefore s'' is an isomorphism in $\mathscr{C}$, so it suffices to check that there exists a unique morphism $r' : (j'^o, Y') \to (i^o, P)$ in $\mathscr{C}$ with $q_t \circ r' = r_t \circ s''$ for $t = 1, 2$. Thus, we may assume that r_t is the class of a morphism $(\rho_t^o, f_t) : (j^o, Y) \to (i^o, X_t)$ for $t = 1, 2$ (notation of (3.1.18), so here $\rho_t : i \to j$ is a morphism of I and $f_t : Y \to \mathscr{C}_{\rho_t} X_t$ is a morphism of $\mathscr{C}_j$ for $t = 1, 2$). Since I is filtered, we may then find a morphism $\rho' : j \to j'$ in I such that $\rho' \circ \rho_1 = \rho' \circ \rho_2$, and since $(\rho'^o, \mathbf{1}_{\mathscr{C}_\rho} Y) : (j'^o, \mathscr{C}_\rho Y) \to (j^o, Y)$ is an isomorphism in $\mathscr{C}$, we may further replace (ρ_t^o, f_t) by its composition with $(\rho'^o, \mathbf{1}_{\mathscr{C}_\rho} Y)$ for $t = 1, 2$, and assume as well that $\rho := \rho_1 = \rho_2$. In this situation set $P' = \mathscr{C}_\rho P$ and $X'_t := \mathscr{C}_\rho X_t$, $p'_t := \mathscr{C}_\rho(p_t)$ for $t = 1, 2$; we notice that r_t factors through a morphism $r'_t := (\mathbf{1}_{j^o}, f'_t) : (j^o, Y) \to (j^o, X'_t)$ and the cartesian morphism $(\rho, \mathbf{1}_{X'_t}) : (j^o, X'_t) \to (i^o, X_t)$, and moreover

$$(\rho, \mathbf{1}_{X'_t}) \circ (\mathbf{1}_{j^o}, p'_t) = (\mathbf{1}_{i^o}, p_t) \circ (\rho, \mathbf{1}_{P'}) \qquad \text{in } \mathscr{F}ib(\mathscr{C}_\bullet) \text{ for } t = 1, 2.$$

Since $\mathscr{C}_\rho$ is left exact, the cone $(p'_t : P' \to X'_t \mid t = 1, 2)$ is universal in $\mathscr{C}_j$, so there exists a unique morphism $g' : Y \to P'$ in $\mathscr{C}_j$ such that $p'_t \circ g' = f'_t$ for $t = 1, 2$. It follows that $r := (\rho, g') : (j^o, Y) \to (i^o, P)$ fulfills the condition of (4.3.22). It remains to check the uniqueness of r. Thus, let $r' : (j^o, Y) \to (i^o, P)$ be another morphism of $\mathscr{C}$ that verifies the identities of (4.3.22). We have $r' = g' \circ s'^{-1}$ for some morphisms g' and s' of $\mathscr{F}ib(\mathscr{C}_\bullet)$ with s' cartesian, and we are reduced to checking that $g' = r \circ s'$. We may then assume that r' is the class of a morphism (ρ'^o, g'') of $\mathscr{F}ib(\mathscr{C}_\bullet)$. Moreover, the identities (4.3.22) for r' mean that there exist cartesian morphisms $v_t : (k_t^o, Y'_t) \to (j^o, Y)$ such that $q_t \circ r' \circ v_t = r_t \circ v_t$ in $\mathscr{F}ib(\mathscr{C}_\bullet)$ for $t = 1, 2$. By (CF3) we may find an object (k^o, Y'') of $\mathscr{F}ib(\mathscr{C}_\bullet)$ and morphisms $v'_t : (k^o, Y'') \to (k_t^o, Y'_t)$ in $\mathscr{F}ib(\mathscr{C}_\bullet)$ with v'_1 cartesian, and such that $v'' := v_1 \circ v'_1 = v_2 \circ v'_2$. Since v'' is an isomorphism in $\mathscr{C}$, it suffices to check that $r \circ v'' = r' \circ v''$, and we may therefore assume that the identities (4.3.22) hold already in the category $\mathscr{F}ib(\mathscr{C}_\bullet)$, for both r and r'.

We may next find a morphism $\nu : j' \to j$ in I such that $\nu \circ \rho = \nu \circ \rho'$, and it suffices to check that $(\nu^o, \mathbf{1}_{\mathscr{C}_\nu Y}) \circ r = (\nu^o, \mathbf{1}_{\mathscr{C}_\nu Y}) \circ r'$. We may therefore assume that $\rho = \rho'$. Then both r and r' factor uniquely through morphisms $(\mathbf{1}_{j^o}, u), (\mathbf{1}_{j^o}, u') : (j^o, Y) \to (j^o, P')$ and the cartesian morphism $(\rho^o, \mathbf{1}_{P'}) : (j^o, P') \to (i^o, P)$, and we have

$$(\mathbf{1}_{j^o}, p'_t) \circ (\mathbf{1}_{j^o}, u) = r'_t = (\mathbf{1}_{j^o}, p'_t) \circ (\mathbf{1}_{j^o}, u') \qquad \text{in } \mathscr{F}ib(\mathscr{C}_\bullet) \text{ for } t = 1, 2.$$

Since P' represents $X'_1 \times X'_2$ in $\mathscr{C}_j$, we conclude that $u = u'$, whence $r = r'$ as required. $\lozenge$

Claim 4.3.23. All equalizers are representable in $\mathscr{C}$.

Proof of the claim. Let $(i_1^o, X_1), (i_2^o, X_2)$ be objects of $\mathscr{C}$ and $f_1, f_2 : (i_1^o, X_1) \to (i_2^o, X_2)$ morphisms of $\mathscr{C}$. We need to represent the equalizer of f_1 and f_2, and arguing as in the foregoing we may assume that $i := i_1 = i_2$, and also that f_t is the class of a morphism (φ_t, g_t) of $\mathscr{F}ib(\mathscr{C}_\bullet)$ for $t = 1, 2$. Then we may further reduce to the case where $\varphi := \varphi_1 = \varphi_2$. After replacing (i_2^o, X_2) by $(i_1^o, \mathscr{C}_\varphi X_2)$, we may then assume that $f_t = (\mathbf{1}_{i^o}, g_t) : (i^o, X_1) \to (i^o, X_2)$ for $t = 1, 2$. By assumption, the equalizer of g_1 and g_2 in $\mathscr{C}_i$ is representable by some $E \in \mathrm{Ob}(\mathscr{C}_i)$, and the universal cone for this equalizer amounts to a morphism $u : E \to X_1$ in $\mathscr{C}_i$ such that $g_1 \circ u = g_2 \circ u$. We claim that (i^o, E) represents the equalizer of f_1 and f_2, and that the morphism $(\mathbf{1}_{i^o}, u) : (i^o, E) \to (i^o, X_1)$ yields a universal cone for this equalizer. Indeed, let (j^o, Y) be an object of $\mathscr{C}$ and $h : (j^o, Y) \to (i^o, X_1)$ a morphism in $\mathscr{C}$ such that $f_1 \circ h = f_2 \circ h$; we need to show that h factors uniquely through u and a morphism $h' : (j^o, Y) \to (i^o, E)$. We reduce easily to the case where h is the class of a morphism $(\psi^o, k) : (j^o, Y) \to (i^o, X_1)$ of $\mathscr{F}ib(\mathscr{C}_\bullet)$ such that $(\mathbf{1}_{i^o}, g_1) \circ (\psi^o, k) = (\mathbf{1}_{i^o}, g_2) \circ (\psi^o, k)$ in $\mathscr{F}ib(\mathscr{C}_\bullet)$. In this situation, set $E' := \mathscr{C}_\psi E$, $u' := \mathscr{C}_\psi u$, $X'_t := \mathscr{C}_\psi X_t$ and $g'_t := \mathscr{C}_\psi g_t$ for $t = 1, 2$. Then (ψ^o, k) factors uniquely through the morphism $(\mathbf{1}_{j^o}, k) : (j^o, Y) \to (j^o, X'_1)$ and the cartesian morphism $(\psi^o, \mathbf{1}_{X'_1}) : (j^o, X'_1) \to (i^o, X_1)$, and we have $(\mathbf{1}_{j^o}, g'_1) \circ (\mathbf{1}_{j^o}, k) = (\mathbf{1}_{j^o}, g'_2) \circ (\mathbf{1}_{j^o}, k)$. Since $\mathscr{C}_\psi$ is left exact, E' represents the equalizer of g'_1 and g'_2 in $\mathscr{C}_j$, so there exists a unique morphism $k' : Y \to E'$ in $\mathscr{C}_j$ such that $u' \circ k' = k$. Then $h := (\psi, u')$ is the sought morphism. The verification of the uniqueness of h is analogous to that of the corresponding assertion in the proof of claim 4.3.21 : the details shall be left to the reader. $\lozenge$

Lastly, the constructions in the proof of claims 4.3.21 and 4.3.23 show that the functors π_i commute with finite products and equalizers, so they are left exact, by proposition 1.3.22(i). $\square$

4.3.24. Let now I be a small cofiltered category and $C_\bullet : I \to (\mathsf{U}, \mathsf{U}')\text{-lex}.\mathbf{Site}$ any pseudo-functor. For every $i \in \mathrm{Ob}(I)$, say that $C_i = (\mathscr{C}_i, J_i)$, and denote by $\mathscr{C}_\bullet : I^o \to \mathsf{U}'\text{-}\mathbf{Cat}$ the composition of $C_\bullet^o$ with the forgetful functor ${}^o(\mathsf{U}, \mathsf{U}')\text{-lex}.\mathbf{Site}^o \to \mathsf{U}'\text{-}\mathbf{Cat}$. We let $(\mathscr{C}, \pi_\bullet)$ be a strong 2-colimit of the pseudo-functor $\mathscr{C}_\bullet$ (theorem 3.3.9), and we endow $\mathscr{C}$ with the coarsest topology J such that all the functors $\pi_i : \mathscr{C}_i \to \mathscr{C}$ are continuous for the topologies J_i and J. In light of example 4.3.12 and lemma 4.3.20, we see that $C := (\mathscr{C}, J)$ is a lex-site, and we obtain a well defined pseudo-cone $\pi_\bullet^o : \mathsf{F}_C \Rightarrow C_\bullet$.

Proposition 4.3.25. *In the situation of* (4.3.24)*, let $F \in \mathrm{Ob}(\mathscr{C}^\wedge_{\mathsf{U}})$ be any presheaf. We have :*

(i) *The pair $(C, \pi_\bullet^o)$ is a strong 2-limit of $C_\bullet$.*
(ii) *F is a sheaf on C if and only if $\pi_i^\wedge(F)$ is a sheaf on C_i for every $i \in \mathrm{Ob}(I)$.*

Proof. (i): Let $D := (\mathscr{D}, J_{\mathscr{D}})$ be any finitely complete U-site, and $\varphi_\bullet : \mathsf{F}_D \Rightarrow C_\bullet$ any pseudo-cone. Hence for each $i \in \mathrm{Ob}(I)$ the functor $\varphi_i : \mathscr{C}_i \to \mathscr{D}$ is left exact and continuous for the topologies J_i and $J_{\mathscr{D}}$. There results a pseudo-cocone $\varphi_\bullet^a : \mathscr{C}_\bullet \Rightarrow \mathsf{F}_{\mathscr{D}}$, whence a unique functor $\varphi : \mathscr{C} \to \mathscr{D}$ such that $\mathsf{F}_\varphi \odot \pi_\bullet^o = \varphi^a$. The assertion then follows immediately from :

Claim 4.3.26. (i) The functor φ is left exact.

(ii) The functor φ is a morphism of sites $C \to D$.
(iii) C is a U-site.

Proof of the claim. (i): It suffices to check that φ commutes with all finite products and all equalizers of $\mathscr{C}$ (proposition 1.3.22(i)). Thus, let X_1, X_2 be two objects of $\mathscr{C}$, and P another object that represents $X_1 \times X_2$; pick also a universal cone for this product, given by a pair of morphisms $p_\bullet := (p_t : P \to X_t \mid t = 1, 2)$. By inspecting the proof of claim 4.3.21, we see that there exist $i \in \mathrm{Ob}(I)$ and objects $P', X'_1, X'_2 \in \mathrm{Ob}(\mathscr{C}_i)$ such that P' represents $X'_1 \times X'_2$ and the cone $p_\bullet$ is isomorphic to $\mathscr{C}_i * p'_\bullet$, where $p'_\bullet := (p'_t : P' \to X'_t \mid t = 1,2)$ is a universal cone in $\mathscr{C}_i$. Then $\varphi * p_\bullet$ is isomorphic to $\varphi_i * p'_\bullet$, and the latter is universal, since φ_i is left exact. This shows that φ commutes with binary products, and hence with all finite products. Similarly, by inspecting the proof of claim 4.3.23 we see that the universal cone of every equalizer in $\mathscr{C}$ is isomorphic to $\mathscr{C}_i * \tau$, for some $i \in \mathrm{Ob}(I)$ and with τ the universal cone of some equalizer in $\mathscr{C}_i$; then again we deduce easily that φ commutes with equalizers.

(ii): In view of (i) and example 4.3.12, it suffices to show that φ is continuous for the topologies J and $J_{\mathscr{D}}$. To this aim, let J' be the topology on $\mathscr{C}$ induced by $J_{\mathscr{D}}$ via φ; it suffices to check that $J \subset J'$. However, from lemma 4.2.4 we see that J is the coarsest topology on $\mathscr{C}$ such that for every $i \in \mathrm{Ob}(I)$, every $X \in \mathrm{Ob}(\mathscr{C}_i)$ and every covering family $(g_\lambda : X_\lambda \to X \mid \lambda \in \Lambda)$ for the topology J_i, the sieve generated by the family $(\pi_i(g_\lambda) \mid \lambda \in \Lambda)$ lies in $J(\pi_i(X))$. Thus, consider any such covering family $(g_\lambda \mid \lambda \in \Lambda)$; according to lemma 4.2.17(ii.b) we are further reduced to checking that the family $(\varphi \circ \pi_i(g_\lambda) \mid \lambda \in \Lambda)$ covers $\varphi \circ \pi_i(X) = \varphi_i(X)$ for the topology $\mathscr{D}$. Since $\varphi \circ \pi_i$ is continuous and $\mathscr{C}_i$ is finitely complete, the latter follows again from lemma 4.2.4.

(iii): For every $i \in \mathrm{Ob}(I)$ pick an essentially small topologically generating family $G_i \subset \mathrm{Ob}(\mathscr{C}_i)$ for the site C_i. It suffices to show that $G := \bigcup_{i \in \mathrm{Ob}(I)} \pi_i(G_i)$ is a topologically generating family for the site C. Thus, let $X \in \mathrm{Ob}(\mathscr{C})$ be any object; by inspecting the construction of $\mathscr{C}$ we find $i \in \mathrm{Ob}(I)$ and $X_i \in \mathrm{Ob}(\mathscr{C}_i)$ such that $X = \pi_i(X_i)$, and by assumption the family $\mathscr{F} := \bigcup_{Y \in G_i} \mathrm{Hom}_{\mathscr{C}_i}(Y, X_i)$ covers X_i for the topology J_i. Hence $(\pi_i(g) \mid g \in \mathscr{F}) \subset \bigcup_{Z \in G} \mathrm{Hom}_{\mathscr{C}}(Z, X)$ covers X for the topology J, whence the contention. ◇

(ii): The condition is obviously necessary. For the converse, we consider the finest topology J'' on $\mathscr{C}$ such that F is a sheaf on the site $(\mathscr{C}, J'')$ (see remark 4.1.8(iii)); it then suffices to show that $J \subset J''$. In view of remark 4.1.8(iii), we come down to checking the following. For every $i \in \mathrm{Ob}(I)$, every $X \in \mathrm{Ob}(\mathscr{C}_i)$, every covering family $(g_\lambda : X_\lambda \to X \mid \lambda \in \Lambda)$ for the topology J_i and every morphism $f : Y \to \pi_i X$ in $\mathscr{C}$, the family

$$(Y \times_{\pi_i X} \pi_i(g_\lambda) : Y \times_{\pi_i X} \pi_i X_\lambda \to Y \mid \lambda \in \Lambda)$$

generates a sieve of 2-descent for the fibration $\mathscr{F}ib(F) \to \mathscr{C}$ of (3.1.15). But recall that $\mathscr{C} = \mathscr{F}ib(\mathscr{C}_\bullet)[\Sigma^{-1}]$, where Σ denotes the set of cartesian morphisms of $\mathscr{F}ib(\mathscr{C}_\bullet)$. With this notation, we have $\pi_i X = (i^o, X)$ and $Y = (j^o, Y') = \pi_j Y'$ for some $j \in \mathrm{Ob}(I)$ and $Y' \in \mathrm{Ob}(\mathscr{C}_j)$. We are then easily reduced to the case where f is the class of a morphism $(\varphi^o, t) : (j^o, Y') \to (i^o, X)$ of $\mathscr{F}ib(\mathscr{C}_\bullet)$, for some morphism $\varphi : i \to j$ in I and $t : Y' \to \mathscr{C}_\varphi X$ in $\mathscr{C}_j$. Notice now the commutative diagrams :

$$\begin{array}{ccc} \pi_j(\mathscr{C}_\varphi X_\lambda) & \xrightarrow{\pi_j(\mathscr{C}_\varphi g_\lambda)} & \pi_j(\mathscr{C}_\varphi X) \\ {\scriptstyle (\varphi, \mathbf{1}_{\mathscr{C}_\varphi X_\lambda})}\downarrow & & \downarrow{\scriptstyle (\varphi, \mathbf{1}_{\mathscr{C}_\varphi X})} \\ \pi_i(X_\lambda) & \xrightarrow{\pi_i(g_\lambda)} & \pi_i(X) \end{array} \qquad \text{for every } \lambda \in \Lambda$$

whose vertical arrows are cartesian morphisms of $\mathscr{F}ib(\mathscr{C}_\bullet)$, so they are isomorphisms in $\mathscr{C}$. Moreover, the continuity of $\mathscr{C}_\varphi$ implies that the family $(\mathscr{C}_\varphi(g_\lambda) \mid \lambda \in \Lambda)$ covers $\mathscr{C}_\varphi X$ for the topology J_j. Furthermore, we have $(\varphi^o, t) = (\mathbf{1}_{j^o}, t) \circ (\varphi, \mathbf{1}_{\mathscr{C}_\varphi X})$. Thus, we may replace $(g_\lambda \mid \lambda \in \Lambda)$ by $(\mathscr{C}_\varphi(g_\lambda) \mid \lambda \in \Lambda)$ and assume that $i = j$ and $\varphi = \mathbf{1}_i$. In this situation, since π_i is left exact, we have natural identifications for every $\lambda, \mu \in \Lambda$:

$$Y_\lambda := Y \times_{\pi_i X} \pi_i X_\lambda \xrightarrow{\sim} \pi_i(Y' \times_X X_\lambda) \qquad Y_\lambda \times_Y Y_\mu \xrightarrow{\sim} \pi_i(Y' \times_X X_\lambda \times_X X_\mu).$$

So finally we are reduced to checking that the natural map from $F(\pi_i Y')$ to the equalizers of the restriction maps

$$\prod_{\lambda\in\Lambda} F\circ\pi_i(Y'\times_X X_\lambda) \rightrightarrows \prod_{\lambda,\mu\in\Lambda} F\circ\pi_i(Y'\times_X X_\lambda\times_X X_\mu)$$

is a bijection. But this is clear, since by assumption $F\circ\pi_i$ is a sheaf on C_i. □

Remark 4.3.27. In the situation of (4.3.24), we also claim that $C^\sim$ represents the 2-limit of the induced pseudo-functor

$$C^\sim_\bullet : I \to \mathsf{U}'\text{-}\mathbf{Cat} \qquad i\mapsto C^\sim_{i,\mathsf{U}} \qquad (\varphi : i\to j)\mapsto (C^\sim_\varphi : C^\sim_{i,\mathsf{U}} \to C^\sim_{j,\mathsf{U}}).$$

Indeed, on the one hand, the natural functor (see definition 2.5.1(ii))

$$\mathsf{PsFun}(\mathscr{C}, \mathbf{Set}^o) \to \mathsf{PsNat}(\mathscr{C}_\bullet, \mathsf{F}_{\mathbf{Set}^o}) \qquad (G : \mathscr{C}\to\mathbf{Set}^o)\mapsto \mathsf{F}_G\odot\pi_\bullet$$

is an equivalence; but we have $\mathsf{PsFun}(\mathscr{C}, \mathbf{Set}^o) = \mathscr{C}^\wedge$, and a pseudo-cocone $G_\bullet : \mathscr{C}_\bullet \Rightarrow \mathsf{F}_{\mathbf{Set}^o}$ is the datum of a system of presheaves G_i on $\mathscr{C}_i$ for every $i\in \mathrm{Ob}(I)$, with isomorphisms

$$\tau^G_\varphi : G_i \xrightarrow{\sim} \mathscr{C}^\wedge_\varphi G_j \qquad \text{in } \mathscr{C}^\wedge_i \text{ for every morphism } \varphi : i\to j \text{ in } I$$

fulfilling the usual coherence axioms. Under these identifications, a presheaf G on $\mathscr{C}$ corresponds to the datum $(G_\bullet, \tau^G_\bullet)$ with $G_i := \pi^\wedge_i G$ for every $i\in\mathrm{Ob}(I)$, and with

$$\tau^G_\varphi := (\tau^\pi_\varphi)^\wedge_G : \pi^\wedge_i G \xrightarrow{\sim} \mathscr{C}^\wedge_\varphi\circ\pi^\wedge_j G$$

where $(\tau^\pi_\varphi)^\wedge : \pi^\wedge_i \xrightarrow{\sim} \mathscr{C}^\wedge_\varphi\circ\pi^\wedge_j$ is the isomorphism of functors induced by the coherence constraint $\tau^\pi_\varphi : \pi_j \xrightarrow{\sim} \mathscr{C}_\varphi\circ\pi_i$ of the universal pseudo-cone $\pi_\bullet : \mathsf{F}_\mathscr{C} \Rightarrow \mathscr{C}_\bullet$, for every morphism $\varphi : i\to j$ in I. On the other hand, by proposition 4.3.25(ii), the presheaf G is a sheaf on C if and only if G_i is a presheaf on C_i for every $i\in\mathrm{Ob}(I)$. Summing up, we obtain a natural equivalence between $C^\sim$ and the category of pairs $(G_\bullet, \tau^G_\bullet)$ as above, such that G_i is a sheaf on C_i for every $i\in\mathrm{Ob}(I)$. But the latter is just a description of the category

$$\mathsf{Cart}_{I^o}(I^o, \mathscr{F}ib(C^\sim_\bullet))$$

which represents the 2-limit of $C^\sim_\bullet$, by the proof of theorem 3.3.9.

4.4. **Topoi.** Let U, U' be a pair of universes with $\mathsf{U}\in\mathsf{U}'$. A (U,U')-*topos* is a U'-small category with U-small Hom-sets which is equivalent to the category of sheaves on a U-small site. We shall often omit the explicit mention of U', and call "U-topos" such a category, or even just "topos" unless this may give rise to ambiguities.

Remark 4.4.1. Let T be any U-topos.

(i) Remark 4.1.19(i,ii) can be summarized by saying that T is a complete and cocomplete, well-powered and co-well-powered category. Moreover, every epimorphism in T is universal effective, every colimit is universal, and all filtered colimits in T commute with finite limits.

(ii) It follows from (i) that a small family $(F_i\to F \mid i\in I)$ of morphisms of T is a covering family for the canonical topology if and only if the induced morphism $\amalg_{i\in I}F_i \to F$ is an epimorphism in T.

(iii) Say that $T = C^\sim$, for a small site $C := (\mathscr{C}, J)$; then the set $\{h^a_X \mid X\in\mathrm{Ob}(\mathscr{C})\}$ is a small topologically generating family for the canonical site $\mathrm{Can}(T)$ (see remark 4.1.8(v)). Indeed, by virtue of remark 4.1.19(iii) we may find for every $F\in\mathrm{Ob}(T)$ a family $(X_i \mid i\in I)$ of objects of $\mathscr{C}$ with an epimorphism $\amalg_{i\in I}h^a_{X_i}\to F$ in the category $\mathscr{C}^\sim$, so the resulting family $(h^a_{X_i}\to F \mid i\in I)$ covers F for the canonical topology, by (ii). Especially, $\mathrm{Can}(T)$ is a U-site.

(iv) On the other hand, by proposition 4.3.18(ii), if C is a U-site, then $C^\sim$ is isomorphic to a U-topos.

Theorem 4.4.2. *Let $C := (\mathscr{C}, J)$ be a site such that $\mathscr{C}$ has small* Hom*-sets. We have :*

(i) *The Yoneda embedding $\mathscr{C} \to \mathscr{C}^\wedge$ is a morphism of sites $h_\mathscr{C} : C^\wedge \to C$ (notation of remark* 4.1.31(ii)*) and is cocontinuous for the topologies J and $J^\wedge$. Moreover the topology J is induced by $J^\wedge$ via the functor $h_\mathscr{C}$.*

(ii) *Suppose that C is a U-site. Then also the forgetful functor $i : C^\sim \to \mathscr{C}^\wedge$, its left adjoint $(-)^a : \mathscr{C}^\wedge \to C^\sim$ and the Yoneda embedding $h^a_C : \mathscr{C} \to C^\sim$ are morphisms of sites $C^\wedge \to \mathrm{Can}(C^\sim)$, $\mathrm{Can}(C^\sim) \to C^\wedge$ and $\mathrm{Can}(C^\sim) \to C$ respectively.*

(iii) *Moreover, in the situation of* (ii) *the functors $(-)^a$ and h^a_C are cocontinuous for the topologies J, $J^\wedge$ and $\mathrm{Can}_{C^\sim}$, and the topologies J and $J^\wedge$ are induced by the canonical topology of $C^\sim$, via the functors $h^a_C : \mathscr{C} \to C^\sim$ and $(-)^a$ respectively. Furthermore, for every universe V containing U, the functors i, $h_\mathscr{C}$, h^a_C, $(-)^a$ induce equivalences :*

$$\begin{array}{ccc} \mathrm{Can}(C^\sim)^\sim_\mathsf{V} & \overset{((-)^a)^\sim_*}{\underset{\widetilde{\imath}_*}{\rightleftarrows}} & (C^\wedge)^\sim_\mathsf{V} \\ & {\scriptstyle \widetilde{h}^a_{C*}} \searrow \quad \swarrow {\scriptstyle \widetilde{h}_{\mathscr{C}*}} & \\ & C^\sim_\mathsf{V}. & \end{array}$$

(iv) *For every topos T, the Yoneda embedding induces an equivalence*

$$h_T : T \to \mathrm{Can}(T)^\sim \qquad F \mapsto h_F.$$

Proof. (i): Pick a universe V containing U, and such that $C^\wedge$ is a V-site; denote by J' the topology on $\mathscr{C}$ induced by $J^\wedge$ on $\mathscr{C}$ via $h_\mathscr{C}$. Notice that condition (a) of theorem 4.3.13 holds for the functor $h_\mathscr{C}$, by virtue of remark 4.4.1(iii), and conditions (b) and (c) hold as well, since $h_\mathscr{C}$ is fully faithful (remark 4.3.17(i)). By theorem 4.3.13(iii) it follows that a family of morphisms $(Y_i \to Y \mid i \in I)$ in $\mathscr{C}$ covers Y in the topology J' if and only if the family $(h_{Y_i} \to h_Y \mid i \in I)$ covers h_Y in the topology $J^\wedge$. Combining with remark 4.1.29(i) we deduce that $J' = J$. Moreover, by theorem 4.3.13(ii), the functor $(h_\mathscr{C})^\sim_{\mathsf{V}*} : (C^\wedge)^\sim_\mathsf{V} \to C^\sim_\mathsf{V}$ is an equivalence, so the same holds for its left adjoint $(h^*_\mathscr{C})^\sim_\mathsf{V}$, and especially, $h_\mathscr{C}$ is a morphism of sites, as stated.

(ii): Let us remark :

Claim 4.4.3. The functor $(-)^a : \mathscr{C}^\wedge \to C^\sim$ satisfies conditions (a),(b),(c) of theorem 4.3.13 for the topology $\mathrm{Can}_{C^\sim}$.

Proof of the claim. Indeed, to check condition (a) it suffices to remark that $(iF)^a$ is isomorphic to F, for every $F \in \mathrm{Ob}(C^\sim)$. Next, let $F, G \in \mathrm{Ob}(\mathscr{C}^\wedge)$ and $\varphi : F^a \to G^a$ a morphism in $C^\sim$; we may find a family $(f_j : h_{X_j} \to F \mid j \in I)$ of morphisms in $\mathscr{C}^\wedge$ such that the induced morphism $\mu : \amalg_{j\in I} h_{X_j} \to F$ is an epimorphism (lemma 1.4.8). The composition

$$\varphi_j : h_{X_j} \xrightarrow{f_j} F \to F^a \xrightarrow{\varphi} G^a \qquad \text{for every } j \in J$$

corresponds to a section $\sigma_j \in G^a(X_j)$. Then we may find for every $j \in I$ a covering family $(f_{jk} : X_{jk} \to X_j \mid k \in I_j)$ for the topology J, and a section $\sigma_{jk} : G(X_{jk})$ for every $k \in I_j$, whose image in $G^a(X_{jk})$ agrees with $G^a(f_{jk})(\sigma_j)$. The section σ_{jk} corresponds to a morphism $\psi_{jk} : h_{X_{jk}} \to G$ in $\mathscr{C}^\wedge$, whose composition with the natural morphism $G \to G^a$ agrees with $\varphi_j \circ h_{f_{jk}}$. Moreover, for every $j \in J$ the resulting morphism $\nu_j : \amalg_{k\in I_j} h_{X_{jk}} \to h_{X_j}$ of $\mathscr{C}^\wedge$ induces an epimorphism ν^a in $C^\sim$ (remark 4.1.29(i) and corollary 4.1.30(i)); likewise μ^a is an epimorphism in $C^\sim$, by virtue of proposition 1.3.25(v). Thus, the family $(f^a_j \circ h^a_{f_{jk}} : h^a_{X_{jk}} \to F^a \mid j \in I,\ k \in I_j)$ covers F^a for the topology $\mathrm{Can}_{C^\sim}$ (remark 4.4.1(ii)), and by construction we have $\varphi \circ (f_j \circ h_{f_{jk}})^a = \psi^a_{jk}$ for every $j \in I$ and every $k \in I_j$. This shows that condition (b) holds. Lastly, let $\varphi, \varphi' : F \to G$ be two morphisms of $\mathscr{C}^\wedge$ such that $\varphi^a = \varphi'^a$, and choose again a family $(f_j : h_{X_j} \to F \mid j \in I)$ as in the foregoing; each f_j corresponds to a section $\sigma_j \in FX_j$, and by assumption the images of $\varphi_{X_j}(\sigma_j)$ and $\varphi'_{X_j}(\sigma_j)$ agree in $G^a(X_j)$. Then we may find for every $j \in I$ a covering family $(f_{jk} : X_{jk} \to X_j \mid k \in I_j)$ such that

$$\varphi_{X_{jk}} \circ (Ff_{jk})(\sigma_j) = (Gf_{jk}) \circ \varphi_{X_j}(\sigma_j) = (Gf_{jk}) \circ \varphi'_{X_j}(\sigma_j) = \varphi'_{X_{jk}} \circ (Ff_{jk})(\sigma_j)$$

for every $k \in I_j$. Again, the family $(f_j^a \circ h^a_{f_{jk}} : h^a_{X_{jk}} \to F^a \mid j \in I,\ k \in I_j)$ covers F^a for the topology $\mathrm{Can}_{C^\sim}$, and $\varphi \circ f_j \circ h_{f_{jk}} = \varphi' \circ f_j \circ h_{f_{jk}}$ for every $j \in I$ and every $k \in I_j$. This shows that also condition (c) holds. ♢

Let J' be the topology on $\mathscr{C}^\wedge$ induced by $\mathrm{Can}_{C^\sim}$ via the functor $(-)^a$; since $C^\sim$ is isomorphic to a U-site (remark 4.4.1(iv)), theorem 4.3.13 and claim 4.4.3 imply that $(-)^a : (\mathscr{C}^\wedge, J') \to \mathrm{Can}(C^\sim)$ is continuous and cocontinuous, and a family $(F_j \to F \mid j \in I)$ of morphisms in $\mathscr{C}^\wedge$ generates a covering sieve for the topology J' if and only if the family $(F_j^a \to F^a \mid j \in I)$ generates a covering sieve for the topology $\mathrm{Can}_{C^\sim}$. But the latter condition holds if and only if the family $(F_j \to F \mid j \in I)$ generates a covering sieve for the topology $J^\wedge$, by virtue of remark 4.1.31(i). In other words, $J' = J^\wedge$, and thus theorem 4.3.13 also says that $(-)^a$ induces an equivalence $((-)^a)^\sim_{\mathsf{V}*} : \mathrm{Can}(C^\sim)^\sim_{\mathsf{V}} \xrightarrow{\sim} (\mathscr{C}^\wedge, J')^\sim_{\mathsf{V}}$ for every universe V containing U. Then also the left adjoint of $((-)^a)^\sim_{\mathsf{V}*}$ is an equivalence, and especially $(-)^a : \mathrm{Can}(C^\sim) \to C^\wedge$ is a morphism of sites. Lastly, since $(-)^a \circ i = \mathbf{1}_{C^\sim}$, we see that $\tilde{\imath}_{\mathsf{V}*} : (C^\wedge)^\sim_{\mathsf{V}} \to \mathrm{Can}(C^\sim)^\sim_{\mathsf{V}}$ is an equivalence as well, and so is its left adjoint (which is $((-)^a)^\sim_{\mathsf{V}*}$); especially, $i : C^\wedge \to \mathrm{Can}(C^\sim)$ is a morphism of sites.

Claim 4.4.4. The Yoneda embedding $h^a_C : \mathscr{C} \to C^\sim$ fulfills conditions (a),(b),(c) of theorem 4.3.13, for the canonical topology on $C^\sim$.

Proof of the claim. Indeed, condition (a) has already been noticed in remark 4.4.1(iii). Next, let $X, Y \in \mathrm{Ob}(\mathscr{C})$ and $\varphi : h^a_X \to h^a_Y$ any morphism of $C^\sim$; then φ corresponds to a section $s \in h^a_Y(X)$, which in turn is represented by the datum of a covering family $(f_i : X_i \to X \mid i \in I)$ for the topology J and a compatible system of morphisms $(s_i : X_i \to Y \mid i \in I)$ in $\mathscr{C}$. With this notation, $h^a_{f_i}(s)$ is the image of s in $h^a_Y(X_i)$, so we get $h^a_{s_i} = \varphi \circ h^a_{f_i}$ for every $i \in I$. Lastly, the family $(h^a_{f_i} : h^a_{X_i} \to h^a_X \mid i \in I)$ covers $h^a_{X_i}$ in the canonical topology, by remarks 4.4.1(ii) and 4.1.29(i) and corollary 4.1.30(i).

To check condition (c), consider morphisms $\varphi, \psi : X \to Y$ in $\mathscr{C}$ such that $h^a_\varphi = h^a_\psi$. The latter means that the images $\overline{\varphi}$ and $\overline{\psi}$ of φ and ψ agree in $h^a_Y(X)$, so there exists a covering family $(f_i : X_i \to X \mid i \in I)$ such that $h^a_Y(f_i)(\overline{\varphi}) = h^a_Y(f_i)(\overline{\psi})$ for every $i \in I$. This in turn translates as the identity $\varphi \circ f_i = \psi \circ f_i$ for every $i \in I$. To conclude, we observe as in the foregoing that the family $(h^a_{f_i} : h^a_{X_i} \to h^a_X \mid i \in I)$ covers h^a_X in the topology $\mathrm{Can}_{C^\sim}$. ♢

Let J' be the topology on $\mathscr{C}$ induced by $\mathrm{Can}_{C^\sim}$ via $h^a_{\mathscr{C}}$; since we have already noticed that $C^\sim$ is isomorphic to a U-site, claim 4.4.4 and theorem 4.3.13 imply that $h^a_{\mathscr{C}}$ induces an equivalence $H_{\mathsf{V}} := (h^a_C)^\sim_{\mathsf{V}*} : \mathrm{Can}(C^\sim)^\sim_{\mathsf{V}} \xrightarrow{\sim} (\mathscr{C}, J')^\sim_{\mathsf{V}}$ for every universe V containing U. Moreover, arguing as in the foregoing we deduce that $J' = J$. Thus, we get an equivalence $H : \mathrm{Can}(C^\sim)^\sim \xrightarrow{\sim} C^\sim$, which as usual implies that $h^a_C : C^\sim \to C$ is a morphism of sites, concluding the proof of (ii) and (iii). Lastly, by a simple inspection we see that $H \circ h_{C^\sim}$ is naturally isomorphic to $\mathbf{1}_{C^\sim}$, whence (iv). □

Definition 4.4.5. (i) Let S and T be two topoi. A *morphism of topoi* $f : T \to S$ is a datum

$$(f^*, f_*, \eta) \qquad \text{where} \qquad f_* : T \to S \qquad f^* : S \to T$$

are two functors such that f^* is left exact and left adjoint to f_*, and $\eta : \mathbf{1}_S \Rightarrow f_*f^*$ is a unit of an adjunction ϑ for the pair (f^*, f_*) (these are sometimes called *geometric morphisms* : see [30, Def.2.12.1]). We shall say that η and ϑ are respectively *the unit and the adjunction of* f.

(ii) Let $f := (f^*, f_*, \eta_f) : T'' \to T'$ and $g := (g^*, g_*, \eta_g) : T' \to T$ be two morphisms of topoi. The composition $g \circ f$ is the morphism

$$(f^* \circ g^*, g_* \circ f_*, (g_* * \eta_f * g^*) \odot \eta_g) : T'' \to T$$

(see remark 1.1.17(i)). To check the associativity of this composition law, it suffices to notice that $(g_* * \eta_f * g^*) \odot \eta_g$ is the unit of the composition of the adjunctions determined by η_f and η_g, as defined in remark 1.1.17(i). For every topos T, let $i_T : \mathbf{1}_T \stackrel{\sim}{\to} \mathbf{1}_T$ be the identity automorphism of $\mathbf{1}_T$; the datum

$$(\mathbf{1}_T, \mathbf{1}_T, , i_T)$$

yields a morphism of topoi $T \to T$ which is neutral for left and right compositions relative to the foregoing composition law.

(iii) Let $f, g : T \to S$ be two morphisms of topoi. A *natural transformation* $\tau : f \Rightarrow g$ is just a natural transformation of functors $\tau_* : f_* \Rightarrow g_*$. Notice that, in view of remark 1.1.17(ii), the datum of τ_* is the same as the datum of a natural transformation $\tau^* : g^* \Rightarrow f^*$.

(iv) Let U' be a universe with $\mathsf{U} \in \mathsf{U}'$; the $(\mathsf{U}, \mathsf{U}')$-topoi are the objects of the 2-category

$$(\mathsf{U}, \mathsf{U}')\text{-}\mathbf{Topos}$$

whose 1-cells are the morphisms of topoi, and whose 2-cells are the natural transformations between such morphisms. Obviously this category depends on the choice of U', but two different choices are related by a 2-equivalence of 2-categories, so we shall usually omit mentioning an explicit choice of U', and write U-$\mathbf{Topos}$ for this 2-category. Moreover, when the choice of U is clear from the context, we shall likewise drop the mention of U and write just $\mathbf{Topos}$.

Example 4.4.6. Let $T_\bullet := (T_i \mid i \in I)$ be any U-small family of $(\mathsf{U}, \mathsf{U}')$-topoi.

(i) The product category $T := \prod_{i \in I} T_i$ is a topos (see example 1.2.25(i)). Indeed, pick for each $i \in I$ a small site $C_i := (\mathscr{C}_i, J_i)$ with an equivalence $C_i^\sim \stackrel{\sim}{\to} T_i$; according to example 1.2.25(i), the coproduct of the family of categories $(\mathscr{C}_i \mid i \in I)$, is represented by a category $\mathscr{C}$ whose set of objects is the disjoint union $\amalg_{i \in I} \mathrm{Ob}(\mathscr{C}_i)$. We endow $\mathscr{C}$ with the topology J such that $J(i, X) := J_i(X)$ for every $(i, X) \in \mathrm{Ob}(\mathscr{C})$. Then it is easily seen that there is a natural isomorphism of categories

$$(\mathscr{C}, J)^\sim \stackrel{\sim}{\to} \prod_{i \in I} C_i^\sim .$$

However, the product of the categories $C_i^\sim$ also represents the 2-product, in the 2-category U'-$\mathbf{Cat}$, of the same family of categories, and the standard universal cone is also a universal pseudo-cone (see example 3.3.12); there follows an equivalence of categories

$$(\mathscr{C}, J)^\sim \stackrel{\sim}{\to} T$$

whence the contention.

(ii) Moreover, T represents the 2-coproduct of the family $T_\bullet$ in the 2-category $\mathbf{Topos}$. Indeed, for every $i \in I$ let $\pi_i : T \to T_i$ be the projection functor. We define a right adjoint $e_i : T_i \to T$ for π_i by the rule : $X \mapsto (X_j \mid j \in I)$ for every $X \in \mathrm{Ob}(T_i)$, where $X_i := X$ and $X_j := 1_{T_j}$ is a fixed choice of a final object in T_j, for every $j \in I \setminus \{i\}$. It is easily seen that π_i is left exact, and there is a natural choice of adjunction for the pair (π_i, e_i), whence a well defined morphism of topoi

$$\varphi_i : T_i \to T \qquad \text{such that } \varphi_i^* = \pi_i \text{ and } \varphi_{i*} := e_i.$$

Next, let S be another topos, and $(\psi_i : T_i \to S \mid i \in I)$ a given family of morphisms of topoi. By the universal property of the product, there exists a unique functor $\lambda^* : S \to T$ such that

$$\varphi_i^* \circ \lambda^* = \psi_i^* \qquad \text{for every } i \in I \tag{4.4.7}$$

and it is easily seen that λ^* is left exact. We construct a right adjoint $\lambda_* : T \to S$ for λ^* by the following rule. Given any object $X_\bullet := (X_i \mid i \in I)$ of T, let $\lambda_*(X_\bullet)$ be any object of S representing the product $\prod_{i \in I} \psi_{i*} X_i$, and fix as well a universal cone $(p_i^X : \lambda_*(X_\bullet) \to$

$\psi_{i*}X_i \mid i \in I)$. Then, for every morphism $f_\bullet : X_\bullet \to Y_\bullet$ in T there is a unique morphism $\lambda_*(f_\bullet) : \lambda_*(X_\bullet) \to \lambda_*(Y_\bullet)$ that makes commute the diagram

$$\begin{array}{ccc} \lambda_*(X_\bullet) & \xrightarrow{\lambda_*(f_\bullet)} & \lambda_*(Y_\bullet) \\ {\scriptstyle p_i^X}\downarrow & & \downarrow{\scriptstyle p_i^Y} \\ \psi_{i*}X_i & \xrightarrow{\psi_{i*}f_i} & \psi_{i*}Y_i \end{array} \qquad \text{for every } i \in I.$$

Thus, the rules $X_\bullet \mapsto \lambda_*(X_\bullet)$ and $f_\bullet \mapsto \lambda_*(f_\bullet)$ yield a well defined functor as sought, unique up to unique isomorphism. Then it is easily seen that λ^* is left adjoint to λ_*, and there is even a natural choice of adjunction for the pair (λ^*, λ_*) (details left to the reader), so the latter yields a well defined morphism of topoi $\lambda : T \to S$. Lastly, (4.4.7) implies that $\lambda_* \circ \varphi_*$ is right adjoint to ψ_i^*, whence an isomorphism of functors

$$\lambda_* \circ \varphi_* \xrightarrow{\sim} \psi_{i*} \qquad \text{for every } i \in I$$

which shows that the system $(\varphi_i \mid i \in I)$ is a universal pseudo-cocone, whence the contention.

Proposition 4.4.8. *Let T, T' be two topoi, $f : T \to T'$ a left exact functor, φ a morphism of T.*

- (i) *If f commutes with all small colimits in T, the following holds :*
  - (a) *f is continuous for the canonical topologies on T and T'.*
  - (b) *There exists a morphism of topoi $F : T' \to T$, unique up to unique isomorphism, such that $F^* = f$.*
- (ii) *f is conservative if and only if it reflects epimorphisms.*
- (iii) *If f is exact, the natural morphism $f(\mathrm{Im}\,\varphi) \to \mathrm{Im}(f\varphi)$ is an isomorphism.*

Proof. (i.a): Let $S := \{g_i : X_i \to X \mid i \in I\}$ be a small covering family of morphisms in Can_T; by lemma 4.2.4, it suffices to show that $fS := \{f(g_i) \mid i \in I\}$ is a covering family in $\mathrm{Can}_{T'}$. However, our assumption implies that the induced morphism $\amalg_{i \in I} X_i \to X$ is an epimorphism, hence the same holds for the induced morphism $\amalg_{i \in I} fX_i \to fX$ in T' (by proposition 1.3.18(ii)). But all epimorphisms are universal effective in T' (remark 4.4.1(i)), hence $(f(g_i) \mid i \in I)$ is a universal effective family, as required.

(i.b): By corollary 4.3.19(i.a,i.c), the functor f gives rise to a morphism of topoi

$$(\widetilde{f}^*, \widetilde{f}_*, \eta) : \mathrm{Can}(T')^\sim \to \mathrm{Can}(T)^\sim$$

(where η is any choice of unit of adjunction), and it suffices to show that $\widetilde{f}^*$ is isomorphic to f, under the natural identifications $T \xrightarrow{\sim} \mathrm{Can}(T)^\sim$, $T' \xrightarrow{\sim} \mathrm{Can}(T')^\sim$ given by the Yoneda embeddings. The latter follows from corollary 4.3.19(i.b).

(iii) follows easily from remark 4.1.25.

(ii) follows from propositions 4.1.24(i) and 1.3.21. □

Remark 4.4.9. (i) Let us consider the 2-category $(\mathsf{U}, \mathsf{U}')$-$\mathbf{Topos}^*$ whose objects are the same as those of the 2-category $(\mathsf{U}, \mathsf{U}')$-$\mathbf{Topos}$, and such that for every two U-topoi T and S, the 1-cells $f : T \to S$ are the left exact functors $f : S \to T$ that commute with all small colimits. For any pair of such functors $f, f' : S \to T$, the 2-cells $\beta : f \Rightarrow f'$ of $(\mathsf{U}, \mathsf{U}')$-$\mathbf{Topos}^*$ are the natural transformations $\beta : f' \Rightarrow f$. As usual, we shall often write U-$\mathbf{Topos}^*$ or just $\mathbf{Topos}^*$ for this 2-category. We have a natural strict pseudo-functor

$$(-)^* : \mathbf{Topos} \to \mathbf{Topos}^*$$

that is the identity on objects, and assigns to every morphism $f := (f^*, f_*, \eta_f) : T \to S$ of topoi the functor $f^* : S \to T$, and to every natural transformation $\beta : (f^*, f_*, \eta_f) \Rightarrow (g^*, g_*, \eta_g)$ the adjoint transformation $(\beta, \eta_f, \eta_g)^\dagger : g^* \Rightarrow f^*$. Then proposition 4.4.8(i.b) can be restated by saying that $(-)^*$ is a strict 2-equivalence of 2-categories.

(ii) Notice also the strict pseudo-functors

$$\mathsf{U}'\text{-}\mathbf{Cat}^o \xleftarrow{(-)^*} \mathbf{Topos} \xrightarrow{(-)_*} \mathsf{U}'\text{-}\mathbf{Cat}$$

defined as follows. The pseudo-functor $(-)_*$ is the identity map on objects : $T \mapsto T$, and is given on morphisms by the rule : $(F_*, F^*, \eta_F) \mapsto F_*$ (notation of definition 4.4.5(i)). On natural transformations it is also the identity : $(\tau : (F^*, F_*, \eta_F) \Rightarrow (G^*, G_*, \eta_G)) \mapsto (\tau : F_* \Rightarrow G_*)$.

The pseudo-functor $(-)^*$ is given on objects by the rule : $T \mapsto T^o$, and on morphisms by the rule : $(F^*, F_*, \eta_F) \mapsto (F^*)^o$. On natural transformations it is defined by the rule :

$$(\tau : (F^*, F_*, \eta_F) \Rightarrow (G^*, G_*, \eta_G)) \mapsto ((\tau, \vartheta_F, \vartheta_G)^{\dagger o} : F^{*o} \Rightarrow G^{*o})$$

where ϑ^F and ϑ^G are the adjunctions for the pairs (F^*, F_*) and respectively (G^*, G_*) whose units are η_F and respectively η_G, and $(\tau, \vartheta_F, \vartheta_G)^\dagger$ is the adjoint to τ, as in remark 1.1.17(ii). The pseudo-functoriality of $(-)^*$ follows straightforwardly from remark 1.1.17(iv,v) : the details shall be left to the reader.

Example 4.4.10. Let T be a topos.

(i) Say that $T = (\mathscr{C}, J)^\sim$ for a small site $(\mathscr{C}, J)$. According to remark 4.1.8(vi), the category $\mathscr{C}^\wedge$ is also a topos, and since the functor $F \mapsto F^a$ of theorem 4.1.13 is left exact, it determines a morphism of topoi $T \to \mathscr{C}^\wedge$. (On the other hand, the category $T^\wedge$ is too large to be a U-topos.)

(ii) Let $f := (f^*, f_*, \eta) : T \to S$ be any morphism of topoi; we have an essentially commutative diagram whose horizontal arrows are the Yoneda embeddings :

$$\begin{array}{ccc} T & \xrightarrow{h_T} & \mathrm{Can}(T)^\sim \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle (f^*)^\sim_*} \\ S & \xrightarrow{h_S} & \mathrm{Can}(S)^\sim. \end{array}$$

Indeed, an isomorphism $(f^*)^\sim_* \circ h_T \xrightarrow{\sim} h_S \circ f_*$ is given explicitly as follows. Let ϑ be the adjunction for the pair (f^*, f_*) resulting from the unit η. So, ϑ consists of a system of bijections:

$$h_X \circ f^*(Y) = \mathrm{Hom}_T(f^*Y, X) \xrightarrow{\vartheta_{Y,X}} \mathrm{Hom}_S(Y, f_*X) = h_{f_*X}(Y)$$

natural in $X \in \mathrm{Ob}(T)$ and $Y \in \mathrm{Ob}(S)$. The naturality with respect to morphisms $Y' \to Y$ in S then implies that the rule $Y \mapsto \vartheta_{Y,X}$ is an isomorphism of sheaves $\vartheta_{\bullet,X} : h_X \circ f^* \xrightarrow{\sim} h_{f_*X}$ for every $X \in \mathrm{Ob}(T)$. Lastly, the naturality with respect to morphisms $X \to X'$ in T says that the rule $X \mapsto \vartheta_{\bullet,X}$ yields an isomorphism of functors as sought : the details are left to the reader.

4.4.11. *Global section functors.* In view of theorem 4.4.2(iv), the objects of any topos T may be thought of as sheaves on $\mathrm{Can}(T)$, and one uses often the suggestive notation

$$X(S) := \mathrm{Hom}_T(S, X) \qquad \text{for every } X, S \in \mathrm{Ob}(T).$$

The elements of $X(S)$ are also called the *S-sections* of X. If 1_T is any final object of T, one defines the *global sections functor* $\Gamma : T \to \mathbf{Set}$ by the rule :

$$U \mapsto \Gamma(T, U) := U(1_T).$$

Moreover Γ admits a left adjoint :

$$(-)_T : \mathbf{Set} \to T \ : \ S \mapsto S_T := S \times 1_T$$

(the coproduct of S copies of 1_T) and one calls S_T the *constant sheaf with value S*.

Proposition 4.4.12. *With the notation of* (4.4.11)*, we have :*

(i) *The functor $(-)_T$ is exact, hence the pair $((-)_T, \Gamma)$ yields a morphism of topoi*

$$\Gamma_T : T \to \mathbf{Set}.$$

(ii) *For every pair of morphisms of topoi* $u, v : T \to \mathbf{Set}$ *there exists a unique natural transformation* $\beta : u_* \Rightarrow v_*$*, and* β *is an isomorphism of functors.*

(iii) *For every morphism* $f : T \to S$ *of topoi, there exists a unique isomorphism :*

$$\Gamma_S \circ f \stackrel{\sim}{\to} \Gamma_T.$$

Proof. (See [8, Exp.IV, §4.3]). For the proof of (i), we may assume that $T = C^\sim$ for a small site $C := (\mathscr{C}, J)$. Then, for every (small) set S, let $c_S : \mathscr{C}^o \to \mathbf{Set}$ be the constant presheaf on $\mathscr{C}$ with value S; since the colimits in $\mathscr{C}^o$ are computed argumentwise (corollary 1.4.3(ii)), we have $c_S = \coprod_{s \in S} c_{\{s\}}$, whence $S_T = c_S^a$, since the functor $(-)^a$ commutes with small colimits. Since $(-)^a$ is an exact functor (theorem 4.1.13(ii)), it suffices to check that the same holds for the functor $\mathbf{Set} \to \mathscr{C}^o : S \mapsto c_S$. But more precisely, the latter commutes with all small limits and all small colimits, again because the latter are computed argumentwise.

(ii): Since u^* is exact, $u^*(\{\varnothing\})$ is a final object of T, and we get natural bijections :

$$u_* X \stackrel{\sim}{\to} \mathrm{Hom}_{\mathbf{Set}}(\{\varnothing\}, u_* X) \stackrel{\sim}{\to} \mathrm{Hom}_T(u^*(\{\varnothing\}), X) \qquad \text{for every } X \in \mathrm{Ob}(T)$$

i.e. the presheaf $u_* : T \to \mathbf{Set}$ on the category T^o is represented by the initial object $u^*(\{\varnothing\})^o$. The same applies to v_*, and then Yoneda's lemma yields natural bijections

$$\mathrm{Hom}_{(T^o)^\wedge}(u_*, v_*) \stackrel{\sim}{\to} \mathrm{Hom}_T(v^*(\{\varnothing\}), u^*(\{\varnothing\})) \stackrel{\sim}{\to} \{\varnothing\}$$

whence the contention.

(iii) is an immediate consequence of (ii). □

Remark 4.4.13. (i) If T is a U-topos, $C := (\mathscr{C}, J)$ is a U-lex-site, and $g : \mathrm{Can}(T) \to C$ is a morphism of sites, then the underlying functor $g : \mathscr{C} \to T$ is left exact. Indeed, by corollary 4.3.19(i.b) we have an essentially commutative diagram of functors whose vertical arrows are the Yoneda embeddings :

(4.4.14)
$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{g} & T \\ {\scriptstyle h^a_{\mathscr{C}}}\downarrow & & \downarrow{\scriptstyle h_T} \\ C^\sim & \xrightarrow{\widetilde{g}^*} & \mathrm{Can}(T)^\sim. \end{array}$$

Then, h_T is an equivalence (theorem 4.4.2(iv)) and $\widetilde{g}^*$ is left exact, since g is a morphism of sites; it suffices therefore to show that $h^a_{\mathscr{C}}$ is left exact. But the functor $(-)^a : \mathscr{C}^\wedge \to C^\sim$ is left exact by theorem 4.1.13(ii), so the assertion follows from corollary 1.4.3(vi).

(ii) Every morphism of U-sites $u : C' \to C$ induces a morphism of topoi

$$\widetilde{u} := (\widetilde{u}^*, \widetilde{u}_*, \widetilde{\eta}^u) : C'^\sim \to C^\sim$$

after choosing a unit $\widetilde{\eta}^u$ for the adjoint pair $(\widetilde{u}^*, \widetilde{u}_*)$ (corollary 4.3.19(i)).

(iii) Conversely, a morphism $(f^*, f_*, \eta) : T \to S$ of topoi determines a morphism of sites

$$f^* : \mathrm{Can}(T) \to \mathrm{Can}(S).$$

Indeed, $g := f^* : S \to T$ is continuous for the canonical topologies, by virtue of proposition 4.4.8(i), and by corollary 4.3.19(i.b) we have an essentially commutative diagram of functors as in (4.4.14), with $\mathscr{C} := S$ and $C = \mathrm{Can}(S)$. In this case, both vertical arrows of (4.4.14) are equivalences (theorem 4.4.2(iv)). Since g is left exact, it then follows that the same holds for $\widetilde{g}^*$, whence the assertion.

(iv) Likewise, for every pair of U-sites $C := (\mathscr{C}, J), C' := (\mathscr{C}', J')$, for every cocontinuous functor $v : \mathscr{C} \to \mathscr{C}'$, it is clear that $\breve{v}^*$ is an exact functor, so v induces a morphism of topoi

$$\breve{v} := (\breve{v}^*, \breve{v}_*, \breve{\eta}^v) : C^\sim \to C'^\sim$$

(corollary 4.3.19(ii)), after choosing a unit $\breve{\eta}^v$ for the adjoint pair $(\breve{v}^*, \breve{v}_*)$.

(v) In the situation of (iv), suppose that v admits a right adjoint u. Then u is continuous for the topologies of the sites C and C', and $\widetilde{u}^*$ is isomorphic to $\breve{v}^*$ (lemma 4.2.14). Especially, $\widetilde{u}^*$ is exact, so u is a morphism of sites, and we have an isomorphism of morphisms of topoi :

$$\widetilde{u} \stackrel{\sim}{\to} \breve{v}.$$

4.4.15. We define as follows a pair of pseudo-functors

$$\mathbf{Site} \underset{\mathrm{Can}}{\overset{\mathsf{T}}{\rightleftarrows}} \mathbf{Topos}.$$

Let C be a U-site; by remark 4.4.1(iv) we may find an isomorphism of topoi

$$\omega_C : C^\sim \stackrel{\sim}{\to} \mathsf{T}(C)$$

with a U-topos $\mathsf{T}(C)$, and it is easily seen that we may take this topos to be also U'-small. Then the rule : $C \mapsto \mathsf{T}(C)$ defines the pseudo-functor T on objects. Next, let $g : C \to C'$ be any morphism of U-sites; by remark 4.4.13(ii), any choice of a unit η^g for the adjoint pair $(\widetilde{g}^*, \widetilde{g}_*)$ yields a morphism of topoi $\widetilde{g} : C^\sim \to C'^\sim$. There follows easily a unique morphism of U-topoi

$$\mathsf{T}(g) := (\mathsf{T}(g)^*, \mathsf{T}(g)_*, \eta^{\mathsf{T}(g)}) : \mathsf{T}(C) \to \mathsf{T}(C')$$

that is identified with $(\widetilde{g}^*, \widetilde{g}_*, \eta^g)$ via the isomorphisms ω_C and $\omega_{C'}$ (details left to the reader). Lastly, let $\beta : g \Rightarrow g'$ be a natural transformation between morphisms of sites $g, g' : C \to C'$; there follows a natural transformation $\beta^\wedge : g'^\wedge \Rightarrow g^\wedge$ (notation of (1.2.2)) that induces by restriction a natural transformation

$$\beta^\sim_* : \widetilde{g}'_* \Rightarrow \widetilde{g}_*$$

which is in turn identified via ω_C and $\omega_{C'}$ with a unique natural transformation

$$\mathsf{T}(\beta) : \mathsf{T}(g')_* \Rightarrow \mathsf{T}(g)_*.$$

If $g'' : C \to C'$ is a third morphism of sites and $\beta' : g' \Rightarrow g''$ is another natural transformation, it is clear that

$$\mathsf{T}(\beta' \odot \beta) = \mathsf{T}(\beta) \odot \mathsf{T}(\beta').$$

Likewise, if $h, h' : C' \to C''$ is another pair of morphisms of sites and $\alpha : h \Rightarrow h'$ is another natural transformation, we deduce easily from (1.2.2) the identity :

$$\mathsf{T}(\alpha * \beta) = \mathsf{T}(\beta) * \mathsf{T}(\alpha).$$

For a composable pair $C \xrightarrow{g} C' \xrightarrow{g'} C''$ of morphisms of U-sites, it is easily seen that $\mathsf{T}(g')_* \circ \mathsf{T}(g)_* = \mathsf{T}(g' \circ g)_*$, and the coherence constraint $\gamma^{\mathsf{T}}_{g,g'}$ of T is given by $\mathbf{1}_{\mathsf{T}(g' \circ g)_*}$. Likewise, $\mathsf{T}(\mathbf{1}_C)_* = \mathbf{1}_{\mathsf{T}(C)}$ for every U-site C, and the coherence constraint δ^{T}_C is given by the identity automorphism of $\mathbf{1}_{\mathsf{T}(C)}$. Then the required coherence axioms are trivially satified; however, notice that T is *not* a strict pseudo-functor, since we do not have necessarily $\mathsf{T}(g)^* \circ \mathsf{T}(g')^* = \mathsf{T}(g' \circ g)^*$. This completes the construction of T.

The pseudo-functor Can assigns to every U-topos T the U-site $\mathrm{Can}(T)$ and to every morphism of U-topoi $f : T \to S$ the morphism of sites $f^* : \mathrm{Can}(T) \to \mathrm{Can}(S)$ (see remark 4.4.13(iii)). Lastly, let $(f^*, f_*, \eta^f), (g^*, g_*, \eta^g) : T \to S$ be two morphisms of topoi and $\beta : f_* \Rightarrow g_*$ a natural transformation; then we let $\mathrm{Can}(\beta) := (\beta, \eta^f, \eta^g)^\dagger : g^* \Rightarrow f^*$, the adjoint transformation of β, as defined in remark 1.1.17(ii). Taking into account remark 1.1.17(iv,v,vi), we easily see that these rules define a strict pseudo-functor Can as sought.

Remark 4.4.16. Notice that the pseudo-functor Can factors through the inclusion strict pseudo-functor $\mathsf{lex}.\mathbf{Site} \to \mathbf{Site}$. We then get as well a pair of pseudo-functors :

$$\mathsf{lex}.\mathbf{Site} \underset{\mathsf{lex}.\mathrm{Can}}{\overset{\mathsf{lex}.\mathsf{T}}{\rightleftarrows}} \mathbf{Topos}.$$

where $\mathsf{lex.T}$ is the restriction of T.

Theorem 4.4.17. (i) *The pseudo-functor* T *is right* 2*-adjoint to the pseudo-functor* Can.
(ii) *The pseudo-functor* $\mathsf{lex.T}$ *is right* 2*-adjoint to the pseudo-functor* $\mathsf{lex.}\mathrm{Can}$.
(iii) *The pseudo-functors* Can *and* $\mathsf{lex.}\mathrm{Can}$ *are fully faithful.*

Proof. (i): We construct as follows a pseudo-natural equivalence $\mathsf{h}_\bullet : \mathbf{1}_{\mathbf{Topos}} \Rightarrow \mathsf{T}\circ\mathrm{Can}$. For every U-topos T, we choose a quasi-inverse $\mathsf{h}^*_T : \mathsf{T}\circ\mathrm{Can}(T) \to T$ for the composition $\mathsf{h}_{T*} : T \to \mathsf{T}\circ\mathrm{Can}(T)$ of the Yoneda embedding $h_T : T \to \mathrm{Can}(T)^\sim$ with the isomorphism $\omega_{\mathrm{Can}(T)}$ of (4.4.15). We also fix an isomorphism of functors $\eta^{\mathsf{h}_T} : \mathbf{1}_{\mathsf{T}\circ\mathrm{Can}(T)} \xrightarrow{\sim} \mathsf{h}_{T*}\circ\mathsf{h}^*_T$, and we consider the morphism of U-topoi :

$$\mathsf{h}_T := (\mathsf{h}^*_T, \mathsf{h}_{T*}, \eta^{\mathsf{h}_T}) : T \to \mathsf{T}\circ\mathrm{Can}(T).$$

In order to show that the rule $T \mapsto \mathsf{h}_T$ defines the sought pseudo-natural equivalence, we need to explicit its coherence constraint; the latter amounts to an isomorphism of functors

$$\tau^{\mathsf{h}}_f : \mathsf{T}(f^*)_*\circ\mathsf{h}_{T*} \xrightarrow{\sim} \mathsf{h}_{S*}\circ f_*$$

for every morphism of U-topoi $(f^*, f_*, \eta^f) : T \to S$. However, example 4.4.10(ii) yields an isomorphism of functors $\tau^h_f : (f^*)^\sim_* \circ h_T \xrightarrow{\sim} h_S\circ f_*$, so we can take :

$$\tau^{\mathsf{h}}_f := \omega_{(S,\mathrm{Can}_S)} * \tau^h_f.$$

To check the coherence axioms for τ^{h}, notice first that T and Can are unital pseudo-functors, so that the first coherence axiom amounts to the identity : $\tau^{\mathsf{h}}_{\mathbf{1}_T} = \mathbf{1}_{\mathsf{h}_{T*}}$ for every U-topos T. The latter follows from the identity : $\tau^h_{\mathbf{1}_T} = \mathbf{1}_{h_T}$, which is clear from the construction of τ^h.

Next, for two morphisms of U-topoi $f := (f^*, f_*, \eta^f) : T \to T'$, $g := (g^*, g_*, \eta^g) : T' \to T''$, we come down to checking that

$$(\eta^h_g * f_*) \odot ((g^*)^\sim_* * \eta^h_f) = \eta^h_{g\circ f}. \tag{4.4.18}$$

Now, recall that η^h_f is induced by the unique adjunction ϑ^f for the pair (f^*, f_*) whose unit is η^f, and likewise for η^h_g and $\eta^h_{g\circ f}$; on the other hand, $\eta^h_{g\circ f}$ is precisely the unit of the adjunction $\vartheta^g\circ\vartheta^f$ (notation of remark 1.1.17(i)). From this, the identity (4.4.18) follows straightforwardly.

Likewise, let $f, f' : T \to S$ be two morphisms of topoi, and $\beta : f \Rightarrow f'$ a natural transformation; in order to check the naturality of τ^{h} we come down to showing that :

$$\tau^h_{f'} \odot (\beta^\dagger)^\sim_* = (h_S * \beta) \odot \tau^h_f.$$

The latter follows by inspecting the characterization of $\beta^\dagger$ furnished by remark 1.1.17(iii), and this completes the construction of $\mathsf{h}_\bullet$. From the pseudo-natural equivalence $\mathsf{h}_\bullet$ we construct the sought 2-adjunction as the composition of the pseudo-natural transformation

$$H_{\mathsf{T}} * (\mathrm{Can}^o \times \mathbf{1}_{\mathbf{Site}}) : \mathrm{Hom}_{\mathbf{Site}}(\mathrm{Can}, -) \Rightarrow \mathrm{Hom}_{\mathbf{Topos}}(\mathsf{T}\circ\mathrm{Can}, \mathsf{T})$$

with the pseudo-natural transformation

$$H_{\mathbf{Topos}} * (\mathsf{h}^o_\bullet \times \mathbf{1}_{\mathbf{Topos}}) : \mathrm{Hom}_{\mathbf{Topos}}(\mathsf{T}\circ\mathrm{Can}, \mathsf{T}) \Rightarrow \mathrm{Hom}_{\mathbf{Topos}}(\mathbf{1}_{\mathbf{Topos}}, \mathsf{T}).$$

This composition assigns to every U-topos T and every U-site C a functor

$$\Phi_{T,C} : \mathrm{Hom}_{\mathbf{Site}}(\mathrm{Can}(T), C) \to \mathrm{Hom}_{\mathbf{Topos}}(T, \mathsf{T}(C))$$

as required, and it remains to check that this functor is an equivalence for every such T and C.

Explicitly, $\Phi_{T,C}$ assigns to every morphism of sites $g : \mathrm{Can}(T) \to C$ the morphism of topoi $\mathsf{T}(g)\circ\mathsf{h}_T : T \to \mathsf{T}(C)$, and to every 2-cell $\beta : g \Rightarrow g'$ of $\mathbf{Site}$ the natural transformation $\mathsf{T}(\beta) * \mathsf{h}_T : \mathsf{T}(g)\circ\mathsf{h}_T \Rightarrow \mathsf{T}(g')\circ\mathsf{h}_T$. Let us show the essential surjectivity : thus, say that $C = (\mathscr{C}, J)$, and let $f := (f^*, f_*, \eta^f) : T \to \mathsf{T}(C)$ be a morphism of topoi; from theorem 4.4.2(ii) we see that $g := f^*\circ\omega_C\circ h^a_{\mathscr{C}} : \mathscr{C} \to T$ is a morphism of sites $g : \mathrm{Can}(T) \to C$, and we have an isomorphism of functors $h_T \circ g \xrightarrow{\sim} \widetilde{g}^* \circ h^a_{\mathscr{C}}$ (corollary 4.3.19(i.b)). We deduce an isomorphism :

$$\mathsf{h}_{T*} \circ f^* \circ \omega_C \circ h^a_{\mathscr{C}} \xrightarrow{\sim} \omega_{\mathrm{Can}(T)} \circ \widetilde{g}^* \circ h^a_{\mathscr{C}}. \tag{4.4.19}$$

Claim 4.4.20. Let D be any category, $k, k' : C^\sim \to D$ two functors, $\lambda : k \circ h^a_{\mathscr{C}} \xrightarrow{\sim} k' \circ h^a_{\mathscr{C}}$ an isomorphism of functors, and suppose that k commutes with all small colimits. Then there exists an isomorphism of functors $\lambda' : k \xrightarrow{\sim} k'$.

Proof of the claim. Let $G \subset \mathrm{Ob}(\mathscr{C})$ be an essentially small topologically generating subset, $\mathscr{G} \subset \mathscr{C}$ the full subcategory with $\mathrm{Ob}(\mathscr{G}) = G$; endow $\mathscr{G}$ with the topology $J_{\mathscr{G}}$ induced by J via the inclusion functor $u : \mathscr{G} \to \mathscr{C}$, and set $G := (\mathscr{G}, J_{\mathscr{G}})$. Then $\widetilde{u}_* : C^\sim \to G^\sim$ is an equivalence (proposition 4.3.18), hence the same holds for its left adjoint $\widetilde{u}^* : G^\sim \to C^\sim$. According to corollary 4.3.19(i.b), we have an isomorphism of functors $\mu : \widetilde{u}^* \circ h^a_{\mathscr{G}} \xrightarrow{\sim} h^a_{\mathscr{C}} \circ i$. There follows a unique isomorphism $\lambda' : k \circ \widetilde{i}^* \circ h^a_{\mathscr{G}} \xrightarrow{\sim} k' \circ \widetilde{i}^* \circ h^a_{\mathscr{G}}$ such that

$$(k' * \mu) \odot \lambda' = (\lambda * i) \odot (k * \omega).$$

Now, if $k \circ \widetilde{i}^*$ and $k' \circ \widetilde{i}^*$ are isomorphic, the same follows for k and k', since $\widetilde{i}^*$ is an equivalence (details left to the reader). Thus, we may replace C by G and λ by λ', and assume from start that C is a small site. Next, let $F \in \mathrm{Ob}(C^\sim)$; we have a universal cocone τ^F given by the rule :

$$(X, s) \mapsto (\tau^F_{(X,s)} : h^a_X \to F) \qquad \text{for every } (X, s) \in \mathrm{Ob}(\mathscr{F}ib(F))$$

where $\tau^F_{(X,s)}$ is characterized as the morphism of sheaves whose composition with the natural morphism of presheaves $h_X \to h^a_X$ is the unique morphism of presheaves $t^F_{(X,s)} : h_X \to F$ such that $(t^F_{(X,s)})_X(\mathbf{1}_X) = s$ (remark 4.1.19(iii)). By assumption, the cocone $k * \tau^F$ is still universal, hence there exists a unique isomorphism $\lambda'_F : kF \xrightarrow{\sim} k'F$ that makes commute the diagram :

$$\begin{array}{ccc} k(h^a_X) & \xrightarrow{k(\tau^F_{(X,s)})} & kF \\ {\scriptstyle \lambda_X}\downarrow & & \downarrow{\scriptstyle \lambda'_F} \\ k'(h^a_X) & \xrightarrow{k'(\tau^F_{(X,s)})} & k'F \end{array} \qquad \text{for every } (X, s) \in \mathrm{Ob}(\mathscr{F}ib(F))$$

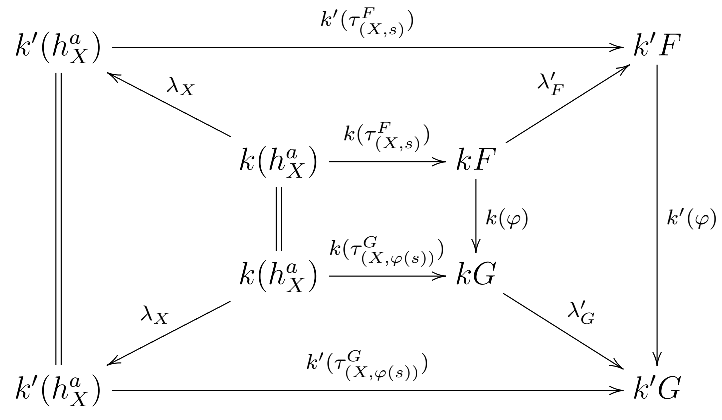
and we are reduced to checking that the rule : $F \mapsto \lambda'_F$ is a natural transformation $k \Rightarrow k'$. Thus, let $\varphi : F \to G$ be any morphism of sheaves on C; recall that φ induces a functor

$$\mathscr{F}ib(\varphi) : \mathscr{F}ib(F) \to \mathscr{F}ib(G) \qquad (X, s) \mapsto (X, \varphi_X(s))$$

and the foregoing characterizations of τ^F and τ^G easily imply that :

$$\varphi \circ \tau^F_{(X,s)} = \tau^G_{(X,\varphi_X(s))} \qquad \text{for every } (X, s) \in \mathrm{Ob}(\mathscr{F}ib(F)). \tag{4.4.21}$$

Consider now for every $(X, s) \in \mathrm{Ob}(\mathscr{F}ib(F))$ the diagram :



We have to prove the commutativity of the right trapezoidal subdiagram, and by (4.4.21), the inner and outer square subdiagrams both commute; moreover, also the upper and lower trapezoidal subdiagrams commute, by construction. Now, since by assumption the cocone $k * \tau^F$ is still universal, it suffices to check that $k'(\varphi) \circ \lambda'_F \circ k(\tau^F_{(X,s)}) = \lambda'_G \circ k(\varphi) \circ k(\tau^F_{(X,s)})$. The latter follows from a straightforward diagram chase, left to the reader. ◊

Since $\widetilde{g}^*$ commutes with small colimits, claim 4.4.20 and (4.4.19) yield an isomorphism :

$$\mathsf{h}_{T*} \circ f^* \circ \omega_C \stackrel{\sim}{\to} \omega_{\mathrm{Can}(T)} \circ \widetilde{g}^*.$$

From this, we further deduce an isomorphism of functors :

$$f^* \stackrel{\sim}{\to} \mathsf{h}_T^* \circ \mathsf{T}(g)^*$$

whence, finally, an isomorphism of their respective right adjoint functors : $\mathsf{T}(g)_* \circ \mathsf{h}_{T*} \stackrel{\sim}{\to} f_*$.

Next, we check that $\Phi_{T,C}$ is faithful : let $g, g' : \mathrm{Can}(T) \to C$ be two morphisms of sites, and $\beta, \beta' : g \Rightarrow g'$ two 2-cells of $\mathbf{Site}$ such that $\mathsf{T}(\beta)_* * \mathsf{h}_T = \mathsf{T}(\beta')_* * \mathsf{h}_T$; we need to show that $\beta = \beta'$. However, the condition means that $\widetilde{\beta}_* \circ h_T = \widetilde{\beta}'_* \circ h_T$; *i.e.* for every $X \in \mathrm{Ob}(T)$, every $A \in \mathrm{Ob}(\mathscr{C})$ and every morphism $\varphi : gA \to X$ in T, we have $\varphi \circ \beta_A = \varphi \circ \beta'_A$. Letting $X := gA$ and $\varphi := \mathbf{1}_{gA}$, we get $\beta_A = \beta'_A$, for every $A \in \mathrm{Ob}(\mathscr{C})$, as required.

Lastly, in order to check that $\Phi_{T,C}$ is full, let $\alpha : \mathsf{T}(g) \circ \mathsf{h}_T \Rightarrow \mathsf{T}(g') \circ \mathsf{h}_T$ be any natural transformation, with g and g' as in the foregoing. This is the same as a natural transformation $\alpha' : \widetilde{g}_* \circ h_T \Rightarrow \widetilde{g}'_* \circ h_T$. The latter assigns to every $X \in \mathrm{Ob}(T)$ a morphism of sheaves $\alpha'_X : h_X \circ g^o \to h_X \circ g'^o$; in particular, for every $A \in \mathrm{Ob}(\mathscr{C})$, we deduce a map

$$(\alpha'_{gA})_A : \mathrm{Hom}_T(gA, gA) \to \mathrm{Hom}_T(g'A, gA) \qquad \text{and we set :} \qquad \beta_A := (\alpha'_{gA})_A(\mathbf{1}_{gA}).$$

Let us check that the rule : $A \mapsto \beta_A$ yields a natural transformation $\beta : g' \to g$. Indeed, let $\varphi : A \to B$ be any morphism of $\mathscr{C}$; we need to show that $g(\varphi) \circ \beta_A = \beta_B \circ g'(\varphi)$. However, the naturality of the rule $X \mapsto \alpha'_X$ for every $X \in \mathrm{Ob}(T)$ implies the identity :

$$g(\varphi) \circ (\alpha'_{gA})_A(\mathbf{1}_{gA}) = (\alpha'_{gB})_A(g(\varphi))$$

and the naturality of the rule : $Y \mapsto (\alpha'_{gB})_Y$ for every $Y \in \mathrm{Ob}(\mathscr{C})$ yields the identity :

$$(\alpha'_{gB})_A(g(\varphi)) = (\alpha'_{gB})_B(\mathbf{1}_{gB}) \circ g'(\varphi)$$

whence the contention. It remains to check that $\mathsf{T}(\beta)_* * \mathsf{h}_T = \alpha$, or equivalently, that $\widetilde{\beta}_* * h_T = \alpha'$. The latter comes down to the identity : $(\alpha'_X)_A(\varphi) = \varphi \circ \beta_A$ for every $X \in \mathrm{Ob}(T)$, every $A \in \mathrm{Ob}(\mathscr{C})$ and every morphism $\varphi : gA \to X$ in T. This identity in turn follows easily from the naturality of the rule : $X \mapsto \alpha'_X$ (details left to the reader).

(ii): Notice that – by virtue of example 4.3.12 – the same rules defining $\mathsf{h}_\bullet$ also yield a pseudo-natural transformation $\mathbf{1_{Topos}} \Rightarrow \mathsf{lex.T} \circ \mathsf{lex.Can}$. We use this pseudo-natural transformation as in the foregoing to construct the sought 2-adjunction for the pair $(\mathsf{lex.Can}, \mathsf{lex.T})$; the details shall be left to the reader.

(iii): Since $\mathsf{h}_\bullet$ is a pseudo-natural equivalence, the assertion for Can follows from corollary 2.4.29. The same arguments applies to $\mathsf{lex.Can}$. □

Remark 4.4.22. As an application of theorem 4.4.17 we deduce the representability of the 2-limit of any cofiltered system of topoi. Indeed, let I be a small cofiltered category, and $T_\bullet : I \to (\mathsf{U}, \mathsf{U}')\text{-}\mathbf{Topos}$ any pseudo-functor. By proposition 4.3.25, the pseudo-functor $\mathsf{lex.Can} \circ T_\bullet : I \to \mathsf{lex.}\mathbf{Site}$ admits a 2-limit $(C, \pi_\bullet)$. By theorem 4.4.17(ii) and proposition 2.5.9(i), the pair $(\mathsf{lex.T}(C), \mathsf{lex.T} * \pi_\bullet)$ is a 2-limit of the pseudo-functor $\mathsf{lex.T} \circ \mathsf{lex.Can} \circ T_\bullet : I \to (\mathsf{U}, \mathsf{U}')\text{-}\mathbf{Topos}$. But by theorem 4.4.17(iii) and corollary 2.4.29, we have a pseudo-natural equivalence $\omega : \mathsf{lex.T} \circ \mathsf{lex.Can} \stackrel{\sim}{\to} \mathbf{1_{Topos}}$, so $(\mathsf{lex.T}(C), (\omega * T_\bullet) \odot (\mathsf{lex.T} * \pi_\bullet))$ is a 2-limit for $T_\bullet$ (lemma

2.5.3). Combining with remark 4.3.27, we see that the 2-limit of $T_\bullet$ is also a 2-limit of the induced pseudo-functor

$$I \to \mathsf{U}'\text{-}\mathbf{Cat} \qquad i \mapsto T_i \qquad (\varphi : i \to j) \mapsto (T_{\varphi *} : T_i \to T_j).$$

4.5. **Fibred sites.** The formalism developed in this section and the next one shall allow us to deal with families of sites or of topoi, indexed by arbitrary categories. Especially, we will explain how to combine the members of such a family into a single *total site* or *total topos*.

Definition 4.5.1. (i) Let $\mathscr{C}$ be a category and V a universe. A *fibred site with* V*-small fibres over* $\mathscr{C}$ is a datum $(\mathscr{A}, p, J_\bullet)$ consisting of a fibration with V-small fibre categories

$$p : \mathscr{A} \to \mathscr{C}$$

and of a topology J_X on the fibre category $\mathscr{A}_X$, for every $X \in \mathrm{Ob}(\mathscr{C})$, such that the following holds. For every cleavage $\boldsymbol{\lambda}$ of p and every morphism $\varphi : X \to Y$ in $\mathscr{C}$, the functor $\mathsf{c}_\varphi : \mathscr{A}_Y \to \mathscr{A}_X$ is a morphism of sites $(\mathscr{A}_X, J_X) \to (\mathscr{A}_Y, J_Y)$, where $\mathsf{c} : \mathscr{C}^o \to \mathsf{V}\text{-}\mathbf{Cat}$ is the pseudo-functor associated with $\boldsymbol{\lambda}$.

(ii) A datum $(\mathscr{A}, p, J_\bullet)$ as in (i) is a *fibred lex-site with* V*-small fibres* if $(\mathscr{A}_X, J_X)$ is a lex-site for every $X \in \mathrm{Ob}(\mathscr{C})$ and the functors c_φ are left exact for every morphism φ of $\mathscr{C}$ and every cleavage $\boldsymbol{\lambda}$ with associated pseudo-functor c. As usual, we shall mostly omit mentioning the universe V, unless the omission would be source of ambiguities.

(iii) Let $(\mathscr{A}, p, J_\bullet)$ be a fibred site over the category $\mathscr{C}$. We endow $\mathscr{A}$ with the coarsest topology $J_{\mathscr{A}}$ such that the inclusion functor $\mathscr{A}_X \to \mathscr{A}$ is continuous for the topologies J_X and $J_{\mathscr{A}}$, for every $X \in \mathrm{Ob}(\mathscr{C})$. The *total site* of $(\mathscr{A}, p, J_\bullet)$ is the resulting site

$$(\mathscr{A}, J_{\mathscr{A}}).$$

(iv) Let $(\mathscr{A}, p, J_\bullet)$ and $(\mathscr{A}', p', J'_\bullet)$ be two fibred sites over $\mathscr{C}$. A *morphism of fibred sites*

$$\varphi : (\mathscr{A}, p, J_\bullet) \to (\mathscr{A}', p', J'_\bullet)$$

is a $\mathscr{C}$-cartesian functor $\varphi : \mathscr{A}' \to \mathscr{A}$ whose restriction $\varphi_X : \mathscr{A}'_X \to \mathscr{A}_X$ is a morphism of sites $(\mathscr{A}_X, J_X) \to (\mathscr{A}'_X, J'_X)$, for every $X \in \mathrm{Ob}(\mathscr{C})$. If $(\mathscr{A}, p, J_\bullet)$ and $(\mathscr{A}', p', J'_\bullet)$ are fibred lex-sites, we say that φ is a *morphism of fibred lex-sites*, if it is $\mathscr{C}$-cartesian, and the restriction $\varphi_X : (\mathscr{A}_X, J_X) \to (\mathscr{A}'_X, J'_X)$ is a morphism of lex-sites, for every $X \in \mathrm{Ob}(\mathscr{C})$.

Remark 4.5.2. (i) With the notation of definition 4.5.1, it is clear that if there exists a cleavage of p with associated pseudo-functor c, such that c_φ is a morphism of sites $(\mathscr{A}_X, J_X) \to (\mathscr{A}_Y, J_Y)$ for every morphism $\varphi : X \to Y$, then the same holds for every cleavage of p (and in this case, p is a fibred site). Likewise, if c_φ is also left exact for every such φ, then the same holds for every cleavage of p (and then p is a fibred lex-site).

(ii) Let $(\mathscr{A}, p, J_\bullet)$ and $(\mathscr{A}, J_{\mathscr{A}})$ be as in definition 4.5.1(iii). By lemma 4.2.4, we have $\mathscr{S} \in J_{\mathscr{A}}(A)$ for every $A \in \mathrm{Ob}(\mathscr{A})$ and every sieve $\mathscr{S} \subset \mathscr{A}/A$ such that

$$\mathscr{S} \cap (\mathscr{A}_{pA}/A) \in J_{pA}(A). \tag{4.5.3}$$

(iii) In the situation of (ii), suppose moreover that $\mathscr{C}$ is small and $(\mathscr{A}_X, J_X)$ is a U-site for every $X \in \mathrm{Ob}(\mathscr{C})$; under these assumptions, the fibre category $\mathscr{A}_X$ has small Hom-sets for every such X, but the same does not necessarily hold for the category $\mathscr{A}$. However, for every $A, A' \in \mathrm{Ob}(\mathscr{A})$ the set $\mathrm{Hom}_{\mathscr{A}}(A, A')$ is essentially small : indeed, let $\boldsymbol{\lambda}$ be a cleavage for $\mathscr{A}$ and c its associated pseudo-functor; since every morphism $f : A' \to A$ in $\mathscr{A}$ factors as a morphism $A' \to \mathsf{c}_{p(f)}A$ of $\mathscr{A}_{pA'}$ and the cartesian morphism $\boldsymbol{\lambda}(A, p(f)) : \mathsf{c}_{p(f)}A \to A$, the assertion follows easily. Thus, after replacing $\mathscr{A}$ by an isomorphic category, we may assume that $\mathscr{A}$ has small Hom-sets, and in this case we claim that $(\mathscr{A}, J_{\mathscr{A}})$ is a U-site. Indeed, choose for every $X \in \mathrm{Ob}(\mathscr{C})$ an essentially small topologically generating family $G_X \subset \mathrm{Ob}(p^{-1}X)$

for $(\mathscr{A}_X, J_X)$; then it is clear from (ii) that $\bigcup_{X\in\mathrm{Ob}(\mathscr{C})} G_X$ is an essentially small topologically generating family for $(\mathscr{A}, J_{\mathscr{A}})$.

(iv) Clearly the fibred sites (resp. the fibred lex-sites) over $\mathscr{C}$ with V-small fibres are the objects of a 2-category :

$$\mathsf{V}\text{-fib}.\mathbf{Site}(\mathscr{C}) \qquad \text{(resp. } \mathsf{V}\text{-fib.lex}.\mathbf{Site})$$

whose 1-cells are the morphisms of fibred sites (resp. of fibred lex-sites), and whose 2-cells $g \Rightarrow g'$ are the natural $\mathscr{C}$-transformations $g' \Rightarrow g$ between the underlying functor of such morphisms, with the obvious composition laws for 1-cells and 2-cells. We have an obvious forgetful strict pseudo-functor :

$$\mathsf{V}\text{-fib}.\mathbf{Site}(\mathscr{C}) \to {}^o(\mathsf{V}\text{-}\mathsf{Fib}(\mathscr{C}))^o \qquad (\mathscr{A}, p, J_\bullet) \mapsto (\mathscr{A}, p).$$

(v) Let $\mathscr{C}$ be a category with V-small Hom-sets and whose set of objects is essentially V-small. Taking into account theorem 3.1.24, it is easily seen that the pseudo-functor $\mathscr{F}ib_{\mathscr{C}}$ induces strict and strong 2-equivalences of 2-categories :

$$\underline{\mathscr{F}ib_{\mathscr{C}}} : \mathsf{PsFun}(\mathscr{C}, \mathsf{V}\text{-}\mathbf{Site}) \to \mathsf{V}\text{-fib}.\mathbf{Site}(\mathscr{C})$$
$$\mathsf{lex}.\underline{\mathscr{F}ib_{\mathscr{C}}} : \mathsf{PsFun}(\mathscr{C}, \mathsf{V}\text{-lex}.\mathbf{Site}) \to \mathsf{V}\text{-fib.lex}.\mathbf{Site}(\mathscr{C}).$$

Namely, we have an obvious forgetful strict pseudo-functor

$$\Phi : \mathsf{V}\text{-}\mathbf{Site} \to {}^o(\mathsf{V}\text{-}\mathbf{Cat})^o \qquad (\mathscr{A}, J) \mapsto \mathscr{A}$$

whence a pseudo-functor $\mathsf{PsFun}(\mathscr{C}, \Phi) : \mathsf{PsFun}(\mathscr{C}, \mathsf{V}\text{-}\mathbf{Site}) \to \mathsf{PsFun}(\mathscr{C}, {}^o(\mathsf{V}\text{-}\mathbf{Cat})^o)$ (see remark 2.2.17(i)). Then, since ${}^o\mathscr{C}^o = \mathscr{C}^o$, we have a strict isomorphism of 2-categories :

$$\mathsf{PsFun}(\mathscr{C}, {}^o(\mathsf{V}\text{-}\mathbf{Cat})^o) \xrightarrow{\sim} {}^o\mathsf{PsFun}(\mathscr{C}^o, \mathsf{V}\text{-}\mathbf{Cat})^o$$

(see (2.2.13)) and a strict and strong 2-equivalence

$${}^o\mathscr{F}ib^o_{\mathscr{C}} : {}^o\mathsf{PsFun}(\mathscr{C}^o, \mathsf{V}\text{-}\mathbf{Cat})^o \xrightarrow{\sim} {}^o(\mathsf{V}\text{-}\mathsf{Fib}(\mathscr{C}))^o.$$

The composition $\mathsf{PsFun}(\mathscr{C}, \mathsf{V}\text{-}\mathbf{Site}) \to {}^o(\mathsf{V}\text{-}\mathsf{Fib}(\mathscr{C}))^o$ of these pseudo-functors assigns to every pseudo-functor

$$\underline{\mathscr{A}} : \mathscr{C} \to \mathsf{V}\text{-}\mathbf{Site} \qquad X \mapsto (\mathscr{A}_X, J_X)$$

the fibration $\mathscr{A}' := \mathscr{F}ib({}^o(\Phi \circ \underline{\mathscr{A}})^o)$ whose fibre category $\mathscr{A}'_X$ over every $X \in \mathrm{Ob}(\mathscr{C})$ is naturally identified with $\mathscr{A}_X$. Then $\underline{\mathscr{F}ib}(\underline{\mathscr{A}})$ is the fibred site $(\mathscr{A}', J'_\bullet)$ where J'_X is the topology on $\mathscr{A}'_X$ corresponding to J_X under this identification $\mathscr{A}'_X \xrightarrow{\sim} \mathscr{A}_X$, for every $X \in \mathrm{Ob}(\mathscr{C})$.

(vi) By remark 2.4.31, the pseudo-functor $\underline{\mathscr{F}ib_{\mathscr{C}}}$ admits a strict and strong pseudo-inverse

$$\underline{\mathsf{c}}^\bullet : \mathsf{V}\text{-fib}.\mathbf{Site} \to \mathsf{PsFun}(\mathscr{C}, \mathsf{V}\text{-}\mathbf{Site}) \qquad \underline{\mathscr{A}} := (\mathscr{A}, p, J_\bullet) \mapsto \underline{\mathsf{c}}^{\underline{\mathscr{A}}}$$

which, to every fibred site $(\mathscr{A}, p, J_\bullet)$ over $\mathscr{C}$, assigns the pseudo-functor $\mathsf{c}^{\mathscr{A}} : \mathscr{C}^o \to \mathsf{V}\text{-}\mathbf{Cat}$ associated with a cleavage of $\mathscr{A}$, and for every $X \in \mathrm{Ob}(\mathscr{C})$, endows the category $\mathsf{c}^{\mathscr{A}}_X = \mathscr{A}_X$ with the topology J_X. Obviously $\underline{\mathsf{c}}^\bullet$ restricts to a strict and strong pseudo-inverse for $\mathsf{lex}.\underline{\mathscr{F}ib_{\mathscr{C}}}$.

4.5.4. Let $\mathscr{C}$ be a category, $(\mathscr{A}, p, J_\bullet)$ a fibred lex-site over $\mathscr{C}$, and $(\mathscr{A}, J_{\mathscr{A}})$ its total site. For every $X \in \mathrm{Ob}(\mathscr{C})$ denote by $i_X : \mathscr{A}_X \to \mathscr{A}$ the inclusion functor. We have :

Proposition 4.5.5. *In the situation of* (4.5.4)*, the following holds :*

(i) *The functor i_X commutes with fibre products and equalizers for every $X \in \mathrm{Ob}(\mathscr{C})$.*

(ii) *For every $A \in \mathrm{Ob}(\mathscr{A})$ and every sieve $\mathscr{S} \subset \mathscr{A}/A$, we have $\mathscr{S} \in J_{\mathscr{A}}(A)$ if and only if $\mathscr{S}$ satisfies condition* (4.5.3)*.*

(iii) *The functor i_X is both continuous and cocontinuous for the topologies J_X and $J_{\mathscr{A}}$, for every $X \in \mathrm{Ob}(\mathscr{C})$.*

(iv) *A presheaf F on $\mathscr{A}$ is a sheaf for the topology $J_{\mathscr{A}}$ if and only if $i^\wedge_X F$ is a sheaf on $(\mathscr{A}_X, J_X)$ for every $X \in \mathrm{Ob}(\mathscr{C})$.*

(v) *Suppose moreover that $\mathscr{C}$ is small, and $(\mathscr{A}_X, J_X)$ is a* U*-site for every $X \in \mathrm{Ob}(\mathscr{C})$. Then we have an essentially commutative diagram of categories :*

$$\begin{array}{ccc} \mathscr{A}^\wedge & \xrightarrow{i_X^\wedge} & \mathscr{A}_X^\wedge \\ {\scriptstyle (-)^a}\downarrow & & \downarrow{\scriptstyle (-)^a} \\ (\mathscr{A}, J_{\mathscr{A}})^\sim & \xrightarrow{i_{X*}^\sim} & (\mathscr{A}_X, J_X)^\sim \end{array} \qquad \textit{for every } X \in \mathrm{Ob}(\mathscr{C}).$$

Proof. Choose a cleavage $\boldsymbol{\lambda}$ for p and let $\mathsf{c} : \mathscr{C}^o \to \mathsf{V}\text{-}\mathbf{Cat}$ be the associated pseudo-functor (for some universe V such that $\mathscr{A}_X$ is a V-small category for every $X \in \mathrm{Ob}(\mathscr{C})$).

(i): Let $A' \xrightarrow{f'} A \xleftarrow{f''} A''$ be two morphisms in $\mathscr{A}_X$; pick $P \in \mathrm{Ob}(\mathscr{A}_X)$ representing $A' \times_A A''$ in $\mathscr{A}_X$, and let $A' \xleftarrow{q'} P \xrightarrow{q''} A''$ be a universal cone. Let then $B \in \mathrm{Ob}(\mathscr{A})$ be any object and $A' \xleftarrow{g'} B \xrightarrow{g''} A''$ two morphisms in $\mathscr{A}$ such that $f' \circ g' = f'' \circ g''$. It follows easily that $\varphi := p(g') = p(g'')$, so g' (resp. g'') factors through a morphism $h' : B \to \mathsf{c}_\varphi A'$ (resp. $h'' : B \to \mathsf{c}_\varphi A''$) in $\mathscr{A}_{pB}$ and the cartesian morphism $\boldsymbol{\lambda}(A', \varphi) : \mathsf{c}_\varphi A' \to A'$ (resp. $\boldsymbol{\lambda}(A'', \varphi) : \mathsf{c}_\varphi A'' \to A''$), and since c_φ is left exact, we get a unique morphism $h : B \to \mathsf{c}_\varphi P$ in $\mathscr{A}_{pB}$ such that $\mathsf{c}_\varphi(q') \circ h = h'$ and $\mathsf{c}_\varphi(q'') \circ h = h''$. Then $k := \boldsymbol{\lambda}(\varphi, P) \circ h : B \to P$ is the unique morphism in $\mathscr{A}$ such that $q' \circ k = g'$ and $q'' \circ k = g''$. This shows that P represents the fibre product of A' and A'' over A in $\mathscr{A}$. Similarly one shows the assertion for equalizers : the details shall be left to the reader.

(ii): In view of (i) and lemma 4.2.4, it suffices to check that the family $\mathscr{J}$ of sieves verifying condition (4.5.3) yields a topology on $\mathscr{A}$. Now, condition (c) of definition 4.1.1(i) obviously holds for $\mathscr{J}$. Next, let $\mathscr{S} \subset \mathscr{A}/A$ be a sieve fulfilling condition (4.5.3), and $f : A' \to A$ a morphism of $\mathscr{A}$; we set $\varphi := p(f)$ and factor f through a morphism $f' : A' \to A'' := \mathsf{c}_\varphi A$ in $p^{-1}A'$ and the cartesian morphism $f'' := \boldsymbol{\lambda}(A, \varphi) : A'' \to A$. Then let $\mathscr{S}' := \mathscr{S} \times_A f''$, and notice that if $(g_\lambda : B_\lambda \to A \mid \lambda \in \Lambda)$ generates $\mathscr{S}_{|A} := \mathscr{S} \cap (\mathscr{A}_{pA}/A)$, then the family $(\mathsf{c}_\varphi(g_\lambda) : \mathsf{c}_\varphi B_\lambda \to A'' \mid \lambda \in \Lambda)$ lies in $\mathscr{S}'$, and the latter family covers A'' for the topology $J_{pA''}$, since c_φ is continuous for the topologies J_{pA} and $J_{pA''}$, and since $\mathscr{S}_{|A}$ covers A for the topology J_{pA}. Then $\mathscr{S} \times_A f = \mathscr{S}' \times_{A''} f'$ contains the family $(A' \times_{A''} \mathsf{c}_\varphi B_\lambda \to A' \mid \lambda \in \Lambda)$, which covers A' for the topology $J_{pA'}$. This shows that condition (a) of definition 4.1.1(i) holds for $\mathscr{J}$. Lastly, let $\mathscr{S} \subset \mathscr{A}/A$ be an element of $\mathscr{J}$, and consider another sieve $\mathscr{T} \subset \mathscr{A}/A$ such that for every $(f : A' \to A) \in \mathrm{Ob}(\mathscr{S})$ the sieve $\mathscr{T} \times_A f$ lies in $\mathscr{J}$. Let $\mathscr{S}_{|A} \subset \mathscr{A}_{pA}/A$ be as in the foregoing, and define likewise $\mathscr{T}_{|A}$; then $\mathscr{S}_{|A} \in J_{pA}(A)$ and it is easily seen that

$$\mathscr{T}_{|A} \times_A f = (\mathscr{T} \times_A f) \cap (\mathscr{A}_{pA'}/A') \qquad \text{for every } (f : A' \to A) \in \mathrm{Ob}(\mathscr{S}_{|A}).$$

Hence $\mathscr{T}_{|A} \in J_{pA}(A)$, so $\mathscr{T} \in \mathscr{J}$; this shows that $\mathscr{J}$ fulfills condition (b) of definition 4.1.1(i).

(iii) follows immediately from (ii).

(v): Pick a universe V containing U, and such that $\mathscr{A}$ is V-small; in view of remarks 4.1.23(i) and 4.5.2(iii) we are easily reduced to checking the essential commutativity of the diagram :

$$\begin{array}{ccc} \mathscr{A}_{\mathsf{V}}^\wedge & \xrightarrow{i_{X,\mathsf{V}}^\wedge} & \mathscr{A}_{X,\mathsf{V}}^\wedge \\ {\scriptstyle (-)^+}\downarrow & & \downarrow{\scriptstyle (-)^+} \\ \mathscr{A}_{\mathsf{V}}^\wedge & \xrightarrow{i_X^\wedge} & \mathscr{A}_{X,\mathsf{V}}^\wedge \end{array} \qquad \text{for every } X \in \mathrm{Ob}(\mathscr{C})$$

where the functors $(-)^+$ are defined as in (4.1.11). We may then replace U by V and assume that $\mathscr{A}$ is a small category. For every $X \in \mathrm{Ob}(\mathscr{C})$ and every $A \in \mathrm{Ob}(\mathscr{A}_X)$ we endow $J_X(A)$ and $J_{\mathscr{A}}(A)$ with the ordering induced by inclusion of sieves, and consider the map

$$J_X(A) \to J_{\mathscr{A}}(A) \qquad \mathscr{S} \mapsto \mathscr{S}^*$$

that assigns to every $\mathscr{S} \in J_X(A)$ the sieve $\mathscr{S}^* \in J_{\mathscr{A}}(A)$ generated by $\mathrm{Ob}(\mathscr{S}) \subset \mathrm{Ob}(\mathscr{A}/A)$. This map is obviously injective, and according to (ii), its image is a cofinal subset in the opposite ordered set $J_{\mathscr{A}}(A)^o$. Moreover, for every morphism $f : A' \to A$ in $\mathscr{A}_X$, clearly we have

$$(\mathscr{S} \times_A f)^* = \mathscr{S}^* \times_A f \qquad \text{for every } \mathscr{S} \in J_X(A).$$

Furthermore, for every such $\mathscr{S}$ and every presheaf F on $\mathscr{A}_X$, we have a natural identification of $F(\mathscr{S}) := \mathrm{Hom}_{\mathscr{A}_X^\wedge}(h_{\mathscr{S}}, F)$ with the equalizer of the two natural maps (see (4.1.6))

$$\mathrm{Equal}\Big(\prod_{(B\to A)\in\mathrm{Ob}(\mathscr{S})} F(B) \rightrightarrows \prod_{(B\to A\leftarrow B')\in\mathrm{Ob}(\mathscr{S})\times\mathrm{Ob}(\mathscr{S})} F(B\times_A B')\Big) \tag{4.5.6}$$

and for every inclusion $\mathscr{S}' \subset \mathscr{S}$ of sieves covering A, the map $F(\mathscr{S}) \to F(\mathscr{S}')$ corresponds, under this identification, to the restriction of the projection

$$\prod_{(B\to A)\in\mathrm{Ob}(\mathscr{S})} F(B) \to \prod_{(B\to A)\in\mathrm{Ob}(\mathscr{S}')} F(B). \tag{4.5.7}$$

Recall that

$$F^+(A) := \operatorname*{colim}_{\mathscr{S}\in J_X(A)^o} F(\mathscr{S})$$

and for every morphism $f : A' \to A$ in $\mathscr{A}_X$, the map $F^+(f) : F^+(A) \to F^+(A')$ is the colimit of the maps $F(\mathscr{S}) \to F(\mathscr{S} \times_A f) \to F^+(A')$ induced by the projection $\mathscr{S} \times_A f \to \mathscr{S}$, for every $\mathscr{S} \in J_X(A)^o$. Now, if $F = i_X^\wedge G$ for a presheaf G on $\mathscr{A}$, it follows from (4.1.6) that the equalizer (4.5.6) is also naturally identified with $G(\mathscr{S}^*)$, and likewise, for every inclusion of sieves $\mathscr{S}' \subset \mathscr{S}$ in $J_X(A)$, the map $G(\mathscr{S}^*) \to G(\mathscr{S}'^*)$ corresponds to the restriction of (4.5.7), under this same identification. After taking colimits, we thus get an isomorphism

$$(i_X^\wedge G)^+(A) \xrightarrow{\sim} G^+(A)$$

natural in both the presheaf G and the object $A \in \mathrm{Ob}(\mathscr{A}_X)$. The assertion follows.

(iv): The condition is obviously necessary. For the converse, let F be a presheaf on $\mathscr{A}$ such that $i_X^\wedge(F)$ is a sheaf on $(\mathscr{A}_X, J_X)$ for every $X \in \mathrm{Ob}(\mathscr{C})$; denote by F^a the sheaf on $(\mathscr{A}, J_{\mathscr{A}})$ associated with F, and by $j : F \to F^a$ the natural map of presheaves. For every $X \in \mathrm{Ob}(\mathscr{C})$ we get a commutative diagram of presheaves on $\mathscr{A}_X$:

$$\begin{array}{ccc} i_X^\wedge(F) & \xrightarrow{i_X^\wedge(j)} & i_X^\wedge(F^a) \\ \downarrow & & \downarrow \\ (i_X^\wedge F)^a & \xrightarrow{(i_X^\wedge(j))^a} & (i_X^\wedge F^a)^a \end{array}$$

whose vertical arrows are isomorphisms, by assumption. On the other hand, in view of (v), the morphism $(i_X^\wedge(j))^a$ is naturally identified with $i_X^\wedge(j^a) : i_X^\wedge(F^a) \to i_X^\wedge((F^a)^a)$; now, j^a is an isomorphism, hence the same holds for $i_X^\wedge(j^a)$, and then also for $i_X^\wedge(j)$, for every $X \in \mathrm{Ob}(\mathscr{C})$. But this means that j is already an isomorphism, *i.e.* F is a sheaf on $(\mathscr{A}, J_{\mathscr{A}})$. □

Example 4.5.8. In the situation of (4.5.4), suppose that $\mathscr{C}$ admits a final object X_0. Then the functor i_{X_0} is a morphism of sites $(\mathscr{A}, J_{\mathscr{A}}) \to (\mathscr{A}_{X_0}, J_X)$. Indeed, let A_0 be any final object of $\mathscr{A}_{X_0}$; it is easily seen that A_0 is also a final object of $\mathscr{A}$, and taking into account propositions 1.3.22(i) and 4.5.5(i) we deduce that i_{X_0} is left exact. Then the assertion follows from proposition 4.5.5(iii) and example 4.3.12.

Proposition 4.5.9. *Let $u : (\mathscr{A}', p', J'_\bullet) \to (\mathscr{A}, p, J_\bullet)$ be a morphism of fibred lex-sites over a category $\mathscr{C}$, and denote by $(\mathscr{A}, J_{\mathscr{A}})$, $(\mathscr{A}', J_{\mathscr{A}'})$ the corresponding total sites. Then u is a morphism of sites $(\mathscr{A}', J_{\mathscr{A}'}) \to (\mathscr{A}, J_{\mathscr{A}})$.*

Proof. Proposition 4.5.5(i,ii) easily implies that u is continuous for the topologies $J_{\mathscr{A}}$ and $J_{\mathscr{A}'}$, so it remains only to check that the fibration $\mathsf{s} : \mathscr{A}'/u\mathscr{A} \to \mathscr{A}'$ is locally cofiltered for the topology $J_{\mathscr{A}'}$ (proposition 4.3.9). Thus, let $X \in \mathrm{Ob}(\mathscr{C})$ and $A' \in \mathrm{Ob}(\mathscr{A}'_X)$; the fibration $\mathsf{s}_{(X)} : \mathscr{A}'_X/u_X\mathscr{A}_X \to \mathscr{A}'_X$ is locally cofiltered, by proposition 4.3.9, hence there exists a covering family $g_\bullet := (g_\lambda : A'_\lambda \to A' \mid \lambda \in \Lambda)$ relative to the topology J'_X such that $\mathrm{Ob}(\mathsf{s}^{-1}_{(X)}A'_\lambda) \neq \varnothing$ for every $\lambda \in \Lambda$; but the family $g_\bullet$ covers A' also relative to the topology $J_{\mathscr{A}'}$ (proposition 4.5.5(ii)), so s fulfills condition (a) of definition 4.3.3. To check condition (b), consider morphisms $f_i : A' \to uA_i$ in $\mathscr{A}'$ for $i = 1, 2$, and let $\varphi_i : X \to Y_i$ be the image of f_i in $\mathscr{C}$ for $i = 1, 2$. Pick any cleavage for $\mathscr{A}$, and let c be its associated pseudo-functor; then $f_i = u(f''_i) \circ f'_i$ where $f'_i : A' \to u(\mathsf{c}_{\varphi_i}A_i)$ is a morphism in $\mathscr{A}'_X$ and $f''_i : \mathsf{c}_{\varphi_i}A_i \to A_i$ is a cartesian morphism of $\mathscr{A}$, for $i = 1, 2$. Since $\mathsf{s}_{(X)}$ is locally cofiltered, we may then find a covering family $g_\bullet$ as in the foregoing, and for every $\lambda \in \Lambda$ a morphism $f_\lambda : A'_\lambda \to uA_\lambda$ in $\mathscr{A}'_X$ and morphisms $h_{i,\lambda} : A_\lambda \to \mathsf{c}_{\varphi_1}A_1$ in $\mathscr{A}_X$ for $i = 1, 2$, such that $u(h_{i,\lambda}) \circ f_\lambda = f'_i \circ g_\lambda$. Then we get morphisms in $\mathscr{A}/u\mathscr{A}'$

$$(A'_\lambda/f''_i \circ h_{i,\lambda}) : (A'_\lambda, f_\lambda) \to (A'_\lambda, f_i \circ g_\lambda) \qquad \text{for } i = 1, 2 \text{ and every } \lambda \in \Lambda$$

whence the contention. Lastly, let $(f_i : A' \to uA_i \mid i = 1, 2)$ be a pair of morphisms in $\mathscr{A}'$, and $h_1, h_2 : A_1 \to A_2$ two morphisms of $\mathscr{A}$ with $u(h_i) \circ f_1 = f_2$ for $i = 1, 2$. Let $\varphi_i : X \to Y_i$ be the image of f_i in $\mathscr{C}$ for $i = 1, 2$; there exist morphisms $h'_1, h'_2 : \mathsf{c}_{\varphi_1}A_1 \to \mathsf{c}_{\varphi_2}A_2$ in $\mathscr{A}_X$ and cartesian morphisms $f'_i : \mathsf{c}_{\varphi_i}A_i \to A_i$ in $\mathscr{A}$ such that $h_i \circ f'_1 = f'_2 \circ h'_i$ for $i = 1, 2$. Since $\mathsf{s}_{(X)}$ is locally cofiltered, we may then find $g_\bullet$ as in the foregoing, and for every $\lambda \in \Lambda$ a morphism $f_\lambda : A'_\lambda \to uA_\lambda$ in $\mathscr{A}'_X$ and a morphism $h_\lambda : A_\lambda \to \mathsf{c}_{\varphi_1}A_1$ in A_X such that $h'_1 \circ h_\lambda = h'_2 \circ h_\lambda$. It follows that $h_1 \circ f'_1 \circ h_\lambda = h_2 \circ f'_1 \circ h_\lambda$, which shows that condition (c) holds as well for s. □

Remark 4.5.10. Let V be any universe. Proposition 4.5.9 says that the rule that assigns to every fibred lex-site over $\mathscr{C}$ with V-small fibres its total site defines a strict pseudo-functor

$$\mathsf{totSite} : \mathsf{V}\text{-}\mathsf{fib.lex.}\mathbf{Site}(\mathscr{C}) \to \mathsf{V}\text{-}\mathbf{Site}$$

that assigns to every morphism of lex-sites the underlying functor, and is the identity on 2-cells.

4.5.11. Let $(\mathscr{A}, p, J_\bullet)$ be a fibred site over a category $\mathscr{C}$, and $u : \mathscr{C}' \to \mathscr{C}$ any functor. Set $\mathscr{A}' := \mathsf{Fib}(u)^*\mathscr{A}$, and recall that for every $X \in \mathrm{Ob}(\mathscr{C}')$ the natural projection $\pi : \mathscr{A}' \to \mathscr{A}$ restricts to an isomorphism of categories $\pi_X : \mathscr{A}'_X \xrightarrow{\sim} \mathscr{A}_{uX}$. Then we get a fibred site denoted

$$\mathscr{C}' \times_{(\mathscr{C},u)} (\mathscr{A}, p, J_\bullet) \qquad \text{or more simply} \qquad \mathscr{C}' \times_{\mathscr{C}} (\mathscr{A}, p, J_\bullet)$$

whose underlying fibration is $p' : \mathscr{A}' \to \mathscr{C}'$ and where the topology J_X on $\mathscr{A}'_X$ is induced by J_X via the isomorphism π_X, for every $X \in \mathrm{Ob}(\mathscr{C}')$. Obviously, if $(\mathscr{A}, p, J_\bullet)$ is a fibred lex-site, the same holds for $\mathscr{C}' \times_{\mathscr{C}} (\mathscr{A}, p, J_\bullet)$, and in this case, proposition 4.5.5(ii,iv) easily implies that π is a continuous and cocontinuous functor for the respective total sites $(\mathscr{A}, J_{\mathscr{A}}), (\mathscr{A}', J_{\mathscr{A}'})$.

4.5.12. In the situation of (4.5.11), let $v : \mathscr{C}' \to \mathscr{C}$ be another functor, and $\beta : u \Rightarrow v$ any natural transformation. After choosing a cleavage $\boldsymbol{\lambda}$ for the fibration $\mathscr{A}$, we attach to β a $\mathscr{C}'$-cartesian functor $\mathsf{Fib}(\beta)^*_{\mathscr{A}} : \mathsf{Fib}(v)^*\mathscr{A} \to \mathsf{Fib}(u)^*\mathscr{A}$, as in (3.1.29). Explicitly, let c be the pseudo-functor associated with $\boldsymbol{\lambda}$; then for every $X \in \mathrm{Ob}(\mathscr{C}')$ the restriction $\mathscr{A}_{vX} \to \mathscr{A}_{uX}$ of $\mathsf{Fib}(\beta)^*_{\mathscr{A}}$ is given by the functor c_{β_X}. Hence, $\mathsf{Fib}(\beta)^*_{\mathscr{A}}$ is a morphism of fibred sites denoted :

$$\beta \times_{\mathscr{C}} (\mathscr{A}, p, J_\bullet) : \mathscr{C}' \times_{(\mathscr{C},u)} (\mathscr{A}, p, J_\bullet) \to \mathscr{C}' \times_{(\mathscr{C},v)} (\mathscr{A}, p, J_\bullet).$$

It is easily seen that $\beta \times_{\mathscr{C}} (\mathscr{A}, p, J_\bullet)$ is independent, up to isomorphisms of morphisms of fibred sites, of the choice of cleavage $\boldsymbol{\lambda}$: the details shall be left to the reader.

4.5.13. Let $\mathscr{C}, \mathscr{D}$ be two small categories, $t : \mathscr{D} \to \mathscr{C}$ a functor, and $u : (\mathscr{A}_0, p_0, J_{0,\bullet}) \to (\mathscr{A}_1, p_1, J_{1,\bullet})$ a morphism of fibred lex sites over $\mathscr{C}$. Set

$$(\mathscr{B}_i, q_i, J'_{i,\bullet}) := \mathscr{D} \times_{\mathscr{C}} (\mathscr{A}_i, p_i, J_{i,\bullet}) \qquad \text{for } i = 0, 1$$

(notation of (4.5.11)). Clearly u induces a morphism of fibred lex sites

$$v := \mathscr{D} \times_{\mathscr{C}} u : (\mathscr{B}_0, q_0, J'_{0,\bullet}) \to (\mathscr{B}_1, q_1, J'_{1,\bullet}).$$

Then u and v are also morphisms of the respective total sites $u : (\mathscr{A}_0, J_0) \to (\mathscr{A}_1, J_1)$ and $v : (\mathscr{B}_0, J'_0) \to (\mathscr{B}_1, J'_1)$ (proposition 4.5.9). For $i = 0, 1$, let $\pi_i : \mathscr{B}_i \to \mathscr{A}_i$ be the projection. We deduce a commutative diagram of categories :

$$\mathscr{E} \quad : \quad \begin{array}{ccc} \mathscr{A}_0^{\sim} & \xrightarrow{\widetilde{u}_*} & \mathscr{A}_1^{\sim} \\ {\scriptstyle \widetilde{\pi}_{0*}} \downarrow & & \downarrow {\scriptstyle \widetilde{\pi}_{1*}} \\ \mathscr{B}_0^{\sim} & \xrightarrow{\widetilde{v}_*} & \mathscr{B}_1^{\sim} \end{array}$$

and notice that each functor of the diagram $\mathscr{E}$ admits a left adjoint; after choosing an adjunction for each of the resulting adjoint pairs of functors, we can view $\mathscr{E}$ as a square of links, oriented by the identity $\mathbf{1}_{\widetilde{\pi}_{1*} \circ \widetilde{u}_*}$ (see (2.3.8)).

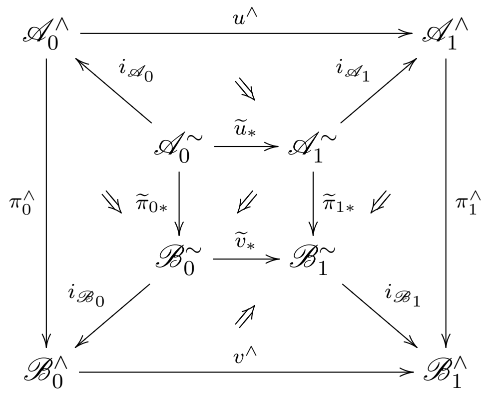
Proposition 4.5.14. *In the situation of* (4.5.13)*, suppose that* $(\mathscr{A}_{i,X}, J_{i,X})$ *is a* U*-site for every* $X \in \mathrm{Ob}(\mathscr{C})$ *and* $i = 0, 1$*. Then the base change transformation*

$$\Upsilon(\mathscr{E}) : \widetilde{v}^* \circ \widetilde{\pi}_{1*} \to \widetilde{\pi}_{0*} \circ \widetilde{u}^*$$

is an isomorphism of functors.

Proof. We consider first the oriented diagram of categories :

$\mathscr{E}'$:



whose diagonal arrows are the inclusion functors, and all whose orientations are identities. Moreover, each functor in $\mathscr{E}'$ still admits a left adjoint, so as usual $\mathscr{E}'$ can be regarded as an oriented diagram of links. By adding a further identity orientation for the "front face" we obtain a cubical diagram as in (2.3.21), which obviously commutes on 2-cells, except that the orientations do not agree with those of the corresponding diagram in (2.3.21), nor with those of its variant considered in remark 2.3.22(iii). However, the induced diagram ${}^\dagger\mathscr{E}'$ of adjoint squares as in (2.3.8) still commutes on 2-cells, due to proposition 2.3.4, and moreover its orientations agree with those of the variant from remark 2.3.22(iii). More precisely, the left and right faces of ${}^\dagger\mathscr{E}'$ are oriented by ${}^o(\mathbf{1}_{i_{\mathscr{B}_1} \circ \widetilde{v}_*})^\dagger$ and respectively ${}^o(\mathbf{1}_{i_{\mathscr{A}_1} \circ \widetilde{u}_*})^\dagger$, and the top and bottom faces are oriented by ${}^o(\mathbf{1}_{i_{\mathscr{B}_1} \circ \widetilde{\pi}_{1*}})^\dagger$ and respectively ${}^o(\mathbf{1}_{i_{\mathscr{B}_0} \circ \widetilde{\pi}_{0*}})^\dagger$.

Claim 4.5.15. The base change transformations

$$\Upsilon(\mathbf{1}_{i_{\mathscr{B}_1} \circ \widetilde{\pi}_{1*}}) : (-)^a_{\mathscr{B}_1} \circ \pi_1^\wedge \to \widetilde{\pi}_{1*} \circ (-)^a_{\mathscr{A}_1} \qquad \text{and} \qquad \Upsilon(\mathbf{1}_{i_{\mathscr{B}_0} \circ \widetilde{\pi}_{0*}}) : (-)^a_{\mathscr{B}_0} \circ \pi_0^\wedge \to \widetilde{\pi}_{0*} \circ (-)^a_{\mathscr{A}_0}$$

are isomorphisms of functors.

Proof of the claim. We check the assertion for $\Upsilon(\mathbf{1}_{i_{\mathscr{B}_0}\circ\widetilde{\pi}_{0*}})$; the same argument shall apply also to $\Upsilon(\mathbf{1}_{i_{\mathscr{B}_1}\circ\widetilde{\pi}_{1*}})$. Since $i_{\mathscr{B}_0}$ is a fully faithful functor, the counit $\varepsilon^{\mathscr{B}_0}$ of the chosen adjunction for the pair $((-)^a_{\mathscr{B}_0}, i_{\mathscr{B}_0})$ is an isomorphism (proposition 1.1.20(iii)), so we are reduced to checking that $((-)^a_{\mathscr{B}_0}\circ\pi_0^\wedge)*\eta^{\mathscr{A}_0}$ is an isomorphism, where $\eta^{\mathscr{A}_0}$ and $\varepsilon^{\mathscr{A}_0}$ are the unit and counit of an adjunction for the pair $((-)^a_{\mathscr{A}_0}, i_{\mathscr{A}_0})$. But recall that there exists an isomorphism of functors : $(-)^a_{\mathscr{B}_0}\circ\pi_0^\wedge \xrightarrow{\sim} \widetilde{\pi}_{0*}\circ(-)^a_{\mathscr{A}_0}$ (corollary 4.3.19(iii)), so it suffices to show that $(-)^a_{\mathscr{A}_0}*\eta^{\mathscr{A}_0}$ is an isomorphism. However, $\varepsilon^{\mathscr{A}_0}$ is an isomorphism, since $i_{\mathscr{A}_0}$ is fully faithful, hence the contention follows from the triangular identities of (1.1.13). ◊

In light of claim 4.5.15, proposition 2.3.10, and remark 2.3.22(i,iii), we are then reduced to showing that the base change transformation

$$\Upsilon(\mathbf{1}_{\pi_1^\wedge\circ u^\wedge}) : v_!\circ\pi_1^\wedge \to \pi_0^\wedge\circ u_!$$

is an isomorphism of functors. To this aim, recall that for every $A_0\in\mathrm{Ob}(\mathscr{A}_0)$ and every presheaf F on $\mathscr{A}_1$, the set $u_!F(A_0)$ represents the colimit of the functor

$$F\circ\mathsf{t}^o_{A_0} : (A_0/u\mathscr{A}_1)^o \to \mathbf{Set}$$

where t_{A_0} denotes the usual target functor as in (1.1.27). Thus, $u_!F(A_0)$ is the set of equivalence classes $[s,\varphi]_u$ of pairs (s,φ), where $\varphi : A_0\to uA_1$ is a morphism of $\mathscr{A}_0$, and $s\in FA_1$. For every morphism $\psi : A_0\to A'_0$ in $\mathscr{A}_0$, the map $u_!F(\psi)$ is given by the rule : $[s,\varphi]_u\mapsto[s,\varphi\circ\psi]_u$ for every such $[s,\varphi]_u$. Moreover, for every morphism $f : F\to F'$ of presheaves on $\mathscr{A}_0$, the morphism $u_!f : u_!F\to u_!F'$ is given by the rule : $[s,\varphi]_u\mapsto[f_{A_1}(s),\varphi]_u$ for every such $[s,\varphi]_u$. We have a standard adjunction for the pair $(u_!, u^\wedge)$ whose unit and counit (η^u,ε^u) are given by the following rules, for every presheaf F on $\mathscr{A}_1$ and F' on $\mathscr{A}_0$:

$$\eta^u_{F,A_1}(s) := [s,\mathbf{1}_{uA_1}]_u \in u^\wedge u_!F(A_1) \qquad \text{for every } A_1\in\mathrm{Ob}(\mathscr{A}_1) \text{ and every } s\in FA_1$$
$$\varepsilon^u_{F',A_0}[s, A_0\xrightarrow{\varphi} uA'_1]_u := F'\varphi(s)\in F'A_0 \qquad \text{for every } s\in F(uA'_1).$$

A similar description applies to the functor $v_!$ and the unit and counit of its standard adjunction. Thus, for every such F, the morphism $(v_!\pi_1^\wedge*\eta^u)_F : v_!\pi_1^\wedge F\to v_!\pi_1^\wedge u^\wedge u_!F$ is given by the rule:

$$[s, B_0\xrightarrow{\varphi} vB_1]_v \mapsto [[s,\mathbf{1}_{u\pi_1B_1}]_u,\varphi]_v \qquad \text{for every } s\in F\pi_1B_1$$

and the morphism $(\varepsilon^v*\pi_0^\wedge u_!)_F : v_!v^\wedge\pi_0^\wedge u_!F\to\pi_0^\wedge u_!F$ is given by the rule :

$$[[t,\pi_0vA'_1\xrightarrow{\psi}uA''_1]_u, A_0\xrightarrow{\varphi}vA'_1]_v \mapsto \pi_0^\wedge u_!F(\varphi)[t,\pi_0vA'_1\xrightarrow{\psi}uA''_1]_u = [t,\psi\circ\pi_0(\varphi)]_u$$

for every $t\in FA''_1$. Summing up, $\Upsilon(\mathbf{1}_{\pi_1^\wedge\circ u^\wedge})$ is then given by the rule :

$$[s, B_0\xrightarrow{\varphi}vB_1]_v \mapsto [s,\pi_0(\varphi)]_u \qquad \text{for every } s\in F\pi_1B_1.$$

Consider then the functor

$$\rho^{B_0} : (B_0/v\mathscr{B}_1)^o \to (\pi_0B_0/u\mathscr{A}_1)^o \qquad (B_0\xrightarrow{\varphi}vB_1)\mapsto(\pi_0B_0\xrightarrow{\pi_0(\varphi)}\pi_0vB_1 = u\pi_1B_1)$$

and notice that $\mathsf{t}^o_{\pi_0B_0}\circ\rho^{B_0} = \pi_1^o\circ\mathsf{t}^o_{B_0}$, so that ρ^{B_0} induces a morphism of presheaves on the category $\mathbf{Set}^o$, as in (the dual of) remark 1.2.11(i) :

$$\operatorname*{colim}_{(\pi_0B_0/u\mathscr{A}_1)^o} F\circ\mathsf{t}^o_{\pi_0B_0} \to \operatorname*{colim}_{(\mathscr{B}_0/v\mathscr{B}_1)^o} F\circ\pi_1^o\circ\mathsf{t}^o_{B_0}$$

which corresponds to a map $\omega : v_!\pi_1^\wedge F(B_0)\to\pi_0^\wedge u_!F(B_0)$, and by a direct inspection it is easily seen that $\omega = \Upsilon(\mathbf{1}_{\pi_1^\wedge\circ u^\wedge})_{F,B_0}$. In view of proposition 1.5.2, we are then reduced to showing :

Claim 4.5.16. For every $X\in\mathrm{Ob}(\mathscr{B}_0)$ we have :
(i) The categories $X/v\mathscr{B}_1$ and $\pi_0X/u\mathscr{A}_1$ are cofiltered.
(ii) The functor ρ^X is cofinal.

Proof of the claim. (i): We show that $X/v\mathscr{B}_1$ is cofiltered; the same argument shall apply to $\pi_0 X/u\mathscr{A}_1$ as well. Indeed, let us check that $X/v\mathscr{B}_1$ is not the empty category : to this aim, let $v_{q_0X} : \mathscr{B}_{1,q_0X} \to \mathscr{B}_{0,q_0X}$ be the restriction of v to the fibre categories over q_0X, so it suffices to check that $X/v_{q_0X}\mathscr{B}_{1,q_0X}$ is not empty; but $\mathscr{B}_{1,q_0X}$ is finitely complete, hence it admits a final object Y_0, and vY_0 is a final object of $\mathscr{B}_{0,q_0X}$, since v is left exact. The unique morphism $X \to vY_0$ is then an object of $X/v_{q_0X}\mathscr{B}_{1,q_0X}$.

Next, consider two objects $(Y_1, \varphi_1), (Y_2, \varphi_2)$ of $X/v\mathscr{B}_1$; for $i = 1, 2$ we may find cartesian morphisms $\beta_i : Y'_i \to Y_i$ in $\mathscr{B}_1$ such that $q_1(\beta_i) = q_0(\varphi_i)$, and then there exists a unique morphism $\varphi'_i : X \to vY'_i$ in the fibre category $\mathscr{B}_{0,q_0X}$ such that $v(\beta_i) \circ \varphi'_i = \varphi_i$. The pair (Y'_i, φ'_i) is an object of $X/v\mathscr{B}_1$, and β_i is a morphism $(Y'_i, \varphi'_i) \to (Y_i, \varphi_i)$ for $i = 1, 2$. Since $\mathscr{B}_{1,q_0X}$ is finitely complete, the product $Z := Y'_1 \times Y'_2$ is representable, and since v_{q_0X} is left exact, vZ represents $vY'_1 \times vY'_2$, whence a morphism $\psi' : X \to vZ$ in $\mathscr{B}_{0,q_0X}$ whose composition with the projection $v(p_i) : vZ \to vY'_i$ agrees with φ'_i, for $i = 1, 2$. Thus, $(Z, \psi') \in \mathrm{Ob}(X/v\mathscr{B}_1)$ and we have morphisms $(Z, \psi') \xrightarrow{X/p_i} (Y'_i, \varphi'_i) \xrightarrow{X/\beta_i} (Y_i, \varphi_i)$ for $i = 1, 2$.

Lastly, let $X/\alpha_1, X/\alpha_2 : (Y_1, \varphi_1) \to (Y_2, \varphi_2)$ be two morphisms of $X/v\mathscr{B}_1$. We find cartesian morphisms $\beta_i : Y'_i \to Y_i$ in $\mathscr{B}_1$ such that $q_1(\beta_i) = q_0(\varphi_i)$ for $i = 1, 2$, and notice that $q_1(\alpha_1 \circ \beta_1) = q_0(\varphi_2) = q_1(\alpha_2 \circ \beta_1)$. Then there exist unique morphisms $\alpha'_i : Y'_1 \to Y'_2$ in $\mathscr{B}_{1,q_0X}$ such that $\beta_2 \circ \alpha'_i = \alpha_i \circ \beta_1$ for $i = 1, 2$. Since $\mathscr{B}_{1,q_0X}$ is finitely complete, the equalizer of the pair (α'_1, α'_2) is representable in $\mathscr{B}_{1,q_0X}$ by a morphism $\lambda' : E \to Y'_1$ in $\mathscr{B}_{1,q_0X}$. Set $\lambda := \beta_1 \circ \lambda'$; it follows that $\alpha_1 \circ \lambda = \alpha_2 \circ \lambda$. Moreover, for $i = 1, 2$ we may find a morphism $\varphi'_i : X \to vY'_i$ in $\mathscr{B}_{0,q_0X}$ such that $v(\beta_i) \circ \varphi'_i = \varphi_i$. We compute :

$$v(\beta_2 \circ \alpha'_1) \circ \varphi'_1 = v(\alpha_1 \circ \beta_1) \circ \varphi'_1 = v(\alpha_1) \circ \varphi_1 = \varphi_2 = v(\alpha_2) \circ \varphi_1 = v(\alpha_2 \circ \beta_1) \circ \varphi'_1 = v(\beta_2 \circ \alpha'_2) \circ \varphi'_1$$

whence $v(\alpha'_1) \circ \varphi'_1 = v(\alpha'_2) \circ \varphi'_1$, since $v(\beta_2)$ is cartesian. However, $v(\lambda)$ is the equalizer of the pair $(v(\alpha'_1), v(\alpha'_2))$, since v_{q_0X} is left exact; so finally φ'_1 factors through a unique morphism $\varphi''_1 : X \to vE$ and $v(\lambda') : vE \to vY'_1$. Thus, we get a morphism $X/\lambda : (E, \varphi''_1) \to (Y_1, \varphi_1)$ in $X/v\mathscr{B}_1$ such that $(X/\alpha_1) \circ (X/\lambda) = (X/\alpha_2) \circ (X/\lambda)$, whence the contention.

(ii): In view of (i), it suffices to show that conditions (a) and (b) of lemma 1.5.7(i) hold for ρ^X. To this aim, let $u_{q_0X} : \mathscr{A}_{1,q_0X} \to \mathscr{A}_{0,q_0X}$ be the restriction of u to the fibre categories, and (Y, φ) any object of $(\pi_0 X/u\mathscr{A}_1)^o$. Arguing as in the proof of (i), we find $(Y', \varphi') \in \mathrm{Ob}((\pi_0 X/u_{q_0X}\mathscr{A}_{1,q_0X})^o)$ and a morphism $(Y, \varphi) \to (Y', \varphi')$ in $(\pi_0 X/u\mathscr{A}_1)^o$. But notice that ρ^X restricts to an isomorphism of categories $(X/v_{q_0X}\mathscr{B}_{1,q_0X})^o \xrightarrow{\sim} (\pi_0 X/u_{q_0X}\mathscr{A}_{1,q_0X})^o$. Condition (a) is an immediate consequence. Lastly, let $(Y, \varphi) \in \mathrm{Ob}(X/v\mathscr{B}_1)$, $(Z, \varphi') \in \mathrm{Ob}(\pi_0 X/u\mathscr{A}_1)$, and a pair of morphisms in $\pi_0 X/u\mathscr{A}_1$:

$$\pi_0 X/\beta_i : \rho^X(Y, \varphi) = (\pi_1 Y, \pi_0(\varphi)) \to (Z, \varphi') \qquad i = 1, 2.$$

Hence, $\beta_1, \beta_2 : \pi_1 Y \to Z$ are morphisms of $\mathscr{B}_1$ with $u(\beta_1) \circ \pi_0(\varphi) = u(\beta_2) \circ \pi_0(\varphi)$. Arguing as in the proof of (i) we find a cartesian morphism $\beta : Y' \to Y$ in $\mathscr{B}_1$ and a morphism $\psi : X \to vY'$ in $\mathscr{B}_{0,q_0X}$ with $v(\beta) \circ \psi = \varphi$. Notice that $p_1(\beta_1 \circ \pi_1(\beta)) = p_0(\varphi') = p_1(\beta_2 \circ \pi_1(\beta))$. Then we may find $Z'' \in \mathrm{Ob}(\mathscr{A}_{1,q_0X})$ and a cartesian morphism $\gamma : Z'' \to Z$ in $\mathscr{A}_1$ such that $p_1(\gamma) = p_0(\varphi')$, and for $i = 1, 2$, a morphism $\beta''_i : \pi_1 Y' \to Z'$ in $\mathscr{A}_{1,q_0X}$ such that $\beta_1 \circ \pi_1(\beta) = \gamma \circ \beta''_i$. Since π_1 restricts to an isomorphism $\mathscr{B}_{1,q_0X} \xrightarrow{\sim} \mathscr{A}_{1,q_0X}$, there exist $Z' \in \mathrm{Ob}(\mathscr{B}_{1,q_0X})$ with $\pi_1 Z' = Z''$, and unique morphisms $\beta'_i : Y' \to Z'$ in $\mathscr{B}_{1,q_0X}$ such that $\pi_1(\beta'_i) = \beta''_i$ for $i = 1, 2$. We compute :

$$\begin{aligned} u(\gamma) \circ u\pi_1(\beta'_1) \circ \pi_0(\psi) = u(\beta_1 \circ \pi_1(\beta)) \circ \pi_0(\psi) &= u(\beta_1) \circ \pi_0(\varphi) \\ &= u(\beta_2) \circ \pi_0(\varphi) \\ &= u(\beta_2 \circ \pi_1(\beta)) \circ \pi_0(\psi) \\ &= u(\gamma) \circ u\pi_1(\beta'_2) \circ \pi_0(\psi). \end{aligned}$$

Since $u(\gamma)$ is cartesian and π_0 restricts to an isomorphism $\mathscr{B}_{0,q_0X} \xrightarrow{\sim} \mathscr{A}_{0,q_0X}$, we conclude that $v(\beta'_1) \circ \psi = v(\beta'_2) \circ \psi$. Let $\lambda : E \to Y'$ be the equalizer of the pair (β'_1, β'_2) in $\mathscr{B}_{1,q_0X}$; since v_{q_0X} is left exact, $v(\lambda) : vE \to vY'$ represents the equalizer of $(v(\beta'_1), v(\beta'_2))$ in $\mathscr{A}_{1,q_0X}$, so finally ψ factor through $v(\lambda)$ and a unique morphism $\mu : X \to vE$ in $\mathscr{A}_{1,q_0X}$. We have $(E, \mu) \in \mathrm{Ob}(X/v\mathscr{B}_1)$, and a morphism $X/(\beta \circ \lambda) : (E, \mu) \to (Y, \varphi)$ in $X/v\mathscr{B}_1$ such that $\rho^X(X/(\beta \circ \lambda))$ equalizes π_0X/β_1 and π_0X/β_2 in $(\pi_0X/u\mathscr{A}_1)^o$. This concludes the proof of condition (b). □

4.6. **Fibred topoi.** For every small category I, we consider next the 2-category

$$\mathsf{PsFun}(I, \mathbf{Topos})$$

whose objects $T : I \to \mathbf{Topos}$ shall be called the *fibred topoi over* I. The 1-cells, that is the pseudo-natural transformations $\omega : T \Rightarrow S$ shall be called *morphisms of fibred topoi over* I, and denoted by a simple arrow $T \to S$. The 2-cells, *i.e.* the modifications $\omega \rightsquigarrow \omega'$ in $\mathsf{PsFun}(I, \mathbf{Topos})$ shall be called also *transformations* of morphisms of fibred topoi over I.

Remark 4.6.1. (i) Recall that the definition of the 2-category $\mathbf{Topos}$ involves the choice of two universes U, U' with $\mathsf{U} \in \mathsf{U}'$, so that every object of $\mathbf{Topos}$ is both a U-topos and a small U'-category (see definition 4.4.5(iv)). Then, consider any pseudo-functor

$$F : I \to \mathsf{U}'\text{-}\mathbf{Cat}$$

such that (a): F_i is an U-topos for every $i \in \mathrm{Ob}(I)$, and (b): for every morphism $\varphi : i \to j$ of I there exists a morphism of topoi $f_\varphi : F_i \to F_j$ with $F_\varphi = f_{\varphi*}$. Then we get a fibred topos T over I such that $T_i := F_i$ for every $i \in \mathrm{Ob}(I)$ and $T_\varphi := f_\varphi$ for every morphism φ of I. This fibred topos depends on the choice of the morphisms of topoi f_φ, but any two such choices yield isomorphic fibred topoi over I.

(ii) In particular, any *functor* $F : I \to \mathsf{U}'\text{-}\mathbf{Cat}$ fulfilling (a) and (b) can be regarded as a fibred topos over I; in this case, the coherence constraint (δ^T, γ^T) of the resulting T shall be given by identity automorphisms of functors :

$$\delta^T_i = \mathbf{1}_{T_i} \qquad \gamma^T_{\varphi,\psi} = \mathbf{1}_{T_\varphi \circ T_\psi} \qquad \text{in } \mathsf{U}'\text{-}\mathbf{Cat}$$

which are however *not necessarily* identity 2-cells of the 2-category $\mathbf{Topos}$, since one does not necessarily have $f^*_\psi \circ f^*_\varphi = f^*_{\psi\circ\varphi}$, nor $f^*_{\mathbf{1}_i} = \mathbf{1}_{T_i}$. Hence, even when it is given by an actual functor, T will only in general be a (non-strict) pseudo-functor, when regarded as a fibred topos.

(iii) From the pseudo-functors of remark 4.4.9(ii) we deduce two strict pseudo-functors :

$$\mathsf{PsFun}(I^o, \mathsf{U}'\text{-}\mathbf{Cat})^o \xleftarrow{[-]^*} \mathsf{PsFun}(I, \mathbf{Topos}) \xrightarrow{[-]_*} \mathsf{PsFun}(I, \mathsf{U}'\text{-}\mathbf{Cat}).$$

- Namely, $[-]_*$ assigns to every fibred topos $T : I \to \mathbf{Topos}$ over I the composition

$$T_* := (-)_* \circ T : I \to \mathsf{U}'\text{-}\mathbf{Cat} \qquad i \mapsto T_i \qquad (i \xrightarrow{\varphi} j) \mapsto (T_{\varphi*} : T_i \to T_j)$$

and to every morphism of fibred topoi $\omega : T \to S$ the pseudo-natural transformation $\omega_* := (-)_* * \omega : T_* \Rightarrow S_*$. To every transformation $\Xi : \omega \rightsquigarrow \omega'$ it assigns the modification

$$\Xi_* := (-)_* \circ \Xi : \omega_* \rightsquigarrow \omega'_*.$$

- Likewise, $[-]^*$ assigns to every fibred topos T over I the pseudo-functor

$$T^* := ((-)^* \circ T)^o : I^o \to \mathsf{U}'\text{-}\mathbf{Cat} \qquad i^o \mapsto T^o_i \qquad (j^o \xrightarrow{\varphi^o} i^o) \mapsto (T^{*o}_\varphi : T^o_j \to T^o_i)$$

and to every morphism of fibred topoi $\omega : T \to S$ the pseudo-natural transformation $\omega^* := ((-)^* * \omega)^o : S^* \Rightarrow T^*$. To every transformation $\Xi : \omega \rightsquigarrow \omega'$ it assigns the modification

$$\Xi^* := ((-)^* \circ \Xi)^o : \omega^* \rightsquigarrow \omega'^*.$$

4.6.2. There are several useful ways of attaching a fibration to any fibred topos T over I : namely, with the notation of (3.1.18) and remark 4.6.1(iii), we can consider the fibrations

$$\pi : \mathscr{F}ib(T_*) \to I^o \qquad \text{and} \qquad \pi' : \mathscr{F}ib(T^*) \to I.$$

More precisely, we have strict pseudo-functors :

$$\mathsf{U}'\text{-}\mathsf{Fib}(I)^o \xleftarrow{\mathscr{F}ib_I^o \circ [-]^*} \mathsf{PsFun}(I, \mathbf{Topos}) \xrightarrow{\mathscr{F}ib_{I^o} \circ [-]_*} \mathsf{U}'\text{-}\mathsf{Fib}(I^o).$$

• We have also a natural pseudo-functor from fibred topoi over I to fibred sites over I : namely, the composition

$$\underline{\mathrm{Can}} : \mathsf{PsFun}(I, \mathbf{Topos}) \xrightarrow{\mathsf{PsFun}(I, \mathrm{Can})} \mathsf{PsFun}(I, \mathsf{U}'\text{-}\mathbf{Site}) \xrightarrow{\underline{\mathscr{F}ib}_I} \mathsf{U}'\text{-fib.}\mathbf{Site}(I)$$

where $\mathrm{Can} : \mathbf{Topos} \to \mathbf{Site}$ is the pseudo-functor of (4.4.15), and $\underline{\mathscr{F}ib}_I$ is the 2-equivalence of remark 4.5.2(v). Explicitly, $\underline{\mathrm{Can}}(T)$ is given as follows. Since I is a usual category, we have ${}^oI^o = I^o$, so T^* induces a pseudo-functor

$$I^o \xrightarrow{{}^oT^*} {}^o(\mathsf{U}'\text{-}\mathbf{Cat}) \xrightarrow{(-)^o} \mathsf{U}'\text{-}\mathbf{Cat} \qquad i^o \mapsto T_i \qquad (j^o \xrightarrow{\varphi} i^o) \mapsto (T^*_\varphi : T_j \to T_i)$$

where $(-)^o$ denotes the strict isomorphism of example 2.1.24. The fibration

$$p : \underline{\mathrm{Can}}(T) := \mathscr{F}ib((-)^o \circ {}^oT^*) \to I$$

carries a natural structure of fibred lex-site : namely, the category $p^{-1}(i)$ is naturally identified with T_i for every $i \in \mathrm{Ob}(I)$, and inherits the latter's canonical topology; also, p admits a natural cleavage whose associated pseudo-functor c is identified with $(-)^o \circ {}^oT^*$, so for every morphism $\varphi : i \to j$ of I the functor $\mathsf{c}_\varphi : p^{-1}(j) \to p^{-1}(i)$ is in turn identified with the left exact functor $T^*_\varphi : T_j \to T_i$, which is continuous for the canonical topologies (remark 4.4.13(iii)).

• Conversely, to every fibred site we may attach a fibred topos, via the pseudo-functors

$$\underline{\mathsf{T}} : \mathsf{U}'\text{-fib.}\mathbf{Site}(I) \xrightarrow{\underline{\mathsf{c}}^\bullet} \mathsf{PsFun}(I, \mathsf{U}'\text{-}\mathbf{Site}) \xrightarrow{\mathsf{PsFun}(I, \mathsf{T})} \mathsf{PsFun}(I, \mathbf{Topos})$$

$$\underline{\mathsf{lex.T}} : \mathsf{U}'\text{-fib.lex.}\mathbf{Site}(I) \xrightarrow{\mathsf{lex.}\underline{\mathsf{c}}^\bullet} \mathsf{PsFun}(I, \mathsf{U}'\text{-lex.}\mathbf{Site}) \xrightarrow{\mathsf{PsFun}(I, \mathsf{lex.T})} \mathsf{PsFun}(I, \mathbf{Topos})$$

where $\mathsf{T} : \mathbf{Site} \to \mathbf{Topos}$ and $\mathsf{lex.T} : \mathsf{lex.}\mathbf{Site} \to \mathbf{Topos}$ are as in (4.4.15), and $\underline{\mathsf{c}}^\bullet$ (resp. $\mathsf{lex.}\underline{\mathsf{c}}^\bullet$) is a strict and strong pseudo-inverse for $\underline{\mathscr{F}ib}_I$ (resp. for $\mathsf{lex.}\underline{\mathscr{F}ib}_I$: see remark 4.5.2(vi)).

Lemma 4.6.3. *With the notation of* (4.6.2)*, there exists a natural isomorphism of I-categories*

$$\mathscr{F}ib(T^*) \xrightarrow{\sim} \mathscr{F}ib(T_*)^o.$$

Proof. For every morphism $\varphi : i \to j$ of I, let $(\eta_\varphi, \varepsilon_\varphi)$ be the unit and counit for the adjunction for the pair $(T^*_\varphi, T_{\varphi*})$ that defines the morphism of topoi $T_\varphi : T_i \to T_j$. This adjunction assigns to every morphism $f : T^*_\varphi Y \to X$ (for any $X \in \mathrm{Ob}(T_i)$ and $Y \in \mathrm{Ob}(T_j)$) an adjoint morphism

$$f^\dagger := T_{\varphi*}(f) \circ \eta_{\varphi,Y} : Y \to T_{\varphi*}X.$$

Notice that $\mathscr{F}ib(T^*)$ and $\mathscr{F}ib(T_*)$ have the same set of objects, and the morphisms of $\mathscr{F}ib(T^*)$ (resp. $\mathscr{F}ib(T_*)$) are the pairs (φ, f) where $\varphi : i \to j$ is a morphism of I and $f : T^*_\varphi Y \to X$ (resp. $f : Y \to T_{\varphi*}X$) is a morphism of T_i (resp. of T_j). We claim that the rules

$$(i, X) \mapsto (i, X) \qquad (\varphi, f) \mapsto (\varphi, f^\dagger)$$

yield the sought isomorphism of categories. Indeed, denote also by (δ, γ) the coherence constraint of the pseudo-functor T^*; hence

$$\gamma_{(\varphi,\psi)} : T^*_{\psi\varphi} \xrightarrow{\sim} T^*_\varphi T^*_\psi \qquad \text{and} \qquad \delta_i : T^*_{\mathbf{1}_i} \xrightarrow{\sim} \mathbf{1}_{T_i}$$

are isomorphisms of functors, for every $i \in \mathrm{Ob}(I)$ and every pair of morphisms $\varphi : i \to j$ and $\psi : j \to k$ in I, and according to remark 1.1.17(ii), there follow adjoint isomorphisms

$$\gamma^\dagger_{(\varphi,\psi)} : T_{\psi*}T_{\varphi*} \xrightarrow{\sim} T_{\psi\circ\varphi*} \qquad \text{and} \qquad \delta^\dagger_i : \mathbf{1}_{T_i} \xrightarrow{\sim} T_{\mathbf{1}_i*}.$$

With this notation, we need to check the identity

$$(f \circ T^*_\varphi g \circ \gamma_{(\varphi,\psi),Z})^\dagger = \gamma^\dagger_{(\varphi,\psi),X} \circ T_{\psi*}(f^\dagger) \circ g^\dagger$$

for every pair of morphisms $(\varphi, f : T^*_\varphi Y \to X)$ and $(\psi, g : T^*_\psi Z \to Y)$ of $\mathscr{F}ib(T^*)$. However, on the one hand we have

$$(f \circ T^*_\varphi(g) \circ \gamma_{(\varphi,\psi),Z})^\dagger = T_{\psi\varphi*}(f \circ T^*_\varphi g \circ \gamma_{(\varphi,\psi),Z}) \circ \eta_{\psi\varphi,Z}$$

and on the other hand :

$$\begin{aligned}\gamma^\dagger_{(\varphi,\psi),X} \circ T_{\psi*}(f^\dagger) \circ g^\dagger &= \gamma^\dagger_{(\varphi,\psi),X} \circ T_{\psi*}T_{\varphi*}(f) \circ T_{\psi*}(\eta_{\varphi,Y}) \circ T_{\psi*}(g) \circ \eta_{\psi,Z} \\ &= T_{\psi\varphi*}(f) \circ \gamma^\dagger_{(\varphi,\psi),T^*_\varphi Y} \circ T_{\psi*}T_{\varphi*}T^*_\varphi(g) \circ T_{\psi*}(\eta_{\varphi,T^*_\psi Z}) \circ \eta_{\psi,Z} \\ &= T_{\psi\varphi*}(f) \circ \gamma^\dagger_{(\varphi,\psi),T^*_\varphi Y} \circ T_{\psi*}T_{\varphi*}T^*_\varphi(g) \circ \eta_{(\varphi,\psi),Z}\end{aligned}$$

where $(\eta_{(\varphi,\psi)}, \varepsilon_{(\varphi,\psi)})$ denotes the unit and counit of the adjunction that defines the composition $T_\psi \circ T_\varphi : T_i \to T_k$, by virtue of remark 1.1.17(i). Thus, we are reduced to checking that :

$$T_{\psi\varphi*}(T^*_\varphi g \circ \gamma_{(\varphi,\psi),Z}) \circ \eta_{\psi\varphi,Z} = \gamma^\dagger_{(\varphi,\psi),T^*_\varphi Y} \circ T_{\psi*}T_{\varphi*}T^*_\varphi(g) \circ \eta_{(\varphi,\psi),Z}.$$

But we have $\gamma^\dagger_{(\varphi,\psi),T^*_\varphi Y} \circ T_{\psi*}T_{\varphi*}T^*_\varphi(g) = T_{\psi\varphi*}T^*_\varphi(g) \circ \gamma^\dagger_{(\varphi,\psi),T^*_\varphi T^*_\psi Z}$, so we are further reduced to showing the identity :

$$T_{\psi\varphi*}(\gamma_{(\varphi,\psi),Z}) \circ \eta_{\psi\varphi,Z} = \gamma^\dagger_{(\varphi,\psi),T^*_\varphi T^*_\psi Z} \circ \eta_{(\varphi,\psi),Z}.$$

But using the explicit expressions of remark 1.1.17(ii) we see that :

$$\gamma^\dagger_{(\varphi,\psi)} * T^*_\varphi T^*_\psi = (T_{\psi\varphi*} * \varepsilon_{(\varphi,\psi)} * T^*_\varphi T^*_\psi) \odot (T_{\psi\varphi*} * \gamma_{(\varphi,\psi)} * T_{\psi*}T_{\varphi*}T^*_\varphi T^*_\psi) \odot (\eta_{\psi\varphi} * T_{\psi*}T_{\varphi*}T^*_\varphi T^*_\psi)$$

and on the other hand

$$(\eta_{\psi\varphi} * T_{\psi*}T_{\varphi*}T^*_\varphi T^*_\psi) \odot \eta_{(\varphi,\psi)} = (T_{\psi\varphi*}T^*_{\psi\varphi} * \eta_{(\varphi,\psi)}) \odot \eta_{\psi\varphi}$$

so it suffices to check that :

$$\gamma_{(\varphi,\psi)} = (\varepsilon_{(\varphi,\psi)} * T^*_\varphi T^*_\psi) \odot (\gamma_{(\varphi,\psi)} * T_{\psi*}T_{\varphi*}T^*_\varphi T^*_\psi) \odot (T^*_{\psi\varphi} * \eta_{(\varphi,\psi)}).$$

The latter follows from the identity :

$$(\gamma_{(\varphi,\psi)} * T_{\psi*}T_{\varphi*}T^*_\varphi T^*_\psi) \odot (T^*_{\psi\varphi} * \eta_{(\varphi,\psi)}) = (T^*_\varphi T^*_\psi * \eta_{(\varphi,\psi)}) \odot \gamma_{(\varphi,\psi)}$$

together with the triangular identities for $(\eta_{(\varphi,\psi)}, \varepsilon_{(\varphi,\psi)})$. □

Definition 4.6.4. (i) For any two categories $\mathscr{A}, \mathscr{B}$ and any functor $F : \mathscr{A} \to \mathscr{B}$ we denote

$$\Sigma(\mathscr{A}/\mathscr{B})$$

the *category of sections of* F, which is the subcategory of $\mathsf{Fun}(\mathscr{B}, \mathscr{A})$ whose objects are the functors $G : \mathscr{B} \to \mathscr{A}$ such that $F \circ G = \mathbf{1}_\mathscr{B}$ and whose morphisms are the natural transformations $\beta : G \Rightarrow G'$ with $F * \beta = i_\mathscr{B}$ (where $i_\mathscr{B}$ is the identity automorphism of $\mathbf{1}_\mathscr{B}$).

(ii) Let T be a fibred topos over I, and π, π' its associated fibrations as in (4.6.2); we let :

$$\mathsf{Top}(T)_* := \Sigma(\mathscr{F}ib(T_*)/I^o) \qquad \text{and} \qquad \mathsf{Top}(T) := \Sigma(\mathscr{F}ib(T^*)/I)^o.$$

By virtue of lemma 4.6.3 and remark 1.1.19(i) we have a natural isomorphism of categories :

$$\mathsf{Top}(T) \xrightarrow{\sim} \mathsf{Top}(T)_*. \qquad (4.6.5)$$

4.6.6. Let $\omega : T \to S$ be any morphism of fibred topoi over I. From the pseudo-natural transformations $\omega^* : S^* \Rightarrow T^*$ and $\omega_* : T_* \Rightarrow S_*$ we get the functors

$$\mathscr{F}ib(\omega^*) : \mathscr{F}ib(S^*) \to \mathscr{F}ib(T^*)$$
$$\mathscr{F}ib(\omega_*) : \mathscr{F}ib(T_*) \to \mathscr{F}ib(S_*)$$

(notation of (3.1.19)); whence functors

$$\mathsf{Top}(\omega)^* : \mathsf{Top}(S) \to \mathsf{Top}(T) \qquad E \mapsto \mathscr{F}ib(\omega^*) \circ E$$
$$\mathsf{Top}(\omega)_* : \mathsf{Top}(T)_* \to \mathsf{Top}(S)_* \qquad E \mapsto \mathscr{F}ib(\omega_*) \circ E.$$

By virtue of (4.6.5) we may then identify $\mathsf{Top}(\omega)_*$ with a functor which we shall also denote

$$\mathsf{Top}(\omega)_* : \mathsf{Top}(T) \to \mathsf{Top}(S).$$

Moreover, ω also induces a morphism of fibred sites

$$\underline{\mathrm{Can}}(\omega) := \mathscr{F}ib((-)^o * {}^o\omega^*) : \underline{\mathrm{Can}}(T) \to \underline{\mathrm{Can}}(S).$$

Let $\omega : T \to S$ and $\nu : S \to U$ be two morphisms of fibred topoi over I. A simple inspection of the definition shows that

$$\begin{aligned} \mathsf{Top}(\omega)^* \circ \mathsf{Top}(\nu)^* &= \mathsf{Top}(\nu \circ \omega)^* \\ \mathsf{Top}(\nu)_* \circ \mathsf{Top}(\omega)_* &= \mathsf{Top}(\nu \circ \omega)_* \\ \underline{\mathrm{Can}}(\nu) \circ \underline{\mathrm{Can}}(\omega) &= \underline{\mathrm{Can}}(\nu \circ \omega). \end{aligned} \tag{4.6.7}$$

4.6.8. Let $\omega, \omega' : T \to S$ be two morphisms of fibred topoi over I, and $\Xi : \omega \rightsquigarrow \omega'$ a transformation. Then Ξ induces natural transformations

$$\mathscr{F}ib(\Xi_*) : \mathscr{F}ib(\omega_*) \Rightarrow \mathscr{F}ib(\omega'_*) \qquad \mathscr{F}ib(\Xi^*) : \mathscr{F}ib(\omega^*) \Rightarrow \mathscr{F}ib(\omega'^*)$$

whence natural transformations :

$$\mathsf{Top}(\Xi)^* : \mathsf{Top}(\omega')^* \Rightarrow \mathsf{Top}(\omega)^* \qquad E \mapsto \mathscr{F}ib(\Xi^*) * E$$
$$\mathsf{Top}(\Xi)_* : \mathsf{Top}(\omega)_* \Rightarrow \mathsf{Top}(\omega')_* \qquad E \mapsto \mathscr{F}ib(\Xi_*) * E.$$

Moreover, Ξ induces a natural transformation of morphisms of fibred sites :

$$\underline{\mathrm{Can}}(\Xi) := \mathscr{F}ib((-)^o \circ {}^o\Xi^*) : \underline{\mathrm{Can}}(\omega') \to \underline{\mathrm{Can}}(\omega).$$

Remark 4.6.9. (i) Let T be a fibred topos over I with coherence constraint (δ^T, γ^T), and $T^u : I \to \mathbf{Topos}$ the unital pseudo-functor associated with T (proposition 2.4.3); by (4.6.7) and remark 2.2.14(i), the pseudo-natural isomorphism $\alpha^T : T \stackrel{\sim}{\to} T^u$ induces an isomorphism of categories $\mathsf{Top}(\alpha^T)^* : \mathsf{Top}(T^u) \stackrel{\sim}{\to} \mathsf{Top}(T)$. Then the following description of $\mathsf{Top}(T^u)$ – obtained by direct inspection – also applies to $\mathsf{Top}(T)$, up to natural isomorphism of categories:

- the objects of $\mathsf{Top}(T^u)$ are the systems $E_\bullet := ((E_i, E_\varphi) \mid i \in \mathrm{Ob}(I), \varphi \in \mathrm{Morph}(I))$ with $E_i \in \mathrm{Ob}(T_i)$ for every $i \in \mathrm{Ob}(I)$ and where

  $$E_\varphi : T^*_\varphi E_j \to E_i \qquad \text{for every } \varphi : i \to j \text{ in } I$$

  is a morphism of T_i, such that the following diagram commutes :

  $$\begin{array}{ccc} T^*_\psi T^*_\varphi E_k & \xrightarrow{T^*_\psi E_\varphi} & T^*_\psi E_j \\ {\scriptstyle (\gamma^{T^u}_{(\varphi,\psi)})^\dagger_{E_k}} \uparrow & & \downarrow {\scriptstyle E_\varphi} \\ T^*_{\psi\circ\varphi} E_k & \xrightarrow{E_{\psi\circ\varphi}} & E_i \end{array}$$

  for every pair of morphisms $i \xrightarrow{\varphi} j \xrightarrow{\psi} k$ of I, and $E_{\mathbf{1}_i} = \mathbf{1}_{E_i}$ for every $i \in \mathrm{Ob}(I)$.

- the morphisms $f_\bullet : E_\bullet \to F_\bullet$ are the systems $(f_i \mid i \in \mathrm{Ob}(I))$ where $f_i : E_i \to F_i$ is a morphism of T_i for every $i \in \mathrm{Ob}(I)$, such that the following diagram commutes :

$$\begin{array}{ccc} T^*_\varphi E_j & \xrightarrow{T^*_\varphi f_j} & T^*_\varphi F_j \\ {\scriptstyle E_\varphi}\downarrow & & \downarrow{\scriptstyle F_\varphi} \\ E_i & \xrightarrow{f_i} & F_i \end{array} \qquad \text{for every morphism } \varphi : i \to j \text{ of } I.$$

- the composition of morphisms $f_\bullet : E_\bullet \to F_\bullet$ and $g_\bullet : F_\bullet \to G_\bullet$ is defined by the obvious rule : $(g_\bullet \circ f_\bullet)_i := g_i \circ f_i$ for every $i \in \mathrm{Ob}(I)$.

(ii) Likewise, notice that for every unital fibred topos T over I, the objects of $\mathsf{Top}(T)_*$ are the systems $F_\bullet := (F_i, F_\varphi \mid i \in \mathrm{Ob}(I),\ \varphi \in \mathrm{Morph}(I))$ with $F_i \in \mathrm{Ob}(T_i)$ for every $i \in \mathrm{Ob}(I)$ and $F_\varphi : F_j \to T_{\varphi*}F_i$ a morphism of T_j, for every morphism $\varphi : i \to j$ of I. This datum is required to satisfy the identity

$$\gamma^T_{(\varphi,\psi),F_i} \circ (T_{\varphi*}F_\varphi) \circ F_\psi = F_{\psi\circ\varphi}$$

for every pair $i \xrightarrow{\varphi} j \xrightarrow{\psi} k$ of morphisms of I, and $F_{\mathbf{1}_i} = \mathbf{1}_{F_i}$ for every $i \in \mathrm{Ob}(I)$, where, as usual, γ^T denotes the coherence constraint of T. The morphisms $f_\bullet : F_\bullet \to F'_\bullet$ are the systems $(f_i : F_i \to F'_i \mid i \in \mathrm{Ob}(I))$ where f_i is a morphism of T_i for every $i \in \mathrm{Ob}(I)$ and

$$F'_\varphi \circ f_j = (T_{\varphi*}f_i) \circ F_\varphi \qquad \text{for every morphism } \varphi : i \to j \text{ of } I.$$

(iii) To every morphism $\omega : T \to S$ of fibred topoi over I there corresponds a morphism of unital fibred topoi $\omega^u : T^u \to S^u$ (see remark 2.4.2(i)); thus, in order to describe $\mathsf{Top}(\omega)^*$, we may assume without loss of generality that T and S are unital. In this case, we get the following explicit description. By definition, the coherence constraint of ω assigns to every morphism $\varphi : i \to j$ of I a natural transformation

$$\begin{array}{ccc} S_j & \xrightarrow{\omega^*_j} & T_j \\ {\scriptstyle S^*_\varphi}\downarrow & \swArrow_{\tau^{\omega\dagger}_\varphi} & \downarrow{\scriptstyle T^*_\varphi} \\ S_i & \xrightarrow[\omega^*_i]{} & T_i. \end{array}$$

Then $\mathsf{Top}(\omega)^*$ is given by the rule : $E_\bullet \mapsto ((\omega^*_i(E_i), E^\omega_\varphi) \mid i \in \mathrm{Ob}(I), \varphi \in \mathrm{Morph}(I))$, with

$$E^\omega_\varphi : T^*_\varphi \circ \omega^*_j(E_j) \xrightarrow{\tau^{\omega\dagger}_{\varphi,E_j}} \omega^*_i \circ S^*_\varphi(E_j) \xrightarrow{\omega^*_i(E_\varphi)} \omega^*_i(E_i)$$

and by assigning to every morphism $g_\bullet : E_\bullet \to E'_\bullet$ of $\mathsf{Top}(S)$ the morphism

$$(\omega^*_i(g_i) : \omega^*_i(E_i) \to \omega^*_i(E'_i) \mid i \in \mathrm{Ob}(I)).$$

Proposition 4.6.10. *The functor* $\mathsf{Top}(\omega)^*$ *is left adjoint to* $\mathsf{Top}(\omega)_*$.

Proof. Arguing as in remark 4.6.9(i), we reduce easily to the case where T and S are unital. In light of the identification (4.6.5), it suffices to exhibit an adjunction for the pair of functors

$$\mathsf{Top}(S)_* \underset{\Omega_*}{\overset{\Omega^*}{\rightleftarrows}} \mathsf{Top}(T)$$

resulting from the pair $(\mathsf{Top}(\omega)^*, \mathsf{Top}(\omega)_*)$ via these identifications. Explicitly, Ω^* assigns to every object $F_\bullet$ of $\mathsf{Top}(S)_*$ (see remark 4.6.9(ii)) the following object of $\mathsf{Top}(T)$:

$$(\omega^*_i(F_i), F^\Omega_\varphi := \omega^*_i(F^\dagger_\varphi) \circ (\tau^\omega_\varphi)^\dagger_{F_j} \mid i \in \mathrm{Ob}(I),\ \varphi \in \mathrm{Morph}(I))$$

where $F^\dagger_\varphi : T^*_\varphi F_j \to F_i$ is the morphism of T_i corresponding to F_φ under the adjunction of T_φ, *i.e.* $F^\dagger_\varphi = \varepsilon^{T_\varphi}_{E_i} \circ (T^*_\varphi F_\varphi)$, where ε^{T_φ} is the counit of the adjunction of T_φ. For every morphism $f_\bullet : F_\bullet \to F'_\bullet$ of $\mathsf{Top}(S)_*$ we have $\Omega^*(f_\bullet) = (\omega^*_i f_i \mid i \in \mathrm{Ob}(I))$.

Likewise, Ω_* assigns to every object $E_\bullet$ of $\mathsf{Top}(T)$ the following object of $\mathsf{Top}(S)_*$:

$$(\omega_{i*}(E_i), E^\Omega_\varphi := (\tau^\omega_\varphi)^{-1}_{E_i} \circ \omega_{j*}(E^\dagger_\varphi) \mid i \in \mathrm{Ob}(I),\ \varphi \in \mathrm{Morph}(I))$$

where $E^\dagger_\varphi : E_j \to T_{\varphi*}E_i$ is the morphism of T_j corresponding to E_φ under the adjunction of T_φ, *i.e.* $E^\dagger_\varphi = (T_{\varphi*}E_\varphi) \circ \eta^{T_\varphi}_{E_j}$, where η^{T_φ} is the unit of T_φ. For every morphism $g_\bullet : E_\bullet \to E'_\bullet$ of $\mathsf{Top}(T)$ we have $\Omega_*(g_\bullet) = (\omega_{i*}g_i \mid i \in \mathrm{Ob}(I))$.

Now, let $\beta : \Omega^*(F_\bullet) \to E_\bullet$ be a morphism of $\mathsf{Top}(T)$; thus, β is a system of morphisms $(\beta_i : \omega^*_i(F_i) \to E_i \mid i \in \mathrm{Ob}(I))$ such that the following diagram commutes :

(4.6.11)
$$\begin{array}{ccc} T^*_\varphi \omega^*_j F_j & \xrightarrow{T^*_\varphi \beta_j} & T^*_\varphi E_j \\ {\scriptstyle F^\Omega_\varphi}\downarrow & & \downarrow{\scriptstyle E_\varphi} \\ \omega^*_i F_i & \xrightarrow{\beta_i} & E_i \end{array} \qquad \text{for every morphism } i \xrightarrow{\varphi} j \text{ of } I.$$

Our candidate adjunction assigns to β the system of morphisms

$$\beta^\dagger := (\beta^\dagger_i := \omega_{i*}(\beta_i) \circ \eta^{\omega_i}_{F_i} : F_i \to \omega_{i*}E_i \mid i \in \mathrm{Ob}(I))$$

where η^{ω_i} is the unit of $\omega_i : T_i \to S_i$, so $\beta^\dagger_i$ is the morphism corresponding to β_i under the adjunction of ω_i. Thus, we need to show the commutativity of the diagram :

$$\begin{array}{ccc} F_j & \xrightarrow{\beta^\dagger_j} & \omega_{j*}E_j \\ {\scriptstyle F_\varphi}\downarrow & & \downarrow{\scriptstyle E^\Omega_\varphi} \\ S_{\varphi*}F_i & \xrightarrow{S_{\varphi*}(\beta^\dagger_i)} & S_{\varphi*}\omega_{i*}E_i \end{array} \qquad \text{for every morphism } i \xrightarrow{\varphi} j \text{ of } I.$$

Set $R := \omega_j \circ T_\varphi$, and let η^R and ε^R be the unit and counit of R; we consider the diagram :

$$\begin{array}{ccccccc} F_j & \xrightarrow{\eta^R_{F_j}} & R_*R^*F_j & \xrightarrow{R_*(F^\Omega_\varphi)} & R_*\omega^*_i F_i & \xrightarrow{(\tau^\omega_\varphi)^{-1}_{\omega^*_i F_i}} & S_{\varphi*}\omega_{i*}\omega^*_i F_i \\ {\scriptstyle \beta^\dagger_j}\downarrow & & \downarrow{\scriptstyle R_*T^*_\varphi(\beta_j)} & & \downarrow{\scriptstyle R_*(\beta_i)} & & \downarrow{\scriptstyle S_{\varphi*}\omega_{i*}(\beta_i)} \\ \omega_{j*}E_j & \xrightarrow[\omega_{j*}(\eta^{T_\varphi}_{E_j})]{} & R_*T^*_\varphi E_j & \xrightarrow[R_*(E_\varphi)]{} & R_*E_i & \xrightarrow[(\tau^\omega_\varphi)^{-1}_{E_i}]{} & S_{\varphi*}\omega_{i*}E_i. \end{array}$$

Recalling that $\eta^R_{F_j} = \omega_{j*}(\eta^{T_\varphi}_{\omega^*_j F_j}) \circ \eta^{\omega_j}_{F_j}$, we see that the left square commutes, by naturality of η^{T_φ}. The commutativity of the central square follows trivially from that of (4.6.11), and that of the right square follows from the naturality of τ^ω_φ. Notice now that the composition of the three bottom horizontal arrows equals E^Ω_φ. We are therefore reduced to checking the identity :

$$(\tau^\omega_\varphi)^{-1}_{\omega^*_i F_i} \circ R_*(F^\Omega_\varphi) \circ \eta^R_{F_j} = S_{\varphi*}(\eta^{\omega_i}_{F_i}) \circ F_\varphi.$$

We compute :

$$\begin{aligned}
R_*(F^\Omega_\varphi)\circ\eta^R_{F_j} &= R_*(\omega_i^*(F^\dagger_\varphi)\circ\tau^{\omega\dagger}_{\varphi,F_j})\circ\eta^R_{F_j} \\
&= R_*\omega_i^*(F^\dagger_\varphi)\circ R_*(\varepsilon^R_{\omega_i^*S^*_\varphi F_j})\circ R_*R^*(\tau^\omega_{\varphi,\omega_i^*S^*_\varphi F_j}\circ\eta^{S_\varphi\circ\omega_i}_{F_j})\circ\eta^R_{F_j} \\
&= R_*\omega_i^*(F^\dagger_\varphi)\circ R_*(\varepsilon^R_{\omega_i^*S^*_\varphi F_j})\circ\eta^R_{R_*\omega_i^*S^*_\varphi F_j}\circ\tau^\omega_{\varphi,\omega_i^*S^*_\varphi F_j}\circ\eta^{S_\varphi\circ\omega_i}_{F_j} \\
&= R_*\omega_i^*(F^\dagger_\varphi)\circ\tau^\omega_{\varphi,\omega_i^*S^*_\varphi F_j}\circ\eta^{S_\varphi\circ\omega_i}_{F_j} \\
&= \tau^\omega_{\varphi,\omega_i^*F_i}\circ S_{\varphi*}\omega_{i*}\omega_i^*(F^\dagger_\varphi)\circ\eta^{S_\varphi\circ\omega_i}_{F_j} \\
&= \tau^\omega_{\varphi,\omega_i^*F_i}\circ S_{\varphi*}\omega_{i*}\omega_i^*(\varepsilon^{S_\varphi}_{E_i})\circ S_{\varphi*}\omega_{i*}\omega_i^*S^*_\varphi(F_\varphi)\circ\eta^{S_\varphi\circ\omega_i}_{F_j} \\
&= \tau^\omega_{\varphi,\omega_i^*F_i}\circ S_{\varphi*}\omega_{i*}\omega_i^*(\varepsilon^{S_\varphi}_{E_i})\circ\eta^{S_\varphi\circ\omega_i}_{S_{\varphi*}F_i}\circ F_\varphi.
\end{aligned}$$

So we are further reduced to showing that :

$$S_{\varphi*}\omega_{i*}\omega_i^*(\varepsilon^{S_\varphi}_{E_i})\circ\eta^{S_\varphi\circ\omega_i}_{S_{\varphi*}F_i} = S_{\varphi*}(\eta^{\omega_i}_{F_i}).$$

But we have

$$\begin{aligned}
S_{\varphi*}\omega_{i*}\omega_i^*(\varepsilon^{S_\varphi}_{E_i})\circ\eta^{S_\varphi\circ\omega_i}_{T_{\varphi*}F_i} &= S_{\varphi*}\omega_{i*}\omega_i^*(\varepsilon^{S_\varphi}_{E_i})\circ S_{\varphi*}(\eta^{\omega_i}_{S^*_\varphi S_{\varphi*}F_i})\circ\eta^{S_\varphi}_{S_{\varphi*}F_i} \\
&= S_{\varphi*}(\omega_{i*}\omega_i^*(\varepsilon^{S_\varphi}_{E_i})\circ\eta^{\omega_i}_{S^*_\varphi S_{\varphi*}F_i})\circ\eta^{S_\varphi}_{S_{\varphi*}F_i} \\
&= S_{\varphi*}(\eta^{\omega_i}_{F_i})\circ S_{\varphi*}(\varepsilon^{S_\varphi}_{F_i})\circ\eta^{S_\varphi}_{S_{\varphi*}F_i}
\end{aligned}$$

so it suffices to invoke the triangular identities for the pair $(\eta^{S_\varphi},\varepsilon^{S_\varphi})$ to conclude.

Likewise one shows that for every morphism $\lambda : F_\bullet \to \Omega_*E_\bullet$ in $\mathsf{Top}(S)_*$, the system

$$(\lambda_i^\dagger := \varepsilon^{\omega_i}_{E_i}\circ\omega_i^*(\lambda_i) : \omega_i^*F_i \to E_i \mid i\in\mathrm{Ob}(I))$$

is a morphism $\Omega^*F_\bullet \to E_\bullet$ in $\mathsf{Top}(T)$: the verification shall be left to the reader. In view of remark 1.1.17(ii), it is then clear that these two rules yield mutually inverse bijections between $\mathrm{Hom}_{\mathsf{Top}(T)}(\Omega^*(F_\bullet),E_\bullet)$ and $\mathrm{Hom}_{\mathsf{Top}(S)_*}(F_\bullet,\Omega_*(E_\bullet))$. Lastly, the naturality of the rule $(\beta : \Omega^*(F_\bullet)\to E_\bullet)\mapsto\beta^\dagger$ with respect to both $F_\bullet$ and $E_\bullet$ follows directly from the same property for the adjunction of each morphism ω_i (details left to the reader). □

Remark 4.6.12. In the situation of proposition 4.6.10, denote by $\lambda_S : \mathsf{Top}(S) \xrightarrow{\sim} \mathsf{Top}(S)_*$ the natural isomorphism of (4.6.5). Then we have $\mathsf{Top}(\omega)_*\circ\mathsf{Top}(\omega)^* = \lambda_S^{-1}\circ\Omega_*\circ\Omega^*\circ\lambda_S$, and $\mathsf{Top}(\omega)^*\circ\mathsf{Top}(\omega)_* = \Omega^*\circ\Omega_*$. It follows that from the proof of proposition 4.6.10 we can extract an adjunction for the pair $(\mathsf{Top}(\omega)^*,\mathsf{Top}(\omega)_*)$, whose unit $\eta^{\mathsf{Top}(\omega)}$ and counit $\varepsilon^{\mathsf{Top}(\omega)}$ are given explicitly by the rules :

$$F_\bullet\mapsto(\eta^{\omega_i}_{F_i}\mid i\in\mathrm{Ob}(I)) \qquad E_\bullet\mapsto(\varepsilon^{\omega_i}_{E_i}\mid i\in\mathrm{Ob}(I))$$

where $(\eta^{\omega_i},\varepsilon^{\omega_i})$ are the unit and counit for the adjoint pair (ω_i^*,ω_{i*}), for every $i\in\mathrm{Ob}(I)$.

4.6.13. Let $\pi : \mathscr{A}\to I$ and $\rho : J\to I$ be any two functors; we have an obvious functor

$$\Sigma(\mathscr{A}/\rho)^* : \Sigma(\mathscr{A}/I)\to\Sigma(J\times_I\mathscr{A}/J) \qquad (G : I\to\mathscr{A})\mapsto J\times_I G$$

that assigns to every morphism $\alpha : G\Rightarrow G'$ in $\Sigma(\mathscr{A}/I)$ the natural transformation $J\times_I\alpha : J\times_I G\Rightarrow J\times_I G'$. Especially, if I and J are small categories, and T is any fibred topos over I, recalling the natural identification $\mathscr{F}ib((T\circ\rho)^*)\xrightarrow{\sim}J\times_I\mathscr{F}ib(T^*)$ of remark 3.1.5(ii), we deduce a functor

$$\mathsf{Top}(T/\rho)^* := \Sigma(\mathscr{F}ib(T^*)/\rho)^{*o} : \mathsf{Top}(T)\to\mathsf{Top}(T\circ\rho).$$

Explicitly, $\mathsf{Top}(T/\rho)^*$ assigns to every $E_\bullet$ as in remark 4.6.9(i) the datum $(E_{\rho(j)},E_{\rho(\varphi)}\mid j\in\mathrm{Ob}(J),\ \varphi\in\mathrm{Morph}(J))$, and to every morphism $f_\bullet : E_\bullet\to F_\bullet$ of $\mathsf{Top}(T)$, the morphism of

$\mathsf{Top}(T \circ \rho)$ given by the system $(f_{\rho(j)} : E_{\rho(j)} \to F_{\rho(j)} \mid j \in \mathrm{Ob}(J))$. We wish next to exhibit left and right adjoints for $\mathsf{Top}(T/\rho)^*$. To this aim, we observe more generally :

Proposition 4.6.14. *Let $\rho : J \to I$ be a functor between small categories, U' a universe containing U, and $\mathsf{c} : I^o \to \mathsf{U}'$-$\mathbf{Cat}$ any pseudo-functor such that*

(a) *The category c_i is complete for every $i \in \mathrm{Ob}(I)$.*

(b) *The functor $\mathsf{c}_f : \mathsf{c}_i \to \mathsf{c}_{i'}$ commutes with small limits, for every morphism $i' \xrightarrow{f} i$ in I.*

Then $\Sigma(\mathscr{F}ib(\mathsf{c})/\rho)^* : \Sigma(\mathscr{F}ib(\mathsf{c})/I) \to \Sigma(\mathscr{F}ib(\mathsf{c} \circ \rho^o)/J)$ admits a right adjoint.

Proof. By proposition 2.4.3 we may assume that c is unital. Now, set $\mathscr{A} := \mathscr{F}ib(\mathsf{c})$, denote by $\pi : \mathscr{A} \to I$ the projection, and let $F_\bullet : J \to J \times_I \mathscr{A}$ be any object of $\Sigma(J \times_I \mathscr{A}/J)$; hence $F_\bullet$ assigns to every $j \in \mathrm{Ob}(J)$ an object $F_j \in \mathsf{c}_{\rho(j)}$, and to every morphism $\psi : j \to j'$ in J a morphism $F_\psi : F_j \to \mathsf{c}_{\rho(\psi)} F_{j'}$ of $\mathsf{c}_{\rho(j)}$. We set

$$\Lambda_i(j, \varphi) := \mathsf{c}_\varphi F_j \qquad \text{for every } i \in \mathrm{Ob}(I) \text{ and } (j, \varphi : i \to \rho(j)) \in \mathrm{Ob}(i/\rho J).$$

For every morphism $(j, \varphi) \xrightarrow{i/\psi} (j', \varphi')$ of $i/\rho J$ we define $\Lambda_i(i/\psi) : \Lambda_i(j, \varphi) \to \Lambda_i(j', \varphi')$ as the composition

$$\mathsf{c}_\varphi F_j \xrightarrow{\mathsf{c}_\varphi F_\psi} \mathsf{c}_\varphi \mathsf{c}_{\rho(\psi)} F_{j'} \xrightarrow{(\gamma^{\mathsf{c}}_{\varphi, \rho(\psi)})_{F_{j'}}} \mathsf{c}_{\varphi'} F_{j'}$$

where γ^{c} is the coherence constraint of c. Also, for every morphism $f : i' \to i$ in I, let us set

$$\Lambda_f(j, \varphi) := (\gamma^{\mathsf{c}}_{f, \varphi})^{-1}_{F_j} : \Lambda_{i'}(j, \varphi \circ f) \to \mathsf{c}_f(\Lambda_i(j, \varphi)) \qquad \text{for every } (j, \varphi) \in \mathrm{Ob}(i/\rho J).$$

Claim 4.6.15. (i) The rules : $(j, \varphi) \mapsto \Lambda_i(j, \varphi)$ and $i/\psi \mapsto \Lambda(i/\psi)$ for every object (j, φ) and morphism i/ψ of $i/\rho J$ define a functor

$$\Lambda_i^{F_\bullet} : i/\rho J \to \pi^{-1}(i) \qquad \text{for every } i \in \mathrm{Ob}(I).$$

(ii) Let the functor $f/\rho J : i/\rho J \to i'/\rho J$ be as in (1.1.27). The rule : $(j, \varphi) \mapsto \Lambda_f(j, \varphi)$ defines a natural transformation

$$\Lambda_f^{F_\bullet} : \Lambda_{i'}^{F_\bullet} \circ (f/\rho J) \Rightarrow \mathsf{c}_f \circ \Lambda_i^{F_\bullet}.$$

Proof of the claim. (i): Since c is unital, a simple inspection shows that $\Lambda_i(\mathbf{1}_{(j,\varphi)}) = \mathbf{1}_{\Lambda(j,\varphi)}$ for every $(j, \varphi) \in \mathrm{Ob}(\rho J/i)$. Next consider a pair of morphisms $(j, \varphi) \xrightarrow{i/\psi} (j', \varphi') \xrightarrow{i/\psi'} (j'', \varphi'')$ of $i/\rho J$. We have

$$\begin{aligned}
\Lambda_i(i/\psi') \circ \Lambda_i(i/\psi) &= (\gamma^{\mathsf{c}}_{\varphi', \rho(\psi')})_{F_{j''}} \circ \mathsf{c}_{\varphi'} F_{\psi'} \circ (\gamma^{\mathsf{c}}_{\varphi, \rho(\psi)})_{F_{j'}} \circ \mathsf{c}_\varphi F_\psi \\
&= (\gamma^{\mathsf{c}}_{\varphi', \rho(\psi')})_{F_{j''}} \circ (\gamma^{\mathsf{c}}_{\varphi, \rho(\psi)} * \mathsf{c}_{\rho(\psi')})_{F_{j'}} \circ \mathsf{c}_\varphi \mathsf{c}_{\rho(\psi)}(F_{\psi'}) \circ \mathsf{c}_\varphi F_\psi \\
&= (\gamma^{\mathsf{c}}_{\varphi', \rho(\psi')})_{F_{j''}} \circ (\gamma^{\mathsf{c}}_{\varphi, \rho(\psi)} * \mathsf{c}_{\rho(\psi')})_{F_{j'}} \circ \mathsf{c}_\varphi((\gamma^{\mathsf{c}\,-1}_{\rho(\psi), \rho(\psi')})_{F_{j''}} \circ F_{\psi' \circ \psi}) \\
&= (\gamma^{\mathsf{c}}_{\varphi, \rho(\psi' \circ \psi)})_{F_{j''}} \circ \mathsf{c}_\varphi F_{\psi' \circ \psi} \\
&= \Lambda_i(i/\psi' \circ \psi)
\end{aligned}$$

whence the contention.

(ii): It suffices to remark the commutativity of the diagram :

$$\begin{array}{ccccc}
\mathsf{c}_{\varphi \circ f} F_j & \xrightarrow{\mathsf{c}_{\varphi \circ f} F_\psi} & \mathsf{c}_{\varphi \circ f} \mathsf{c}_{\rho(\psi)} F_{j'} & \xrightarrow{(\gamma^{\mathsf{c}}_{\varphi \circ f, \rho(\psi)})_{F_{j'}}} & \mathsf{c}_{\rho(\psi) \circ \varphi \circ f} F_{j'} \\
\big\downarrow {\scriptstyle (\gamma^{\mathsf{c}}_{f,\varphi})^{-1}_{F_j}} & & \big\downarrow {\scriptstyle (\gamma^{\mathsf{c}}_{f,\varphi})^{-1}_{\mathsf{c}_{\rho(\psi)} F_{j'}}} & & \big\downarrow {\scriptstyle (\gamma^{\mathsf{c}}_{f,\varphi'})^{-1}_{F_{j'}}} \\
\mathsf{c}_f \mathsf{c}_\varphi F_j & \xrightarrow[\mathsf{c}_f \mathsf{c}_\varphi F_\psi]{} & \mathsf{c}_f \mathsf{c}_\varphi \mathsf{c}_{\rho(\psi)} F_{j'} & \xrightarrow[\mathsf{c}_f(\gamma^{\mathsf{c}}_{\varphi, \rho(\psi)})_{F_{j'}}]{} & \mathsf{c}_f \mathsf{c}_{\rho(\psi) \circ \varphi} F_{j'}
\end{array}$$

for every morphism $(i/\psi) : (j, \varphi) \to (j', \varphi')$ of $i/\rho J$. $\Diamond$

In view of claim 4.6.15(i), for every $i \in \mathrm{Ob}(I)$ we choose $\lambda(F_\bullet, i) \in \mathrm{Ob}(\mathsf{c}_i)$ representing the limit of $\Lambda_i^{F_\bullet}$, and a universal cone $\tau^{F_\bullet,i} : c_{\lambda(F_\bullet,i)} \Rightarrow \Lambda_i^{F_\bullet}$. In view of claim 4.6.15(ii) there follows a cone

$$\Upsilon_f^{F_\bullet} := \Lambda_f^{F_\bullet} \odot (\tau^{F_\bullet,i'} * (f/\rho J)) : c_{\lambda(F_\bullet,i')} \Rightarrow \mathsf{c}_f \circ \Lambda_i^{F_\bullet}.$$

On the other hand, notice that the cone $\mathsf{c}_f * \tau^{F_\bullet,i} : c_{\mathsf{c}_f\lambda(F_\bullet,i)} \Rightarrow \mathsf{c}_f \circ \Lambda_i^{F_\bullet}$ is still universal, since c_f commutes with small limits; hence there exists a unique morphism in $\mathsf{c}_{i'}$

$$\lambda(F_\bullet, f) : \lambda(F_\bullet, i') \to \mathsf{c}_f\lambda(F_\bullet, i) \qquad \text{such that} \qquad (\mathsf{c}_f * \tau^{F_\bullet,i}) \odot c_{\lambda(F_\bullet,f)} = \Upsilon_f^{F_\bullet}.$$

Claim 4.6.16. The rules : $i \mapsto \lambda(F_\bullet, i)$ and $f \mapsto \lambda(F_\bullet, f)$ for every $i \in \mathrm{Ob}(I)$ and every morphism f of I define a functor

$$\lambda(F_\bullet) : I \to \mathscr{A}.$$

Proof of the claim. Since c is unital, a simple inspection shows that $\Lambda_{\mathbf{1}_i}^{F_\bullet} = \mathbf{1}_{\Lambda_i^{F_\bullet}}$ for every $i \in \mathrm{Ob}(I)$, whence $\lambda(F_\bullet, \mathbf{1}_i) = \mathbf{1}_{\lambda(F_\bullet,i)}$. Next, let $i'' \xrightarrow{f'} i' \xrightarrow{f} i$ be two morphisms of I; we need to check that $\lambda(F_\bullet, f) \circ \lambda(F_\bullet, f') = \lambda(F_\bullet, f \circ f')$, and by the universality of $\mathsf{c}_{f\circ f'} * \tau^{F_\bullet,i}$ it suffices to show that $X := (\mathsf{c}_{f\circ f'} * \tau^{F_\bullet,i}) \odot c_{\lambda(F_\bullet,f)\circ\lambda(F_\bullet,f')} = (\mathsf{c}_{f\circ f'} * \tau^{F_\bullet,i}) \odot c_{\lambda(F_\bullet,f\circ f')}$. We compute :

$$\begin{aligned}
X &= (\mathsf{c}_{f\circ f'} * \tau^{F_\bullet,i}) \odot c_{(\gamma^T_{f',f})_{\lambda(F_\bullet,i)}} \odot c_{\mathsf{c}_{f'}\lambda(F_\bullet,f)} \odot c_{\lambda(F_\bullet,f')} \\
&= (\gamma^{\mathsf{c}}_{f,f'} * \Lambda_i^{F_\bullet}) \odot (\mathsf{c}_{f'}\mathsf{c}_f * \tau^{F_\bullet,i}) \odot c_{\mathsf{c}_{f'}\lambda(F_\bullet,f)} \odot c_{\lambda(F_\bullet,f')} \\
&= (\gamma^{\mathsf{c}}_{f,f'} * \Lambda_i^{F_\bullet}) \odot \mathsf{c}_{f'} * (\Lambda_f^{F_\bullet} \odot (\tau^{F_\bullet,i'} * (f/\rho J))) \odot c_{\lambda(F_\bullet,f')} \\
&= (\gamma^{\mathsf{c}}_{f,f'} * \Lambda_i^{F_\bullet}) \odot (\mathsf{c}_{f'} * \Lambda_f^{F_\bullet}) \odot ((\Lambda_{f'}^{F_\bullet} \odot (\tau^{F_\bullet,i''} * (f'/\rho J))) * (f/\rho J)) \\
&= (\gamma^{\mathsf{c}}_{f,f'} * \Lambda_i^{F_\bullet}) \odot (\mathsf{c}_{f'} * \Lambda_f^{F_\bullet}) \odot (\Lambda_{f'}^{F_\bullet} * (f/\rho J)) \odot (\tau^{F_\bullet,i''} * (f \circ f'/\rho J)).
\end{aligned}$$

So we are reduced to checking that

$$(\gamma^{\mathsf{c}}_{f,f'} * \Lambda_i^{F_\bullet}) \odot (\mathsf{c}_{f'} * \Lambda_f^{F_\bullet}) \odot (\Lambda_{f'}^{F_\bullet} * (f/\rho J)) = \Lambda_{f\circ f'}^{F_\bullet}.$$

But the latter follows directly from the coherence axioms for γ^{c}. ◊

It is clear that the functor $\lambda(F_\bullet)$ of claim 4.6.16 is a section of the projection $\mathscr{A} \to I$. Next, let $\beta_\bullet : F_\bullet \to F'_\bullet$ be any morphism of $\Sigma(J \times_I \mathscr{A}/J)$; we deduce easily a natural transformation

$$\Lambda_i^{\beta_\bullet} : \Lambda_i^{F_\bullet} \Rightarrow \Lambda_i^{F'_\bullet} \qquad (j, \varphi) \mapsto (\mathsf{c}_\varphi\beta_j : \mathsf{c}_\varphi F_j \to \mathsf{c}_\varphi F'_j) \qquad \text{for every } i \in \mathrm{Ob}(I)$$

such that

$$(\mathsf{c}_f\Lambda_i^{\beta_\bullet}) \odot \Lambda_f^{F_\bullet} = \Lambda_f^{F'_\bullet} \odot (\Lambda_{i'}^{\beta_\bullet} * (f/\rho J)) \qquad \text{for every morphism } f : i' \to i \text{ of } I \tag{4.6.17}$$

whence a unique morphism in c_i

$$\lambda(\beta_\bullet, i) : \lambda(F_\bullet, i) \to \lambda(F'_\bullet, i) \qquad \text{such that} \qquad \tau^{F'_\bullet,i} \odot c_{\lambda(\beta_\bullet,i)} = \Lambda_i^{\beta_\bullet} \odot \tau^{F_\bullet,i}.$$

Claim 4.6.18. The rule $i \mapsto \lambda(\beta_\bullet, i)$ defines a natural transformation

$$\lambda(\beta_\bullet) : \lambda(F_\bullet) \Rightarrow \lambda(F'_\bullet).$$

Proof of the claim. Let $f : i' \to i$ be any morphism of I; as usual, we reduce to checking the identity : $X := (\mathsf{c}_f * \tau^{F'_\bullet,i}) \odot c_{\lambda(F'_\bullet,f)} \odot c_{\lambda(\beta_\bullet,i')} = Y := (\mathsf{c}_f * \tau^{F'_\bullet,i}) \odot (\mathsf{c}_f * c_{\lambda(\beta_\bullet,i)}) \odot c_{\lambda(F_\bullet,f)}$. We compute :

$$\begin{aligned}
X &= \Upsilon_f^{F'_\bullet} \odot c_{\lambda(\beta_\bullet,i')} = \Lambda_f^{F'_\bullet} \odot (\Lambda_{i'}^{\beta_\bullet} \odot \tau^{F_\bullet,i'}) * (f/\rho J) \\
Y &= \mathsf{c}_f(\Lambda_i^{\beta_\bullet} \odot \tau^{F_\bullet,i}) \odot c_{\lambda(F_\bullet,f)} = \mathsf{c}_f\Lambda_i^{\beta_\bullet} \odot \Upsilon_f^{F_\bullet}
\end{aligned}$$

so the assertion follows from (4.6.17). ◊

Let $\beta_\bullet : F_\bullet \to F'_\bullet$ and $\beta'_\bullet : F'_\bullet \to F''_\bullet$ be two morphisms of $\Sigma(J \times_I \mathscr{A}/J)$; by a simple inspection we see that $\Lambda_i^{\beta'_\bullet \circ \beta_\bullet} = \Lambda_i^{\beta'_\bullet} \odot \Lambda_i^{\beta_\bullet}$ for every $i \in \mathrm{Ob}(I)$, whence $\lambda(\beta'_\bullet \circ \beta_\bullet) = \lambda(\beta'_\bullet) \odot \lambda(\beta_\bullet)$. It is also easily seen that $\lambda(\mathbf{1}_{F_\bullet}) = \mathbf{1}_{\lambda(F_\bullet)}$ for every object $F_\bullet$ of $\Sigma(J \times_I \mathscr{A}/J)$, so we have finally obtained a functor

$$\Sigma(\mathscr{A}/\rho)_* : \Sigma(J \times_I \mathscr{A}/J) \to \Sigma(\mathscr{A}/I) \qquad F_\bullet \mapsto \lambda(F_\bullet) \qquad (\beta_\bullet : F_\bullet \to F'_\bullet) \mapsto \lambda(\beta_\bullet).$$

To see that $\Sigma(\mathscr{A}/\rho)_*$ is the sought right adjoint, consider any objects $E_\bullet$ of $\Sigma(\mathscr{A}/I)$ and $F_\bullet$ of $\Sigma(J \times_I \mathscr{A}/J)$, and a morphism $\beta_\bullet : E_\bullet \to \lambda(F_\bullet)$ in $\Sigma(\mathscr{A}/I)$. It is easily seen that the rule :

$$j \mapsto \beta_j^\dagger : E_{\rho(j)} \xrightarrow{\beta_{\rho(j)}} \lambda(F_\bullet, \rho(j)) \xrightarrow{\tau^{F_\bullet,\rho(j)}_{(j,\mathbf{1}_{\rho(j)})}} F_j \qquad \text{for every } j \in \mathrm{Ob}(J)$$

defines a morphism $\beta_\bullet^\dagger : J \times_I E_\bullet \to F_\bullet$ in $\Sigma(J \times_I \mathscr{A}/J)$ (details left to the reader). Conversely, let $\alpha_\bullet : J \times_I E_\bullet \to F_\bullet$ be a morphism in $\Sigma(J \times_I \mathscr{A}/J)$; for every $i \in \mathrm{Ob}(I)$ and $(j, \varphi) \in \mathrm{Ob}(i/\rho J)$ we let $\tau^{\alpha_\bullet,i}_{(j,\varphi)} := \mathsf{c}_\varphi \alpha_j \circ E_\varphi : E_i \to \mathsf{c}_\varphi F_j$. It is easily seen that the rule : $(j, \varphi) \mapsto \tau^{\alpha_\bullet,i}_{(j,\varphi)}$ defines a cone $\tau^{\alpha_\bullet,i} : c_{E_i} \Rightarrow \Lambda_i^{F_\bullet}$ (details left to the reader); there follows a unique morphism

$$\alpha_i^\dagger : E_i \to \lambda(F_\bullet, i) \qquad \text{in } \mathsf{c}_i \text{ such that} \qquad \tau^{F_\bullet,i} \odot c_{\alpha_i^\dagger} = \tau^{\alpha_\bullet,i}.$$

Claim 4.6.19. The rule $i \mapsto \alpha_i^\dagger$ defines a morphism $\alpha_\bullet^\dagger : E_\bullet \to \lambda(F_\bullet)$ in $\Sigma(\mathscr{A}/I)$.

Proof of the claim. As usual we reduce to checking that

$$X := (\mathsf{c}_f * \tau^{F_\bullet,i}) \odot c_{\mathsf{c}_f \alpha_i^\dagger} \odot c_{E_f} = Y := (\mathsf{c}_f * \tau^{F_\bullet,i}) \odot c_{\lambda(F_\bullet,f)} \odot c_{\alpha_{i'}^\dagger}$$

for every morphism $f : i' \to i$ in I. However, we have :

$$X = (\mathsf{c}_f * \tau^{\alpha,i}) \odot c_{E_f} \qquad Y = \Upsilon_f^{F_\bullet} \odot c_{\alpha_{i'}^\dagger} = \Lambda_f^{F_\bullet} \odot (\tau^{\alpha,i'} * (f/\rho J)).$$

So we come down to checking that

$$(\mathsf{c}_f \mathsf{c}_\varphi \alpha_j) \circ (\mathsf{c}_f E_\varphi) \circ E_f = (\gamma^{\mathsf{c}}_{f,\varphi})^{-1}_{F_j} \circ \mathsf{c}_{\varphi \circ f} \alpha_j \circ E_{\varphi \circ f} \qquad \text{for every } (j, \varphi) \in \mathrm{Ob}(i/\rho J).$$

To this aim, it suffices to notice that we have : $(\gamma^{\mathsf{c}}_{f,\varphi})_{F_j} \circ (\mathsf{c}_f \mathsf{c}_\varphi \alpha_j) = \mathsf{c}_{\varphi \circ f} \alpha_j \circ (\gamma^{\mathsf{c}}_{f,\varphi})_{E_{\rho(j)}}$ and $(\gamma^{\mathsf{c}}_{f,\varphi})_{E_{\rho(j)}} \circ (\mathsf{c}_f E_\varphi) \circ E_f = E_{\varphi \circ f}$. ◊

Claim 4.6.20. For every morphism $\alpha_\bullet : J \times_I E_\bullet \to F_\bullet$ in $\Sigma(J \times_I \mathscr{A}/J)$ and $\beta_\bullet : E_\bullet \to \lambda(F_\bullet)$ in $\Sigma(\mathscr{A}/I)$ we have $(\alpha_\bullet^\dagger)^\dagger = \alpha_\bullet$ and $(\beta_\bullet^\dagger)^\dagger = \beta_\bullet$.

Proof of the claim. The assertion concerning $\alpha_\bullet$ follows by direct inspection. Next, let us consider for every $i \in \mathrm{Ob}(I)$ and every $(j, \varphi) \in \mathrm{Ob}(i/\rho J)$ the diagram of morphisms in c_i

$$\mathscr{D}_{(j,\varphi)} \quad : \quad \begin{array}{ccccc} E_i & \xrightarrow{E_\varphi} & \mathsf{c}_\varphi E_{\rho(j)} & \xrightarrow{\mathsf{c}_\varphi \beta_{\rho(j)}} & \mathsf{c}_\varphi \lambda(F_\bullet, \rho(j)) \\ & \searrow^{\beta_i} & & \nearrow_{\lambda(F_\bullet,\varphi)} & \downarrow{\scriptstyle \mathsf{c}_\varphi \tau^{F_\bullet,\rho(j)}_{(j,\mathbf{1}_{\rho(j)})}} \\ & & \lambda(F_\bullet, i) & \xrightarrow[\tau^{F_\bullet,i}_{(j,\varphi)}]{} & \mathsf{c}_\varphi F_j. \end{array}$$

The assertion is equivalent to the commutativity of $\mathscr{D}_{(j,\varphi)}$ for every such i and (j, φ). However, notice that $(\Lambda_\varphi^{F_\bullet})_{(j,\mathbf{1}_{\rho(j)})} = \mathbf{1}_{\mathsf{c}_\varphi F_j}$, since c is unital; this implies that the lower triangular subdiagram commutes. The commutativity of the upper triangular subdiagram is clear. ◊

Lastly, it is easily seen that the rule : $\beta_\bullet \mapsto \beta_\bullet^\dagger$ is natural in both $E_\bullet$ and $F_\bullet$ (details left to the reader); in view of claim 4.6.20, this rule then establishes the sought adjunction between $\Sigma(\mathscr{A}/\rho)^*$ and $\Sigma(\mathscr{A}/\rho)_*$. □

Corollary 4.6.21. *In the situation of* (4.6.13)*, the functor* $\mathsf{Top}(T/\rho)^*$ *admits both a left and a right adjoint, denoted respectively :*

$$\mathsf{Top}(T/\rho)_! : \mathsf{Top}(T \circ \rho) \to \mathsf{Top}(T) \qquad \textit{and} \qquad \mathsf{Top}(T/\rho)_* : \mathsf{Top}(T \circ \rho) \to \mathsf{Top}(T).$$

Proof. It is easily seen that the isomorphism of categories (4.6.5) (and the corresponding one for $T \circ \rho$) identifies the functor $\mathsf{Top}(T/\rho)^*$ with $\Sigma(\mathscr{F}ib(T_*)/\rho^o)^*$, hence the existence of the right adjoint $\mathsf{Top}(T/\rho)_*$ follows from proposition 4.6.14. On the other hand, by applying the same proposition to $\Sigma(\mathscr{F}ib(T^*)/\rho)^* : \Sigma(\mathscr{F}ib(T^*)/I) \to \Sigma(\mathscr{F}ib(T \circ \rho)^*/J)$ we see that the latter admits as well a right adjoint, hence its opposite functor $\mathsf{Top}(T/\rho)^*$ admits a left adjoint. □

Remark 4.6.22. (i) Let $\rho : J \to I$ be any functor between small categories and T any fibred topos over I. Corollary 4.6.21 implies that the functor $\mathsf{Top}(T/\rho)^*$ commutes with all small limits and all small colimits (proposition 1.3.25(iii,iv)).

(ii) Let $\mathbb{1}$ be a final object of the category $\mathbf{Cat}$ (*i.e.* a category with one object and one morphism); for every $t \in \mathrm{Ob}(I)$ we have a unique functor $\rho^t : \mathbb{1} \to I$ that sends the unique object of $\mathbb{1}$ to t. Clearly $\mathsf{Top}(T \circ \rho^t) = T_t$ and the induced functor

$$\mathsf{Top}(T/\rho^t)^* : \mathsf{Top}(T) \to T_t$$

assigns to every datum $E_\bullet$ as in remark 4.6.9(i) the object E_t and to every morphism $f_\bullet : E_\bullet \to F_\bullet$ of $\mathsf{Top}(T)$ the morphism $f_t : E_t \to F_t$ of T_t. The assertion that $\mathsf{Top}(T/\rho^t)^*$ commutes with limits and colimits for every $t \in \mathrm{Ob}(I)$ then means that *the small limits and colimits in the category* $\mathsf{Top}(T)$ *are computed fibrewise*.

(iii) Moreover, the observation of (ii) determines the limits and colimits in $\mathsf{Top}(T)$ up to unique isomorphism. Indeed, let $F : \Lambda \to \mathsf{Top}(T)$ be any functor from a small category Λ, and let $(L_\bullet, \tau)$ be a pair consisting of an object $L_\bullet$ of $\mathsf{Top}(T)$ representing the colimit of F, and a universal cocone $\tau : F \Rightarrow c_{L_\bullet}$. By (ii) we know that $L_t \in \mathrm{Ob}(T_t)$ represents the colimit of $F^t := \mathsf{Top}(T/\rho^t)^* \circ F : \Lambda \to T_t$ and $\tau^t := \mathsf{Top}(T/\rho^t)^* * \tau : F^t \Rightarrow c_{L_t}$ is a universal cocone for every $t \in \mathrm{Ob}(I)$. Now, let $\varphi : s \to t$ be any morphism in I; we get a natural transformation

$$F^\varphi : T^*_\varphi \circ F^t \Rightarrow F^s \qquad \lambda \mapsto ((F\lambda)_\varphi : T^*_\varphi(F\lambda)_t \to (F\lambda)_s).$$

Since T^*_φ admits a right adjoint, the cocone $T^*_\varphi * \tau^t : T^*_\varphi \circ F^t \Rightarrow c_{T^*_\varphi L_t}$ is still universal (proposition 1.3.25(iv)), hence there exists a unique morphism

$$L'_\varphi : T^*_\varphi L_t \to L_s \qquad \text{in } T_s \text{ such that} \qquad \tau^s \odot F^\varphi = c_{L'_\varphi} \odot (T^*_\varphi * \tau^t).$$

But then we must have $L'_\varphi = L_\varphi$ necessarily, so $L_\bullet$ is completely determined by a choice of universal cocones $(\tau^t \mid t \in \mathrm{Ob}(I))$. Likewise we see that the limit of F is completely determined by the choice of a system of pairs $(M_t, \mu^t : F^t \Rightarrow c_{M_t} \mid t \in \mathrm{Ob}(I)$ with $M_t \in \mathrm{Ob}(T_t)$ representing the limit of F^t, and μ_t a universal cone, for every $t \in \mathrm{Ob}(I)$.

(iv) By direct inspection, we see that the functor

$$\mathsf{Top}(T/\rho^t)_! : T_t \to \mathsf{Top}(T)$$

assigns to every $E \in \mathrm{Ob}(T_t)$ the datum $(E_i, E_\varphi \mid i \in \mathrm{Ob}(I), \varphi \in \mathrm{Morph}(I))$ such that :

$$E_i := \coprod_{f : i \to t} T^*_f E \qquad \text{for every } i \in \mathrm{Ob}(I)$$

where the f ranges over all the morphisms $i \to t$ in I. For every morphism $\varphi : i \to j$ of I, the morphism $E_\varphi : T^*_\varphi E_j \to E_i$ is the unique one fitting in the commutative diagram :

$$\begin{array}{ccc} T^*_\varphi T^*_f E & \xrightarrow{\gamma^{T^*}_{\varphi,f}} & T^*_{f\circ\varphi} E \\ \downarrow & & \downarrow \\ E_j & \xrightarrow{E_\varphi} & E_i \end{array} \qquad \text{for every morphism } f : j \to t \text{ in } I$$

where γ^{T^*} is the coherence constraint of T^*, and where the vertical arrows are induced by the chosen universal cocones for the representatives E_i and E_j of the foregoing coproducts. For every morphism h in T_t, the morphism $\mathsf{Top}(T/\rho^t)_!(h)$ is the natural transformation that assigns to every $i \in \mathrm{Ob}(I)$ the coproduct of morphisms $\coprod_{f:i\to t} T^*_f(h)$: details left to the reader.

4.6.23. Let T be a fibred topos over I; we consider the associated fibred lex-site $\pi : \underline{\mathrm{Can}}(T) \to I$ as in (4.6.2), and its total site $(\underline{\mathrm{Can}}(T), J)$. Also, for every $t \in \mathrm{Ob}(I)$, we denote by $i_t : T_t \to \underline{\mathrm{Can}}(T)$ the inclusion functor.

Theorem 4.6.24. (i) *With the notation of* (4.6.23)*, we have a natural equivalence of categories:*

$$a_T : \mathsf{Top}(T) \xrightarrow{\sim} (\underline{\mathrm{Can}}(T), J)^\sim_{\mathsf{U}}$$

and a morphism of sites :

$$b_T : \mathrm{Can}(\mathsf{Top}(T)) \to (\underline{\mathrm{Can}}(T), J)$$

fitting into an essentially commutative diagram (notation of remark 4.6.22(ii)*) :*

$$\begin{array}{ccccc} \mathrm{Can}(\mathsf{Top}(T))^\sim_{\mathsf{U}} & \xleftarrow{h_{\mathsf{Top}(T)}} & \mathsf{Top}(T) & \xrightarrow{\mathsf{Top}(T/\rho^t)^*} & T_t \\ & \searrow^{b^\sim_{T*}} & \downarrow{a_T} & & \downarrow{h_{T_t}} \\ & & (\underline{\mathrm{Can}}(T), J)^\sim_{\mathsf{U}} & \xrightarrow{(i_t)^\sim_{\mathsf{U}*}} & \mathrm{Can}(T_t)^\sim_{\mathsf{U}} \end{array} \qquad \textit{for every } t \in \mathrm{Ob}(I).$$

(ii) *The category* $\mathsf{Top}(T)$ *is an* U*-topos, which we call the* total topos *of the fibred topos* T.

Proof. Let us remark more generally :

Claim 4.6.25. For every pseudo-functor $\mathsf{c} : I \to \mathsf{lex.Site}$ there exists a natural isomorphism of categories (notation of (4.4.15) and remarks 4.5.10 and 4.5.2(v)) :

$$\alpha_{\mathsf{c}} : \mathsf{Top}(\mathsf{T} \circ \mathsf{c}) \xrightarrow{\sim} \mathsf{T}(\mathsf{totSite} \circ \underline{\mathscr{F}ib}_I(\mathsf{c})).$$

Proof of the claim. By proposition 2.4.3, we may assume that c is unital, and we let γ^{c} be the coherence constraint of c. By proposition 4.5.5(iv), a presheaf F on the total site of $\underline{\mathscr{F}ib}_I(\mathsf{c})$ is a sheaf if and only if its restriction F_i to each fibre site c_i is a sheaf. Hence, such a sheaf F is the datum $F_\bullet := (F_i, F_\varphi : F_j \to (\mathsf{c}_\varphi)^\sim_* F_i \mid i \in \mathrm{Ob}(I),\ (\varphi : i \to j) \in \mathrm{Morph}(I))$ of a system of sheaves and morphisms of sheaves that make commute the diagrams

$$\begin{array}{ccc} F_k & \xrightarrow{F_{\psi\circ\varphi}} & (\mathsf{c}_{\psi\circ\varphi})^\sim_* F_i \\ \downarrow{F_\psi} & & \uparrow{(\gamma^{\mathsf{c}}_{\varphi,\psi})^\sim_{*,F_i}} \\ (\mathsf{c}_\psi)^\sim_* F_j & \xrightarrow{(\mathsf{c}_\psi)^\sim_* F_\varphi} & (\mathsf{c}_\psi)^\sim_* \circ (\mathsf{c}_\varphi)^\sim_* F_i \end{array}$$

for every pair of morphisms $i \xrightarrow{\varphi} j \xrightarrow{\psi} k$ of I. Namely, $F_{\varphi,X} := F(\varphi, \mathbf{1}_{\mathsf{c}_\varphi X}) : F_j X \to F_i(\mathsf{c}_\varphi X)$ for every such φ, and every $X \in \mathrm{Ob}(T_j)$. Recall now that, for every site $(\mathscr{C}, J)$, the topos $\mathsf{T}(\mathscr{C}, J)$ is isomorphic to $(\mathscr{C}, J)^\sim$; especially, $\mathsf{T}(\mathsf{c}_i)$ is isomorphic to the category of sheaves on the site c_i, for every $i \in \mathrm{Ob}(I)$. Under this identification, we then see that a datum $F_\bullet$ as in the foregoing corresponds precisely to an object of the category $\mathsf{Top}(\mathsf{T} \circ \mathsf{c})_*$, and likewise it is easily seen that the morphisms of $\mathsf{Top}(\mathsf{T} \circ \mathsf{c})_*$ correspond naturally to the morphisms of sheaves on the total site of $\underline{\mathscr{F}ib}_I(\mathsf{c})$. Then the sought isomorphism is the composition of this natural identification with the isomorphism $\mathsf{Top}(\mathsf{T} \circ \mathsf{c})_* \xrightarrow{\sim} \mathsf{Top}(\mathsf{T} \circ \mathsf{c})_*$ of lemma 4.6.3. ◊

Recall now that the unit $\eta : \mathbf{1_{Topos}} \to \mathsf{T} \circ \mathrm{Can}$ of the 2-adjoint pair $(\mathrm{Can}, \mathsf{T})$ is a pseudo-natural equivalence (theorem 4.4.17(iii)); there follows a pseudo-natural equivalence $\eta_T : T \xrightarrow{\sim} \mathsf{T} \circ \mathrm{Can} \circ T$. Then the sought equivalence a_T is the composition of $\mathsf{Top}(\eta_T) : \mathsf{Top}(T) \to \mathsf{Top}(\mathsf{T} \circ \mathrm{Can} \circ T)$ with the isomorphism $\alpha_{\mathrm{Can}\circ T}$ of claim 4.6.25. The essential commutativity of the square subdiagram of the diagram of (i) follows by direct inspection.

Now, $(\underline{\mathrm{Can}}(T), J)$ is isomorphic to an U-site (remark 4.5.2(iii)), hence $(\underline{\mathrm{Can}}(T), J)^\sim_{\mathsf{U}}$ is isomorphic to an U-topos (remark 4.4.1(iv)), whence (ii). Especially, for every pseudo-functor $\mathsf{c} : I \to \mathsf{lex}.\mathbf{Site}$ the category $\mathsf{Top}(\mathsf{T} \circ \mathsf{c})$ is an U-topos, and then the 2-adjunction of theorem 4.4.17 assigns to the isomorphism α_{c} of claim 4.6.25 a morphism of sites :

$$\beta_{\mathsf{c}} : \mathrm{Can}(\mathsf{Top}(\mathsf{T} \circ \mathsf{c})) \to \mathsf{totSite} \circ \underline{\mathscr{F}ib}_I(\mathsf{c})$$

so that we may let $b_T := \beta_{\mathrm{Can}\circ T} \circ \mathrm{Can}(\mathsf{Top}(\eta_T))$. The essential commutativity of the triangular subdiagram of (i) follows from the explicit construction of the unit of this 2-adjunction, provided by the proof of theorem 4.4.17. □

Remark 4.6.26. (i) By unwinding the constructions in the proof of theorem 4.6.24, we see that the equivalence a_T assigns to every object $E := ((E_i, E_\varphi) \mid i \in \mathrm{Ob}(I), \varphi \in \mathrm{Morph}(I))$ of $\mathsf{Top}(T)$ the datum $(h_{E_i}, g_{E_\varphi} \mid i \in \mathrm{Ob}(I), \varphi \in \mathrm{Morph}(I))$, consisting of the sheaf h_{E_i} on T_i represented by E_i for every $i \in \mathrm{Ob}(I)$, and the morphism

$$g_{E_\varphi} : h_{E_j} \xrightarrow{h_{E^\dagger_\varphi}} h_{T_{\varphi *}E_i} \xrightarrow{\sim} (T^*_\varphi)^\sim_* h_{E_i} \qquad \text{for every } (i \xrightarrow{\varphi} j) \in \mathrm{Morph}(I).$$

Here $E^\dagger_\varphi$ is the adjoint to the morphism E_φ, relative to the adjunction for the pair $(T^*_\varphi, T_{\varphi *})$ that defines the morphism of topoi $T_\varphi : T_i \to T_j$. The natural identification $h_{T_{\varphi *}E_i} \xrightarrow{\sim} (T^*_\varphi)^\sim_* h_{E_i}$ is deduced as well from the same adjunction, so the morphism of sheaves g_{E_φ} turns out be given by the rule : $(f : X \to E_j) \mapsto (E_\varphi \circ T^*_\varphi f)$ for every $X \in \mathrm{Ob}(S_j)$ and every $f \in h_{E_j}(X)$.

(ii) On the other hand, the morphism of sites b_T of theorem 4.6.24 is a composition

$$\underline{\mathrm{Can}}(T) \xrightarrow{h^a_{\underline{\mathrm{Can}}(T)}} (\underline{\mathrm{Can}}(T), J)^\sim \xrightarrow{a_T^{-1}} \mathsf{Top}(T)$$

where we have denoted by a_T^{-1} any choice of a quasi-inverse of a_T. To describe b_T more explicitly, consider any $(i, X) \in \mathrm{Ob}(\underline{\mathrm{Can}}(T))$; hence $i \in \mathrm{Ob}(I)$ and $X \in \mathrm{Ob}(T_i)$; the proof of theorem 4.6.24 assigns to (i, X) the system $(F(i, X)_j, F(i, X)_\varphi \mid j \in \mathrm{Ob}(I), \varphi \in \mathrm{Morph}(I))$ where $F(i, X)_j$ is the restriction to $\mathrm{Can}(T_j)$ of the sheaf $h^a_{(i,X)}$ on $(\underline{\mathrm{Can}}(T), J)$. Now, let $G(i, X)_j$ be the restriction of $h_{(i,X)}$ to T_j; for every $Y \in \mathrm{Ob}(T_j)$ we have

$$G(i, X)_j(Y) = \{(\varphi, f) \mid \varphi \in \mathrm{Hom}_I(j, i), f \in \mathrm{Hom}_{T_j}(Y, T^*_\varphi X)\}.$$

For every morphism $h : Y' \to Y$ in T_j, the map $G(i, X)_j(h)$ is given by the rule : $(\varphi, f) \mapsto (\varphi, f \circ h)$. Then set

$$[i, X, j] := \coprod_{\varphi : j \to i} T^*_\varphi X \in \mathrm{Ob}(T_j)$$

and fix a universal cocone $(e_{X,\varphi} : T^*_\varphi X \to [i, X, j] \mid j \xrightarrow{\varphi} i)$. Notice the morphism of presheaves

$$G(i, X)_j \to h_{[i,X,j]} \qquad (\varphi, f : Y \to T^*_\varphi X) \mapsto (e_{X,\varphi} \circ f : Y \to [i, X, j]). \tag{4.6.27}$$

Let us check that (4.6.27) is bicovering. Indeed, (4.6.27) is clearly a monomorphism, hence it suffices to show that it is a covering morphism. Thus, let $f : Y \to [i, X, j]$ be any morphism on T_j and for every $\varphi \in \mathrm{Hom}_I(j, i)$ set $Y_\varphi := Y \times_{[i,X,j]} T^*_\varphi$. The induced cocone

$(e_\varphi : Y_\varphi \to Y \mid j \xrightarrow{\varphi} i)$ is still universal (remark 4.4.1(i)) hence it defines a covering family for Y in the canonical topology of T_j (remark 4.4.1(ii)). But clearly $f \circ e_\varphi$ lies in the image of the map $G(i,X)_j(Y_\varphi) \to h_{[i,X,j]}(Y_\varphi)$ for every $\varphi : j \to i$, whence the contention. Hence, (4.6.27) induces an isomorphism $G(i,X)^a_j \xrightarrow{\sim} h_{[i,X,j]}$, from the sheaf $G(i,X)^a_j$ on $\mathrm{Can}(T_j)$ associated with the presheaf $G(i,X)_j$; the latter is naturally isomorphic to $F(i,X)_j$, by proposition 4.5.5(v). Summing up, we have exhibited a natural isomorphism of sheaves on $\mathrm{Can}(T_j)$:

$$F(i,X)_j \xrightarrow{\sim} h_{[i,X,j]} \qquad \text{for every } j \in \mathrm{Ob}(I).$$

(iii) Next, according to the proof of theorem 4.6.24, for every morphism $\psi : j \to k$ in I, the morphism of sheaves $F(i,X)_\psi : F(i,X)_k \to (T^*_\psi)^\sim_* F(i,X)_j$ assigns to every $Y \in \mathrm{Ob}(T_k)$ the map $h^a_{(i,X)}(\psi, \mathbf{1}_{T^*_\psi Y}) : h^a_{(i,X)}(k,Y) \to h^a_{(i,X)}(j, T^*_\psi Y)$. Then, for every such φ and Y let

$$G(i,X)_{\psi,Y} := h_{(i,X)}(\psi, \mathbf{1}_{T^*_\psi Y}) : G(i,X)_k(Y) \to (T^*_\psi)^\wedge G(i,X)_j(Y).$$

Explicitly, if γ^{T^*} denotes the coherence constraint of the pseudo-functor T^*, we get :

$$G(i,X)_{\psi,Y}(\varphi, f) = (\varphi \circ \psi, \gamma^{T^*}_{(\psi,\varphi),X} \circ T^*_\varphi f) \qquad \text{for every } (\varphi, f) \in G(i,X)_k(Y).$$

The rule : $Y \mapsto G(i,X)_{\psi,Y}$ defines a morphism $G(i,X)_\psi : G(i,X)_k(Y) \to (T^*_\psi)^\wedge G(i,X)_j$ of presheaves, and $G(i,X)^a_\psi = F(i,X)_\psi$ for every morphism ψ of I. Now, denote by $[i,X,\psi] : T^*_\psi[i,X,k] \to [i,X,j]$ the unique morphism in T_j fitting into the commutative diagrams :

$$\begin{array}{ccc} T^*_\psi T^*_\varphi X & \xrightarrow{\gamma^{T^*}_{(\psi,\varphi),X}} & T^*_{\varphi\circ\psi} X \\ {\scriptstyle T^*_\psi(e_{X,\varphi})}\downarrow & & \downarrow{\scriptstyle e_{X,\varphi\circ\psi}} \\ T^*_\psi[i,X,k] & \xrightarrow{[i,X,\psi]} & [i,X,j] \end{array} \qquad \text{for every } \varphi : k \to i.$$

The foregoing discussion easily implies that we have a commutative diagram of sheaves :

$$\begin{array}{ccc} F(i,X)_k & \xrightarrow{\sim} & h_{[i,X,k]} \\ {\scriptstyle F(i,X)_\psi}\downarrow & & \downarrow{\scriptstyle h^\dagger_{[i,X,\psi]}} \\ (T^*_\psi)^\sim_* F(i,X)_j & \xrightarrow{\sim} & (T^*_\psi)^\sim_* h_{[i,X,j]} \end{array}$$

whose horizontal arrows are the isomorphisms constructed in (ii), and the adjoint $h^\dagger_{[i,X,\psi]}$ of $h_{[i,X,\psi]}$ is described explicitly as in (i). Summing up, we find that (up to natural isomorphism) the functor b_T is given by the rule :

$$(i,X) \mapsto ([i,X,j], [i,X,\psi] \mid j \in \mathrm{Ob}(I), \psi \in \mathrm{Morph}(I)) \qquad \text{for every } (i,X) \in \mathrm{Ob}(\underline{\mathrm{Can}}(T)).$$

Corollary 4.6.28. *Let I, J be two small categories, and T, S two fibred topoi over I. We have :*

(i) *For every morphism $\omega : T \to S$ of fibred topoi over I, let $\eta^{\mathsf{Top}(\omega)}$ be the unit of the adjunction for the pair of functors $(\mathsf{Top}(\omega)^*, \mathsf{Top}(\omega)_*)$ provided by remark 4.6.12. Then $(\mathsf{Top}(\omega)^*, \mathsf{Top}(\omega)_*, \eta^{\mathsf{Top}(\omega)})$ is a morphism of topoi :*

$$\mathsf{Top}(\omega) : \mathsf{Top}(T) \to \mathsf{Top}(S).$$

(ii) *For every functor $\rho : J \to I$, let $\eta^{\mathsf{Top}(T/\rho)}$ be the unit of any adjunction for the pair $(\mathsf{Top}(T/\rho)^*, \mathsf{Top}(T/\rho)_*)$. Then $(\mathsf{Top}(T/\rho)^*, \mathsf{Top}(T/\rho)_*, \eta^{\mathsf{Top}(T/\rho)})$ is a morphism of topoi :*

$$\mathsf{Top}(T/\rho) : \mathsf{Top}(T \circ \rho) \to \mathsf{Top}(T).$$

(iii) *For every pair $\omega : T \to S$, $\nu : S \to U$ of morphisms of fibred topoi over I, we have :*

$$\mathsf{Top}(\nu \circ \omega) = \mathsf{Top}(\nu) \circ \mathsf{Top}(\omega).$$

Proof. Assertion (ii) is clear from remark 4.6.22(i).

(i): In view of proposition 4.6.10 and theorem 4.6.24, there remains only to check that $\mathsf{Top}(\omega)^*$ is left exact. Arguing as in remark 4.6.9(i,iii) we may assume that T and S are unital; then a functor $E_\bullet : J \to \mathsf{Top}(S)$ assigns to every $j \in \mathrm{Ob}(J)$ a datum $E_{\bullet,j} := (E_{ij}, E_{\varphi,j} \mid i \in \mathrm{Ob}(I),\ \varphi \in \mathrm{Morph}(I))$ as in remark 4.6.9(i), and to every morphism $f : j \to j'$ a system of morphisms $(E_{i,f} : E_{ij} \to E_{ij'} \mid i \in \mathrm{Ob}(I))$ such that

$$E_{\varphi,j'} \circ S^*_\varphi E_{i',f} = E_{i,f} \circ E_{\varphi,j} \qquad \text{for every morphism } \varphi : i \to i' \text{ of } I.$$

The limit of $E_\bullet$ in the category $\mathsf{Top}(S)$ is the same as the colimit of $E^o_\bullet$ in the category $\Sigma(\mathscr{F}ib(S^*)/I)$, and we construct the latter following remark 4.6.22(iii). Thus, for every $i \in \mathrm{Ob}(I)$ we consider the functor $(E^o_\bullet)^i : J \to \mathscr{F}ib(S^*)$ given by the rules : $j \mapsto E_{ij}$ for every $j \in \mathrm{Ob}(J)$ and $f \mapsto E^o_{i,f}$ for every morphism f of J. Then $(E^o_\bullet)^i$ factors through the inclusion functor $S^o_i \to \mathscr{F}ib(S^*)$, and the colimit of the resulting functor $(\overline{E}{}^o_\bullet)^i : J \to S^o_i$ in S^o_i is the limit of the opposite functor $\overline{E}{}^i_\bullet : J^o \to S_i$, so we pick $L(i) \in \mathrm{Ob}(S_i)$ representing the latter limit and a universal cone $\tau^i : c_{L(i)} \Rightarrow \overline{E}{}^i_\bullet$. Next, every morphism $\varphi : i \to i'$ in I induces a natural transformation

$$\overline{E}{}^\varphi_\bullet : S^*_\varphi \circ \overline{E}{}^{i'}_\bullet \Rightarrow \overline{E}{}^i_\bullet \qquad j \mapsto E_{\varphi,j} : S^*_\varphi E_{i'j} \to E_{ij}.$$

Then there exists a unique morphism $L(\varphi) : S^*_\varphi L(i') \to L(i)$ of S_i fitting into the commutative diagram :

$$\mathscr{D} \quad : \quad \begin{array}{ccc} S^*_\varphi \circ c_{L(i')} & \xRightarrow{\ c_{L(\varphi)}\ } & c_{L(i)} \\ {\scriptstyle S^*_\varphi * \tau^{i'}} \Downarrow & & \Downarrow {\scriptstyle \tau^i} \\ S^*_\varphi \circ \overline{E}{}^{i'}_\bullet & \xRightarrow{\ \overline{E}{}^\varphi_\bullet\ } & \overline{E}{}^i_\bullet \end{array}$$

and remark 4.6.22(iii) shows that $L_\bullet := (L(i), L(\varphi) \mid i \in \mathrm{Ob}(I)\ \varphi \in \mathrm{Morph}(I))$ is an object of $\mathsf{Top}(S)$ representing the limit of $E_\bullet$, and the system of cones $(\tau^i \mid i \in \mathrm{Ob}(I))$ adds up to a universal cone $\tau : c_{L_\bullet} \Rightarrow E_\bullet$. Now, set

$$F_\bullet := \mathsf{Top}(\omega)^* \circ E_\bullet : J \to \mathsf{Top}(T).$$

We define likewise the functors $\overline{F}{}^i_\bullet : J \to T_i$ for every $i \in \mathrm{Ob}(I)$ and the natural transformations $\overline{F}{}^\varphi_\bullet : T^*_\varphi \circ \overline{F}{}^{i'}_\bullet \Rightarrow \overline{F}{}^i_\bullet$ for every morphism $\varphi : i \to i'$ of I; then by a simple inspection we deduce from $\mathscr{D}$ the commutative diagram

$$\mathscr{D}^\omega \quad : \quad \begin{array}{ccc} T^*_\varphi \circ c_{\omega^*_{i'} L(i')} & \xRightarrow{\ c_{L(\varphi)^\omega}\ } & c_{\omega^*_i L(i)} \\ {\scriptstyle T^*_\varphi * \omega^*_{i'} * \tau^{i'}} \Downarrow & & \Downarrow {\scriptstyle \omega^*_i * \tau^i} \\ T^*_\varphi \circ \overline{F}{}^{i'}_\bullet & \xRightarrow{\ \overline{F}{}^\varphi_\bullet\ } & \overline{F}{}^i_\bullet \end{array}$$

with $(\omega^*_i L(i), L(\varphi)^\omega \mid i \in \mathrm{Ob}(I),\ \varphi \in \mathrm{Morph}(I)) := \mathsf{Top}(\omega)^* L_\bullet$ as in remark 4.6.9(ii). Lastly, if J is a finite category, then by assumption $\omega^*_i * \tau^i$ is still a universal cone for every $i \in \mathrm{Ob}(I)$, and therefore the commutativity of $\mathscr{D}^\omega$ determines again $L(\varphi)^\omega$ uniquely. In this case, remark 4.6.22(iii) shows that $\mathsf{Top}(\omega)^* L_\bullet$ represents the limit of $F_\bullet$ and $\mathsf{Top}(\omega)^* * \tau$ is a universal cone.

(iii) follows easily from the explicit description of $\eta^{\mathsf{Top}(\omega)}, \eta^{\mathsf{Top}(\nu)}$ and $\eta^{\mathsf{Top}(\nu\circ\omega)}$ given by remark 4.6.12. □

Remark 4.6.29. Let I be any small category.

(i) From corollary 4.6.28(i,iii) it follows easily that the rules : $T \mapsto \mathsf{Top}(T)$, $\omega \mapsto \mathsf{Top}(\omega)$ and $\Xi \mapsto \mathsf{Top}(\Xi)_*$ for every fibred topos T over I, every morphism ω of such fibred topoi, and every transformation Ξ of such morphisms, define a strict pseudo-functor

$$\mathsf{Top} : \mathsf{PsFun}(I, \mathbf{Topos}) \to \mathbf{Topos}.$$

(ii) It is also easily seen that the isomorphisms of claim 4.6.25 yield a pseudo-natural isomorphism of pseudo-functors :

$$\alpha_\bullet : \mathsf{Top} \circ \mathsf{PsFun}(I, \mathsf{lex.T}) \stackrel{\sim}{\to} \mathsf{T} \circ \mathsf{totSite} \circ \mathsf{lex.}\underline{\mathscr{F}ib}_I.$$

Moreover, let $\underline{\mathsf{c}}^\bullet$: $\mathsf{fib.lex.}\mathbf{Site}(I) \to \mathsf{PsFun}(I, \mathsf{lex.}\mathbf{Site})$ be a strict and strong pseudo-inverse for the 2-equivalence $\mathsf{lex.}\underline{\mathscr{F}ib}_I$ (see remark 4.5.2(v)); then from $\alpha_\bullet * \underline{\mathsf{c}}^\bullet$ we deduce a pseudo-natural isomorphism of pseudo-functors (notation of (4.6.2)) :

$$\mathsf{a}_\bullet : \mathsf{Top} \circ \underline{\mathsf{lex.T}} \stackrel{\sim}{\to} \mathsf{T} \circ \mathsf{totSite}.$$

By 2-adjunction, from $\alpha_\bullet$ and $\mathsf{a}_\bullet$ we then deduce as well pseudo-natural transformations

$$\beta_\bullet : \mathrm{Can} \circ \mathsf{Top} \circ \mathsf{PsFun}(I, \mathsf{lex.T}) \to \mathsf{totSite} \circ \mathsf{lex.}\underline{\mathscr{F}ib}_I \qquad \mathsf{c} \mapsto \beta_{\mathsf{c}}$$
$$\mathsf{b}_\bullet : \mathrm{Can} \circ \mathsf{Top} \circ \underline{\mathsf{lex.T}} \to \mathsf{totSite}$$

where β_{c} is defined as in the proof of theorem 4.6.24 : the details are left to the reader.

(iii) Lastly, the rule $T \mapsto a_T$ of theorem 4.6.24(i) defines the pseudo-natural equivalence :

$$a_\bullet := \mathsf{PsFun}(I, \eta) \odot (\mathsf{a}_\bullet * \underline{\mathrm{Can}}) : \mathsf{Top} \stackrel{\sim}{\to} \mathsf{T} \circ \mathsf{totSite} \circ \underline{\mathrm{Can}}$$

where $\eta : \mathbf{1_{Topos}} \to \mathsf{T} \circ \mathrm{Can}$ is the unit of the 2-adjoint pair $(\mathrm{Can}, \mathsf{T})$. Again by 2-adjunction, the rule : $T \mapsto b_T$ yields likewise a pseudo-natural transformation

$$b_\bullet : \mathrm{Can} \circ \mathsf{Top} \to \mathsf{totSite} \circ \underline{\mathrm{Can}}.$$

4.7. **Localization and points of a topos.** Let $\mathscr{C}$ be a category, and $f : Y \to X$ any morphism of $\mathscr{C}$. We consider the source functor $\mathsf{s}_X : \mathscr{C}/X \to \mathscr{C}$ as in (1.1.24), and the induced functor

$$(\mathsf{s}_X)_{|f} : (\mathscr{C}/X)/f \to \mathscr{C}/(\mathsf{s}_X f) = \mathscr{C}/Y$$

as in (1.1.26), which is obviously an isomorphism of categories, hence it induces a bijection :

$$\{\text{sieves of } \mathscr{C}/Y\} \stackrel{\sim}{\to} \{\text{sieves of } (\mathscr{C}/X)/f\} \qquad \mathscr{S} \mapsto (\mathsf{s}_X)^{-1}_{|f}\mathscr{S}$$

(notation of definition 3.5.1(iii)). Fix a universe V with $\mathsf{U} \subset \mathsf{V}$ and such that $\mathscr{C}$ is V-small, so that the functor $(\mathsf{s}_X)_{\mathsf{V}!} : (\mathscr{C}/X)^\wedge_{\mathsf{V}} \to \mathscr{C}^\wedge_{\mathsf{V}}$ is well defined. We notice that for every subobject R of the presheaf h_f on $\mathscr{C}/X$, the presheaf $(\mathsf{s}_X)_{\mathsf{V}!}R$ is a subobject of h_Y : more precisely, in light of (1.4.11) we see that for a sieve $\mathscr{S}$ of $(\mathscr{C}/X)/f$ and a sieve $\mathscr{T}$ of $\mathscr{C}/Y$, we have :

$$h_{\mathscr{T}} = (\mathsf{s}_X)_{\mathsf{V}!}h_{\mathscr{S}} \quad \Leftrightarrow \quad \mathscr{S} = (\mathsf{s}_X)^{-1}_{|f}\mathscr{T}. \tag{4.7.1}$$

Let now J be a topology on $\mathscr{C}$; endow $\mathscr{C}/X$ with the topology J_X induced by J via s_X, and set

$$C := (\mathscr{C}, J) \qquad C/X := (\mathscr{C}/X, J_X).$$

Since $(\mathsf{s}_X)_{\mathsf{V}!}$ commutes with fibre products (proposition 1.4.13(vi.c)), lemma 4.2.17(ii.a) says that a subobject $R \subset h_f$ covers f for the topology J_X if and only if the induced morphism $(\mathsf{s}_X)_{\mathsf{V}!}R \to (\mathsf{s}_X)_{\mathsf{V}!}(h_f) = h_Y$ covers Y for the topology J. In other words, a family of morphisms $(h_\lambda/X : (f_\lambda : Y_\lambda \to Y) \to f \mid \lambda \in \Lambda)$ covers f for the topology J_X if and only if the family $(h_\lambda : Y_\lambda \to Y \mid \lambda \in \Lambda)$ covers Y for the topology J. In view of (4.7.1), it follows easily that s_X is both continuous and cocontinuous for the topologies J and J_X.

4.7.2. Suppose now that C is a U-site; then the same holds for C/X as well : indeed, if $G \subset \mathrm{Ob}(\mathscr{C})$ is an essentially small topologically generating family for C, then $G/X \subset \mathrm{Ob}(\mathscr{C}/X)$ is an essentially small topologically generating family for C/X (notation of definition 4.1.20(i)). By virtue of corollary 4.3.19, the functor $\widetilde{\mathsf{s}}_{X*} \simeq \check{\mathsf{s}}^*_X : C^\sim \to (C/X)^\sim$ admits therefore both a left adjoint $\widetilde{\mathsf{s}}^*_X$ and a right adjoint $\check{\mathsf{s}}_{X*}$. We introduce a special notation and terminology for these functors :

- The functor $\widetilde{\mathsf{s}}_{X*}$ shall be also denoted j^*_X, and called the functor of *restriction to* X.
- The functor $\check{\mathsf{s}}_{X*}$ shall be denoted j_{X*}, and called the *direct image* functor.

- The functor $\widetilde{\mathsf{s}}_X^*$ shall be denoted $j_{X!}$, and called the functor of *extension by empty*.

Thus, j_X^* is right adjoint to $j_{X!}$, and left adjoint to j_{X*}. However, a left adjoint for j_X^* can be exhibited alternatively by a more explicit construction, which will allow us to extract some useful additional properties of the category $(C/X)^\sim$. Indeed, even though $\mathscr{C}/X$ is not necessarily U-small, we can invoke proposition 1.4.13(vi.a) and remark 4.1.23(ii), in order to define a functor by the same rule as in (4.2.8), namely

$$j_{X!} : (C/X)^\sim_{\mathsf{U}} \to C^\sim_{\mathsf{U}} \qquad F \mapsto (\mathsf{s}_{X!} \circ i_{C/X}F)^a$$

where $i_{C/X} : (C/X)^\sim \to (\mathscr{C}/X)^\wedge$ is the forgetful functor. Moreover, remark 4.1.19(v) (and again remark 4.1.23(ii)) implies that this functor is (isomorphic to) the restriction of $(\widetilde{\mathsf{s}}_X)^*_{\mathsf{V}}$, and since the inclusion functor $C^\sim_{\mathsf{U}} \to C^\sim_{\mathsf{V}}$ is fully faithful, we deduce that it is also a left adjoint to j_X^*. Furthermore, recall that $\mathsf{s}_{X!}$ factors through the source functor $\mathsf{s}_{h_X} : \mathscr{C}^\wedge/h_X \to \mathscr{C}^\wedge$ and an equivalence $e_X : (\mathscr{C}/X)^\wedge \xrightarrow{\sim} \mathscr{C}^\wedge/h_X$ (proposition 1.4.13(vi.b)); consequently, $j_{X!}$ factors through the source functor $\mathsf{s}_{h_X^a} : C^\sim/h_X^a \to C^\sim$ and the composition

$$\widetilde{e}_X : (C/X)^\sim \xrightarrow{i_{C/X}} (\mathscr{C}/X)^\wedge \xrightarrow{e_X} \mathscr{C}^\wedge/h_X \xrightarrow{(-)^a_{|h_X}} C^\sim/h_X^a$$

where $(-)^a_{|h_X}$ is induced by $(-)^a : \mathscr{C}^\wedge \to C^\sim$ and the object $h_X \in \mathrm{Ob}(\mathscr{C}^\wedge)$, as in (1.1.26).

Remark 4.7.3. (i) Let $g : Y \to Z$ be any morphism in $\mathscr{C}$; then we have the sites C/Y and C/Z as in (4.7), as well as the functor $g_* : \mathscr{C}/Y \to \mathscr{C}/Z$ of (1.1.25). The isomorphism of categories $(\mathsf{s}_Z)_{|g} : (\mathscr{C}/Z)/g \xrightarrow{\sim} \mathscr{C}/Y$ identifies g_* with the source functor $\mathsf{s}_g : (\mathscr{C}/Z)/g \to \mathscr{C}/Z$. Hence, the discussion of (4.7.2) applies to g_* as well, and since $\mathsf{s}_Z \circ g_* = \mathsf{s}_Y$, we see that J_Y is the topology induced by J_Z via g_* on $\mathscr{C}/Y$; so, g_* is continuous and cocontinuous for the topologies J_Y and J_Z. Moreover, if C is a U-site, then $(\widetilde{g}_*)_* : (C/Z)^\sim \to (C/Y)^\sim$ admits a left adjoint $(\widetilde{g}_*)^*$ and a right adjoint $(\breve{g}_*)_*$. In agreement with the foregoing, we let

$$j_g^* := (\widetilde{g}_*)_* \qquad j_{g*} := (\breve{g}_*)_* \qquad j_{g!} := (\widetilde{g}_*)^*.$$

Clearly $j_g^* \circ j_Z^* = j_Y^*$, and we have isomorphisms of functors : $j_{Z*} \circ j_{g*} \xrightarrow{\sim} j_{Y*}$ and $j_{Z!} \circ j_{g!} \xrightarrow{\sim} j_{Y!}$.

(ii) By propositions 4.4.8(i) and 1.3.25(iii,iv), we also see that the pairs (j_X^*, j_{X*}) and (j_g^*, j_{g*}) determine morphisms of topoi, unique up to unique isomorphism :

$$j_X : (C/X)^\sim \to C^\sim \qquad j_g : (C/Y)^\sim \to (C/Z)^\sim.$$

(iii) Suppose that all finite products of $\mathscr{C}$ are representable. Then for every $X \in \mathrm{Ob}(\mathscr{C})$, the source functor s_X admits a right adjoint

$$p_X : \mathscr{C} \to \mathscr{C}/X \qquad Y \mapsto (p_{X,Y} : X \times Y \to X)$$

where $X \times Y$ is any choice of a representative for the product of X and Y, and $p_{X,Y}$ is the corresponding natural projection : see proposition 1.4.13(iii). Since s_X is cocontinuous for the topologies J and J_X, the functor p_X is a morphism of sites $C/X \to C$ (remark 4.4.13(v)), and we have an isomorphism of morphisms of topoi :

$$j_X \xrightarrow{\sim} \widetilde{p}_X.$$

Proposition 4.7.4. *With the notation of* (4.7.2)*, the following holds :*

(i) *The functor $\widetilde{e}_X$ is an equivalence.*

(ii) *We have essentially commutative diagrams :*

$$\begin{array}{ccc} (C/X)^\sim & \xrightarrow{i_{C/X}} & (\mathscr{C}/X)^\wedge \\ {\scriptstyle \widetilde{e}_X}\downarrow & & \downarrow{\scriptstyle e_X} \\ C^\sim/h_X^a & \xrightarrow{(i_C)_{|h_X}} & \mathscr{C}^\wedge/h_X \end{array} \qquad \begin{array}{ccc} (\mathscr{C}/X)^\wedge & \xrightarrow{(-)^a} & (C/X)^\sim \\ {\scriptstyle e_X}\downarrow & & \downarrow{\scriptstyle \widetilde{e}_X} \\ \mathscr{C}^\wedge/h_X & \xrightarrow{(-)^a_{|h_X}} & C^\sim/h_X^a \end{array}$$

where $(i_C)_{|h_X}$ *and* $(-)^a_{|h_X}$ *are the functors attached to the adjoint pair* $((-)^a, i_C)$ *and the object* $h_X \in \mathrm{Ob}(\mathscr{C}^\wedge)$*, as in example* 1.2.28(i).

Proof. Let $\eta_F : h_X \to h_X^a$ be the natural morphism in $\mathscr{C}^\wedge$. We remark :

Claim 4.7.5. For every presheaf F on $\mathscr{C}/X$ there exists a cartesian diagram in $\mathscr{C}^\wedge$:

$$\begin{array}{ccc} \mathsf{s}_{X!}(F^a) & \longrightarrow & (\mathsf{s}_{X!}F)^a \\ {\scriptstyle e_X(F^a)}\downarrow & & \downarrow{\scriptstyle e_X(F)^a} \\ h_X & \xrightarrow{\eta_X} & h_X^a. \end{array}$$

Proof of the claim. As already pointed out in the proof of proposition 1.4.13(vi.b), the presheaf $\mathbb{1}_{\mathscr{C}/X} := h_{\mathbf{1}_X}$ is a final object of $(\mathscr{C}/X)^\wedge$, and the equivalence e_X assigns to every presheaf F on $\mathscr{C}/X$ the h_X-presheaf $e_X(F) : \mathsf{s}_{X!}F \to \mathsf{s}_{X!}(\mathbb{1}_{\mathscr{C}/X}) \xrightarrow{\sim} h_X$ deduced from the unique morphism $F \to \mathbb{1}_{\mathscr{C}/X}$. Moreover, e_X admits the quasi-inverse

$$e'_X : \mathscr{C}^\wedge/h_X \to (\mathscr{C}/X)^\wedge \qquad (\varphi : G \to h_X) \mapsto \mathsf{s}_X^\wedge(G) \times_{\mathsf{s}_X^\wedge(h_X)} \mathbb{1}_{\mathscr{C}/X}$$

where the fibre product is defined via the unit of adjunction $\eta_{\mathbb{1}} : \mathbb{1}_{\mathscr{C}/X} \to \mathsf{s}_X^\wedge(\mathsf{s}_{X!}\mathbb{1}_{\mathscr{C}/X}) \xrightarrow{\sim} \mathsf{s}_X^\wedge(h_X)$ and the morphism $\mathsf{s}_X^\wedge(\varphi)$. Thus, we have a natural isomorphism

$$F \xrightarrow{\sim} \mathsf{s}_X^\wedge(\mathsf{s}_{X!}F) \times_{\mathsf{s}_X^\wedge(h_X)} \mathbb{1}_{\mathscr{C}/X} \qquad \text{in } (\mathscr{C}/X)^\wedge.$$

On the other hand, the functor $\mathsf{s}_X^\wedge$ commutes with the functor $G \mapsto G^a$ (corollary 4.3.19(iii)), so there follows an isomorphism

$$F^a \xrightarrow{\sim} \mathsf{s}_X^\wedge(\mathsf{s}_{X!}F)^a \times_{\mathsf{s}_X^\wedge(h_X^a)} \mathbb{1}_{\mathscr{C}/X} \qquad \text{in } (C/X)^\sim.$$

But it is easily seen that the morphism $\eta_{\mathbb{1}}^a : \mathbb{1}_{\mathscr{C}/X} \to \mathsf{s}_X^\wedge(h_X^a)$ is the composition of $\eta_{\mathbb{1}}$ and the natural morphism $\mathsf{s}_X^\wedge(h_X) \to \mathsf{s}_X^\wedge(h_X^a)$, so we get a commutative diagram with cartesian squares:

$$\begin{array}{ccccc} F^a & \xrightarrow{\alpha} & \mathsf{s}_X^\wedge((\mathsf{s}_{X!}F)^a \times_{h_X^a} h_X) & \longrightarrow & \mathsf{s}_X^\wedge(\mathsf{s}_{X!}F)^a \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{1}_{\mathscr{C}/X} & \longrightarrow & \mathsf{s}_X^\wedge(h_X) & \longrightarrow & \mathsf{s}_X^\wedge(h_X^a). \end{array}$$

Especially, α induces an isomorphism $F^a \xrightarrow{\sim} e'_X((\mathsf{s}_{X!}F)^a \times_{h_X^a} h_X)$, whence an isomorphism

$$e_X(F^a) \xrightarrow{\sim} ((\mathsf{s}_{X!}F)^a \times_{h_X^a} h_X \to h_X)$$

and the claim follows. ◊

(i): Let $\varphi : G \to h_X^a$ be any object of $C^\sim/h_X^a$; set $G' := G \times_{h_X^a} h_X$, and let $\varphi' : G' \to h_X$ be the induced projection. Let also $F := e'_X(\varphi')$, which is a presheaf on $\mathscr{C}/X$. We claim that F is a sheaf for the topology J_X. Indeed, by claim 4.7.5 we have an isomorphism in $\mathscr{C}^\wedge/h_X$:

$$\mathsf{s}_{X!}(F^a) \xrightarrow{\sim} (\mathsf{s}_{X!}F)^a \times_{h_X^a} h_X \xrightarrow{\sim} G'^a \times_{h_X^a} h_X \xrightarrow{\sim} G'$$

and since e'_X is quasi-inverse to e_X, we have an isomorphism of h_X-presheaves $\mathsf{s}_{X!}(F) \xrightarrow{\sim} G'$. Thus, $F \xrightarrow{\sim} F^a$ in $(\mathscr{C}/X)^\wedge$, as required. Next we claim that the resulting functor

$$\widetilde{e}'_X : C^\sim/h_X^a \to (C/X)^\sim \qquad (\varphi : G \xrightarrow{\varphi} h_X^a) \mapsto e'_X(G \times_{h_X^a} h_X \xrightarrow{\varphi \times_{h_X^a} h_X} h_X)$$

is a quasi-inverse for $\widetilde{e}_X$. Indeed, for every φ as in the foregoing we have an isomorphism

$$\widetilde{e}_X \circ \widetilde{e}'_X(\varphi) \xrightarrow{\sim} (\varphi \times_{h_X^a} h_X)^a \xrightarrow{\sim} \varphi \qquad \text{in } C^\sim/h_X^a$$

which is natural with respect to φ. Lastly, for every sheaf F on C/X we have $\widetilde{e}'_X \circ \widetilde{e}_X(F) = e'_X((e_X(F)^a \times_{h_X^a} h_X)$, which is isomorphic to $e'_X(e_X(F)) \simeq F$, by claim 4.7.5.

(ii): For every sheaf F on C/X, claim 4.7.5 yields an isomorphism :

$$e_X(i_{C/X}F) \stackrel{\sim}{\to} (\widetilde{e}_X(F) \times_{h^a_X} h_X \to h_X).$$

The essential commutativity of the left square diagram in (ii) is an immediate consequence; since e_X and $\widetilde{e}_X$ are both equivalences, and since $(-)^a_{|h_X}$ is left adjoint to $(i_C)_{|h_X}$, we deduce the essential commutativity also for the right square diagram. □

Remark 4.7.6. (i) Under the equivalence $\widetilde{e}_X$ of proposition 4.7.4, the functor j^*_X is identified with the functor :

$$C^\sim \to C^\sim/h^a_X \qquad F \mapsto (F \times h^a_X \to h^a_X)$$

and $j_{X!}$ is identified with the source functor $\mathsf{s}_{h^a_X} : C^\sim/h^a_X \to C^\sim$ of (1.1.24).

(ii) Likewise, the functor j^*_g of remark 4.7.3(i) is identified with the functor

$$C^\sim/h^a_Z \to C^\sim/h^a_Y \qquad (F \to h^a_Z) \mapsto (F \times_{h^a_Z} h^a_Y \mapsto h^a_Y)$$

and $j_{g!}$ is identified with the functor $(h^a_g)_* : C^\sim/h^a_Y \to C^\sim/h^a_Z$ induced by $h^a_g : h^a_Y \to h^a_Z$.

Proposition 4.7.7. *Let $C := (\mathscr{C}, J)$ and $C' := (\mathscr{C}'; J')$ be two sites, $u : \mathscr{C} \to \mathscr{C}'$ a continuous functor, X any object of $\mathscr{C}$. Then we have :*

(i) *The functor $u_{|X} : \mathscr{C}/X \to \mathscr{C}'/uX$ is continuous for the sites C/X and C'/uX.*

(ii) *We have essentially commutative diagrams :*

$$\begin{array}{ccc} (C'/uX)^\sim & \xrightarrow{\widetilde{u_{|X*}}} & (C/X)^\sim \\ {\scriptstyle \widetilde{e}_{uX}}\downarrow & & \downarrow{\scriptstyle \widetilde{e}_X} \\ C'^\sim/h^a_{uX} & \xrightarrow{(\widetilde{u}_*)_{|h^a_X}} & C^\sim/h^a_X \end{array} \qquad \begin{array}{ccc} (C/X)^\sim & \xrightarrow{\widetilde{u_{|X}}^*} & (C'/uX)^\sim \\ {\scriptstyle \widetilde{e}_X}\downarrow & & \downarrow{\scriptstyle \widetilde{e}_{uX}} \\ C^\sim/h^a_X & \xrightarrow{(\widetilde{u}^*)_{|h^a_X}} & C'^\sim/h^a_{uX}. \end{array}$$

(iii) *If u is a morphism of sites $C' \to C$, then $u_{|X}$ is a morphism of sites $C'/uX \to C/X$.*

Proof. (i): Combining propositions 4.7.4(ii) and 1.4.19(i) with example 1.2.28(iii) we get an essentially commutative diagram, whose vertical arrows are equivalences :

$$\begin{array}{ccc} (C'/uX)^\sim & \xrightarrow{(u_{|X})^\wedge \circ i_{C'/X}} & (\mathscr{C}/X)^\wedge \\ {\scriptstyle \widetilde{e}_{uX}}\downarrow & & \downarrow{\scriptstyle e_X} \\ C'^\sim/h^a_{uX} & \xrightarrow{(u^\wedge \circ i_{C'})_{|h_X}} & \mathscr{C}^\wedge/h_X. \end{array}$$

Now, let F be a sheaf on C'/uX, and denote by $\varphi : G \to h^a_{uX}$ the h^a_{uX}-sheaf $\widetilde{e}_{uX}(F)$. By proposition 4.7.4(ii), it suffices to show that $(u^\wedge \circ i_{C'})_{|h_X}(\varphi) \simeq (u^\wedge G \times_{u^\wedge h^a_{uX}} h_X \to h_X)$ lies in the essential image of $(i_C)_{|h_X}$. But the morphism $h_X \to u^\wedge h^a_{uX}$ is the composition of the natural morphisms $h_X \to h^a_X$ and $h^a_X \to u^\wedge h^a_{uX}$, hence $u^\wedge G \times_{u^\wedge h^a_{uX}} h_X \stackrel{\sim}{\to} G' \times_{h^a_X} h_X$, with $G' := u^\wedge G \times_{u^\wedge h^a_{uX}} h^a_X$, and the latter is a sheaf on C, since u is continuous, whence the contention.

(ii): Recall that we have a natural isomorphism $\widetilde{u}^*(h^a_X) \stackrel{\sim}{\to} h^a_{uX}$ in $C'^\sim$ (lemma 4.2.11(ii)). By example 1.2.28(i), we deduce an adjoint pair of functors :

$$(\widetilde{u}^*)_{|h^a_X} : C^\sim/h^a_X \to C'^\sim/h^a_{uX} \qquad (\widetilde{u}_*)_{|h^a_X} : C'^\sim/h^a_{uX} \to C^\sim/h^a_X.$$

Now, the proof of (i) yields a natural isomorphism of functors :

$$\begin{aligned} \widetilde{e}_X \circ (u_{|X})^\sim_* &\stackrel{\sim}{\to} (-)^a_{|h_X} \circ e_X \circ i_{C/X} \circ (u_{|X})^\sim_* \\ &\stackrel{\sim}{\to} (-)^a_{|h_X} \circ e_X \circ (u_{|X})^\wedge \circ i_{C'/uX} \\ &\stackrel{\sim}{\to} (-)^a_{|h_X} \circ (u^\wedge \circ i_{C'})_{|h_X} \circ \widetilde{e}_{uX} \end{aligned}$$

and a direct inspection of the constructions shows that $(-)^a_{|h_X} \circ (u^\wedge \circ i_{C'})_{|h_X}$ is isomorphic to the functor $(\widetilde{u}_*)_{|h^a_X}$, so that the left square diagram of (ii) is essentially commutative. Since $\widetilde{e}_X$ and $\widetilde{e}_{uX}$ are equivalences, we deduce that the same holds also for the right square diagram.

(iii): By assumption, the functor $\widetilde{u}^*$ is left exact, so the same holds for $(\widetilde{u}^*)_{|h^a_X}$ (proposition 1.4.19(ii)), and by (ii) the same follows for $\widetilde{u_{|X}}^*$, whence the contention. □

Example 4.7.8. (i) Let T be a topos, and $U \in \mathrm{Ob}(T)$ any object. By proposition 4.7.4(i) and remark 4.7.3(iii), we have a natural equivalence

$$(\mathrm{Can}(T)/U)^\sim \xrightarrow{\sim} \mathrm{Can}(T)^\sim/h_U$$

identifying the functor $j^*_U : \mathrm{Can}(T)^\sim \to (\mathrm{Can}(T)/U)^\sim$ with $p_{h_U} : \mathrm{Can}(T)^\sim \to \mathrm{Can}(T)^\sim/h_U$. The source functor $\mathsf{s}_{h_U} : \mathrm{Can}(T)^\sim/h_U \to \mathrm{Can}(T)^\sim$ is a *left* adjoint for p_{h_U}, but the latter admits also a *right* adjoint, which is naturally identified with $j_{U*} : (\mathrm{Can}(T)/U)^\sim \to \mathrm{Can}(T)^\sim$. On the other hand, the Yoneda equivalence $T \xrightarrow{\sim} \mathrm{Can}(T)^\sim$ induces an equivalence $T/U \xrightarrow{\sim} \mathrm{Can}(T)^\sim/h_U$, which shows that T/U is a topos, and identifies p_{h_U} in turn with the functor

$$p_U : T \to T/U \qquad X \mapsto X_{|U} := (X \times U \to U).$$

Hence, the latter is a morphisms of U-sites $\mathrm{Can}(T) \to \mathrm{Can}(T/U)$, and we denote again by

$$j_U := (j^*_U, j_{U*}, \eta^U) : T/U \to T$$

the corresponding morphism of topoi. Moreover, the source functor s_U is a left adjoint for j^*_U, and shall be denoted also by $j_{U!} : T/U \to T$. Likewise, any morphism $f : U' \to U$ in T determines, up to unique isomorphism, a morphism of topoi

$$j_f := (j^*_f, j_{f*}, \eta^f) : T/U' \to T/U$$

with an isomorphism $j_U \circ j_f \xrightarrow{\sim} j_{U'}$ of morphisms of topoi.

(ii) Recall that the target functor $\mathsf{t} : \mathsf{Morph}(T) \to T$ is a fibration (example 3.1.3(iii)); pick a cleavage for t, and let $\mathsf{c} : T^o \to \mathsf{V}\text{-}\mathbf{Cat}$ be its associated pseudo-functor (for a universe V such that T is V-small). We may choose the cleavage so that for every morphism $f : U' \to U$, the functor $\mathsf{c}_f : T/U \to T/U'$ agrees with j^*_f (the details are left to the reader). Notice then that $\mathsf{c}^o : T \to \mathsf{V}\text{-}\mathbf{Cat}^o$ factors through the forgetful strict pseudo-functor

$$\mathsf{Link}(\mathsf{V}\text{-}\mathbf{Cat}^o) \to \mathsf{V}\text{-}\mathbf{Cat}^o \qquad \mathscr{A} \mapsto \mathscr{A} \qquad (F, G, \eta, \varepsilon) \mapsto F$$

and a pseudo-functor

$$T \to \mathsf{Link}(\mathsf{V}\text{-}\mathbf{Cat}^o) \qquad U \mapsto T/U \qquad (U' \xrightarrow{f} U) \mapsto (j^*_f, j_{f*}, \eta^f, \varepsilon^f)$$

where ε^f is the counit for the adjunction of the pair (j^*_f, j_{f*}), whose unit is η^f. We compose the latter with the strict isomorphism of 2-categories $\mathsf{Link}(\mathsf{V}\text{-}\mathbf{Cat}^o) \xrightarrow{\sim} {}^o\mathsf{Link}(\mathsf{V}\text{-}\mathbf{Cat})$ provided by proposition 2.3.4, and notice that the resulting pseudo-functor maps T to the strong sub-2-category ${}^o\mathbf{Topos}$ of ${}^o\mathsf{Link}(\mathsf{V}\text{-}\mathbf{Cat})$. Thus, we obtain a well defined fibred topos over T :

$$T/- : T = {}^oT \to \mathbf{Topos} \qquad U \mapsto T/U \qquad (U' \xrightarrow{f} U) \mapsto j_f.$$

(iii) In case U is a subobject of the final object 1_T of T, the morphism j_U of (i) is called an *open subtopos* of T. In this case we denote by $\mathsf{C}U$ the full subcategory of T such that

$$\mathrm{Ob}(\mathsf{C}U) = \{X \in \mathrm{Ob}(T) \mid j^*_U X = 1_{T/U}\}.$$

Then $\mathsf{C}U$ is a topos, called the *complement of* U *in* T, and the inclusion functor $i_* : \mathsf{C}U \to T$ admits a left adjoint $i^* : T \to \mathsf{C}U$, namely, the functor which assigns to every $X \in \mathrm{Ob}(T)$ the push-out $X_{|\complement U}$ in the cocartesian diagram :

$$\begin{array}{ccc} X \times U & \xrightarrow{p_X} & X \\ {\scriptstyle p_U}\downarrow & & \downarrow \\ U & \longrightarrow & X_{|\complement U} \end{array}$$

where p_X and p_U are the natural projections. Moreover, i^* is an exact functor, hence the adjoint pair (i^*, i_*) defines a morphism of topoi $\complement U \to T$, unique up to unique isomorphism. (See [8, Exp.IV, Prop.9.3.4].)

Remark 4.7.9. (i) Let $f : T' \to T$ be a morphism of topoi, $U \in \mathrm{Ob}(T)$, and let us fix final objects 1_T of T and $1_{T'}$ of T'. For every $Y \in \mathrm{Ob}(T')$, let also $u_Y : Y \to 1_{T'}$ be the unique morphism in T'. Suppose that $s : T' \to T/U$ is a morphism of topoi with an isomorphism $\omega : f \xrightarrow{\sim} j_U \circ s$. Hence, we have a functorial isomorphism (notation of remark 1.1.17(ii))

$$\omega^\dagger_X : s^*(j^*_U X) \xrightarrow{\sim} f^*X \qquad \text{for every } X \in \mathrm{Ob}(T).$$

So, $s^*\mathbf{1}_U \xrightarrow{\sim} f^*1_T$ is a final object of T', hence $u_{s^*\mathbf{1}_U} : s^*\mathbf{1}_U \to 1_{T'}$ is an isomorphism. Let $\Delta_U : \mathbf{1}_U \to (U \times U \xrightarrow{j^*_U(U)} U)$ be the diagonal morphism; then

$$\sigma(s, \omega) := \omega^\dagger_U \circ s^*(\Delta_U) \circ u^{-1}_{s^*\mathbf{1}_U} : 1_{T'} \to f^*U$$

is an element of $\Gamma(T', f^*U)$ (notation (4.4.11)). Moreover, for every object $\varphi : X \to U$ of T/U, we have a cartesian diagram in T/U :

$$\mathscr{D} \quad : \quad \begin{array}{ccccc} X & & \xrightarrow{\Gamma_\varphi} & & X \times U \\ & \searrow^{\varphi} & & \swarrow^{j^*_U X} & \\ {\scriptstyle \varphi/U}\downarrow & & U & & \downarrow{\scriptstyle j^*_U \varphi} \\ & \nearrow_{\mathbf{1}_U} & & \nwarrow_{j^*_U U} & \\ U & & \xrightarrow[\Delta_U]{} & & U \times U \end{array}$$

where Γ_φ is the graph of φ, and $s^*\mathscr{D}$ is isomorphic to the cartesian diagram (in T') :

$$\mathscr{E} \quad : \quad \begin{array}{ccc} s^*\varphi & \xrightarrow{\omega^\dagger_X \circ s^*\Gamma_\varphi} & f^*X \\ {\scriptstyle u_{s^*\mathbf{1}_U} \circ s^*(\varphi/U)}\downarrow & & \downarrow{\scriptstyle f^*\varphi} \\ 1_{T'} & \xrightarrow{\sigma(s,\omega)} & f^*U. \end{array}$$

This shows that s^* – and therefore also s – is determined, up to isomorphism, by $\sigma(s, \omega)$.

(ii) Conversely, if $\sigma \in \Gamma(T', f^*U)$ is any global section, we may define a functor $s^* : T/U \to T'$ by means of the cartesian diagram $\mathscr{E}$: this amounts to choosing a representative in T' for the fibre product $1_{T'} \times_{(\sigma, f^*\varphi)} f^*X$, for every object $\varphi : X \to U$ of T/U. We claim that there exists an isomorphism of functors

$$s^* \circ j^*_U \xrightarrow{\sim} f^*.$$

Indeed, recall that $j^*_U X = (q_X : X \times U \to U)$, where $X \times U$ is a representative of the product of X and U in T, and q_X is the natural projection; let also $p_X : X \times U \to X$ be the other projection. Likewise, pick a representative $f^*X \times f^*U$ for the fibre product of f^*X and f^*U in T', and let $p_{f^*X} : f^*X \times f^*U \to f^*X$ and $q_{f^*X} : f^*X \times f^*U \to f^*U$ be the projections.

Since f^* is left exact, there exists a unique isomorphism $\lambda_X : f^*X \times f^*U \xrightarrow{\sim} f^*(X \times U)$ in T' such that $f^*(p_X) \circ \lambda_X = p_{f^*X}$ and $f^*(q_X) \circ \lambda_X = q_{f^*X}$. We deduce a commutative diagram :

$$\begin{array}{ccccc} f^*X & \xrightarrow{\mathbf{1}_{f^*X} \times (\sigma \circ u_{f^*X})} & f^*X \times f^*U & \xrightarrow{\lambda} & f^*(X \times U) \\ {\scriptstyle u_{f^*X}} \downarrow & & \downarrow {\scriptstyle q_{f^*X}} & \swarrow {\scriptstyle f^*j_U^*X} & \\ 1_{T'} & \xrightarrow{\sigma} & f^*U & & \end{array}$$

whose square subdiagram is cartesian (the details are left to the reader). The assertion follows straightforwardly. Let us also define a functor $t : T' \to T'/f^*U$, by the rule :

$$tY := \sigma \circ u_Y \qquad \text{for every } Y \in \mathrm{Ob}(T').$$

By inspecting the diagram $\mathscr{E}$, we deduce natural bijections :

$$\mathrm{Hom}_{T'}(Y, s^*\varphi) \xrightarrow{\sim} \mathrm{Hom}_{T'/f^*U}(tY, f^*\varphi) \qquad \text{for every } Y \in \mathrm{Ob}(T') \text{ and } \varphi \in \mathrm{Ob}(T/U).$$

Since f^* is exact, it follows easily that s^* is left exact (indeed, s^* commutes with all the limits with which f^* commutes). Moreover, since all colimits are universal in T (see (4.1.19)(i)), it is easily seen that s^* commutes with all colimits. Then, by proposition 4.4.8(i.b), the functor s^* determines a morphism of topoi $s : T' \to T/U$, unique up to unique isomorphism, with an isomorphism $f \xrightarrow{\sim} j_U \circ s$.

(iii) Next, let $s, s' : T' \to T/U$ be two morphisms of topoi, $\omega : f \xrightarrow{\sim} j_U \circ s$ and $\omega' : f \xrightarrow{\sim} j_U \circ s'$ two isomorphisms, and $\beta : s \Rightarrow s'$ any natural transformation such that

$$(j_U * \beta) \odot \omega = \omega'. \tag{4.7.10}$$

By remark 1.1.17(iv,vi), it follows that $\omega'^\dagger = \omega^\dagger \odot (\beta^\dagger * j_U^*)$. We may then compute :

$$\begin{aligned} \sigma(s', \omega') = \omega'^\dagger_U \circ s'^*(\Delta_U) \circ u^{-1}_{s'^*\mathbf{1}_U} &= \omega^\dagger_U \circ \beta^\dagger_{j_U^*U} \circ s'^*(\Delta_U) \circ u^{-1}_{s'^*\mathbf{1}_U} \\ &= \omega^\dagger_U \circ s^*(\Delta_U) \circ \beta^\dagger_{\mathbf{1}_U} \circ u^{-1}_{s'^*\mathbf{1}_U} \\ &= \sigma(s, \omega). \end{aligned}$$

By considering the cartesian diagram $\mathscr{E}$ of (i), we then deduce that for every object $\varphi : X \to U$ of T/U there exists a unique isomorphism

$$\lambda_\varphi : s'^*\varphi \xrightarrow{\sim} s^*\varphi \qquad \text{such that} \qquad \omega^\dagger_X \circ s^*\Gamma_\varphi \circ \lambda_\varphi = \omega'^\dagger_X \circ s'^*\Gamma_\varphi.$$

Since $\omega^\dagger_X$ is an isomorphism, the latter identity yields :

$$s^*\Gamma_\varphi \circ \lambda_\varphi = s^*\Gamma_\varphi \circ \beta^\dagger_\varphi.$$

However, Γ_φ is a monomorphism, hence the same holds for $s^*\Gamma_\varphi$ (proposition 1.3.18(i)), whence $\lambda_\varphi = \beta^\dagger_\varphi$. Hence, β is *the unique isomorphism of functors* $s \xrightarrow{\sim} s'$ verifying (4.7.10).

(iv) Lastly, consider a morphism $\psi : U \to V$ of T, and $\sigma \in \Gamma(T', f^*U)$. Let $s^U : T/U \to T'$ be the morphism of topoi deduced from σ as in (ii), with its associated isomorphism of functors $\omega^U : s^{U*} \circ j_U^* \xrightarrow{\sim} f^*$. Likewise, let $s^V : T/V \to T'$ be the morphism of topoi and $\omega^V : s^{V*} \circ j_V^* \xrightarrow{\sim} f^*$ the isomorphism deduced, again as in (ii), from $f^*(\psi) \circ \sigma \in \Gamma(T', f^*V)$. A direct inspection of the constructions yields a natural isomorphism of functors

$$\beta : s^{V*} \xrightarrow{\sim} s^{U*} \circ j_\psi^* \qquad \text{such that} \qquad \omega^V = \omega^U \odot (s^{U*} * \gamma^{\dagger -1}_{\psi,V}) \odot (\beta * j_V^*)$$

where $\gamma_{\psi,V} : j_V \circ j_\psi \xrightarrow{\sim} j_U$ is the coherence constraint of the fibred topos $T/-$ of example 4.7.8(ii). In view of (iii), it follows easily that for every pair (s, ω) as in (i) we have :

$$\sigma(j_\psi \circ s, (\gamma^{-1}_{\psi,V} * s) \odot \omega) = f^*(\psi) \circ \sigma(s, \omega)$$

Definition 4.7.11. Let T be a topos.

(i) A *point* of T (or a T*-point*) is a morphism of topoi $\mathbf{Set} \to T$. If $\xi = (\xi^*, \xi_*)$ is a point, and F is any object of T, the set $\xi^* F$ is usually denoted by F_ξ.

(ii) If ξ is a T-point, and $f : T \to S$ is any morphism of topoi, we denote by $f(\xi)$ the S-point $f \circ \xi$. If $\beta : \xi \to \xi'$ is any morphism of T-points, we let likewise $f(\beta) := f_* * \beta$.

(iii) A *neighborhood* of ξ is a pair (U, a), where $U \in \mathrm{Ob}(T)$, and $a \in U_\xi$. A morphism of neighborhoods $(U, a) \to (U', a')$ is a morphism $f : U \to U'$ in T such that $f_\xi(a) = a'$. The category of all neighborhoods of ξ shall be denoted $\mathbf{Nbd}(\xi)$.

(iv) A family $(f_\lambda : T_\lambda \to T \mid \lambda \in \Lambda)$ of morphisms of topoi is *conservative*, if the functor

$$T \to \prod_{\lambda \in \Lambda} T_\lambda \qquad F \mapsto (f_\lambda^* F \mid \lambda \in \Lambda)$$

is conservative (definition 1.1.4(ii)) (the product $\prod_{\lambda \in \Lambda} T_\lambda$ is formed in a sufficiently large universe containing U). We say that T *has enough points*, if T has a conservative set of points.

(v) Consider a graph Γ (see definition 1.6.1(i)) and a system $(C_S \mid S \in \mathrm{Ob}(\mathbf{Topos}))$, where C_S is a given set of morphisms of graphs $\Gamma \to S$, for every topos S, such that $f^*\varphi \in C_{S'}$, for every $\varphi \in C_S$ and every morphism $f : S' \to S$ of topoi. Let $T_\bullet := (T_\lambda \xrightarrow{f_\lambda} T \mid \lambda \in \Lambda)$ be a conservative family of morphisms of topoi, and $\mathbf{P}(S, \varphi)$ a property defined on elements $\varphi \in C_S$, for every topos S; we say that $\mathbf{P}$ *can be checked on* $T_\bullet$, if for every $\varphi \in C_T$ we have :

- $\mathbf{P}(T, \varphi)$ holds if and only if $\mathbf{P}(T_\lambda, f_\lambda^* \circ \varphi)$ holds for every $\lambda \in \Lambda$.

Moreover, we say that $\mathbf{P}$ *can be checked on stalks*, if for every topos S and every conservative set Ω of S-points, the property $\mathbf{P}$ can be checked on Ω.

Example 4.7.12. Let T be a topos, and $(U_\lambda \to 1_T \mid \lambda \in \Lambda)$ any family of morphisms of T covering the final object 1_T (for the canonical topology of T); then the induced family of morphisms of topoi $(j_{U_\lambda} : T/U_\lambda \to T \mid \lambda \in \Lambda)$ is conservative. For the proof, we easily reduce to the case where Λ is a small set, by lemma 4.1.22; then it suffices to recall that, by remark 4.4.1(i), every $X \in \mathrm{Ob}(T)$ is the coequalizer of the induced pair of morphisms of T

$$\coprod_{\lambda,\mu \in \Lambda} U_\lambda \times U_\mu \times X \rightrightarrows \coprod_{\lambda \in \Lambda} U_\lambda \times X$$

(the details are left to the reader).

Remark 4.7.13. (i) With the notation of definition 4.7.11, notice that

$$\mathbf{Nbd}(\xi)^o = \mathscr{F}ib(\xi^*)$$

where the fibration $\pi : \mathscr{F}ib(\xi^*) \to T^o$ is defined as in (3.1.15).

(ii) Moreover, the category $\mathbf{Nbd}(\xi)$ is finitely complete. Indeed, let Λ be any finite category, and $F : \Lambda \to \mathbf{Nbd}(\xi)$ any functor; say that $F\lambda = (U_\lambda, a_\lambda)$ for every $\lambda \in \mathrm{Ob}(\Lambda)$. Then it is easily seen that the limit of F is represented by (U, a), where $U \in \mathrm{Ob}(T)$ represents the limit of the functor $\pi^o \circ F : \Lambda \to T$, and $a \in U_\xi = \lim_\Lambda \xi^* \circ \pi^o \circ F$ is the unique element whose image under the induced projection $U_\xi \to U_{\lambda,\xi}$ agrees with a_λ, for every $\lambda \in \mathrm{Ob}(\Lambda)$.

(iii) Let ξ be a point of the topos T; from (ii) it follows that the category $\mathbf{Nbd}(\xi)$ is cofiltered.

(iv) In view of (i) and (1.4.7), for every $F \in \mathrm{Ob}(T)$ there is a natural cocone :

$$\mathbf{Nbd}(\xi)^o \underset{c_{F_\xi}}{\overset{\mathrm{Hom}_T(\pi, F)}{\underset{\Downarrow \tau_{\xi,F}}{\rightrightarrows}}} \mathbf{Set}$$

assigning to every neighborhood (U, a) of ξ, the map $\tau_{\xi,F}(U, a) : \mathrm{Hom}_T(U, F) \to F_\xi$ such that $s \mapsto s_\xi(a)$ for every $s : U \to F$. Then lemma 1.4.8 says that $\tau_{\xi,F}$ induces a natural bijection

$$\text{(4.7.14)} \qquad \operatorname*{colim}_{\mathbf{Nbd}(\xi)^o} \mathrm{Hom}_T(\pi, F) \xrightarrow{\sim} F_\xi \qquad \text{for every } F \in \mathrm{Ob}(T).$$

(v) It follows from remark 4.7.9(iii), that a neighborhood (U, a) of ξ is the same as the datum of an isomorphism class of a point ξ_U of the topos T/U which lifts ξ, *i.e.* such that $\xi \simeq j_U(\xi_U)$. Moreover, say that $\xi_{U'}$ is another lifting of ξ, corresponding to a neighborhood (U', a') of ξ; then, by inspecting the constructions of remark 4.7.9(i,ii) we see that, under this identification, a morphism $(U, a) \to (U', a')$ of neighborhoods of ξ corresponds to the datum of :

$$\text{a morphism } \varphi : U \to U' \text{ in } T \text{ and an isomorphism } j_\varphi(\xi_U) \xrightarrow{\sim} \xi_{U'} \text{ of } T\text{-points}$$

where j_φ is the morphism of topoi $T/U \to T/U'$ induced by φ.

Proposition 4.7.15. *Let T be a topos with enough points; we have :*

(i) *For every object U of T, the topos T/U has enough points.*

(ii) *More precisely, let Ω be a conservative set of T-points, and denote by $j_U^{-1}\Omega$ the set of all T/U-points ξ_U such that there exists $\xi \in \Omega$ with an isomorphism $\xi \xrightarrow{\sim} j_U(\xi_U)$; then $j_U^{-1}\Omega$ is a conservative set of T/U-points.*

Proof. Let $\varphi : X \to Y$ be a morphism in T/U such that φ_ξ is an epimorphism for every $\xi \in j_U^{-1}\Omega$; by proposition 1.3.21(ii), it suffices to show that φ is an epimorphism, and the latter will follow, if we show that the same holds for $j_{U!}\varphi$. Thus suppose by way of contradiction, that $j_{U!}\varphi$ is not an epimorphism; then there exist $\xi \in \Omega$ and $y \in (j_{U!}Y)_\xi$, such that y does not lie in the image of $(j_{U!}\varphi)_\xi$. Let $a := (j_{U!}\pi_Y)_\xi(y) \in U_\xi$, where $\pi_Y : Y \to 1_{T/U} = \mathbf{1}_U$ is the unique morphism in T/U. By remark 4.7.13(v), we may find a lifting $(\xi_U : \mathbf{Set} \to T/U, \omega_U)$ of ξ such that

$$\sigma(\xi_U, \omega_U) = a \tag{4.7.16}$$

where $\sigma(\xi_U, \omega_U)$ is defined as in remark 4.7.9(i). After replacing ξ by $j_U(\xi_U)$, we may assume that $\omega_U = \mathbf{1}_\xi$, in which case (4.7.16) means that $a = \xi_U^*\Delta_U$, where $\Delta_U : 1_{T/U} \to j_U^* j_{U!} 1_{T/U}$ is the unit of adjunction (*i.e* the diagonal $U \to U \times U$). We have a commutative diagram of sets :

$$\begin{array}{ccc} \xi_U^* X & \xrightarrow{\xi_U^*\varphi} & \xi_U^* Y \\ {\scriptstyle \xi_U^*\Delta_X}\downarrow & & \downarrow{\scriptstyle \xi_U^*\Delta_Y} \\ (j_{U!}X)_\xi & \xrightarrow{(j_{U!}\varphi)_\xi} & (j_{U!}Y)_\xi = \xi_U^*(j_U^* j_{U!} Y) \end{array}$$

where $\Delta_X : X \to j_U^* j_{U!} X$ is the unit of adjunction, and likewise for Δ_Y. More plainly, $\Delta_Y : Y \times 1_{T/U} \to Y \times (j^* j_{U!} 1_{T/U})$ is the product $\mathbf{1}_Y \times \Delta_U$, and under this identification, $\xi_U^*\Delta_Y$ is the mapping $\xi_U^* Y \to \xi_U^* Y \times (j^* j_{U!} 1_{T/U})$ given by the rule : $z \mapsto (z, a)$ for every $z \in \xi_U^* Y$. Especially, we see that y lies in the image of $\xi_U^*\Delta_Y$. But by assumption, the map $\xi_U^*\varphi$ is surjective, hence y lies in the image of $(j_{U!}\xi)_\xi$, a contradiction. □

4.7.17. Remark 4.7.13(v) prompts the following construction. Let us consider the fibred topos

$$T/- : T \to \mathbf{Topos} \qquad U \mapsto T/U \qquad (U' \xrightarrow{\varphi} U) \mapsto (T/U' \xrightarrow{j_\varphi} T/U)$$

of example 4.7.8(ii). Denote also by 1_T a fixed final object of T, and for every $U \in \mathrm{Ob}(T)$ let $\varphi_U : U \to 1_T$ be the unique morphism in T; the source functor yields a natural isomorphism $T/1_T \xrightarrow{\sim} T$ that identifies $j_{\varphi_U} : T/U \to T/1_T$ with $j_U : T/U \to T$. Moreover, the rule : $U \mapsto \varphi_U$ clearly defines a natural transformation

$$\varphi_\bullet : \mathbf{1}_T \Rightarrow c_{1_T}$$

where $c_{1_T} : T \to T$ is the constant functor with value 1_T. For a sufficiently large universe V, example 2.2.7(i) yields a strict pseudo-functor $H_{\mathbf{Topos}} : \mathbf{Topos}^o \times \mathbf{Topos} \to \mathsf{V}\text{-}\mathbf{Cat}$ which restricts to a strict pseudo-functor

$$\mathrm{Pt} : \mathbf{Topos} \to \mathsf{V}\text{-}\mathbf{Cat} \qquad S \mapsto \mathrm{Pt}(S) := \mathrm{Hom}_{\mathbf{Topos}}(\mathbf{Set}, S)$$

assigning to every topos S the category of S-points. We consider the composition

$$\mathrm{Pt}_T := \mathrm{Pt} \circ (T/-) : T \to \mathsf{V}\text{-}\mathbf{Cat} \qquad U \mapsto \mathrm{Pt}(T/U)$$

as well as the pseudo-natural transformation

$$\mathrm{Pt}_T * \varphi_\bullet : \mathrm{Pt}_T \Rightarrow \mathsf{F}_{\mathrm{Pt}(T)}$$

from Pt_T to the constant pseudo-functor $\mathsf{F}_{\mathrm{Pt}(T)}$ with value $\mathrm{Pt}(T)$. There follows a cartesian functor of fibrations over T^o :

$$\mathscr{F}ib(\mathrm{Pt}_T * \varphi_\bullet) : \mathscr{F}ib(\mathrm{Pt}_T) \to T^o \times \mathrm{Pt}(T).$$

Let also $\sigma_\xi : T^o \to T^o \times \mathrm{Pt}(T)$ be the cartesian section such that $\sigma_\xi(U) := (U, \xi)$ for every $U \in \mathrm{Ob}(T)$. Lastly, we consider the 2-cartesian diagram in the category $\mathsf{Fib}(T^o)$:

$$\begin{array}{ccc} \mathscr{L}(\xi) := \mathscr{F}ib(\mathrm{Pt}_T) \overset{2}{\underset{T^o \times \mathrm{Pt}(T)}{\times}} T^o & \longrightarrow & \mathscr{F}ib(\mathrm{Pt}_T) \\ \big\downarrow & & \big\downarrow {\scriptstyle \mathscr{F}ib(\mathrm{Pt}_T * \varphi_\bullet)} \\ T^o & \xrightarrow{\ \sigma_\xi\ } & T^o \times \mathrm{Pt}(T) \end{array}$$

and we set

$$\mathbf{Lift}(\xi) := \mathscr{L}(\xi)^o.$$

Taking into account theorem 3.3.25(ii), we see that for every $U \in \mathrm{Ob}(T)$, the fibre $\mathscr{L}(\xi)_U$ of the fibration $\mathscr{L}(\xi) \to T^o$ is the category whose objects are the pairs (ξ_U, ω_U), where ξ_U is a T/U-point, and $\omega_U : \xi \overset{\sim}{\to} j_U(\xi_U)$ is an isomorphism of T-points. The morphisms $(\xi_U, \omega_U) \to (\xi'_U, \omega'_U)$ are the morphisms of T/U-points

$$\beta : \xi_U \to \xi'_U \qquad \text{such that} \qquad \omega_{U'} = j_U(\beta) \odot \omega_U$$

(with the obvious composition law). Hence, the objects of $\mathbf{Lift}(\xi)$ are the triples (U, ξ_U, ω_U) with $U \in \mathrm{Ob}(T)$ and $(\xi_U, \omega_U) \in \mathrm{Ob}(\mathscr{L}(\xi)_U)$. The morphisms $(U, \xi_U, \omega_U) \to (V, \xi_V, \omega_V)$ are the pairs (φ, β), where $\varphi : U \to V$ is a morphism in T, and $\beta : \xi_V \to j_\varphi(\xi_U)$ is a morphism of T/V-points, such that

$$(\gamma_{\varphi,V} * \xi_U) \odot j_V(\beta) \odot \omega_V = \omega_U$$

where $\gamma_{\varphi,V} : j_V \circ j_\varphi \overset{\sim}{\to} j_U$ is the coherence constraint of $T/-$. Then, remark 4.7.9 shows that $\mathscr{L}(\xi)$ is a fibration in groupoids, and we have an equivalence of categories :

(4.7.18) $$N_\xi : \mathbf{Lift}(\xi) \overset{\sim}{\to} \mathbf{Nbd}(\xi) \qquad (U, \xi_U, \omega_U) \mapsto (U, \sigma(\xi_U, \omega_U))$$

where $\sigma(\xi_U, \omega_U) \in U_\xi$ is defined as in remark 4.7.9(i).

4.7.19. Notice next that, since $\mathscr{L}(\xi)$ is a fibration in groupoids, the natural projection $\mathscr{L}(\xi) \to \mathscr{F}ib(\mathrm{Pt}_T)$ factors through the inclusion pseudo-functor $\mathscr{F}ib(\mathrm{Pt}_T^\times) = \mathscr{F}ib(\mathrm{Pt}_T)^\times \to \mathscr{F}ib(\mathrm{Pt}_T)$ (notation of remark 3.4.8(iii)). To every small set X, we wish now to attach a functor

$$\Xi^X : \mathscr{F}ib(\mathrm{Pt}^\times) \to \mathbf{Set} \qquad (\zeta : \mathbf{Set} \to S) \mapsto \zeta^* \zeta_* X.$$

To this aim, according to remark 3.1.23(i), it will suffice to exhibit a lax-natural transformation

$$\alpha^X : {}^o\mathrm{Pt}^\times \Rightarrow {}^o\mathsf{F}_{\mathbf{Set}}$$

where $\mathsf{F}_{\mathbf{Set}}$ is the constant pseudo-functor $\mathbf{Topos} \to \mathsf{V}\text{-}\mathbf{Cat}$ with value $\mathbf{Set}$. We need therefore to give for every $S \in \mathrm{Ob}(\mathbf{Topos})$ a functor $\alpha^X_S : \mathrm{Pt}(S)^\times \to \mathbf{Set}$, together with a coherence constraint for the resulting system $(\alpha^X_S \mid S \in \mathrm{Ob}(\mathbf{Topos}))$. Now, let us set

$$\alpha^X_S(\xi) := \zeta^* \zeta_* X \qquad \text{for every } \zeta \in \mathrm{Ob}(\mathrm{Pt}(S)).$$

Next, let $\beta : \zeta \stackrel{\sim}{\to} \zeta'$ be any morphism in $\mathrm{Pt}(S)^\times$; *i.e.* $\beta : \zeta_* \stackrel{\sim}{\to} \zeta'_*$ is an isomorphism of functors. Then we define $\alpha^X_S(\beta) : \zeta^*\zeta_*X \to \zeta'^*\zeta'_*X$ as the composition

$$\zeta^*\zeta_*X \xrightarrow{(\zeta^**\beta)_X} \zeta^*\zeta'_*X \xrightarrow{(\beta^\dagger*\zeta'_*)^{-1}_X} \zeta'^*\zeta'_*X$$

where $\beta^\dagger : \zeta'^* \stackrel{\sim}{\to} \zeta^*$ is the transpose of the natural transformation β (see remark 1.1.17(ii)). From remark 1.1.17(vi) we see already that $\alpha^X_S(\mathbf{1}_\zeta) = \mathbf{1}_{\zeta^*\zeta_*X}$. Next, let $\beta' : \zeta' \to \zeta''$ be another morphism of $\mathrm{Pt}(S)^\times$; we compute :

$$\begin{aligned}\alpha^X_S(\beta')\circ\alpha^X_S(\beta) &= ((\beta'^\dagger * \zeta''_*)^{-1}\odot(\zeta'^* * \beta')\odot(\beta^\dagger * \zeta'_*)^{-1}\odot(\zeta^* * \beta))_X \\ &= ((\beta'^\dagger * \zeta''_*)^{-1}\odot(\beta^\dagger * \zeta''_*)^{-1}\odot(\zeta^* * \beta')\odot(\zeta^* * \beta))_X \\ &= (((\beta'\odot\beta)^\dagger * \zeta''_*)^{-1}\odot(\zeta^* * (\beta'\odot\beta)))_X \\ &= \alpha^X_S(\beta'\odot\beta)\end{aligned}$$

where the second equality follows from (1.1.11), and the third follows from remark 1.1.17(vi). Next, let $f : S \to S'$ be any morphism of topoi, and $\varepsilon^f : f^*\circ f_* \Rightarrow \mathbf{1}_S$ the counit of the adjoint pair (f^*, f_*); our candidate coherence constraint for $\alpha^X_\bullet$ assigns to every S-point ζ the map

$$\tau^X_{f,\zeta} : f(\zeta)^*f(\zeta)_*X = \zeta^*\circ f^*\circ f_*\circ\zeta_*X \xrightarrow{(\zeta^**\varepsilon^f*\zeta_*)_X} \zeta^*\zeta_*X$$

(see example 2.2.6(ii)). We need to check that the rule $\zeta \mapsto \tau^X_{f,\zeta}$ yields a natural transformation

$$\tau^X_f : \alpha^X_{S'}\circ\mathrm{Pt}(f)^\times \Rightarrow \alpha^X_S.$$

Thus, let $\beta : \zeta_* \stackrel{\sim}{\to} \zeta'_*$ be any morphism of $\mathrm{Pt}(S)^\times$; the assertion comes down to showing the commutativity of the diagram :

$$\begin{array}{ccccc} \zeta^*f^*f_*\zeta_*X & \xrightarrow{(\zeta^*f^*f_**\beta)_X} & \zeta^*f^*f_*\zeta'_*X & \xrightarrow{((f_**\beta)^\dagger*f_*\zeta'_*)^{-1}_X} & \zeta'^*f^*f_*\zeta'_*X \\ {\scriptstyle (\zeta^**\varepsilon^f*\zeta_*)_X}\downarrow & & {\scriptstyle (\zeta^**\varepsilon^f*\zeta'_*)_X}\downarrow & & \downarrow{\scriptstyle (\zeta'^**\varepsilon^f*\zeta'_*)_X} \\ \zeta^*\zeta_*X & \xrightarrow{(\zeta^**\beta)_X} & \zeta^*\zeta'_*X & \xrightarrow{(\beta^\dagger*\zeta'_*)^{-1}_X} & \zeta'^*\zeta'_*X \end{array}$$

However, the commutativity of the left square subdiagram follows from the naturality of $\zeta^* * \varepsilon^f$. Next, recall that $(f_* * \beta)^\dagger = \beta^\dagger * f^*$ (remark 1.1.17(vi)); then the commutativity of the right square subdiagram follows by applying (1.1.11) to the natural transformations $\beta^\dagger$ and $\varepsilon^f * \zeta'_*$.

Lastly, we have to check the coherence axioms for $\tau^X_\bullet$. Since ${}^o\mathrm{Pt}^\times$ and ${}^o\mathsf{F}_{\mathbf{Set}}$ are both strict pseudo-functors, we come down to showing that

$$\tau^X_{\mathbf{1}_S} = \mathbf{1}_{\alpha^X_S} \qquad\text{and}\qquad \tau^X_g \boxminus \tau^X_f = \tau^X_{g\circ f}$$

for every topos S and every pair of morphisms of topoi $S \xrightarrow{f} S' \xrightarrow{g} S''$ (see remark 2.2.5(i)). These follow easily by direct inspection. This completes the construction of α^X, and thus also of $\Xi^X := \mathscr{F}ib(\alpha^X)$. After composing with the projection $\mathscr{L}(\xi) \to \mathscr{F}ib(\mathrm{Pt}^\times_T) \to \mathscr{F}ib(\mathrm{Pt}^\times)$, we deduce a functor

$$\Xi^{X,\xi} : \mathbf{Lift}(\xi)^o \to \mathbf{Set} \qquad (U, \xi_U, \omega_U) \mapsto \xi^*_U\xi_{U*}X.$$

Lemma 4.7.20. (i) *For every $(S,\zeta) \in \mathrm{Ob}(\mathscr{F}ib(\mathrm{Pt}^\times))$ and every set X, let $\varepsilon^{(S,\zeta)}_X : \zeta^*\zeta_*X \to X$ be the counit of adjunction. Then the rule : $(S,\zeta) \mapsto \varepsilon^{(S,\zeta)}_X$ defines a natural cocone*

$$\Xi^X \Rightarrow c_X.$$

(ii) *The cocone of* (i) *induces a natural bijection :*

$$\operatorname*{colim}_{\mathbf{Lift}(\xi)^o} \Xi^{X,\xi} \stackrel{\sim}{\to} X \qquad \textit{for every (small) set } X.$$

Proof. (i): Let $\nu^X : {}^o\mathrm{Pt}^\times \Rightarrow {}^o\mathsf{F}_{\mathbf{Set}}$ be the strict pseudo-natural transformation such that

$$\nu^X_S : \mathrm{Pt}(S)^\times \to \mathbf{Set}$$

is the constant functor with value X, for every topos S (*i.e.* $\nu^X_S(\zeta) := X$ for every S-point ζ, and $\nu^X_S(\beta) = \mathbf{1}_X$ for every invertible morphism β of S-points). For every such S, let also

$$\varepsilon^{(S,\bullet)}_X : \mathrm{Ob}(\mathrm{Pt}(S)) \to \mathrm{Morph}(\mathbf{Set})$$

be the map that associates with every S-point ζ, the map of sets $\varepsilon^{(S,\zeta)}_X$. We notice :

Claim 4.7.21. The rule : $S \mapsto \varepsilon^{(S,\bullet)}_X$ for every topos S, defines a modification :

$$\varepsilon^{(\bullet,\bullet)}_X : \nu^X \rightsquigarrow \alpha^X.$$

Proof of the claim. First, we need to check that $\varepsilon^{(S,\bullet)}_X$ is a natural transformation $\alpha^X_S \Rightarrow \nu^X_S$, for every topos S. The latter assertion amounts to the commutativity of the diagram :

$$\begin{array}{ccc} \zeta^*\zeta_*X & \xrightarrow{\varepsilon^{(S,\zeta)}_X} & X \\ {\scriptstyle \alpha^X_S(\beta)}\downarrow & & \| \\ \zeta'^*\zeta'_*X & \xrightarrow{\varepsilon^{(S,\zeta')}_X} & X \end{array}$$

for every isomorphism $\beta : \zeta \stackrel{\sim}{\to} \zeta'$ of S-points. Since $(\beta^\dagger * \zeta'_*)^{-1}_X = ((\beta^{-1})^\dagger * \zeta'_*)_X$ (remark 1.1.17(iv)), this is in turn the same as the identity :

$$\varepsilon^{(S,\zeta')}_X \circ ((\beta^{-1})^\dagger * \zeta'_*)_X = \varepsilon^{(S,\zeta)}_X \circ (\zeta^* * \beta^{-1})_X$$

which follows from remark 1.1.17(iii). Next, we need to check the compatibility condition for $\varepsilon^{(S,\bullet)}_X$, which comes down to the identity :

$$\varepsilon^{(S,\zeta)}_X \odot \tau^X_f = \varepsilon^{(S,f(\zeta))}_X$$

for every morphism of topoi $f : S \to S'$ and every S-point ζ, where τ^X_f denotes the coherence constraint of α^X. The latter follows by direct inspection. ◊

By remark 3.1.23(ii), the sought cocone is then $\mathscr{F}ib(\varepsilon^{(\bullet,\bullet)}_X) : \Xi^X \Rightarrow \mathscr{F}ib(\nu^X) = c_X$.

(ii): Let $\pi : \mathscr{F}ib(\mathrm{Pt}^\times) \to \mathbf{Topos}^o$ be the fibration arising from the strict pseudo-functor $\mathrm{Pt}^\times$. We let $\mathscr{C}$ be the fibre product in the cartesian square of categories :

$$\begin{array}{ccccc} \mathsf{Morph}(\mathscr{F}ib(\mathrm{Pt}^\times)) & \xrightarrow{p} & \mathscr{C} & \xrightarrow{\mathsf{s}'} & \mathscr{F}ib(\mathrm{Pt}^\times) \\ & {\scriptstyle \mathsf{Morph}(\pi)}\searrow & {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ & & \mathsf{Morph}(\mathbf{Topos}^o) & \xrightarrow{\mathsf{s}} & \mathbf{Topos}^o \end{array}$$

where s and $\mathsf{s}' \circ p$ are the source functors. By remark 3.1.27(ii), we have a natural identification

$$\mathscr{C} \stackrel{\sim}{\to} \mathscr{F}ib(\mathrm{Pt}^\times \circ \mathsf{s}^o)$$

and recall that $\mathsf{Morph}(\pi)$ is a fibration as well, by example 3.1.9. To every morphism of topoi $f : S' \to S$, we attach the functor

$$\Sigma^X_f : \pi'^{-1}(f) = \mathrm{Pt}(S)^\times \to \mathbf{Set} \qquad \zeta \mapsto \Gamma(S', f^*\zeta_*X)$$

where $\Gamma(S', -) : S' \to \mathbf{Set}$ is defined as in (4.4.11), after fixing a final object $1_{S'}$ of S'. To every isomorphism $\alpha : \zeta_1 \stackrel{\sim}{\to} \zeta_2$ of S-points, the functor Σ^X_f assigns the map

$$\Sigma^X_f(\alpha) := \Gamma(S', f^*\alpha_X) : \Gamma(S', f^*\zeta_{1*}X) \to \Gamma(S', f^*\zeta_{2*}X).$$

Next, notice that if $(S'_1 \xrightarrow{f_1} S_1)^o$ and $(S'_2 \xrightarrow{f_2} S_2)^o$ are two objects of $\mathsf{Morph}(\mathbf{Topos}^o)$, then a morphism $f_1^o \to f_2^o$ in this category is a commutative diagram of morphisms of topoi :

$$\mathscr{D}_{g,g'} \qquad : \qquad \begin{array}{ccc} S'_2 & \xrightarrow{f_2} & S_2 \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ S'_1 & \xrightarrow{f_1} & S_1. \end{array}$$

The (split) cleavage $\mathrm{Pt}^\times \circ \mathsf{s}^o$ of $\mathscr{C}$ associates with $\mathscr{D}_{g,g'}$ the functor

$$\mathrm{Pt}(g)^\times : \mathrm{Pt}(S_2)^\times \to \mathrm{Pt}(S_1)^\times \qquad \zeta \mapsto g(\zeta)$$

that assigns to every morphism α of $\mathrm{Pt}(S_2)^\times$ the morphism $g(\alpha)$ of $\mathrm{Pt}(S_1)^\times$. To the diagram $\mathscr{D}_{g,g'}$ and every object ζ of $\mathrm{Pt}(S_2)^\times$ we attach as well the map :

$$\tau_\zeta^{g,g'} : \Gamma(S'_1, f_1^* g_* \zeta_* X) \xrightarrow{\Gamma(S'_1, \Upsilon(\mathscr{D}_{g,g'}) * \zeta_*)_X} \Gamma(S'_1, g'_* f_2^* \zeta_* X) \xrightarrow{\omega^{g'}_{f_2^* \zeta_* X}} \Gamma(S'_2, f_2^* \zeta_* X)$$

where $\Upsilon(\mathscr{D}_{g,g'})$ is the base change transformation associated with the diagram $\mathscr{D}_{g,g'}$, viewed as an oriented square diagram of links, whose orientation is the identity (see (2.3.8)). The second map in this composition is induced by the unique isomorphism of morphisms of topoi $\omega^{g'} : \Gamma_{S'_1} \circ g'_* \xrightarrow{\sim} \Gamma_{S'_2}$ (proposition 4.4.12(ii)).

Claim 4.7.22. (i) For every diagram $\mathscr{D}_{g,g'}$, the rule : $\zeta \mapsto \tau_\zeta^{g,g'}$ defines a natural transformation

$$\begin{array}{ccc} \mathrm{Pt}(S_2)^\times & \xrightarrow{\Sigma^X_{f_2}} & \mathbf{Set} \\ {\scriptstyle \mathrm{Pt}(g)^\times}\downarrow & \tau^{g,g'} \Uparrow & \| \\ \mathrm{Pt}(S_1)^\times & \xrightarrow[\Sigma^X_{f_1}]{} & \mathbf{Set}. \end{array}$$

(ii) The rule : $\mathscr{D}_{g,g'} \mapsto \tau^{g,g'}$ provides the coherence constraint for a lax-natural transformation

$$\Sigma^X_\bullet : {}^o(\mathrm{Pt}^\times \circ \mathsf{s}^o) \Rightarrow {}^o\mathsf{F}_{\mathbf{Set}}.$$

Proof of the claim. (i): For every object $(S' \xrightarrow{f} S)^o$ of $\mathsf{Morph}(\mathbf{Topos}^o)$, consider the functor

$$\sigma_f : \mathrm{Pt}(S)^\times \to S \qquad \zeta \mapsto \zeta_* X$$

that assigns to every isomorphism α as in the foregoing, the morphism α_X of S. With this notation, $\tau^{g,g'}$ is the composition of the following oriented squares :

$$\begin{array}{ccccccc} \mathrm{Pt}(S_2)^\times & \xrightarrow{\sigma_{f_2}} & S_2 & \xrightarrow{f_2^*} & S'_2 & \xrightarrow{\Gamma_{S'_2}} & \mathbf{Set} \\ {\scriptstyle \mathrm{Pt}(g)^\times}\downarrow & \mathbf{1}_{g_* \circ \sigma_{f_2}} \Uparrow & {\scriptstyle g_*}\downarrow & \Upsilon(\mathscr{D}_{g,g'}) \Uparrow & \downarrow{\scriptstyle g'_*} & \omega^{g'} \Uparrow & \| \\ \mathrm{Pt}(S_1)^\times & \xrightarrow[\sigma_{f_1}]{} & S_1 & \xrightarrow[f_1^*]{} & S'_2 & \xrightarrow[\Gamma_{S'_1}]{} & \mathbf{Set}. \end{array}$$

(ii): Clearly $\tau^{\mathbf{1}_S, \mathbf{1}_{S'}} = \mathbf{1}_{\Sigma^X_f}$, for every morphism of topoi $f : S' \to S$. According to remark 2.2.5(i), it remains to check the identity

$$\tau^{g,g'} \boxminus \tau^{h,h'} = \tau^{g \circ h, g' \circ h'}$$

for every composable pair of diagrams $\mathscr{D}_{g,g'} : f_1^o \to f_2^o$ and $\mathscr{D}_{h,h'} : f_2^o \to f_3^o$. However, by proposition 2.1.9 we have :

$$\tau^{g,g'} \boxminus \tau^{h,h'} = (\mathbf{1}_{g_* \circ \sigma_{f_2}} \boxminus \mathbf{1}_{h_* \circ \sigma_{f_3}}) \boxplus (\Upsilon(\mathscr{D}_{g,g'}) \boxminus \Upsilon(\mathscr{D}_{h,h'})) \boxplus (\omega^{g'} \boxminus \omega^{h'}).$$

Clearly $\mathbf{1}_{g_*\circ\sigma_{f_2}} \boxminus \mathbf{1}_{h_*\circ\sigma_{f_3}} = \mathbf{1}_{(g\circ h)_*\circ\sigma_{f_3}}$, and we have $\omega^{g'} \boxminus \omega^{h'} = \omega^{g'\circ h'}$, by the uniqueness properties of the isomorphisms $\omega^\bullet$. Lastly, $\Upsilon(\mathscr{D}_{g,g'}) \boxminus \Upsilon(\mathscr{D}_{h,h'}) = \Upsilon(\mathscr{D}_{g,g'} \boxbar \mathscr{D}_{h,h'})$, by proposition 2.3.12(i), whence the claim. $\diamondsuit$

From claim 4.7.22(ii) and remark 3.1.23(i) we deduce a functor $\mathscr{F}ib(\Sigma^X_\bullet) : \mathscr{C} \to \mathbf{Set}$.

Next, notice that, for every object $(S' \xrightarrow{f} S)^o$ of $\mathsf{Morph}(\mathbf{Topos}^o)$, the objects of the fibre category $\mathsf{Morph}(\pi)^{-1}(f)$ are the triples (ζ, ζ', β), where ζ is an S-point, ζ' is an S'-point, and $\beta : \zeta \xrightarrow{\sim} f(\zeta')$ is an isomorphism of S-points. The morphisms $(\zeta_1, \zeta'_1, \beta_1) \to (\zeta_2, \zeta'_2, \beta_2)$ are the pairs (α, α') where $\alpha : \zeta_1 \xrightarrow{\sim} \zeta_2$ and $\alpha' : \zeta'_1 \xrightarrow{\sim} \zeta'_2$ are isomorphisms of S-points and respectively S'-points, such that

$$(f_* * \alpha') \odot \beta_1 = \beta_2 \odot \alpha. \tag{4.7.23}$$

If $(\gamma, \gamma') : (\zeta_2, \zeta'_2, \beta_2) \to (\zeta_3, \zeta'_3, \beta_3)$ is another such morphism, we have

$$(\gamma, \gamma') \circ (\alpha, \alpha') = (\gamma \odot \alpha, \gamma' \odot \alpha') \qquad \text{in } \mathsf{Morph}(\pi)^{-1}(f).$$

The (split) cleavage c of $\mathsf{Morph}(\mathscr{F}ib(\mathrm{Pt}^\times))$ attaches to the diagram $\mathscr{D}_{g,g'}$, the functor

$$\mathsf{c}_{g,g'} : \mathsf{Morph}(\pi)^{-1}(f_2) \to \mathsf{Morph}(\pi)^{-1}(f_1) \qquad (\zeta, \zeta', \beta) \mapsto (g(\zeta), g'(\zeta'), g(\beta))$$

that assigns to every morphism (α, α') of $\mathsf{Morph}(\pi)^{-1}(f_2)$ the morphism $(g(\alpha), g'(\alpha'))$. With this notation, p is the (cartesian) functor associated with the strict pseudo-natural transformation

$$t : \mathsf{c} \Rightarrow \mathrm{Pt}^\times \circ \mathsf{s}^o$$

that assigns to every morphims of topoi $f : S' \to S$ the functor

$$\mathsf{Morph}(\pi)^{-1}(f) \to \mathrm{Pt}(S) \qquad (\zeta, \zeta', \beta) \mapsto \zeta \qquad (\alpha, \alpha') \mapsto \alpha.$$

Now, to every morphism of topoi $f : S' \to S$, and every $(\zeta, \zeta', \beta) \in \mathrm{Ob}(\mathsf{Morph}(\pi)^{-1}(f))$, let us attach the map $b_{f,(\zeta,\zeta',\beta)} : \Gamma(S', f^*\zeta_* X) \to \zeta^*\zeta_* X$ defined as the composition :

$$\Gamma(S', f^*\zeta_*X) \xrightarrow{\Gamma(S', (\eta^{\zeta'} * f^*\zeta_*)_X)} \Gamma(S', \zeta'_*\zeta'^* f^*\zeta_*X) \xrightarrow{\omega^{\zeta'}_{\zeta'^* f^*\zeta_* X}} \zeta'^* f^*\zeta_*X \xrightarrow{(\beta^\dagger * \zeta_*)_X} \zeta^*\zeta_*X$$

where $\eta^{\zeta'} : \mathbf{1}_{S'} \Rightarrow \zeta'_*\zeta'^*$ is the unit of adjunction, and $\omega^{\zeta'} : \Gamma_{S'} \circ \zeta' \xrightarrow{\sim} \mathbf{1}_{\mathbf{Set}}$ is the unique isomorphism of morphisms of topoi provided by proposition 4.4.12(ii).

Claim 4.7.24. (i) For every morphism of topoi $f : S' \to S$, the rule : $(\zeta, \zeta', \beta) \mapsto b_{f,(\zeta,\zeta',\beta)}$ defines a natural transformation

$$b_f : \Sigma^X_f \circ t_f \Rightarrow \alpha^X_S \circ t_f.$$

(ii) The rule : $f \mapsto b_f$ defines a modification

$$b_\bullet : (\alpha^X * \mathsf{s}^o) \odot {}^o t \rightsquigarrow \Sigma^X_\bullet \odot {}^o t.$$

Proof of the claim. (i): Let $(\alpha, \alpha') : (\zeta_1, \zeta'_1, \beta_1) \to (\zeta_2, \zeta'_2, \beta_2)$ be a morphism in $\mathsf{Morph}(\pi)^{-1}(f)$, and set $U := (\zeta'_{2*} * \alpha'^\dagger * f^*\zeta_{2*})^{-1}_X \circ (\zeta'_{2*} * \alpha' * f^*\zeta_{2*})_X$. We notice that :

$$\begin{aligned}(\eta^{\zeta'_2} * f^*\zeta_{2*})_X \circ (f^*(\alpha_X) &= U \circ (\eta^{\zeta'_1} * f^*\zeta_{2*})_X \circ (f^*(\alpha_X) \\ &= U \circ (\zeta'_{1*}\zeta'^*_1 f^* * \alpha)_X \circ (\eta^{\zeta'_1} * f^*\zeta_{1*})_X\end{aligned}$$

where the first equality holds by remark 1.1.17(iii), and the second one follows from (1.1.11). We are then reduced to checking the commutativity of the diagram :

$$\begin{array}{ccccc}
\Gamma(S',\zeta'_{1*}\zeta'^*_1 f^*\zeta_{1*}X) & \xrightarrow{\omega^{\zeta'_1}_{\zeta'^*_1 f^*\zeta_{1*}X}} & \zeta'^*_1 f^*\zeta_{1*}X & \xrightarrow{(\beta^\dagger_1 * \zeta_{1*})_X} & \zeta^*_1\zeta_{1*}X \\
\downarrow{\scriptstyle \Gamma(S',\zeta'_{1*}\zeta'^*_1 f^*\alpha_X)} & & \downarrow{\scriptstyle \zeta'^*_1 f^*\alpha_X} & & \downarrow{\scriptstyle \zeta^*_1\alpha_X} \\
\Gamma(S',\zeta'_{1*}\zeta'^*_1 f^*\zeta_{2*}X) & \xrightarrow{\omega^{\zeta'_1}_{\zeta'^*_1 f^*\zeta_{2*}X}} & \zeta'^*_1 f^*\zeta_{2*}X & \xrightarrow{(\beta^\dagger_1 * \zeta_{2*})_X} & \zeta^*_1\zeta_{2*}X \\
\downarrow{\scriptstyle \Gamma(S',(\alpha' * \zeta'^*_1 f^*\zeta_{2*})_X)} & & \| & & \| \\
\Gamma(S',\zeta'_{2*}\zeta'^*_1 f^*\zeta_{2*}X) & \xrightarrow{\omega^{\zeta'_2}_{\zeta'^*_1 f^*\zeta_{2*}X}} & \zeta'^*_1 f^*\zeta_{2*}X & \xrightarrow{(\beta^\dagger_1 * \zeta_{2*})_X} & \zeta^*_1\zeta_{2*}X \\
\downarrow{\scriptstyle \Gamma(S',(\zeta'_{2*} * \alpha'^\dagger * f^*\zeta_{2*})^{-1}_X)} & & \downarrow{\scriptstyle (\alpha'^\dagger * f^*\zeta_{2*})^{-1}_X} & & \downarrow{\scriptstyle (\alpha^\dagger * \zeta_{2*})^{-1}_X} \\
\Gamma(S',\zeta'_{2*}\zeta'^*_2 f^*\zeta_{2*}X) & \xrightarrow{\omega^{\zeta'_2}_{\zeta'^*_2 f^*\zeta_{2*}X}} & \zeta'^*_2 f^*\zeta_{2*}X & \xrightarrow{(\beta^\dagger_2 * \zeta_{2*})_X} & \zeta^*_2\zeta_{2*}X.
\end{array}$$

However, the commutativity of the top two squares and of the bottom square on the left is clear; the commutativity of the bottom square on the right follows from remark 1.1.17(iv,v) and (4.7.23). Lastly, the commutativity of the left square on the central row follows immediately from the uniqueness properties of the isomorphisms $\omega^{\zeta'_1}$ and $\omega^{\zeta'_2}$ (proposition 4.4.12(ii)), whence the contention.

(ii): Consider a diagram $\mathscr{D}_{g,g'}$ as in the foregoing, and an object (ζ,ζ',β) of $\mathsf{Morph}(\pi)^{-1}(f_2)$. We have to check the compatibility condition :

$$b_{f_2,(\zeta,\zeta',\beta)} \circ \tau^{g,g'}_\zeta = \tau^X_{g,\zeta} \circ b_{f_1,(g(\zeta),g(\zeta'),g(\beta))}.$$

To this aim, notice that $\omega^{g'(\zeta')} = \omega^{\zeta'} \odot (\omega^{g'} * \zeta'_*)$, due to proposition 4.4.12(ii); then it suffices to show the commutativity of the diagram :

$$\begin{array}{ccccc}
\Gamma(S'_1, f^*_1 g_*\zeta_*X) & \xrightarrow{\Gamma(S'_1,\Upsilon(\mathscr{D}_{g,g'})*\zeta_*)_X} & \Gamma(S'_1, g'_*f^*_2\zeta_*X) & \xrightarrow{\omega^{g'}_{f^*_2\zeta_*X}} & \Gamma(S'_2, f^*_2\zeta_*X) \\
\downarrow{\scriptstyle \Gamma(S'_1,(\eta^{g'(\zeta')}*f^*_1g_*\zeta_*)_X)} & & & & \\
\Gamma(S'_1, g'(\zeta')_*g'(\zeta')^*f^*_1g_*\zeta_*X) & & & & \downarrow{\scriptstyle \Gamma(S'_2,(\eta^{\zeta'}*f^*_2\zeta_*)_X)} \\
\downarrow{\scriptstyle \omega^{g'}_{\zeta'_*\zeta'^*g'^*f^*_1g_*\zeta_*X}} & & & & \\
\Gamma(S'_2, \zeta'_*g'(\zeta')^*f^*_1g_*\zeta_*X) & & \xrightarrow{\Gamma(S'_2,(\zeta'^*f^*_2*\varepsilon^g*\zeta_*)_X)} & & \Gamma(S'_2, \zeta'_*\zeta'^*f^*_2\zeta_*X) \\
\downarrow{\scriptstyle \omega^{\zeta'}_{\zeta'^*g'^*f^*_1g_*\zeta_*X}} & & & & \downarrow{\scriptstyle \omega^{\zeta'}_{\zeta'^*f^*_2\zeta_*X}} \\
\zeta'^*g'^*f^*_1g_*\zeta_*X & = & \zeta'^*f^*_2g^*g_*\zeta_*X & \xrightarrow{(\zeta'^*f^*_2*\varepsilon^g*\zeta_*)_X} & \zeta'^*f^*_2\zeta_*X \\
\downarrow{\scriptstyle (g(\beta)^\dagger*g(\zeta)_*)_X} & & & & \downarrow{\scriptstyle (\beta^\dagger*\zeta_*)_X} \\
g(\zeta)^*g(\zeta)_*X & & \xrightarrow{(\zeta^**\varepsilon^g*\zeta_*)_X} & & \zeta^*\zeta_*X.
\end{array}$$

However, the commutativity of the bottom square follows from (1.1.11) and remark 1.1.17(vi), and that of the central square is clear, by the naturality of $\omega^{\zeta'}$. In order to show the commutativity of the top square, set $A := \zeta_*X$, and recall that

$$\Upsilon(\mathscr{D}_{g,g'}) = (g'_*f^*_2 * \varepsilon^g) \odot (\eta^{g'} * f^*_1 g_*) \qquad \text{and} \qquad \eta^{g'(\zeta')} = (g'_* * \eta^{\zeta'} * g'^*) \odot \eta^{g'}$$

(proposition 2.3.10). We are then reduced to checking the commutativity of the diagram :

$$\begin{array}{ccccc}
 & & \Gamma(S'_1, g'_*g'^*f_1^*g_*A) & \xrightarrow{\Gamma(S'_1,(g'_*f_2^* * \varepsilon^g)_A)} & \Gamma(S'_1, g'_*f_2^*A) \\
 & \swarrow^{\Gamma(S'_1,(g'_* * \eta^{\zeta'} * g'^* f_1^*)_A)} & \downarrow{\scriptstyle \omega^{g'}_{g'^*f_1^*g_*A}} & & \downarrow{\scriptstyle \omega^{g'}_{f_2^*A}} \\
\Gamma(S'_1, g'(\zeta')_*g'(\zeta')^*f_1^*g_*A) & & \Gamma(S'_2, g'^*f_1^*g_*A) & \xrightarrow{\Gamma(S'_2,(f_2^* * \varepsilon^g *)_A)} & \Gamma(S'_2, f_2^*A) \\
 & \searrow_{\omega^{g'}_{\zeta'_*g'(\zeta')^*f_1^*g_*A}} & \downarrow{\scriptstyle \Gamma(S'_2,(\eta^{\zeta'} * g'^*f_1^*g_*)_A)} & & \downarrow{\scriptstyle \Gamma(S'_2,(\eta^{\zeta'} * f_2^*)_A)} \\
 & & \Gamma(S'_2, \zeta'_*g'(\zeta')^*f_1^*g_*A) & \xrightarrow{\Gamma(S'_2,(\zeta'_*f_2^* * \varepsilon^g)_A)} & \Gamma(S'_2, \zeta'_*\zeta'^*f_2^*A).
\end{array}$$

However, it is easily seen that the two square subdiagrams and the triangular subdiagram commute (details left to the reader), whence the contention. ◊

From claim 4.7.24(ii) and remark 3.1.23(ii) we get the natural transformation

$$\mathscr{F}ib(b_\bullet) : \mathscr{F}ib(\Sigma^X_\bullet) \circ p \Rightarrow \Xi^X \circ \mathsf{s}' \circ p.$$

Lastly, let $q : \mathscr{L}(\xi) \to \mathscr{F}ib(\mathrm{Pt}^\times)$ be the natural projection, $\mathsf{s}_{\mathscr{L}(\xi)} : \mathsf{Morph}(\mathscr{L}(\xi)) \to \mathscr{L}(\xi)$ the source functor, and set $\mathsf{G} := \mathscr{F}ib(\Sigma^X_\bullet) \circ p \circ \mathsf{Morph}(q)$; we deduce the natural transformation

$$\beta : \mathscr{F}ib(b_\bullet) * \mathsf{Morph}(q) : \mathsf{G} \Rightarrow \Xi^{X,\xi} \circ \mathsf{s}_{\mathscr{L}(\xi)}.$$

By theorem 1.3.4, the latter induces a natural transformation

$$\int^{\mathsf{s}_{\mathscr{L}(\xi)}} \mathsf{G} \Rightarrow \Xi^{X,\xi}. \tag{4.7.25}$$

Notice that, under the natural identification $\mathsf{Morph}(\mathscr{L}(\xi)) \xrightarrow{\sim} \mathsf{Morph}(\mathbf{Lift}(\xi))^o$, the functor $\mathsf{s}_{\mathscr{L}(\xi)}$ correspond to the functor $\mathsf{t}^o_{\mathbf{Lift}(\xi)}$, where $\mathsf{t}_{\mathbf{Lift}(\xi)} : \mathsf{Morph}(\mathbf{Lift}(\xi)) \to \mathbf{Lift}(\xi)$ is the target functor. Moreover, $\mathsf{t}_{\mathbf{Lift}(\xi)}$ is a fibration, by virtue of remark 4.7.13(ii) and example 3.1.3(iii). Furthermore, for every $(U, \xi_U, \omega_U) \in \mathrm{Ob}(\mathbf{Lift}(\xi))$, the fibre category $\mathsf{t}^{-1}(U, \xi_U, \omega_U)$ is

$$\mathbf{Lift}(\xi)/(U, \xi_U, \omega_U) \xrightarrow{\sim} \mathbf{Lift}(\xi_U).$$

Hence, the restriction of β to the fibre category $\mathsf{t}^{-1}(U, \xi_U, \omega_U)$ is a cocone

$$\mathbf{Lift}(\xi_U)^o \underset{c_{\xi_U^*\xi_{U*}X}}{\overset{\mathsf{G}}{\underset{\Downarrow}{\rightrightarrows}}} \mathbf{Set}. \tag{4.7.26}$$

By a direct inspection of the construction, we see that (4.7.26) is the cocone $\tau_{\xi,\xi_{U*}X} * N^o_{\xi_U}$, where $N_{\xi_U} : \mathbf{Lift}(\xi_U) \xrightarrow{\sim} \mathbf{Nbd}(\xi_U)$ is the equivalence (4.7.18), and $\tau_{\xi,\xi_{U*}X}$ is the universal cocone of remark 4.7.13(iv). Hence, (4.7.26) is a universal cocone, for every $(U, \xi_U, \omega_U) \in \mathrm{Ob}(\mathbf{Lift}(\xi))$; taking into account example 3.1.14(iv), we deduce that (4.7.25) is an isomorphism of functors, and we get a natural isomorphism :

$$\operatorname*{colim}_{\mathsf{Morph}(\mathbf{Lift}(\xi))^o} \mathsf{G} \xrightarrow{\sim} \operatorname*{colim}_{\mathbf{Lift}(\xi)^o} \Xi^{X,\xi}.$$

Now, notice that the functor

$$\mathbf{Lift}(\xi)^o \to \mathsf{Morph}(\mathbf{Lift}(\xi))^o \qquad (U, \xi_U, \omega_U) \mapsto \mathbf{1}_{(U,\xi_U,\omega_U)} \tag{4.7.27}$$

is cofinal. A simple inspection reveals that the composition of G with (4.7.27) is the constant functor with value X, whence the contention. □

4.7.28. Suppose now that $T = C^\sim$ for some small site $C := (\mathscr{C}, J)$. For a point ξ of T, we may define another category $\mathbf{Nbd}(\xi, C)$, whose objects are the pairs (U, a) where $U \in \mathrm{Ob}(\mathscr{C})$ and $a \in (h^a_U)_\xi$; the morphisms $(U, a) \to (U', a')$ are the morphisms $f : U \to U'$ in $\mathscr{C}$ such that $(h^a_f)_\xi(a) = a'$. (Notation of remark 4.1.19(iii).) The rule $(U, a) \mapsto (h^a_U, a)$ defines a functor

$$\mathbf{Nbd}(\xi, C) \to \mathbf{Nbd}(\xi). \tag{4.7.29}$$

Proposition 4.7.30. *In the situation of* (4.7.28)*, we have :*

(i) *The category* $\mathbf{Nbd}(\xi, C)$ *is cofiltered.*
(ii) *The functor* (4.7.29) *is cofinal.*

Proof. To begin with, since ξ^* commutes with all colimits, remark 4.1.19(iii) implies easily that, for every object (U, a) of $\mathbf{Nbd}(\xi)$ there exist an object (V, b) of $\mathbf{Nbd}(\xi, C)$ and a morphism $(h^a_V, b) \to (U, a)$ in $\mathbf{Nbd}(\xi)$. Hence, it suffices to show (i).

Thus, let (U_1, a_1) and (U_2, a_2) be two objects of $\mathbf{Nbd}(\xi, C)$; since $\mathbf{Nbd}(\xi)$ is cofiltered, we may find an object (F, b) of $\mathbf{Nbd}(\xi)$ and morphisms $\varphi_i : (F, b) \to (h^a_{U_i}, a_i)$ (for $i = 1, 2$). By the foregoing, we may also assume that $(F, b) = (h^a_V, b)$ for some object (V, b) of $\mathbf{Nbd}(\xi, C)$, in which case $\varphi_i \in h^a_{U_i}(V)$ for $i = 1, 2$. By remark 4.1.19(iv), we may find a sieve $\mathscr{S}$ covering V such that $\varphi_i \in \mathrm{Hom}_{\mathscr{C}^\wedge}(h_\mathscr{S}, h^{\mathrm{sep}}_{U_i})$ for both $i = 1, 2$.

On the other hand, (3.5.3) and proposition 4.1.26 yield a natural isomorphism :

$$\operatorname*{colim}_{\mathscr{S}} h^a \circ \mathsf{s} \xrightarrow{\sim} h^a_V.$$

Therefore, since ξ^* commutes with all colimits, we may find $(f : S \to V) \in \mathrm{Ob}(\mathscr{S})$ and $c \in (h^a_S)_\xi$ such that $h^a_f : (h^a_S, c) \to (h^a_V, b)$ is a morphism of neighborhoods of ξ.

For $i = 1, 2$, denote by $\overline{\varphi}_{S,i} \in h^{\mathrm{sep}}_U(S)$ the image of φ_i (under the map induced by the natural morphism $h_S \to h_\mathscr{S}$ coming from (3.5.3)). Pick any $\varphi_{S,i} \in \mathrm{Hom}_\mathscr{C}(S, V)$ in the preimage of $\overline{\varphi}_{S,i}$; then $\varphi_{S,i}$ defines a morphism $(S, c) \to (U_i, a_i)$ in $\mathbf{Nbd}(\xi, C)$.

Next, suppose that $\varphi_1, \varphi_2 : (U, a) \to (U', a')$ are two morphisms in $\mathbf{Nbd}(\xi, C)$; arguing as in the foregoing, we may find an object (V, b) of $\mathbf{Nbd}(\xi, C)$, and $\psi \in h^{\mathrm{sep}}_U(V) = \mathrm{Hom}_{\mathscr{C}^\wedge}(h^{\mathrm{sep}}_V, h^{\mathrm{sep}}_U)$ whose image in $h^a_U(V)$ yields a morphism $\psi^a : (h^a_V, b) \to (h^a_U, a)$ in $\mathbf{Nbd}(\xi)$, and such that $\varphi^{\mathrm{sep}}_1 \circ \psi = \varphi^{\mathrm{sep}}_2 \circ \psi$ in $h^{\mathrm{sep}}_{U'}(V)$. We may then find a covering subobject $i : R \to h_V$, such that $\varphi_1 \circ (\psi \circ i) = \varphi_2 \circ (\psi \circ i)$ in $\mathrm{Hom}_{\mathscr{C}^\wedge}(R, h'_U)$. Again, by combining (3.5.3) and proposition 4.1.26, we deduce that there exists a morphism $\beta : (V', b') \to (V, b)$ in $\mathbf{Nbd}(\xi, C)$, such that $\varphi_1 \circ (\psi \circ \beta) = \varphi_2 \circ (\psi \circ \beta)$. This completes the proof of (i). □

As a corollary of proposition 4.7.30 and of remark (4.7.13)(iv), we deduce, for every sheaf F on C, a natural isomorphism :

$$\operatorname*{colim}_{\mathbf{Nbd}(\xi,C)^o} F \circ \iota^o_C \xrightarrow{\sim} F_\xi$$

where $\iota_C : \mathbf{Nbd}(\xi, C) \to \mathscr{C}$ is the functor such that $(U, a) \mapsto U$ for every object (U, a).

4.8. **Algebra on a topos.** Let T be any topos, and endow T with the structure of tensor category as explained in example 3.7.10 (so, the tensor functor is given by fixed choices of products for every pairs of objects of T, and any final object 1_T can be taken for unit object of $(T, \otimes)$). We notice that $(T, \otimes)$ admits an internal Hom functor (see remark 3.7.12(ii)). Indeed, let X and X' be any two objects of T. It is easily seen that the presheaf on T :

$$U \mapsto \mathrm{Hom}_{T/U}(X'_{|U}, X_{|U}) = \mathrm{Hom}_T(X' \times U, X)$$

is actually a sheaf on (T, C_T) (notation of example 4.7.8(i)), so it is an object of T, denoted :

$$\mathscr{H}\!om_T(X', X).$$

The functor :

$$T \to T \quad : \quad X \mapsto \mathscr{H}\!om_T(X', X)$$

is right adjoint to the functor $T \to T : Y \mapsto Y \times X'$, so it is an internal Hom functor for X'.

If $f : T \to S$ is a morphism of topoi, and $Y \in \mathrm{Ob}(S)$, we have a natural isomorphism in S :

$$\mathscr{H}om_S(Y, f_*X) \xrightarrow{\sim} f_*\mathscr{H}om_T(f^*Y, X) \tag{4.8.1}$$

which, on every $U \in \mathrm{Ob}(S)$, induces the natural bijection :

$$\mathrm{Hom}_S(Y \times U, f_*X) \xrightarrow{\sim} \mathrm{Hom}_T(f^*Y \times f^*U, X)$$

given by the adjunction (f^*, f_*). By general nonsense, from (4.8.1) we derive a natural morphism in S :

$$f_*\mathscr{H}om_T(X', X) \to \mathscr{H}om_S(f_*X', f_*X)$$

and in T :

$$f^*\mathscr{H}om_S(Y', Y) \xrightarrow{\vartheta_f} \mathscr{H}om_T(f^*Y', f^*Y) \qquad \text{for any } Y, Y' \in \mathrm{Ob}(S).$$

Moreover, if $g : U \to T$ is another morphism of topoi, the diagram :

$$\begin{array}{ccc} g^*f^*\mathscr{H}om_S(Y', Y) & \xrightarrow{g^*\vartheta_f} & g^*\mathscr{H}om_T(f^*Y', f^*Y) \\ & {\scriptstyle\vartheta_{f\circ g}}\searrow \quad \swarrow{\scriptstyle\vartheta_g} & \\ & \mathscr{H}om_U(g^*f^*Y', g^*f^*Y) & \end{array} \tag{4.8.2}$$

commutes, up to a natural isomorphism.

4.8.3. Let A be any object of T, and (X, μ_X) a left A-module for the tensor category structure on T as in (4.8); for every object U of T we obtain a left $A_{|U}$-module (on the topos T/U : see example 4.7.8(i)), by the rule :

$$(X, \mu_X)_{|U} := (X_{|U}, \mu_X \times \mathbf{1}_U).$$

If $(X', \mu_{X'})$ is another left A-module, it is easily seen that the presheaf on T :

$$U \mapsto \mathrm{Hom}_{A_{|U}\text{-}\mathbf{Mod}_l}((X, \mu_X)_{|U}, (X', \mu_{X'})_{|U})$$

is actually a sheaf for the canonical topology, so it is an object of T, denoted :

$$\mathscr{H}om_{A_l}((X, \mu_X), (X', \mu_{X'}))$$

(or just $\mathscr{H}om_{A_l}(X, X')$, if the notation is not ambiguous). The same considerations can be repeated for the sets of morphisms of right B-modules, and of (A, B)-bimodules, so one gets objects $\mathscr{H}om_{B_r}(X, X')$ and $\mathscr{H}om_{(A,B)}(X, X')$. By a simple inspection, we see that these objects are naturally isomorphic to the objects denoted in the same way in (3.7.14), so the notation is not in conflict with *loc.cit.*; it also follows that $\mathscr{H}om_{A_l}(X, X')$ is the equalizer of two morphisms in T :

$$\mathscr{H}om_T(X, X') \rightrightarrows \mathscr{H}om_T(A \times X, X') .$$

In the same vein, let $A, B, C \in \mathrm{Ob}(T)$ be any three objects, S an (A, B)-bimodule, S' a (C, B)-bimodule, and S'' a (C, A)-bimodule. Then the (C, B)-bimodule $\mathscr{H}om_{B_r}(S, S')$ and the (C, B)-bimodule $S'' \otimes_A S$ (see (3.7.17)) are the sheaves on (T, C_T) associated with the presheaves given by the rules : $U \mapsto \mathrm{Hom}_{B_{|U,r}}(S_{|U}, S'_{|U})$, and respectively : $U \mapsto S'(U) \otimes_{M(U)} S(U)$ for every object U of T. Furthermore, the general theory of monoids, their modules and their tensor products, developed in section 3.7 is available in the present situation, so we have a well defined notion of T-monoid (see example 3.7.21(ii) and remark 3.7.24). Via the equivalence of theorem (4.4.2), a T-monoid $\underline{M}$ is also the same as a sheaf of monoids M on the site (T, C_T), and a left (resp. right, resp. bi-) $\underline{M}$-module is the same as the datum of a sheaf S in (T, C_T), such that $S(U)$ is a left (resp. right, resp. bi-) $M(U)$-module, for every object U of T.

4.8.4. Let $f : T_1 \to T_2$ be a morphism of topoi, A_1 an object of T_1, and (X, μ_X) a left A_1-module. Since f_* is left exact, we have a natural isomorphism : $f_*(A_1 \times X) \stackrel{\sim}{\to} f_*A_1 \times f_*X$, so we obtain a left f_*A_1-module :

$$f_*(X, \mu_X) := (f_*X, f_*\mu_X)$$

which we denote just f_*X, unless the notation is ambiguous. Likewise, since f^* is left exact, from any object A_2 of T_2, and any left A_2-module (Y, μ_Y), we obtain a left f^*A_2-module :

$$f^*(Y, \mu_Y) := (f^*Y, f^*\mu_Y).$$

The same considerations apply of course, also to right modules and to bimodules. Furthermore, let A_i, B_i, C_i be three objects of T_i (for $i = 1, 2$); since f_* is left exact, for any (A_1, B_1)-bimodule X and any (C_1, A_1)-bimodule X' we have a natural morphism of (f_*C_1, f_*B_1)-bimodules :

$$f_*X' \otimes_{f_*A_1} f_*X \to f_*(X' \otimes_{A_1} X)$$

and since f^* is exact, for any (A_2, B_2)-bimodule Y and any (C_2, A_2)-bimodule Y' we have a natural isomorphism :

$$f^*Y' \otimes_{f^*A_2} f^*Y \stackrel{\sim}{\to} f^*(Y' \otimes_{A_2} Y)$$

of (f^*C_2, f^*B_2)-bimodules.

4.8.5. The constructions of the previous paragraphs also apply to presheaves on T : this can be seen, *e.g.* as follows. Pick a universe V such that T is V-small; then $T^\wedge_\mathsf{V}$ is a V-topos. Hence, if A is any V-presheaf on T, and $X, X' \in \mathrm{Ob}(T^\wedge_\mathsf{V})$ two left A-modules, we may construct $\mathscr{H}om_{A_l}(X, X')$ as an object in $T^\wedge_\mathsf{V}$. Now, if A, X, X' lie in the full subcategory $T^\wedge_\mathsf{U}$ of $T^\wedge_\mathsf{V}$, it is easily seen that also $\mathscr{H}om_{A_l}(X, X')$ lies in $T^\wedge_\mathsf{U}$. Likewise, if A, B, C are two U-presheaves on T, we may define $X' \otimes_A X$ in $T^\wedge_\mathsf{U}$, for any (A, B)-bimodule X and (C, A)-bimodule X', and this tensor product will still be left adjoint to the $\mathscr{H}om$-functor for presheaves on T.

Moreover, we have a natural morphism of topoi $i_\mathsf{V} : (T, C_T)^\sim_\mathsf{V} \to T^\wedge_\mathsf{V}$, given by the forgetful functor and its left adjoint i^*_V, which is given by the rule : $F \mapsto F^a$: see example 4.4.10(i). The restriction of i^*_V to the full subcategory $T^\wedge_\mathsf{U}$ is isomorphic to a functor that factors through the Yoneda embedding $T \to (T, C_T)^\sim_\mathsf{V}$, therefore the discussion of (4.8.4) specializes to show that, for every U-presheaf A on T, and every A-module $X \in \mathrm{Ob}(T^\wedge_\mathsf{U})$, the object $X^a \in \mathrm{Ob}(T)$ is naturally an A^a-module. Also, for any (A, B)-bimodule X and (C, A)-bimodule X', such that A, B, C, X, X' are U-small, we have a natural isomorphism of (C^a, B^a)-bimodules :

$$X'^a \otimes_{A^a} X^a \stackrel{\sim}{\to} (X' \otimes_A X)^a.$$

The following definition gathers some further notions – specific to monoids over a topos – which shall be used in this work.

Definition 4.8.6. Let T be a topos, $\underline{M}$ a T-monoid, S a left (resp. right, resp. bi-) $\underline{M}$-module.

(i) S is said to be *of finite type*, if there exist a covering family $(U_\lambda \to 1_T \mid \lambda \in \Lambda)$ of the final object of T, and for every $\lambda \in \Lambda$ an integer $n_\lambda \in \mathbb{N}$ and an epimorphism of left (resp. right, resp. bi-) $\underline{M}_{|U_\lambda}$-modules : $\underline{M}^{\oplus n_\lambda}_{|U_\lambda} \to S_{|U_\lambda}$.

(ii) S is *finitely presented*, if there exist a covering family $(U_\lambda \to 1_T \mid \lambda \in \Lambda)$ of the final object of T, and for every $\lambda \in \Lambda$ integers $m_\lambda, n_\lambda \in \mathbb{N}$ and morphisms f_λ, g_λ : $\underline{M}^{\oplus m_\lambda}_{|U_\lambda} \to \underline{M}^{\oplus n_\lambda}_{|U_\lambda}$ whose coequalizer – in the category of left (resp. right, resp. bi-) $\underline{M}_{|U_\lambda}$-modules – is isomorphic to $S_{|U_\lambda}$.

(iii) S is said to be *coherent*, if it is of finite type, and for every object U in T, every submodule of finite type of $S_{|U}$ is finitely presented.

(iv) S is said to be *invertible*, if there exist a covering family $(U_\lambda \to 1_T \mid \lambda \in \Lambda)$, and for every $\lambda \in \Lambda$, an isomorphism $\underline{M}_{|U_\lambda} \stackrel{\sim}{\to} S_{|U_\lambda}$ of left (resp. right, resp. bi-) $\underline{M}_{|U}$-modules. (Thus, every invertible module is finitely presented.)

(v) An ideal $I \subset \underline{M}$ is said to be *invertible*, (resp. *of finite type*, resp. *finitely presented*, resp. *coherent*) if it is such, when regarded as an $\underline{M}$-bimodule.

Example 4.8.7. (i) Take again $T = \mathbf{Set}$. Then an M-module S is of finite type if and only if there exists a finite subset $\Sigma \subset S$, such that $S = M \cdot \Sigma$, with obvious notation. In this case, we say that Σ is a *finite system of generators* of S (and we also say that S is *finitely generated*; likewise, an ideal of finite type is also called finitely generated). We say that S is *cyclic*, if $S = M \cdot s$ for some $s \in S$.

(ii) If S and S' are two M-modules, the coproduct $S \oplus S'$ is the disjoint union of S and S', with scalar multiplication given by the disjoint union of the laws μ_S and $\mu_{S'}$. The product $S \times S'$ is the cartesian product of the underlying sets, with scalar multiplication given by the rule : $x \cdot (s, s') := (x \cdot s, x \cdot s')$ for every $x \in M$, $s, \in S$, $s' \in S'$.

For future use, let us also make the :

Definition 4.8.8. Let T be any topos, $\underline{P}$ a T-monoid, and $(N, +, 0)$ any monoid.

(i) We say that $\underline{P}$ is N-*graded*, if it admits a morphism of monoids $\pi : \underline{P} \to N_T$, where N_T is the constant sheaf of monoids arising from N (the coproduct of copies of the final object 1_T indexed by N). For every $n \in N$ we let $\underline{P}_n := \pi^{-1}(n_T)$, the preimage of the global section corresponding to n. Then

$$\underline{P} = \coprod_{n \in N} \underline{P}_n$$

the coproduct of the objects $\underline{P}_n$, and the multiplication law of $\underline{P}$ restricts to a map $\underline{P}_n \times \underline{P}_m \to \underline{P}_{n+m}$, for every $n, m \in N$. Especially, each $\underline{P}_n$ is a $\underline{P}_0$-module, and $\underline{P}$ is also the direct sum of the $\underline{P}_n$, in the category of $\underline{P}_0$-modules. The morphism π is called the *grading* of $\underline{P}$.

(ii) In the situation of (i), let S be a left (resp. right, resp. bi-) $\underline{P}$-module. We say that S is N-*graded*, if it admits a morphism of $\underline{P}$-modules $\pi_S : S \to N_T$, where N_T is regarded as a $\underline{P}$-bimodule via the grading π of $\underline{P}$. Then S is the coproduct $S = \coprod_{n \in N} S_n$, where $S_n := \pi_S^{-1}(n_T)$, and the scalar multiplication of S restricts to morphisms $\underline{P}_n \times S_m \to S_{n+m}$, for every $n, m \in N$. The morphism π_S is called the *grading* of S.

(iii) A morphism $\underline{P} \to \underline{Q}$ of N-graded T-monoids is a morphism of monoids that respects the gradings, with obvious meaning. Likewise one defines morphisms of N-graded $\underline{P}$-modules.

Example 4.8.9. Take $T = \mathbf{Set}$, and let M be any commutative monoid. Then we claim that the only invertible object in the tensor category M-$\mathbf{Mod}_l$ is M; *i.e.* if S and S' are any two (M, M)-bimodules, then $S \otimes_M S' \simeq M$ if and only if S and S' are both isomorphic to M.

Indeed, let $\varphi : S \otimes_M S' \xrightarrow{\sim} M$ be an isomorphism, and choose $s_0 \in S$, $s'_0 \in S'$ such that $\varphi(s_0 \otimes s'_0) = 1$. Consider the morphisms of left M-modules :

$$M \xrightarrow{\alpha} S \xrightarrow{\beta} M \qquad M \xrightarrow{\alpha'} S' \xrightarrow{\beta'} M$$

such that :

$$\alpha(m) = m \cdot s_0 \quad \beta(s) = \varphi(s \otimes s'_0) \quad \alpha'(m) = m \cdot s'_0 \quad \beta'(s') = \varphi(s_0 \otimes s')$$

for every $m \in M$, $s \in S$, $s' \in S'$; we notice that $\beta \circ \alpha = \mathbf{1}_M = \beta' \circ \alpha'$. There follows natural morphisms :

$$S' \xrightarrow{\gamma} M \otimes_M S' \xrightarrow{\alpha \otimes_M S'} S \otimes_M S' \xrightarrow{\beta \otimes_M S'} M \otimes_M S' \xrightarrow{\gamma^{-1}} S'$$

whose composition is the identity $\mathbf{1}_{S'}$. However, it is easily seen that $\varphi \circ (\alpha \otimes_M S') \circ \gamma = \beta'$ and $\gamma^{-1} \circ (\beta \otimes_M S') \circ \varphi^{-1} = \alpha'$, thus $\alpha' \circ \beta' = \mathbf{1}_{S'}$, hence both α' and β' are isomorphisms, and the same holds for α and β.

Example 4.8.10. Let $\underline{M}$ be a T-monoid, and $\mathscr{L}$ a $\underline{M}$-bimodule. For every $n \in \mathbb{N}$, let $\mathscr{L}^{\otimes n} := \mathscr{L} \otimes_{\underline{M}} \cdots \otimes_{\underline{M}} \mathscr{L}$, the n-fold tensor power of $\mathscr{L}$. The $\mathbb{N}$-graded $\underline{M}$-bimodule

$$\mathrm{Tens}^{\bullet}_{\underline{M}}\mathscr{L} := \coprod_{n \in \mathbb{N}} \mathscr{L}^{\otimes n}$$

is naturally a $\mathbb{N}$-graded T-monoid, with composition law induced by the natural morphisms $\mathscr{L}^{\otimes n} \otimes_{\underline{M}} \mathscr{L}^{\otimes m} \xrightarrow{\sim} \mathscr{L}^{\otimes n+m}$, for every $n, m \in \mathbb{N}$. (Here we set $\mathscr{L}^{\otimes 0} := \underline{M}$.) If $\mathscr{L}$ is invertible, $\mathrm{Tens}^{\bullet}_{\underline{M}}\mathscr{L}$ is a commutative $\mathbb{N}$-graded T-monoid, which we also denote $\mathrm{Sym}^{\bullet}_{\underline{M}}\mathscr{L}$.

Remark 4.8.11. (i) Let $f : T \to S$ be a morphism of topoi, $\underline{M} := (M, \mu_M)$ a T-semigroup, and $\underline{N} := (N, \mu_N)$ a S-semigroup. Then clearly $f_*\underline{M} := (f_*M, f_*\mu_M)$ is a S-semigroup, and $f^*\underline{N} := (f^*N, f^*\mu_N)$ is a T-semigroup.

(ii) Furthermore, if $1_M : 1_T \to M$ (resp. $1_N : 1_S \to N$) is a unit for M (resp. for N), then notice that $f_*1_T = 1_S$ (resp. $f^*1_S = 1_T$), since the final object is the empty product; it follows that f_*1_M (resp. f^*1_S) is a unit for $f_*\underline{M}$ (resp. for $f^*\underline{N}$).

(iii) Obviously, if $\underline{M}$ (resp. $\underline{N}$) is commutative, the same holds for $f_*\underline{M}$ (resp. $f^*\underline{N}$).

(iv) If X is a left $\underline{M}$-module, then f_*X is a left $f_*\underline{M}$-module, and if Y is a left $\underline{N}$-module, then f^*Y is a left $f^*\underline{N}$-module. The same holds for right modules and bimodules.

(v) Moreover, let $\varepsilon_M : f^*f_*\underline{M} \to \underline{M}$ (resp. $\eta_N : \underline{N} \to f_*f^*\underline{N}$) be the counit (resp. unit) of adjunction. Then the counit (resp. unit) :

$$\varepsilon_X : f^*f_*X \to X_{(\varepsilon_M)} \qquad (\text{resp. } \eta_Y : Y \to f_*f^*Y_{(\eta_N)})$$

is a morphism of $f^*f_*\underline{M}$-modules (resp. of $\underline{N}$-modules) (notation of (3.7.26)). (Details left to the reader.)

(vi) Let $\varphi : f^*\underline{N} \to \underline{M}$ be a morphism of T-monoids. Then the functor

$$\underline{N}\text{-}\mathbf{Mod}_l \to \underline{M}\text{-}\mathbf{Mod}_l \quad : \quad Y \mapsto M \otimes_{f^*N} f^*Y$$

is left adjoint to the functor :

$$\underline{M}\text{-}\mathbf{Mod}_l \to \underline{N}\text{-}\mathbf{Mod}_l \quad : \quad X \mapsto f_*X_{(\eta_N)}.$$

(And likewise for right modules and bimodules : details left to the reader.)

(vii) The considerations of (4.8.5) also apply to monoids : we get that, for any presheaf of monoids $\underline{M} := (M, \mu_M, 1_M)$ on T, the datum $\underline{M}^a := (M^a, \mu^a_M, 1^a_M)$ is a T-monoid, and we have a well defined functor :

$$\underline{M}\text{-}\mathbf{Mod}_l \to \underline{M}^a\text{-}\mathbf{Mod}_l \qquad X \mapsto X^a.$$

(And as usual, the same applies to right modules and bimodules.)

4.8.12. Let T be a topos, U any object of T, and $\underline{M}$ a T-monoid. As a special case of remark 4.8.11(i), we have the T/U-monoid $j^*_U\underline{M} = \underline{M}_{|U}$, and if we take $\varphi := \mathbf{1}_{j^*_U\underline{M}}$ in remark 4.8.11(vi), we deduce that the functor

$$j^*_U : \underline{M}\text{-}\mathbf{Mod}_l \to \underline{M}_{|U}\text{-}\mathbf{Mod}_l \qquad Y \mapsto Y_{|U}$$

admits the right adjoint j_{U*}. Now, suppose that $X \to U$ is any left $\underline{M}_{|U}$-module. The scalar multiplication of X is a U-morphism $\mu_X : M \times X \to X$ and $j_{U!}\mu_X$ is the same morphism, seen as a morphism in T (notation of example 4.7.8(i)). In other words, $j_{U!}$ induces a faithful functor on left modules, also denoted :

$$j_{U!} : \underline{M}_{|U}\text{-}\mathbf{Mod}_l \to \underline{M}\text{-}\mathbf{Mod}_l.$$

It is easily seen that this functor is left adjoint to the foregoing functor j^*_U. Especially, this functor is right exact; it is not generally left exact, since it does not preserve the final object (unless $U = 1_T$). However, it does commute with fibre products, and therefore transforms monomorphisms into monomorphisms. All this holds also for right modules and bimodules.

4.8.13. Let T be any category as in example 3.7.21(i), denote by 1_T a final object of T, and by $\underline{M}$ any T-monoid. A *pointed left* $\underline{M}$*-module* is a datum

$$(S, 0_S)$$

consisting of a left $\underline{M}$-module S and a morphism of $\underline{M}$-modules $0_S : 1_T \to S$, where 0 is the final object of M-$\mathbf{Mod}_l$. Often we shall write S instead of $(S, 0_S)$, unless this may give rise to ambiguities. As usual, a morphism $\varphi : S \to T$ of pointed modules is a morphism of M-modules, such that $0_T = \varphi \circ 0_S$. In other words, the resulting category is just $0/M$-$\mathbf{Mod}_l$, and shall be denoted $\underline{M}$-$\mathbf{Mod}_{l\circ}$.

Likewise one may define the category $\underline{M}$-$\mathbf{Mod}_{r\circ}$ of right $\underline{M}$-modules, and $(\underline{M}, \underline{N})$-$\mathbf{Mod}_\circ$ of pointed bimodules, for given T-monoids $\underline{M}$ and $\underline{N}$.

Remark 4.8.14. Let T be a category as in remark 3.7.24.

(i) For reasons that will become readily apparent, for many purposes the categories of pointed modules are more useful than the non-pointed variant of (3.7.22). In any case, we have a faithful functor :

$$\underline{M}\text{-}\mathbf{Mod}_l \to \underline{M}\text{-}\mathbf{Mod}_{l\circ} \qquad S \mapsto S_\circ := (S \oplus 0, 0_S) \tag{4.8.15}$$

where $0_S : 0 \to S \oplus 0$ is the obvious inclusion map. Thus, we may – and often will, without further comment – regard any $\underline{M}$-module as a pointed module, in a natural way. (The same can of course be repeated for right modules and bimodules.)

(ii) In turn, when dealing with pointed M-modules, things often work out nicer if $\underline{M}$ itself is a *pointed* T*-monoid*. The latter is the datum $(\underline{M}, 0_M)$ of a T-monoid $\underline{M}$ and a morphism of $\underline{M}$-modules $0_M : 0 \to M$. A morphism of pointed T-monoids is of course just a morphism $f : \underline{M} \to \underline{M}'$ of T-monoids, such that $f \circ 0_M = 0_{M'}$. As customary, we shall often just write $\underline{M}$ instead of $(\underline{M}, 0_M)$, unless we wish to stress that $\underline{M}$ is pointed.

(iii) Let $(\underline{M}, 0_M)$ be a pointed T-monoid; a *pointed left* $(\underline{M}, 0_M)$*-module* is a pointed left $\underline{M}$-module S, such that $0 \cdot s = 0$ for every $s \in S$. A morphism of pointed left $(\underline{M}, 0_M)$-modules is just a morphism of pointed left $\underline{M}$-modules. As usual, these gadgets form a category $(\underline{M}, 0_M)$-$\mathbf{Mod}_{l\circ}$. Similarly we have the right and bi-module variant of this definition.

(iv) The forgetful functor from the category of pointed T-monoids to the category of T-monoids, admits a left adjoint :

$$\underline{M} \mapsto (\underline{M}_\circ, 0_{M_\circ}).$$

Namely, $M_\circ$ is the $\underline{M}$-module $M \oplus 0$, the zero map $0_{M_\circ} : 0 \to M \oplus 0$ is the obvious inclusion, and the scalar multiplication $M \times M_\circ \to M_\circ$ is extended to a multiplication law $\mu : M_\circ \times M_\circ \to M_\circ$ in the unique way for which $(\underline{M}_\circ, \mu, 0_{M_0})$ is a pointed monoid. The unit of adjunction $\underline{M} \to \underline{M}_\circ$ is the obvious inclusion map.

(v) If $\underline{M}$ is a (non-pointed) monoid, the restriction of scalars

$$(\underline{M}_\circ, 0_{M_\circ})\text{-}\mathbf{Mod}_{l\circ} \to \underline{M}\text{-}\mathbf{Mod}_{l\circ}$$

is an isomorphism of categories. Namely, any pointed left $\underline{M}$-module S is naturally a pointed left $\underline{M}_\circ$-module : the given scalar multiplication $M \times S \to S$ extends to a scalar multiplication $M_\circ \times S \to S$ whose restriction $0 \times S \to S$ factors through the zero section 0_S (and likewise for right modules and bimodules).

(vi) Let T be a topos. The notions introduced thus far for non-pointed T-monoids, also admit pointed variants. Thus, a pointed module $(S, 0_S)$ is said to be *of finite type* if the same holds for S, and S is *finitely presented* if, locally on T, it is the coequalizer of two morphisms between free $\underline{M}$-modules of finite type.

Example 4.8.16. Take $T := \mathbf{Set}$, and let M be any monoid; then a pointed left M-module is just a left M-module S endowed with a distinguished *zero element* $0 \in S$, such that $m \cdot 0 = 0$

for every $m \in M$. A morphism $\varphi : S \to S'$ of pointed left M-modules is just a morphism of left M-modules such that $\varphi(0) = 0$ (and similarly for right modules and bimodules.)

Likewise, a pointed monoid is endowed with a distinguished *zero element*, denoted 0 as usual, such that $0 \cdot x = 0$ for every $x \in M$.

Remark 4.8.17. Let T be a category as in remark 3.7.24, and $\underline{M}$ a T-monoid.

(i) Regardless of whether $\underline{M}$ is pointed or not, the category $\underline{M}\text{-}\mathbf{Mod}_{l\circ}$ is also complete and cocomplete; for instance, if $(S, 0_S)$ and $(S', 0_{S'})$ are two pointed modules, the coproduct $(S'', 0_{S''}) := (S, 0_S) \oplus (S', 0_{S'})$ is defined by the push-out (in the category $\underline{M}\text{-}\mathbf{Mod}_l$) of the cocartesian diagram :

$$\begin{array}{ccc} 0 \oplus 0 & \xrightarrow{0_S \oplus 0_{S'}} & S \oplus S' \\ \downarrow & & \downarrow \\ 0 & \xrightarrow{0_{S''}} & S''. \end{array}$$

Likewise, if $\varphi' : S' \to S$ and $\varphi'' : S'' \to S$ are two morphisms in $\underline{M}\text{-}\mathbf{Mod}_{l\circ}$, the fibre product $S' \times_S S''$ in the category $\underline{M}\text{-}\mathbf{Mod}_l$ is naturally pointed, and represents the fibre product in the category of pointed modules. All this holds also for right modules and bimodules.

(ii) The forgetful functor $\underline{M}\text{-}\mathbf{Mod}_{l\circ} \to T_\circ := 1_T/T$ to the category of pointed objects of T, commutes with all limits, since it is a right adjoint; it also commutes with all colimits. This forgetful functor admits a left adjoint, that assigns to any $\Sigma \in \mathrm{Ob}(T)$ the *free pointed* $\underline{M}$*-module* $\underline{M}^{(\Sigma)\circ}$. If $\underline{M}$ is pointed, the latter is defined as the push-out in the cocartesian diagram

$$\begin{array}{ccc} 1_T \times \Sigma & \longrightarrow & M \times \Sigma \\ \downarrow & & \downarrow \\ 1_T & \longrightarrow & \underline{M}^{(\Sigma)\circ} \end{array}$$

and if $\underline{M}$ is not pointed, one defines it via the equivalence of remark 4.8.14(v) : by a simple inspection we find that in this case $\underline{M}^{(\Sigma)\circ} = (\underline{M}^{(\Sigma)})_\circ$, where $\underline{M}^{(\Sigma)}$ is the free (unpointed) $\underline{M}$-module, as in remark 3.7.24(iii).

Notice as well that the forgetful functors $T_\circ \to T$ and $\underline{M}\text{-}\mathbf{Mod}_{l\circ} \to \underline{M}\text{-}\mathbf{Mod}_l$ both commute with all connected colimits, hence the same also holds for the forgetful functor $\underline{M}\text{-}\mathbf{Mod}_{l\circ} \to T$. (See definition 1.2.19(vii).) The same can be repeated for right modules and bimodules.

(iii) Moreover, if $\varphi : S \to S'$ is any morphism in $\underline{M}\text{-}\mathbf{Mod}_{l\circ}$, we may define $\mathrm{Ker}\,\varphi$ and $\mathrm{Coker}\,\varphi$ (in the category $\underline{M}\text{-}\mathbf{Mod}_{l\circ}$); namely, the kernel is the limit of the diagram $S \xrightarrow{\varphi} S' \leftarrow 0$ and the cokernel is the colimit of the diagram $0 \leftarrow S \xrightarrow{\varphi} S'$. Especially, if S is a submodule of S', we have a well defined quotient S'/S of pointed left $\underline{M}$-modules. Furthermore, we say that a sequence of morphisms of pointed left $\underline{M}$-modules :

$$0 \to S' \xrightarrow{\varphi} S \xrightarrow{\psi} S'' \to 0$$

is *right exact*, if ψ induces an isomorphism $\mathrm{Coker}\,\varphi \xrightarrow{\sim} S''$; we say that it is *left exact*, if φ induces an isomorphism $S' \xrightarrow{\sim} \mathrm{Ker}\,\psi$, and it is *short exact* if it is both left and right exact. (Again, all this can be repeated also for right modules and bimodules.)

Example 4.8.18. Take $T = \mathbf{Set}$, and let M be a pointed or not-pointed monoid. Then the argument from example 3.7.27 can be repeated for the free pointed M-modules : if Σ is any set, we have

$$M^{(\Sigma)\circ} \otimes_M \{1\} \xrightarrow{\sim} \{1\}^{(\Sigma)\circ} = \Sigma_\circ$$

where $\Sigma_\circ$ is the disjoint union of Σ and the final object of $\mathbf{Set}$ (a set with one element). Hence, the cardinality of Σ is an invariant, called the *rank* of the free pointed M-module $M^{(\Sigma)\circ}$, and denoted $\mathrm{rk}^\circ_M M^{(\Sigma)\circ}$.

4.8.19. Let T be a topos, $(\underline{M}, 0_M)$, $(\underline{N}, 0_N)$ and $(\underline{P}, 0_P)$ three pointed T-monoids, S, (resp. S') a pointed $(\underline{M}, \underline{N})$-bimodule (resp. $(\underline{P}, \underline{N})$-bimodule); we denote

$$\mathrm{Hom}_{(\underline{N}, 0_N)_r}(S, S')$$

the set of all morphisms of pointed right $\underline{N}$-modules $S \to S'$. As usual, the presheaf

$$\mathscr{H}om_{(\underline{N}, 0_N)_r}(S, S') \quad : \quad U \mapsto \mathrm{Hom}_{(\underline{N}, 0_N)_{r|U}}(S_{|U}, S'_{|U})$$

(with obvious notation) is a sheaf on (T, C_T), hence it is represented by an object of T. Indeed, this object is also the fibre product in the cartesian diagram :

$$\begin{array}{ccc} \mathscr{H}om_{(\underline{N}, 0_N)_r}(S, S') & \longrightarrow & \mathscr{H}om_{N_r}(S, S') \\ \downarrow & & \downarrow {\scriptstyle 0_S^*} \\ \mathscr{H}om_{N_r}(0, 0) & \xrightarrow{0_{S'*}} & \mathscr{H}om_{N_r}(0, S'). \end{array}$$

Especially, $\mathscr{H}om_{(\underline{N}, 0_N)_r}(S, S')$ is naturally a $(\underline{P}, \underline{M})$-bimodule, and moreover, it is pointed : its zero section represents the unique morphism $S \to S'$ which factors through 0.

Notice also that, for every pointed $(\underline{P}, \underline{M})$-bimodule S'', the tensor product $S'' \otimes_M S$ is naturally pointed, and as in the non-pointed case, the functor

$$(4.8.20) \qquad (\underline{P}, \underline{M})\text{-}\mathbf{Mod}_\circ \to (\underline{P}, \underline{N})\text{-}\mathbf{Mod}_\circ \quad : \quad S'' \mapsto S'' \otimes_M S$$

is left adjoint to the functor

$$(\underline{P}, \underline{N})\text{-}\mathbf{Mod}_\circ \to (\underline{P}, \underline{M})\text{-}\mathbf{Mod}_\circ \quad : \quad S' \mapsto \mathscr{H}om_{(\underline{N}, 0_N)_r}(S, S').$$

By general nonsense, the functor (4.8.20) is right exact; especially, for any right exact sequence $T' \to T \to T'' \to 0$ of pointed $(\underline{P}, \underline{M})$-bimodules, the induced sequence

$$T' \otimes_M S \to T \otimes_M S \to T'' \otimes_M S \to 0$$

is again right exact.

Remark 4.8.21. Suppose $\underline{M}$, $\underline{N}$ and $\underline{P}$ are non-pointed T-monoids, S is a $(\underline{M}, \underline{N})$-bimodule and S'' a $(\underline{P}, \underline{M})$-bimodule.

(i) If S and S'' are pointed, one may define a tensor product $S'' \otimes_M S$ in the category $(\underline{P}, \underline{N})$-$\mathbf{Mod}_\circ$, if one regards S as a pointed $(\underline{M}_\circ, \underline{N}_\circ)$-bimodule, and S'' as a $(\underline{P}_\circ, \underline{M}_\circ)$-bimodule as in remark 4.8.14(v); then one sets simply $S'' \otimes_M S := S'' \otimes_{M_\circ} S$, which is then viewed as a pointed $(\underline{P}, \underline{N})$-bimodule. In this way one obtains a left adjoint to the corresponding internal Hom-functor $\mathscr{H}om_N$ from pointed $(\underline{P}, \underline{N})$-bimodules to pointed $(\underline{P}, \underline{M})$-bimodules (details left to the reader).

(ii) Lastly, if neither S nor S'' is pointed, notice the natural isomorphism :

$$(S'' \otimes_M S)_\circ \xrightarrow{\sim} S''_\circ \otimes_{M_\circ} S_\circ \qquad \text{in the category } (\underline{P}, \underline{N})\text{-}\mathbf{Mod}_\circ.$$

Definition 4.8.22. In the situation of (4.8.19), let $P = N := (1_T)_\circ$, and notice that – with these choices of P and N – a pointed $(\underline{P}, \underline{M})$-bimodule (resp. a pointed $(\underline{M}, \underline{N})$-bimodule) is just a right pointed $\underline{M}$-module (resp. a left pointed $\underline{M}$-module), and a pointed $(\underline{P}, \underline{N})$-module is just a pointed object of T.

(i) We say that S is a *flat* pointed left $\underline{M}$-module (or briefly, that S is $\underline{M}$*-flat*), if the functor (4.8.20) transforms short exact sequences of right pointed $\underline{M}$-modules, into short exact sequences of pointed T-objects. Likewise, we define flat pointed right $\underline{M}$-modules.

(ii) Let $\varphi : \underline{M} \to \underline{M}'$ be a morphism of pointed T-monoids. We say that φ is *flat*, if $\underline{M}'$ is a flat left $\underline{M}$-module, for the module structure induced by φ.

Remark 4.8.23. (i) In the situation of remark 4.8.11(i), suppose that $\underline{M} := (M, 0_M)$ is a pointed T-monoid and $\underline{N} := (N, 0_N)$ a pointed S-monoid. By arguing as in remark 4.8.11(ii), we see that $f^*\underline{N} := (f^*N, f^*0_N)$ is a pointed T-monoid, and $f_*\underline{M} := (f_*M, f_*0_M)$ is a pointed S-monoid.

(ii) Likewise, if $(X, 0_X)$ is a pointed left $\underline{M}$-module, and $(Y, 0_Y)$ a pointed left $\underline{N}$-module, the $f_*(X, 0_X) := (f_*X, f_*0_X)$ is a pointed $f_*\underline{M}$-module, and $f^*(Y, 0_Y) := (f^*Y, f^*0_Y)$ is a pointed $f^*\underline{N}$-module (and likewise for right modules and bimodules).

(iii) Also, just as in remark 4.8.11(vii), the associated sheaf functor $F \mapsto F^a$ transforms a presheaf $\underline{M}$ of pointed monoids on T, into a pointed T-monoid $\underline{M}^a$, and sends pointed left (resp. right, resp. bi-) $\underline{M}$-modules to pointed left (resp. right, resp. bi-) $\underline{M}^a$-modules.

(iv) Moreover, if $\varphi : f^*\underline{N} \to \underline{M}$ is a morphism of pointed T-monoids, then – in view of the discussion of (4.8.19) – the adjunction of remark 4.8.11(vi) extends to pointed modules : we leave the details to the reader.

(v) Furthermore, in the situation of (4.8.12), we may also define a functor

$$j_{U!} : \underline{M}_{|U}\text{-}\mathbf{Mod}_{l\circ} \to \underline{M}\text{-}\mathbf{Mod}_{l\circ}$$

which will be a left adjoint to j_U^*. Indeed, let $(X, 0_X)$ be a left pointed $\underline{M}_{|U}$-module; the functor from (4.8.12) yields a morphism $j_{U!}0_X$ of (non-pointed) $\underline{M}$-modules, and we define $j_{U!}(X, 0_X)$ to be the push-out (in the category $\underline{M}\text{-}\mathbf{Mod}_l$) of the diagram $0 \leftarrow j_{U!}0_X \xrightarrow{j_{U!}0_X} j_{U!}X$. The latter is endowed with a natural morphism $0 \to j_{U!}(X, 0_X)$, so we have a well defined pointed left $\underline{M}$-module. We leave to the reader the verification that the resulting functor, called *extension by zero*, is indeed left adjoint to the restriction functor.

(vi) It is convenient to extend definition 4.8.22 to non-pointed modules and monoids : namely, if S is a non-pointed left $\underline{M}$-module, we shall say that S is *flat*, if the same holds for the pointed left $\underline{M}_\circ$-module $S_\circ$. Likewise, we say that a morphism $\varphi : \underline{M} \to \underline{N}$ of non-pointed T-monoids is *flat*, if the same holds for $\varphi_\circ$.

Lemma 4.8.24. *Let T be a topos, U any object of T, and denote by $i_* : \complement U \to T$ the inclusion functor of the complement of U in T (see example* 4.7.8(iii)*). Let also $\underline{M}$, $\underline{N}$, $\underline{P}$ be three pointed T-monoids. Then the following holds :*

(i) *The functor $j_{U!}$ of extension by zero is faithful, and transforms exact sequences of pointed left $\underline{M}_{|U}$-modules, into exact sequences of pointed left $\underline{M}$-modules (and likewise for right modules and bimodules).*

(ii) *For every pointed $(\underline{M}, \underline{N})$-bimodule S and every pointed $(\underline{P}_{|U}, \underline{M}_{|U})$-bimodule S', the natural morphism of pointed $(\underline{P}, \underline{N})$-modules*

$$j_{U!}(S' \otimes_{M_{|U}} S_{|U}) \to j_{U!}S' \otimes_M S$$

is an isomorphism.

(iii) *If S is flat pointed left $\underline{M}_{|U}$-module, then $j_{U!}S$ is a flat pointed left $\underline{M}$-module (and likewise for right modules).*

(iv) *For every pointed $(\underline{M}, \underline{N})$-bimodule S, and every pointed $(i^*\underline{P}, i^*\underline{M})$-bimodule S', the natural morphism of pointed $(\underline{P}, \underline{N})$-bimodules*

$$i_*S' \otimes_M S \to i_*(S' \otimes_{i^*M} i^*S)$$

is an isomorphism.

(v) *If S is a flat pointed left $i^*\underline{M}$-module, then i_*S is a flat left pointed $\underline{M}$-module (and likewise for right modules).*

(vi) *If S is a flat pointed left $\underline{M}$-module, then $S_{|U}$ is a flat left pointed $\underline{M}_{|U}$-module.*

Proof. (i): Let us show first that $j_{U!}$ is faithful. Indeed, suppose that $\varphi, \psi : S \to S'$ are two morphisms of left pointed $\underline{M}_{|U}$-modules, such that $j_{U!}\varphi = j_{U!}\psi$. We need to show that $\varphi = \psi$.

Let $p : S' \to S''$ be the coequalizer of φ and ψ; then $j_{U!}p$ is the coequalizer of $j_{U!}\varphi$ and $j_{U!}\psi$ (since $j_{U!}$ is right exact); hence we are reduced to showing that a morphism $p : S' \to S''$ is an isomorphism if and only if the same holds for $j_{U!}p$. This follows from remark 3.7.24(ii) and the following more general :

Claim 4.8.25. Let $\varphi : X \to X'$, $A \to X$, $A \to B$ be three morphisms in T. Then φ is a monomorphism (resp. an epimorphism) if and only if the same holds for the induced morphism $\varphi \amalg_A B : X \amalg_A B \to X' \amalg_A B$.

Proof of the claim. We may assume that $T = C^\sim$ for some small site $C := (\mathscr{C}, J)$. Then $\varphi \amalg_A B = (i\varphi \amalg_{iA} iB)^a$, where $i : C^\sim \to \mathscr{C}^\wedge$ is the forgetful functor. Since the functor $F \mapsto F^a$ is exact, we are reduced to the case where $T = \mathscr{C}^\wedge$, and in this case the assertion can be checked argumentwise, *i.e.* we may assume that $T = \mathbf{Set}$, where the claim is obvious. $\Diamond$

Next, we already know that $j_{U!}$ transforms right exact sequences into right exact sequences. To conclude, it suffices then to check that $j_{U!}$ transforms monomorphisms into monomorphisms.To this aim, we apply again remark 3.7.24(ii) and claim 4.8.25.

(ii) is proved by general nonsense, and (iii) is an immediate consequence of (i) and (ii) : we leave the details to the reader.

(iv): By (4.8.4), we have $j_U^*(i_*S' \otimes_M S) \simeq 0 \otimes_{j_U^*M} j_U^*S \simeq 1_{T/U}$, hence $i_*S' \otimes_M S \in \mathrm{Ob}(\mathsf{C}U)$. Notice now that, for every object X of $\mathsf{C}U$, the counit of adjunction $i^*i_*X \to X$ is an isomorphism (proposition 1.1.20(iii)); by the triangular identities of (1.1.13), it follows that the same holds for the unit of adjunction $i_*X \to i_*i^*i_*X$. Especially, the natural morphism :

$$i_*S' \otimes_M S \to i_*i^*(i_*S' \otimes_M S) \xrightarrow{\sim} i_*(i^*i_*S' \otimes_{i^*M} i^*S) \xrightarrow{\sim} i_*(S' \otimes_{i^*M} i^*S).$$

is an isomorphism. The latter is the morphism of assertion (iv).

(v) follows easily from (iv) and its proof.

(vi): In view of (i), it suffices to show that the functor $S' \mapsto j_{U!}(S' \otimes_{M_{|U}} S_{|U})$ transforms exact sequences into exact sequences. The latter follows easily from (ii). $\square$

Proposition 4.8.26. *Let $\mathbf{P}(T, \underline{M}, S)$ be the property : "S is a flat pointed left $\underline{M}$-module" (for a monoid $\underline{M}$ on a topos T). Then $\mathbf{P}$ can be checked on stalks. (See definition* 4.7.11(v).*)*

Proof. Suppose first that S_ξ is a flat left $\underline{M}_\xi$-module for every ξ in a conservative set of T-points; let $\varphi : X \to X'$ be a monomorphism of pointed right $\underline{M}$-modules; by (4.8.4) we have a natural isomorphism

$$(\varphi \otimes_M S)_\xi \xrightarrow{\sim} \varphi_\xi \otimes_{M_\xi} S_\xi$$

in the category of pointed sets, and our assumption implies that these morphisms are monomorphisms. Since an arbitrary product of monomorphisms is a monomorphism, remark 1.2.21(ii) shows that $\varphi \otimes_M S$ is also a monomorphism, whence the contention.

Next, suppose that S is a flat pointed left $\underline{M}$-module. We have to show that the functor

$$S' \mapsto S' \otimes_{M_\xi} S_\xi \qquad (4.8.27)$$

from pointed right $\underline{M}_\xi$-modules to pointed sets, preserves monomorphisms.

However, let (U, ξ_U, ω_U) be any lifting of ξ (see (4.7.17)); in view of (4.8.4), we have

$$P_U(S') := (\xi_U^*\xi_{U*}S') \otimes_{A_\xi} S_\xi \simeq (\xi_U^*\xi_{U*}S') \otimes_{\xi_U^*A_{|U}} \xi_U^*S_{|U} \simeq \xi_U^*(\xi_{U*}S' \otimes_{A_{|U}} S_{|U})$$

and then lemma 4.8.24(vi) implies that the functor $S' \to P_U(S')$ preserves monomorphisms. By lemma 4.7.20, remark 4.7.13(ii) and (4.7.18), the functor (4.8.27) is a filtered colimit of such functors P_U, hence it preserves monomorphisms as well. $\square$

4.8.28. We wish now to introduce a few notions that pertain to the special class of commutative T-monoids. When $T = \mathbf{Set}$, these notions are well known, and we wish to explain quickly that they generalize without problems, to arbitrary topoi.

To begin with, for every category T as in example 3.7.21(i), we denote by $\mathbf{Mnd}_T$ (resp. $\mathbf{Mnd}_{T\circ}$) the category of commutative non-pointed (resp. pointed) T-monoids; in case $T = \mathbf{Set}$, we shall usually drop the subcript, and write just $\mathbf{Mnd}$ (resp. $\mathbf{Mnd}_\circ$). Notice that if $\underline{M}$ is any (pointed or not pointed) commutative T-monoid, every left or right $\underline{M}$-module is a $\underline{M}$-bimodule in a natural way, hence we shall denote indifferently by $\underline{M}$-$\mathbf{Mod}$ (resp. $\underline{M}$-$\mathbf{Mod}_\circ$) the category of non-pointed (resp. pointed) left or right $\underline{M}$-modules.

The following lemma is a special case of a result that holds more generally, for every "algebraic theory" in the sense of [29, Def.3.3.1] (see [29, Prop.3.4.1, Prop.3.4.2]).

Lemma 4.8.29. *Let T be a topos. We have :*

(i) *The category $\mathbf{Mnd}_T$ admits arbitrary limits and colimits.*
(ii) *In the category $\mathbf{Mnd}_T$, filtered colimits commute with all finite limits.*
(iii) *The forgetful functor $\iota : \mathbf{Mnd}_T \to T$ that assigns to a monoid its underlying object of T, commutes with all limits, and with all filtered colimits.*

Proof. (iii): Commutation with limits holds because ι admits a left adjoint : namely, to an object Σ of T one assigns the *free monoid* $\mathbb{N}_T^{(\Sigma)}$ generated by Σ, defined as the sheaf associated with the presheaf of monoids

$$U \mapsto \mathbb{N}^{(\Sigma(U))} \qquad \text{for every } U \in \mathrm{Ob}(T)$$

where $\mathbb{N}$ is the additive monoid of natural numbers (see remark 4.8.11(vii)). One verifies easily that this T-monoid represents the functor

$$\underline{M} \mapsto \mathrm{Hom}_T(\Sigma, \underline{M}) \qquad \mathbf{Mnd}_T \to \mathbf{Set}.$$

Moreover, if I is any small category, and $F : I \to \mathbf{Mnd}_T$ any functor, one checks easily that the limit of $\iota \circ F$ can be endowed with a unique composition law (indeed, the limit of the composition laws of the monoids F_i), such that the resulting monoid represents the limit of F.

A similar argument also shows that $\mathbf{Mnd}_T$ admits arbitrary filtered colimits, and that ι commutes with filtered colimits. It is likewise easy to show that the product of two T-monoids $\underline{M}$ and $\underline{N}$ is also the coproduct of $\underline{M}$ and $\underline{N}$. To complete the proof of (i), it suffices therefore to show that any two maps $f, g : \underline{M} \to \underline{N}$ admit a coequalizer; the latter is obtained as the coequalizer $\underline{N}'$ (in the category T) of the two morphisms :

$$\underline{M} \times \underline{N} \underset{\mu_N \circ (g \times \mathbf{1}_N)}{\overset{\mu_N \circ (f \times \mathbf{1}_N)}{\rightrightarrows}} \underline{N}.$$

We leave to the reader the verification that the composition law of N descends to a (necessarily unique) composition law on $\underline{N}'$.

(ii) follows from (iii) and the fact that the same assertion holds in T (remark 4.4.1(i)). □

Example 4.8.30. (i) For instance, if $T = \mathbf{Set}$, the product $M_1 \times M_2$ of any two commutative monoids is representable in $\mathbf{Mnd}$; its underlying set is the cartesian product of M_1 and M_2, and the composition law is the obvious one.

(ii) As usual, the kernel $\mathrm{Ker}\,\varphi$ (resp. cokernel $\mathrm{Coker}\,\varphi$) of a map of T-monoids $\varphi : \underline{M} \to \underline{N}$ is defined as the fibre product (resp. push-out) of the diagram of T-monoids

$$\underline{M} \xrightarrow{\varphi} \underline{N} \leftarrow \underline{1}_T \qquad (\text{resp. } \underline{1}_T \leftarrow \underline{M} \xrightarrow{\varphi} \underline{N}).$$

Especially, if $\underline{M} \subset \underline{N}$, one defines in this way the quotient $\underline{N}/\underline{M}$.

(iii) Also, if $T = \mathbf{Set}$, and $\varphi_1 : M \to M_1$, $\varphi_2 : M \to M_2$ are two maps in $\mathbf{Mnd}$, the push-out $M_1 \amalg_M M_2$ can be described as follows. As a set, it is the quotient $(M_1 \times M_2)/\!\sim$, where $\sim$ denotes the minimal equivalence relation such that

$$(m_1, m_2 \cdot \varphi_2(m)) \sim (m_1 \cdot \varphi_1(m), m_2) \qquad \text{for every } m \in M, m_1 \in M_1, m_2 \in M_2$$

and the composition law is the unique one such that the projection $M_1 \times M_2 \to M_1 \amalg_M M_2$ is a map of monoids. We deduce the following :

Lemma 4.8.31. *Let G be an abelian group. The following holds :*

(i) *If $\varphi : M \to N$ and $\psi : M \to G$ are two morphisms of monoids (in the topos $T = \mathbf{Set}$), $G \amalg_M N$ is the quotient $(G \times N)/\!\approx$, where $\approx$ is the equivalence relation such that :*

$$(g, n) \approx (g', n') \quad \Leftrightarrow \quad (\psi(a) \cdot g, \varphi(b) \cdot n) = (\psi(b) \cdot g', \varphi(a) \cdot n') \text{ for some } a, b \in M.$$

(ii) *If $\varphi : G \to M$ and $\psi : G \to N$ are two morphisms of monoids, the set underlying $M \amalg_G N$ is the set-theoretic quotient $(M \times N)/G$ for the G-action defined via (φ, ψ^{-1}).*

(iii) *Especially, if M is a monoid and G is a submonoid of M, then the set underlying M/G is the set-theoretic quotient of M by the translation action of G.*

Proof. (i): One checks easily that the relation $\approx$ thus defined is transitive. Let $\sim$ be the equivalence relation defined as in example 4.8.30(iii). Clearly :

$$(g, n \cdot \psi(m)) \approx (g \cdot \varphi(m), n) \qquad \text{for every } g \in G, n \in N \text{ and } m \in M$$

hence $(g, n) \sim (g', n')$ implies $(g, n) \approx (g', n')$. Conversely, suppose that $(\psi(a) \cdot g, \varphi(b) \cdot n) = (\psi(b) \cdot g', \varphi(a) \cdot n')$ for some $g \in G$, $n \in N$ and $a, b \in M$. Then :

$$(g, n) = (g, \varphi(a) \cdot \varphi(a)^{-1} \cdot n) \sim (\psi(a) \cdot g, \varphi(a)^{-1} \cdot n) = (\psi(b) \cdot g', \varphi(a)^{-1} \cdot n)$$

as well as : $(g', n') = (g', \varphi(b) \cdot \varphi(a)^{-1} \cdot n) \sim (\psi(b) \cdot g', \varphi(a)^{-1} \cdot n)$. Hence $(g, n) \sim (g', n')$ and the claim follows.

(ii) follows directly from example 4.8.30(iii), and (iii) is a special case of (ii). □

4.8.32. Let T be a topos. For any T-ring $\underline{R}$, we let $\underline{R}$-$\mathbf{Mod}$ be the category of $\underline{R}$-modules (defined in the usual way); especially, we may consider the T-ring $\mathbb{Z}_T$ (the constant sheaf with value $\mathbb{Z}$: see (4.4.11)). Then $\mathbb{Z}_T$-$\mathbf{Mod}$ is the category of abelian T-groups. The forgetful functor $\mathbb{Z}_T$-$\mathbf{Mod} \to \mathbf{Mnd}_T$ admits a right adjoint :

$$\mathbf{Mnd}_T \to \mathbb{Z}_T\text{-}\mathbf{Mod} \quad : \quad \underline{M} \mapsto \underline{M}^\times.$$

The latter can be defined as the fibre product in the cartesian diagram :

$$\begin{array}{ccc} \underline{M}^\times & \longrightarrow & \underline{M} \times \underline{M} \\ \big\downarrow & & \big\downarrow{\scriptstyle \mu_M} \\ 1_T & \xrightarrow{1_M} & \underline{M}. \end{array}$$

For $i = 1, 2$, let $p_i : \underline{M} \times \underline{M} \to \underline{M}$ be the projections, and $p'_i : \underline{M}^\times \to \underline{M}$ the restriction of p_i; for every $U \in \mathrm{Ob}(T)$, the image of $p'_i(U) : \underline{M}^\times(U) \to \underline{M}(U)$ consists of all sections x which are *invertible*, *i.e.* for which there exists $y \in \underline{M}(U)$ such that $\mu_M(x, y) = 1_M$. It is easily seen that such inverse is unique, hence p'_i is a monomorphism, p'_1 and p'_2 define the same subobject of $\underline{M}$, and this subobject $\underline{M}^\times$ is the largest abelian T-group contained in $\underline{M}$. We say that $\underline{M}$ is *sharp*, if $\underline{M}^\times = 1_T$. The inclusion functor, from the full subcategory of sharp T-monoids, to $\mathbf{Mnd}_T$, admits a left adjoint

$$\underline{M} \mapsto \underline{M}^\sharp := \underline{M}/\underline{M}^\times.$$

We call $\underline{M}^\sharp$ the *sharpening* of $\underline{M}$.

4.8.33. Let $\underline{S}$ be a submonoid of a commutative T-monoid $\underline{M}$, and $F_S : \mathbf{Mnd}_T \to \mathbf{Set}$ the functor that assigns to any commutative T-monoid $\underline{N}$ the set of all morphisms $f : \underline{M} \to \underline{N}$ such that $f(\underline{S}) \subset \underline{N}^\times$. We claim that F_S is representable by a T-monoid $\underline{S}^{-1}\underline{M}$.

In case $T = \mathbf{Set}$, one may realize $\underline{S}^{-1}\underline{M}$ as the quotient $(\underline{S} \times \underline{M})/\sim$ for the equivalence relation such that $(s_1, x_1) \sim (s_2, x_2)$ if and only if there exists $t \in \underline{S}$ such that $ts_1x_2 = ts_2x_1$. The composition law of $\underline{S}^{-1}\underline{M}$ is the obvious one; then the class of a pair (s, x) is denoted naturally by $s^{-1}x$. This construction can be repeated on a general topos : letting $X := \underline{S} \times \underline{M}$, the foregoing equivalence relation can be encoded as the equalizer R of two maps $X \times X \times S \to \underline{M}$, and the quotient under this equivalence relation shall be represented by the coequalizer of two other maps $R \to X$; the reader may spell out the details, if he wishes. Equivalently, $\underline{S}^{-1}\underline{M}$ can be realized as the sheaf on (T, C_T) associated with the presheaf :

$$T \to \mathbf{Mnd} \quad : \quad U \mapsto \underline{S}(U)^{-1}\underline{M}(U)$$

(see remark 4.8.11(vii)). The natural morphism $\underline{M} \to \underline{S}^{-1}\underline{M}$ is called the *localization map*. For $T = \mathbf{Set}$, and $f \in M$ any element, we shall also use the standard notation :

$$M_f := S_f^{-1}M \qquad \text{where } S_f := \{f^n \mid n \in \mathbb{N}\}.$$

Lemma 4.8.34. *Let $f_1 : \underline{M} \to \underline{N}_1$ and $f_2 : \underline{M} \to \underline{N}_2$ be morphisms of T-monoids, $\underline{S} \subset \underline{M}$, $\underline{S}_i \subset \underline{N}_i$ ($i = 1, 2$) three submonoids, such that $f_i(\underline{S}) \subset \underline{S}_i$ for $i = 1, 2$. Then the natural morphism :*

$$(\underline{S}_1 \cdot \underline{S}_2)^{-1}(\underline{N}_1 \amalg_{\underline{M}} \underline{N}_2) \to \underline{S}_1^{-1}\underline{N}_1 \amalg_{\underline{S}^{-1}\underline{M}} \underline{S}_2^{-1}\underline{N}_2$$

is an isomorphism.

Proof. One checks easily that both these T-monoids represent the functor $\mathbf{Mnd}_T \to \mathbf{Set}$ that assigns to any T-monoid $\underline{P}$ the pairs of morphisms (g_1, g_2) where $g_i : \underline{N}_i \to \underline{P}$ satisfies $g_i(\underline{S}_i) \subset \underline{P}^\times$, for $i = 1, 2$, and $g_1 \circ f_1 = g_2 \circ f_2$. The details are left to the reader. □

4.8.35. The forgetful functor $\mathbb{Z}\text{-}\mathbf{Mod}_T \to \mathbf{Mnd}_T$ from abelian T-groups to commutative T-monoids, admits a left adjoint

$$\underline{M} \mapsto \underline{M}^{\mathrm{gp}} := \underline{M}^{-1}\underline{M}.$$

A commutative T-monoid $\underline{M}$ is said to be *integral* if the unit of adjunction $\underline{M} \to \underline{M}^{\mathrm{gp}}$ is a monomorphism. The functor $\underline{M} \mapsto \underline{M}^{\mathrm{gp}}$ commutes with all colimits, since all left adjoints do; it does not commute with arbitrary limits (see example 4.8.36(v)).

We denote by $\mathbf{Int}.\mathbf{Mnd}_T$ the full subcategory of $\mathbf{Mnd}_T$ consisting of all integral monoids; when $T = \mathbf{Set}$, we omit the subscript, and write just $\mathbf{Int}.\mathbf{Mnd}$. The natural inclusion ι : $\mathbf{Int}.\mathbf{Mnd}_T \to \mathbf{Mnd}_T$ admits a left adjoint :

$$\mathbf{Mnd}_T \to \mathbf{Int}.\mathbf{Mnd}_T \quad : \quad \underline{M} \mapsto \underline{M}^{\mathrm{int}}.$$

Namely, $\underline{M}^{\mathrm{int}}$ is the image (in the category T) of the unit of adjunction $\underline{M} \to \underline{M}^{\mathrm{gp}}$. It follows easily that the category $\mathbf{Int}.\mathbf{Mnd}_T$ is cocomplete, since the colimit of a family $(\underline{M}_\lambda \mid \lambda \in \Lambda)$ of integral monoids is represented by

$$(\operatorname*{colim}_{\lambda \in \Lambda} \iota \underline{M}_\lambda)^{\mathrm{int}}.$$

Likewise, $\mathbf{Int}.\mathbf{Mnd}_T$ is complete, and limits commute with the forgetful functor to T; to check this, it suffices to show that

$$L := \lim_{\lambda \in \Lambda} \iota(\underline{M}_\lambda)$$

is integral. However, by lemma 4.8.29(iii) we have $L \subset \prod_{\lambda \in \Lambda} \underline{M}_\lambda \subset \prod_{\lambda \in \Lambda} \underline{M}_\lambda^{\mathrm{gp}}$, whence the claim.

Example 4.8.36. (i) Take $T = \mathbf{Set}$; if M is any monoid, and $a \in M$ is any element, we say that a is *regular*, if the map $M \to M$ given by the rule : $x \mapsto a \cdot x$ is injective. It is easily seen that M is integral if and only if every element of M is regular.

(ii) For an arbitrary topos T, notice that the T-monoid $\underline{G}^a$ associated with a presheaf of groups $\underline{G}$ on T, is a T-group : indeed, the condition $\underline{G}^\times = \underline{G}$ implies $(\underline{G}^a)^\times = \underline{G}^a$, since the functor $F \mapsto F^a$ is exact. More precisely, for every presheaf $\underline{M}$ of monoids on T, we have a natural isomorphism :

$$(\underline{M}^{\mathrm{gp}})^a \xrightarrow{\sim} (\underline{M}^a)^{\mathrm{gp}} \qquad \text{for every } T\text{-monoid } \underline{M}$$

since both functors are left adjoint to the forgetful functor from T-groups to presheaves of monoids on T.

(iii) It follows from (ii) that a T-monoid $\underline{M}$ is integral if and only if $\underline{M}(U)$ is an integral monoid, for every $U \in \mathrm{Ob}(T)$. Indeed, if $\underline{M}$ is integral, then $\underline{M}(U) \subset \underline{M}^{\mathrm{gp}}(U)$ for every such U, so $\underline{M}(U)$ is integral. Conversely, by definition $\underline{M}^{\mathrm{gp}}$ is the sheaf associated with the presheaf $U \mapsto \underline{M}(U)^{\mathrm{gp}}$; now, if $\underline{M}(U)$ is integral, we have $\underline{M}(U) \subset \underline{M}(U)^{\mathrm{gp}}$, and consequently $\underline{M} \subset \underline{M}^{\mathrm{gp}}$, since the functor $F \mapsto F^a$ is exact.

(iv) We also deduce from (ii) that the functor $\underline{M} \mapsto \underline{M}^a$ sends presheaves of integral monoids, to integral T-monoids. Therefore we have a natural isomorphism :

$$(\underline{M}^{\mathrm{int}})^a \xrightarrow{\sim} (\underline{M}^a)^{\mathrm{int}} \tag{4.8.37}$$

as both functors are left adjoint to the forgetful functor from integral T-monoids, to presheaves of monoids on T. In the same vein, it is easily seen that the forgetful functor $\mathbf{Int}.\mathbf{Mnd}_T \to T$ commutes with filtered colimits : indeed, (4.8.37) and lemma 4.8.29(iii) reduce the assertion to showing that the colimit of a filtered system of presheaves of integral monoids is integral, which can be verified directly.

(v) Take $T = \mathbf{Set}$, and let $\varphi : M \to N$ be an injective map of monoids; if N (hence M) is integral, one sees easily that the induced map $\varphi^{\mathrm{gp}} : M^{\mathrm{gp}} \to N^{\mathrm{gp}}$ is also injective. This may fail, when N is not integral : for instance, if M is any integral monoid, and $N := M_\circ$ is the pointed monoid associated with M as in remark 4.8.14(iv), then for the natural inclusion $i : M \to M_\circ$ we have $i^{\mathrm{gp}} = 0$, since $(M_\circ)^{\mathrm{gp}} = \{1\}$.

Lemma 4.8.38. *Let T be a topos with enough points, $\underline{M}$ an integral T-monoid, and $\underline{N} \subset \underline{M}$ a T-submonoid. Then $\underline{M}/\underline{N}$ is an integral T-monoid.*

Proof. In light of example 4.8.36(iv), we are reduced to the case where $T = \mathbf{Set}$. Moreover, since the natural morphism $\underline{M}/\underline{N} \to \underline{N}^{-1}\underline{M}/\underline{N}^{\mathrm{gp}}$ is an isomorphism, we may assume that $\underline{N}$ is an abelian group. Now, notice that $(\underline{M}/\underline{N})^{\mathrm{gp}} = \underline{M}^{\mathrm{gp}}/\underline{N}$ since the functor $P \mapsto P^{\mathrm{gp}}$ commutes with colimits. On the other hand, $\underline{M}/\underline{N}$ is the set-theoretic quotient of $\underline{M}$ by the translation action of $\underline{N}$ (lemma 4.8.31(iii)). This shows that the unit of adjunction $\underline{M}/\underline{N} \to (\underline{M}/\underline{N})^{\mathrm{gp}}$ is injective, as required. ∎

4.8.39. Let M be an integral monoid. Classically, one says that M is *saturated*, if we have :

$$M = \{a \in M^{\mathrm{gp}} \mid a^n \in M \text{ for some integer } n > 0\}.$$

In order to globalize the class of saturated monoid to arbitrary topoi, we make the following :

Definition 4.8.40. Let T be a topos, $\varphi : \underline{M} \to \underline{N}$ a morphism of integral T-monoids.

(i) We say that φ is *exact* if the diagram of commutative T-monoids

$$\mathscr{D}_\varphi \quad : \quad \begin{array}{ccc} \underline{M} & \xrightarrow{\varphi} & \underline{N} \\ \downarrow & & \downarrow \\ \underline{M}^{\mathrm{gp}} & \xrightarrow{\varphi^{\mathrm{gp}}} & \underline{N}^{\mathrm{gp}} \end{array}$$

is cartesian (where the vertical arrows are the natural morphisms).

(ii) For any integer $k > 0$, the *k-Frobenius map* of $\underline{M}$ is the endomorphism $\boldsymbol{k}_M$ of $\underline{M}$ given by the rule : $x \mapsto x^k$ for every $U \in \mathrm{Ob}(T)$ and every $x \in \underline{M}(U)$. We say that $\underline{M}$ is *k-saturated*, if $\boldsymbol{k}_M$ is an exact morphism.

(iii) We say that $\underline{M}$ is *saturated*, if $\underline{M}$ is integral and k-saturated for every integer $k > 0$.

We denote by $\mathbf{Sat}.\mathbf{Mnd}_T$ the full subcategory of $\mathbf{Int}.\mathbf{Mnd}_T$ whose objects are the saturated T-monoids. As usual, when $T = \mathbf{Set}$, we shall drop the subscript, and just write $\mathbf{Sat}.\mathbf{Mnd}$ for this category. The above definition (and several of the related results in section 6.2) is borrowed from [159].

Remark 4.8.41. (i) Clearly, when $T = \mathbf{Set}$, definition 4.8.40(iii) recovers the classical notion of saturated monoid. Again, for usual monoids, it is easily seen that the forgetful functor $\mathbf{Sat}.\mathbf{Mnd} \to \mathbf{Int}.\mathbf{Mnd}$ admits a left adjoint, that assigns to any integral monoid M its *saturation* M^{sat}. The latter is the monoid consisting of all elements $x \in M^{\mathrm{gp}}$ such that $x^k \in M$ for some integer $k > 0$; especially, the torsion subgroup of M^{gp} is always contained in M^{sat}. The easy verification is left to the reader. Clearly, M is saturated if and only if $M = M^{\mathrm{sat}}$. More generally, the unit of adjunction $M \to M^{\mathrm{sat}}$ is just the inclusion map.

(ii) For a general topos T, and a morphism φ as in definition 4.8.40(i), notice that φ is exact if and only if the induced map of monoids $\varphi(U) : \underline{M}(U) \to \underline{N}(U)$ is exact for every $U \in \mathrm{Ob}(T)$. Indeed, if $\mathscr{D}_\varphi$ is cartesian, then the same holds for the induced diagram $\mathscr{D}_\varphi(U)$ of monoids; since the natural map $\underline{M}(U)^{\mathrm{gp}} \to \underline{M}^{\mathrm{gp}}(U)$ is injective (and likewise for $\underline{N}$), it follows easily that the diagram of monoids $\mathscr{D}_{\varphi(U)}$ is cartesian, *i.e.* $\varphi(U)$ is exact. For the converse, notice that $\mathscr{D}_\varphi$ is of the form $(h\mathscr{D}_\varphi)^a$, where $h : T \to T^\wedge$ is the Yoneda embedding, and $F \mapsto F^a$ denotes the associated sheaf functor $T^\wedge \to (T, C_T)^\sim = T$; the assumption means that $h\mathscr{D}$ is a cartesian diagram in $T^\wedge$, hence $\mathscr{D}$ is exact in T, since the associated sheaf functor is exact.

(iii) Example 4.8.36(iii) and (ii) imply that a T-monoid $\underline{M}$ is saturated, if and only if $\underline{M}(U)$ is a saturated monoid, for every $U \in \mathrm{Ob}(T)$. We also remark that, in view of example 4.8.36(ii), the functor $F \mapsto F^a$ takes presheaves of k-saturated (resp. saturated) monoids, to k-saturated (resp. saturated) T-monoids : indeed, if $\eta : \underline{M} \to \underline{M}^{\mathrm{gp}}$ is the unit of adjunction for a presheaf of monoids $\underline{M}$, then $\eta^a : \underline{M}^a \to (\underline{M}^{\mathrm{gp}})^a = (\underline{M}^a)^{\mathrm{gp}}$ is the unit of adjunction for the associated T-monoid, hence it is clear the functor $F \mapsto F^a$ preserves exact morphisms.

(iv) It follows easily that the inclusion functor $\mathbf{Sat}.\mathbf{Mnd}_T \to \mathbf{Int}.\mathbf{Mnd}_T$ admits a left adjoint, namely the functor

$$\mathbf{Int}.\mathbf{Mnd}_T \to \mathbf{Sat}.\mathbf{Mnd}_T \quad : \quad \underline{M} \mapsto \underline{M}^{\mathrm{sat}}$$

that assigns to $\underline{M}$ the sheaf associated with the presheaf $U \to \underline{M}'(U) := \underline{M}(U)^{\mathrm{sat}}$ on T (notice that the functor $\underline{M} \mapsto \underline{M}'$ from presheaves of integral monoids, to presheaves of saturated monoids, is left adjoint to the inclusion functor). Just as in example 4.8.36(iv), we deduce a natural isomorphism

$$(\underline{M}^{\mathrm{sat}})^a \xrightarrow{\sim} (\underline{M}^a)^{\mathrm{sat}} \qquad \text{for every } T\text{-monoid } \underline{M} \tag{4.8.42}$$

since both functors are left adjoint to the forgetful functor from $\mathbf{Sat}.\mathbf{Mnd}_T$, to presheaves of integral monoids on T.

(v) By the usual general nonsense, the saturation functor commutes with all colimits. Moreover, the considerations of (4.8.35) can be repeated for saturated monoids : first, the category $\mathbf{Sat}.\mathbf{Mnd}_T$ is cocomplete, and arguing as in example 4.8.36(iv), one checks that filtered colimits commute with the forgetful functor $\mathbf{Sat}.\mathbf{Mnd}_T \to T$; next, if $F : \Lambda \to \mathbf{Sat}.\mathbf{Mnd}_T$ is a functor from a small category Λ, then for each integer $k > 0$, the induced diagram of integral monoids

$$\lim_\Lambda \mathscr{D}_{\boldsymbol{k}_F} \quad : \quad \begin{array}{ccc} \lim_\Lambda F & \xrightarrow{\lim_\Lambda \boldsymbol{k}_F} & \lim_\Lambda F \\ \downarrow & & \downarrow \\ \lim_\Lambda F^{\mathrm{gp}} & \xrightarrow{\lim_\Lambda \boldsymbol{k}_F^{\mathrm{gp}}} & \lim_\Lambda F^{\mathrm{gp}} \end{array}$$

is cartesian; since the natural morphism

$$(\lim_\Lambda F)^{\mathrm{gp}} \to \lim_\Lambda F^{\mathrm{gp}}$$

is a monomorphism, it follows easily that the limit of F is saturated, hence $\mathbf{Sat}.\mathbf{Mnd}_T$ is complete, and furthermore all limits commute with the forgetful functor to T.

4.8.43. In view of remark 4.8.11(i,ii,iii), a morphism of topoi $f : T \to S$ induces functors :

(4.8.44) $$f_* : \mathbf{Mnd}_T \to \mathbf{Mnd}_S \qquad f^* : \mathbf{Mnd}_S \to \mathbf{Mnd}_T$$

and one verifies easily that (4.8.44) is an adjoint pair of functors.

Lemma 4.8.45. *Let $f : T \to S$ be a morphism of topoi, $\underline{M}$ an S-monoid. We have :*

(i) *If $\underline{M}$ is integral (resp. saturated), $f^*\underline{M}$ is an integral (resp. saturated) T-monoid.*

(ii) *More precisely, there is a natural isomorphism :*

$$f^*(\underline{M}^{\mathrm{int}}) \xrightarrow{\sim} (f^*\underline{M})^{\mathrm{int}} \qquad \textit{(resp. } f^*(\underline{M}^{\mathrm{sat}}) \xrightarrow{\sim} (f^*\underline{M})^{\mathrm{sat}}, \ \textit{if } \underline{M} \textit{ is integral).}$$

(iii) *If φ is an exact morphism of integral S-monoids, then $f^*\varphi$ is an exact morphism of integral T-monoids.*

Proof. To begin with, notice that the adjoint pair (f^*, f_*) of (4.8.44) restricts to a corresponding adjoint pair of functors between the categories of abelian T-groups and abelian S-groups (since the condition $\underline{G} = \underline{G}^\times$ for monoids, is preserved by any left exact functor).

There follows a natural isomorphism :

$$(f^*\underline{M})^{\mathrm{gp}} \xrightarrow{\sim} f^*(\underline{M}^{\mathrm{gp}}) \qquad \text{for every } S\text{-monoid } \underline{M}$$

since both functors are left adjoint to the functor f_* from abelian T-groups to S-monoids. Now, if $\underline{M}$ is an integral S-module, and $\eta : \underline{M} \to \underline{M}^{\mathrm{gp}}$ is the unit of adjunction, it is easily seen that $f^*\eta : f^*\underline{M} \to (f^*\underline{M})^{\mathrm{gp}}$ is also the unit of adjunction. From this and proposition 4.4.8(iii), we deduce the assertion concerning $f^*(\underline{M}^{\mathrm{int}})$.

By the same token, we get assertion (iii) of the lemma. Especially, if $\underline{M}$ is saturated, then the same holds for $f^*\underline{M}$. The assertion concerning $f^*(\underline{M}^{\mathrm{sat}})$ follows by the usual argument. □

Lemma 4.8.46. (i) *The functor f^* of* (4.8.44) *commutes with all finite limits and all colimits.*

(ii) *Let $\mathbf{P}(T, M)$ be the property "M is an integral (resp. saturated) T-monoid". Then $\mathbf{P}$ can be checked on every conservative set of morphisms of topoi (see definition* 4.7.11(v).*)*

Proof. (i): Concerning finite limits, in light of lemma 4.8.29(iii) we are reduced to the assertion that $f^* : S \to T$ is left exact, which holds by definition. Next f^* commutes with colimits, because it is a left adjoint.

(ii): A T-monoid $\underline{M}$ is integral if and only if the unit of adjunction $\eta : \underline{M} \to \underline{M}^{\mathrm{int}}$ is an isomorphism. However, $f^*(\underline{M}^{\mathrm{int}}) \xrightarrow{\sim} (f^*\underline{M})^{\mathrm{int}}$, by lemma 4.8.45(ii), and $f^*\eta : \underline{M} \to (f^*\underline{M})^{\mathrm{int}}$ is the unit of adjunction. The assertion is an immediate consequence. The same argument applies as well to saturated T-monoids. □

Example 4.8.47. (i) For instance, the unique morphism of topoi $\Gamma : T \to \mathbf{Set}$ (see proposition 4.4.12(i,ii)) induces a pair of adjoint functors :

(4.8.48) $$\mathbf{Mnd}_T \to \mathbf{Mnd} \ : \ M \mapsto \Gamma(T, M) \qquad \text{and} \qquad \mathbf{Mnd} \to \mathbf{Mnd}_T \ : \ P \mapsto P_T$$

where P_T is the constant sheaf of monoids on (T, C_T) with value P.

(ii) Specializing lemma 4.8.45(ii) to this adjoint pair, we obtain natural isomorphisms :

$$(M_T)^{\mathrm{int}} \xrightarrow{\sim} (M^{\mathrm{int}})_T \qquad (M_T)^{\mathrm{sat}} \xrightarrow{\sim} (M^{\mathrm{sat}})_T \tag{4.8.49}$$

of functors $\mathbf{Mnd} \to \mathbf{Int}.\mathbf{Mnd}_T$ and $\mathbf{Int}.\mathbf{Mnd} \to \mathbf{Sat}.\mathbf{Mnd}_T$. Especially, if M is an integral (resp. saturated) monoid, then the constant T-monoid M_T is integral (resp. saturated).

(iii) If ξ is any T-point, notice also that the stalk $M_{T,\xi}$ is isomorphic to M, since ξ is a section of $\Gamma : T \to \mathbf{Set}$.

4.8.50. Let T be a topos, $\underline{R}$ a T-ring. We have a forgetful functor $\underline{R}\text{-}\mathsf{Alg} \to \mathbf{Mnd}_T$ that assigns to a (unital, commutative) $\underline{R}$-algebra $(\underline{A}, +, \cdot, 1_A)$ its multiplicative T-monoid $(\underline{A}, \cdot)$. If $T = \mathbf{Set}$, this functor admits a left adjoint $\mathbf{Mnd} \to R\text{-}\mathsf{Alg} : M \mapsto R[M]$. Explicitly, $R[M] = \bigoplus_{x \in M} xR$, and the multiplication law is uniquely determined by the rule :

$$xa \cdot yb := (x \cdot y)ab \qquad \text{for every } x, y \in M \text{ and } a, b \in R.$$

For a general topos T, the above construction globalizes to give a left adjoint

$$\mathbf{Mnd}_T \to \underline{R}\text{-}\mathsf{Alg} \quad : \quad \underline{M} \mapsto \underline{R}[\underline{M}]. \tag{4.8.51}$$

The latter is the sheaf on (T, C_T) associated with the presheaf $U \mapsto \underline{R}(U)[\underline{M}(U)]$, for every $U \in \mathrm{Ob}(T)$. The functor (4.8.51) commutes with arbitrary colimits (since it is a left adjoint); especially, if $\underline{M} \to \underline{M}_1$ and $\underline{M} \to \underline{M}_2$ are two morphisms of monoids, we have a natural identification :

$$\underline{R}[\underline{M}_1 \amalg_{\underline{M}} \underline{M}_2] \xrightarrow{\sim} \underline{R}[\underline{M}_1] \otimes_{\underline{R}[\underline{M}]} \underline{R}[\underline{M}_2]. \tag{4.8.52}$$

By inspecting the universal properties, we also get a natural isomorphism :

$$\underline{S}^{-1}\underline{R}[\underline{M}] \xrightarrow{\sim} \underline{R}[\underline{S}^{-1}\underline{M}] \tag{4.8.53}$$

for every monoid $\underline{M}$ and every submonoid $\underline{S} \subset \underline{M}$.

4.8.54. Likewise, if $\underline{M}$ is any T-monoid, let $\underline{R}[\underline{M}]$-$\mathbf{Mod}$ denote as usual the category of modules over the T-ring $\underline{R}[\underline{M}]$; we have a forgetful functor $\underline{R}[\underline{M}]\text{-}\mathbf{Mod} \to \underline{M}\text{-}\mathbf{Mod}$. When $T = \mathbf{Set}$, this functor admits a left adjoint $M\text{-}\mathbf{Mod} \to R[M]\text{-}\mathbf{Mod} : S \mapsto R[S]$. Explicitly, $R[S]$ is the free R-module with basis given by S, and the $R[M]$-module structure on $R[S]$ is determined by the rule:

$$xa \cdot sb := \mu_S(x, s)ab \qquad \text{for every } x \in M,\ s \in S \text{ and } a, b \in R.$$

For a general topos T, this construction globalizes to give a left adjoint

$$\underline{M}\text{-}\mathbf{Mod} \to \underline{R}[\underline{M}]\text{-}\mathbf{Mod} \quad : \quad (S, \mu_S) \mapsto \underline{R}[S]$$

which is defined as the sheaf associated with the presheaf $U \mapsto \underline{R}(U)[S(U)]$ in $T^\wedge$.

Example 4.8.55. (i) Take $T = \mathbf{Set}$, let $(M, \mu_M, 1_M)$ be any monoid, and Λ any set. Then the coproduct $M^{(\Lambda)}$ of Λ copies of M is representable in the category of monoids, as the subset of M^Λ consisting of all sequences $a_\bullet := (a_\lambda \mid \lambda \in \Lambda)$ such that $\{\lambda \in \Lambda \mid a_\lambda \neq 0\}$ is a finite set; the composition law on $M^{(\Lambda)}$ is the obvious one :

$$a_\bullet \cdot b_\bullet := (\mu_M(a_\lambda, b_\lambda) \mid \lambda \in \Lambda) \qquad \text{for every } a_\bullet, b_\bullet \in M^{(\Lambda)}.$$

(ii) Let also R be any ring, and $(\mathbb{N}, +, 0)$ the monoid of non-negative integers; then the ring

$$R[\mathbb{N}^{(\Lambda)}]$$

is the free polynomial R-algebra $R[T_\lambda \mid \lambda \in \Lambda]$ on the set of variables $(T_\lambda \mid \lambda \in \Lambda)$.

(iii) We shall later encounter also the following variant. Let p be a prime integer, and $(\mathbb{N}[1/p], +, 0)$ the additive monoid of all non-negative rationals of the form a/p^n, for every $a, n \in \mathbb{N}$. Notice that $\mathbb{N}[1/p]$ is the colimit of the direct system of monoids

$$\mathbb{N} \to \mathbb{N} \to \mathbb{N} \cdots$$

with transition maps given by multiplication by p. Then, for every set Λ, the ring

$$R[T_\lambda^{1/p^\infty} \mid \lambda \in \Lambda] := R[\mathbb{N}[1/p]^{(\Lambda)}]$$

is the colimit of the system of R-algebras $R[\mathbb{N}^{(\Lambda)}] \to R[\mathbb{N}^{(\Lambda)}] \to R[\mathbb{N}^{(\Lambda)}] \to \cdots$ with transition maps : $R[T_\lambda \mid \lambda \in \Lambda] \to R[T_\lambda \mid \lambda \in \Lambda]$ such that $T_\lambda \mapsto T_\lambda^p$ for all $\lambda \in \Lambda$. Thus, the system $(T_\lambda^{1/p^n} \mid \lambda \in \Lambda, n \in \mathbb{N})$ is a family of generators for this R-algebras, and the free polynomial R-algebra $R[T_\lambda \mid \lambda \in \Lambda]$ is naturally an R-subalgebra of $R[T_\lambda^{1/p^\infty} \mid \lambda \in \Lambda]$.

4.9. **Torsors on a topos.** In this section we discuss torsors over (not necessarily abelian) group objects of a topos. Then we explain some basic notions concerning the points of the étale and Zariski topoi of a scheme, and we conclude with the proof of Hilbert's theorem 90 (lemma 4.9.27(iv)).

Definition 4.9.1. Let T be a topos, and G a T-group.

(i) A *left G-torsor* is a left G-module (X, μ_X), inducing an isomorphism

$$(\mu_X, p_X) : G \times X \to X \times X$$

(where $p_X : G \times X \to X$ is the natural projection) and such that there exists a covering morphism $U \to 1_T$ in T for which $X(U) \neq \varnothing$. This is the same as saying that the unique morphism $X \to 1_T$ is an epimorphism.

(ii) A *morphism of left G-torsors* is just a morphism of the underlying G-modules. Likewise, we define right G-torsors, G-bitorsors, and morphisms between them. We let:

$$H^1(T, G)$$

be the set of isomorphism classes of right G-torsors.

(iii) A (left or right or bi-) G-torsor (X, μ_X) is said to be *trivial*, if $\Gamma(T, X) \neq \varnothing$.

Remark 4.9.2. (i) In the situation of definition 4.9.1, notice that $H^1(T, G)$ always contains a distinguished element, namely the class of the trivial G-torsor (G, μ_G).

(ii) Conversely, suppose that (X, μ_X) is a trivial left G-torsor, and say that $\sigma \in \Gamma(T, X)$; then we have a cartesian diagram :

$$\begin{array}{ccc} G & \xrightarrow{\mu_\sigma} & X \\ \big\downarrow & & \big\downarrow{\scriptstyle \mathbf{1}_X \times \sigma} \\ G \times X & \xrightarrow{(\mu_X, p_X)} & X \times X \end{array}$$

which shows that (X, μ_X) is isomorphic to (G, μ_G) (and likewise for right G-torsors).

(iii) Notice that every morphism $f : (X, \mu_X) \to (X', \mu_{X'})$ of G-torsors is an isomorphism. Indeed, the assertion can be checked locally on T (*i.e.*, after pull-back by a covering morphism $U \to 1_T$). Then we may assume that X admits a global section $\sigma \in \Gamma(T, X)$, in which case $\sigma' := \sigma \circ f \in \Gamma(T, X')$. Then, arguing as in (ii), we get a commutative diagram :

$$\begin{array}{ccc} & G & \\ {\scriptstyle \mu_\sigma}\swarrow & & \searrow{\scriptstyle \mu_{\sigma'}} \\ X & \xrightarrow{f} & X' \end{array}$$

where both μ_σ and $\mu_{\sigma'}$ are isomorphisms, and then the same holds for f.

(iv) The tensor product of a G-bitorsor and a left G-bitorsor is a left G-torsor. Indeed the assertion can be checked locally on T, so we are reduced to checking that the tensor product of a trivial G-bitorsors and a trivial left G-torsor is the trivial left G-torsor, which is obvious.

(v) Likewise, if $G_1 \to G_2$ is any morphism of T-groups, and X is a left G_1-torsor, it is easily seen that the base change $G_2 \otimes_{G_1} X$ yields a left G_2-torsor (and the same holds for right torsors and bitorsors). Hence the rule $G \mapsto H^1(T,G)$ is a functor from the category of T-groups, to the category of pointed sets. One can check that $H^1(T,G)$ is an essentially small set (see [78, Chap.III, §3.6.6.1]).

(vi) Let $f : T \to S$ be a morphism of topoi; if X is a left G-torsor, the f_*G-module f_*X is not necessarily a f_*G-torsor, since we may not be able to find a covering morphism $U \to 1_S$ such that $f_*X(U) \neq \varnothing$. On the other hand, if H a S-group and Y a left H-torsor, then it is easily seen that f^*Y is a left f^*H-torsor.

4.9.3. Let $f : T' \to T$ be a morphism of topoi, and G a T'-monoid; we define a U-presheaf $R^1 f_*^\wedge G$ on T, by the rule :

$$U \mapsto H^1(T'/f^*U, G_{|f^*U}).$$

(More precisely, since this set is only essentially small, we replace it by an isomorphic small set). If $\varphi : U \to V$ is any morphism in T, and X is any $G_{|f^*V}$-torsor, then $X \times_{f^*V} f^*U$ is a $G_{|f^*U}$-torsor, whose isomorphism class depends only on the isomorphism class of X; this defines the map $R^1 f_*^\wedge G(\varphi)$, and it is clear that $R^1 f_*^\wedge G(\varphi \circ \psi) = R^1 f_*^\wedge G(\psi) \circ R^1 f_*^\wedge G(\varphi)$, for any other morphism $\psi : W \to U$ in T. Lastly, we denote by :

$$R^1 f_* G$$

the sheaf on (T, C_T) associated with the presheaf $R^1 f_*^\wedge G$. Notice that the object $R^1 f_* G$ is *pointed*, *i.e.* it is endowed with a natural global section :

$$\tau_{f,G} : 1_T \to R^1 f_* G$$

namely, the morphism associated with the morphism of presheaves $1_T \to R^1 f_*^\wedge G$ which, for every $U \in \mathrm{Ob}(T)$, singles out the isomorphism class $\tau_{f,G}(U) \in R^1 f_*^\wedge G(U)$ of the trivial $G_{|f^*U}$-torsor.

4.9.4. Let $g : T'' \to T'$ be another morphism of topoi, and G a T''-group. Notice that :

$$f^{*\wedge} R^1 g_*^\wedge G = R^1 (f \circ g)_*^\wedge G$$

hence the natural morphism (in $T'^\wedge$) $R^1 f_*^\wedge G \to R^1 f_* G$ induces a morphism $R^1(f \circ g)_*^\wedge G \to f^{*\wedge} R^1 g_* G$ in $T^\wedge$, which yields, after taking associated sheaves, a morphism in T :

$$R^1 (f \circ g)_* G \to f_* R^1 g_* G. \tag{4.9.5}$$

One sees easily that this is a *morphism of pointed objects* of T, *i.e.* the image of the global section $\tau_{f \circ g, G}$ under this map, is the global section $f_* \tau_{g,G}$.

Next, suppose that $U \in \mathrm{Ob}(T)$ and X is any $g_*G_{|f^*U}$-torsor (on T'/f^*U); we may form the $g^*g_*G_{|g^*f^*U}$-torsor g^*X, and then base change along the natural morphism $g^*g_*G \to G$, to obtain the $G_{|g^*f^*U}$-torsor $G \otimes_{g^*g_*G} g^*X$. This rule yields a map $R^1 f_*^\wedge(g_*G) \to R^1(f \circ g)_*^\wedge G$, and after taking associated sheaves, a natural morphism of pointed objects :

$$R^1 f_*(g_* G) \to R^1 (f \circ g)_* G. \tag{4.9.6}$$

Remark 4.9.7. As a special case, let $h : S' \to S$ be a morphism of topoi, H a S'-group. If we take $T'' := S'$, $T' := S$, $g := h$ and $f : S \to \mathbf{Set}$ the (essentially) unique morphism of topoi, (4.9.6) and (4.9.5) boil down to maps of pointed sets :

$$H^1(S, h_* H) \to H^1(S', H) \to \Gamma(S, R^1 h_* H). \tag{4.9.8}$$

These considerations are summarized in the following :

Theorem 4.9.9. *In the situation of* (4.9.4)*, there exists a natural* exact sequence of pointed objects *of* T *:*

$$1_T \to R^1 f_*(g_*G) \to R^1(f \circ g)_*G \to f_*R^1g_*G.$$

Proof. The assertion means that (4.9.6) identifies $R^1f_*(g_*G)$ with the subobject :

$$R^1(f \circ g)_*G \times_{f_*R^1g_*G} f_*\tau_{g,G}$$

(briefly : the preimage of the trivial global section). We begin with the following :

Claim 4.9.10. In the situation of remark 4.9.7, the sequence of maps (4.9.8) identifies the pointed set $H^1(S, h_*H)$ with the preimage of the trivial global section $\tau_{h,H}$ of R^1h_*H.

Proof of the claim. Notice first that a global section Y of $R^1h_*^\wedge H$ maps to the trivial section $\tau_{h,H}$ of R^1h_*H if and only if there exists a covering morphism $U \to 1_S$ in (S, C_S), such that $Y(h^*U) \neq \varnothing$. Thus, let X be a right h_*H-torsor; the image in $H^1(S', H)$ of its isomorphism class is the class of the H-torsor $Y := h^*X \otimes_{h^*h_*H} H$. The latter defines a global section of R^1h_*H. However, by definition there exists a covering morphism $U \to 1_S$ such that $X(U) \neq \varnothing$, hence also $h^*X(h^*U) \neq \varnothing$, and therefore $Y(h^*U) \neq \varnothing$. This shows that the image of $H^1(S, h_*H)$ lies in the preimage of $\tau_{h,H}$.

Moreover, notice that $h_*Y(U) \neq \varnothing$, hence h_*Y is a h_*H-torsor. Now, let $\varepsilon_H : h^*h_*H \to H$ (resp. $\eta_{h_*H} : h_*H \to h_*h^*h_*H$) be the counit (resp. unit of adjunction); we have a natural morphism $\alpha : h^*X \to Y_{(\varepsilon_H)}$ of h^*h_*H-modules, whence a morphism :

$$h_*\alpha : h_*h^*X \to h_*Y_{(h_*\varepsilon_H)}$$

of $h_*h^*h_*H$-modules. On the other hand, the unit of adjunction $\eta_X : X \to h_*h^*X_{(\eta_{h_*H})}$ is a morphism of h_*H-modules (remark 4.8.11(v)). Since $h_*\varepsilon_H \circ \eta_{h_*H} = \mathbf{1}_{h_*H}$ (see (1.1.13)), the composition $h_*\alpha \circ \eta_X$ is a morphism of h_*H-modules, hence it is an isomorphism, by remark 4.9.2(iii). This implies that the first map of (4.9.8) is injective.

Conversely, suppose that the class of a H-torsor X' gets mapped to $\tau_{h,H}$; we need to show that the class of X' lies in the image of $H^1(S, h_*H)$. However, the assumption means that there exists a covering morphism $U \to 1_S$ such that $X'(h^*U) \neq \varnothing$; by adjunction we deduce that $h_*X'(U) \neq \varnothing$, hence h_*X' is a h_*H-torsor. In order to conclude, it suffices to show that the image in $H^1(S', H)$ of the class of h_*X' is the class of X.

Now, the counit of adjunction $h^*h_*X' \to X'$ is a morphism of h^*h_*H-modules (remark 4.8.11(v)); by adjunction it induces a map $h^*h_*X' \otimes_{h^*h_*H} H \to H$ of H-torsors, which must be an isomorphism, according to remark 4.9.2(iii). ◊

If we apply claim 4.9.10 with $S := T'/f^*U$, $S' := T''/(g^*f^*U)$ and $h := g/(g^*f^*U)$, for U ranging over the objects of T, we deduce an exact sequence of presheaves of pointed sets :

$$1_T \to R^1f_*^\wedge(g_*H) \to R^1(f \circ g)_*^\wedge G \to f^{*\wedge}R^1g_*G$$

from which the theorem follows, after taking associated sheaves. □

4.9.11. Let $f : T' \to T$ be a morphism of topoi, U an object of T, G a T'-group, $p : X \to U$ a right $G_{|U}$-torsor. Then, for every object V of T we have an induced sequence of maps of sets

$$X(f^*V) \xrightarrow{p_*} U(f^*V) \xrightarrow{\partial} R^1f_*^\wedge G(V)$$

where p_* is deduced from p, and for every $\sigma \in U(f^*V)$, we let $\partial(\sigma)$ be the isomorphism class of the right $G_{|f^*V}$-torsor $(X \times_U f^*V \to f^*V)$. Clearly the image of p_* is precisely the preimage of (the isomorphism class of) the trivial $G_{|f^*V}$-torsor. After taking associated sheaves, we deduce a natural sequence of morphisms in T :

$$f_*X \xrightarrow{p} f_*U \xrightarrow{\partial} R^1f_*G \qquad (4.9.12)$$

such that the preimage of the global section $\tau_{f,G}$ is precisely the image in $\Gamma(T', U)$ of the set of global sections of X.

4.9.13. A *ringed topos* is a pair $(T, \mathscr{O}_T)$ consisting of a topos T and a (unital, associative) T-ring $\mathscr{O}_T$, called the *structure ring* of T. A morphism $f : (T, \mathscr{O}_T) \to (S, \mathscr{O}_S)$ of ringed topoi is the datum of a morphism of topoi $f : T \to S$ and a morphism of T-rings :

$$f^\natural : f^*\mathscr{O}_S \to \mathscr{O}_T.$$

We denote, as usual, by $\mathscr{O}_T^\times \subset \mathscr{O}_T$ the subobject representing the invertible sections of $\mathscr{O}_T$. For every object U of T, and every $s \in \mathscr{O}_T(U)$, let $D(s) \subset U$ be the subobject such that :

$$\mathrm{Hom}_T(V, D(s)) := \{\varphi \in U(V) \mid \varphi^* s \in \mathscr{O}_T^\times(V)\}.$$

We say that $(T, \mathscr{O}_T)$ is *locally ringed*, if $D(0) = \varnothing_T$ (the initial object of T), and moreover

$$D(s) \cup D(1-s) = U \qquad \text{for every } U \in \mathrm{Ob}(T), \text{ and every } s \in \mathscr{O}_T(f).$$

A morphism of locally ringed topoi $f : (T, \mathscr{O}_T) \to (S, \mathscr{O}_S)$ is a morphism of ringed topoi such that

$$f^*D(s) = D(f^\natural(U)(f^*s)) \qquad \text{for every } U \in \mathrm{Ob}(S) \text{ and every } s \in \mathscr{O}_S(U).$$

If T has enough points, then $(T, \mathscr{O}_T)$ is locally ringed if and only if the stalks $\mathscr{O}_{T,\xi}$ of the structure ring at all the points ξ of T are local rings. Likewise, a morphism $f : (T, \mathscr{O}_T) \to (S, \mathscr{O}_S)$ of ringed topoi is locally ringed if and only if, for every T-point ξ, the induced map $\mathscr{O}_{S,f(\xi)} \to \mathscr{O}_{T,\xi}$ is a local ring homomorphism.

4.9.14. In the rest of this section we present a few results concerning the special case of topologies on a scheme. Hence, for any scheme X, we shall denote by $X_{\text{ét}}$ (resp. by X_{Zar}) the small étale (resp. the small Zariski) site on X. It is clear that X_{Zar} is a small site, and it is not hard to show that $X_{\text{ét}}$ is a U-site ([9, Exp.VII, §1.7]). The inclusion of underlying categories :

$$u_X : X_{\mathrm{Zar}} \to X_{\text{ét}}$$

is a continuous functor (see definition 4.2.1(i)) commuting with finite limits, whence a morphism of topoi :

$$\widetilde{u}_X := (\widetilde{u}_X^*, \widetilde{u}_{X*}) : X_{\text{ét}}^\sim \to X_{\mathrm{Zar}}^\sim.$$

such that the diagram of functors :

$$\begin{array}{ccc} X_{\mathrm{Zar}} & \xrightarrow{u_X} & X_{\text{ét}} \\ \downarrow & & \downarrow \\ X_{\mathrm{Zar}}^\sim & \xrightarrow{\widetilde{u}_X^*} & X_{\text{ét}}^\sim \end{array}$$

commutes, where the vertical arrows are the Yoneda embeddings (lemma 4.2.11).

The topoi $X_{\mathrm{Zar}}^\sim$ and $X_{\text{ét}}^\sim$ are locally ringed in a natural way, and by faithfully flat descent, we see easily that $\widetilde{u}_{X*}\mathscr{O}_{X_{\text{ét}}} = \mathscr{O}_{X_{\mathrm{Zar}}}$. By inspection, $\widetilde{u}_X$ is a morphism of locally ringed topoi.

4.9.15. For any ring R (of our fixed universe U), denote by Sch/R the category of R-schemes, and by $\mathsf{Sch}/R_{\mathrm{Zar}}$ (resp. $\mathsf{Sch}/R_{\text{ét}}$) the big Zariski (resp. étale) site on Sch/R. For $R = \mathbb{Z}$, we shall usually just write $\mathsf{Sch}_{\mathrm{Zar}}$ and $\mathsf{Sch}_{\text{ét}}$ for these sites. The morphisms u_X of (4.9.13) are actually restrictions of a single morphism of sites :

$$u : \mathsf{Sch}_{\mathrm{Zar}} \to \mathsf{Sch}_{\text{ét}}$$

which, for every universe V such that $\mathsf{U} \in \mathsf{V}$, induces a morphism of V-topoi :

$$\widetilde{u}_{\mathsf{V}} : (\mathsf{Sch}_{\text{ét}})_{\mathsf{V}}^\sim \to (\mathsf{Sch}_{\mathrm{Zar}})_{\mathsf{V}}^\sim.$$

4.9.16. Let X be scheme; a *geometric point* of X is a morphism of schemes $\xi : \mathrm{Spec}\,\kappa \to X$, where κ is an arbitrary separably closed field. Notice that both the Zariski and étale topoi of $\mathrm{Spec}\,\kappa$ are equivalent to the category $\mathbf{Set}$, so ξ induces a topos-theoretic point $\xi^{\sim}_{\text{ét}} : (\mathrm{Spec}\,\kappa)^{\sim}_{\text{ét}} \to X^{\sim}_{\text{ét}}$ of $X^{\sim}_{\text{ét}}$ (and likewise for $X^{\sim}_{\mathrm{Zar}}$). A basic feature of both the Zariski and étale topologies, is that every point of $X^{\sim}_{\mathrm{Zar}}$ and $X^{\sim}_{\text{ét}}$ arise in this way.

More precisely, we say that two geometric points ξ and ξ' of X are *equivalent*, if there exists a third such point ξ'' which factors through both ξ and ξ'. It is easily seen that this is an equivalence relation on the set of geometric points of X, and two topos-theoretic points $\xi^{\sim}_{\text{ét}}$ and $\xi'^{\sim}_{\text{ét}}$ are isomorphic if and only if the same holds for the points $\xi^{\sim}_{\mathrm{Zar}}$ and $\xi'^{\sim}_{\mathrm{Zar}}$, if and only if ξ is equivalent to ξ'.

Definition 4.9.17. Let X be a scheme, x a point of X, and $\overline{x} : \mathrm{Spec}\,\kappa \to X$ a geometric point.

(i) We let $\kappa(x)$ be the residue field of the local ring $\mathscr{O}_{X,x}$, and set

$$|x| := \mathrm{Spec}\,\kappa(x) \qquad \kappa(\overline{x}) := \kappa \qquad |\overline{x}| := \mathrm{Spec}\,\kappa(\overline{x})$$

If $\{x\} \subset X$ is the image of $\overline{x}$, we say that $\overline{x}$ is *localized at* x, and that x is the *support* of $\overline{x}$. We say that $\overline{x}$ is *strict* if $\kappa(\overline{x})$ is a separable closure of $\kappa(x)$.

(ii) We associate with $\overline{x}$ a strict geometric point $\overline{x}^{\mathrm{st}}$, as follows. Let $\kappa(\overline{x}^{\mathrm{st}})$ be the separable closure of $\kappa(x)$ inside $\kappa(\overline{x})$; the inclusion $\kappa(\overline{x}^{\mathrm{st}}) \subset \kappa(\overline{x})$ defines a morphism of schemes :

$$|\overline{x}| \to |\overline{x}^{\mathrm{st}}| := \mathrm{Spec}\,\kappa(\overline{x}^{\mathrm{st}}) \tag{4.9.18}$$

and $\overline{x}$ is the composition of (4.9.18) and a unique strict geometric point localized at x

$$\overline{x}^{\mathrm{st}} : |\overline{x}^{\mathrm{st}}| \to X.$$

(iii) The *localization of* X *at* x is the local scheme

$$X(x) := \mathrm{Spec}\,\mathscr{O}_{X,x}.$$

The *strict henselization of* X *at* $\overline{x}$ is the strictly local scheme

$$X(\overline{x}) := \mathrm{Spec}\,\mathscr{O}_{X,\overline{x}}$$

where $\mathscr{O}_{X,\overline{x}}$ denotes the strict henselization of $\mathscr{O}_{X,x}$ relative to the geometric point $\overline{x}$ ([66, Ch.IV, Déf.18.8.7]) (recall that a local ring is called *strictly local*, if it is henselian with separably closed residue field; a scheme is called *strictly local*, if it is the spectrum of a strictly local ring: see [66, Ch.IV, Déf.18.8.2]). By definition, the geometric point $\overline{x}$ lifts to a unique geometric point of $X(\overline{x})$, which shall be denoted again by $\overline{x}$.

(iv) Moreover, we shall denote by

$$i_x : X(x) \to X \qquad i_{\overline{x}} : X(\overline{x}) \to X(x)$$

the natural morphisms of schemes, and if $\mathscr{F}$ is any sheaf on X_{Zar} (resp. $X_{\text{ét}}$), we let

$$\mathscr{F}(x) := i_x^*\mathscr{F} \qquad \mathscr{F}(\overline{x}) := i_{\overline{x}}^*\mathscr{F}(x)$$

so $\mathscr{F}(x)$ is a sheaf on $X(x)_{\mathrm{Zar}}$ (resp. $X(x)_{\text{ét}}$) and $\mathscr{F}(\overline{x})$ is a sheaf on $X(\overline{x})_{\mathrm{Zar}}$ (resp. $X(\overline{x})_{\text{ét}}$).

(v) If $f : Y \to X$ is any morphism of schemes, we let

$$f^{-1}(x) := Y \times_X |x| \quad f^{-1}(\overline{x}) := Y \times_X |\overline{x}| \qquad Y(x) := Y \times_X X(x) \quad Y(\overline{x}) := Y \times_X X(\overline{x}).$$

Also, if $\overline{y}$ is any geometric point of Y, we define $f(\overline{y})$ as the geometric point $f \circ \overline{y}$ of X, and we call $f(\overline{y})^{\mathrm{st}}$ the *strict image* of $\overline{y}$ in X. Notice that the natural identification $|\overline{y}| = |f(\overline{y})|$ induces a morphism of schemes :

$$|\overline{y}^{\mathrm{st}}| \to |f(\overline{y})^{\mathrm{st}}|.$$

4.9.19. Many discussions concerning the Zariski or étale site of a scheme, only make appeal to general properties of these two topologies, and therefore apply indifferently to either of them, with only minor verbal changes. For this reason, to avoid tiresome repetitions, the following notational device is often useful. Namely, instead of referring each time to X_{Zar} and $X_{\text{ét}}$ in the course of an argument, we shall write just X_τ, with the convention that $\tau \in \{\mathrm{Zar}, \text{ét}\}$ has been chosen arbitrarily at the beginning of the discussion. In the same manner, a τ-point of X will mean a point of the topos $X_\tau^\sim$, and a τ-open subset of X will be any object of the site X_τ. With this convention, a Zar-point is a usual point of X, whereas an ét-point shall be a geometric point. Likewise, if ξ is a given τ-point of X, the localization $X(\xi)$ makes sense for both topologies : if $\tau = \text{ét}$, then $X(\xi)$ is the strict henselization as in definition 4.9.17(ii); if $\tau = \mathrm{Zar}$, then $X(\xi)$ is the usual localization of X at the (Zariski) point ξ. If $\tau = \text{ét}$, the support of ξ is given by definition 4.9.17(i); if $\tau = \mathrm{Zar}$, then the support of ξ is just ξ itself (and correspondingly, in this case ξ is localized at ξ). Furthermore, $\mathscr{O}_{X,\xi}$ is a local ring if $\tau = \mathrm{Zar}$, and it is a strictly local ring, in case $\tau = \text{ét}$.

4.9.20. Let $f : X \to Y$ be a morphism of schemes, $\overline{x}$ a geometric point of X, and set $\overline{y} := f(\overline{x})$. The natural morphism $f_x : X(x) \to Y(y)$ induces a unique local morphism of strictly local schemes

$$f_{\overline{x}} : X(\overline{x}) \to Y(\overline{y})$$

([66, Ch.IV, Prop.18.8.8(ii)]) that fits in a commutative diagram :

$$\begin{array}{ccccccc}
|\overline{x}| & \xrightarrow{\overline{x}} & X(\overline{x}) & \xrightarrow{i_{\overline{x}}} & X(x) & \xrightarrow{i_x} & X \\
\Vert & & \downarrow{\scriptstyle f_{\overline{x}}} & & \downarrow{\scriptstyle f_x} & & \downarrow{\scriptstyle f} \\
|\overline{y}| & \xrightarrow{\overline{y}} & Y(\overline{y}) & \xrightarrow{i_{\overline{y}}} & Y(y) & \xrightarrow{i_y} & Y.
\end{array}$$

Let now $\mathscr{F}$ be any sheaf on $Y_{\text{ét}}$; there follows a natural isomorphism :

$$f_{\overline{x}}^*\mathscr{F}(\overline{y}) \xrightarrow{\sim} (f^*\mathscr{F})(\overline{x}).$$

Notice also the natural bijections :

$$\begin{aligned}
\mathscr{F}_{\overline{y}} &\xrightarrow{\sim} \Gamma(Y(\overline{y}), \mathscr{F}(\overline{y})) \xrightarrow{\sim} \Gamma(|\overline{y}|, \overline{y}^*\mathscr{F}(\overline{y})) \\
f^*\mathscr{F}_{\overline{x}} &\xrightarrow{\sim} \Gamma(X(\overline{x}), f^*\mathscr{F}(\overline{x})) \xrightarrow{\sim} \Gamma(|\overline{x}|, \overline{x}^* f^*\mathscr{F}(\overline{x}))
\end{aligned} \tag{4.9.21}$$

which induce a natural identification :

$$\mathscr{F}_{\overline{y}} \xrightarrow{\sim} f^*\mathscr{F}_{\overline{x}} \quad : \quad \sigma \mapsto f_{\overline{x}}^*(\sigma). \tag{4.9.22}$$

4.9.23. Let X be a scheme, $x, x' \in X$ any two points, such that x is a specialization of x'. Choose a geometric point $\overline{x}$ localized at x. The localization map $\mathscr{O}_{X,x} \to \mathscr{O}_{X,x'}$ induces a natural *specialization morphism* of X-schemes :

$$X(x') \to X(x).$$

Set $W := X(\overline{x}) \times_{X(x)} X(x')$. The natural map $g : W \to X(x')$ is faithfully flat, and is the limit of a cofiltered system of étale morphisms; hence we may find $w \in W$ lying over the closed point of $X(x')$, and the induced map $\kappa(x') \to \kappa(w)$ is algebraic and separable. Choose also a geometric point $\overline{w}$ of W localized at w, and set $\overline{x}' := g(\overline{w})$. Then g induces an isomorphism $g_{\overline{w}} : W(\overline{w}) \xrightarrow{\sim} X(\overline{x}')$, whence a unique morphism

$$X(\overline{x}') \to X(\overline{x}) \tag{4.9.24}$$

which makes commute the diagram :

$$\begin{array}{ccccc} W(\overline{w}) & \xrightarrow{g_{\overline{w}}} & X(\overline{x}') & \xrightarrow{i_{\overline{x}'}} & X(x') \\ {\scriptstyle i_{\overline{w}}}\downarrow & & \downarrow & & \downarrow \\ W(w) & \longrightarrow & X(\overline{x}) & \xrightarrow{i_{\overline{x}}} & X(x) \end{array}$$

where the left bottom arrow is the natural projection, and the right-most vertical arrow is the specialization map. In this situation, we say that $\overline{x}$ is a *specialization* of $\overline{x}'$ (and that $\overline{x}'$ is a *generization* of $\overline{x}$), and we call (4.9.24) a *strict specialization morphism.* Combining with (4.9.21), we obtain the *strict specialization map induced by* (4.9.24)

$$\mathscr{G}_{\overline{x}} \to \mathscr{G}_{\overline{x}'} \qquad (4.9.25)$$

for every sheaf $\mathscr{G}$ on $X_{\text{ét}}$.

Remark 4.9.26. (i) In the situation of (4.9.20), suppose that $\mathscr{G} = f^*\mathscr{F}$ for a sheaf $\mathscr{F}$ on $Y_{\text{ét}}$. Then (4.9.25) is a map $\mathscr{F}_{f(\overline{x})} \to \mathscr{F}_{f(\overline{x}')}$. By inspecting the definition, it is easily seen that the latter agrees with the strict specialization map for $\mathscr{F}$ induced by a unique strict specialization morphism $Y(f(\overline{x}')) \to Y(f(\overline{x}))$.

(ii) Notice that (4.9.24) and (4.9.25) depend not only on the choice of w (which may not be unique, when $X(x)$ is not unibranch) but also on the geometric point $\overline{w}$. Indeed, the group of automorphisms of the $X(x')$-scheme $X(\overline{x}')$ is naturally isomorphic to the Galois group $\mathrm{Gal}(\kappa(x')^{\mathrm{s}}/\kappa(x'))$ ([66, Ch.IV, (18.8.8.1)]).

Lemma 4.9.27. *Let X be a scheme, $\mathscr{F}$ a sheaf on $X_{\text{ét}}$. We have :*

(i) *The counit of the adjunction $\varepsilon_{\mathscr{F}} : \widetilde{u}_X^* \circ \widetilde{u}_{X*}\mathscr{F} \to \mathscr{F}$ is a monomorphism.*
(ii) *Suppose there exist a sheaf $\mathscr{G}$ on X_{Zar}, and an epimorphism $f : \widetilde{u}^*\mathscr{G} \to \mathscr{F}$ (resp. a monomorphism $f : \mathscr{F} \to \widetilde{u}^*\mathscr{G}$). Then $\varepsilon_{\mathscr{F}}$ is an isomorphism.*
(iii) *The functor $\widetilde{u}_X^*$ is fully faithful.*
(iv) (Hilbert 90) $R^1\widetilde{u}_{X*}\mathscr{O}^\times_{X_{\text{ét}}} = 1_{X_{\text{Zar}}}$.

Proof. (i): The assertion can be checked on the stalks. Hence, let ξ be any geometric point of X; we have to show that the natural map $(\widetilde{u}_{X*}\mathscr{F})_\xi \to \mathscr{F}_\xi$ is injective. To this aim, say that $s, s' \in (\widetilde{u}_{X*}\mathscr{F})_\xi$, and suppose that the image of s in $\mathscr{F}_\xi$ agrees with the image of s'; we may find an open neighborhood U of ξ in X_{Zar}, such that s and s' lie in the image of $\mathscr{F}(U)$, and by assumption, there exists an étale morphism $f : V \to U$ such that the images of s and s' coincide in $\mathscr{F}(V)$. However, $f(V) \subset U$ is an open subset ([64, Ch.IV, Th.2.4.6]), and the induced map $V \to f(V)$ is a covering morphism in $X_{\text{ét}}$; it follows that the images of s and s' agree already in $\mathscr{F}(f(V))$, therefore also in $(\widetilde{u}_{X*}\mathscr{F})_\xi$.

(iii): According to proposition 1.1.20(iii), it suffices to show that the unit of the adjunction $\eta_{\mathscr{G}} : \mathscr{G} \to \widetilde{u}_{X*} \circ \widetilde{u}_X^*\mathscr{G}$ is an isomorphism, for every $\mathscr{G} \in \mathrm{Ob}(X^\sim_{\text{Zar}})$. However, we have morphisms :

$$\widetilde{u}_X^*\mathscr{G} \xrightarrow{\widetilde{u}_X^*(\eta_{\mathscr{G}})} \widetilde{u}_X^* \circ \widetilde{u}_{X*} \circ \widetilde{u}_X^*\mathscr{G} \xrightarrow{\varepsilon_{\widetilde{u}_X^*\mathscr{G}}} \widetilde{u}_X^*\mathscr{G}.$$

whose composition is the identity of $\widetilde{u}_X^*\mathscr{G}$ (see (1.1.13)); also, (i) says that $\varepsilon_{\widetilde{u}_X^*\mathscr{G}}$ is a monomorphism, and then it follows formally that it is actually an isomorphism (*e.g.* from the dual of [28, Prop.1.9.3]). Hence the same holds for $\widetilde{u}_X^*(\eta_{\mathscr{G}})$, and by considering the stalks of the latter, we conclude that also $\eta_{\mathscr{G}}$ is an isomorphism, as required.

(ii): Suppose first that $f : \widetilde{u}^*\mathscr{G} \to \mathscr{F}$ is an epimorphism. We have just seen that $\eta_{\mathscr{G}}$ is an isomorphism, therefore we have a morphism $\widetilde{u}^* \circ \widetilde{u}_* f : \widetilde{u}^*\mathscr{G} \to \widetilde{u}^* \circ \widetilde{u}_*\mathscr{F}$ whose composition with $\varepsilon_{\mathscr{F}}$ is f; especially, $\varepsilon_{\mathscr{F}}$ is an epimorphism, so the assertion follows from (i).

In the case of a monomorphism $f : \mathscr{F} \to \widetilde{u}^*\mathscr{G}$, set $\mathscr{H} := \widetilde{u}^*\mathscr{G} \amalg_{\mathscr{F}} \widetilde{u}^*\mathscr{G}$; we may represent f as the equalizer of the two natural maps $j_1, j_2 : \widetilde{u}^*\mathscr{G} \to \mathscr{H}$. However, the natural morphism

$\widetilde{u}^*(\mathscr{G} \amalg \mathscr{G}) \to \mathscr{H}$ is an epimorphism, hence the counit $\varepsilon_{\mathscr{H}}$ is an isomorphism, by the previous case. Then (iii) implies that $j_i = \widetilde{u}^* j'_i$ for morphisms $j'_i : \mathscr{G} \to \widetilde{u}_* \mathscr{H}$ ($i = 1, 2$). Let $\mathscr{F}'$ be the equalizer of j'_1 and j'_2; then $\widetilde{u}^*\mathscr{F}' \simeq \mathscr{F}$, and since we have already seen that the unit of adjunction is an isomorphism, the assertion follows from the triangular identitities of (1.1.13).

(iv): The assertion can be checked on the stalks. To ease notation, set $\mathscr{F} := R^1\widetilde{u}_{X*}\mathscr{O}^\times_{X_{\text{ét}}}$. Let ξ be any geometric point of X, and say that $s \in \mathscr{F}_\xi$; pick a (Zariski) open neighborhood $U \subset X$ of ξ such that s lies in the image of $\mathscr{F}(U)$. We may then find a Zariski open covering $(U_\lambda \to U \mid \lambda \in \Lambda)$ of U, such that the image of s in $\mathscr{F}(U_i)$ is represented by a $\mathscr{O}^\times_{U_\lambda,\text{ét}}$-torsor on $U_{\lambda,\text{ét}}$, for every $\lambda \in \Lambda$. After replacing U by any U_λ containing the support of ξ, we may assume that s is the image of the isomorphism class of some $\mathscr{O}^\times_{U_{\text{ét}}}$-torsor $X_{\text{ét}}$ on $U_{\text{ét}}$. By faithfully flat descent, there exist a $\mathscr{O}^\times_{U_{\text{Zar}}}$-torsor X on U_{Zar}, and an isomorphism of $\mathscr{O}^\times_{U_{\text{ét}}}$-torsors :

$$X_{\text{ét}} \xrightarrow{\sim} \mathscr{O}^\times_{U_{\text{ét}}} \otimes_{\widetilde{u}^*_U \mathscr{O}^\times_{U_{\text{Zar}}}} \widetilde{u}^*_U X.$$

However, after replacing U by a smaller open neighborhood of ξ, we may suppose that $X(U) \neq \varnothing$, therefore $X_{\text{ét}}(U) \neq \varnothing$ as well, *i.e.* s is the image of the trivial section of $\mathscr{F}(U)$. □

5. Stacks

Let $(\mathscr{C}, J)$ be any site, and consider the rule that assigns to every $U \in \text{Ob}(\mathscr{C})$ the category $(U, J_U)^\sim$ of sheaves on $\mathscr{C}/U$, for the topology J_U induced by J via the source functor $\mathscr{C}/U \to \mathscr{C}$. Every morphism $g : U' \to U$ of $\mathscr{C}$ induces a pull-back functor $\widetilde{g}^* : (U, J_U)^\sim \to (U', J_{U'})^\sim$, but if $g' : U'' \to U'$ is another such morphism, the composition $\widetilde{g}'^* \circ \widetilde{g}^*$ is not usually equal, but only *isomorphic* to the pull-back functor induced by $g \circ g'$. Moreover, given a covering family $(U_i \to U \mid i \in I)$, we can describe the category $(U, J_U)^\sim$ only *up to equivalence* in terms of the corresponding categories on the U_i and their intersections; this "gluing up to equivalence" is a 2-categorical manipulation of descent data relative to the given covering, so it involves not only double intersections $U_i \times U_j$, but also triple intersections. Summing up, the rule $U \mapsto (U, J_U)^\sim$ does not quite define a sheaf of categories on C, but rather a kind of 2-categorical analogue of the latter : it is the first example of a structure that is often encountered in algebraic geometry, and whose systematic study is the subject of the *theory of stacks* developed in this chapter.

5.1. **Prestacks and stacks on a site.**

As usual, U will denote a chosen universe, and small will be synonymous with U-small, throughout this section.

Definition 5.1.1. Let $C := (\mathscr{C}, J)$ be a site, $\varphi : \mathscr{A} \to \mathscr{C}$ a fibration, and $i \leq 2$ an integer. We say that φ is an *i-separated prestack* if for every $X \in \text{Ob}(\mathscr{C})$, every $\mathscr{S} \in J(X)$ is a sieve of φ-i-descent (see definition 3.5.10). A 2-separated prestack is called a *stack*. For every universe V, we denote by

$$\mathsf{V}\text{-}i\text{-}\mathsf{PreStack}(C) \qquad (\text{resp. } \mathsf{V}\text{-}\mathsf{Stack}(C)\ )$$

the full 2-subcategory of $\mathsf{V}\text{-}\mathsf{Fib}(\mathscr{C})$ whose objects are the i-separated prestacks (resp. the stacks) on C. Hence $\mathsf{V}\text{-}(-1)\text{-}\mathsf{PreStack}(C) = \mathsf{V}\text{-}\mathsf{Fib}(\mathscr{C})$ and $\mathsf{V}\text{-}2\text{-}\mathsf{PreStack}(C) = \mathsf{V}\text{-}\mathsf{Stack}(C)$. As usual, if $\mathsf{V} = \mathsf{U}$ we often drop the mention of the universe from this notation.

Example 5.1.2. Let $(\mathscr{C}, J)$ be any site, F any presheaf on $\mathscr{C}$, and $\mathsf{s}_F : \mathscr{F}ib(F) \to \mathscr{C}$ the fibration attached to F, as in (3.1.15). Let also $X \in \text{Ob}(\mathscr{C})$ be any object, and $\mathscr{S}$ any sieve of $\mathscr{C}/X$; by (3.1.15) and remark 3.5.2(ii) we have a commutative diagram

$$\begin{array}{ccc} FX = \text{Hom}_{\mathscr{C}^\wedge}(h_X, F) & \longrightarrow & \text{Hom}_{\mathscr{C}^\wedge}(h_{\mathscr{S}}, F) \\ \downarrow & & \downarrow \\ \mathscr{F}ib(F)(X) & \longrightarrow & \mathsf{Cart}_{\mathscr{C}}(\mathscr{S}, \mathscr{F}ib(F)) \end{array}$$

whose vertical arrows are bijections, and whose top (resp. bottom) horizontal arrow is induced by the inclusion $h_{\mathscr{S}} \to h_X$ (resp. $\mathscr{S} \subset \mathscr{C}/X$). We conclude that $\mathscr{F}ib(F)$ is a 0-separated prestack; by the same token, $\mathscr{F}ib(F)$ is a 1-separated prestack (resp. a stack) if and only if F is a separated presheaf (resp. a sheaf). Thus, we get a commutative diagram of functors

$$(5.1.3)\qquad \begin{array}{ccc} (\mathscr{C}, J)^{\sim} & \longrightarrow & \mathsf{Stack}(\mathscr{C}, J) \\ \downarrow & & \downarrow \\ \mathscr{C}^{\wedge} & \xrightarrow{\mathscr{F}ib} & 0\text{-}\mathsf{PreStack}(\mathscr{C}, J) \end{array}$$

(whose vertical arrows are the inclusion functors) which allows us to regard $(\mathscr{C}, J)^{\sim}$ as a full subcategory of $\mathsf{Stack}(\mathscr{C}, J)$.

5.1.4. We can characterize 0-separated and 1-separated prestacks on a small site $C := (\mathscr{C}, J)$ by means of the following construction. Let $\mathscr{A} \to \mathscr{C}$ be any fibration with small fibres. To every $X \in \mathrm{Ob}(\mathscr{C})$ and every pair of cartesian sections $\sigma, \sigma' \in \mathscr{A}(X)$ we attach the presheaf

$$\mathscr{C}art(\sigma, \sigma') : (\mathscr{C}/X)^o \to \mathbf{Set} \qquad (Y \xrightarrow{f} X) \mapsto \mathsf{Cart}_{\mathscr{C}}(\sigma \circ f_*, \sigma' \circ f_*)$$

which assigns to every morphism $(Z \xrightarrow{g} X) \xrightarrow{h/X} (Y \xrightarrow{f} X)$ of $\mathscr{C}/X$ the map

$$\mathscr{C}art(\sigma, \sigma')(f) \to \mathscr{C}art(\sigma, \sigma')(g) \qquad \beta \mapsto \beta * h_*$$

(with $f_* : \mathscr{C}/Y \to \mathscr{C}/X$ and $h_* : \mathscr{C}/Z \to \mathscr{C}/Y$ as in (1.1.25)). Notice that :

$$(5.1.5)\qquad (f_*)^{\wedge}\mathscr{C}art(\sigma, \sigma') = \mathscr{C}art(\sigma \circ f_*, \sigma' \circ f_*) \qquad \text{for every } (Y \xrightarrow{f} X) \in \mathrm{Ob}(\mathscr{C}/X).$$

Moreover, for every $\sigma, \sigma', \sigma'' \in \mathscr{A}(X)$, the system of composition maps

$$\mathsf{Cart}_{\mathscr{C}}(\sigma \circ f_*, \sigma' \circ f_*) \times \mathsf{Cart}_{\mathscr{C}}(\sigma' \circ f_*, \sigma'' \circ f_*) \to \mathsf{Cart}_{\mathscr{C}}(\sigma \circ f_*, \sigma'' \circ f_*) \qquad (\beta, \beta') \mapsto \beta' \odot \beta$$

clearly defines a morphism of presheaves on $\mathscr{C}/X$:

$$(5.1.6)\qquad \mathscr{C}art(\sigma, \sigma') \times \mathscr{C}art(\sigma', \sigma'') \to \mathscr{C}art(\sigma, \sigma'').$$

Furthermore, every pair of isomorphisms of cartesian sections : $\mu : \sigma \xrightarrow{\sim} \tau$, $\mu' : \sigma' \xrightarrow{\sim} \tau'$ induces an isomorphism of presheaves

$$(5.1.7)\quad \mathscr{C}art(\sigma, \sigma') \xrightarrow{\sim} \mathscr{C}art(\tau, \tau') \qquad (\beta : \sigma \circ f_* \Rightarrow \sigma' \circ f_*) \mapsto (\mu' * f_*) \odot \beta \odot (\mu^{-1} * f_*).$$

Lastly, every cartesian functor $F : \mathscr{A} \to \mathscr{A}'$ of $\mathscr{C}$-fibrations yields a morphism of presheaves :

$$(5.1.8)\qquad \mathscr{C}art(\sigma, \sigma') \to \mathscr{C}art(F \circ \sigma, F \circ \sigma') \qquad \beta \mapsto F * \beta.$$

Lemma 5.1.9. *With the notation of* (5.1.4)*, the following conditions are equivalent :*

(a) *The fibration $\mathscr{A} \to \mathscr{C}$ is 0-separated (resp. 1-separated).*
(b) *For every $X \in \mathrm{Ob}(\mathscr{C})$ and every pair of cartesian sections $\sigma, \sigma' \in \mathscr{A}(X)$, the presheaf $\mathscr{C}art(\sigma, \sigma')$ is separated (resp. is a sheaf) on the site C/X (notation of* (4.7)*).*

Proof. Let $X \in \mathrm{Ob}(\mathscr{C})$ be any object, and $\mathscr{S} \subset \mathscr{C}/X$ any covering sieve; choose a generating family $(f_i : Y_i \to X \mid i \in I)$ for $\mathscr{S}$, and for every $i, j \in I$ set $\mathscr{C}/Y_{ij} := \mathscr{C}/Y_i \times_{\mathscr{C}/X} \mathscr{C}/Y_j$. There follows a commutative diagram :

$$\begin{array}{ccccc} \mathscr{A}(X) & \xrightarrow{\mathsf{Cart}_{\mathscr{C}}(j, \mathscr{A})} & \mathsf{Cart}_{\mathscr{C}}(\mathscr{S}, \mathscr{A}) & & \\ & \searrow^{\rho} & \downarrow{\scriptstyle \varepsilon^*} & & \\ & & \prod_{i \in I} \mathscr{A}(Y_i) & \overset{\partial_0^*}{\underset{\partial_1^*}{\rightrightarrows}} & \prod_{i,j \in I} \mathsf{Cart}_{\mathscr{C}}(\mathscr{C}/Y_{ij}, \mathscr{A}) \end{array}$$

where $j : \mathscr{S} \to \mathscr{C}/X$ is the inclusion functor, ρ is the unique functor whose composition with the projection onto the factor $\mathscr{A}(Y_i)$ agrees with $\mathsf{Cart}_{\mathscr{C}}(f_{i*}, \mathscr{A})$, for every $i \in I$, and the functors ε^*, ∂_0^* and ∂_1^* are as in remark 3.5.11. Clearly ε^* is faithful; it follows that ρ is faithful if and only if the same holds for $\mathsf{Cart}_{\mathscr{C}}(j, \mathscr{A})$. On the other hand, the family $((f_i/X) : f_i \to \mathbf{1}_X \mid i \in I)$ of morphisms of $\mathscr{C}/X$ covers $\mathbf{1}_X$, according to (4.7). Taking into account (5.1.5), we conclude easily that $\mathscr{A}$ is 0-separated if and only if all the presheaves $\mathscr{C}art(\sigma, \sigma')$ are separated. Next, by remark 3.5.11 the functor ε^* induces an isomorphism of $\mathsf{Cart}_{\mathscr{C}}(\mathscr{S}, \mathscr{A})$ onto the equalizer E of ∂_0^* and ∂_1^*, and clearly ρ factors through a unique functor $\rho' : \mathscr{A}(X) \to E$. It follows that ρ' is fully faithful if and only if the same holds for $\mathsf{Cart}_{\mathscr{C}}(j, \mathscr{A})$. Again, this means that $\mathscr{A}$ is 1-separated if and only if all the presheaves $\mathscr{C}art(\sigma, \sigma')$ are sheaves. □

Theorem 5.1.10. *Let $C := (\mathscr{C}, J)$ be any small site. The inclusion strict pseudo-functor* $1\text{-}\mathsf{PreStack}(C) \to \mathsf{Fib}(\mathscr{C})$ *admits a strict and strong left 2-adjoint pseudo-functor :*

$$\mathsf{Fib}(\mathscr{C}) \to 1\text{-}\mathsf{PreStack}(C) \qquad \mathscr{A} \mapsto \mathscr{A}^{\mathrm{sep}}.$$

Proof. Let $\mathscr{A} \to \mathscr{C}$ be any fibration with small fibre categories; pick a unital cleavage $\boldsymbol{\lambda}$ for $\mathscr{A}$, and let c be its associated unital pseudo-functor. For every $X \in \mathrm{Ob}(\mathscr{C})$ consider the functor

$$\beta_X^{\boldsymbol{\lambda}} : \mathscr{A}_X \to \mathscr{A}(X) \qquad A \mapsto \beta_{X,A}^{\boldsymbol{\lambda}}$$

defined as in the proof of theorem 3.2.7. For every $A, A' \in \mathrm{Ob}(\mathscr{A}_X)$, consider as well the map

$$\omega_{A,A'} : \mathrm{Hom}_{\mathscr{A}_X}(A, A') \to \mathscr{C}art(\beta_{X,A}^{\boldsymbol{\lambda}}, \beta_{X,A'}^{\boldsymbol{\lambda}})(\mathbf{1}_X) \qquad (f : A \to A') \mapsto \beta_{X,f}^{\boldsymbol{\lambda}}.$$

Let $\mathscr{H}_{A,A'}$ be the sheaf on C/X associated with the presheaf $\mathscr{C}art(\beta_{X,A}^{\boldsymbol{\lambda}}, \beta_{X,A'}^{\boldsymbol{\lambda}})$, and denote

$$\widetilde{\omega}_{A,A'} : \mathrm{Hom}_{\mathscr{A}_X}(A, A') \to H_{A,A'} := \mathscr{H}_{A,A'}(\mathbf{1}_X)$$

the composition of $\omega_{A,A'}$ with the natural map $\mathscr{C}art(\beta_{X,A}^{\boldsymbol{\lambda}}, \beta_{X,A'}^{\boldsymbol{\lambda}})(\mathbf{1}_X) \to H_{A,A'}$. For every $A, A', A'' \in \mathrm{Ob}(\mathscr{A}_X)$, we have morphisms of presheaves as in (5.1.6)

$$\mathscr{C}art(\beta_{X,A}^{\boldsymbol{\lambda}}, \beta_{X,A'}^{\boldsymbol{\lambda}}) \times \mathscr{C}art(\beta_{X,A'}^{\boldsymbol{\lambda}}, \beta_{X,A''}^{\boldsymbol{\lambda}}) \to \mathscr{C}art(\beta_{X,A}^{\boldsymbol{\lambda}}, \beta_{X,A''}^{\boldsymbol{\lambda}})$$

whence a morphism of sheaves $\mathscr{H}_{A,A'} \times \mathscr{H}_{A',A''} \to \mathscr{H}_{A,A''}$, which induces a composition map :

$$H_{A,A'} \times H_{A',A''} \to H_{A,A''}.$$

It is easily seen that this composition law is associative, for every $A, A', A'', A''' \in \mathrm{Ob}(\mathscr{A}_X)$, and $\widetilde{\omega}_{A,A}(\mathbf{1}_A)$ is neutral for left and right composition. Hence, we have a category $\mathscr{A}_X^{\mathrm{sep}}$ whose set of objects is $\mathrm{Ob}(\mathscr{A}_X)$, and such that $\mathrm{Hom}_{\mathscr{A}_X^{\mathrm{sep}}}(A, A') := H_{A,A'}$ for every $A, A' \in \mathrm{Ob}(\mathscr{A}_X)$, with the foregoing composition law. Furthermore, the system of maps $\widetilde{\omega}_{\bullet\bullet}$ yields a functor

$$\widetilde{\omega}^X : \mathscr{A}_X \to \mathscr{A}_X^{\mathrm{sep}}$$

which is the identity map on objects. Next, notice that for every morphism $g : Y \to X$ in $\mathscr{C}$ and every $A \in \mathrm{Ob}(\mathscr{A}_X)$ we have $\beta_{X,A}^{\boldsymbol{\lambda}}(g) = \beta_{Y,\mathsf{c}_g A}^{\boldsymbol{\lambda}}(\mathbf{1}_Y) = \mathsf{c}_g A$; it follows that there exists a unique isomorphism of cartesian sections

$$\beta_{X,A}^{\boldsymbol{\lambda}} \circ g_* \xrightarrow{\sim} \beta_{Y,\mathsf{c}_g A}^{\boldsymbol{\lambda}} \qquad \text{such that} \qquad \mathbf{1}_Y \mapsto \mathbf{1}_{\mathsf{c}_g A}. \tag{5.1.11}$$

By virtue of (5.1.5) and (5.1.7), the isomorphisms (5.1.11) induce an isomorphism of presheaves

$$\mu_{(A,A')}^g : g_*^{\wedge}\mathscr{C}art(\beta_{X,A}^{\boldsymbol{\lambda}}, \beta_{X,A'}^{\boldsymbol{\lambda}}) \xrightarrow{\sim} \mathscr{C}art(\beta_{Y,\mathsf{c}_g A}^{\boldsymbol{\lambda}}, \beta_{Y,\mathsf{c}_g A'}^{\boldsymbol{\lambda}}) \qquad \text{for every } A, A' \in \mathrm{Ob}(\mathscr{A}_X).$$

Since g_* is continuous and cocontinuous for the topologies of C/X and C/Y (remark 4.7.3(i)), combining with lemma 4.2.15(ii) we deduce a natural isomorphism of sheaves on C/Y :

$$\widetilde{\mu}_{(A,A')}^g : j_g^* \mathscr{H}_{A,A'} \xrightarrow{\sim} \mathscr{H}_{\mathsf{c}_g A, \mathsf{c}_g A'}$$

whence a map $\mathsf{d}^g_{A,A'} : H_{A,A'} \xrightarrow{\mathscr{H}_{A,A'}(g/X)} \mathscr{H}_{A,A'}(g) \xrightarrow{\widetilde{\mu}^g_{(A,A'),\mathbf{1}_Y}} H_{\mathsf{c}_g A,\mathsf{c}_g A'}$ and it is then easily seen that the rules : $A \mapsto \mathsf{c}_g A$ for every $A \in \mathrm{Ob}(\mathscr{A}_X)$ and $f \mapsto \mathsf{d}^g_{A,A'}(f)$ for every morphism $f : A \to A'$ of $\mathscr{A}_X^{\mathrm{sep}}$ define a functor

$$\mathsf{d}_g : \mathscr{A}_X^{\mathrm{sep}} \to \mathscr{A}_Y^{\mathrm{sep}}.$$

Furthermore, a direct inspection yields a commutative diagram :

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathscr{A}_X}(A, A') & \xrightarrow{\widetilde{\omega}_{A,A'}} & H_{A,A'} \\
{\scriptstyle \mathsf{c}_{g,(A,A')}}\downarrow & & \downarrow{\scriptstyle \mathsf{d}^g_{A,A'}} \\
\mathrm{Hom}_{\mathscr{A}_Y}(\mathsf{c}_g A, \mathsf{c}_g A') & \xrightarrow{\widetilde{\omega}_{\mathsf{c}_g A,\mathsf{c}_g A'}} & H_{\mathsf{c}_g A,\mathsf{c}_g A'}
\end{array}$$

where $\mathsf{c}_{g,(A,A')}$ is the map given by the functor $\mathsf{c}_g : \mathscr{A}_X \to \mathscr{A}_Y$. In other words :

$$\mathsf{d}_g \circ \widetilde{\omega}^X = \widetilde{\omega}^Y \circ \mathsf{c}_g.$$

Next, let $Y' \xrightarrow{g'} Y \xrightarrow{g} X$ be two morphisms of $\mathscr{C}/X$; for every $A \in \mathrm{Ob}(\mathscr{A}_X)$ there exists a unique isomorphism of cartesian sections

$$(5.1.12) \qquad \beta^{\boldsymbol{\lambda}}_{Y',\mathsf{c}_{g'}\mathsf{c}_g A} \xrightarrow{\sim} \beta^{\boldsymbol{\lambda}}_{Y',\mathsf{c}_{gg'} A} \qquad \text{such that} \qquad \mathbf{1}_{Y'} \mapsto (\gamma^{\mathsf{c}}_{(g,g'),A} : \mathsf{c}_{g'}\mathsf{c}_g A \xrightarrow{\sim} \mathsf{c}_{gg'} A)$$

where $\gamma^{\mathsf{c}}_{(\bullet,\bullet)}$ denotes the coherence constraint of c. Denote by

$$\gamma^{\mathsf{d}}_{(g,g'),A} \in \mathrm{Hom}_{\mathscr{A}^{\mathrm{sep}}_{Y'}}(\mathsf{c}_g\mathsf{c}_{g'} A, \mathsf{c}_{gg'} A)$$

the image of (5.1.12), where the latter is seen as an element of $\mathscr{C}art(\beta^{\boldsymbol{\lambda}}_{Y',\mathsf{c}_{g'}\mathsf{c}_g A}, \beta^{\boldsymbol{\lambda}}_{Y',\mathsf{c}_{gg'} A})(\mathbf{1}_{Y'})$. A direct inspection of the construction yields a commutative diagram :

$$\begin{array}{ccc}
(g \circ g')^\wedge_* \mathscr{C}art(\beta^{\boldsymbol{\lambda}}_{X,A}, \beta^{\boldsymbol{\lambda}}_{X,A'}) & \xrightarrow{g'^\wedge_*(\mu^g_{(A,A')})} & g'^\wedge_* \mathscr{C}art(\beta^{\boldsymbol{\lambda}}_{Y,\mathsf{c}_g A}, \beta^{\boldsymbol{\lambda}}_{Y,\mathsf{c}_g A'}) \\
{\scriptstyle \mu^{g\circ g'}_{(A,A')}}\downarrow & & \downarrow{\scriptstyle \mu^{g'}_{(\mathsf{c}_g A,\mathsf{c}_g A')}} \\
\mathscr{C}art(\beta^{\boldsymbol{\lambda}}_{Y',\mathsf{c}_{gg'} A}, \beta^{\boldsymbol{\lambda}}_{Y',\mathsf{c}_{gg'} A'}) & \xrightarrow{\delta^{g,g'}_{(A,A')}} & \mathscr{C}art(\beta^{\boldsymbol{\lambda}}_{Y',\mathsf{c}_{g'}\mathsf{c}_g A}, \beta^{\boldsymbol{\lambda}}_{Y',\mathsf{c}_{g'}\mathsf{c}_g A'})
\end{array}$$

where $\delta^{g,g'}_{(A,A')}$ is the isomorphism of presheaves (5.1.7) induced by the isomorphisms (5.1.12). Let $(\delta^{g,g'}_{(A,A')})^a$ be the isomorphism of sheaves on C/X associated with $\delta^{g,g'}_{(A,A')}$; there follows a commutative diagram :

$$\begin{array}{ccc}
H_{A,A'} & \xrightarrow{\mathsf{d}^g_{A,A'}} & H_{\mathsf{c}_g A,\mathsf{c}_g A'} \\
{\scriptstyle \mathsf{d}^{gg'}_{A,A'}}\downarrow & & \downarrow{\scriptstyle \mathsf{d}^{g'}_{\mathsf{c}_g A,\mathsf{c}_g A'}} \\
H_{\mathsf{c}_{gg'} A,\mathsf{c}_{gg'} A} & \xrightarrow{\mathsf{d}^{g,g'}_{(A,A')}} & H_{\mathsf{c}_{g'}\mathsf{c}_g A,\mathsf{c}_{g'}\mathsf{c}_g A'}
\end{array}$$

with $\mathsf{d}^{g,g'}_{(A,A')} := (\delta^{g,g'}_{(A,A')})^a_{\mathbf{1}_{Y'}}$. This translates as the commutativity of the diagram :

$$\begin{array}{ccc}
\mathsf{c}_{g'}\mathsf{c}_g A & \xrightarrow{\mathsf{d}_{g'}\circ\mathsf{d}_g(f)} & \mathsf{c}_{g'}\mathsf{c}_g A' \\
{\scriptstyle \gamma^{\mathsf{d}}_{(g,g'),A}}\downarrow & & \downarrow{\scriptstyle \gamma^{\mathsf{d}}_{(g,g'),A'}} \\
\mathsf{c}_{gg'} A & \xrightarrow{\mathsf{d}_{gg'}(f)} & \mathsf{c}_{gg'} A'
\end{array} \qquad \text{for every } f \in \mathrm{Hom}_{\mathscr{A}_X^{\mathrm{sep}}}(A, A')$$

which means that the rule : $A \mapsto \gamma^{\mathsf{d}}_{(g,g'),A}$ yields an isomorphism of functors $\mathsf{d}_{g'} \circ \mathsf{d}_g \xrightarrow{\sim} \mathsf{d}_{gg'}$. Then it is easily seen that the rules : $X \mapsto \mathscr{A}_X^{\mathrm{sep}}$ and $g \mapsto \mathsf{d}_g$ for every $X \in \mathrm{Ob}(\mathscr{C})$ and every morphism g of $\mathscr{C}$ yield a unital pseudo-functor whose coherence constraint is the system of isomorphisms $\gamma^{\mathsf{d}}_{\bullet\bullet}$: the detailed verification shall be left to the reader. Likewise, it follows easily that the rule $X \mapsto \widetilde{\omega}^X$ defines a strict pseudo-natural transformation $\widetilde{\omega}^\bullet : \mathsf{c} \Rightarrow \mathsf{d}$. Set

$$\mathscr{A}^{\mathrm{sep}} := \mathscr{F}ib(\mathsf{d})$$

and let $\boldsymbol{\lambda}^*$ be the distinguished cleavage of $\mathscr{A}^{\mathrm{sep}}$, whose associated pseudo-functor is naturally identified with d (see remark 3.1.27(i)). Then we have a unique cartesian functor

$$\widetilde{\omega}_{\mathscr{A}} : \mathscr{A} \to \mathscr{A}^{\mathrm{sep}}$$

restricting to $\widetilde{\omega}^X : \mathscr{A}_X \to \mathscr{A}^{\mathrm{sep}}_X$ for every $X \in \mathrm{Ob}(\mathscr{C})$. A direct inspection then shows that

$$\beta^{\boldsymbol{\lambda}^*}_{X,A} = \widetilde{\omega}_{\mathscr{A}} \circ \beta^{\boldsymbol{\lambda}}_{X,A} \qquad \text{for every } X \in \mathrm{Ob}(\mathscr{C}) \text{ and } A \in \mathrm{Ob}(\mathscr{A}_X)$$

and we have a natural isomorphism of presheaves :

$$\mathscr{H}_{A,A'} \stackrel{\sim}{\to} \mathscr{C}art(\beta^{\boldsymbol{\lambda}^*}_{X,A}, \beta^{\boldsymbol{\lambda}^*}_{X,A'}) \qquad \text{for every } A, A' \in \mathrm{Ob}(\mathscr{A}_X)$$

which identifies the morphism of presheaves induced by $\widetilde{\omega}_{\mathscr{A}}$ as in (5.1.8) :

$$\mathscr{C}art(\beta^{\boldsymbol{\lambda}}_{X,A}, \beta^{\boldsymbol{\lambda}}_{X,A'}) \to \mathscr{C}art(\widetilde{\omega}_{\mathscr{A}} \circ \beta^{\boldsymbol{\lambda}}_{X,A}, \widetilde{\omega}_{\mathscr{A}} \circ \beta^{\boldsymbol{\lambda}}_{X,A'}) = \mathscr{C}art(\beta^{\boldsymbol{\lambda}^*}_{X,A}, \beta^{\boldsymbol{\lambda}^*}_{X,A'})$$

with the natural bicovering morphism of presheaves $\mathscr{C}art(\beta^{\boldsymbol{\lambda}}_{X,A}, \beta^{\boldsymbol{\lambda}}_{X,A'}) \to \mathscr{H}_{A,A'}$. Especially, $\mathscr{C}art(\beta^{\boldsymbol{\lambda}^*}_{X,A}, \beta^{\boldsymbol{\lambda}^*}_{X,A'})$ is a sheaf on C/X for every such X, A, A', so $\mathscr{A}^{\mathrm{sep}}$ is 1-separated over C, by lemma 5.1.9. Consider now any 1-separated fibration $\mathscr{A}'$ over C, and any cartesian functor $F : \mathscr{A} \to \mathscr{A}'$. For every $X \in \mathrm{Ob}(\mathscr{C})$ and $A, A' \in \mathscr{A}_X$, the morphism of presheaves

$$\mathscr{C}art(\beta^{\boldsymbol{\lambda}}_A, \beta^{\boldsymbol{\lambda}}_{A'}) \to \mathscr{C}art(F \circ \beta^{\boldsymbol{\lambda}}_A, F \circ \beta^{\boldsymbol{\lambda}}_{A'}) \tag{5.1.13}$$

as in (5.1.8) factors uniquely through $\mathscr{H}_{A,A'}$, since $\mathscr{C}art(F \circ \beta^{\boldsymbol{\lambda}}_A, F \circ \beta^{\boldsymbol{\lambda}}_{A'})$ is a sheaf on C/X (lemma 5.1.9). There follows a map $F^{\mathrm{sep}}_{A,A'} : H_{A,A'} \to \mathrm{Hom}_{\mathscr{A}'}(FA, FA')$, whose composition with $\widetilde{\omega}_{A,A'}$ equals $F_{A,A'} : \mathrm{Hom}_{\mathscr{A}}(A, A') \to \mathrm{Hom}_{\mathscr{A}'}(FA, FA')$. It is easily seen that the rules : $A \mapsto F^{\mathrm{sep}}A := FA$ and $(A, A') \mapsto F^{\mathrm{sep}}_{A,A'}$ yield a well defined functor $F^{\mathrm{sep}}_{|X} : \mathscr{A}^{\mathrm{sep}}_X \to \mathscr{A}'_X$ such that $F^{\mathrm{sep}}_{|X} \circ \widetilde{\omega}^X$ agrees with the restriction $\mathscr{A}_X \to \mathscr{A}'_X$ of F.

Let $\boldsymbol{\lambda}'$ be a unital cleavage for $\mathscr{A}'$, and c' its associated pseudo-functor; then F corresponds to a pseudo-natural transformation $\alpha : \mathsf{c} \Rightarrow \mathsf{c}'$, and denote by $\tau^\alpha_\bullet$ the coherence constraint of α. For every $X \in \mathrm{Ob}(\mathscr{C})$ and $A \in \mathrm{Ob}(\mathscr{A}_X)$ we have also a unique isomorphism

$$\beta^{\boldsymbol{\lambda}'}_{FA} \stackrel{\sim}{\to} F \circ \beta^{\boldsymbol{\lambda}}_A \qquad \text{such that} \qquad \mathbf{1}_X \mapsto \mathbf{1}_{FA}. \tag{5.1.14}$$

Explicitly, (5.1.14) assigns to every $(g : Y \to X) \in \mathrm{Ob}(\mathscr{C}/X)$ the isomorphism $\tau^\alpha_{g,A} : \mathsf{c}'_g FA \stackrel{\sim}{\to} F\mathsf{c}_g A$ of $\mathscr{A}'_Y$. On the other hand, we have a morphism of presheaves

$$\mathscr{C}art(\beta^{\boldsymbol{\lambda}}_A, \beta^{\boldsymbol{\lambda}}_A) \to \mathscr{C}art(\beta^{\boldsymbol{\lambda}'}_{FA}, \beta^{\boldsymbol{\lambda}'}_{FA}) \tag{5.1.15}$$

described explicitly as follows. For every $(g : Y \to X) \in \mathrm{Ob}(\mathscr{C}/X)$, every morphism of cartesian sections $t_\bullet : \beta^{\boldsymbol{\lambda}}_A \circ g_* \Rightarrow \beta^{\boldsymbol{\lambda}}_{A'} \circ g_*$ is determined by $t_{\mathbf{1}_Y}$: namely, $t_h = \mathsf{c}_h(t_{\mathbf{1}_Y})$ for every $h \in \mathrm{Ob}(\mathscr{C}/Y)$; then (5.1.15) maps every such $t_\bullet$ to the morphism $t'_\bullet : \beta^{\boldsymbol{\lambda}'}_{FA} \circ g_* \Rightarrow \beta^{\boldsymbol{\lambda}'}_{FA'} \circ g_*$ with $t'_h := \mathsf{c}'_h(Ft)$ for every $h \in \mathrm{Ob}(\mathscr{C}/Y)$. With this notation, the condition that τ^α_g is a natural isomorphism $\mathsf{c}'_g \circ F \stackrel{\sim}{\to} F \circ \mathsf{c}_g$ for every $g \in \mathrm{Ob}(\mathscr{C}/X)$ translates as the commutativity of the diagram of presheaves on $\mathscr{C}/X$:

$$\begin{array}{ccc}
\mathscr{C}art(\beta^{\boldsymbol{\lambda}}_A, \beta^{\boldsymbol{\lambda}}_A) & \longrightarrow & \mathscr{C}art(\beta^{\lambda'}_{FA}, \beta^{\boldsymbol{\lambda}'}_{FA'}) \\
\downarrow & & \downarrow \\
\mathscr{C}art(F \circ \beta^{\boldsymbol{\lambda}}_A, F \circ \beta^{\boldsymbol{\lambda}}_A) & \longrightarrow & \mathscr{C}art(F \circ \beta^{\boldsymbol{\lambda}}_A, \beta^{\lambda'}_{FA})
\end{array}$$

whose top horizontal (resp. left vertical) arrow is (5.1.15) (resp. (5.1.13)) and whose other two arrows are the morphisms (5.1.7) induced by the isomorphisms (5.1.14) (and by the identities of $\beta^{\lambda'}_{FA}$ and of $F\circ\beta^{\boldsymbol\lambda}_A$). After taking associated sheaves, we obtain a similar diagram, in which $\mathscr{C}art(\beta^{\boldsymbol\lambda}_A,\beta^{\boldsymbol\lambda}_A)$ is replaced by $\mathscr{H}_{A,A'}$. Unwinding the definitions, we find that the commutativity of the latter diagram yields the commutativity of the following diagram of morphisms of $\mathscr{A}_Y$:

$$\begin{array}{ccc} \mathsf{c}'_g FA & \xrightarrow{\mathsf{c}'_g F^{\mathrm{sep}} f} & \mathsf{c}'_g FA' \\ {\scriptstyle\tau^\alpha_{g,A}}\downarrow & & \downarrow{\scriptstyle\tau^\alpha_{g,A'}} \\ F\mathsf{c}_g A & \xrightarrow{F^{\mathrm{sep}}\mathsf{d}_g f} & F\mathsf{c}_g A' \end{array} \qquad \text{for every } f\in H_{A,A'}.$$

Thus, τ^α_g extends to a natural isomorphism $F^{\mathrm{sep}}_{|Y}\circ\mathsf{c}_g \xrightarrow{\sim} \mathsf{c}'_g\circ F^{\mathrm{sep}}_{|X}$, and then it is clear that the rule $X\mapsto F^{\mathrm{sep}}_{|X}$ yields a pseudo-natural transformation $\alpha' : \mathsf{d}\Rightarrow\mathsf{c}'$ whose coherence constraint is again $\tau^\alpha_\bullet$. Finally, α' corresponds to a unique cartesian functor $F^{\mathrm{sep}} : \mathscr{A}^{\mathrm{sep}}\to\mathscr{A}'$ whose restriction $\mathscr{A}^{\mathrm{sep}}_X\to\mathscr{A}'_X$ agrees with $F^{\mathrm{sep}}_{|X}$ for every $X\in\mathrm{Ob}(\mathscr{C})$. It is then clear that F^{sep} is the unique such cartesian functor whose composition with $\widetilde\omega_{\mathscr{A}}$ equals F. This universal property then easily implies that the rules : $\mathscr{A}\mapsto\mathscr{A}^{\mathrm{sep}}$ and $F\mapsto F^{\mathrm{sep}}$ yield a left adjoint $(-)^{\mathrm{sep}}$ for the inclusion functor $1\text{-}\mathsf{PreStack}(C)\mathsf{Fib}(\mathscr{C})$: the details shall be left to the reader.

Lastly, let $\mu : F\Rightarrow G$ be any natural $\mathscr{C}$-transformation of cartesian functors $F,G:\mathscr{A}\to\mathscr{A}'$ between $\mathscr{C}$-fibrations. Then for every $X\in\mathrm{Ob}(\mathscr{C})$ and every $A\in\mathscr{A}_X$, the morphism $\mu_A : FA\to GA$ determines a unique natural transformation $\mu^*_A : F\circ\beta^{\boldsymbol\lambda}_A\Rightarrow G\circ\beta^{\boldsymbol\lambda}_A$ such that $\mathbf{1}_X\mapsto\mu_A$. There follows for every $A,A'\in\mathrm{Ob}(\mathscr{A})$ a commutative diagram of presheaves :

$$\begin{array}{ccc} \mathscr{C}art(\beta^{\boldsymbol\lambda}_A,\beta^{\boldsymbol\lambda}_{A'}) & \longrightarrow & \mathscr{C}art(F\circ\beta^{\boldsymbol\lambda}_A, F\beta^{\boldsymbol\lambda}_{A'}) \\ \downarrow & & \downarrow \\ \mathscr{C}art(G\circ\beta^{\boldsymbol\lambda}_A, G\circ\beta^{\boldsymbol\lambda}_{A'}) & \longrightarrow & \mathscr{C}art(F\circ\beta^{\boldsymbol\lambda}_A, G\circ\beta^{\boldsymbol\lambda}_{A'}) \end{array}$$

whose top horizontal and left vertical arrows are the morphisms (5.1.13), and whose other two arrows are the morphisms (5.1.7) induced by μ^*_A and $\mu^*_{A'}$. After taking associated sheaves on C/X, we deduce a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathscr{A}^{\mathrm{sep}}}(A,A') & \longrightarrow & \mathrm{Hom}_{\mathscr{B}^{\mathrm{sep}}}(FA,FB) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathscr{A}^{\mathrm{sep}}}(GA,GA') & \longrightarrow & \mathrm{Hom}_{\mathscr{A}^{\mathrm{sep}}}(FA,GA') \end{array}$$

which shows that μ is also a natural transformation $F^{\mathrm{sep}}\Rightarrow G^{\mathrm{sep}}$. Clearly the rule $\mu\mapsto\mu^{\mathrm{sep}}$ is inverse to the rule $(\nu : F^{\mathrm{sep}}\Rightarrow G^{\mathrm{sep}})\mapsto\nu*\widetilde\omega_{\mathscr{A}}$, so $(-)^{\mathrm{sep}}$ is the sought strong left 2-adjoint of the inclusion pseudo-functor. □

5.1.16. Let $(\mathscr{C},J)$ be a small site. Our next task is to construct the analogue for stacks of the functor $F\mapsto F^+$ on presheaves (see (4.1.11)). To this aim, recall that for every $X\in\mathrm{Ob}(\mathscr{C})$ the set $J(X)$ of sieves covering X is cofiltered, and if $\mathscr{S}'\subset\mathscr{S}$ are two such sieves and $F:\mathscr{E}\to\mathscr{C}$ any fibration, the inclusion functor $i:\mathscr{S}'\to\mathscr{S}$ induces a functor

$$\mathsf{Cart}_{\mathscr{C}}(i,\mathscr{E}) : \mathsf{Cart}_{\mathscr{C}}(\mathscr{S},\mathscr{E})\to\mathsf{Cart}_{\mathscr{C}}(\mathscr{S}',\mathscr{E})$$

whence a well defined filtered system of small categories $C_X := (\mathsf{Cart}_{\mathscr{C}}(\mathscr{S},\mathscr{E})\mid\mathscr{S}\in J(X)^o)$ associated with every object X of $\mathscr{C}$. If $\mathscr{E}$ has small fibres, we may then consider the functor

$$\mathsf{c}^+_F : \mathscr{C}^o\to\mathbf{Cat} \qquad X\mapsto \operatorname*{colim}_{\mathscr{S}\in J(X)^o}\mathsf{Cart}_{\mathscr{C}}(\mathscr{S},\mathscr{E})$$

where the colimit is explicitly given by the construction detailed in example 1.5.10. For any morphism $f : Y \to X$ in $\mathscr{C}$, the corresponding functor $\mathsf{c}^+_F(f) : \mathsf{c}^+_F(X) \to \mathsf{c}^+_F(Y)$ is obtained as follows. For every $\mathscr{S} \in J(X)$, the functor $f_{*|\mathscr{S}} : \mathscr{S} \times_X f \to \mathscr{S}$ induces a functor

$$\mathsf{Cart}_{\mathscr{C}}(f_{*|\mathscr{S}}, \mathscr{E}) : \mathsf{Cart}_{\mathscr{C}}(\mathscr{S}, \mathscr{E}) \to \mathsf{Cart}(\mathscr{S} \times_X f, \mathscr{E}) \to \mathsf{c}^+_F(Y)$$

(see remark 3.5.2(v)) and the system of such functors obviously forms a cocone of vertex $\mathsf{c}^+_F(Y)$ and basis $C_X : J(X)^o \to \mathbf{Cat}$. There follows a unique functor $\mathsf{c}^+_F(f) : \mathsf{c}^+_F(X) \to \mathsf{c}^+_F(Y)$ whose composition with the natural functor $\mathsf{Cart}_{\mathscr{C}}(\mathscr{S}, \mathscr{E}) \to \mathsf{c}^+_F(X)$ agrees with $\mathsf{Cart}_{\mathscr{C}}(f_{*|\mathscr{S}}, \mathscr{E})$ for every $\mathscr{S} \in J(X)$. It is then easily seen that

$$\mathsf{c}^+_F(f \circ g) = \mathsf{c}^+_F(g) \circ \mathsf{c}^+_F(f) \qquad \text{for every pair of morphisms } Z \xrightarrow{g} Y \xrightarrow{f} X \text{ in } \mathscr{C}.$$

With the notation of (3.1.18), we set

$$\mathscr{E}^+ := \mathscr{F}ib(\mathsf{c}^+_F)$$

and let $F^+ : \mathscr{E}^+ \to \mathscr{C}$ be the structure functor of $\mathscr{E}^+$. Recall also that $\mathscr{E}^+$ is endowed with a distinguished split cleavage $\boldsymbol{\lambda}^+_F$ whose associated pseudo-functor is naturally identified with c^+_F (see remark 3.1.27(i)). Thus, the objects of $\mathscr{E}^+$ are represented by the pairs

$$[X, \psi : \mathscr{S} \to \mathscr{E}]$$

with $X \in \mathrm{Ob}(\mathscr{C})$ and ψ is a $\mathscr{C}$-cartesian functor defined on some covering sieve $\mathscr{S} \in J(X)$. A morphism $[f, \sigma] : [X', \psi' : \mathscr{S}' \to \mathscr{E}] \to [X, \psi : \mathscr{S} \to \mathscr{E}]$ is represented by the datum of a morphism $f : X' \to X$ in $\mathscr{C}$, and a natural $\mathscr{C}$-transformation $\sigma : \psi'_{|\mathscr{S}''} \Rightarrow (\psi \circ f_*)_{|\mathscr{S}''}$, where $\mathscr{S}'' \subset \mathscr{S}' \cap (\mathscr{S} \times_X f)$ is a sieve covering X'.

5.1.17. Next, let $F' : \mathscr{E}' \to \mathscr{C}$ be another fibration with small fibres, and $G : \mathscr{E} \to \mathscr{E}'$ any $\mathscr{C}$-cartesian functor; the system of induced functors

$$\mathsf{Cart}_{\mathscr{C}}(\mathscr{S}, G) : \mathsf{Cart}_{\mathscr{C}}(\mathscr{S}, \mathscr{E}) \to \mathsf{Cart}_{\mathscr{C}}(\mathscr{S}, \mathscr{E}') \qquad \text{for every } X \in \mathrm{Ob}(\mathscr{C}) \text{ and } \mathscr{S} \in J(X)$$

yields, after taking colimits, a strict pseudo-natural transformation

$$\mathsf{c}^+_G : \mathsf{c}^+_F \Rightarrow \mathsf{c}^+_{F'} \qquad \text{and then a } \mathscr{C}\text{-cartesian functor :} \qquad G^+ := \mathscr{F}ib(\mathsf{c}^+_G) : \mathscr{E}^+ \to \mathscr{E}'^+.$$

Moreover, if $\omega : G \Rightarrow G'$ is any natural $\mathscr{C}$-transformation of $\mathscr{C}$-cartesian functors $G, G' : \mathscr{E} \to \mathscr{E}'$, we get a system of natural transformations

$$\mathsf{Cart}_{\mathscr{C}}(\mathscr{S}, \omega) : \mathsf{Cart}_{\mathscr{C}}(\mathscr{S}, G) \Rightarrow \mathsf{Cart}_{\mathscr{C}}(\mathscr{S}, G') \qquad \text{for every } X \in \mathrm{Ob}(\mathscr{C}) \text{ and } \mathscr{S} \in J(X)$$

whence again, after taking colimits, a modification

$$\mathsf{c}^+_\omega : \mathsf{c}^+_G \rightsquigarrow \mathsf{c}^+_{G'} \qquad \text{whence a natural transformation :} \qquad \omega^+ := \mathsf{Fib}(\mathsf{c}^+_\omega) : G^+ \Rightarrow G'^+.$$

It is then clear that the rules : $(F : \mathscr{E} \to \mathscr{C}) \mapsto F^+$, $(G : \mathscr{E} \to \mathscr{E}') \mapsto G^+$ and $(\omega : G \Rightarrow G') \mapsto \omega^+$ yield a well defined strict pseudo-functor

$$(-)^+ : \mathsf{Fib}(\mathscr{C}) \to \mathsf{Fib}(\mathscr{C}).$$

Moreover, since $\mathscr{C}/X \in J(X)$ for every $X \in \mathrm{Ob}(\mathscr{C})$, we have an obvious strict pseudo-natural transformation of strict pseudo-functors :

$$\mathsf{j}_{\mathscr{E}} : \mathsf{Cart}_{\mathscr{C}}(\mathscr{C}/-, \mathscr{E}) \Rightarrow \mathsf{c}^+_{\mathscr{E}}$$

(notation of (3.2.4)), whence a $\mathscr{C}$-cartesian functor

$$j_{\mathscr{E}} := \mathsf{Fib}(\mathsf{j}_F) : \mathsf{C}(\mathscr{E}) \to \mathscr{E}^+$$

and it is clear that the rule $(F : \mathscr{E} \to \mathscr{C}) \mapsto j_{\mathscr{E}}$ yields a strict pseudo-natural transformation of strict pseudo-functors.

Remark 5.1.18. Let $F : \mathscr{E} \to \mathscr{C}$ be any fibration with small fibres, $X \in \mathrm{Ob}(\mathscr{C})$ any object, and $\mathscr{S}$ any sieve of $\mathscr{C}/X$ covering X. By unwinding the definitions, and taking into account lemma 3.1.20(i), we see that a $\mathscr{C}$-cartesian functor $\mathscr{S} \to \mathscr{E}^+$ is the datum of :

- for every object $Y \xrightarrow{f} X$ of $\mathscr{S}$, a sieve $\mathscr{S}^{(f)}$ covering Y and a $\mathscr{C}$-cartesian functor
$$\sigma^{(f)} : \mathscr{S}^{(f)} \to \mathscr{E}$$
- for every morphism $h/X : (Y' \xrightarrow{f'} X) \to (Y \xrightarrow{f} X)$ of $\mathscr{S}$, a sieve $\mathscr{S}^{(h/X)} \subset \mathscr{S}^{(f')} \cap (\mathscr{S}^{(f)} \times_Y h)$ covering Y' and a natural isomorphism of $\mathscr{C}$-functors
$$\sigma^{(h/X)} : \sigma^{(f')}_{|\mathscr{S}^{(h/X)}} \xrightarrow{\sim} (\mathsf{c}^+_F(h)(\sigma^{(f)}))_{|\mathscr{S}^{(h/X)}} = (\sigma^{(f)} \circ h_*)_{|\mathscr{S}^{(h/X)}}$$
where $h_* : \mathscr{S}^{(f)} \times_Y h \to \mathscr{S}^{(f)}$ is the functor $(Z \xrightarrow{g} Y') \mapsto (Z \xrightarrow{h\circ g} Y)$
- for every h/X as in the foregoing and every other morphism $h'/X : (Y'' \xrightarrow{f''} X) \to (Y' \xrightarrow{f} X)$ of $\mathscr{S}_\sigma$, a covering sieve $\mathscr{S}^{(h,h'/X)} \subset \mathscr{S}^{(h'/X)} \cap (\mathscr{S}^{(h/X)} \times_{Y'} h')$ such that

$$(\sigma^{(h/X)} * h'_*)_{|\mathscr{S}^{(h,h'/X)}} \odot \sigma^{(h'/X)}_{|\mathscr{S}^{(h,h'/X)}} = \sigma^{(h\circ h'/X)}_{|\mathscr{S}^{(h,h'/X)}}. \tag{5.1.19}$$

Lemma 5.1.20. *Let $i \leq 2$ be an integer, $F : \mathscr{E} \to \mathscr{C}$ an i-separated prestack, $X \in \mathrm{Ob}(\mathscr{C})$. Then every inclusion $\mathscr{T} \subset \mathscr{S}$ of sieves of $\mathscr{C}/X$ covering X induces an i-faithful functor*
$$j : \mathsf{Cart}_{\mathscr{C}}(\mathscr{S}, \mathscr{E}) \to \mathsf{Cart}_{\mathscr{C}}(\mathscr{T}, \mathscr{E}).$$

Proof. It is an immediate consequence of lemma 3.5.35(i). □

Lemma 5.1.21. *Let $\mathscr{C}$ be a small category, $F : \mathscr{E} \to \mathscr{C}$ a fibration with small fibres. We have :*

(i) *The prestack $F^+ : \mathscr{E}^+ \to \mathscr{C}$ is 0-separated.*
(ii) *If F is a 0-separated prestack, then F^+ is a 1-separated prestack.*
(iii) *If F is a 1-separated prestack, then F^+ is a stack.*
(iv) *If F is a stack, the functor $j_{\mathscr{E}} : \mathsf{C}(\mathscr{E}) \to \mathscr{E}^+$ is an equivalence.*
(v) *There exists an essentially commutative diagram of strict pseudo-functors (notation of definition* 4.1.5(ii)*) :*

$$\begin{array}{ccc} \mathscr{C}^\wedge & \xrightarrow{\mathscr{F}ib} & 0\text{-}\mathsf{PreStack}(\mathscr{C}, J) \\ {\scriptstyle (-)^+}\downarrow & & \downarrow{\scriptstyle (-)^+} \\ (\mathscr{C}, J)^{\mathrm{sep}} & \xrightarrow{\mathscr{F}ib} & 1\text{-}\mathsf{PreStack}(\mathscr{C}, J). \end{array}$$

Proof. Let $X \in \mathrm{Ob}(\mathscr{C})$ be any object, and $\mathscr{T}$ any covering sieve of $\mathscr{C}/X$; let also $[\varphi], [\psi] \in (F^+)^{-1}X$, and $\mathscr{S}$ a sieve covering X such that $[\varphi]$ and $[\psi]$ are represented by $\mathscr{C}$-cartesian functors $\varphi, \psi : \mathscr{S} \to \mathscr{E}$.

(i): Let $[\alpha], [\beta] : [\varphi] \to [\psi]$ be morphisms of $(F^+)^{-1}X$ whose images agree under the functor
$$j : (F^+)^{-1}X \to \mathsf{Cart}_{\mathscr{C}}(\mathscr{T}, \mathscr{E}^+).$$
deduced from the split cleavage c^+_F. We need to check that $\alpha = \beta$. To this aim, may assume that $[\alpha]$ and $[\beta]$ are represented by natural $\mathscr{C}$-transformations $\alpha, \beta : \varphi \Rightarrow \psi$; then, for every $(Y \xrightarrow{f} X) \in \mathrm{Ob}(\mathscr{T})$ there exists a sieve $\mathscr{T}^{(f)} \subset \mathscr{S} \times_X f$ covering Y, such that
$$(\alpha * f_*)_{|\mathscr{T}^{(f)}} = (\beta * f_*)_{|\mathscr{T}^{(f)}}.$$
Now, let $\mathscr{S}' \subset \mathscr{S}$ be the sieve generated by $\bigcup_{f \in \mathrm{Ob}(\mathscr{T})} f_*(\mathrm{Ob}(\mathscr{T}^{(f)}))$; by remark 4.1.3(iii), the sieve $\mathscr{S}'$ covers X, and clearly $\alpha_{|\mathscr{S}'} = \beta_{|\mathscr{S}'}$, whence the contention.

(ii): Let $\alpha : j([\varphi]) \Rightarrow j([\psi])$ be a given natural $\mathscr{C}$-transformation; we need to check that $\alpha = j(\alpha')$ for some sieve $\mathscr{S}' \subset \mathscr{S}$ covering X and some natural $\mathscr{C}$-transformation $\alpha' : \varphi_{|\mathscr{S}'} \Rightarrow \psi_{|\mathscr{S}'}$. However, α is described by a datum as follows :

- For every $(Y \xrightarrow{f} X) \in \mathrm{Ob}(\mathscr{T})$, a sieve $\mathscr{T}^{(f)} \subset \mathscr{S} \times_X f$ covering Y and a natural $\mathscr{C}$-transformation $\alpha^{(f)} : (\varphi \circ f_*)_{|\mathscr{T}^{(f)}} \Rightarrow (\psi \circ f_*)_{|\mathscr{T}^{(f)}}$
- for every morphism $(h/X) : (Y' \xrightarrow{f'} X) \to (Y \xrightarrow{f} X)$ in $\mathscr{T}$, a sieve $\mathscr{T}^{(h/X)}$ covering Y' and contained in $\mathscr{T}^{(f')} \cap (\mathscr{T}^{(f)} \times_Y h)$ such that

$$\alpha^{(f')}_{|\mathscr{T}^{(h/X)}} = (\alpha^{(f)} * h_*)_{|\mathscr{T}^{(h/X)}}.$$

But then, in view of lemma 5.1.20, we may take already $\mathscr{T}^{(h/X)} := \mathscr{T}^{(f')} \cap (\mathscr{T}^{(f)} \times_Y h)$. With this notation, let $\mathscr{S}'$ be the sieve generated by $\bigcup_{f \in \mathrm{Ob}(\mathscr{T})} f_*(\mathrm{Ob}(\mathscr{T}^{(f)}))$; then $\mathscr{S}'$ covers X, by remark 4.1.3(iii). For every $(Y \xrightarrow{g} X) \in \mathrm{Ob}(\mathscr{S}')$, pick any $f \in \mathrm{Ob}(\mathscr{T})$ such that $g = g' \circ f$ for some $g' \in \mathrm{Ob}(\mathscr{T}^{(f)})$, and set

$$\alpha'_g := \alpha^{(f)}_{g'} : \varphi(g) \to \psi(g).$$

Let us check that this definition is independent of the choices of f and g'. Indeed, suppose that $g = g'' \circ f'$ for some other $f' \in \mathrm{Ob}(\mathscr{T})$ and some $g'' \in \mathrm{Ob}(\mathscr{T}^{(f')})$; then we have

$$(\alpha^{(f)} * g'_*)_{|\mathscr{T}'} = (\alpha^{(g)})_{|\mathscr{T}'} = (\alpha^{(f')} * g''_*)_{|\mathscr{T}'} \quad \text{with } \mathscr{T}' := \mathscr{T}^{(g)} \cap (\mathscr{T}^{(f)} \times_Y g') \cap (\mathscr{T}^{(f')} \times_Y g'')$$

and by invoking again lemma 5.1.20, we deduce that

$$(\alpha^{(f)} * g'_*)_{|\mathscr{T}''} = (\alpha^{(f')} * g''_*)_{|\mathscr{T}''} \qquad \text{with } \mathscr{T}'' := (\mathscr{T}^{(f)} \times_Y g') \cap (\mathscr{T}^{(f')} \times_Y g'').$$

Especially, we get : $\alpha^{(f)}_{g'} = (\alpha^{(f)} * g'_*)_{\mathbf{1}_Y} = (\alpha^{(f')} * g''_*)_{\mathbf{1}_Y} = \alpha^{(f')}_{g''}$. It is then clear that α' is a well defined natural $\mathscr{C}$-transformation as sought.

(v) follows now from (ii) and a direct inspection of the construction of the functors $(-)^+$ for presheaves and for prestacks.

(iii): Let $\sigma : \mathscr{T} \to \mathscr{E}^+$ be a $\mathscr{C}$-cartesian functor; hence, this is a datum

$$\Big((\mathscr{S}^{(f)}, \sigma^{(f)} \mid f \in \mathrm{Ob}(\mathscr{T})), (\mathscr{S}^{(h/X)}, \sigma^{(h/X)} \mid f' \xrightarrow{h} f \text{ in } \mathscr{T}), (\mathscr{S}^{(h,h'/X)} \mid f'' \xrightarrow{h'} f' \xrightarrow{h} f \text{ in } \mathscr{T})\Big)$$

as in remark 5.1.18. But then, lemma 5.1.20 says that we may even take :

$$\mathscr{S}^{(h/X)} := \mathscr{S}^{(f')} \cap (\mathscr{S}^{(f)} \times_Y h) \qquad \mathscr{S}^{(h,h'/X)} := \mathscr{S}^{(f'')} \cap (\mathscr{S}^{(f')} \times_{Y'} h') \cap (\mathscr{S}^{(f)} \times_Y (h \circ h'))$$

for every $f \in \mathrm{Ob}(\mathscr{T})$, every morphism $f' \xrightarrow{h} f$ in $\mathscr{T}$, and every pair $f'' \xrightarrow{h'} f' \xrightarrow{h} f$ of morphisms in $\mathscr{T}$. Let $\mathscr{T}' \subset \mathscr{C}/X$ be the sieve generated by $\bigcup_{f \in \mathrm{Ob}(\mathscr{T})} f_*(\mathrm{Ob}(\mathscr{S}^{(f)}))$, which covers X, by remark 4.1.3(iii). We consider the functor

$$G_{\mathscr{T}'} : \mathscr{T}' \to \mathbf{Cat}/(\mathscr{C}/X) \qquad (Y \xrightarrow{g} X) \mapsto (\mathscr{C}/X)/g = \mathscr{C}/Y$$

defined as in (3.5.8), and we construct a pseudo-cocone

$$\psi_\bullet : G_{\mathscr{T}'} \Rightarrow \mathsf{F}_{\mathscr{E}}$$

as follows. For every $(Y \xrightarrow{g} X) \in \mathrm{Ob}(\mathscr{T}')$, choose $g' \in \mathrm{Ob}(\mathscr{T})$ and $g'' \in \mathrm{Ob}(\mathscr{S}^{(g')})$ with

$$g = g' \circ g'' \qquad \text{and set :} \qquad \psi_g := \sigma^{(g')} \circ g''_* : \mathscr{C}/Y \to \mathscr{E}.$$

Next, let $g_1, g_2 \in \mathrm{Ob}(\mathscr{T}')$ and $h/X : (Z_1 \xrightarrow{g_1} X) \to (Z_2 \xrightarrow{g_2} X)$ a morphism of $\mathscr{T}'$; with this notation, we have the $\mathscr{C}$-cartesian functors

$$\sigma^{(g'_1)} \circ g''_{1*}, \sigma^{(g'_2)} \circ (g''_2 \circ h)_* : \mathscr{C}/Z_1 \to \mathscr{E}$$

and two isomorphisms of $\mathscr{C}$-functors

$$(\sigma^{(g'_1)} \circ g''_{1*})_{|\mathscr{S}^{(g_1)}} \xLeftarrow{\sigma^{(g''_1/X)}} \sigma^{(g_1)} \xRightarrow{\sigma^{(g''_2 \circ h/X)}} (\sigma^{(g'_2)} \circ (g''_2 \circ h)_*)_{|\mathscr{S}^{(g_1)}}.$$

Since F is 1-separated, it follows that the composition $\sigma^{(g''_2\circ h/X)}_{|\mathscr{S}(g_1)} \odot (\sigma^{(g''_1/X)})^{-1}_{|\mathscr{S}(g_1)}$ extends uniquely to an isomorphism of $\mathscr{C}$-functors

$$\tau^{\psi}_{h/X} : \psi_{g_1} \xrightarrow{\sim} \psi_{g_2} \circ h_*.$$

We claim that the rule $g \mapsto \psi_g$ yields a pseudo-cocone as sought, with coherence constraint given by the system of isomorphisms $\tau^{\psi}_{\bullet}$. Indeed, let $h'/X : (Z_0 \xrightarrow{g_0} X) \to (Z_1 \xrightarrow{g_1} X)$ be another morphism of $\mathscr{T}'$; then we have as well the isomorphisms of $\mathscr{C}$-functors

$$\begin{array}{ccccc} (\sigma^{(g'_0)} \circ g''_{0*})_{|\mathscr{S}(g_0)} & \xLeftarrow{\sigma^{(g''_0/X)}} & \sigma^{(g_0)} & \xRightarrow{\sigma^{(g''_1\circ h'/X)}} & (\sigma^{(g'_1)} \circ (g''_1 \circ h')_*)_{|\mathscr{S}(g_0)} \\ & & \Big\Downarrow{\scriptstyle \sigma^{(g''_2\circ h'\circ h/X)}} & & \\ & & (\sigma^{(g'_2)} \circ (g''_2 \circ h \circ h')_*)_{|\mathscr{S}(g_0)} & & \end{array}$$

and with this notation, we have :

$$(\tau^{\psi}_{h'/X})_{|\mathscr{S}(g_0)} = \sigma^{(g''_1\circ h'/X)} \odot (\sigma^{(g''_0/X)})^{-1} \qquad (\tau^{\psi}_{h\circ h'/X})_{|\mathscr{S}(g_0)} = (\sigma^{(g''_2\circ h\circ h'/X)}) \odot (\sigma^{(g''_0/X)})^{-1}.$$

Set $\mathscr{S} := \mathscr{S}^{(g''_2\circ h, h'/X)} \cap \mathscr{S}^{(g''_1, h'/X)} \subset \mathscr{S}^{(g_0)}$. We claim that

$$(\tau^{\psi}_{h\circ h'/X})_{|\mathscr{S}} = (\tau^{\psi}_{h/X} * h'_*)_{|\mathscr{S}} \odot (\tau^{\psi}_{h'/X})_{|\mathscr{S}}.$$

Indeed, the latter identity follows by combining the following ones :

$$\sigma^{(g''_2\circ h\circ h'/X)}_{|\mathscr{S}} = (\sigma^{(g''_2\circ h/X)} * h'_*)_{|\mathscr{S}} \odot \sigma^{(h'/X)}_{|\mathscr{S}}$$
$$\sigma^{(g''_1\circ h'/X)}_{|\mathscr{S}} = (\sigma^{(g''_1/X)} * h'_*)_{|\mathscr{S}} \odot \sigma^{(h'/X)}_{|\mathscr{S}}$$

that are provided by remark 5.1.18. Since $\mathscr{S}$ covers Z_0 and F is 1-separated, it follows that

$$\tau^{\psi}_{h\circ h'/X} = (\tau^{\psi}_{h/X} * h'_*) \odot \tau^{\psi}_{h'/X}$$

which shows that τ^{ψ} satisfies the required coherence axioms. By lemma 3.5.9, we deduce that there exist a $\mathscr{C}$-functor $\psi : \mathscr{T}' \to \mathscr{E}$ and an invertible modification $\Xi : \mathsf{F}_\psi \odot \widehat{\varepsilon}_{\mathscr{T}'} \rightsquigarrow \psi_\bullet$, where $\widehat{\varepsilon}_{\mathscr{T}'} : G_{\mathscr{T}'} \Rightarrow \mathsf{F}_{\mathscr{T}'}$ is the universal pseudo-cocone defined as in (3.5.8); explicitly, this amounts to a system of oriented squares with invertible 2-cells :

$$\begin{array}{ccc} \mathscr{C}/Y & \xrightarrow{g_*} & \mathscr{T}' \\ {\scriptstyle g''_*}\big\downarrow & \not\swarrow \Xi_g & \big\downarrow{\scriptstyle \psi} \\ \mathscr{S}^{(g')} & \xrightarrow[\sigma^{(g')}]{} & \mathscr{E} \end{array} \qquad \text{for every } g \in \mathrm{Ob}(\mathscr{T}')$$

from which it follows easily that ψ is $\mathscr{C}$-cartesian (here $g = g' \circ g''$ is the factorisation used to define ψ_g). The compatibility condition for Ξ amounts to the identity :

(5.1.22) $\Xi_{g_2} * h_* = \tau^{\psi}_{h/X} \odot \Xi_{g_1}$ for every morphism $(Z_1 \xrightarrow{g_1} X) \xrightarrow{h/X} (Z_2 \xrightarrow{g_2} X)$ of $\mathscr{T}'$.

Thus, the functor ψ represents an object $[\psi]$ of the fibre category $\mathscr{E}^+(X)$, and to conclude, we need to check that $j([\psi])$ is isomorphic to σ. However, since we know already from (ii) that $\mathscr{E}^+$ is a 1-separated prestack, by lemma 5.1.20 it suffices to find an isomorphism between the images of $j([\psi])$ and σ in $\mathsf{Cart}_{\mathscr{C}}(\mathscr{T}', \mathscr{E}^+)$. This comes down to exhibiting isomorphisms of functors

$$\vartheta_g : \psi \circ g_* \xrightarrow{\sim} \sigma^{(g)} \qquad \text{for every } g \in \mathrm{Ob}(\mathscr{T}')$$

(where $g_* : \mathscr{S}^{(f)} \to \mathscr{T}'$ is the usual functor with $t \mapsto g \circ t$ for every $t \in \mathrm{Ob}(\mathscr{S}^{(f)})$) such that

(5.1.23) $\sigma^{(h/X)} \odot \vartheta_{g_1|\mathscr{S}(h/X)} = (\vartheta_{g_2} * h_*)_{|\mathscr{S}(h/X)}$ for every morphism $g_1 \xrightarrow{h/X} g_2$ of $\mathscr{T}'$.

To this aim, we set :

$$\vartheta_g := (\sigma^{(g''/X)})^{-1} \odot (\Xi_g)_{|\mathscr{S}^{(g)}}.$$

Since $\mathscr{E}$ is 1-separated, the identities (5.1.23) can be checked after restriction to any covering sieve contained in $\mathscr{S}^{(h/X)}$; but on such a suitable sieve we may apply the identities (5.1.19), and combining with (5.1.22) we conclude easily : the details shall be left to the reader.

(iv): It suffices to check that $j_{\mathscr{E}}$ is a fibrewise equivalence when $\mathscr{E}$ is a stack (corollary 3.1.28(i)), and this follows by direct inspection of the definitions. □

Theorem 5.1.24. *For every small site $(\mathscr{C}, J)$, the inclusion strict pseudo-functor*

$$\mathsf{F} : \mathsf{Stack}(\mathscr{C}, J) \to \mathsf{Fib}(\mathscr{C})$$

admits a left 2-adjoint

$$\mathsf{Fib}(\mathscr{C}) \to \mathsf{Stack}(\mathscr{C}, J) \qquad (F : \mathscr{E} \to \mathscr{C}) \mapsto (F^a : \mathscr{E}^a \to \mathscr{C})$$

which assigns to every fibration over $\mathscr{C}$ its associated stack. *Moreover, we have a pseudo-commutative diagram of pseudo-functors (see remark* 2.4.10*) :*

(5.1.25)
$$\begin{array}{ccc} \mathscr{C}^\wedge & \xrightarrow{\mathscr{F}ib} & \mathsf{Fib}(\mathscr{C}) \\ {\scriptstyle (-)^a}\downarrow & & \downarrow{\scriptstyle (-)^a} \\ (\mathscr{C}, J)^\sim & \xrightarrow{\mathscr{F}ib} & \mathsf{Stack}(\mathscr{C}, J) \end{array}$$

whose left vertical arrow is the usual left adjoint to the inclusion functor for presheaves.

Proof. Recall first that the counit of the 2-adjoint pair (F, C) of (3.2.4) is a $\mathscr{C}$-equivalence $\mathsf{ev}^{\mathscr{E}} : \mathsf{C}(\mathscr{E}) \to \mathscr{E}$ for every fibration $F : \mathscr{E} \to \mathscr{C}$ (theorem 3.2.7). Moreover, the unit of the same 2-adjoint pair assigns to every split fibration $(F, \boldsymbol{\lambda})$ a $\mathscr{C}$-equivalence $\eta_{\mathscr{E}} : \mathscr{E} \to \mathsf{C}(\mathscr{E})$, deduced from the strict pseudo-natural equivalence $\beta^{\boldsymbol{\lambda}}$ defined in the proof of theorem 3.2.7 (see remark 3.2.9(i)). There follows a $\mathscr{C}$-equivalence :

$$\omega_{\mathscr{E}} :=: \mathsf{C}(\mathscr{E})^+ \xrightarrow{(\mathsf{ev}^{\mathscr{E}})^+} \mathscr{E}^+ \xrightarrow{\eta_{\mathscr{E}^+}} \mathsf{C}(\mathscr{E}^+).$$

By inspecting the definitions, we see that $\omega_{\mathscr{E}}$ assigns to every object $[X, \varphi : \mathscr{S} \to \mathsf{C}(\mathscr{E})]$ of $\mathsf{C}(\mathscr{E})^+$ the cartesian section $\omega_{\mathscr{E}}([X, \varphi]) \in \mathscr{E}^+(X)$ given by the rule :

$$(f : Y \to X) \mapsto (\mathscr{S} \times_X f \xrightarrow{f_*} \mathscr{S} \xrightarrow{\mathsf{ev}^{\mathscr{E}} \circ \varphi} \mathscr{E})$$

and notice that $\mathsf{ev}^{\mathscr{E}} \circ \varphi \circ f_*(h) = \varphi_{f \circ h}(\mathbf{1}_Z)$ for every $(h : Z \to Y) \in \mathrm{Ob}(\mathscr{S} \times_X f)$. For every morphism $g/X : (f' : Y' \to X) \to (f : Y \to X)$ in $\mathscr{C}/X$, the induced natural transformation $\omega_{\mathscr{E}}([X, \varphi])(g) : \omega_{\mathscr{E}}([X, \varphi])(f') \Rightarrow \omega_{\mathscr{E}}([X, \varphi])(f)$ is the identity $\mathsf{ev}^{\mathscr{E}} \circ \varphi \circ f'_* \xrightarrow{\sim} \mathsf{ev}^{\mathscr{E}} \circ \varphi \circ f_* \circ g_*$.

Claim 5.1.26. (i) For every two $\mathscr{C}$-fibrations $\mathscr{A} \xrightarrow{\varphi} \mathscr{C} \xleftarrow{\varphi'} \mathscr{A}'$, the functor

$$\mathsf{C}_{\mathscr{A}, \mathscr{A}'} : \mathsf{Cart}_{\mathscr{C}}(\mathscr{A}, \mathscr{A}') \to \mathsf{Cart}_{\mathscr{C}}(\mathsf{C}(\mathscr{A}), \mathsf{C}(\mathscr{A}')) \qquad F \mapsto \mathsf{C}(F) \qquad (\alpha : F \Rightarrow F') \mapsto \mathsf{C}(\alpha)$$

is an equivalence.

(ii) For every $\mathscr{C}$-fibration $\mathscr{A} \to \mathscr{C}$ the following diagram of functors commutes:

$$\begin{array}{ccc} \mathsf{Cart}_{\mathscr{C}}(\mathsf{C}(\mathscr{E}), \mathscr{A}) & \xrightarrow{(-)^+} & \mathsf{Cart}_{\mathscr{C}}(\mathsf{C}(\mathscr{E})^+, \mathscr{A}^+) \\ {\scriptstyle \mathsf{C}_{\mathsf{C}(\mathscr{E}), \mathscr{A}}}\downarrow & & \downarrow{\scriptstyle \mathsf{Cart}_{\mathscr{C}}(j_{\mathsf{C}(\mathscr{E})}, \mathscr{A}^+)} \\ \mathsf{Cart}_{\mathscr{C}}(\mathsf{C}(\mathsf{C}(\mathscr{E})), \mathsf{C}(\mathscr{A})) & \xrightarrow{\mathsf{Cart}_{\mathscr{C}}(\mathsf{C}(\mathsf{C}(\mathscr{E})), j_{\mathscr{A}})} & \mathsf{Cart}_{\mathscr{C}}(\mathsf{C}(\mathsf{C}(\mathscr{E})), \mathscr{A}^+). \end{array}$$

Proof of the claim. Assertion (ii) follows by direct inspection. For (i) let us notice the commutative diagram :

$$\begin{array}{ccc} \mathsf{Cart}_{\mathscr{C}}(\mathscr{A},\mathscr{A}') & \xrightarrow{\mathsf{C}_{\mathscr{A},\mathscr{A}'}} & \mathsf{Cart}_{\mathscr{C}}(\mathsf{C}(\mathscr{A}),\mathsf{C}(\mathscr{A}')) \\ & {\scriptstyle \mathsf{Cart}_{\mathscr{C}}(\mathsf{ev}^{\mathscr{A}},\mathscr{A}')} \searrow \quad \swarrow {\scriptstyle \mathsf{Cart}_{\mathscr{C}}(\mathscr{A},\mathsf{ev}^{\mathscr{A}'})} & \\ & \mathsf{Cart}_{\mathscr{C}}(\mathsf{C}(\mathscr{A}),\mathscr{A}') & \end{array}$$

where all arrows except possibly $\mathsf{C}_{\mathscr{A},\mathscr{A}}$ are equivalences; then the same must hold for $\mathsf{C}_{\mathscr{A},\mathscr{A}'}$. ◇

Claim 5.1.27. For every fibration $\mathscr{A} \to \mathscr{C}$ the diagram of functors :

$$\begin{array}{ccc} \mathsf{Cart}_{\mathscr{C}}(\mathscr{E}^+,\mathscr{A}) & \xrightarrow{\mathsf{Cart}_{\mathscr{C}}(j_{\mathscr{E}},\mathscr{A})} & \mathsf{Cart}_{\mathscr{C}}(\mathsf{C}(\mathscr{E}),\mathscr{A}) \\ {\scriptstyle \mathsf{C}_{\mathscr{E}^+,\mathscr{A}}}\downarrow & & \downarrow{\scriptstyle (-)^+} \\ \mathsf{Cart}_{\mathscr{C}}(\mathsf{C}(\mathscr{E}^+),\mathsf{C}(\mathscr{A})) & \xrightarrow{\mathsf{Cart}_{\mathscr{C}}(\omega_{\mathscr{E}},j_{\mathscr{A}})} & \mathsf{Cart}_{\mathscr{C}}(\mathsf{C}(\mathscr{E})^+,\mathscr{A}^+) \end{array}$$

is essentially commutative (notation of claim 5.1.26(i)).

Proof of the claim. Let $G : \mathscr{E}^+ \to \mathscr{A}$ be any $\mathscr{C}$-cartesian functor; the $\mathscr{C}$-cartesian functor $G' := (G \circ j_{\mathscr{E}})^+ : \mathsf{C}(\mathscr{E})^+ \to \mathscr{A}^+$ can be described as follows. Let $[X, \psi : \mathscr{S} \to \mathsf{C}(\mathscr{E})]$ be any object of $\mathsf{C}(\mathscr{E})^+$; then $G'([X,\psi]) = [X, G \circ j_{\mathscr{E}} \circ \psi : \mathscr{S} \to \mathscr{A}]$, and notice that $j_{\mathscr{E}} \circ \psi$ is the $\mathscr{C}$-cartesian functor given by the rule :

$$(f : Y \to X) \mapsto [Y, \psi_f : \mathscr{C}/Y \to \mathscr{E}] \qquad \text{for every } f \in \mathrm{Ob}(\mathscr{S}).$$

On the other hand, the $\mathscr{C}$-cartesian functor $G'' : j_{\mathscr{A}} \circ \mathsf{C}(G) \circ \omega_{\mathscr{E}} : \mathsf{C}(\mathscr{E})^+ \to \mathscr{A}^+$ is described as follows. For any $[X,\psi]$ as in the foregoing, $G''([X,\psi]) = [X, G \circ \psi' : \mathscr{C}/X \to \mathscr{A}]$, where $\psi' : \mathscr{S} \to \mathscr{E}^+$ is the $\mathscr{C}$-cartesian functor given by the rule :

$$(f : Y \to X) \mapsto [Y, \mathsf{ev}^{\mathscr{E}} \circ \psi \circ f_*] \qquad \text{for every } f \in \mathrm{Ob}(\mathscr{C}/X)$$

and notice that $\mathsf{ev}^{\mathscr{E}} \circ \psi \circ f_* : \mathscr{S} \times_X f \to \mathscr{E}$ is in turn given by the rule :

$$(g : Z \to Y) \mapsto \psi_{f \circ g}(\mathbf{1}_Z) \qquad \text{for every } g \in \mathrm{Ob}(\mathscr{S} \times_X f).$$

Let $\psi'' : \mathscr{S} \to \mathscr{A}$ be the restriction of ψ'; then $[X,\psi'] = [X,\psi'']$ in $\mathscr{E}^+$, and $\mathscr{S} \times_X f = \mathscr{C}/Y$ for every $(f : Y \to X) \in \mathrm{Ob}(\mathscr{S})$. For such f, we need to compare the cartesian functors $\psi_f, \mathsf{ev}^{\mathscr{E}} \circ \psi \circ f_* : \mathscr{C}/Y \to \mathscr{E}$. However, $\psi_f(\mathbf{1}_Y) = \mathsf{ev}^{\mathscr{E}} \circ \psi \circ f_*(\mathbf{1}_Y)$, so there exists a unique isomorphism of functors

$$\tau_{\psi,f} : \mathsf{ev}^{\mathscr{E}} \circ \psi \circ f_* \xrightarrow{\sim} \psi_f \qquad \text{such that} \qquad (\tau_{\psi,f})_{\mathbf{1}_Y} = \mathbf{1}_{\psi_f(\mathbf{1}_Y)}.$$

Explicitly, every object $(g : Z \to Y)$ of $\mathscr{C}/Y$ yields a morphism $g/Y : g \to \mathbf{1}_Y$ in $\mathscr{C}/Y$, and therefore $(\tau_{\psi,f})_g$ is determined by the commutativity of the diagram :

$$\begin{array}{ccc} \psi_{f\circ g}(\mathbf{1}_Z) & \xrightarrow{(\tau_{\psi,f})_g} & \psi_f(g) \\ {\scriptstyle (\mathsf{ev}^{\mathscr{E}}\circ\psi * f_*)_{g/Y}}\downarrow & & \downarrow{\scriptstyle \psi_f(g/Y)} \\ \psi_f(\mathbf{1}_Y) & = & \psi_f(\mathbf{1}_Y) \end{array}$$

However, $(\mathsf{ev}^{\mathscr{E}} \circ \psi \circ f_*)_{g/Y} = \psi_f(g/Y) \circ \psi_{g/X,\mathbf{1}_Z}$, so that

$$(\tau_{\psi,f})_g = \psi_{g/X,\mathbf{1}_Z} \qquad \text{for every } (g : Z \to Y) \in \mathrm{Ob}(\mathscr{C}/Y).$$

We claim that we get an isomorphism of $\mathscr{C}$-cartesian functors

$$G([\tau_\psi]) : G''([X,\psi]) \xrightarrow{\sim} G'([X,\psi]) \quad f \mapsto G([\mathbf{1}_Y, \tau_{\psi,f}]) \quad \text{for every } (f : Y \to X) \in \mathrm{Ob}(\mathscr{S}).$$

Indeed, let $g/X : (f' : Y' \to X) \to (f : Y \to X)$ be any morphism in $\mathscr{S}$; the assertion comes down to the commutativity of the diagram :

$$\begin{array}{ccc} G([Y', \mathsf{ev}^{\mathscr{E}} \circ \psi \circ f'_*]) & = & G([Y', \mathsf{ev}^{\mathscr{E}} \circ \psi \circ f_* \circ g_*]) \\ \downarrow{\scriptstyle G([\mathbf{1}_{Y'}, \tau_{\psi,f'}])} & & \downarrow{\scriptstyle G([\mathbf{1}_{Y'}, \tau_{\psi,f*g_*}])} \\ G([Y', \psi_{f'}]) & \xrightarrow{G([\mathbf{1}_{Y'}, \psi_{g/X}])} & G([Y', \psi_f \circ g_*]). \end{array}$$

The latter in turn follows by applying G to the system of identities :

$$\psi_{g/X,h} \circ \psi_{h/X,\mathbf{1}_Z} = \psi_{g\circ h/X,\mathbf{1}_Z} \qquad \text{for every } (h : Z \to Y') \in \mathrm{Ob}(\mathscr{C}/Y')$$

which hold by functoriality of ψ, applied to the morphisms $f' \circ h \xrightarrow{h/X} f' \xrightarrow{g/X} f$ of $\mathscr{S}$.

Next, we check the naturality of the rule $\psi \mapsto G([\tau_\psi])$. Hence, let $[X_i, \psi_i : \mathscr{S}_i \to \mathsf{C}(\mathscr{E})]$ for $i = 1, 2$ be two objects of $\mathsf{C}(\mathscr{E})^+$, and $[t, \sigma] : [X_2, \psi_2] \to [X_1, \psi_1]$ a morphism; *i.e.* $t : X_2 \to X_1$ is a morphism of $\mathscr{C}$ and $\sigma : \psi_{2|\mathscr{S}_3} \Rightarrow (\psi_1 \circ t_*)_{|\mathscr{S}_3}$ is a natural $\mathscr{C}$-transformation defined on a sieve $\mathscr{S}_3 \subset \mathscr{S}_2 \cap (\mathscr{S}_1 \times_{X_1} t)$. We have

$$G'([t, \sigma]) = [t, (G \circ j_{\mathscr{E}}) * \sigma : G \circ j_{\mathscr{E}} \circ \psi_{2|\mathscr{S}_3} \Rightarrow G \circ j_{\mathscr{E}} \circ (\psi_1 \circ t_*)_{|\mathscr{S}_3}].$$

On the other hand, notice that $\omega_{\mathscr{E}}([t, \sigma]) : \omega_{\mathscr{E}}([X_2, \psi_2]) \to \omega_{\mathscr{E}}([X_1, \psi_1])$ is the natural transformation $\sigma' : \psi'_2 \Rightarrow \psi'_1 * t_*$, where ψ'_i is the functor such that $(f : Y \to X_i) \mapsto [Y, \mathsf{ev}^{\mathscr{E}} \circ \psi_i \circ f_*]$ for every $f \in \mathrm{Ob}(\mathscr{C}/X_i)$, and $\sigma'_f := \mathsf{ev}^{\mathscr{E}} * \sigma * f_*$, with $f_* : \mathscr{S}_3 \times_{X_2} f \to \mathscr{S}_3$ for every $f \in \mathrm{Ob}(\mathscr{C}/X_2)$. Explicitly, σ'_f assigns to every $(g : Z \to Y) \in \mathrm{Ob}(\mathscr{S}_3 \times_{X_2} f)$ the morphism

$$\sigma_{f\circ g,\mathbf{1}_Z} : \psi_{2,f\circ g}(\mathbf{1}_Z) \to \psi_{1,t\circ f\circ g}(\mathbf{1}_Z).$$

Thus, we need to check the commutativity of the diagram :

$$\begin{array}{ccc} [X_2, G \circ \psi'_2] & \xrightarrow{G([\tau_{\psi_2}])} & [X_2, G \circ j_{\mathscr{E}} \circ \psi_2] \\ \downarrow{\scriptstyle [t, G*\sigma']} & & \downarrow{\scriptstyle [t, (G\circ j_{\mathscr{E}})*\sigma]} \\ [X_1, G \circ \psi'_1] & \xrightarrow{G([\tau_{\psi_1}])} & [X_1, G \circ j_{\mathscr{E}} \circ \psi_1]. \end{array}$$

which in turn follows from the commutativity of the diagram :

$$\begin{array}{ccc} \psi_{2,f\circ g}(\mathbf{1}_Z) & \xrightarrow{\sigma_{f\circ g,\mathbf{1}_Z}} & \psi_{1,t\circ f\circ g}(\mathbf{1}_Z) \\ \downarrow{\scriptstyle \psi_{2,g/X_2,\mathbf{1}_Z}} & & \downarrow{\scriptstyle \psi_{1,g/X_1,\mathbf{1}_Z}} \\ \psi_{2,f}(g) & \xrightarrow{\sigma_{f,g}} & \psi_{1,t\circ f}(g) \end{array}$$

for every $f \in \mathrm{Ob}(\mathscr{C}/X_2)$ and every $g \in \mathrm{Ob}(\mathscr{S}_3 \times_{X_2} f)$. Since $g/X_1 = t_*(g/X_2)$ in $\mathscr{S}_1$, the assertion holds by naturality of σ. Lastly, the naturality with respect to natural $\mathscr{C}$-transformations $G \Rightarrow G'$ is immediate from the definitions. ◊

Now, let $\mathscr{A} \to \mathscr{C}$ be a stack; then $j_{\mathscr{A}}$ is an equivalence, by lemma 5.1.21(iv), and it follows easily that $\mathsf{Cart}_{\mathscr{C}}(\omega_{\mathscr{E}}, j_{\mathscr{A}})$ is an equivalence as well. The same holds for $\mathsf{C}_{\mathscr{E}^+,\mathscr{A}}$ and $\mathsf{C}_{\mathsf{C}(\mathscr{E}),\mathscr{A}}$, by claim 5.1.26(i). We deduce that the functor $(-)^+$ appearing in the diagrams of claims 5.1.27 and 5.1.26(ii) admits both left and right quasi-inverse functors, so it is an equivalence. By claim 5.1.27, we finally conclude that $\mathsf{Cart}_{\mathscr{C}}(j_{\mathscr{E}}, \mathscr{A})$ is an equivalence as well. Now, set

$$\mathscr{E}^a := \mathscr{E}^{+++}.$$

The composition $\mathsf{Cart}_{\mathscr{C}}(j_{\mathscr{E}}, \mathscr{A}) \circ \mathsf{Cart}_{\mathscr{C}}(j_{\mathscr{E}^+}, \mathscr{A}) \circ \mathsf{Cart}_{\mathscr{C}}(j_{\mathscr{E}^{++}}, \mathscr{A})$ is an equivalence

$$\lambda_{\mathscr{E},\mathscr{A}} : \mathsf{Cart}_{\mathscr{C}}(\mathscr{E}^a, \mathscr{A}) \xrightarrow{\sim} \mathsf{Cart}_{\mathscr{C}}(\mathsf{C}(\mathsf{C}(\mathsf{C}(\mathscr{E}))), \mathscr{A}).$$

The rules $(\mathscr{E},\mathscr{A})\mapsto \mathsf{Cart}_{\mathscr{C}}(\mathscr{E}^a,\mathscr{A})$ and $(\mathscr{E},\mathscr{A})\mapsto \mathsf{Cart}_{\mathscr{C}}(\mathsf{C}(\mathsf{C}(\mathsf{C}(\mathscr{E}))),\mathscr{A})$ yield strict pseudo-functors

$$\mathsf{Cart}_{\mathscr{C}}((-)^a,-),\mathsf{Cart}_{\mathscr{C}}(\mathsf{C}\circ\mathsf{C}\circ\mathsf{C},-):\mathsf{Fib}(\mathscr{C})^o\times\mathsf{Stack}(\mathscr{C},J)\to\mathbf{Cat}$$

and clearly the rule $(\mathscr{E},\mathscr{A})\mapsto\lambda_{\mathscr{E},\mathscr{A}}$ defines a strict pseudo-natural equivalence of functors

$$\lambda:\mathsf{Cart}_{\mathscr{C}}((-)^a,-)\Rightarrow\mathsf{Cart}_{\mathscr{C}}(\mathsf{C}\circ\mathsf{C}\circ\mathsf{C},-).$$

On the other hand, by composing evaluation functors we get as well a strict pseudo-natural equivalence $\mathsf{C}\circ\mathsf{C}\circ\mathsf{C}\Rightarrow\mathbf{1}_{\mathsf{Fib}(\mathscr{C})}$, whence an induced strict pseudo-natural equivalence

$$\lambda':\mathsf{Cart}_{\mathscr{C}}(-,\mathsf{F})\stackrel{\sim}{\to}\mathsf{Cart}_{\mathscr{C}}(\mathsf{C}\circ\mathsf{C}\circ\mathsf{C},-)$$

where $\mathsf{F}:\mathsf{Stack}(\mathscr{C},J)\to\mathsf{Fib}(\mathscr{C})$ is the (forgetful) inclusion strict pseudo-functor. After choosing a quasi-inverse for λ (see theorem 2.4.12) and composing with λ', we obtain the required (non-strict) pseudo-natural equivalence

$$\mathsf{Cart}_{\mathscr{C}}(-,\mathsf{F})\stackrel{\sim}{\to}\mathsf{Cart}_{\mathscr{C}}((-)^a,-).$$

Lastly, the essential commutativity of (5.1.25) follows directly from lemma 5.1.21(v). □

5.2. **Covering morphisms of prestacks.** In this section we wish to characterize the i-faithful functors of prestacks (for $i=0,1,2$), by means of certain conditions that are analogous to those given in definition 4.1.28 for presheaves; to this aim, we make the following :

Definition 5.2.1. Let $(\mathscr{C},J)$ be a site, $F:\mathscr{E}\to\mathscr{C}$ and $F':\mathscr{E}'\to\mathscr{C}$ two fibrations, and $\varphi:\mathscr{E}'\to\mathscr{E}$ a $\mathscr{C}$-cartesian functor.

(i) φ is a 0-*covering* if for every $Y\in\mathrm{Ob}(\mathscr{E})$ there exist a covering family $(f_i:X_i\to FY\mid i\in I)$ and for every $i\in I$ an object $Y'_i\in\mathrm{Ob}(\mathscr{E}'_{X_i})$ and a $\mathscr{C}$-cartesian morphism $h_i:\varphi Y'_i\to Y$ in $\mathscr{E}$ such that $F(h_i)=f_i$.

(ii) φ is a 1-*covering* if for every $X\in\mathrm{Ob}(\mathscr{C})$, every $Y_1,Y_2\in\mathrm{Ob}(\mathscr{E}'_X)$ and every morphism $g:\varphi Y_1\to\varphi Y_2$ in $\mathscr{E}_X$ there exist a covering family $(f_i:X_i\to X\mid i\in I)$, and morphisms $Y_2\stackrel{g_i}{\leftarrow}Z_i\stackrel{h_i}{\to}Y_1$ in $\mathscr{E}'$, where h_i is $\mathscr{C}$-cartesian for every $i\in I$, such that

$$\varphi(g_i)=g\circ\varphi(h_i)\qquad\text{and}\qquad F'(h_i)=f_i.$$

(iii) φ is a 2-*covering* if for every $X\in\mathrm{Ob}(\mathscr{C})$, every $Y_1,Y_2\in\mathrm{Ob}(\mathscr{E}'_X)$, and every pair $g_1,g_2:Y_1\to Y_2$ of morphisms in $\mathscr{E}'_X$ with $\varphi(g_1)=\varphi(g_2)$ there exist a covering family $(f_i:X_i\to X\mid i\in I)$, and for every $i\in I$ a $\mathscr{C}$-cartesian morphism

$$h_i:Z_i\to Y_1\quad\text{in }\mathscr{E}'\text{ such that}\quad g_1\circ h_i=g_2\circ h_i\quad\text{and}\quad F'(h_i)=f_i.$$

Remark 5.2.2. Suppose that $(\mathscr{E}\xrightarrow{F}\mathscr{C},\boldsymbol{\lambda})$ and $(\mathscr{E}'\xrightarrow{F'}\mathscr{C},\boldsymbol{\lambda}')$ are split fibrations over $\mathscr{C}$, and $\varphi:(\mathscr{E}',\boldsymbol{\lambda}')\to(\mathscr{E},\boldsymbol{\lambda})$ is a split cartesian functor. Let c and c' be the strict pseudo-functors associated with $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}'$. Then we can rephrase the conditions of definition 5.2.1 as follows :

(i) φ is 0-covering if and only if for every $Y\in\mathrm{Ob}(\mathscr{E})$ there exist a covering family $(f_i:X_i\to FY\mid i\in I)$ and for every $i\in I$ an object $Y'_i\in\mathrm{Ob}(\mathscr{E}'_{X_i})$ and an isomorphism $\varphi Y'_i\stackrel{\sim}{\to}\mathsf{c}_{f_i}Y$ in $\mathscr{E}_{X_i}$.

(ii) φ is 1-covering if and only if for every $X\in\mathrm{Ob}(\mathscr{C})$, every $Y_1,Y_2\in\mathrm{Ob}(\mathscr{E}'_X)$ and every morphism $g:\varphi Y_1\to\varphi Y_2$ in $\mathscr{E}_X$ we have a covering family $(f_i:X_i\to X\mid i\in I)$, and for every $i\in I$ a morphism $g_i:\mathsf{c}'_{f_i}Y_1\to\mathsf{c}'_{f_i}Y_2$ in $\mathscr{E}'_{X_i}$ such that $\varphi(g_i)=\mathsf{c}_{f_i}(g)$.

(iii) φ is a 2-*covering* if for every $X\in\mathrm{Ob}(\mathscr{C})$, every $Y_1,Y_2\in\mathrm{Ob}(\mathscr{E}'_X)$, and every pair $g_1,g_2:Y_1\to Y_2$ of morphisms in $\mathscr{E}'_X$ with $\varphi(g_1)=\varphi(g_2)$ there exists a covering family $(f_i:X_i\to X\mid i\in I)$ such that $\mathsf{c}'_{f_i}(g_1)=\mathsf{c}'_{f_i}(g_2)$.

Lemma 5.2.3. *Consider a site $(\mathscr{C}, J)$, and an essentially commutative diagram of the 2-category $\mathsf{Fib}(\mathscr{C})$, whose vertical arrows are equivalences of categories :*

$$\begin{array}{ccc} \mathscr{E}' & \xrightarrow{\varphi} & \mathscr{E} \\ {\scriptstyle\omega'}\downarrow & & \downarrow{\scriptstyle\omega} \\ \mathscr{F}' & \xrightarrow{\psi} & \mathscr{F}. \end{array}$$

Let also $i \in \{0, 1, 2\}$. Then φ is i-covering if and only if the same holds for ψ.

Proof. Denote $\mathscr{E} \xrightarrow{F} \mathscr{C} \xleftarrow{F'} \mathscr{E}'$ and $\mathscr{F} \xrightarrow{G} \mathscr{C} \xleftarrow{G'} \mathscr{F}'$ the respective fibrations. The assumption means that there exists an isomorphism of functors $\beta : \psi \circ \omega' \xrightarrow{\sim} \omega \circ \varphi$ such that $G * \beta = \mathbf{1}_{F'}$.

Suppose that φ is 0-covering, let $Y \in \mathrm{Ob}(\mathscr{F})$ be any object, and set $X := GY$; by assumption there exists $Z \in \mathrm{Ob}(\mathscr{E}_X)$ with an isomorphism $t : \omega Z \xrightarrow{\sim} Y$ in $\mathscr{F}_X$ (corollary 3.1.28(i)). Then there exist a covering family $(f_j : X_j \to X \mid j \in I)$ and for every $j \in I$ a cartesian morphism $h_j : \varphi Z'_j \to Z$ with $Fh_j = f_j$. It follows that

$$h'_j := t \circ \omega(h_j) \circ \beta_{Z'_j} : \psi(\omega' Z'_j) \to Y$$

is cartesian and $Gh'_j = f_j$ for every $j \in I$, so ψ is 0-covering.

Conversely, suppose that ψ is 0-covering; let $Y \in \mathrm{Ob}(\mathscr{E})$ and $X := FY$. By assumption there exist a covering family $(f_j : X_j \to X \mid j \in I)$ and for every $j \in I$ a cartesian morphism $h_j : \psi Z_j \to \omega Y$ with $Gh_j = f_j$. Then we find for every $j \in I$ an isomorphism $t_j : \omega' Z'_j \xrightarrow{\sim} Z_j$ in $\mathscr{F}'_{X_j}$; it follows that $h_j := h \circ \psi(t_j) \circ \beta^{-1}_{Z'_j} : \omega(\varphi Z'_j) \to \omega Y$ is cartesian, and since ω is an equivalence, there exists a unique cartesian morphism $h'_j : \varphi Z'_j \to Y$ in $\mathscr{E}$ with $\omega(h'_j) = h_j$, for every $j \in I$ (details left to the reader); by construction, $Fh'_j = f_j$, so φ is 0-covering.

Next, if φ is 1-covering, let $X \in \mathrm{Ob}(\mathscr{C})$ and $g : \psi Y_1 \to \psi Y_2$ any morphism in $\mathscr{F}_X$. We find isomorphisms $t_i : \omega' Z_i \xrightarrow{\sim} Y_i$ in $\mathscr{F}'_X$ $(i = 1, 2)$, and a morphism $g' : \psi(\omega' Z_1) \to \psi(\omega' Z_2)$ such that $\psi(t_2) \circ g' = g \circ \psi(t_1)$; then $\beta_{Z_2} \circ g' = \omega(g'') \circ \beta_{Z_1}$ for a morphism $g'' : \varphi Z_1 \to \varphi Z_2$ in $\mathscr{E}_X$. By assumption, there exist a covering family $(f_j : X_j \to X \mid j \in I)$ and morphisms $Z_2 \xleftarrow{g_j} W_j \xrightarrow{h_j} Z_1$ in $\mathscr{E}'$ for every $j \in I$, such that h_j is cartesian with $F' h_j = f_j$, and $\varphi(g_j) = g'' \circ \varphi(h_j)$. Set $h'_j := t_1 \circ \omega'(h_j)$ and $g'_j := t_2 \circ \omega'(g_j)$; then h'_j is cartesian with $G'(h'_j) = f_j$, and $\psi(g'_j) = g \circ \psi(h'_j)$ for every $j \in I$. This shows that ψ is 1-covering.

Conversely, suppose that ψ is 1-covering, and let $g : \varphi Y_1 \to \varphi Y_2$ be a morphism in $\mathscr{E}_X$. Then there exist a covering family $(f_j : X_j \to X \mid j \in I)$ and morphisms $\omega' Y_2 \xleftarrow{g_j} Z_j \xrightarrow{h_j} \omega' Y_1$ in $\mathscr{F}'$ such that h_j is cartesian with $G' h_j = f_j$ and $\beta_{Y_2} \circ \psi(g_j) = \omega(g) \circ \beta_{Y_1} \circ \psi(h_j)$ for every $j \in I$. We pick also an isomorphism $t_j : \omega' Z'_j \xrightarrow{\sim} Z_j$ in $\mathscr{F}'_{X_j}$ for every $j \in I$; then there exist a unique cartesian morphism $h'_j : Z'_j \to Y_1$ with $\omega'(h'_j) = h_j \circ t_j$ and a unique morphism $g'_j : Z'_j \to Y_2$ with $\omega'(g'_j) = g_j \circ t_j$. A direct computation shows that $\omega(g \circ \varphi(h'_j)) = \omega(\varphi(g'_j))$, whence $g \circ \varphi(h'_j) = \varphi(g'_j)$ for every $j \in I$. Thus, φ is 1-cartesian.

Lastly, suppose that φ is 2-covering, and let $g_1, g_2 : Y_1 \to Y_2$ be two morphisms in $\mathscr{F}'_X$ such that $\psi(g_1) = \psi(g_2)$. Pick isomorphisms $t_i : \omega' Z_i \xrightarrow{\sim} Y_i$ for $i = 1, 2$; then there exists for $i = 1, 2$ a unique morphism $g'_i : Z_1 \to Z_2$ such that $t_2 \circ \omega'(g'_i) = g_i \circ t_1$. It follows that

$$\begin{aligned} \psi(t_2) \circ \beta^{-1}_{Z_2} \circ \omega(\varphi(g'_1)) &= \psi(t_2) \circ \psi(\omega' g_1) \circ \beta^{-1}_{Z_1} \\ &= \psi(g_1 \circ t_1) \circ \beta^{-1}_{Z_1} \\ &= \psi(g_2 \circ t_1) \circ \beta^{-1}_{Z_1} \\ &= \psi(t_2) \circ \beta^{-1}_{Z_2} \circ \omega(\varphi(g'_2)) \end{aligned}$$

whence $\varphi(g_1') = \varphi(g_2')$. Then we have a covering family $(f_j : X_j \to X \mid j \in I)$ and for every $j \in I$ a cartesian morphism $h_j : W_j \to Z_1$ such that $g_1' \circ h_j = g_2' \circ h_j$ and $Fh_j = f_j$. Hence $h_j' := t_1 \circ \omega'(h_j) : \omega' W_j \to Y_1$ is cartesian with $G'h_j' = f_j$ and $g_1 \circ h_j' = g_2 \circ h_j'$. This proves that ψ is 2-covering. Conversely, suppose that ψ is 2-covering, and let $g_1, g_2 : Y_1 \to Y_2$ be two morphisms in $\mathscr{E}'_X$ such that $\varphi(g_1) = \varphi(g_2)$. Set $g_i' := \omega'(g_i)$ for $i = 1, 2$; then $\psi(g_1') = \psi(g_2')$, so there exist a covering family $(f_j : X_j \to X \mid j \in I)$ and for every $j \in I$ a cartesian morphism $h_j : Z_j \to \omega' Y_1$ such that $g_1' \circ h_j = g_2' \circ h_j$ and $G'h_j = f_j$. For every $j \in I$ pick an isomorphism $t_j : \omega' Z_j' \stackrel{\sim}{\to} Z_j$ in $G'^{-1}X$; there exists a unique cartesian morphism $h_j' : Z_j' \to Y_1$ such that $\omega'(h_j') = h_j \circ t_j$. It is easily seen that $\omega'(g_1 \circ h_j') = \omega'(g_2 \circ h_j')$, whence $g_1 \circ h_j' = g_2 \circ h_j'$, and $F'h_j' = f_j$ for every $j \in I$. Thus, φ is 2-covering. □

Lemma 5.2.4. *Let $(\mathscr{C}, J)$ be a site, $F : \mathscr{E} \to \mathscr{C}$, $F' : \mathscr{E}' \to \mathscr{C}$, $F'' : \mathscr{E}'' \to \mathscr{C}$ three fibrations, $\varphi : \mathscr{E}'' \to \mathscr{E}'$ and $\psi : \mathscr{E}' \to \mathscr{E}$ two cartesian functors, and $i \in \{0, 1, 2\}$. We have:*

(i) *If φ and ψ are i-covering, the same holds for $\psi \circ \varphi$.*
(ii) *If $\psi \circ \varphi$ is i-covering, and φ is j-covering for every $j < i$, then ψ is i-covering.*
(iii) *If $\psi \circ \varphi$ is i-covering and ψ is j-covering for every $j > i$, then φ is i-covering.*

Proof. In view of lemma 5.2.3 and claim 3.2.8 we may replace φ and ψ by $\mathsf{C}(\varphi)$ and $\mathsf{C}(\psi)$, and assume from start that φ and ψ are split cartesian functors. We then denote by c, c' and c'' the strict pseudo-functors associated with the split cleavages of $\mathscr{E}$, $\mathscr{E}'$ and $\mathscr{E}''$ respectively.

(i): Suppose first that $i = 0$, let $Y \in \mathrm{Ob}(\mathscr{E})$ be any object, and set $X := FY$; by assumption there exist a covering family $(f_j : X_j \to X \mid j \in I)$ and for every $j \in I$ an isomorphism $h_j : \psi Y_j' \stackrel{\sim}{\to} \mathsf{c}_{f_j} Y$ in $\mathscr{E}_{X_j}$. Likewise, for every $j \in I$ there exist a covering family $(f_{j\lambda} : X_{j\lambda} \to X_j \mid \lambda \in \Lambda_j)$ and for every $\lambda \in \Lambda_j$ an isomorphism $h_{j\lambda} : \varphi Y''_{j\lambda} \stackrel{\sim}{\to} \mathsf{c}'_{f_{j\lambda}} Y_j'$ in $\mathscr{E}'_{X_{j\lambda}}$. Then the family $(f'_{j\lambda} := f_j \circ f_{j\lambda} \mid j \in I,\ \lambda \in \Lambda_j)$ covers X and $\mathsf{c}_{f_{j\lambda}}(h_j) \circ \psi(h_{j\lambda}) : \psi \circ \varphi(Y''_{j\lambda}) \to \mathsf{c}_{f'_{j\lambda}} Y$ is an isomorphism in $\mathscr{E}_{X_{j\lambda}}$ for every $j \in I$ and $\lambda \in \Lambda_j$, whence the contention, in this case.

Next, suppose that $i = 1$. Let $X \in \mathrm{Ob}(\mathscr{C})$, $Y_1'', Y_2'' \in \mathrm{Ob}(\mathscr{E}''_X)$, and $g : \psi \circ \varphi Y_1'' \to \psi \circ \varphi Y_2''$ any morphism in $\mathscr{E}_X$. By assumption there exist a covering family $(f_j : X_j \to X \mid j \in I)$ and for every $j \in I$ a morphism

$$g_j : \varphi(\mathsf{c}''_{f_j} Y_1'') = \mathsf{c}'_{f_j}(\varphi Y_1'') \to \mathsf{c}'_{f_j}(\varphi Y_2'') = \varphi(\mathsf{c}''_{f_j} Y_2'') \quad \text{in } \mathscr{E}'_{X_j} \text{ such that} \quad \psi(g_j) = \mathsf{c}_{f_j}(g).$$

Since φ is 1-covering, we find for every $j \in I$ a covering family $(f_{j\lambda} : X_{j\lambda} \to X_j \mid \lambda \in \Lambda_j)$ and for every $\lambda \in \Lambda_j$ a morphism

$$g_{j\lambda} : \mathsf{c}''_{f_{j\lambda}}(\mathsf{c}''_{f_j} Y_1'') \to \mathsf{c}''_{f_{j\lambda}}(\mathsf{c}''_{f_j} Y_2'') \qquad \text{in } \mathscr{E}'_{X_j} \text{ such that} \qquad \varphi(g_{j\lambda}) = \mathsf{c}'_{f_{j\lambda}}(g_j).$$

Then the family $(f'_{j\lambda} := f_j \circ f_{j\lambda} \mid j \in I,\ \lambda \in \Lambda_j)$ covers X, and $\psi \circ \varphi(g_{j\lambda}) = \mathsf{c}_{f'_{j\lambda}}(g)$ for every $j \in I$ and every $\lambda \in \Lambda_j$. It follows that $\psi \circ \varphi$ is 1-covering.

Lastly, let $i = 2$; consider morphisms $g_1, g_2 : Y_1'' \to Y_2''$ in $\mathscr{E}''_X$ with $\psi \circ \varphi(g_1) = \psi \circ \varphi(g_2)$. By assumption, there exists a covering family $(f_j : X_j \to X \mid j \in I)$ such that

$$\varphi(\mathsf{c}''_{f_j}(g_1)) = \mathsf{c}'_{f_j}(\varphi(g_1)) = \mathsf{c}'_{f_j}(\varphi(g_2)) = \varphi(\mathsf{c}''_{f_j}(g_2)) \qquad \text{for every } j \in I.$$

We may then find for every $j \in I$ a covering family $(f_{j\lambda} : X_{j\lambda} \to X_j \mid \lambda \in \Lambda_j)$ such that $\mathsf{c}''_{f_{j\lambda}} \circ \mathsf{c}''_{f_j}(g_1) = \mathsf{c}''_{f_{j\lambda}} \circ \mathsf{c}''_{f_j}(g_2)$ for every $\lambda \in \Lambda_j$. We conclude that $\psi \circ \varphi$ is 2-covering.

(ii): Again, we suppose first that $i = 0$, in which case the assumption means that $\psi \circ \varphi$ is 0-covering (and there is no condition on φ), and we need to check that ψ is 0-covering. However, the assertion follows straightforwardly from the definitions (details left to the reader).

Next, suppose that $i = 1$, in which case $\psi \circ \varphi$ is 1-covering and φ is 0-covering. Let $g : \psi Y_1' \to \psi Y_2'$ be a morphism in $\mathscr{E}_X$, for some $X \in \mathrm{Ob}(\mathscr{C})$; by assumption there exist for $t = 1, 2$ a covering family $(f_{tj} : X_{tj} \to X \mid j \in I_t)$ and isomorphisms $h_{tj} : \varphi Y''_{tj} \stackrel{\sim}{\to} \mathsf{c}'_{f_j} Y_t'$ in $F'^{-1} X_j$ for every $j \in I_t$. Notice then that for every morphism $f' : X' \to X_{ti}$ in $\mathscr{C}$ we get an isomorphism $\mathsf{c}'_{f'}(h_{tj}) : \varphi(\mathsf{c}''_{f'}Y''_{tj}) \stackrel{\sim}{\to} \mathsf{c}'_{f_j\circ f'}Y'_t$ in $\mathscr{E}'_{X'}$. We deduce that, after replacing the covering families $f_{1\bullet}$ and $f_{2\bullet}$ by a common refinement, we may assume that $I := I_1 = I_2$ and $f_j := f_{1j} = f_{2j}$ for every $j \in I$. In this situation, for every $j \in I$ there exists a morphism

$$g_j : \psi \circ \varphi(Y''_{1j}) \to \psi \circ \varphi(Y''_{2j}) \qquad \text{in } \mathscr{E}_{X_j} \text{ such that} \qquad \psi(h_{2j}) \circ g_j = \mathsf{c}_{f_j}(g) \circ \psi(h_{1j}).$$

By our assumption on $\psi \circ \varphi$, for every $j \in I$ we then find a covering family $(f_{j\lambda} : X_{j\lambda} \to X_j \mid \lambda \in \Lambda_j)$ and for every $\lambda \in \Lambda_j$ a morphism

$$g_{j\lambda} : Y''_{1j} \to Y''_{1j} \qquad \text{in } \mathscr{E}''_{X_{j\lambda}} \qquad \text{such that} \qquad \psi \circ \varphi(g_{j\lambda}) = \mathsf{c}_{f_{j\lambda}}(g_j).$$

It follows that $\psi(\mathsf{c}'_{f_{j\lambda}}h_{2j}) \circ \psi(\varphi(g_{j\lambda})) = \mathsf{c}_{f_j\circ f_{j\lambda}}(g) \circ \psi(\mathsf{c}'_{f_j}h_{1j})$ for every $j \in I$ and $\lambda \in \Lambda_j$. Since the family $(f_i \circ f_{i\lambda} \mid i \in I,\ \lambda \in \Lambda_i)$ covers X, we conclude that ψ is 1-covering.

Next, suppose that $i = 2$, so $\psi \circ \varphi$ is 2-covering and φ is j-covering for $j = 0, 1$. Let $X \in \mathrm{Ob}(\mathscr{C})$ and $g_1, g_2 : Y'_1 \to Y'_2$ two morphisms of $\mathscr{E}'_X$ such that $\psi(g_1) = \psi(g_2)$. Arguing as in the previous case, we find a covering family $(f_j : X_j \to X \mid j \in I)$ and isomorphisms $h_{tj} : \varphi Y''_{tj} \stackrel{\sim}{\to} \mathsf{c}'_{f_j}Y'_t$ in $\mathscr{E}'_{X_j}$ for $t = 1, 2$ and every $j \in I$. In this situation, for every $j \in I$ and $t = 1, 2$ there exists a unique morphism $g_{tj} : \varphi Y''_{1i} \to \varphi Y''_{2i}$ in $\mathscr{E}'_{X_j}$ such that

$$h_{2j} \circ g_{tj} = \mathsf{c}_{f_j}(g_t) \circ h_{1j}. \tag{5.2.5}$$

Notice that $\psi(h_{2j}) \circ \psi(g_{1j}) = \mathsf{c}'_{f_j}(\psi(g_1)) = \mathsf{c}'_{f_j}(\psi(g_2)) = \psi(h_{2j}) \circ \psi(g_{2j})$, whence

$$\psi(g_{1j}) = \psi(g_{2j}) \qquad \text{for every } j \in I.$$

Then, since φ is 1-covering, for every $j \in I$ and $t = 1, 2$ we find a covering family $f_{tj\bullet} := (f_{tj\lambda} : X_{tj\lambda} \to X_j \mid \lambda \in \Lambda_{tj})$ and morphisms

$$g_{tj\lambda} : \mathsf{c}''_{f_{tj\lambda}}Y''_{1j} \to \mathsf{c}''_{f_{tj\lambda}}Y''_{2j} \qquad \text{in } \mathscr{E}''_{X_{tj\lambda}} \qquad \text{such that} \qquad \varphi(g_{tj\lambda}) = \mathsf{c}'_{f_{tj\lambda}}(g_{tj})$$

and arguing as in the foregoing, we may replace $f_{1j\bullet}$ and $f_{2j\bullet}$ by a common refinement, and assume that $\Lambda_j := \Lambda_{1j} = \Lambda_{2j}$ for every $j \in I$, and $f_{j\lambda} := f_{1j\lambda} = f_{2j\lambda}$, for every $j \in I$ and every $\lambda \in \Lambda_j$. We deduce that

$$\psi \circ \varphi(g_{1j\lambda}) = \mathsf{c}''_{f_{j\lambda}}(\psi(g_{1j})) = \mathsf{c}''_{f_{j\lambda}}(\psi(g_{2j})) = \psi \circ \varphi(g_{2j\lambda}) \qquad \text{for every } j \in I \text{ and } \Lambda \in \Lambda_j.$$

Since $\psi \circ \varphi$ is 2-covering, there exists therefore a covering family $(f_{j\lambda\lambda'} : X_{j\lambda\lambda'} \to X_{j\lambda} \mid \lambda' \in \Lambda_{j\lambda})$ such that $\mathsf{c}''_{f_{j\lambda\lambda'}}(g_{1j\lambda}) = \mathsf{c}''_{f_{j\lambda\lambda'}}(g_{2j\lambda})$ for every $\lambda' \in \Lambda_{j\lambda}$. Therefore $\mathsf{c}'_{f_{j\lambda\lambda'}} \circ \mathsf{c}'_{f_{j\lambda}}(g_{1j}) = \varphi(\mathsf{c}''_{f_{j\lambda\lambda'}}(g_{1j\lambda})) = \varphi(\mathsf{c}''_{f_{j\lambda\lambda'}}(g_{2j\lambda})) = \mathsf{c}'_{f_{j\lambda\lambda'}} \circ \mathsf{c}'_{f_{j\lambda}}(g_{2j})$ for every $j \in I$, $\lambda \in \Lambda_j$ and $\lambda' \in \Lambda_{j\lambda}$. Set $f'_{j\lambda\lambda'} := f_j \circ f_{j\lambda} \circ f_{j\lambda\lambda'}$ for every such j, λ, λ'; combining with (5.2.5), we deduce easily that $\mathsf{c}_{f'_{j\lambda\lambda'}}(g_1) = \mathsf{c}_{f'_{j\lambda\lambda'}}(g_2)$, and we conclude that ψ is 2-covering, as sought.

(iii): The case where $i = 2$ is immediate from the definitions.

Say that $i = 1$, so $\psi \circ \varphi$ is 1-covering and ψ is 2-covering; let $X \in \mathrm{Ob}(\mathscr{C})$ and $g : \varphi Y''_1 \to \varphi Y''_2$ a morphism in $\mathscr{E}'_X$. By assumption, there exist a covering family $(f_j : X_j \to \ \mid j \in I)$ and a morphism $g_j : \mathsf{c}_{f_j}Y''_1 \to \mathsf{c}_{f_j}Y''_2$ in $\mathscr{E}''_{X_j}$ for every $j \in I$ such that

$$\psi \circ \varphi(g_j) = \mathsf{c}_{f_j}(\psi(g)) = \psi(\mathsf{c}'_{f_j}(g)).$$

Then, since ψ is 2-covering, we find for every $j \in I$ a covering family $(f_{j\lambda} : X_{j\lambda} \to X_j \mid \lambda \in \Lambda_j)$ such that $\mathsf{c}'_{f_{j\lambda}}(\varphi(g_j)) = \mathsf{c}'_{f_{j\lambda}} \circ \mathsf{c}'_{f_j}(g)$ for every $\lambda \in \Lambda_j$, *i.e.* $\varphi(\mathsf{c}''_{f_{j\lambda}}(g_j)) = \mathsf{c}'_{f_j\circ f_{j\lambda}}(g)$. This shows that φ is 1-covering.

In order to deal with the case where $i = 0$, let us remark :

Claim 5.2.6. Suppose ψ is j-covering for $j = 1, 2$. Let $X \in \mathrm{Ob}(\mathscr{C})$ and $Z, Z' \in \mathrm{Ob}(\mathscr{E}'_X)$ such that ψZ and $\psi Z'$ are isomorphic in $\mathscr{E}_X$. Then there exists a covering family $(f_j : X_j \to X \mid j \in I)$ such that $\mathsf{c}'_{f_j}Z$ and $\mathsf{c}'_{f_j}Z'$ are isomorphic in $\mathscr{E}'_{X_j}$ for every $j \in I$.

Proof of the claim. Let $l : \psi Z \xrightarrow{\sim} \psi Z'$ be an isomorphism in $\mathscr{E}_X$; since ψ is 1-covering, there exist a covering family $(f'_j : X'_j \to X \mid j \in I)$ and for every $j \in I$ a morphism $l_j : \mathsf{c}'_{f'_j} Z \to \mathsf{c}'_{f'_j} Z'$ such that $\psi(l_j) = \mathsf{c}_{f'_j}(l)$. Likewise, there exist a covering family $(f''_j : X'_j \to X \mid j \in I')$ and for every $j \in I'$ a morphism $l'_j : \mathsf{c}'_{f''_j} Z' \to \mathsf{c}'_{f''_j} Z$ such that $\psi(l'_i) = \mathsf{c}_{f''_j}(l^{-1})$. After replacing $(f'_j \mid j \in I)$ and $(f''_j \mid j \in I')$ by a common refinement, we may then assume that $I = I'$ and $f_j := f'_j = f''_j$ for every $j \in I$ (details left to the reader). It follows that $\varphi(k_j \circ l_j) = \mathsf{c}_{f_j}(\mathbf{1}_Z) = \varphi(\mathbf{1}_{\mathsf{c}'_{f_j} Z})$ and $\varphi(l_j \circ k_j) = \mathsf{c}_{f_j}(\mathbf{1}_{Z'}) = \varphi(\mathbf{1}_{\mathsf{c}'_{f_j} Z'})$ for every $j \in I$. Then, since ψ is 2-covering, for every $j \in I$ there exists a covering family $(f_{j\lambda} : X_{j\lambda} \to X_j \mid \lambda \in \Lambda_j)$ such that $\mathsf{c}'_{f_{j\lambda}}(k_j \circ l_j) = \mathsf{c}'_{f_{j\lambda}}(\mathbf{1}_{\mathsf{c}'_{f_j} Z})$; likewise, for every $j \in I$ there exists a covering family $(f'_{j\lambda} : X'_{j\lambda} \to X_j \mid \lambda \in \Lambda'_j)$ such that $\mathsf{c}'_{f'_{j\lambda}}(l_j \circ k_j) = \mathsf{c}'_{f'_{j\lambda}}(\mathbf{1}_{\mathsf{c}'_{f_j} Z'})$. After replacing these families by a common refinement, we may assume that $\Lambda_j = \Lambda'_j$ for every $j \in I$, and $f_{j\lambda} = f'_{j\lambda}$ for every $j \in I$ and every $\lambda \in \Lambda_j$. Then, set $h_{j\lambda} := f_j \circ f_{j\lambda}$ for every $j \in I$ and every $\lambda \in \Lambda_j$; we deduce easily that $\mathsf{c}'_{f'_{j\lambda}}(l_j)$ is an isomorphism $\mathsf{c}'_{h_{j\lambda}} Z \xrightarrow{\sim} \mathsf{c}'_{h_{j\lambda}} Z'$, whence the claim. ◊

Now, suppose that $\psi \circ \varphi$ is 0-covering and ψ is j-covering for $j = 1, 2$, and let $X \in \mathrm{Ob}(\mathscr{C})$, $Y' \in \mathrm{Ob}(\mathscr{E}'_X)$; by assumption there exist a covering family $(f_j : X_j \to X \mid j \in I)$ and for every $j \in I$ an isomorphism $\psi \circ \varphi Y''_j \xrightarrow{\sim} \mathsf{c}_{f_j} \psi Y' = \psi(\mathsf{c}'_{f_j} Y')$ in $\mathscr{E}_X$. By claim 5.2.6, there exists for every $j \in I$ a covering family $(f_{j\lambda} : X_{j\lambda} \to X_j \mid j \in I)$ such that $\mathsf{c}'_{f_{j\lambda}}(\varphi Y''_j) = \varphi(\mathsf{c}''_{f_{j\lambda}} Y''_j)$ is isomorphic to $\mathsf{c}'_{f_j \circ f_{j\lambda}} Y'$ in $\mathscr{E}'_{X_i}$ for every $\lambda \in \Lambda_j$. This shows that φ is 0-covering, and concludes the proof of the lemma. □

Lemma 5.2.7. *Let $C := (\mathscr{C}, J)$ be a site, $\mathscr{E} \xrightarrow{F} \mathscr{C} \xleftarrow{F'} \mathscr{E}'$ two fibrations, and $\varphi : \mathscr{E} \to \mathscr{E}'$ a $\mathscr{C}$-cartesian functor. The following conditions are equivalent :*

(i) *φ is 1-covering (resp. 2-covering).*

(ii) *For every $X \in \mathrm{Ob}(\mathscr{C})$ and every $\sigma, \sigma' \in \mathscr{E}(X)$, the induced morphism :*

$$\mathscr{C}art(\sigma, \sigma')^a \to \mathscr{C}art(\varphi \circ \sigma, \varphi \circ \sigma')^a \tag{5.2.8}$$

of sheaves on C/X is an epimorphism (resp. a monomorphism).

Proof. Recall that – as described in (5.1.8) – for every X, σ, σ' as in (ii) we have a morphism of presheaves $\varphi^*_{\sigma,\sigma'} : \mathscr{C}art(\sigma, \sigma') \to \mathscr{C}art(\varphi \circ \sigma, \varphi \circ \sigma')$ on $\mathscr{C}/X$, and (5.2.8) is the morphism of associated sheaves $(\varphi^*_{\sigma,\sigma'})^a$ on the site C/X. Then the assertion follows easily by inspecting the definitions, and taking into account corollary 4.1.30(i,ii) and remark 4.1.29(ii,iii). □

Proposition 5.2.9. *Let $(\mathscr{C}, J)$ be a site, $\mathscr{E} \xrightarrow{F} \mathscr{C} \xleftarrow{F'} \mathscr{E}'$ two fibrations, $\varphi : \mathscr{E} \to \mathscr{E}'$ a $\mathscr{C}$-cartesian functor, and $i \in \{0, 1, 2\}$ such that $\mathscr{E}$ is i-separated and $\mathscr{E}'$ is $(i-1)$-separated. Then φ is i-faithful if and only if it is j-covering for every $j \geq 2 - i$.*

Proof. We consider first the case where $i = 0$, so by assumption $\mathscr{E}$ is a 0-separated prestack (and no conditions on $\mathscr{E}'$); the assertion is that φ is faithful if and only if it is 2-covering. However, if φ is faithful, it is obviously 2-covering. Thus, suppose that φ is 2-covering, and let $X \in \mathrm{Ob}(\mathscr{C})$ and $g_1, g_2 : Y \to Z$ two morphisms in $\mathscr{E}_X$ such that $\varphi(g_1) = \varphi(g_2)$; by assumption there exist a covering family $(f_j : X_j \to X \mid j \in I)$ and for every $j \in I$ a cartesian morphism $h_j : Y_j \to Y$ in $\mathscr{E}$ such that $Fh_j = f_j$ and $g_1 \circ h_j = g_2 \circ h_j$. We may then find two cartesian sections $\psi_Y, \psi_Z \in \mathscr{E}(X)$ such that $\psi_Y(\mathbf{1}_X) = Y$ and $\psi_Z(\mathbf{1}_X) = Z$, and for $i = 1, 2$ there exists a unique natural $\mathscr{C}$-transformation $\sigma_i : \psi_Y \Rightarrow \psi_Z$ such that $\sigma_{i,\mathbf{1}_X} = g_i$. It follows that $(\sigma_1)_{|\mathscr{S}} = (\sigma_2)_{|\mathscr{S}}$, so $\sigma_1 = \sigma_2$, since $\mathscr{E}$ is 0-separated; hence $g_1 = g_2$, so φ is faithful.

In case $i = 1$, the prestack $\mathscr{E}$ is 1-separated and $\mathscr{E}'$ is 0-separated, and we need to check that φ is fully faithful if and only if it is j-covering for $j = 1, 2$. Again, if φ is fully faithful, obviously it is j-covering for $j = 1, 2$. Conversely, if the latter condition holds, by the foregoing case we know already that φ is faithful, so it remains only to check that φ is full. Thus, let $X \in \mathrm{Ob}(\mathscr{C})$ and $g : \varphi Y_1 \to \varphi Y_2$ any morphism in $\mathscr{E}'_X$; by assumption, there exist a covering family $(f_j : X_j \to X \mid j \in I)$ and morphisms $Y_2 \xleftarrow{g_j} Z_j \xrightarrow{h_j} Y_1$ with h_j cartesian, such that $Fh_j = f_j$ and $\varphi(g_j) = g \circ \varphi(h_j)$ for every $j \in I$. As in the foregoing, we pick for $i = 1, 2$ a cartesian section $\psi_i \in \mathscr{E}(X)$ such that $\psi_i(\mathbf{1}_X) = Y_i$, and then there exists a unique natural $\mathscr{C}$-transformation $\sigma : \varphi \circ \psi_1 \Rightarrow \varphi \circ \psi_2$ such that $\sigma_{\mathbf{1}_X} = g$. Since $\psi_1(f_j/X) : \psi_1(f_j) \to Y_1$ is cartesian, there exists moreover a unique isomorphism in $F^{-1}X_j$

$$h'_j : \psi_1(f_j) \xrightarrow{\sim} Z_j \qquad \text{such that} \qquad h_j \circ h'_j = \psi_1(f_j/X).$$

Likewise, since $\psi_2(f_j/X) : \psi_2(f_j) \to Y_2$ is cartesian, there exists a unique morphism in $F^{-1}X_j$

$$g'_j : \psi_1(f_j) \to \psi_2(f_j) \qquad \text{such that} \qquad g_j \circ h'_j = \psi_2(f_j/X) \circ g'_j.$$

We get therefore for every $j \in I$ a unique natural $\mathscr{C}$-transformation

$$\sigma_j : \psi_1 \circ f_{j*} \Rightarrow \psi_2 \circ f_{j*} \qquad \text{such that} \qquad \sigma_{j,\mathbf{1}_{X_j}} = g'_j.$$

Claim 5.2.10. $\varphi * \sigma_j = \sigma * f_j$ for every $j \in I$.

Proof of the claim. It suffices to check that $\varphi(\sigma_{j,\mathbf{1}_{X_j}}) = (\sigma * f_j)_{\mathbf{1}_{X_j}}$, *i.e.* $\varphi(g'_j) = \sigma_{f_j}$ for every $j \in I$. However : $\varphi(\psi_2(f_j/X)) \circ \varphi(g') = \varphi(g_j) \circ \varphi(h'_j) = g \circ \varphi(h_j) \circ \varphi(h'_j) = g \circ \varphi(\psi_1(f_j/X)) = \varphi(\psi_2(f_j/X)) \circ \sigma_{f_j}$, whence the assertion, since $\varphi(\psi_2(f_j/X))$ is cartesian. ◊

Recall now that the proof of theorem 3.2.7 yields a natural equivalence $\eta_{\mathsf{C}\mathscr{E})} : \mathsf{C}(\mathscr{E}) \xrightarrow{\sim} \mathsf{C}(\mathsf{C}(\mathscr{E}))$, induced by the strict pseudo-natural equivalence $\beta^{\boldsymbol{\lambda}}$ associated with the natural split cleavage $\boldsymbol{\lambda}$ of $\mathsf{C}(\mathscr{E})$. We set $\psi^*_i := \eta_{\mathsf{C}(\mathscr{E})}(\psi_i)$ for $i = 1, 2$. Explicitly, $\psi^*_i(f) = \psi_i \circ f_*$ for every $f \in \mathrm{Ob}(\mathscr{C}/X)$, and $\psi^*_{i,g/X} = \mathbf{1}_{\psi^*_i(f')}$ for every morphism $g/X : f' \to f$ in $\mathscr{C}/X$. Let $\mathscr{S} \subset \mathscr{C}/X$ be the sieve generated by $(f_j \mid j \in I)$; we construct as follows a natural $\mathscr{C}$-transformation

$$\sigma^\dagger : \psi^*_{|\mathscr{S}} \Rightarrow \psi^*_{|\mathscr{S}}.$$

For every $(f : X' \to X) \in \mathrm{Ob}(\mathscr{S})$, pick $j \in J$ such that there exists a factorisation $f = f_j \circ f'$ for some morphism $f' : X \to X_j$ of $\mathscr{C}$, and set $\sigma^\dagger_f := \sigma_j * f'_*$. Let us check that $\sigma^\dagger_f$ is independent of the choice of factorisation : indeed, if $f = f'' \circ f_k$ for some other $k \in I$ and some morphism $f'' : X' \to X_k$, we have $\varphi * (\sigma_j * f') = \sigma * f_j * f' = \sigma * f_k * f'' = \varphi * (\sigma_k * f'')$ by claim 5.2.10, whence the contention, since φ is faithful. Likewise, the naturality of the rule $f \mapsto \sigma^\dagger_f$ can be checked after composition with φ, where it follows easily (details left to the reader).

Now, since $\mathscr{E}$ is 1-separated, the same holds for $\mathsf{C}(\mathscr{E})$, hence $\sigma^\dagger$ extends to a natural $\mathscr{C}$-transformation $\sigma^* : \psi^*_1 \Rightarrow \psi^*_2$. Then there exists a unique natural $\mathscr{C}$-transformation $\sigma' : \psi_1 \Rightarrow \psi_2$ such that $\eta_{\mathsf{C}(\mathscr{E})}(\sigma') = \sigma^*$. We claim that

$$\varphi * \sigma' = \sigma.$$

Indeed, it suffices to show that $\eta_{\mathsf{C}(\mathscr{E})}(\varphi * \sigma') = \eta_{\mathsf{C}(\mathscr{E})}(\sigma)$, *i.e.* that $\mathsf{C}(\varphi) * \sigma^* = \eta_{\mathsf{C}(\mathscr{E})}(\sigma)$, and since $\mathscr{E}'$ is 0-separated, we are reduced to checking that $(\mathsf{C}(\varphi) * \sigma^*)_{|\mathscr{S}} = \eta_{\mathsf{C}(\mathscr{E})}(\sigma)_{|\mathscr{S}}$, *i.e.* that $\mathsf{C}(\varphi) * \sigma^\dagger = \eta_{\mathsf{C}(\mathscr{E})}(\sigma)_{|\mathscr{S}}$. Then we are further reduced to showing that $\varphi * \sigma^\dagger_{f_j} = \sigma * f_{j*}$ for every $j \in I$. But the latter identities hold by claim 5.2.10. Lastly, set $g' := \sigma'_{\mathbf{1}_X}$; we deduce that $\varphi(g') = g$, and this shows that φ is full, as required.

In case $i = 2$, the prestack $\mathscr{E}'$ is 1-separated and $\mathscr{E}$ is a stack, and we need to check that φ is an equivalence if and only if it is j-covering for $j = 0, 1, 2$. Again, if φ is an equivalence, clearly it is j-covering for all j; conversely, if this condition holds, by the foregoing we know already that φ is fully faithful, so it remains only to check that φ is essentially surjective. To this aim, let $X \in \mathrm{Ob}(\mathscr{C})$ and $Y' \in \mathrm{Ob}(\mathscr{E}'_X)$; by assumption, there exist a covering family $(f_j : X_j \to X \mid j \in I)$ and for every $j \in I$ a cartesian morphism $h_j : \varphi Y_j \to Y'$ with

$F'h_j = f_j$. Pick, for every $j \in I$ a cartesian section $\psi_j \in \mathscr{E}(X_j)$ with $\psi_j(\mathbf{1}_{X_j}) = Y_j$. Now, for every $j,k \in I$ consider the category $\mathscr{C}/X_{jk} := \mathscr{C}/X_j \times_{(f_{j*},f_{k*})} \mathscr{C}/X_k$ (see (3.5.4)). The objects of $\mathscr{C}/X_{jk}$ are the pairs of morphisms $X_j \xleftarrow{f'_j} X' \xrightarrow{f'_k} X_k$ such that $f' := f_j \circ f'_j = f_k \circ f'_k$. We have two natural projections $\mathscr{C}/X_j \xleftarrow{\pi^0_{jk}} \mathscr{C}/X_{jk} \xrightarrow{\pi^1_{jk}} \mathscr{C}/X_k$ that send every such pair (f'_j, f'_k) to f'_j and respectively f'_k. For every such pair (f'_j, f'_k) and $s = j,k$ we get cartesian morphisms $\psi_s(f'_s/X_s) : \psi_s(f'_s) \to Y_s$ such that $F\psi_s(f'_s/X_s) = f'_s$. There follows a unique isomorphism

$$\tau : \varphi \circ \psi_j(f'_j) \xrightarrow{\sim} \varphi \circ \psi_k(f'_k) \qquad \text{such that} \qquad h_k \circ \varphi(\psi_k(f'_k/X_k)) \circ \tau = h_j \circ \varphi(\psi_j(f'_j/X_j))$$

and since φ is fully faithful, we get a unique isomorphism

$$\omega^{ij}_{(f'_j,f'_k)} : \psi_j(f'_j) \xrightarrow{\sim} \psi_k(f'_k) \qquad \text{such that} \qquad \varphi(\omega^{ij}_{(f'_j,f'_k)}) = \tau$$

and we claim that the rule $(f'_j, f'_k) \mapsto \omega^{ij}_{(f'_j,f'_k)}$ yields an isomorphism of functors

$$\omega^{ij} : \psi_j \circ \pi^0_{jk}(f'_j, f'_k) \xrightarrow{\sim} \psi_k \circ \pi^1_{jk}(f'_j, f'_k).$$

Indeed, let $(X_j \xleftarrow{f''_j} X'' \xrightarrow{f''_k} X_k)$ be another object of $\mathscr{C}/X_{jk}$; a morphism $(f''_j, f''_k) \to (f'_j, f'_k)$ is a morphism $t : X'' \to X'$ in $\mathscr{C}$ such that $f'_j \circ t = f''_j$ and $f'_k \circ t = f''_k$, and the assertion amounts to the identity :

$$A := \psi_k(t/X_k) \circ \omega^{jk}_{(f''_j,f''_k)} = B := \omega^{jk}_{(f'_j,f'_k)} \circ \psi_j(t/X_j).$$

However, we have :

$$\begin{aligned} h_k \circ (\varphi \circ \psi_k(f'_k/X_k)) \circ \varphi(A) &= h_k \circ \varphi(\psi_k(f''_k/X_k) \circ \omega^{jk}_{(f''_j,f''_k)}) \\ &= h_j \circ \varphi(\psi_j(f''_j/X_j)) \\ &= h_j \circ \varphi(\psi_j(f'_j/X_j)) \circ \varphi(\psi_j(t/X_j)) \\ &= h_k \circ (\varphi \circ \psi_k(f'_k/X_k)) \circ \varphi(B) \end{aligned}$$

whence the contention, since $h_k \circ (\varphi \circ \psi_k(f'_k/X_k))$ is cartesian and φ is faithful. Next, for every $j,k,l \in I$ set $\mathscr{C}/X_{jkl} := \mathscr{C}/X_{jk} \times_{\mathscr{C}/X} \mathscr{C}/X_l$; the objects of this category are the triples $(f'_j : X' \to X_j, f'_k : X' \to X_k, f'_l : X' \to X_l)$ such that $f_j \circ f'_j = f_k \circ f'_k = f_l \circ f'_l$. We have obvious projection functors

$$\pi^0_{jkl} : \mathscr{C}/X_{jkl} \to \mathscr{C}/X_{kl} \qquad \pi^1_{jkl} : \mathscr{C}/X_{jkl} \to \mathscr{C}/X_{jl} \qquad \pi^2_{jk} : \mathscr{C}/X_{jkl} \to \mathscr{C}/X_{jk}.$$

With this notation, the uniqueness properties of ω^{ij} easily imply the cocycle identity :

$$(\omega^{jk} * \pi^2_{jkl}) \odot (\omega^{kl} * \pi^0_{jkl}) = \omega^{jl} * \pi^1_{jkl} \qquad \text{for every } j,k,l \in I$$

(details left to the reader). Let $\mathscr{S} \subset \mathscr{C}/X$ be the sieve generated by the family $(f_j \mid j \in I)$; in view of proposition 3.5.15, we deduce that there exist a cartesian functor $\psi : \mathscr{S} \to \mathscr{E}$ and a system of isomorphisms $(\eta_j : \psi_j \xrightarrow{\sim} \psi * f_{j*} \mid j \in I)$ fulfilling the compatibility condition

$$(\eta_k * \pi^1_{jk}) \odot \omega^{jk} = \eta_j * \pi^0_{jk} \qquad \text{for every } j,k \in I.$$

Since $\mathscr{E}$ is stack, we may then find a cartesian section $\psi' \in \mathscr{E}(X)$ with an isomorphism $\psi'_{|\mathscr{S}} \xrightarrow{\sim} \psi$. Set $Y := \psi'(\mathbf{1}_X)$; to conclude the proof, it will suffice to exhibit an isomorphism $Y' \xrightarrow{\sim} \varphi Y$ in $\mathscr{E}'$. To this aim, pick as well a cartesian section $\widetilde{\psi} \in \mathscr{E}'(X)$ such that $\widetilde{\psi}(\mathbf{1}_X) = Y'$; we set $\widetilde{\psi}_j := \widetilde{\psi} \circ f_{j*} : \mathscr{C}/X_j \to \mathscr{E}'$ for every $j \in I$. Notice that $\widetilde{\psi}_j \circ \pi^0_{jk} = \widetilde{\psi}_k \circ \pi^1_{jk}$ for every $j,k \in I$.

Since h_j is cartesian, there exists a unique isomorphism $h'_j : \widetilde{\psi}(f_j) \stackrel{\sim}{\to} \varphi Y_j$ in $F'^{-1}X_j$ such that $h_j \circ h'_j = \widetilde{\psi}(f_j/X)$. There follows an isomorphism of cartesian functors

$$\widetilde{\eta}_j : \widetilde{\psi}_j \stackrel{\sim}{\to} \varphi \circ \psi_j \qquad \text{such that} \qquad \widetilde{\eta}_{j,\mathbf{1}_{X_j}} = h'_j \qquad \text{for every } j \in I.$$

Explicitly, for every $(f' : X' \to X_j) \in \mathrm{Ob}(\mathscr{C}/X_j)$, the isomorphism $\widetilde{\eta}_{j,f'} : \widetilde{\psi}_j(f') \stackrel{\sim}{\to} \varphi\circ\psi_j(f')$ is characterized by the identity :

$$\varphi(\psi_j(f'/X_j)) \circ \widetilde{\eta}_{j,f'} = h'_j \circ \widetilde{\psi}_j(f'/X_j) \qquad \text{in } \mathscr{E}'.$$

We claim that the following diagram commutes for every $j, k \in I$:

$$\begin{array}{ccc} \widetilde{\psi}_j \circ \pi^0_{jk} & = & \widetilde{\psi}_k \circ \pi^1_{jk} \\ {\scriptstyle \widetilde{\eta}_j * \pi^0_{jk}} \Downarrow & & \Downarrow {\scriptstyle \widetilde{\eta}_k * \pi^1_{jk}} \\ \varphi \circ \psi_j \circ \pi^0_{jk} & \xRightarrow{\varphi * \omega^{ij}} & \varphi \circ \psi_k \circ \pi^1_{jk}. \end{array}$$

Indeed, let (f'_j, f'_k) be any object of $\mathscr{C}/X_{jk}$; we compute :

$$\begin{aligned} h_k \circ \varphi(\psi_k(f'_k/X_k)) \circ (\widetilde{\eta}_k * \pi^1_{jk})_{(f'_j,f'_k)} &= h_k \circ h'_k \circ \widetilde{\psi}_k(f'_k/X_k) \\ &= \widetilde{\psi}(f_k/X) \circ \widetilde{\psi}(f'_k/X) \\ &= \widetilde{\psi}(f_k \circ f'_k/X) \\ &= \widetilde{\psi}(f_j \circ f'_j/X) \\ &= h_j \circ h'_j \circ \widetilde{\psi}_j(f'_j/X_j) \\ &= h_j \circ \varphi(\psi_j(f'_j/X_j)) \circ \widetilde{\eta}_{j,f'} \\ &= h_k \circ \varphi(\psi_k(f'_k/X_k)) \circ \varphi(\omega^{ij}_{(f'_j,f'_k)}) \circ \widetilde{\eta}_{j,f'} \end{aligned}$$

whence the claim, as usual, since $h_k \circ \varphi(\psi_k(f'_k/X_k))$ is cartesian. In view of proposition 3.5.15, we deduce that there exists a unique isomorphism of cartesian functors

$$\widetilde{\psi}_{|\mathscr{S}} \stackrel{\sim}{\to} \varphi \circ \psi = (\varphi \circ \psi')_{|\mathscr{S}}.$$

But since $\mathscr{E}'$ is 1-separated, the latter comes from a unique isomorphism $\widetilde{\psi} \stackrel{\sim}{\to} \varphi \circ \psi'$, whence the sought isomorphism $Y' \stackrel{\sim}{\to} \varphi Y$. □

Proposition 5.2.11. *Let $(\mathscr{C}, J)$ be a small site, $F : \mathscr{E} \to \mathscr{C}$ a fibration, and $j \in \{0, 1, 2\}$. Then:*

(i) *The unit of adjunction $\eta_{\mathscr{E}} : \mathscr{E} \to \mathscr{E}^a$ is i-covering for $i = 0, 1, 2$.*
(ii) *$\mathscr{E}$ is j-separated if and only if $\eta_{\mathscr{E}}$ is j-faithful.*
(iii) *A cartesian functor $\varphi : \mathscr{E} \to \mathscr{E}'$ of $\mathscr{C}$-fibrations is j-covering if and only if the same holds for the induced functor $\varphi^a : \mathscr{E}^a \to \mathscr{E}'^a$.*

Proof. (i): Quite generally, let $G : \mathscr{A} \to \mathscr{B}$ be a $\mathscr{C}$-cartesian functor of fibrations, and $H : \mathscr{A}' \stackrel{\sim}{\to} \mathscr{A}$ a $\mathscr{C}$-equivalence of categories; then it is easily seen that G is i-covering for some $i \in \{0, 1, 2\}$ if and only if the same holds for $G \circ H$. Taking into account lemma 5.2.4(i), we are therefore reduced to checking :

Claim 5.2.12. The functor $j_{\mathscr{E}} : \mathsf{C}(\mathscr{E}) \to \mathscr{E}^+$ is i-covering for $i = 0, 1, 2$.

Proof of the claim. Let $[X, \psi : \mathscr{S} \to \mathscr{E}]$ be any object of $\mathscr{E}^+$, and pick a generating family $(f_j : X_j \to X \mid j \in I)$ for $\mathscr{S}$; then $(X_j, \psi \circ f_j : \mathscr{C}/X_j \to \mathscr{E})$ is an object of $\mathsf{C}(\mathscr{E})$ and f_j induces a cartesian morphism $j_{\mathscr{E}}(X_j, \psi \circ f_{j*}) \to [X, \psi]$ for every $j \in I$. Thus, $j_{\mathscr{E}}$ is 0-covering.

Next, let $[t,\sigma] : [X',\psi' : \mathscr{C}/X' \to \mathscr{E}] \to [X,\psi : \mathscr{C}/X \to \mathscr{E}]$ be a morphism in $\mathscr{E}^+$. By definition, $t : X' \to X$ is a morphism of $\mathscr{C}$, and $\sigma : \psi'_{|\mathscr{S}} \Rightarrow (\psi \circ t_*)_{|\mathscr{S}}$ is a natural $\mathscr{C}$-transformation defined on a covering sieve $\mathscr{S} \subset \mathscr{C}/X'$. Again, we pick a generating family $(f_j : X'_i \to X' \mid j \in I)$ for $\mathscr{S}$, and denote by $(f_j, h_j) : (X'_j, \psi' \circ f_{j*}) \to (X', \psi')$ the cartesian morphism in $\mathsf{C}(\mathscr{E})$ induced by f_j. Then $\sigma * f_{j*} : \psi' \circ f_{j*} \Rightarrow \psi \circ (t \circ f_j)_*$ defines a morphism $(t \circ f_j, g_j) : (X'_j, \psi' \circ f_{j*}) \to (X, \psi)$ in $\mathsf{C}(\mathscr{E})$ such that $j_{\mathscr{E}}(t \circ f_j, g_j) = [t,\sigma] \circ j_{\mathscr{E}}(f_j, h_j)$ for every $j \in I$. This proves that $j_{\mathscr{E}}$ is 1-covering.

Lastly, let $(t_1,\sigma_1),(t_2,\sigma_2) : (X',\psi' : \mathscr{C}/X' \to \mathscr{E}) \to (X,\psi : \mathscr{C}/X \to \mathscr{E})$ be two morphisms in $\mathsf{C}(\mathscr{E})$ such that $j_{\mathscr{E}}(t_1,\sigma_1) = j_{\mathscr{E}}(t_2,\sigma_2)$. Especially, this means that $t := t_1 = t_2$, and there exists a covering sieve $\mathscr{S} \subset \mathscr{C}/X'$ such that $(\sigma_1)_{|\mathscr{S}} = (\sigma_2)_{|\mathscr{S}}$. We pick a generating family $(f_j : X'_j \to X' \mid j \in I)$ for $\mathscr{S}$, and let $(f_j, h_j) : (X'_j, \psi' \circ f_{j*}) \to (X', \psi')$ be the cartesian morphism induced by f_j; then $(t,\sigma_1) \circ (f_j,h_j) = (t,\sigma_2) \circ (f_j,h_j)$ for every $j \in I$, so $j_{\mathscr{E}}$ is 2-covering. ♢

(ii): From (i) and proposition 5.2.9 we see already that if $\mathscr{E}$ is j-separated, then $\eta_{\mathscr{E}}$ is j-faithful. Suppose then that $\eta_{\mathscr{E}}$ is j-faithful; let $X \in \mathrm{Ob}(\mathscr{C})$ be any object, and $\mathscr{S} \subset \mathscr{C}/X$ any covering sieve; we consider the induced commutative diagram of categories :

$$\begin{array}{ccc} \mathscr{E}(X) & \longrightarrow & \mathsf{Cart}_{\mathscr{C}}(\mathscr{S},\mathscr{E}) \\ {\scriptstyle \mathsf{Cart}_{\mathscr{C}}(\mathscr{C}/X,\eta_{\mathscr{E}})}\downarrow & & \downarrow{\scriptstyle \mathsf{Cart}_{\mathscr{C}}(\mathscr{S},\eta_{\mathscr{E}})} \\ \mathscr{E}^a(X) & \longrightarrow & \mathsf{Cart}_{\mathscr{C}}(\mathscr{S},\mathscr{E}^a) \end{array}$$

whose bottom horizontal arrow is an equivalence, since $\mathscr{E}^a$ is a stack, and whose vertical arrows are j-faithful, by virtue of corollary 3.1.28(ii.a). It follows that the top horizontal arrow is j-faithful as well, which means that $\mathscr{E}$ is j-separated.

(iii): We have an essentially commutative diagram of cartesian functors

$$\begin{array}{ccc} \mathscr{E} & \xrightarrow{\varphi} & \mathscr{E}' \\ \downarrow & & \downarrow \\ \mathscr{E}^a & \xrightarrow{\varphi^a} & \mathscr{E}'^a \end{array}$$

whose vertical arrows are i-covering for $i = 0,1,2$, by (i). Then the assertion follows immediately from lemma 5.2.4(i,ii). □

5.3. **Local calculus of fractions.** Let $C := (\mathscr{C}, J)$ be a site, $\mathscr{A} \to \mathscr{C}$ a fibration; choose a unital cleavage $\boldsymbol{\lambda}$ for $\mathscr{A}$, and let $\mathsf{c} : \mathscr{C}^o \to \mathbf{Cat}$ be the associated unital pseudo-functor. To ease notation, for every morphism $f : X' \to X$ in $\mathscr{C}$ we let $f^* := \mathsf{c}_f : \mathscr{A}_X \to \mathscr{A}_{X'}$, and denote by $\gamma^{\mathsf{c}}_{\bullet,\bullet}$ the coherence constraint of c. Consider a system of sets $\Sigma_\bullet := (\Sigma_X \mid X \in \mathrm{Ob}(\mathscr{C}))$ with :

- $\Sigma_X \subset \mathrm{Morph}(\mathscr{A}_X)$ for every $X \in \mathrm{Ob}(\mathscr{C})$
- $f^*(\Sigma_X) \subset \Sigma_{X'}$ for every morphism $f : X' \to X$ of $\mathscr{C}$.

Proposition 5.3.1. *In the situation of* (5.3)*, there exist a* 1*-separated prestack* $\mathscr{A}\{\Sigma_\bullet^{-1}\}$ *over* C*, and a cartesian functor*

$$L_{\{\Sigma\}} : \mathscr{A} \to \mathscr{A}\{\Sigma_\bullet^{-1}\}$$

with the following properties :

(i) $L_{\{\Sigma\}} f$ *is an isomorphism in* $\mathscr{A}\{\Sigma_\bullet^{-1}\}$*, for every* $f \in \Sigma := \bigcup_{f \in \mathrm{Ob}(\mathscr{C})} \Sigma_X$.

(ii) *For every* 1*-separated prestack* $\mathscr{E}$ *on* C*, and every cartesian functor* $F : \mathscr{A} \to \mathscr{E}$ *such that* Ff *is invertible in* $\mathscr{E}$ *for every* $f \in \Sigma$*, there exists a unique cartesian functor* $F' : \mathscr{A}\{\Sigma_\bullet^{-1}\} \to \mathscr{B}$ *such that* $F = F' \circ L_{\{\Sigma\}}$.

Proof. According to theorem 1.6.9, for every morphism $f : X' \to X$ of $\mathscr{C}$ the functor f^* extends uniquely to a functor $f^*_\Sigma : \mathscr{A}_X[\Sigma_X^{-1}] \to \mathscr{A}_{X'}[\Sigma_{X'}^{-1}]$, and $\gamma^{\mathsf{c}}_{f,g} : f^* \circ g^* \xrightarrow{\sim} (g \circ f)^*$ extends uniquely to an isomorphism of functors $\gamma^{\Sigma}_{f,g} : f^*_\Sigma \circ g^*_\Sigma \xrightarrow{\sim} (g \circ f)^*_\Sigma$ for every composable pair (f, g) of morphisms of $\mathscr{C}$ (corollary 1.6.11). It is then easily seen that the rules : $X \mapsto \mathscr{A}_X[\Sigma_X^{-1}]$ for every $X \in \mathrm{Ob}(\mathscr{C})$ and $f \mapsto f^*_\Sigma$ for every morphism f of $\mathscr{C}$ yield a unital pseudo-functor $\mathsf{c}[\Sigma_\bullet^{-1}] : \mathscr{C}^o \to \mathbf{Cat}$ with coherence constraint $\gamma^{\Sigma}_{\bullet\bullet}$, and we set

$$\mathscr{A}[\Sigma_\bullet^{-1}] := \mathscr{F}ib(\mathsf{c}[\Sigma_\bullet^{-1}]).$$

From theorem 1.6.9 it follows easily that the system of localization functors $\mathscr{A}_X \to \mathscr{A}_X[\Sigma_X^{-1}]$ yields a unique cartesian functor $L' : \mathscr{A} \to \mathscr{A}[\Sigma_\bullet^{-1}]$ such that the following holds. For every fibration $\mathscr{F}$ over $\mathscr{C}$, and every cartesian functor $F : \mathscr{A} \to \mathscr{F}$ such that Ff is invertible in $\mathscr{F}$ for every $f \in \Sigma$, there exists a unique cartesian functor $F' : \mathscr{A}[\Sigma_\bullet^{-1}] \to \mathscr{F}$ such that $F = F' \circ L'$. By theorem 5.1.10 we conclude that the composition of L' with the natural cartesian functor $\mathscr{A}[\Sigma_\bullet^{-1}] \to \mathscr{A}\{\Sigma_\bullet^{-1}\} := \mathscr{A}[\Sigma_\bullet^{-1}]^{\mathrm{sep}}$ fulfills the stated conditions. $\square$

Definition 5.3.2. In the situation of (5.3), we shall say that the system $\Sigma_\bullet$ *admits a right local calculus of fractions* if the following conditions hold for every $X \in \mathrm{Ob}(\mathscr{C})$:

(LCF1) The set Σ_X contains the isomorphisms of $\mathscr{A}_X$.
(LCF2) The set Σ_X fulfills condition (CF2) of definition 1.6.14.
(LCF3) For every morphism $f : A \to B$ in $\mathscr{A}_X$ and every $s : C \to B$ in Σ_X, there exist a covering family $(\varphi_\lambda : X_\lambda \to X \mid \lambda \in \Lambda)$ in the site C, and for every $\lambda \in \Lambda$ a morphism $g_\lambda : D_\lambda \to \varphi^*_\lambda C$ in $\mathscr{A}_{X_\lambda}$ and $t_\lambda : D_\lambda \to \varphi^*_\lambda A$ in Σ_{X_λ} such that $\varphi^*_\lambda(f) \circ t_\lambda = \varphi^*_\lambda(s) \circ g_\lambda$.
(LCF4) If $f, g : A \to B$ are any two morphisms in $\mathscr{A}_X$ such that $s \circ f = s \circ g$ for some $s : B \to C$ in Σ_X, then there exist a covering family $(\varphi_\lambda : X_\lambda \to X \mid \lambda \in \Lambda)$ in the site C and for every $\lambda \in \Lambda$ an element $t_\lambda : D_\lambda \to \varphi^*_\lambda A$ in Σ_{X_λ} such that $\varphi^*(f) \circ t_\lambda = \varphi^*_\lambda(g) \circ t_\lambda$.

5.3.3. In the situation of (5.3), for every $X \in \mathrm{Ob}(\mathscr{C})$ and $A \in \mathrm{Ob}(\mathscr{A}_X)$ let $\Sigma_{X/A}$ be the full subcategory of $\mathscr{A}_X/A$ whose objects are the elements of Σ_X with target equal to A. For every morphism $(Y' \xrightarrow{\varphi'} X) \xrightarrow{\psi/X} (Y \xrightarrow{\varphi} X)$ of $\mathscr{C}/X$ notice that $\psi^* : \mathscr{A}_Y \to \mathscr{A}_{Y'}$ induces a functor

$$(\psi/X)^* : \mathscr{A}_Y/\varphi^*A \to \mathscr{A}_{Y'}/\varphi'^*A \qquad (I \xrightarrow{f} \varphi^*A) \mapsto (\psi^*I \xrightarrow{\gamma^{\mathsf{c}}_{(\psi,\varphi),A} \circ \psi^*(f)} \varphi'^*A)$$

Suppose now that $\Sigma_\bullet$ fulfills condition (LCF1) of definition 5.3.2; then $(\psi/X)^*$ restricts to

$$(\psi/X)^* : \Sigma_{Y/\varphi^*A} \to \Sigma_{Y'/\varphi'^*A}.$$

Moreover, if $(\psi'/X) : (Y'' \xrightarrow{\varphi''} X) \to (Y' \xrightarrow{\varphi'} X)$ is another morphism of $\mathscr{C}/X$, we have an isomorphism in $\Sigma_{Y''/\varphi''^*A}$ for every $(s : I \to \varphi^*A) \in \mathrm{Ob}(\Sigma_{Y/\varphi^*A})$

$$(\gamma^{\mathsf{c}}_{(\psi',\psi),I}/X) : (\psi'/X)^* \circ (\psi/X)^*(s) \xrightarrow{\sim} (\psi \circ \psi'/X)^*(s).$$

Clearly the rule $s \mapsto (\gamma^{\mathsf{c}}_{(\psi',\psi),I}/X)$ yields an isomorphism of functors $\gamma^{\Sigma}_{\psi',\psi} : (\psi'/X)^* \circ (\psi/X)^* \xrightarrow{\sim} (\psi \circ \psi'/X)^*$, and then it easily seen that the rules : $\varphi \mapsto \Sigma_{Y/\varphi^*A}$ for every $(\varphi : Y \to X) \in \mathrm{Ob}(\mathscr{C}/X)$ and $\psi/X \mapsto (\psi/X)^*$ for every morphism ψ/X of $\mathscr{C}/X$ define a unital pseudo-functor $\Sigma_{\bullet/A} : (\mathscr{C}/X)^o \to \mathbf{Cat}$, whose coherence constraint is given by the system of isomorphisms $\gamma^{\Sigma}_{\psi',\psi}$. There follows a fibration

$$S_A := \mathscr{F}ib(\Sigma_{\bullet/A}) \to \mathscr{C}/X$$

and we denote by $\boldsymbol{\lambda}_A$ its natural unital cleavage. Next, for $A, B \in \mathrm{Ob}(\mathscr{A}_X)$ consider the functor

$$\mathscr{H}_{A,B} : S^o_A \to \mathbf{Set} \qquad (Y \xrightarrow{\varphi} X, I \xrightarrow{s} \varphi^*A) \mapsto \{(s, f) \mid f \in \mathrm{Hom}_{\mathscr{A}_Y}(I, \varphi^*B)\}$$

which assigns to every morphism $(\psi/X, h/\varphi^*A) : (\varphi, s) \to (\varphi', s')$ of S_A the map

$$\mathscr{H}_{A,B}(\psi/X, h/\varphi^*A) : \mathscr{H}_{A,B}(\varphi', s') \to \mathscr{H}_{A,B}(\varphi, s) \qquad (s', f) \mapsto (s, ((\psi/X)^*f) \circ h).$$

Lemma 5.3.4. *With the notation of* (5.3.3)*, suppose that the system $\Sigma_\bullet$ admits a right local calculus of fraction. Then the fibration S_A is locally cofiltered over the site C/X.*

Proof. Condition (LCF1) implies immediately condition (a) of definition 4.3.3. Next, let $(\varphi : Y \to X) \in \mathrm{Ob}(\mathscr{C}/X)$, and $(\varphi, s : I \to \varphi^*A)$ and $(\varphi, s' : I' \to \varphi^*A)$ be two objects of Σ_{Y/φ^*A}. By (LCF3) we may find a covering $(\psi_\lambda : Y_\lambda \to Y \mid \lambda \in \Lambda)$ for the topology J, and for every $\lambda \in \Lambda$ a morphism $f_\lambda : J \to \psi_\lambda^* I$ in $\mathscr{A}_{Y_\lambda}$ and an element $t_\lambda : J \to \psi_\lambda^* I'$ of Σ_{Y_λ} such that $s''_\lambda := \psi^*_\lambda(s) \circ f_\lambda = \psi^*_\lambda(s') \circ t_\lambda$. By (LCF1) and (LCF2), we have $u_\lambda := \gamma_{(\varphi,\psi_\lambda),A} \circ s''_\lambda \in \Sigma_{Y_\lambda/(\varphi\circ\psi_\lambda)^*A}$ for every $\lambda \in \Lambda$, and it follows that t_λ and f_λ yield morphisms $u_\lambda \to (\psi/X)^*(s')$ and respectively $u_\lambda \to (\psi/X)^*(s)$. Hence S_A fulfills condition (b) of definition 4.3.3.

Lastly, consider two morphisms $(g/\varphi^*A), (g'/\varphi^*) : (s : I \to \varphi^*A) \to (s' : I \to \varphi^*A)$ in Σ_{Y/φ^*A}. In other words, $g, g' : I \to I'$ are morphisms in $\mathscr{A}_Y$ with $s' \circ g = s' \circ g' = s$. By (LCF3) there exist then a covering family $(\psi_\lambda : Y_\lambda \to Y \mid \lambda \in \Lambda)$ and for every $\lambda \in \Lambda$ an element $h_\lambda : J \to \psi^*_\lambda I$ of $\Sigma_{\mathscr{A}_{Y_\lambda}}$ with $t_\lambda := \psi^*_\lambda(g) \circ h_\lambda = \psi^*_\lambda(g') \circ h_\lambda$. Notice that $u_\lambda := \psi^*_\lambda(s') \circ t_\lambda = \psi^*_\lambda(s) \circ h_\lambda$ lies in $\Sigma_{Y_\lambda,(\psi_\lambda)^*\varphi^*A}$. Hence h_λ yields a morphism $(h/(\varphi\circ\psi_\lambda)^*A) : \gamma_{(\varphi,\psi_\lambda),A} \circ u_\lambda \to (\psi/X)^*(s)$ in $\Sigma_{Y_\lambda,(\varphi\circ\psi_\lambda)^*A}$ with $(\psi_\lambda/X)^*(g) \circ (h/(\varphi\circ\psi_\lambda)^*A) = (\psi_\lambda/X)^*(g') \circ (h/(\varphi\circ\psi_\lambda)^*A)$. This shows that condition (c) holds as well. ☐

5.3.5. With the notation of (4.3.4), consider the presheaf on $\mathscr{C}/X$ and the sheaf on C/X :

$$M_{A,B} := \int^{\boldsymbol{\lambda}_A} \mathscr{H}_{A,B} \qquad \text{and} \qquad H_{A,B} := \int_a^{\boldsymbol{\lambda}_A} \mathscr{H}_{A,B}.$$

Recall that for every $(\varphi : Y \to X) \in \mathrm{Ob}(\mathscr{C}/X)$, the set $M_{AB}(\varphi)$ consists of the equivalence classes of $[s, f]$ of pairs with $s \in \mathrm{Ob}(\Sigma_{Y/\varphi^*A})$ and $(s, f) \in \mathscr{H}_{AB}(\varphi, s)$. We shall denote by $\{s, f\} \in H_{A,B}(\varphi)$ the image of such a class $[s, f]$. Taking into account lemmata 4.3.5 and 5.3.4 we see that if $\Sigma_\bullet$ admits a right local calculus of fraction, the following holds :

(a) For every $\sigma \in H_{AB}(\varphi)$ we have a covering family $(\varphi_\lambda : Y_\lambda \to Y \mid \lambda \in \Lambda)$ in the site C, and for every $\lambda \in \Lambda$ a morphism $g_\lambda : C_\lambda \to \varphi^*_\lambda B$ in $\mathscr{A}_{Y_\lambda}$ and an element $s_\lambda : C_\lambda \to \varphi^*_\lambda A$ of Σ_{Y_λ} such that $H_{AB}(\varphi_\lambda/X)(\sigma)$ is the image of $[s_\lambda, g_\lambda]$ in $H_{AB}(\varphi \circ \varphi_\lambda/X)$.

(b) For every two pairs $(s : I \to \varphi^*A, f), (s' : I' \to \varphi^*A, f') \in \mathscr{H}_{AB}(\varphi, s)$ with $[s, f] = [s', f']$ in $M_{AB}(\varphi, s)$ there exist a covering family $(\varphi_\lambda : Y_\lambda \to Y \mid \lambda \in \Lambda)$ in the site C, and for every $\lambda \in \Lambda$ two morphisms in $\mathscr{A}_{Y_\lambda}$

$$\varphi^*_\lambda I' \xleftarrow{t'_\lambda} J_\lambda \xrightarrow{t_\lambda} \varphi^*_\lambda I \qquad \text{with} \qquad (\varphi_\lambda/X)^*(f) \circ t_\lambda = (\varphi_\lambda/X)^*(f') \circ t'_\lambda$$

and such that $(\varphi_\lambda/X)^*(s) \circ t_\lambda = (\varphi_\lambda/X)^*(s') \circ t'_\lambda$ lies in $\Sigma_{Y_\lambda,(\varphi\circ\varphi_\lambda)^*A}$.

Let $\boldsymbol{\lambda}_\Sigma$ and $\boldsymbol{\lambda}_{\{\Sigma\}}$ be the cleavages of $\mathscr{A}[\Sigma_\bullet^{-1}]$ and respectively $\mathscr{A}\{\Sigma_\bullet^{-1}\}$ deduced from the cleavage $\boldsymbol{\lambda}$ of $\mathscr{A}$, as in the proofs of proposition 5.3.1 and theorem 5.1.10. For every $A \in \mathrm{Ob}(\mathscr{A}\{\Sigma_\bullet^{-1}\}_X) = \mathrm{Ob}(\mathscr{A}_X[\Sigma_\bullet^{-1}]) = \mathrm{Ob}(\mathscr{A}_X)$ define the cartesian sections $\beta_A^{\boldsymbol{\lambda}_\Sigma} : \mathscr{C}/X \to \mathscr{A}[\Sigma_\bullet^{-1}]$ and $\beta_{X,A}^{\boldsymbol{\lambda}_{\{\Sigma\}}} : \mathscr{C}/X \to \mathscr{A}\{\Sigma_\bullet^{-1}\}$ as in the proof of claim 3.2.8. Let also $L_\Sigma : \mathscr{A} \to \mathscr{A}[\Sigma_\bullet^{-1}]$ be the localization functor; we have a natural morphism of presheaves on $\mathscr{C}/X$:

$$M_{A,B} \to \mathscr{C}art(\beta_A^{\boldsymbol{\lambda}_\Sigma}, \beta_B^{\boldsymbol{\lambda}_\Sigma}) \qquad \text{for every } A, B \in \mathrm{Ob}(\mathscr{A}_X)$$

which assigns to every $[s, f] \in M_{A,B}(\varphi)$ the unique natural $\mathscr{C}$-transformation $\beta_A^{\boldsymbol{\lambda}_\Sigma} \circ \varphi_* \to \beta_B^{\boldsymbol{\lambda}_\Sigma} \circ \varphi_*$ such that $\mathbf{1}_Y \mapsto L_\Sigma(f) \circ L_\Sigma(s)^{-1}$, for every $\varphi \in \mathrm{Ob}(\mathscr{C}/X)$. We deduce a morphism $H_{A,B} \to \mathscr{C}art(\beta_A^{\boldsymbol{\lambda}_\Sigma}, \beta_B^{\boldsymbol{\lambda}_\Sigma})^a$ of associated sheaves on C/X. But recall that the proof of theorem 5.1.10 yields as well a natural identification

$$\mathscr{C}art(\beta_A^{\boldsymbol{\lambda}_\Sigma}, \beta_B^{\boldsymbol{\lambda}_\Sigma})^a \xrightarrow{\sim} \mathscr{C}art(\beta_A^{\boldsymbol{\lambda}\{\Sigma\}}, \beta_B^{\boldsymbol{\lambda}\{\Sigma\}}).$$

The composition of the latter two morphisms is a morphism of sheaves on C/X :

$$H_{AB} \to \mathscr{C}art(\beta^{\boldsymbol{\lambda}\{\Sigma\}}_{X,A}, \beta^{\boldsymbol{\lambda}\{\Sigma\}}_{X,B}) \qquad \{s,f\} \mapsto L_{\{\Sigma\}}(f) \circ L_{\{\Sigma\}}(s)^{-1}. \tag{5.3.6}$$

Proposition 5.3.7. *Suppose that the system $\Sigma_\bullet$ admits a right local calculus of fraction. Then* (5.3.6) *is an isomorphism of sheaves for every $A, B \in \mathrm{Ob}(\mathscr{A}_X)$.*

Proof. We define for every $X \in \mathrm{Ob}(\mathscr{C})$ a category

$$\mathscr{L}_X$$

whose set of objects is $\mathrm{Ob}(\mathscr{A}_X)$, and such that $\mathrm{Hom}_{\mathscr{L}_X}(A,B) := H_{AB}(\mathbf{1}_X)$ for every $A, B \in \mathrm{Ob}(\mathscr{A}_X)$. For every $A, B, C \in \mathrm{Ob}(\mathscr{A}_X)$, the composition law

$$H_{AB}(\mathbf{1}_X) \times H_{BC}(\mathbf{1}_X) \to H_{AC}(\mathbf{1}_X) \tag{5.3.8}$$

is obtained as follows. Let $(\varphi : Y \to X) \in \mathrm{Ob}(\mathscr{C}/X)$ and $[s : I \to \varphi^* A, f] \in M_{AB}(\varphi)$ and $[s' : I' \to \varphi^* B, f'] \in M_{BC}(\varphi)$ any two sections; by virtue of (LCF3) we may find a covering family $(\varphi_\lambda : Y_\lambda \to Y \mid \lambda \in \Lambda)$ and for every $\lambda \in \Lambda$, morphisms in $\mathscr{A}_{Y_\lambda}$:

$$\varphi^*_\lambda I \xleftarrow{s_\lambda} J_\lambda \xrightarrow{f_\lambda} \varphi^*_\lambda I' \qquad \text{such that} \qquad (\varphi_\lambda/X)^*(f) \circ s_\lambda = (\varphi_\lambda/X)^*(s') \circ f_\lambda \quad \text{and} \quad s_\lambda \in \Sigma_{Y_\lambda}.$$

Set $t_\lambda := (\varphi_\lambda/X)^*(s) \circ s_\lambda : J_\lambda \to (\varphi \circ \varphi_\lambda)^* A$ and $g_\lambda := (\varphi_\lambda/X)^*(f') \circ f_\lambda : J_\lambda \to (\varphi \circ \varphi_\lambda)^* B$ for every λ. For $\lambda, \mu \in \Lambda$, consider morphisms $Y_\lambda \xleftarrow{\psi_\lambda} Z \xrightarrow{\psi_\mu} Y_\mu$ such that $\psi := \varphi_\lambda \circ \psi_\lambda = \varphi_\mu \circ \psi_\mu$.

Claim 5.3.9. $H_{AB}(\psi_\lambda/X)\{t_\lambda, g_\lambda\} = H_{AB}(\psi_\mu/X)\{t_\mu, g_\mu\}$.

Proof of the claim. We need to show that $\{(\psi_\lambda/X)^* t_\lambda, (\psi_\lambda/X)^* g_\lambda\} = \{(\psi_\mu/X)^* t_\mu, (\psi_\mu/X)^* g_\mu\}$. However, by (LCF3) we may find a covering family $(\rho_\nu : Z_\nu \to Z \mid \nu \in \Lambda')$ and for every $\nu \in \Lambda'$, morphisms $\rho^*_\nu \psi^*_\mu J_\mu \xleftarrow{s_\nu} J_\nu \xrightarrow{s'_\nu} \rho^*_\nu \psi^*_\lambda J_\lambda$ in $\mathscr{A}_{Z_\nu}$ with $s_\nu \in \Sigma_{Z_\nu}$, and such that

$$(\rho_\nu/Y)^*(\psi_\mu/Y)^*(s_\mu) \circ s_\nu = (\rho_\nu/Y)^*(\psi_\lambda/Y)^*(s_\lambda) \circ s'_\nu.$$

We compute :

$$\begin{aligned}
(\rho_\nu/Y)^*((\psi/X)^*(s') \circ (\psi_\mu/Y)^*(f_\mu)) \circ s_\nu &= (\rho_\nu/Y)^*((\psi_\mu/X)^*((\varphi_\mu/X)^*(s') \circ f_\mu)) \circ s_\nu \\
&= (\rho_\nu/Y)^*((\psi_\mu/X)^*((\varphi_\mu/X)^*(f) \circ s_\lambda)) \circ s_\nu \\
&= (\rho_\nu/Y)^*((\psi/X)^*(f) \circ (\psi_\mu/Y)^*(s_\mu)) \circ s_\nu \\
&= (\rho_\nu/Y)^*((\psi/X)^*(f) \circ (\psi_\lambda/Y)^*(s_\lambda)) \circ s'_\nu \\
&= (\rho_\nu/Y)^*((\psi/X)^*(s') \circ (\psi_\lambda/Y)^*(f_\lambda)) \circ s'_\nu.
\end{aligned}$$

By (LCF4) it follows that for every $\nu \in \Lambda'$ there exist a covering family $(\rho_{\nu\nu'} : Z_{\nu\nu'} \to Z_\nu \mid \nu' \in \Lambda'_\nu)$ and for every $\nu' \in \Lambda'_\nu$ an element $t_{\nu\nu'} : J_{\nu\nu'} \to \rho^*_{\nu\nu'} J_\nu$ of $\Sigma_{Z_{\nu\nu'}}$ such that

$$(\rho_{\nu\nu'}/Y)^*(\rho_\nu/Y)^*((\psi_\mu/Y)^*(f_\mu) \circ s_\nu) \circ t_{\nu\nu'} = (\rho_{\nu\nu'}/Y)^*(\rho_\nu/Y)^*((\psi_\lambda/Y)^*(f_\lambda) \circ s'_\nu) \circ t_{\nu\nu'}.$$

Set $\rho'_{\nu\nu'} := \rho_\nu \circ \rho_{\nu\nu'}$ for every $\nu' \in \Lambda'_\nu$, and

$$T_{\lambda\nu\nu'} := (\psi_\lambda \circ \rho'_{\nu\nu'}/X)^* t_\lambda \qquad G_{\lambda\nu\nu'} := (\psi_\lambda \circ \rho'_{\nu\nu'}/X)^* g_\lambda \qquad \gamma_{\lambda\nu\nu'} := \gamma^{\mathsf{c}}_{(\rho'_{\nu\nu'},\psi_\lambda),J_\lambda}$$

and define likewise $T_{\mu\nu\nu'}$, $G_{\mu\nu\nu'}$ and $\gamma_{\mu\nu\nu'}$. We compute :

$$\begin{aligned}
[T_{\lambda\nu\nu'}, G_{\lambda\nu\nu'}] &= [T_{\lambda\nu\nu'} \circ \gamma_{\lambda\nu\nu'} \circ (\rho_{\nu\nu'}/Z)^*(s'_\nu) \circ t_{\nu\nu'}, G_{\lambda\nu\nu'} \circ \gamma_{\lambda\nu\nu'} \circ (\rho'_{\nu\nu'}/Z)^*(s'_\nu) \circ t_{\nu\nu'}] \\
&= [T_{\mu\nu\nu'} \circ \gamma_{\mu\nu\nu'} \circ (\rho_{\nu\nu'}/Z)^*(s_\nu) \circ t_{\nu\nu'}, G_{\mu\nu\nu'} \circ \gamma_{\mu\nu\nu'} \circ (\rho_{\nu\nu'}/Z)^*(s_\nu) \circ t_{\nu\nu'}] \\
&= [T_{\mu\nu\nu'}, G_{\mu\nu\nu'}]
\end{aligned}$$

whence the contention. ◊

Claim 5.3.9 implies that the system $(\{t_\lambda, g_\lambda\} \mid \lambda \in \Lambda)$ defines a unique section of $H_{AC}(\varphi)$ that we denote $\{s', f'\} \circ \{s, f\}$. Notice that the argument proves as well that the construction of $\{s', f'\} \circ \{s, f\}$ is independent of all auxiliary choices. A direct inspection shows then that we obtain a well defined morphism of presheaves

$$M_{AB} \times M_{BC} \to H_{AC} \qquad ([s, f], [s', f']) \mapsto \{s', f'\} \circ \{s, f\}$$

which therefore induces a unique morphism of sheaves on C/X :

$$H_{AB} \times H_{BC} \to H_{AC} \qquad \text{such that} \qquad (\{s, f\}, \{s', f'\}) \mapsto \{s', f'\} \circ \{s, f\}.$$

Then the sought map (5.3.8) is the map of sets obtained by evaluating the foregoing morphism on the object $\mathbf{1}_X \in \mathrm{Ob}(\mathscr{C}/X)$. Let us check that $\{\mathbf{1}_{\varphi^*B}, \mathbf{1}_{\varphi^*B}\} \circ \sigma = \sigma = \sigma \circ \{\mathbf{1}_{\varphi^*A}, \mathbf{1}_{\varphi^*A}\}$ for every $A, B \in \mathrm{Ob}(\mathscr{A}_X)$ and every $\varphi \in \mathrm{Ob}(\mathscr{C}/X)$: indeed, it suffices to check these identities for $\sigma = \{s, f\}$, for any $(s, f) \in \mathscr{H}_{A,B}(\varphi)$, and the latter follow by simple inspection. Lastly, in order to show the associativity property for our composition law, consider $A, B, C, D \in \mathrm{Ob}(\mathscr{A}_X)$ and $(s : I \to \varphi^*A, f), (s' : I' \to \varphi^*B, f'), (s'' : I'' \to \varphi^*C, f'') \in \mathscr{H}_{AB}(\varphi)$, for some $(\varphi : Y \to X) \in \mathrm{Ob}(\mathscr{C}/X)$. By (LCF3) we may find a covering family $(\varphi_\lambda : Y_\lambda \to Y \mid \lambda \in \Lambda)$ and for every $\lambda \in \Lambda$, morphisms in $\mathscr{A}_{Y_\lambda}$:

$$\varphi_\lambda^* I \xleftarrow{t_\lambda} J_\lambda \xrightarrow{g_\lambda} \varphi_\lambda^* I' \xleftarrow{t'_\lambda} J'_\lambda \xrightarrow{g'_\lambda} \varphi_\lambda^* I'' \qquad \text{with} \qquad t_\lambda, t'_\lambda \in \Sigma_{Y_\lambda}$$

such that $(\varphi_\lambda/X)^*(f) \circ t_\lambda = (\varphi_\lambda/X)^* s' \circ g_\lambda$ and $(\varphi_\lambda/X)^*(f') \circ t'_\lambda = (\varphi_\lambda/X)^*(s'') \circ g'_\lambda$. Then we may also find for every $\lambda \in \Lambda$ a covering family $(\varphi_{\lambda\mu} : Y_{\lambda\mu} \to Y_\lambda \mid \mu \in \Lambda_\lambda)$ and for every $\mu \in \Lambda_\lambda$, morphisms in $\mathscr{A}_{Y_{\lambda\mu}}$:

$$\varphi_{\lambda\mu}^* J_\lambda \xleftarrow{t_{\lambda\mu}} J_{\lambda\mu} \xrightarrow{g_{\lambda\mu}} \varphi_{\lambda\mu}^* J'_\lambda \qquad \text{such that} \qquad (\varphi_{\lambda\mu}/Y)^*(g_\lambda) \circ t_{\lambda\mu} = (\varphi_{\lambda\mu}/Y)^*(t_\lambda) \circ g_{\lambda\mu}$$

and with $t_{\lambda\mu} \in \Sigma_{Y_{\lambda\mu}}$. For every such λ and μ set $\psi_{\lambda\mu} := \varphi_\lambda \circ \varphi_{\lambda\mu}$ and

$$(t'_{\lambda\mu}, g'_{\lambda\mu}) := ((\psi_{\lambda\mu}/X)^*(s) \circ (\varphi_{\lambda\mu}/Y)^*(t_\lambda) \circ t_{\lambda\mu}, (\psi_{\lambda\mu}/X)^*(f'') \circ (\varphi_{\lambda\mu}/Y)^*(g_\lambda) \circ g_{\lambda\mu}).$$

It then follows easily that the system $(\{t'_{\lambda\mu}, g'_{\lambda\mu}\} \mid \lambda \in \Lambda, \mu \in \Lambda_\lambda)$ represents both $\{s'', f''\} \circ (\{s', f'\} \circ \{s, f\})$ and $(\{s'', f''\} \circ \{s', f'\}) \circ \{s, f\}$. The sought associativity property is a straightforward consequence. This concludes the construction of $\mathscr{L}_X$.

Next, notice that for every morphism $\varphi : Y \to X$ in $\mathscr{C}$ and every $A, B \in \mathrm{Ob}(\mathscr{A}_X)$ we have a natural isomorphism of presheaves on $\mathscr{C}/Y$:

$$M_{\varphi^*A, \varphi^*B} \xrightarrow{\sim} (\varphi_*)^\wedge M_{A,B}$$

Namely, for every $(\psi : Z \to Y) \in \mathrm{Ob}(\mathscr{C}/Y)$, the isomorphism is given by the map :

$$M_{\varphi^*A, \varphi^*B}(\psi) \xrightarrow{\sim} M_{A,B}(\varphi \circ \psi) \qquad [s, f] \mapsto [\gamma^{\mathsf{c}}_{(\psi,\varphi),A} \circ s, \gamma^{\mathsf{c}}_{(\psi,\varphi),B} \circ f]$$

Since φ_* is continuous and cocontinuous for the sites C/X and C/Y (remark 4.7.3(i)), combining with lemma 4.2.15(ii) there follows a natural isomorphism of sheaves on C/Y :

$$H_{\varphi^*A, \varphi^*B} \xrightarrow{\sim} j_\varphi^* H_{A,B}. \tag{5.3.10}$$

We deduce a map

$$\mathsf{d}_{\varphi,(A,B)} : \mathrm{Hom}_{\mathscr{L}_X}(A, B) \to H_{A,B}(\varphi) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{L}_Y}(\varphi^*A, \varphi^*B)$$

namely the composition of the restriction map $H_{A,B}(\varphi/X) : H_{A,B}(\mathbf{1}_X) \to H_{A,B}(\varphi)$ with the evaluation at $\mathbf{1}_Y \in \mathrm{Ob}(\mathscr{C}/Y)$ of the inverse of the foregoing isomorphism of sheaves. A simple inspection shows that the rules : $A \mapsto \varphi^*A$ and $\sigma \mapsto \mathsf{d}_{\varphi,(A,B)}(\sigma)$ for every $A, B \in \mathscr{A}_X$ and every $\sigma \in \mathrm{Hom}_{\mathscr{L}_X}(A, B)$ yield a well defined functor

$$\mathsf{d}_\varphi : \mathscr{L}_X \to \mathscr{L}_Y.$$

Furthermore, for every pair $Z \xrightarrow{\psi} Y \xrightarrow{\varphi} X$ of morphisms of $\mathscr{C}$, the natural isomorphism $\gamma^{\mathsf{c}}_{\psi,\varphi}$: $\psi^* \circ \varphi^* \xrightarrow{\sim} (\varphi \circ \psi)^*$ induces a natural isomorphism

$$\gamma^{\mathsf{d}}_{\psi,\varphi} : \mathsf{d}_\psi \circ \mathsf{d}_\varphi \xrightarrow{\sim} \mathsf{d}_{\varphi\circ\psi} \qquad A \mapsto \{\mathbf{1}_{\psi^*\varphi^*A}, \gamma^{\mathsf{c}}_{(\psi,\varphi),A}\}.$$

Clearly the rules : $X \mapsto \mathscr{L}_X$ and $\varphi \mapsto \mathsf{d}_\varphi$ for every $X \in \mathrm{Ob}(\mathscr{C})$ and every morphism φ of $\mathscr{C}$ define a unital pseudo-functor $\mathsf{d} : \mathscr{C}^o \to \mathbf{Cat}$, with coherence constraint $\gamma^{\mathsf{d}}_{\bullet,\bullet}$. We set :

$$\mathscr{L} := \mathscr{F}ib(\mathsf{d})$$

and we denote by $\boldsymbol{\lambda}_{\mathscr{L}}$ the natural cleavage of $\mathscr{L}$. Then, for every $X \in \mathrm{Ob}(\mathscr{C})$ and every $A \in \mathrm{Ob}(\mathscr{L}_X)$ let $\beta^{\boldsymbol{\lambda}_{\mathscr{L}}}_{X,A} : \mathscr{C}/X \to \mathscr{L}$ be the cartesian section defined as in the proof of claim 3.2.8; by inspecting the constructions, we see that for every $A, B \in \mathrm{Ob}(\mathscr{L}_X)$, the system of isomorphisms (5.3.10) induces a natural isomorphism of presheaves on $\mathscr{C}/X$:

$$H_{AB} \xrightarrow{\sim} \mathscr{C}art(\beta^{\boldsymbol{\lambda}_{\mathscr{L}}}_{X,A}, \beta^{\boldsymbol{\lambda}_{\mathscr{L}}}_{X,B}).$$

Especially, $\mathscr{C}art(\sigma, \sigma')$ is a sheaf on C/X for every pair of cartesian sections $\sigma, \sigma' \in \mathscr{L}(X)$, and therefore $\mathscr{L}$ is a 1-separated prestack (lemma 5.1.9). We have a natural functor

$$F_X : \mathscr{A}_X \to \mathscr{L}_X \qquad \text{for every } X \in \mathrm{Ob}(\mathscr{C})$$

that is the identity on objects and is given by the rule : $(f : A \to B) \mapsto \{\mathbf{1}_A, f\}$ on morphisms f of $\mathscr{A}_X$. Clearly the rule $X \mapsto F_X$ defines a strict pseudo-natural transformation $F_\bullet : \mathsf{c} \Rightarrow \mathsf{d}$, whence a cartesian functor

$$L := \mathscr{F}ib(F_\bullet) : \mathscr{A} \to \mathscr{L}.$$

By construction, $L(s)$ is an isomorphism in $\mathscr{L}$, for every $s \in \bigcup_{X\in\mathrm{Ob}(\mathscr{C})} \Sigma_X$, hence L factors uniquely through a cartesian functor $L' : \mathscr{A}\{\Sigma_\bullet^{-1}\} \to \mathscr{L}$ and the universal cartesian functor $L_{\{\Sigma\}}$ of proposition 5.3.1. To construct conversely a functor $\mathscr{L} \to \mathscr{A}\{\Sigma_\bullet^{-1}\}$, we begin by evaluating at $\mathbf{1}_X$ the morphism of sheaves (5.3.6), to get a map

$$L''_{X,A,B} : \mathrm{Hom}_{\mathscr{L}_X}(A, B) \to \mathrm{Hom}_{\mathscr{A}\{\Sigma_\bullet^{-1}\}_X}(A, B).$$

Then, a direct inspection of the constructions yields a functor $L''_X : \mathscr{L}_X \to \mathscr{A}\{\Sigma_\bullet^{-1}\}_X$ that is the identity on objects, and is given on morphisms by the rule : $\sigma \mapsto L''_{X,A,B}(\sigma)$ for every $A, B \in \mathrm{Ob}(\mathscr{L}_X)$, and every $\sigma \in \mathrm{Hom}_{\mathscr{L}_X}(A, B)$: the details shall be left to the reader.

Lastly, if $\mathsf{c}_{\{\Sigma\}}$ denotes the pseudo-functor associated with the cleavage $\boldsymbol{\lambda}_{\{\Sigma\}}$, we easily see that the rule : $X \mapsto \mathscr{L}''_X$ for every $X \in \mathrm{Ob}(\mathscr{C})$ defines a strict pseudo-natural $\mathscr{C}$-transformation $L''_\bullet : \mathsf{d} \Rightarrow \mathsf{c}_{\{\Sigma\}}$, whence a cartesian functor

$$L'' := \mathscr{F}ib(L''_\bullet) : \mathscr{L} \to \mathscr{A}\{\Sigma_\bullet^{-1}\}.$$

A direct inspection shows that $L' \circ L'' = \mathbf{1}_{\mathscr{L}}$. In order to show that $L'' \circ L' = \mathbf{1}_{\mathscr{A}\{\Sigma_\bullet^{-1}\}}$, it suffices to check that $L'' \circ L' \circ L_{\{\Sigma\}} = L_{\{\Sigma\}}$, due to the universal property of $L_{\{\Sigma\}}$. Thus, we come down to the proving that $L'' \circ L = L_{\{\Sigma\}}$, which again holds by direct inspection. Summing up, we have proven that L'' is an isomorphism of categories; the proposition follows at once. □

5.4. **Functorial properties of the categories of stacks.** In section 4.2 we have attached to every continuous or cocontinuous functor between sites certain natural functors on the corresponding categories of sheaves. Hereafter we shall likewise associate with such functors certain natural pseudo-functors on the corresponding 2-categories of stacks.

Proposition 5.4.1. *Let $(\mathscr{C}, J)$ and $(\mathscr{C}', J')$ be two sites, $u : \mathscr{C} \to \mathscr{C}'$ a cocontinuous functor, and $i \in \{0, 1, 2\}$. The following holds :*

(i) *For every i-covering cartesian functor $\mathscr{E}_1 \xrightarrow{\varphi} \mathscr{E}_2$ of $\mathscr{C}'$-fibrations $\mathscr{E}_1 \xrightarrow{F_1} \mathscr{C}' \xleftarrow{F_2} \mathscr{E}_2$, the functor* $\mathsf{Fib}(u)^*(\varphi) : \mathscr{C} \times_{\mathscr{C}'} \mathscr{E}_1 \to \mathscr{C} \times_{\mathscr{C}'} \mathscr{E}_2$ *is i-covering (notation of remark* 3.1.5(i)*).*

(ii) *For every i-separated fibration $\mathscr{F} \to \mathscr{C}$, the fibration* $\mathsf{Fib}(u)_*(\mathscr{F})$ *is i-separated.*

Proof. (i): Recall that the objects of $\mathscr{C} \times_{\mathscr{C}'} \mathscr{E}_j$ for $j = 1, 2$ are the pairs (X, Y) with $X \in \mathrm{Ob}(\mathscr{C})$, $Y \in \mathrm{Ob}(\mathscr{E}_j)$, such that $uX = F_j Y$. The morphisms $(X, Y) \to (X', Y')$ are the pairs (f, g) where $f : X \to X'$ is a morphism in $\mathscr{C}$, $g : Y \to Y'$ is a morphism in $\mathscr{E}_j$, and $u(f) = F_j(g)$.

Suppose first that $i = 0$, and let $(X, Y) \in \mathrm{Ob}(\mathscr{C} \times_{\mathscr{C}'} \mathscr{E}_2)$ be any object. By assumption, there exist a covering family $(f'_j : X'_j \to uX \mid j \in I)$ and for every $j \in I$ a cartesian morphism $h_j : \varphi Y_j \to Y$ in $\mathscr{E}_2$ with $F_2 h_j = f'_j$. Let $\mathscr{S} \subset \mathscr{C}'/uX$ be the sieve generated by $(f'_j \mid j \in I)$; since u is cocontinuous, there exists a covering family $(f_k : X_k \to X \mid k \in I')$ such that $u(f_k) \in \mathrm{Ob}(\mathscr{S})$ for every $k \in I'$. Then, for every $k \in I'$ pick $j(k) \in I$ such that $u(f_k) = f'_{j(k)} \circ f''_k$ for some morphism $f''_k : uX_k \to X'_{j(k)}$ in $\mathscr{C}'$, and choose a cartesian morphism $h'_k : Z_k \to Y_{j(k)}$ in $\mathscr{E}_1$ such that $F_1 h'_k = f''_k$. Then $h''_k := h_{j(k)} \circ \varphi(h'_k) : \varphi Z_k \to Y$ is cartesian and $F_2 h''_k = u(f_k)$; therefore $(f_k, h''_k) : (X_k, \varphi Z_k) \to (X, Y)$ is a cartesian morphism in $\mathscr{C} \times_{\mathscr{C}'} \mathscr{E}_2$ for every $j \in I$. This proves that $\mathsf{Fib}(u)^*(\varphi)$ is 0-covering.

Next, suppose that $i = 1$, and let $(\mathbf{1}_X, g) : (X, \varphi Y_1) \to (X, \varphi Y_2)$ be a morphism in $\mathscr{C} \times_{\mathscr{C}'} \mathscr{E}_2$. By assumption there exist a covering family $(f'_j : X'_j \to uX \mid j \in I)$ in $\mathscr{C}'$ and morphisms $Y_2 \xleftarrow{g_j} Z_j \xrightarrow{h_j} Y_1$ in $\mathscr{E}_1$ with h_j cartesian, such that $\varphi(g_j) = g \circ \varphi(h_j)$ and $F_1 h_j = f'_j$ for every $j \in I$. Then, choose a covering family $(f_k : X_k \to X \mid k \in I')$ as in the foregoing, so that for every $k \in I'$ there exist $j(k) \in I$ and a factorization $u(f_k) = f'_{j(k)} \circ f''_k$ for some morphism $f''_k : uX_k \to X'_{j(k)}$ in $\mathscr{C}'$. Choose also for every $k \in I'$ a cartesian morphism $h'_k : Z'_k \to Z_{j(k)}$ with $F_1 h'_k = f''_k$. Then $h''_k := h_{j(k)} \circ h'_k : Z'_k \to Y_1$ is cartesian and $F_1 h''_k = u(f_k)$, so we get a cartesian morphism $(f_k, h''_k) : (X_k, Z'_k) \to (X, Y_1)$ such that $(\mathbf{1}_X, g) \circ (f_k, \varphi(h''_k)) = (f_k, \varphi(g_{j(k)} \circ h'_k))$ for every $k \in I'$. This shows that $\mathsf{Fib}(u)^*(\varphi)$ is 1-covering.

Lastly, say that $i = 2$, and let $(\mathbf{1}_X, g_1), (\mathbf{1}_X, g_2) : (X, Y_1) \to (X, Y_2)$ be two morphisms in $\mathscr{C} \times_{\mathscr{C}'} \mathscr{E}_1$ with $\varphi(g_1) = \varphi(g_2)$. By assumption there exist a covering family $(f'_j : X'_j \to uX \mid j \in I)$ and a cartesian morphism $h_j : Z_j \to Y_1$ in $\mathscr{E}_1$ such that $g_1 \circ h_j = g_2 \circ h_j$ and $F_1 h_j = f'_j$ for every $j \in I$. We pick again a covering family $(f_k : X_k \to X \mid k \in I')$ in $\mathscr{C}$ such that for every $k \in I'$ there exist $j(k) \in I$ and a factorization $u(f_k) = f'_{j(k)} \circ f''_k$, and we choose for every $k \in I'$ a cartesian morphism $h'_k : Z'_k \to Z_{j(k)}$ such that $F_1 h'_k = f''_k$. Then $h''_k := h_{j(k)} \circ h'_k : Z'_k \to Y_1$ is cartesian and $F_1 h''_k = u(f_k)$; therefore $(f_k, h''_k) : (X_k, Z'_k) \to (X, Y_1)$ is cartesian morphism in $\mathscr{C} \times_{\mathscr{C}'} \mathscr{E}_1$ such that $(\mathbf{1}_X, g_1) \circ (f_k, h''_k) = (\mathbf{1}_X, g_2) \circ (f_k, h''_k)$ for every $k \in I'$. This proves that $\mathsf{Fib}(u)^*(\varphi)$ is 2-covering.

(ii): See (3.3.14) for the definition of the pseudo-functor $\mathsf{Fib}(u)_*$. Let $X \in \mathrm{Ob}(\mathscr{C})$ be any object, and $\mathscr{S} \subset \mathscr{C}/X$ a covering sieve; we consider the essentially commutative diagram :

$$\begin{array}{ccccc}
\mathsf{Fib}(u)_*\mathscr{F}(X) & \longrightarrow & \mathsf{Cart}_{\mathscr{C}}(\mathsf{Fib}(u)^*(\mathscr{C}/X), \mathscr{F}) & \longrightarrow & \mathsf{Cart}_{\mathscr{C}}((\mathsf{Fib}(u)^*(\mathscr{C}/X))^a, \mathscr{F}^a) \\
\downarrow & & \downarrow & & \downarrow \\
\mathsf{Cart}_{\mathscr{C}}(\mathscr{S}, \mathsf{Fib}(u)_*\mathscr{F}) & \longrightarrow & \mathsf{Cart}_{\mathscr{C}}(\mathsf{Fib}(u)^*(\mathscr{S}), \mathscr{F}) & \longrightarrow & \mathsf{Cart}_{\mathscr{C}}((\mathsf{Fib}(u)^*(\mathscr{S}))^a, \mathscr{F}^a).
\end{array}$$

We need to show that the left vertical arrow is i-faithful, and we know that the horizontal arrows of the left square subdiagram are equivalences. We are thus reduced to checking that the central vertical arrow is i-faithful. But since $\mathscr{F}$ is i-separated, the unit of adjunction $\mathscr{F} \to \mathscr{F}^a$ is i-faithful (proposition 5.2.11(ii)), so the same holds for the horizontal arrows of the right square subdiagram (corollary 3.1.28(ii.a)), and we are further reduced to checking that the right vertical arrow is an equivalence. To this aim, it suffices to check that the inclusion functor $t : \mathscr{S} \to \mathscr{C}/X$ induces an equivalence $(\mathsf{Fib}(u)^*(\mathscr{S}))^a \xrightarrow{\sim} (\mathsf{Fib}(u)^*(\mathscr{C}/X))^a$ (corollary 3.1.28(ii.b)). Now, recall that $\mathscr{C}/X$ is isomorphic to $\mathscr{F}ib(h_X)$, where h_X is the presheaf on $\mathscr{C}$ represented by X; likewise, $\mathscr{S}$ is isomorphic to $\mathscr{F}ib(h_{\mathscr{S}})$, for a covering subobject $s : h_{\mathscr{S}} \to$

h_X. We deduce natural isomorphisms :

$$(\mathsf{Fib}(u)^*(\mathscr{S}))^a \xrightarrow{\sim} \mathscr{F}ib(u^\wedge h_{\mathscr{S}})^a \xrightarrow{\sim} \mathscr{F}ib((u^\wedge h_{\mathscr{S}})^a)$$

and likewise $(\mathsf{Fib}(u)^*(\mathscr{C}/X))^a \xrightarrow{\sim} \mathscr{F}ib((u^\wedge h_X)^a)$, by theorem 5.1.24 and remark 3.2.9(ii). Clearly, these isomorphisms identify s with $\mathscr{F}ib((u^\wedge s)^a)$; the latter is an isomorphism, according to lemma 4.2.9, whence the contention. □

Definition 5.4.2. Let $C := (\mathscr{C}, J)$ and $C' := (\mathscr{C}', J')$ be two sites, and $u : \mathscr{C} \to \mathscr{C}'$ a functor. We say that $u : C' \to C$ is a *weak morphism of sites* if for every universe V and every V-stack $\mathscr{E}$ on C', the fibration $\mathsf{V}\text{-}\mathsf{Fib}(u)^*\mathscr{E}$ is a V-stack on C. Then, for every such V, the pseudo-functor $\mathsf{V}\text{-}\mathsf{Fib}(u)^*$ induces by restriction a (strict) pseudo-functor

$$\mathsf{V}\text{-}\mathsf{St}(u)_* : \mathsf{V}\text{-}\mathsf{Stack}(C') \to \mathsf{V}\text{-}\mathsf{Stack}(C).$$

Remark 5.4.3. (i) From example 5.1.2, it follows that every weak morphism of sites $C' \to C$ is a continuous functor for the topologies of C and C'.

(ii) We shall see hereafter that every morphism of sites is a weak morphism of sites.

Proposition 5.4.4. *Let $C := (\mathscr{C}, J)$ and $C' := (\mathscr{C}', J')$ be two sites, $u : \mathscr{C} \to \mathscr{C}'$ a functor, and V a universe such that $\mathscr{C}$ is V-small and $\mathscr{C}'$ has V-small Hom-sets. The following conditions are equivalent :*

(a) *For every V-stack $\mathscr{E}$ on C', the fibration $\mathsf{V}\text{-}\mathsf{Fib}(u)^*\mathscr{E}$ is a V-stack on C.*
(b) *For every morphism $\varphi : \mathscr{E} \to \mathscr{F}$ in $\mathsf{V}\text{-}\mathsf{Fib}(\mathscr{C})$ that is i-covering for $i = 0, 1, 2$, the morphism $\mathsf{V}\text{-}\mathsf{Fib}(u)_!\varphi : \mathsf{V}\text{-}\mathsf{Fib}(u)_!\mathscr{E} \to \mathsf{V}\text{-}\mathsf{Fib}(u)_!\mathscr{F}$ is i-covering for $i = 0, 1, 2$.*
(c) *For every $X \in \mathrm{Ob}(\mathscr{C})$ and every covering sieve $\mathscr{S} \subset \mathscr{C}/X$, the induced morphism of $\mathscr{C}'$-fibrations $\mathsf{V}\text{-}\mathsf{Fib}(u)_!\mathscr{S} \to \mathsf{V}\text{-}\mathsf{Fib}(u)_!(\mathscr{C}/X)$ is i-covering for $i = 0, 1, 2$.*
(d) *$u : C' \to C$ is a weak morphism of sites.*

Proof. (a)⇒(b): In view of remark 3.3.15(v), we may replace V by a larger universe, and assume that both $\mathscr{C}$ and $\mathscr{C}'$ are V-small. Then, according to propositions 5.2.9 and 5.2.11(iii), the functor $\varphi^a : \mathscr{E}^a \to \mathscr{F}^a$ is an equivalence, and we need to check that $(\mathsf{V}\text{-}\mathsf{Fib}(u)_!\varphi)^a$ is an equivalence. To this aim, for every V-stack $\mathscr{A}$ on C', we consider the diagram :

$$\begin{array}{ccccc}
\mathsf{Cart}_{\mathscr{C}'}(\mathsf{Fib}(u)_!(\mathscr{F})^a, \mathscr{A}) & \longrightarrow & \mathsf{Cart}_{\mathscr{C}'}(\mathsf{Fib}(u)_!(\mathscr{F}), \mathscr{A}) & \longrightarrow & \mathsf{Cart}_{\mathscr{C}}(\mathscr{F}, \mathsf{Fib}(u)^*\mathscr{A}) \\
\downarrow{\scriptstyle \mathsf{Cart}_{\mathscr{C}'}(\mathsf{Fib}(u)_!(\varphi)^a, \mathscr{A})} & & \downarrow{\scriptstyle \mathsf{Cart}_{\mathscr{C}'}(\mathsf{Fib}(u)_!(\varphi), \mathscr{A})} & & \downarrow{\scriptstyle \mathsf{Cart}_{\mathscr{C}}(\varphi, \mathsf{Fib}(u)^*\mathscr{A})} \\
\mathsf{Cart}_{\mathscr{C}'}(\mathsf{Fib}(u)_!(\mathscr{E})^a, \mathscr{A}) & \longrightarrow & \mathsf{Cart}_{\mathscr{C}'}(\mathsf{Fib}(u)_!(\mathscr{E}), \mathscr{A}) & \longrightarrow & \mathsf{Cart}_{\mathscr{C}}(\mathscr{E}, \mathsf{Fib}(u)^*\mathscr{A})
\end{array}$$

whose left square subdiagram is induced by the natural cartesian functors

$$\mathsf{Fib}(u)_!(\mathscr{F}) \to \mathsf{Fib}(u)_!(\mathscr{F})^a \qquad \text{and} \qquad \mathsf{Fib}(u)_!(\mathscr{E}) \to \mathsf{Fib}(u)_!(\mathscr{E})^a$$

and whose right square subdiagram is given by the coherence constraint of the 2-adjunction for the pair $(\mathsf{V}\text{-}\mathsf{Fib}(u)_!, \mathsf{V}\text{-}\mathsf{Fib}(u)^*)$. Thus, both subdiagrams are essentially commutative, and the same then holds for their composition. Moreover, all horizontal arrows are equivalences, so the left-most vertical arrow is an equivalence if and only if the same holds for the right-most one, and in light of lemma 2.1.13(iii) we are reduced to checking that $\mathsf{Cart}_{\mathscr{C}'}(\varphi, \mathsf{Fib}(u)^*\mathscr{A})$ is an equivalence for every such $\mathscr{A}$. But by assumption, $\mathsf{V}\text{-}\mathsf{Fib}(u)^*\mathscr{A}$ is a stack on C, so we have as well the pseudo-commutative diagram

$$\begin{array}{ccc}
\mathsf{Cart}_{\mathscr{C}}(\mathscr{F}^a, \mathsf{Fib}(u)^*\mathscr{A}) & \longrightarrow & \mathsf{Cart}_{\mathscr{C}}(\mathscr{F}, \mathsf{Fib}(u)^*\mathscr{A}) \\
{\scriptstyle \mathsf{Cart}_{\mathscr{C}}(\varphi^a, \mathsf{Fib}(u)^*\mathscr{A})}\downarrow & & \downarrow{\scriptstyle \mathsf{Cart}_{\mathscr{C}}(\varphi, \mathsf{Fib}(u)^*\mathscr{A})} \\
\mathsf{Cart}_{\mathscr{C}}(\mathscr{E}^a, \mathsf{Fib}(u)^*\mathscr{A}) & \longrightarrow & \mathsf{Cart}_{\mathscr{C}}(\mathscr{E}, \mathsf{Fib}(u)^*\mathscr{A})
\end{array}$$

whose horizontal arrows are equivalences. Hence it suffices to show that $\mathsf{Cart}_{\mathscr{C}}(\varphi^a, \mathsf{Fib}(u)^*\mathscr{A})$ is an equivalence for every V-stack $\mathscr{A}$ on C', which follows by invoking again lemma 2.1.13(iii).

(b)⇒(c): This is clear from propositions 5.2.9 and 5.2.11(iii), after inspecting the pseudo-commutative diagram 5.1.25.

(c)⇒(d): Let V′ be any universe; we need to check that $\mathsf{V}'\text{-}\mathsf{Fib}(u)^*\mathscr{E}$ is a V′-stack on C, for every V′-stack $\mathscr{E}$ on C'. To this aim, we can replace V′ by a larger universe, and assume that $\mathsf{V} \subset \mathsf{V}'$, and both $\mathscr{C}$ and $\mathscr{C}'$ are V′-small. Moreover, in light of remark 3.3.15(v) and our assumption (c), we see that for every $X \in \mathrm{Ob}(\mathscr{C})$ and every covering sieve $\mathscr{S} \subset \mathscr{C}/X$, the induced morphism of $\mathscr{C}'$-fibrations $\mathsf{V}'\text{-}\mathsf{Fib}(u)_!\mathscr{S} \to \mathsf{V}'\text{-}\mathsf{Fib}(u)_!(\mathscr{C}/X)$ is i-covering for $i = 0, 1, 2$. Now, the coherence constraint for the 2-adjoint pair $(\mathsf{V}'\text{-}\mathsf{Fib}(u)_!, \mathsf{V}'\text{-}\mathsf{Fib}(u)^*)$ yields an essentially commutative diagram :

$$\begin{array}{ccc}
\mathscr{C}art_{\mathscr{C}}(\mathscr{C}/X, \mathsf{V}'\text{-}\mathsf{Fib}(u)^*\mathscr{E}) & \longrightarrow & \mathscr{C}art_{\mathscr{C}}(\mathscr{S}, \mathsf{V}'\text{-}\mathsf{Fib}(u)^*\mathscr{E}) \\
\downarrow & & \downarrow \\
\mathscr{C}art_{\mathscr{C}'}(\mathsf{V}'\text{-}\mathsf{Fib}(u)_!(\mathscr{C}/X), \mathscr{E}) & \longrightarrow & \mathscr{C}art_{\mathscr{C}'}(\mathsf{V}'\text{-}\mathsf{Fib}(u)_!\mathscr{S}, \mathscr{E})
\end{array}$$

whose vertical arrow are equivalences. Thus, we are reduced to checking that the bottom horizontal arrow is an equivalence. But since $\mathscr{E}$ is a stack, we have as well the following pseudo-commutative commutative diagram, whose vertical arrows are again equivalences :

$$\begin{array}{ccc}
\mathscr{C}art_{\mathscr{C}'}(\mathsf{V}'\text{-}\mathsf{Fib}(u)_!(\mathscr{C}/X)^a, \mathscr{E}) & \longrightarrow & \mathscr{C}art_{\mathscr{C}'}(\mathsf{V}'\text{-}\mathsf{Fib}(u)_!(\mathscr{S})^a, \mathscr{E}) \\
\downarrow & & \downarrow \\
\mathscr{C}art_{\mathscr{C}'}(\mathsf{V}'\text{-}\mathsf{Fib}(u)_!(\mathscr{C}/X), \mathscr{E}) & \longrightarrow & \mathscr{C}art_{\mathscr{C}'}(\mathsf{V}'\text{-}\mathsf{Fib}(u)_!\mathscr{S}, \mathscr{E})
\end{array}$$

Hence, it suffices to check that the top horizontal arrow of this latter diagram is an equivalence. But the functor $\mathsf{V}'\text{-}\mathsf{Fib}(u)_!(\mathscr{S})^a \to \mathsf{V}'\text{-}\mathsf{Fib}(u)_!(\mathscr{C}/X)^a$ is an equivalence, by propositions 5.2.9 and 5.2.11(iii); the contention is an immediate consequence.

(d)⇒(a) is trivial. □

5.4.5. Let $C := (\mathscr{C}, J)$ and $C' := (\mathscr{C}', J')$ be two sites; we wish to give some useful criteria for a functor $u : \mathscr{C} \to \mathscr{C}'$ to be a weak morphism of sites $C' \to C$. To this aim, we consider the following conditions :

- (C0) For every $X \in \mathrm{Ob}(\mathscr{C})$ and every covering family $(X_i \to X \mid i \in I)$ for the topology J, the family $(uX_i \to uX \mid i \in I)$ covers uX for the topology J'.
- (C1) u is continuous for the topologies J and J'.
- (C2) Condition (C0) holds, and for every $X \in \mathrm{Ob}(\mathscr{C})$, every covering family $(X_i \to X \mid i \in I)$ for the topology J admits a refinement $(X'_i \to X \mid i \in I')$ that still covers X, such that the fibre products $X'_i \times_X X'_j$ and $X'_i \times_X X'_j \times_X X'_k$ are representable in $\mathscr{C}$ for every $i, j, k \in I$ and u commutes with these fibre products.
- (C3) Condition (C0) holds, and there exists a universe V containing U such that $\mathscr{C}$ and $\mathscr{C}'$ are V-small and the functor $u^a_{\mathsf{V}!} : \mathscr{C}^\wedge_{\mathsf{V}} \to C'^{\sim}_{\mathsf{V}}$ commutes with fibre products.
- (C4) $u : C' \to C$ is a morphism of sites.

Remark 5.4.6. (i) Recall that (C1) means that $u^\wedge$ transforms sheaves on C' into sheaves on C. Likewise, according to lemma 4.2.4, condition (C0) holds if and only if $u^\wedge$ transforms separated presheaves on C' into separated presheaves on C.

(ii) We have (C4)⇒(C3)⇒(C1)⇒(C0), by lemma 4.2.4. By inspecting the proof of lemma 4.2.4 we also easily see that (C2)⇒(C1).

(iii) In the situation of (5.4.5), suppose that C is a lex-site and u is left exact. Then example 4.3.12 says that u fulfills condition (C4) if and only if it fullfils condition (C0).

5.4.7. In the situation of (5.4.5), let $\mathscr{A}' \to \mathscr{C}'$ be a fibration, and $X \in \mathrm{Ob}(\mathscr{C})$. Set $\mathscr{A} := \mathsf{Fib}(u)^*\mathscr{A}' \to \mathscr{C}$ and let $\pi : \mathscr{A} \to \mathscr{A}'$ be the natural projection. For every cartesian section $\sigma \in \mathscr{A}'(uX)$ there exists a unique cartesian section

$$u^*_{|X}(\sigma) \in \mathscr{A}(X) \qquad \text{such that} \qquad \pi \circ u^*_{|X}(\sigma) = \sigma \circ u_{|X}.$$

Also, for every pair of cartesian sections $\sigma, \tau \in \mathscr{A}'(uX)$ and every natural $\mathscr{C}$-transformation $\beta : \sigma \Rightarrow \tau$ there exists a unique natural $\mathscr{C}$-transformation

$$u^*_{|X}(\beta) : u^*_{|X}(\sigma) \Rightarrow u^*_{|X}(\tau) \qquad \text{such that} \qquad \pi * u^*_{|X}(\beta) = \beta * u_{|X}.$$

This characterization easily implies that the rules : $\sigma \mapsto u^*_{|X}$ and $\beta \mapsto u^*_{|X}(\beta)$ for every such σ and β, define a functor $u^*_{|X} : \mathscr{A}'(uX) \to \mathscr{A}(X)$ fitting into the commutative diagram :

$$\begin{array}{ccc} \mathscr{A}'(uX) & \xrightarrow{u^*_{|X}} & \mathscr{A}(X) \\ {\scriptstyle \mathsf{ev}^{\mathscr{A}'}_{uX}} \downarrow & & \downarrow {\scriptstyle \mathsf{ev}^{\mathscr{A}}_{X}} \\ \mathscr{A}'_{uX} & \longrightarrow & \mathscr{A}_X \end{array}$$

whose bottom horizontal arrow is the isomorphism of categories induced by π, and whose vertical arrows are the evaluation functors of (3.2.5). Since the latter are equivalences, it follows that the same holds for $u^*_{|X}$. Notice that for every $(f : Y \to X) \in \mathrm{Ob}(\mathscr{C})$ and every cartesian section $\sigma \in \mathscr{A}'(uX)$ we have :

$$u^*_{|Y}(\sigma \circ (uf)_*) = u^*_{|X}(\sigma) \circ f_*$$

(detail left to the reader). Hence, for every pair of cartesian sections $\sigma, \tau \in \mathscr{A}'(uX)$ and every $(f : Y \to X) \in \mathrm{Ob}(\mathscr{C}/X)$ we deduce a bijection

$$\mathsf{Cart}(\sigma, \tau)(uf) \xrightarrow{\sim} \mathsf{Cart}(u^*_{|X}\sigma, u^*_{|X}\tau)(f) \qquad \beta \mapsto u^*_{|Y}\beta.$$

Finally, it is easily seen that this system of bijections amounts to an isomorphism of presheaves

$$u^{\wedge}_{|X}\mathscr{C}art(\sigma, \tau) \xrightarrow{\sim} \mathscr{C}art(u^*_{|X}\sigma, u^*_{|X}\tau).$$

Proposition 5.4.8. *In the situation of* (5.4.5)*, let $F : \mathscr{E} \to \mathscr{C}'$ be any fibration. We have :*

(i) *If $\mathscr{E}$ is* 0*-separated, and* (C0) *holds, then* $\mathsf{Fib}(u)^*(\mathscr{E})$ *is* 0*-separated.*
(ii) *If $\mathscr{E}$ is* 1*-separated, and* (C1) *holds, then* $\mathsf{Fib}(u)^*(\mathscr{E})$ *is* 1*-separated.*
(iii) *If* (C2) *holds, then u is a weak morphism of sites $C' \to C$.*

Proof. To ease notation, set $\mathscr{F} := \mathsf{Fib}(u)^*(\mathscr{E})$, and let $\pi : \mathscr{F} \to \mathscr{E}$ be the natural projection.

(i): Let $X \in \mathrm{Ob}(\mathscr{C})$ be any object, and $\sigma, \tau \in \mathscr{F}(X)$ two cartesian sections; according to lemma 5.1.9 it suffices to show that the presheaf $\mathscr{C}art(\sigma, \tau)$ is separated on the site C/X. However, the discussion of (5.4.7) yields cartesian sections $\sigma', \tau' \in \mathscr{E}(uX)$ with isomorphisms $u^*_{|X}\sigma' \xrightarrow{\sim} \sigma$ and $u^*_{|X}\tau' \xrightarrow{\sim} \tau$, whence isomorphisms of presheaves

$$\mathscr{C}art(\sigma, \tau) \xrightarrow{\sim} \mathscr{C}art(u^*_{|X}\sigma', u^*_{|X}\tau') \xrightarrow{\sim} u^{\wedge}_{|X}\mathscr{C}art(\sigma', \tau').$$

Since $\mathscr{E}$ is 0-separated, lemma 5.1.9 says that $\mathscr{C}art(\sigma', \tau')$ is a separated presheaf on the site C'/uX. On the other hand, the explicit description of the covering sieves of the sites C/X and C'/uX furnished by (4.7) easily implies that the functor $u_{|X}$ also fulfills condition (C0), relative to these sites (details left to the reader). Then the assertion follows from remark 5.4.6(i).

(ii) is similar : arguing as in the foregoing, we are reduced to checking that $u^{\wedge}_{|X}(\mathscr{C}art(\sigma', \tau'))$ is a sheaf on C/X. However, $\mathscr{C}art(\sigma', \tau')$ is a sheaf on C'/uX, by lemma 5.1.9, and $u_{|X}$ is continuous on the sites C/X and C'/uX, by proposition 4.7.7, whence the assertion.

(iii): Suppose that (C2) holds, and that $\mathscr{E}$ is a stack on C'; let $X \in \mathrm{Ob}(\mathscr{C})$ be any object, $\mathscr{S} \subset \mathscr{C}/X$ a sieve covering X, and $\mathscr{S}' \subset \mathscr{S}$ a refinement covering X, generated by a family $f_\bullet := (f_i : X'_i \to X \mid i \in I)$ with the properties of condition (C2). We get restriction functors :

$$\mathscr{F}(X) \xrightarrow{\rho} \mathsf{Cart}_{\mathscr{C}}(\mathscr{S}, \mathscr{F}) \xrightarrow{\rho'} \mathsf{Cart}_{\mathscr{C}}(\mathscr{S}', \mathscr{F})$$

and we need to prove that ρ is an equivalence. We check first that ρ is fully faithful. To this aim, notice that $\mathscr{F}$ is 0-separated by virtue of (i), hence ρ' is faithful (lemma 5.1.20), and we are then easily reduced to showing that $\rho' \circ \rho$ is fully faithful. Now, choose a cleavage c for $\mathscr{E}$, so that $\mathsf{c} \circ u^o$ is a cleavage for $\mathscr{F}$. Then $\mathsf{Cart}_{\mathscr{C}}(\mathscr{S}', \mathscr{F})$ is equivalent to the category of descent data $\mathsf{Desc}(\mathscr{F}, f_\bullet, \mathsf{c} \circ u^o)$ as in (3.5.22). However, π induces an isomorphism of categories $\mathscr{F}_X \xrightarrow{\sim} \mathscr{E}_{uX}$, and since u commutes with the fibre products $X_i \times_X X_j$ and $X_i \times_X X_j \times_X X_k$, a simple inspection shows that the natural functor

$$\mathscr{F}_X \to \mathsf{Desc}(\mathscr{F}, f_\bullet, \mathsf{c} \circ u^o)$$

is naturally identified with the corresponding functor

$$\mathscr{E}_{uX} \to \mathsf{Desc}(\mathscr{E}, u(f_\bullet), \mathsf{c}).$$

The latter is an equivalence, since $\mathscr{E}$ is a stack. Thus $\rho' \circ \rho$ is even an equivalence, and therefore ρ is fully faithful, as stated. Since X and $\mathscr{S}$ are arbitrary, we have thus proved that $\mathscr{F}$ is 1-separated; but then ρ' is fully faithful, again by lemma 5.1.20, and it follows easily that ρ is an equivalence if and only if the same holds for $\rho' \circ \rho$. For the latter, the assertion has just been shown, so finally $\mathscr{F}$ is a stack, as required. □

Lemma 5.4.9. *In the situation of* (5.4.5), *let* $\varphi : \mathscr{E}_1 \to \mathscr{E}_2$ *be a* $\mathscr{C}'$*-cartesian functor between two fibrations over* $\mathscr{C}'$*. Suppose that* (C0) *holds, and moreover that for every* $X \in \mathrm{Ob}(\mathscr{C}')$ *there exists a covering family* $(uY_j \to X \mid j \in I)$ *for the topology* J'*. Let also* $i \in \{0, 1, 2\}$ *such that* $\mathsf{Fib}(u)^*(\varphi)$ *is* i*-covering for the topology* J*. Then* φ *is* i*-covering for the topology* J'*.*

Proof. In view of lemma 5.2.3, we may replace φ by $\mathsf{C}(\varphi)$, and reduce to the case where $\mathscr{E}_1$ and $\mathscr{E}_2$ are split $\mathscr{C}'$-fibrations, and φ is a split cartesian functor. For $t = 1, 2$ let $\mathsf{c}_t : \mathscr{C}' \to \mathbf{Cat}$ be the strict pseudo-functor associated with the split cleavage of $\mathscr{E}_t$. Suppose first that $i = 0$, and let $X \in \mathrm{Ob}(\mathscr{C}')$ and $E \in \mathrm{Ob}(\mathscr{E}_{2,X})$. By assumption, we may find a covering family $(f_j : uY_j \to X \mid j \in I)$ and we set $E_j := \mathsf{c}_{2,f_j} E \in \mathrm{Ob}(\mathscr{E}_{2,uY_j})$ for every $j \in I$. Since $\mathsf{Fib}(u)^*(\varphi)$ is 0-covering, we may find for every $j \in I$ a covering family $(f_{jj'} : Y_{jj'} \to Y_j \mid j' \in \Lambda_j)$ and for every $j' \in \Lambda_j$ an object $E'_{jj'} \in \mathrm{Ob}(\mathscr{E}_{1,uY_{jj'}})$ with an isomorphism $\varphi E'_{jj'} \xrightarrow{\sim} \mathsf{c}_{2,u(f_{jj'})} E_j$. Set $h_{jj'} := f_j \circ u(f_{jj'}) : uY_{jj'} \to X$ for every $j \in I$ and $j' \in \Lambda_j$; since (C0) holds for u, the family $(h_{jj'} \mid j \in I,\ j' \in \Lambda_j)$ covers X, and $\varphi E'_{jj'} \xrightarrow{\sim} \mathsf{c}_{2,h_{jj'}} E$ for every $j \in I$ and $j' \in \Lambda_j$. This shows that φ is 0-covering.

Next, suppose that $i = 1$, and let $X \in \mathrm{Ob}(\mathscr{C}')$ and $g : \varphi E_1 \to \varphi E_2$ a morphism in $\mathscr{E}_{2,X}$. We pick a covering family $(f_j : uY_j \to X \mid j \in I)$, and set $E_{tj} := \mathsf{c}_{1,f_j} E_t$ for $t = 1, 2$ and every $j \in I$; also, let $g_j := \mathsf{c}_{2,f_j}(g) : \varphi E_{1j} \to \varphi E_{2j}$ for every $j \in J$. Since $\mathsf{Fib}(u)^*(\varphi)$ is 1-covering, for every $j \in I$ there exist a covering family $(f_{jj'} : Y_{jj'} \to Y_j \mid j' \in \Lambda_j)$ and for every $j' \in \Lambda_j$ a morphism $g_{jj'} : \mathsf{c}_{1,u(f_{jj'})} E_{1j} \to \mathsf{c}_{1,u(f_{jj'})} E_{2j}$ with $\varphi(g_{jj'}) = \mathsf{c}_{2,u(f_{jj'})}(g_j) = \mathsf{c}_{2,u(f_{jj'}) \circ f_j}(g)$. Arguing as in the foregoing, we easily deduce that φ is 1-covering (details left to the reader).

Lastly, suppose that $i = 2$, and let $X \in \mathrm{Ob}(\mathscr{C}')$ and $g_1, g_2 : E_1 \to E_2$ two morphisms in $\mathscr{E}_{1,X}$ such that $\varphi(g_1) = \varphi(g_2)$. Pick again a covering family $(f_j : uY_j \to X \mid j \in I)$ and set $g_{tj} := \mathsf{c}_{1,f_j}(g_t)$ for $t = 1, 2$ and every $j \in I$. Then $\varphi(g_{1j}) = \varphi(g_{2j})$ for every $j \in I$, and since $\mathsf{Fib}(u)^*(\varphi)$ is 2-covering, there exists for every such j a covering family $(f_{jj'} : Y_{jj'} \to Y_j \mid j' \in \Lambda_j)$ such that $\mathsf{c}_{1,u(f_{jj'})}(g_{1j}) = \mathsf{c}_{1,u(f_{jj'})}(g_{2j})$ for every $j' \in \Lambda_j$. Arguing as in the foregoing, it follows easily that φ is 2-covering. □

5.4.10. Let $C := (\mathscr{C}, J)$ be a small site and define the site $C^{\wedge}_{\mathsf{U}} := (\mathscr{C}^{\wedge}_{\mathsf{U}}, J^{\wedge})$ as in remark 4.1.31(ii). By theorem 4.4.2(i), the Yoneda embedding is a cocontinuous morphism of sites $h_{\mathscr{C}} : C^{\wedge}_{\mathsf{U}} \to C$. Thus, for every universe V containing U and every i-separated V-prestack $\varphi : \mathscr{E} \to \mathscr{C}$ on $\mathscr{C}$, the fibration

$$\mathsf{V}\text{-}\mathsf{Fib}(h_{\mathscr{C}})_*(\mathscr{E}) \to \mathscr{C}^{\wedge}_{\mathsf{U}}$$

is an i-separated prestack for the topology $J^{\wedge}$ (proposition 5.4.1(ii)). Moreover, since $h_{\mathscr{C}}$ is fully faithful, remark 3.3.15(iv) and corollary 2.4.29 imply that the counit of 2-adjunction

$$\mathsf{V}\text{-}\mathsf{Fib}(h_{\mathscr{C}})^* \circ \mathsf{V}\text{-}\mathsf{Fib}(h_{\mathscr{C}})_*(\mathscr{E}) \to \mathscr{E}$$

is an equivalence of categories. Thus, $\mathsf{V}\text{-}\mathsf{Fib}(h_{\mathscr{C}})_*(\mathscr{E})$ is a natural extension of $\mathscr{E}$ to the site $C^{\wedge}$ which contains C as a full subcategory with the topology induced from $J^{\wedge}$.

Remark 5.4.11. (i) In the situation of (5.4.10), we can describe more explicitly the fibration $\mathsf{V}\text{-}\mathsf{Fib}(h_{\mathscr{C}})_*(\mathscr{E})$, as follows. We consider the functor

$$\mathsf{c}^{\triangledown}_{\varphi} : (\mathscr{C}^{\wedge}_{\mathsf{U}})^o \to \mathsf{V}\text{-}\mathbf{Cat} \qquad F \mapsto \mathsf{Cart}_{\mathscr{C}}(\mathscr{F}ib(F), \mathscr{E})$$

which assigns to every morphism of presheaves $\beta : F \to F'$, the functor $\mathsf{Cart}_{\mathscr{C}}(\mathscr{F}ib(\beta), \mathscr{E}) : \mathsf{c}^{\triangledown}_{\varphi}(F') \to \mathsf{c}^{\triangledown}_{\varphi}(F)$. Then, combining remark 3.3.15(ii) and example 3.1.16(iii) we get a natural equivalence of $\mathscr{C}^{\wedge}_{\mathsf{U}}$-categories :

$$\mathsf{V}\text{-}\mathsf{Fib}(h_{\mathscr{C}})_*(\mathscr{E}) \xrightarrow{\sim} \mathscr{F}ib(\mathsf{c}^{\triangledown}_{\varphi}).$$

(ii) Let $i : C^{\sim}_{\mathsf{U}} \to \mathscr{C}^{\wedge}_{\mathsf{U}}$ be the inclusion functor, and $(-)^a : \mathscr{C}^{\wedge}_{\mathsf{U}} \to C^{\sim}_{\mathsf{U}}$ its left adjoint. We have a pseudo-natural equivalence of pseudo-functors :

$$\mathsf{V}\text{-}\mathsf{Fib}((-)^a)_* \xrightarrow{\sim} \mathsf{V}\text{-}\mathsf{Fib}(i)^*.$$

Indeed, let $\mathscr{E}$ be any fibration on $\mathscr{C}^{\wedge}_{\mathsf{U}}$; in view of remark 3.3.15(ii), the fibration $\mathsf{V}\text{-}\mathsf{Fib}((-)^a)_*(\mathscr{E})$ is naturally equivalent to the fibration associated with the strict pseudo-functor

$$F \mapsto \mathsf{Cart}_{\mathscr{C}^{\wedge}_{\mathsf{U}}}((-)^a\mathscr{C}^{\wedge}_{\mathsf{U}}/F, \mathscr{E}) \qquad \text{for every } F \in \mathrm{Ob}(C^{\sim}_{\mathsf{U}}).$$

On the other hand, $\mathsf{C}(\mathsf{Fib}(i)^*(\mathscr{E}))$ is the fibration associated with the strict pseudo-functor

$$F \mapsto \mathsf{Cart}_{\mathscr{C}^{\wedge}_{\mathsf{U}}}(\mathscr{C}^{\wedge}_{\mathsf{U}}/iF, \mathscr{E}) \qquad \text{for every } F \in \mathrm{Ob}(C^{\sim}_{\mathsf{U}}).$$

It suffices then to notice that $(-)^a\mathscr{C}^{\wedge}_{\mathsf{U}}/F = \mathscr{C}^{\wedge}_{\mathsf{U}}/iF$ for every sheaf F on the site C.

Theorem 5.4.12. *Let $C := (\mathscr{C}, J)$ be a small site, $h_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}^{\wedge}_{\mathsf{U}}$ the Yoneda embedding and $i : C^{\sim}_{\mathsf{U}} \to \mathscr{C}^{\wedge}_{\mathsf{U}}$ the inclusion functor. For every universe V containing U we have :*

(i) *The strict pseudo-functor $\mathsf{V}\text{-}\mathsf{Fib}(h_{\mathscr{C}})^*$ restricts to a 2-equivalence ;*

$$\mathsf{V}\text{-}\mathsf{Stack}(C^{\wedge}_{\mathsf{U}}) \xrightarrow{\sim} \mathsf{V}\text{-}\mathsf{Stack}(C).$$

(ii) *Likewise, i and its left adjoint $(-)^a : \mathscr{C}^{\wedge}_{\mathsf{U}} \to C^{\sim}_{\mathsf{U}}$ induce 2-equivalences :*

$$\mathsf{V}\text{-}\mathsf{Stack}(C^{\wedge}_{\mathsf{U}}) \underset{\mathsf{V}\text{-}\mathsf{Fib}((-)^a)^*}{\overset{\mathsf{V}\text{-}\mathsf{Fib}(i)^*}{\rightleftarrows}} \mathsf{V}\text{-}\mathsf{Stack}(\mathrm{Can}(C^{\sim}_{\mathsf{U}})).$$

Proof. (i): Let η and ε be the unit and counit of a 2-adjunction for the pair of pseudo-functors $(\mathsf{V}\text{-}\mathsf{Fib}(h_{\mathscr{C}})^*, \mathsf{V}\text{-}\mathsf{Fib}(h_{\mathscr{C}})_*)$. Since $h_{\mathscr{C}}$ is a cocontinuous morphism of sites $C^{\wedge} \to C$ (theorem 4.4.2(i)), we know already that $\mathsf{V}\text{-}\mathsf{Fib}(h_{\mathscr{C}})_*$ sends V-stacks on C to V-stacks on $C^{\wedge}$ (proposition 5.4.1(ii)); according to corollary 2.4.32(i), it then suffices to show that for every V-stack $\mathscr{E}$ on $C^{\wedge}$ and $\mathscr{E}'$ on C, the fibration $\mathsf{V}\text{-}\mathsf{Fib}(h_{\mathscr{C}})^*(\mathscr{E})$ is a V-stack on C, and $\eta_{\mathscr{E}}$ and $\varepsilon_{\mathscr{E}'}$ are equivalences. The assertion for $\varepsilon_{\mathscr{E}'}$ has already been noticed in (5.4.10). Next, to ease notation, let us drop the prefix V, and set $\mathscr{F} := \mathsf{Fib}(h_{\mathscr{C}})^*(\mathscr{E})$. Let also $i_{\mathscr{F}} : \mathscr{F} \to \mathscr{F}^a$ be the natural morphism of fibrations on $\mathscr{C}$, and denote by

$$\beta_{\mathscr{E}} : \mathscr{E} \to \mathsf{Fib}(h_{\mathscr{C}})_*(\mathscr{F}^a)$$

the composition of $\eta_{\mathscr{E}}$ and $\mathsf{Fib}(h_{\mathscr{C}})_*(i_{\mathscr{F}})$. Then $\varepsilon_{\mathscr{F}^a} \circ (\mathsf{Fib}(h_{\mathscr{C}})^* \beta_{\mathscr{E}})$ is isomorphic to

$$\varepsilon_{\mathscr{F}^a} \circ (\mathsf{Fib}(h_{\mathscr{C}})^* \circ \mathsf{Fib}(h_{\mathscr{C}})_*(i_{\mathscr{F}})) \circ (\mathsf{Fib}(h_{\mathscr{C}})^* \eta_{\mathscr{E}})$$

which is in turn isomorphic to $i_{\mathscr{F}} \circ \varepsilon_{\mathscr{F}} \circ (\mathsf{Fib}(h_{\mathscr{C}})^* \eta_{\mathscr{E}})$, and the latter is isomorphic to $i_{\mathscr{F}}$, by virtue of the triangular modifications associated with the pair (η, ε) (theorem 2.4.24(i)). Since $\varepsilon_{\mathscr{F}^a}$ is an equivalence, and $i_{\mathscr{F}}$ is j-covering for $j = 0, 1, 2$, it follows that $\mathsf{Fib}(h_{\mathscr{C}})^*(\beta_{\mathscr{E}})$ is j-covering for $j = 0, 1, 2$. Since the functor $h_{\mathscr{C}}$ verifies condition (C0) of (5.4.5), lemma 5.4.9 implies that $\beta_{\mathscr{E}}$ is also j-covering for $j = 0, 1, 2$. But $\mathsf{Fib}(h_{\mathscr{C}})_*(\mathscr{E})$ is a stack (proposition 5.4.1(ii)), hence $\beta_{\mathscr{E}}$ is an equivalence (proposition 5.2.9). Therefore, $\mathsf{Fib}(h_{\mathscr{C}})^*(\beta)$ is an equivalence as well, and by the foregoing, the same then holds for $i_{\mathscr{F}}$. The latter means that $\mathsf{Fib}(h_{\mathscr{C}})^*(\mathscr{E})$ is a stack, and it also follows that $\eta_{\mathscr{E}}$ is an equivalence, as sought.

(ii): By theorem 4.4.2(iii), the functor $(-)^a$ is cocontinuous for the topologies $J^\wedge$ and $\mathrm{Can}_{C^\sim}$; taking into account remark 5.4.11(ii) and proposition 5.4.1(ii), it follows that i is a weak morphism of sites $C^\wedge_{\mathsf{U}} \to \mathrm{Can}(C^\sim_{\mathsf{U}})$. Also, $(-)^a$ is a weak morphism of sites $\mathrm{Can}(C^\sim_{\mathsf{U}}) \to C^\wedge_{\mathsf{U}}$, due to proposition 5.4.8(iii). Moreover, since $(-)^a \circ i$ is isomorphic to $\mathbf{1}_{C^\sim}$, we have a pseudo-natural isomorphism of pseudo-functors:

$$\mathsf{V}\text{-}\mathsf{Fib}(i)^* \circ \mathsf{V}\text{-}\mathsf{Fib}((-)^a)^* \xrightarrow{\sim} \mathbf{1}_{\mathsf{V}\text{-}\mathsf{Stack}(\mathrm{Can}(C^\sim))}.$$

Next, let $\mathscr{E}$ be any stack on $C^\wedge$ with V-small fibres; recall that $\mathscr{E}$ is naturally equivalent to the split fibration associated with the strict pseudo-functor $\mathsf{Cart}_{\mathscr{C}^\wedge}(\mathscr{C}^\wedge/-, \mathscr{E})$. Likewise, $\mathsf{V}\text{-}\mathsf{Fib}((-)^a)^* \circ \mathsf{V}\text{-}\mathsf{Fib}(i)^*(\mathscr{E})$ is naturally equivalent to the split fibration associated with the strict pseudo-functor

$$\mathscr{C}^\wedge \to \mathsf{V}\text{-}\mathbf{Cat} \qquad F \mapsto \mathsf{Cart}_{\mathscr{C}^\wedge}(\mathscr{C}^\wedge/F^a, \mathscr{E}).$$

By virtue of corollary 2.4.32(i), we are thus reduced to checking that for every $F \in \mathrm{Ob}(\mathscr{C}^\wedge)$, the natural morphism $j_F : F \to F^a$ induces an equivalence of categories $\mathsf{Cart}_{\mathscr{C}^\wedge}(\mathscr{C}^\wedge/F^a, \mathscr{E}) \xrightarrow{\sim} \mathsf{Cart}_{\mathscr{C}^\wedge}(\mathscr{C}^\wedge/F, \mathscr{E})$. To this aim, we notice :

Claim 5.4.13. For every stack $\mathscr{E}$ on $C^\wedge$ and every bicovering morphism $f : G \to G'$ of $(\mathscr{C}^\wedge)^\wedge_{\mathsf{V}}$ (for the topology $J^\wedge$ of $\mathscr{C}^\wedge$), the functor $\mathsf{Cart}_{\mathscr{C}^\wedge}(\mathscr{F}ib(f), \mathscr{E})$ is an equivalence.

Proof of the claim. Indeed, according to theorem 5.1.24 we have an essentially commutative diagram :

$$\begin{array}{ccc} \mathsf{Cart}_{\mathscr{C}}(\mathscr{F}ib(G'^a), \mathscr{E}) & \xrightarrow{\mathsf{Cart}_{\mathscr{C}}(\mathscr{F}ib(f^a), \mathscr{E})} & \mathsf{Cart}_{\mathscr{C}}(\mathscr{F}ib(G^a), \mathscr{E}) \\ \downarrow & & \downarrow \\ \mathsf{Cart}_{\mathscr{C}}(\mathscr{F}ib(G'), \mathscr{E}) & \xrightarrow{\mathsf{Cart}_{\mathscr{C}}(\mathscr{F}ib(f), \mathscr{E})} & \mathsf{Cart}_{\mathscr{C}}(\mathscr{F}ib(G), \mathscr{E}) \end{array}$$

whose vertical arrows are equivalences. By assumption, $f^a : G^a \to G'^a$ is an isomorphism of sheaves on $(C^\wedge)^\sim_{\mathsf{V}}$, hence the top horizontal arrow is an isomorphism of categories, and therefore the bottom horizontal arrow is an equivalence, as claimed. ◊

In view of claim 5.4.13, and recalling the natural identifications

$$\mathscr{C}^\wedge/F \xrightarrow{\sim} \mathscr{F}ib(h_F) \qquad \mathscr{C}^\wedge/F^a \xrightarrow{\sim} \mathscr{F}ib(h_{F^a})$$

(example 3.1.16(i)), we are then further reduced to showing :

Claim 5.4.14. j_F induces a bicovering morphism $h_{j_F} : h_F \to h_{F^a}$ of $(\mathscr{C}^\wedge)^\wedge_{\mathsf{V}}$.

Proof of the claim. Consider any V-presheaf G on $\mathscr{C}^\wedge$, denote by $G^a \in (C^\wedge)^\sim_{\mathsf{V}}$ the sheaf associated with G, and let $j_G : G \to G^a$ be the natural morphism; since $h_{\mathscr{C}}$ is continuous and cocontinuous for the topologies J and $J^\wedge$ (theorem 4.4.2(i)), lemma 4.2.15(ii) implies that $h^\wedge_{\mathscr{C}}(G^a)$ is a V-sheaf on C, and the morphism of V-presheaves $h^\wedge_{\mathscr{C}}(G) \to h^\wedge_{\mathscr{C}}(G^a)$ on $\mathscr{C}$ is bicovering. Taking $G := h_F$, we get $h^\wedge_{\mathscr{C}}(h_F) = F$, whence an isomorphism $t : h^\wedge_{\mathscr{C}}(h^a_F) \xrightarrow{\sim} F^a$

that identifies $h^\wedge_{\mathscr{C}}(j_G)$ with j_F. Likewise, we have $h^\wedge_{\mathscr{C}}(h_{F^a}) = F^a$, and since $h^\wedge_{\mathscr{C}}$ restricts to an equivalence $(C^\wedge)^\sim_{\mathsf{V}} \stackrel{\sim}{\to} C^\sim_{\mathsf{V}}$ (theorem 4.4.2(iii)), there follows a unique isomorphism $s : h^a_F \stackrel{\sim}{\to} h_{F^a}$ such that $h^\wedge_{\mathscr{C}}(s) = t$. Summing up, we deduce that $h^\wedge_{\mathscr{C}}(s \circ j_G) = j_F = h^\wedge_{\mathscr{C}}(h_{j_F})$; since the functor $h^\wedge_{\mathscr{C}^\wedge} : \mathscr{C}^\wedge \to (\mathscr{C}^\wedge)^\wedge_{\mathsf{V}}$ is fully faithful, we get $s \circ j_G = h_{j_F}$, and since s is an isomorphism, the claim follows. □

Corollary 5.4.15. *In the situation of* (5.4.5)*, suppose that condition* (C3) *holds for some universe* V. *Then u is a weak morphism of sites $C' \to C$.*

Proof. By remark 4.2.18(iii) we may assume that also $\mathscr{C}^\wedge$ and $\mathscr{C}'^\wedge$ are V-small. We consider the essentially commutative diagram :

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{\quad u \quad} & \mathscr{C}' \\ {\scriptstyle h_{\mathscr{C}}}\downarrow & & \downarrow{\scriptstyle h^a_{C'}} \\ \mathscr{C}^\wedge_{\mathsf{V}} & \xrightarrow{\quad u^a_{\mathsf{V}!} \quad} & C'^\sim_{\mathsf{V}} \end{array}$$

and we endow $\mathscr{C}^\wedge_{\mathsf{V}}$ with the topology $J^\wedge$ as in remark 4.1.31, and $C'^\sim_{\mathsf{V}}$ with its canonical topology. We notice that, with these topologies, the functor $u^a_{\mathsf{V}!}$ satisfies condition (C2) : indeed, condition (C0) holds for $u^a_{\mathsf{V}!}$ by definition of the topology $J^\wedge$; also, all fibre products of $\mathscr{C}^\wedge_{\mathsf{V}}$ are representable, and by assumption $u^a_{\mathsf{V}!}$ commutes with fibre products. Let V' be a universe containing V and such that $\mathsf{C}^\wedge_{\mathsf{V}}$ and $C'^\sim_{\mathsf{V}}$ are V'-small; by the foregoing case, we deduce that V'-$\mathsf{Fib}(u^a_{\mathsf{V}!})^*$ sends stacks on $\mathrm{Can}(C'^\sim_{\mathsf{V}})$ to stacks on $(\mathscr{C}^\wedge_{\mathsf{V}}, J^\wedge)$. On the other hand, theorem 5.4.12 implies that V'-$\mathsf{Fib}(h^a_{C'})^*$ restricts to a 2-equivalence from the 2-category of V'-stacks on $\mathrm{Can}(C'^\sim_{\mathsf{V}})$ to that of V'-stacks on $(\mathscr{C}', J')$, so we may assume that $\mathscr{E} = \mathsf{V}'$-$\mathsf{Fib}(h^a_{C'})^*(\mathscr{E}')$ for some V'-stack on $\mathrm{Can}(C'^\sim_{\mathsf{V}})$, and we set $\mathscr{E}'' := \mathsf{V}'$-$\mathsf{Fib}(u^a_{\mathsf{V}!})^*(\mathscr{E}')$. Then $\mathsf{Fib}(u)^*(\mathscr{E})$ is equivalent to V'-$\mathsf{Fib}(h_{\mathscr{C}})^*(\mathscr{E}'')$, and the latter is a stack, by theorem 5.4.12(i). □

Example 5.4.16. Let V be a universe, $C := (\mathscr{C}, J)$ a V-small site, and $X \in \mathrm{Ob}(\mathscr{C})$. Define the site C/X as in (4.7), and recall that the source functor $\mathsf{s}_X : \mathscr{C}/X \to \mathscr{C}$ is continuous and cocontinuous for the topologies of C and C/X. Moreover, the functor $\mathsf{s}_{\mathsf{V}!}$ commutes with fibre products, by proposition 1.4.13(vi.c), hence condition (C3) holds for s_X, and by corollary 5.4.15 we conclude that s_X is a weak morphism of sites $C \to C/X$.

Proposition 5.4.17. *Let $C := (\mathscr{C}, J)$ and $C' := (\mathscr{C}', J')$ be two* U*-sites, and $u : C \to C'$ a morphism of sites such that :*

(a) *u induces an equivalence of categories $\widetilde{u}_* : C^\sim \stackrel{\sim}{\to} C'^\sim$.*

(b) *The subset $u(\mathrm{Ob}(\mathscr{C}')) \subset \mathrm{Ob}(\mathscr{C})$ is a topologically generating family for the site C.*

Then, for every universe V *such that C is a* V*-site, the following holds :*

(i) *u induces an equivalence of categories $\widetilde{u}_{\mathsf{V}*} : C^\sim_{\mathsf{V}} \stackrel{\sim}{\to} C'^\sim_{\mathsf{V}}$.*

(ii) *u induces a 2-equivalence of 2-categories* V-$\mathsf{Fib}(u)^* : \mathsf{V}$-$\mathsf{Stack}(C) \stackrel{\sim}{\to} \mathsf{V}$-$\mathsf{Stack}(C')$.

Proof. (i): To ease notation, set $v := \widetilde{u}_*$. Recall that $\mathrm{Can}(C^\sim)$ and $\mathrm{Can}(C'^\sim)$ are isomorphic to U-sites (remark 4.4.1(iv)). By corollary 4.3.19(i.b) and theorem 4.4.2(iii) there follows, for every universe V containing U, an essentially commutative diagram

$$\begin{array}{ccc} \mathrm{Can}(C^\sim)^\sim_{\mathsf{V}} & \xrightarrow{\quad \widetilde{v}_{\mathsf{V}*} \quad} & \mathrm{Can}(C'^\sim)^\sim_{\mathsf{V}} \\ \downarrow & & \downarrow \\ C^\sim_{\mathsf{V}} & \xrightarrow{\quad \widetilde{u}_{\mathsf{V}*} \quad} & C'^\sim_{\mathsf{V}} \end{array}$$

whose vertical arrows are equivalences. Since v is an equivalence, the same holds for $\widetilde{v}_*$, and then also for $\widetilde{u}_*$. Next, let V be any universe such that C is a V-site, and pick another universe

V' containing V and U. We deduce a commutative diagram

$$\begin{array}{ccc} C^\sim_{\mathsf{V}} & \xrightarrow{\widetilde{u}_{\mathsf{V}*}} & C'^\sim_{\mathsf{V}} \\ \downarrow & & \downarrow \\ C^\sim_{\mathsf{V}'} & \xrightarrow{\widetilde{u}_{\mathsf{V}'*}} & C'^\sim_{\mathsf{V}'} \end{array}$$

whose vertical arrows are the fully faithful inclusion functors and whose bottom horizontal arrow is an equivalence, by the foregoing case. It follows already that $\widetilde{u}_{\mathsf{V}*}$ is fully faithful. We may now argue as in the proof of theorem 4.3.13 : let F be any V-sheaf on C'; we know already that there exists a V'-sheaf G on C such that $\widetilde{u}_{*\mathsf{V}'}G$ is isomorphic to F, and we need to check that GX is essentially V-small for every $X \in \mathrm{Ob}(\mathscr{C})$. But by condition (b) we may find a covering family $(uY_i \to X \mid i \in I)$, and by claim 4.3.16 we may assume that I is V-small. Then the induced map $GX \to \prod_{i\in I} G(uY_i)$ is injective, and by assumption $G(uY_i) \simeq FY_i$ is essentially V-small for every $i \in I$, whence the contention.

(ii): Suppose first that V contains U and both $\mathscr{C}^\wedge$ and $\mathscr{C}'^\wedge$ are V-small; notice that u fulfills condition (C3) of (5.4.5) (remark 5.4.6(ii)), and therefore $\mathsf{V}\text{-}\mathsf{Fib}(u)^*$ sends V-stacks on C to V-stacks on C' (corollary 5.4.15). Likewise, $\mathsf{V}\text{-}\mathsf{Fib}(v)^*$ sends V-stacks on $\mathrm{Can}(C^\sim)$ to V-stacks on $\mathrm{Can}(C'^\sim)$. By corollary 4.3.19(i.b) and theorem 5.4.12(ii) we then get a pseudo-commutative diagram

$$\begin{array}{ccc} \mathsf{V}\text{-}\mathsf{Stack}(\mathrm{Can}(C^\sim)) & \xrightarrow{\mathsf{V}\text{-}\mathsf{Fib}(v)^*} & \mathsf{V}\text{-}\mathsf{Stack}(\mathrm{Can}(C'^\sim)) \\ \downarrow & & \downarrow \\ \mathsf{V}\text{-}\mathsf{Stack}(C) & \xrightarrow{\mathsf{V}\text{-}\mathsf{Fib}(u)^*} & \mathsf{V}\text{-}\mathsf{Stack}(C') \end{array}$$

whose vertical arrows are 2-equivalences. Since v is an equivalence, $\mathsf{V}\text{-}\mathsf{Fib}(v)^*$ is also a 2-equivalence, and then the same holds for $\mathsf{V}\text{-}\mathsf{Fib}(u)^*$. Lastly, let V be any universe such that C is a V-small site, and pick a universe V' containing V and U, and such that $\mathscr{C}^\wedge$ and $\mathscr{C}'^\wedge$ are V'-small; we deduce a commutative diagram

$$\begin{array}{ccc} \mathsf{V}\text{-}\mathsf{Stack}(C) & \xrightarrow{\mathsf{V}\text{-}\mathsf{Fib}(u)^*} & \mathsf{V}\text{-}\mathsf{Stack}(C') \\ \downarrow & & \downarrow \\ \mathsf{V}'\text{-}\mathsf{Stack}(C) & \xrightarrow{\mathsf{V}'\text{-}\mathsf{Fib}(u)^*} & \mathsf{V}'\text{-}\mathsf{Stack}(C') \end{array}$$

whose vertical arrows are the fully faithful inclusion strict pseudo-functors, and whose bottom horizontal arrow is a 2-equivalence, by the foregoing case. It follows already that $\mathsf{V}\text{-}\mathsf{Fib}(u)^*$ is fully faithful. We remark :

Claim 5.4.18. Let $T \subset \mathrm{Ob}(\mathscr{C})$ be an essentially V-small topologically generating family of the site C, and $F : \mathscr{E} \to \mathscr{C}$ a V'-stack on C such that $F^{-1}X$ is an essentially V-small category for every $X \in T$. Then $\mathscr{E}$ is equivalent to a V-stack on C.

Proof of the claim. Denote by $\mathscr{T} \subset \mathscr{C}$ the full subcategory with $\mathrm{Ob}(\mathscr{T}) = T$, let $j : \mathscr{T} \to \mathscr{C}$ be the inclusion functor, and endow $\mathscr{T}$ with the topology $J_{\mathscr{T}}$ induced by J via the functor j. Let also $\eta_{\mathscr{E}} : \mathscr{E} \to \mathscr{F} := \mathsf{V}'\text{-}\mathsf{Fib}(j)_* \circ \mathsf{V}'\text{-}\mathsf{Fib}(j)^*(\mathscr{E})$ be the unit of adjunction. By proposition 4.3.18(ii), the functor j is a morphism of V'-sites $C \to (\mathscr{T}, J_{\mathscr{T}})$, hence $\mathscr{E}' := \mathsf{V}'\text{-}\mathsf{Fib}(j)^*(\mathscr{E})$ is a V'-stack on $(\mathscr{T}, J_{\mathscr{T}})$ (remark 5.4.6(ii) and corollary 5.4.15), and by assumption its fibres are V-small, *i.e.* it is a V-stack on $(\mathscr{T}, J_{\mathscr{T}})$. Moreover, since j is cocontinuous for the topologies J and $J_{\mathscr{T}}$ (proposition 4.3.18(i)), we also see that $\mathscr{F}$ is a V'-stack on C (proposition 5.4.1(ii)). Furthermore, since j is fully faithful, the same holds for $\mathsf{V}'\text{-}\mathsf{Fib}(j)_*$ (remark 3.3.15(iv)), so the counit of adjunction $\varepsilon_{\mathscr{E}'} : \mathsf{V}'\text{-}\mathsf{Fib}(j)^*(\mathscr{F}) \to \mathscr{E}'$ is an equivalence. By virtue of the triangular modifications for the pair (η, ε) (theorem 2.4.24(i)), it follows that $\mathsf{V}'\text{-}\mathsf{Fib}(j)^*(\eta_{\mathscr{E}})$ is an equivalence as well, and then it is i-covering for $i = 0, 1, 2$ (proposition 5.2.9). By invoking lemma 5.4.9 we conclude finally that $\eta_{\mathscr{E}}$ is i-covering for $i = 0, 1, 2$, so it is an equivalence, again by proposition 5.2.9. Summing up, we are reduced to checking that if $\mathscr{E}'$ is a V-stack on $(\mathscr{T}, J_{\mathscr{T}})$, then $\mathscr{F}' := \mathsf{V}'\text{-}\mathsf{Fib}(j)_*(\mathscr{E}')$ is a V-stack on C. To this aim, pick any cleavage $\mathsf{c} : \mathscr{T}^o \to \mathsf{V}\text{-}\mathbf{Cat}$ for $\mathscr{E}'$; by construction, the fibre of $\mathscr{F}'$ over any $X \in \mathrm{Ob}(\mathscr{C})$ is the 2-limit of the pseudo-functor $\mathsf{c} \circ \mathsf{t}_{X^o} : X^o/j^o\mathscr{T}^o \to \mathsf{V}\text{-}\mathbf{Cat}$, and it suffices to remark that $X^o/j^o\mathscr{T}^o$ is a V-small category. ◊

Now, by assumption $\mathscr{C}$ admits a V-small topologically generating family T, and by assumption (b), for every $X \in T$ we may find a covering family $(uY_i \to X \mid i \in I_X)$. By claim 4.3.16, for every $X \in T$ there exists then a V-small subset $I'_X \subset I_X$ such that $(uY_i \to X \mid i \in I'_X)$ is still a covering family; set $T' := \{uY_i \mid X \in T,\ i \in I_X\}$. Clearly T' is a V-small topologically generating family for C. Lastly, let $\mathscr{E}'$ be any V-stack on C'; we know already that there exists a V'-stack $\mathscr{E}$ on C such that $\mathsf{V}'\text{-}\mathsf{Fib}(u)^*(\mathscr{E})$ is equivalent to $\mathscr{E}'$. But the latter condition implies that the fibre of $\mathscr{E}$ over each $X \in T'$ is a V-small category; by claim 5.4.18 we deduce that $\mathscr{E}$ is equivalent to a V-stack on C, and this shows that $\mathsf{V}\text{-}\mathsf{Fib}(u)^*$ is a 2-equivalence, as stated. □

Corollary 5.4.19. *For every* U*-site* $C := (\mathscr{C}, J)$ *the inclusion pseudo-functor* $\mathsf{Stack}(C) \to \mathsf{Fib}(\mathscr{C})$ *admits a left* 2*-adjoint*

$$\mathsf{Fib}(\mathscr{C}) \to \mathsf{Stack}(C) \qquad \mathscr{E} \mapsto \mathscr{E}^a.$$

Proof. Let $T \subset \mathrm{Ob}(\mathscr{C})$ be an essentially small topologically generating family, $\mathscr{T} \subset \mathscr{C}$ the full subcategory with $\mathrm{Ob}(\mathscr{T}) = T$, and $j : \mathscr{T} \to \mathscr{C}$ the inclusion functor. Endow $\mathscr{T}$ with the topology $J_{\mathscr{T}}$ induced by J via j. Let also $\mathscr{E} \to \mathscr{C}$ be any fibration with essentially small fibres, and set $\mathscr{E}' := \mathsf{Fib}(j)^*(\mathscr{E})$. Let $\mathscr{E}'^a$ be the stack associated with $\mathscr{E}'$ (theorem 5.1.24), and denote by $i : \mathscr{E}' \to \mathscr{E}'^a$ the natural morphism of fibrations over $\mathscr{T}$. Choose any universe V containing U and such that $\mathscr{C}$ is V-small, and consider the composition

$$\beta_{\mathscr{E}} : \mathscr{E} \xrightarrow{\eta_{\mathscr{E}}} \mathsf{V}\text{-}\mathsf{Fib}(j)_*(\mathscr{E}') \xrightarrow{\mathsf{V}\text{-}\mathsf{Fib}(j)_*(i)} \mathscr{E}^* := \mathsf{V}\text{-}\mathsf{Fib}(j)_*(\mathscr{E}'^a)$$

where $\eta_{\mathscr{E}}$ is the unit of adjunction. By arguing as in the proof of claim 5.4.18, we see that $\mathscr{E}^*$ is a stack with essentially small fibres on C and that $\mathsf{V}\text{-}\mathsf{Fib}(j)^*(\eta_{\mathscr{E}})$ is an equivalence of categories. Moreover, since j is fully faithful, the counit of adjunction $\mathsf{V}\text{-}\mathsf{Fib}(j)^* \circ \mathsf{V}\text{-}\mathsf{Fib}(j)_* \Rightarrow \mathbf{1}_{\mathsf{V}\text{-}\mathsf{Fib}(\mathscr{T})}$ is a pseudo-natural equivalence, so we have an essentially commutative diagram of functors

$$\begin{array}{ccc}
\mathsf{V}\text{-}\mathsf{Fib}(j)^* \circ \mathsf{V}\text{-}\mathsf{Fib}(j)_*(\mathscr{E}') & \xrightarrow{\mathsf{V}\text{-}\mathsf{Fib}(j)^* \circ \mathsf{V}\text{-}\mathsf{Fib}(j)_*(i)} & \mathsf{V}\text{-}\mathsf{Fib}(j)^*(\mathscr{E}^*) \\
\downarrow & & \downarrow \\
\mathscr{E}' & \xrightarrow{\quad i \quad} & \mathscr{E}'^a
\end{array}$$

whose vertical arrows are equivalences of categories. Since i is l-covering for $l = 0, 1, 2$ (proposition 5.2.11(i)), we conclude that the same holds for $\mathsf{V}\text{-}\mathsf{Fib}(j)^*(\beta_{\mathscr{E}})$, and then also for $\beta_{\mathscr{E}}$, by virtue of lemma 5.4.9. Let us denote by $\mathscr{E}^a_{\mathsf{V}}$ the V-stack associated with $\mathscr{E}$ (a priori, its fibres are only essentially V-small), and $i_{\mathsf{V}} : \mathscr{E} \to \mathscr{E}^a_{\mathsf{V}}$ the unit of adjunction; there exists a morphism of fibrations $\omega_{\mathscr{E}} : \mathscr{E}^a_{\mathsf{V}} \to \mathscr{E}^*$, pseudo-natural in $\mathscr{E}$, such that the functor $\omega_{\mathscr{E}} \circ i_{\mathsf{V}}$ is isomorphic to $\beta_{\mathscr{E}}$. Since i_{V} is l-covering for $l = 0, 1, 2$ (proposition 5.2.11(i)), it follows that the same holds for $\omega_{\mathscr{E}}$ (lemma 5.2.4(ii)), and then the latter is an equivalence of categories (proposition 5.2.9). Summing up, the rule : $\mathscr{E} \mapsto \omega_{\mathscr{E}}$ yields a pseudo-natural equivalence of pseudo-functors, so the pseudo-functor given by the rule : $\mathscr{E} \mapsto \mathscr{E}^*$ is the sought left 2-adjoint. □

5.4.20. Let now $C := (\mathscr{C}, J)$ and $C' := (\mathscr{C}', J')$ be two U-sites, $u : C \to C'$ a weak morphism of sites (definition 5.4.2); then, for every universe V with $\mathsf{U} \subset \mathsf{V}$, the pseudo-functor $\mathsf{V}\text{-}\mathsf{St}(u)_*$ admits a left 2-adjoint

$$\mathsf{V}\text{-}\mathsf{St}(u)^* : \mathsf{V}\text{-}\mathsf{Stack}(C') \to \mathsf{V}\text{-}\mathsf{Stack}(C)$$

defined as follows. We choose an essentially U-small topologically generating subset $G \subset \mathrm{Ob}(\mathscr{C}')$ and let $\mathscr{G} \subset \mathscr{C}'$ be the full subcategory with $\mathrm{Ob}(\mathscr{G}) = G$. We endow $\mathscr{G}$ with the topology $J_\mathscr{G}$ induced by J' via the inclusion functor $j : \mathscr{G} \to \mathscr{C}'$. Recall that j is a morphism of sites $C' \to (\mathscr{G}, J_\mathscr{G})$ and $\widetilde{\jmath}_* : C'^\sim \xrightarrow{\sim} (\mathscr{G}, J_\mathscr{G})^\sim$ is an equivalence (proposition 4.3.18(ii)). Then $\mathsf{V}\text{-}\mathsf{St}(j)_*$ is well defined (remark 5.4.6(ii)) and it is a 2-equivalence (proposition 5.4.17(ii)), and $\mathsf{V}\text{-}\mathsf{St}(u)^*$ is given by the composition :

$$\mathsf{V}\text{-}\mathsf{Stack}(C') \xrightarrow{\mathsf{V}\text{-}\mathsf{St}(j)_*} \mathsf{V}\text{-}\mathsf{Stack}(\mathscr{G}, J_\mathscr{G}) \xrightarrow{\mathsf{V}\text{-}\mathsf{Fib}(u\circ j)_!} \mathsf{V}\text{-}\mathsf{Fib}(\mathscr{C}) \xrightarrow{(-)^a} \mathsf{V}\text{-}\mathsf{Stack}(C)$$

where $(-)^a$ is the pseudo-functor of corollary 5.4.19. It is then easily seen that the resulting pseudo-functor is a left 2-adjoint for $\mathsf{V}\text{-}\mathsf{St}(u)_*$; especially, it is independent, up to pseudo-natural equivalence, of the choice of G (remark 2.4.28(iii)); the details shall be left to the reader. As usual, we shall often omit mentioning V, in case $\mathsf{V} = \mathsf{U}$.

Example 5.4.21. By remark 5.4.6(ii), every morphism of sites $u : C := (\mathscr{C}, J') \to C' := (\mathscr{C}', J)$ fulfills (C3), hence it is a weak morphism of sites (corollary 5.4.15), and so it induces a pseudo-functor $\mathsf{V}\text{-}\mathsf{St}(u)_*$. If C and C' are U-sites, we have also the pseudo-functor $\mathsf{V}\text{-}\mathsf{St}(u)^*$.

Proposition 5.4.22. *Let $C := (\mathscr{C}, J)$ and $C' := (\mathscr{C}', J')$ be two small sites, $u : C \to C'$ a morphism of sites, $i \in \{0, 1, 2\}$, and $f : \mathscr{E} \to \mathscr{F}$ an i-covering cartesian functor of $\mathscr{C}'$-fibrations. We have :*

(i) *The functor $\mathsf{Fib}(u)_!(f) : \mathsf{Fib}(u)_!(\mathscr{E}) \to \mathsf{Fib}(u)_!(\mathscr{F})$ is i-covering.*
(ii) *If $\mathscr{E}$ and $\mathscr{F}$ are stacks on C', then $\mathsf{St}(u)^*(f) : \mathsf{St}(u)^*(\mathscr{E}) \to \mathsf{St}(u)^*(\mathscr{F})$ is i-covering.*

Proof. We show first the following special case :

Claim 5.4.23. The proposition holds if u is a morphism of lex-sites.

Proof of the claim. By proposition 5.2.11(iii), it suffices to show assertion (i) of the proposition. To this aim, we consider the essentially commutative diagram

$$\begin{array}{ccc} \mathsf{Fib}(u)_!(\mathsf{C}(\mathscr{E})) & \xrightarrow{\mathsf{Fib}(u)_!(\mathsf{C}(f))} & \mathsf{Fib}(u)_!(\mathsf{C}(\mathscr{F})) \\ {\scriptstyle \mathsf{Fib}(u)_!(\mathsf{ev}^\mathscr{E})}\downarrow & & \downarrow{\scriptstyle \mathsf{Fib}(u)_!(\mathsf{ev}^\mathscr{F})} \\ \mathsf{Fib}(u)_!(\mathscr{E}) & \xrightarrow{\mathsf{Fib}(u)_!(f)} & \mathsf{Fib}(u)_!(\mathscr{F}) \end{array}$$

where $\mathsf{ev}^\bullet$ is the pseudo-natural equivalence described in (3.2.5). Then both vertical arrows of the diagram are equivalences (lemma 2.4.11(ii)), so $\mathsf{Fib}(u)_!(f)$ is i-covering if and only if the same holds for $\mathsf{Fib}(u)_!(\mathsf{C}(f))$ (lemma 5.2.3), and by the same token, f is i-covering if and only if the same holds for $\mathsf{C}(f)$. Thus, we may replace f by $\mathsf{C}(f)$, and assume from start that f is a split cartesian functor $(\mathscr{E}, \boldsymbol{\lambda}^\mathscr{E}) \to (\mathscr{F}, \boldsymbol{\lambda}^\mathscr{F})$. In this case, let $\mathsf{c}^\mathscr{E}$ and $\mathsf{c}^\mathscr{F}$ be the strict pseudo-functors associated with the cleavages $\boldsymbol{\lambda}^\mathscr{E}$ and $\boldsymbol{\lambda}^\mathscr{F}$; then $\mathsf{Fib}(u)_!(\mathscr{E})$ is equivalent to the fibration associated with the pseudo-functor

$$\mathsf{d}^\mathscr{E} : \mathscr{C}^o \to \mathbf{Cat} \qquad X \mapsto \underset{(X/u\mathscr{C}')^o}{2\text{-colim}}\, \mathsf{c}^\mathscr{E} \circ \mathsf{t}_X^o$$

where $\mathsf{t}_X : X/u\mathscr{C}' \to \mathscr{C}'$ is the target functor. Hence, $\mathsf{d}_X^\mathscr{E} = \mathscr{F}ib(\mathsf{c}^\mathscr{E} \circ \mathsf{t}_X^o)[\Sigma_{\mathscr{E},X}^{-1}]$, where $\Sigma_{\mathscr{E},X}$ is the set of cartesian morphisms of $\mathscr{F}ib(\mathsf{c}^\mathscr{E} \circ \mathsf{t}_X^o)$ (see the proof of theorem 3.3.9). The universal pseudo-cocone $\pi^{\mathscr{E},X} : \mathsf{c}^{\mathscr{E}} \circ \mathsf{t}^o_X \Rightarrow \mathsf{F}_{\mathsf{d}^{\mathscr{E}}_X}$ is the strict pseudo-natural transformation that assigns to every object $(X', \psi : X \to uX')$ of $(X/u\mathscr{C}')^o$ the functor

$$\pi^{\mathscr{E},X}_{(X',\psi)} : \mathsf{c}^{\mathscr{E}}_{X'} \to \mathsf{d}^{\mathscr{E}}_X \qquad E \mapsto ((X', \psi), E).$$

To every morphism $\varphi : X \to Y$ of $\mathscr{C}$ we then attach the functor

$$\mathsf{d}^{\mathscr{E}}_\varphi : \mathsf{d}^{\mathscr{E}}_Y \to \mathsf{d}^{\mathscr{E}}_X \qquad \text{such that} \qquad \pi^{\mathscr{E},X} * \varphi^{*o} = \mathsf{F}_{\mathsf{d}^{\mathscr{E}}_\varphi} \odot \pi^{\mathscr{E},Y}$$

where $\varphi^* : Y/u\mathscr{C}' \to X/u\mathscr{C}'$ is the functor induced by φ (see example 2.2.8(ii)); unwinding the definitions, we find that $\mathsf{d}^{\mathscr{E}}_\varphi$ is given by the rule :

$$((X', \psi : Y \to uX'), E) \mapsto ((X', \psi \circ \varphi), E) \qquad \text{for every } ((X', \psi), E) \in \mathrm{Ob}(\mathsf{d}^{\mathscr{E}}_Y).$$

Likewise we describe $\mathsf{Fib}(u)_!(\mathscr{F})$ up to equivalence; next, the split cartesian functor f corresponds to a strict pseudo-natural transformation $\mathsf{c}^f : \mathsf{c}^{\mathscr{E}} \Rightarrow \mathsf{c}^{\mathscr{F}}$, and under the foregoing identifications, the cartesian functor $\mathsf{Fib}(u)_!(f)$ corresponds to the strict pseudo-natural transformation $\mathsf{d}^f : \mathsf{d}^{\mathscr{E}} \Rightarrow \mathsf{d}^{\mathscr{F}}$ assigning to every $X \in \mathrm{Ob}(\mathscr{C})$ the functor :

$$\mathsf{d}^f_X : \mathsf{d}^{\mathscr{E}}_X \to \mathsf{d}^{\mathscr{F}}_X \qquad ((X', \psi), E) \mapsto ((X', \psi), \mathsf{c}^f_{X'} E).$$

The objects of $\mathscr{F}ib(\mathsf{d}^{\mathscr{E}})$ are then the data (X, X', ψ, E) with $X \in \mathrm{Ob}(\mathscr{C})$ and $((X', \psi), E) \in \mathrm{Ob}(\mathsf{d}^{\mathscr{E}}_X)$, and likewise for the objects of $\mathscr{F}ib(\mathsf{d}^{\mathscr{F}})$. After this preparation, suppose that f is 0-covering, and let us check that the same holds for $\mathsf{Fib}(u)_!(f)$, where the latter is naturally identified with $\mathscr{F}ib(\mathsf{d}^f)$. Thus, consider any object $(X, X', \psi : X \to uX', F)$ of $\mathscr{F}ib(\mathsf{d}^{\mathscr{F}})$; according to remark 5.2.2, we need to find a covering family $(\varphi_\lambda : Y_\lambda \to X \mid \lambda \in \Lambda)$ and for every $\lambda \in \Lambda$ an object $\underline{E}_\lambda := ((Y'_\lambda, \psi_\lambda), E) \in \mathsf{d}^{\mathscr{E}}_{Y_\lambda}$ with an isomorphism

$$((Y'_\lambda, \psi_\lambda), \mathsf{c}^f_{Y'_\lambda} E_\lambda) = \mathsf{d}^f_{Y_\lambda}((Y'_\lambda, \psi_\lambda), E_\lambda) \xrightarrow{\sim} \mathsf{d}^{\mathscr{F}}_{\varphi_\lambda}((X', \psi), F) = ((X', \psi \circ \varphi_\lambda), F) \qquad \text{in } \mathsf{d}^{\mathscr{F}}_{Y_\lambda}.$$

But since f is 0-covering, there exist a covering $(\tau_\lambda : Y'_\lambda \to X' \mid \lambda \in \Lambda)$ for the topology J' and an object $E_\lambda \in \mathrm{Ob}(\mathsf{c}^{\mathscr{E}}_{Y'_\lambda})$ with an isomorphism $\omega_\lambda : \mathsf{c}^f_{Y'_\lambda} E_\lambda \xrightarrow{\sim} \mathsf{c}^{\mathscr{F}}_{\tau_\lambda} F$ in $\mathsf{c}^{\mathscr{F}}_{Y'_\lambda}$ for every $\lambda \in \Lambda$. Thus, let us set $Y_\lambda := X \times_{uX'} uY'_\lambda$, and denote by $\varphi_\lambda : Y_\lambda \to X$ the induced projection, for every $\lambda \in \Lambda$. Since u is continuous, the family $(u(\tau_\lambda) \mid \lambda \in \Lambda)$ covers uX' for the topology J, and therefore $(\varphi_\lambda \mid \lambda \in \Lambda)$ covers X. Lastly, the sought isomorphism shall be the composition of the isomorphism $((Y'_\lambda, \psi_\lambda), \mathsf{c}^f_{Y'_\lambda} E_\lambda) \xrightarrow{\sim} ((Y'_\lambda, \psi_\lambda), \mathsf{c}^{\mathscr{F}}_{\tau_\lambda} F)$ of $\mathscr{F}ib(\mathsf{c}^{\mathscr{F}} \circ \mathsf{t}^o_{Y_\lambda})$ induced by ω_λ, and the cartesian morphism $((Y'_\lambda, \psi_\lambda), \mathsf{c}^{\mathscr{F}}_{\tau_\lambda} F) \to ((X', \psi \circ \varphi_\lambda), F)$ of $\mathscr{F}ib(\mathsf{c}^{\mathscr{F}} \circ \mathsf{t}^o_{Y_\lambda})$.

Next, suppose that f is 1-covering, and let $((X'_i, \psi_i : X \to uX'_i), E_i)$ for $i = 1, 2$ be two objects of $\mathsf{d}^{\mathscr{E}}_X$ with a morphism $g : ((X'_1, \psi_1), \mathsf{c}^f_{X'_1} E_1) \to ((X'_2, \psi_2), \mathsf{c}^f_{X'_2} E_2)$ in $\mathsf{d}^{\mathscr{F}}_X$. Notice that $(Z/u\mathscr{C}')^o$ is a filtered category for every $Z \in \mathrm{Ob}(\mathscr{C})$ (example 1.3.16(i)), hence $\Sigma_{\mathscr{F},Z}$ admits a right calculus of fraction (example 3.3.13(ii)); moreover, every cartesian morphism $(\sigma, s) : ((Z'', \psi'), F') \to ((Z', \psi), F)$ in $\Sigma_{\mathscr{F},Z}$ admits the factorization :

$$(\sigma, s) = \boldsymbol{\lambda}^{\mathscr{F}}(F, \sigma) \circ (\mathbf{1}_{Z''}, s) \qquad \text{in } \mathscr{F}ib(\mathsf{c}^{\mathscr{F}} \circ \mathsf{t}^o_Z) \tag{5.4.24}$$

where $(\mathbf{1}_{X''}, s)$ is an isomorphism of $\mathsf{c}^{\mathscr{F}}_{X''}$ (lemma 3.1.20(i)), so g can be represented by a pair

$$((X'_1, \psi_1), \mathsf{c}^f_{X'_1} E_1) \xleftarrow{\boldsymbol{\lambda}^{\mathscr{F}}(\mathsf{c}^f_{X'_1} E_1, \rho)} ((X''_1, \psi'_1), \mathsf{c}^{\mathscr{F}}_\rho \circ \mathsf{c}^f_{X'_1} E_1) \xrightarrow{g'} ((X'_2, \psi_2), \mathsf{c}^f_{X'_2} E_2)$$

where $\rho : X''_1 \to X'_1$ is a morphism of $\mathscr{C}'$ and $\psi'_1 : X \to uX''_1$ is a morphism of $\mathscr{C}$ such that $u(\rho) \circ \psi'_1 = \psi_1$. Also, g' is a morphism in $\mathscr{F}ib(\mathsf{c}^{\mathscr{F}} \circ \mathsf{t}^o_X)$. Then in turn, g' is a pair $(X/\mu, t)$, where X/μ is a morphism $(X''_1, \psi'_1) \to (X'_2, \psi_2)$ in $X/u\mathscr{C}'$, *i.e.* a morphism $\mu : X''_1 \to X'_2$ in $\mathscr{C}'$ with $u(\mu) \circ \psi_1 = \psi_2$, and g' is a morphism in $\mathsf{c}^{\mathscr{F}}_{X''_1}$

$$g' : \mathsf{c}^f_{X''_1} \circ \mathsf{c}^{\mathscr{E}}_\rho E_1 = \mathsf{c}^{\mathscr{F}}_\rho \circ \mathsf{c}^f_{X'_1} E_1 \to \mathsf{c}^{\mathscr{F}}_\mu \circ \mathsf{c}^f_{X'_2} E_2 = \mathsf{c}^f_{X''_1} \circ \mathsf{c}^{\mathscr{E}}_\mu E_2.$$

According to remark 5.2.2, we need to exhibit a covering family $(\varphi_\lambda : Y_\lambda \to X \mid \lambda \in \Lambda)$, and for every $\lambda \in \Lambda$ a morphism $g_\lambda : ((X'_1, \psi_1 \circ \varphi_\lambda), E_1) \to ((X'_2, \psi_2 \circ \varphi_\lambda), E_2)$ in $\mathsf{d}^{\mathscr{E}}_{Y_\lambda}$ such that $\mathsf{d}^f_{X'_1}(g_\lambda) = \mathsf{d}^{\mathscr{F}}_{\varphi_\lambda}(g)$. Now, since f is 1-covering, there exist a covering family $(\tau_\lambda : Y''_\lambda \to X''_1 \mid \lambda \in \Lambda)$ and for every $\lambda \in \Lambda$ a morphism

$$g'_\lambda : \mathsf{c}^{\mathscr{E}}_{\tau_\lambda} \circ \mathsf{c}^{\mathscr{E}}_\rho E_1 \to \mathsf{c}^{\mathscr{E}}_{\tau_\lambda} \circ \mathsf{c}^{\mathscr{E}}_\mu E_2 \qquad \text{in } \mathsf{c}^{\mathscr{E}}_{Y''_\lambda} \text{ such that} \qquad \mathsf{c}^f_{Y''_\lambda}(g'_\lambda) = \mathsf{c}^{\mathscr{F}}_{\tau_\lambda}(g').$$

We set $Y_\lambda := X \times_{uX''_1} uY''_\lambda$, and we let $\varphi_\lambda : Y_\lambda \to X$ and $\psi''_\lambda : Y_\lambda \to uY''_\lambda$ be the induced projections for every $\lambda \in \Lambda$; arguing as in the foregoing it is easily seen that the family $(g_\lambda \mid \lambda \in \Lambda)$ covers X, and we define g_λ as the morphism of $\mathsf{d}^{\mathscr{E}}_{Y_\lambda}$ represented by the pair of morphisms :

$$((X'_\lambda, \psi_1 \circ \varphi_\lambda), E_1) \xleftarrow{\boldsymbol{\lambda}^{\mathscr{E}}(E_1, \rho\circ\tau_\lambda)} ((Y''_\lambda, \psi''_\lambda), \mathsf{c}^{\mathscr{E}}_{\rho\circ\tau_\lambda} E_1) \xrightarrow{(Y_\lambda/\mu\circ\tau_\lambda, g'_\lambda)} ((X'_2, \psi_2 \circ \varphi_\lambda), E_2).$$

In view of remark 3.2.2(ii) we see that $\mathsf{d}^f_{Y_\lambda}(g'_\lambda)$ is represented by the pair

$$((X'_\lambda, \psi_1\varphi_\lambda), \mathsf{c}^f_{X'_1} E_1) \xleftarrow{\boldsymbol{\lambda}^{\mathscr{F}}(\mathsf{c}^f_{X'_1} E_1, \rho\tau_\lambda)} ((Y''_\lambda, \psi''_\lambda), \mathsf{c}^f_{Y''_\lambda}\mathsf{c}^{\mathscr{E}}_{\rho\tau_\lambda} E_1) \xrightarrow{(Y_\lambda/\mu\tau_\lambda, \mathsf{c}^{\mathscr{F}}_{\tau_\lambda}(g'))} ((X'_2, \psi_2\varphi_\lambda), \mathsf{c}^f_{X'_2} E_2)$$

and by remark 3.2.2(i) we have :

$$\boldsymbol{\lambda}^{\mathscr{F}}(\mathsf{c}^f_{X'_1} E_1, \rho\tau_\lambda) = \boldsymbol{\lambda}^{\mathscr{F}}(\mathsf{c}^f_{X'_1} E_1, \rho) \circ \boldsymbol{\lambda}^{\mathscr{F}}(\mathsf{c}^{\mathscr{F}}_\rho \mathsf{c}^f_{X'_1} E_1, \tau_\lambda).$$

On the other hand, $(Y_\lambda/\mu \circ \tau_\lambda, \mathsf{c}^{\mathscr{F}}_{\tau_\lambda}(g'))$ is the composition :

$$((Y''_\lambda,\psi''_\lambda), \mathsf{c}^f_{Y''_\lambda}\mathsf{c}^{\mathscr{E}}_{\rho\tau_\lambda} E_1) \xrightarrow{\boldsymbol{\lambda}^{\mathscr{F}}(\mathsf{c}^{\mathscr{F}}_\rho \mathsf{c}^f_{X'_1} E_1, \tau_\lambda)} ((X''_\lambda, \psi'_1\varphi_\lambda), \mathsf{c}^{\mathscr{F}}_\rho \mathsf{c}^f_{X'_1} E_1) \xrightarrow{\mathsf{d}^{\mathscr{F}}_{\varphi_\lambda}(X/\mu, g')} ((X'_2, \psi_2\varphi_\lambda), \mathsf{c}^f_{X'_2} E_1)$$

so $\mathsf{d}^f_{Y_\lambda}(g'_\lambda)$ is also represented by the pair

$$((X'_\lambda, \psi_1\varphi_\lambda), \mathsf{c}^f_{X'_1} E_1) \xleftarrow{\boldsymbol{\lambda}^{\mathscr{F}}(\mathsf{c}^f_{X'_1} E_1, \rho)} ((X''_\lambda, \psi'_1\varphi_\lambda), \mathsf{c}^{\mathscr{F}}_\rho \mathsf{c}^f_{X'_1} E_1) \xrightarrow{\mathsf{d}^{\mathscr{F}}_{\varphi_\lambda}(X/\mu, g')} ((X'_2, \psi_2\varphi_\lambda), \mathsf{c}^f_{X'_2} E_1)$$

which represents $\mathsf{d}^{\mathscr{F}}_{\varphi_\lambda}(g)$ as well, as required.

Lastly, suppose that f is 2-covering, and consider objects $((X'_j, \psi_j), E_j)$ of $\mathsf{d}^{\mathscr{E}}_X$ for $j = 1, 2$, and two morphisms $g_1, g_2 : ((X'_1, \psi_1), E_1) \to ((X'_2, \psi_2), E_2)$ in $\mathsf{d}^{\mathscr{E}}_X$ such that

$$\mathsf{d}^f_X(g_1) = \mathsf{d}^f_X(g_2). \tag{5.4.25}$$

We need to exhibit a covering family $(\varphi_\lambda : Y_\lambda \to X \mid \lambda \in \Lambda)$ such that $\mathsf{d}^{\mathscr{E}}_{\varphi_\lambda}(g_1) = \mathsf{d}^{\mathscr{E}}_{\varphi_\lambda}(g_2)$ for every $\lambda \in \Lambda$. We reduce easily to the case where g_1 and g_2 are two classes of morphisms of $\mathscr{F}ib(\mathsf{c}^{\mathscr{E}} \circ \mathsf{t}^o_X)$. In this case, (5.4.25) means that there exists a cartesian morphism $h : ((X''_1, \psi'_1), F) \to ((X'_1, \psi_1), \mathsf{c}^f_{X'_1} E_1)$ in $\mathscr{F}ib(\mathsf{c}^{\mathscr{F}} \circ \mathsf{t}^o_X)$ such that $\mathscr{F}ib(\mathsf{c}^f \circ \mathsf{t}^o_X)(g_1) \circ h = \mathscr{F}ib(\mathsf{c}^f \circ \mathsf{t}^o_X)(g_2) \circ h$. However, we have a factorization $h = \boldsymbol{\lambda}^{\mathscr{F}}(\mathsf{c}^f_{X'_1} E_1, \mu) \circ (\mathbf{1}_{X''_1}, h')$ as in (5.4.24), where $(\mathbf{1}_{X''_1}, h')$ is an isomorphism of $\mathsf{c}^{\mathscr{F}}_{X''_1}$, so we may assume that $h = \boldsymbol{\lambda}^{\mathscr{F}}(\mathsf{c}^f_{X'_1} E_1, \mu) = \mathscr{F}ib(\mathsf{c}^f \circ \mathsf{t}^o_X)(\boldsymbol{\lambda}^{\mathscr{E}}(E_1, \mu))$. Then we may as well replace g_i by $g_i \circ \boldsymbol{\lambda}^{\mathscr{E}}(E_1, \mu)$ for $i = 1, 2$, and assume from start that

$$\mathscr{F}ib(\mathsf{c}^f \circ \mathsf{t}^o_X)(g_1) = \mathscr{F}ib(\mathsf{c}^f \circ \mathsf{t}^o_X)(g_2).$$

Especially, there exist a morphism $\rho : X'_1 \to X'_2$ in $\mathscr{C}'$ such that $u(\rho) \circ \psi_1 = \psi_2$, and morphisms $g'_i : E_1 \to \mathsf{c}^{\mathscr{E}}_\rho E_2$ such that $g_i = (\rho, g'_i)$ for $i = 1, 2$, and $\mathsf{c}^f_{X'_1}(g'_1) = \mathsf{c}^f_{X'_1}(g'_2)$. Then, since f is 2-covering, we find a covering family $(\tau_\lambda : Y'_\lambda \to X'_1 \mid \lambda \in \Lambda)$ such that

$$\mathsf{c}^{\mathscr{E}}_{\tau_\lambda}(g'_1) = \mathsf{c}^{\mathscr{E}}_{\tau_\lambda}(g'_2) \qquad \text{for every } \lambda \in \Lambda. \tag{5.4.26}$$

We set again $Y_\lambda := X \times_{uX'_1} uY'_\lambda$, and we let $\varphi_\lambda : Y_\lambda \to X$ and $\psi_\lambda : Y_\lambda \to Y'_\lambda$ be the induced projections, for every $\lambda \in \Lambda$. We claim that the resulting covering $(\varphi_\lambda \mid \lambda \in \Lambda)$ of X will do.

Indeed, $\mathsf{d}^{\mathscr{E}}_{\varphi_\lambda}(g_i) : ((X'_1, \psi_1 \circ \varphi_\lambda), E_1) \to ((X'_2, \psi_2 \circ \varphi_\lambda), E_2)$ is represented by the morphism $(Y_\lambda/\rho, g'_i)$ of $\mathscr{F}ib(\mathsf{c}^{\mathscr{E}} \circ \mathsf{t}^o_{Y_\lambda})$ for $i = 1, 2$, and (5.4.26) implies easily that

$$(Y_\lambda/\rho, g'_1) \circ \boldsymbol{\lambda}^{\mathscr{E}}(E_1, \tau_\lambda) = (Y_\lambda/\rho, g'_2) \circ \boldsymbol{\lambda}^{\mathscr{E}}(E_1, \tau_\lambda)$$

where $\boldsymbol{\lambda}^{\mathscr{E}}(E_1, \tau_\lambda) : ((Y'_\lambda, \psi_\lambda), \mathsf{c}^{\mathscr{E}}_{\tau_\lambda} E_1) \to ((X'_1, \psi_1 \circ \varphi_\lambda), E_1)$ is a cartesian morphism of the fibration $\mathscr{F}ib(\mathsf{c}^{\mathscr{E}} \circ \mathsf{t}^o_{Y_\lambda})$, whence the contention. ◊

We consider now the essentially commutative diagram

$$\begin{array}{ccc} \mathsf{Fib}(u)_!(\mathscr{E}) & \xrightarrow{\mathsf{Fib}(u)_!(f)} & \mathsf{Fib}(u)_!(\mathscr{F}) \\ {\scriptstyle \mathsf{Fib}(u)_!(\eta_{\mathscr{E}})}\downarrow & & \downarrow{\scriptstyle \mathsf{Fib}(u)_!(\eta_{\mathscr{F}})} \\ \mathsf{Fib}(u)_!(\mathscr{E}^a) & \xrightarrow{\mathsf{Fib}(u)_!(f^a)} & \mathsf{Fib}(u)_!(\mathscr{F}^a) \end{array}$$

where $\eta_{\mathscr{E}} : \mathscr{E} \to \mathscr{E}^a$ and $\eta_{\mathscr{F}} : \mathscr{F} \to \mathscr{F}^a$ are the natural morphisms, and notice that $\mathsf{Fib}(u)_!(\eta_{\mathscr{E}})^a$ and $\mathsf{Fib}(u)_!(\eta_{\mathscr{F}})^a$ are equivalences, by virtue of proposition 5.4.4. Then $\mathsf{Fib}(u)_!(f)^a$ is i-covering if and only if the same holds for $\mathsf{Fib}(u)_!(f^a)^a$ (lemma 5.2.3), and finally $\mathsf{Fib}(u)_!(f)$ is i-covering if and only if the same holds for $\mathsf{Fib}(u)_!(f^a)$ (proposition 5.2.11(iii)). Summing up, we see that, in order to prove the proposition, we may assume that $\mathscr{E}$ and $\mathscr{F}$ are stacks on C', and moreover, in this case it suffices to show assertion (ii).

Now, by lemma 4.2.11(ii) we have the essentially commutative diagram :

$$\begin{array}{ccc} \mathscr{C}' & \xrightarrow{u} & \mathscr{C} \\ {\scriptstyle h^a_{C'}}\downarrow & & \downarrow{\scriptstyle h^a_C} \\ C'^{\sim}_{\mathsf{U}} & \xrightarrow{\widetilde{u}^*} & C^{\sim}_{\mathsf{U}} \end{array}$$

whose vertical arrows are the Yoneda embeddings. Especially, for every universe V containing U and such that $\mathscr{C}^\wedge_{\mathsf{U}}$ is V-small, $\mathsf{V}\text{-}\mathsf{St}(h^a_{C'})_*$ is a 2-equivalence from the 2-category of stacks on $\mathrm{Can}(C'^{\sim}_{\mathsf{U}})$ to that of stacks on C (theorem 5.4.12), and therefore there exists a cartesian functor $f' : \mathscr{E}' \to \mathscr{F}'$ of V-stacks on $\mathrm{Can}(C'^{\sim}_{\mathsf{U}})$ with an essentially commutative diagram :

$$\mathscr{D} \quad : \quad \begin{array}{ccc} \mathscr{E} & \xrightarrow{\qquad f \qquad} & \mathscr{F} \\ \downarrow & & \downarrow \\ \mathsf{V}\text{-}\mathsf{St}(h^a_{C'})_*(\mathscr{E}') & \xrightarrow{\mathsf{V}\text{-}\mathsf{St}(h^a_{C'})_*(f')} & \mathsf{V}\text{-}\mathsf{St}(h^a_{C'})_*(\mathscr{F}') \end{array}$$

whose vertical arrows are equivalences. We notice :

Claim 5.4.27. Let $C := (\mathscr{C}, J)$ be a small site, $h^a_C : \mathscr{C} \to C^{\sim}$ the Yoneda embedding, $\varphi : \mathscr{A} \to \mathscr{B}$ a morphism of $\mathsf{V}\text{-}\mathsf{Fib}(C^{\sim})$, and $j \in \{0, 1, 2\}$. Then φ is j-covering for the canonical topology of $C^{\sim}$ if and only if $\mathsf{V}\text{-}\mathsf{Fib}(h^a_C)^*(\varphi)$ is j-covering for the topology J.

Proof of the claim. Suppose that φ is j-covering; since h^a_C is cocontinuous for the topologies J (theorem 4.4.2(iii)) and $\mathrm{Can}_{C^{\sim}}$, proposition 5.4.1(i) says that $\mathsf{V}\text{-}\mathsf{Fib}(h^a_C)^*(\varphi)$ is j-covering. The converse follows from lemma 5.4.9. ◊

Since f is i-covering, the same holds for $\mathsf{V}\text{-}\mathsf{St}(h^a_{C'})_*(f')$, by lemma 5.2.3, and then also for f', by claim 5.4.27. Moreover, considering the essentially commutative diagram $\mathsf{V}\text{-}\mathsf{St}(u)^*(\mathscr{D})$ and applying again lemma 5.2.3, we are reduced to checking that $\mathsf{V}\text{-}\mathsf{St}(u)^* \circ \mathsf{V}\text{-}\mathsf{St}(h^a_{C'})_*(f')$ is i-covering. To ease notation, set $S := \mathsf{V}\text{-}\mathsf{St}(h^a_C)_* \circ \mathsf{V}\text{-}\mathsf{St}(h^a_C)^* \circ \mathsf{V}\text{-}\mathsf{St}(u)^* \circ \mathsf{V}\text{-}\mathsf{St}(h^a_{C'})_*$; recalling that $\mathsf{V}\text{-}\mathsf{St}(h^a_C)^*$ is a pseudo-inverse for the 2-equivalence $\mathsf{V}\text{-}\mathsf{St}(h^a_C)_*$, we get an essentially commutative diagram

$$\begin{array}{ccc}\mathsf{V\text{-}St}(u)^* \circ \mathsf{V\text{-}St}(h^a_{C'})_*(\mathscr{E}') & \xrightarrow{\mathsf{V\text{-}St}(u)^*\circ\mathsf{V\text{-}St}(h^a_{C'})_*(f')} & \mathsf{V\text{-}St}(u)^* \circ \mathsf{V\text{-}St}(h^a_{C'})_*(\mathscr{F}') \\ \downarrow & & \downarrow \\ S(\mathscr{E}) & \xrightarrow{S(f')} & S(\mathscr{F})\end{array}$$

whose vertical arrows are equivalences; invoking once again lemma 5.2.3, we are then reduced to checking that $S(f)$ is i-covering. However, we have pseudo-natural equivalences :

$$S \stackrel{\sim}{\to} \mathsf{V\text{-}St}(h^a_C)_* \circ \mathsf{V\text{-}St}(\widetilde{u}^*)^* \circ \mathsf{V\text{-}St}(h^a_{C'})^* \circ \mathsf{V\text{-}St}(h^a_{C'})_* \stackrel{\sim}{\to} \mathsf{V\text{-}St}(h^a_C)_* \circ \mathsf{V\text{-}St}(\widetilde{u}^*)^*$$

which, after applying once more lemma 5.2.3, further reduces to checking that the functor $\mathsf{V\text{-}St}(h^a_C)_* \circ \mathsf{V\text{-}St}(\widetilde{u}^*)^*(f')$ is i-covering. The latter follows from claims 5.4.23 and 5.4.27. □

5.4.28. Let $C := (\mathscr{C}, J)$ and $C' := (\mathscr{C}', J')$ be two small sites, and $u : \mathscr{C} \to \mathscr{C}'$ a cocontinuous functor; by proposition 5.4.1(ii), the pseudo-functor $\mathsf{Fib}(u)_*$ restricts to a pseudo-functor

$$\mathsf{St}(\breve{u})_* : \mathsf{Stack}(C) \to \mathsf{Stack}(C')$$

and it is easily seen that the latter admits a left 2-adjoint

$$\mathsf{St}(\breve{u})^* : \mathsf{Stack}(C') \xrightarrow{i} \mathsf{Fib}(\mathscr{C}') \xrightarrow{\mathsf{Fib}(u)^*} \mathsf{Fib}(\mathscr{C}) \xrightarrow{(-)^a_C} \mathsf{Stack}(C)$$

with i the inclusion pseudo-functor. On the other hand, by remark 4.4.13(iii,iv), the functor u induces a morphism of sites $\breve{u}^* : \mathrm{Can}(C^\sim) \to \mathrm{Can}(C'^\sim)$, whence a 2-adjoint pair of pseudo-functors $(\mathsf{St}(\breve{u}^*)^*, \mathsf{St}(\breve{u}^*)_*)$. Define also $h^a_C : \mathscr{C} \to C^\sim$, $h^a_{C'} : \mathscr{C}' \to C'^\sim$ as in remark 4.1.19(iii).

Proposition 5.4.29. (i) *In the situation of* (5.4.28)*, we have pseudo-commutative diagrams :*

$$\begin{array}{ccc}\mathsf{Stack}(\mathrm{Can}\, C'^\sim) & \xrightarrow{\mathsf{St}(\breve{u}^*)^*} & \mathsf{Stack}(\mathrm{Can}\, C^\sim) \\ {\scriptstyle \mathsf{St}(h^a_{C'})_*}\downarrow & & \downarrow{\scriptstyle \mathsf{St}(h^a_C)_*} \\ \mathsf{Stack}(C') & \xrightarrow{\mathsf{St}(\breve{u})^*} & \mathsf{Stack}(C)\end{array} \qquad \begin{array}{ccc}\mathsf{Stack}(\mathrm{Can}\, C^\sim) & \xrightarrow{\mathsf{St}(\breve{u}^*)_*} & \mathsf{Stack}(\mathrm{Can}\, C'^\sim) \\ {\scriptstyle \mathsf{St}(h^a_C)_*}\downarrow & & \downarrow{\scriptstyle \mathsf{St}(h^a_{C'})_*} \\ \mathsf{Stack}(C) & \xrightarrow{\mathsf{St}(\breve{u})_*} & \mathsf{Stack}(C').\end{array}$$

(ii) *If u is also a weak morphism of sites $C' \to C$, we have a pseudo-natural equivalence*

$$\mathsf{St}(\breve{u})^* \stackrel{\sim}{\to} \mathsf{St}(u)_*.$$

(iii) *If u admits a right adjoint $v : \mathscr{C}' \to \mathscr{C}$ that is a weak morphism of sites $C \to C'$, we have pseudo-natural equivalences :*

$$\mathsf{St}(v)_* \stackrel{\sim}{\to} \mathsf{St}(\breve{u})_* \qquad \mathsf{St}(v)^* \stackrel{\sim}{\to} \mathsf{St}(\breve{u})^*.$$

Proof. (i): Since $\mathsf{St}(h^a_C)_*$ and $\mathsf{St}(h^a_{C'})_*$ are 2-equivalences (theorem 5.4.12(i,ii)), it suffices to show the pseudo-commutativity of the left square diagram. To this aim, consider the sites $C^\wedge$ and $C'^\wedge$ on the categories $\mathscr{C}^\wedge$ and $\mathscr{C}'^\wedge$ defined in remark 4.1.31(ii), and notice that since $u^\wedge : \mathscr{C}^\wedge \to \mathscr{C}'^\wedge$ commutes with all limits and all colimits, it induces a morphism of topoi $C'^\wedge \to C^\wedge$ (proposition 4.4.8(i)), whence a morphism of sites $u^\wedge : C'^\wedge) \to C^\wedge)$ (remark 4.4.13(iii)); combining with lemma 4.2.9 and theorem 4.4.2(ii) we deduce an essentially commutative diagram of sites:

$$\begin{array}{ccc}\mathrm{Can}(C'^\sim) & \xrightarrow{\breve{u}^*} & \mathrm{Can}(C^\sim) \\ {\scriptstyle (-)^a_{C'}}\downarrow & & \downarrow{\scriptstyle (-)^a_C} \\ C'^\wedge & \xrightarrow{u^\wedge} & C^\wedge.\end{array}$$

Notice as well that $\mathsf{St}((-)^a_C)_* : \mathsf{Stack}(\mathrm{Can}(C^\sim)) \to \mathsf{Stack}(C^\wedge)$ is a 2-equivalence (proposition 5.4.17(ii) and theorem 4.4.2(iii)), and likewise for $\mathsf{St}((-)^a_{C'})_*$. There follows a pseudo-commutative diagram (details left to the reader) :

$$\begin{array}{ccc} \mathsf{Stack}(\mathrm{Can}(C'^\sim)) & \xrightarrow{\mathsf{St}(\breve{u}^*)^*} & \mathsf{Stack}(\mathrm{Can}(C^\sim)) \\ {\scriptstyle \mathsf{St}((-)^a_{C'})_*}\downarrow & & \downarrow{\scriptstyle \mathsf{St}((-)^a_C)_*} \\ \mathsf{Stack}(C'^\wedge) & \xrightarrow{\mathsf{St}(u^\wedge)^*} & \mathsf{Stack}(C^\wedge). \end{array}$$

Thus, we are reduced to checking that the following diagram pseudo-commutes :

$$\begin{array}{ccccc} \mathsf{Fib}(\mathscr{C}'^\wedge) & \xrightarrow{\mathsf{Fib}(u^\wedge)_!} & \mathsf{Fib}(\mathscr{C}^\wedge) & \xrightarrow{(-)^a_{C^\wedge}} & \mathsf{Stack}(C^\wedge) \\ {\scriptstyle \mathsf{Fib}(h_{\mathscr{C}'})^*}\downarrow & & {\scriptstyle \mathsf{Fib}(h_{\mathscr{C}})^*}\downarrow & & \downarrow{\scriptstyle \mathsf{St}(h_{\mathscr{C}})_*} \\ \mathsf{Fib}(\mathscr{C}') & \xrightarrow{\mathsf{Fib}(u)^*} & \mathsf{Fib}(\mathscr{C}) & \xrightarrow{(-)^a_C} & \mathsf{Stack}(C). \end{array}$$

However, the pseudo-commutativity of the left square subdiagram follows from proposition 3.3.19. For the right subdiagram, it suffices to observe that $h_{\mathscr{C}}$ is cocontinuous for the sites C and $C^\wedge$ (theorem 4.4.2(i)) and invoke proposition 5.4.1(i).

(ii) follows by inspecting the definitions, and the first pseudo-natural equivalence of (iii) follows from remark 3.3.15(vi); then the second pseudo-natural equivalence of (iii) follows from the first one and from remark 2.4.28(iii). □

5.4.30. Let $C := (\mathscr{C}, J)$ and $C' := (\mathscr{C}', J')$ be two U-sites, and $u : C' \to C$ a weak morphism of sites. We have a commutative diagram of 2-categories :

$$\mathscr{D}_u \quad : \quad \begin{array}{ccc} \mathsf{Stack}(C') & \xrightarrow{i_{C'}} & \mathsf{Fib}(\mathscr{C}') \\ {\scriptstyle \mathsf{St}(u)_*}\downarrow & & \downarrow{\scriptstyle \mathsf{Fib}(u)^*} \\ \mathsf{Stack}(C) & \xrightarrow{i_C} & \mathsf{Fib}(\mathscr{C}) \end{array}$$

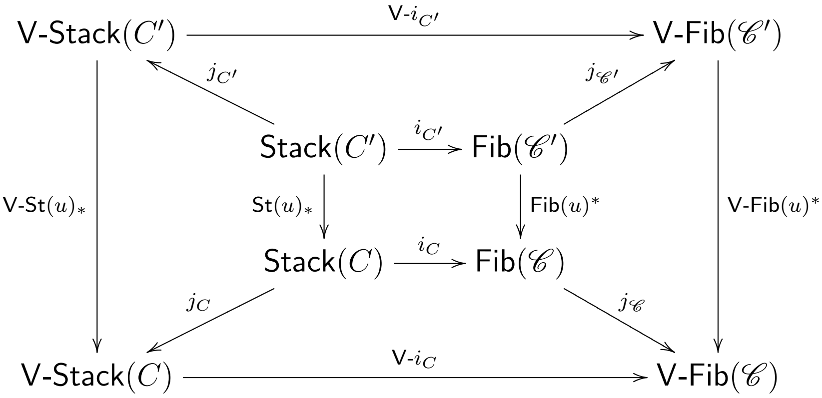
where the inclusion pseudo-functors i_C and $i_{C'}$ admit left 2-adjoints $(-)^a_C : \mathsf{Fib}(\mathscr{C}) \to \mathsf{Stack}(C)$ and $(-)^a_{C'} : \mathsf{Fib}(\mathscr{C}') \to \mathsf{Stack}(C')$ (corollary 5.4.19). After fixing an adjunction for these two pairs of 2-adjoint pseudo-functors, we may then regard $\mathscr{D}_u$ as a square of weak links as in (2.3.19), oriented by the identity pseudo-natural transformation. If we similarly fix adjunctions also for the pairs $(\mathsf{Fib}(u)_!, \mathsf{Fib}(u)^*)$ and $(\mathsf{St}(u)^*, \mathsf{St}(u)_*)$, the diagram $\mathscr{D}_u$ becomes even an oriented square of links.

Corollary 5.4.31. *In the situation of* (5.4.30)*, suppose that the functor $u : \mathscr{C} \to \mathscr{C}'$ is also cocontinuous for the topologies J and J'. Then the base change transformation*

$$\Upsilon(\mathscr{D}_u) : (-)^a_C \circ \mathsf{Fib}(u)^* \to \mathsf{St}(u)_* \circ (-)^a_{C'}$$

is a pseudo-natural equivalence.

Proof. Let V be a universe with $\mathsf{U} \subset \mathsf{V}$, and such that $\mathscr{C}$ and $\mathscr{C}'$ are V-small; we get a commutative diagram of 2-categories :



whose horizontal arrows are the inclusion pseudo-functors; since the latter all admit left 2-adjoints, we may regard this diagram as an oriented cubical diagram of weak links as in 2.3.21, whose orientations are given by identities. Especially, the lower and upper trapezoidal subdiagrams are oriented respectively by $\mathbf{1}_{j_{\mathscr{C}} \circ i_C}$ and $\mathbf{1}_{j_{\mathscr{C}'} \circ i_{C'}}$, and we remark :

Claim 5.4.32. $\Upsilon(\mathbf{1}_{j_{\mathscr{C}} \circ i_C})$ and $\Upsilon(\mathbf{1}_{j_{\mathscr{C}'} \circ i_{C'}})$ are pseudo-natural equivalences.

Proof of the claim. We show the assertion for $\Upsilon(\mathbf{1}_{j_{\mathscr{C}} \circ i_C})$: the same argument shall apply to $\Upsilon(\mathbf{1}_{j_{\mathscr{C}'} \circ i_{C'}})$ as well. Let (η, ε) (resp. $(\mathsf{V}\text{-}\eta, \mathsf{V}\text{-}\varepsilon)$) be the unit and counit of a 2-adjunction for the 2-adjoint pair $((-)^a_C, i_C)$ (resp. for the 2-adjoint pair $(\mathsf{V}\text{-}(-)^a_C, \mathsf{V}\text{-}i_C)$); since $\mathsf{V}\text{-}i_C$ is fully faithful, $\mathsf{V}\text{-}\varepsilon$ is a pseudo-natural equivalence, so it suffices to check that the same holds for $(\mathsf{V}\text{-}(-)^a_C \circ j_{\mathscr{C}}) * \eta$. But in view of the pseudo-natural equivalence $\mathsf{V}\text{-}(-)^a_C \circ j_{\mathscr{C}} \overset{\sim}{\to} j_C \circ (-)^a_C$, we are further reduced to checking that $(-)^a_C * \eta$ is a pseudo-natural equivalence; the latter follows from the triangular identities for the pair (η, ε), since ε is a pseudo-natural equivalence. ◇

Denote by $\mathsf{V}\text{-}\mathscr{D}_u$ the "front face" of the foregoing diagram (with its identity orientation); from claim 5.4.32 and remark 2.3.22(i) we see that if $\Upsilon(\mathsf{V}\text{-}\mathscr{D}_u)$ is a pseudo-natural equivalence, the same holds for $\Upsilon(\mathscr{D}_u)$. Thus, we may replace U by V, and assume from start that $\mathscr{C}$ and $\mathscr{C}'$ are small. Let now $\mathscr{F}$ be any fibration on $\mathscr{C}'$; the functor $\Upsilon(\mathscr{D}_u)_{\mathscr{F}} : (\mathsf{Fib}(u)^*\mathscr{F})^a \to \mathsf{St}(u)_*(\mathscr{F}^a)$ is obtained explicitly as follows. First, let $\eta_{\mathscr{F}} : \mathscr{F} \to \mathscr{F}^a$ be the unit of the chosen 2-adjunction for the pair $((-)^a_{C'}, i_{C'})$. Then $\Upsilon(\mathscr{D}_u)_{\mathscr{F}}$ is the composition of the induced functor

$$(\mathsf{Fib}(u)^*\eta_{\mathscr{F}})^a : (\mathsf{Fib}(u)^*\mathscr{F})^a \to (\mathsf{St}(u)_*(\mathscr{F}^a))^a$$

with the counit of adjunction $\varepsilon_{\mathscr{F}^a} : (\mathsf{St}(u)_*(\mathscr{F}^a))^a \to \mathsf{St}(u)_*(\mathscr{F}^a)$. We need to check that $\Upsilon(\mathscr{D}_u)_{\mathscr{F}}$ is an equivalence; however, $\varepsilon_{\mathscr{F}^a}$ is an equivalence, so it suffices to check that the same holds for $(\mathsf{Fib}(u)^*\eta_{\mathscr{F}})^a$. By propositions 5.2.9 and 5.2.11(iii) we are reduced to showing that $\mathsf{Fib}(u)^*(\eta_{\mathscr{F}}) : \mathsf{Fib}(u)^*\mathscr{F} \to \mathsf{St}(u)_*(\mathscr{F}^a)$ is i-covering for $i = 0, 1, 2$. Since u is cocontinuous, the latter assertion follows from propositions 5.4.1 and 5.2.11(i). □

5.5. **Sheaves of categories.** We wish next to elucidate the relationship between stacks and the related notion of sheaf of categories on a given site. We begin by recalling the following :

Definition 5.5.1. Let $C := (\mathscr{C}, J)$ be any site, and $\mathscr{A}$ any other category.

(i) A *presheaf on* $\mathscr{C}$ *with values in* $\mathscr{A}$ is any object of the category (notation of (1.1.10))

$$(\mathscr{C}, \mathscr{A})^\wedge := \mathsf{Fun}(\mathscr{C}^o, \mathscr{A}).$$

(ii) Let V be a universe such that $\mathscr{A}$ has V-small Hom-sets; following [59, Ch.0, §3.1], we say that such a presheaf $\mathscr{F}$ is a *sheaf on* C *with values in* $\mathscr{A}$ if for every $T \in \mathrm{Ob}(\mathscr{A})$, the rule :

$$U \mapsto \mathscr{F}_T(U) := \mathrm{Hom}_{\mathscr{A}}(T, \mathscr{F}(U)) \qquad \text{for every } U \in \mathrm{Ob}(\mathscr{C})$$

defines a V-sheaf (of sets) $\mathscr{F}_T$ on C. This condition is obviously independent of the choice of V. We denote by

$$(C,\mathscr{A})^\sim$$

the full subcategory of $(\mathscr{C},\mathscr{A})^\wedge$ whose objects are the sheaves on C with values in $\mathscr{A}$.

Remark 5.5.2. (i) Let $C := (\mathscr{C}, J)$ be a site and $\mathscr{F}$ a presheaf on $\mathscr{C}$ with values in a category $\mathscr{A}$. According to (4.1.6), the presheaf $\mathscr{F}$ is a sheaf on C with values in $\mathscr{A}$ if and only if, for every $T \in \mathrm{Ob}(\mathscr{A})$, every $U \in \mathrm{Ob}(\mathscr{C})$, and every sieve $\mathscr{S} \in J(U)$ the natural map

$$\mathscr{F}_T(U) \to \lim_{\mathscr{S}^o} \mathscr{F}_T \circ \mathsf{s}^o_{\mathscr{S}}$$

is bijective (where the limit is taken in a sufficiently large universe V). The latter in turns is equivalent to the following. For every U and $\mathscr{S}$ as in the foregoing, the induced cone :

$$(\mathscr{F}(f) : \mathscr{F}(U) \to \mathscr{F}(U') \mid (f : U' \to U) \in \mathrm{Ob}(\mathscr{S}))$$

is universal in $\mathscr{A}$. Especially, a sheaf on C with values in $\mathbf{Set}$ is just a usual sheaf (of sets).

In case $\mathscr{C}$ is small, $\mathscr{A}$ is complete, and all the fibre products in $\mathscr{C}$ are representable, arguing similarly we see that $\mathscr{F}$ is a sheaf on C with values in $\mathscr{A}$ if and only if, for every $U \in \mathrm{Ob}(\mathscr{C})$ and every small family $(U_i \to U \mid i \in I)$ of objects of $\mathscr{C}/U$ that generate a sieve covering U, the restriction morphisms $\mathscr{F}(U) \to \mathscr{F}(U_i)$ induce an isomorphism

$$\mathscr{F}(U) \xrightarrow{\sim} \mathrm{Equal}\Big(\prod_{i\in I} \mathscr{F}(U_i) \rightrightarrows \prod_{(i,j)\in I\times I} \mathscr{F}(U_i \times_U U_j)\Big) \qquad \text{in } \mathscr{A}.$$

(ii) Let $\mathscr{A}'$ be another category, and $F : \mathscr{A} \to \mathscr{A}'$ a functor that commutes with limits; it follows from (i) that $(\mathscr{C}, F)^\wedge := \mathsf{Fun}(\mathscr{C}^o, F) : (\mathscr{C},\mathscr{A})^\wedge \to (\mathscr{C},\mathscr{A}')^\wedge$ restricts to a functor

$$(C, F)^\sim : (C,\mathscr{A})^\sim \to (C,\mathscr{A}')^\sim.$$

(iii) Suppose that $\mathscr{A}$ is complete and V-small for a universe V such that $\mathsf{U} \subset \mathsf{V}$. Consider a functor

$$\mathscr{F}_\bullet : \Lambda \to (C,\mathscr{A})^\sim \qquad \lambda \mapsto \mathscr{F}_\lambda$$

from a small category Λ. Recall that the limit L of $\mathscr{F}_\bullet$ in $(\mathscr{C},\mathscr{A})^\wedge$ is computed argumentwise, *i.e.* $L(U)$ represents the limit of the induced functor

$$\mathscr{F}_\bullet(U) : \Lambda \to \mathscr{A} \qquad \lambda \mapsto \mathscr{F}_\lambda(U)$$

for every $U \in \mathrm{Ob}(\mathscr{C})$ (corollary 1.4.1(ii)). We claim that L is a sheaf on C with values in $\mathscr{A}$, and therefore it represents the limit of $\mathscr{F}_\bullet$ in $(C,\mathscr{A})^\sim$; especially, the latter category is complete. Indeed, for any $T \in \mathrm{Ob}(\mathscr{A})$ we have natural identifications

$$\mathrm{Hom}_{\mathscr{A}}(T, L(U)) \xrightarrow{\sim} \lim_{\lambda\in\mathrm{Ob}(\Lambda)} \mathrm{Hom}_{\mathscr{A}}(T, \mathscr{F}_\lambda(U)) = \lim_{\lambda\in\mathrm{Ob}(\Lambda)} \mathscr{F}_{\lambda,T}(U) \xrightarrow{\sim} \Big(\lim_{\lambda\in\mathrm{Ob}(\Lambda)} \mathscr{F}_{\lambda,T}\Big)(U)$$

and the limit of the functor $\mathscr{F}_{\bullet,T}$ from Λ to the category of V-presheaves (of sets) on $\mathscr{C}$ is a V-sheaf, since by assumption every $\mathscr{F}_{\lambda,T}$ is a V-sheaf (remark 4.1.19(i)). This shows that L_T is a V-sheaf on C for every $T \in \mathrm{Ob}(\mathscr{A})$, as required.

(iv) Let $C' := (\mathscr{C}', J')$ be another site, and $u : \mathscr{C} \to \mathscr{C}'$ a continuous functor for the topologies J and J'. Then u induces a functor

$$(u,\mathscr{A})^\wedge := \mathsf{Fun}(u^o,\mathscr{A}) : (\mathscr{C}',\mathscr{A})^\wedge \to (\mathscr{C},\mathscr{A})^\wedge$$

and notice that for every presheaf $\mathscr{F}'$ on $\mathscr{C}'$ with values in $\mathscr{A}$, and every $T \in \mathrm{Ob}(\mathscr{A})$ we have

$$u^\wedge(\mathscr{F}'_T) = ((u,\mathscr{A})^\wedge \mathscr{F}')_T. \tag{5.5.3}$$

It follows that $(u,\mathscr{A})^\wedge$ restricts to a functor

$$(\widetilde{u},\mathscr{A})_* : (C',\mathscr{A})^\sim \to (C,\mathscr{A})^\sim.$$

Likewise, every natural transformation $\alpha : u \Rightarrow v$ between such continuous functors $u, v : \mathscr{C} \to \mathscr{C}'$ induces a natural transformation

$$(\alpha, \mathscr{A})^\wedge := \mathsf{Fun}(\alpha^o, \mathscr{A}) : (v, \mathscr{A})^\wedge \Rightarrow (u, \mathscr{A})^\wedge$$

which yields by restriction a natural transformation

$$(\widetilde{\alpha}, \mathscr{A})_* : (v, \mathscr{A})^\sim_* \Rightarrow (u, \mathscr{A})^\sim_*.$$

(v) For every $U \in \mathrm{Ob}(\mathscr{C})$, the topology J induces a topology J_U on $\mathscr{C}/U$ (see (4.7.2)), and we let $C/U := (\mathscr{C}/U, J_U)$. The source functor $\mathsf{s}_U : \mathscr{C}/U \to \mathscr{C}$ is continuous for the topologies J and J_U, hence (iv) yields a well defined functor

$$(\widetilde{\mathsf{s}}_U, \mathscr{A})_* : (C, \mathscr{C})^\sim \to (C/U, \mathscr{C})^\sim \qquad \mathscr{F} \mapsto \mathscr{F}_{|U}.$$

Likewise, if $g : U \to V$ is any morphism of $\mathscr{C}$, the corresponding functor $g_* : \mathscr{C}/U \to \mathscr{C}/V$ is continuous for the topologies J_U and J_V, so it restricts to a functor

$$(\widetilde{g}, \mathscr{A})_* : (C/V, \mathscr{A})^\sim \to (C/U, \mathscr{A})^\sim$$

generalizing remark 4.7.3(i). We may then consider the category

$$(C/\bullet, \mathscr{A})^\sim$$

whose objects are the pairs $(U, \mathscr{F})$ consisting of objects $U \in \mathrm{Ob}(\mathscr{C})$ and $\mathscr{F} \in \mathrm{Ob}(C/U, \mathscr{A})^\sim$. A morphism $(U, \mathscr{F}) \to (V, \mathscr{G})$ is the datum of a morphism $g : U \to V$ of $\mathscr{C}$ and a morphism $\mathscr{F} \to (\widetilde{g}, \mathscr{A})_* \mathscr{G}$ in $(C/U, \mathscr{A})^\sim$. It follows easily from (5.5.3) that the resulting functor

$$(C/\bullet, \mathscr{A})^\sim \to \mathscr{C} \qquad (U, \mathscr{F}) \mapsto U$$

is a fibration. Moreover, for every $U \in \mathrm{Ob}(\mathscr{C})$, every covering family $(U_i \to U \mid i \in I)$ for J_U generates a sieve of 1-descent for this fibration, which is even of 2-descent, if $\mathscr{A}$ is complete and $\mathscr{C}$ is small.

5.5.4. In this section we are mainly interested in presheaves and sheaves with values in the category $\mathbf{Cat}$ of small categories, but the following auxiliary construction shall also be useful. For every category $\mathscr{B}$ *whose fibred products are representable* we consider the 2-category

$$\mathbf{Cat}^*(\mathscr{B})$$

of *category objects* of $\mathscr{B}$, whose objects are the data $\mathscr{C}^* := (O, M, \mathsf{s}, \mathsf{t}, \mathbb{1}, \mathsf{c})$ such that $O, M \in \mathrm{Ob}(\mathscr{B})$, and

(5.5.5) $$M \overset{\mathsf{s}}{\underset{\mathsf{t}}{\rightrightarrows}} O \qquad \mathbb{1} : O \to M \qquad M \times_{(\mathsf{t},\mathsf{s})} M \xrightarrow{\mathsf{c}} M$$

are morphisms in $\mathscr{B}$ called respectively the *source*, *target*, *identity* and *composition laws* of $\mathscr{C}^*$, fulfilling the identities :

$$\mathsf{s} \circ \mathbb{1} = \mathbf{1}_O = \mathsf{t} \circ \mathbb{1}$$

and making commute the diagrams of morphisms of $\mathscr{B}$:

$$\begin{array}{ccc} M \times_{(\mathsf{t},\mathsf{s})} M \times_{(\mathsf{t},\mathsf{s})} M & \xrightarrow{\mathbf{1}_M \times \mathsf{c}} & M \times_{(\mathsf{t},\mathsf{s})} M \\ {\scriptstyle \mathsf{c} \times \mathbf{1}_M} \downarrow & & \downarrow {\scriptstyle \mathsf{c}} \\ M \times_{(\mathsf{t},\mathsf{s})} M & \xrightarrow{\mathsf{c}} & M \end{array} \qquad \begin{array}{ccc} M & \xrightarrow{(\mathbf{1}_M, \mathbb{1} \circ \mathsf{t})} & M \times_{(\mathsf{t},\mathsf{s})} M \\ {\scriptstyle (\mathbb{1} \circ \mathsf{s}, \mathbf{1}_M)} \downarrow & \searrow^{\mathbf{1}_M} & \downarrow {\scriptstyle \mathsf{c}} \\ M \times_{(\mathsf{t},\mathsf{s})} M & \xrightarrow{\mathsf{c}} & M. \end{array}$$

Notice that the datum of c includes the choice of a representative for the fibre product $M \times_{(\mathsf{t},\mathsf{s})} M$, and we (implicitly) fix as well a universal cone $M \leftarrow M \times_{(\mathsf{t},\mathsf{s})} M \to M$ for this fibre product.

The 1-cells $(F_1, F_2) : \mathscr{C}^* := (O, M, \mathsf{s}, \mathsf{t}, \mathbb{1}, \mathsf{c}) \to \mathscr{C}'^* := (O', M', \mathsf{s}', \mathsf{t}', \mathbb{1}', \mathsf{c}')$ in $\mathbf{Cat}^*(\mathscr{B})$ are the pairs of morphisms $(F_1 : O \to O', F_2 : M \to M')$ of $\mathscr{B}$ such that

$$\mathsf{s}' \circ F_2 = F_1 \circ \mathsf{s} \qquad \mathsf{t}' \circ F_2 = F_1 \circ \mathsf{t} \qquad \mathbb{1}' \circ F_1 = F_2 \circ \mathbb{1} \qquad \mathsf{c}' \circ (F_2 \times_{(\mathsf{t},\mathsf{s})} F_2) = F_2 \circ \mathsf{c}.$$

The composition law of 1-cells is given by the obvious rule :

$(F'_1, F'_2) \circ (F_1, F_2) := (F'_1 \circ F_1, F'_2 \circ F_2)$ for every pair of 1-cells $\mathscr{C}^* \xrightarrow{(F_1,F_2)} \mathscr{C}'^* \xrightarrow{(F'_1,F'_2)} \mathscr{C}''^*$.

For every pair of category objects $\mathscr{C}^*$ and $\mathscr{C}'^*$ of $\mathscr{B}$, and every pair of 1-cells $(F_1, F_2), (G_1, G_2) : \mathscr{C}^* \to \mathscr{C}'^*$, the 2-cells $\beta : (F_1, F_2) \Rightarrow (G_1, G_2)$ are the morphisms $\beta : O \to M'$ of $\mathscr{B}$ such that

$$(5.5.6) \qquad \mathsf{s}' \circ \beta = F_1 \qquad \mathsf{t}' \circ \beta = G_1 \qquad \mathsf{c}' \circ (\beta \circ \mathsf{s}, G_2) = \mathsf{c}' \circ (G_1, \beta \circ \mathsf{t})$$

(where $(\beta \circ \mathsf{s}, G_2)$ and $(G_1, \beta \circ \mathsf{t})$ are morphisms $M \to M' \times_{(\mathsf{t}',\mathsf{s}')} M'$).

Remark 5.5.7. (i) We have a natural strict and strong 2-equivalence of 2-categories :

$$(5.5.8) \qquad \mathsf{V}\text{-}\mathbf{Cat} \xrightarrow{\sim} \mathbf{Cat}^*(\mathsf{V}\text{-}\mathbf{Set}) \qquad \text{for every universe } \mathsf{V}.$$

Namely, to each V-small category $\mathscr{C}$ we assign the object $[\mathscr{C}] := (\mathrm{Ob}(\mathscr{C}), \mathrm{Morph}(\mathscr{C}), \mathsf{s}, \mathsf{t}, \mathbb{1}, \mathsf{c})$ where $\mathsf{s}, \mathsf{t}, \mathbb{1}, \mathsf{c}$ encode the source, target, identity and composition laws for $\mathscr{C}$ in the obvious way. Then any functor $F : \mathscr{C} \to \mathscr{C}'$ induces an obvious 1-cell $[F] : [\mathscr{C}] \to [\mathscr{C}']$, and every natural transformation $\beta : F \Rightarrow F'$ induces a 2-cell $[\beta] : [F_1] \Rightarrow [F_2]$. The quasi-inverse assigns to every datum $\mathscr{C}^* := (O, M, \mathsf{s}, \mathsf{t}, \mathbb{1}, \mathsf{c})$ the category $[\mathscr{C}^*]$ with $\mathrm{Ob}([\mathscr{C}^*]) := O$ and $\mathrm{Hom}_{[\mathscr{C}^*]}(X, Y) := \mathsf{s}^{-1}(X) \cap \mathsf{t}^{-1}(Y)$ for every $X, Y \in O$, with the composition law induced by c in the obvious way, and with $\mathbf{1}_X := \mathbb{1}(X)$ for every $X \in O$. Then every 1-cell $(F_1, F_2) : \mathscr{C}^* \to \mathscr{C}'^*$ induces a functor $[F_1, F_2] : [\mathscr{C}^*] \to [\mathscr{C}'^*]$ and every 2-cell $\beta : (F_1, F_2) \Rightarrow (G_1, G_2)$ induces a natural transformation $[\beta] : [F_1, F_2] \Rightarrow [G_1, G_2]$, in the obvious fashion.

(ii) We have obvious *objects* and *morphisms* functors :

$$\mathsf{V}\text{-}\mathbf{Cat} \underset{\mathrm{Morph}}{\overset{\mathrm{Ob}}{\rightrightarrows}} \mathsf{V}\text{-}\mathbf{Set} \qquad \text{for every universe } \mathsf{V}$$

which – in terms of the 2-equivalence (5.5.8) – translate as the functors $\mathbf{Cat}^*(\mathsf{V}\text{-}\mathbf{Set}) \to \mathsf{V}\text{-}\mathbf{Set}$ that extract from each datum $(O, M, \mathsf{s}, \mathsf{t}, \mathbb{1}, \mathsf{c})$ the set O and respectively the set M, and that assign to every morphism (F_1, F_2) of $\mathbf{Cat}^*(\mathsf{V}\text{-}\mathbf{Set})$ the map F_1 and respectively the map F_2.

(iii) Notice that both Ob and Morph are representable functors. Indeed, let $\mathbb{1}$ be the category with one object and one morphism, and let $\mathbb{2}$ be the category with exactly two objects a and b, and a single morphism from a to b (and no morphisms from b to a). Then it is easily seen that $\mathbb{1}$ (resp. $\mathbb{2}$) represents the presheaf Ob (resp. Morph) on $\mathsf{V}\text{-}\mathbf{Cat}^o$: the details shall be left to the reader. Especially, Ob and Morph commute with all the limits of $\mathsf{V}\text{-}\mathbf{Cat}$ (example 1.3.15(ii)).

Moreover, obviously a functor $F : \mathscr{A} \to \mathscr{A}'$ is an isomorphism in $\mathsf{V}\text{-}\mathbf{Cat}$ if and only if both $\mathrm{Ob}(F) : \mathrm{Ob}(\mathscr{A}) \to \mathrm{Ob}(\mathscr{A}')$ and $\mathrm{Morph}(F) : \mathrm{Morph}(\mathscr{A}) \to \mathrm{Morph}(\mathscr{A}')$ are bijections.

(iv) Taking into account (iii) and remark 5.5.2(ii), it follows immediately that a presheaf of V-small categories F on $\mathscr{C}$ is a sheaf of categories on C if and only if $\mathrm{Ob}(F) := \mathrm{Ob} \circ F$ and $\mathrm{Morph}(F) := \mathrm{Morph} \circ F$ are sheaves (of sets) on C.

(v) For every site $C := (\mathscr{C}, J)$, the categories $(\mathscr{C}, \mathsf{V}\text{-}\mathbf{Cat})^\wedge$ and $(C, \mathsf{V}\text{-}\mathbf{Cat})^\sim$ inherit from $\mathsf{V}\text{-}\mathbf{Cat}$ a natural structure of 2-category. Namely, $(\mathscr{C}, \mathsf{V}\text{-}\mathbf{Cat})^\wedge$ can be described as the sub-2-category of of $\mathsf{PsFun}(\mathscr{C}, \mathsf{V}\text{-}\mathbf{Cat})$ whose objects are the strict pseudo-functors, whose 1-cells are all the strict pseudo-natural transformations (where $\mathscr{C}$ is regarded as usual, as a 2-category with trivial 2-cells), and whose 2-cells are all the modifications $\beta \rightsquigarrow \beta'$, for every pair of 1-cells $\beta, \beta' : F \Rightarrow F'$ of $(\mathscr{C}, \mathsf{V}\text{-}\mathbf{Cat})^\wedge$. Similarly we describe $(C, \mathsf{V}\text{-}\mathbf{Cat})^\sim$ as the strong sub-2-category of $(\mathscr{C}, \mathsf{V}\text{-}\mathbf{Cat})^\wedge$ whose objects are the sheaves of V-small categories on the site C (see definition 2.4.9(iii)). Then (5.5.8) generalizes to strict and strong 2-equivalences

$$(5.5.9) \qquad (\mathscr{C}, \mathsf{V}\text{-}\mathbf{Cat})^\wedge \xrightarrow{\sim} \mathbf{Cat}^*(\mathscr{C}^\wedge_{\mathsf{V}}) \qquad (C, \mathsf{V}\text{-}\mathbf{Cat})^\sim \xrightarrow{\sim} \mathbf{Cat}^*(C^\sim_{\mathsf{V}}).$$

(vi) Let $\mathscr{B}$ and $\mathscr{B}'$ be any two categories whose fibre products are representable, and $u : \mathscr{B} \to \mathscr{B}'$ any functor *that commutes with fibre products*. Then u induces a strict pseudo-functor

$$\mathbf{Cat}^*(u) : \mathbf{Cat}^*(\mathscr{B}) \to \mathbf{Cat}^*(\mathscr{B}') \qquad (O, M, \mathsf{s}, \mathsf{t}, \mathbb{1}, \mathsf{c}) \mapsto (u(0), u(M), u(\mathsf{s}), u(\mathbb{1}), u(\mathsf{c})).$$

Indeed, we can regard $u(\mathsf{c})$ as a morphism $u(M) \times_{(u(\mathsf{t}),u(\mathsf{s}))} u(M) \to u(M)$ (since u commutes with fibre products), and then it is clear that the rule defining $\mathbf{Cat}^*(u)$ takes objects of $\mathbf{Cat}^*(\mathscr{B})$ to objects of $\mathbf{Cat}^*(\mathscr{B}')$. To every 1-cell (F_1, F_2) and every 2-cell β, the pseudo-functor $\mathbf{Cat}^*(u)$ assigns likewise respectively the 1-cell $(u(F_1), u(F_2))$ and the 2-cell $u(\beta)$ of $\mathbf{Cat}^*(\mathscr{B}')$.

If $v : \mathscr{B} \to \mathscr{B}'$ is another functor commuting with fibre products, and $\alpha : u \Rightarrow v$ is any natural transformation, we get a strict pseudo-natural transformation

$$\mathbf{Cat}^*(\alpha) : \mathbf{Cat}^*(u) \Rightarrow \mathbf{Cat}^*(v) \qquad (O, M, \mathsf{s}, \mathsf{t}, \mathbb{1}, \mathsf{c}) \mapsto (\alpha_O, \alpha_M).$$

Indeed, notice that $u(M \times_{(\mathsf{t},\mathsf{s})} M)$ represents $u(M) \times_{(u(\mathsf{t}),u(\mathsf{s}))} u(M)$ since u commutes with fibre products, and likewise for $v(M \times_{(\mathsf{t},\mathsf{s})} M)$, and under these identifications, the morphism $\alpha_{M\times_{(\mathsf{t},\mathsf{s})}M}$ corresponds to $\alpha_M \times_{(u(\mathsf{t}),u(\mathsf{s}))} \alpha_M$, whence it is easily seen that (α_O, α_M) is a 1-cell of $\mathbf{Cat}^*(\mathscr{B}')$, and the strict pseudo-functoriality of $\mathbf{Cat}^*(\alpha)$ is then immediate from the definition.

(vii) Clearly the rules $u \mapsto \mathbf{Cat}^*(u)$ and $\alpha \mapsto \mathbf{Cat}^*(\alpha)$ define a strict pseudo-functor from the sub-2-category of $\mathbf{Cat}$ whose objects are all the small categories whose fibre products are representable, whose 1-cells are the functor commuting with fibre products and whose 2-cells are the natural transformations, to the 2-category whose objects are the small 2-categories, whose 1-cells are the strict pseudo-functors, and whose 2-cells are the strict pseudo-natural transformations.

5.5.10. Let now $C := (\mathscr{C}, J)$ be a U-site; then we deduce that the strongly faithful inclusion pseudo-functor $(C, \mathsf{V}\text{-}\mathbf{Cat})^\sim \to (\mathscr{C}, \mathsf{V}\text{-}\mathbf{Cat})^\wedge$ admits a strong left 2-adjoint :

$$(\mathscr{C}, \mathsf{V}\text{-}\mathbf{Cat})^\wedge \to (C, \mathsf{V}\text{-}\mathbf{Cat})^\sim \qquad F \mapsto F^a \tag{5.5.11}$$

for every universe V with $\mathsf{U} \subset \mathsf{V}$. Namely, by remark 5.5.7(vi,vii), the exact left adjoint $(-)^a : \mathscr{C}^\wedge_\mathsf{V} \to C^\sim_\mathsf{V}$ provided by remark 4.1.23(ii) induces first a strong left 2-adjoint $\mathbf{Cat}^*((-)^a) : \mathbf{Cat}^*(\mathscr{C}^\wedge_\mathsf{V}) \to \mathbf{Cat}^*(C^\sim_\mathsf{V})$ to the inclusion of 2-categories $\mathbf{Cat}^*(C^\sim_\mathsf{V}) \to \mathbf{Cat}^*(C^\wedge_\mathsf{V})$. Then, combining with the strong and strict 2-equivalences (5.5.9) we get the sought strong left 2-adjoint. It is also easily seen that (5.5.11) is an exact functor on the underlying categories.

5.5.12. Let $C := (\mathscr{C}, J)$ and $C' := (\mathscr{C}', J')$ be two U-sites, $u : \mathscr{C} \to \mathscr{C}'$ a functor, U' a universe with $\mathsf{U} \subset \mathsf{U}'$ and such that $\mathscr{C}$ and $\mathscr{C}'$ are U'-small and the functor $u^a_{\mathsf{U}'!} : \mathscr{C}^\wedge_{\mathsf{U}'} \to C'^\sim_{\mathsf{U}'}$ commutes with fibre products. Arguing as in (5.5.10), and in light of remark 4.2.18(iii), we deduce that for every universe V with $\mathsf{U} \subset \mathsf{V}$, the pseudo-functor $u^a_{\mathsf{V}!}$ induces a pseudo-functor

$$(u, \mathsf{V}\text{-}\mathbf{Cat})^a_! : (\mathscr{C}, \mathsf{V}\text{-}\mathbf{Cat})^\wedge \to (C', \mathsf{V}\text{-}\mathbf{Cat})^\sim.$$

In case u is continuous for the topologies J and J', corollary 4.3.19(i) and remark 5.5.7(vi,vii) easily imply that $(u, \mathsf{V}\text{-}\mathbf{Cat})^a_!$ restricts to a strong and strict left 2-adjoint

$$(\widetilde{u}, \mathsf{V}\text{-}\mathbf{Cat})^* : (C, \mathsf{V}\text{-}\mathbf{Cat})^\sim \to (C', \mathsf{V}\text{-}\mathbf{Cat})^\sim$$

for the strict pseudo-functor $(\widetilde{u}, \mathsf{V}\text{-}\mathbf{Cat})_*$ of remark 5.5.2(iv).

5.5.13. Let $C := (\mathscr{C}, J)$ and $C' := (\mathscr{C}', J')$ be two U-sites, $u : \mathscr{C} \to \mathscr{C}'$ a cocontinuous functor, and V a universe with $\mathsf{U} \subset \mathsf{V}$. Arguing as in (5.5.10) we see that the adjoint pair of left exact functors $(\breve{u}^*_\mathsf{V}, \breve{u}_{\mathsf{V}*})$ provided by corollary 4.3.19(ii) induces a strict and strong 2-adjoint pair of strict pseudo-functors

$$(C', \mathsf{V}\text{-}\mathbf{Cat})^\sim \underset{(\breve{u}, \mathsf{V}\text{-}\mathbf{Cat})_*}{\overset{(\breve{u}, \mathsf{V}\text{-}\mathbf{Cat})^*}{\rightleftarrows}} (C, \mathsf{V}\text{-}\mathbf{Cat})^\sim.$$

Remark 5.5.14. (i) In the situation of (5.5.12), suppose that u is a continuous functor and $\widetilde{u}_{\mathsf{V}*}$ is an equivalence, so that $\widetilde{u}^*_\mathsf{V}$ is its quasi-inverse. From the explicit description of the functors $(\widetilde{u}, \mathsf{V}\text{-}\mathbf{Cat})_*$ and $(\widetilde{u}, \mathsf{V}\text{-}\mathbf{Cat})^*$, we then deduce straightforwardly that the latter are mutually quasi-inverse equivalences and strong 2-equivalences as well.

(ii) Combining with theorem 4.4.2(iii), we deduce especially that for every U-site $C := (\mathscr{C}, J)$, the Yoneda embedding $h^a_{\mathscr{C}} : \mathscr{C} \to C^{\sim}_{\mathsf{U}}$ induces a strong 2-equivalence of 2-categories :

$$(\widetilde{h}^a_{\mathscr{C}}, \mathsf{V}\text{-}\mathbf{Cat})_* : (\mathrm{Can}(C^{\sim}_{\mathsf{U}}), \mathsf{V}\text{-}\mathbf{Cat})^{\sim} \xrightarrow{\sim} (C, \mathsf{V}\text{-}\mathbf{Cat})^{\sim}.$$

(iii) Let $C := (\mathscr{C}, J)$ and $C' := (\mathscr{C}', J')$ be two sites such that $\mathscr{C}$ is small and $\mathscr{C}'$ has small Hom-sets, and let $u : \mathscr{C} \to \mathscr{C}'$ be any continuous functor for the topologies J and J'. Recall that for every universe V with $\mathsf{U} \subset \mathsf{V}$, the functor $(u, \mathsf{V}\text{-}\mathbf{Cat})^\wedge$ admits a left adjoint

$$(u, \mathsf{V}\text{-}\mathbf{Cat})_! : (\mathscr{C}, \mathsf{V}\text{-}\mathbf{Cat})^\wedge \to (\mathscr{C}', \mathsf{V}\text{-}\mathbf{Cat})^\wedge$$

computed by left Kan extensions : see theorem 1.3.4. If C' is a U-site, we can compose this functor with the functor (5.5.11), to get a functor $(u, \mathsf{V}\text{-}\mathbf{Cat})^a_! : (\mathscr{C}, \mathsf{V}\text{-}\mathbf{Cat})^\wedge \to (C', \mathsf{V}\text{-}\mathbf{Cat})^{\sim}$, whose restriction to $(\mathscr{C}, \mathsf{V}\text{-}\mathbf{Cat})^{\sim}$ is a left adjoint for $(\widetilde{u}, \mathsf{V}\text{-}\mathbf{Cat})_*$. If C is no longer a small site, but only a U-site, we can pick as usual an essentially small topologically generating full subcategory $\mathscr{G} \subset \mathscr{C}$, endowed with the topology $J_{\mathscr{G}}$ induced by J via the inclusion functor $\jmath : \mathscr{G} \to \mathscr{C}$, so that $(\widetilde{\jmath}, \mathsf{V}\text{-}\mathbf{Cat})_* : (C, \mathsf{V}\text{-}\mathbf{Cat})^{\sim} \to ((\mathscr{G}, J_{\mathscr{G}}), \mathsf{V}\text{-}\mathbf{Cat})^{\sim}$ is an equivalence, by (i); then $(\widetilde{u}, \mathsf{V}\text{-}\mathbf{Cat})_*$ admits the left adjoint $(\widetilde{u \circ \jmath}, \mathsf{V}\text{-}\mathbf{Cat})^* \circ (\widetilde{\jmath}, \mathsf{V}\text{-}\mathbf{Cat})_*$. This yields another construction for the functor underlying the strict pseudo-functor $(\widetilde{u}, \mathsf{V}\text{-}\mathbf{Cat})^*$ of (5.5.12), which is available under weaker assumptions, but is not obviously pseudo-functorial for the 2-category structures of $(C, \mathsf{V}\text{-}\mathbf{Cat})^{\sim}$ and $(C', \mathsf{V}\text{-}\mathbf{Cat})^{\sim}$.

(iv) On the other hand, in the situation of (iii), recall that the strict pseudo-functor

$$\mathsf{PsFun}(u^o, \mathsf{V}\text{-}\mathbf{Cat}) : \mathsf{PsFun}(\mathscr{C}'^o, \mathsf{V}\text{-}\mathbf{Cat}) \to \mathsf{PsFun}(\mathscr{C}^o, \mathsf{V}\text{-}\mathbf{Cat})$$

admits a left 2-adjoint that can be computed by the strong left 2-Kan extension $2\text{-}\int^{u^o}$ along u^o (remark 2.6.18(ii) and theorem 3.3.9). The latter restricts to a strict pseudo-functor

$$(u, \mathsf{V}\text{-}\mathbf{Cat})_! : (\mathscr{C}, \mathsf{V}\text{-}\mathbf{Cat})^\wedge \to (\mathscr{C}', \mathsf{V}\text{-}\mathbf{Cat})^\wedge.$$

Moreover, the 2-adjunction for the pair $(2\text{-}\int^{u^o}, \mathsf{PsFun}(u^o, \mathsf{V}\text{-}\mathbf{Cat}))$ restricts to a natural functor

$$(-)^\dagger_{F,G} : \mathrm{Hom}_{(\mathscr{C}, \mathsf{V}\text{-}\mathbf{Cat})^\wedge}(G \circ u^o, F) \to \mathrm{Hom}_{(\mathscr{C}, \mathsf{V}\text{-}\mathbf{Cat})^\wedge}\Big(G, 2\text{-}\int^{u^o} F\Big)$$

for every presheaf of categories F on $\mathscr{C}$ and G on $\mathscr{C}'$ (remark 2.6.24). But in this generality, though such functors are fully faithful, it is not clear whether they are equivalences of categories.

(v) However, *suppose now additionally that the category $X/u\mathscr{C}$ is cofiltered for every* $X \in \mathrm{Ob}(\mathscr{C}')$ (this condition is fulfilled, notably, in case *C and C' are two lex-sites, and u is a morphism of lex-sites $C' \to C$*, by virtue of example 1.3.16(i)). Recall that for every presheaf of V-small categories F on $\mathscr{C}$ and every $X \in \mathrm{Ob}(\mathscr{C}')$, the value $2\text{-}\int^{u^o} F(X)$ represents the 2-colimit of the strict pseudo-functor $F \circ \mathrm{t}^o_X : (X/u\mathscr{C})^o \to \mathsf{V}\text{-}\mathbf{Cat}$; but then such 2-colimit is represented by the colimit of the same functor, and every universal cocone for such colimit is also a universal pseudo-cocone (example 3.3.13(iv)). This means that the restriction to $(\mathscr{C}, \mathsf{V}\text{-}\mathbf{Cat})^\wedge$ of the strong left 2-Kan extension $2\text{-}\int^{u^o}$ agrees with the left Kan extension $\int^{u^o}$ of theorem 1.3.4; moreover, a direct inspection shows that the functor $(-)^\dagger_{F,G}$ agrees on objects with the standard adjunction for the pair $(\int^{u^o}, (u, \mathsf{V}\text{-}\mathbf{Cat})^\wedge)$, provided by the proof of theorem 1.3.4. *I.e.*, under these assumptions, $(u, \mathsf{V}\text{-}\mathbf{Cat})_!$ is indeed a strong and strict left 2-adjoint for $(u, \mathsf{V}\text{-}\mathbf{Cat})^\wedge$, and we get an essentially commutative diagram of 2-categories :

$$\begin{array}{ccc}
(\mathscr{C}, \mathsf{V}\text{-}\mathbf{Cat})^\wedge & \xrightarrow{(u, \mathsf{V}\text{-}\mathbf{Cat})_!} & (\mathscr{C}', \mathsf{V}\text{-}\mathbf{Cat})^\wedge \\
\downarrow & & \downarrow \\
\mathsf{PsFun}(\mathscr{C}^o, \mathsf{V}\text{-}\mathbf{Cat}) & \xrightarrow{2\text{-}\int^{u^o}} & \mathsf{PsFun}(\mathscr{C}'^o, \mathsf{V}\text{-}\mathbf{Cat})
\end{array}$$

whose vertical arrows are the inclusion strict pseudo-functors (with a different choice of left 2-Kan extension, this diagram is then still at least pseudo-commutative). Lastly, under these same assumptions, we obtain as well a pseudo-commutative diagram :

$$\begin{array}{ccc} (\mathscr{C}, \mathsf{V}\text{-}\mathbf{Cat})^\wedge & \xrightarrow{(u,\mathsf{V}\text{-}\mathbf{Cat})_!} & (\mathscr{C}', \mathsf{V}\text{-}\mathbf{Cat})^\wedge \\ {\scriptstyle \mathscr{F}ib_{\mathscr{C}}}\downarrow & & \downarrow{\scriptstyle \mathscr{F}ib_{\mathscr{C}'}} \\ \mathsf{V}\text{-}\mathsf{Fib}(\mathscr{C}) & \xrightarrow{\mathsf{V}\text{-}\mathsf{Fib}(u)_!} & \mathsf{V}\text{-}\mathsf{Fib}(\mathscr{C}'). \end{array}$$

Arguing as in (iii), we get, again under the current assumptions, an alternative construction of the strict pseudo-functor $(\widetilde{u}, \mathsf{V}\text{-}\mathbf{Cat})^*$, as the restriction to $(\mathscr{C}, \mathsf{V}\text{-}\mathbf{Cat})^\sim$ of the composition of the foregoing pseudo-functor $(u, \mathsf{V}\text{-}\mathbf{Cat})_!$ with the pseudo-functor (5.5.11).

Lemma 5.5.15. *Let $C := (\mathscr{C}, J)$ be any site, and consider a presheaf of categories on $\mathscr{C}$*

$$\mathscr{A}_\bullet : \mathscr{C}^o \to \mathbf{Cat} \qquad X \mapsto \mathscr{A}_X \qquad (f : Y \to X) \mapsto (\mathscr{A}_f : \mathscr{A}_X \to \mathscr{A}_Y).$$

The following holds :

(i) *If $\mathscr{A}_\bullet$ is a sheaf on C, the associated prestack $\mathscr{F}ib(\mathscr{A}_\bullet)$ on C is 1-separated.*
(ii) *Suppose that C is a U-site, and that $\mathscr{F}ib(\mathscr{A}_\bullet)$ is a stack on C, and let $j : \mathscr{A}_\bullet \to \mathscr{A}_\bullet^a$ be the unit of adjunction (notation of (5.5.10)). Then the induced morphism of prestacks $\mathscr{F}ib(j) : \mathscr{F}ib(\mathscr{A}_\bullet) \to \mathscr{F}ib(\mathscr{A}_\bullet^a)$ is an equivalence of categories.*

Proof. (i): By lemma 5.1.9, it suffices to show that for every $X \in \mathrm{Ob}(\mathscr{C})$ and every pair of cartesian sections $\sigma, \sigma' : \mathscr{C}/X \to \mathscr{F}ib(\mathscr{A}_\bullet)$, the presheaf $\mathscr{C}art(\sigma, \sigma')$ is a sheaf on the site C/X. Set $A := \sigma(\mathbf{1}_X)$ and $A' := \sigma'(\mathbf{1}_X)$; it is easily seen that $\mathscr{C}art(\sigma, \sigma')$ is isomorphic to the presheaf

$$H_{AA'} : (\mathscr{C}/X)^o \to \mathbf{Set} \qquad (f : Y \to X) \mapsto \mathrm{Hom}_{\mathscr{A}_Y}(\mathscr{A}_f(A), \mathscr{A}_f(A'))$$

that assigns to every morphism $(h/X) : (Y \xrightarrow{f} X) \to (Y' \xrightarrow{f'} X)$ of $\mathscr{C}/X$ the induced map

$$\mathrm{Hom}_{\mathscr{A}'_Y}(\mathscr{A}_{f'}(A), \mathscr{A}_{f'}(A')) \to \mathrm{Hom}_{\mathscr{A}_Y}(\mathscr{A}_f(A), \mathscr{A}_f(A')) \qquad g \mapsto \mathscr{A}_h(g).$$

Now, let $\mathbb{1}_{\mathscr{C}/X}$ be a final object of $(\mathscr{C}/X)^\wedge$ (see example 1.2.16(v)); it is easily seen that $\mathbb{1}_{\mathscr{C}/X}$ is a sheaf on C/X. Recall that the source functor $\mathsf{s}_X : \mathscr{C}/X \to \mathscr{C}$ is continuous for the topologies of C and C/X. Denote also by $(O_{\mathscr{A}}, M_{\mathscr{A}}, \mathsf{s}_{\mathscr{A}}, \mathsf{t}_{\mathscr{A}}, \mathbb{1}_{\mathscr{A}})$ the object of $(\mathscr{C}, \mathbf{Cat}^*)^\wedge$ corresponding to $\mathscr{A}_\bullet$; by remark 5.5.7(iv), both $O_{\mathscr{A}}$ and $M_{\mathscr{A}}$ are sheaves on C, hence we get sheaves $\widetilde{\mathsf{s}}_{X*}(O_{\mathscr{A}})$ and $\widetilde{\mathsf{s}}_{X*}(M_{\mathscr{A}})$ on C/X, as well as source and target morphisms of sheaves $\widetilde{\mathsf{s}}_{X*}(\mathsf{s}_{\mathscr{A}}), \widetilde{\mathsf{s}}_{X*}(\mathsf{t}_{\mathscr{A}}) : \widetilde{\mathsf{s}}_{X*}(M_{\mathscr{A}}) \to \widetilde{\mathsf{s}}_{X*}(O_{\mathscr{A}})$. There follows a cartesian diagram of presheaves :

$$\begin{array}{ccc} H_{AA'} & \longrightarrow & \widetilde{\mathsf{s}}_{X*}(M_{\mathscr{A}}) \\ \downarrow & & \downarrow{\scriptstyle (\widetilde{\mathsf{s}}_{X*}(\mathsf{s}_{\mathscr{A}}), \widetilde{\mathsf{s}}_{X*}(\mathsf{t}_{\mathscr{A}}))} \\ \mathbb{1}_{\mathscr{C}/X} & \xrightarrow{\tau} & \widetilde{\mathsf{s}}_{X*}(O_{\mathscr{A}} \times O_{\mathscr{A}}) \end{array} \qquad \text{on } \mathscr{C}/X$$

where $\tau_f : \{\mathbb{1}\} = \mathbb{1}_{\mathscr{C}/X}(X) \to \mathrm{Ob}(\mathscr{A}_Y \times \mathscr{A}_Y)$ is the map such that $\mathbb{1} \mapsto (\mathscr{A}_f(A), \mathscr{A}_f(A'))$, for every morphism $f : Y \to X$ of $\mathscr{C}$. But then $H_{AA'}$ is a sheaf on C/X, as required.

(ii): By assumption, $\mathscr{F}ib(\mathscr{A}_\bullet)$ is 2-separated, and $\mathscr{F}ib(\mathscr{A}_\bullet^a)$ is 1-separated, by (i); in view of proposition 5.2.9, it then suffices to check that $\mathscr{F}ib(j)$ is i-covering for $i = 0, 1, 2$. The latter follows easily from remark 5.2.2, taking into account the explicit description of $\mathscr{A}_\bullet^a$ from (5.5.10). □

5.5.16. Fix a universe V with $\mathsf{U}\subset\mathsf{V}$, and a U-site $C := (\mathscr{C}, J_{\mathscr{C}})$. Consider the pseudo-functor

$$\mathscr{F}ib^a_C : (C, \mathsf{V}\text{-}\mathbf{Cat})^\sim \to \mathsf{V}\text{-}\mathsf{Stack}(C) \qquad \mathscr{A}_\bullet \mapsto \mathscr{F}ib(\mathscr{A}_\bullet)^a$$

composition of the restriction of the pseudo-functor $\mathscr{F}ib_{\mathscr{C}}$ of theorem 3.1.24 with $(-)^a$ of corollary 5.4.19 (where $(C, \mathsf{V}\text{-}\mathbf{Cat})^\sim$ is a 2-category as in remark 5.5.7(v)). Let also $C' := (\mathscr{C}', J_{\mathscr{C}'})$ be another U-site and $u : C' \to C$ a weak morphism of sites (definition 5.4.2); we wish to attach an orientation to the induced diagram :

$$\begin{array}{ccc} (C', \mathsf{V}\text{-}\mathbf{Cat})^\sim & \xrightarrow{(\widetilde{u}, \mathsf{V}\text{-}\mathbf{Cat})_*} & (C, \mathsf{V}\text{-}\mathbf{Cat})^\sim \\ {\scriptstyle \mathscr{F}ib^a_{C'}}\downarrow & & \downarrow{\scriptstyle \mathscr{F}ib^a_C} \\ \mathsf{V}\text{-}\mathsf{Stack}(C') & \xrightarrow{\mathsf{V}\text{-}\mathsf{St}(u)_*} & \mathsf{V}\text{-}\mathsf{Stack}(C) \end{array} \tag{5.5.17}$$

i.e. a pseudo-natural transformation

$$\Delta^u : \mathscr{F}ib^a_C \circ (\widetilde{u}, \mathsf{V}\text{-}\mathbf{Cat})_* \Rightarrow \mathsf{V}\text{-}\mathsf{St}(u)_* \circ \mathscr{F}ib^a_{C'}.$$

To this aim, notice that remark 3.1.27(ii) yields an oriented diagram of 2-categories :

$$\begin{array}{ccc} (C', \mathsf{V}\text{-}\mathbf{Cat})^\sim & \xrightarrow{(\widetilde{u}, \mathsf{V}\text{-}\mathbf{Cat})_*} & (C, \mathsf{V}\text{-}\mathbf{Cat})^\sim \\ {\scriptstyle \mathscr{F}ib_{\mathscr{C}'}}\downarrow & \Downarrow \perp^u & \downarrow{\scriptstyle \mathscr{F}ib_{\mathscr{C}}} \\ \mathsf{V}\text{-}\mathsf{Fib}(\mathscr{C}') & \xrightarrow[\mathsf{V}\text{-}\mathsf{Fib}(u)^*]{} & \mathsf{V}\text{-}\mathsf{Fib}(\mathscr{C}) \end{array} \tag{5.5.18}$$

whose orientation $\perp^u$ is a strict pseudo-natural isomorphism of strict pseudo-functors. On the other hand, the square of links $\mathscr{D}_u$ of (5.4.30), oriented by the identity pseudo-natural transformation, yields a base change square :

$$\begin{array}{ccc} \mathsf{V}\text{-}\mathsf{Fib}(\mathscr{C}') & \xrightarrow{\mathsf{V}\text{-}\mathsf{Fib}(u)^*} & \mathsf{V}\text{-}\mathsf{Fib}(\mathscr{C}) \\ {\scriptstyle (-)^a_{C'}}\downarrow & \Downarrow \Upsilon(\mathscr{D}_u) & \downarrow{\scriptstyle (-)^a_C} \\ \mathsf{V}\text{-}\mathsf{Stack}(C') & \xrightarrow[\mathsf{V}\text{-}\mathsf{St}(u)_*]{} & \mathsf{V}\text{-}\mathsf{Stack}(C) \end{array} \tag{5.5.19}$$

and then we take $\Delta^u := \perp^u \boxdot \Upsilon(\mathscr{D}_u)$. In case we wish to emphasize the dependence on V, we shall also write $\mathsf{V}\text{-}\Delta^u$ for this orientation of (5.5.17).

Remark 5.5.20. In the situation of (5.5.16), suppose that u is a morphism of sites; in this case, the pseudo-functors $(\widetilde{u}, \mathsf{V}\text{-}\mathbf{Cat})_*$ and $\mathsf{V}\text{-}\mathsf{St}(u)_*$ admit left 2-adjoints $(\widetilde{u}, \mathsf{V}\text{-}\mathbf{Cat})^*$ and $\mathsf{V}\text{-}\mathsf{St}(u)^*$, and after choosing units and counits for these 2-adjoint pairs, and for the 2-adjoint pair $(\mathsf{V}\text{-}\mathsf{Fib}(u)_!, \mathsf{V}\text{-}\mathsf{Fib}(u)^*)$ we then get well defined links

$$(\widetilde{u}, \mathsf{V}\text{-}\mathbf{Cat}) := ((\widetilde{u}, \mathsf{V}\text{-}\mathbf{Cat})^*, (\widetilde{u}, \mathsf{V}\text{-}\mathbf{Cat})_*, \eta^{(\widetilde{u}, \mathsf{V}\text{-}\mathbf{Cat})}, \varepsilon^{(\widetilde{u}, \mathsf{V}\text{-}\mathbf{Cat})}) : (C', \mathsf{V}\text{-}\mathbf{Cat})^\sim \to (C, \mathsf{V}\text{-}\mathbf{Cat})^\sim$$

$$\mathsf{V}\text{-}\mathsf{St}(u) := (\mathsf{V}\text{-}\mathsf{St}(u)^*, \mathsf{V}\text{-}\mathsf{St}(u)_*, \eta^{\mathsf{V}\text{-}\mathsf{St}(u)}, \varepsilon^{\mathsf{V}\text{-}\mathsf{St}(u)}) : \mathsf{V}\text{-}\mathsf{Stack}(C') \to \mathsf{V}\text{-}\mathsf{Stack}(C)$$

$$\mathsf{V}\text{-}\mathsf{Fib}(u) := (\mathsf{V}\text{-}\mathsf{Fib}(u)_!, \mathsf{V}\text{-}\mathsf{Fib}(u)^*, \eta^{\mathsf{V}\text{-}\mathsf{Fib}(u)}, \varepsilon^{\mathsf{V}\text{-}\mathsf{Fib}(u)}) : \mathsf{V}\text{-}\mathsf{Fib}(\mathscr{C}') \to \mathsf{V}\text{-}\mathsf{Fib}(\mathscr{C})$$

of the 2-category $\mathsf{V}\text{-}\overline{2\text{-}\mathbf{Cat}}$, and the data

$$((-)^a_{C'}, (-)^a_C, \Upsilon(\mathscr{D}_u)) \qquad \text{and} \qquad (\mathscr{F}ib_{\mathscr{C}'}, \mathscr{F}ib_{\mathscr{C}}, \perp^u)$$

can be regarded as 1-cells $\mathsf{V}\text{-}\mathsf{Fib}(u) \to \mathsf{V}\text{-}\mathsf{St}(u)$ and $(\widetilde{u}, \mathsf{V}\text{-}\mathbf{Cat}) \to \mathsf{V}\text{-}\mathsf{Fib}(u)$ in the 2-category $\mathsf{wLink}(\mathsf{V}\text{-}\overline{2\text{-}\mathbf{Cat}})$ (see (2.3.19)) whose composition is the 1-cell

$$(\mathscr{F}ib^a_{C'}, \mathscr{F}ib^a_C, \Delta^u) : (\widetilde{u}, \mathsf{V}\text{-}\mathbf{Cat}) \to \mathsf{V}\text{-}\mathsf{St}(u).$$

5.5.21. Let V be a universe with $\mathsf{U} \in \mathsf{V}$; we shall denote by

$$(\mathsf{U}, \mathsf{V})\text{-}\mathbf{wSite}$$

the sub-2-category of (U, V)-$\mathbf{Site}$ (see definition 4.3.1(ii)) whose objects are all the V-small U-sites, whose 1-cells are the weak morphisms of sites, and whose 2-cells $g \Rightarrow g'$ are the natural transformations of functors $g' \Rightarrow g$. As in remark 4.3.2(i), we shall usually drop the mention of U and V, and write simply $\mathbf{wSite}$ for this 2-category. We have a strict pseudo-functor

$$(-, \mathsf{V}\text{-}\mathbf{Cat})^{\sim} : \mathbf{wSite} \to \mathsf{V}\text{-}\overline{2\text{-}\mathbf{Cat}} \qquad C \mapsto (C, \mathsf{V}\text{-}\mathbf{Cat})^{\sim}$$

(notation of remark 2.2.16) that assigns to every morphism of U-sites u the strict pseudo-functor $(\widetilde{u}, \mathsf{V}\text{-}\mathbf{Cat})_*$, and to every natural transformation $\beta : u \Rightarrow v$ the 2-cell $(\widetilde{\beta}, \mathsf{V}\text{-}\mathbf{Cat})_* : (\widetilde{v}, \mathsf{V}\text{-}\mathbf{Cat})_* \Rightarrow (\widetilde{u}, \mathsf{V}\text{-}\mathbf{Cat})_*$. Likewise, we get a pseudo-functor

$$\mathsf{V}\text{-}\mathsf{Stack} : \mathbf{wSite} \to \mathsf{V}\text{-}\overline{2\text{-}\mathbf{Cat}} \qquad C \mapsto \mathsf{V}\text{-}\mathsf{Stack}(C) \qquad (u : C' \to C) \mapsto \mathsf{V}\text{-}\mathsf{St}(u)_*.$$

To define the coherence constraints $(\delta^{\mathsf{V}\text{-}\mathsf{Stack}}, \gamma^{\mathsf{V}\text{-}\mathsf{Stack}})$ of V-Stack, and the action of V-Stack on natural transformations $\beta : u \Rightarrow v$ between weak morphisms of sites $u, v : C' \to C$, notice that $\mathsf{V}\text{-}\mathsf{Stack}(\mathscr{C}, J) \subset \mathsf{V}\text{-}\mathsf{Fib}(\mathscr{C})$ for every U-site $(\mathscr{C}, J)$, and $\mathsf{V}\text{-}\mathsf{St}(u)_*$ is the restriction of $\mathsf{V}\text{-}\mathsf{Fib}(u)^*$, for every morphism of sites u; we may then simply define $(\delta^{\mathsf{V}\text{-}\mathsf{Stack}}, \gamma^{\mathsf{V}\text{-}\mathsf{Stack}})$ as the restriction of the coherence constraint $(\delta^{\mathsf{V}\text{-}\mathsf{Fib}}, \gamma^{\mathsf{V}\text{-}\mathsf{Fib}})$ of the pseudo-functor V-Fib of (3.1.29), and for any β as above, let $\mathsf{V}\text{-}\mathsf{St}(\beta)_* : \mathsf{V}\text{-}\mathsf{St}(v) \Rightarrow \mathsf{V}\text{-}\mathsf{St}(u)_*$ be the restriction of $\mathsf{V}\text{-}\mathsf{Fib}(\beta)^*$.

With this notation, it would be tempting to state that the rule : $C \mapsto \mathscr{F}ib^a_C$ of (5.5.16) defines a pseudo-natural transformation $\mathscr{F}ib^a_\bullet : (-, \mathsf{V}\text{-}\mathbf{Cat})^{\sim} \Rightarrow \mathsf{V}\text{-}\mathsf{Stack}$ whose coherence constraint is given by the orientations Δ^u. However, notice first that Δ^u points in the direction opposite to the one which is required for such a coherence constraint. This can be fixed, by stating instead that $\mathscr{F}ib^a_\bullet$ should be a pseudo-natural transformation ${}^o(-, \mathsf{V}\text{-}\mathbf{Cat})^{\sim} \Rightarrow {}^o\mathsf{V}\text{-}\mathsf{Stack}$. But since *the orientations Δ^u are not, in general, pseudo-natural equivalences*, we have rather :

Proposition 5.5.22. *There exists a lax-natural transformation*

$${}^o\mathscr{F}ib^a_\bullet : {}^o(-, \mathsf{V}\text{-}\mathbf{Cat})^{\sim} \Rightarrow {}^o\mathsf{V}\text{-}\mathsf{Stack} \qquad C \mapsto {}^o\mathscr{F}ib^a_C \qquad \textit{for every } \mathsf{U}\textit{-site } C$$

whose coherence constraint attaches to every weak morphism of U-sites $u : C' \to C$ the square ^{o}(5.5.17), with its orientation ${}^o\Delta^u$.

Proof. Consider the forgetful strict pseudo-functor

$$\Phi : \mathbf{wSite} \to {}^o\mathsf{V}\text{-}\mathbf{Cat}^o \qquad (\mathscr{C}, J) \mapsto \mathscr{C}$$

that assigns to every weak morphism of sites $u : (\mathscr{C}', J') \to (C, J)$ the functor $u : \mathscr{C} \to \mathscr{C}'$, and to every 2-cell $\beta : u \Rightarrow v$ of $\mathbf{wSite}$ the natural transformation $\beta : v \Rightarrow u$. From corollary 3.1.36 we deduce a pseudo-natural transformation of pseudo-functors :

$${}^o\mathscr{F}ib_\bullet : {}^o(-, \mathsf{V}\text{-}\mathbf{Cat})^{\sim} \Rightarrow {}^o\mathsf{V}\text{-}\mathsf{Fib} \circ {}^o\Phi \qquad (\mathscr{C}, J) \mapsto {}^o\mathscr{F}ib_{\mathscr{C}} \qquad \text{for every } \mathsf{U}\text{-site } (\mathscr{C}, J)$$

whose coherence constraint assigns to every morphism of sites u the oriented square diagram (5.5.18). It then suffices to exhibit a lax-natural transformation

$${}^o(-)^a_\bullet : {}^o\mathsf{V}\text{-}\mathsf{Fib} \circ {}^o\Phi \Rightarrow {}^o\mathsf{V}\text{-}\mathsf{Stack} \qquad C \mapsto {}^o(-)^a_C \qquad \text{for every } \mathsf{U}\text{-site } C$$

whose coherence constraint assigns to every weak morphism of U-sites $u : C' \to C$ the oriented square $\Upsilon(\mathscr{D}_u)$. Indeed, we will then define ${}^o\mathscr{F}ib^a_\bullet := {}^o(-)^a_\bullet \odot {}^o\mathscr{F}ib_\bullet$. Now, notice that the pseudo-functors V-Stack and V-Fib factor through well-defined pseudo-functors

$$\mathsf{V}\text{-}\mathsf{LStack} : \mathbf{wSite} \to \mathsf{Link}(\mathsf{V}\text{-}\overline{2\text{-}\mathbf{Cat}}) \qquad \mathsf{V}\text{-}\mathsf{LFib} : {}^o\mathsf{V}\text{-}\mathbf{Cat}^o \to \mathsf{Link}(\mathsf{V}\text{-}\overline{2\text{-}\mathbf{Cat}}).$$

Namely, V-LStack assigns to every site C the 2-category $\mathsf{V}\text{-}\mathsf{Stack}(C)$ and to every weak morphism of sites $u : C' \to C$ the link $\mathsf{V}\text{-}\mathsf{St}(u)$ defined as in remark 5.5.20; likewise, V-LFib assigns to every category $\mathscr{C}$ the 2-category $\mathsf{V}\text{-}\mathsf{Fib}(\mathscr{C})$, and to every functor $u : \mathscr{C} \to \mathscr{C}'$ the link

$\mathsf{V}\text{-}\mathsf{Fib}(u)$. Next, let $i_C : \mathsf{V}\text{-}\mathsf{Stack}(C) \to \mathsf{V}\text{-}\mathsf{Fib}(\mathscr{C})$ be the inclusion strict pseudo-functor; notice that the rule : $C \mapsto i_C$ for every site C yields a pseudo-natural transformation

$$i_\bullet : \mathsf{V}\text{-}\mathsf{LStack} \Rightarrow \mathsf{V}\text{-}\mathsf{LFib} \circ \Phi$$

whose coherence constraints are given by the oriented diagrams $\mathscr{D}_u$ (notice that $i_\bullet$ is not strict, even though $\mathscr{D}_u$ is oriented by the identity pseudo-natural transformation). According to remark 2.2.5(ii), the pseudo-natural transformation $i_\bullet$ corresponds to a pseudo-functor $\widetilde{\imath}_\bullet : \mathbf{wSite} \to \mathscr{M} := 2\text{-}\mathsf{Morph}(\mathsf{Link}(\mathsf{V}\text{-}\overline{2\text{-}\mathbf{Cat}}))$, and we consider the composition :

$$\Psi : \mathbf{wSite} \xrightarrow{\widetilde{\imath}_\bullet} \mathscr{M} \xrightarrow{\Upsilon^o_{\mathsf{V}\text{-}\overline{2\text{-}\mathbf{Cat}}}} (2\text{-}\mathsf{Morph}(\mathsf{V}\text{-}\overline{2\text{-}\mathbf{Cat}}^o))^o \xrightarrow{\sim} {}^o2\text{-}\mathsf{Morph}({}^o\mathsf{V}\text{-}\overline{2\text{-}\mathbf{Cat}})$$

with the pseudo-functor $\Upsilon^o_{\mathsf{V}\text{-}\overline{2\text{-}\mathbf{Cat}}}$ of theorem 2.3.17 and the strict isomorphism of 2-categories given by example 2.1.23(i). Let $\mathsf{s}, \mathsf{t} : 2\text{-}\mathsf{Morph}({}^o\mathsf{V}\text{-}\overline{2\text{-}\mathbf{Cat}}) \to {}^o\mathsf{V}\text{-}\overline{2\text{-}\mathbf{Cat}}$ be the source and target strict pseudo-functors; by definition, ${}^o\Psi$ is a lax-natural transformation $\mathsf{s} \circ {}^o\Psi \Rightarrow \mathsf{t} \circ {}^o\Psi \Rightarrow$. Lastly, a direct inspection shows that $\mathsf{s} \circ {}^o\Psi = \mathsf{LStack}$ and $\mathsf{t} \circ {}^o\Psi \Rightarrow = \mathsf{LFib} \circ \Phi$, so we may let ${}^o(-)^a_\bullet := {}^o\Psi$. □

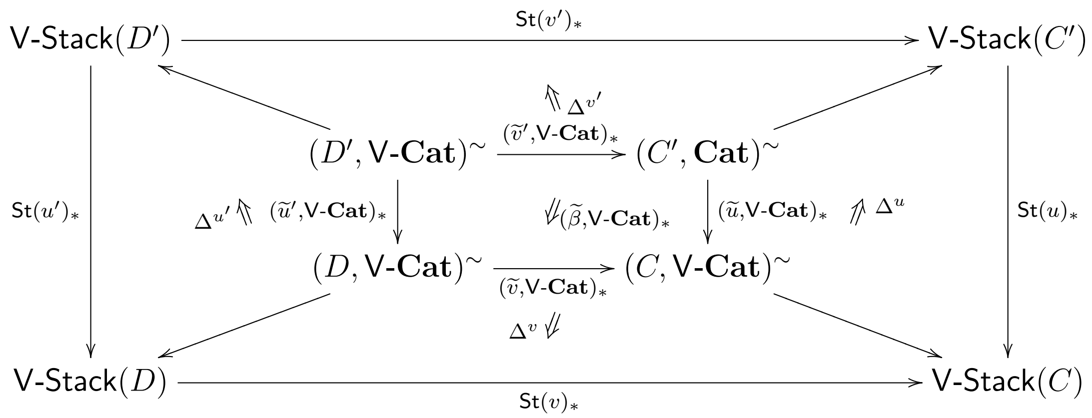
5.5.23. We consider now an oriented square :

$$\begin{array}{ccc} D' := (\mathscr{D}', J_{\mathscr{D}'}) & \xrightarrow{v'} & C' := (\mathscr{C}', J_{\mathscr{C}'}) \\ {\scriptstyle u'}\downarrow & \Downarrow \beta & \downarrow{\scriptstyle u} \\ D := (\mathscr{D}, J_{\mathscr{D}}) & \xrightarrow{v} & C := (\mathscr{C}, J_{\mathscr{C}}) \end{array}$$

where v and v' are morphisms of U-sites, and u and u' are weak morphisms of U-sites. In case also u and u' are morphisms of sites, we get an induced oriented diagram of topoi :

$$(5.5.24) \qquad \begin{array}{ccc} \widetilde{D}' & \xrightarrow{\widetilde{v}'} & \widetilde{C}' \\ {\scriptstyle \widetilde{u}'}\downarrow & \Downarrow \widetilde{\beta}_* & \downarrow{\scriptstyle \widetilde{u}} \\ \widetilde{D} & \xrightarrow{\widetilde{v}} & \widetilde{C}. \end{array}$$

Even when u and u' are only weak morphisms of sites, both $\widetilde{u}_*$ and $\widetilde{u}'_*$ admit left adjoint functors, hence (5.5.24) may still be regarded as an oriented square of links in the 2-category $\mathsf{V}\text{-}\mathbf{Cat}$. On the other hand, we deduce a diagram of 2-categories :



whose diagonal arrows are the pseudo-functors $\mathscr{F}ib^a$. We complete it by adding the orientation

$$\mathsf{St}(\beta)^\gamma_* := (\gamma^{\mathsf{Stack}}_{u',v})^{-1} \odot \mathsf{St}(\beta)_* \odot \gamma^{\mathsf{Stack}}_{v',u} : \mathsf{St}(u)_* \circ \mathsf{St}(v')_* \Rightarrow \mathsf{St}(v)_*\mathsf{St}(u')_*$$

for the external square subdiagram. In light of proposition 5.5.22, we then see that the resulting cubical diagram commutes on 2-cells, in the sense of remark 2.3.22(iii).

5.5.25. In the situation of (5.5.23), we associate with v the links $(\widetilde{v}, \mathsf{V}\text{-}\mathbf{Cat})$, $\mathsf{V}\text{-}\mathsf{St}(v)$ and $\mathsf{V}\text{-}\mathsf{Fib}(v)$ of the 2-category $\mathsf{V}\text{-}\overline{2\text{-}\mathbf{Cat}}$, as in remark 5.5.20, as well as 1-cells

$$((-)^a_D, (-)^a_C, \Upsilon(\mathscr{D}^v)) : \mathsf{V}\text{-}\mathsf{Fib}(v) \to \mathsf{V}\text{-}\mathsf{St}(v) \qquad (\mathscr{F}ib_{\mathscr{D}}, \mathscr{F}ib_{\mathscr{C}}, \bot^v) : (\widetilde{v}, \mathsf{V}\text{-}\mathbf{Cat}) \to \mathsf{V}\text{-}\mathsf{Fib}(v)$$

in the 2-category $\mathsf{wLink}(\mathsf{V}\text{-}\overline{2\text{-}\mathbf{Cat}})$, whose composition is the 1-cell :

$$(\mathscr{F}ib^a_D, \mathscr{F}ib^a_C, \Delta^v) : (\widetilde{v}, \mathsf{V}\text{-}\mathbf{Cat}) \to \mathsf{V}\text{-}\mathsf{St}(v)$$

(and likewise for v'). Additionally, we get two more 1-cells of $\mathsf{wLink}(\mathsf{V}\text{-}\overline{2\text{-}\mathbf{Cat}})$:

$$((\widetilde{u}', \mathbf{Cat})_*, (\widetilde{u}, \mathbf{Cat})_*, (\widetilde{\beta}, \mathbf{Cat})_*) : (\widetilde{v}', \mathbf{Cat}) \to (\widetilde{v}, \mathbf{Cat})$$
$$(\mathsf{St}(u')_*, \mathsf{St}(u)_*, \mathsf{St}(\beta)^{\gamma}_*) : \mathsf{St}(v') \to \mathsf{St}(v).$$

Corollary 5.5.26. *With the notation of* (5.5.25)*, let* $\mathscr{A}_\bullet \in \mathrm{Ob}((C', \mathbf{Cat})^\sim)$. *We have :*

(i) *If* $\mathscr{F}ib_{\mathscr{C}'}(\mathscr{A}_\bullet)$ *is a stack on* C'*, then* $\Delta^u_{\mathscr{A}_\bullet}$ *is an equivalence of categories.*
(ii) *If* u *is also cocontinuous, then* Δ^u *is a pseudo-natural equivalence.*
(iii) $\Upsilon(\mathscr{F}ib^a_D, \mathscr{F}ib^a_C, \Delta^v)$ *is a pseudo-natural equivalence.*
(iv) *In the situation of* (5.5.23)*, suppose that both* $\mathscr{F}ib_{\mathscr{C}'}(\mathscr{A}_\bullet)$ *and* $\mathscr{F}ib_{\mathscr{D}'}((\widetilde{v}', \mathbf{Cat})^*\mathscr{A}_\bullet)$ *are stacks. Then* $\mathscr{F}ib^a_D(\Upsilon((\widetilde{\beta}, \mathbf{Cat})_*)_{\mathscr{A}_\bullet})$ *is an equivalence if and only if the same holds for* $\Upsilon(\mathsf{St}(\beta)^{\gamma}_*)_{\mathscr{F}ib(\mathscr{A}_\bullet)}$.

Proof. (i): It suffices to show that $\Upsilon(\mathscr{D}^u)_{\mathscr{E}}$ is an equivalence if $\mathscr{E}$ is a stack on C'. Thus, let (η^C, ε^C) be the unit and counit for the 2-adjoint pair $(()^a_C, i_C)$ (notation of (5.4.30)) and define likewise $(\eta^{C'}, \varepsilon^{C'})$. Since i_C is fully faithful, ε^C is a pseudo-natural equivalence, and likewise for $\varepsilon^{C'}$. From the triangular identities of theorem 2.4.24(i) we deduce that also $\eta^C * i_C$ and $\eta^{C'} * i_{C'}$ are pseudo-natural equivalences; the assertion follows directly.

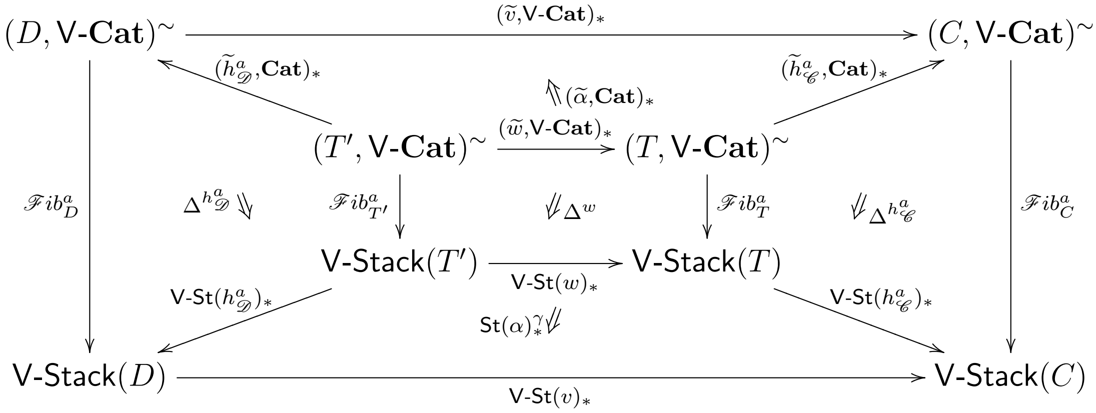
(ii): Again, it suffices to check that $\Upsilon(\mathscr{D}^u)$ is a pseudo-natural equivalence if u is cocontinuous; this is known by corollary 5.4.31.

Claim 5.5.27. In order to prove (iii), we may assume that v is a morphism of small lex-sites.

Proof of the claim. Let T (resp. T') be the site whose underlying category is $C^\sim_{\mathsf{U}}$ (resp $D^\sim_{\mathsf{U}}$), with its canonical topology. Pick a universe V' with $\mathsf{V} \subset \mathsf{V}'$, such that T and T' are V'-small. By direct inspection, we see that $\mathsf{V}\text{-}\Delta^v$ is the restriction of $\mathsf{V}'\text{-}\Delta^v$; hence if $\Upsilon(\mathsf{V}'\text{-}\Delta^v)$ is a pseudo-natural equivalence, the same holds for $\Upsilon(\mathsf{V}\text{-}\Delta^v)$. We may then replace V by V', and suppose that C, D, T, T' are V-small; notice that $w := \widetilde{u}^* : T' \to T$ is a morphism of lex-sites (remark 4.4.13(ii)). We consider the induced essentially commutative diagram of morphisms of sites :

$$\begin{array}{ccc} T' & \xrightarrow{\ w\ } & T \\ {\scriptstyle h^a_{\mathscr{D}}}\downarrow & \swarrow_{\alpha} & \downarrow{\scriptstyle h^a_{\mathscr{C}}} \\ D & \xrightarrow{\ v\ } & C \end{array}$$

(lemma 4.2.11(ii)) which, according to (5.5.23), induces a diagram :

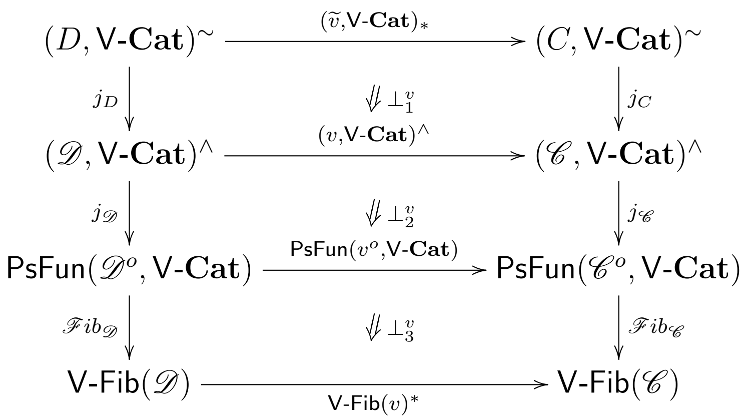


completed by adding the orientation Δ^v for the external square subdiagram. Indeed, this diagram is obtained by rotating suitably the corresponding diagram of (5.5.23), and notice that, after such rotation, the diagram still commutes on 2-cells, but now in the sense of (2.3.21). The four diagonal arrows of the diagrams are 2-equivalences, by theorem 4.4.2(iii), proposition 5.4.17(ii), and remark 5.5.14(ii). Then $\Upsilon((\widetilde{\alpha}, \mathbf{Cat})_*)$ and $\Upsilon(\mathsf{St}(\alpha)^\gamma_*)$ are pseudo-natural equivalences, by remark 2.3.11. Combining with (ii), theorem 4.4.2(iii) and remark 2.3.22(i), we conclude that if $\Upsilon(\Delta^w)$ is a pseudo-natural equivalence, the same holds for $\Upsilon(\Delta^v)$, whence the claim. ◊

(iii): By claim 5.5.27, we shall henceforth suppose that v is a morphism of small lex-sites. By remark 2.3.20 we have $\Upsilon(\Upsilon(\mathscr{D}^v)) = \mathbf{1}^\dagger_{i_C \circ \text{V-St}(v)_*}$, and since

$$\mathbf{1}_{i_C \circ \text{V-St}(v)_*} : \mathsf{Fib}(v) \circ ((-)^a_{C'}, i_{C'}, \eta^C, \varepsilon^C) \Rightarrow ((-)^a_C, i_C, \eta^C, \varepsilon^C) \circ \mathsf{St}(v)$$

is an invertible 2-cell in the category $\mathsf{Link}(\text{V-}\overline{2\text{-}\mathbf{Cat}})$, the strict isomorphisms of proposition 2.3.4 show that $\mathbf{1}^\dagger_{i_C \circ \text{V-St}(v)_*}$ is invertible as well. On the other hand, $\Upsilon(\perp^v)$ is not necessarily a pseudo-natural equivalence, but in order to conclude the proof of (iii) it will suffice to show that $(-)^a_D * \Upsilon(\perp^v)$ is a pseudo-natural equivalence. To this aim, notice that (5.5.18) (for the morphism v) can be further decomposed as a chain of three oriented squares :



where j_C, $j_{\mathscr{C}}$, j_D and $j_{\mathscr{D}}$ are the inclusion pseudo-functors, and where now $\mathscr{F}ib_{\mathscr{C}}$ and $\mathscr{F}ib_{\mathscr{D}}$ are strong 2-equivalences (theorem 3.1.24). The orientation $\perp^v_3$ is still a pseudo-natural equivalence, and both orientations $\perp^v_1$ and $\perp^v_2$ are identities. After choosing as usual a unit and counit (η, ε) for the 2-adjoint pair $((v, \text{V-}\mathbf{Cat})_!, (v, \text{V-}\mathbf{Cat})^\wedge)$ and a unit and counit (η', ε') for the 2-adjoint pair $(2\text{-}\int^{v^o}, \mathsf{PsFun}(v^o, \text{V-}\mathbf{Cat}))$, we may regard the three squares as oriented squares of weak links. Then $\Upsilon(\perp^v_3)$ is well defined, and is a pseudo-natural equivalence (remark 2.3.11). Next, let $\mathscr{A}_\bullet$ be any presheaf of categories on $\mathscr{C}$, and set $\mathscr{B}_\bullet := (v, \mathsf{V}\text{-}\mathbf{Cat})^\wedge\mathscr{A}$; by definition we have

$$\Upsilon(\perp_2^v)_{\mathscr{A}_\bullet} = \varepsilon'_{\mathscr{B}_\bullet} \odot 2\text{-}\int^{v^o} \eta_{\mathscr{A}_\bullet}.$$

But the discussion of remark 5.5.14(v) shows that

$$\varepsilon'_{\mathscr{B}_\bullet} = \varepsilon_{\mathscr{B}_\bullet} \qquad \text{and} \qquad 2\text{-}\int^{v^o} \eta_{\mathscr{A}_\bullet} = (v, \mathsf{V}\text{-}\mathbf{Cat})_!(\eta_{\mathscr{A}_\bullet}).$$

Hence, the triangular identities of (1.1.13) yield $\Upsilon(\perp_2^v)_{\mathscr{A}_\bullet} = \mathbf{1}_{\mathscr{B}_\bullet}$. Thus, we are further reduced to checking that $\mathscr{F}ib^a_D * \Upsilon(\perp_1^v)$ is a pseudo-natural equivalence, or equivalently, that the same holds for ${}^o(\mathscr{F}ib^a_D) * {}^o\Upsilon(\perp_1^v)$. However, let $[-]^a_C$ (resp. $[-]^a_D$) be the left 2-adjoint of j_C (resp. of j_D), and (η^C, ε^C) (resp. (η^D, ε^D)) the unit and counit of the 2-adjoint pair $([-]^a_C, j_C)$ (resp. $([-]^a_D, j_D)$); by proposition 2.3.10 we have

$${}^o\Upsilon(\perp_1^v) = (\Upsilon({}^o(\perp_1^v)^\dagger) = {}^o((j_D \circ (\widetilde{v}, \mathsf{V}\text{-}\mathbf{Cat})^*) * \varepsilon^C) \odot (j_D * \perp_1^{v\dagger} * j_C) \odot (\eta^D * ((v, \mathsf{V}\text{-}\mathbf{Cat})_! \circ j_C)))$$

and notice that ε^C is a pseudo-natural equivalence, since j_C is fully faithful (corollary 2.4.29). Also $\perp_1^{v\dagger}$ is a pseudo-natural equivalence, by virtue of proposition 2.3.4. Therefore, let $\mathscr{B}_\bullet$ be any sheaf of categories on C, and set $\mathscr{B}'_\bullet := (v, \mathsf{V}\text{-}\mathbf{Cat})_!\mathscr{B}_\bullet$; we are reduced to checking that the cartesian functor $\mathscr{F}ib^a_D(\eta^C_{\mathscr{B}'_\bullet})$ is an equivalence of categories. But remark 5.2.2 easily implies that $\mathscr{F}ib_{\mathscr{D}}(\eta^C_{\mathscr{B}'_\bullet})$ is t-covering for $t = 0, 1, 2$, so the assertion follows from propositions 5.2.11(iii) and 5.2.9.

(iv): Under the stated assumptions, (i) says that both $\Delta^u_{\mathscr{A}_\bullet}$ and $\Delta^{u'}_{(\widetilde{v}', \mathbf{Cat})^*\mathscr{A}}$ are equivalences, and both $\Upsilon(\Delta^v)$ and $\Upsilon(\Delta^{v'})$ are pseudo-natural equivalences, by virtue of (iii). Then the assertion follows by arguing as in remark 2.3.22(i). □

Example 5.5.28. (i) Let $C := (\mathscr{C}, J)$ be a U-site, and A any small category. The *constant presheaf of categories on* $\mathscr{C}$ *with value* A is the presheaf $A_{\mathscr{C}}$ such that $A_{\mathscr{C}}(X) := A$ for every $X \in \mathrm{Ob}(\mathscr{C})$, and $A_{\mathscr{C}}(f) := \mathbf{1}_A$ for every morphism f of $\mathscr{C}$. The associated sheaf $A^a_{\mathscr{C}}$ is then the *constant sheaf of categories on* C *with value* A. Likewise, we say that $\mathscr{F}ib(A_{\mathscr{C}})$ is the *constant fibration on* $\mathscr{C}$ *with value* A, and $\mathscr{F}ib(A_{\mathscr{C}})^a$ is the *constant stack on* C *with value* A.

(ii) Now, let $C' := (C', J')$ be another U-site, and $u : C' \to C$ a morphism of sites; we notice that $(\widetilde{u}, \mathbf{Cat})^* A_{\mathscr{C}}$ is isomorphic to the constant sheaf of categories on C' with value A. Indeed, for the proof we are easily reduced to the corresponding assertion for constant sheaves of sets, which is known by remark 4.3.2(iii).

(iii) In light of (i) and lemma 5.5.15(iii), we deduce that the stack $\mathsf{St}(u)^*(\mathscr{F}ib(A_{\mathscr{C}})^a)$ is equivalent to the constant stack on C' with value A.

5.5.29. In section (5.7) we will prove two *base change theorems*, asserting that, for suitable oriented diagrams of sites as in (5.5.23), the *base change map* $\Upsilon(\mathsf{St}(\beta)^\gamma_*)$ is a pseudo-natural equivalence. To this aim we shall apply corollary 5.5.26(iv), thereby reducing the assertion to the corresponding one for $\Upsilon((\widetilde{\beta}, \mathbf{Cat})_*)$, *i.e.* we shall deduce a base change theorem for stacks from one for sheaves of categories. The latter in turn can be reduced to the corresponding assertion for $\Upsilon(\beta^\sim_*)$, as we explain hereafter. Indeed, by remark 5.5.7(v,vi,vii) we deduce from

(5.5.23) a diagram of 2-categories whose four diagonal arrows are 2-equivalences :

$$\begin{array}{ccccccc}
(D',\mathbf{Cat})^\sim & & & \xrightarrow{\quad(\widetilde{v}',\mathbf{Cat})_*\quad} & & & (C',\mathbf{Cat})^\sim \\
 & \nwarrow & & & & \nearrow & \\
 & & \mathbf{Cat}^*(D'^\sim) & \xrightarrow{\mathbf{Cat}^*(\widetilde{v}'_*)} & \mathbf{Cat}^*(C'^\sim) & & \\
{\scriptstyle (\widetilde{u}',\mathbf{Cat})_*}\Big\downarrow & & {\scriptstyle \mathbf{Cat}^*(\widetilde{u}'_*)}\Big\downarrow & \swarrow_{\mathbf{Cat}^*(\widetilde{\beta}_*)} & \Big\downarrow{\scriptstyle \mathbf{Cat}^*(\widetilde{u}_*)} & & \Big\downarrow{\scriptstyle (\widetilde{u},\mathbf{Cat})_*} \\
 & & \mathbf{Cat}^*(D^\sim) & \xrightarrow[\mathbf{Cat}^*(\widetilde{v}_*)]{} & \mathbf{Cat}^*(C^\sim) & & \\
 & \swarrow & & & & \searrow & \\
(D,\mathbf{Cat})^\sim & & & \xrightarrow{\quad(\widetilde{v},\mathbf{Cat})_*\quad} & & & (C,\mathbf{Cat})^\sim
\end{array}$$

whose four external trapezoidal subdiagrams are strictly commutative, and can therefore be oriented by adding in the respective identity pseudo-natural transformations. We further complete the diagram by inserting the orientation

$$(\widetilde{\beta},\mathbf{Cat})_* : (\widetilde{u},\mathbf{Cat})_* \circ (\widetilde{v}',\mathbf{Cat})_* \Rightarrow (\widetilde{v},\mathbf{Cat})_* \circ (\widetilde{u}',\mathbf{Cat})_*$$

for the "front face" of the resulting cubical diagram $\mathscr{D}$. We regard $\mathscr{D}$ as a diagram of 1-cells and 2-cells in V-$\overline{2\text{-}\mathbf{Cat}}$ (for a suitable universe V); then it is clear that $\mathscr{D}$ commutes on 2-cells in the sense of (2.3.21). Moreover, all the horizontal arrows in $\mathscr{D}$ admit left 2-adjoint pseudo-functors, which we regard as left adjoint 1-cells, in the 2-category V-$\overline{2\text{-}\mathbf{Cat}}$; after fixing such a system of left adjoints, we may regard $\mathscr{D}$ as a diagram oriented squares of links in V-$\overline{2\text{-}\mathbf{Cat}}$, and especially, the 2-cells $\Upsilon((\widetilde{\beta},\mathbf{Cat})_*)$ and $\Upsilon(\mathbf{Cat}^*(\widetilde{\beta}_*))$ are then well defined. Invoking again remarks 2.3.11), and 2.3.22(i) we deduce that $\Upsilon((\widetilde{\beta},\mathbf{Cat})_*)$ is invertible if and only if the same holds for $\Upsilon(\mathbf{Cat}^*(\widetilde{\beta}_*))$ (recall that the invertible 2-cells of V-$\overline{2\text{-}\mathbf{Cat}}$ are the equivalence classes of pseudo-natural equivalences of pseudo-functors). According to proposition 2.3.24, we have $\Upsilon(\mathbf{Cat}^*(\widetilde{\beta}_*)) = \mathbf{Cat}^*(\Upsilon(\beta^\sim_*))$, where $\Upsilon(\beta^\sim_*)$ is the base change transformation associated with the oriented square of links (5.5.24); so, if $\Upsilon(\beta^\sim_*)$ is invertible, the same holds for $\Upsilon((\widetilde{\beta},\mathbf{Cat})_*)$.

5.5.30. Here is a first illustration of the method explained in (5.5.29). Consider a morphism of sites $u : C' := (\mathscr{C}', J') \to C := (\mathscr{C}, J)$; we assume that all finite products are representable in $\mathscr{C}$ and $\mathscr{C}'$, and that u commutes with such products. For every $X, Y \in \mathrm{Ob}(\mathscr{C})$ we choose a representative $X \times Y \in \mathrm{Ob}(\mathscr{C})$ for the product of X and Y, and let $X \xleftarrow{q_{X,Y}} X \times Y \xrightarrow{p_{X,Y}} Y$ be the universal projections. For every $X \in \mathrm{Ob}(\mathscr{C})$ we get a morphism of sites $p_X : C/X \to C$ that assigns to every $Y \in \mathrm{Ob}(\mathscr{C})$ the projection $p_{X,Y}$: see remark 4.7.3(iii). Likewise we define the morphism of sites $p_{uX} : C'/uX \to C'$. For every $Y \in \mathrm{Ob}(\mathscr{C})$, there is by assumption a unique isomorphism $\beta_Y : u(X \times Y) \xrightarrow{\sim} uX \times uY$ in $\mathscr{C}'$ such that $p_{uX,uY} \circ \beta_Y = u(p_{X,Y})$, and $q_{uX,uY} \circ \beta_Y = u(q_{X,Y})$, where $uX \xleftarrow{q_{uX,uY}} uX \times uY \xrightarrow{p_{uX,uY}} uY$ are likewise the universal projections. Especially, $\beta_Y/X : u_{|X}(p_{X,Y}) \xrightarrow{\sim} p_{uX,uY}$ is an isomorphism in $\mathscr{C}'/uX$ (notation of (1.1.26)). Thus, we get an oriented square diagram of sites :

$$\begin{array}{ccc}
C'/uX & \xrightarrow{p_{uX}} & C' \\
{\scriptstyle u_{|X}}\Big\downarrow & \swarrow_{\beta} & \Big\downarrow{\scriptstyle u} \\
C/X & \xrightarrow{p_X} & C.
\end{array}$$

Proposition 5.5.31. *In the situation of* (5.5.30)*, the base change transformation* $\Upsilon(\mathsf{St}(\beta)^\gamma_*)$ *is a pseudo-natural equivalence.*

Proof. To begin with, we remark :

Claim 5.5.32. Let $\mathscr{A}_\bullet \in \mathrm{Ob}((C', \mathbf{Cat})^\sim)$ such that $\mathscr{E} := \mathscr{F}ib_{\mathscr{C}'}(\mathscr{A}_\bullet)$ is a stack on C'. Then the fibration $\mathscr{F}ib_{\mathscr{C}'/uX}(\widetilde{p}_{uX}, \mathbf{Cat})^*\mathscr{A}_\bullet$ is a stack on C'/uX.

Proof of the claim. The isomorphism of functors $\widetilde{p}^*_{uX} \simeq \widetilde{\mathsf{s}}_{uX*}$ of remark 4.7.3(iii) induces an isomorphism of pseudo-functors $(\widetilde{p}_{uX}, \mathbf{Cat})^* \stackrel{\sim}{\to} (\widetilde{\mathsf{s}}_{uX}, \mathbf{Cat})_*$, hence it suffices to show that $\mathscr{F} := \mathscr{F}ib_{\mathscr{C}'/uX}(\widetilde{\mathsf{s}}_{uX}, \mathbf{Cat})_*\mathscr{A}_\bullet = \mathscr{F}ib_{\mathscr{C}'/uX}(\mathscr{A}_\bullet \circ \mathsf{s}^o_{uX}) \simeq \mathscr{C}'/uX \times_{\mathscr{C}'} \mathscr{E}$ is a stack. Thus, let $(Z \xrightarrow{h} uX) \in \mathrm{Ob}(\mathscr{C}'/uX)$ and $\mathscr{S}' \subset (\mathscr{C}'/uX)/h$ a sieve covering h for the site C'/uX; under the natural identification of categories

$$(\mathscr{C}'/uX)/h \stackrel{\sim}{\to} \mathscr{C}'/Z$$

the sieve $\mathscr{S}'$ corresponds to a sieve $\mathscr{S} \subset \mathscr{C}'/Z$ covering Z for the site C (see (4.7)). Then the restriction functor

$$\mathscr{F}(h) := \mathsf{Cart}_{\mathscr{C}'/uX}((\mathscr{C}'/uX)/h, \mathscr{F}) \to \mathsf{Cart}_{\mathscr{C}'/uX}(\mathscr{S}', \mathscr{F})$$

is naturally identified with the restriction functor

$$\mathscr{E}(Z) := \mathsf{Cart}_{\mathscr{C}'}(\mathscr{C}'/Z, \mathscr{E}) \to \mathsf{Cart}_{\mathscr{C}'}(\mathscr{S}, \mathscr{E})$$

whence the contention. ◊

By claim 5.5.32, lemma 5.5.15(ii), corollary 5.5.26(iv) and the discussion of (5.5.29), we are reduced to showing that the base change transformation $\Upsilon(\beta^\sim_*) : \widetilde{p}^*_X \circ \widetilde{u}_* \to \widetilde{u}_{|X*} \circ \widetilde{p}^*_{uX}$ is an isomorphism of functors. Now, recall that the source functor $\mathsf{s}_X : \mathscr{C}/X \to \mathscr{C}$ is continuous for the sites C and C/X, and is left adjoint to p_X; more precisely, we have an explicit adjunction :

$$\vartheta_{h,Z} : \mathrm{Hom}_{\mathscr{C}}(Y, Z) \stackrel{\sim}{\to} \mathrm{Hom}_{\mathscr{C}/X}(h, p_{X,Z}) \quad \text{for every } (Y \xrightarrow{h} X) \in \mathrm{Ob}(\mathscr{C}/X) \text{ and } Z \in \mathrm{Ob}(\mathscr{C})$$

that assigns to every morphism $f : Y \to Z$ of $\mathscr{C}$ the unique morphism $f^*/X : h \to p_{X,Z}$ of $\mathscr{C}/X$ such that $q_{X,Z} \circ f^* = f$. The adjunction $\vartheta_{h,Z}$ induces a corresponding adjunction for the pair $(\widetilde{\mathsf{s}}_{X*}, \widetilde{p}_{X*})$, as described in remark 1.1.19(iii,iv). Explicitly, the unit of this induced adjunction assigns to every sheaf $\mathscr{F}$ on C the morphism of sheaves :

$$\eta^X_{\mathscr{F}} : \mathscr{F} \to \mathscr{F} \circ \mathsf{s}^o_X \circ p^o_X \qquad Y \mapsto (\mathscr{F}(q_{X,Y}) : \mathscr{F}(Y) \to \mathscr{F}(X \times Y))$$

and the counit assigns to every sheaf $\mathscr{G}$ on C/X the morphism of sheaves :

$$\varepsilon^X_{\mathscr{G}} : \mathscr{G} \circ p^o_X \circ \mathsf{s}^o_X \to \mathscr{G} \qquad (Y \xrightarrow{h} X) \mapsto (\mathscr{G}(h^*/X) : \mathscr{G}(p_{X,Y}) \to \mathscr{G}(h)).$$

The same description applies to the adjunction for the pair $(\widetilde{\mathsf{s}}_{uX*}, \widetilde{p}_{uX*})$, and we get therefore isomorphisms of links

$$\begin{aligned} \mathscr{L}_X &:= (\widetilde{\mathsf{s}}_{X*}, \widetilde{p}_{X*}, \eta^X, \varepsilon^X) \stackrel{\sim}{\to} \widetilde{p}_X := (\widetilde{p}^*_X, \widetilde{p}_{X*}, \eta^{p_X}, \varepsilon^{p_X}) \\ \mathscr{L}_{uX} &:= (\widetilde{\mathsf{s}}_{uX*}, \widetilde{p}_{uX*}, \eta^{uX}, \varepsilon^{uX}) \stackrel{\sim}{\to} \widetilde{p}_{uX} := (\widetilde{p}^*_{uX}, \widetilde{p}_{uX*}, \eta^{p_{uX}}, \varepsilon^{p_{uX}}). \end{aligned}$$

Hence, it suffices to check that the base change transformation for the oriented diagram of links:

$$\mathscr{D} \quad : \quad \begin{array}{ccc} (C'/uX)^\sim & \xrightarrow{\mathscr{L}_{uX}} & C'^\sim \\ {\scriptstyle \widetilde{u}_{|X}}\big\downarrow & \Downarrow \widetilde{\beta}_* & \big\downarrow{\scriptstyle \widetilde{u}} \\ (C/X)^\sim & \xrightarrow[\mathscr{L}_X]{} & C^\sim \end{array}$$

is an isomorphism of functors $\Upsilon(\mathscr{D}) : \widetilde{\mathsf{s}}_{X*} \circ \widetilde{u}_* \stackrel{\sim}{\to} \widetilde{u}_{|X*} \circ \widetilde{\mathsf{s}}_{uX*}$. However, a direct computation that we leave to the reader easily shows that $\Upsilon(\mathscr{D})$ is the identity automorphism of $\widetilde{\mathsf{s}}_{X*} \circ \widetilde{u}_* = \widetilde{u}_{|X*} \circ \widetilde{\mathsf{s}}_{uX*}$ □

5.6. **Stacks in groupoids and ind-finite stacks.** Let $C := (\mathscr{C}, J)$ be any site; for every universe V denote by

$$\mathsf{V}\text{-}\mathsf{Stack}^\times(C) \qquad \text{and} \qquad \mathsf{V}\text{-}\mathsf{StGpd}(C)$$

respectively the strong 2-subcategory of $\mathsf{V}\text{-}\mathsf{Fib}^\times(\mathscr{C})$ whose objects are the V-stacks on C and the strong 2-subcategory of $\mathsf{V}\text{-}\mathsf{Stack}(C)$ whose objects are the V-*stacks in groupoids* on C, *i.e.* the V-stacks on C that are fibrations in groupoids. For every $\mathscr{C}$-fibration $F : \mathscr{E} \to \mathscr{C}$, every $X \in \mathrm{Ob}(\mathscr{C})$, and every sieve $\mathscr{S} \subset \mathscr{C}/X$, the inclusion $\mathscr{E}^\times \to \mathscr{E}$ yields a commutative diagram

$$\begin{array}{ccc} \mathscr{E}(X)^\times & \longrightarrow & \mathscr{E}^\times(X) \\ \downarrow & & \downarrow \\ \mathsf{Cart}_\mathscr{C}(\mathscr{S}, \mathscr{E})^\times & \longrightarrow & \mathsf{Cart}_\mathscr{C}(\mathscr{S}, \mathscr{E}^\times) \end{array}$$

whose horizontal arrows are isomorphisms of categories, due to remark 3.4.8(i); notice that the source functor $\mathsf{s} : \mathscr{S} \to \mathscr{C}$ is a fibration in groupoids, by remark 3.5.2(ii). Suppose now that $\mathscr{E}$ is i-separated on C for some $i \in \{0, 1, 2\}$; from remark 3.4.1(ii) we deduce that the left vertical arrow is an i-faithful functor whenever $\mathscr{S} \in J(X)$, and then the same holds for the right vertical arrow, so finally $\mathscr{E}^\times$ is also i-separated on C. Especially, the pseudo-functor $(-)^\times_\mathscr{C}$ restricts to a pseudo-functor

$$(-)^\times_C : \mathsf{V}\text{-}\mathsf{Stack}^\times(C) \to \mathsf{V}\text{-}\mathsf{StGpd}(C).$$

Next, if C is a small site, taking into account remark 3.4.1(iii) we get a natural identification:

$$(\mathscr{E}^+_X)^\times \xrightarrow{\sim} \operatorname*{colim}_{\mathscr{S} \in J(X)} \mathsf{Cart}_\mathscr{C}(\mathscr{S}, \mathscr{E})^\times \xrightarrow{\sim} \operatorname*{colim}_{\mathscr{S} \in J(X)} \mathsf{Cart}_\mathscr{C}(\mathscr{S}, \mathscr{E}^\times) = ((\mathscr{E}^\times)^+)_X$$

whence natural isomorphisms of $\mathscr{C}$-fibrations :

$$(\mathscr{E}^+)^\times \xrightarrow{\sim} (\mathscr{E}^\times)^+ \qquad (\mathscr{E}^a)^\times \xrightarrow{\sim} (\mathscr{E}^\times)^a. \tag{5.6.1}$$

If $C' := (\mathscr{C}', J')$ is another site, and $u : C \to C'$ a weak morphism of sites, it is also clear that $\mathsf{V}\text{-}\mathsf{St}(u)_*$ restricts to pseudo-functors

$$\mathsf{V}\text{-}\mathsf{St}^\times(u)_* : \mathsf{V}\text{-}\mathsf{Stack}^\times(C) \to \mathsf{V}\text{-}\mathsf{Stack}^\times(C') \qquad \mathsf{V}\text{-}\mathsf{StGpd}(u)_* : \mathsf{V}\text{-}\mathsf{StGpd}(C) \to \mathsf{V}\text{-}\mathsf{StGpd}(C')$$

that make commute the diagram of 2-categories :

$$\begin{array}{ccc} \mathsf{V}\text{-}\mathsf{Stack}^\times(C) & \xrightarrow{\mathsf{V}\text{-}\mathsf{St}^\times(u)_*} & \mathsf{V}\text{-}\mathsf{Stack}^\times(C') \\ {\scriptstyle (-)^\times_C}\downarrow & & \downarrow{\scriptstyle (-)^\times_{C'}} \\ \mathsf{V}\text{-}\mathsf{StGpd}(C) & \xrightarrow{\mathsf{V}\text{-}\mathsf{StGpd}(u)_*} & \mathsf{V}\text{-}\mathsf{StGpd}(C'). \end{array} \tag{5.6.2}$$

5.6.3. Consider any U-site $C := (\mathscr{C}, J)$, and let $j : \mathscr{T} \to \mathscr{C}$ be the inclusion functor of an essentially small subcategory whose set of objects is a topologically generating family for C; we endow as usual $\mathscr{T}$ with the topology $J_\mathscr{T}$ induced by J via j, so that j is a continuous functor for J and $J_\mathscr{T}$. Say that $\mathscr{C}$ is U'-small for some universe U', and recall that the rule : $\mathscr{A} \mapsto \mathscr{A}^a := \mathsf{U}'\text{-}\mathsf{Fib}(j)_*(\mathsf{U}'\text{-}\mathsf{Fib}(j)^*(\mathscr{A})^a)$ for every fibration $\mathscr{A}$ with small fibres on $\mathscr{C}$ yields a left 2-adjoint for the inclusion pseudo–functor $\mathsf{Stack}(C) \to \mathsf{Fib}(\mathscr{C})$ (see the proof of corollary 5.4.19). Combining with (3.4.5) and (3.4.6) we deduce a pseudo-commutative diagram :

$$\begin{array}{ccc} \mathsf{V}\text{-}\mathsf{Fib}^\times(\mathscr{C}) & \xrightarrow{(-)^a} & \mathsf{V}\text{-}\mathsf{Stack}^\times(C) \\ {\scriptstyle (-)^\times_\mathscr{C}}\downarrow & & \downarrow{\scriptstyle (-)^\times_C} \\ \mathsf{V}\text{-}\mathsf{Gpd}(\mathscr{C}) & \xrightarrow{(-)^a} & \mathsf{V}\text{-}\mathsf{StGpd}(C) \end{array}$$

for every universe V containing U.

Proposition 5.6.4. *For every morphism of* U*-sites* $u : C' := (\mathscr{C}', J') \to C := (\mathscr{C}, J)$*, and every universe* V *containing* U*, the pseudo-functor* $\mathsf{V}\text{-}\mathsf{St}(u)^*$ *restricts to pseudo-functors*

$$\mathsf{V}\text{-}\mathsf{St}^\times(u)^* : \mathsf{V}\text{-}\mathsf{Stack}^\times(C) \to \mathsf{V}\text{-}\mathsf{Stack}^\times(C') \qquad \mathsf{V}\text{-}\mathsf{StGpd}(u)^* : \mathsf{V}\text{-}\mathsf{StGpd}(C) \to \mathsf{V}\text{-}\mathsf{StGpd}(C')$$

and we have a pseudo-commutative diagram :

$$\begin{array}{ccc} \mathsf{V}\text{-}\mathsf{Stack}^\times(C) & \xrightarrow{\mathsf{V}\text{-}\mathsf{St}^\times(u)^*} & \mathsf{V}\text{-}\mathsf{Stack}^\times(C') \\ {\scriptstyle (-)^\times_C}\downarrow & & \downarrow{\scriptstyle (-)^\times_{C'}} \\ \mathsf{V}\text{-}\mathsf{StGpd}(C) & \xrightarrow{\mathsf{V}\text{-}\mathsf{StGpd}(u)^*} & \mathsf{V}\text{-}\mathsf{StGpd}(C'). \end{array}$$

Proof. Endow the topoi $T := C^\sim_{\mathsf{U}}$ and $T' := C'^\sim_{\mathsf{U}}$ with their canonical topologies; we get an essentially commutative diagram of morphisms of sites :

$$\begin{array}{ccc} T' & \xrightarrow{u'} & T \\ {\scriptstyle h^a_{C'}}\downarrow & & \downarrow{\scriptstyle h^a_C} \\ C' & \xrightarrow{u} & C \end{array} \qquad \text{with } u' := \widetilde{u}^*$$

(lemma 4.2.11(ii)). After replacing U by a larger universe, we may assume that T and T' are small; notice also that u' is a morphism of lex-sites (remark 4.4.13(ii)). Especially, the proposition is already known for u', by virtue of (3.4.7) and (5.6.1).

Claim 5.6.5. The proposition holds for h^a_C and $h^a_{C'}$.

Proof of the claim. It suffices to check the assertion for h^a_C, since the same argument will work for $h^a_{C'}$. Now, let $\mathscr{A}$ be a stack in groupoids on C; we need to check that $\mathsf{St}(h^a_C)^*(\mathscr{A})$ is a stack of groupoids on T. However, since $\mathsf{St}(h^a_C)_*$ is a 2-equivalence (proposition 5.4.17(ii)), we may assume that $\mathscr{A} = \mathsf{St}(h^a_C)_*\mathscr{B}$ for a stack $\mathscr{B}$ on T, and we may moreover assume that $\mathscr{B}$ is a stack in groupoids, by (5.6.1). But since $\mathsf{St}(h^a_C)^*(\mathscr{A})$ is equivalent to $\mathscr{B}$, the assertion follows. This shows that the pseudo-functor $\mathsf{StGpd}(h^a_C)^*$ is well defined, and then it is clearly a pseudo-inverse for $\mathsf{StGpd}(h^a_C)_*$. But then the required pseudo-commutativity of the resulting diagram follows easily from the commutativity of (5.6.2) : details left to the reader. ◊

Now, let $\mathscr{A}$ be a stack in groupoids on C; we need to check that $\mathscr{B} := \mathsf{St}(u)^*(\mathscr{A})$ is a stack in groupoids on C', and by claim 5.6.5 it suffices to show that $\mathsf{St}(h^a_{C'})^*(\mathscr{B})$ is a stack in groupoids on T'; but the latter is equivalent to $\mathsf{St}(u')^* \circ \mathsf{St}(h^a_C)^*(\mathscr{A})$, and the proposition is already known for both u' and h^a_C. This shows that $\mathsf{StGpd}(u)^*$ is well defined.

Lastly, we consider the diagram of 2-categories :

$$\begin{array}{ccccccc} \mathsf{StGpd}(T) & & & \xrightarrow{\mathsf{StGpd}(u')^*} & & & \mathsf{StGpd}(T') \\ & \nwarrow{\scriptstyle (-)^\times_T} & & & & \nearrow{\scriptstyle (-)^\times_{T'}} & \\ & & \mathsf{Stack}^\times(T) & \xrightarrow{\mathsf{St}^\times(u')^*} & \mathsf{Stack}^\times(T') & & \\ {\scriptstyle \mathsf{StGpd}(h^a_C)^*}\downarrow & & {\scriptstyle \mathsf{St}^\times(h^a_C)^*}\downarrow & & \downarrow{\scriptstyle \mathsf{St}^\times(h^a_{C'})^*} & & \downarrow{\scriptstyle \mathsf{StGpd}(h^a_{C'})^*} \\ & & \mathsf{Stack}^\times(C) & \xrightarrow{\mathsf{St}^\times(u)^*} & \mathsf{Stack}^\times(C') & & \\ & \swarrow{\scriptstyle (-)^\times_C} & & & & \searrow{\scriptstyle (-)^\times_{C'}} & \\ \mathsf{StGpd}(C) & & & \xrightarrow{\mathsf{StGpd}(u)^*} & & & \mathsf{StGpd}(C') \end{array}$$

whose inner and outer square subdiagrams are pseudo-commutative. Moreover, by the foregoing, we know that also the left, the right, and the top trapezoidal subdiagrams pseudo-commute.

Since $\mathsf{St}^\times(h^a_C)$ is a 2-equivalence, a little diagram chase shows easily that the bottom trapezoidal subdiagram pseudo-commutes as well, whence the proposition. □

5.6.6. Let $C := (\mathscr{C}, J)$ be any U-site; according to example 5.1.2, for every universe V containing U, the pseudo-functor $\mathscr{F}ib_{\mathscr{C}}$ of (3.4.9) restricts to a (strict) pseudo-functor :

$$\mathscr{F}ib_C : C^\sim_{\mathsf{V}} \to \mathsf{V}\text{-}\mathsf{StGpd}(C)$$

and then it is clear that $\pi_0^{\mathscr{C}}$ induces a left 2-adjoint pseudo-functor

$$\pi_0^C : \mathsf{V}\text{-}\mathsf{StGpd}(C) \to C^\sim_{\mathsf{V}} \qquad \mathscr{E} \mapsto (\pi_0^{\mathscr{C}}(\mathscr{E}))^a.$$

Lemma 5.6.7. *In the situation of* (5.6.6)*, let* $\varphi : \mathscr{E} \to \mathscr{F}$ *be a cartesian functor of fibrations in groupoids over* $\mathscr{C}$*. The following holds :*

(i) φ *is* 0*-covering (for the topology of* C*) if and only if* $\pi_0^{\mathscr{C}}(\varphi)^a : \pi_0^{\mathscr{C}}(\mathscr{E})^a \to \pi_0^{\mathscr{C}}(\mathscr{F})^a$ *is an epimorphism of sheaves on* C*.*

(ii) *If* φ *is* i*-covering for* $i = 0, 1$*, then* $\pi_0^{\mathscr{C}}(\varphi)^a$ *is an isomorphism.*

Proof. Both assertions follow by direct inspection of the definitions, taking into account corollary 4.1.30 and remark 4.1.29(ii,iii). □

5.6.8. Let $u : C' := (\mathscr{C}', J') \to C := (\mathscr{C}, J)$ be a morphism of U-sites; then for every universe V containing U the induced diagram of 2-categories

$$\begin{array}{ccc}
\mathsf{V}\text{-}\mathsf{StGpd}(C) & \xrightarrow{\mathsf{V}\text{-}\mathsf{StGpd}(u)^*} & \mathsf{V}\text{-}\mathsf{StGpd}(C') \\
{\scriptstyle \pi_0^C}\downarrow & & \downarrow{\scriptstyle \pi_0^{C'}} \\
C^\sim & \xrightarrow{\quad \widetilde{u}^* \quad} & C'^\sim
\end{array}$$

is essentially commutative. Indeed, lemma 5.6.7(ii) and proposition 5.2.11(i) imply that for every fibration $\mathscr{E}$ over $\mathscr{C}'$, the unit of adjunction $\mathscr{E} \to \mathscr{E}^a$ induces an isomorphism $\pi_0^{\mathscr{C}'}(\mathscr{E})^a \xrightarrow{\sim} \pi_0^{C'}(\mathscr{E}^a)$ of sheaves on C'. The assertion follows easily, taking into account the essential commutativity of (3.4.10) : the details are left to the reader.

5.6.9. *Ind-finite stacks.* It is easy to say when a sheaf of groups G on a topological space $\mathscr{T}$ is ind-finite, when $\mathscr{T}$ admits a basis of quasi-compact open subsets : in which case, one simply asks that the group of U-sections $G(U)$ is a filtered union of finite groups, for every quasi-compact open subset U of $\mathscr{T}$. This definition is however not suitable for more general topological spaces, nor of course for arbitrary sites. Before we explain a definition that is appropriate for the general case, let us introduce a notion of quasi-compactness for such context:

Definition 5.6.10. Let $(\mathscr{C}, J)$ be any site, and $X \in \mathrm{Ob}(\mathscr{C})$. We say that X is *quasi-compact for the topology* J, if for every covering sieve $\mathscr{S} \in J(X)$ there exists a finite subset $S \subset \mathscr{S}$ that generates a sieve covering X.

Obviously this definition recovers the standard one, in the case of the site of open subsets of any topological space.

Definition 5.6.11. Let $C := (\mathscr{C}, J)$ be a site, and G a sheaf of groups on C. We say that G is *ind-finite* if the following holds. For every $X \in \mathrm{Ob}(\mathscr{C})$ and every finite subset $\Sigma \subset GX$, there exists a covering family $(f_i : X_i \to X \mid i \in I)$ for the topology J such that the set $Gf_i(\Sigma) := \{Gf_i(\sigma) \mid \sigma \in \Sigma\}$ generates a finite subgroup of GX_i, for every $i \in I$.

Remark 5.6.12. The first observation is that if G is ind-finite on the site $(\mathscr{C}, J)$, then for every quasi-compact $X \in \mathrm{Ob}(\mathscr{C})$ the group GX is *ind-finite*, *i.e.* it is a filtered union of finite groups. Indeed, let $\Sigma \subset GX$ be any finite subset; by assumption we have a covering family $(X_i \to X \mid i \in I)$ such that $Gf_i(\Sigma)$ generates a finite group H_i for every $i \in I$, and since X is quasi-compact, we may assume that I is a finite set. But the natural map $GX \to \prod_{i\in I} GX_i$ is injective, and maps the subgroup H generated by Σ into the finite group $\prod_{i\in I} H_i$, so H is finite, whence the claim.

5.6.13. For every group G and every subset $\Sigma \subset G$, let us write $\langle\Sigma\rangle \subset G$ for the subgroup generated by Σ. Definition 5.6.11 prompts the following construction. Let $C := (\mathscr{C}, J)$ be any site, and H any presheaf of groups on $\mathscr{C}$. For every $n \in \mathbb{N}$ and every $X \in \mathrm{Ob}(\mathscr{C})$ we set :

$$(H^n)_{\mathsf{f}}(X) := \{(\sigma_1, \dots, \sigma_n) \in H^n(X) \mid \langle\sigma_1, \dots, \sigma_n\rangle \text{ is a finite group}\}.$$

Clearly the rule : $X \mapsto (H^n)_{\mathsf{f}}(X)$ yields a sub-presheaf H^n_{f} of the *presheaf of sets* H^n, for every $n \in \mathbb{N}$. If H is a sheaf on C, $(H^n)_{\mathsf{f}}$ is not necessarily a sheaf, hence we consider the sheaf

$$(H^n)_{\mathsf{lf}}$$

defined as the smallest subsheaf of sets of H^n containing $(H^n)_{\mathsf{f}}$. The latter is also (naturally isomorphic to) the sheaf of sets $(H^n)^a_{\mathsf{f}}$ on C associated with the presheaf $(H^n)_{\mathsf{f}}$. Clearly every morphism $\varphi : H \to K$ of presheaves of groups on $\mathscr{C}$ induces a morphism of presheaves of sets

$$(\varphi^n)_{\mathsf{f}} : (H^n)_{\mathsf{f}} \to (K^n)_{\mathsf{f}} \qquad \text{for every } n \in \mathbb{N}.$$

If H and K are sheaves of groups on C, we get also an induced morphism of sheaves of sets

$$(\varphi^n)_{\mathsf{lf}} : (H^n)_{\mathsf{lf}} \to (K^n)_{\mathsf{lf}} \qquad \text{for every } n \in \mathbb{N}.$$

Thus, in the notation of definition 5.5.1, we get by these rules two well defined functors

$$(-)^n_{\mathsf{f}} : (\mathscr{C}, \mathsf{V}\text{-}\mathbf{Grp})^\wedge \to \mathscr{C}^\wedge_{\mathsf{V}} \qquad (-)^n_{\mathsf{lf}} : (C, \mathsf{V}\text{-}\mathbf{Grp})^\sim \to C^\sim_{\mathsf{V}} \qquad \text{for every } n \in \mathbb{N}$$

for every universe V.

Remark 5.6.14. Let $C := (\mathscr{C}, J)$ be a site, and H a sheaf of groups on C.

(i) Notice that $(H^n)_{\mathsf{f}}$ is a separated presheaf for every $n \in \mathbb{N}$, since it is a sub-presheaf of the sheaf H^n; hence $(H^n)_{\mathsf{lf}} = (H^n)^+_{\mathsf{f}}$ (claim 4.1.16(ii)). This means that for every $X \in \mathrm{Ob}(\mathscr{C})$, the set $(H^n)_{\mathsf{lf}}(X)$ consists of all $(\sigma_1, \dots, \sigma_n) \in H^n(X)$ for which there exists a covering family $(f_i : X_i \to X \mid i \in I)$ for the topology J such that $\langle Gf_i(\sigma_1), \dots, Gf_i(\sigma_n)\rangle$ is a finite subgroup of GX_i, for every $i \in I$.

(ii) We deduce easily from (i) that H is ind-finite if and only if $H^n = (H^n)_{\mathsf{lf}}$ for every $n \in \mathbb{N}$.

(iii) It also follows from (i) that $(H^n)_{\mathsf{f}}(X) = (H^n)_{\mathsf{lf}}(X)$ for every quasi-compact object X of $\mathscr{C}$. Indeed, let $\underline{\sigma} := (\sigma_1, \dots, \sigma_n) \in (H^n)_{\mathsf{lf}}(X)$, and pick a covering family $(f_i : X_i \to X \mid i \in I)$ verifying the condition of (i) relative to $\underline{\sigma}$; since X is quasi-compact, we may assume that I is a finite set, and then arguing as in remark 5.6.12 we deduce that $\underline{\sigma} \in (H^n)_{\mathsf{f}}(X)$.

Lemma 5.6.15. *Let $(\mathscr{C}, J)$ be a site, and $\varphi : H \to K$ a bicovering morphism of presheaves of groups on $\mathscr{C}$. Then $(\varphi^n)_{\mathsf{f}} : (H^n)_{\mathsf{f}} \to (K^n)_{\mathsf{f}}$ is bicovering for every $n \in \mathbb{N}$.*

Proof. Let $X \in \mathrm{Ob}(\mathscr{C})$ and $\underline{\sigma} := (\sigma_1, \dots, \sigma_n) \in (K^n)_{\mathsf{f}}(X)$; according to remark 4.1.29(ii) we have to exhibit a covering family $(f_i : X_i \to X \mid i \in I)$ such that $(Kf_i(\sigma_1), \dots, Kf_i(\sigma_n))$ lies in the image of the map $(\varphi^n)_{\mathsf{f},X_i} : (H^n)_{\mathsf{f}}(X_i) \to (H^n)_{\mathsf{f}}(X_i)$ for every $i \in I$. But by assumption, for every $j = 1, \dots, n$ there exists a covering sieve $\mathscr{S}_j \subset \mathscr{C}/X$ such that $Kf(\sigma_j)$ lies in the image of the map $\varphi_Y : HY \to KY$ for every $f : Y \to X$ in $\mathscr{S}_j$. Then the sieve $\mathscr{S} := \mathscr{S}_1 \cap \dots \cap \mathscr{S}_n$ still covers X (remark 4.1.3(i)), and for every $g : Y \to X$ in $\mathscr{S}$ we see that $K^n g(\underline{\sigma}) = \varphi^n_Y(\underline{\tau})$ for some $\underline{\tau} := (\tau_1, \dots, \tau_n) \in H^n(Y)$. To conclude, it then suffices to exhibit for every such g and τ a covering sieve $\mathscr{S}' \subset \mathscr{S}/Y$ such that $H^n g'(\underline{\tau}) \in (H^n)_{\mathsf{f}}(Y')$ for every $g' : Y' \to Y$ in $\mathscr{S}'$. However, by construction we have $\varphi^n_Y(\underline{\tau}) \in (K^n)_{\mathsf{f}}(Y)$; the latter means that the subgroup $\langle\varphi_Y(\tau_1), \dots, \varphi_Y(\tau_n)\rangle \subset KY$ is finite. This in turn means that there exists an integer $N \geq 2$, such that for every $\underline{j} \in \{1, \dots, n\}^N$ we may find an integer $M(\underline{j}) < N$ and $\underline{j}' \in \{1, \dots, n\}^{M(\underline{j})}$ with $\prod_{k=1}^N \varphi_Y(\tau_{j_k}) = \prod_{k=1}^{M(\underline{j})} \varphi_Y(\tau_{j'_k})$. By remark 4.1.29(iii), this implies that there exists a covering sieve $\mathscr{S}'_{\underline{j}} \subset \mathscr{C}/Y$ such that $\prod_{k=1}^{N} Hf(\tau_{j_k}) = \prod_{k=1}^{M(\underline{j})} Hf(\tau_{j'_k})$ for every $f : Y' \to Y$ in $\mathscr{S}'_{\underline{j}}$. Then the sieve $\mathscr{S}' := \bigcap_{\underline{j}\in\{1,\dots,n\}^N} \mathscr{S}'_{\underline{j}}$ covers Y, and we deduce that the sequence $H^n f(\underline{\tau})$ generates a finite subgroup of $H^n Y'$, for every $f : Y' \to Y$ in $\mathscr{S}'$; *i.e.* $H^n f(\underline{\tau}) \in (H^n)_{\mathsf{f}}(Y')$, as required.

Lastly, let $\underline{\sigma}, \underline{\sigma}' \in (H^n)_{\mathsf{f}}(X)$ be two sections such that $\varphi^n_X(\underline{\sigma}) = \varphi^n_X(\underline{\sigma}')$; according to remark 4.1.29(iii) we need to exhibit a covering sieve $\mathscr{S} \subset \mathscr{C}/X$ such that $H^n f(\underline{\sigma}) = H^n f(\underline{\sigma}')$ for every $f : Y \to X$ in $\mathscr{S}$. But by assumption, for $i = 1, \dots, n$ there exists a sieve $\mathscr{S}_i$ such that $Hf(\sigma_i) = Hf(\sigma'_i)$ for every f in $\mathscr{S}_i$; clearly $\mathscr{S} := \mathscr{S}_1 \cap \cdots \cap \mathscr{S}_n$ will do. $\square$

5.6.16. Let now $C := (\mathscr{C}, J)$ and $C' := (\mathscr{C}', J')$ be two sites, $u : \mathscr{C}' \to \mathscr{C}$ a functor, and H a presheaf of groups on $\mathscr{C}$. Clearly $u^\wedge H$ is a presheaf of groups on $\mathscr{C}'$, and we have :

$$u^\wedge((H^n)_{\mathsf{f}}) = ((u^\wedge H)^n)_{\mathsf{f}} \qquad \text{for every } n \in \mathbb{N}.$$

Proposition 5.6.17. *In the situation of* (5.6.16)*, suppose that u is cocontinuous for the topologies J and J', and H is a sheaf of groups on C. Then the following holds :*

(i) *$\breve{u}^* H$ is a sheaf of groups on C', and $\breve{u}^*((H^n)_{\mathsf{lf}}) = ((\breve{u}^* H)^n)_{\mathsf{lf}}$ for every $n \in \mathbb{N}$.*
(ii) *If H is ind-finite, the same holds for $\breve{u}^* H$.*

Proof. (i): Since the functor $(-)^a$ is exact, it is clear that the group law $K \times K \to K$ of every presheaf of groups K on $\mathscr{C}'$ yields a group law $K^a \times K^a \to K^a$ for the sheaf K^a : details left to the reader; the first assertion of the proposition is an immediate consequence.

To check the sought identity, recall that by definition $\breve{u}^*((H^n)_{\mathsf{lf}}) = (u^\wedge((H^n)^a_{\mathsf{f}}))^a$, and

$$(u^\wedge((H^n)^a_{\mathsf{f}}))^a = (u^\wedge((H^n)_{\mathsf{f}}))^a$$

by lemma 4.2.9 and corollary 4.1.30(iii). Next, $(u^\wedge((H^n)_{\mathsf{f}}))^a = (((u^\wedge H)^n)_{\mathsf{f}})^a$, by (5.6.16). Lastly, $(((u^\wedge H)^n)_{\mathsf{f}})^a = ((((u^\wedge H)^a)^n)_{\mathsf{f}})^a$, by lemma 5.6.15, whence the contention.

(ii) follows immediately from (i) and remark 5.6.14(ii). $\square$

Lemma 5.6.18. *In the situation of* (5.6.16)*, suppose that the category $X/u\mathscr{C}'$ is cofiltered for every $X \in \mathrm{Ob}(\mathscr{C})$, and let K be a presheaf of groups on $\mathscr{C}'$. Then $u_!K$ is a presheaf of groups on $\mathscr{C}$, and we have a natural isomorphism of presheaves :*

$$u_!((K^n)_{\mathsf{f}}) \xrightarrow{\sim} ((u_!K)^n)_{\mathsf{f}} \qquad \textit{for every } n \in \mathbb{N}.$$

Proof. Under the stated assumption, the functor $u_!$ is exact (*cp.* the proof of corollary 1.5.19(i)), so we easily deduce that $u_!K$ is a presheaf of groups, as in the proof of proposition 5.6.17(i).

Next, we get a natural morphism of presheaves $\omega : u_!((K^n)_{\mathsf{f}}) \to ((u_!K)^n)_{\mathsf{f}}$ as follows. First, the unit of adjunction is a morphism $\eta_K : K \to u^\wedge u_! K$, which induces a morphism $(\eta^n_K)_{\mathsf{f}} : (K^n)_{\mathsf{f}} \to ((u^\wedge u_! K)^n)_{\mathsf{f}}$. Then, by (5.6.16) we have $((u^\wedge u_! K)^n)_{\mathsf{f}} = u^\wedge(((u_!K)^n)_{\mathsf{f}})$, and the resulting morphism $(K^n)_{\mathsf{f}} \to u^\wedge(((u_!K)^n)_{\mathsf{f}})$ yields by adjunction the sought morphism.

Explicitly, for every $X \in \mathrm{Ob}(\mathscr{C})$, every element of $u_!((K^n)_{\mathsf{f}})(X)$ is the class $[\underline{\sigma}]$ of some $\underline{\sigma} := (\sigma_1, \dots, \sigma_n) \in K^n(Y)$, for an object $f : X \to uY$ of $X/u\mathscr{C}'$, and $\omega_X([\underline{\sigma}])$ is the section $([\sigma_1], \dots, [\sigma_n]) \in ((u_!K)^n)_{\mathsf{f}}(X)$, where $[\sigma_i] \in u_!K(X)$ is the class of σ_i, for $i = 1, \dots, n$.

To show the injectivity of ω_X, let $[\underline{\sigma}], [\underline{\tau}] \in u_!((K^n)_{\mathsf{f}})(X)$ such that $\omega_X([\underline{\sigma}]) = \omega_X([\underline{\tau}])$; we may assume that $\underline{\sigma}, \underline{\tau} \in K^n Y$ for some morphism $f : X \to uY$ of $\mathscr{C}'$, and the assumption means that for $i = 1, \dots, n$ there exists a morphism $X/h_i : (g_i : X \to uY_i) \to f$ in $X/u\mathscr{C}'$ such that $Kh_i(\sigma_i) = Kh_i(\tau_i)$ in KY_i. Since $X/u\mathscr{C}'$ is cofiltered, we may then find an object $g' : X \to uY'$ of $X/u\mathscr{C}'$ and morphisms $X/h'_i : g' \to g_i$ of $X/u\mathscr{C}'$ with $(X/h_i) \circ (X/h'_i) = (X/h_j) \circ (X/h'_j)$ for every $i, j = 1, \dots, n$. Recall that $h_i : Y_i \to Y$ and $h'_i : Y' \to Y_i$ are morphisms in $\mathscr{C}'$, and the foregoing identity means that $h := h_i \circ h'_i = h_j \circ h'_j$ for every such i, j. We conclude that $Kh(\sigma_i) = Kh(\tau_i)$ for $i = 1, \dots, n$, whence $[\underline{\sigma}] = [\underline{\tau}]$, as required.

To check the surjectivity of ω_X, let $([\sigma_1],\dots,[\sigma_n]) \in ((u_!K)^n)_{\mathfrak{f}}(X)$; hence, for $i=1,\dots,n$ there exists an object $f_i : X \to uY_i$ of $X/u\mathscr{C}'$ such that $\sigma_i \in KY_i$. Arguing as in the foregoing, we easily reduce to the case where $Y := Y_1 = \cdots = Y_n$ and $f := f_1 = \cdots = f_n$. For every morphism $X/h : (g : X \to uY') \to f$ in $X/u\mathscr{C}'$, let $G_h := \langle Kh(\sigma_1),\dots,Kh(\sigma_n)\rangle \subset KY'$, and set $G := \langle[\sigma_1],\dots,[\sigma_n]\rangle \subset u_!K(X)$. The source morphism $(X/u\mathscr{C}')/f \to X/u\mathscr{C}'$ is coinitial (example 1.5.9(i)), and a direct inspection shows that the group G is the colimit of the functor

$$G_\bullet : ((X/u\mathscr{C}')/f)^o \to \mathbf{Grp} \qquad h \mapsto G_h.$$

To every pair of objects $X/h : (g : X \to uY') \to f$ and $X/h : (g : X \to uY'') \to f$ of $(X/u\mathscr{C}')/f$, and every morphism $(X/l)/f : (X/h') \to (X/h)$ in $(X/u\mathscr{C}')/f$, the functor $G_\bullet$ assigns the group homomorphism $G_h \to G_{h'}$ given by the restriction of $Kh : KY' \to KY''$. By assumption, G is a finite group; let us then remark more generally :

Claim 5.6.19. Let $\Gamma_\bullet : I \to \mathbf{Grp}$ be a functor from a filtered category I, such that :

(a) Γ_i is a finitely generated group for every $i \in \mathrm{Ob}(I)$
(b) Γ_φ is a surjective group homomorphism for every morphism φ of I
(c) The colimit G of $\Gamma_\bullet$ is a finite group.

Let also $\tau_\bullet : \Gamma_\bullet \Rightarrow c_G$ be a universal cocone. Then there exists a cofinal functor $\psi : J \to I$ such that $\tau * \psi$ is an isomorphism of functors $\Gamma_\bullet \circ \psi \xrightarrow{\sim} c_G$.

Proof of the claim. By virtue of proposition 1.5.21(ii) we may assume that $(I,\leq)$ is a filtered partially ordered set. Since I is filtered, the colimit of $\Gamma_\bullet$ commutes with the forgetful functor $\Phi : \mathbf{Grp} \to \mathbf{Set}$. Especially, for every $x \in G$ we may find $i(x) \in \mathrm{Ob}(I)$ and $x' \in \Gamma_{i(x)}$ such that $\tau_{i(x)}(x') = x$. Since I is filtered and G is finite, we may then find $i \in \mathrm{Ob}(I)$ with $i \geq i(x)$ for every $x \in G$. Hence the rule : $x \mapsto \Gamma_{i(x),i}(x')$ defines a set-theoretic section $\sigma_i : G \to \Gamma_i$ of τ_i. Then for every $j \geq i$, the composition $\sigma_j := \Gamma_{ij} \circ \sigma_i$ is a set-theoretic section of τ_j. After replacing I by the cofinal subset $\{j \in I \mid j \geq i\}$ we may therefore assume that there exists a cone $\sigma_\bullet : c_G \Rightarrow \Phi \circ \Gamma_\bullet$ such that $\tau_i \circ \sigma_i = \mathbf{1}_G$ for every $i \in I$. Next, for every $x, y \in G$ we may find $i(x,y) \in I$ such that

$$\sigma_{i(x,y)}(xy) = \sigma_{i(x,y)}(x) \cdot \sigma_{i(x,y)}(y)$$

(details left to the reader). Again, we may then pick $i_0 \in I$ such that $i_0 \geq i(x,y)$ for every $x, y \in G$, in which case it is easily seen that $\sigma_{i_0}(xy) = \sigma_{i_0}(x) \cdot \sigma_{i_0}(y)$ for every $x, y \in G$, *i.e.* σ_{i_0} is a group homomorphism. Then clearly σ_j is a group homomorphism for every $j \geq i_0$, so after replacing I by the cofinal subset of elements $\geq i_0$, we may assume that $\sigma_\bullet$ is even a cone $c_G \Rightarrow \Gamma_\bullet$, and that I admits an initial element i_0. Now, let $g_1,\dots,g_k$ be a finite system of generators for Γ_{i_0}. For $t = 1,\dots,k$ and every $l \in I$ we have $\tau_l \circ \sigma_l \circ \tau_l(g_t) = \tau_l \circ \Gamma_{i_0,l}(g_t)$; hence we may find $j_t \in I$ such that

$$\sigma_{j_t} \circ \tau_{j_t}(g_t) = \Gamma_{i_0,j_t}(g_t).$$

Then pick again $j \in I$ such that $j \geq j_t$ for every $t = 1,\dots,k$, and set $g'_t := \Gamma_{i_0,j}(g_t)$ for every such t. It follows that for every $t = 1,\dots,k$ we have :

$$\begin{aligned} \sigma_j \circ \tau_j(g'_t) &= \sigma_j \circ \tau_j \circ \Gamma_{i_0,j}(g_t) \\ &= \sigma_j \circ \tau_{j_t} \circ \Gamma_{i_0,j_t}(g_t) \\ &= \Gamma_{j_t,j} \circ \sigma_{j_t} \circ \tau_{j_t} \circ \Gamma_{i_0,j_t}(g_t) \\ &= \Gamma_{j_t,j} \circ \Gamma_{i_0,j_t}(g_t) \\ &= g'_t. \end{aligned}$$

Notice that $g'_1,\dots,g'_k$ is a generating system for Γ_j, since $\Gamma_{i_0,j}$ is a surjective map. We conclude that $\sigma_j \circ \tau_j = \mathbf{1}_{\Gamma_j}$, so τ_j and σ_j are mutually inverse group isomorphisms. After replacing I

by the cofinal subset $\{l \in I \mid l \geq j\}$, we may then assume that $j = i_0$ is the initial element of I. Then, for every $i \in I$ the map $\sigma_i = \Gamma_{i_0,i} \circ \sigma_{i_0}$ is surjective; but σ_i is also injective by construction, so σ_i is an isomorphism for every $i \in I$, and finally, the same follows for τ_i. $\lozenge$

From claim 5.6.19 we deduce that there exists a morphism $X/h : (X \to uY') \to f$ in $X/u\mathscr{C}'$ such that G_h is a finite group, and therefore $\underline{\sigma}' := K^n h(\sigma_1, \dots, \sigma_n) \in (K^n)_{\mathsf{f}}(Y')$. Then the class $[\underline{\sigma}'] \in u_!(K^n)(X)$ lies in $u_!((K^n)_{\mathsf{f}})(X)$ and $\omega_X([\underline{\sigma}']) = ([\sigma_1], \dots, [\sigma_n])$ as required. $\square$

Theorem 5.6.20. *In the situation of* (5.6.16)*, suppose that* $u : C \to C'$ *is a morphism of sites, and let* K *be a sheaf of groups* K *on* C'*. Then* $\widetilde{u}^* K$ *is a sheaf of groups on* C*, and we have a natural isomorphism of sheaves :*

$$\omega^{(n)}_{u,K} : \widetilde{u}^*((K^n)_{\mathsf{lf}}) \xrightarrow{\sim} ((\widetilde{u}^* K)^n)_{\mathsf{lf}} \qquad \text{for every } n \in \mathbb{N}.$$

Proof. By assumption, for every universe V such that C and C' are V-small, the functor $\widetilde{u}^* : C'^{\sim} \to C^{\sim}$ is exact; as in the proof of proposition 5.6.17(i), we deduce that $\widetilde{u}^* K$ is a sheaf of groups. We get therefore for every such V a well defined functor

$$(\widetilde{u}, \mathsf{V}\text{-}\mathbf{Grp})^* : (C', \mathbf{Grp})^{\sim} \to (C, \mathsf{V}\text{-}\mathbf{Grp})^{\sim} \qquad K \mapsto \widetilde{u}^* K.$$

Next, the unit of adjunction $\eta_K : K \to \widetilde{u}_* \widetilde{u}^* K$ induces a morphism of presheaves $(\eta^n_K)_{\mathsf{f}} : (K^n)_{\mathsf{f}} \to ((\widetilde{u}_* \widetilde{u}^* K)^n)_{\mathsf{f}}$. By (5.6.16), we have $((\widetilde{u}_* \widetilde{u}^* K)^n)_{\mathsf{f}} = u^\wedge(((\widetilde{u}^* K)^n)_{\mathsf{f}})$, so by adjunction $(\eta^n_K)_{\mathsf{f}}$ corresponds to a morphism of presheaves $u_!((K^n)_{\mathsf{f}}) \to ((\widetilde{u}^* K)^n)_{\mathsf{f}}$. After taking associated sheaves, we get a morphism of sheaves $\omega : (u_!((K^n)_{\mathsf{f}}))^a \to ((\widetilde{u}^* K)^n)_{\mathsf{lf}}$. Lastly, the inclusion of presheaves $(K^n)_{\mathsf{f}} \to (K^n)_{\mathsf{lf}}$ induces an isomorphism $(u_!((K^n)_{\mathsf{f}}))^a \xrightarrow{\sim} \widetilde{u}^*((K^n)_{\mathsf{lf}})$, by lemma 4.2.11(ii), and $\omega^{(n)}_{u,K}$ is the composition of ω with the inverse of this isomorphism. With the notation of (5.6.13), we have therefore a natural transformation :

$$\omega^{(n)}_{u,\bullet} : \widetilde{u}^* \circ (-)^n_{\mathsf{lf}} \Rightarrow (-)^n_{\mathsf{lf}} \circ (\widetilde{u}, \mathbf{Grp})^* \qquad K \mapsto \omega^{(n)}_{u,K}$$

and it remains to check that $\omega^{(n)}_{u,K}$ is an isomorphism. We begin with the following more explicit description of the map $(\omega^{(n)}_{u,K})_X : \widetilde{u}^*((K^n)_{\mathsf{lf}})(X) \to ((\widetilde{u}^* K)^n)_{\mathsf{lf}}(X)$, for every $X \in \mathrm{Ob}(\mathscr{C})$. First, by definition every element of $\widetilde{u}^*((K^n)_{\mathsf{lf}})(X)$ is the equivalence classes $[\underline{\sigma}]_f$ of a sequence $\underline{\sigma} := (\sigma_1, \dots, \sigma_n) \in K^n Y$, for a morphism $f : X \to uY$ in $\mathscr{C}$, such that for some covering sieve $\mathscr{S} \subset \mathscr{C}'/Y$, the subgroup $\langle Kg(\sigma_1), \dots, Kg(\sigma_n) \rangle \subset KY'$ is finite for every $(g : Y' \to Y) \in \mathrm{Ob}(\mathscr{S})$. Likewise, every element of $((\widetilde{u}^* K)^n)_{\mathsf{lf}}(X)$ is a sequence $([\tau_1]_{f_1}, \dots, [\tau_n]_{f_n})$ such that $[\tau_i]_{f_i} \in \widetilde{u}^* K(X)$ for every $i = 1, \dots, n$ is the equivalence class of a section $\tau_i \in KY_i$, for a morphism $f_i : X \to uY_i$ in $\mathscr{C}$, and there exists a covering sieve $\mathscr{T} \subset \mathscr{C}/X$ such that the subgroup $\langle [\tau_1]_{f_1 \circ h}, \cdots, [\tau_n]_{f_n \circ h} \rangle \subset (\widetilde{u}^* K)(X')$ is finite for every $(h : X' \to X) \in \mathrm{Ob}(\mathscr{T})$.

Now, given such class $[\underline{\sigma}]_f$ and covering sieve $\mathscr{S}$, the sieve $u(\mathscr{S}) \subset \mathscr{C}/uY$ generated by $\{u(g) \mid g \in \mathrm{Ob}(\mathscr{S})\}$ covers uY (lemma 4.2.4), and $\mathscr{T} := f \times_{uY} u(\mathscr{S})$ covers X for the topology J. For every $(h : X' \to X) \in \mathrm{Ob}(\mathscr{T})$ there exist $(g : Y' \to Y) \in \mathrm{Ob}(\mathscr{S})$ and a morphism $f' : X' \to uY'$ in $\mathscr{C}$ with $f \circ h = u(g) \circ f'$. It follows that

$$[\sigma_i]_{f \circ h} = [\sigma_i]_{u(g) \circ f'} = [Kg(\sigma_i)]_{f'} \qquad \text{for } i = 1, \dots, n$$

and therefore $\langle [\sigma_1]_{f \circ h}, \dots, [\sigma_n]_{f \circ h} \rangle$ is a finite subgroup of $(\widetilde{u}^* K)(X')$. Then, by unwinding the definition, we find that $(\omega^{(n)}_{u,K})_X$ is given by the rule :

$$[\underline{\sigma}]_f \mapsto ([\sigma_1]_f, \dots, [\sigma_n]_f).$$

Next, endow $T := C^\sim$ and $T' := C'^\sim$ with their canonical topologies, and consider the essentially commutative diagram of sites provided by lemma 4.2.11(ii) :

$$\begin{array}{ccc} T & \xrightarrow{u'} & T' \\ {\scriptstyle v}\downarrow & \not\swarrow_\beta & \downarrow{\scriptstyle v'} \\ C & \xrightarrow[u]{} & C' \end{array} \qquad \text{with } u' := \widetilde{u}^*, \ \ v' := h^a_{C'} \text{ and } v := h^a_C.$$

After replacing U by a larger universe, we may assume that T and T' are small; we notice :

Claim 5.6.21. The theorem holds for u' and every sheaf H on T'.

Proof of the claim. Recall that u' is a morphism of lex-sites (remark 4.4.13(ii)), and in particular u' fulfills the condition of lemma 5.6.18, so we get an isomorphism of presheaves $\omega : u'_!((H^n)_{\mathsf{f}}) \xrightarrow{\sim} ((u'_! H)^n)_{\mathsf{f}}$. Taking into account lemma 4.2.11(ii), we deduce an isomorphism of sheaves $\omega^a : \widetilde{u}'^*((H^n)_{\mathsf{lf}}) \xrightarrow{\sim} ((\widetilde{u}'^* H)^n)_{\mathsf{lf}}$. But a direct inspection of the construction shows that the latter agrees with $\omega^{(n)}_{u',H}$. ◊

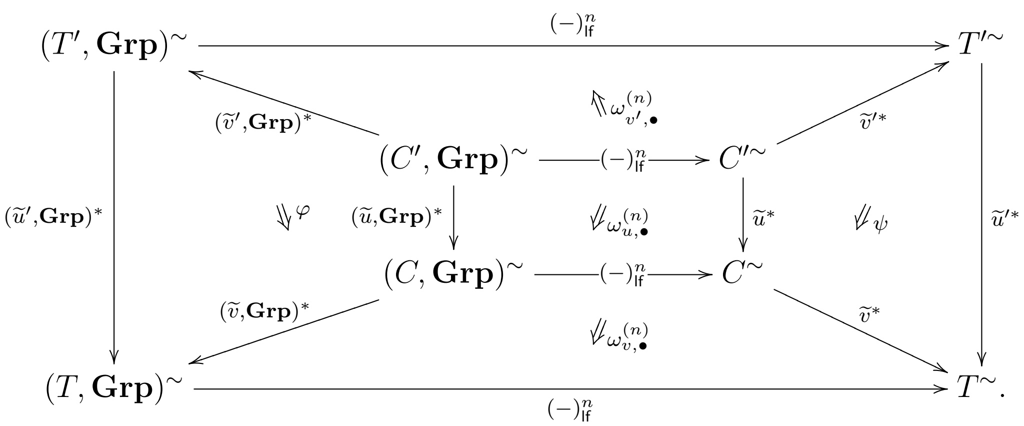
Claim 5.6.22. The theorem holds as well for v and v'.

Proof of the claim. It suffices to show the claim for v, since the same argument will work for v' as well. Recall now that $\breve{v}^* = \widetilde{v}_* : C^\sim \to T^\sim$ is an equivalence (theorem 4.4.2(iii)), and $\widetilde{v}^*$ is its quasi-inverse. It follows that $(\widetilde{v}, \mathbf{Grp})_* : (C, \mathbf{Grp})^\sim \to (T, \mathbf{Grp})^\sim$ is also an equivalence, and its inverse is $(\widetilde{v}, \mathbf{Grp})^*$. Then, let H be any sheaf of groups on C; in order to check that $\omega_{v,H}$ is an isomorphism, we may assume that $H = \widetilde{v}_* L$ for a sheaf of groups L on T, and it suffices to check the commutativity of the diagram :

$$\begin{array}{ccc} \widetilde{v}^*((\widetilde{v}_* L^n)_{\mathsf{lf}}) & \xrightarrow{\omega^{(n)}_{v,\widetilde{v}_* L}} & ((\widetilde{v}^*\widetilde{v}_* L)^n)_{\mathsf{lf}} \\ \Vert & & \downarrow{\scriptstyle ((\varepsilon_L)^n)_{\mathsf{lf}}} \\ \widetilde{v}^*\widetilde{v}_*((L^n)_{\mathsf{lf}}) & \xrightarrow{\varepsilon_{(L^n)_{\mathsf{lf}}}} & (L^n)_{\mathsf{lf}} \end{array}$$

where $\varepsilon_\bullet : \widetilde{v}^*\widetilde{v}_* \xrightarrow{\sim} \mathbf{1}_{T^\sim}$ is the counit for the adjoint $(\widetilde{v}^*, \widetilde{v}_*)$. However, for every sheaf F on T, and every $X \in \mathrm{Ob}(T)$ the elements of $\widetilde{v}^*\widetilde{v}_* F(X)$ are the equivalence classes $[\sigma]_f$, where $f : X \to vY$ is a morphism in T, and $\sigma \in \widetilde{v}_* F(Y) = F(vY)$; with this notation, the map $\varepsilon_{F,X} : \widetilde{v}^*\widetilde{v}_* F(X) \xrightarrow{\sim} FX$ is given by the rule : $[\sigma]_f \mapsto Ff(\sigma)$. Then the assertion follows after simple inspection, by combining with the foregoing explicit description of $\omega^{(n)}_{v,\widetilde{v}_* L}$. ◊

We deduce an oriented diagram of categories :



which we complete by adding the orientation $\omega^{(n)}_{u',\bullet}$ for the "front face". Here φ and ψ denote the identifications induced by β :

$$(\widetilde{u}', \mathbf{Grp})^* \circ (\widetilde{v}', \mathbf{Grp})^* \stackrel{\sim}{\to} (\widetilde{v}, \mathbf{Grp})^* \circ (\widetilde{u}, \mathbf{Grp})^* \qquad \text{and} \qquad \widetilde{u}'^* \circ \widetilde{v}'^* \stackrel{\sim}{\to} \widetilde{v}^* \circ \widetilde{u}^*.$$

Explicitly, for every sheaf F' on C' and every $X \in \mathrm{Ob}(T)$, the elements of $\widetilde{u}'^* \circ \widetilde{v}'^* F(X)$ are the equivalences classes $[[\tau]_g]_f$, where $f : X \to u'Y$ is a morphism of T, $g : Y \to v'Z$ is a morphism of T', and $\tau \in FZ$; likewise one describes the elements of $\widetilde{v}^* \circ \widetilde{u}^* F(X)$, and notice that $\beta_Z \circ u'(g) \circ f$ is a morphism $X \to vuZ$ in T. Then $\psi_{F,X} : \widetilde{u}'^* \circ \widetilde{v}'^* F(X) \stackrel{\sim}{\to} \widetilde{v}^* \circ \widetilde{u}^* F(X)$ is given by the rule :

$$[[\tau]_g]_f \mapsto [[\tau]_{\beta_Z \circ u'(g) \circ f}]_{\mathbf{1}_{uZ}}.$$

If F is a sheaf of groups on T, the same rule defines the group isomorphism $\varphi_{F,X}$. Now, a simple inspection shows that this oriented diagram commutes on 2-cells, in the sense of (2.3.21). By claims 5.6.21 and 5.6.22 we know that $\omega^{(n)}_{u',\bullet}$, $\omega^{(n)}_{v,\bullet}$ and $\omega^{(n)}_{v',\bullet}$ are isomorphisms; we conclude that the same holds for $\omega^{(n)}_{u,\bullet}$, as stated. □

Corollary 5.6.23. *In the situation of theorem* 5.6.20*, the following holds :*

(i) *If K is ind-finite, the same holds for $\widetilde{u}^* K$.*
(ii) *If $\widetilde{u}^*$ is a conservative functor and $\widetilde{u}^* K$ is ind-finite, then K is ind-finite.*

Proof. The assertions are immediate consequences of theorem 5.6.20 and remark 5.6.14(ii). □

Proposition 5.6.24. *In the situation of* (5.6.16)*, suppose that u is continuous and there exists a topologically generating family $G \subset \mathrm{Ob}(\mathscr{C}')$ such that uY is quasi-compact for the topology J, for every $Y \in G$. Then, if H is an ind-finite sheaf on C, the sheaf $\widetilde{u}_* H$ is ind-finite on C'.*

Proof. Let $X \in \mathrm{Ob}(\mathscr{C}')$ be any object, and $\Sigma \subset \widetilde{u}_* H(X) = H(uX)$ any finite set. Pick a covering family $(f_i : Y_i \to X \mid i \in I)$ with $Y_i \in G$ for every $i \in I$; by remark 5.6.12, the group $\widetilde{u}_* H(Y_i) = H(uY_i)$ is ind-finite, hence $Hf_i(\Sigma)$ generates a finite subgroup of $H(uY_i)$ for every $i \in I$, whence the assertion. □

5.6.25. $C := (\mathscr{C}, J)$ be a site, and $\mathscr{A} \to \mathscr{C}$ a fibration; for every $X \in \mathrm{Ob}(\mathscr{C})$ and every cartesian section $\sigma \in \mathrm{Ob}(\mathscr{A}(X))$ we define the presheaf on $\mathscr{C}/X$:

$$\mathscr{C}art^\times(\sigma) := \mathscr{C}art(\sigma^\times, \sigma^\times)$$

(notation of (3.4.2)). Clearly, for every $f \in \mathrm{Ob}(\mathscr{C}/X)$, the set $\mathscr{C}art^\times(\sigma)(f)$ carries a natural group structure, given by composition of automorphisms of the cartesian section $(\sigma \circ f_*)^\times$ of $\mathscr{A}^\times$, so $\mathscr{C}art^\times(\sigma)$ is a presheaf of groups on $\mathscr{C}/X$.

Definition 5.6.26. With the notation of (5.6.25), we say that $\mathscr{A}$ is an *ind-finite prestack* on C, if $\mathscr{C}art^\times(\sigma)$ an ind-finite sheaf of groups on C/X, for every $X \in \mathrm{Ob}(\mathscr{C})$, and every $\sigma \in \mathrm{Ob}(\mathscr{A}(X))$ (notation of (4.7)). We say that $\mathscr{A}$ is an *ind-finite stack* on C, if it is both an ind-finite prestack and a stack on C.

5.6.27. Let $u : C := (\mathscr{C}, J) \to C' := (\mathscr{C}', J')$ be a morphism of small sites, $\mathscr{E}$ a stack on C'; let also $X \in \mathrm{Ob}(\mathscr{C}')$, and $\sigma, \tau \in \mathrm{Ob}(\mathscr{E}(X))$. Set $\mathscr{E}' := \mathsf{St}(u)^* \mathscr{E}$, and denote

$$\eta_{\mathscr{E}} : \mathscr{E} \to \mathsf{St}(u)_* \mathscr{E}'$$

the unit of the 2-adjunction for the 2-adjoint pair $(\mathsf{St}(u)^*, \mathsf{St}(u)_*)$. According to (5.4.7), the natural projection $\pi : \mathsf{St}(u)_* \mathscr{E}' \to \mathscr{E}'$ induces an equivalence of categories :

$$u^*_{|X} : \mathscr{E}'(uX) \stackrel{\sim}{\to} \mathsf{St}(u)_* \mathscr{E}'(X).$$

We may then find $\sigma', \tau' \in \mathrm{Ob}(\mathscr{E}'(uX))$ and isomorphisms

$$\omega_1 : \eta_{\mathscr{E}} \circ \sigma \stackrel{\sim}{\to} u^*_{|X}(\sigma') \qquad \omega_2 : \eta_{\mathscr{E}} \circ \tau \stackrel{\sim}{\to} u^*_{|X}(\tau').$$

Taking into account lemma 5.1.9 we deduce isomorphisms of sheaves on C'/X :

$$\mathscr{C}art(\eta_{\mathscr{E}} \circ \sigma, \eta_{\mathscr{E}} \circ \tau) \xrightarrow{\sim} \mathscr{C}art(u^*_{|X}\sigma', u^*_{|X}\tau') \xrightarrow{\sim} \widetilde{u}_{|X*}\mathscr{C}art(\sigma', \tau')$$

namely the composition of the isomorphism (5.1.7) induced by ω_1 and ω_2, with the isomorphism of presheaves given in (5.4.7). Then (5.1.8) yields as well a morphism of sheaves :

$$\mathscr{C}art(\sigma, \tau) \to \mathscr{C}art(\eta_{\mathscr{E}} \circ \sigma, \eta_{\mathscr{E}} \circ \tau) \qquad \beta \mapsto \eta_{\mathscr{E}} * \beta.$$

By adjunction, the composition of these morphisms induces a morphism of sheaves on C/uX :

$$\widetilde{u}^*_{|X}\mathscr{C}art(\sigma, \tau) \to \mathscr{C}art(\sigma', \tau'). \tag{5.6.28}$$

Remark 5.6.29. (i) Let $(f : Y \to uX) \in \mathrm{Ob}(\mathscr{C}/uX)$. The evaluation at f of the morphism (5.6.28) can be described explicitly as follows. For every element $[\alpha] \in \widetilde{u}^*_{|X}\mathscr{C}art(\sigma, \tau)(f)$ we may find a covering family $(k_\lambda : Y_\lambda \to Y \mid \lambda \in \Lambda)$ and for every $\lambda \in \Lambda$ a morphism $h_\lambda : Y_\lambda \to uZ_\lambda$ in $\mathscr{C}$ and a morphism $g_\lambda : Z_\lambda \to X$ in $\mathscr{C}'$ with $f \circ k_\lambda = u(g_\lambda) \circ h_\lambda$, such that $[\alpha]$ is represented by a system of morphisms of cartesian sections $(\alpha_\lambda : \sigma \circ g_{\lambda*} \Rightarrow \tau \circ g_{\lambda*})$. Set

$$\alpha^*_\lambda := \pi(\omega_{2,g_\lambda} \circ \eta_{\mathscr{E}}(\alpha_{\lambda,\mathbf{1}_{Z_\lambda}}) \circ \omega^{-1}_{1,g_\lambda}) : \sigma'(ug_\lambda) \to \tau'(ug_\lambda) \qquad \text{for every } \lambda \in \Lambda.$$

Then for every $\lambda \in \Lambda$ there exists a unique morphism $\alpha^{**}_\lambda : \sigma'(f \circ k_\lambda) \to \tau'(f \circ k_\lambda)$ in $\mathscr{E}'_{Y_\lambda}$ that makes commute the following diagram in $\mathscr{E}'$:

$$\begin{array}{ccc} \sigma'(f \circ k_\lambda) & \xrightarrow{\alpha^{**}_\lambda} & \tau'(f \circ k_\lambda) \\ {\scriptstyle \sigma'(h_\lambda/uX)} \downarrow & & \downarrow {\scriptstyle \tau'(h_\lambda/uX)} \\ \sigma'(ug_\lambda) & \xrightarrow{\alpha^*_\lambda} & \tau'(ug_\lambda) \end{array}$$

and there exists a unique morphism of cartesian sections $\alpha^{**} : \sigma' \circ f_* \Rightarrow \tau' \circ f*$ such that $\alpha^{**}_{k_\lambda} = \alpha^{**}_\lambda$ for every $\lambda \in \Lambda$. Then the map (5.6.28) is given by the rule : $[\alpha] \mapsto \alpha^{**}$.

(ii) The morphism (5.6.28) depends obviously on the choice of σ', τ', ω_1 and ω_2. However, say that $\sigma'', \tau'' \in \mathrm{Ob}(\mathscr{E}'(uX))$ are two other cartesian sections with isomorphisms $\omega'_1 : \eta_{\mathscr{E}} \circ \sigma \xrightarrow{\sim} u^*_{|X}(\sigma'')$ and $\omega'_2 : \eta_{\mathscr{E}} \circ \tau \xrightarrow{\sim} u^*_{|X}(\tau'')$. Then there exists unique isomorphisms $\rho_1 : \sigma' \xrightarrow{\sim} \sigma''$ and $\rho_2 : \tau' \xrightarrow{\sim} \tau''$ in $\mathscr{E}'(uX)$ such that

$$\omega'_i = u^*_{|X}(\rho_i) \circ \omega_i \qquad \text{for } i = 1, 2$$

and in light of (i) it is easily seen that we get a commutative diagram :

$$\begin{array}{ccc} & \mathscr{C}art(\sigma, \tau) & \\ \swarrow & & \searrow \\ \mathscr{C}art(\sigma', \tau') & \xrightarrow{\sim} & \mathscr{C}art(\sigma'', \tau'') \end{array}$$

whose downward arrows are the morphisms (5.6.28) relative to the choices (ω_1, ω_2) and respectively (ω'_1, ω'_2), and whose horizontal arrow is the isomorphism (5.1.7) induced by (ρ_1, ρ_2).

Proposition 5.6.30. *The morphism* (5.6.28) *is an isomorphism.*

Proof. Let us check that (5.6.28) is a monomorphism. Thus, let $(f : Y \to uX) \in \mathrm{Ob}(\mathscr{C}/uX)$, and $[\alpha], [\beta] \in \widetilde{u}^*_{|X}\mathscr{C}art(\sigma, \tau)(f)$ whose images agree in $\mathscr{C}art(\sigma', \tau')(f)$; we need to show that $[\alpha] = [\beta]$. We may find a covering family $(k_\lambda : Y_\lambda \to Y \mid \lambda \in \Lambda)$ such that the restrictions $[\alpha_\lambda], [\beta_\lambda] \in \widetilde{u}^*_{|X}\mathscr{C}art(\sigma, \tau)(f \circ k_\lambda)$ of $[\alpha]$ and $[\beta]$ are in the image of the natural map $u_{|X!}\mathscr{C}art(\sigma, \tau)(f \circ k_\lambda) \to \widetilde{u}^*_{|X}\mathscr{C}art(\sigma, \tau)(f \circ k_\lambda)$, for every $\lambda \in \Lambda$. Then the images of

$[\alpha_\lambda]$ and $[\beta_\lambda]$ agree in $\mathscr{C}art(\sigma', \tau')(f \circ k_\lambda)$, and it suffices to check that $[\alpha_\lambda] = [\beta_\lambda]$ for every such λ. Thus, we may assume from start that $[\alpha], [\beta]$ are in the image of the natural map $u_{|X!}\mathscr{C}art(\sigma, \tau)(f) \to \widetilde{u}^*_{|X}\mathscr{C}art(\sigma, \tau)(f)$. This means that there exists a commutative diagram

$$\begin{array}{ccccc} uZ' & \xleftarrow{h'} & Y & \xrightarrow{h} & uZ \\ & {\scriptstyle u(g')}\searrow & \downarrow{\scriptstyle f} & \swarrow{\scriptstyle u(g)} & \\ & & uX & & \end{array}$$

in $\mathscr{C}$ such that $[\alpha]$ and $[\beta]$ are represented by natural $\mathscr{C}$-transformations

$$\alpha : \sigma \circ g_* \Rightarrow \tau \circ g_* \qquad \beta : \sigma \circ g'_* \Rightarrow \tau \circ g'_*.$$

By proposition 4.3.9, the fibration $\mathsf{s} : \mathscr{C}/u\mathscr{C}' \to \mathscr{C}$ is locally cofiltered relative to the topology J of C, hence there exist a covering family $(k_\lambda : Y_\lambda \to Y \mid \lambda \in \Lambda)$ and for every $\lambda \in \Lambda$ a morphism $h''_\lambda : Y_\lambda \to uZ_\lambda$ in $\mathscr{C}$, and two morphisms $l_\lambda : Z_\lambda \to Z$, $l'_\lambda : Z_\lambda \to Z'$ such that

$$h \circ k_\lambda = u(l_\lambda) \circ h''_\lambda \qquad \text{and} \qquad h' \circ k_\lambda = u(l'_\lambda) \circ h''_\lambda \qquad \text{for every } \lambda \in \Lambda.$$

It suffices to check that the images of $[\beta]$ and $[\alpha]$ in $\mathscr{C}art(\sigma, \tau)(f \circ k_\lambda)$ coincide for every $\lambda \in \Lambda$; but these images are the classes of $[\alpha * l_\lambda]$ and respectively $[\beta * l'_\lambda]$. Hence, we may assume from start that $Z = Z'$, $h = h'$ and $g = g'$. With the notation of (5.4.7), the condition on $[\alpha]$ and $[\beta]$ amounts then to the identity :

$$u^*_{|Z}(\eta_{\mathscr{E}} * \alpha) * h_* = u^*_{|Z}(\eta_{\mathscr{E}} * \beta) * h_* \qquad \text{in } \mathscr{E}'(Y). \tag{5.6.31}$$

Recall that the evaluation functor $\mathsf{ev}^{\mathscr{E}}_Z : \mathscr{E}(Z) \to \mathscr{E}_Z$ is an equivalence of categories for every $Z \in \mathrm{Ob}(\mathscr{C})$, natural with respect to morphism in $\mathscr{C}$ (claim 3.2.8); this equivalence assigns to the cartesian sections $\sigma \circ g_*$ and $\tau \circ g_*$ their evaluations $\sigma_g := \sigma(g)$ and $\tau_g := \tau(g)$ in c_X, and to α and β the morphisms $\alpha_0 := \alpha_{\mathbf{1}_Z} : \sigma_g \to \tau_g$ and $\beta_0 := \beta_{\mathbf{1}_Z} : \sigma_g \to \tau_g$ in $\mathscr{E}_Z$. We wish next to similarly interpret the identity (5.6.31) as an equality of morphisms in the fibre category $\mathscr{E}'_Y$. To this aim, let us pick a unital cleavage for $\mathscr{E}$, and let c be its associated unital pseudo-functor; denote also by γ^{c} the coherence constraint of c. Recall that for every $Y \in \mathrm{Ob}(\mathscr{C})$, the fibre category $\mathsf{Fib}(u)_!\mathscr{E}_Y$ of the $\mathscr{C}$-fibration $\mathsf{Fib}(u)_!\mathscr{E}$ represents the 2-colimit of the pseudo-functor

$$F_Y := \mathsf{c} \circ \mathsf{t}^o_Y : (Y/u\mathscr{C}')^o \to \mathbf{Cat} \qquad (g : Y \to uZ) \mapsto \mathsf{c}_Z = \mathscr{E}_Z.$$

The latter is realized by the localization $\mathscr{F}ib(F_Y)[\Sigma^{-1}_Y]$, where Σ_Y denotes the set of cartesian morphisms of $\mathscr{F}ib(F_Y)$. Recall that the objects of $\mathscr{F}ib(F_Y)[\Sigma^{-1}_Y]$ are the data

$$(Z, h : Y \to uZ, T) \qquad \text{with } (Z, h) \in \mathrm{Ob}(Y/u\mathscr{C}') \text{ and } T \in \mathrm{Ob}(\mathscr{E}_Z).$$

The morphisms are the pairs $[Y/g, t] : (Z, h, T) \to (Z', h', T')$ where $Y/g : h \to h'$ is a morphism in $Y/u\mathscr{C}'$ and $t : T \to \mathsf{c}_g T'$ is a morphism in $\mathscr{E}_Z$. Moreover, every morphism $k : Y \to Y'$ in $\mathscr{C}$ induces a $\mathscr{C}$-cartesian functor

$$k^* : \mathscr{F}ib(F_{Y'}) \to \mathscr{F}ib(F_Y) \qquad (Z, h, T) \mapsto (Z, h \circ k, T) \qquad [Y'/g, t] \mapsto [Y/g, t]$$

which therefore extends uniquely to a functor

$$\mathscr{F}ib(F_{Y'})[\Sigma^{-1}_{Y'}] \to \mathscr{F}ib(F_Y)[\Sigma^{-1}_Y]$$

and the system of such functors provides a natural split cleavage for $\mathsf{Fib}(u)_!\mathscr{E}$. Notice then the rules : $Y \mapsto \mathscr{F}ib(F_Y)$ and $k \mapsto k^*$ for every $Y \in \mathrm{Ob}(\mathscr{C})$ and every morphism k of $\mathscr{C}$, define a pseudo-functor $\mathsf{d} : \mathscr{C}^o \to \mathbf{Cat}$, and the foregoing description amounts to saying that $\mathsf{Fib}(u)_!\mathscr{E}$ is represented by the fibration which in the proof of proposition 5.3.1 is denoted

$$\mathscr{F}ib(\mathsf{d})[\Sigma^{-1}_\bullet] \to \mathscr{C}.$$

By proposition 5.3.1, the natural functor $j : \mathsf{Fib}(u)_!\mathscr{E} \to \mathscr{E}'$ factors then as a composition

$$\mathsf{Fib}(u)_!\mathscr{E} \xrightarrow{L_{\{\Sigma\}}} \mathscr{F}ib(\mathsf{d})\{\Sigma_\bullet^{-1}\} \xrightarrow{j'} \mathscr{E}'.$$

Furthermore, for every $Z \in \mathrm{Ob}(\mathscr{C}')$ we have a natural identification $u^*_{|uZ} : \mathsf{St}(u)_*\mathscr{E}'_Z \xrightarrow{\sim} \mathscr{E}'_{uZ}$. Hence, $\eta_\mathscr{E}$ restricts to a functor of fibre categories $\eta_{\mathscr{E},Z} : \mathscr{E}_Z \to \mathscr{E}'_{uZ}$ for every such Z. However, $\mathscr{E}_Z$ is also the fibre category over $\mathbf{1}_{uZ} \in \mathrm{Ob}(uZ/u\mathscr{C}')$ of the $(uZ/u\mathscr{C}')$-fibration $\mathscr{F}ib(F_{uZ})$; in terms of the foregoing realization of $\mathsf{Fib}(u)_!\mathscr{E}_{uZ}$, the functor $\eta_{\mathscr{E},Z}$ is then just given by the rule :

$$\mathscr{E}_Z \to \mathscr{F}ib(F_{uZ})[\Sigma^{-1}_{uZ}] \xrightarrow{j_{uZ}} \mathscr{E}'_{uZ} \qquad T \mapsto j_{uZ}(Z, \mathbf{1}_{uZ}, T)$$

where j_{uZ} is the restriction of j. Summing up, we find that the objects $\eta_\mathscr{E}(\sigma_g)$ and $\eta_\mathscr{E}(\tau_g)$ of $\mathscr{E}'_{uZ}$ are respectively $j_{uZ}(Z, \mathbf{1}_{uZ}, \sigma_g)$ and $j_{uZ}(Z, \mathbf{1}_{uZ}, \tau_g)$, and (5.6.31) translates as the identity:

$$j_{uZ}[Y/\mathbf{1}_Z, \alpha_0] = j_{uZ}[Y/\mathbf{1}_Z, \beta_0].$$

But since j is 2-covering (proposition 5.2.11(i)), we deduce that there exists a covering family $(k_\lambda : Y_\lambda \to Y \mid \lambda \in \Lambda)$ in $\mathscr{C}$ such that $[Y_\lambda/\mathbf{1}_Z, \alpha_0] = [Y_\lambda/\mathbf{1}_Z, \beta_0]$ in $\mathscr{F}ib(F_{Y_\lambda})[\Sigma^{-1}_{Y_\lambda}]$ for every $\lambda \in \Lambda$. Since it suffices to check that the images of $[\alpha]$ and $[\beta]$ coincide in $\mathscr{C}art(\sigma, \tau)(f \circ k_\lambda)$ for every λ, we may then assume from start that $[Y/\mathbf{1}_Z, \alpha_0] = [Y/\mathbf{1}_Z, \beta_0]$ in $\mathscr{F}ib(F_Y)[\Sigma_Y^{-1}]$.

Claim 5.6.32. The system $\Sigma_\bullet := (\Sigma_Y \mid Y \in \mathrm{Ob}(\mathscr{C}))$ admits a right local calculus of fractions.

Proof of the claim. Conditions (LCF1) and (LCF2) of definition 5.3.2 obviously hold for $\Sigma_\bullet$. Next, let $(Y/k', l) : (h', T') \to (h, T)$ be a morphism in $\mathscr{F}ib(F_Y)$ and $(Y/k'', t) : (h'', T'') \to (h, T)$ an element of Σ_Y; *i.e.* we have a commutative diagram of morphisms of $\mathscr{C}$:

$$\begin{array}{ccccc} uZ' & \xleftarrow{h'} & Y & \xrightarrow{h''} & uZ'' \\ & {\scriptstyle u(k')}\searrow & \downarrow{\scriptstyle h} & \swarrow{\scriptstyle u(k'')} & \\ & & uZ & & \end{array}$$

together with a morphism $l : T' \to \mathsf{c}_{k'}T$ and an isomorphism $t : T'' \xrightarrow{\sim} \mathsf{c}_{k''}T$ in $\mathscr{A}_Y$. By proposition 4.3.9, the fibration $\mathsf{s} : \mathscr{C}/u\mathscr{C}' \to \mathscr{C}$ is locally cofiltered relative to the topology J of C, hence there exist a covering family $(g_\lambda : Y_\lambda \to Y \mid \lambda \in \Lambda)$ and for every $\lambda \in \Lambda$ a morphism $h_\lambda : Y_\lambda \to uZ_\lambda$ in $\mathscr{C}$, and two morphisms $k'_\lambda : Z_\lambda \to Z'$, $k''_\lambda : Z_\lambda \to Z''$ such that

$$h' \circ g_\lambda = u(k'_\lambda) \circ h_\lambda \qquad h'' \circ g_\lambda = u(k''_\lambda) \circ h_\lambda \qquad k' \circ k'_\lambda = k'' \circ k''_\lambda \qquad \text{for every } \lambda \in \Lambda$$

whence a well defined morphism in $\mathscr{A}_{Y_\lambda}$, for every $\lambda \in \Lambda$:

$$l_\lambda : \mathsf{c}_{k'_\lambda}T' \xrightarrow{\mathsf{c}_{k'_\lambda}l} \mathsf{c}_{k'_\lambda}\mathsf{c}_{k'}T \xrightarrow{\gamma^{\mathsf{c}}_{(k'_\lambda,k'),T}} \mathsf{c}_{k'\circ k'_\lambda}T \xrightarrow{\gamma^{\mathsf{c}\,-1}_{(k''_\lambda,k''),T}} \mathsf{c}_{k''_\lambda}\mathsf{c}_{k''}T \xrightarrow{(\mathsf{c}_{k''_\lambda}t)^{-1}} \mathsf{c}_{k''_\lambda}T''$$

which then yield a commutative diagram in $\mathscr{F}ib(F_{Y_\lambda})$ for every such λ :

$$\begin{array}{ccc} (h_\lambda, \mathsf{c}_{k'_\lambda}T') & \xrightarrow{(Y_\lambda/k'_\lambda, t_\lambda)} & g^*_\lambda(h', T') \\ {\scriptstyle (Y_\lambda/k''_\lambda, l_\lambda)}\downarrow & & \downarrow{\scriptstyle g^*_\lambda(Y/k', l)} \\ g^*_\lambda(h'', T'') & \xrightarrow{g^*_\lambda(Y/k'', t)} & g^*_\lambda(Y/h, T) \end{array}$$

where we take for t_λ the identity of $\mathsf{c}_{k'_\lambda}T'$. This shows that condition (LCF3) holds for $\Sigma_\bullet$. Lastly, suppose that we have a commutative diagram in $\mathscr{F}ib(\mathscr{A}_Y)$:

$$(h, T) \overset{(Y/k,l)}{\underset{(Y/k',l')}{\rightrightarrows}} (h', T') \xrightarrow{(Y/k'', t)} (h'', T'')$$

with $(Y/k'', t) \in \Sigma_Y$. This means that we have a commutative diagram in $\mathscr{C}$:

$$\begin{array}{ccccc} & & Y & & \\ & \swarrow^{h} & \downarrow{\scriptstyle h'} & {}^{h''}\searrow & \\ uZ & \underset{u(k')}{\overset{u(k)}{\rightrightarrows}} & uZ' & \xrightarrow{u(k'')} & uZ'' \end{array}$$

and morphisms $l : T \to \mathsf{c}_k T'$, $l' : T \to \mathsf{c}_{k'} T'$ and an isomorphism $t : T' \xrightarrow{\sim} \mathsf{c}_{k''} T''$ in $\mathscr{A}_Y$ with

$$k'' \circ k = k'' \circ k' \qquad \text{and} \qquad \gamma^{\mathsf{c}}_{(k,k''),T''} \circ \mathsf{c}_k(t) \circ l = \gamma^{\mathsf{c}}_{(k',k''),T''} \circ \mathsf{c}_{k'}(t) \circ l'.$$

Since the fibration $\mathsf{s} : \mathscr{C}/u\mathscr{C}' \to \mathscr{C}$ is locally cofiltered, there exist a covering family $(g_\lambda : Y_\lambda \to Y \mid \lambda \in \Lambda)$ and for every $\lambda \in \Lambda$, morphisms $h_\lambda : Y_\lambda \to uZ_\lambda$ in $\mathscr{C}$ and $k_\lambda : Z_\lambda \to Z$ in $\mathscr{C}'$ such that $h \circ g_\lambda = u(k_\lambda) \circ h_\lambda$ in $\mathscr{C}$ and $k \circ k_\lambda = k' \circ k_\lambda$ in $\mathscr{C}'$. We compute :

$$\begin{aligned}
\gamma^{\mathsf{c}}_{(k\circ k_\lambda,k''),T''} \circ \mathsf{c}_{k\circ k_\lambda}(t) \circ \gamma^{\mathsf{c}}_{(k,k_\lambda),T'} \circ \mathsf{c}_{k_\lambda}(l) &= \gamma^{\mathsf{c}}_{(k\circ k_\lambda,k''),T''} \circ \gamma^{\mathsf{c}}_{(k,k_\lambda),\mathsf{c}_{k''}T''} \circ \mathsf{c}_{k_\lambda}\mathsf{c}_k(t) \circ \mathsf{c}_{k_\lambda}(l) \\
&= \gamma^{\mathsf{c}}_{(k_\lambda,k''\circ k),T''} \circ \mathsf{c}_{k_\lambda}(\gamma^{\mathsf{c}}_{(k,k''),T''}) \circ \mathsf{c}_{k_\lambda}\mathsf{c}_k(t) \circ \mathsf{c}_{k_\lambda}(l) \\
&= \gamma^{\mathsf{c}}_{(k_\lambda,k''\circ k'),T''} \circ \mathsf{c}_{k_\lambda}(\gamma^{\mathsf{c}}_{(k,k''),T''} \circ \mathsf{c}_k(t) \circ l) \\
&= \gamma^{\mathsf{c}}_{(k_\lambda,k''\circ k'),T''} \circ \mathsf{c}_{k_\lambda}(\gamma^{\mathsf{c}}_{(k',k''),T''} \circ \mathsf{c}_{k'}(t) \circ l') \\
&= \gamma^{\mathsf{c}}_{(k_\lambda,k''\circ k'),T''} \circ \mathsf{c}_{k_\lambda}(\gamma^{\mathsf{c}}_{(k',k''),T''}) \circ \mathsf{c}_{k_\lambda}\mathsf{c}_{k'}(t) \circ \mathsf{c}_{k_\lambda}(l') \\
&= \gamma^{\mathsf{c}}_{(k'\circ k_\lambda,k''),T''} \circ \gamma^{\mathsf{c}}_{(k',k_\lambda),\mathsf{c}_{k''}T''} \circ \mathsf{c}_{k_\lambda}\mathsf{c}_{k'}(t) \circ \mathsf{c}_{k_\lambda}(l') \\
&= \gamma^{\mathsf{c}}_{(k'\circ k_\lambda,k''),T''} \circ \mathsf{c}_{k'\circ k_\lambda}(t) \circ \gamma^{\mathsf{c}}_{(k',k_\lambda),T'} \circ \mathsf{c}_{k_\lambda}(l')
\end{aligned}$$

whence $\gamma^{\mathsf{c}}_{(k,k_\lambda),T'} \circ \mathsf{c}_{k_\lambda}(l) = \gamma^{\mathsf{c}}_{(k',k_\lambda),T'} \circ \mathsf{c}_{k_\lambda}(l')$. We deduce a commutative diagram in $\mathscr{F}ib(F_{Y_\lambda})$

$$(h_\lambda, \mathsf{c}_{k_\lambda}T) \xrightarrow{(Y/k_\lambda,t_\lambda)} g^*_\lambda(h,T) \underset{g^*_\lambda(Y/k',l')}{\overset{g^*_\lambda(Y/k,l)}{\rightrightarrows}} g^*_\lambda(h',T') \qquad \text{for every } \lambda \in \Lambda$$

where t_λ is the identity of $\mathsf{c}_{k_\lambda}T$. This shows that (LFC4) holds for $\Sigma_\bullet$. ◊

To ease notation, set $A := L_{\{\Sigma\}}(Z, h, \sigma_g)$ and $B := L_{\{\Sigma\}}(Z, h, \tau_g)$, and define the presheaf $M_{A,B}$ on $\mathscr{C}/uZ$ and the sheaf $H_{A,B}$ on C/uZ as in (5.3.5). Then

$$L_{\{\Sigma\}}[Y/\mathbf{1}_Z, \alpha] = L_{\{\Sigma\}}[Y/\mathbf{1}_Z, \beta] \in H_{A,B}(h)$$

so there exists a covering family $(k_\lambda : Y_\lambda \to Y \mid \lambda \in \Lambda)$ such that the images of $[Y_\lambda/\mathbf{1}_Z, \alpha]$ and $[Y_\lambda/\mathbf{1}_Z, \beta]$ agree in $M_{A,B}(h \circ g_\lambda)$. Hence as usual we may assume from start that the images of $[Y/\mathbf{1}_Z, \alpha_0]$ and $[Y/\mathbf{1}_Z, \beta_0]$ agree in $M_{A,B}(h)$. Then, in view of claim 5.6.32, condition (b) of (5.3.5) tells us that there exist a covering family $(k_\lambda : Y_\lambda \to Y \mid \lambda \in \Lambda)$, and for every $\lambda \in \Lambda$ a morphism $h_\lambda : Y_\lambda \to uZ_\lambda$ in $\mathscr{C}$ and a morphism $l_\lambda : Z_\lambda \to Z$ in $\mathscr{C}'$ such that $h \circ k_\lambda = u(l_\lambda) \circ h_\lambda$ and with $\mathsf{c}_{l_\lambda}(\alpha_0) = \mathsf{c}_{l_\lambda}(\beta_0)$. This means that the images of $[\alpha]$ and $[\beta]$ agree in $\widetilde{u}^*_{|X}\mathscr{C}art(\sigma, \tau)(f \circ k_\lambda)$ for every $\lambda \in \Lambda$, so $[\alpha] = [\beta]$, as required.

Next, let us check that (5.6.28) is an epimorphism. To this aim, set $\mathscr{E}'' := \mathsf{Fib}(u)_!\mathscr{E}$, and let $\eta'_{\mathscr{E}} : \mathscr{E} \to \mathsf{Fib}(u)^*\mathscr{E}''$ be the unit of the 2-adjunction for the 2-adjoint pair $(\mathsf{Fib}(u)_!, \mathsf{Fib}(u)^*)$; as in(5.6.27), the projection $\pi' : \mathsf{Fib}(u)^*\mathscr{E}'' \to \mathscr{E}''$ induces an equivalence of categories

$$u^*_{|X} : \mathscr{E}''(uX) \xrightarrow{\sim} \mathsf{Fib}(u)^*\mathscr{E}''(X)$$

so we may find $\sigma'', \tau'' \in \mathrm{Ob}(\mathscr{E}''(uX))$ with isomorphisms :

$$\eta'_{\mathscr{E}} \circ \sigma \xrightarrow{\sim} u^*_{|X}(\sigma'') \qquad \eta'_{\mathscr{E}} \circ \tau \xrightarrow{\sim} u^*_{|X}(\tau'').$$

Since $\mathsf{Fib}(u)^*(j) \circ \eta'_{\mathscr{E}} : \mathscr{E} \to \mathsf{St}(u)_*\mathscr{E}'$ is isomorphic to $\eta_{\mathscr{E}}$, we deduce isomorphisms

$$\sigma' \xrightarrow{\sim} j \circ \sigma'' \qquad \tau' \xrightarrow{\sim} j \circ \tau''$$

and it suffices to show that the corresponding morphism of sheaves on C/uX

$$\widetilde{u}^*_{|X}\mathscr{C}art(\sigma,\tau) \to \mathscr{C}art(j \circ \sigma'', j \circ \tau'')$$

is an epimorphism. Thus, let $(f : Y \to X) \in \mathrm{Ob}(\mathscr{C}/X)$ and $\mu' \in \mathscr{C}art(j \circ \sigma'', j \circ \tau'')(f)$; we need to show that $\mu' : j \circ \sigma'' \circ f_* \Rightarrow j \circ \tau'' \circ f_*$ is the image of some $[\mu] \in \widetilde{u}^*_{|X}\mathscr{C}art(\sigma,\tau)(f)$. Set $\sigma''_0 := \sigma''(f)$, $\tau''_0 := \tau''(f)$, and recall that μ' is determined by its evaluation

$$\mu'_0 := \mu'_{\mathbf{1}_Y} : j(\sigma''_0) \to j(\tau''_0).$$

Since j is 1-covering (proposition 5.2.11(i)), there exist a covering family $(k_\lambda : Y_\lambda \to Y \mid \lambda \in \Lambda)$, and for every $\lambda \in \Lambda$ a morphism $\mu''_\lambda : \sigma''(f \circ k_\lambda) \to \tau''(f \circ k_\lambda)$ in $\mathscr{E}''_{Y_\lambda}$ such that $\mu'_{k_\lambda} = j(\mu''_\lambda)$. Now, suppose that for every $\lambda \in \Lambda$ there exists $\mu_\lambda \in \widetilde{u}^*_{|X}\mathscr{C}art(\sigma,\tau)(f \circ k_\lambda)$ whose image in $\mathscr{C}art(\sigma',\tau')(f \circ k_\lambda)$ equals $\mu' * k_\lambda$. Then, for every $\lambda, \lambda' \in \Lambda$, and every pair of morphisms $Y_{\lambda'} \xleftarrow{k''_{\lambda\lambda'}} Y_{\lambda\lambda'} \xrightarrow{k'_{\lambda\lambda'}} Y_\lambda$ in $\mathscr{C}$ such that $k_{\lambda\lambda'} := k_\lambda \circ k'_{\lambda\lambda'} = k_{\lambda'} \circ k''_{\lambda\lambda'}$, let $\mu_{\lambda\lambda'}$ and $\mu_{\lambda'\lambda}$ be the images of μ_λ and respectively $\mu_{\lambda'}$ in $\widetilde{u}^*_{|X}\mathscr{C}art(\sigma,\tau)(k_{\lambda\lambda'} \circ f)$; it follows that $\mu_{\lambda\lambda'}$ and $\mu_{\lambda'\lambda}$ are both mapped to $\mu' * k_{\lambda\lambda'}$ by the morphism (5.6.28). Since we already know that the latter is a monomorphism, we deduce that $\mu_{\lambda\lambda'} = \mu_{\lambda'\lambda}$ for every such pair of morphisms $k'_{\lambda\lambda'}, k''_{\lambda\lambda'}$. Then the system $(\mu_\lambda \mid \lambda \in \Lambda)$ determines a unique $\mu \in \widetilde{u}^*_{|X}\mathscr{C}art(\sigma,\tau)(f)$ whose image in $\mathscr{C}art(\sigma',\tau')(f)$ equals μ', as sought. Thus, we may replace μ by $\mu * k_\lambda$ for every $\lambda \in \Lambda$, and assume from start that there exists a morphism

$$\mu''_0 : L_{\{\Sigma\}}\sigma''_0 \to L_{\{\Sigma\}}\tau''_0 \qquad \text{in } \mathscr{F}ib(\mathsf{d})\{\Sigma^{-1}_\bullet\}_Y \text{ such that} \qquad \mu'_0 = j'(\mu''_0).$$

Set $\sigma''' := L_{\{\Sigma\}}\sigma''$ and $\tau''' := L_{\{\Sigma\}}\tau''$; then μ''_0 corresponds to a unique morphism of cartesian sections $\mu'' : \sigma''' \circ f_* \Rightarrow \tau''' \circ f_*$ with $j' * \mu'' = \mu'$. Recall that σ''_0 (resp. τ''_0) is a datum (Z, h, T) (resp. (Z', h', T')). Now, according to condition (a) of (5.3.5), we may find a covering family $(k_\lambda : Y_\lambda \to Y \mid \lambda \in \Lambda)$, and for every $\lambda \in \Lambda$, an object $(Z_\lambda, h_\lambda : Y_\lambda \to uZ_\lambda, T_\lambda)$ of $\mathscr{F}ib(F_{Y_\lambda})$, and morphisms $(Z, h \circ k_\lambda, T) \xleftarrow{(Y_\lambda/l_\lambda, s_\lambda)} (Z_\lambda, h_\lambda, T_\lambda) \xrightarrow{(Y_\lambda/l'_\lambda, t_\lambda)} (Z', h' \circ k_\lambda, T')$ in $\mathscr{F}ib(F_{Y_\lambda})$, where $s_\lambda : T_\lambda \to \mathsf{c}_{l_\lambda}T$ is an isomorphism in $\mathscr{E}_{Z_\lambda}$, and such that we have a commutative diagram in $\mathscr{F}ib(\mathsf{d})\{\Sigma^{-1}_\bullet\}_{Y_\lambda}$:

$$\begin{array}{ccc}
(Z, h \circ k_\lambda, T) & \xrightarrow{L_{\{\Sigma\}}(Y_\lambda/l'_\lambda, t_\lambda) \circ L_{\{\Sigma\}}(Y_\lambda/l_\lambda, s_\lambda)^{-1}} & (Z', h' \circ k_\lambda, T) \\
\downarrow & & \downarrow \\
\sigma'''(f \circ k_\lambda) & \xrightarrow{\quad \mu''_{k_\lambda} \quad} & \tau'''(f \circ k_\lambda)
\end{array}$$

whose vertical arrows are isomorphisms. The latter is equivalent to a commutative diagram :

$$\begin{array}{ccc}
(Z_\lambda, h_\lambda, \mathsf{c}_{l_\lambda}T) & \xrightarrow{L_{\{\Sigma\}}(Y_\lambda/\mathbf{1}_{Z_\lambda}, t_\lambda \circ s_\lambda^{-1})} & (Z_\lambda, h_\lambda, \mathsf{c}_{l'_\lambda}T') \\
\downarrow & & \downarrow \\
\sigma'''(f \circ k_\lambda) & \xrightarrow{\quad \mu''_{k_\lambda} \quad} & \tau'''(f \circ k_\lambda)
\end{array}$$

whose vertical arrows are again isomorphisms. After replacing σ''' and τ''' by isomorphic cartesian sections, we may therefore assume that $\mu''_{k_\lambda} = L_{\{\Sigma\}}(Y_\lambda/\mathbf{1}_{Z_\lambda}, t_\lambda \circ s_\lambda^{-1})$ for every $\lambda \in \Lambda$.

Arguing as in the foregoing, we may then assume that there exist a morphism $h : Y \to uZ$ in $\mathscr{C}$ and a morphism $t : T \to T'$ in $\mathscr{E}_Z$ such that $\mu'_0 = j(Y/\mathbf{1}_Z, t) : j(Z, h, T) \to j(Z, h, T')$.

Next, since the fibration $\mathsf{s} : \mathscr{C}/u\mathscr{C}' \to \mathscr{C}$ is locally cofiltered over the site C, we may find a covering family $(k_\lambda : Y_\lambda \to Y \mid \lambda \in \Lambda)$, and for every $\lambda \in \Lambda$ a morphism $h_\lambda : Y_\lambda \to uZ_\lambda$ in $\mathscr{C}$ and two morphisms $X \xleftarrow{g_\lambda} Z_\lambda \xrightarrow{g'_\lambda} Z$ such that $f \circ k_\lambda = g_\lambda \circ h_\lambda$ and $h'' \circ k_\lambda = g'_\lambda \circ h_\lambda$. Then again, we may replace σ' and τ' by isomorphic cartesian sections, and assume that $\mu'_{k_\lambda} =$

$j(Y_\lambda/\mathbf{1}_{Z_\lambda}, \mathsf{c}_{g'_\lambda}t) : (Z_\lambda, h_\lambda, \mathsf{c}_{g'_\lambda}T) \to (Z_\lambda, h_\lambda, \mathsf{c}_{g'_\lambda}T')$. Arguing once again as in the foregoing, we may finally assume that we have still $\mu'_0 = j(Y/\mathbf{1}_Z, t) : j(Z, h, T) \to j(Z, h, T')$, and moreover there exists a morphism $g : Z \to X$ in $\mathscr{C}$ such that $u(g) \circ h = f$. But then, t corresponds to a morphism of cartesian sections $\mu : \sigma \circ f_* \to \tau \circ f_*$, and the pair $(Y/g : f \to h, \mu)$ yields the sought section $[\mu]$. □

Theorem 5.6.33. *In the situation of* (5.6.27)*, the following holds :*

(i) *If $\mathscr{E}$ is ind-finite, then the same holds for $\mathscr{E}'$.*
(ii) *If $\widetilde{u}^*$ is a conservative functor and $\mathscr{E}'$ is ind-finite, then $\mathscr{E}$ is ind-finite.*

Proof. (i): In view of proposition 5.6.4, we may replace $\mathscr{E}$ by $\mathscr{E}^\times$, and assume from start that $\mathscr{E}$ is a stack in groupoids, in which case we need to show that for every $Y \in \mathrm{Ob}(\mathscr{C})$ and every $\sigma' \in \mathrm{Ob}(\mathscr{E}'(Y))$, the sheaf of groups $\mathscr{C}\!art(\sigma', \sigma')$ is ind-finite on C/Y.

Since the natural functor $j : \mathscr{E}'' := \mathsf{Fib}(u)_!\mathscr{E} \to \mathscr{E}'$ is 0-covering (proposition 5.2.11(i)), there exist a covering family $(\varphi_\lambda : Y_\lambda \to Y \mid \lambda \in \Lambda)$, and for every $\lambda \in \Lambda$ a cartesian section $\sigma''_\lambda \in \mathrm{Ob}(\mathscr{E}''(Y_\lambda))$ with an isomorphism $j \circ \sigma''_\lambda \xrightarrow{\sim} \sigma' \circ \varphi_{\lambda*}$. Clearly it suffices to show that $\mathscr{C}\!art(\sigma' \circ \varphi_{\lambda*}, \sigma' \circ \varphi_{\lambda*})$ is ind-finite for every $\lambda \in \Lambda$. We may thus replace Y by Y_λ for every such λ, and assume from start that $\sigma' = j \circ \sigma''$ for some $\sigma'' \in \mathrm{Ob}(\mathscr{E}''(Y))$. Recall that $\sigma''(\mathbf{1}_Y) \in \mathrm{Ob}(\mathscr{E}''_Y)$ is a datum $(X, f : Y \to uX, T)$ where $(X, f) \in \mathrm{Ob}(Y/u\mathscr{C}')$ and $T \in \mathrm{Ob}(\mathscr{E}_X)$. Let then $\tau : \mathscr{C}'/X \to \mathscr{E}$ and $\tau' : \mathscr{C}/uX \to \mathscr{E}''$ be cartesian sections with $\tau_{\mathbf{1}_X} = T$ and $\tau'(\mathbf{1}_{uX}) := (X, \mathbf{1}_{uX}, T)$. As explained in the proof of of proposition 5.6.30, we have an isomorphism :

$$\eta'_{\mathscr{E}} \circ \tau \xrightarrow{\sim} u^*_{|X}(\tau')$$

where $\eta'_{\mathscr{E}} : \mathscr{E} \to \mathsf{Fib}(u)^*\mathscr{E}''$ is the unit of a 2-adjunction for the 2-adjoint pair $(\mathsf{Fib}(u)_!, \mathsf{Fib}(u)^*)$. Since $\mathsf{Fib}(u)^*(j) \circ \eta'_{\mathscr{E}} : \mathscr{E} \to \mathsf{St}(u)_*\mathscr{E}'$ is isomorphic to the unit $\eta_{\mathscr{E}} : \mathscr{E} \to \mathsf{St}(u)_*\mathscr{E}'$ of a 2-adjunction for the 2-adjoint pair $(\mathsf{Fib}(u)_!, \mathsf{Fib}(u)^*)$, there follows an isomorphism

$$\eta_{\mathscr{E}} \circ \tau \xrightarrow{\sim} u^*_{|X}(j \circ \tau').$$

From proposition 5.6.30 and corollary 5.6.23(i) we conclude that $\mathscr{C}\!art(j \circ \tau', j \circ \tau')$ is ind-finite on C/uX. On the other hand, we have $\tau' \circ f_* \simeq \sigma''$, whence $j \circ \tau' \circ f_* \simeq \sigma'$. Combining with (5.1.5), there follows an isomorphism of sheaves $\mathscr{C}\!art(\sigma', \sigma') \xrightarrow{\sim} (f_*)^\sim_*\mathscr{C}\!art(j \circ \tau', j \circ \tau')$. But recall that f_* is both continuous and cocontinuous for the topologies of the sites C/Y and C/uX (remark 4.7.3(i)); then the assertion follows from proposition 5.6.17(ii) and lemma 4.2.15(i).

(ii): We may reduce as in (i) to the case where $\mathscr{E}$ is a stack in groupoids. Next notice that if $\widetilde{u}^*$ is conservative, $\widetilde{u}^*_{|X}$ is conservative for every $X \in \mathrm{Ob}(\mathscr{C}')$, by virtue of proposition 4.7.7(ii). Then the assertion follows immediately from proposition 5.6.30 and corollary 5.6.23(ii). □

Proposition 5.6.34. *In the situation of* (5.6.16)*, suppose that u is a weak morphism of sites, and moreover there exists a topologically generating family $G \subset \mathrm{Ob}(\mathscr{C}')$ such that uY is quasi-compact for the topology J, for every $Y \in G$. Then, for every ind-finite stack $\mathscr{E}$ on C, the stack $\mathsf{St}(u)_*\mathscr{E}$ is ind-finite on C'.*

Proof. In view of the pseudo-commutative diagram (5.6.2) we may assume that $\mathscr{E}$ is a stack in groupoids, in which case the same holds for $\mathscr{E}' := \mathsf{St}(u)_*\mathscr{E}$. Now, let $X \in \mathrm{Ob}(\mathscr{C}')$ and $\sigma, \tau \in \mathscr{E}'(X)$. Recall that u induces a continuous functor $u_{|X} : \mathscr{C}'/X \to \mathscr{C}/uX$ (proposition 4.7.7(i)), and moreover $G/X \subset \mathrm{Ob}(\mathscr{C}'/X)$ is a topologically generating family, according to (4.7.2); then it is easily seen that for every $(Y \xrightarrow{g} X) \in \mathrm{Ob}(G/X)$ the object $uY \xrightarrow{u(g)} uX$ of $\mathscr{C}/uX$ is quasi-compact. On the other hand, arguing as in the proof of proposition 5.4.8(i), we find $\sigma', \tau' \in \mathscr{E}(uX)$ and isomorphisms of sheaves

$$\mathscr{C}\!art(\sigma, \tau) \xrightarrow{\sim} \widetilde{u}_{|X*}\mathscr{C}\!art(\sigma', \tau').$$

The assertion then follows from proposition 5.6.24. □

5.6.35. *Stalk of a stack over a T-point.* Let T be a topos, and $\xi := (\xi^*, \xi_*) : \mathbf{Set} \to T$ a T-point (see definition 4.7.11(i)). For every stack $\mathscr{E}$ on the canonical site $\mathrm{Can}(T)$ we set

$$\mathscr{E}_\xi := \mathsf{St}(\xi^*)^*\mathscr{E}.$$

Likewise, for every morphism $\varphi : \mathscr{E} \to \mathscr{F}$ of stacks on $\mathrm{Can}(T)$, we let

$$\varphi_\xi := \mathsf{St}(\xi^*)^*(\varphi) : \mathscr{E}_\xi \to \mathscr{F}_\xi.$$

Remark 5.6.36. (i) Recall that $\mathbf{Set}$ is equivalent to $(\mathbb{1}, J)^\sim$, where $\mathbb{1}$ is the category with one object and one morphism, and J is the unique topology on $\mathbb{1}$. It follows easily that a stack on the canonical site $\mathrm{Can}(\mathbf{Set})$ of $\mathbf{Set}$ amounts to the datum of a category, and a morphism of stacks on $\mathrm{Can}(\mathbf{Set})$ is just an arbitrary functor.

(ii) Likewise, a functor $F : \mathscr{A} \to \mathscr{B}$ between categories is 0-covering (resp. 1-covering, resp. 2-covering) when regarded as a morphism of stacks on $\mathrm{Can}(\mathbf{Set})$, if and only if it is essentially surjective (resp. full, resp. faithful) : the details shall be left to the reader.

Proposition 5.6.37. *With the notation of* (5.6.35)*, let Ω be a conservative set of T-points (see definition* 4.7.11(iv)*). Then for every $i = 0, 1, 2$ the following conditions are equivalent :*

(a) *φ is i-covering.*
(b) *The morphism φ_ξ is i-covering for every $\xi \in \Omega$.*

Proof. (a)⇒(b) follows from proposition 5.4.22(ii).

(b)⇒(a): For $i = 1$ and $i = 2$, the assertion follows easily from lemma 5.2.7 and proposition 5.6.30. Suppose then that φ_ξ is 0-covering for every $\xi \in \Omega$; by lemma 5.6.7(i), it follows that $\pi_0(\varphi_\xi) : \pi_0(\mathscr{E}_\xi) \to \pi_0(\mathscr{F}_\xi)$ is a surjection. But according to (5.6.8), $\pi_0(\mathscr{E}_\xi)$ is naturally identified with the stalk $\pi_0^T(\mathscr{E})_\xi$ of the sheaf $\pi_0^T(\mathscr{E})$ on $\mathrm{Can}(T)$, and likewise for $\pi_0(\mathscr{F}_\xi)$. Thus, the morphism of sheaves $\pi_0^T(\varphi) : \pi_0^T(\mathscr{E}) \to \pi_0^T(\mathscr{F})$ is an epimorphism, and therefore φ is 0-covering, again by lemma 5.6.7(i). □

5.7. **Stacks on fibred sites and fibred topoi.** Let $\mathscr{C}$ and $\mathscr{C}'$ be two categories, $u : \mathscr{C}' \to \mathscr{C}$ any functor, $(\mathscr{A}, p, J_\bullet)$ a fibred site over $\mathscr{C}$, and $(\mathscr{A}', p'_\bullet, J'_\bullet) := \mathscr{C}' \times_{\mathscr{C}} (\mathscr{A}, p, J_\bullet)$ the induced fibred site as in (4.5.11); denote also by $(\mathscr{A}, J_{\mathscr{A}})$ and $(\mathscr{A}', J_{\mathscr{A}'})$ the respective fibred sites. Then it follows easily from proposition 4.5.5(ii,iv) that the projection $\pi : \mathscr{A}' \to \mathscr{A}$ fulfills condition (C2) of (5.4.5), relative to the topologies $J_{\mathscr{A}}$ and $J_{\mathscr{A}'}$, so it is a weak morphism of sites $\pi : (\mathscr{A}, J_{\mathscr{A}}) \to (\mathscr{A}', J_{\mathscr{A}'})$, and we get a well defined pseudo-functor

$$\mathsf{St}(\pi)_* : \mathsf{Stack}(\mathscr{A}, J_{\mathscr{A}}) \to \mathsf{Stack}(\mathscr{A}', J_{\mathscr{A}'}).$$

If $\mathscr{C}$ and $\mathscr{C}'$ are small and $(\mathscr{A}_X, J_X)$ is a U-site for every $X \in \mathrm{Ob}(\mathscr{C})$, then $(\mathscr{A}, J_{\mathscr{A}})$ and $(\mathscr{A}', J_{\mathscr{A}'})$ are also U-sites (remark 4.5.2(iii)), and in this case also the left 2-adjoint $\mathsf{St}(\pi)^*$ of $\mathsf{St}(\pi)_*$ is well defined; moreover, π is cocontinuous (see (4.5.11)), so we have a pseudo-natural equivalence $\mathsf{St}(\pi)_* \xrightarrow{\sim} \mathsf{St}(\breve{\pi})^*$ (proposition 5.4.29(ii)).

As a special case, for every $X \in \mathrm{Ob}(\mathscr{C})$ the inclusion functor $i_X : \mathscr{A}_X \to \mathscr{A}$ is a weak morphism of sites $(\mathscr{A}, J_{\mathscr{A}}) \to (\mathscr{A}_X, J_X)$, and we get a well defined pseudo-functor

$$\mathsf{St}(i_X)_* : \mathsf{Stack}(\mathscr{A}, J_{\mathscr{A}}) \to \mathsf{Stack}(\mathscr{A}_X, J_X).$$

If $\mathscr{C}$ is small and $(\mathscr{A}_X, J_X)$ is a U-site for every $X \in \mathrm{Ob}(\mathscr{C})$, also $\mathsf{St}(i_X)^*$ is well defined.

Proposition 5.7.1. *In the situation of* (5.7)*, we have :*

(i) *Let $\varphi : \mathscr{E} \to \mathscr{E}'$ be a cartesian functor of fibrations over $\mathscr{A}$, and $i \in \{0, 1, 2\}$. The following conditions are equivalent :*
   (a) *φ is i-covering for the topology $J_{\mathscr{A}}$.*
   (b) *$\mathsf{Fib}(i_X)^*(\varphi)$ is i-covering for the topology J_X, for every $X \in \mathrm{Ob}(\mathscr{C})$.*

(ii) *Suppose that $\mathscr{C}$ is small, and $(\mathscr{A}_X, J_X)$ is a* U*-site for every $X \in \mathrm{Ob}(\mathscr{C})$. Then the following diagram of 2-categories is pseudo-commutative :*

$$\begin{array}{ccc} \mathsf{Fib}(\mathscr{A}) & \xrightarrow{\mathsf{Fib}(i_X)^*} & \mathsf{Fib}(\mathscr{A}_X) \\ {\scriptstyle (-)^a}\downarrow & & \downarrow{\scriptstyle (-)^a} \\ \mathsf{Stack}(\mathscr{A}, J_{\mathscr{A}}) & \xrightarrow{\mathsf{St}(i_X)_*} & \mathsf{Stack}(\mathscr{A}_X, J_X) \end{array} \qquad \textit{for every } X \in \mathrm{Ob}(\mathscr{C}).$$

Proof. (i.a)⇒(i.b): This follows directly from propositions 5.4.1(i) and 4.5.5(iii).

(i.b)⇒(i.a): Suppose first that $i = 0$, and let $X \in \mathrm{Ob}(\mathscr{A})$ and $E' \in \mathrm{Ob}(\mathscr{E}'_X)$; pick cleavages for $\mathscr{E}$ and $\mathscr{E}'$, and let c and c' be the corresponding associated pseudo-functors. By assumption, there exist a covering family $f_\bullet : (f_i : X_i \to X \mid i \in I)$ for the topology J_{pX}, and for every $i \in I$ an object $E_i \in \mathrm{Ob}(\mathscr{E}_{X_i})$ such that φE_i is isomorphic to $\mathsf{c}'_{f_i} E'$ in $\mathscr{E}'_{X_i}$. Since $f_\bullet$ is also a covering family for the topology $J_{\mathscr{A}}$, this shows that φ is 0-covering, as required. The cases where $i = 1, 2$ are similar : the details shall be left to the reader.

(ii): The assertion is a special case of corollary 5.4.31. □

Corollary 5.7.2. *Let $j \leq 2$ be any integer. The following holds :*

(i) *In the situation of* (4.3.24)*, let $\mathscr{E}$ be a fibration on $\mathscr{C}$. Then $\mathscr{E}$ is a j-separated prestack on C if and only if* $\mathsf{Fib}(\pi_i)^*(\mathscr{E})$ *is a j-separated prestack on C_i for every $i \in \mathrm{Ob}(I)$.*

(ii) *In the situation of* (5.7)*, let $\mathscr{E}$ be a fibration on $\mathscr{A}$. Then $\mathscr{E}$ is a j-separated prestack on $(\mathscr{A}, J_{\mathscr{A}})$ if and only if* $\mathsf{Fib}(i_X)^*(\mathscr{E})$ *is j-separated on $(\mathscr{A}_X, J_X)$ for every $X \in \mathrm{Ob}(\mathscr{C})$.*

Proof. (i): The condition is necessary, by virtue of corollary 5.4.15 and proposition 5.4.8(i,ii). For the converse, we argue as in the proof of proposition 4.3.25(ii). Indeed, suppose that $\mathsf{Fib}(\pi_i)^*(\mathscr{E})$ is a j-separated prestack on C_i for every $i \in \mathrm{Ob}(I)$; according to remark 4.1.8(ii) there exists a topology J' on $\mathscr{C}$ such that, for every $X \in \mathrm{Ob}(\mathscr{C})$, the set $J'(X)$ consists of the sieves $\mathscr{S} \subset \mathscr{C}/X$ of universal $\mathscr{E}$-j-descent. It then suffices to prove that $J \subset J'$. We come down to checking the following. For every $i \in \mathrm{Ob}(I)$, every $X \in \mathrm{Ob}(\mathscr{C}_i)$, every covering family $g_\bullet := (g_\lambda : X_\lambda \to X \mid \lambda \in \Lambda)$ for the topology J_i, and every morphism $f : Y \to \pi_i X$ in $\mathscr{C}$, the family $(Y \times_{\pi_i X} \pi_i(g_\lambda) : Y \times_{\pi_i X} \pi_i X_\lambda \to Y \mid \lambda \in \Lambda)$ generates a sieve of $\mathscr{E}$-j-descent. Arguing as in the proof of proposition 4.3.25(ii), we are first reduced to the case where $Y = \pi_i Y'$ for some $Y' \in \mathrm{Ob}(\mathscr{C}_i)$ and $f = \pi_i(t)$ for some morphism $t : Y' \to X$ of $\mathscr{C}_i$, and then we may even replace the family $g_\bullet$ by $(Y' \times_X g_\lambda \mid \lambda \in \Lambda)$ and reduce further to the case where $t = \mathbf{1}_X$. Finally, pick a cleavage c for $\mathscr{E}$; we need to show that the natural functor

$$\mathscr{E}(\pi_i X) \to \mathsf{Desc}(\mathscr{E}, \pi_i(g_\bullet), \mathsf{c})$$

is j-faithful (notation of (3.5.22)). But since π_i is left exact, this functor is naturally identified with the corresponding functor for $\mathscr{E}_i := \mathsf{Fib}(\pi_i)^*(\mathscr{E})$:

$$\mathscr{E}_i(X) \to \mathsf{Desc}(\mathscr{E}_i, g_\bullet, \mathsf{c} \circ \pi_i^o)$$

and the latter is j-faithful, since $\mathscr{E}_i$ is a j-separated prestack on C_i.

(ii): Again, the condition is necessary, by corollary 5.4.15 and proposition 5.4.8(i,ii). For the converse we argue as in the proof of proposition 4.5.5. Namely, we let J' be the topology on $\mathscr{A}$ such that for every $A \in \mathrm{Ob}(\mathscr{A})$ the set $J'(A)$ consists of the sieves $\mathscr{S} \subset \mathscr{A}/A$ of universal $\mathscr{E}$-j-descent, and we check that $J_{\mathscr{A}} \subset J'$, assuming that $\mathscr{E}_X := \mathsf{Fib}(i_X)^*(\mathscr{E})$ is a j-separated prestack on $(\mathscr{A}_X, J_X)$ for every $X \in \mathrm{Ob}(\mathscr{C})$. To this aim, we consider any morphism $f : A' \to A$ in $\mathscr{A}$ and any family $(g_\lambda : B_\lambda \to A \mid \lambda \in \Lambda)$ of morphisms in $\mathscr{A}_{pA}$ covering A for the topology J_{pA}; after choosing a cleavage for $\mathscr{A}$, we then construct the family $g'_\bullet := (g'_\lambda : B'_\lambda \to A' \mid \lambda \in \Lambda)$ of morphisms of $\mathscr{A}_{pA'}$ as in the proof of proposition 4.5.5. Let $i_{pA'} : \mathscr{A}_{pA'} \to \mathscr{A}$ be the inclusion functor; taking into account lemma 3.5.35(ii), we reduce to checking that the sieve of $\mathscr{A}/A'$

generated by the family $i_{pA'}(g'_\bullet) := (i_{pA'}(g_\lambda) \mid \lambda \in \Lambda)$ is of $\mathscr{E}$-j-descent. Now, choose a cleavage c for $\mathscr{E}$, and set

$$B'_{\lambda\mu} := B'_\lambda \times_{A'} B'_\mu \qquad \text{and} \qquad B'_{\lambda\mu\nu} := B'_{\lambda\mu} \times_{A'} B'_\nu \qquad \text{for every } \lambda, \mu, \nu \in \Lambda$$

and notice that $i_{pA'}B'_{\lambda\mu}$ represents the fibre product $i_{pA'}B'_\lambda \times_{i_{pA'}A'} B'_\mu$, and likewise $i_{pA'}B'_{\lambda\mu\nu}$ represents $i_{pA'}B'_{\lambda\mu} \times_{i_{pA'}A'} i_{pA'}B'_\nu$ (remark 4.5.2(ii)), so that the category $\mathsf{Desc}(\mathscr{E}, i_{pA'}(g'_\bullet), \mathsf{c})$ is well defined, and we are reduced to checking that the induced map

$$\mathscr{E}(i_{pA'}A') \to \mathsf{Desc}(\mathscr{E}, i_{pA'}(g'_\bullet), \mathsf{c})$$

is j-faithful. But the latter is naturally identified with the analogous map

$$\mathscr{E}_{pA'}(A') \to \mathsf{Desc}(\mathscr{E}_{pA'}, g'_\bullet, \mathsf{c} \circ i^o_{pA'})$$

which is j-faithful by assumption. □

Proposition 5.7.3. *In the situation of* (4.5.13)*, suppose that* $(\mathscr{A}_{i,X}, J_{i,X})$ *is a* U*-site for every* $X \in \mathrm{Ob}(\mathscr{C})$ *and* $i = 0, 1$. *Then the induced base change transformation :*

$$\Upsilon(\mathsf{St}(\mathbf{1}_{u\circ\pi_1})^\gamma_*) : \mathsf{St}(v)^* \circ \mathsf{St}(\pi_1)_* \to \mathsf{St}(\pi_0)_* \circ \mathsf{St}(u)^*$$

is a pseudo-natural equivalence.

Proof. Recall that the projections π_0 and π_1 satisfy condition (C2) from (5.4.5) : see (5.7); in particular, they are weak morphisms of sites, and moreover they are cocontinuous (see (4.5.11)). We are then in the situation contemplated in (5.5.23), and by corollary 5.5.26(ii), both Δ^{π_0} and Δ^{π_1} are pseudo-natural equivalences, and the same holds for $\Upsilon(\Delta^u)$ and $\Upsilon(\Delta^v)$, by virtue of corollary 5.5.26(iii). Hence, arguing as in remark 2.3.22(i), we are reduced to checking that $\Upsilon((\mathbf{1}^\sim_{u\circ\pi_1}, \mathbf{Cat})_*)$ is a pseudo-natural equivalence. The latter assertion follows from proposition 4.5.14 and the discussion of (5.5.29). □

Definition 5.7.4. Let $(\mathscr{A}, p, J_\bullet)$ and $(\mathscr{A}', p', J'_\bullet)$ be two fibred sites over a category $\mathscr{C}$. A *weak morphism of fibred sites* $\varphi : (\mathscr{A}, p, J_\bullet) \to (\mathscr{A}', p', J'_\bullet)$ is a functor $\varphi : \mathscr{A}' \to \mathscr{A}$ with $p \circ \varphi = p'$, whose restriction $\varphi_X : \mathscr{A}'_X \to \mathscr{A}_X$ is a weak morphism of sites $(\mathscr{A}_X, J_X) \to (\mathscr{A}'_X, J'_X)$, for every $X \in \mathrm{Ob}(\mathscr{C})$.

Proposition 5.7.5. *Let* $(\mathscr{A}, p, J_\bullet)$ *and* $(\mathscr{A}', p', J'_\bullet)$ *be two fibred lex-sites over a category* $\mathscr{C}$, *and* $\varphi : (\mathscr{A}, p, J_\bullet) \to (\mathscr{A}', p', J'_\bullet)$ *a weak morphism of fibred sites. Then* φ *is a weak morphism of the respective total sites* $(\mathscr{A}, J_\mathscr{A}) \to (\mathscr{A}', J_{\mathscr{A}'})$.

Proof. For every $X \in \mathrm{Ob}(\mathscr{C})$, let $i_x : \mathscr{A}_X \to \mathscr{A}$ and $i'_X : \mathscr{A}'_X \to \mathscr{A}'$ be the inclusion functors, and $\varphi_X : \mathscr{A}'_X \to \mathscr{A}_X$ the restriction of φ; in light of corollary 5.7.2(ii), it suffices to check that for every stack $\mathscr{E}$ over $(\mathscr{A}, J_\mathscr{A})$, the fibration $\mathscr{F} := \mathsf{Fib}(i'_X)^* \circ \mathsf{Fib}(\varphi)^*(\mathscr{E})$ is a stack on $(\mathscr{A}'_X, J'_X)$. But we have a natural identification $\mathscr{F} \xrightarrow{\sim} \mathsf{Fib}(\varphi_X)^* \circ \mathsf{Fib}(i_X)^*(\mathscr{E})$, and $\mathsf{Fib}(i_X)^*(\mathscr{E})$ is a stack on $(\mathscr{A}_X, J_X)$, again by corollary 5.7.2(ii), whence the contention. □

5.7.6. Let $(\mathscr{A}, p, J_\bullet)$ be a small fibred lex-site over a small category $\mathscr{C}$; pick a unital cleavage $\boldsymbol{\lambda}$ for p, and let $\mathsf{c} : \mathscr{C}^o \to \mathbf{Cat}$ be the associated pseudo-functor. Clearly c^o factors through a pseudo-functor

$$\widetilde{\mathsf{c}} : \mathscr{C} \to \mathsf{lex}.\mathbf{Site} \qquad X \mapsto (\mathscr{A}_X, J_X)$$

and the forgetful functor $\mathsf{lex}.\mathbf{Site} \to \mathbf{Cat}^o$. Let $\Sigma_\mathscr{A}$ be the set of cartesian morphisms of $\mathscr{A}$. By remark 3.3.13(i), we know that $\mathscr{A}[\Sigma^{-1}_\mathscr{A}]$ represents the strong 2-colimit of c :

$$\text{2-}\operatorname*{colim}_{\mathscr{C}^o} \mathsf{c} \xrightarrow{\sim} \mathscr{A}[\Sigma^{-1}_\mathscr{A}].$$

Suppose now that $\mathscr{C}$ *is cofiltered.* Then, the construction of (4.3.24) endows $\mathscr{A}[\Sigma^{-1}_\mathscr{A}]$ with a natural topology $J^*_\mathscr{A}$ such that $(\mathscr{A}[\Sigma^{-1}_\mathscr{A}], J^*_\mathscr{A})$ is a strong 2-limit of $\widetilde{\mathsf{c}}$, and again by direct inspection we see that the localization functor $L_{\mathscr{A}} : \mathscr{A} \to \mathscr{A}[\Sigma_{\mathscr{A}}^{-1}]$ is continuous for the topology $J^*_{\mathscr{A}}$ and the topology $J_{\mathscr{A}}$ of the total site $(\mathscr{A}, J_{\mathscr{A}})$. Moreover, for every $X \in \mathrm{Ob}(X)$, the composition $\mathscr{A}_X \to \mathscr{A}[\Sigma_{\mathscr{A}}^{-1}]$ of the inclusion functor $i_X : \mathscr{A}_X \to \mathscr{A}$ and $F_{\mathscr{A}}$, is a morphism of sites

$$L_{\mathscr{A}} \circ i_X : (\mathscr{A}[\Sigma_{\mathscr{A}}^{-1}], J^*_{\mathscr{A}}) \to (\mathscr{A}_X, J_X)$$

and the rule : $X \mapsto L_{\mathscr{A}} \circ i_X$ defines both a universal pseudo-cone $\widetilde{\tau} : \mathsf{F}_{(\mathscr{A}[\Sigma_{\mathscr{A}}^{-1}], J^*_{\mathscr{A}})} \Rightarrow \widetilde{\mathsf{c}}$ and a universal pseudo-cocone $\tau : \mathsf{c} \Rightarrow \mathsf{F}_{\mathscr{A}[\Sigma_{\mathscr{A}}^{-1}]}$. Recall as well that $\Sigma_{\mathscr{A}}$ admits a right calculus of fraction (example 3.3.13(ii)), so that the category $X/L_{\mathscr{A}}\mathscr{A}$ is cofiltered for every $X \in \mathrm{Ob}(\mathscr{A}[\Sigma_{\mathscr{A}}^{-1}])$ (proposition 1.6.25(i)). Then the source fibration $\mathsf{s} : \mathscr{A}[\Sigma_{\mathscr{A}}^{-1}]/L_{\mathscr{A}}\mathscr{A} \to \mathscr{A}[\Sigma_{\mathscr{A}}^{-1}]$ is trivially locally cofiltered, and from proposition 4.3.9 we see that $L_{\mathscr{A}}$ is a morphism of sites

$$L_{\mathscr{A}} : (\mathscr{A}[\Sigma_{\mathscr{A}}^{-1}], J^*_{\mathscr{A}}) \to (\mathscr{A}, J_{\mathscr{A}}).$$

Proposition 5.7.7. *In the situation of* (5.7.6)*, the following holds :*

(i) *The functor $L^{\sim}_{\mathscr{A}*} : (\mathscr{A}[\Sigma_{\mathscr{A}}^{-1}], J^*_{\mathscr{A}})^{\sim} \to (\mathscr{A}, J_{\mathscr{A}})^{\sim}$ is fully faithful.*

(ii) *Suppose that every object of $\mathscr{A}$ is quasi-compact for the topology $J_{\mathscr{A}}$ (see definition* 5.6.10*), and let F and $\mathscr{E}$ be respectively a sheaf and a stack on $(\mathscr{A}, J_{\mathscr{A}})$. Then:*

(a) *$L_{\mathscr{A}!}F$ is a sheaf on $(\mathscr{A}[\Sigma_{\mathscr{A}}^{-1}], J^*_{\mathscr{A}})$.*

(b) *$\mathsf{Fib}(L_{\mathscr{A}})_!\mathscr{E}$ is a stack on $(\mathscr{A}[\Sigma_{\mathscr{A}}^{-1}], J^*_{\mathscr{A}})$.*

Proof. (i) follows easily from corollary 1.6.11.

Next, let $F \in \mathrm{Ob}(\mathscr{A}^\wedge)$ and $A \in \mathrm{Ob}(\mathscr{A}[\Sigma_{\mathscr{A}}^{-1}])$; recall that $L_{\mathscr{A}!}F(A)$ represents the colimit of

$$F \circ \mathsf{t}^o_A : (A/L_{\mathscr{A}}\mathscr{A})^o \to \mathbf{Set} \qquad (A \to L_{\mathscr{A}}B) \mapsto FB.$$

We define as follows a functor $\Phi_A : \mathscr{C}/pA \to A/L_{\mathscr{A}}\mathscr{A}$. For every $\varphi \in \mathrm{Ob}(\mathscr{C}/pA)$ we let

$$\Phi_A(\varphi) := (\mathsf{c}_\varphi A, (L_{\mathscr{A}}\boldsymbol{\lambda}(A, \varphi))^{-1} : A \to L_{\mathscr{A}}(\mathsf{c}_\varphi A))$$

(see (3.1.6)). For every morphism $\psi/pA : \varphi \to \varphi'$ of $\mathscr{C}/pA$ we have $\varphi = \varphi' \circ \psi$, and we set

$$\Phi_A(\psi/pA) := A/L_{\mathscr{A}}(\lambda(\mathbf{1}_A, \psi)) : \Phi_A(\varphi) \to \Phi_A(\varphi').$$

Claim 5.7.8. The functor Φ_A is coinitial.

Proof of the claim. Since $\mathscr{C}$ is cofiltered, the same holds for $\mathscr{C}/A$ (example 1.5.9(i)), hence we may apply the criterion of (the dual of) lemma 1.5.7(i). First, for every object $(B, f : A \to L_{\mathscr{A}}B)$ of $A/L_{\mathscr{A}}\mathscr{A}$ we may find a morphism $\varphi : pB \to pA$ of $\mathscr{C}$ and a morphism $f' : \mathsf{c}_\varphi A \to B$ in $p^{-1}(pB)$ such that $f = L_{\mathscr{A}}(f') \circ L_{\mathscr{A}}(\lambda(A, \varphi))^{-1}$ (details left to the reader), so that f' yields a morphism $\Phi_A(\varphi) \to (B, f)$ in $A/L_{\mathscr{A}}\mathscr{A}$. Next, let $\varphi : X \to pA$ be any object of $\mathscr{C}/pA$ and $A/g, A/h : \Phi_A(\varphi) \to (B, f : A \to L_{\mathscr{A}}B)$ two morphisms of $A/L_{\mathscr{A}}\mathscr{A}$; by definition, we have $g = f \circ L_{\mathscr{A}}(\lambda(A, \varphi)) = h$, so condition (b) of lemma 1.5.7(i) is trivially verified. ◊

By claim 5.7.8, the set $L_{\mathscr{A}!}F(A)$ also represents the colimit of the functor

$$F \circ (\mathsf{t}_A \circ \Phi_A)^o : (\mathscr{C}/pA)^o \to \mathbf{Set} \qquad (\varphi : X \to pA) \mapsto F(\mathsf{c}_\varphi A).$$

(ii.a): Notice first that, under the stated assumptions, for every $X \in \mathrm{Ob}(\mathscr{C})$, every $B \in \mathrm{Ob}(\mathscr{A}_X)$ is quasi-compact for the topology J_X : indeed, if $\mathscr{S} \in J_X(B)$, let $\mathscr{S}'$ be the sieve of $\mathscr{A}/B$ generated by $\mathscr{S}$; we have $\mathscr{S}' \in J_{\mathscr{A}}(B)$, so by assumption, there exists a finite subset $\{f_i : B_i \to B \mid i \in I\} \subset \mathscr{S}'$ that generates a sieve $\mathscr{S}''$ covering B in $(\mathscr{A}, J_{\mathscr{A}})$. But for every $i \in I$ there exist $(f'_i : C_i \to B) \in \mathscr{S}$ and a morphism $g_i : B_i \to C_i$ in $\mathscr{A}$ such that $f_i = f'_i \circ g_i$; the family $(f'_i \mid i \in I)$ still generates a covering sieve for $J_{\mathscr{A}}$, and then the same family covers B also in the site $(\mathscr{A}_X, J_X)$. Next we remark :

Claim 5.7.9. Let $A \in \mathrm{Ob}(\mathscr{A}[\Sigma_{\mathscr{A}}^{-1}]) = \mathrm{Ob}(\mathscr{A})$, and $\mathscr{S}$ a sieve of $\mathscr{A}[\Sigma_{\mathscr{A}}^{-1}]/A$. Then $\mathscr{S} \in J^*_{\mathscr{A}}(A)$ if and only if there exist a cartesian morphism $h : A' \to A$ in $\mathscr{A}$ and a finite covering family $(f_i : A'_i \to A')$ in $(\mathscr{A}_{pA'}, J_{pA'})$ with $L_{\mathscr{A}}(h \circ f_i) \in \mathscr{S}$ for every $i \in I$.

Proof of the claim. Since $L_{\mathscr{A}}$ is continuous and $L_{\mathscr{A}}h$ is an isomorphism in $\mathscr{A}[\Sigma_{\mathscr{A}}^{-1}]$, the stated condition implies that $\mathscr{S} \in J^*_{\mathscr{A}}(A)$. For the converse, recall that $J^*_{\mathscr{A}}$ is the coarsest topology such that $L_{\mathscr{A}} \circ i_X$ is a continuous functor for the topology J_X and $J^*_{\mathscr{A}}$, for every $X \in \mathrm{Ob}(\mathscr{C})$. Thus, for every $B \in \mathrm{Ob}(\mathscr{A})$, denote by $J^{**}_{\mathscr{A}}(B)$ the set of all sieves of $\mathscr{A}[\Sigma_{\mathscr{A}}^{-1}]/B$ fulfilling the condition of the claim; it suffices then to check that $J^{**}_{\mathscr{A}} := (J^{**}_{\mathscr{A}}(B) \mid B \in \mathrm{Ob}(A))$ is a topology on $\mathscr{A}[\Sigma_{\mathscr{A}}^{-1}]$, and that $L_{\mathscr{A}} \circ i_X$ is a continuous functor for the topologies J_X and $J^{**}_{\mathscr{A}}$, for every $X \in \mathrm{Ob}(\mathscr{C})$. To check stability under base change for $J^{**}_{\mathscr{A}}$ (see definition 4.1.1(i)), consider any morphism $g : B \to A$ in $\mathscr{A}[\Sigma_{\mathscr{A}}^{-1}]$; we need to show that $\mathscr{S} \times_A g \in J^{**}_{\mathscr{A}}(B)$. However, since $\Sigma_{\mathscr{A}}$ admits a right calculus of fraction, there exist a cartesian morphism $h' : B' \to B$ in $\mathscr{A}$ and a morphism $k : B' \to A$ in $\mathscr{A}$ such that $g = L_{\mathscr{A}}(k) \circ L_{\mathscr{A}}(h')^{-1}$. Suppose now that $\mathscr{S}' := \mathscr{S} \times_A L_{\mathscr{A}}(k) \in J^{**}_{B'}$; this means that there exist a cartesian morphism $h'' : B'' \to B'$ and a finite covering family $(f'_i : B''_i \to B')$ in $(\mathscr{A}_{pB'}, J_{pB'})$ such that $L_{\mathscr{A}}(h'' \circ f'_i) \in \mathscr{S}'$ for every $i \in I$. Therefore $L_{\mathscr{A}}(h' \circ h'' \circ f'_i) \in \mathscr{S}' \times_{B'} L_{\mathscr{A}}(h')^{-1} = \mathscr{S} \times_A g$, and since $h' \circ h''$ is cartesian, we get $\mathscr{S} \times_A g \in J^{**}_{\mathscr{A}}$. Thus, we may replace B by B', and assume from start that $g = L_{\mathscr{A}}(k)$ for some morphism $k : B \to A$ in $\mathscr{A}$. In this case, since $\mathscr{C}$ is cofiltered, we can find $X \in \mathrm{Ob}(\mathscr{C})$ with morphisms $\varphi : X \to pB$ and $\psi : X \to pA'$ such that $p(g) \circ \varphi = p(h) \circ \psi$; then we may also find a morphism $k' : B' \to A''$ in $p^{-1}X$, and cartesian morphisms $h' : B' \to B$ and $h'' : A'' \to A'$ in $\mathscr{A}$ such that $p(h') = \varphi$, $p(h'') = \psi$, and $k \circ h' = h \circ h'' \circ k'$ in $\mathscr{A}$. For every $i \in I$ there exist a morphism $f'_i : A''_i \to A''$ in $p^{-1}X$ and a cartesian morphism $h''_i : A''_i \to A'_i$ such that $f_i \circ h''_i = h'' \circ f'_i$. Since c_ψ is continuous for the topologies $J_{pA'}$ and J_X, it follows easily that $(f'_i \mid i \in I)$ is a covering family in $(\mathscr{A}_X, J_X)$ (details left to the reader), and by construction $L_{\mathscr{A}}(h \circ h'' \circ f'_i) \in \mathscr{S}$ for every $i \in I$. Since $h \circ h''$ is cartesian, we may therefore replace A' by A'', h by $h \circ h''$ and the family $(f_i \mid i \in I)$ by $(f'_i \mid i \in I)$, and assume from start that there exist a morphism $k' : B' \to A'$ in $\mathscr{A}_{pA'}$ and a cartesian morphism $h' : B' \to B$ in $\mathscr{A}$ such that $h \circ k' = k \circ h'$. Let $(f'_i : A'_i \times_{A'} B' \to B' \mid i \in I)$ be the induced finite covering family of B' in $\mathscr{A}_{pA'}$, and for every $i \in I$ let also $k'_i : A'_i \times_{A'} B'' \to A'_i$ be the second projection. It suffices to check that $L_{\mathscr{A}}(h' \circ f'_i) \in \mathscr{S} \times_A g$ for every $i \in I$, *i.e.* that $L_{\mathscr{A}}(k \circ h' \circ f'_i) \in \mathscr{S}$ for every such i. However, $L_{\mathscr{A}}(k \circ h' \circ f'_i) = L_{\mathscr{A}}(h \circ k' \circ f'_i) = L_{\mathscr{A}}(h \circ f_i \circ k'_i)$, whence the contention. To check the local character of $J^{**}_{\mathscr{A}}$, suppose that $\mathscr{T}$ is another sieve of $\mathscr{A}[\Sigma_{\mathscr{A}}^{-1}]/A$ such that for every $(t : A' \to A) \in \mathscr{S}$ we have $\mathscr{T} \times_A t \in J^{**}_{\mathscr{A}}(A')$; we need to show that $\mathscr{T} \in J^{**}(A)$. In particular, for every $i \in I$ there exist a cartesian morphism $h_i : B_i \to A'_i$ in $\mathscr{A}$ and a finite covering family $(g_{i\lambda} : B_{i\lambda} \to B_i \mid \lambda \in \Lambda_i)$ in $(\mathscr{A}_{pB_i}, J_{pB_i})$ such that $L_{\mathscr{A}}(h_i \circ g_{i\lambda}) \in \mathscr{T} \times_A L_{\mathscr{A}}(h \circ f_i)$ for every $\lambda \in \Lambda_i$, *i.e.* $L_{\mathscr{A}}(h \circ f_i \circ h_i \circ g_{i\lambda}) \in \mathscr{T}$ for every such λ. Now, since $\mathscr{C}$ is cofiltered, we may find a morphism $\varphi : Y \to pA'$ in $\mathscr{C}$ such that for every $i \in I$ there exists a morphism $\varphi_i : Y \to pB_i$ with $\varphi = p(h_i) \circ \varphi_i$. Pick then any cartesian morphism $k : C \to A'$ with $p(k) = \varphi$, and a cartesian morphism $k_i : C_i \to B_i$ such that $p(k_i) = \varphi_i$, for every $i \in I$. With this notation, for every $i \in I$ there exists a unique morphism $f'_i : C_i \to C$ in $\mathscr{A}_Y$ such that $k \circ f'_i = f_i \circ h_i \circ k_i$. Since c_φ is continuous for the topologies J_Y and $J_{pA'}$, it follows easily that $(f'_i : C_i \to C \mid i \in I)$ is a finite covering family in $(\mathscr{A}_Y, J_Y)$ (details left to the reader). Lastly, for every $i \in I$ and $\lambda \in \Lambda_i$ pick a cartesian morphism $k_{i\lambda} : C_{i\lambda} \to B_{i\lambda}$ with $p(k_{i\lambda}) = \varphi_i$. Then for every such i and λ there exists a unique morphism $g'_{i\lambda} : C_{i\lambda} \to C_i$ in $\mathscr{A}_Y$ such that $g_{i\lambda} \circ k_{i\lambda} = k_i \circ g'_{i\lambda}$, and arguing as in the foregoing, we easily see that $(g'_{i\lambda} \mid \lambda \in \Lambda_i)$ is a finite covering family in $(\mathscr{A}_Y, J_Y)$, for every $i \in I$. We conclude that $(f'_{i\lambda} := f'_i \circ g'_{i,\lambda} : C_{i\lambda} \to C \mid i \in I, \lambda \in \Lambda_i)$ is a finite covering family in $(\mathscr{A}_Y, J_Y)$ as well. But we have $k \circ f'_{i\lambda} = f_i \circ h_i \circ g_{i\lambda} \circ k_{i\lambda}$ for every $i \in I$ and $\lambda \in \Lambda_i$; since $h \circ k$ is a cartesian morphism, the assertion follows. Thus, $J^{**}_{\mathscr{A}}$ is indeed a topology on $\mathscr{A}[\Sigma_{\mathscr{A}}^{-1}]$. In order to prove that $L_{\mathscr{A}} \circ i_X$ is continuous for $J^{**}_{\mathscr{A}}$ and J_X, it suffices now to check that for every covering family $(f_\lambda : B_\lambda \to B \mid \lambda \in \Lambda)$ in $\mathscr{A}_X$, the family $(L_{\mathscr{A}}(f_\lambda) \mid \lambda \in \Lambda)$ covers B for the topology $J^{**}_{\mathscr{A}}$ (lemma 4.2.4). Since, as we have already observed, B is quasi-compact for the topology J_X, we are easily reduced to the case where Λ is a finite set (details left to the reader); but then the assertion follows immediately from the definition of $J^{**}_{\mathscr{A}}(B)$. $\Diamond$

We shall show that the natural morphism of presheaves $L_{\mathscr{A}!}F \to (L_{\mathscr{A}!}F)^+$ is an isomorphism (notation of (4.1.11)); the lemma will follow immediately. To this aim, claim 5.7.9 reduces to checking the following assertion. For every $A \in \mathrm{Ob}(\mathscr{A})$ and every finite covering family $(f_i : A_i \to A \mid i \in I)$ in $(\mathscr{A}_{pA}, J_{pA})$, the natural map :

$$L_{\mathscr{A}!}FA \to E := \mathrm{Equal}\Big(\prod_{i\in I} L_{\mathscr{A}!}F(A_i) \rightrightarrows \prod_{(i,j)\in I\times I} L_{\mathscr{A}!}F(A_i \times_A A_j)\Big)$$

is bijective. Thus, let $\mathscr{I}$ be the finite category whose set of objects is the disjoint union of I and $I\times I$, and with two morphisms $i \leftarrow (i,j) \to j$ for every $(i,j) \in I\times I$ (and of course, the identity morphism of each object); the covering family $(f_i \mid i \in I)$ induces a functor $A_\bullet : \mathscr{I}^o \to \mathscr{A}_{pA}$ given by the rule : $i \mapsto A_i$ and $(i,j) \mapsto A_{ij} := A_i \times_A A_j$ for every $i,j \in I$, and which assigns to every morphism $l : (i,j) \to k$ of $\mathscr{I}$ the projection $\pi_l : A_{ij} \to A_k$. We consider the functor

$$\Psi : \mathscr{I} \times (\mathscr{C}/pA) \to \mathscr{A}$$

that assigns to every pair $(t, \varphi/pA) \in \mathrm{Ob}(\mathscr{I}) \times \mathrm{Ob}(\mathscr{C}/pA)$ the object $\mathsf{c}_\varphi(A_t)$, and to every morphism $(l,\psi) : (t,\varphi/pA) \to (t',\varphi'/pA)$ the composition :

$$\mathsf{c}_\varphi(A_t) \xrightarrow{\boldsymbol{\lambda}(\mathbf{1}_{A_t},\psi)} \mathsf{c}_{\varphi'}(A_t) \xrightarrow{\mathsf{c}_{\varphi'}(\pi_l)} \mathsf{c}_{\varphi'}(A_{t'}).$$

For every $(t,\varphi) \in \mathrm{Ob}(\mathscr{I} \times (\mathscr{C}/pA))$, let also $[t]$ (resp. $[\varphi]$) be the subcategory of $\mathscr{I}$ (resp. of $\mathscr{C}/pA$) with $\mathrm{Ob}([t]) := \{t\}$ (resp. with $\mathrm{Ob}([\varphi]) = \{\varphi\}$), and denote by $\Psi_t : \mathscr{C}/pA \to \mathscr{A}$ (resp $\Psi^\varphi : \mathscr{I} \to \mathscr{A}$) the restriction of Ψ to $[t] \times (\mathscr{C}/pA) \xrightarrow{\sim} \mathscr{C}/pA$ (resp. to $\mathscr{I} \times [\varphi] \xrightarrow{\sim} \mathscr{I}$). With this notation, $L_{\mathscr{A}!}F(A_t)$ represents the colimit of $F \circ \Psi_t^o$, for every $t \in \mathrm{Ob}(\mathscr{I})$, and since $\mathscr{I}$ is finite and $(\mathscr{C}/pA)^o$ is filtered, we get natural identifications :

$$E = \lim_{t\in\mathrm{Ob}(\mathscr{I})} \operatorname*{colim}_{(\mathscr{C}/pA)^o} F \circ \Psi_t^o \xrightarrow{\sim} \operatorname*{colim}_{\varphi\in\mathrm{Ob}(\mathscr{C}/pA)} \lim_{\mathscr{I}} F \circ (\Psi^\varphi)^o.$$

But since c_φ is left exact and continuous for the topologies J_X and J_Y, for every morphism $\varphi : X \to Y$ of $\mathscr{C}$, and since the restriction of F to $\mathscr{A}_X$ is a sheaf for the topology J_X, for every $X \in \mathrm{Ob}(\mathscr{C})$, we see that $F(\mathsf{c}_\varphi A)$ represents the limit of the functor $F \circ (\Psi^\varphi)^o$, for every such φ, so the functor $(\mathscr{C}/pA)^o \to \mathbf{Set}$ given by the rule $\varphi \mapsto \lim_{\mathscr{I}} F \circ (\Psi^\varphi)^o$ is naturally identified with $F \circ (\mathsf{t}_A \circ \Phi_A)^o$. Summing up, we conclude that E represents the colimit of $F \circ (\mathsf{t}_A \circ \Phi_A)^o$, whence (i).

(ii.b): By claim 3.2.8, we may assume that $\mathscr{E} = \mathscr{F}ib(E_\bullet)$ for a presheaf of categories $E_\bullet$ on $\mathscr{A}$. Then recall that $\mathscr{F} := \mathsf{Fib}(L_{\mathscr{A}})_!\mathscr{E}$ is the fibration associated with the strict pseudo-functor computed by the strong left 2-Kan extension :

$$F_\bullet := 2\text{-}\!\int^{L^o_{\mathscr{A}}} E_\bullet \qquad A \mapsto F_A := \operatorname*{2\text{-}colim}_{(A/L_{\mathscr{A}}\mathscr{A})^o} E_\bullet \circ \mathsf{t}^o_A.$$

However, since $(A/L_{\mathscr{A}}\mathscr{A})^o$ is filtered, the 2-colimit of $E_\bullet \circ \mathsf{t}^o_A$ is represented by the colimit of the same functor, for every $A \in \mathrm{Ob}(\mathscr{A})$ (example 3.3.13(iv)), and in view of claim 5.7.8, this is also the colimit of the functor $E_\bullet \circ (\mathsf{t}_A \circ \Phi_A)^o : (\mathscr{C}/pA)^o \to \mathbf{Cat}$, for every such A.

It suffices to show that the natural functor $\mathsf{C}(\mathscr{F}) \to \mathscr{F}^+$ (see (5.1.17)) is an equivalence of categories, and in view of claim 5.7.9 (and of corollary 3.1.28(i)) we are reduced to checking the following assertion. For every $A \in \mathrm{Ob}(\mathscr{A})$ and every finite covering family $f_\bullet := (f_i : A_i \to A \mid i \in I)$ in $(\mathscr{A}_{pA}, J_{pA})$, the natural functor

$$F_A \xrightarrow{\sim} \mathscr{F}(A) \to \mathsf{Desc}(\mathscr{F}, L_{\mathscr{A}}(f_\bullet))$$

is an equivalence. Now, every morphism $\psi/A : \varphi \to \varphi'$ of $\mathscr{C}/pA$ induces a functor

$$\delta_{\psi/A} : \mathsf{Desc}(\mathscr{E}, \mathsf{c}_{\varphi'}(f_\bullet)) \to \mathsf{Desc}(\mathscr{E}, \mathsf{c}_\varphi(f_\bullet))$$

as follows. Notice that, since c_φ is left exact, a descent datum for $\mathscr{E}$ relative to the covering $\mathsf{c}_\varphi(f_\bullet) := (\mathsf{c}_\varphi(f_i) \mid i \in I)$ is a pair $(e_\bullet, \omega_{\bullet\bullet})$, where $e_\bullet := (e_i \mid i \in I)$ is a system of objects $e_i \in \mathrm{Ob}(E_{\mathsf{c}_\varphi A})$ for every $i \in I$, and $\omega_{\bullet\bullet}$ is a system of isomorphisms $\omega_{ij} : E_{\mathsf{c}_\varphi(\pi_{ij})}(e_i) \xrightarrow{\sim} E_{\mathsf{c}_\varphi(\pi_{ji})}(e_j)$ fulfilling the usual cocycle condition (here we denote by $\pi_{ij} : A_{ij} \to A_i$ and $\pi_{ji} : A_{ij} \to A_j$ the projections). A morphism $h_\bullet : (e_\bullet, \omega_{\bullet\bullet}) \to (e_\bullet, \omega_{\bullet\bullet})$ of such descent data is a system of morphisms $(h_i : e_i \to e'_i \mid i \in I)$ compatible in the obvious fashion with ω_{ij} and ω'_{ij}. Then $\delta_{\psi/A}$ is given by the rules :

$$\begin{aligned}(e_\bullet, \omega_{\bullet\bullet}) &\mapsto ((E_{\boldsymbol{\lambda}(\mathbf{1}_{A_i}, \psi)}(e_i) \mid i \in I), (E_{\boldsymbol{\lambda}(\mathbf{1}_{A_{ij}}, \psi)}(\omega_{ij}) \mid i, j \in I)) \\ h_\bullet &\mapsto ((E_{\boldsymbol{\lambda}(\mathbf{1}_{A_i}, \psi)}(h_i) \mid i \in I)\end{aligned}$$

for every object $(e_\bullet, \omega_{\bullet\bullet})$ and every morphism $h_\bullet$ of $\mathsf{Desc}(\mathscr{E}, \mathsf{c}_{\varphi'}(f_\bullet))$. We thus obtain a functor $\delta : (\mathscr{C}/pA)^o \to \mathbf{Cat}$ that assigns to every $\varphi \in \mathrm{Ob}(\mathscr{C}/pA)$ the category $\mathsf{Desc}(\mathscr{E}, \mathsf{c}_\varphi(f_\bullet))$, and to every morphism ψ/A the functor $\delta_{\psi/A}$, and a direct inspection of the definitions yields a natural identification

$$\mathsf{Desc}(\mathscr{F}, L_\mathscr{A}(f_\bullet)) \xrightarrow{\sim} \operatorname*{colim}_{(\mathscr{C}/pA)^o} \delta.$$

Lastly, since $\mathscr{E}$ is a stack on $(\mathscr{A}[\Sigma_\mathscr{A}^{-1}], J^*_\mathscr{A})$, the natural functor $E_{\mathsf{c}_\varphi A} \to \mathsf{Desc}(\mathscr{E}, \mathsf{c}_\varphi(f_\bullet))$ is an equivalence for every $\varphi \in \mathrm{Ob}(\mathscr{C}/pA)$, and we get an induced equivalence

$$\operatorname*{2\text{-}colim}_{(\mathscr{C}/pA)^o} E_\bullet \circ (\mathsf{t} \circ \Phi_A)^o \xrightarrow{\sim} \operatorname*{2\text{-}colim}_{(\mathscr{C}/pA)^o} \delta.$$

But again, these 2-colimits are represented by the colimits of the same functors, whence the contention. □

5.7.10. Keep the notation of (5.7.6), and let now $\varphi : (\mathscr{A}, p, J_\bullet) \to (\mathscr{A}', p', J'_\bullet)$ be a morphism of small fibred lex-sites over the small category $\mathscr{C}$. We have $\varphi(\Sigma_{\mathscr{A}'}) \subset \Sigma_\mathscr{A}$, whence a commutative diagram of categories, for every $X \in \mathrm{Ob}(\mathscr{C})$:

$$\begin{array}{ccccc}\mathscr{A}'_X & \xrightarrow{i'_X} & \mathscr{A}' & \xrightarrow{L_{\mathscr{A}'}} & \mathscr{A}'[\Sigma_{\mathscr{A}'}^{-1}] \\ \downarrow{\scriptstyle \varphi_X} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi_\Sigma} \\ \mathscr{A}_X & \xrightarrow{i_X} & \mathscr{A} & \xrightarrow{L_\mathscr{A}} & \mathscr{A}[\Sigma_\mathscr{A}^{-1}]\end{array}$$

where i_X and i'_X are the inclusion functors. Let as well $\boldsymbol{\lambda}'$ be a unital cleavage for p', and c' its associated pseudo-functor, which factors likewise through a pseudo-functor $\widetilde{\mathsf{c}}' : \mathscr{C} \to \mathsf{lex}.\mathbf{Site}$; then φ corresponds to a unique pseudo-natural transformation $\omega : \mathsf{c}' \Rightarrow \mathsf{c}$, and the latter in turn can also be regarded as a 2-cell $\widetilde{\omega} : \widetilde{\mathsf{c}} \Rightarrow \widetilde{\mathsf{c}}'$ of $\mathsf{lex}.\mathbf{Site}$. Now, since $\mathscr{A}[\Sigma_\mathscr{A}^{-1}]$ is a strong 2-colimit for c, there exists a unique functor $\psi : \mathscr{A}'[\Sigma_{\mathscr{A}'}^{-1}] \to \mathscr{A}[\Sigma_\mathscr{A}^{-1}]$ such that $\tau \odot \mathsf{F}_\psi = \omega$; clearly we must then have $\psi = \varphi_\Sigma$. On the other hand, since $\mathscr{A}[\Sigma_\mathscr{A}^{-1}]$ is a strong 2-limit for $\widetilde{\mathsf{c}}$, there exists a unique morphism of sites $\widetilde{\psi} : (\mathscr{A}[\Sigma_\mathscr{A}^{-1}], J^*_\mathscr{A}) \to (\mathscr{A}'[\Sigma_{\mathscr{A}'}^{-1}], J^*_{\mathscr{A}'})$ such that $\mathsf{F}_{\widetilde{\psi}} \odot \widetilde{\tau} = \widetilde{\omega}$; thus, we must have $\widetilde{\psi} = \psi$, and this shows that φ_Σ is a morphism of sites as well, so we get a commutative diagram of morphisms of sites :

$$\begin{array}{ccc}(\mathscr{A}[\Sigma_\mathscr{A}^{-1}], J^*_\mathscr{A}) & \xrightarrow{L_\mathscr{A}} & (\mathscr{A}, J_\mathscr{A}) \\ \downarrow{\scriptstyle \varphi_\Sigma} & & \downarrow{\scriptstyle \varphi} \\ (\mathscr{A}'[\Sigma_{\mathscr{A}'}^{-1}], J^*_{\mathscr{A}'}) & \xrightarrow{L_{\mathscr{A}'}} & (\mathscr{A}', J_{\mathscr{A}'})\end{array}$$

which we orient by the identity transformation $\mathbf{1}_{\varphi\circ L_{\mathscr{A}}}$. Following (5.5.23), the resulting oriented diagram then induces an oriented square of 2-categories :

$$\begin{array}{ccc} \mathsf{Stack}(\mathscr{A}[\Sigma_{\mathscr{A}}^{-1}], J^*_{\mathscr{A}}) & \xrightarrow{\mathsf{St}(L_{\mathscr{A}})_*} & \mathsf{Stack}(\mathscr{A}, J_{\mathscr{A}}) \\ {\scriptstyle \mathsf{St}(\varphi_\Sigma)_*}\downarrow & \Downarrow \mathsf{St}(\mathbf{1}_{\varphi\circ L_{\mathscr{A}}})^{\gamma}_* & \downarrow{\scriptstyle \mathsf{St}(\varphi)_*} \\ \mathsf{Stack}(\mathscr{A}'[\Sigma_{\mathscr{A}'}^{-1}], J^*_{\mathscr{A}'}) & \xrightarrow[\mathsf{St}(L_{\mathscr{A}'})_*]{} & \mathsf{Stack}(\mathscr{A}', J_{\mathscr{A}'}) \end{array}$$

which in turn can be regarded as an oriented square of links of the 2-category $\mathsf{V}\text{-}\overline{2\text{-}\mathbf{Cat}}$, for a suitable universe V : see (5.5.25).

Proposition 5.7.11. *In the situation of* (5.7.10)*, suppose that every object of* $\mathscr{A}$ *is quasi-compact for the topology* $J_{\mathscr{A}}$*. Then* $\Upsilon(\mathsf{St}(\mathbf{1}_{\varphi\circ L_{\mathscr{A}}})^{\gamma}_*)$ *is a pseudo-natural equivalence.*

Proof. Let $\mathscr{E}$ be any stack on $(\mathscr{A}, J_{\mathscr{A}})$; as usual, we may assume that $\mathscr{E} = \mathscr{F}ib(E_\bullet)$ for a presheaf of categories $E_\bullet$ on $\mathscr{A}$, and by virtue of lemma 5.5.15(ii) we may also assume that $E_\bullet$ is a sheaf of categories on $(\mathscr{A}, J_{\mathscr{A}})$. According to corollary 5.5.26(iv), and the discussion of (5.5.29), we are then reduced to checking :

Claim 5.7.12. (i) $\mathscr{F}ib((L^{\sim}_{\mathscr{A}}, \mathbf{Cat})^* E_\bullet)$ is a stack on $(\mathscr{A}[\Sigma_{\mathscr{A}}^{-1}], J^*_{\mathscr{A}})$.
(ii) $\Upsilon((\mathbf{1}_{\varphi\circ L_{\mathscr{A}}})^{\sim}_*)$ is an isomorphism of functors.

Proof of the claim. (i): On one hand the natural morphism $(L_{\mathscr{A}}, \mathbf{Cat})_! E_\bullet \to (L^{\sim}_{\mathscr{A}}, \mathbf{Cat})^* E_\bullet$ is an isomorphism, by proposition 5.7.7(ii.a); on the other hand, from claim 5.7.8 and lemma 1.5.7(ii) we see that the category $A/L_{\mathscr{A}}\mathscr{A}$ is cofiltered for every $A \in \mathrm{Ob}(\mathscr{A})$, hence remark 5.5.14(v) yields an equivalence of categories $\mathscr{F}ib((L_{\mathscr{A}}, \mathbf{Cat})_! E_\bullet) \overset{\sim}{\to} \mathsf{Fib}(L_{\mathscr{A}})_!(\mathscr{E})$. Lastly, $\mathsf{Fib}(L_{\mathscr{A}})_!(\mathscr{E})$ is a stack, by proposition 5.7.7(ii.b), whence the assertion.

(ii): By definition, we have :

$$\Upsilon((\mathbf{1}_{\varphi\circ L_{\mathscr{A}}})^{\sim}_*) = (\varepsilon^{L_{\mathscr{A}'}} * \widetilde{\varphi}_{\Sigma *}\widetilde{L}^*_{\mathscr{A}}) \odot (\widetilde{L}^*_{\mathscr{A}'}\widetilde{\varphi}_* * \eta^{L_{\mathscr{A}}})$$

where $\eta^{L_{\mathscr{A}}}$ (resp. $\varepsilon^{L_{\mathscr{A}'}}$) is the unit (resp. the counit) of an adjunction for the pair $(\widetilde{L}^*_{\mathscr{A}}, \widetilde{L}_{\mathscr{A}*})$ (resp. $(\widetilde{L}^*_{\mathscr{A}'}, \widetilde{L}_{\mathscr{A}'*})$). However, $\widetilde{L}_{\mathscr{A}'*}$ is fully faithful (proposition 5.7.7(i)), hence $\varepsilon^{L_{\mathscr{A}'}}$ is an isomorphism (proposition 1.1.20(iii)). Thus, we are reduced to checking that $\widetilde{L}^*_{\mathscr{A}'}\widetilde{\varphi}_* * \eta^{L_{\mathscr{A}}}$ is an isomorphism. However, since also $\varepsilon^{L_{\mathscr{A}}}$ is an isomorphism, we know already that $\widetilde{L}^*_{\mathscr{A}} * \eta^{L_{\mathscr{A}}}$ is an isomorphism, due to the triangular identities of (1.1.13). Thus, we are further reduced to exhibiting an *arbitrary* isomorphism of functors :

$$\widetilde{L}^*_{\mathscr{A}'} \circ \widetilde{\varphi}_* \overset{\sim}{\to} \widetilde{\varphi}_{\Sigma *} \circ \widetilde{L}^*_{\mathscr{A}}.$$

Hence, let F be any sheaf on $(\mathscr{A}, J_{\mathscr{A}})$, and $A' \in \mathrm{Ob}(\mathscr{A}')$; from the proof of proposition 5.7.7(i) we see that $\widetilde{L}^*_{\mathscr{A}'} \circ \widetilde{\varphi}_* F(A')$ represents the colimit of the functor

$$\Psi : (\mathscr{C}/p'A')^o \to \mathbf{Set} \qquad (X \xrightarrow{f} p'A') \mapsto F\varphi(\mathsf{c}'_f A') \qquad (f \xrightarrow{g/p'A'} f') \mapsto F\varphi(\boldsymbol{\lambda}'(\mathbf{1}_{A'}, g))$$

whereas $\widetilde{\varphi}_{\Sigma *} \circ \widetilde{L}^*_{\mathscr{A}} F(A')$ represents the colimit of the functor

$$\Psi' : (\mathscr{C}/p'A')^o \to \mathbf{Set} \qquad (X \xrightarrow{f} p'A') \mapsto F\mathsf{c}_f(\varphi A') \qquad (f \xrightarrow{g/p'A'} f') \mapsto F(\boldsymbol{\lambda}'(\mathbf{1}_{\varphi A'}, g)).$$

Now, let τ^ω denote the coherence constraint of the pseudo-natural transformation ω associated with φ as in (5.7.10), so that $\omega_X : \mathscr{A}_X \to \mathscr{A}'_X$ is the restriction of φ, for every $X \in \mathrm{Ob}(\mathscr{C})$; we claim that the following system of maps yields an isomorphism of functors $\Psi \overset{\sim}{\to} \Psi'$:

(5.7.13) $$(F(\tau^\omega_{f,A'}) : F\varphi(\mathsf{c}'_f A') \overset{\sim}{\to} F\mathsf{c}_f(\varphi A') \mid f \in \mathrm{Ob}(\mathscr{C}/p'A')).$$

More precisely, we show that for every two objects $f : X \to p'A'$ and $f' : X' \to p'A'$ and every morphism $g/p'A' : f \to f'$ of $\mathscr{C}/p'A'$ the following diagram commutes :

$$\begin{array}{ccc} \mathsf{c}_f(\varphi A') & \xrightarrow{\tau^\omega_{f,A'}} & \varphi(\mathsf{c}'_f A') \\ {\scriptstyle \boldsymbol{\lambda}(\mathbf{1}_{\varphi A'}, g/p'A')} \downarrow & & \downarrow {\scriptstyle \varphi(\boldsymbol{\lambda}'(\mathbf{1}_{A'}, g/p'A'))} \\ \mathsf{c}_{f'}(\varphi A') & \xrightarrow{\tau^\omega_{f',A'}} & \varphi(\mathsf{c}'_{f'} A'). \end{array} \tag{5.7.14}$$

To this aim, denote by $\gamma^{\mathsf{c}'}$ the coherence constraint of c', and recall that, under the natural identification $\mathscr{A}' \xrightarrow{\sim} \mathscr{F}ib(\mathsf{c}')$, the morphism $\boldsymbol{\lambda}'(\mathbf{1}_{A'}, g/p'A')$ of $\mathscr{A}'$ corresponds to the morphism

$$(g, \gamma^{\mathsf{c}'\,-1}_{(g,f'),A'} : \mathsf{c}'_f A' \to \mathsf{c}'_g \mathsf{c}'_{f'} A') : (X, \mathsf{c}'_f A') \to (X', \mathsf{c}'_{f'} A').$$

Likewise we describe $\boldsymbol{\lambda}(\mathbf{1}_{\varphi A'}, g)$ in terms of the coherence constraint γ^{c} of c; since $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}'$ are unital, the assertion comes down to the commutativity of the diagram :

$$\begin{array}{ccccc} \mathsf{c}_g \mathsf{c}_{f'}(\varphi A') & & \xrightarrow{\gamma^{\mathsf{c}}_{(g,f'),\varphi A'}} & & \mathsf{c}_f(\varphi A') \\ {\scriptstyle \mathsf{c}_g(\tau^\omega_{f',A'})} \downarrow & & & & \downarrow {\scriptstyle \tau^\omega_{f,A'}} \\ \mathsf{c}_g \circ \varphi(\mathsf{c}'_{f'} A') & \xrightarrow{\tau^\omega_{g,\mathsf{c}'_{f'}A'}} & \varphi(\mathsf{c}'_g \mathsf{c}'_{f'} A') & \xrightarrow{\omega_X(\gamma^{\mathsf{c}'}_{(g,f'),A'})} & \varphi(\mathsf{c}'_f A') \end{array}$$

which in turn follows from the coherence axiom for τ^ω. So finally, the colimit of the system (5.7.13) yields a bijection $t_{F,A'} : \widetilde{L}^*_{\mathscr{A}'} \circ \widetilde{\varphi}_* F(A') \xrightarrow{\sim} \widetilde{\varphi}_{\Sigma *} \circ \widetilde{L}^*_{\mathscr{A}} F(A')$ for every $A' \in \mathrm{Ob}(\mathscr{A}')$, and we need to check that the rule : $A' \mapsto t_{F,A'}$ defines a morphism of sheaves t_F. Thus, let

$$(\beta^A_{F,f} : F(\mathsf{c}_f A) \to \widetilde{L}^*_{\mathscr{A}} F(A) \mid f \in \mathrm{Ob}(\mathscr{C}/pA)) \qquad \text{for every } A \in \mathrm{Ob}(\mathscr{A})$$

be a fixed system of universal cocones, and $h : B \to A$ any morphism in $\mathscr{A}[\Sigma^{-1}_{\mathscr{A}}]$; for a given object $f : X \to pA$ of $\mathscr{C}/pA$, we may find a morphism $g : Y \to pB$ in $\mathscr{C}$ and a morphism $h_{g,f} : \mathsf{c}_g B \to \mathsf{c}_f A$ in $\mathscr{A}$ such that $L_{\mathscr{A}}(\boldsymbol{\lambda}(A, f))^{-1} \circ h = L_{\mathscr{A}}(h_{g,f}) \circ L_{\mathscr{A}}(\boldsymbol{\lambda}(B, g))^{-1}$, and by unraveling the definitions, it is easily seen that :

$$\widetilde{L}^*_{\mathscr{A}} F(h) \circ \beta^A_{F,f} = \beta^B_{F,g} \circ F h_{g,f}.$$

Likewise, fix a system of universal cocones :

$$(\beta'^{A'}_{F,f} : F\varphi(\mathsf{c}'_f A') \to \widetilde{L}^*_{\mathscr{A}'} \widetilde{\varphi}_* F(A') \mid f \in \mathrm{Ob}(\mathscr{C}/p'A')) \qquad \text{for every } A' \in \mathrm{Ob}(\mathscr{A}').$$

If $h' : B' \to A'$ is any morphism of $\mathscr{A}'[\Sigma^{-1}_{\mathscr{A}'}]$, and $f : X \to p'A'$ any object of $\mathscr{C}/p'A'$, we may find a morphism $g : Y \to p'B'$ in $\mathscr{C}$ and a morphism $h'_{g,f} : \mathsf{c}'_g B' \to \mathsf{c}'_f A'$ of $\mathscr{A}'$ such that $L_{\mathscr{A}'}(\boldsymbol{\lambda}'(A, f))^{-1} \circ h' = L_{\mathscr{A}'}(h'_{g,f}) \circ L_{\mathscr{A}'}(\boldsymbol{\lambda}'(B, g))^{-1}$, and again it is easily seen that :

$$\widetilde{L}^*_{\mathscr{A}'} \widetilde{\varphi}_* F(h') \circ \beta'^{A'}_{F,f} = \beta'^{B'}_{F,g} \circ F\varphi(h'_{g,f}).$$

Now, for every such h' we need to check the identity :

$$t_{F,B'} \circ \widetilde{L}^*_{\mathscr{A}'} \widetilde{\varphi}_* F(h') = \widetilde{\varphi}_{\Sigma *} \widetilde{L}^*_{\mathscr{A}} F(h') \circ t_{F,A'} \tag{5.7.15}$$

and it suffices to check that the sides of (5.7.15) agree after composition with $\beta'^{A'}_{F,f}$, for every $f \in \mathrm{Ob}(\mathscr{C}/p'A')$. Notice that if $g : Y \to p'B'$ is a suitable choice for constructing $h'_{g,f}$, then the same choice can be used for constructing $(\varphi_\Sigma h')_{g,f}$ (details left to the reader). We compute:

$$t_{F,B'} \circ \widetilde{L}^*_{\mathscr{A}'} \widetilde{\varphi}_* F(h') \circ \beta'^{A'}_{F,f} = t_{F,B'} \circ \beta'^{B'}_{F,g} \circ F\varphi(h'_{g,f}) = \beta^{\varphi B'}_{F,g} \circ F(\tau^\omega_{g,B'}) \circ F\varphi(h'_{g,f})$$
$$\widetilde{\varphi}_{\Sigma *} \widetilde{L}^*_{\mathscr{A}} F(h') \circ t_{F,A'} \circ \beta'^{A'}_{F,f} = \widetilde{\varphi}_{\Sigma *} \widetilde{L}^*_{\mathscr{A}} F(h') \circ \beta^{\varphi A'}_{F,f} \circ F(\tau^\omega_{f,A'}) = \beta^{\varphi B'}_{F,g} \circ F(\varphi_\Sigma h')_{g,f} \circ F(\tau^\omega_{f,A'}).$$

Thus, we are reduced to checking that we may choose $(\varphi_\Sigma h')_{g,f}$ such that :

$$\varphi(h'_{g,f}) \circ \tau^\omega_{g,B'} = \tau^\omega_{f,A'} \circ (\varphi_\Sigma h')_{g,f}.$$

In other words, we have to check the identity :

$$(5.7.16) \quad L_{\mathscr{A}}((\tau^\omega_{f,A'})^{-1} \circ \varphi(h'_{g,f}) \circ \tau^\omega_{g,B'}) \circ L_{\mathscr{A}}(\boldsymbol{\lambda}(\varphi B', g))^{-1} = L_{\mathscr{A}}(\boldsymbol{\lambda}(\varphi A', f))^{-1} \circ (\varphi_\Sigma h').$$

However, if we regard g as a morphism $g/p'B' : g \to \mathbf{1}_{p'B'}$ of $\mathscr{C}/p'B'$, we get

$$\boldsymbol{\lambda}'(\mathbf{1}_{B'}, g/p'B') = \boldsymbol{\lambda}'(B', g) \qquad \text{and} \qquad \boldsymbol{\lambda}(\mathbf{1}_{\varphi B'}, g/p'B') = \boldsymbol{\lambda}(\varphi B', g).$$

Likewise, we have corresponding identities for $\boldsymbol{\lambda}'(A', g)$ and $\boldsymbol{\lambda}(\varphi A', g)$, by regarding f as a morphism $f/p'A' : f \to \mathbf{1}_{p'A'}$ of $\mathscr{C}/p'A'$. Then (5.7.14) yields the identities

$$\varphi(\boldsymbol{\lambda}'(A', f)) \circ \tau^\omega_{f,A'} = \boldsymbol{\lambda}(\varphi A', f) \qquad \varphi(\boldsymbol{\lambda}'(B', g)) \circ \tau^\omega_{g,B'} = \boldsymbol{\lambda}(\varphi B', g).$$

(notice that $\tau^\omega_{\mathbf{1}_{A'},A'} = \mathbf{1}_{\varphi A'}$ and $\tau^\omega_{\mathbf{1}_{B'},B'} = \mathbf{1}_{\varphi B'}$, since c and c' are unital : see remark 2.4.2(ii)). Summing up, we deduce that (5.7.16) is equivalent to the identity :

$$L_{\mathscr{A}}(\varphi(\boldsymbol{\lambda}'(A', f)) \circ \varphi(h'_{g,f})) \circ L_{\mathscr{A}}(\varphi(\boldsymbol{\lambda}'(B', g)))^{-1} = \varphi_\Sigma h'$$

which follows immediately from the definition of $h'_{g,f}$ and the identity $\varphi_\Sigma \circ L_{\mathscr{A}'} = L_{\mathscr{A}} \circ \varphi$.

This conclude the construction of the morphism of sheaves $t_F : \widetilde{L}^*_{\mathscr{A}'} \circ \widetilde{\varphi}_* F \xrightarrow{\sim} \widetilde{\varphi}_{\Sigma *} \circ \widetilde{L}^*_{\mathscr{A}} F$. To conclude the proof, it remains to check that the rule $F \mapsto t_F$ yields the sought isomorphism of functors. Thus, let $\nu : F \to G$ be a morphism of sheaves on $(\mathscr{A}, J_{\mathscr{A}})$; we come down to checking the identity :

$$(5.7.17) \qquad \widetilde{\varphi}_{\Sigma *} \widetilde{L}^*_{\mathscr{A}}(\nu)_{A'} \circ t_{F,A'} = t_{G,A'} \circ \widetilde{L}^*_{\mathscr{A}} \widetilde{\varphi}_*(\nu) \qquad \text{for every } A' \in \mathrm{Ob}(\mathscr{A}')$$

and again, it suffices to check that the two sides of (5.7.17) agree after composition with $\beta'^{A'}_{F,f}$, for every $f \in \mathrm{Ob}(\mathscr{C}/p'A')$. We compute :

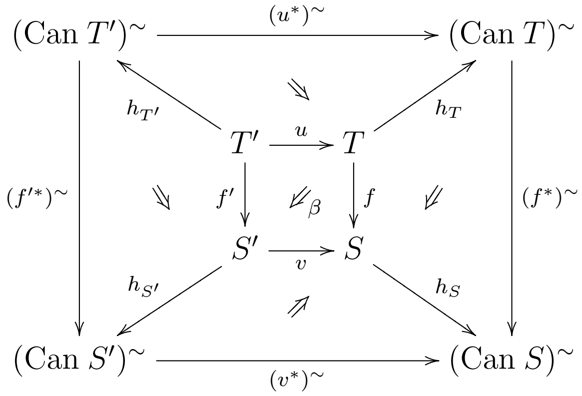
$$\begin{aligned} \widetilde{\varphi}_{\Sigma *} \widetilde{L}^*_{\mathscr{A}}(\nu)_{A'} \circ t_{F,A'} \circ \beta'^{A'}_{F,f} &= \widetilde{\varphi}_{\Sigma *} \widetilde{L}^*_{\mathscr{A}}(\nu)_{A'} \circ \beta^{\varphi A'}_{F,f} \circ F(\tau^\omega_{f,A'}) = \beta^{\varphi A'}_{G,f} \circ \nu_{\mathsf{c}_f(\varphi A')} \circ F(\tau^\omega_{f,A'}) \\ t_{G,A'} \circ \widetilde{L}^*_{\mathscr{A}} \widetilde{\varphi}_*(\nu) \circ \beta'^{A'}_{F,f} &= t_{G,A'} \circ \beta'^{A'}_{G,f} \circ \nu_{\varphi(\mathsf{c}'_f A')} = \beta^{\varphi A'}_{G,f} \circ G(\tau^\omega_{f,A'}) \circ \nu_{\varphi(\mathsf{c}'_f A')}. \end{aligned}$$

So we come down to showing that $\nu_{\mathsf{c}_f(\varphi A')} \circ F(\tau^\omega_{f,A'}) = G(\tau^\omega_{f,A'}) \circ \nu_{\varphi(\mathsf{c}'_f A')}$. The latter is clear, since ν is a morphism of sheaves. □

5.7.18. *Stacks on fibred topoi.* Recall that the pseudo-functor T of (4.4.15) assigns to every site C a topos $\mathsf{T}(C)$ with an isomorphism $\omega_C : C^\sim \xrightarrow{\sim} \mathsf{T}(C)$, and the unit of the 2-adjoint pair $(\mathrm{Can}, \mathsf{T})$ exhibited in the proof of theorem 4.4.17 assigns to every topos T an equivalence $\mathsf{h}_{T*} :\xrightarrow{\sim} \mathsf{T} \circ \mathrm{Can}(T)$ whose composition with $\omega^{-1}_{\mathrm{Can}(T)}$ equals the Yoneda imbedding $h_T : T \xrightarrow{\sim} T^\sim$. Also, for every natural transformation $\beta : u \Rightarrow v$ of morphisms of topoi $u, v : T' \to T$, the isomorphisms $\omega_{\mathrm{Can}(T)}$ and $\omega_{\mathrm{Can}(T')}$ identify $\mathsf{T} \circ \mathrm{Can}(\beta)$ with $(\beta^\dagger)^\sim_*$, where $\beta^\dagger$ is the adjoint transformation to β. Now, consider any oriented square of morphisms of topoi :

$$\begin{array}{ccc} T' & \xrightarrow{u} & T \\ {\scriptstyle f'}\downarrow & \swarrow_\beta & \downarrow{\scriptstyle f} \\ S' & \xrightarrow[v]{} & S \end{array}$$

(*i.e.* β is a natural transformation $f_* \circ u_* \Rightarrow v_* \circ f'_*$). There follows a diagram of oriented squares of topoi :



whose unmarked orientations are identified, via the isomorphisms $\omega_\bullet$, with the coherence constraints of the pseudo-natural transformation $\mathsf{h}_{\bullet *}$. We complete it by inserting the orientation $(\beta^\dagger)^\sim_*$ for the external square. Then the resulting cubical diagram commutes on 2-cells, in the sense of (2.3.21). Since the four diagonal arrows are equivalences (theorem 4.4.2(iv)) it follows that $\Upsilon(\beta)$ is an isomorphism of functors if and only if the same holds for $\Upsilon((\beta^\dagger)^\sim_*)$ (remark 2.3.22(ii)).

5.7.19. Let $\omega : T \to S$ be a morphism of fibred topoi over a small category I; let also $t \in \mathrm{Ob}(I)$ be any object, and define the functor $\rho^t : \mathbb{1} \to I$ as in remark 4.6.22(ii). By direct inspection, we see that :

$$\Psi^*_t := \mathsf{Top}(T/\rho^t)^* \circ \mathsf{Top}(\omega)^* = \omega^*_t \circ \mathsf{Top}(S/\rho^t)^*.$$

Indeed, both functors attach to every object $E_\bullet : I \to \mathscr{F}ib(S^*)$ of $\mathsf{Top}(S)$ the object $\omega^*_t(E_t) \in \mathrm{Ob}(T_t)$, and to every morphism $\beta_\bullet : E_\bullet \to E'_\bullet$, the morphism $\omega^*_t(\beta_t)$. Similarly, we get :

$$\Phi_t := \mathsf{Top}(S/\rho^t)^* \circ \mathsf{Top}(\omega)_* = \omega_{t*} \circ \mathsf{Top}(T/\rho^t)^*.$$

The adjoint of $\mathbf{1}_{\Psi^*_t}$ yields therefore an orientation for the following diagram of topoi :

$$\begin{array}{ccc} T_t & \xrightarrow{\mathsf{Top}(T/\rho^t)} & \mathsf{Top}(T) \\ {\scriptstyle \omega_t}\downarrow & \swarrow \mathbf{1}^\dagger_{\Psi^*_t} & \downarrow{\scriptstyle \mathsf{Top}(\omega)} \\ S_t & \xrightarrow[\mathsf{Top}(S/\rho^t)]{} & \mathsf{Top}(S) \end{array}$$

which we regard as an oriented square of links in the 2-category of categories (see (2.3.8)).

Lemma 5.7.20. *With the notation of* (5.7.19)*, we have :* $\Upsilon(\mathbf{1}^\dagger_{\Psi^*_t}) = \mathbf{1}_{\Phi_t}$.

Proof. We may as usual assume that T and S are unital; then we may replace $\mathsf{Top}(S)$ by the isomorphic category $\mathsf{Top}(S)_*$, and $\mathsf{Top}(\omega)$ by the adjunction $\Omega := (\Omega^*, \Omega_*, \eta^\Omega)$ described in the proof of proposition 4.6.10. Then $\mathsf{Top}(S/\rho^t)^*$ is replaced by $\Sigma(\mathscr{F}ib(S_*)/\rho^{to})^*$ (see the proof of corollary 4.6.21), and we come down to considering the oriented square of links :

$$\begin{array}{ccc} T_t & \xrightarrow{\mathsf{Top}(T/\rho^t)} & \mathsf{Top}(T) \\ {\scriptstyle \omega_t}\downarrow & \swarrow \mathbf{1}^\dagger_{\Lambda^*_t} & \downarrow{\scriptstyle \Omega} \\ S_t & \xrightarrow[\Sigma(\mathscr{F}ib(S_*)/\rho^{to})]{} & \mathsf{Top}(S)_* \end{array} \qquad \text{where } \Lambda^*_t := \mathbf{Top}(T/\rho^t)^* \circ \Omega^*$$

whose bottom horizontal arrow is given by $\Sigma(\mathscr{F}ib(S_*)/\rho^{to})^*$, its right adjoint, and a unit for this adjoint pair. Set $\Phi'_t := \Sigma(\mathscr{F}ib(S_*)/\rho^{to})^* \circ \Omega_*$; by proposition 2.3.10, it suffices to show that $\Upsilon({}^o\mathbf{1}_{\Lambda_t^*}) = {}^o\mathbf{1}_{\Phi_t'}$. However, let $(\eta^{\omega_i}, \varepsilon^{\omega_i})$ be the unit and counit of the adjunction for the pair $(\omega_i^*, \omega_{i*})$ defining the morphism of topoi ω_i, for every $i \in \mathrm{Ob}(I)$; by definition we have :

$$\Upsilon({}^o\mathbf{1}_{\Lambda_t^*}) = (\varepsilon^{{}^o\omega_t} * {}^o\Sigma(\mathscr{F}ib(S_*)/\rho^{to})^* * {}^o\Omega_*) \odot ({}^o\omega_{t*} * {}^o\mathsf{Top}(T/\rho^t)^* * \eta^{{}^o\Omega})$$

and recall that $\varepsilon^{{}^o\omega_t} = {}^o\eta^{\omega_t}$; likewise, $\eta^{{}^o\Omega} = {}^o\varepsilon^{\Omega}$, where ε^{Ω} denotes the counit for the adjoint pair Ω. Thus, we get :

$${}^o\Upsilon({}^o\mathbf{1}_{\Lambda_t^*}) = (\omega_{t*} * \mathsf{Top}(T/\rho^t)^* * \varepsilon^{\Omega}) \odot (\eta^{\omega_t} * \Sigma(\mathscr{F}ib(S_*)/\rho^{to})^* * \Omega_*) = (\Phi_t * \varepsilon^{\Omega}) \odot (\eta^{\omega_t} * \Phi_t).$$

But by inspecting the proof of proposition 4.6.10 we see that ε^{Ω} is the natural transformation that assigns to every object $E_\bullet : I \to \mathscr{F}ib(T^*)$ of $\mathsf{Top}(T)$ the system of morphisms $(\varepsilon^{\omega_i} : \omega_i^* \circ \omega_{i*}E_i \to E_i \mid i \in \mathrm{Ob}(I))$. Thus, finally, the natural transformation $(\Phi_t * \varepsilon^{\Omega}) \odot (\eta^{\omega_t} * \Phi_t)$ attaches to every such $E_\bullet$ the morphism $(\omega_{t*} * \varepsilon^{\omega_t}) \odot (\eta^{\omega_t} * \omega_{t*})$, which is indeed $\mathbf{1}_{\omega_{t*}E_t}$, by the triangular identities of (1.1.13). □

5.7.21. Keep the situation of (5.7.19), and to ease notation set

$$v_t := \mathsf{Top}(S/\rho^t)^* \qquad v_t' := \mathsf{Top}(T/\rho^t)^* \qquad \mathsf{S} := \mathrm{Can} \circ \mathsf{Top} : \mathsf{PsFun}(I, \mathbf{Topos}) \to \mathbf{Site}.$$

Arguing as in (5.5.23), we deduce an essentially commutative diagram :

$$\begin{array}{ccc} \mathsf{Stack}(\mathrm{Can}\, T_t) & \xrightarrow{\mathsf{St}(v_t')_*} & \mathsf{Stack}(\mathsf{S}(T)) \\ {\scriptstyle \mathsf{St}(\omega_t^*)_*}\downarrow & \Downarrow \mathsf{St}(\mathbf{1}_{\Psi_t^*})_*^{\gamma} & \downarrow{\scriptstyle \mathsf{St}(\mathsf{Top}(\omega)^*)_*} \\ \mathsf{Stack}(\mathrm{Can}\, S_t) & \xrightarrow[\mathsf{St}(v_t)_*]{} & \mathsf{Stack}(\mathsf{S}(S)) \end{array}$$

which we regard as an oriented square in $\mathsf{Link}(\mathsf{V}\text{-}\overline{2\text{-}\mathbf{Cat}})$ (see (5.5.25)); then we may state :

Proposition 5.7.22. $\Upsilon(\mathsf{St}(\mathbf{1}_{\Psi_t^*})_*^{\gamma})$ *is a pseudo-natural equivalence.*

Proof. Let $\mathscr{E}$ be any stack on the canonical site of $\mathsf{Top}(T)$; we need to check that $\Upsilon(\mathsf{St}(\mathbf{1}_{\Psi_t^*})_*^{\gamma})_{\mathscr{E}}$ is an equivalence of categories. To this aim, in view of claim 3.2.8 we may replace $\mathscr{E}$ by the split fibration $\mathsf{C}(\mathscr{E})$, so that $\mathscr{E} = \mathscr{F}ib(\mathscr{A}_\bullet)$ for a presheaf of categories $\mathscr{A}_\bullet$ on $\mathsf{Top}(T)$; then by lemma 5.5.15(ii) we may as well replace $\mathscr{A}_\bullet$ by $\mathscr{A}_\bullet^a$, and assume that $\mathscr{A}_\bullet$ is a sheaf of categories on $\mathsf{Top}(T)$.

Claim 5.7.23. $\Upsilon((\mathbf{1}_{\Psi_t^*}^{\sim}, \mathbf{Cat})_*)$ is a pseudo-natural equivalence.

Proof of the claim. As explained in (5.5.29), it suffices to check that $\Upsilon((\mathbf{1}_{\Psi_t^*})_*^{\sim})$ is an isomorphism of functors. By (5.7.18), the latter holds if and only if $\Upsilon(\mathbf{1}_{\Psi_t^*}^{\dagger})$ is an isomorphism of functors. This in turn is clear from lemma 5.7.20. ◊

In light of claim 5.7.23 and corollary 5.5.26(iv), it suffices to check that $\mathscr{F}ib((\widetilde{v}_t', \mathbf{Cat})^*\mathscr{A}_\bullet)$ is a stack on the canonical site of T_t. For every site C, let

$$\omega_C : \mathbf{Cat}^*(C^{\sim}) \xrightarrow{\sim} (C, \mathbf{Cat})^{\sim}$$

be the strict and strong 2-equivalence of (5.5.9). We may assume that $\mathscr{A}_\bullet = \omega_{\mathsf{Top}(T)}(\mathscr{A}^*)$ for a category object $\mathscr{A}^*$ of $\mathrm{Can}(\mathsf{Top}(T))^{\sim}$ and then we need to check that $\mathscr{F}ib(\omega_{T_t} \circ \mathbf{Cat}^*(\widetilde{v}_t'^*)(\mathscr{A}^*))$ is a stack on $\mathrm{Can}(T_t)$. Since $h_{\mathsf{Top}(T)} : \mathsf{Top}(T) \to \mathrm{Can}(\mathsf{Top}(T))^{\sim}$ is an equivalence, we may assume that $\mathscr{A}^* = \mathbf{Cat}^*(h_{\mathsf{Top}(T)})(\mathscr{B}^*)$ for some $\mathscr{B}^* \in \mathrm{Ob}(\mathbf{Cat}^*(\mathsf{Top}(T)))$; by corollary 4.3.19(i.b), it then suffices to check that $\mathscr{F}ib(\omega_{T_t} \circ \mathbf{Cat}^*(h_{T_t}) \circ \mathbf{Cat}^*(v_t')(\mathscr{B}^*))$ is a stack. However, theorem 4.6.24(i) yields an equivalence of categories $a : \mathsf{Top}(T) \xrightarrow{\sim} (\mathrm{Can}(T), J)^{\sim}$ and a morphism of sites $b : \mathrm{Can}(\mathsf{Top}(T)) \to (\mathrm{Can}(T), J)$ with isomorphisms of functors :

$$h_{T_t} \circ v_t' \xrightarrow{\sim} \widetilde{\imath}_{t*} \circ a \qquad a \xrightarrow{\sim} \widetilde{b}_* \circ h_{\mathsf{Top}(T)}$$

where $i_t : T_t \to \mathrm{Can}(T)$ is the inclusion functor. There follow isomorphisms of fibrations :

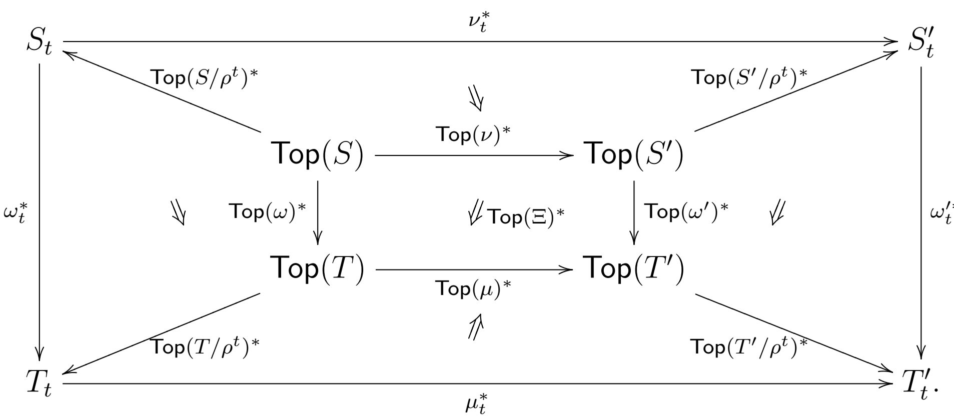
$$\begin{aligned}\mathscr{F}ib(\omega_{T_t} \circ \mathbf{Cat}^*(h_{T_t}) \circ \mathbf{Cat}^*(v'_t)\mathscr{B}^*) &\xrightarrow{\sim} \mathscr{F}ib(\omega_{T_t} \circ \mathbf{Cat}^*(\widetilde{\imath}_{t*}) \circ \mathbf{Cat}^*(a)\mathscr{B}^*) \\ &\xrightarrow{\sim} \mathscr{F}ib((\widetilde{\imath}_t, \mathbf{Cat})_* \circ \omega_{(\mathrm{Can}(T),J)} \circ \mathbf{Cat}^*(a)\mathscr{B}^*) \\ &\xrightarrow{\sim} \mathsf{Fib}(i_t)^*(\mathscr{F}ib(\omega_{(\mathrm{Can}(T),J)} \circ \mathbf{Cat}^*(a)\mathscr{B}^*))\end{aligned}$$

and by corollary 5.7.2, it then suffices to check that $\mathscr{F}ib(\omega_{(\mathrm{Can}(T),J)} \circ \mathbf{Cat}^*(a)\mathscr{B}^*)$ is a stack on $(\mathrm{Can}(T), J)$. We are then further reduced to showing that $\mathscr{F}ib((\widetilde{b}, \mathbf{Cat})_*\mathscr{A}_\bullet)$ is a stack on $(\mathrm{Can}(T), J)$, or equivalently, that the same holds for $\mathsf{Fib}(b)^*(\mathscr{F}ib(\mathscr{A}_\bullet)) = \mathsf{Fib}(b)^*(\mathscr{E})$. The latter holds, by corollary 5.4.15. $\square$

5.7.24. Consider now an oriented diagram of fibred topoi over a small category I :

$$\begin{CD} T' @>{\mu}>> T \\ @V{\omega'}VV \swarrow_{\Xi} @VV{\omega}V \\ S' @>>{\nu}> S \end{CD}$$

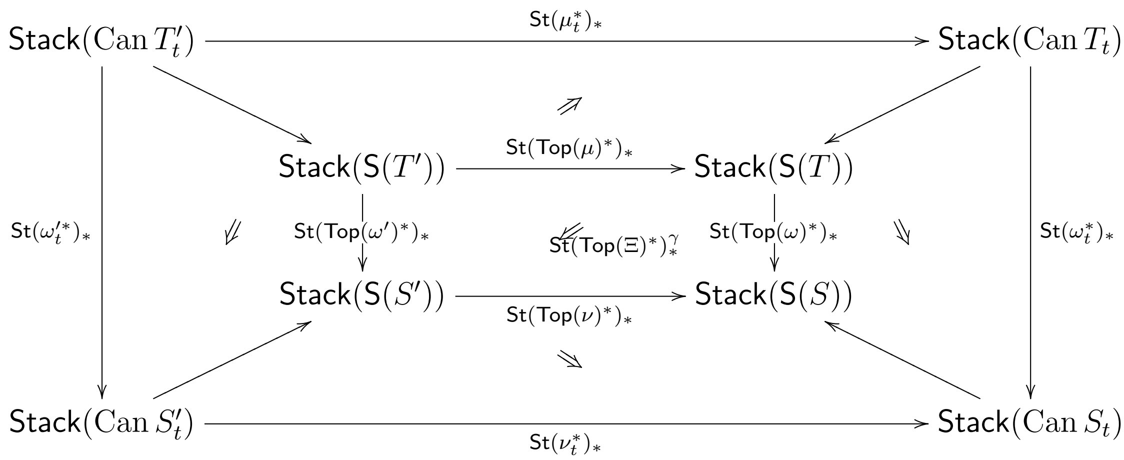
i.e. $\Xi : \omega \circ \mu \rightsquigarrow \nu \circ \omega'$ is a modification. For every $t \in \mathrm{Ob}(I)$ we get an oriented diagram :



whose four unmarked 2-cells are identities. We complete it by adding the orientation

$$\Xi_t^\dagger := (\Xi_t, \eta^{(\omega\circ\mu)_t}, \eta^{(\nu\circ\omega')_t})^\dagger : (\nu \circ \omega')_t^* \Rightarrow (\omega \circ \mu)_t^*$$

for the front face (notation of remark 1.1.17(ii)). Then a direct inspection shows that the resulting cubical diagram commutes both on 1-cells and 2-cells. As in (5.7.21), there follows an essentially commutative oriented diagram of 2-categories :



whose four unmarked 2-cells are defined as in the diagram of (5.7.21), and which we complete to a cubical diagram that commutes on 2-cells, by adding the orientation for the front face

$$\mathsf{St}(\Xi_t^\dagger)_*^\gamma : \mathsf{St}(\omega_t^*)_* \circ \mathsf{St}(\mu_t^*)_* \Rightarrow \mathsf{St}(\nu_t^*)_* \circ \mathsf{St}(\omega_t'^*)_*.$$

Thus, let $\mathscr{E}$ be any stack on the site $\mathrm{Can}(\mathsf{Top}(T))$, and set $\mathscr{E}_t := \mathsf{St}(\mathsf{Top}(T/\rho^t)^*)^*\mathscr{E}$; in light of proposition 5.7.22 and remark 2.3.22(i), we conclude that :

$$\mathsf{St}(\mathsf{Top}(S'/\rho^t)^*)^* * \Upsilon(\mathsf{St}(\mathsf{Top}(\Xi)^*)_*^\gamma)_{\mathscr{E}} \text{ is an equivalence} \Leftrightarrow \Upsilon(\mathsf{St}(\Xi_t^\dagger)_*^\gamma)_{\mathscr{E}_t} \text{ is an equivalence.}$$

5.7.25. Next, consider an oriented diagram of fibred lex-sites over I :

$$\mathscr{D} \quad : \quad \begin{array}{ccc} \underline{\mathscr{A}} & \xrightarrow{\varphi'} & \underline{\mathscr{A}}' \\ {\scriptstyle\psi'}\downarrow & \swarrow_\beta & \downarrow{\scriptstyle\psi} \\ \underline{\mathscr{B}} & \xrightarrow[\varphi]{} & \underline{\mathscr{B}}'. \end{array}$$

For every $t \in \mathrm{Ob}(I)$ the restriction $\mathscr{D}_t$ of the diagram $\mathscr{D}$ to the fibre categories over t yields another oriented square of 2-categories

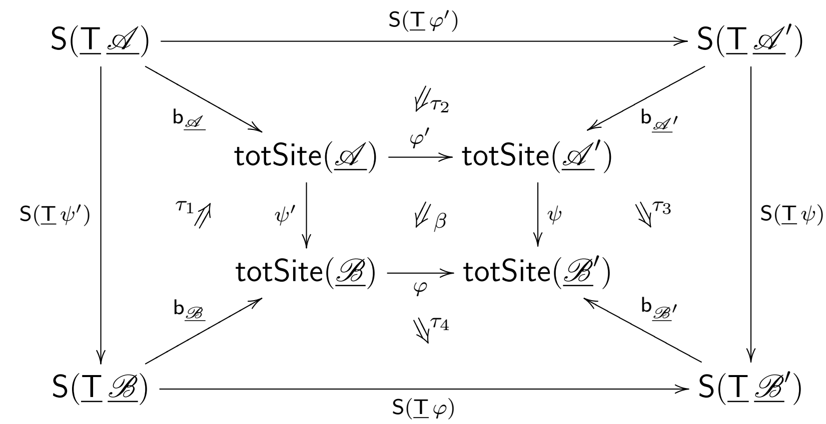
$$\mathsf{St}(\mathscr{D}_t)_* \quad : \quad \begin{array}{ccc} \mathsf{Stack}(\mathscr{A}_t) & \xrightarrow{\mathsf{St}(\varphi_t')_*} & \mathsf{Stack}(\mathscr{A}_t') \\ {\scriptstyle\mathsf{St}(\psi_t')_*}\downarrow & \Downarrow \mathsf{St}(\beta_t)_*^\gamma & \downarrow{\scriptstyle\mathsf{St}(\psi_t)_*} \\ \mathsf{Stack}(\mathscr{B}_t) & \xrightarrow[\mathsf{St}(\varphi_t)_*]{} & \mathsf{Stack}(\mathscr{B}_t'). \end{array}$$

For every fibred site $\underline{\mathscr{A}} := (\mathscr{A}, p, J_\bullet)$ over I, let also $i_{\mathscr{A},t} : \mathscr{A}_t \to \mathscr{A}$ be the inclusion functor.

Corollary 5.7.26. *In the situation of* (5.7.25)*, let $\mathscr{E}$ be any stack on the total site of $\underline{\mathscr{A}}'$, and set $\mathscr{E}_t := \mathsf{St}(i_{\mathscr{A}',t})_*\mathscr{E}$ for every $t \in \mathrm{Ob}(I)$. The following conditions are equivalent :*

(a) $\Upsilon(\mathsf{St}(\beta)_*^\gamma)_{\mathscr{E}}$ *is an equivalence.*
(b) $\Upsilon(\mathsf{St}(\beta_t)_*^\gamma)_{\mathscr{E}_t}$ *is an equivalence for every $t \in \mathrm{Ob}(I)$.*

Proof. By invoking the pseudo-natural transformation $\mathsf{b}_\bullet : \mathsf{S} \circ \underline{\mathsf{lex.T}} \to \mathsf{totSite}$ of remark 4.6.29(ii), we get an oriented cubical diagram of fibred sites :



whose front face is oriented by $\mathsf{S}(\underline{\mathsf{T}}\beta) = \mathsf{Top}(\underline{\mathsf{T}}\beta)^*$, and where the orientations $\tau_1, \dots, \tau_4$ are given by the coherence constraints of $\mathsf{b}_\bullet$; again, this cubical diagram commutes on 2-cells. After applying termwise as usual the pseudo-functor $\mathsf{St}(-)_*$, we deduce a similar cubical diagram whose front and back faces are oriented by $\mathsf{St}(\mathsf{Top}(\underline{\mathsf{T}}\beta)^*)_*^\gamma$ and respectively $\mathsf{St}(\beta)_*^\gamma$, and whose other four faces are oriented by $\mathsf{St}(\tau_i)_*^\gamma$ for $i = 1, \dots, 4$.

Claim 5.7.27. $\mathsf{St}(\mathsf{b}_{\underline{\mathscr{A}}})_*$ is a 2-equivalence for every fibred lex-site $\underline{\mathscr{A}}$ over I.

Proof of the claim. By construction, $\mathsf{b}_{\underline{\mathscr{A}}}$ is the composition of an isomorphism of canonical sites $\mathsf{S}(\underline{\mathsf{T}}\,\underline{\mathscr{A}}) \xrightarrow{\sim} \mathrm{Can} \circ \mathsf{T}(\mathsf{totSite}\,\underline{\mathscr{A}})$ with the counit $\varepsilon_{\mathsf{totSite}(\underline{\mathscr{A}})} : \mathrm{Can} \circ \mathsf{T}(\mathsf{totSite}\,\underline{\mathscr{A}}) \to \mathsf{totSite}(\underline{\mathscr{A}})$. Thus, it suffices to check that $\mathsf{St}(\varepsilon_{\mathsf{totSite}(\underline{\mathscr{A}})})_*$ is a 2-equivalence. But the natural isomorphism $\mathsf{T}(\mathsf{totSite}\,\underline{\mathscr{A}}) \xrightarrow{\sim} \mathsf{totSite}(\underline{\mathscr{A}})^\sim$ identifies $\varepsilon_{\mathsf{totSite}(\underline{\mathscr{A}})}$ with the Yoneda morphism $h^a_{\mathsf{totSite}(\underline{\mathscr{A}})} : \mathsf{totSite}(\underline{\mathscr{A}})^\sim \to \mathsf{totSite}(\underline{\mathscr{A}})$, so the assertion follows from theorem 5.4.12. ◊

Claim 5.7.28. For every fibred lex-site $\underline{\mathscr{A}}$ over I we have pseudo-natural equivalences :

$$\mathsf{St}(\omega^*_{\mathscr{A}_t})^* \circ \mathsf{St}(\mathsf{Top}(\underline{\mathsf{T}}(\underline{\mathscr{A}})/\rho^t)^*)^* \circ \mathsf{St}(\mathsf{b}_{\underline{\mathscr{A}}})^* \xrightarrow{\sim} \mathsf{St}(h^a_{\mathscr{A}_t})^* \circ \mathsf{St}(i_{\mathscr{A},t})_* \qquad \text{for every } t \in \mathrm{Ob}(I)$$

where $\omega_{\mathscr{A}_t} : \mathscr{A}_t^\sim \xrightarrow{\sim} \mathsf{T}(\mathscr{A}_t)$ is the isomorphism of topoi as in (4.4.15).

Proof of the claim. Recall that $\mathsf{St}(h^a_{\mathscr{A}_t})^*$ and its 2-adjoint $\mathsf{St}(h^a_{\mathscr{A}_t})_*$ are 2-equivalences (theorem 5.4.12); then, from proposition 5.4.29(i,ii) we get pseudo-natural equivalences :

$$\mathsf{St}(h^a_{\mathscr{A}_t})^* \circ \mathsf{St}(i_{\mathscr{A},t})_* \xrightarrow{\sim} \mathsf{St}(h^a_{\mathscr{A}_t})^* \circ \mathsf{St}(\breve{\imath}_{\mathscr{A},t})^* \xrightarrow{\sim} \mathsf{St}(\breve{\imath}^*_{\mathscr{A},t})^* \circ \mathsf{St}(h^a_{\mathsf{totSite}(\underline{\mathscr{A}})})^*.$$

We are thus reduced to checking the essential commutativity of the following diagram of sites :

$$\begin{array}{ccc}
\mathrm{Can}(\mathscr{A}_t^\sim) & \xrightarrow{\ \breve{\imath}^*_{\underline{\mathscr{A}},t}\ } & \mathrm{Can}(\mathsf{totSite}\,\underline{\mathscr{A}})^\sim \\
{\scriptstyle \mathsf{Top}(\underline{\mathsf{T}}(\underline{\mathscr{A}})/\rho^t)^*\circ\omega^*_{\mathscr{A}_t}}\big\downarrow & & \big\downarrow{\scriptstyle h^a_{\mathsf{totSite}(\underline{\mathscr{A}})}} \\
\mathrm{Can} \circ \mathsf{Top}(\underline{\mathsf{T}}\,\underline{\mathscr{A}}) & \xrightarrow{\ \mathsf{b}_{\underline{\mathscr{A}}}\ } & \mathsf{totSite}(\underline{\mathscr{A}}).
\end{array}$$

The latter follows by direct inspection. ◊

Now, on the one hand, it is clear that condition (a) holds if and only if $\mathsf{St}(i_{\mathscr{B},t})_* * \Upsilon(\mathsf{St}(\beta)^\gamma_*)_{\mathscr{E}}$ is an equivalence for every $t \in \mathrm{Ob}(I)$. By claim 5.7.28, the latter holds if and only if :

(c) $(\mathsf{St}(\mathsf{Top}(\underline{\mathsf{T}}(\underline{\mathscr{B}})/\rho^t)^*)^* \circ \mathsf{St}(\mathsf{b}_{\underline{\mathscr{B}}})^*) * \Upsilon(\mathsf{St}(\beta)^\gamma_*)_{\mathscr{E}}$ is an equivalence for every $t \in \mathrm{Ob}(I)$.

Set $\mathscr{E}' := \mathsf{St}(\mathsf{b}_{\underline{\mathscr{A}}'})^*(\mathscr{E})$. By claim 5.7.27 and remarks 2.3.11 and 2.3.22(i), condition (c) in turn holds if and only if :

(d) $\mathsf{St}(\mathsf{Top}(\underline{\mathsf{T}}(\underline{\mathscr{B}})/\rho^t)^*)^* * \Upsilon(\mathsf{St}(\mathsf{Top}(\underline{\mathsf{T}}\beta)^*)^\gamma_*)_{\mathscr{E}'}$ is an equivalence for every $t \in \mathrm{Ob}(I)$.

Then, the discussion of (5.7.24) shows that (d) holds if and only if :

(e) $\Upsilon(\mathsf{St}((\mathsf{T}\beta_t)^\dagger)^\gamma_*)_{\mathsf{St}(\mathsf{Top}(\underline{\mathsf{T}}(\underline{\mathscr{A}}')/\rho^t)^*)^*\mathscr{E}'}$ is an equivalence for every $t \in \mathrm{Ob}(I)$.

By invoking again claim 5.7.28, we see that (e) in turn holds if and only if :

(f) $\Upsilon(\mathsf{St}((\beta_t^\sim)^\dagger)^\gamma_*)_{\mathsf{St}(h^a_{\mathscr{A}'_t})^*\mathscr{E}_t}$ is an equivalence for every $t \in \mathrm{Ob}(I)$.

However, from the counit $\varepsilon : \mathrm{Can} \circ \mathsf{T} \Rightarrow \mathbf{1}_{\mathbf{Site}}$ of the 2-adjoint pair $(\mathrm{Can}, \mathsf{T})$ and the system of isomorphisms $\omega_C : C^\sim \xrightarrow{\sim} \mathsf{T}(C)$ of (4.4.15) we get as usual an oriented cubical diagram

$$\begin{array}{ccccccc}
\mathsf{Stack}(\mathrm{Can}\,\mathscr{A}_t^\sim) & & & \xrightarrow{\ \mathsf{St}(\widetilde{\varphi}'^*_t)_*\ } & & & \mathsf{Stack}(\mathrm{Can}\,\mathscr{A}'^\sim_t) \\
& \searrow^{\mathsf{St}(h^a_{\mathscr{A}_t})_*} & & \Uparrow & & {}^{\mathsf{St}(h^a_{\mathscr{A}'_t})_*}\swarrow & \\
& & \mathsf{Stack}(\mathscr{A}_t) & \xrightarrow{\ \mathsf{St}(\varphi'_t)_*\ } & \mathsf{Stack}(\mathscr{A}'_t) & & \\
{\scriptstyle \mathsf{St}(\widetilde{\psi}'^*_t)_*}\big\downarrow & \Downarrow & {\scriptstyle \mathsf{St}(\psi'_t)_*}\big\downarrow & \Downarrow{\scriptstyle \mathsf{St}(\beta_t)^\gamma_*} & \big\downarrow{\scriptstyle \mathsf{St}(\psi_t)_*} & \Downarrow & \big\downarrow{\scriptstyle \mathsf{St}(\widetilde{\psi}^*_t)_*} \\
& & \mathsf{Stack}(\mathscr{B}_t) & \xrightarrow[\ \mathsf{St}(\varphi_t)_*\ ]{} & \mathsf{Stack}(\mathscr{B}'_t) & & \\
& \nearrow^{\mathsf{St}(h^a_{\mathscr{B}_t})_*} & & \Downarrow & & {}^{\mathsf{St}(h^a_{\mathscr{B}'_t})_*}\nwarrow & \\
\mathsf{Stack}(\mathrm{Can}\,\mathscr{B}_t^\sim) & & & \xrightarrow[\ \mathsf{St}(\widetilde{\varphi}^*_t)_*\ ]{} & & & \mathsf{Stack}(\mathrm{Can}\,\mathscr{B}'^\sim_t).
\end{array}$$

commuting on 2-cells, whose four unmarked orientations are pseudo-natural equivalences, and whose front face is oriented by $\mathsf{St}((\beta_t^\sim)^\dagger)^\gamma_*$. Combining with remarks 2.3.11 and 2.3.22(i) we finally conclude that (f)⇔(b). □

Example 5.7.29. Let $u : C' := (\mathscr{C}', J') \to C := (\mathscr{C}, J)$ be a morphism of lex-sites.

(i) Recall that the target functor $\mathsf{t}_\mathscr{C} : \mathsf{Morph}(\mathscr{C}) \to \mathscr{C}$ is a fibration whose fibre category $\mathsf{t}_\mathscr{C}^{-1}X$ is naturally identified with $\mathscr{C}/X$, for every $X \in \mathrm{Ob}(\mathscr{C})$ (example 3.1.3(iii)). Fix a unital cleavage for this fibration, and let c be the associated pseudo-functor; it is easily seen that for every morphism $\varphi : X \to Y$ in $\mathscr{C}$, the functor $\mathsf{c}_\varphi : \mathscr{C}/Y \to \mathscr{C}/X$ is right adjoint to $\varphi_* : \mathscr{C}/X \to \mathscr{C}/Y$, hence it is left exact : explicitly, c_φ assigns to every object $Z \xrightarrow{h} Y$ of $\mathscr{C}/Y$ the induced projection $(X \times_Y Z \to X) \in \mathrm{Ob}(\mathscr{C}/X)$, where $X \times_Y Z$ is a choice of representative for the fibre product of X and Z over Y (detail left to the reader). This description implies easily that c_φ is a morphism of lex-sites $C/X \to C/Y$, for every such φ (notation of (4.7)). The collection of sites $(C/X \mid X \in \mathrm{Ob}(\mathscr{C}))$ then endows the fibration $\mathsf{t}_\mathscr{C}$ with a well defined structure of fibred lex-site $(\mathsf{Morph}(\mathscr{C}), \mathsf{t}_\mathscr{C}, J^\mathscr{C}_\bullet)$.

(ii) The functor $u : \mathscr{C} \to \mathscr{C}'$ induces a commutative diagram of categories :

$$\begin{array}{ccc} \mathsf{Morph}(\mathscr{C}) & \xrightarrow{\mathsf{Morph}(u)} & \mathsf{Morph}(\mathscr{C}') \\ {\scriptstyle \mathsf{t}_\mathscr{C}}\downarrow & & \downarrow{\scriptstyle \mathsf{t}_{\mathscr{C}'}} \\ \mathscr{C} & \xrightarrow{\quad u \quad} & \mathscr{C}'. \end{array}$$

Let now $\mathscr{B}$ be any category, and $F : \mathscr{B} \to \mathscr{C}$ a given functor; we set

$$\mathscr{A} := \mathsf{Fib}(F)^*(\mathsf{Morph}(\mathscr{C})) \qquad \mathscr{A}' := \mathsf{Fib}(F \circ u)^*(\mathsf{Morph}(\mathscr{C}'))$$

and let

$$\mathsf{Morph}(\mathscr{C}) \xleftarrow{\pi} \mathscr{A} \xrightarrow{\varphi} \mathscr{B} \xleftarrow{\varphi'} \mathscr{A}' \xrightarrow{\pi'} \mathsf{Morph}(\mathscr{C}')$$

be the natural projections. There is a unique functor

$$g : \mathscr{A} \to \mathscr{A}' \qquad \text{such that } \varphi' \circ g = \varphi \text{ and } \pi' \circ g = \mathsf{Morph}(u) \circ \pi$$

and since u is left exact, it is easily seen that g is cartesian (details left to the reader). Moreover, the restriction $\varphi^{-1}B \to \varphi'^{-1}B$ of g is naturally identified with $u_{|FB} : \mathscr{C}/FB \to \mathscr{C}'/uFB$, which is a morphism of lex-sites $C'/uFB \to C/FB$. Thus, g is a morphism of fibred lex-sites

$$\mathscr{B} \times_{\mathscr{C}'} (\mathsf{Morph}(\mathscr{C}'), \mathsf{t}_{\mathscr{C}'}, J^{\mathscr{C}'}_\bullet) \to \mathscr{B} \times_\mathscr{C} (\mathsf{Morph}(\mathscr{C}), \mathsf{t}_\mathscr{C}, J^\mathscr{C}_\bullet)$$

and therefore, g is also a morphism of the induced total sites (proposition 4.5.9)

$$g : (\mathscr{A}', J_{\mathscr{A}'}) \to (\mathscr{A}, J_\mathscr{A}).$$

(iii) Suppose moreover that $\mathscr{B}$ is cofiltered, in which case the pseudo-functor c^o factors through a pseudo-functor $\widetilde{\mathsf{c}} : \mathscr{B} \to \mathsf{lex}.\mathbf{Site}$ as in (5.7.6), whose strong 2-limit is represented by the localization $\mathscr{A}[\Sigma_\mathscr{A}^{-1}]$, endowed with a certain topology $J^*_\mathscr{A}$, where $\Sigma_\mathscr{A}$ denotes the set of cartesian morphisms of $\mathscr{A}$. Likewise, the corresponding localization $\mathscr{A}'[\Sigma_{\mathscr{A}'}^{-1}]$ of $\mathscr{A}'$ carries a natural topology $J^*_{\mathscr{A}'}$, and for every $B \in \mathrm{Ob}(\mathscr{B})$ the composition of the inclusion functor $i_{FB} : \mathscr{C}/FB \to \mathscr{A}$ (resp. $i_{uFB} : \mathscr{C}'/uFB \to \mathscr{A}'$) with the localization functor $L_\mathscr{A} : \mathscr{A} \to \mathscr{A}[\Sigma_\mathscr{A}^{-1}]$ (resp. $L_{\mathscr{A}'} : \mathscr{A}' \to \mathscr{A}'[\Sigma_{\mathscr{A}'}^{-1}]$) is a morphism of sites

$$l_{FB} : (\mathscr{A}[\Sigma_\mathscr{A}^{-1}], J^*_\mathscr{A}) \to C/FB \qquad (\text{resp. } l_{uFB} : (\mathscr{A}'[\Sigma_{\mathscr{A}'}^{-1}], J^*_{\mathscr{A}'}) \to C'/uFB\ ).$$

Hence, for every $B \in \mathrm{Ob}(\mathscr{B})$ we get an oriented square of sites :

$$\begin{array}{ccc} (\mathscr{A}'[\Sigma^{-1}_{\mathscr{A}'}], J^*_{\mathscr{A}'}) & \xrightarrow{l_{uFB}} & C'/uFB \\ {\scriptstyle g_\Sigma}\downarrow & \Downarrow & \downarrow{\scriptstyle u_{|FB}} \\ (\mathscr{A}[\Sigma^{-1}_{\mathscr{A}}], J^*_{\mathscr{A}}) & \xrightarrow{l_{FB}} & C/FB \end{array}$$

whose orientation is the identity $\mathbf{1}_{l_{FB} \circ g_\Sigma}$, and where g_Σ is the localization of g (see (5.7.10)). With this notation, we have the following :

Corollary 5.7.30. *In the situation of example* 5.7.29(iii)*, suppose furthermore that $\mathscr{B}$ admits a final object B_0, and that every object of $\mathscr{C}'$ is quasi-compact for the topology J'. Then the base change transformation*

$$\Upsilon(\mathsf{St}(\mathbf{1}_{l_{FB_0} \circ g_\Sigma})^\gamma_*) : \mathsf{St}(l_{FB_0})^* \circ \mathsf{St}(u_{|FB_0})_* \to \mathsf{St}(g_\Sigma)_* \circ \mathsf{St}(l_{uFB_0})^*$$

is a pseudo-natural equivalence.

Proof. We regard C/FB_0 as a fibred lex-site over the category $\mathbb{1}$ with one object and one morphism; then we may define the fibred lex-site $\mathscr{B} \times_{\mathbb{1}} C/FB_0$ over $\mathscr{B}$, as in (4.5.11) : its underlying category is $\mathscr{B} \times \mathscr{C}/FB_0$, and its fibre categories are all naturally identified with $\mathscr{C}/FB_0$, and endowed with the topology of C/FB_0. Likewise we define the fibred lex-site $\mathscr{B} \times_{\mathbb{1}} C'/uFB_0$, and we endow as well the underlying categories with their respective total site topologies. According to example 4.5.8, the inclusion functors $i_{FB_0} : \mathscr{C}/FB_0 \to \mathscr{B} \times \mathscr{C}/FB_0$ and $i_{uFB_0} : \mathscr{C}'/uFB_0 \to \mathscr{B} \times \mathscr{C}'/uFB_0$ are then morphisms of sites :

$$i_{FB_0} : \mathscr{B} \times_{\mathbb{1}} C/FB_0 \to C/FB_0 \qquad i_{uFB_0} : \mathscr{B} \times_{\mathbb{1}} C'/uFB_0 \to C'/uFB_0.$$

For every $B \in \mathrm{Ob}(\mathscr{B})$, let $t_B : B \to B_0$ be the unique morphism in $\mathscr{B}$; we have a natural isomorphism of categories

$$(\mathscr{C}/FB_0)/Ft_B \xrightarrow{\sim} \mathscr{C}/FB$$

that identifies the source functor $\mathsf{s}_{Ft_B} : (\mathscr{C}/FB_0)/Ft_B \to \mathscr{C}/FB_0$ with the functor $(Ft_B)_* : \mathscr{C}/FB \to \mathscr{C}/FB_0$, and likewise for the category $(\mathscr{C}'/uFB_0)/uFt_B$. According to (5.5.30), for every such B we have then an oriented square of sites :

$$\begin{array}{ccc} C'/uFB & \xrightarrow{p_{uFt_B}} & C'/uFB_0 \\ {\scriptstyle u_{|FB}}\downarrow & \Downarrow \beta_B & \downarrow{\scriptstyle u_{|FB_0}} \\ C/FB & \xrightarrow{p_{Ft_b}} & C/FB_0. \end{array}$$

It is easily seen that the system of orientations $(\beta_B \mid B \in \mathrm{Ob}(\mathscr{B}))$ amounts to an orientation β for the central square subdiagram in the following diagram of oriented squares of sites :

$$\begin{array}{ccccccc} (\mathscr{A}'[\Sigma^{-1}_{\mathscr{A}'}], J^*_{\mathscr{A}'}) & \xrightarrow{L_{\mathscr{A}'}} & (\mathscr{A}', J_{\mathscr{A}'}) & \xrightarrow{p_{\mathscr{A}'}} & \mathscr{B} \times_{\mathbb{1}} C'/uFB_0 & \xrightarrow{i_{uFB_0}} & C'/uFB_0 \\ {\scriptstyle g_\Sigma}\downarrow & \Downarrow & \downarrow{\scriptstyle g} & \Downarrow \beta & \downarrow{\scriptstyle \mathscr{B} \times u_{|FB_0}} & \Downarrow & \downarrow{\scriptstyle u_{|FB_0}} \\ (\mathscr{A}[\Sigma^{-1}_{\mathscr{A}}], J^*_{\mathscr{A}}) & \xrightarrow{L_{\mathscr{A}}} & (\mathscr{A}, J_{\mathscr{A}}) & \xrightarrow{p_{\mathscr{A}}} & \mathscr{B} \times_{\mathbb{1}} C/FB_0 & \xrightarrow{i_{FB_0}} & C/FB_0. \end{array}$$

Here $p_{\mathscr{A}}$ is the cartesian functor whose restriction to fibre categories $\mathscr{C}/FB_0 \to \mathscr{C}/FB$ agrees with p_{Ft_B}, for every $B \in \mathrm{Ob}(\mathscr{B})$, and the orientations of the left and right square are identities. Under our assumptions, it is easily seen that every object of $\mathscr{A}$ is quasi-compact for the topology $J_{\mathscr{A}'}$; then the base change transformation associated with the left square subdiagram is a pseudo-natural equivalence, by virtue of proposition 5.7.11. The same holds for the base change transformation associated with the central square, by proposition 5.5.31 and corollary

5.7.26. A simple inspection shows that the composition of the three top (resp. bottom) horizontal arrows equals l_{uFB_0} (resp. l_{FB_0}). It remains therefore only to check that the base change transformation

$$\mathsf{St}(i_{FB_0})^*\mathsf{St}(u_{|FB_0})_* \to \mathsf{St}(\mathscr{B} \times u_{|FB_0})_*\mathsf{St}(i_{uFB_0})^*$$

associated with the right square is a pseudo-natural equivalence. To this aim, we remark :

Claim 5.7.31. Let $\mathscr{F}_\bullet$ be any sheaf of categories on C'/uFB_0 such that $\mathscr{F}ib(\mathscr{F}_\bullet)$ is a stack on C'/uFB_0. Then $\mathscr{F}ib((\widetilde{\imath}^*_{uFB_0}, \mathbf{Cat})^*\mathscr{F}_\bullet)$ is a stack on $\mathscr{B} \times_{\mathbb{1}} C'/uFB_0$.

Proof of the claim. notice first that, since B_0 is a final object in $\mathscr{B}$, the projection $q_{uFB_0} : \mathscr{B} \times \mathscr{C}'/uFB_0 \to \mathscr{C}'/uFB_0$ is a left adjoint for the functor i_{uFB_0}. On the other hand, proposition 4.5.5(iv) easily implies that q_{uFB_0} is continuous for the sites C'/uFB_0 and $\mathscr{B} \times_{\mathbb{1}} C'/uFB_0$. Combining with lemma 4.2.14(ii), we deduce a natural isomorphism of functors :

$$\widetilde{\imath}^*_{uFB_0} \xrightarrow{\sim} \widetilde{q}_{uFB_0*}. \tag{5.7.32}$$

Thus, it suffices to check that $\mathscr{F}ib((\widetilde{q}_{uFB_0*}, \mathbf{Cat})_*\mathscr{F}_\bullet)$ is a stack. The latter follows easily from corollary 5.7.2(ii). ◊

In light of claim 5.7.31, corollary 5.5.26(iv), and the discussion of (5.5.29), we are then reduced to checking that the base change transformation

$$\widetilde{\imath}^*_{FB_0} \circ (u_{|FB_0})^\sim_* \to (\mathscr{B} \times u_{|FB_0})^\sim_* \circ \widetilde{\imath}^*_{uFB_0}$$

is an isomorphism of functors. But notice that i_{uFB_0} is fully faithful, and it is both continuous and cocontinuous (proposition 4.5.5(iii)); it follows that $\widetilde{\imath}^*_{uFB_0}$ is fully faithful (lemma 4.2.15(iv)), hence the unit of the adjoint pair $(\widetilde{\imath}^*_{uFB_0}, \widetilde{\imath}_{uFB_0*})$ is an isomorphism. Thus, let η and ε be the unit and counit for the adjoint pair $(\widetilde{\imath}^*_{FB_0}, \widetilde{\imath}_{FB_0*})$; we are reduced to checking that $\varepsilon * ((\mathscr{B} \times u_{|FB_0})^\sim_* \circ \widetilde{\imath}^*_{uFB_0})$ is an isomorphism. However, let $q_{FB_0} : \mathscr{B} \times \mathscr{C}/FB_0 \to \mathscr{C}/FB_0$ be the projection; arguing as in the proof of claim 5.7.31 we get an isomorphism of functors

$$\widetilde{\imath}^*_{FB_0} \xrightarrow{\sim} \widetilde{q}_{FB_0*}. \tag{5.7.33}$$

From (5.7.32) and (5.7.33) there follow isomorphisms of functors :

$$(\mathscr{B} \times u_{|FB_0})^\sim_* \circ \widetilde{\imath}^*_{uFB_0} \xrightarrow{\sim} (\mathscr{B} \times u_{|FB_0})^\sim_* \circ \widetilde{q}_{uFB_0*} \xrightarrow{\sim} \widetilde{q}_{FB_0*} \circ (u_{|FB_0})^\sim_* \xrightarrow{\sim} \widetilde{\imath}^*_{FB_0} \circ (u_{|FB_0})^\sim_*.$$

So it suffices to check that $\varepsilon * (\widetilde{\imath}^*_{FB_0} \circ (u_{|FB_0})^\sim_*)$ is an isomorphism; but again, arguing as in the foregoing we see that $\widetilde{\imath}^*_{FB_0}$ is fully faithful, so η is an isomorphism, and finally the assertion follows, taking into account the triangular identities of (1.1.13). □

Remark 5.7.34. In the situation of example 5.7.29(ii), the functors $\varphi : \mathscr{A} \to \mathscr{B}$ and $\mathsf{s}_{\mathscr{C}} \circ \pi : \mathscr{A} \to \mathscr{C}$ induce a functor

$$s : \mathscr{A} \to \mathscr{B} \times \mathscr{C} \qquad (B, f : X \to Y) \mapsto (B, X)$$

(namely, s is the unique functor whose composition with the projections $\mathscr{B} \leftarrow \mathscr{B} \times \mathscr{C} \to \mathscr{C}$ equals respectively φ and $\mathsf{s}_{\mathscr{C}} \circ \pi$). The functor s admits a right adjoint :

$$q : \mathscr{B} \times \mathscr{C} \to \mathscr{A}$$

that assigns to every $(B, X) \in \mathrm{Ob}(\mathscr{B} \times \mathscr{C})$ the object $(B, p_{X,B} : X \times FB \to FB) \in \mathrm{Ob}(\mathscr{A})$, where $p_{X,B}$ is the natural projection. To every morphism $(\beta, f) : (B, X) \to (B', X')$ of $\mathscr{B} \times \mathscr{C}$, the functor q assigns the morphism $q(\beta, f) := (\beta, \mathscr{D})$ of $\mathscr{A}$, with $\mathscr{D}$ the commutative square :

$$\begin{array}{ccc} X \times FB & \xrightarrow{p_{X,B}} & FB \\ {\scriptstyle f \times F\beta}\downarrow & & \downarrow{\scriptstyle F\beta} \\ X' \times FB' & \xrightarrow{p_{X',B'}} & FB'. \end{array}$$

Let us regard $(\mathscr{C}, J)$ as a fibred site over the category $\mathbb{1}$ with one object and one morphism; then $\mathscr{B} \times_{\mathbb{1}} (\mathscr{C}, J)$ is a fibred category over $\mathscr{B}$, and taking into account remark 4.7.3(iii), it is easily seen that q is a morphism of fibred sites

$$q : \mathscr{B} \times_{\mathscr{C}} (\mathsf{Morph}(\mathscr{C}), \mathsf{t}_{\mathscr{C}}, J_{\bullet}^{\mathscr{C}}) \to \mathscr{B} \times_{\mathbb{1}} (\mathscr{C}, J)$$

and therefore it is as well a morphism of the respective total sites

$$q : (\mathscr{A}, J_{\mathscr{A}}) \to (\mathscr{B} \times \mathscr{C}, J_{\mathscr{B}\times\mathscr{C}}).$$

It follows that s is cocontinuous for the topologies $J_{\mathscr{A}}$ and $J_{\mathscr{B}\times\mathscr{C}}$ (lemma 4.2.14(i)); moreover, we easily deduce from example 5.4.16 that s is a weak morphism of fibred sites

$$s : \mathscr{B} \times_{\mathbb{1}} (\mathscr{C}, J) \to \mathscr{B} \times_{\mathscr{C}} (\mathsf{Morph}(\mathscr{C}), \mathsf{t}_{\mathscr{C}}, J_{\bullet}^{\mathscr{C}})$$

(see definition 5.7.4); therefore it is as well a weak morphism of the respective total sites

$$s : (\mathscr{B} \times \mathscr{C}, J_{\mathscr{B}\times\mathscr{C}}) \to (\mathscr{A}, J_{\mathscr{A}})$$

(proposition 5.7.5). Combining with proposition 5.4.29(ii,iii), we deduce a pseudo-natural equivalence of pseudo-functors :

$$\mathsf{St}(s)_* \xrightarrow{\sim} \mathsf{St}(q)^*.$$

6. Monoids and polyhedra

Unless explicitly stated otherwise, *every monoid encountered in this chapter shall be commutative*. For this reason, we shall usually economize adjectives, and write just "monoid" when referring to commutative monoids.

6.1. **Monoids.** If M is any monoid, we shall usually denote the composition law of M by multiplicative notation: $(x, y) \mapsto x \cdot y$ (so 1 is the neutral element). However, sometimes it is convenient to be able to switch to an additive notation; to allow for that, we shall denote by $(\log M, +)$ the monoid with additive composition law, whose underlying set is the same as for the given monoid $(M, \cdot)$, and such that the identity map is an isomorphism of monoids (then, the neutral element of $\log M$ is denoted by 0). For emphasis, we may sometimes denote by $\log : M \xrightarrow{\sim} \log M$ the identity map, so that one has the tautological identities :

$$\log 1 = 0 \qquad \text{and} \qquad \log(x \cdot y) = \log x + \log y \quad \text{for every } x, y \in M.$$

Conversely, if $(M, +)$ is a given monoid with additive composition law, we may switch to a multiplicative notation by writing $(\exp M, \cdot)$, in the same way.

6.1.1. For any monoid M, and any two subsets $S, S' \subset M$, we let :

$$S \cdot S' := \{s \cdot s' \mid s \in S, s' \in S'\}$$

and S^a is defined recursively for every $a \in \mathbb{N}$, by the rule :

$$S^0 := \{1\} \qquad \text{and} \qquad S^a := S \cdot S^{a-1} \quad \text{if } a > 0.$$

Notice that the pair $(\mathscr{P}(M), \cdot)$ consisting of the set of all subsets of M, together with the composition law just defined, is itself a monoid : the neutral element is the subset $\{1\}$. In the same vein, the exponential notation for subsets of M becomes a multiplicative notation in the monoid $(\log \mathscr{P}(M), +) = (\mathscr{P}(\log M), +)$, *i.e.* we have the tautological identity : $\log S^a = a \cdot \log S$, for every $S \in \mathscr{P}(M)$ and every $a \in \mathbb{N}$.

Furthermore, for any two monoids M and N, the set $\mathrm{Hom}_{\mathbf{Mnd}}(M, N)$ is naturally a monoid. The composition law assigns to any two morphisms $\varphi, \psi : M \to N$ their product $\varphi \cdot \psi$, given by the rule : $\varphi \cdot \psi(m) := \varphi(m) \cdot \psi(m)$ for every $m \in M$.

Basic examples of monoids are the set $(\mathbb{N}, +)$ of natural numbers, and the non-negative real (resp. rational) numbers $(\mathbb{R}_+, +)$ (resp. $(\mathbb{Q}_+, +)$), with their standard addition laws.

6.1.2. Given a surjection $X \to Y$ of monoids, it may not be possible to express Y as a quotient of X – a problem relevant to the construction of *presentations* for given monoids, in terms of free monoids. For instance, consider the monoid $(\mathbb{Z}, \odot)$ consisting of the set $\mathbb{Z}$ with the composition law $\odot$ such that :

$$x \odot y := \begin{cases} x + y & \text{if either } x, y \geq 0 \text{ or } x, y \leq 0 \\ \max(x, y) & \text{otherwise} \end{cases}$$

for every $x, y \in \mathbb{Z}$. Define a surjective map $\varphi : \mathbb{N}^{\oplus 2} \to (\mathbb{Z}, \odot)$ by the rule $(n, m) \mapsto n \odot -m$, for every $n \in \mathbb{N}$. Then one verifies easily that $\operatorname{Ker} \varphi = \{0\}$, and nevertheless φ is not an isomorphism. The right way to proceed is indicated by the following :

Lemma 6.1.3. *Every surjective map of monoids is an effective epimorphism (in the category* $\mathbf{Mnd}$*).*

Proof. (See example 4.1.9(v) for the notion of effective epimorphism.) Let $\pi : M \to N$ be a surjection of monoids. For every monoid X, we have a natural diagram of sets :

$$\operatorname{Hom}_{\mathbf{Mnd}}(N, X) \xrightarrow{j} \operatorname{Hom}_{\mathbf{Mnd}}(M, X) \overset{p_1^*}{\underset{p_2^*}{\rightrightarrows}} \operatorname{Hom}_{\mathbf{Mnd}}(M \times_N M, X)$$

where $p_1, p_2 : M \times_N M \to M$ are the two natural projections, and we have to show that the map j identifies $\operatorname{Hom}_{\mathbf{Mnd}}(N, X)$ with the equalizer of p_1^* and p_2^*. First of all, the surjectivity of π easily implies that j is injective. Hence, let $\varphi : M \to X$ be any map such that $\varphi \circ p_1 = \varphi \circ p_2$; we have to show that φ factors through π. To this aim, it suffices to show that the map of sets underlying φ factors as a composition $\varphi' \circ \pi$, for some map of sets $\varphi' : N \to X$, since φ' will then be necessarily a morphism of monoids. However, the forgetful functor $F : \mathbf{Mnd} \to \mathbf{Set}$ commutes with fibre products (lemma 4.8.29(iii)), and $F(\pi)$ is an effective epimorphism, since in the category $\mathbf{Set}$ all surjections are effective epimorphisms. The assertion follows. □

6.1.4. Lemma 6.1.3 allows to construct presentations for an arbitrary monoid M, as follows. First, we choose a surjective map of monoids $F := \mathbb{N}^{(S)} \to M$, for some set S. Then we choose another set T and a surjection of monoids $\mathbb{N}^{(T)} \to F \times_M F$. Composing with the natural projections $p_1, p_2 : F \times_M F \to F$, we obtain a diagram :

$$\mathbb{N}^{(T)} \overset{q_1}{\underset{q_2}{\rightrightarrows}} \mathbb{N}^{(S)} \longrightarrow M \qquad (6.1.5)$$

which, in view of lemma 6.1.3, identifies M to the coequalizer of q_1 and q_2.

Definition 6.1.6. Let M be a monoid, $\Sigma \subset M$ a subset.

(i) Let $(e_\sigma \mid \sigma \in \Sigma)$ be the natural basis of the free monoid $\mathbb{N}^{(\Sigma)}$. We say that Σ is a *system of generators* for M, if the map of monoids $\mathbb{N}^{(\Sigma)} \to M$ such that $e_\sigma \mapsto \sigma$ for every $\sigma \in \Sigma$, is a surjection.
(ii) M is said to be *finitely generated* if it admits a finite system of generators.
(iii) M is said to be *fine* if it is integral and finitely generated.
(iv) A *finite presentation* for M is a diagram such as (6.1.5) that identifies M to the coequalizer of q_1 and q_2, and such that, moreover, S and T are finite sets.
(v) We say that a morphism of monoids $\varphi : M \to N$ is *finite*, if N is a finitely generated M-module, for the M-module structure induced by φ.

Lemma 6.1.7. (i) *Every finitely generated monoid admits a finite presentation.*

(ii) *Let M be a finitely generated monoid, and $(N_i \mid i \in I)$ a filtered family of monoids. Then the natural map :*

$$\operatorname*{colim}_{i \in I} \operatorname{Hom}_{\mathbf{Mnd}}(M, N_i) \to \operatorname{Hom}_{\mathbf{Mnd}}(M, \operatorname*{colim}_{i \in I} N_i)$$

is a bijection.

Proof. (i): Let M be a finitely generated monoid, and choose a surjection $\pi : \mathbb{N}^{(S)} \to M$ with S a finite set. We have seen that M is the coequalizer of the two projections $p_1, p_2 : P := \mathbb{N}^{(S)} \times_M \mathbb{N}^{(S)} \to \mathbb{N}^{(S)}$. For every finitely generated submonoid $N \subset P$, let $p_{1,N}, p_{2,N} : N \to \mathbb{N}^{(S)}$ be the restrictions of p_1 and p_2, and denote by C_N the coequalizer of $p_{1,N}$ and $p_{2,N}$. By the universal property of C_N, the map π factors through a map of monoids $\pi_N : C_N \to M$, and since π is surjective, the same holds for π_N. It remains to show that π_N is an isomorphism, for N large enough. We apply the functor $M \mapsto \mathbb{Z}[M]$ of (4.8.50), and we derive that $\mathbb{Z}[M]$ is the coequalizer of the two maps $\mathbb{Z}[p_1], \mathbb{Z}[p_2] : \mathbb{Z}[P] \to \mathbb{Z}[S]$, *i.e.* $\mathbb{Z}[M] \simeq \mathbb{Z}[S]/I$, where I is the ideal generated by $\mathrm{Im}(\mathbb{Z}[p_1] - \mathbb{Z}[p_2])$. Clearly I is the colimit of the filtered system of analogous ideals I_N generated by $\mathrm{Im}(\mathbb{Z}[p_{1,N}] - \mathbb{Z}[p_{2,N}])$, for N ranging over the filtered family $\mathscr{F}$ of finitely generated submonoids of P. By noetherianity, there exists $N \in \mathscr{F}$ such that $I = I_N$, therefore $\mathbb{Z}[M]$ is the coequalizer of $\mathbb{Z}[p_{1,N}]$ and $\mathbb{Z}[p_{2,N}]$. But the latter coequalizer is also the same as $\mathbb{Z}[C_N]$, whence the contention.

(ii): This is a standard consequence of (i). Indeed, say that $f_1, f_2 : M \to N_i$ are two morphisms whose compositions with the natural map $N_i \to N := \mathrm{colim}_{i \in I} N_i$ agree, and pick a finite set of generators $x_1, \dots, x_n$ for M. For any morphism $\varphi : i \to j$ in the filtered category I, denote by $g_\varphi : N_i \to N_j$ the corresponding morphism; then we may find such a morphism φ, so that $g_\varphi \circ f_1(x_k) = g_\varphi \circ f_2(x_k)$ for every $k \leq n$, whence the injectivity of the map in (ii). Next, let $f : M \to N$ be a given morphism, and pick a finite presentation (6.1.5); we deduce a morphism $g : \mathbb{N}^{(S)} \to N$, and since S is finite, it is clear that g factors through a morphism $g_i : \mathbb{N}^{(S)} \to N_i$ for some $i \in I$. Set $g'_i := g_i \circ q_1$ and $g''_i := g_i \circ q_2$; by assumption, after composing g'_i and of g''_i with the natural map $N_i \to N$, we obtain the same map, so by the foregoing there exists a morphism $\varphi : i \to j$ in I such that $g_\varphi \circ g'_i = g_\varphi \circ g''_i$. It follows that $g_\varphi \circ g_i$ factors through M, whence the surjectivity of the map in (ii). □

Definition 6.1.8. Let M be a monoid, $I \subset M$ an ideal.

(i) We say that I is *principal*, if it is cyclic, when regarded as an M-module.
(ii) The *radical* of I is the ideal $\mathrm{rad}(I)$ consisting of all $x \in M$ such that $x^n \in I$ for every sufficiently large $n \in \mathbb{N}$. If $I = \mathrm{rad}(I)$, we also say that I is a *radical ideal*.
(iii) A *face* of M is a submonoid $F \subset M$ with the following property. If $x, y \in M$ are any two elements, and $xy \in F$, then $x, y \in F$.
(iv) Notice that the complement of a face is always an ideal. We say that I is a *prime ideal* of M, if $M \backslash I$ is a face of M.

Proposition 6.1.9. *Let M be a finitely generated monoid, and S a finitely generated M-module. Then we have :*

(i) *Every submodule of S is finitely generated.*
(ii) *Especially, every ideal of M is finitely generated.*

Proof. Of course, (ii) is a special case of (i). To show (i), let $S' \subset S$ be an M-submodule, $\Sigma \subset S'$ any system of generators. Let $\mathscr{P}'(\Sigma)$ be the set of all finite subsets of Σ, and for every $A \in \mathscr{P}'(\Sigma)$, denote by $S'_A \subset S'$ the submodule generated by A; clearly S' is the filtered union of the family $(S'_A \mid A \in \mathscr{P}'(\Sigma))$, hence $\mathbb{Z}[S']$ is the filtered union of the family of $\mathbb{Z}[M]$-submodules $(\mathbb{Z}[S'] \mid S \in \mathscr{P}'(\Sigma))$. Since $\mathbb{Z}[M]$ is noetherian and $\mathbb{Z}[S]$ is a finitely generated $\mathbb{Z}[M]$-module, it follows that $\mathbb{Z}[S'_A] = \mathbb{Z}[S']$ for some finite subset $A \subset \Sigma$, whence the contention. □

Definition 6.1.10. (i) Let M be a monoid. The *spectrum* of M is the set :

$$\mathrm{Spec}\, M$$

consisting of all prime ideals of M. It has a natural partial ordering, given by inclusion of prime ideals; the minimal element of $\mathrm{Spec}\, M$ is the empty ideal $\varnothing \subset M$, and the maximal element is:

$$\mathfrak{m}_M := M \backslash M^\times.$$

(ii) The *dimension* of M, denoted $\dim M \in \mathbb{N} \cup \{+\infty\}$, is defined as the supremum of all $r \in \mathbb{N}$ such that there exists a chain of strict inclusions of prime ideals of M :

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_r.$$

(iii) The *height* of a prime ideal $\mathfrak{p} \in \mathrm{Spec}\, M$ is defined as $\mathrm{ht}\, \mathfrak{p} := \dim M_\mathfrak{p}$.

(iv) A *facet* of M is the complement of a prime ideal of M of height one.

(v) Any morphism $\varphi : M \to N$ of monoids induces a natural map :

$$\varphi^* : \mathrm{Spec}\, N \to \mathrm{Spec}\, M \qquad \mathfrak{p} \mapsto \varphi^{-1}\mathfrak{p}$$

of partially ordered sets. We say that φ is *local*, if $\varphi(\mathfrak{m}_M) \subset \mathfrak{m}_N$.

Remark 6.1.11. Let M be any monoid.

(i) It is clear from definition 6.1.10(i) that $\dim M = 0$ if and only if $M = M^\times$.

(ii) Moreover, if $(I_\lambda \mid \lambda \in \Lambda)$ is any collection of ideals of M then it is easily seen that both $\bigcup_{\lambda\in\Lambda} I_\lambda$ and $\bigcap_{\lambda\in\Lambda} I_\lambda$ are ideals of M. If $(\mathfrak{p}_\lambda \mid \lambda \in \Lambda)$ is any family of prime ideals of M, then $\bigcup_{\lambda\in\Lambda} \mathfrak{p}_\lambda$ is a prime ideal of M.

Corollary 6.1.12. *Let M be any fine and sharp monoid. The set $\mathfrak{m}_M \backslash \mathfrak{m}_M^2$ is finite, and is the unique minimal system of generators of M.*

Proof. It is easily seen that any system of generators of M must contain $\Sigma := \mathfrak{m}_M \backslash \mathfrak{m}_M^2$, hence the latter must be a finite set. On the other hand, suppose that there exists an element $x_0 \in M$ which is not contained in the submonoid M' generated by Σ. Then we may write $x_0 = x_1 y_1$ for some $x_1, y_1 \in \mathfrak{m}_M$, with $x_1 \notin M'$, so x_1 admits a similar decomposition. Proceeding in this way, we obtain a sequence of elements $(x_n \mid n \in \mathbb{N})$ with the property that $Mx_n \subset Mx_{n+1}$ for every $n \in \mathbb{N}$. We claim that $Mx_n \neq Mx_{n+1}$ for every $n \in \mathbb{N}$. Indeed, if the inequality fails for some $n \in \mathbb{N}$, we may write $x_{n+1} = ax_n$ for some $a \in M$, and on the other hand, we have by construction $x_n = yx_{n+1}$ for some $y \in \mathfrak{m}_M$; summing up, we get $x_n = yax_n$, whence $ya = 1$, since M is integral, therefore $y \in M^\times$, a contradiction.

Thus, from the given x_0, we have produced an infinite strictly ascending chain of ideals of M, which is ruled out by virtue of proposition 6.1.9(ii). This means that x_0 cannot exist, and the corollary follows. □

Lemma 6.1.13. *Let $f_1 : M \to N_1$ and $f_2 : M \to N_2$ be two local morphisms of monoids. If N_1 and N_2 are sharp, then $N_1 \amalg_M N_2$ is sharp.*

Proof. Let $(a, b) \in N_1 \times N_2$, and suppose there exist $c \in M$, $a' \in N_1$, $b' \in N_2$ such that $(a, b) = (a' f_1(c), b')$ and $(1, 1) = (a', f_2(c)b')$; since N_2 is sharp, we deduce $f_2(c) = b' = 1$, so $b = 1$. Then, since f_2 is local, we get $c \in M^\times$, hence $f_1(c) = 1$ and $a = a' = 1$. One argues symmetrically in case $(1, 1) = (a' f_1(c), b')$ and $(a, b) = (a', f_2(c)b')$. We conclude that (a, b) represents the unit class in $N_1 \amalg_M N_2$ if and only if $a = b = 1$. Now, suppose that the class of (a, b) is invertible in $N_1 \amalg_M N_2$; it follows that there exists (c, d) such that $ac = 1$ and $bd = 1$, which implies that $a = 1$ and $b = 1$, whence the contention. □

Lemma 6.1.14. *Let $S \subset M$ be any submonoid. The localization $j : M \to S^{-1}M$ induces an injective map $j^* : \mathrm{Spec}\, S^{-1}M \to \mathrm{Spec}\, M$ which identifies $\mathrm{Spec}\, S^{-1}M$ with the subset of $\mathrm{Spec}\, M$ consisting of all prime ideals $\mathfrak{p}$ such that $\mathfrak{p} \cap S = \varnothing$.*

Proof. For every $\mathfrak{p} \in \mathrm{Spec}\, M$, denote by $S^{-1}\mathfrak{p}$ the ideal of $S^{-1}M$ generated by the image of $\mathfrak{p}$. We claim that $\mathfrak{p} = j^*(S^{-1}\mathfrak{p})$ for every $\mathfrak{p} \in \mathrm{Spec}\, M$ such that $\mathfrak{p} \cap S = \varnothing$. Indeed, clearly

$\mathfrak{p} \subset j^*(S^{-1}\mathfrak{p})$; next, if $f \in j^*(S^{-1}\mathfrak{p})$, there exist $s \in S$ and $g \in \mathfrak{p}$ such that $s^{-1}g = f$ in $S^{-1}M$; therefore there exists $t \in S$ such that $tg = tsf$ in M, especially $tsf \in \mathfrak{p}$, hence $f \in \mathfrak{p}$, since $t, s \notin \mathfrak{p}$. Likewise, one checks easily that $S^{-1}\mathfrak{p}$ is a prime ideal if $\mathfrak{p} \cap S = \varnothing$, and $\mathfrak{q} = S^{-1}(j^*\mathfrak{q})$ for every $\mathfrak{q} \in \operatorname{Spec} S^{-1}M$, whence the contention. □

Remark 6.1.15. (i) If we take $S_\mathfrak{p} := M \backslash \mathfrak{p}$, the complement of a prime ideal $\mathfrak{p}$ of M, we obtain the monoid

$$M_\mathfrak{p} := S_\mathfrak{p}^{-1} M$$

and $\operatorname{Spec} M_\mathfrak{p} \subset \operatorname{Spec} M$ is the subset consisting of all prime ideals $\mathfrak{q}$ of M contained in $\mathfrak{p}$.

(ii) Likewise, if $\mathfrak{p} \subset M$ is any prime ideal, the spectrum $\operatorname{Spec}(M \backslash \mathfrak{p})$ is naturally identified with the subset of $\operatorname{Spec} M$ consisting of all prime ideals $\mathfrak{q}$ containing $\mathfrak{p}$ (details left to the reader).

(iii) Let $S \subset M$ be any submonoid. Then there exists a smallest face F of M containing S (namely, the intersection of all the faces that contain S). It is easily seen that F is the subset of all $x \in M$ such that $xM \cap S \neq \varnothing$. From this characterization, it is clear that $S^{-1}M = F^{-1}M$. In other words, every localization of M is of the type $M_\mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Spec} M$.

Lemma 6.1.16. *Let M be a monoid, and $I \subset M$ any ideal. Then $\operatorname{rad}(I)$ is the intersection of all the prime ideals of M containing I.*

Proof. It is easily seen that a prime ideal containing I also contains $\operatorname{rad}(I)$. Conversely, say that $f \in M \setminus \operatorname{rad}(I)$; let $\varphi : M \to M_f$ be the localization map. Denote by $\mathfrak{m}$ the maximal ideal of M_f. We claim that $I \subset \varphi^{-1}\mathfrak{m}$, Indeed, otherwise there exist $g \in I$, $h \in M$ and $n \in \mathbb{N}$ such that $g^{-1} = f^{-n}h$ in M_f; this means that there exists $m \in \mathbb{N}$ such that $f^{m+n} = f^m gh$ in M, hence $f^{m+n} \in I$, which contradicts the assumption on f. On the other hand, obviously $f \notin \varphi^{-1}\mathfrak{m}$. □

Lemma 6.1.17. (i) *Let M be a monoid, and $G \subset M^\times$ a subgroup.*

(a) *The map given by the rule : $I \mapsto I/G$ establishes a natural bijection from the set of ideals of M onto the set of ideals of M/G.*

(b) *Especially, the natural projection $\pi : M \to M/G$ induces a bijection :*

$$\pi^* : \operatorname{Spec} M/G \to \operatorname{Spec} M$$

(ii) *Let $(M_i \mid i \in I)$ be any finite family of monoids, and for each $j \in I$, denote by $\pi_j : \prod_{i\in I} M_i \to M_j$ the natural projection. The induced map*

$$\prod_{i\in I} \operatorname{Spec} M_i \to \operatorname{Spec} \prod_{i\in I} M_i \quad : \quad (\mathfrak{p}_i \mid i \in I) \mapsto \bigcup_{i\in I} \pi_i^* \mathfrak{p}_i$$

is a bijection.

(iii) *Let $(M_i \mid i \in I)$ be any filtered system of monoids. The natural map*

$$\operatorname{Spec} \operatorname*{colim}_{i\in I} M_i \to \lim_{i\in I} \operatorname{Spec} M_i$$

is a bijection.

Proof. (i): By lemma 4.8.31(iii), M/G is the set-theoretic quotient of M by the translation action of G. By definition, any ideal I of M is stable under the G-action, hence the quotient I/G is well defined, and one checks easily that it is an ideal of M/G. Moreover, if $\mathfrak{p} \subset M$ is a prime ideal, it is easily seen that $\mathfrak{p}/G$ is a prime ideal of M/G. Assertions (a) and (b) are straightforward consequences.

(ii): The assertion can be rephrased by saying that every face F of $\prod_{i\in I} M_i$ is a product of faces $F_i \subset M_i$. However, if $\underline{m} := (m_i \mid i \in I) \in F$, then, for each $i \in I$ we can write $\underline{m} = \underline{m}(i) \cdot \underline{n}(i)$, where, for each $j \in I$, the j-th-component of $\underline{m}(i)$ (resp. of $\underline{n}(i)$) equals 1 (resp. m_j), unless $j = i$, in which case it equals m_i (resp. 1). Thus, $\underline{m}(i) \in F$ for every $i \in I$, and the contention follows easily.

(iii): Denote by M the colimit of the system $(M_i \mid i \in I)$, and $\varphi_i : M_i \to M$ the natural morphisms of monoids, as well as $\varphi_f : M_i \to M_j$ the transition maps, for every morphism $f : i \to j$ in I. Recall that the set underlying M is the colimit of the system of sets $(M_i \mid i \in I)$ (lemma 4.8.29(iii)). Let now $\mathfrak{p}_\bullet := (\mathfrak{p}_i \mid \ \in I)$ be a compatible system of prime ideals, *i.e.* such that $\mathfrak{p}_i \in \operatorname{Spec} M_i$ for every $i \in I$, and $\varphi_f^{-1}\mathfrak{p}_j = \mathfrak{p}_i$ for every $f : i \to j$. We let $\beta(\mathfrak{p}_\bullet) := \bigcup_{i\in I} \varphi_i(\mathfrak{p}_i)$. We claim that $\beta(\mathfrak{p}_\bullet)$ is a prime ideal of M. Indeed, suppose that $x, y \in M$ and $xy \in \beta(\mathfrak{p}_\bullet)$; since I is filtered, we may find $i \in I$, $x_i, y_i \in M_i$, and $z_i \in \mathfrak{p}_i$, such that $x = \varphi_i(x_i)$, $y = \varphi_i(y_i)$, and $xy = \varphi_i(z_i)$. Especially, $\varphi_i(z_i) = \varphi_i(x_i y_i)$, so there exists a morphism $f : i \to j$ such that $\varphi_f(z_i) = \varphi_f(x_i y_i)$. But $\varphi_f(z_i) \in \mathfrak{p}_j$, so either $\varphi_f(x_i) \in \mathfrak{p}_j$ or $\varphi_f(y_i) \in \mathfrak{p}_j$, and finally either $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$, as required.

Let $\mathfrak{p} \subset M$ be any prime ideal; it is easily seen that $\beta(\varphi_i^{-1}\mathfrak{p} \mid i \in I) = \mathfrak{p}$. To conclude, it suffices to show that $\mathfrak{p}_i = \varphi_i^{-1}\beta(\mathfrak{p}_\bullet)$, for every compatible system $\mathfrak{p}_\bullet$ as above, and every $i \in I$. Hence, fix $i \in I$ and pick $x_i \in M_i$ such that $\varphi_i(x_i) \in \beta(\mathfrak{p}_\bullet)$; then there exist $j \in I$ and $x_j \in \mathfrak{p}_j$ such that $\varphi_i(x_i) = \varphi_j(x_j)$. Since I is filtered, we may find $k \in I$ and morphisms $f : i \to k$ and $g : j \to k$ such that $\varphi_f(x_i) = \varphi_g(x_j)$, so $\varphi_f(x_i) \in \mathfrak{p}_k$, and finally $x_i \in \mathfrak{p}_i$, as sought. □

Remark 6.1.18. In case $(M_i \mid i \in I)$ is an infinite family of monoids, the natural map of lemma 6.1.17(ii) is still injective, but it is not necessarily surjective. For instance, let I be any infinite set, and let $\mathscr{U} \subset \mathscr{P}(I)$ be a non-principal ultrafilter; denote by ${}^*\mathbb{N}$ the quotient of $\mathbb{N}^I$ under the equivalence relation $\sim_{\mathscr{U}}$ such that $(a_i \mid i \in I) \sim_{\mathscr{U}} (b_i \mid i \in I)$ if and only if there exists $U \in \mathscr{U}$ such that $a_i = b_i$ for every $i \in U$. It is clear that the composition law on $\mathbb{N}^I$ descends to ${}^*\mathbb{N}$; the resulting structure $({}^*\mathbb{N}, +)$ is called the monoid of hypernatural numbers. Denote by $\pi : \mathbb{N}^I \to {}^*\mathbb{N}$ the projection, and let $\underline{0} \in \mathbb{N}^I$ be the unit; then it is easily seen that $\{\pi(0)\}$ is a face of ${}^*\mathbb{N}$, but $\pi^{-1}(\pi(0))$ is not a product of faces $F_i \subset \mathbb{N}$.

Remark 6.1.19. (i) Notice that not all epimorphisms in **Mnd** are surjections on the underlying sets; for instance, every localization map $M \to S^{-1}M$ is an epimorphism.

(ii) If $\varphi : N \to M$ is a map of fine monoids (see definition 6.1.6(vi)), then it will follow from corollary 6.4.2 that $N \times_M N$ is also finitely generated. If M is not integral, then this fails in general : a counter-example is provided by the morphism φ constructed in (6.1.2).

(iii) Let Σ be any set; it is easily seen that a free monoid $M \simeq \mathbb{N}^{(\Sigma)}$ admits a unique minimal system of generators, in natural bijection with Σ. Especially, the cardinality of Σ is determined by the isomorphism class of M; this invariant is called the *rank* of the free monoid M. This is the same as the rank of M as an $\mathbb{N}$-module (see example 3.7.27).

(iv) A submonoid of a finitely generated monoid is not necessarily finitely generated. For instance, consider the submonoid $M \subset \mathbb{N}^{\oplus 2}$, with $M := \{(0,0)\} \cup \{(a,b) \mid a > 0\}$. However, the following result shows that a face of a finitely generated monoid is again finitely generated.

Lemma 6.1.20. *Let $f : M \to N$ be a map of monoids, $F \subset N$ a face of N, and $\Sigma \subset N$ a system of generators for N. Then :*

(i) *$N^\times$ is a face of N, and $f^{-1}F$ is a face of M.*
(ii) *$\Sigma \cap F$ is a system of generators for F.*
(iii) *If N is finitely generated,* $\operatorname{Spec} N$ *is a finite set, and* $\dim N$ *is finite.*
(iv) *If N is finitely generated (resp. fine) then the same holds for $F^{-1}N$.*

Proof. (i) and (ii) are left to the reader, and (iii) is an immediate consequence of (ii). To show (iv), notice that – in view of (ii) – the set $\Sigma \cup \{f^{-1} \mid f \in F \cap \Sigma\}$ is a system of generators of $F^{-1}N$. □

Definition 6.1.21. (i) If M is a pointed or not pointed monoid (see remark 4.8.14(ii)), and S is a pointed M-module, we say that S is *integral*, if for every $x, y \in M$ and every $s \in S$ such that

$xs = ys \neq 0$, we have $x = y$. The *annihilator ideal* of S is the ideal

$$\mathrm{Ann}_M(S) := \{m \in M \mid ms = 0 \text{ for every } s \in S\}.$$

If $s \in S$ is any element, we also write $\mathrm{Ass}_M(s) := \mathrm{Ass}_M(Ms)$. The *support* of S is the subset :

$$\mathrm{Supp}\, S := \{\mathfrak{p} \in \mathrm{Spec}\, M \mid S_\mathfrak{p} \neq 0\}.$$

(ii) A pointed monoid $(M, 0_M)$ is called *integral*, if it is integral when regarded as a pointed M-module; it is called *fine*, if it is finitely generated and integral in the above sense.

(iii) The forgetful functor $\mathbf{Mnd}_\circ \to \mathbf{Set}$ (notation of (4.8.28)) admits a left adjoint, which assigns to any set Σ the *free pointed monoid* $\mathbb{N}^{(\Sigma)}_\circ := (\mathbb{N}^{(\Sigma)})_\circ$.

(iv) A morphism $\varphi : M \to N$ of pointed monoids is *local*, if both $M, N \neq 0$, and φ is local when regarded as a morphism of non-pointed monoids.

Remark 6.1.22. (i) Quite generally, a (non-pointed) monoid M is finitely generated (resp. free, resp. integral, resp. fine) if and only if $M_\circ$ has the corresponding property for pointed monoids. However, there exist integral pointed monoids which are not of the form $M_\circ$ for any non-pointed monoid M.

(ii) Let $(M, 0_M)$ be a pointed monoid. An *ideal* of $(M, 0_M)$ is a pointed submodule $I \subset M$. Just as for non-pointed monoids, we say that I is a *prime ideal*, if $M \setminus I$ is a (non-pointed) submonoid of M, and a non-pointed submonoid which is the complement of a prime ideal, is called a *face* of M. Hence the smallest ideal is $\{0\}$. Notice though, that $\{0\}$ is not necessarily a prime ideal, hence the spectrum $\mathrm{Spec}\,(M, 0_M)$ does not always admit a least element. However, if $M = N_\circ$ for some non-pointed monoid N, the natural morphism of monoids $N \to N_\circ$ induces a bijection :

$$\mathrm{Spec}\,(N_\circ, 0_{N_\circ}) \to \mathrm{Spec}\, N.$$

(iii) Let $I \subset M$ be any ideal; the inclusion map $I \to M$ can be regarded as a morphism of pointed M-modules (if M is not pointed, this is achieved via the faithful embedding (4.8.15)), whence a pointed M-module M/I, with a natural morphism $M \to M/I$. The latter map is also a morphism of monoids, for the obvious monoid structure on M/I. One checks easily that if M is integral, M/I is an integral pointed monoid.

(iv) Let M be a (pointed or not pointed) monoid, $\mathfrak{p} \subset M$ a prime ideal. Then the natural morphism of monoids $M \to M/\mathfrak{p}$ induces a bijection :

$$\mathrm{Spec}\, M/\mathfrak{p} \xrightarrow{\sim} \{\mathfrak{q} \in \mathrm{Spec}\, M \mid \mathfrak{p} \subset \mathfrak{q}\} = \mathrm{Spec}\, M \backslash \mathfrak{p}.$$

(v) Let M be a (pointed or not pointed) monoid, and $S \neq 0$ a pointed M-module. Then the support of S contains at least the maximal ideal of M. This trivial observation shows that a pointed M-module is 0 if and only if its support is empty.

(vi) Let M be a pointed monoid, and $\Sigma \subset M$ a *non-pointed* submonoid. The localization $\Sigma^{-1}M$ (defined in the category of monoids, as in (4.8.33)) is actually a pointed monoid : its zero element $0_{\Sigma^{-1}M}$ is the image of 0_M.

(vii) If M is a (pointed or not pointed) monoid, $\Sigma \subset M$ any *non-pointed* submonoid, and S a pointed M-module, we let as usual $\Sigma^{-1}S := \Sigma^{-1}M \otimes_M S$ (see remark 4.8.21(i), if M is not pointed). The resulting functor $M\text{-}\mathbf{Mod}_\circ \to \Sigma^{-1}M\text{-}\mathbf{Mod}_\circ$ is exact. Indeed, it is right exact, since it is left adjoint to the restriction of scalars arising from the localization map $M \to \Sigma^{-1}M$ (see (3.7.26)), and one verifies directly that it commutes with finite limits. Also, it is clear that

$$\mathrm{Supp}\, \Sigma^{-1}S = \mathrm{Supp}\, S \cap \mathrm{Spec}\, \Sigma^{-1}M.$$

(viii) Let M be a pointed monoid, and $N \subset M$ a pointed submonoid. Since the final object 1 of the category of pointed monoids is not isomorphic to the initial object $1_\circ$, the push-out of the diagram $1 \leftarrow N \to M$ is not an interesting object (it is always isomorphic to 1). Even if we form the quotient M/N in the category of non-pointed monoids, we still get always 1, since $0_M \in N$, and therefore in the quotient M/N the images of 0_M and of the unit of M coincide.

The only case that may give rise to a non-trivial quotient, is when N is *non-pointed*; in this situation we may form M/N *in the category of non-pointed monoids*, and then remark that the image of 0_M yields a zero element $0_{M/N}$ for M/N, so the latter is a pointed monoid.

Example 6.1.23. (i) Let M be a (pointed or not pointed) monoid, $G \subset M^\times$ a subgroup, and S a pointed M-module. Then $M/G \otimes_M S = S/G$ is the set of orbits of S under the induced G-action.

(ii) In the situation of (i), notice that the functor

$$M\text{-}\mathbf{Mod} \to M/G\text{-}\mathbf{Mod} \quad : \quad S \mapsto S/G$$

is exact, hence M/G is a flat M-module. (See definition 4.8.22(i).)

(iii) Likewise, if $\Sigma \subset M$ is a non-pointed submonoid, then the localization $\Sigma^{-1}M$ is a flat M-module, due to remark 6.1.22(vii).

(iv) Suppose that S is an integral pointed M-module (with M either pointed or not pointed), and let $\Sigma \subset M$ be a non-pointed submonoid. Then $\Sigma^{-1}S$ is also an integral pointed $\Sigma^{-1}M$-module. Indeed, suppose that the identity

$$(s^{-1}a) \cdot (s'^{-1}b) = (s^{-1}a) \cdot (s''^{-1}c) \neq 0 \tag{6.1.24}$$

holds for some $a, b, c \in S$ and $s, s', s'' \in \Sigma$; we need to check that $s'^{-1}b = s''^{-1}c$, or equivalently, that $s''b = s'c$ in $\Sigma^{-1}S$. However, (6.1.24) is equivalent to $s''ab = s'ac$ in $\Sigma^{-1}S$, and the latter holds if and only if there exists $t \in \Sigma$ such that $ts''ab = ts'ac$ in S. The two sides in the latter identity are $\neq 0$, as the same holds for the two sides of the identity (6.1.24); therefore $s''b = s'c$ holds already in S, and the contention follows.

(v) Likewise, in the situation of (iv), $S/\Sigma := S \otimes_M M/\Sigma$ is an integral pointed M/Σ-module. Indeed, notice the natural identification $S/\Sigma = \Sigma^{-1}S \otimes_{\Sigma^{-1}M} (\Sigma^{-1}M)/\Sigma^{\mathrm{gp}}$ which – in view of (iv) – reduces the proof to the case where Σ is a subgroup of $M^\times$. Then the assertion is easily verified, taking into account (i).

Especially, if M is an integral pointed monoid, and $\Sigma \subset M$ is any non-pointed submonoid, then both $\Sigma^{-1}M$ and M/Σ are integral pointed monoids (this generalizes lemma 4.8.38).

(vi) Let G be any abelian group, $\varphi : M \to G$ a morphism of non-pointed monoids. Then $G_\circ$ is a flat $M_\circ$-module. For the proof, we may – in light of (iii) – replace M by M^{gp}, thereby reducing to the case where M is a group. Next, by (ii), we may assume that φ is injective, in which case G is a free M-module with basis G/M.

Remark 6.1.25. (i) Let $M \to N$ and $M \to N'$ be morphisms of pointed monoids; N and N' can be regarded as pointed M-modules in an obvious way, hence we may form the tensor product $N'' := N \otimes_M N'$; the latter is endowed with a unique monoid structure such that the maps $e : N \to N''$ and $e' : N' \to N''$ given by the rule $n \mapsto n \otimes 1$ for all $n \in N$ (resp. $n' \mapsto 1 \otimes n'$ for all $n' \in N'$) are morphisms of monoids. Just as for usual ring homomorphisms, the monoid N'' is a coproduct of N and N' over M, *i.e.* there is a unique isomorphism of pointed monoids:

$$N \otimes_M N' \xrightarrow{\sim} N \amalg_M N' \tag{6.1.26}$$

that identifies e and e' to the natural morphisms $N \to N \amalg_M N'$ and $N' \to N \amalg_M N'$. As usual all this extends to non-pointed monoids. (Details left to the reader.)

(ii) Especially, if we take $M = \{1\}_\circ$, the initial object in $\mathbf{Mnd}_\circ$, we obtain an explicit description of the pointed monoid $N \oplus N'$: it is the quotient $(N \times N')/\!\sim$, where $\sim$ denotes the minimal equivalence relation such that $(x, 0) \sim (0, x')$ for every $x \in N$, $x' \in N'$. From this, a direct calculation shows that a direct sum of pointed integral monoids is again a pointed integral monoid.

Remark 6.1.27. (i) Clearly every pointed M-module S is the colimit of the filtered family of its finitely generated submodules. Moreover, S is the colimit of a filtered family of finitely presented pointed M-modules. Recall the standard argument : pick a countable set I, and let $\mathscr{C}$ be the (small) full subcategory of the category $M\text{-}\mathbf{Mod}_\circ$ whose objects are the coequalizers of every pair of maps of pointed M-modules $p, q : M^{(I_1)} \to M^{(I_2)}$, for every finite sets $I_1, I_2 \subset I$ (this means that, for every such pair p, q we pick one representative for this coequalizer). Then there is a natural isomorphism of pointed M-modules :

$$\operatorname*{colim}_{i\mathscr{C}/S} \iota_S \xrightarrow{\sim} S$$

where $i : \mathscr{C} \to M\text{-}\mathbf{Mod}_\circ$ is the inclusion functor, and ι_S is the functor as in (1.1.27).

(ii) If S is finitely generated, we may find a finite filtration of S by submodules $0 = S_0 \subset S_1 \subset \cdots \subset S_n = S$ such that S_{i+1}/S_i is a cyclic M-module, for every $i = 1, \dots, n$.

(iii) Notice that if S is integral and $S' \subset S$ is any submodule, then S/S' is again integral. Moreover, if S is integral and cyclic, we have a natural isomorphism of M-modules :

$$S \xrightarrow{\sim} M/\mathrm{Ann}_M(S).$$

(Details left to the reader.)

(iv) Suppose furthermore, that $M^\sharp$ is finitely generated, and S is any pointed M-module. Lemma 6.1.17(i.a) and proposition 6.1.9(ii) easily imply that every ascending chain

$$I_0 \subset I_1 \subset I_2 \subset \cdots$$

of ideals of M is stationary; especially, the set $\{\mathrm{Ann}_M(s) \mid s \in S\setminus\{0\}\}$ admits maximal elements. Let I be a maximal element in this set; a standard argument as in commutative algebra shows that I is a prime ideal : indeed, say that $xy \in I = \mathrm{Ann}_M(s)$ and $x \notin I$; then $xs \neq 0$, hence $y \in \mathrm{Ann}_M(xs) = I$, by the maximality of I. Now, if S is also finitely generated, it follows that we may find a finite filtration of S as in (ii) such that, additionally, each quotient S_{i+1}/S_i is of the form $M/\mathfrak{p}$, for some prime ideal $\mathfrak{p} \subset M$.

These properties make the class of integral pointed modules especially well behaved : essentially, the full subcategory $M\text{-}\mathbf{Int}.\mathbf{Mod}_\circ$ of $M\text{-}\mathbf{Mod}_o$ consisting of these modules mimics closely a category of A-modules for a ring A, familiar from standard linear algebra. This shall be amply demonstrated henceforth. For instance, we point out the following combinatorial version of Nakayama's lemma :

Proposition 6.1.28. *Let M be a (pointed or not pointed) monoid, S a finitely generated integral pointed M-module, and $S' \subset S$ a pointed submodule, such that*

$$S = S' \cup \mathfrak{m}_M \cdot S.$$

Then $S = S'$.

Proof. After replacing S by S/S', we may assume that $\mathfrak{m}_M S = S$, in which case we have to check that $S = 0$. Suppose then, that $S \neq 0$; from remark 6.1.27(ii,iii) it follows that S admits a (pointed) submodule $T \subset S$ such that $S/T \simeq M/\mathfrak{m}_M$. Especially, $\mathfrak{m}_M \cdot (S/T) = 0$, *i.e.* $\mathfrak{m}_M S \subset T$, and therefore $S = T$, which contradicts the choice of T. The contention follows. □

Remark 6.1.29. (i) The integrality assumption cannot be omitted in proposition 6.1.28. Indeed, take $M := \mathbb{N}$ and $S := 0_\circ$, where 0 denotes the final $\mathbb{N}$-module. Then $S \neq 0$, but $\mathfrak{m}_M S = S$.

(ii) Let us say that an element of the M-module S is *primitive*, if it does not lie in $\mathfrak{m}_M S$. We deduce from proposition 6.1.28, the following :

Corollary 6.1.30. *Let M be a sharp (pointed or not pointed) monoid, S a finitely generated integral pointed monoid. Then the set $S \setminus \mathfrak{m}_M S$ of primitive elements of S is finite, and is the unique minimal system of generators of S.*

Proof. Indeed, it is easily seen that every system of generators of S must contain all the primitive elements, so $S \setminus \mathfrak{m}_M S$ must be finite. On the other hand, let $S' \subset S$ be the submodule generated by the primitive elements; clearly $S' \cup \mathfrak{m}_M S = S$, hence $S' = S$, by proposition 6.1.28. □

6.1.31. Let R be any ring, M a non-pointed monoid. Notice that the M-module underlying any $R[M]$-module is naturally pointed, whence a forgetful functor $R[M]\text{-}\mathbf{Mod} \to M\text{-}\mathbf{Mod}_\circ$. The latter admits a left adjoint

$$M\text{-}\mathbf{Mod}_\circ \to R[M]\text{-}\mathbf{Mod} \quad : \quad (S, 0_S) \mapsto R\langle S\rangle := \mathrm{Coker}\, R[0_S].$$

Likewise, the monoid $(A, \cdot)$ underlying any (commutative unital) R-algebra A is naturally pointed, whence a forgetful functor $R\text{-}\mathsf{Alg} \to \mathbf{Mnd}_\circ$, which again admits a left adjoint

$$\mathbf{Mnd}_\circ \to R\text{-}\mathsf{Alg} \quad : \quad (M, 0_M) \mapsto R\langle M\rangle := R[M]/(0_M)$$

where $(0_M) \subset R[M]$ denotes the ideal generated by the image of 0_M.

If $(M, 0_M)$ is a pointed monoid, and S is a pointed $(M, 0_M)$-module, then notice that $R\langle S\rangle$ is actually a $R\langle M\rangle$-module, so we have as well a natural functor

$$(M, 0_M)\text{-}\mathbf{Mod}_\circ \to R\langle M\rangle\text{-}\mathbf{Mod} \qquad S \mapsto R\langle S\rangle$$

which is again left adjoint to the forgetful functor.

For instance, let $I \subset M$ be an ideal; from the foregoing, it follows that $R\langle M/I\rangle$ is naturally an $R[M]$-algebra, and we have a natural isomorphism :

$$R\langle M/I\rangle \xrightarrow{\sim} R[M]/IR[M].$$

Explicitly, for any $x \in F := M\setminus I$, let $\overline{x} \in R\langle M/I\rangle$ be the image of x; then $R\langle M/I\rangle$ is a free R-module, with basis $(\overline{x} \mid x \in F)$. The multiplication law of $R\langle M/I\rangle$ is determined as follows. Given $x, y \in F$, then $\overline{x} \cdot \overline{y} = \overline{xy}$ if $xy \in F$, and otherwise it equals zero.

Notice that if I_1 and I_2 are two ideals of M, we have a natural identification :

$$R\langle M/(I_1 \cap I_2)\rangle \xrightarrow{\sim} R\langle M/I_1\rangle \times_{R\langle M/(I_1\cup I_2)\rangle} R\langle M/I_2\rangle.$$

These algebras will play an important role in section 12.5. As a special case, suppose that $\mathfrak{p} \subset M$ is a prime ideal; then the inclusion $M\setminus\mathfrak{p} \subset M$ induces an isomorphism of R-algebras :

$$R[M\setminus\mathfrak{p}] \xrightarrow{\sim} R\langle M/\mathfrak{p}\rangle.$$

Furthermore, general nonsense yields a natural isomorphism of R-modules :

$$R\langle S \otimes_M S'\rangle \xrightarrow{\sim} R\langle S\rangle \otimes_{R\langle M\rangle} R\langle S'\rangle \qquad \text{for all pointed } M\text{-modules } S \text{ and } S'. \tag{6.1.32}$$

6.1.33. Let A be a commutative ring with unit, and $f : M \to (A, \cdot)$ a morphism of pointed monoids. Then f induces (forgetful) functors :

$$A\text{-}\mathbf{Mod} \to M\text{-}\mathbf{Mod}_\circ \qquad A\text{-}\mathsf{Alg} \to M/\mathbf{Mnd}_\circ$$

(notation of (1.1.24)) which admit left adjoints :

$$\begin{aligned} M\text{-}\mathbf{Mod}_\circ &\to A\text{-}\mathbf{Mod} & S &\mapsto S \otimes_M A := \mathbb{Z}\langle S\rangle \otimes_{\mathbb{Z}\langle M\rangle} A \\ M/\mathbf{Mnd}_\circ &\to A\text{-}\mathsf{Alg} & N &\mapsto N \otimes_M A := \mathbb{Z}\langle N\rangle \otimes_{\mathbb{Z}\langle M\rangle} A. \end{aligned}$$

Sometimes we may also use the notation :

$$S \otimes_M N := \mathbb{Z}\langle S\rangle \otimes_{\mathbb{Z}\langle M\rangle} N \quad \text{and} \quad S \overset{\mathbf{L}}{\otimes}_M K_\bullet := \mathbb{Z}\langle S\rangle \overset{\mathbf{L}}{\otimes}_{\mathbb{Z}\langle M\rangle} K_\bullet$$

for any pointed M-module S, any A-module N and any object $K_\bullet$ of $\mathsf{D}^-(A\text{-}\mathbf{Mod})$. The latter derived tensor product is obtained by tensoring with a flat $\mathbb{Z}\langle M\rangle$-flat resolution of $\mathbb{Z}\langle S\rangle$. (Such resolutions can be constructed combinatorially, starting from a simplicial resolution of S.) All the verifications are standard, and shall be left to the reader.

Moreover, for any ideal $I \subset M$ and any A-module N, we shall write $f(I)N$, or sometimes just IN, for the A-submodule of N generated by the system $(f(x) \cdot y \mid x \in I,\ y \in N)$.

Definition 6.1.34. Let M be a pointed monoid, A a commutative ring with unit, $\varphi : M \to (A, \cdot)$ a morphism of pointed monoids, N an A-module. We say that N is *φ-flat* (or just M-flat, if no ambiguity is likely to arise), if the functor

$$M\text{-}\mathbf{Int.Mod}_\circ \to A\text{-}\mathbf{Mod} \quad : \quad S \mapsto S \otimes_M N$$

is *exact*, in the sense that it sends exact sequences of pointed integral M-modules, to exact sequences of A-modules. We say that N is *faithfully φ-flat* if this functor is exact in the above sense, and we have $S \otimes_M N = 0$ if and only if $S = 0$.

Remark 6.1.35. (i) Notice that the functor $S \mapsto S \otimes_M N$ of definition 6.1.34 is right exact in the categorical sense (*i.e.* it commutes with finite colimits), since it is a right adjoint. However, even when N is faithfully flat, this functor is not always left exact in the categorical sense : it does not commute with finite products, nor with equalizers, in general.

(ii) Let M and S be as in definition 6.1.34, and let R be any non-zero commutative unital ring. Denote by $\varphi : M \to (R\langle M \rangle, \cdot)$ the natural morphism of pointed monoids. In light of (6.1.32), it is clear that if $R\langle S \rangle$ is a flat $R\langle M \rangle$-module, then S is a flat pointed M-module, and the latter condition implies that $R\langle S \rangle$ is φ-flat.

(iii) Let P be a pointed monoid, A a ring, $\varphi : P \to (A, \cdot)$ a morphism of monoids, $f : A \to B$ a ring homomorphism. If A is φ-flat and f is flat, then B is $(f \circ \varphi)$-flat. Conversely, if B is $(f \circ \varphi)$-flat and f is faithfully flat, then A is φ-flat : the verifications are standard, and shall be left to the reader.

Lemma 6.1.36. *Let M be a monoid, A a ring, $\varphi : M \to (A, \cdot)$ a morphism of monoids, and assume that φ is local and A is φ-flat. Then A is faithfully φ-flat.*

Proof. Let S be an integral pointed M-module, and suppose that $S \otimes_M A = \{0\}$; we have to show that $S = \{0\}$. Say that $s \in S$, and let $Ms \subset S$ be the M-submodule generated by s. Since A is φ-flat, it follows easily that $Ms \otimes_M A = \{0\}$, hence we are reduced to the case where S is cyclic. By remark 6.1.27(iii), we may then assume that $S = M/I$ for some ideal $I \subset M$. It follows that $S \otimes_M A = A/\varphi(I)A$, so that $\varphi(I)$ generates A. Since φ is local, this implies that $I = M$, whence the contention. □

Lemma 6.1.37. *Let M be an integral pointed monoid, $I, J \subset M$ two ideals, A a ring, $\alpha : M \to (A, \cdot)$ a morphism of monoids, N an α-flat A-module, and S a flat M-module. Then :*

$$IS \cap JS = (I \cap J)S$$
$$\alpha(I)N \cap \alpha(J)N = \alpha(I \cap J)N.$$

Proof. We consider the commutative ladder of pointed M-modules, with exact rows and injective vertical arrows :

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I \cap J & \longrightarrow & I & \longrightarrow & I/(I \cap J) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & J & \longrightarrow & M & \longrightarrow & M/(I \cup J) & \longrightarrow & 0 \end{array} \tag{6.1.38}$$

By assumption, the ladder of A-modules (6.1.38)$\otimes_M N$ has still exact rows and injective vertical arrows. Then, the snake lemma gives the following short exact sequence involving the cokernels of the vertical arrows :

$$0 \to JN/(I \cap J)N \to N/IN \xrightarrow{p} N/(IN + JN) \to 0$$

(where we have written JN instead of $\alpha(J)N$, and likewise for the other terms). However $\mathrm{Ker}\, p = JN/(IN \cap JN)$, whence the second stated identity. The first stated identity can be deduced from the second, by virtue of remark 6.1.35(ii). □

Remark 6.1.39. By inspection of the proof, we see that the first identity of lemma 6.1.37 holds, more generally, whenever $\mathbb{Z}\langle S\rangle$ is φ-flat, where $\varphi : M \to \mathbb{Z}\langle M\rangle$ is the natural morphism of pointed monoids.

Proposition 6.1.40. *Let M be a pointed integral monoid, A a ring, $\varphi : M \to (A, \cdot)$ a morphism of monoids, N an A-module. Then we have :*

- (i) *The following conditions are equivalent :*
  - (a) *N is φ-flat.*
  - (b) $\mathrm{Tor}_i^{\mathbb{Z}\langle M\rangle}(\mathbb{Z}\langle T\rangle, N) = 0$ *for every $i > 0$ and every pointed integral M-module T.*
  - (c) $\mathrm{Tor}_1^{\mathbb{Z}\langle M\rangle}(\mathbb{Z}\langle M/I\rangle, N) = 0$ *for every ideal $I \subset M$.*
  - (d) *The natural map $I \otimes_M N \to N$ is injective for every ideal $I \subset M$.*
- (ii) *If moreover, $M^\sharp$ is finitely generated, then the conditions* (a)-(d) *of* (i) *are equivalent to either of the following two conditions :*
  - (e) $\mathrm{Tor}_1^{\mathbb{Z}\langle M\rangle}(\mathbb{Z}\langle M/\mathfrak{p}\rangle, N) = 0$ *for every prime ideal $\mathfrak{p} \subset M$.*
  - (f) *The natural map $\mathfrak{p} \otimes_M N \to N$ is injective for every prime ideal $\mathfrak{p} \subset M$.*

Proof. Clearly (a)⇒(b)⇒(c)⇒(e). Next, by considering the short exact sequence of pointed integral M-modules $0 \to I \to M \to M/I \to 0$ we easily see that (c)⇔(d) and (e)⇔(f).

(c)⇒(a) : Let $\Sigma := (0 \to S' \to S \to S'' \to 0)$ be a short exact sequence of pointed integral M-modules; we need to show that the induced map $S' \otimes_M N \to S \otimes_M N$ is injective. Since the sequence $\mathbb{Z}\langle\Sigma\rangle$ is still exact, the long Tor-exact sequence reduces to showing that $\mathrm{Tor}_1^{\mathbb{Z}\langle M\rangle}(\mathbb{Z}\langle T\rangle, N) = 0$ for every pointed integral M-module T. In view of remark 6.1.27(i), we are easily reduced to the case where T is finitely generated; next, using remark 6.1.27(ii,iii), the long exact Tor-sequence, and an easy induction on the number of generators of T, we may assume that $T = M/I$, whence the contention.

Lastly, if $M^\sharp$ is finitely generated, then remark 6.1.27(iv) shows that, in the foregoing argument, we may further reduce to the case where $T = M/\mathfrak{p}$ for a prime ideal $\mathfrak{p} \subset M$; this shows that (e)⇒(a). □

Lemma 6.1.41. *Let M be a pointed monoid, S a pointed M-module, and suppose that the following conditions hold for S :*

- (F1) *If $s \in S$ and $a \in \mathrm{Ann}_M(s)$, then there exists $b \in \mathrm{Ann}_M(a)$ such that $s \in bS$.*
- (F2) *If $a_1, a_2 \in M$ and $s_1, s_2 \in S$ satisfy the identity $a_1 s_1 = a_2 s_2 \neq 0$, then there exist $b_1, b_2 \in M$ and $t \in S$ such that $s_i = b_i t$ for $i = 1, 2$ and $a_1 b_1 = a_2 b_2$.*

Then the natural map $I \otimes_M S \to S$ is injective for every ideal $I \subset M$.

Proof. Let I be an ideal, and suppose that two elements $a_1 \otimes s_1$ and $a_2 \otimes s_2$ of $I \otimes_M S$ are mapped to the same element of S. If $a_i s_i = 0$ for $i = 1, 2$, then (F1) says that there exist $b_1, b_2 \in M$ and $t_1, t_2 \in S$ such that $a_i b_i = 0$ and $s_i = b_i t_i$ for $i = 1, 2$; thus $a_i \otimes s_i = a_i \otimes b_i t_i = a_i b_i \otimes s_i = 0$ in $I \otimes_M S$. In case $a_i s_i \neq 0$, pick $b_1, b_2 \in M$ and $t \in S$ as in (F2); we conclude that $a_1 \otimes s_1 = a_1 \otimes b_1 t = a_1 b_1 \otimes t = a_2 b_2 \otimes t = a_2 \otimes s_2$ in $I \otimes_M S$, whence the contention. □

Theorem 6.1.42. *Let M be an integral pointed monoid, S a pointed M-module. The following conditions are equivalent :*

- (a) *S is M-flat.*
- (b) *For every morphism $M \to P$ of pointed monoids, $P \otimes_M S$ is P-flat.*
- (c) *For every short exact sequence Σ of integral pointed M-modules, the sequence $\Sigma \otimes_M S$ is again short exact.*

(d) *Conditions* (F1) *and* (F2) *of lemma* 6.1.41 *hold for* S.

Proof. Clearly (b)⇒(a)⇒(c).

(c)⇒(d): To show (F1), set $I := Ma$, and denote by $i : I \to M$ the inclusion; (c) implies that the induced map $i \otimes_M S : I \otimes_M S \to S$ is injective. However, we have a natural isomorphism $I \stackrel{\sim}{\to} M/\mathrm{Ann}_M(a)$ of M-modules (remark 6.1.27(iii)), whence an isomorphism : $I \otimes_M S \stackrel{\sim}{\to} S/\mathrm{Ann}_M(a)S$, and under this identification, $i \otimes_M S$ is induced by the map $S \to S : s \mapsto as$. Thus, multiplication by a maps the subset $S \setminus \mathrm{Ann}_M(a)S$ injectively into itself, which is the claim.

For (F2), notice that $\mathbb{Z}\langle S\rangle$ is φ-flat under condition (c), for $\varphi : M \to \mathbb{Z}\langle M\rangle$ the natural morphism. Now, say that $a_1s_1 = a_2s_2 \neq 0$ in S; set $I := Ma_1$, $J := Ma_2$; the assumption means that $a_1s_1 \in IS \cap JS$, in which case remark 6.1.39 shows that there exist $t \in S$ and $b_1, b_2 \in M$ such that $a_1b_1 = a_2b_2$, and $a_1s_1 = a_1b_1t$, hence $a_2s_2 = a_2b_2t$. Since we have seen that multiplication by a_1 maps $S \setminus \mathrm{Ann}_M(a_1)S$ injectively into itself, we deduce that $s_1 = b_1t$, and likewise we get $s_2 = b_2t$.

To prove that (d)⇒(b), we observe :

Claim 6.1.43. Let P be a pointed monoid, Λ a small locally directed category (see definition 1.2.19(iv)), and $S_\bullet : \Lambda \to P\text{-}\mathbf{Mod}_\circ$ a functor, such that S_λ fulfills conditions (F1) and (F2), for every $\lambda \in \mathrm{Ob}(\Lambda)$. Then the colimit of $S_\bullet$ also fulfills conditions (F1) and (F2).

Proof of the claim. In light of remark 1.2.21(ii), we may assume that Λ is either discrete or connected. Suppose first that Λ is connected; then remark 4.8.17(ii) allows to check directly that conditions (F1) and (F2) hold for the colimit of $S_\bullet$, since they hold for every S_λ. If Λ is discrete, the assertion is that conditions (F1) and (F2) are preserved by arbitrary (small) direct sums, which we leave as an exercise for the reader. ◊

To a given pointed M-module S, we attach the small category S^*, such that :

$$\mathrm{Ob}(S^*) = S\setminus\{0\} \qquad \text{and} \qquad \mathrm{Hom}_{S^*}(s', s) = \{a \in M \mid as = s'\}.$$

The composition of morphisms is induced by the composition law of M, in the obvious way. Notice that S^* is locally directed if and only if S satisfies condition (F2).

We define a functor $F : S^* \to M\text{-}\mathbf{Mod}_\circ$ as follows. For every $s \in \mathrm{Ob}(S^*)$ we let $F(s) := M$, and for every morphism $a : s' \to s$ we let $F(a) := a \cdot \mathbf{1}_M$. We have a natural transformation $\tau : F \Rightarrow c_S$, where $c_S : S^* \to M\text{-}\mathbf{Mod}_\circ$ is the constant functor associated with S; namely, for every $s \in \mathrm{Ob}(S^*)$, we let $\tau_s : M \to S$ be the map given by the rule $a \mapsto as$ for all $a \in M$. There follows a morphism of pointed M-modules :

$$\operatorname*{colim}_{S^*} F \to S \tag{6.1.44}$$

Claim 6.1.45. If S fulfills conditions (F1), the map (6.1.44) is an isomorphism.

Proof of the claim. Indeed, we have a natural decomposition of S^* as coproduct of a family $(S^*_i \mid i \in I)$ of connected subcategories (for some small set I : see remark 1.2.21(ii)); especially we have $\mathrm{Hom}_{S^*}(s, s') = \varnothing$ if $s \in \mathrm{Ob}(S^*_i)$ and $s' \in \mathrm{Ob}(S^*_j)$ for some $i \neq j$ in I.

For each $i \in I$, let $F_i : S^*_i \to M\text{-}\mathbf{Mod}_\circ$ be the restriction of F. There follows a natural isomorphism :

$$\bigoplus_{i\in I} \operatorname*{colim}_{S^*_i} F_i \stackrel{\sim}{\to} \operatorname*{colim}_{S^*} F.$$

Since the colimit of F_i commutes with the forgetful functor to sets, an inspection of the definitions yields the following explicit description of the colimit T_i of F_i. Every element of T_i is represented by some pair (s, a) where $s \in \mathrm{Ob}(S^*_i) \subset S\setminus\{0\}$ and $a \in M$; such pair is mapped to as by (6.1.44), and two such pairs (s, a), (s', a') are identified in T_i if there exists $b \in M$ such that $bs = s'$ and $ba' = a$.

Hence, denote by S_i the image under (6.1.44) of T_i; we deduce first, that $S_i \cap S_j = \{0\}$ if $i \neq j$. Indeed, say that $t \in S_i \cap S_j$; by the foregoing, there exist $s_i \in \mathrm{Ob}(S^*_i)$, $s_j \in \mathrm{Ob}(S^*_j)$, and $a_i, a_j \in M$, such that $a_i s_i = t = a_j s_j$. If $t \neq 0$, we get morphisms $a_i : t \to s_i$ and $a_j : t \to s_j$ in S^*; say that $t \in S^*_k$ for some $k \in I$; it then follows that $S^*_i = S^*_k = S^*_j$, a contradiction. Next, it is clear that (6.1.44) is surjective. It remains therefore only to show that each T_i maps injectively onto S_i. Hence, say that (s_1, a_1) and (s_2, a_2) represent two elements of T_i with $t := a_1 s_1 = a_2 s_2$. If $t \neq 0$, we get, as before, morphisms $a_1 : t \to s_1$ and $a_2 : t \to s_2$ in S^*_i, and the two pairs are identified in T_i to the pair $(t, 1)$. Lastly, if $t = 0$, condition (F1) yields $b \in M$ and $s' \in S$ such that $a_1 b = 0$ and $s_1 = bs'$, whence a morphism $b : s_1 \to s'$ in S^*_i, and $F_i(b)(a_1, s_1) = (0, s')$ which represents the zero element of T_i. The same argument applies as well to (a_2, s_2), and the claim follows. ♢

Claim 6.1.46. Let $M \to P$ be any morphism of pointed monoids, and S a pointed M-module fulfilling conditions (F1) and (F2). Then the natural map $I \otimes_M S \to P \otimes_M S$ is injective, for every ideal $I \subset P$.

Proof of the claim. From claim 6.1.45 we deduce that $P \otimes_M S$ is the locally directed colimit of the functor $P \otimes_M F$, and notice that the pointed P-module P fulfills conditions (F1) and (F2); by claim 6.1.43 we deduce that $P \otimes_M S$ also fulfills the same conditions, so the claim follows from lemma 6.1.41. ♢

After these preliminaries, suppose that conditions (F1) and (F2) hold for S, and let $\Sigma := (0 \to T' \to T \to T'' \to 0)$ be a short exact sequence of pointed M-modules. We wish to show that $\Sigma \otimes_M S$ is still short exact. However, if $U'' \subset T''$ is any M-submodule, let $U \subset T$ be the preimage of U'', and notice that the induced sequence $0 \to T' \to U \to U'' \to 0$ is still short exact. Since a filtered colimit of short exact sequences is short exact, remark 6.1.27(i) allows to reduce to the case where T'' is finitely generated.

We shall argue by induction on the number n of generators of T''. Hence, suppose first that T'' is cyclic, and let $t \in T$ be any element whose image in T'' is a generator. Set $C := Mt \subset T$, and let $C' \subset T'$ be the preimage of C. We obtain a cocartesian (and cartesian) diagram of pointed M-modules :

$$\mathscr{D} \quad : \quad \begin{array}{ccc} C' & \longrightarrow & C \\ \downarrow & & \downarrow \\ T' & \longrightarrow & T. \end{array}$$

The induced diagram $\mathscr{D} \otimes_M S$ is still cocartesian, hence the same holds for the diagram of sets underlying $\mathscr{D} \otimes_M S$ (remark 4.8.17(ii)). Especially, if the induced map $C' \otimes_M S \to C \otimes_M S$ is injective, the same will hold for the map $T' \otimes_M S \to T \otimes_M S$. We may thus replace T' and T by respectively C' and C, which allows to assume that also T is cyclic. In this case, pick a generator $u \in T$; we claim that there exists a unique multiplication law μ_T on T, such that the surjection $p : M \to T : a \mapsto au$ is a morphism of pointed monoids. Indeed, for every $t, t' \in T$, write $t = au$ for some $a \in M$, and set $\mu_T(t, t') := at'$. Using the linearity of p we easily check that $\mu_T(t, t')$ does not depend on the choice of a, and the resulting composition law μ_T is commutative and associative. Then T' is an ideal of T, so claim 6.1.46 tells us that the map $T' \otimes_M S \to T \otimes_M S$ is injective, as required.

Lastly, suppose that $n > 1$, and the assertion is already known whenever T'' is generated by at most $n - 1$ elements. Let $U'' \subset T''$ be a pointed M-submodule, such that U'' is generated by at most $n - 1$ elements, and T''/U'' is cyclic. Denote by $U \subset T$ the preimage of U''; we deduce short exact sequences $\Sigma' := (0 \to T' \to U \to U'' \to 0)$ and $\Sigma'' := (0 \to U \to T \to T''/U'' \to 0)$, and by inductive assumption, both $\Sigma' \otimes_M S$ and $\Sigma'' \otimes_M S$ are short exact. Therefore the natural map $T' \otimes_M S \to T \otimes_M S$ is the composition of two injective maps, hence it is injective, as stated. □

Remark 6.1.47. In the situation of remark 6.1.35(ii), suppose that M is pointed integral. Then theorem 6.1.42 implies that S is a flat pointed M-module if and only if $R\langle S\rangle$ is φ-flat.

Corollary 6.1.48. *Let M be an integral pointed monoid, S a pointed M-module. Then*

(i) *The following conditions are equivalent :*
  (a) *S is M-flat.*
  (b) *For every ideal $I \subset M$, the induced map $I \otimes_M S \to S$ is injective.*

(ii) *If moreover $M^\sharp$ is finitely generated, then these conditions are equivalent to :*
  (c) *For every prime ideal $\mathfrak{p} \subset M$, the induced map $\mathfrak{p} \otimes_M S \to S$ is injective.*

Proof. (i): Indeed, by remark 6.1.47 and proposition 6.1.40(i) (together with (6.1.32)), both (a) and (b) hold if and only if $\mathbb{Z}\langle S\rangle$ is φ-flat, for $\varphi : M \to \mathbb{Z}\langle M\rangle$ the natural morphism.

(ii): This follows likewise from proposition 6.1.40(ii). □

Corollary 6.1.49. *Let $\varphi : M \to N$ be a morphism of pointed monoids, $G \subset M^\times$, $H \subset N^\times$ two subgroups such that $\varphi(G) \subset H$, and denote by $\overline{\varphi} : M/G \to N/H$ the induced morphism. Let also S be any N-module. We have :*

(i) *If $S_{(\varphi)}$ is a flat M-module, then $S/H_{(\overline{\varphi})}$ is a flat M/G-module (notation of* (3.7.26)*).*

(ii) *If moreover, M is a pointed integral monoid and S is a pointed integral H-module, then also the converse of* (i) *holds.*

Proof. (i): We have a natural isomorphism

$$(S/H)_{(\overline{\varphi})} \xrightarrow{\sim} N/H \otimes_{N/\varphi G} S_{(\varphi)}/\varphi G.$$

However, by example 6.1.23(ii), N/H is a flat $N/\varphi G$-module, and $S_{(\varphi)}/\varphi G$ is a flat M/G-module, whence the contention.

(ii): By theorem 6.1.42, it suffices to check that conditions (F1) and (F2) of lemma 6.1.41 hold in $S_{(\varphi)}$, and notice that, by the same token, both conditions hold for the M/G-module $S/H_{(\overline{\varphi})} = S_{(\varphi)}/G$, since M/G is pointed integral (example 6.1.23(v)).

Hence, say that $\varphi(a)s = 0$ for some $a \in M$ and $s \in S$; we may then find $b \in M$, $t' \in S$ and $g \in G$ such that $\varphi(gb)t = s$ and $ab = 0$. Setting $t := \varphi(g)t'$, we deduce that (F1) holds.

Next, say that $\varphi(a_1)s_1 = \varphi(a_2)s_2 \neq 0$ for some $a_1, a_2 \in M$ and $s_1, s_2 \in S$; then we may find $g \in G$, $h_1, h_2 \in H$, $b_1, b_2 \in M$ and $t' \in S$ such that $ga_1b_1 = a_2b_2$ and

$$\varphi(b_i)h_it' = s_i \qquad \text{for } i = 1, 2. \tag{6.1.50}$$

After replacing b_1 by gb_1 and h_1 by $h_1\varphi(g^{-1})$, we reduce to the case where $a_1b_1 = a_2b_2$. From (6.1.50) we deduce that

$$\varphi(a_1b_1)h_1t' = \varphi(a_1)s_1 = \varphi(a_2)s_2 = \varphi(a_2b_2)h_2t' = \varphi(a_1b_1)h_2t'$$

whence $h_1 = h_2$, since S is a pointed integral H-module. Setting $t := h_1t'$, we see that (F2) holds. □

Corollary 6.1.51. *Let $\varphi : M \to N$ be a flat morphism of pointed monoids, with M pointed integral, and let $\mathfrak{p} \subset N$ be any prime ideal. Then the morphism $M/\varphi^{-1}\mathfrak{p} \to N/\mathfrak{p}$ induced by φ is also flat.*

Proof. Let $F := N\setminus\mathfrak{p}$; by theorem 6.1.42, it suffices to check that conditions (F1) and (F2) of lemma 6.1.41 hold for the $\varphi^{-1}F$-module F. However, since $0 \notin F$, condition (F1) holds trivially. Moreover, by assumption these two conditions hold for the M-module N; hence, say that $\varphi(a_1) \cdot s_1 = \varphi(a_2) \cdot s_2$ in F, for some $a_1, a_2 \in \varphi^{-1}F$ and $s_1, s_2 \in F$. It follows that there exist $b_1, b_2 \in M$ and $t \in N$ such that $a_1b_1 = a_2b_2$ in M, and $\varphi(b_i) \cdot t = s_i$ for $i = 1, 2$. Since F is a face, this implies that $\varphi(b_1), \varphi(b_2), t \in F$, so (F2) holds for F, as stated. □

Another corollary is the following analogue of a well known criterion due to Lazard.

Proposition 6.1.52. *Let M be an integral pointed monoid, S an integral pointed M-module, R a non-zero commutative ring with unit. The following conditions are equivalent :*

(a) *S is M-flat.*
(b) *S is the colimit of a filtered system of free pointed M-modules (see remark* 4.8.17(ii)*).*
(c) *$R\langle S\rangle$ is a flat $R\langle M\rangle$-module.*

Proof. Obviously (b)⇒(a) and (b)⇒(c).

(c)⇒(a) has already been observed in remark 6.1.35(ii).

(a)⇒(b): It suffices to prove that if S is flat, the category S^* attached to S as in the proof of theorem 6.1.42, is pseudo-filtered. However, a simple inspection of the construction shows that S^* is pseudo-filtered if and only if S satisfies both condition (F2) of lemma 6.1.41, and the following further condition. For every $a, b \in M$ and $s \in S$ such that $as = bs \neq 0$, there exist $t \in S$ and $c \in M$ such that $ac = bc$ and $ct = s$. This condition is satisfied by every integral pointed M-module, whence the contention. □

6.1.53. Consider now, a cartesian diagram of integral pointed monoids :

$$\mathscr{D}(P_0, I, P_1) \quad : \quad \begin{array}{ccc} P_0 & \longrightarrow & P_1 \\ \downarrow & & \downarrow \\ P_2 & \longrightarrow & P_3 \end{array}$$

where P_2 (resp. P_3) is a quotient P_0/I (resp. P_1/IP_1) for some ideal $I \subset P_0$, and the vertical arrows of $\mathscr{D}(P_0, I, P_1)$ are the natural surjections. In this situation, it is easily seen that the induced map $I \to IP_1$ is bijective; especially, I is both a P_0-module and a P_1-module. Let $\varphi_i : P_i \to \mathbb{Z}\langle P_i\rangle$ be the units of adjunction, for $i = 0, 1, 2$. Let also M be any $\mathbb{Z}\langle P_0\rangle$-module.

Lemma 6.1.54. *In the situation of* (6.1.53)*, we have :*

(i) *Let $J \subset P_0$ be any ideal. Then $S := P_0/J$ admits a three-step filtration*
$$0 \subset \mathrm{Fil}_0 S \subset \mathrm{Fil}_1 S \subset \mathrm{Fil}_2 S = S$$
such that $\mathrm{Fil}_0 S$ and $\mathrm{gr}_2 S$ are P_2-modules, and $\mathrm{gr}_1 S$ is a P_1-module.
(ii) *The following conditions are equivalent :*
 (a) *M is φ_0-flat.*
 (b) *$M \otimes_{P_0} P_i$ is φ_i-flat and $\mathrm{Tor}_1^{\mathbb{Z}\langle P_0\rangle}(M, \mathbb{Z}\langle P_i\rangle) = 0$, for $i = 1, 2$.*
(iii) *The following conditions are equivalent :*
 (a) *$\mathrm{Tor}_1^{\mathbb{Z}\langle P_0\rangle}(M, \mathbb{Z}\langle P_i\rangle) = 0$ for $i = 1, 2, 3$.*
 (b) *$\mathrm{Tor}_1^{\mathbb{Z}\langle P_i\rangle}(M \otimes_{P_0} P_i, \mathbb{Z}\langle P_3\rangle) = 0$ for $i = 1, 2$.*
(iv) *Suppose moreover, that $P_3 \neq 0$ is a free pointed P_2-module. Then the $\mathbb{Z}\langle P_0\rangle$-module M is φ_0-flat if and only if the $\mathbb{Z}\langle P_1\rangle$-module $M \otimes_{P_0} P_1$ is φ_1-flat.*

Proof. (i): Define $\mathrm{Fil}_1 S := \mathrm{Ker}(S \to P_0/(I \cup J)) = (I \cup J)/J$. Then it is already clear that $S/\mathrm{Fil}_1 S$ is a P_2-module. Next, let $\mathrm{Fil}_0 S := \mathrm{Ker}(\mathrm{Fil}_1 S \to (I \cup JP_1)/JP_1) = JP_1/J$. Since $IP_1 = I$, we see that $\mathrm{Fil}_0 S$ is a P_2-module, and $(I \cup JP_1)/JP_1$ is a P_1-module.

(ii): Clearly (a)⇒(b), hence suppose that (b) holds, and let us prove (a). By proposition 6.1.40(i), it suffices to show that $\mathrm{Tor}_1^{\mathbb{Z}\langle P_0\rangle}(M, \mathbb{Z}\langle P_0/I\rangle) = 0$ for every ideal $J \subset P_0$. In view of (i), we are then reduced to showing that $\mathrm{Tor}_1^{\mathbb{Z}\langle P_0\rangle}(M, \mathbb{Z}\langle S\rangle) = 0$, whenever S is a P_i-module, for $i = 1, 2$. However, for such P_i-module S, we have a change of rings spectral sequence

$$E^2_{pq} : \mathrm{Tor}_p^{\mathbb{Z}\langle P_i\rangle}(\mathrm{Tor}_q^{\mathbb{Z}\langle P_0\rangle}(M, \mathbb{Z}\langle P_i\rangle), \mathbb{Z}\langle S\rangle) \Rightarrow \mathrm{Tor}_{p+q}^{\mathbb{Z}\langle P_0\rangle}(M, \mathbb{Z}\langle S\rangle).$$

Under assumption (b), we deduce : $\mathrm{Tor}_1^{\mathbb{Z}\langle P_0\rangle}(M, \mathbb{Z}\langle S\rangle) = \mathrm{Tor}_1^{\mathbb{Z}\langle P_i\rangle}(M \otimes_{P_0} P_i, \mathbb{Z}\langle S\rangle) = 0.$

(iii): Notice that the induced diagram of rings $\mathbb{Z}\langle\mathscr{D}(P_0, I, P_1)\rangle$ is still cartesian. Then, this is a special case of [75, Lemma 3.4.15].

(iv): Suppose that $M \otimes_{P_0} P_1$ is φ_1-flat, and P_3 is a free pointed P_2-module, say $P_3 \simeq P_2^{(\Lambda)\circ}$, for some set $\Lambda \neq \varnothing$; then the $\mathbb{Z}\langle P_2\rangle$-module $\mathbb{Z}\langle P_3\rangle$ is isomorphic to $\mathbb{Z}\langle P_2\rangle^{(\Lambda)}$, especially it is faithfully flat, and we deduce that $\mathrm{Tor}_1^{\mathbb{Z}\langle P_0\rangle}(M, \mathbb{Z}\langle P_i\rangle) = 0$ for $i = 1, 2$, by (iii). On the other hand, $M \otimes_{P_0} P_3$ is φ_3-flat, so it also follows that $M \otimes_{P_0} P_2$ is φ_2-flat, by proposition 6.1.52. Summing up, this shows that M fulfills condition (ii.b), hence also (ii.a), as sought. □

6.1.55. Let now $P \to Q$ be an injective morphism of integral pointed monoids, and suppose that $P^\sharp$ and $Q^\sharp$ are finitely generated monoids, and Q is a finitely generated P-module. Denote $\varphi_P : P \to \mathbb{Z}\langle P\rangle$ and $\varphi_Q : Q \to \mathbb{Z}\langle Q\rangle$ the usual units of adjunction.

Theorem 6.1.56. *In the situation of* (6.1.55), *let M be a $\mathbb{Z}\langle P\rangle$-module, and suppose that the $\mathbb{Z}\langle Q\rangle$-module $M \otimes_P Q$ is φ_Q-flat. Then M is φ_P-flat.*

Proof. Using lemma 6.1.54(iv), and an easy induction, it suffices to show that there exists a finite chain

$$P = Q_0 \subset Q_1 \subset \cdots \subset Q_n = Q \tag{6.1.57}$$

of inclusions of integral pointed monoids, and for every $j = 0, \dots, n-1$ an ideal $I_j \subset Q_j$, and a cartesian diagram of integral pointed monoids $\mathscr{D}(Q_j, I_j, Q_{j+1})$ as in (6.1.53), and such that $Q_{j+1}/I_j \neq 0$ is a free pointed Q_j/I_j-module. (Notice that each $Q_i^\sharp$ is a quotient of $Q_i/P^\times$, and the latter is a submodule of $Q/P^\times$, hence $Q_i^\sharp$ is still a finitely generated monoid, by proposition 6.1.9(i).) If $P = Q$, there is nothing to prove; so we may assume that P is strictly contained in Q, and – invoking again proposition 6.1.9(i) – we further reduce to showing that there exist a monoid $Q_1 \subset Q$ strictly containing P, and an ideal $I \subset P$, such that the diagram $\mathscr{D}(P, I, Q_1)$ fulfills the above conditions.

Suppose first that the support of Q/P contains only the maximal ideal $\mathfrak{m}_P$ of P, and let $x_1, \dots, x_r$ be a finite system of generators for $\mathfrak{m}_P$ (proposition 6.1.9(ii)). For each $i = 1, \dots, r$, the localization $(Q/P)_{x_i}$ is a P_{x_i}-module with empty support, hence $(Q/P)_{x_i} = 0$ (remark 6.1.22(v)). It follows that every element of Q/P is annihilated by some power of x_i, and since Q/P is finitely generated, we may find $N \in \mathbb{N}$ large enough, such that $x_i^N Q/P = 0$ for $i = 1, \dots, r$. After replacing N by some possibly larger integer, we get $\mathfrak{m}_P^N \cdot Q/P = 0$, and we may assume that N is the least integer with this property. If $N = 0$, there is nothing to prove; hence suppose that $N > 0$, and set $Q_1 := P \cup \mathfrak{m}_P^{N-1} Q$. Notice that Q_1 is a monoid, and $Q_1/P \neq 0$ is annihilated by $\mathfrak{m}_P$. Especially, $\mathfrak{m}_P Q_1 = \mathfrak{m}_P$. Moreover, the induced map $P/\mathfrak{m}_P \to Q_1/\mathfrak{m}_P$ is injective and Q_1 is an integral pointed module, therefore the group $P^\times$ acts freely on $Q_1 \setminus \mathfrak{m}_P$, *i.e.* $Q_1/\mathfrak{m}_P$ is a free pointed $P/\mathfrak{m}_P$-module. It follows easily that if we take $I := \mathfrak{m}_P$, we do obtain a diagram $\mathscr{D}(P, \mathfrak{m}_P, Q_1)$ with the sought properties, in this case.

For the general case, let $\mathfrak{p} \subset P$ be a minimal element of $\mathrm{Supp}\, Q/P$ (for the ordering given by inclusion). Then the induced morphism $P_\mathfrak{p} \to Q_\mathfrak{p}$ still satisfies the conditions of (6.1.55) (lemma 6.1.20(iv)). Moreover, $\mathrm{Supp}\, Q_\mathfrak{p}/P_\mathfrak{p} = \{\mathfrak{p}P_\mathfrak{p}\}$ by remark 6.1.22(vii). By the previous case, we deduce that there exists a chain of inclusions of integral pointed monoids $P_\mathfrak{p} \subset Q'_1 \subset Q_\mathfrak{p}$, such that the resulting diagram $\mathscr{D}(P_\mathfrak{p}, \mathfrak{p}P_\mathfrak{p}, Q'_1)$ is cartesian, and $Q'_1/\mathfrak{p}P_\mathfrak{p} \neq 0$ is a free pointed $P_\mathfrak{p}/\mathfrak{p}P_\mathfrak{p}$-module. Let $\overline{e}_1, \dots, \overline{e}_d$ be a basis of the latter $P_\mathfrak{p}/\mathfrak{p}P_\mathfrak{p}$-module, with $\overline{e}_1 = 1$. Hence, $\overline{e}_i \in Q_\mathfrak{p} \setminus \mathfrak{p}P_\mathfrak{p}$ for every $i = 1, \dots, d$, and after multiplying $\overline{e}_2, \dots, \overline{e}_d$ by a suitable element of $P \setminus \mathfrak{p}$, we may assume that each $\overline{e}_i$ is the image in $Q_\mathfrak{p}$ of an element $e_i \in Q$. Moreover, for every $i, j \leq d$, either $\overline{e}_i\overline{e}_j = 0$, or else there exist $a_{ij} \in P_\mathfrak{p}$ and $k(i,j) \leq d$ such that $\overline{e}_i\overline{e}_j = a_{ij}\overline{e}_{k(i,j)}$. Furthermore, fix a system of generators $x_1, \dots, x_r$ for $\mathfrak{p}$; then, for every $i \leq r$ and every $j \leq d$ we have $x_i\overline{e}_j \in \mathfrak{p}P_\mathfrak{p}$. Again, after multiplying $\overline{e}_2, \dots, \overline{e}_d$ by some

$c \in P\backslash\mathfrak{p}$, we may assume that $a_{ij} \in P$ for every $i, j \leq d$, and moreover that $x_i\bar{e}_j$ lies in the image of $\mathfrak{p}$ for every $i \leq r$ and $j \leq d$.

And if we multiply yet again $e_2, \dots, e_d$ by a suitable element of $P\backslash\mathfrak{p}$, we may finally reach a system of elements $e_1, \dots, e_d \in Q$ such that $e_1 = 1$ and :

- For every $i, j \leq d$, we have either $e_ie_j = 0$ or else $e_ie_j = a_{ij}e_{k(i,j)}$.
- $x_ie_j \in \mathfrak{p}$ for every $i \leq r$ and $j \leq d$.

Clearly these elements span a P-module Q_1 which is a monoid containing P and contained in Q; moreover, by construction we have $\mathfrak{p}Q_1 = \mathfrak{p}$, hence the resulting diagram $\mathscr{D}(P, \mathfrak{p}, Q_1)$ is cartesian. Notice also that $(Q_1/\mathfrak{p})_\mathfrak{p} \simeq Q'_1/\mathfrak{p}P_\mathfrak{p}$, and that $P/\mathfrak{p} = P'_\circ$, where $P' := P\backslash\mathfrak{p}$ is an integral (non-pointed) monoid. To conclude it suffices now to apply the following

Claim 6.1.58. Let P' be an integral non-pointed monoid, S a pointed $P'_\circ$-module, and $\underline{e} := (e_1, \dots, e_d)$ a system of generators for S. Suppose that $S \otimes_{P'_\circ} P'^{\mathrm{gp}}_\circ$ is a free pointed $P'^{\mathrm{gp}}_\circ$-module, and the image of $\underline{e}$ is a basis for this module. Then S is a free pointed $P'_\circ$-module, with basis $\underline{e}$.

Proof of the claim. If $\underline{e}$ is not a basis, we have a relation in S of the type $a_1e_1 = a_2e_2$, for some $a_1, a_2 \in P'$. This relation must persist in $S \otimes_{P'_\circ} P'^{\mathrm{gp}}_\circ$, and implies that $a_1 = a_2 = 0$ in $P'^{\mathrm{gp}}_\circ$. However, under the stated assumptions the localization map $P'_\circ \to P'^{\mathrm{gp}}_\circ$ is injective, a contradiction. □

6.2. **Integral monoids.** We begin presently the study of a special class of monoids, the integral non-pointed monoids, and the subclass of saturated monoids (see definition 4.8.40(iii)). Later, we shall complement this section with further results on fine monoids (see sections 6.4 and 12.5). *Throughout this section, all the monoids under consideration shall be non-pointed.*

Definition 6.2.1. Let $\varphi : M \to N$ be a morphism of monoids.

(i) φ is said to be *integral* if, for any integral monoid M', and any morphism $M \to M'$, the push-out $N \otimes_M M'$ is integral.

(ii) φ is said to be *strongly flat* (resp. *strongly faithfully flat*) if the induced morphism $\mathbb{Z}[\varphi] : \mathbb{Z}[M] \to \mathbb{Z}[N]$ is flat (resp. faithfully flat).

Lemma 6.2.2. *Let $f : M \to N$ and $g : N \to P$ be two morphisms of monoids.*

(i) *If f and g are integral (resp. strongly flat), the same holds for $g \circ f$.*

(ii) *If f is integral (resp. strongly flat), and $M \to M'$ is any other morphism, then the morphism $\mathbf{1}_{M'} \otimes_M f : M' \to M' \otimes_M N$ is integral (resp. strongly flat).*

(iii) *If f is integral, and $S \subset M$ and $T \subset N$ are any two submonoids such that $f(S) \subset T$, then the induced morphism $S^{-1}M \to T^{-1}N$ is integral.*

(iv) *If $S \subset M$ is any submonoid, the natural map $M \to S^{-1}M$ is strongly flat.*

(v) *If f is integral, then the same holds for f^{int}, and the natural map $M^{\mathrm{int}} \otimes_M N \to N^{\mathrm{int}}$ is an isomorphism.*

(vi) *The unit of adjunction $M \to M^{\mathrm{int}}$ is an integral morphism.*

Proof. (i) is obvious. Assertion (ii) for integral maps is likewise clear, and (ii) for strongly flat morphisms follows from (4.8.52). Assertion (iv) follows immediately from (4.8.53).

(iii): Let $S^{-1}M \to M'$ be any map of integral monoids; in view of lemma 4.8.34, we have $P := M' \otimes_{S^{-1}M} T^{-1}N \simeq T^{-1}(M' \otimes_M N)$, hence P is integral, which is the claim.

(v): The second assertion follows easily by comparing the universal properties (details left to the reader); using this and and (ii), we deduce that f^{int} is integral.

(vi) is left to the reader. □

Theorem 6.2.3. *Let $\varphi : M \to N$ be a morphism of integral monoids. Consider the following conditions :*

(a) *φ is integral.*
(b) *φ is flat (see remark* 4.8.23(vi)*).*
(c) *φ is flat and injective.*
(d) *φ is strongly flat.*
(e) *For every field k, the induced map $k[\varphi] : k[M] \to k[N]$ is flat.*

Then : (e) $\Leftrightarrow$ (d) $\Leftrightarrow$ (c) $\Rightarrow$ (b) $\Leftrightarrow$ (a).

Proof. This result appears in [111, Prop.4.1(1)], with a different proof.

(a)$\Rightarrow$(b): Let $I \subset M$ be any ideal; we consider the M-module :

$$\mathsf{R}(M,I) := \bigoplus_{n\in\mathbb{N}} I^n$$

where I^n denotes, for each $n \in \mathbb{N}$, the n-th power of I in the monoid $(\mathscr{P}(M), \cdot)$ of (6.1.1). Then there exists an obvious multiplication law on $\mathsf{R}(M,I)$, such that the latter is a $\mathbb{N}$-graded integral monoid, and the inclusion $M \to \mathsf{R}(M,I)$ in degree zero is a morphism of monoids. We call $\mathsf{R}(M,I)$ the *Rees monoid* associated with M and I.

Denote also by $j : \mathsf{R}(M,I) \to M \times \mathbb{N}$ the natural inclusion map. By assumption, the monoid $\mathsf{R}(M,I) \otimes_M N$ is integral. However, the natural map $\mathsf{R}(M,I) \otimes_M N \to (\mathsf{R}(M,I) \otimes_M N)^{\mathrm{gp}}$ factors through $j \otimes_M N$. The latter means that, for every $n \in \mathbb{N}$, the induced map $I^n \otimes_M N \to N$ is injective. Then the assertion follows from proposition 6.1.40 and remark 4.8.21(ii).

(b)$\Rightarrow$(a): Let $M \to M'$ be any morphism of integral monoids; we need to show that the natural map $M' \otimes_M N \to M'^{\mathrm{gp}} \otimes_{M^{\mathrm{gp}}} N^{\mathrm{gp}}$ is injective (see remark 6.1.25(i)). The latter factors through the morphism $M' \otimes_M N \to M'^{\mathrm{gp}} \otimes_M N$, which is injective by theorem 6.1.42 and remark 4.8.21(ii). We are thus reduced to proving the injectivity of the natural map :

$$M'^{\mathrm{gp}} \otimes_{M^{\mathrm{gp}}} (M^{\mathrm{gp}} \otimes_M N) \to M'^{\mathrm{gp}} \otimes_{M^{\mathrm{gp}}} N^{\mathrm{gp}}.$$

By comparing the respective universal properties, it is easily seen that $M^{\mathrm{gp}} \otimes_M N$ is the localization $\varphi(M)^{-1}N$, which of course injects into N^{gp}. Then the contention follows from the following general :

Claim 6.2.4. Let G be a group, $T \to T'$ an injective morphism of monoids, $G \to P$ a morphism of monoids. Then the natural map $P \otimes_G T \to P \otimes_G T'$ is injective.

Proof of the claim. This follows easily from remark 6.1.25(i) and lemma 4.8.31(ii). ◊

(d)$\Rightarrow$(e) and (c)$\Rightarrow$(b) are trivial.

(e)$\Rightarrow$(c): The flatness of φ has already been noticed in remark 6.1.35(ii). To show that φ is injective, let $a_1, a_2 \in M$ and let k be any field. Under assumption (e), the image in $k[N]$ of the annihilator $\mathrm{Ann}_{k[M]}(a_1 - a_2)$ generates the ideal $\mathrm{Ann}_{k[N]}(\varphi(a_1) - \varphi(a_2))$. Set $b := a_1 a_2^{-1}$; it follows that $\mathrm{Ann}_{k[M^{\mathrm{gp}}]}(1-b)$ generates $\mathrm{Ann}_{k[N^{\mathrm{gp}}]}(1-\varphi(b))$. However, one checks easily that the annihilator of $1-b$ in $k[M^{\mathrm{gp}}]$ is either 0 if b is not a torsion element of M^{gp}, or else is generated (as an ideal) by $1 + b + \cdots + b^{n-1}$, where n is the order of b in the group M^{gp}. Now, if $\varphi(a_1) = \varphi(a_2)$, we have $\varphi(b) = 1$, hence the annihilator of $1-b$ cannot be 0, and in fact $k[N^{\mathrm{gp}}] = \mathrm{Ann}_{k[N^{\mathrm{gp}}]}(1-\varphi(b)) = nk[N^{\mathrm{gp}}]$. Since k is arbitrary, it follows that $n = 1$, *i.e.* $a_1 = a_2$.

(c)$\Rightarrow$(d): since φ is injective, $N_\circ$ is an integral pointed $M_\circ$-module, so the assertion is a special case of proposition 6.1.52. □

Remark 6.2.5. (i) Let G be a group; then every morphism of monoids $M \to G$ is integral. Indeed, lemma 6.2.2(i,vi) reduces the assertion to the case where M is integral, in which case it is an immediate consequence of theorem 6.2.3 and example 6.1.23(vi).

(ii) Let M be an integral monoid, and S a flat pointed $M_\circ$-module. From theorem 6.1.42 we see that $T := S \setminus \{0\}$ is an M-submodule of S, hence $S = T_\circ$.

(iii) Let $\varphi : M \to N$ be a morphism of integral monoids, and set $\Gamma := \mathrm{Coker}\,\varphi^{\mathrm{gp}}$; then the natural map $\pi : N \to \Gamma$ defines a grading on N (see definition 4.8.8(i)), which we call the *φ-grading*. As usual, we shall write N_γ instead of $\pi^{-1}(\gamma)$, for every $\gamma \in \Gamma$. We shall use the additive notation for the composition law of Γ; especially, the neutral element shall be denoted by 0. Clearly φ factors through a morphism of monoids $M \to N_0$, and each graded summand N_γ is naturally a M-module.

(iv) With the notation of (iii), we claim that the induced morphism $\overline{\varphi} : \varphi(M) \to N$ is flat (hence strongly flat, by theorem 6.2.3), if and only if N_γ is a filtered union of cyclic M-modules, for every $\gamma \in \Gamma$. Indeed, notice that a cyclic M-submodule of N_γ is a free $\varphi(M)$-module of rank one (since N is integral), hence the condition implies that N_γ is a flat $\varphi(M)$-module, hence $\overline{\varphi}$ flat. Conversely, suppose that $\overline{\varphi}$ is flat, and let $n_1, n_2 \in N_\gamma$ (for some $\gamma \in \Gamma$); this means that there exist $a_1, a_2 \in M$ such that $\varphi(a_1)n_1 = \varphi(a_2)n_2$ in N. Then, condition (F2) of theorem 6.1.42 says that there exist $n' \in N$ and $b_1, b_2 \in M$ such that $n_i = \varphi(b_i)n'$ for $i = 1, 2$; especially, $n' \in N_\gamma$, which shows that N_γ is a filtered union of cyclic M-modules.

(v) With the notation of (iii), notice as well, that a morphism $\varphi : M \to N$ of integral monoids is exact (see definition 4.8.40(i)) if and only if $\mathrm{Ker}\,\varphi^{\mathrm{gp}} \subset M$ and φ induces an isomorphism $M/\mathrm{Ker}\,\varphi^{\mathrm{gp}} \xrightarrow{\sim} N_0$. (Details left to the reader.)

Theorem 6.2.6. *Let $M \to N$ be a finite, injective morphism of integral monoids, and S a pointed $M_\circ$-module. Then S is $M_\circ$-flat if and only if $N_\circ \otimes_{M_\circ} S$ is $N_\circ$-flat.*

Proof. In light of theorem 6.1.42, we may assume that $N_\circ \otimes_{M_\circ} S$ is $N_\circ$-flat, and we shall show that S is $M_\circ$-flat. To this aim, let 1 denote the trivial monoid (the initial and final object in the category of monoids); any pointed $M_\circ$-module X is a pointed $1_\circ$-module by restriction of scalars, and if X and Y are any two pointed $M_\circ$-modules, we define a $M_\circ$-module structure on $X \otimes_{1_\circ} Y$ by the rule

$$a \cdot (x \otimes y) := ax \otimes ay \qquad \text{for every } a \in M_\circ,\ x \in X \text{ and } y \in Y.$$

With this notation, we remark :

Claim 6.2.7. Let $\varphi : M \to G$ be a morphism of monoids, where G is a group and M is integral. Then a pointed $M_\circ$-module S is $M_\circ$-flat if and only if the same holds for the $M_\circ$-module $S \otimes_{1_\circ} G_\circ$.

Proof of the claim. Suppose that S is $M_\circ$-flat; then $S = T_\circ$ for some M-module T (remark 6.2.5(ii)), and it is easily seen that $S \otimes_{1_\circ} G_\circ = (T \times G)_\circ$. By theorem 6.1.42, it suffices to check that conditions (F1) and (F2) of lemma 6.1.41 hold for $(T \times G)_\circ$.

Hence, suppose that $a \in M_\circ$ and $h \in (T \times G)_\circ$ satisfy the identity $ah = 0$; in this case, a simple inspection shows that either $a = 0$ or $h = 0$; condition (F1) follows straightforwardly. To check (F2), say that $a_1 \cdot (s_1, g) = a_2 \cdot (s_2, g_2) \neq 0$, for some $a_i \in M$, $s_i \in S$, $g_i \in G$ $(i = 1, 2)$; since (F2) holds for S, we may find $b_1, b_2 \in M$ and $t \in S$ such that $a_1 b_1 = a_2 b_2$ and $b_i t = s_i$ $(i = 1, 2)$. Notice that $g := \varphi(b_1)^{-1} g_1 = \varphi(b_2)^{-1} g_2$ in G, therefore $b_i \cdot (t, g) = (s_i, g_i)$ for $i = 1, 2$, whence the contention.

Conversely, suppose that $S \otimes_{1_\circ} G_\circ$ is $M_\circ$-flat; we wish to show that (F1) and (F2) hold for S. However, suppose that $as = 0$ for some $a \in M_\circ$ and $s \in S$ with $s \neq 0$; then $a \cdot (s \otimes e) = 0$ (where $e \in G$ is the neutral element); since (F1) holds for $S \otimes_{1_\circ} G_\circ$, we deduce that there exist $b \in M_\circ$ and $t \otimes g \in S \otimes_{1_\circ} G_\circ$ such that $ba = 0$ and $s \otimes e = b \cdot (t \otimes g) = bt \otimes \varphi(b)g$; this implies that $bt = s$, so (F1) holds for S, as sought.

Lastly, suppose that $a_1 s_1 = a_2 s_2 \neq 0$ for some $a_i \in M$ and $s_i \in S$ $(i = 1, 2)$. It follows that $a_1 \cdot (s_1 \otimes \varphi(a_2)) = a_2 \cdot (s_2 \otimes \varphi(a_1))$. By applying condition (F2) to this identity in $S \otimes_{1_\circ} G_\circ$, we deduce that the same condition holds also for S. $\Diamond$

Next, we observe that there is a natural isomorphism of $N_\circ$-modules :

$$(6.2.8)\quad N_\circ \otimes_{M_\circ} (S \otimes_{1_\circ} N_\circ^{\mathrm{gp}}) \xrightarrow{\sim} (N_\circ \otimes_{M_\circ} S) \otimes_{1_\circ} N_\circ^{\mathrm{gp}} \qquad n \otimes (s \otimes g) \mapsto (n \otimes s) \otimes \varphi(n)g$$

whose inverse is given by the rule : $(n \otimes s) \otimes g \mapsto n \otimes (s \otimes \varphi(n)^{-1}g)$ for every $n \in N$, $s \in S$, $g \in N_\circ^{\mathrm{gp}}$. We leave to the reader the verification that these maps are well defined, and they are inverse to each other. In view of (6.2.8) and claim 6.2.7, we may then replace S by $S \otimes_{1_\circ} N_\circ^{\mathrm{gp}}$, which allows to assume that S is an integral pointed $M_\circ$-module and $N_\circ \otimes_{M_\circ} S$ is an integral pointed $N_\circ$-module. In this case, in view of proposition 6.1.52 we know that $\mathbb{Z}\langle N_\circ \otimes_{M_\circ} S\rangle$ is a flat $\mathbb{Z}\langle N_\circ\rangle$-module, and it suffices to show that $\mathbb{Z}\langle S\rangle$ is a flat $\mathbb{Z}\langle M_\circ\rangle$-module.

However, under our assumptions, the ring homomorphism $\mathbb{Z}\langle M_\circ\rangle \to \mathbb{Z}\langle N_\circ\rangle$ is finite and injective, so the assertion follows from [86, Part II, Th.1.2.4] and (6.1.32). □

Lemma 6.2.9. *Let M be an integral monoid, and $S \subset M$ a submonoid. We have :*

(i) *The natural map $S^{-1}M^{\mathrm{sat}} \to (S^{-1}M)^{\mathrm{sat}}$ is an isomorphism.*
(ii) *If M is saturated, then M/S is saturated, and if S is a group, also the converse holds.*
(iii) *The inclusion map $M \to M^{\mathrm{sat}}$ is a local morphism.*
(iv) *The inclusion $M \subset M^{\mathrm{sat}}$ induces a natural bijection :*

$$\mathrm{Spec}\, M^{\mathrm{sat}} \xrightarrow{\sim} \mathrm{Spec}\, M.$$

Proof. (i) and (iii) are left to the reader. For (ii), the natural isomorphism $S^{-1}M/S^{\mathrm{gp}} \xrightarrow{\sim} M/S$, together with (i), reduces to the case where S is a group, in which case it suffices to remark that $(M/S)^{\mathrm{sat}} = M^{\mathrm{sat}}/S$. Lastly, to show (iv) it suffices to prove that, for any face $F \subset M$, the submonoid $F^{\mathrm{sat}} \subset M^{\mathrm{sat}}$ is a face, and $F^{\mathrm{sat}} \cap M = F$, and that every face of M^{sat} is of this form. These assertions are easy exercises, which we leave as well to the reader. □

Lemma 6.2.10. *Let M be a saturated monoid such that $M^\sharp$ is fine. Then there exists an isomorphism of monoids :*

$$M \xrightarrow{\sim} M^\sharp \times M^\times$$

and if M is fine, $M^\times$ is a finitely generated abelian group. Moreover, the projection $M \to M^\sharp$ induces a bijection :

$$\mathrm{Spec}\, M^\sharp \xrightarrow{\sim} \mathrm{Spec}\, M \quad : \quad \mathfrak{p} \mapsto \mathfrak{p} \times M^\times.$$

Proof. Under the stated assumptions, $G := M^{\mathrm{gp}}/M^\times$ is a free abelian group of finite rank, hence the projection $M^{\mathrm{gp}} \to G$ admits a splitting $\sigma : G \to M^{\mathrm{gp}}$. Set $M_0 := M \cap \sigma(G)$; it is easily seen that $M = M_0 \times M^\times$, whence $M_0 \simeq M^\sharp$. If M is fine, M^{gp} is a finitely generated abelian group, hence the same holds for its direct factor $M^\times$. The last assertion can be proven directly, or can be regarded as a special case of lemma 6.1.17(i.b). □

Definition 6.2.11. Let $\varphi : M \to N$ be a morphism of integral monoids.

(i) We say that φ is *k-saturated* (for some integer $k > 0$), if the push-out $P \otimes_M N$ is integral and k-saturated, for every morphism $M \to P$ with P integral and k-saturated.
(ii) We say that φ is *saturated*, if the following holds. For every morphism of monoids $M \to P$ such that P is integral and saturated, the monoid $P \otimes_M N$ is also integral and saturated.

Clearly, if φ is k-saturated for every integer $k > 0$, then φ is saturated.

Lemma 6.2.12. *Let $\varphi : M \to N$ be a morphism of integral monoids, and $S \subset M$, $T \subset N$ two submonoids such that $\varphi(S) \subset T$. The following holds :*

(i) *The localization map $M \to S^{-1}M$ is saturated.*
(ii) *If φ is saturated, the same holds for the morphisms $S^{-1}M \to T^{-1}N$ and $M/S \to N/T$ induced by φ.*

(iii) *If S and T are two groups, then φ is saturated if and only if the same holds for the induced morphism $M/S \to N/T$.*
(iv) *If φ is saturated, then the natural map $N \otimes_M M^{\mathrm{sat}} \to N^{\mathrm{sat}}$ is an isomorphism.*

Proof. (i) follows from the standard isomorphism : $S^{-1}M \otimes_M N \stackrel{\sim}{\to} \varphi(S)^{-1}N$, together with lemma 6.2.9(i). Next, let $M/S \to P$ and $S^{-1}M \to Q$ be morphisms of monoids. Then

$$P \otimes_{M/S} N/T \simeq (P \otimes_M N)/T \qquad \text{and} \qquad Q \otimes_{S^{-1}M} T^{-1}N \simeq T^{-1}(Q \otimes_M N)$$

(lemma 4.8.34) so assertions (ii) and (iii) follow from lemma 6.2.9(i,ii).
(iv) follows by comparing the universal properties. □

Example 6.2.13. (i) Let M be an integral monoid, $I \subset M$ an ideal, and consider again the Rees monoid $\mathsf{R}(M,I)$ of the proof of theorem 6.2.3. Clearly $\mathsf{R}(M,I)$ is an integral monoid. However, easy examples show that, even when M is saturated, $\mathsf{R}(M,I)$ is not generally saturated. More precisely, the following holds. For every ideal $J \subset M$, set

$$J^{\mathrm{sat}} := \{a \in M^{\mathrm{gp}} \mid a^n \in J^n \text{ for some integer } n > 0\}$$

where J^n denotes the n-th power of J in the monoid $(\mathscr{P}, \cdot)$ of (6.1.1). Then J^{sat} is an ideal of M^{sat}. With this notation, we have the identity :

$$\mathsf{R}(M,I)^{\mathrm{sat}} = \bigoplus_{n\in\mathbb{N}} (I^n)^{\mathrm{sat}}.$$

(Verification left to the reader.)
(ii) For instance, take $M := \mathbb{Q}_+^{\oplus 2}$, and let $I \subset M$ be the ideal consisting of all pairs (x,y) such that $x + y > 1$. Then $I^n = \{(x,y) \in \mathbb{Q}_+^{\oplus 2} \mid x + y > n\}$, and it is easily seen that $I^n = (I^n)^{\mathrm{sat}}$ for every $n \in \mathbb{N}$, hence in this case $\mathsf{R}(M,I)$ is saturated, in view of (ii). It is easily seen that $\mathsf{R}(M,I)$ does not fulfill condition (F2) of lemma 6.1.41, hence the natural inclusion map $i : M \to \mathsf{R}(M,I)$ is not flat (theorem 6.1.42), hence it is not an integral morphism, according to theorem 6.2.3. On the other hand, we have :

Lemma 6.2.14. *With the notation of example* 6.2.13(ii)*, the morphism i is saturated.*

Proof. We prefer to work with the multiplicative notation, so we shall argue with the monoid $(\exp \mathbb{Q}_+^{\oplus 2}, \cdot)$ (see (6.1)). Indeed, let $\varphi : M \to P$ be any morphism of monoids, with P saturated. Clearly $\mathsf{R}(M,I)$ is the direct sum of the P-modules $P \otimes_M I^n$, for all $n \in \mathbb{N}$.

Claim 6.2.15. The natural map $P \otimes_M I^n \to I^n P$ is an isomorphism, for every $n \in \mathbb{N}$

Proof of the claim. Indeed, the map is obviously surjective. Hence, suppose that $a_1 x_1 = a_2 x_2$ for some $a_1, a_2 \in I^n$, $x_1, x_2 \in P$ such that $x_1 \otimes a_1 = x_2 \otimes a_2$. For every $\vartheta \in \mathbb{Q}$ with $0 \leq \vartheta \leq 1$, set $a_\vartheta := a_1^\vartheta \cdot a_2^{1-\vartheta}$ and notice that

$$x_\vartheta := a_1 a_\vartheta^{-1} x_1 = a_2 a_\vartheta^{-1} x_2 \in P^{\mathrm{gp}}$$

and if $N \in \mathbb{N}$ is large enough, so that $N\vartheta \in \mathbb{N}$, then $x_\vartheta^N = x_1^{N\vartheta} x_2^{N\vartheta} \in P$, hence $x_\vartheta \in P$.
Next, for any $(a,b), (a',b') \in \mathbb{Q}_+^{\oplus 2}$, set $(a,b) \vee (a',b') := (\min(a,a'), \min(b,b'))$. Choose an increasing sequence $0 := \vartheta_0 < \vartheta_1 < \cdots < \vartheta_n := 1$ of rational numbers, such that

$$b_i := a_{\vartheta_i} \vee a_{\vartheta_{i+1}} \in I^n \qquad \text{for every } i = 0, \dots, n-1.$$

Then there exist $c_i, d_i \in \mathbb{Q}_+^{\oplus 2}$ such that $a_{\vartheta_i} = b_i c_i$ and $a_{\vartheta_{i+1}} = b_i d_i$ for $i = 0, \dots, n-1$. We may then compute in $I^n \otimes_M P$:

$$a_{\vartheta_i} \otimes x_{\vartheta_i} = b_i c_i \otimes x_{\vartheta_i} = b_i \otimes c_i x_{\vartheta_i} \quad \text{and likewise} \quad a_{\vartheta_{i+1}} \otimes x_{\vartheta_{i+1}} = b_i \otimes d_i x_{\vartheta_{i+1}}.$$

By construction, we have $c_i x_{\vartheta_i} = d_i x_{\vartheta_{i+1}}$ for $i = 0, \dots, n-1$, whence the contention. ◊

In view of claim 6.2.15, we are reduced to showing that $\mathsf{R}(P, IP)$ is saturated, and by example 6.2.13, this comes down to proving that $(I^n P)^{\mathrm{sat}} = I^n P$ for every $n \in \mathbb{N}$. However, say that $x \in P^{\mathrm{gp}}$, and $x^n = a_1 x_1 \cdots a_n x_n$ for some $a_i \in I$ and $x_i \in P$; set $a := (a_1 \cdots a_n)^{1/n}$, and notice that $a \in I$. Then $x^n = a^n \cdot x_1 \cdots x_n$, so that $xa^{-1} \in P$, and finally $x \in IP$, as required. □

Lemma 6.2.14 shows that a saturated morphism is not necessarily integral. Notwithstanding, we shall see later that integrality holds for an important class of saturated morphisms (corollary 6.4.6). Now we wish to globalize the class of saturated morphisms, to an arbitrary topos. Of course, we could define the notion of saturated morphism of T-monoids, just by repeating word by word definition 6.2.11(ii). However, it is not clear that the resulting condition would be of a type which can be checked on stalks, in the sense of definition 4.7.11(v). For this reason, we prefer to proceed as in Tsuji's work [159].

Lemma 6.2.16. *Let T be a topos, $\varphi : \underline{M} \to \underline{N}$ and $\psi : \underline{N} \to \underline{P}$ two morphisms of integral T-monoids. We have:*

(i) *If φ and ψ are exact, the same holds for $\psi \circ \varphi$.*
(ii) *If $\psi \circ \varphi$ is exact, the same holds for φ.*
(iii) *Consider a commutative diagram of integral T-monoids :*

$$(6.2.17) \qquad \begin{array}{ccc} \underline{M} & \xrightarrow{\varphi} & \underline{N} \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\psi'} \\ \underline{M}' & \xrightarrow{\varphi'} & \underline{N}'. \end{array}$$

Then the following holds :
(a) *If* (6.2.17) *is a cartesian diagram and φ' is exact, then φ is exact.*
(b) *If* (6.2.17) *is cocartesian (in the category* $\mathbf{Int.Mnd}_T$*) and φ is exact, then φ' is exact.*

Proof. For all these assertions, remark 4.8.41(ii) easily reduces to the case where $T = \mathbf{Set}$, which therefore we assume from start. Now, (i) and (ii) are left to the reader.

(iii.a): Let $x \in M^{\mathrm{gp}}$ such that $\varphi^{\mathrm{gp}}(x) \in N$; hence $(\varphi' \circ \psi)^{\mathrm{gp}}(x) = \psi'(\varphi^{\mathrm{gp}}(x)) \in N'$, and therefore $\psi^{\mathrm{gp}}(x) \in M'$, since φ' is exact. It follows that $x \in M$, so φ is exact.

(iii.b): Let $x \in (M')^{\mathrm{gp}}$ such that $(\varphi')^{\mathrm{gp}}(x) \in N'$; then we may write $(\varphi')^{\mathrm{gp}}(x) = \varphi'(y) \cdot \psi'(z)$ for some $y \in M'$ and $z \in N$. Therefore, $(\varphi')^{\mathrm{gp}}(xy^{-1}) = \psi'(z)$; since the functor $P \mapsto P^{\mathrm{gp}}$ commutes with colimits, it follows that we may find $w \in M^{\mathrm{gp}}$ such that $\psi^{\mathrm{gp}}(w) = xy^{-1}$ and $\varphi^{\mathrm{gp}}(w) = z$. Since φ is exact, we deduce that $w \in M$, therefore $xy^{-1} \in M'$, and finally $x \in M'$, whence the contention. □

6.2.18. Let T be a topos; for two morphisms $\underline{P} \leftarrow \underline{M} \to \underline{N}$ of integral T-monoids, we set

$$\underline{N} \overset{\mathrm{int}}{\otimes}_M \underline{P} := (\underline{N} \otimes_M \underline{P})^{\mathrm{int}}$$

which is the push-out of these morphisms, in the category $\mathbf{Int.Mnd}_T$. Notice that, for every morphism $f : T' \to T$ of topoi, the natural morphism of T'-monoids

$$(6.2.19) \qquad f^*(\underline{N} \overset{\mathrm{int}}{\otimes}_M \underline{P}) \to f^*\underline{N} \overset{\mathrm{int}}{\otimes}_{f^*M} f^*\underline{P}$$

is an isomorphism (by lemmata 4.8.45(i) and 4.8.46(i)).

Let now $\varphi : \underline{M} \to \underline{N}$ be a morphism of integral T-monoids. For any integer $k > 0$, let $\boldsymbol{k}_M$ and $\boldsymbol{k}_N$ be the k-Frobenius maps of M and N (definition 4.8.40(ii)), and consider the cocartesian diagram :

$$\begin{array}{ccc} \underline{M} & \xrightarrow{\ \boldsymbol{k}_M\ } & \underline{M} \\ {\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle\varphi'} \\ \underline{N} & \xrightarrow{\boldsymbol{k}_M \overset{\mathrm{int}}{\otimes}_M \mathbf{1}_N} & \underline{P}. \end{array} \tag{6.2.20}$$

The endomorphism $\boldsymbol{k}_N$ factors through $\boldsymbol{k}_M \overset{\mathrm{int}}{\otimes}_M \mathbf{1}_N$ and a unique morphism $\beta : \underline{P} \to \underline{N}$ such that $\beta \circ \varphi' = \varphi$. A simple inspection shows that $\boldsymbol{k}_P = (\boldsymbol{k}_M \otimes_M \mathbf{1}_N) \circ \beta$. Now, if $\boldsymbol{k}_M$ is exact, then the same holds for $\boldsymbol{k}_M \overset{\mathrm{int}}{\otimes}_M \mathbf{1}_N$, in view of lemma 6.2.16(iii.b). Hence, if $\underline{P}$ is k-saturated, then β is exact (lemma 6.2.16(ii)), and if $\underline{M}$ is k-saturated, also the converse holds. Likewise, if $\underline{M}$ is k-saturated and β is exact, then $\underline{N}$ is k-quasi-saturated.

These considerations motivate the following :

Definition 6.2.21. Let T be a topos, $\varphi : \underline{M} \to \underline{N}$ a morphism of integral T-monoids.

(i) A commutative diagram of integral T-monoids :

$$\begin{array}{ccc} \underline{M} & \xrightarrow{\ \varphi\ } & \underline{N} \\ \big\downarrow & & \big\downarrow \\ \underline{M}' & \xrightarrow{\ \varphi'\ } & \underline{N}' \end{array}$$

is called an *exact square*, if the induced morphism $\underline{M}' \overset{\mathrm{int}}{\otimes}_M \underline{N} \to \underline{N}'$ is exact.

(ii) φ is said to be *k-quasi-saturated* if the commutative diagram :

$$\begin{array}{ccc} \underline{M} & \xrightarrow{\ \varphi\ } & \underline{N} \\ {\scriptstyle\boldsymbol{k}_M}\big\downarrow & & \big\downarrow{\scriptstyle\boldsymbol{k}_N} \\ \underline{M} & \xrightarrow{\ \varphi\ } & \underline{N} \end{array} \tag{6.2.22}$$

is an exact square (the vertical arrows are the k-Frobenius maps).

(iii) φ is said to be *quasi-saturated* if it is k-quasi-saturated for every integer $k > 0$.

Proposition 6.2.23. *Let T be a topos.*

(i) *If* (6.2.17) *is an exact square, and $\underline{M} \to \underline{P}$ is any morphism of integral T-monoids, the square $\underline{P} \overset{\mathrm{int}}{\otimes}_M$* (6.2.17) *is exact.*

(ii) *Consider a commutative diagram of integral T-monoids :*

$$\begin{array}{ccccc} \underline{M} & \xrightarrow{\ \varphi_1\ } & \underline{N} & \xrightarrow{\ \varphi_2\ } & \underline{P} \\ {\scriptstyle\psi}\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle\psi'} \\ \underline{M}' & \xrightarrow{\ \varphi'_1\ } & \underline{N}' & \xrightarrow{\ \varphi'_2\ } & \underline{P}' \end{array} \tag{6.2.24}$$

and suppose that the left and right square subdiagrams of (6.2.24) *are exact. Then the same holds for the square diagram :*

$$\begin{array}{ccc} \underline{M} & \xrightarrow{\ \varphi_2 \circ \varphi_1\ } & \underline{P} \\ {\scriptstyle\psi}\big\downarrow & & \big\downarrow{\scriptstyle\psi'} \\ \underline{M}' & \xrightarrow{\ \varphi'_2 \circ \varphi'_1\ } & \underline{P}'. \end{array}$$

Proof. (i): We have a commutative diagram :

$$\begin{array}{ccc}
(\underline{P} \overset{\mathrm{int}}{\otimes}_M \underline{M}') \overset{\mathrm{int}}{\otimes}_M \underline{N} & \xrightarrow{\ \sigma\ } & \underline{P} \overset{\mathrm{int}}{\otimes}_M (\underline{M}' \overset{\mathrm{int}}{\otimes}_M \underline{N}) \\
{\scriptstyle\omega}\downarrow & & \downarrow{\scriptstyle \mathbf{1}_P \overset{\mathrm{int}}{\otimes}_M \alpha} \\
(\underline{P} \overset{\mathrm{int}}{\otimes}_M \underline{M}') \overset{\mathrm{int}}{\otimes}_P (\underline{P} \overset{\mathrm{int}}{\otimes}_M \underline{N}) & \xrightarrow{\ \beta\ } & \underline{P} \overset{\mathrm{int}}{\otimes}_M \underline{N}'
\end{array}$$

where ω, σ are the natural isomorphisms, and β and $\alpha : \underline{M}' \overset{\mathrm{int}}{\otimes}_M \underline{N} \to \underline{N}'$ are the natural maps. By assumption, α is exact, hence the same holds for $\mathbf{1}_P \overset{\mathrm{int}}{\otimes}_M \alpha$, by lemma 6.2.16(iii.b). So β is exact, which is the claim.

(ii): Let $\alpha : \underline{M}' \overset{\mathrm{int}}{\otimes}_M \underline{N} \to \underline{N}'$ and $\beta : \underline{N}' \overset{\mathrm{int}}{\otimes}_N \underline{P} \to \underline{P}'$ be the natural maps; by assumptions, these are exact morphisms. However, we have a natural commutative diagram :

$$\begin{array}{ccc}
(\underline{M}' \overset{\mathrm{int}}{\otimes}_M \underline{N}) \overset{\mathrm{int}}{\otimes}_N \underline{P} & \xrightarrow{\ \omega\ } & \underline{M}' \overset{\mathrm{int}}{\otimes}_M \underline{P} \\
{\scriptstyle \alpha \overset{\mathrm{int}}{\otimes}_N \mathbf{1}_P}\downarrow & & \downarrow{\scriptstyle\gamma} \\
\underline{N}' \overset{\mathrm{int}}{\otimes}_N \underline{P} & \xrightarrow{\ \beta\ } & \underline{P}'
\end{array}$$

where ω is the natural isomorphism, and γ is the map deduced from ψ' and $\varphi'_2 \circ \varphi'_1$. Then the assertion follows from lemma 6.2.16(i,iii.b). □

Corollary 6.2.25. *Let T be a topos, $\varphi : \underline{M} \to \underline{N}$, $\psi : \underline{N} \to \underline{P}$ two morphisms of integral T-monoids, and $h, k > 0$ any two integers. The following holds :*

- (i) *If φ is both h-quasi-saturated and k-quasi-saturated, then φ is hk-quasi-saturated.*
- (ii) *If φ and ψ are k-quasi-saturated, the same holds for $\psi \circ \varphi$.*
- (iii) *If $\underline{M} \to \underline{P}$ is any map of integral T-monoids, and φ is k-quasi-saturated (resp. quasi-saturated), then the same holds for $\varphi \overset{\mathrm{int}}{\otimes}_M \mathbf{1}_P : \underline{P} \to \underline{N} \overset{\mathrm{int}}{\otimes}_M \underline{P}$.*
- (iv) *Let $\underline{S} \subset \varphi^{-1}(\underline{N}^\times)$ be a T-submonoid. Then φ is k-quasi-saturated (resp. quasi-saturated) if and only if the same holds for the induced map $\varphi_S : \underline{S}^{-1}\underline{M} \to \underline{N}$.*
- (v) *Let $\underline{G} \subset \operatorname{Ker} \varphi$ be a subgroup. Then φ is k-quasi-saturated (resp. quasi-saturated) if and only if the same holds for the induced map $\overline{\varphi} : \underline{M}/\underline{G} \to \underline{N}$.*
- (vi) *φ is quasi-saturated if and only if it is p-quasi-saturated for every prime number p.*
- (vii) *$\underline{M}$ is k-saturated (resp. saturated) if and only if the unique morphism of T-monoids $\{1\} \to \underline{M}$ is k-quasi-saturated (resp. quasi-saturated).*
- (viii) *If φ is k-quasi-saturated (resp. quasi-saturated), and $\underline{M}$ is k-saturated (resp. saturated), then $\underline{N}$ is k-saturated (resp. saturated).*
- (ix) *If $\underline{M}$ is integral, and $\underline{N}$ is a T-group, φ is quasi-saturated.*

Proof. (i) and (ii) are straightforward consequences of proposition 6.2.23(ii).

To show (iii), set $\underline{P}' := \underline{N} \overset{\mathrm{int}}{\otimes}_M \underline{P}$ and let us remark that we have a commutative diagram

$$\begin{array}{ccccc}
\underline{P} & \longrightarrow & \underline{P}_1 & \longrightarrow & \underline{P} \\
{\scriptstyle \varphi \overset{\mathrm{int}}{\otimes}_M \mathbf{1}_P}\downarrow & & \downarrow & & \downarrow{\scriptstyle \varphi \overset{\mathrm{int}}{\otimes}_M \mathbf{1}_P} \\
\underline{P}' & \longrightarrow & \underline{P}_2 & \longrightarrow & \underline{P}'
\end{array}$$

such that :

- the composition of the top (resp. bottom) arrows is the k-Frobenius map
- the left square subdiagram is (6.2.22) $\overset{\mathrm{int}}{\otimes}_M \underline{P}$
- the right square subdiagram is cocartesian (hence exact).

then the assertion follows from proposition 6.2.23.

(iv): Suppose that φ is k-quasi-saturated. Then the same holds for $\underline{S}^{-1}\varphi : \underline{S}^{-1}M \to \underline{S}^{-1}M \overset{\mathrm{int}}{\otimes}_M \underline{N}$, according to (iii). However, we have a natural isomorphism $\underline{S}^{-1}\underline{M} \overset{\mathrm{int}}{\otimes}_M \underline{N} \overset{\sim}{\to} \varphi(\underline{S})^{-1}\underline{N} = N$, so φ_S is k-quasi-saturated.

Conversely, suppose that φ_S is k-quasi-saturated. By (ii), in order to prove that the same holds for φ, it suffices to show that the localisation map $\underline{M} \to \underline{S}^{-1}\underline{M}$ is k-quasi-saturated. But this is clear, since (6.2.22) becomes cocartesian if we take for φ the localisation map.

(v): To begin with, $\underline{M}/\underline{G}$ is an integral monoid, by lemma 4.8.38. Suppose that φ is k-quasi-saturated. Arguing as in the proof of (iv) we see that the natural map $\underline{N} \to (\underline{M}/\underline{G}) \overset{\mathrm{int}}{\otimes}_M \underline{N}$ is an isomorphism, hence $\overline{\varphi}$ is k-quasi-saturated, by (iii).

Conversely, suppose that $\overline{\varphi}$ is k-quasi-saturated. By (ii), in order to prove that φ is saturated, it suffices to show that the same holds for the projection $\underline{M} \to \underline{M}/\underline{G}$. By (iii), we are further reduced to showing that the unique map $\underline{G} \to \{1\}$ is k-quasi-saturated, which is trivial.

(vi) is a straightforward consequence of (i). Assertion (vii) can be verified easily on the definitions. Next, suppose that φ is quasi-saturated and $\underline{M}$ is saturated. By (ii) and (vii) it follows that the unique morphism $\{1\} \to \underline{N}$ is quasi-saturated, hence (vii) implies that $\underline{N}$ is saturated, so (viii) holds.

(ix): In view of (iv), we are reduced to the case where $\underline{M}$ is also a group, in which case the assertion is obvious. □

Proposition 6.2.26. *Let $\varphi : M \to N$ be an integral morphism of integral monoids, $k > 0$ an integer. The following conditions are equivalent :*

(a) *φ is k-quasi-saturated.*

(b) *φ is k-saturated.*

(c) *The push-out P of the cocartesian diagram* (6.2.20)*, is k-saturated.*

Proof. (a)⇒(b) by virtue of corollary 6.2.25(iii,viii), and trivially (b)⇒(c). Lastly, the implication (c)⇒(a) was already remarked in (6.2.18). □

Corollary 6.2.27. *Let $\varphi : M \to N$ be an integral morphism of integral monoids. Then φ is quasi-saturated if and only if it is saturated.* □

Proposition 6.2.26 motivates the following :

Definition 6.2.28. Let T be a topos, $\varphi : \underline{M} \to \underline{N}$ a morphism of integral T-monoids, $k > 0$ an integer. We say that φ is *k-saturated* (resp. *saturated*) if φ is integral and k-quasi-saturated (resp. and quasi-saturated).

In case $T = \mathbf{Set}$, we see that an integral morphism of integral monoids is saturated in the sense of definition 6.2.28, if and only if it is saturated in the sense of the previous definition 6.2.11. We may now state :

Corollary 6.2.29. *Let $\mathbf{P}(T, \varphi)$ be the property "φ is an integral (resp. exact, resp. k-saturated, resp. saturated) morphism of integral T-monoids" (for a topos T). Then $\mathbf{P}$ can be checked on stalks. (See definition* 4.7.11(v).*)*

Proof. The fact that integrality of a T-monoid can be checked on stalks, has already been established in lemma 4.8.46(ii). For the property "φ is an integral morphism" (between integral T-monoids), it suffices to apply theorem 6.2.3 and proposition 4.8.26.

For the property "φ is exact" one applies lemma 4.8.46(iii). From this, and from (6.2.19) one deduces that also the properties of k-saturation and saturation can be checked on stalks. □

Lemma 6.2.30. *Let $f : M \to N$ be a morphism of integral monoids. We have :*

(i) *If f is exact, then f is local.*

(ii) *Conversely, if f is an integral and local morphism, then f is exact.*

Proof. (i) is left to the reader as an exercise.

(ii): Suppose $x \in M^{\mathrm{gp}}$ is an element such that $b := f_S(x) \in N$. Write $x = z^{-1}y$ for certain $y, z \in M$; therefore $b \cdot f(z) = f(y)$ holds in N, and theorems 6.2.3, 6.1.42 imply that there exist $c \in N$ and $a_1, a_2 \in M$, such that $1 = c \cdot f(a_1)$, $b = c \cdot f(a_2)$ and $ya_1 = za_2$. Since f is local, we deduce that $a_1 \in M^\times$, hence $x = a_1^{-1}a_2$ lies in M. □

Proposition 6.2.31. *Let $f : M \to N$ be an integral map of integral monoids, $k > 0$ an integer, and $N = \bigoplus_{\gamma\in\Gamma} N_\gamma$ the f-grading of N (see remark* 6.2.5(iii)*). Then the following conditions are equivalent :*

(a) *f is k-quasi-saturated.*
(b) *$N_{k\gamma} = N_\gamma^k$ for every $\gamma \in \Gamma$. (Here the k-th power of subsets of N is taken in the monoid $\mathscr{P}(N)$, as in* (6.1.1)*.)*

Proof. (a)⇒(b): Let $\pi : N^{\mathrm{gp}} \to \Gamma$ be the projection, suppose that $y \in N_{k\gamma}$ for some $\gamma \in \Gamma$, and pick $x \in N^{\mathrm{gp}}$ such that $\pi(x) = \gamma$. This means that $y = x^k \cdot f^{\mathrm{gp}}(z)$ for some $z \in M^{\mathrm{gp}}$. By (a), it follows that we may find a pair $(a, b) \in M \times N$ and an element $w \in M^{\mathrm{gp}}$, such that $(aw^{-k}, bw) = (z, x)$ in $M^{\mathrm{gp}} \times N^{\mathrm{gp}}$. Especially, $b, b \cdot f(a) \in N_\gamma$, and consequently $y = b^{k-1} \cdot (b \cdot f(a)) \in N_{k\gamma}$, as stated.

(b)⇒(a): Let $S := f^{-1}(N^\times)$; by lemmata 6.2.30(ii) and 6.2.2(iii), the induced map $f_S : S^{-1}M \to N$ is exact and integral. Moreover, by corollary 6.2.25(iv), f is k-quasi-saturated if and only if the same holds for f_S, and clearly the f-grading of N agrees with the f_S-grading. Hence we may replace f by f_S and assume from start that f is exact. In this case, $G := \mathrm{Ker}\, f^{\mathrm{gp}} \subset M$, and corollary 6.2.25(v) says that f is k-quasi-saturated if and only if the same holds for the induced map $\overline{f} : M/G \to N$; moreover $\overline{f}$ is still integral, since it is deduced from f by push-out along the map $M \to M/G$. Hence we may replace f by $\overline{f}$, thereby reducing to the case where f is injective. Also, f is flat, by theorem 6.2.3. The assertion boils down to the following. Suppose that $(x, y) \in M^{\mathrm{gp}} \times N^{\mathrm{gp}}$ is a pair such that $a := f^{\mathrm{gp}}(x) \cdot y^k \in N$; we have to show that there exists a pair $(m, n) \in M \times N$ whose class in the push-out P of the diagram

$$N \xleftarrow{\ f\ } M \xrightarrow{\ \boldsymbol{k}_M\ } M$$

agrees with the image of (x, y) in P^{gp}. However, set $\gamma := \pi(y)$, and notice that $a \in N_{k\gamma}$, hence we may write $a = n_1 \cdots n_k$ for certain $n_1, \dots, n_k \in N_\gamma$. Then, according to remark 6.2.5(iv), we may find $n \in N_\gamma$ and $x_1, \dots, x_k \in M$ such that $n_i = n \cdot f(x_i)$ for every $i = 1, \dots, k$. It follows that $y \cdot n^{-1} \in f^{\mathrm{gp}}(M^{\mathrm{gp}})$, say $y = n \cdot f(z)$ for some $z \in M^{\mathrm{gp}}$. Set $m := x_1 \cdots x_k$; then $(x, y) = (x, n \cdot f(z))$ represents the same class as $(x \cdot z^k, n)$ in P^{gp}. Especially, $f(x \cdot z^k) \cdot n^k = a = f(m) \cdot n^k$, hence $m = x \cdot n^k$, since f is injective. The claim follows. □

Corollary 6.2.32. *Let $f : M \to N$ be an integral and local morphism of integral and sharp monoids. Then :*

(i) *f is exact and injective.*
(ii) *If furthermore, f is saturated, then* $\mathrm{Coker}\, f^{\mathrm{gp}}$ *is a torsion-free abelian group.*

Proof. (i): It follows from lemma 6.2.31 that f is exact. Next, suppose that $f(a_1) = f(a_2)$ for some $a_1, a_2 \in M$. By theorem 6.2.3 and 6.1.42, it follows that there exist $b_1, b_2 \in M$ and $t \in N$ such that $1 = f(b_1)t = f(b_2)t$ and $a_1b_1 = a_2b_2$. Since N is sharp, we deduce that $f(b_1) = f(b_2) = 1$, and since f is local and M is sharp, we get $b_1 = b_2 = 1$, thus $a_1 = a_2$, whence $x = 1$, which is the contention.

(ii): Let us endow N with its f-grading. Suppose now that $g \in G$ is a torsion element, and say that $g^k = 1$ in G for some integer $k > 0$; by propositions 6.2.31 and 6.2.26, we then have

$N_1 = N_g^k$. Especially, there exist $a_1, \dots, a_k \in N_g$, such that $a_1 \cdots a_k = 1$. Since N is sharp, we must have $a_i = 1$ for $i = 1, \dots, k$, hence $g = 1$. $\square$

Corollary 6.2.33. *Let $\varphi : M \to N$ be a morphism of integral monoids, $F \subset N$ any face, and $\varphi_F : \varphi^{-1}F \to F$ the restriction of φ.*

(i) *If φ is flat, then the same holds for φ_F, and the induced map $\mathrm{Coker}\, \varphi_F^{\mathrm{gp}} \to \mathrm{Coker}\, \varphi^{\mathrm{gp}}$ is injective.*
(ii) *If φ is saturated, the same holds for φ_F.*

Proof. (i): The fact that φ_F is flat, is a special case of corollary 6.1.51. The assertion about cokernels boils down to showing that the induced diagram of abelian groups :

$$\begin{array}{ccc} (\varphi^{-1}F)^{\mathrm{gp}} & \xrightarrow{\varphi_F^{\mathrm{gp}}} & F^{\mathrm{gp}} \\ \downarrow & & \downarrow \\ M^{\mathrm{gp}} & \xrightarrow{\varphi^{\mathrm{gp}}} & N^{\mathrm{gp}} \end{array}$$

is cartesian. However, say that $\varphi^{\mathrm{gp}}(a_1 a_2^{-1}) = f_1 f_2^{-1}$ for some $a_1, a_2 \in M$ and $f_1, f_2 \in F$. This means that $\varphi(a_1) f_2 = \varphi(a_2) f_1$ in N; by condition (F2) of theorem 6.1.42, we deduce that there exist $b_1, b_2 \in M$ and $t \in N$ such that $b_2 a_1 = b_1 a_2$ and $f_i = t \cdot \varphi(b_i)$ $(i = 1, 2)$. Since F is a face, it follows that $b_1, b_2 \in \varphi^{-1}F$, hence $a_1 a_2^{-1} = b_1 b_2^{-1} \in (\varphi^{-1}F)^{\mathrm{gp}}$, as required.

(ii): By (i), the morphism φ_F is integral, hence it suffices to show that φ_F quasi-saturated (proposition 6.2.26), and to this aim we shall apply the criterion of proposition 6.2.31. Indeed, let $N = \bigoplus_{\gamma \in \Gamma} N_\gamma$ (resp. $F = \bigoplus_{\gamma \in \Gamma_F} F_\gamma$) be the φ-grading (resp. the φ_F-grading); according to (i), the induced map $j : \Gamma_F \to \Gamma$ is injective. This means that $F_\gamma = F \cap N_{j(\gamma)}$ for every $\gamma \in \Gamma_F$. Hence, for every integer $k > 0$ we may compute

$$F_{k\gamma} = F \cap N_{j(\gamma)}^k = (F \cap N_{j(\gamma)})^k = F_\gamma^k$$

where the first identity follows by applying proposition 6.2.31 (and proposition 6.2.26) to the saturated map φ, and the second identity holds because F is a face of N. $\square$

6.3. **Polyhedral cones.** Fine monoids can be studied by means of certain combinatorial objects, which we wish to describe. Part of the material that follows is borrowed from [73]. Again, *all the monoids in this section are non-pointed.*

6.3.1. Quite generally, a *convex cone* is a pair (V, σ), where V is a finite dimensional $\mathbb{R}$-vector space, and $\sigma \subset V$ is a non-empty subset such that :

$$\mathbb{R}_+ \cdot \sigma = \sigma = \sigma + \sigma$$

where the addition is formed in the monoid $(\mathscr{P}(V), +)$ as in (6.1.1), and scalar multiplication by the set $\mathbb{R}_+$ is given by the rule :

$$\mathbb{R}_+ \cdot S := \{r \cdot s \mid r \in \mathbb{R}_+, s \in S\} \qquad \text{for every } S \in \mathscr{P}(V).$$

A subset $S \subset \sigma$ is called a *ray* of σ, if it is of the form $\mathbb{R}_+ \cdot \{s\}$, for some $s \in \sigma \backslash \{0\}$.

We say that (V, σ) is a *closed convex cone* if σ is closed as a subset of V (of course, V is here regarded as a topological space via any choice of isomorphism $V \simeq \mathbb{R}^n$). We denote by $\langle \sigma \rangle \subset V$ the $\mathbb{R}$-vector space generated by σ. To a convex cone (V, σ) one assigns the *dual cone* $(V^\vee, \sigma^\vee)$, where $V^\vee := \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$, the dual of V, and :

$$\sigma^\vee := \{u \in V^\vee \mid u(v) \geq 0 \text{ for every } v \in \sigma\}.$$

Also, the *opposite cone* of σ is the cone

$$-\sigma := \{-v \in V \mid v \in \sigma\}.$$

Notice that $\sigma^\vee$ and $-\sigma$ are always closed cones. Notice as well that the restriction of the addition law of V determines a monoid structure $(\sigma, +)$ on the set σ. A map of cones

$$\varphi : (W, \sigma_W) \to (V, \sigma_V)$$

is an $\mathbb{R}$-linear map $\varphi : W \to V$ such that $\varphi(\sigma_W) \subset \sigma_V$. Clearly, the restriction of φ yields a map of monoids $(\sigma_W, +) \to (\sigma_V, +)$. If $S \subset V$ is any subset, we set :

$$S^\perp := \{u \in V^\vee \mid u(s) = 0 \text{ for every } s \in S\} \subset V^\vee.$$

Lemma 6.3.2. *Let (V, σ) be a closed convex cone, Then, under the natural identification $V \xrightarrow{\sim} (V^\vee)^\vee$, we have $(\sigma^\vee)^\vee = \sigma$.*

Proof. This follows from [40, Ch.II, §5, n.3, Cor.5]. □

6.3.3. A *convex polyhedral cone* is a cone (V, σ) such that σ is of the form :

$$\sigma := \{r_1 v_1 + \cdots + r_s v_s \in V \mid r_i \geq 0 \text{ for every } i \leq s\}$$

for a given set of vectors $v_1, \dots, v_s \in V$, called *generators* for the cone σ. One also says that $\{v_1, \dots, v_s\}$ is a *generating set* for σ, and that $\mathbb{R}_+ \cdot v_1, \dots, \mathbb{R}_+ \cdot v_s$ are *generating rays* for σ. We say that σ is a *simplicial cone*, if it is generated by a system of linearly independent vectors.

Lemma 6.3.4. *Let (V, σ) be a convex polyhedral cone, S a finite generating set for σ. Then :*

(i) *For every $v \in \sigma$ there is a subset $T \subset S$ consisting of linearly independent vectors, such that v is contained in the convex polyhedral cone generated by T.*

(ii) *(V, σ) is a closed convex cone.*

Proof. (i): Let $T \subset S$ be a subset such that v is contained in the cone generated by T; up to replacing T by a subset, we may assume that T is minimal, *i.e.* no proper subset of T generates a cone containing v. We claim that T consists of linearly independent vectors. Otherwise, we may find a linear relation of the form $\sum_{w \in T} a_w \cdot w = 0$, for certain $a_w \in \mathbb{R}$, at least one of which is non-zero; we may then assume that :

(6.3.5) $$a_w > 0 \qquad \text{for at least one vector } w \in T.$$

Say also that $v = \sum_{w \in T} b_w \cdot w$ with $b_w \in \mathbb{R}_+$; by the minimality assumption on T, we must actually have $b_w > 0$ for every $w \in T$. We deduce the identity : $v = \sum_{w \in T}(b_w - t a_w) \cdot w$ for every $t \in \mathbb{R}$; let t_0 be the supremum of the set of positive real numbers t such that $b_w - t a_w \geq 0$ for every $w \in T$. In view of (6.3.5) we have $t_0 \in \mathbb{R}_+$; moreover $b_w - t_0 a_w \geq 0$ for every $w \in T$, and $b_w - t_0 a_w = 0$ for at least one vector w, which contradicts the minimality of T.

(ii): In view of (i), we are reduced to the case where σ is generated by finitely many linearly independent vectors, and for such cones the assertion is clear (details left to the reader). □

6.3.6. A *face* of a convex cone σ is a subset of the form $\sigma \cap \mathrm{Ker}\, u$, for some $u \in \sigma^\vee$. The *dimension* (resp. *codimension*) of a face τ of σ is the dimension of the $\mathbb{R}$-vector space $\langle\tau\rangle$ (resp. of the $\mathbb{R}$-vector space $\langle\sigma\rangle/\langle\tau\rangle$). A *facet* of σ is a face of codimension one. Notice that if σ is a polyhedral cone, and τ is a face of σ, then (V, τ) is also a convex polyhedral cone; indeed if $S \subset V$ is a generating set for σ, then $S \cap \tau$ is a generating set for τ.

Lemma 6.3.7. *Let (V, σ) be a convex polyhedral cone. Then the faces of (V, σ) are the same as the faces of the monoid $(\sigma, +)$.*

Proof. We may assume that $\sigma \neq \{0\}$, and let $v_1, \dots, v_n$ be a system of generators for (V, σ). Let F be a face of the polyhedral cone σ, and pick $u \in \sigma^\vee$ such that $F = \sigma \cap \mathrm{Ker}\, u$. Then $\sigma \setminus F = \{x \in \sigma \mid u(x) > 0\}$, and this is an ideal of the monoid $(\sigma, +)$; so F is a face of $(\sigma, +)$.

Conversely, suppose that F is a face of $(\sigma, +)$. First, let us show that F is a cone in V. Indeed, let $f \in F$, and $r > 0$ any real number; we have to prove that $r \cdot f \in F$. To this aim, it suffices to show that $(r/N) \cdot f \in F$ for some integer $N > 0$, so that, after replacing r by r/N for N large enough, we may assume that $0 < r < 1$. In this case, $(1-r) \cdot f \in \sigma$, and we have $f = r \cdot f + (1-r) \cdot f$, so that $r \cdot f \in F$, since F is a face of $(\sigma, +)$. Notice now that $S := \mathbb{R}_+ v_1 \cup \cdots \cup \mathbb{R}_+ v_n$ is a system of generators for $(\sigma, +)$; therefore $S \cap F$ is a system of generators for F (lemma 6.1.20(ii)), and it is a union of generating rays, since F is a cone. Thus, F is the polyhedral cone spanned by $\{v_1, \dots, v_n\} \cap F$.

Let $N := \sigma + (-F)$; it is a convex polyhedral cone, since the same holds for σ and F. Now, let $u_1, \dots, u_k \in N^\vee$ be a system of generators of the $\mathbb{R}$-subspace of $V^\vee$ spanned by $N^\vee$, and set $u := u_1 + \cdots + u_k$. Notice that $F \subset \operatorname{Ker} u$, since $F \cup (-F) \subset N$. Conversely, let $x \in \sigma \cap \operatorname{Ker} u$; then $-x \in \operatorname{Ker} u_i$ for every $i = 1, \dots, k$, and therefore $-x \in N^{\vee\vee} = N$ (lemmata 6.3.2 and 6.3.4(ii)). Hence, we may write $-x = m - f$ for some $m \in \sigma$ and $f \in F$, or equivalently, $f = m + x$, which shows that $x \in F$. Summing up, $F = \sigma \cap \operatorname{Ker} u$, *i.e.* F is a face of (V, σ). $\square$

Proposition 6.3.8. *Let (V, σ) be a convex polyhedral cone. The following holds :*

(i) *Any intersection of faces of σ is still a face of σ.*
(ii) *If τ is a face of σ, and γ is a face of (V, τ), then γ is a face of σ.*
(iii) *Every proper face of σ is the intersection of the facets that contain it.*

Proof. (i): Say that $\tau_i = \sigma \cap \operatorname{Ker} u_i$, where $u_1, \dots, u_n \in \sigma^\vee$. Then $\bigcap_{i=1}^n \tau_i = \sigma \cap \operatorname{Ker} \sum_{i=1}^n u_i$.

(ii): Say that $\tau = \sigma \cap \operatorname{Ker} u$ and $\gamma = \tau \cap \operatorname{Ker} v$, where $u \in \sigma^\vee$ and $v \in \tau^\vee$. Then, for large $r \in \mathbb{R}_+$, the linear form $v' := v + ru$ is non-negative on any given finite generating set of σ, hence it lies in $\sigma^\vee$, and $\gamma = \sigma \cap \operatorname{Ker} v'$.

(iii): To begin with, we prove the following :

Claim 6.3.9. (i) Every proper face of σ is contained in a facet.

(ii) Every face of σ of codimension 2 is the intersection of exactly two facets.

Proof of the claim. We may assume that $\langle \sigma \rangle = V$. Let τ be a face of σ of codimension at least two, and denote by $\overline{\sigma}$ be the image of σ in the quotient $\overline{V} := V/\langle \tau \rangle$; clearly $(\overline{V}, \overline{\sigma})$ is again a polyhedral cone. Moreover, choose $u \in \sigma^\vee$ such that $\tau = \sigma \cap \operatorname{Ker} u$; the linear form u descends to $\overline{u} \in \overline{\sigma}^\vee$, therefore $\overline{\sigma} \cap \operatorname{Ker} \overline{u} = \{0\}$ is a face of $\overline{\sigma}$.

(ii): If τ has codimension two, $\overline{V}$ has dimension two. Suppose that the assertion is known for $(\overline{V}, \overline{\sigma})$; then we find exactly two facets $\overline{\gamma}_1, \overline{\gamma}_2$ of $\overline{\sigma}$ whose intersection is $\{0\}$. Their preimages in V intersect σ in facets γ_1, γ_2 that satisfy $\gamma_1 \cap \gamma_2 = \tau$. Hence, we may assume from start that $\tau = \{0\}$ and V has dimension two, in which case the verification is easy, and shall be left to the reader.

(i): Arguing by induction on the codimension, it suffices to show that τ is contained in a proper face spanning a larger subspace. To this aim, suppose that the claim is known for $\overline{\sigma}$; since $\{0\}$ is a face of $\overline{\sigma}$ of codimension at least two, it is contained in a proper face $\overline{\gamma}$; the preimage γ of the latter intersects σ in a proper face containing τ. Thus again, we are reduced to the case where $\tau = \{0\}$. Pick $u_0 \in \sigma^\vee$ such that $\sigma \cap \operatorname{Ker} u_0 = \{0\}$; choose also any other $u_1 \in V^\vee$ such that $\sigma \cap \operatorname{Ker} u_1 \neq \{0\}$. Since $\dim_{\mathbb{R}} V^\vee \geq 2$, we may find a continuous map $f : [0,1] \to V^\vee \setminus \{0\}$ with $f(0) = u_0$ and $f(1) = u_1$. Let $\mathbb{P}_+(V)$ be the topological space of rays of V (*i.e.* the topological quotient $V \setminus \{0\}/\sim$ by the equivalence relation such that $v \sim v'$ if and only if v and v' generate the same ray), and define likewise $\mathbb{P}_+(V^\vee)$; let also $Z \subset P := \mathbb{P}_+(V) \times \mathbb{P}_+(V^\vee)$ be the incidence correspondence, *i.e.* the subset of all pairs $(\overline{v}, \overline{u})$ such that $u(v) = 0$, for any representative u of the class $\overline{u}$ and v of the class $\overline{v}$. Finally, let $\mathbb{P}_+(\sigma) \subset \mathbb{P}_+(V)$ be the image of $\sigma \setminus \{0\}$. Then Z (resp. $\mathbb{P}_+(\sigma)$) is a closed subset of P (resp. of $\mathbb{P}_+(V)$), hence $Y := Z \cap (\mathbb{P}_+(\sigma) \times \mathbb{P}_+(V^\vee))$ is a closed subset of P. Since the projection $\pi : P \to \mathbb{P}_+(V^\vee)$ is proper, $\pi(Y)$ is closed in $\mathbb{P}_+(V^\vee)$. Let $\overline{f} : [0,1] \to \mathbb{P}_+(V^\vee)$ be the composition of f and the natural projection $V^\vee \setminus \{0\} \to \mathbb{P}_+(V^\vee)$; then $f^{-1}(\pi(Y))$ is a closed subset of $[0,1]$, hence it admits a smallest element, say a (notice that $a > 0$). Moreover, $u_a \in \sigma^\vee$; indeed, otherwise we may find $v \in \sigma\setminus\{0\}$ such that $u_a(v) < 0$, and since $u_0(v) > 0$, we would have $u_b(v) = 0$ for some $b \in (0, a)$. The latter means that $f(b) \in \pi(Y)$, which contradicts the definition of a. Since by construction, $\sigma \cap \mathrm{Ker}\, u_a \neq \{0\}$, the claim follows. ◇

Let τ be any face of σ; to show that (iii) holds for τ, we argue by induction on the codimension of τ. The case of codimension 2 is covered by claim 6.3.9(ii). If τ has codimension > 2, we apply claim 6.3.9(i) to find a proper face γ containing τ; by induction, τ is the intersection of facets of γ, and each of these is the intersection of two facets in σ (again by claim 6.3.9(ii)), whence the contention. □

6.3.10. Suppose σ spans V (*i.e.* $\langle\sigma\rangle = V$) and let τ be a facet of σ; by definition there exists an element $u_\tau \in \sigma^\vee$ such that $\tau = \sigma \cap \mathrm{Ker}\, u_\tau$, and one sees easily that the ray $R_\tau := \mathbb{R}_+ \cdot u_\tau \subset \sigma^\vee$ is well-defined, independently of the choice of u_τ. Hence the half-space :

$$H_\tau := \{v \in V \mid u_\tau(v) \geq 0\}$$

depends only on τ. Recall that the *interior* (resp. the *topological closure*) of a subset $E \subset V$ is the largest open subset (resp. the smallest closed subset) of V contained in E (resp. containing E). The *topological boundary* of E is the intersection of the topological closures of E and of its complement $V\setminus E$.

Proposition 6.3.11. *Let (V, σ) be a convex polyhedral cone, such that σ spans V. We have:*

(i) *The topological boundary of σ is the union of its facets.*
(ii) *If $\sigma \neq V$, then $\sigma = \bigcap_{\tau\subset\sigma} H_\tau$, where τ ranges over the facets of σ.*
(iii) *The rays R_τ, where τ ranges over the facets of σ, generate the cone $\sigma^\vee$.*

Proof. (i): Notice that σ spans V if and only if the interior of σ is not empty. A proper face τ is the intersection of σ with a hyperplane $\mathrm{Ker}\, u \subset V$ with $u \in \sigma^\vee\setminus\{0\}$; therefore, every neighborhood $U \subset V$ of any point $v \in \tau$ intersects $V\setminus\sigma$. This shows that τ lies in the topological boundary of σ.

Conversely, if v is in the boundary of σ, choose a sequence $(v_i \mid i \in \mathbb{N})$ of points of $V\setminus\sigma$, converging to the point v; by lemma 6.3.2, for every $i \in \mathbb{N}$ there exists $u_i \in \sigma^\vee$ such that $u_i(v_i) < 0$. Up to multiplication by scalars, we may assume that the vectors u_i lie on some sphere in $V^\vee$ (choose any norm on $V^\vee$); hence we may find a convergent subsequence, and we may then assume that the sequence $(u_i \mid i \in \mathbb{N})$ converges to an element $u \in V^\vee$. Necessarily $u \in \sigma^\vee$, therefore v lies on the face $\sigma \cap \mathrm{Ker}\, u$, and the assertion follows from proposition 6.3.8(iii).

(ii): Suppose, by way of contradiction, that v lies in every half-space H_τ, but $v \notin \sigma$. Choose any point v' in the interior of σ, and let $t \in [0,1]$ be the largest value such that $w := tv + (1-t)v' \in \sigma$. Clearly w lies on the boundary of σ, hence on some facet τ, by (i). Say that $\tau = \sigma \cap \mathrm{Ker}\, u$; then $u(v') > 0$ and $u(w) = 0$, so $u(v) < 0$, a contradiction.

(iii): When $\sigma = V$, there is nothing to prove, hence we may assume that $\sigma \neq V$. In this case, suppose that $u \in \sigma^\vee$, and u is not in the cone C generated by the rays R_τ. Applying lemma 6.3.2 to the cone $(V^\vee, C)$, we deduce that there exists a vector $v \in V$ with $v \in H_\tau$ for all the facets τ of σ, and $u(v) < 0$, which contradicts (ii). □

Corollary 6.3.12. *Let (V, σ) and (V, σ') be two convex polyhedral cones. Then :*

(i) (Farkas' theorem) *The dual $(V^\vee, \sigma^\vee)$ is also a convex polyhedral cone.*
(ii) *If τ is a face of σ, then $\tau^* := \sigma^\vee \cap \tau^\perp$ is a face of $\sigma^\vee$ such that $\langle\tau^*\rangle = \langle\tau\rangle^\perp$. Especially:*

$$\dim_{\mathbb{R}}\langle\tau\rangle + \dim_{\mathbb{R}}\langle\tau^*\rangle = \dim_{\mathbb{R}} V. \qquad (6.3.13)$$

The rule $\tau \mapsto \tau^*$ is a bijection from the set of faces of σ to those of $\sigma^\vee$. The smallest face of σ is $\sigma \cap (-\sigma)$.

(iii) $(V, \sigma \cap \sigma')$ *is a convex polyhedral cone, and every face of* $\sigma \cap \sigma'$ *is of the form* $\tau \cap \tau'$*, for some faces* τ *of* σ *and* τ' *of* σ'.

Proof. (i): Set $W := \langle \sigma \rangle \subset V$, and pick a basis $u_1, \ldots, u_k$ of $W^\perp$; by proposition 6.3.11(iii), the assertion holds for the dual $(W^\vee, \sigma^\vee)$ of the cone (W, σ). However, $W^\vee \simeq V^\vee / W^\perp$, hence the dual cone $(V^\vee, \sigma^\vee)$ is generated by lifts of generators of $(W^\vee, \sigma^\vee)$, together with the vectors u_i and $-u_i$, for $i = 1, \ldots, k$.

(ii): Notice first that the faces of $\sigma^\vee$ are exactly the cones $\sigma^\vee \cap \{u\}^\perp$, for $u \in \sigma = (\sigma^\vee)^\vee$. For a given $v \in \sigma$, let τ be the smallest face of σ such that $v \in \tau$; this means that $\tau^\vee \cap \{v\}^\perp = \tau^\perp$ (where $(V^\vee, \tau^\vee)$ is the dual of (V, τ)). Hence $\sigma^\vee \cap \{v\}^\perp = \sigma^\vee \cap (\tau^\vee \cap \{v\}^\perp) = \tau^*$, so every face of $\sigma^\vee$ has the asserted form. The rule $\tau \mapsto \tau^*$ is clearly order-reversing, and from the obvious inclusion $\tau \subset (\tau^*)^*$ it follows that $\tau^* = ((\tau^*)^*)^*$, hence this map is a bijection. It follows from this, that the smallest face of σ is $(\sigma^\vee)^* = \sigma \cap (\sigma^\vee)^\perp = (\sigma^\vee)^\perp = \sigma \cap (-\sigma)$. In particular, we see that $(\sigma \cap (-\sigma))^* = \sigma^\vee$, and furthermore, (6.3.13) holds for $\tau := \sigma \cap (-\sigma)$. Identity (6.3.13) for a general face τ can be deduced by inserting τ in a maximal chain of faces of σ, and comparing with the dual chain of faces of $\sigma^\vee$ (details left to the reader). Finally, it is clear that $\langle \tau \rangle \subset \langle \tau^* \rangle^\perp$; since these spaces have the same dimension, we deduce that $\langle \tau \rangle^\perp = \langle \tau^* \rangle$.

(iii): Indeed, lemma 6.3.2 implies that $\sigma^\vee + \sigma'^\vee$ is the dual of $\sigma \cap \sigma'$, hence (i) implies that $\sigma \cap \sigma'$ is polyhedral. It also follows that every face τ of $\sigma \cap \sigma'$ is the intersection of $\sigma \cap \sigma'$ with the kernel of a linear form $u + u'$, for some $u \in \sigma^\vee$ and $u' \in \sigma'^\vee$. Consequently, $\tau = (\sigma \cap \operatorname{Ker} u) \cap (\sigma' \cap \operatorname{Ker} u')$. □

Corollary 6.3.14. *For a convex polyhedral cone* (V, σ)*, the following conditions are equivalent:*

(a) $\sigma \cap (-\sigma) = \{0\}$.
(b) σ *contains no non-zero linear subspaces.*
(c) *There exists* $u \in \sigma^\vee$ *such that* $\sigma \cap \operatorname{Ker} u = \{0\}$.
(d) $\sigma^\vee$ *spans* $V^\vee$.

Proof. (a) ⇔ (b) since $\sigma \cap (-\sigma)$ is the largest subspace contained in σ. Next, (a) ⇔ (c) since $\sigma \cap (-\sigma)$ is the smallest face of σ. Finally, (a) ⇔ (d) since $\dim_{\mathbb{R}} \langle \sigma \cap (-\sigma) \rangle + \dim_{\mathbb{R}} \langle \sigma^\vee \rangle = n$ (corollary 6.3.12(ii)). □

6.3.15. A convex polyhedral cone fulfilling the equivalent conditions of corollary 6.3.14 is said to be *strictly convex*. Suppose that (V, σ) is strictly convex; then proposition 6.3.11(iii) says that σ is generated by the rays R_τ, where τ ranges over the facets of $\sigma^\vee$. The rays R_τ are uniquely determined by σ, and are called the *extremal rays* of σ. Moreover, these R_τ form *the unique minimal set of generating rays* for σ. Indeed, concerning the minimality : for each facet τ of $\sigma^\vee$, pick $v_\tau \in \sigma$ with $\mathbb{R}_+ \cdot v_\tau = R_\tau$; suppose that $v_{\tau_0} = \sum_{\tau \in S} t_\tau \cdot v_\tau$ for some subset S of the set of facets of $\sigma^\vee$, and appropriate $t_\tau > 0$, for every $\tau \in S$. It follows easily that $u(v_\tau) = 0$ for every $u \in \tau_0$, and every $\tau \in S$. But by definition of R_τ, this implies that $S = \{\tau_0\}$, which is the claim. Concerning uniqueness : suppose that Σ is another system of generating rays; especially, for any facet $\tau \subset \sigma^\vee$, the ray R_τ is in the convex cone generated by Σ; it follows easily that there exists $\rho \in \Sigma$ such that $u(\rho) = 0$ for every $u \in \tau$, in which case $\rho = R_\tau$. This shows that Σ must contain all the extremal rays.

Example 6.3.16. (i) Suppose that $\dim_{\mathbb{R}} V = 2$, and (V, σ) is a strictly convex polyhedral cone, and assume that σ generates V. Then the only face of codimension two of σ is $\{0\}$, so it follows easily from claim 6.3.9(ii) that σ admits exactly two facets, and these are also the extremal rays of σ, especially σ is simplicial. Of course, these assertions are rather obvious; in dimension > 2, the general situation is much more complicated.

(ii) Let (V, σ) be a convex polyhedral cone, and suppose that σ spans V. Let τ be a face of σ. Notice that $(\sigma, +)^{\mathrm{gp}} = (V, +)$, and $(\tau, +)$ is a face of the monoid $(\sigma, +)$, by lemma

6.3.7. Hence we may view the localization $\tau^{-1}\sigma$ naturally as a submonoid of $(V,+)$, and it is easily seen that $\tau^{-1}\sigma$ is a convex cone. By proposition 6.3.11(iii), the polyhedral cone $\sigma^\vee$ is generated by its extremal rays $\mathbb{R}l_1,\dots,\mathbb{R}l_n$, and by proposition 6.3.8(iii), we may assume that $\tau=\sigma\cap\mathrm{Ker}\,(l_1+\cdots+l_k)$ for some $k\leq n$. Clearly $l_i(v)\geq 0$ for every $v\in\tau^{-1}\sigma$ and every $i\leq k$. Conversely, if $l\in(\tau^{-1}\sigma)^\vee$, we must have $\tau\subset\mathrm{Ker}\,l$ and $l\in\sigma^\vee$; if we write $l=\sum_{i=1}^n a_il_i$ for some $a_i\geq 0$, it follows easily that $a_i=0$ for every $i>k$. On the other hand, suppose that $v\in V$ satisfies the inequalities $l_i(v)\geq 0$ for $i=1,\dots,k$; then, for every $i=k+1,\dots,n$ we may find $u_i\in\tau$ such that $l_i(v+u_i)\geq 0$, hence $v+u_{k+1}+\cdots+u_n\in\sigma$, and therefore $v\in\tau^{-1}\sigma$. This shows that $\tau^{-1}\sigma$ is a closed convex cone, and its dual $(\tau^{-1}\sigma)^\vee$ is the convex cone generated by $l_1,\dots,l_k$; especially, it is a convex polyhedral cone, and then the same holds for $\tau^{-1}\sigma$, by virtue of lemma 6.3.2 and corollary 6.3.12(i).

(iii) In the situation of (ii), let $v\in\tau$ be any element that lies in the *relative interior* of τ, *i.e.* in the complement of the union of the facets of τ. Denote by $S_v\subset\tau$ the submonoid generated by v. Then we claim that

$$\tau^{-1}\sigma=S_v^{-1}\sigma.$$

Indeed, let $s\in\sigma$ and $t\in\tau$ be any two element; in view of proposition 6.3.11(i), it is easily seen that there exists an integer $N>0$ such that $v-N^{-1}t\in\tau$, hence $t':=Nv-t\in\tau$. Therefore $s-t=(s+t')-Nv\in S^{-1}\sigma$, and the assertion follows.

Lemma 6.3.17. *Let $f:V\to W$ be a linear map of finite dimensional $\mathbb{R}$-vector spaces, (V,σ) a convex polyhedral cone. The following holds :*

(i) *$(W,f(\sigma))$ is a convex polyhedral cone.*

(ii) *Suppose moreover, that $\sigma\cap\mathrm{Ker}\,f$ does not contain non-zero linear subspaces of V. Then, for every face τ of $f(\sigma)$ there exists a face τ' of σ such that $f(\tau')\subset\tau$, and f restricts to an isomorphism : $\langle\tau'\rangle\xrightarrow{\sim}\langle\tau\rangle$.*

Proof. (i) is obvious. To show (ii) we argue by induction on $n:=\dim_{\mathbb{R}}\mathrm{Ker}\,f$. The assertion is obvious when $n=0$, hence suppose that $n>0$ and that the claim is already known whenever $\mathrm{Ker}\,f$ has dimension $<n$. Let τ be a face of $f(\sigma)$; then $f^{-1}\tau$ is a face of $f^{-1}f(\sigma)$, hence $\sigma\cap f^{-1}\tau$ is a face of $\sigma=\sigma\cap f^{-1}f(\sigma)$. In view of proposition 6.3.8(ii), we may then replace σ by $\sigma\cap f^{-1}\tau$, and therefore assume from start that $\tau=f(\sigma)$. We may as well assume that $V=\langle\sigma\rangle$ and $W=\langle\tau\rangle$. The assumption on σ implies especially that $\sigma\neq V$, hence σ is the intersection of the half-spaces H_γ corresponding to its facets γ (proposition 6.3.11(ii)). For each facet γ of σ, let u_γ be a chosen generator of the ray R_γ (notation of (6.3.10)). Since $\sigma\cap\mathrm{Ker}\,f$ does not contain non-zero linear subspaces, we may find a facet γ such that $V':=\mathrm{Ker}\,u_\gamma$ does not contain $\mathrm{Ker}\,f$. Then, the inductive assumption applies to the restriction $f_{|V'}:V'\to W$ and the convex polyhedral cone $\sigma\cap V'$, and yields a face τ' of the latter, such that f induces an isomorphism $\langle\tau'\rangle\xrightarrow{\sim}\langle f(\sigma\cap V')\rangle$. Finally, $\langle f(\sigma\cap V')\rangle=W$, since V' does not contain $\mathrm{Ker}\,f$. □

Lemma 6.3.18. *Let $f:V\to W$ be a linear map of finite dimensional $\mathbb{R}$-vector spaces, $f^\vee$: $W^\vee\to V^\vee$ the transpose of f, and (W,σ) a convex polyhedral cone. Then :*

(i) *$(V,f^{-1}\sigma)$ is a convex polyhedral cone and $(f^{-1}\sigma)^\vee=f^\vee(\sigma^\vee)$.*

(ii) *For every face δ of $f^{-1}\sigma$, there exists a face τ of σ such that $\delta=f^{-1}\tau$. If furthermore, $\langle\sigma\rangle+f(V)=W$, we may find such a τ so that additionally, f induces an isomorphism:*

$$V/\langle\delta\rangle\xrightarrow{\sim}W/\langle\tau\rangle.$$

(iii) *Conversely, for every face τ of σ, the cone $f^{-1}\tau$ is a face of $f^{-1}\sigma$, and $(f^{-1}\tau)^*$ is the smallest face of $(f^{-1}\sigma)^\vee$ containing $f^\vee(\tau^*)$ (notation of corollary* 6.3.12(ii)*).*

Proof. (i): By corollary 6.3.12(i), $(W^\vee,\sigma^\vee)$ is a convex polyhedral cone, hence we may find $u_1,\dots,u_s\in\sigma^\vee$ such that $\sigma=\bigcap_{i=1}^s u_i^{-1}(\mathbb{R}_+)$. Therefore $f^{-1}\sigma=\bigcap_{i=1}^s(u_i\circ f)^{-1}(\mathbb{R}_+)$. Let

$\gamma \subset V^\vee$ be the cone generated by the set $\{u_1 \circ f, \dots, u_s \circ f\}$; then $f^{-1}\sigma = \gamma^\vee$, and the assertion results from lemma 6.3.2 and a second application of (i).

(iii): For every $u \in \sigma^\vee$, we have : $f^{-1}(\sigma \cap \operatorname{Ker} u) = (f^{-1}\sigma) \cap \operatorname{Ker} u \circ f$. Since we already know that $(f^{-1}\sigma)^\vee = f^\vee(\sigma^\vee)$, we see that the faces of $f^{-1}\sigma$ are exactly the subsets of the form $f^{-1}\tau$, where τ ranges over the faces of σ. Next, for any such τ, the set $f^\vee(\tau^*)$ consists of all $u \in V^\vee$ of the form $u = w \circ f$ for some $w \in \sigma^\vee$ such that $w(\tau) = 0$. From this description it is clear that $f^\vee(\tau^*) \subset (f^{-1}\tau)^*$. To show that $(f^{-1}\tau)^*$ is the smallest face containing $f^\vee(\tau^*)$, it then suffices to prove that $f^\vee(\tau^*)^\perp \cap f^{-1}\sigma \subset f^{-1}\tau$. However, let $v \in f^\vee(\tau^*)^\perp \cap f^{-1}\sigma$; then $w \circ f(v) = 0$ for every $w \in \tau^*$, *i.e.* $f(v) \in (\tau^*)^* = \tau$, whence the contention.

(ii): The first assertion has already been shown; hence, suppose that $\langle\sigma\rangle + f(V) = W$. We deduce that $\sigma^\vee \cap \operatorname{Ker} f^\vee$ does not contain non-zero linear subspaces of $W^\vee$; indeed, if $u \in W^\vee$, and both u and $-u$ lie in $\sigma^\vee$, then u vanishes on $\langle\sigma\rangle$, and if $u \in \operatorname{Ker} f^\vee$, then u vanishes as well on $f(V)$, hence $u = 0$. We may then apply lemma 6.3.17(ii) to find a face γ of $\sigma^\vee$ such that $f^\vee(\gamma) \subset \delta^*$ and $f^\vee$ restricts to an isomorphism : $\langle\gamma\rangle \xrightarrow{\sim} \langle\delta^*\rangle$. Especially, δ^* is the smallest face of $(f^{-1}\sigma)^\vee$ containing $f^\vee(\gamma)$, hence $\delta^* = (f^{-1}\gamma^*)^*$ by (iii), *i.e.* $\delta = f^{-1}\gamma^*$. We also deduce that f induces an isomorphism : $V/\langle\delta^*\rangle^\perp \xrightarrow{\sim} W/\langle\gamma\rangle^\perp$. Since $\langle\delta^*\rangle^\perp = \langle\delta\rangle$ and $\langle\gamma\rangle^\perp = \langle\gamma^*\rangle$, the second assertion holds with $\tau := \gamma^*$. □

Lemma 6.3.19. *Let (V, σ) and (V', σ') be two convex polyhedral cones. Then :*

(i) $(V \oplus V', \sigma \times \sigma')$ *is a convex polyhedral cone.*
(ii) *Every face of $\sigma \times \sigma'$ is of the form $\tau \times \tau'$, for some faces τ of σ and τ' of σ'.*

Proof. Indeed, $\sigma \times \sigma' = (p_1^{-1}\sigma) \cap (p_2^{-1}\sigma')$, where p_1 and p_2 are the natural projections of $V \oplus V'$ onto V and V'. Hence, assertions (i) and (ii) follow from corollary 6.3.12(iii) and lemma 6.3.18(i). □

6.3.20. Let $(L, +)$ be a free abelian group of finite rank, $\sigma \subset L_\mathbb{R} := L \otimes_\mathbb{Z} \mathbb{R}$ a convex polyhedral cone. We say that σ is *L-rational* (or briefly : *rational*, when there is no danger of ambiguity) if it admits a generating set consisting of elements of L. Then it is clear that every face of a rational convex polyhedral cone is again rational (see (6.3.6)). On the other hand, let

$$(M, +) \subset (L, +)$$

be a submonoid of L; we shall denote by $(L_\mathbb{R}, M_\mathbb{R})$ the convex cone generated by M (*i.e.* the smallest convex cone in $L_\mathbb{R}$ containing the image of M). If M is fine, $M_\mathbb{R}$ is a convex polyhedral cone. Later we shall also find useful to consider the subset :

$$M_\mathbb{Q} := \{m \otimes q \mid m \in M, q \in \mathbb{Q}_+\} \subset L_\mathbb{Q} := L \otimes_\mathbb{Z} \mathbb{Q}$$

which is a submonoid of $L_\mathbb{Q}$.

Proposition 6.3.21. *Let $(L, +)$ be a free abelian group of finite rank, with dual*

$$L^\vee := \operatorname{Hom}_\mathbb{Z}(L, \mathbb{Z}).$$

Let also $(L_\mathbb{R}, \sigma)$ and $(L_\mathbb{R}, \sigma')$ be two L-rational convex polyhedral cones. We have :

(i) *The dual $(L^\vee_\mathbb{R}, \sigma^\vee)$ is an $L^\vee$-rational convex polyhedral cone.*
(ii) $(L_\mathbb{R}, \sigma \cap \sigma')$ *is also an L-rational convex polyhedral cone.*
(iii) *Let $g : L' \to L$ (resp. $h : L \to L'$) be a map of free abelian groups of finite rank, and denote by $g_\mathbb{R} : L'_\mathbb{R} \to L_\mathbb{R}$ (resp. $h_\mathbb{R} : L_\mathbb{R} \to L'_\mathbb{R}$) the induced $\mathbb{R}$-linear map. Then, $(L'_\mathbb{R}, g_\mathbb{R}^{-1}\sigma)$ and $(L'_\mathbb{R}, h_\mathbb{R}(\sigma))$ are L'-rational.*
(iv) *Let L' be another free abelian group of finite rank, and $(L'_\mathbb{R}, \sigma')$ an L'-rational convex polyhedral cone. Then $(L_\mathbb{R} \oplus L'_\mathbb{R}, \sigma \times \sigma')$ is $L \oplus L'$-rational.*

Proof. (i) and (ii) follow easily, by inspecting the proof of corollary 6.3.12(i),(iii) : the details shall be left to the reader.

(iii): The assertion concerning $h_\mathbb{R}(\sigma)$ is obvious. To show the assertion for $g_\mathbb{R}^{-1}\sigma$, one argues as in the proof of lemma 6.3.18(i) : by (i), we may find $u_1, \dots, u_s \in L^\vee$ such that $\sigma = \bigcap_{i=1}^s u_{i,\mathbb{R}}^{-1}(\mathbb{R}_+)$. Therefore $g_\mathbb{R}^{-1}\sigma = \bigcap_{i=1}^s (u_i \circ g)_\mathbb{R}^{-1}(\mathbb{R}_+)$. Let $\gamma \subset V^\vee$ be the cone generated by the set $\{(u_1 \circ g)_\mathbb{R}, \dots, (u_s \circ g)_\mathbb{R}\}$; then $g_\mathbb{R}^{-1}\sigma = \gamma^\vee$, and the assertion results from lemma 6.3.2 and a second application of (i).

Lastly, arguing as in the proof of lemma 6.3.19(i), one derives (iii) from (ii) and (iii). □

Parts (i) and (iii) of the following proposition provide the bridge connecting convex polyhedral cones to fine monoids.

Proposition 6.3.22. *Let $(L, +)$ be a free abelian group of finite rank, $(L_\mathbb{R}, \sigma)$ an L-rational convex polyhedral cone, and set $\sigma_L := L \cap \sigma$. Then :*

(i) (Gordan's lemma) *σ_L is a fine and saturated submonoid of L, and $L \cap \langle\sigma\rangle = \sigma_L^{\mathrm{gp}}$.*

(ii) *For every $v \in L_\mathbb{R}$, the subset $L \cap (\sigma - v)$ is a finitely generated σ_L-module.*

(iii) *For any submonoid $M \subset L$, we have : $M_\mathbb{Q} = M_\mathbb{R} \cap L_\mathbb{Q}$ and $M^{\mathrm{sat}} = M_\mathbb{R} \cap L$.*

Proof. (Here $\sigma - v \subset L_\mathbb{R}$ denotes the translate of σ by the vector $-v$, *i.e.* the subset of all $w \in L_\mathbb{R}$ such that $w + v \in \sigma$.) Choose $v_1, \dots, v_s \in L$ that generate σ, and set

$$C_\varepsilon := \left\{ \sum_{i=1}^s t_i v_i \mid t_i \in [0, \varepsilon] \ \text{ for } i = 1, \dots, s \right\} \qquad \text{for every } \varepsilon > 0.$$

(i): Clearly $L \cap \sigma$ is saturated. Since C_1 is compact and L is discrete, $C_1 \cap L$ is a finite set. We claim that $C_1 \cap L$ generates the monoid σ_L. Indeed, if $v \in \sigma_L$, write $v = \sum_{i=1}^s r_i v_i$, with $r_i \geq 0$ for every $i = 1, \dots, s$; hence $r_i = m_i + t_i$ for some $m_i \in \mathbb{N}$ and $t_i \in [0, 1[$, and therefore $v = v' + \sum_{i=1}^s m_i v_i$, where $v', v_1, \dots, v_s \in C_1 \cap L$. Next, it is clear that $\sigma_L^{\mathrm{gp}} \subset L \cap \langle\sigma\rangle$; for the converse, say that $w \in L \cap \langle\sigma\rangle$, and write $w = w_1 - w_2$, for some $w_1, w_2 \in \sigma$. Then $w_1 = \sum_{i=1}^s t_i v_i$ for some $t_i \geq 0$; we pick $t'_i \in \mathbb{N}$ such that $t'_i \geq t_i$ for every $i \leq s$, and we set $w'_i := \sum_{i=1}^s t'_i v_i$. It follows that $w = w'_1 - w'_2$, where $w'_2 := w_2 + (w'_1 - w_1))$, and notice that $w'_1 \in \sigma_L$ and $w'_2 \in \sigma$; then we must have $w'_2 \in \sigma_L$ as well, and therefore $w \in \sigma_L^{\mathrm{gp}}$.

(ii) is similar : from the compactness of C_1 one sees that $L \cap (C_1 - v)$ is a finite set; on the other hand, arguing as in the proof of (i), one checks easily that the latter set generates the σ_L-module $L \cap (\sigma - v)$.

(iii): Let $x \in M_\mathbb{R} \cap L_\mathbb{Q}$; then we may write

$$x = \sum_{i=1}^n r_i \otimes m_i \qquad \text{where } m_i \in M, r_i > 0 \text{ for every } i \leq n. \tag{6.3.23}$$

Claim 6.3.24. In the situation of proposition 6.3.22(iii), let $x \in M_\mathbb{R} \cap L_\mathbb{Q}$, and write x as in (6.3.23). Then, for every $\varepsilon > 0$ there exist $q_1, \dots, q_n \in \mathbb{Q}_+$ with $|r_i - q_i| < \varepsilon$ for every $i = 1, \dots, n$, and such that $x = \sum_{i=1}^n q_i \otimes m_i$.

Proof of the claim. Up to a reordering, we may assume that $m_1, \dots, m_k$ form a basis of the $\mathbb{Q}$-vector space generated by $m_1, \dots, m_n$, therefore $m_{k+i} = \sum_{j=1}^k q_{ij} m_j$ for a matrix

$$A := (q_{ij} \mid i = 1, \dots, n-k; \ j = 1, \dots, k)$$

with entries in $\mathbb{Q}$. Let $\underline{r} := (r_1, \dots, r_k)$ and $\underline{r}' := (r_{k+1}, \dots, r_n)$; since $x \in L_\mathbb{Q}$, we deduce that $\underline{b} := \underline{r} + \underline{r}' \cdot A \in \mathbb{Q}^{\oplus k}$. Moreover, if $\underline{s} := (s_1, \dots, s_k) \in \mathbb{Q}^{\oplus k}$ and $\underline{s}' := (s_{k+1}, \dots, s_n) \in \mathbb{Q}^{\oplus n-k}$ satisfy the identity $\underline{b} = \underline{s} + \underline{s}' \cdot A$, then $\sum_{i=1}^n s_i \otimes m_i = x$. If we choose $\underline{s}'$ very close to $\underline{r}'$, then $\underline{s}$ shall be very close to $\underline{r}$; especially, we can achieve that both $\underline{s}$ and $\underline{s}'$ are vectors with positive coordinates. ◊

Claim 6.3.24 shows that $x \in M_{\mathbb{Q}}$, whence the first stated identity; for the second identity, we are reduced to showing that $M^{\mathrm{sat}} = M_{\mathbb{Q}} \cap L$, which is immediate. □

For various algebraic and geometric applications of the theory of polyhedral cones, one is led to study subdivisions of a given cone, in the sense of the following definition 6.3.25. Later we shall see a more abstract notion of subdivision, in the context of general fans, which however finds its roots and motivation in the intuitive manipulations of polyhedra that we formalize hereafter.

Definition 6.3.25. Let V be a finite dimensional $\mathbb{R}$-vector space.

(i) A *fan* in V is a finite set Δ consisting of convex polyhedral cones of V, such that :
- for every $\sigma \in \Delta$, and every face τ of σ, also $\tau \in \Delta$;
- for every $\sigma, \tau \in \Delta$, the intersection $\sigma \cap \tau$ is also an element of Δ, and is a face of both σ and τ.

(ii) We say that Δ is a *simplicial fan* if all the elements of Δ are simplicial cones.

(iii) Suppose that $V = L \otimes_{\mathbb{Z}} \mathbb{R}$ for some free abelian group L; then we say that Δ is L*-rational* if the same holds for every $\tau \in \Delta$.

(iv) A *refinement* of the fan Δ is a fan Δ' in V with $\bigcup_{\sigma\in\Delta} \sigma = \bigcup_{\tau\in\Delta'} \tau$, and such that every $\tau \in \Delta$ is the union of the $\gamma \in \Delta'$ contained in τ.

(v) A *subdivision* of a convex polyhedral cone (V, σ) is a refinement of the fan Δ_σ consisting of σ and its faces.

Lemma 6.3.26. *Let (V, σ) be any convex polyhedral cone, Δ a subdivision of (V, σ). We let*

$$\Delta^s := \{\tau \in \Delta \mid \langle\tau\rangle = \langle\sigma\rangle\}.$$

Then $\bigcup_{\tau\in\Delta^s} \tau = \sigma$.

Proof. Let $\tau_0 \in \Delta$ be any element. Then $\sigma' := \bigcup_{\tau\neq\tau_0} \tau$ is a closed subset of σ. If $\langle\tau_0\rangle \neq \langle\sigma\rangle$, then $\sigma \setminus \tau_0$ is a dense open subset of σ contained in σ'; it follows that $\sigma' = \sigma$ in this case. Especially, $\tau_0 = \bigcup_{\tau\neq\tau_0}(\tau \cap \tau_0)$; since each $\tau \cap \tau_0$ is a face of τ_0, we see that there must exist $\tau \neq \tau_0$ such that τ_0 is a face of τ. The lemma follows immediately. □

Example 6.3.27. (i) Certain useful subdivisions of a polyhedral cone σ are produced by means of auxiliary real-valued functions defined on σ. Namely, let us say that a continuous function $f : \sigma \to \mathbb{R}$ is a *roof*, if the following holds. There exists a non-empty finite set of $\mathbb{R}$-linear forms $\{l_1, \dots, l_n\}$ on V, such that $f(v) = \min(l_1(v), \dots, l_n(v))$ for every $v \in \sigma$. The concept of roof shall be reintroduced in section 6.6, in a more abstract and general guise; however, in order to grasp the latter, it is useful to keep in mind its more concrete polyhedral incarnation. We attach to f a subdivision of σ, as follows. For every $i, j = 1, \dots, n$ define $l_{ij} := l_i - l_j$, and let $\tau_i \subset V^\vee$ be the polyhedral cone $\sigma^\vee + \mathbb{R}l_{i1} + \cdots + \mathbb{R}l_{in}$. From the identity $l_{ik} = l_{ij} + l_{jk}$ we easily deduce that $\tau_i^\vee \cap \tau_j^\vee$ is a face of both $\tau_i^\vee$ and $\tau_j^\vee$, for every $i, j = 1, \dots, n$. Denote by Θ the smallest subdivision of σ containing all the $\tau_i^\vee$; it is easily seen that

$$\sigma = \bigcup_{i=1}^{n} \tau_i^\vee$$

and the restriction of f to each $\tau_i^\vee$ agrees with l_i.

(ii) Conversely, let $f : \sigma \to \mathbb{R}$ a continuous function; suppose there exist a subdivision Θ of σ, and a system $(l_\tau \mid \tau \in \Theta)$ of $\mathbb{R}$-linear forms on V such that

- $f(v) = l_\tau(v)$ for every $\tau \in \Theta$ and every $v \in \tau$.
- $f(u + v) \geq f(u) + f(v)$ for every $u, v \in \sigma$.

Then we claim that f is a roof on σ. Indeed, let $\Theta^s \subset \Theta$ be the subset of all τ that span $\langle\sigma\rangle$. Notice fist that the system $(l_\tau \mid \tau \in \Theta^s)$ already determines f uniquely, by virtue of lemma 6.3.26. Next, let $\tau, \tau' \in \Theta^s$ be any two elements, and pick an element v of the interior of τ. For any $u \in \tau'$ and any $\varepsilon > 0$ we have, by assumption : $f(v + \varepsilon u) \geq f(v) + f(\varepsilon u)$. If ε is small enough, we have as well $v + \varepsilon u \in \tau$, in which case the foregoing inequality can be written as :

$$l_\tau(v) + \varepsilon \cdot l_\tau(u) = l_\tau(v + \varepsilon u) \geq l_\tau(v) + \varepsilon \cdot l_{\tau'}(u)$$

whence $l_\tau(u) \geq l_{\tau'}(u) = f(u)$ and the assertion follows.

Proposition 6.3.28. *Let $f : V \to W$ be a linear map of finite dimensional $\mathbb{R}$-vector spaces, (V, σ) a convex polyhedral cone, and $h : \sigma \to f(\sigma)$ the restriction of f. Then :*

(i) *There exists a subdivision Δ of $(W, f(\sigma))$ such that :*

$$h^{-1}(a + b) = h^{-1}(a) + h^{-1}(b) \qquad \text{for every } \tau \in \Delta \text{ and every } a, b \in \tau.$$

(ii) *Suppose moreover that $V = L \otimes_{\mathbb{Z}} \mathbb{R}$, $W = L' \otimes_{\mathbb{Z}} \mathbb{R}$ and $f = g \otimes_{\mathbb{Z}} \mathbf{1}_{\mathbb{R}}$ for a map $g : L \to L'$ of free abelian groups. If σ is L-rational, then we may find an L-rational subdivision Δ such that* (i) *holds.*

Proof. Let V_0 be the largest linear subspace contained in $\sigma \cap \mathrm{Ker}\, f$. Notice that, under the assumptions of (ii), we have : $V_0 = \mathbb{R} \otimes_{\mathbb{Z}} \mathrm{Ker}\, g$. One verifies easily that the proposition holds for the given map f and the cone (V, σ), if and only if it holds for the induced map $\overline{f} : V/V_0 \to W/f(V_0)$ and the cone $(V/V_0, \overline{\sigma})$ (where $\overline{\sigma}$ is the image of σ in V/V_0). Hence, we may replace f by $\overline{f}$, and assume from start that $\sigma \cap \mathrm{Ker}\, f$ contains no non-zero linear subspaces. Moreover, we may assume that σ spans V and $f(\sigma)$ spans W.

(i): Let S be the set of faces τ of σ such that f restricts to an isomorphism $\langle\tau\rangle \xrightarrow{\sim} W$.

Claim 6.3.29. Let $\lambda \subset f(\sigma)$ be any ray. Then :

(i) $\lambda' := h^{-1}\lambda$ is a strictly convex polyhedral cone. Especially, λ' is generated by its extremal rays (see (6.3.15)).

(ii) For every extremal ray ρ of λ' with $\rho \not\subset \mathrm{Ker}\, f$, there exists $\tau \in S$ such that $\rho = \tau \cap f^{-1}(\lambda)$.

Proof of the claim. λ' is a convex polyhedral cone by lemma 6.3.18(i) and corollary 6.3.12(iii). To see that λ' is strictly convex, notice that any subspace $L \subset f^{-1}(\lambda)$ lies already in $\mathrm{Ker}\, f$, and if $L \subset \sigma$, we must have $L = \{0\}$ by assumption. Let ρ be an extremal ray of λ' which is not contained in $\mathrm{Ker}\, f$; notice that λ' is the intersection of the polyhedral cones $\lambda_1 := h^{-1}\langle\lambda\rangle$ and $\lambda_2 := f^{-1}\lambda$, hence we can find faces δ_i of λ_i $(i = 1, 2)$ such that $\rho = \delta_1 \cap \delta_2$ (corollary 6.3.12(iii) and lemma 6.3.18(i)). However, the only proper face of λ_2 is $\mathrm{Ker}\, f$ (lemma 6.3.18(ii)), hence $\delta_2 = \lambda_2$. Likewise, $f^{-1}\langle\lambda\rangle$ has no proper faces, hence $\delta_1 = \gamma \cap f^{-1}\langle\lambda\rangle$ for some face γ of σ (again by corollary 6.3.12(iii)). Since λ_2 is a half-space in $f^{-1}\langle\lambda\rangle$, we deduce easily that either $\delta_1 = \rho$ or $\delta_1 = \langle\rho\rangle$. Especially, $\dim_{\mathbb{R}}(f^{-1}\langle\lambda\rangle)/\langle\delta_1\rangle = \dim_{\mathbb{R}} \mathrm{Ker}\, f$. We may then apply lemma 6.3.18(ii) to the embedding $f^{-1}\langle\lambda\rangle \subset V$, to find a face τ of σ such that :

$$\tau \cap f^{-1}\langle\lambda\rangle = \delta_1 \qquad \langle\tau\rangle \cap f^{-1}\langle\lambda\rangle = \langle\rho\rangle \qquad \dim_{\mathbb{R}} V/\langle\tau\rangle = \dim_{\mathbb{R}} \mathrm{Ker}\, f.$$

It follows that $\langle\tau\rangle \cap \mathrm{Ker}\, f = \{0\}$, and therefore $\tau \in S$, as required. ◊

We construct as follows a subdivision of $(W, f(\sigma))$. For every $\tau \in S$, let $F(\tau)$ be the set consisting of the facets of the polyhedral cone $f(\tau)$; set also $F := \bigcup_{\tau \in S} F(\tau)$. Notice that, for every $\gamma \in F$, the subspace $\langle\gamma\rangle$ is a hyperplane of W; we let :

$$U := f(\sigma) \setminus \bigcup_{\gamma \in F} \langle\gamma\rangle.$$

Then U is an open subset of $f(\sigma)$, and the topological closure $\overline{C}$ of every connected component C of U is a convex polyhedral cone. Moreover, if C and D are any two such connected components, the intersection $\overline{C} \cap \overline{D}$ is a face of both $\overline{C}$ and $\overline{D}$. We let Δ be the subdivision of $f(\sigma)$ consisting of the cones $\overline{C}$ – where C ranges over all the connected components of U – together with all their faces.

Claim 6.3.30. For every $\delta \in \Delta$ and every $\tau \in S$, the intersection $\delta \cap f(\tau)$ is a face of δ.

Proof of the claim. Due to proposition 6.3.8(iii), we may assume that δ is the topological closure of a connected component C of U. We may also assume that $f(\tau) \neq W$, otherwise there is nothing to prove; in that case, we have $f(\tau) = \bigcap_{\gamma \in F(\tau)} H_\gamma$, where, for each $\gamma \in F(\tau)$, the half-space H_γ is the unique one that contains both $f(\tau)$ and γ (proposition 6.3.11(ii)). It then suffices to show that $\delta \cap H_\gamma$ is a face of δ for each such H_γ. We may assume that $\delta \not\subset H_\gamma$. Since C is connected and $C \subset W \setminus \langle\gamma\rangle$, it follows that $\delta \subset -H_\gamma$, the topological closure of the complement of H_γ. Hence $(-H_\gamma)^\vee \subset \delta^\vee$ (where $(-H_\gamma)^\vee$ is the dual of the polyhedral cone $(W, -H_\gamma)$), and therefore $\delta \cap H_\gamma = \delta \cap H_\gamma \cap (-H_\gamma) = \delta \cap \langle\gamma\rangle$ is indeed a face of δ. ◊

Next, for every $w \in f(\sigma)$, let $I(w) := \{\tau \in S \mid w \in f(\tau)\}$.

Claim 6.3.31. Let $\delta \in \Delta$, and $w_1, w_2 \in \delta$. Then $I(w_1 + w_2) \subset I(w_1) \cap I(w_2)$.

Proof of the claim. Suppose first that $w_1 + w_2$ is contained in a face δ' of δ; say that $\delta' = \delta \cap \mathrm{Ker}\, u$, for some $u \in \delta^\vee$. This means that $u(w_1 + w_2) = 0$, hence $u(w_1) = u(w_2) = 0$, *i.e.* $w_1, w_2 \in \delta'$. Hence, we may replace δ by δ', and assume that δ is the smallest element of Δ containing $w_1 + w_2$. Thus, suppose that $\tau \in I(w_1 + w_2)$; therefore $w_1 + w_2 \in f(\tau) \cap \delta$. From claim 6.3.30 we deduce that $\delta \subset f(\tau)$, hence $\tau \in I(w_1) \cap I(w_2)$, as claimed. ◊

Finally, we are ready to prove assertion (i). Hence, let $a, b \in f(\sigma)$ be any two vectors that lie in the same element of Δ. Clearly :

$$h^{-1}(a) + h^{-1}(b) \subset h^{-1}(a + b)$$

hence it suffices to show the converse inclusion. However, directly from claim 6.3.29(ii) we derive the identity :

$$h^{-1}(\mathbb{R}_+ \cdot w) = (\sigma \cap \mathrm{Ker}\, f) + \sum_{\tau \in I(w)} (\tau \cap f^{-1}(\mathbb{R}_+ \cdot w)) \qquad \text{for every } w \in f(\sigma).$$

Taking into account claim 6.3.31, we are then reduced to showing that :

$$\tau \cap f^{-1}(\mathbb{R}_+ \cdot (a + b)) \subset (\tau \cap f^{-1}(\mathbb{R}_+ \cdot a)) + (\tau \cap f^{-1}(\mathbb{R}_+ \cdot b)) \qquad \text{for every } \tau \in I(a + b).$$

The latter assertion is obvious, since f restricts to an isomorphism $\langle\tau\rangle \xrightarrow{\sim} W$.

(ii): By inspecting the construction, one verifies easily that the subdivision Δ thus exhibited shall be L-rational, whenever σ is. □

6.3.32. Later we shall also be interested in rational variants of the identities of proposition 6.3.28(i). Namely, consider the following situation. Let $g : L \to L'$ be a map of free abelian groups of finite rank, $g_\mathbb{R} : L_\mathbb{R} \to L'_\mathbb{R}$ the induced $\mathbb{R}$-linear map, and $(L_\mathbb{R}, \sigma)$ an L-rational convex polyhedral cone; set $\tau := g_\mathbb{R}(\sigma)$, and denote by $h_\mathbb{R} : \sigma \to \tau$ (resp. $h_\mathbb{Q} : \sigma \cap L_\mathbb{Q} \to \tau \cap L'_\mathbb{Q}$) the restriction of $g_\mathbb{R}$. We point out, for later reference, the following observation :

Lemma 6.3.33. *In the situation of* (6.3.32)*, suppose that :*

$$h_\mathbb{R}^{-1}(x_1) + h_\mathbb{R}^{-1}(x_2) = h_\mathbb{R}^{-1}(x_1 + x_2) \qquad \textit{for every } x_1, x_2 \in \tau$$

(where the sum is taken in the monoid $(\mathscr{P}(\sigma), +)$). Then we have as well :

$$h_\mathbb{Q}^{-1}(x_1) + h_\mathbb{Q}^{-1}(x_2) = h_\mathbb{Q}^{-1}(x_1 + x_2) \qquad \textit{for every } x_1, x_2 \in \tau \cap L'_\mathbb{Q}.$$

Proof. Let $x_1, x_2 \in \tau \cap L'_{\mathbb{Q}}$ be any two elements, and $v \in h_{\mathbb{Q}}^{-1}(x_1 + x_2)$, so we may write $v = v_1 + v_2$ for some $v_i \in h_{\mathbb{R}}^{-1}(x_i)$ $(i = 1, 2)$. Let also $u_1, \dots, u_k$ be a finite system of generators for $\sigma^\vee$, and set

$$J_i := \{j \leq k \mid u_j(v_i) = 0\} \quad E_i := g_{\mathbb{R}}^{-1}(x_i) \cap \bigcap_{j \in J_i} \mathrm{Ker}\, u_j \qquad (i = 1, 2).$$

Clearly $L_{\mathbb{Q}} \cap E_i$ is a dense subset of E_i for $i = 1, 2$, hence, in any neighborhood of (x_1, x_2) in $L_{\mathbb{R}}^{\oplus 2}$ we may find a solution $(y_1, y_2) \in L_{\mathbb{Q}}^{\oplus 2}$ for the system of equations

$$g_{\mathbb{R}}(y_i) = x_i \qquad u_j(y_i) = 0 \qquad \text{for } i = 1, 2 \text{ and every } j \in J_i.$$

Since $u_j(x_i) > 0$ for every $j \notin J_i$, we will also have $u_j(y_i) > 0$ for every $j \notin J_i$, provided y_i is sufficiently close to x_i. The lemma follows. □

6.3.34. We conclude this section with some considerations that shall be useful later, in our discussion of normalized lengths for model algebras (see (17.1.37)). Keep the notation of proposition 6.3.22, and for every subset $U \subset L_{\mathbb{R}}$, let

$$\mathscr{S}_{L,\sigma}(U) := \{L \cap (\sigma - v) \mid v \in U\} \qquad \text{and set} \qquad \mathscr{S}_{L,\sigma} := \mathscr{S}_{L,\sigma}(L_{\mathbb{R}}).$$

There is a natural L-module structure on $\mathscr{S}_{L,\sigma}$; namely, notice that

$$(L \cap (\sigma - v)) + l = L \cap (\sigma - (v - l)) \qquad \text{for every } v \in L_{\mathbb{R}} \text{ and } l \in L$$

hence the rule $\tau_l : S \mapsto S + l$ defines a bijection of $\mathscr{S}_{L,\sigma}$ onto itself, for every $l \in L$, and clearly $\tau_l \circ \tau_{l'} = \tau_{l+l'}$ for every $l, l' \in L$. Also, for every $S \in \mathscr{S}_{L,\sigma}$ define

$$\Omega(\sigma, S) := \{v \in L_{\mathbb{R}} \mid L \cap (\sigma - v) = S\}$$

and denote by $\overline{\Omega}(\sigma, S)$ the topological closure of $\Omega(\sigma, S)$ in $L_{\mathbb{R}}$. For given $u \in L_{\mathbb{R}}^\vee$ and $r \in \mathbb{R}$, set $H_{u,r} := \{v \in L_{\mathbb{R}} \mid u(v) \geq r\}$. We shall say that a subset of $L_{\mathbb{R}}$ is $\mathbb{Q}$-*linearly constructible*, if it lies in the boolean subalgebra of $\mathscr{P}(L_{\mathbb{R}})$ generated by the subsets $H_{u \otimes_{\mathbb{Q}} \mathbf{1}_{\mathbb{R}}, r}$, for u ranging over all the $\mathbb{Q}$-linear forms $L_{\mathbb{Q}} \to \mathbb{Q}$, and r ranging over all rational numbers.

Proposition 6.3.35. *With the notation of* (6.3.34)*, the following holds :*

(i) *$\mathscr{S}_{L,\sigma}(U)$ is a finite set, for every bounded subset $U \subset L_{\mathbb{R}}$.*
(ii) *$\mathscr{S}_{L,\sigma}$ is a finitely generated L-module.*
(iii) *For every non-empty $S \in \mathscr{S}_{L,\sigma}$, the subset $\Omega(\sigma, S)$ is $\mathbb{Q}$-linearly constructible.*
(iv) *Suppose moreover, that σ spans $L_{\mathbb{R}}$. Then, for every $S \in \mathscr{S}_{L,\sigma}$, the subset $\Omega(\sigma, S)$ is contained in the topological closure of its interior (see* (6.3.10)*).*
(v) *For every $S \in \mathscr{S}_{L,\sigma}$, and every $v \in \overline{\Omega}(\sigma, S)$, we have $S \subset L \cap (\sigma - v)$.*

Proof. (i): Define C_ε as in the proof of proposition 6.3.22; since U is bounded, it is contained in the union of finitely many subsets of $L_{\mathbb{R}}$ of the form $C_1 + l$, for l ranging over a finite subset of L. On the other hand, τ_l induces a bijection

$$\mathscr{S}_{L,\sigma}(C_1) \xrightarrow{\sim} \mathscr{L}_{L,\sigma}(C_1 - l) \qquad \text{for every } l \in L.$$

Hence, it suffices to check the assertion for $U = C_1$. However, the proof of proposition 6.3.22(ii) shows that $L \cap (\sigma - v)$ is generated by $L \cap (C_1 - v)$; if $v \in C_1$, the latter subset is contained in $C' := C_1 \cup (-C_1)$, which is a compact subset of $L_{\mathbb{R}}$. Therefore $L \cap C'$ is a finite set, and the claim follows.

(ii): We have already observed that the L-module $\mathscr{S}_{L,\sigma}$ is generated by $\mathscr{S}_{L,\sigma}(C_1)$, and this is a finite set, by (i).

(iii): Fix a minimal system $S_1, \dots, S_n$ of generators of the L-module $\mathscr{S}_{L,\sigma}$ (*i.e.* the S_i are chosen representatives for the orbits of the L-action on $\mathscr{S}_{L,\sigma}$). After replacing S_i by some translates $S_i + l$ (for an appropriate $l \in L$) we may also assume that either $S_i = \varnothing$, or else $0 \in S_i$, and notice that this implies :

$$S_i \subset \langle\sigma\rangle \cap L = \sigma_L^{\mathrm{gp}} \qquad \text{for every } i = 1, \dots, n \tag{6.3.36}$$

(proposition 6.3.22(i)). Set

$$A_{ij} := \{l \in L \mid S_i \subset S_j - a\} \qquad \text{for every } i, j \leq n$$

and notice that A_{ij} is a σ_L-module, for every $i, j \leq n$.

Claim 6.3.37. If $S_i, S_j \neq \varnothing$, the σ_L-module A_{ij} is finitely generated.

Proof of the claim. Fix $l \in L$ such that $\sigma_L + l \subset S_i$. Next, say that $x_1, \dots, x_t$ is a finite system of generators for the σ_L-module S_j (proposition 6.3.22(ii)); by virtue of (6.3.36), for every $s = 1, \dots, t$, we may write $x_s = a_s - b_s$ for certain $a_s, b_s \in \sigma_L$. Set $l' := b_1 + \cdots + b_t$, and notice that $S_j \subset \sigma_L - l'$. Now, if $S_i \subset S_j - a$, we deduce that $\sigma_L + l \subset \sigma_L - a - l'$, especially $l \in \sigma_L - a - l'$, *i.e.* $a \in \sigma_L - (l + l')$. This shows that A_{ij} is isomorphic to an ideal of σ_L, and then the claim follows from proposition 6.1.9(ii). ◊

Now, let $i, j \leq n$ such that $S_i, S_j \neq \varnothing$. Suppose first that $i \neq j$, and let $A'_{ij} \subset A_{ij}$ be any finite generating system for the σ_L-module A_{ij}. From the construction, it is clear that every element of LS_j that contains S_i, must contain $S_j - l$, for some $l \in A'_{ij}$. To deal with the case where $i = j$, we remark, more generally :

Claim 6.3.38. Let P be any fine and saturated monoid, $M \subset P^{\mathrm{gp}}$ a non-empty finitely generated P-submodule, and $a \in P^{\mathrm{gp}}$ an element such that $aM \subset M$. Then $a \in P$.

Proof of the claim. Pick any $m \in M$, and denote by $M' \subset M$ the submodule generated by $(a^k m \mid k \in \mathbb{N})$. According to proposition 6.1.9(i), there exists $N \geq 0$ such that M' is generated by the finite system $(a^k m \mid k = 0, \dots, N)$. Especially, $a^{N+1}m \in M'$, and therefore there exist $x \in P$ and $i \leq N$ such that $a^{N+1}m = a^i m x$ in M; it follows that $a^{N+1-i} \in P$, and finally $a \in P$, since P is saturated. ◊

From (6.3.36) we see that $A_{ii} \subset \sigma_L^{\mathrm{gp}}$, if $S_i \neq \varnothing$; combining with claim 6.3.38, we deduce that $A_{ii} = \sigma_L$. Moreover, notice as well that if $S_i = S_i - a$ for some $a \in \sigma_L^{\mathrm{gp}}$, then both a and $-a \in A_{ii}$, so that $a \in \sigma_L^{\times}$. Thus, let A'_{ii} be any set of representatives of $\mathfrak{m}_\sigma \setminus \mathfrak{m}_\sigma^2$, where $\mathfrak{m}_\sigma$ denotes the maximal ideal of $\sigma_L^{\sharp}$. If $a \in L$, and $S_i - a$ contains strictly S_i, then a is a non-invertible element of σ_L, and taking into account corollary 6.1.12, we see that A'_{ii} is finite, and there exists $l \in A'_{ii}$ such that $S_i - l \subset S_i - a$. Next, for every $i \leq n$ such that $S_i \neq \varnothing$, set

$$\mathscr{S}^i := \bigcup_j \{S_j + l \mid l \in A'_{ij}\}$$

where $j \leq n$ runs over the indices such that $S_j \neq \varnothing$. Summing up, we conclude that $\mathscr{S}^i$ is a finite set for every $i \leq n$ with $S_i \neq \varnothing$, and if an element of $\mathscr{S}_{L,\sigma}$ contains strictly S_i, then it contains some element of $\mathscr{S}^i$. Lastly, in order to prove assertion (iii), we may assume that $S = S_i$ for some $i \leq n$, and notice that :

$$\Omega(\sigma, S_i) = \{v \in L_{\mathbb{R}} \mid S \subset \sigma - v\} \setminus \bigcup_{S' \in \mathscr{S}^i} \{v \in L_{\mathbb{R}} \mid S' \subset \sigma - v\}. \tag{6.3.39}$$

Since S is finitely generated (proposition 6.3.22(ii)), we reduce to showing that, for every $a \in L$, the subset $\Omega(\sigma, a) := \{v \in L_{\mathbb{R}} \mid a \in \sigma - v\}$ is $\mathbb{Q}$-linearly constructible, which follows easily from proposition 6.3.21(i) and lemma 6.3.2.

(iv): We remark :

Claim 6.3.40. Let C_ε be as in the proof of proposition 6.3.22. Then, for every $S \in \mathscr{S}_{L,\sigma}$ and every $a \in \Omega(\sigma, S)$ there exists $\varepsilon > 0$ such that $a + C_\varepsilon \subset \Omega(\sigma, S)$.

Proof of the claim. Since σ is closed in $L_\mathbb{R}$, for every $a \in L_\mathbb{R}$ and every $b \in L_\mathbb{R} \setminus \Omega(\sigma, a)$ there exists $\varepsilon > 0$ such that $(b + C_\varepsilon) \cap \Omega(\sigma, a) = \varnothing$. Taking into account (6.3.39), the claim follows easily. ◊

If σ span $L_\mathbb{R}$, the subset C_ε has non-empty interior U_ε, for every $\varepsilon > 0$, and the topological closure of U_ε equals C_ε. The assertion is then an immediate consequence of claim 6.3.40.

(v): The assertion follows easily from proposition 6.3.22(ii) : the details shall be left to the reader. □

6.3.41. Let L be as in (6.3.34), and for all integers $n, m > 0$ set

$$\frac{1}{m}L := \{v \in L_\mathbb{Q} \mid mv \in L\} \qquad \frac{1}{m}L[1/n] := \bigcup_{k\geq 0} \frac{1}{n^k m}L.$$

For future reference, let us also point out :

Lemma 6.3.42. *With the notation of* (6.3.41)*, let* $\Omega \subset L_\mathbb{R}$ *be a* $\mathbb{Q}$*-linearly constructible subset. Then we have :*

(i) *The topological closure of* Ω *in* $L_\mathbb{R}$ *is again* $\mathbb{Q}$*-linearly constructible.*
(ii) *There exists an integer* $m > 0$ *such that* $\frac{1}{m}L[1/n] \cap \Omega$ *is dense in* Ω*, for every* $n > 1$.

Proof. (i): Ω is a finite union of non-empty subsets of the form $H_1 \cap \cdots \cap H_k$, where each H_i is either of the form $H_{u\otimes \mathbf{1}_\mathbb{R}, r}$ for some non-zero $\mathbb{Q}$-linear form u of $L_\mathbb{Q}$ and some $r \in \mathbb{Q}$ (and this is a closed subset of $L_\mathbb{R}$), or else is the complement in $L_\mathbb{R}$ of a subset of this type (and then its closure is a half-space $H_{-u\otimes \mathbf{1}_\mathbb{R}, r}$). One verifies that the closure of $H_1 \cap \cdots \cap H_k$ is the intersection of the closures of $H_1, \dots, H_k$, whence the assertion.

(ii): We may assume that $\Omega = \Omega_1 \cap \Omega_2$, where Ω_1 is a finite intersection of rational hyplerplanes, and Ω_2 is a finite intersection of open half-spaces (*i.e.* of complements of closed half-spaces). Suppose that $\frac{1}{m}L[1/n] \cap \Omega_1$ is dense in Ω_1; then clearly $\frac{1}{m}L[1/n] \cap \Omega$ is dense in Ω. Hence, we may further assume that Ω is a non-empty intersection of rational hyperplanes. In this case, Ω is of the form $V_\mathbb{R} + v_0$, where $v_0 \in L_\mathbb{Q}$, and $V_\mathbb{R} = V \otimes_\mathbb{Z} \mathbb{R}$ for some subgroup $V \subset L$. Notice that $L[1/n] \cap V_\mathbb{R}$ is dense in $V_\mathbb{R}$ for every integer $n > 1$. Then, any integer $m > 0$ such that $v_0 \in \frac{1}{m}L$ will do. □

6.4. **Fine and saturated monoids.** This section presents the further developments of the theory of fine and saturated monoids. Again, all the monoids in this section are non-pointed. We begin with a few corollaries of proposition 6.3.22(i,iii).

Corollary 6.4.1. *Let* M *be an integral monoid, such that* $M^\sharp$ *is fine. We have :*

(i) *The inclusion map* $M \to M^\mathrm{sat}$ *is a finite morphism of monoids.*
(ii) *Especially, if* M *is fine, any monoid* N *with* $M \subset N \subset M^\mathrm{sat}$*, is fine.*

Proof. (i): From lemma 6.2.9(ii) we deduce that M^sat is a finitely generated M-module if and only if $(M^\sharp)^\mathrm{sat}$ is a finitely generated $M^\sharp$-module. Hence, we may replace M by $M^\sharp$, and assume that M is fine. Pick a surjective group homomorphism $\varphi : \mathbb{Z}^{\oplus n} \to M^\mathrm{gp}$; it is easily seen that :

$$\varphi^{-1}(M^\mathrm{sat}) = (\varphi^{-1}M)^\mathrm{sat}$$

and clearly it suffices to show that $\varphi^{-1}N$ is finitely generated, hence we may replace M by $\varphi^{-1}M$, and assume throughout that M^gp is a free abelian group of finite rank. In this case, proposition 6.3.22(i,iv) already implies that M^sat is finitely generated. Let $a_1, \dots, a_k \in M^\mathrm{sat}$ be a finite system of generators, and pick integers $n_1, \dots, n_k > 0$ such that $a_i^{n_i} \in M$ for $i = 1, \dots, k$. For every $i = 1, \dots, k$ let $\Sigma_i := \{a_i^j \mid j = 0, \dots, n_i - 1\}$; it is easily seen $\Sigma_1 \cdots \Sigma_k \subset M^\mathrm{sat}$ is a system of generators for the M-module M^sat (where the product of the sets Σ_i is formed in the monoid $\mathscr{P}(M^\mathrm{sat})$ of (6.1.1)).

(ii) follows from (i), in view of proposition 6.1.9(i). □

Corollary 6.4.2. *Let $f : M_1 \to M$ and $g : M_2 \to M$ be two morphisms of monoids, such that M_1 and M_2 are finitely generated, and M is integral. Then the fibre product $M_1 \times_M M_2$ is a finitely generated monoid, and if M_1 and M_2 are fine, the same holds for $M_1 \times_M M_2$.*

Proof. If the monoids M, M_1 and M_2 are integral, $M_1 \times_M M_2$ injects in $M_1^{\mathrm{gp}} \times_{M^{\mathrm{gp}}} M_2^{\mathrm{gp}}$ (lemma 4.8.29(iii)), hence it is integral. To show that the fibre product is finitely generated, choose surjective morphisms $\mathbb{N}^{\oplus a} \to M_1$ and $\mathbb{N}^{\oplus b} \to M_2$, for some $a, b \in \mathbb{N}$; by composition we get maps of monoids $\varphi : \mathbb{N}^{\oplus a} \to M$, $\psi : \mathbb{N}^{\oplus b} \to M$, such that the induced morphism $P := \mathbb{N}^{\oplus a} \times_M \mathbb{N}^{\oplus b} \to M_1 \times_M M_2$ is surjective. Hence it suffices to show that P is finitely generated. To this aim, let $L := \mathrm{Ker}(\varphi^{\mathrm{gp}} - \psi^{\mathrm{gp}} : \mathbb{Z}^{\oplus a+b} \to M^{\mathrm{gp}})$; for every $i = 1, \dots, a+b$, denote also by $\pi_i : \mathbb{Z}^{\oplus a+b} \to \mathbb{Z}$ the projection onto the i-th direct summand. The system $\{\pi_i \mid i = 1, \dots, a+b\}$ generates a rational convex polyhedral cone $\sigma \subset L^\vee \otimes_{\mathbb{Z}} \mathbb{R}$, and one verifies easily that $P = L \cap \sigma^\vee$, so the assertion follows from propositions 6.3.21(i) and 6.3.22(i). □

Corollary 6.4.3. (i) *Let $f : M \to N$ be morphisms of monoids, with M finitely generated. Then* $\mathrm{Ker}\, f$ *is a finitely generated monoid.*

(ii) *Let $f, f' : M \to N$ be two morphisms of monoids, with M finitely generated, and N integral. Then* $\mathrm{Equal}(f, f')$ *is finitely generated.*

Proof. (i): By assumption, there exists $r \in \mathbb{N}$ and a surjective morphism of monoids $g : \mathbb{N}^{\oplus r} \to M$. Clearly $\mathrm{Ker}\, f = g(\mathrm{Ker}\, f \circ g)$, hence it suffices to check that $\mathrm{Ker}\, g \circ f$ is finitely generated, so we may assume that M is a fine monoid. In that case, it is easily seen that $\mathrm{Ker}\, f = (\mathrm{Ker}\, f)^{\mathrm{gp}} \cap M = (\mathrm{Ker}\, f)^{\mathrm{gp}} \times_{M^{\mathrm{gp}}} M$, so the assertion follows from corollary 6.4.2.

(ii): Let $i : M \to M^{\mathrm{gp}}$ be the unit of adjunction; it is easily seen that $\mathrm{Equal}(f, f') = \mathrm{Ker}((f^{\mathrm{gp}} - f'^{\mathrm{gp}}) \circ i)$, so by (i), the monoid $\mathrm{Equal}(f, f')$ is finitely generated. □

Remark 6.4.4. Both corollaries 6.4.2 and 6.4.3 can fail for non-integral monoids. For instance, let $I := \{0, 1\}$, which is a monoid with the composition law such that $0 + 0 = 0$ and $1 + 1 = 1 = 0 + 1$. Consider the surjective morphism of monoids $f : \mathbb{N} \to I$ such that $f(0) = 0$ and $f(n) = 1$ for every $n > 0$. For $i = 1, 2$, let also $\pi_i : N^{\oplus 2} \to \mathbb{N}$ be the projection; then $\mathbb{N} \times_I \mathbb{N} = \mathrm{Equal}(f \circ \pi_1, f \circ \pi_2)$ is not finitely generated.

Corollary 6.4.5. *Let $(\Gamma, +, 0)$ be an integral monoid, M a finitely generated Γ-graded monoid. Then M_0 is a finitely generated monoid, and M_γ is a finitely generated M_0-module, for every $\gamma \in \Gamma$.*

Proof. We have $M_0 = M \times_\Gamma \{0\}$, hence M_0 is finitely generated, by corollary 6.4.2. The given element $\gamma \in \Gamma$ determines a unique morphism of monoids $\mathbb{N} \to \Gamma$ such that $1 \mapsto \gamma$. Let $p_1 : M' := M \times_\Gamma \mathbb{N} \to \mathbb{N}$ and $p_2 : M' \to M$ be the two natural projections; by lemma 4.8.29(iii), we have $M_\gamma = p_2(p_1^{-1}(1))$. In light of corollary 6.4.2, M' is still finitely generated, hence we are reduced to the case where $\Gamma = \mathbb{N}$ and $\gamma = 1$. In this case, pick a finite set of generators S for M. One checks easily that $M_1 \cap S$ generates the M_0-module M_1. □

Corollary 6.4.6. *Let M be an integral monoid, such that $M^\sharp$ is fine, and $\varphi : M \to N$ a saturated morphism of monoids. Then φ is flat.*

Proof. In view of corollary 6.4.1(i) and theorem 6.2.6, it suffices to show that the M^{sat}-module $M^{\mathrm{sat}} \otimes_M N$ is flat. Hence we may replace M by M^{sat}, and assume that M is saturated. Let $I \subset M$ be any ideal, and define $\mathsf{R}(M, I)$ as in the proof of theorem 6.2.3; by assumption, $\mathsf{R}(M, I)^{\mathrm{sat}} \otimes_M N$ is a saturated – especially, integral – monoid, *i.e.* the natural map

$$\mathsf{R}(M, I) \otimes_M N \to \mathsf{R}(M, I)^{\mathrm{gp}} \otimes_{M^{\mathrm{gp}}} N^{\mathrm{gp}}$$

is injective. The latter factors through the morphism $j \otimes_M N$, where $j : \mathsf{R}(M, I) \to M \times \mathbb{N}$ is the obvious inclusion. In light of example 6.2.13(i), we deduce that the induced map $I^{\mathrm{sat}} \otimes_M N \to$

N is injective. Now, if I is a prime ideal, then $I^{\mathrm{sat}} = I$, hence the contention follows from corollary 6.1.48(ii). □

The following corollary generalizes lemma 6.2.10.

Corollary 6.4.7. *Let $f : M \to N$ be a local, flat and saturated morphism of fine monoids, with M sharp. Then there exists an isomorphism of monoids*

$$g : N \xrightarrow{\sim} N^\sharp \times N^\times$$

that fits into a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{f^\sharp} & N^\sharp \\ {\scriptstyle f}\downarrow & & \downarrow \\ N & \xrightarrow{g} & N^\sharp \times N^\times \end{array}$$

whose right vertical arrow is the natural inclusion map.

Proof. From lemma 6.2.30(ii), we know that f is exact, and since M is sharp, we easily deduce that $f(M)^{\mathrm{gp}} \cap N^\times = \{1\}$. Hence, the induced group homomorphism $M^{\mathrm{gp}} \oplus N^\times \to N^{\mathrm{gp}}$ is injective. On the other hand, since f is flat, local and saturated, the same holds for $f^\sharp : M \to N^\sharp$ (lemma 6.2.12(iii) and corollary 6.4.6); then corollary 6.2.32(ii) says that the cokernel of the induced group homomorphism $M^{\mathrm{gp}} \to (N^\sharp)^{\mathrm{gp}} = N^{\mathrm{gp}}/N^\times$ is a free abelian group G (of finite rank). Summing up, we obtain an isomorphism of abelian groups :

$$h : M^{\mathrm{gp}} \oplus N^\times \oplus G \xrightarrow{\sim} N^{\mathrm{gp}}$$

extending the map f^{gp}. Set $N_0 := N \cap h(M^{\mathrm{gp}} \oplus G)$; it follows easily that the natural map $N_0 \times N^\times \to N$ is an isomorphism; especially, the projection $N \to N^\sharp$ maps N_0 isomorphically onto $N^\sharp$, and the contention follows. □

6.4.8. Let $(M, \cdot)$ be a fine (non-pointed) monoid, so that M^{gp} is a finitely generated abelian group. We set $M^{\mathrm{gp}}_{\mathbb{R}} := \log M^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{R}$, and we let $M_{\mathbb{R}}$ be the convex polyhedral cone generated by the image of $\log M$. Then $(M_{\mathbb{R}}, +)$ is a monoid, and we have a natural morphism of monoids

$$\varphi : \log M \to (M_{\mathbb{R}}, +).$$

Proposition 6.4.9. *With the notation of* (6.4.8)*, we have :*

(i) *The rule : $F \mapsto F_{\mathbb{R}}$ establishes a bijection between the set of faces of M and the set of faces of $M_{\mathbb{R}}$.*

(ii) *The map φ induces a bijection*

$$\varphi^* : \operatorname{Spec} M_{\mathbb{R}} \to \operatorname{Spec} M.$$

Proof. Clearly, we may assume that $M \neq \{1\}$. Let $\mathfrak{p} \subset M$ be a prime ideal; we denote by $\mathfrak{p}_{\mathbb{R}}$ the ideal of $(M_{\mathbb{R}}, +)$ generated by all elements of the form $r \cdot \varphi(x)$, where r is any strictly positive real number, and x is any element of $\mathfrak{p}$. We also denote by $(M\backslash\mathfrak{p})_{\mathbb{R}}$ the convex cone of $M^{\mathrm{gp}}_{\mathbb{R}}$ generated by the image of $M\backslash\mathfrak{p}$.

Claim 6.4.10. $M_{\mathbb{R}}$ is the disjoint union of $(M\backslash\mathfrak{p})_{\mathbb{R}}$ and $\mathfrak{p}_{\mathbb{R}}$.

Proof of the claim. To begin with, we show that $M_{\mathbb{R}} = (M\backslash\mathfrak{p})_{\mathbb{R}} \cup \mathfrak{p}_{\mathbb{R}}$. Indeed, let $x \in M_{\mathbb{R}}$; then we may write $x = \sum_{i=1}^h m_i \otimes a_i$ for certain $a_1, \dots, a_h \in \mathbb{R}_+$ and $m_1, \dots, m_h \in M$. We may assume that $a_1, \dots, a_k \in \mathfrak{p}$ and $a_{k+1}, \dots, a_h \in M\backslash\mathfrak{p}$. Now, if $k = 0$ we have $x \in (M\backslash\mathfrak{p})_{\mathbb{R}}$, and otherwise $x \in \mathfrak{p}_{\mathbb{R}}$, which shows the assertion.

It remains to show that $(M\setminus\mathfrak{p})_{\mathbb{R}} \cap \mathfrak{p}_{\mathbb{R}} = \varnothing$. To this aim, suppose by way of contradiction, that this intersection contains an element x; this means that we have finite subsets $S_0 \subset M\setminus\mathfrak{p}$ and $S_1 \subset M$ such that $S_1 \cap \mathfrak{p} \neq \varnothing$, and an identity of the form :

$$x = \sum_{\sigma\in S_0} \sigma \otimes a_\sigma = \sum_{\sigma\in S_1} \sigma \otimes b_\sigma \tag{6.4.11}$$

where $a_\sigma > 0$ for every $\sigma \in S_0$ and $b_\sigma > 0$ for every $\sigma \in S_1$. For every $\sigma \in S_0$, choose a rational number $a'_\sigma \geq a_\sigma$; after adding the summand $\sum_{\sigma\in S_0} \sigma \otimes (a'_\sigma - a_\sigma)$ to both sides of (6.4.11), we may assume that $a_\sigma \in \mathbb{Q}_+$ for every $\sigma \in S_0$. Let $N \subset M$ be the submonoid generated by S_1; it follows that $x \in N_{\mathbb{R}} \cap M_{\mathbb{Q}} = N_{\mathbb{Q}}$ (proposition 6.3.22(iii)), hence we may assume that all the coefficients a_σ and b_σ are rational and strictly positive (see remark 6.3.24). We may further multiply both sides of (6.4.11) by a large integer, to obtain that these coefficients are actually integers. Then, up to further multiplication by some integer, the identity of (6.4.11) lifts to an identity between elements of $\log M$, of the form : $\sum_{\sigma\in S_0} a_\sigma \cdot \sigma = \sum_{\sigma\in S_1} b_\sigma \cdot \sigma$. The latter is absurd, since $S_1 \cap \mathfrak{p} \neq \varnothing$ and $S_0 \cap \mathfrak{p} = \varnothing$. ◊

Claim 6.4.10 implies that $\mathfrak{p}_{\mathbb{R}}$ is a prime ideal of $M_{\mathbb{R}}$, and clearly $\mathfrak{p} \subset \varphi^*(\mathfrak{p}_{\mathbb{R}})$. Since we have as well $M\setminus\mathfrak{p} \subset \varphi^{-1}(M\setminus\mathfrak{p})_{\mathbb{R}}$, we deduce that $\mathfrak{p} = \varphi^*(\mathfrak{p}_{\mathbb{R}})$. Hence the rule $\mathfrak{p} \mapsto \mathfrak{p}_{\mathbb{R}}$ yields a right inverse $\varphi_* : \operatorname{Spec} M \to \operatorname{Spec} M_{\mathbb{R}}$ for the natural map φ^*. To show that φ_* is also a left inverse, let $\mathfrak{q} \subset M_{\mathbb{R}}$ be a prime ideal; by lemma 6.3.7 and proposition 6.3.21(i), the face $M_{\mathbb{R}}\setminus\mathfrak{q}$ is of the form $M_{\mathbb{R}} \cap \operatorname{Ker} u$, for some $u \in M_{\mathbb{R}}^\vee \cap (\log M^{\mathrm{gp}})^\vee$. Then it is easily seen that $M_{\mathbb{R}}\setminus\mathfrak{q}$ is the convex cone generated by $\varphi(M) \cap \operatorname{Ker} u$, in other words, $M_{\mathbb{R}} \setminus \mathfrak{q} = \varphi^{-1}(\operatorname{Ker} u)_{\mathbb{R}}$. Again by claim 6.4.10, it follows that $\mathfrak{q} = (M\setminus\varphi^{-1}(\operatorname{Ker} u))_{\mathbb{R}} = (\varphi^*\mathfrak{q})_{\mathbb{R}}$, as stated. The argument also shows that every face of $M_{\mathbb{R}}$ is of the from $(M\setminus\mathfrak{p})_{\mathbb{R}}$ for a unique prime ideal $\mathfrak{p}$, which settles assertion (i). □

Corollary 6.4.12. *Let M be a fine monoid. We have :*

(i) $\dim M = \mathrm{rk}_{\mathbb{Z}}(M^{\mathrm{gp}}/M^\times)$.
(ii) $\dim(M\setminus\mathfrak{p}) + \operatorname{ht}\mathfrak{p} = \dim M$ *for every* $\mathfrak{p} \in \operatorname{Spec} M$.
(iii) *If $M \neq \{1\}$ is sharp (see* (4.8.32)*), there exists a local morphism $M \to \mathbb{N}$.*
(iv) *If M is sharp and M^{gp} is a torsion-free abelian group of rank r, there exists an injective morphism of monoids $M \to \mathbb{N}^{\oplus r}$.*

Proof. (i): By proposition 6.4.9, the dimension of M can be computed as the length of the longest chain $F_0 \subset F_1 \subset \cdots \subset F_d$ of strict inclusions of faces of $M_{\mathbb{R}}$. On the other hand, given such a maximal chain, denote by r_i the dimension of the $\mathbb{R}$-vector space spanned by F_i; in view proposition 6.3.8(ii),(iii), it is easily seen that $r_{i+1} - r_i = 1$ for every $i = 0, \dots, d-1$. Since $M_{\mathbb{R}} \cap (-M_{\mathbb{R}})$ is the minimal face of $M_{\mathbb{R}}$, we deduce that

$$\dim M = \dim_{\mathbb{R}} M_{\mathbb{R}}^{\mathrm{gp}} - \dim_{\mathbb{R}} M_{\mathbb{R}} \cap (-M_{\mathbb{R}}).$$

Clearly $\dim_{\mathbb{R}} M_{\mathbb{R}}^{\mathrm{gp}} = \mathrm{rk}_{\mathbb{Z}} M^{\mathrm{gp}}$; moreover, by proposition 6.4.9, the face $M_{\mathbb{R}} \cap (-M_{\mathbb{R}})$ is spanned by the image of the face $M^\times$ of M. whence the assertion.

(ii) is similar : again proposition 6.3.8(ii),(iii) implies that, every face F of $M_{\mathbb{R}}$ fits into a maximal strictly ascending chain of faces of $M_{\mathbb{R}}$, and the length of any such maximal chain is $\dim M$, by (i).

(iii): Notice that $\mathrm{rk}_{\mathbb{Z}} M^{\mathrm{gp}} > 0$, by (i). By proposition 6.4.9(i), $M_{\mathbb{R}}$ is strictly convex, therefore, by proposition 6.3.21(i), we may find a non-zero linear map $\varphi : M^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q}$, such that $M_{\mathbb{R}} \cap \operatorname{Ker} \varphi \otimes_{\mathbb{Q}} \mathbb{R} = \{0\}$ and $\varphi(M) \subset \mathbb{Q}_+$. A suitable positive integer of φ will do.

(iv): Under the stated assumption, we may regard M as a submonoid of $M_{\mathbb{R}}$, and the latter contains no non-zero linear subspaces. By corollary 6.3.14 and proposition 6.3.21(i), we may then find r linearly independent forms $u_1, \dots, u_r : M^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q}$ which are positive on M. It follows that $u_1 \otimes_{\mathbb{Q}} \mathbb{R}, \dots, u_r \otimes_{\mathbb{Q}} \mathbb{R}$ generate a polyhedral cone $\sigma^\vee \subset M_{\mathbb{R}}^\vee$, so its dual cone $\sigma \subset M^{\mathrm{gp}}_{\mathbb{R}}$ contains $M_{\mathbb{R}}$. By construction, σ admits precisely r extremal rays, say the rays generated by the vectors $v_1, \dots, v_r$, which we can pick in $M^{\mathrm{gp}}_{\mathbb{Q}}$, in which case they form a basis of the latter $\mathbb{Q}$-vector space. Now, every $x \in M_{\mathbb{R}}$ can be written uniquely in the form $x = \sum_{i=1}^r a_i v_i$ for certain $a_1, \dots, a_r \in \mathbb{Q}_+$; since M is finitely generated, we may find an integer $N > 0$ independent of x, such that $Na_i \in \mathbb{N}$ for every $i = 1, \dots, r$. In other words, M is contained in the monoid generated by $N^{-1}v_1, \dots, N^{-1}v_r$; the latter is isomorphic to $\mathbb{N}^{\oplus r}$. □

6.4.13. For any monoid M, the *dual* of M is the monoid

$$M^\vee := \mathrm{Hom}_{\mathbf{Mnd}}(M, \mathbb{N})$$

(see (6.1.1)). As usual, there is a natural morphism

$$M \to M^{\vee\vee} \quad : \quad m \mapsto (\varphi \mapsto \varphi(m)) \quad \text{for every } m \in M \text{ and } \varphi \in M^\vee.$$

We say that M is *reflexive*, if this morphism is an isomorphism.

Proposition 6.4.14. *Let M be a monoid. We have :*

(i) *$M^\vee$ is integral, saturated and sharp.*

(ii) *If M is finitely generated, $M^\vee$ is fine, and we have a natural identification :*

$$(M^\vee)_{\mathbb{R}} \xrightarrow{\sim} (M_{\mathbb{R}})^\vee$$

(where $(M_{\mathbb{R}})^\vee$ is defined as in (6.3.1)*). Moreover,* $\dim M = \dim M^\vee$.

(iii) *If M is finitely generated and sharp, we have a natural identification :*

$$(M^\vee)^{\mathrm{gp}} \xrightarrow{\sim} (M^{\mathrm{gp}})^\vee.$$

(iv) *If M is fine, sharp and saturated, then M is reflexive.*

Proof. (i): It is easily seen that the natural group homomorphism

$$(M^\vee)^{\mathrm{gp}} \to (M^{\mathrm{gp}})^\vee := \mathrm{Hom}_{\mathbb{Z}}(M^{\mathrm{gp}}, \mathbb{Z}) \tag{6.4.15}$$

is injective. Now, say that $\varphi \in (M^\vee)^{\mathrm{gp}}$ and $N\varphi \in M^\vee$ for some $N \in \mathbb{N}$; we may view φ as group homomorphism $\varphi : M^{\mathrm{gp}} \to \mathbb{Z}$, and the assumption implies that $\varphi(M) \subset \mathbb{Z} \cap \mathbb{Q}_+ = \mathbb{N}$, whence the contention.

(ii): Indeed, let $x_1, \dots, x_n$ be a system of generators of M. Define a group homomorphism $f : (M^{\mathrm{gp}})^\vee \to \mathbb{Z}^{\oplus n}$ by the rule : $\varphi \mapsto (\varphi(x_1), \dots, \varphi(x_n))$ for every $f : M^{\mathrm{gp}} \to \mathbb{Z}$. Then $M^\vee = \varphi^{-1}(\mathbb{N}^{\oplus n})$, and since $(M^{\mathrm{gp}})^\vee$ is fine, corollary 6.4.2 implies that $M^\vee$ is fine as well. Next, the injectivity of (6.4.15) implies especially that $(M^\vee)^{\mathrm{gp}}$ is torsion-free, hence (6.4.15) $\otimes_{\mathbb{Z}} \mathbb{R}$ is still injective; its restriction to $(M^\vee)_{\mathbb{R}}$ factors therefore through an injective map $f : (M^\vee)_{\mathbb{R}} \to (M_{\mathbb{R}})^\vee$. The latter map is determined by the image of $M^\vee$, and by inspecting the definitions, we see that $f(\varphi) := \varphi^{\mathrm{gp}} \otimes 1$ for every $\varphi \in M^\vee$. To prove that f is an isomorphism, it suffices to show that it has dense image. However, say that $\varphi \in (M_{\mathbb{R}})^\vee$; then $\varphi : M^{\mathrm{gp}} \to \mathbb{R}$ is a group homomorphism such that $\varphi(M) \subset \mathbb{R}_+$. Since M is finitely generated, in any neighborhood of φ in $M^{\mathrm{gp}}_{\mathbb{R}}$ we may find some $\varphi' : M^{\mathrm{gp}} \to \mathbb{Q}_+$, and then $N\varphi' \in M^\vee$ for some integer $N \in \mathbb{N}$ large enough. It follows that φ' is in the image of f, whence the contention.

The stated equality follows from the chain of identities :

$$\dim M = \dim M_{\mathbb{R}} = \dim (M_{\mathbb{R}})^\vee = \dim (M^\vee)_{\mathbb{R}} = \dim M^\vee$$

where the first and the last follow from proposition 6.4.9(ii), and the second follows from corollary 6.3.12(ii).

(iii): Let us show first that, under these assumptions, (6.4.15) $\otimes_{\mathbb{Z}} \mathbb{R}$ is an isomorphism. Indeed, if M is sharp, $(M_{\mathbb{R}})^\vee$ spans $(M^{\mathrm{gp}}_{\mathbb{R}})^\vee$ (corollary 6.3.14 and proposition 6.4.9(i)); then the assertion follows from (ii). We deduce that $(M^\vee)^{\mathrm{gp}}$ and $(M^{\mathrm{gp}})^\vee$ are free abelian groups of the same rank, hence we may find a basis $\varphi_1, \dots, \varphi_r$ of $(M^\vee)^{\mathrm{gp}}$ (resp. $\psi_1, \dots, \psi_r$ of $(M^{\mathrm{gp}})^\vee$), and positive integers $N_1, \dots, N_r$ such that (6.4.15) is given by the rule : $\varphi_i \mapsto N_i\psi_i$ for every $i = 1, \dots, r$. But then necessarily we have $N_i = 1$ for every $i \leq r$, and (iii) follows.

(iv): It is easily seen that $M^\vee = (M_\mathbb{R})^\vee \cap (M^{\mathrm{gp}})^\vee$ (notation of (6.4.8)). After dualizing again we find : $M^{\vee\vee} = ((M^\vee)_\mathbb{R})^\vee \cap (M^{\vee\mathrm{gp}})^\vee$. From (ii) we deduce that $((M^\vee)_\mathbb{R})^\vee = (M_\mathbb{R})^{\vee\vee} = M_\mathbb{R}$ (lemma 6.3.2), and from (iii) we get : $(M^{\vee\mathrm{gp}})^\vee = (M^{\mathrm{gp}})^{\vee\vee} = M^{\mathrm{gp}}$. Hence $M^{\vee\vee} = M_\mathbb{R} \cap M^{\mathrm{gp}} = M$ (proposition 6.3.22(iii)). □

Remark 6.4.16. (i) Let M be a sharp and fine monoid. Proposition 6.4.14(iii) implies that the natural map

$$\mathrm{Hom}_{\mathbf{Mnd}}(M, \mathbb{Q}_+)^{\mathrm{gp}} \to \mathrm{Hom}_{\mathbf{Mnd}}(M, \mathbb{Q})$$

is an isomorphism. Indeed, it is easily seen that this map is injective. For the surjectivity, one uses the identification $\mathrm{Hom}_{\mathbf{Mnd}}(M, \mathbb{Q}) \xrightarrow{\sim} (M^\vee)^{\mathrm{gp}} \otimes_\mathbb{Z} \mathbb{Q}$, which follows from *loc.cit.* (Details left to the reader.)

(ii) For $i = 1, 2$, let $N_i \to N$ be two morphisms of monoids. By general nonsense, we have a natural isomorphism :

$$(N_1 \otimes_N N_2)^\vee \xrightarrow{\sim} N_1^\vee \times_{N^\vee} N_2^\vee.$$

(iii) If $f_i : M_i \to M$ ($i = 1, 2$) are morphisms of fine, saturated and sharp monoids, there exists a natural surjection :

$$M_1^\vee \otimes_{M^\vee} M_2^\vee \to (M_1 \times_M M_2)^\vee \qquad (6.4.17)$$

whose kernel is the subgroup of invertible elements. Indeed, set $P := M_1^\vee \otimes_{M^\vee} M_2^\vee$; in view of (ii) and proposition 6.4.14(iv), we have a natural identification $P^{\vee\vee} \xrightarrow{\sim} (M_1 \times_M M_2)^\vee$, and the sought map is its composition with the double duality map $P \to P^{\vee\vee}$. Moreover, clearly P is finitely generated, and it is also integral and saturated, since saturation commutes with colimits. Hence – again by proposition 6.4.14(iv) – the double duality map induces an isomorphism $P/P^\times \xrightarrow{\sim} P^{\vee\vee}$.

(iv) In the situation of (iii), if $f_i : M_i \to M$ ($i = 1, 2$) are epimorphisms, then (6.4.17) is an isomorphism. Indeed, in this case the dual morphisms $f_i^\vee : M^\vee \to M_i^\vee$ are injective, so that P is sharp (lemma 6.1.13), whence the claim.

Theorem 6.4.18. *Let M be a saturated monoid, such that $M^\sharp$ is fine. We have :*

(i) $M = \bigcap_{\mathrm{ht}\,\mathfrak{p}=1} M_\mathfrak{p}$ *(where the intersection runs over the prime ideals of M of height one).*

(ii) *If moreover,* $\dim M = 1$*, then there is an isomorphism of monoids :*

$$M^\times \times \mathbb{N} \xrightarrow{\sim} M.$$

(iii) *Suppose that M^{gp} is a torsion-free abelian group, and let R be any normal domain. Then the group algebra $R[M]$ is a normal domain as well.*

Proof. (i): Pick a decomposition $M = M^\sharp \times M^\times$ as in lemma 6.2.10, and notice that $M^\sharp$ is fine, sharp and saturated. The prime ideals of M are of the form $\mathfrak{p} = \mathfrak{p}_0 \times M^\times$, where $\mathfrak{p}_0$ is a prime ideal of $M^\sharp$. Then it is easily seen that $M_\mathfrak{p} = M^\sharp_{\mathfrak{p}_0} \times M^\times$. Therefore, the sought assertion holds for M if and only if it holds for $M^\sharp$, and therefore we may replace M by $M^\sharp$, which reduces to the case where M is sharp, hence the natural morphism $\varphi : \log M \to M_\mathbb{R}$ is injective. In such situation, we have $M = M_\mathbb{R} \cap M^{\mathrm{gp}}$ and $M_\mathfrak{p} = M_{\mathfrak{p},\mathbb{R}} \cap M^{\mathrm{gp}}$ for every prime ideal $\mathfrak{p} \subset M$ (proposition 6.3.22(iii) and lemma 6.2.9(i)). Thus, we are reduced to showing that

$$M_\mathbb{R} = \bigcap_{\mathrm{ht}\,\mathfrak{p}=1} M_{\mathfrak{p},\mathbb{R}}.$$

However, set $\tau := (M \setminus \mathfrak{p})_{\mathbb{R}}$; by inspecting the definitions, one sees that $M_{\mathfrak{p},\mathbb{R}} = M_{\mathbb{R}} + (-\tau)$, and proposition 6.4.9 shows that τ is a facet of $M_{\mathbb{R}}$, hence $M_{\mathfrak{p},\mathbb{R}}$ is the half-space denoted H_τ in (6.3.10). Then the assertion is a rephrasing of proposition 6.3.11(ii).

(ii): Arguing as in the proof of (i), we may reduce again to the case where M is sharp, in which case $M = M_{\mathbb{R}} \cap M^{\mathrm{gp}}$. The foregoing shows that, in case $\dim M = 1$, the cone $M_{\mathbb{R}}$ is a half-space, whose boundary hyperplane is the only non-trivial face σ of $M_{\mathbb{R}}$. However, σ is generated by the image of the unique non-trivial face of M, *i.e.* by $M^\times = \{1\}$ (proposition 6.4.9(i)), hence $\sigma = \{0\}$, so $M_{\mathbb{R}}$ is a half-line. Now, let $u : M^{\mathrm{gp}}_{\mathbb{R}} \to \mathbb{R}$ be a non-zero linear form, such that $u(M) \geq 0$, and $x_1, \dots, x_n$ a system of non-zero generators for M; say that $u(x_1)$ is the least of the values $u(x_i)$, for $i = 1, \dots, n$. Since M is saturated, it follows easily that every value $u(x_i)$ is an integer multiple of $u(x_1)$ (proposition 6.3.22(iii)), and then x_1 is a generator for M, so $M \simeq \mathbb{N}$.

(iii): To begin with, $R[M] \subset R[M^{\mathrm{gp}}]$, and since M^{gp} is torsion-free, it is clear that $[M^{\mathrm{gp}}]$ is a domain, hence the same holds for $R[M]$. Furthermore, from (i) we derive : $R[M] = \bigcap_{\mathrm{ht}\mathfrak{p}=1} R[M_{\mathfrak{p}}]$, hence it suffices to show that $R[M_{\mathfrak{p}}]$ is normal whenever $\mathfrak{p}$ has height one. However, we have $R[M_{\mathfrak{p}}] \simeq R[M^\times_{\mathfrak{p}}] \otimes_R R[\mathbb{N}]$ in light of (ii), and since $M^\times_{\mathfrak{p}}$ is torsion free, it is a filtered colimit of a family of free abelian groups of finite rank, so everything is clear. □

Example 6.4.19. Let M be a fine, sharp and saturated monoid of dimension 2.

(i) By corollary 6.4.12(i) and example 6.3.16, we see that M admits exactly two facets, which are fine saturated monoids of dimension one; by theorem 6.4.18(ii) each of these facets is generated by an element, say e_i (for $i = 1, 2$). From proposition 6.3.22(iii) it follows that $\mathbb{Q}_+ e_1 \oplus \mathbb{Q}_+ e_2 = M_{\mathbb{Q}}$. Especially, we may find an integer $N > 0$ large enough, such that :

$$\mathbb{N}e_1 \oplus \mathbb{N}e_2 \subset M \subset \mathbb{N}\frac{e_1}{N} \oplus \mathbb{N}\frac{e_2}{N}.$$

(ii) Moreover, clearly e_1 and e_2 are *unimodular* elements of M^{gp} (*i.e.* they generate direct summands of the latter free abelian group of rank 2). We may then find a basis f_1, f_2 of M^{gp} with $e_1 = f_1$, and $e_2 = af_1 + bf_2$, where $a, b \in \mathbb{Z}$ and $(a, b) = 1$. After replacing f_2 by some element of the form $cf_2 + df_1$ with $c \in \{1, -1\}$ and $d \in \mathbb{Z}$, we may assume that $b > 0$ and $0 \leq a < b$. Clearly, such a normalized pair (a, b) determines the isomorphism class of M, since $M_{\mathbb{R}}$ is the strictly convex cone of $M^{\mathrm{gp}}_{\mathbb{R}}$ whose extremal rays are generated by e_1 and e_2, and $M = M^{\mathrm{gp}} \cap M_{\mathbb{R}}$.

(iii) More precisely, suppose that M' is another fine, sharp and saturated monoid of dimension 2, and $\varphi : M \to M'$ an isomorphism. Pick a basis f'_1, f'_2 of M'^{gp} and a normalized pair (a', b') as in (ii), such that $e'_1 := f_1$ and $e'_2 := a' f'_1 + b' f'_2$ generate the two facets of M'. Clearly, φ must send a facet of M onto a facet of M'; we distinguish two possibilities :

- either $\varphi(e_1) = e'_1$ and $\varphi(e_2) = e'_2$, in which case we get $\varphi(f_2) = b^{-1}(a'-a)f'_1 + b^{-1}b' f'_2$; especially, $b', a - a' \in b\mathbb{Z}$. By considering φ^{-1}, we get symmetrically that $b \in b'\mathbb{Z}$, so $b = b'$ and therefore $(a', b) = 1 = (a, b)$ and $0 \leq a, a' < b$, whence $a = a'$
- or else $\varphi(e_1) = e'_2$ and $\varphi(e_2) = e'_1$, in which case we get $\varphi(f_2) = b^{-1}(1 - aa')f_1 + b^{-1}b'af_2$. It follows again that $b' \in b\mathbb{Z}$, so $b = b'$, arguing as in the previous case. Moreover, $0 \leq a' < b$, and $1 - aa' \in b\mathbb{Z}$. In other words, the class of a' in the group $(\mathbb{Z}/b\mathbb{Z})^\times$ is the inverse of the class of a.

Conversely, it is easily seen that if M' is as above, f'_1, f'_2 is a basis of M'^{gp}, and the two facets of M' are generated by f'_1 and $a' f'_1 + b' f'_2$, for a pair (a', b') normalized as in (ii), and such that $aa' \equiv 1 \pmod{b}$, then there exists an isomorphism $M \xrightarrow{\sim} M'$ of monoids (details left to the reader). Hence, set $(\mathbb{Z}/b\mathbb{Z})^\dagger := (\mathbb{Z}/b\mathbb{Z})^\times/\!\sim$, where $\sim$ denotes the smallest equivalence relation such that $[a] \sim [a]^{-1}$ for every $[a] \in (\mathbb{Z}/b\mathbb{Z})^\times$. We conclude that there exists a natural bijection between the set of isomorphism classes of fine, sharp and saturated monoids of dimension 2, and the set of pairs $(b, [a])$, where $b > 0$ is an integer, and $[a] \in (\mathbb{Z}/b\mathbb{Z})^\dagger$.

6.4.20. Let A be a ring, P a monoid. Recall that $\mathfrak{m}_P$ is the maximal (prime) ideal of P (see definition 6.1.10(i)). The *$\mathfrak{m}_P$-adic filtration* of P is the descending sequence of ideals :

$$\cdots \subset \mathfrak{m}_P^3 \subset \mathfrak{m}_P^2 \subset \mathfrak{m}_P$$

where $\mathfrak{m}_P^n$ is the n-th power of $\mathfrak{m}$ in the monoid $\mathscr{P}(P)$ (see (6.1.1)). It induces a *$\mathfrak{m}_P$-adic filtration* $\mathrm{Fil}_\bullet M$ on any P-module M and any $A[P]$-algebra B, defined by letting $\mathrm{Fil}_n M := \mathfrak{m}_P^n M$ and $\mathrm{Fil}_n B := A[\mathfrak{m}_P^n] \cdot B$, for every $n \in \mathbb{N}$.

Lemma 6.4.21. *Suppose that P is fine. We have :*

(i) *The $\mathfrak{m}_P$-adic filtration is separated on P.*
(ii) *If P is sharp, $P \backslash \mathfrak{m}_P^n$ is a finite set, for every $n \in \mathbb{N}$.*

Proof. (i): Indeed, choose A to be a non-zero noetherian ring, set $J := \bigcap_{n \geq 0} A[\mathfrak{m}_P^n]$ and notice that J is generated by $\mathfrak{m}_P^\infty := \bigcap_{n \in \mathbb{N}} \mathfrak{m}_P^n$. On the other hand, J is annihilated by an element of $1 + A[\mathfrak{m}_P]$ ([126, Th.8.9]). Thus, suppose $x \in \mathfrak{m}_P^\infty$, and pick $y \in A[\mathfrak{m}_P]$ such that $(1-y)x = 0$; we may write $y = a_1 t_1 + \cdots + a_r t_r$ for certain $a_1, \dots, a_r \in A$ and $t_1, \dots, t_r \in \mathfrak{m}_P$. Therefore $x = a_1 x t_1 + \cdots + a_r x t_r$ in $A[P]$, which is absurd, since P is integral.

Assertion (ii) is immediate from the definition. □

6.4.22. Keep the assumptions of lemma 6.4.21. It turns out that P can actually be made into a graded monoid, albeit in a non-canonical manner. We proceed as follows. Let $\varepsilon : P \to P^{\mathrm{sat}}$ the inclusion map, $T \subset P^{\mathrm{sat}}$ the torsion subgroup, set $Q := P^{\mathrm{sat}}/T$, and let $\pi : P^{\mathrm{sat}} \to Q$ be the natural surjection. We may regard $\log Q$ as a submonoid of the polyhedral cone $Q_{\mathbb{R}}$, lying in the vector space $Q_{\mathbb{R}}^{\mathrm{gp}}$, as in (6.4.8). Since $Q_{\mathbb{R}}$ is a rational polyhedral cone, the same holds for $Q_{\mathbb{R}}^\vee$, hence we may find a $\mathbb{Q}$-linear form $\gamma : \log Q^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q}$, which is non-negative on $\log Q$, and such that $Q_{\mathbb{R}} \cap \mathrm{Ker}\, \gamma \otimes_{\mathbb{Q}} \mathbb{R}$ is the minimal face of $Q_{\mathbb{R}}$, *i.e.* the $\mathbb{R}$-vector space spanned by the image of $Q^\times$. If we multiply γ by some large positive integer, we may achieve that $\gamma(\log P) \subset \mathbb{N}$. We set :

$$\mathrm{gr}_n^\gamma P := (\gamma \circ \pi \circ \varepsilon)^{-1}(n) \qquad \text{for every } n \in \mathbb{N}.$$

It is clear that $\mathrm{gr}_n^\gamma P \cdot \mathrm{gr}_m^\gamma P \subset \mathrm{gr}_{n+m}^\gamma P$, hence

$$P = \coprod_{n \in \mathbb{N}} \mathrm{gr}_n^\gamma P$$

is a $\mathbb{N}$-graded monoid, and consequently :

$$A[P] = \bigoplus_{n \in \mathbb{N}} A[\mathrm{gr}_n^\gamma P]$$

is a graded A-algebra. Moreover, it is easily seen that $\mathrm{gr}_0^\gamma P = P^\times$. More generally, for $x \in P$, let $\mu(x)$ be the maximal $n \in \mathbb{N}$ such that $x \in \mathfrak{m}_P^n$; then there exists a constant $C \geq 1$ such that :

$$\gamma(x) \geq \mu(x) \geq C^{-1}\gamma(x) \qquad \text{for every } x \in P$$

so that the $\mathfrak{m}_P$-adic filtration and the filtration defined by $\mathrm{gr}_\bullet^\gamma P$, induce the same topology on P and on $A[P]$. As a corollary of these considerations, we may state the following "regularity criterion" for fine monoids :

Proposition 6.4.23. *Let P be an integral monoid such that $P^\sharp$ is fine. Then we have*

$$\mathrm{rk}^\circ_{P^\times} \mathfrak{m}_P/\mathfrak{m}_P^2 \geq \dim P$$

(notation of example 4.8.18*) and the equality holds if and only if $P^\sharp$ is a free monoid.*

Proof. (Notice that $\mathfrak{m}_P/\mathfrak{m}_P^2$ is a free pointed $P^\times$-module, since $P^\times$ obviously acts freely on $\mathfrak{m}_P \setminus \mathfrak{m}_P^2$.) Since $\mathfrak{m}_{P^\sharp}/\mathfrak{m}_{P^\sharp}^2 = (\mathfrak{m}_P/\mathfrak{m}_P^2) \otimes_P P^\sharp$, we may replace P by $P^\sharp$, and assume from start that P is sharp and fine. Then, the rank of $\mathfrak{m}_P/\mathfrak{m}_P^2$ equals the cardinality of the set $\Sigma := \mathfrak{m}_P \setminus \mathfrak{m}_P^2$, which is finite, by lemma 6.4.21. We have a surjective morphism of monoids $\varphi : \mathbb{N}^{(\Sigma)} \to P$, that sends the basis of $\mathbb{N}^{(\Sigma)}$ bijectively onto $\Sigma \subset P$ (corollary 6.1.12). The sought inequality follows immediately, and it is also clear that we have equality, in case P is free. Conversely, if equality holds, φ^{gp} must be a surjective group homomorphism between free abelian group of the same finite rank (corollary 6.4.12(i)), so it is an isomorphism, and then the same holds for φ. □

Proposition 6.4.24. *Let P be a fine and sharp monoid, A a noetherian local ring. Set $S_P := 1 + A[\mathfrak{m}_P]$; then we have :*

$$\dim S_P^{-1}A[P] = \dim A + \dim P.$$

Proof. To begin with, the assumption that P is sharp implies that $S_P^{-1}A[P]$ is local. Next, notice that $A[P]$ is a free A-module, hence the natural map $A \to S_P^{-1}A[P]$ is a flat and local ring homomorphism. Let k be the residue field of A; in view of [126, Th.15.1(ii)], we deduce :

$$\dim S_P^{-1}A[P] = \dim A + \dim S_P^{-1}k[P].$$

Hence we are reduced to showing the stated identity for $A = k$. However, clearly $S_P^{-1}k[P] = k[P]_{\mathfrak{m}}$, where $\mathfrak{m}$ is the maximal ideal generated by the image of $\mathfrak{m}_P$, hence it suffices to apply claim 11.6.37(ii) and corollary 6.4.12(i), to conclude. □

6.4.25. Let A be a ring, and P a fine and sharp monoid. We define :

$$A[[P]] := \lim_{n\in\mathbb{N}} A\langle P/\mathfrak{m}_P^n\rangle.$$

Alternatively, this is the completion of $A[P]$ for its $A[\mathfrak{m}_P]$-adic topology. In view of the finiteness properties of the $\mathfrak{m}_P$-adic filtration (lemma 6.4.21(ii)), one may present $A[[P]]$ as the ring of formal infinite sums $\sum_{\sigma\in P} a_\sigma \cdot \sigma$, with arbitrary coefficients $a_\sigma \in A$, where the multiplication and addition are defined in the obvious way. Moreover, we may use a morphism of monoids $\gamma : \log P \to \mathbb{N}$ as in (6.4.22), to see that :

$$A[[P]] = \prod_{n\in\mathbb{N}} A[\mathrm{gr}_n^\gamma P] \qquad (6.4.26)$$

where $A[\mathrm{gr}_n^\gamma P] \cdot A[\mathrm{gr}_m^\gamma P] \subset A[\mathrm{gr}_{n+m}^\gamma P]$ for every $m, n \in \mathbb{N}$. So any element $x \in A[[P]]$ can be decomposed as an infinite sum

$$x = \sum_{n\in\mathbb{N}} \mathrm{gr}_n^\gamma x.$$

The term $\mathrm{gr}_0^\gamma x \in \mathrm{gr}_0^\gamma A = A$ does not depend on the chosen γ : it is the *constant term* of x, *i.e.* the image of x under the natural projection $A[[P]] \to A$.

Corollary 6.4.27. *Let P be a fine and sharp monoid, A a noetherian local ring. Then :*

(i) *For any local morphism $P \to A$ (see definition* 6.1.10(v)*), we have the inequality:*

$$\dim A \leq \dim A/\mathfrak{m}_P A + \dim P.$$

(ii) $\dim A[P] = \dim A[[P]] = \dim A + \dim P$.

Proof. (i): Set $A_0 := A/\mathfrak{m}_P A$, and $B := S_P^{-1}A_0[P]$, where $S_P \subset A_0[P]$ is the multiplicative subset $1 + A_0[\mathfrak{m}_P]$; if we denote by $\mathrm{gr}_\bullet A$ (resp. $\mathrm{gr}_\bullet B$) the graded A_0-algebra associated to the $\mathfrak{m}_P$-adic filtration on A (resp. on B), we have a natural surjective homomorphism of graded A_0-algebras :

$$\mathrm{gr}_\bullet B \to \mathrm{gr}_\bullet A.$$

Hence $\dim A = \dim \mathrm{gr}_\bullet A \leq \dim \mathrm{gr}_\bullet B = \dim B$, by [126, Th.15.7]. Then the assertion follows from proposition 6.4.24.

(ii): Set $B := S_P^{-1}A[P]$ and let $\mathfrak{m}_A$ (resp $\mathfrak{m}_B$) be the maximal ideal of A (resp. of B); notice that $A[[P]]$ is the $(\mathfrak{m}_P B)$-adic completion of the local noetherian ring B, so $A[[P]]$ is a local noetherian ring as well, with maximal ideal $\mathfrak{n} := \mathfrak{m}_A[[P]] + A[[\mathfrak{m}_P]]$, and the $\mathfrak{n}$-adic completion of $A[[P]]$ is naturally isomorphic to the $\mathfrak{m}_B$-adic completion $B^\wedge$ of B. Hence we get

$$\dim A[[P]] = \dim B^\wedge = \dim B$$

so the second identity follows from proposition 6.4.24. Next, clearly we have $\dim A \geq \dim B$. On the other hand, let $\mathfrak{q} \subset A[P]$ be any prime ideal, set $\mathfrak{p} := \mathfrak{q} \cap A$ and denote by κ the residue field of the local ring $A_\mathfrak{p}$; with this notation, [126, Th.15.1(ii)] says that

$$\dim A[P]_\mathfrak{q} = \dim A_\mathfrak{p} + \dim A[P]_\mathfrak{q} \otimes_A \kappa = \dim A_\mathfrak{p} + \dim \kappa[P]_\mathfrak{q} \leq \dim A + \dim \kappa[P]$$

so the assertion follows from 11.6.37(ii) and corollary 6.4.12(i). □

6.4.28. Now we wish to state and prove the combinatorial versions of the Artin-Rees lemma, and of the local flatness criterion (see [126, Th.22.3]). Namely, let P be a *pointed* monoid, such that $P^\sharp$ is finitely generated; let also $(A, \mathfrak{m}_A)$ be a local noetherian ring, N a finitely generated A-module, and consider a morphism of pointed monoids :

$$\alpha : P \to (A, \cdot)$$

The following is our version of the Artin-Rees lemma :

Lemma 6.4.29. *In the situation of* (6.4.28)*, let $J \subset P$ be an ideal, M a finitely generated P-module, $M_0 \subset M$ a submodule. Then there exists $c \in \mathbb{N}$ such that :*

(6.4.30) $$J^n M \cap M_0 = J^{n-c}(J^c M \cap M_0) \qquad \textit{for every } n > c.$$

Proof. Set $\overline{M} := M/P^\times$, $\overline{M}_0 := M_0/P^\times$ and $\overline{J} := J/P^\times$, the set-theoretic quotients for the respective natural $P^\times$-actions. Notice that $\overline{J}$ is an ideal of $P^\sharp$ and $\overline{M}_0 \subset \overline{M}$ is an inclusion of $\overline{P}$-modules. Moreover, any set of generators of the ideal $\overline{J}$ (resp. of the $P^\sharp$-module $\overline{M}_0$) lifts to a set of generators for J (resp. for the P-module M_0). Furthermore, it is easily seen that (6.4.30) is equivalent to the identity $\overline{J}^n\overline{M} \cap \overline{M}_0 = \overline{J}^{n-c}(\overline{J}^c\overline{M} \cap \overline{M}_0)$. Hence, we are reduced to the case where $P = P^\sharp$ is a finitely generated monoid. Then $\mathbb{Z}[P]$ is noetherian, $\mathbb{Z}[M]$ is a $\mathbb{Z}[P]$-module of finite type, and we notice that :

$$\mathbb{Z}[J^n M \cap M_0] = J^n\mathbb{Z}[M] \cap \mathbb{Z}[M_0] \qquad \mathbb{Z}[J^{n-c}(J^c M \cap M_0)] = J^{n-c}(J^c\mathbb{Z}[M] \cap \mathbb{Z}[M_0]).$$

Thus, the assertion follows from the standard Artin-Rees lemma [126, Th.8.5]. □

Proposition 6.4.31. *In the situation of* (6.4.28)*, suppose moreover that $P^\sharp$ is fine (see remark* 4.8.14(vi)*), and let $\mathfrak{m}_\alpha := \alpha^{-1}\mathfrak{m}_A$. Then the following conditions are equivalent :*

(a) *N is α-flat (see definition* 6.1.34*).*

(b) $\mathrm{Tor}_i^{\mathbb{Z}\langle P\rangle}(N, \mathbb{Z}\langle M\rangle) = 0$ *for every $i > 0$ and every integral pointed P-module M.*

(c) $\mathrm{Tor}_1^{\mathbb{Z}\langle P\rangle}(N, \mathbb{Z}\langle P/\mathfrak{m}_\alpha\rangle) = 0$.

(d) *The natural map :*

$$(\mathfrak{m}_\alpha^n/\mathfrak{m}_\alpha^{n+1}) \otimes_P N \to \mathfrak{m}_\alpha^n N/\mathfrak{m}_\alpha^{n+1}N$$

is an isomorphism of A-modules, for every $n \in \mathbb{N}$ (notation of (6.1.33)*).*

Proof. The assertion (a)⇔(b) is just a restatement of proposition 6.1.40(i), and holds in greater generality, without any assumption on either A or the pointed integral monoid P. Also, obviously (b)⇒(c), As for the remaining assertions, let $S := \alpha^{-1}(A^\times)$; since the localization $P \to S^{-1}P$ is flat, the natural maps :

$$\mathrm{Tor}_i^{\mathbb{Z}\langle P\rangle}(N, \mathbb{Z}\langle M\rangle) \to \mathrm{Tor}_i^{\mathbb{Z}\langle S^{-1}P\rangle}(N, \mathbb{Z}\langle S^{-1}M\rangle)$$

are isomorphisms, for every $i \in \mathbb{N}$ and every P-module M. Also, notice that the two P-modules appearing in (d) are actually $S^{-1}P$-modules (and the natural map is $\mathbb{Z}\langle S^{-1}P\rangle$-linear). Hence, we can replace everywhere P by $S^{-1}P$, which allows to assume that α is local, *i.e.* $\mathfrak{m}_\alpha = \mathfrak{m}_P$.

(c)⇒(d): For every $n \in \mathbb{N}$, we have a short exact sequence of pointed P-modules :

$$0 \to \mathfrak{m}_P^n/\mathfrak{m}_P^{n+1} \to P/\mathfrak{m}_P^{n+1} \to P/\mathfrak{m}_P^n \to 0.$$

It is easily seen that $\mathfrak{m}_P^n/\mathfrak{m}_P^{n+1}$ is a free $P/\mathfrak{m}_P$-module (in the category of pointed modules), so the assumption implies that $\mathrm{Tor}_1^{\mathbb{Z}\langle P\rangle}(N, \mathfrak{m}_P^n/\mathfrak{m}_P^{n+1}) = 0$ for every $n \in \mathbb{N}$. By looking at the induced long Tor-sequences, we deduce that the natural map

$$\mathrm{Tor}_1^{\mathbb{Z}\langle P\rangle}(N, P/\mathfrak{m}_P^{n+1}) \to \mathrm{Tor}_1^{\mathbb{Z}\langle P\rangle}(N, P/\mathfrak{m}_P^n)$$

is injective for every $n \in \mathbb{N}$. Then, a simple induction shows that, under assumption (c), all these modules vanish. The latter means that the natural map :

$$\mathfrak{m}_P^n \otimes_P N \to \mathfrak{m}_P^n N$$

is an isomorphism, for every $n \in \mathbb{N}$. We consider the commutative ladder with exact rows :

$$\begin{array}{ccccccc}
\mathfrak{m}_P^{n+1} \otimes_P N & \longrightarrow & \mathfrak{m}_P^n \otimes_P N & \longrightarrow & (\mathfrak{m}_P^n/\mathfrak{m}_P^{n+1}) \otimes_P N & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathfrak{m}_P^{n+1} N & \longrightarrow & \mathfrak{m}_P^n N & \longrightarrow & \mathfrak{m}_P^n N/\mathfrak{m}_P^{n+1} N & \longrightarrow & 0.
\end{array}$$

By the foregoing, the two left-most vertical arrows are isomorphisms, hence the same holds for the right-most, whence (c).

(d)⇒(c): We have to show that the natural map $u : \mathfrak{m}_P \otimes_P N \to \mathfrak{m}_P N$ is an isomorphism. To this aim, we consider the $\mathfrak{m}_P$-adic filtrations on these two modules; for the associated graded modules one gets :

$$\mathrm{gr}_n(\mathfrak{m}_P N) = \mathfrak{m}_P^n N/\mathfrak{m}_P^{n+1} N \qquad \mathrm{gr}_n(\mathfrak{m}_P \otimes_P N) = (\mathfrak{m}_P^n/\mathfrak{m}_P^{n+1}) \otimes_P N$$

for every $n \in \mathbb{N}$; whence maps of A-modules :

$$\mathrm{gr}_n(\mathfrak{m}_P \otimes_P N) \xrightarrow{\mathrm{gr}_n u} \mathrm{gr}_n(\mathfrak{m}_P N).$$

which are isomorphism by assumption. To conclude, it suffices to show :

Claim 6.4.32. For every ideal $I \subset P$, the the A-module $I \otimes_P N$ is $\mathfrak{m}_P A$-adically separated.

Proof of the claim. Indeed, notice that the ideal $I/P^\times \subset P^\sharp$ is finitely generated (proposition 6.1.9(ii)), hence the same holds for I, so $I \otimes_P N$ is a finitely generated A-module. Then the contention follows from [126, Th.8.10]. ◊

(c)⇒(b): We argue by induction on i. For $i = 1$, suppose first that $M = P/I$ for some ideal $I \subset P$ (notice that any such quotient is an integral pointed P-module); in this case, the assertion to prove is that the natural map $v : I \otimes_P N \to IN$ is an isomorphism. However, consider the $\mathfrak{m}_P$-adic filtration on P/I; for the associated graded module we have :

$$\mathrm{gr}_n(P/I) = (\mathfrak{m}_P^n \cup I)/(\mathfrak{m}^{n+1} \cup I) \qquad \text{for every } n \in \mathbb{N}$$

and it is easily seen that this is a free pointed $P^\times$-module, for every $n \in \mathbb{N}$. Hence, by inspecting the long exact Tor-sequences associated to the exact sequences

$$0 \to \mathrm{gr}_n(P/I) \to P/(\mathfrak{m}_P^{n+1} \cup I) \to P/(\mathfrak{m}_P^n \cup I) \to 0$$

our assumption (c), together with a simple induction yields :

$$\mathrm{Tor}_1^{\mathbb{Z}\langle P\rangle}(N, \mathbb{Z}\langle P/(\mathfrak{m}_P^n \cup I)\rangle) \qquad \text{for every } n \in \mathbb{N}. \tag{6.4.33}$$

Now, fix $n \in \mathbb{N}$; in light of lemma 6.4.29, there exists $k \geq n$ such that $\mathfrak{m}_P^k \cap I \subset \mathfrak{m}_P^n I$. We deduce surjective maps of A-modules :

$$I \otimes_P N \to \frac{I}{\mathfrak{m}_P^k \cap I} \otimes_P N \to \frac{I}{\mathfrak{m}_P^n I} \otimes_P N \xrightarrow{\sim} \frac{I \otimes_P N}{\mathfrak{m}_P^n (I \otimes_P N)}.$$

On the other hand, (6.4.33) says that the natural map $(\mathfrak{m}_P^n \cup I) \otimes_P N \to (\mathfrak{m}_P^n \cup I)N$ is an isomorphism, so the same holds for the induced composed map :

$$\frac{I}{\mathfrak{m}_P^k \cap I} \otimes_P N \xrightarrow{\sim} \frac{\mathfrak{m}_P^k \cup I}{\mathfrak{m}_P^k} \otimes_P N \to \frac{(\mathfrak{m}_P^k \cup I)N}{\mathfrak{m}_P^k N}.$$

Consequently, the kernel of v is contained in $\mathfrak{m}_P^n(I \otimes_P N)$; since n is arbitrary, we are reduced to showing that the $\mathfrak{m}_P$-adic filtration is separated on $I \otimes_P N$, which is claim 6.4.32.

Next, again for $i = 1$, let M be an arbitrary integral P-module. In view of remark 6.1.27(i), we may assume that M is finitely generated; moreover, remark 6.1.27(ii), together with an easy induction further reduces to the case where M is cyclic, in which case, according to remark 6.1.27(iii), M is of the form P/I for some ideal I, so the proof is complete in this case.

Finally, suppose $i > 1$ and assume that the assertion is already known for $i - 1$. We may similarly reduce to the case where $M = P/I$ for some ideal I as in the foregoing; to conclude, we observe that :

$$\mathrm{Tor}_i^{\mathbb{Z}\langle P\rangle}(N, \mathbb{Z}\langle P/I\rangle) \simeq \mathrm{Tor}_{i-1}^{\mathbb{Z}\langle P\rangle}(N, \mathbb{Z}\langle I\rangle).$$

Since obviously I is an integral P-module, the contention follows. □

As a corollary, we have the following combinatorial going-down theorem, which is proved in the same way as its commutative algebra counterpart.

Corollary 6.4.34. *In the situation of* (6.4.28)*, assume that $P^\sharp$ is fine, and that A is α-flat. Let $\mathfrak{p} \subset \mathfrak{q}$ be two prime ideals of P, and $\mathfrak{q}' \subset A$ a prime ideal such that $\mathfrak{q} = \alpha^{-1}\mathfrak{q}'$. Then there exists a prime ideal $\mathfrak{p}' \subset \mathfrak{q}'$ such that $\mathfrak{p} = \alpha^{-1}\mathfrak{p}'$.*

Proof. Let $\beta : P_\mathfrak{q} \to A_{\mathfrak{q}'}$ be the morphism induced by α; it is easily seen that $A_{\mathfrak{q}'}$ is β-flat, and moreover $(P_\mathfrak{p})^\sharp = (P^\sharp)^\sharp_\mathfrak{p}$ is still fine (lemma 6.1.20(iv)). Hence we may replace α by β, which (in view of remark 6.1.15(i)) allows to assume that $\mathfrak{q}$ (resp. $\mathfrak{q}'$) is the maximal ideal of P (resp. of A). Next, let $P_0 := P/\mathfrak{p}$, $A_0 := A/\mathfrak{p}A$ and denote by $\alpha_0 : P_0 \to A_0$ the morphism induced from α; it is easily seen that A_0 is α_0-flat : for instance, the natural map $(\mathfrak{m}_P^n/\mathfrak{m}_P^{n+1}) \otimes_P A_0 \to \mathfrak{m}_P^n A_0/\mathfrak{m}_P^{n+1} A_0$ is of the type $f \otimes_A A_0$, where f is the map in proposition 6.4.31(c), thus if the latter is bijective, so is the former. Moreover $P_0^\sharp$ is a quotient of $P^\sharp$, hence it is again fine. Therefore we may replace P by P_0 and A by A_0, which allows to further assume that $\mathfrak{p} = \{0\}$, and it suffices to show that there exists a prime ideal $\mathfrak{q}' \subset A$, such that $\alpha^{-1}\mathfrak{q}' = \{0\}$. Set $\Sigma := P \setminus \{0\}$; it is easily seen that the natural morphism $P \to \Sigma^{-1}P$ is injective; moreover, its cokernel C (in the category of pointed P-modules) is integral, so that $\mathrm{Tor}_1^{\mathbb{Z}\langle P\rangle}(A, \mathbb{Z}\langle C\rangle) = 0$ by assumption. It follows that the localization map $A \to \Sigma^{-1}A$ is injective, especially $\Sigma^{-1}A \neq \{0\}$, and therefore it contains a prime ideal $\mathfrak{q}''$. The prime ideal $\mathfrak{q}' := \mathfrak{q}'' \cap A$ will do. □

6.4.35. *Fractional ideals.* Let P be an integral monoid. A *fractional ideal* of P is a P-submodule $I \subset P^{\mathrm{gp}}$ such that $I \neq \varnothing$ and $x \cdot I \subset P$ for some $x \in P$. Clearly the union and the intersection of finitely many fractional ideals, are again fractional ideals. We may also define the product of two fractional ideals $I_1, I_2 \subset P^{\mathrm{gp}}$: namely, the subset

$$I_1 I_2 := \{xy \mid x \in I_1, y \in I_2\} \subset P^{\mathrm{gp}}$$

which is again a fractional ideal, by an easy inspection. If I is a fractional ideal of P, we say that I is *finitely generated*, if it is such, when regarded as a P-module. For any two fractional ideals I_1, I_2, we let

$$(I_1 : I_2) := \{x \in P^{\rm gp} \mid x \cdot I_2 \subset I_1\}.$$

It is easily seen that $(I_1 : I_2)$ is a fractional ideal of P (if $x \in I_2$ and $yI_1 \subset P$, then clearly $xy(I_1 : I_2) \subset P$). We set

$$I^{-1} := (P : I) \qquad \text{and} \qquad I^* := (I^{-1})^{-1} \qquad \text{for every fractional ideal } I \subset P^{\rm gp}.$$

Clearly $J^{-1} \subset I^{-1}$, whenever $I \subset J$, and $I \subset I^*$ for all fractional ideals I, J. We say that I is *reflexive* if $I = I^*$. We remark that I^{-1} is reflexive, for every fractional ideal $I \subset P^{\rm gp}$. Indeed, we have $I^{-1} \subset (I^{-1})^*$, and on the other hand $(I^{-1})^* = (I^*)^{-1} \subset I^{-1}$. It follows that I^* is reflexive, for every fractional ideal I. Moreover, $I^* \subset J^*$, whenever $I \subset J$; especially, I^* is the smallest reflexive fractional ideal containing I. Notice furthermore, that $aI^{-1} = (a^{-1}I)^{-1}$ for every $a \in P^{\rm gp}$; therefore, $aI^* = (aI)^*$, for every fractional ideal I and $a \in P^{\rm gp}$.

Lemma 6.4.36. *Let P be any integral monoid, $I, J \subset P^{\rm gp}$ two fractional ideals. Then :*

(i) $(IJ)^* = (I^*J^*)^*$.

(ii) *I^* is the intersection of the invertible fractional ideals of P that contain I (see definition* 4.8.6(iv)*).*

Proof. (i): Since $IJ \subset I^*J^*$, we have $(IJ)^* \subset (I^*J^*)^*$. To show the converse inclusion, it suffices to check that $I^*J^* \subset (IJ)^*$, since $(IJ)^*$ is reflexive, and $(I^*J^*)^*$ is the smallest reflexive fractional ideal containing I^*J^*. Now, let $a \in I$ be eny element; we get $aJ^* = (aJ)^* \subset (IJ)^*$, so $IJ^* \subset (IJ)^*$, and therefore $(IJ^*)^* \subset (IJ)^*$. Lastly, let $b \in J^*$ be any element; we get $bI^* = (bI)^* \subset (IJ^*)^*$, so $I^*J^* \subset (IJ^*)^*$, whence the lemma.

(ii): It suffices to unwind the definitions. Indeed, $a \in P^{\rm gp}$ lies in I^* if and only if $aI^{-1} \subset P$, if and only if $ab \in P$, for every $b \in P^{\rm gp}$ such that $bI \subset P$. In other words, $a \in I^*$ if and only if $a \in b^{-1}P$ for every $b \in P^{\rm gp}$ such that $I \subset b^{-1}P$, which is the contention. □

6.4.37. Let P be any integral monoid. We denote by $\mathrm{Div}(P)$ the set of all reflexive fractional ideals of P. We define a composition law on $\mathrm{Div}(P)$ by the rule :

$$I \odot J := (IJ)^* \qquad \text{for every } I, J \in \mathrm{Div}(P).$$

It follows easily from lemma 6.4.36(i) that $\odot$ is an associative law; indeed we may compute :

$$(I \odot J) \odot K = ((IJ)^*K)^* = (IJK)^* = (I(JK)^*)^* = I \odot (J \odot K)$$

for every $I, J, K \in \mathrm{Div}(P)$. Clearly $I \odot J = J \odot I$ and $P \odot I = I$, for every $I, J \in \mathrm{Div}(P)$, so $(\mathrm{Div}(P), \odot)$ is a commutative monoid. Notice as well that if $I \subset P$, then also $I^* \subset P$ (lemma 6.4.36(ii)), so the subset of all reflexive fractional ideals contained in P is a submonoid $\mathrm{Div}_+(P) \subset \mathrm{Div}(P)$.

Example 6.4.38. Let A be an integral domain, and K the field of fractions of A. Classically, one defines the notion of *fractional ideal* : see *e.g.* [126, p.80] and [34, Ch.VII, §1, no.1], but notice that the definitions in these two references differ slightly, as the zero ideal is a fractional ideal according to the latter, but not according to the former. Additionally, one has a notion of *reflexive fractional ideal* of A, which are also called *divisorial fractional ideals* in [34, Ch.VII, §1, no.1]. In our terminology, these are understood as follows. Set $A' := A \cap K^\times$, and notice that the monoid $(A, \cdot)$ is naturally isomorphic to the integral pointed monoid $A'_\circ$. Then a fractional ideal of A is an $A'_\circ$-submodule of $K^\times_\circ = K$ of the form $I_\circ$, where $I \subset K^\times$ is a fractional ideal of A'. Likewise one may define the reflexive fractional ideals of A. The set $\mathrm{Div}(A)$ of all reflexive fractional ideals of A is then endowed with the unique monoid structure, such that the map $\mathrm{Div}(A') \to \mathrm{Div}(A)$ given by the rule $I \mapsto I_\circ$ is an isomorphism of monoids.

Lemma 6.4.39. *Let P be an integral monoid, and $G \subset P^\times$ a subgroup. We have :*

(i) *The rule $I \mapsto I/G$ induces a bijection from the set of fractional ideals of P to the set of fractional ideals of P/G.*
(ii) *A fractional ideal I of P is reflexive if and only if the same holds for I/G.*
(iii) *The rule $I \mapsto I/G$ defines an isomorphism of monoids*
$$\mathrm{Div}(P) \xrightarrow{\sim} \mathrm{Div}(P/G).$$
(iv) *If $P^\sharp$ is fine, every fractional ideal of P is finitely generated.*

Proof. The first assertion is left to the reader. Next, we remark that $I^{-1}/G = (I/G)^{-1}$ and $(IJ)/G = (I/G) \cdot (J/G)$, for every fractional ideals I, J of P, which imply immediately assertions (ii) and (iii). Lastly, suppose that $P^\sharp$ is finitely generated, and let I be any fractional ideal of P; pick $x \in I^{-1}$; since P is integral, I is finitely generated if and only if the same holds for xI. Hence, in order to show (iv), we may assume that $I \subset P$, in which case the assertion follows from proposition 6.1.9(ii) and lemma 6.1.17(i.a). □

In order to characterize the monoids P such that $\mathrm{Div}(P)$ is a group, we make the following :

Definition 6.4.40. Let P be an integral monoid, and $a \in P^{\mathrm{gp}}$ any element.

(a) We say that a is *power-bounded*, if there exists $b \in P$ such that $a^n b \in P$ for all $n \in \mathbb{N}$.
(b) We say that P is *completely saturated*, if all power-bounded elements of P^{gp} lie in P.

Example 6.4.41. Let $(\Gamma, \leq)$ be an ordered abelian group, and set $\Gamma^+ := \{\gamma \in \Gamma \mid \gamma \leq 1\}$. Then Γ^+ is always a saturated monoid, but it is completely saturated if and only if the convex rank of Γ is ≤ 1 (see [75, Def.6.1.20]). The proof shall be left as an exercise for the reader.

Proposition 6.4.42. *Let P be an integral monoid. We have :*

(i) *$(\mathrm{Div}(P), \odot)$ is an abelian group if and only if P is completely saturated.*
(ii) *If P is fine and saturated, then P is completely saturated.*
(iii) *Let A be a Krull domain, and set $A' := A\backslash\{0\}$. Then $(A', \cdot)$ is a completely saturated monoid.*

Proof. (i): Suppose that $I \in \mathrm{Div}(I)$ admits an inverse J in the monoid $(\mathrm{Div}(P), \odot)$, and notice that $I \odot I^{-1} \subset P$; it follows easily that $I \odot (J \cup I^{-1})^* = P$, hence $I^{-1} \subset J$, by the uniqueness of the inverse. On the other hand, if J strictly contains I^{-1}, then IJ strictly contains P, which is absurd. Thus, we see that $\mathrm{Div}(P)$ is a group if and only if $I \odot I^{-1} = P$ for every $I \in \mathrm{Div}(P)$. Now, suppose first that P is completely saturated. In view of lemma 6.4.36(ii), we are reduced to showing that P is contained in every invertible fractional ideal containing $I^{-1}I$. Hence, say that $I^{-1}I \subset aP$ for some $a \in P^{\mathrm{gp}}$; equivalently, we have $a^{-1}I^{-1}I \subset P$, *i.e.* $a^{-1}I^{-1} \subset I^{-1}$, and then $a^{-k}I^{-1} \subset I^{-1}$ for every integer $k \in \mathbb{N}$. Say that $b \in I^{-1}$ and $c \in I^*$; we conclude that $a^{-k}bc \in P$ for every $k \in \mathbb{N}$, so $a^{-1} \in P$, by assumption, and finally $P \subset aP$, as required.

Conversely, suppose that $\mathrm{Div}(P)$ is a group, and let $a \in P^{\mathrm{gp}}$ be any power-bounded element. By definition, this means that the P-submodule I of P^{gp} generated by $(a^k \mid k \in \mathbb{N})$ is a fractional ideal of P. Then I^{-1} is a reflexive fractional ideal, and by assumption I^{-1} admits an inverse, which must be I^*, by the foregoing. On the other hand, by construction we have $aI \subset I$, hence $aI^* = (aI)^* \subset I^*$. We deduce that $aP = a(I^* \odot I^{-1}) = aI^* \odot I^{-1} \subset I^* \odot I^{-1} = P$, *i.e.* $a \in P$, as stated.

(ii) is a special case of the following :

Claim 6.4.43. Let P be any fine and saturated monoid, $M \subset P^{\mathrm{gp}}$ a non-empty finitely generated P-submodule, and $a \in P^{\mathrm{gp}}$ an element such that $aM \subset M$. Then $a \in P$.

Proof of the claim. Pick any $m \in M$, and denote by $M' \subset M$ the submodule generated by $(a^k m \mid k \in \mathbb{N})$. According to proposition 6.1.9(i), there exists $N \geq 0$ such that M' is generated by the finite system $(a^k m \mid k = 0, \dots, N)$. Especially, $a^{N+1}m \in M'$, and therefore there exist $x \in P$ and $i \leq N$ such that $a^{N+1}m = a^i mx$ in M; it follows that $a^{N+1-i} \in P$, and finally $a \in P$, since P is saturated. ◊

(iii): See [126, §12] for the basic generalities on Krull domains. One is immediately reduced to the case where A is a valuation ring whose value group Γ has rank ≤ 1. Taking into account (i) and lemma 6.4.39, it then suffices to show that the monoid $A'/A^\times$ is completely sataurated. However, the latter is isomorphic to the submonoid Γ^+ of elements ≤ 1 in Γ, so the assertion follows from example 6.4.41. □

6.4.44. Let $\varphi : P \to Q$ be a morphism of integral monoids, and I any fractional ideal of P; notice that $IQ := \varphi^{\mathrm{gp}}(I)Q \subset Q^{\mathrm{gp}}$ is a fractional ideal of Q. Moreover, the identities

$$(I_1 \cup I_2)Q = I_1Q \cup I_2Q \qquad (I_1 I_2)Q = (I_1 Q) \cdot (I_2 Q) \qquad \text{for all fractional ideals } I_1, I_2 \subset P^{\mathrm{gp}}$$

are immediate from the definitions. Likewise, if A an integral domain and $\alpha : P \to (A \setminus \{0\}, \cdot)$ a morphism of monoids, then the A-submodule $IA := \alpha^{\mathrm{gp}}(I)A$ of the field of fractions of A is a fractional ideal of the ring A (in the usual commutative algebraic meaning : see example 6.4.38), and we have corresponding identities :

$$(I_1 \cup I_2)A = I_1A + I_2A \qquad (I_1 I_2)A = (I_1 A) \cdot (I_2 A) \qquad \text{for all fractional ideals } I_1, I_2 \subset P^{\mathrm{gp}}.$$

Lemma 6.4.45. *In the situation of* (6.4.44)*, suppose that φ is flat and A is α-flat, and let $I, J, J' \subset P^{\mathrm{gp}}$ be three fractional ideals, with I finitely generated. Then we have :*

(i) $(J : I)Q = (JQ : IQ)$ *and* $(J : I)A = (J : I)A$.
(ii) *Especially, if I is reflexive, the same holds for IQ and IA (see example* 6.4.38*).*
(iii) *Suppose furthermore that A is local, and α is a local morphism. Then $JA = J'A$ if and only if $J = J'$.*
(iv) *If P is fine, the rule $I \mapsto IQ$ and $I \mapsto IA$ define morphisms of monoids*

$$\mathrm{Div}(\varphi) : \mathrm{Div}(P) \to \mathrm{Div}(Q) \qquad \mathrm{Div}(\alpha) : \mathrm{Div}(P) \to \mathrm{Div}(A)$$

(where $\mathrm{Div}(A)$ *is defined as in example* 6.4.38*), and* $\mathrm{Div}(\alpha)$ *is injective, if α is local and A is a local domain.*

Proof. (i): Say that $I = a_1P \cup \cdots \cup a_nP$ for elements $a_1, \dots, a_n \in P^{\mathrm{gp}}$. Then

$$(J : I) = a_1^{-1}J \cap \cdots \cap a_n^{-1}J \qquad \text{and} \qquad (JQ : IQ) = a_1^{-1}JQ \cap \cdots \cap a_n^{-1}JQ$$

and likewise for $(J : I)A$, hence the assertion follows from an easy induction, and the following

Claim 6.4.46. For any two fractional ideals $J_1, J_2 \subset P$, we have $(J_1 \cap J_2)Q = J_1Q \cap J_2Q$ and $(J_1 \cap J_2)A = J_1A \cap J_2A$.

Proof of the claim. Pick any $x \in P$ such that $xJ_1, xJ_2 \subset P$; since P is an integral monoid, and A is an integral domain, it suffices to show that $x(J_1 \cap J_2)Q = xJ_1Q \cap xJ_2Q$ and likewise for $x(J_1 \cap J_2)A$, and notice that $x(J_1 \cap J_2) = xJ_1 \cap xJ_2$. We may thus assume that J_1 and J_2 are ideals of P, in which case the assertion is lemma 6.1.37. ◊

(ii): Suppose that I is reflexive; from (i) we deduce that $((IA)^{-1})^{-1} = IA$. The assertion is an immediate consequence, once one remarks that, for any fractional ideal $J \subset A$, there is a natural isomorphism of A-modules : $J^{-1} \xrightarrow{\sim} J^\vee := \mathrm{Hom}_A(J, A)$. Indeed, the isomorphism assigns to any $x \in J^{-1}$ the map $\mu_x : J \to A : a \mapsto xa$ for every $a \in J$ (details left to the reader).

(iii): We may assume that $JA = J'A$, and we prove that $J = J'$, and by replacing J' by $J \cup J'$, we may assume that $J \subset J'$. Then the contention follows easily from lemma 6.1.36.

(iv): This is immediate from (i) and (iii). □

Remark 6.4.47. In the situation of lemma 6.4.45(iv), obviously $\mathrm{Div}(\varphi)$ restricts to a morphism of submonoids :

$$\mathrm{Div}_+(\varphi) : \mathrm{Div}_+(P) \to \mathrm{Div}_+(Q).$$

6.4.48. Next, suppose that P is fine and saturated, and let $I \subset P^{\mathrm{gp}}$ be any fractional ideal. Then theorem 6.4.18(i) easily implies that :

$$I^{-1} = \bigcap_{\mathrm{ht}\,\mathfrak{p}=1} (I_\mathfrak{p})^{-1}$$

where the intersection – running over the prime ideals of P of height one – is taken within $\mathrm{Hom}_P(I, P^{\mathrm{gp}})$, which naturally contains all the $(I_\mathfrak{p})^{-1}$. The structure of the fractional ideals of $P_\mathfrak{p}$ when $\mathrm{ht}\,\mathfrak{p} = 1$ is very simple : quite generally, theorem 6.4.18(ii) easily implies that if $\dim P = 1$, then all fractional ideals are cyclic, and then clearly they are reflexive. On the other hand, I is finitely generated, by lemma 6.4.39(iv). We deduce that I is reflexive if and only if :

$$I = \bigcap_{\mathrm{ht}\,\mathfrak{p}=1} I_\mathfrak{p}. \tag{6.4.49}$$

Indeed, suppose that (6.4.49) holds; then we have $I^* = \bigcap_{\mathrm{ht}\,\mathfrak{p}=1} (I_\mathfrak{p}^{-1})^{-1} = \bigcap_{\mathrm{ht}\,\mathfrak{p}=1} I_\mathfrak{p}$, since we have just seen that $I_\mathfrak{p}$ is a reflexive fractional ideal of $P_\mathfrak{p}$, for every prime ideal $\mathfrak{p}$ of height one.

Proposition 6.4.50. *Let P be a fine and saturated monoid, and denote by $D \subset \mathrm{Spec}\,P$ the subset of all prime ideals of height one. Then the mapping :*

$$\mathbb{Z}^{\oplus D} \to \mathrm{Div}(P) \quad : \quad \sum_{\mathrm{ht}\,\mathfrak{p}=1} n_\mathfrak{p}[\mathfrak{p}] \mapsto \bigcap_{\mathrm{ht}\,\mathfrak{p}=1} \mathfrak{m}_{P_\mathfrak{p}}^{n_\mathfrak{p}} \tag{6.4.51}$$

is an isomorphism of abelian groups.

Proof. Here $\mathfrak{m}_{P_\mathfrak{p}} \subset P_\mathfrak{p}$ is the maximal ideal, and for $n \geq 0$, the notation $\mathfrak{m}_{P_\mathfrak{p}}^n$ means the usual n-th power operation in the monoid $\mathscr{P}(P^{\mathrm{gp}})$, which we extend to all integers n, by letting $\mathfrak{m}_{P_\mathfrak{p}}^n := \mathfrak{m}_{P_\mathfrak{p}}^{-n}$ whenever $n < 0$.

In order to show that (6.4.51) is well defined, set $I := \bigcap_{\mathrm{ht}\,\mathfrak{p}=1} \mathfrak{m}_\mathfrak{p}^{n_\mathfrak{p}}$. Pick, for every $\mathfrak{p}$ such that $n_\mathfrak{p} < 0$, an element $x_\mathfrak{p} \in \mathfrak{p}$, and set $y_\mathfrak{p} := x_\mathfrak{p}^{-n_\mathfrak{p}}$; if $n_\mathfrak{p} \geq 0$, set $y_\mathfrak{p} := 1$. Then it is easy to check (using theorem 6.4.18(i)) that $\prod_{\mathrm{ht}\,\mathfrak{p}=1} y_\mathfrak{p}$ lies in I^{-1}, hence I is a fractional ideal. Next, for given $\mathfrak{p}, \mathfrak{p}' \in D$, notice that $(P_\mathfrak{p})_{\mathfrak{p}'} = P^{\mathrm{gp}}$; it follows that

$$I_\mathfrak{p} = \mathfrak{m}_\mathfrak{p}^{n_\mathfrak{p}} \qquad \text{for every } \mathfrak{p} \in D \tag{6.4.52}$$

therefore I is reflexive. Furthermore, it is easily seen (from theorem 6.4.18(ii)), that every reflexive ideal of $P_\mathfrak{p}$ is of the form $\mathfrak{m}_P^n$ for some integer n, and moreover $\mathfrak{m}_P^n = \mathfrak{m}_P^m$ if and only if $n = m$. Then (6.4.49) implies that the mapping (6.4.51) is surjective, and the injectivity follows from (6.4.52). It remains to check that (6.4.51) is a group homomorphism, and to this aim we may assume – in view of lemma 6.4.45(iv) – that $\dim P = 1$, in which case the assertion is immediate. □

6.4.53. A morphism $\varphi : I \to J$ of fractional ideals of P is, by definition, a morphism of P-modules. Let $x, y \in I$ be any two elements; we may find $a, b \in P$ such that $ax = by$ in I, and therefore $a\varphi(x) = \varphi(ax) = \varphi(by) = b\varphi(y)$; thus, $\varphi(y) = (b^{-1}a) \cdot \varphi(x) = (x^{-1}y) \cdot \varphi(x)$. This shows that, for every morphism $\varphi : I \to J$ of fractional ideals, there exists $c \in P^{\mathrm{gp}}$ such that $\varphi(x) = cx$ for every $x \in I$. Especially, $I \simeq J$ if and only if there exists $a \in P^{\mathrm{gp}}$ such that $I = aJ$. Likewise one may characterize the morphisms and isomorphisms of fractional ideals of an integral domain. We denote

$$\overline{\mathrm{Div}}(P)$$

the set of isomorphism classes of reflexive fractional ideals of P. From the foregoing, it is clear that if $I \simeq I'$, we have $I \odot J \simeq I' \odot J$ for every $J \in \mathrm{Div}(P)$; therefore the composition law of $\mathrm{Div}(P)$ descends to a composition law for $\overline{\mathrm{Div}}(P)$, which makes it into a (commutative) monoid, and if P is completely saturated, then $\overline{\mathrm{Div}}(P)$ is an abelian group. We also deduce an exact sequence of monoids

$$1 \to P^\times \to P^{\mathrm{gp}} \xrightarrow{j_P} \mathrm{Div}(P) \to \overline{\mathrm{Div}}(P) \to 1 \qquad (6.4.54)$$

where j_P is given by the rule A: $a \mapsto aP$ for every $a \in P$; especially, j_P restricts to a morphism of monoids

$$j_P^+ : P \to \mathrm{Div}_+(P).$$

Likewise, we define $\overline{\mathrm{Div}}(A)$, for any integral domain A : see example 6.4.38. Moreover, in the situation of (6.4.44), we see from lemma 6.4.45(iv) that if α is local, P is fine, A is α-flat, and φ is flat, then $\mathrm{Div}(\varphi)$ and $\mathrm{Div}(\alpha)$ descend to well defined morphisms of monoids

$$\overline{\mathrm{Div}}(\varphi) : \overline{\mathrm{Div}}(P) \to \overline{\mathrm{Div}}(Q) \qquad \overline{\mathrm{Div}}(\alpha) : \overline{\mathrm{Div}}(P) \to \overline{\mathrm{Div}}(A).$$

Proposition 6.4.55. *Let P be a fine and saturated monoid, $I, J \subset P^{\mathrm{gp}}$ two fractional ideals, A a local integral domain, and $\alpha : P \to (A, \cdot)$ a local morphism of monoids. We have :*

(i) $(I : I) = P$.

(ii) *Suppose that A is α-flat. Then $IA \simeq JA$ if and only if $I \simeq J$. Especially, in this case $\overline{\mathrm{Div}}(\alpha)$ is an injective map.*

Proof. (i): Clearly it suffices to show that $(I : I) \subset P$. Hence, say that $x \in (I : I)$, and pick any $a \in I$; it follows that $x^n a \in P$ for every $n > 0$; in the additive group $\log P^{\mathrm{gp}}$ we have therefore the identity $n \cdot \log(x) + \log(a) \in \log P$, so $\log(x) + n^{-1}\log(a) \in (\log P)_{\mathbb{R}}$ for every $n > 0$. Since $(\log P)_{\mathbb{R}}$ is a convex polyhedral cone in $(\log P^{\mathrm{gp}})_{\mathbb{R}}$, we deduce that $x \in (\log P)_{\mathbb{R}} \cap (\log P^{\mathrm{gp}}) = \log P$ (proposition 6.3.22(iii)), as claimed.

(ii): We may assume that IA is isomorphic to JA, and we show that I is isomorphic to J. Indeed, the assumption means that $a(IA) = JA$ for some $x \in \mathrm{Frac}(A)$; therefore, $a \in (JA : IA)$ and $a^{-1} \in (IA : JA)$, so

$$A = (IA : JA) \cdot (JA : IA) = ((I : J) \cdot (J : I))A$$

by virtue of lemma 6.4.45(i). Since A is local, it follows that there exist $a \in (I : J)$ and $b \in (J : I)$ such that $\alpha(ab) \in A^\times$, whence $ab \in P^\times$, since α is local. It follows easily that $I = aJ$, as asserted. □

Example 6.4.56. (i) Let P be a fine and saturated monoid, and $D \subset \mathrm{Spec}\, P$ the subset of all prime ideals of height one; for every $\mathfrak{p} \in D$, denote

$$v_{\mathfrak{p}} : P \to P_{\mathfrak{p}}^\sharp \xrightarrow{\sim} \mathbb{N}$$

the composition of the localization map, and the natural isomorphism resulting from theorem 6.4.18(ii). A simple inspection shows that the isomorphism (6.4.51) identifies the map j_P of (6.4.54) with the morphism of monoids

$$v_P : P^{\mathrm{gp}} \to \mathbb{Z}^{\oplus D} \qquad x \mapsto (v_{\mathfrak{p}}^{\mathrm{gp}}(x) \mid \mathfrak{p} \in D).$$

With this notation, the isomorphism (6.4.51) is the map given by the rule :

$$k_\bullet \mapsto v_P^{-1}(k_\bullet + \mathbb{N}^{\oplus D}) \qquad \text{for every } k_\bullet \in \mathbb{Z}^{\oplus D}.$$

(ii) Suppose now that P is sharp and $\dim P = 2$, in which case $D = \{\mathfrak{p}_1, \mathfrak{p}_2\}$ contains exactly two elements. According to example 6.4.19(ii), we may find a basis f_1, f_2 of P^{gp}, such that the two facets $P \setminus \mathfrak{p}_1$ and $P \setminus \mathfrak{p}_2$ of P are generated respectively by $e_1 := f_1$ and $e_2 := af_1 + bf_2$, for some $a, b \in \mathbb{N}$, with $a < b$ and $(a,b) = 1$. It follows easily that P is a submonoid of the free monoid

$$Q := \mathbb{N}e'_1 \oplus \mathbb{N}e'_2 \qquad \text{where } e'_1 := b^{-1}e_1 \text{ and } e'_2 := b^{-1}e_2$$

and $Q^{\mathrm{gp}}/P^{\mathrm{gp}} \simeq \mathbb{Z}/b\mathbb{Z}$ (details left to the reader). The induced map $\mathrm{Spec}\, Q \to \mathrm{Spec}\, P$ is a homeomorphism; especially Q admits two prime ideals $\mathfrak{q}_1$, $\mathfrak{q}_2$ of height one, so that $\mathfrak{q}_i \cap P = \mathfrak{p}_i$ for $i = 1, 2$, whence – by proposition 6.4.50 – a natural isomorphism

$$s^* : \mathrm{Div}(Q) \xrightarrow{\sim} \mathrm{Div}(P)$$

and notice that $j_Q : Q^{\mathrm{gp}} \to \mathrm{Div}(Q)$ is the isomorphism given by the rule : $e'_i \mapsto \mathfrak{q}_i$ for $i = 1, 2$. Moreover, we have commutative diagrams of monoids :

$$\begin{array}{ccc} P & \xrightarrow{s} & Q \\ {\scriptstyle v_{\mathfrak{p}_i}}\downarrow & & \downarrow{\scriptstyle v_{\mathfrak{q}_i}} \\ \mathbb{N} & \xrightarrow{t_i} & \mathbb{N} \end{array} \qquad (i = 1, 2).$$

Clearly, $Q \setminus \mathfrak{q}_i$ is the facet generated by e'_i, so $v_{\mathfrak{q}_i}$ is none else than the projection onto the direct factor $\mathbb{N}e'_{3-i}$, for $i = 1, 2$. In order to compute $v_{\mathfrak{p}_i}$, it then suffices to determine t_i, or equivalently t_i^{gp}. However, set $\tau_i := v_{\mathfrak{q}_i}^{\mathrm{gp}} \circ s^{\mathrm{gp}}$; clearly $\tau_1(f_2) = \tau_1(e'_2 - ae'_1) = 1$, so τ_1 is surjective. Also, $\tau_2(f_1) = b$ and $\tau_2(f_2) = -a$, so τ_2 is surjective as well; therefore both t_1 and t_2 are the identity endomorphism of $\mathbb{N}$. Summing up, we find that

$$j_P = s^* \circ j_Q \circ s^{\mathrm{gp}}$$

and the morphism v_P is naturally identified with $s^{\mathrm{gp}} : P^{\mathrm{gp}} \to Q^{\mathrm{gp}}$. Especially, we have obtained a natural isomorphism

$$\overline{\mathrm{Div}}(P) \xrightarrow{\sim} \mathbb{Z}/b\mathbb{Z}.$$

We may then rephrase in more intrinsic terms the classification of example 6.4.19(iii) : namely the isomorphism class of P is completely determined by the datum of $\overline{\mathrm{Div}}(P)$ and the equivalence class of the height one prime ideals of P in the quotient set $\overline{\mathrm{Div}}(P)^\dagger$ defined as in *loc.cit.*

(iii) In the situation of (ii), a simple inspection yields the following explicit description of all reflexive fractional ideals of P. Recall that such ideals are of the form

$$I_{k_1,k_2} := \mathfrak{m}_1^{k_1} \cap \mathfrak{m}_2^{k_2} = \{x \in P^{\mathrm{gp}} \mid v_{\mathfrak{p}_1}(x) \geq k_1,\ v_{\mathfrak{p}_2}(x) \geq k_2\}$$

where $\mathfrak{m}_i$ is the maximal ideal of $P_{\mathfrak{p}_i}$, and $k_i \in \mathbb{Z}$, for $i = 1, 2$. Then

$$I_{k_1,k_2} = \{x_1e_1 + x_2e_2 \mid x_1, x_2 \in b^{-1}\mathbb{Z},\ x_1 \geq b^{-1}k_2,\ x_2 \geq b^{-1}k_1\} \cap P^{\mathrm{gp}} \qquad \text{for all } k_1, k_2 \in \mathbb{Z}.$$

With this notation, the cyclic reflexive ideals are then those of the form

$$(x_1f_1 + x_2f_2)P = I_{x_2b, x_1b - x_2a} \qquad \text{with } x_1, x_2 \in \mathbb{Z}.$$

Especially, we see that the classes of $\mathfrak{p}_1 = I_{0,1}$ and $\mathfrak{p}_2 = I_{1,0}$ are both of order b in $\overline{\mathrm{Div}}(P)$.

Lemma 6.4.57. *Let P be a fine and saturated monoid of dimension 2, and denote by b the order of the finite cyclic group $\overline{\mathrm{Div}}(P)$. We have :*

$$\mathfrak{m}_P^{[b/2]} \subset I \cdot I^{-1} \qquad \textit{for every } I \in \mathrm{Div}(P)$$

(where $[b/2]$ denotes the largest integer $\leq b/2$).

Proof. Notice first that the assertion holds for a given $I \in \mathrm{Div}(P)$, if and only if it holds for xI, for any $x \in P^{\mathrm{gp}}$. If $b = 1$, then $P = \mathbb{N}^{\oplus 2}$, in which case $\overline{\mathrm{Div}}(P) = 0$, so every reflexive fractional ideal of P is isomorphic to P, and the assertion is clear. Hence, assume that $b > 1$; let $\mathfrak{p}_1$, $\mathfrak{p}_2$ be the two prime ideals of height one of P, and define Q, e_1, e_2 and I_{k_1,k_2} for every $k_1, k_2 \in \mathbb{Z}$, as in example 6.4.56(ii,iii). With this notation, notice that

$$\mathfrak{m}_P \setminus \mathfrak{m}_P^2 = \{e_1, e_2\} \cup \Sigma \qquad \text{where } \Sigma \subset \{x_1e_1 + x_2e_2 \mid x_1, x_2 \in b^{-1}\mathbb{Z},\ 0 \leq x_1, x_2 < 1\}.$$

It follows easily that, for every $i \in \mathbb{N}$, every element of $\mathfrak{m}_P^i$ is of the form $x_1e_1 + x_2e_2$ with $x_1, x_2 \in b^{-1}\mathbb{N}$ and $\max(x_1, x_2) \geq b^{-1}i$. Hence, let $I \in \mathrm{Div}(P)$ and $x := x_1e_1 + x_2e_2 \in \mathfrak{m}_P^{[b/2]}$, and say that $bx_1 \geq [b/2]$. According to example 6.4.56(iii), we may assume that $I = \mathfrak{p}_2^j = I_{j,0}$ for some $j \in \{0, \dots, b-1\}$. Moreover, notice that the assertion holds for I if and only if it holds for I^{-1}, whose class in $\overline{\mathrm{Div}}(P)$ agrees with the class of $\mathfrak{p}_2^{b-j}$. Clearly, either $j \leq [b/2]$ or $b - j \leq [b/2]$; hence, we may assume that $j \in \{0, \dots, [b/2]\}$. Thus, $P \subset I^{-1}$, and $I \subset I \cdot I^{-1}$, and clearly $x \in I$, so we are done in this case. The case where $bx_2 \geq [b/2]$ is dealt with in the same way, by writing $I = \mathfrak{p}_1^j$ for some non-negative $j \leq [b/2]$: the details are left to the reader. □

If $f : P \to Q$ is a general morphism of integral monoids, and I a fractional ideal of Q, the P-module $f^{\mathrm{gp}-1}(I)$ is not necessarily a fractional ideal of P (for instance, consider the natural map $P \to P^{\mathrm{gp}}$). One may obtain some positive results, by restricting to the class of morphisms introduced by the following :

Definition 6.4.58. Let $f : P \to Q$ be a morphism of monoids. We say that f is *of Kummer type*, if f is injective, and the induced map $f_\mathbb{Q} : P_\mathbb{Q} \to Q_\mathbb{Q}$ is surjective (notation of (6.3.20)).

Lemma 6.4.59. *Let $f : P \to Q$ be a morphism of monoids of Kummer type, $S_Q \subset Q$ a submonoid, and set $S_P := f^{-1}S_Q$. We have :*

(i) *The map* $\mathrm{Spec}\, f : \mathrm{Spec}\, Q \to \mathrm{Spec}\, P$ *is bijective; especially* $\dim P = \dim Q$.
(ii) *If $Q^\times$ is a torsion-free abelian group, P is the trivial monoid (resp. is sharp) if and only if the same holds for Q.*
(iii) *The induced morphism $S_P^{-1}P \to S_Q^{-1}Q$ is of Kummer type.*
(iv) *If P is integral, the unit of adjunction $P \to P^{\mathrm{sat}}$ is of Kummer type.*
(v) *Suppose that P is integral and saturated. Then $f^\sharp : P^\sharp \to Q^\sharp$ is of Kummer type.*
(vi) *If both P and Q are integral, and P is saturated, then f is exact.*

Proof. (ii) and (iv) are trivial, and (iii) is an exercise for the reader.

(i): Let $F, F' \subset Q$ be two faces such that $f^{-1}F = f^{-1}F'$, and say that $x \in F$. Then $x^n \in f(P)$ for some $n > 0$, so $x^n \in f(f^{-1}F')$, whence $x \in F'$, which implies that $\mathrm{Spec}\, f$ is injective. Next, for a given face F of P, let $F' \subset Q$ be the subset of all $x \in Q$ such that there exists $n > 0$ with $x^n \in f(F)$. It is easily seen that F' is a face of Q, and moreover $f^{-1}F' = F$, which shows that $\mathrm{Spec}\, f$ is also surjective.

(v): Clearly the map $(P^\sharp)_\mathbb{Q} \to (Q^\sharp)_\mathbb{Q}$ is surjective. Now, let $x, y \in P$ such that the images of $f(x)$ and $f(y)$ agree in $Q^\sharp$, *i.e.* there exists $u \in Q^\times$ with $u \cdot f(x) = f(y)$; we may find $n > 0$ such that $u^n, u^{-n} \in f(P)$. Say that $u^n = f(v)$, $u^{-n} = f(w)$; since $f(vw) = 1$, we have $vw = 1$, and moreover $f(vx^n) = f(y^n)$, so $vx^n = y^n$. Therefore $x^ny^{-n}, x^{-n}y^n \in P$, and since P is saturated we deduce that $xy^{-1}, x^{-1}y \in P$, so the images of x and y agree in $P^\sharp$.

(vi): Notice first that f^{gp} is injective, since the same holds for f. Suppose $x \in P^{\mathrm{gp}}$ and $f^{\mathrm{gp}}(x) \in Q$; we may then find an integer $k > 0$ and $y \in P$ such that $f(y) = f(x)^k$. Since P is saturated, it follows that $x \in P$, so f is exact. □

6.4.60. Suppose that $\varphi : P \to Q$ is a morphism of integral monoids of Kummer type, with P saturated, and let $I \subset Q^{\mathrm{gp}}$ be a fractional ideal. Then $\varphi^*I := \varphi^{\mathrm{gp}-1}(I)$ is a fractional ideal of P. Indeed, by assumption there exists $a \in Q$ such that $aI \subset Q$; we may find $k > 0$ and $b \in P$ such that $a^k = \varphi(b)$, so $\varphi(bx) \in \varphi(P)^{\mathrm{gp}} \cap Q = \varphi(P)$ for every $x \in \varphi^*I$, since φ is exact (lemma 6.4.59(v)); therefore $b \cdot \varphi^*(I) \subset P$.

6.4.61. In the situation of (6.4.60), suppose that both P and Q are fine and saturated, and let $\mathrm{gr}_\bullet Q^{\mathrm{gp}}$ be the φ-grading of Q, indexed by $(\Gamma, +) := Q^{\mathrm{gp}}/P^{\mathrm{gp}}$ (see remark 6.2.5(iii)); for every $x \in Q^{\mathrm{gp}}$, denote $\overline{x} \in \Gamma$ the image of x. Let also $I \subset Q^{\mathrm{gp}}$ be any fractional ideal, and denote by

$\mathrm{gr}_\bullet I$ the Γ-grading on I deduced from the φ-grading of Q^{gp}; arguing as in (6.4.60), it is easily seen that, more generally, $\varphi^*(x^{-1}\mathrm{gr}_{\overline{x}}I)$ is a fractional ideal of P, for every $x \in Q$ (details left to the reader). For every prime ideal $\mathfrak{q}$ of height one in Q, we have a commutative diagram of monoids :

(6.4.62)
$$\begin{array}{ccc} P & \xrightarrow{\varphi} & Q \\ {\scriptstyle v_{\varphi^{-1}\mathfrak{q}}}\downarrow & & \downarrow{\scriptstyle v_{\mathfrak{q}}} \\ \mathbb{N} & \xrightarrow{e_{\mathfrak{q}}} & \mathbb{N} \end{array}$$

where $v_{\mathfrak{q}}$ and $v_{\varphi^{-1}\mathfrak{q}}$ are defined as in example 6.4.56(i), and $e_{\mathfrak{q}}$ is the multiplication by a non-zero (positive) integer, which we call the *ramification index of* φ *at* $\mathfrak{q}$, and we denote also $e_{\mathfrak{q}}$.

Lemma 6.4.63. *In the situation of* (6.4.61)*, suppose that I is a reflexive fractional ideal. Then $\varphi^*(a^{-1} \cdot \mathrm{gr}_{\overline{a}}I)$ is a reflexive fractional ideal of P, for every $a \in Q^{\mathrm{gp}}$.*

Proof. Clearly, we may replace I by $a^{-1}I$, and reduce to the case where $a = 1$, in which case we have to check that φ^*I is a reflexive fractional ideal. However, according to example 6.4.56(i), we may write $I = v_Q^{-1}(k_\bullet + \mathbb{N}^{\oplus D})$, where $D \subset \mathrm{Spec}\, Q$ is the subset of the height one prime ideals, and $k_\bullet \in \mathbb{Z}^{\oplus D}$. Set
$$k'_\bullet := ([e_{\mathfrak{q}}^{-1}k_{\mathfrak{q}}] \mid \mathfrak{q} \in D)$$
(where, for a real number x, we let $[x]$ be the smallest integer $\geq x$). Since $\mathrm{Spec}\,\varphi$ is bijective (lemma 6.4.59(i)), the commutative diagrams (6.4.62) imply that $\varphi^*I = v_P^{-1}(k'_\bullet + \mathbb{N}^{\oplus D})$, whence the contention. $\square$

Example 6.4.64. Let P be as in example 6.4.56(ii), set $Q := \mathrm{Div}_+(P)$, and take $\varphi := j_P^+ : P \to Q$ (notation of (6.4.53)). The discussion of *loc.cit.* shows that φ is a morphism of Kummer type, and notice that the φ-grading of Q is indexed by $Q^{\mathrm{gp}}/P^{\mathrm{gp}} = \overline{\mathrm{Div}}(P)$. Now, pick any $x \in Q$, and let $\overline{x} \in \overline{\mathrm{Div}}(P)$ be the equivalence class of x; by lemma 6.4.63, the P-module $\mathrm{gr}_{\overline{x}}Q$ is isomorphic to a reflexive fractional ideal of P. We claim that the isomorphism class of $\mathrm{gr}_{\overline{x}}Q$ is precisely $\overline{x}^{-1}$ (where the inverse is formed in the commutative group $\overline{\mathrm{Div}}(P)$). Indeed, let $a \in P^{\mathrm{gp}}$ be any element; by definition, we have $a \in \varphi^*(x^{-1}\mathrm{gr}_{\overline{x}}Q)$ if and only if $\varphi^{\mathrm{gp}}(a) \in x^{-1}\mathrm{gr}_{\overline{x}}Q$, if and only if $ax \in Q$, if and only if $a \in x^{-1}$, whence the claim. Thus, the family
$$(\mathrm{gr}_\gamma \mathrm{Div}_+(P) \mid \gamma \in \overline{\mathrm{Div}}(P))$$
is a complete system of representatives for the isomorphism classes of the reflexive fractional ideals of a fine, sharp and saturated monoid P of dimension 2.

Remark 6.4.65. Further results on reflexive fractional ideals for monoids, and their divisor class groups can be found in [48].

6.5. **Fans.** According to Kato ([112, §9]), a fan is to a monoid what a scheme is to a ring. More prosaically, the theory of fans is a reformulation of the older theory of rational polyhedral decompositions, developed in [116].

Definition 6.5.1. (i) A *monoidal space* is a datum $(T, \mathscr{O}_T)$ consisting of a topological space T and a sheaf of monoids $\mathscr{O}_T$ on T.

(ii) A *morphism of monoidal spaces* is a datum
$$(f, \log f) : (T, \mathscr{O}_T) \to (S, \mathscr{O}_S)$$
consisting of a continuous map $f : T \to S$, and a morphism $\log f : f^*\mathscr{O}_S \to \mathscr{O}_T$ of sheaves of monoids that is *local*, *i.e.* whose stalk $(\log f)_t : \mathscr{O}_{S,f(t)} \to \mathscr{O}_{T,t}$ is a local morphism, for every $t \in T$. The *strict locus* of $(f, \log f)$ is the subset
$$\mathrm{Str}(f, \log f) \subset T$$

consisting of all $t \in T$ such that $(\log f)_t$ is an isomorphism.

(iii) We say that a monoidal space $(T, \mathscr{O}_T)$ is *sharp*, (resp. *integral*, resp. *saturated*) if $\mathscr{O}_T$ is a sheaf of sharp (resp. integral, resp. integral and saturated) monoids.

(iv) For any monoidal space (resp. integral monoidal space) $(T, \mathscr{O}_T)$, the *sharpening* (resp. the *saturation*) of $(T, \mathscr{O}_T)$ is the sharp monoidal space $(T, \mathscr{O}_T)^\sharp := (T, \mathscr{O}_T^\sharp)$ (resp. $(T, \mathscr{O}_T)^{\mathrm{sat}} := (T, \mathscr{O}_T^{\mathrm{sat}})$).

It is easily seen that the rule $(T, \mathscr{O}_T) \mapsto (T, \mathscr{O}_T)^\sharp$ extends to a functor from the category of monoidal spaces to the full subcategory of sharp monoidal spaces. This functor is right adjoint to the corresponding fully faithful embedding of categories.

Likewise, the functor $(T, \mathscr{O}_T) \mapsto (T, \mathscr{O}_T)^{\mathrm{sat}}$ is right adjoint to the fully faithful embedding of the category of saturated monoidal spaces, into the category of integral monoidal spaces.

6.5.2. Let P be any monoid; for every $f \in P$, let us set

$$D(f) := \{\mathfrak{p} \in \operatorname{Spec} P \mid f \notin \mathfrak{p}\}.$$

Notice that $D(f) \cap D(g) = D(fg)$ for every $f, g \in P$. We endow $\operatorname{Spec} P$ with the topology having a basis consisting of the subsets $D(f)$, for every $f \in P$. Notice that $\mathfrak{m}_P$ is the only closed point of $\operatorname{Spec} P$ (especially, $\operatorname{Spec} P$ is trivially quasi-compact).

By lemma 6.1.14, the localization map $j_f : P \to P_f$ induces an identification $j_f^* : \operatorname{Spec} P_f \xrightarrow{\sim} D(f)$. It is easily seen that $(j_f^*)^{-1} D(fg) = D(j(g)) \subset \operatorname{Spec} P_f$; in other words, the topology of $\operatorname{Spec} P_f$ agrees with the topology induced from $\operatorname{Spec} P$, via j^*.

Next, for every $f \in P$ we set :

$$\mathscr{O}_{\operatorname{Spec} P}(D(f)) := P_f.$$

We claim that $\mathscr{O}_{\operatorname{Spec} P}(D(f))$ depends only on the open subset $D(f)$, up to natural isomorphism (and not on the choice of f). More precisely, say that $D(f) \subset D(g)$ for two given elements $f, g \in P$; it follows that the image of g in P_f lies outside the maximal ideal $\mathfrak{m}_{P_f}$, hence $g \in P_f^\times$, and therefore the localization map $j_f : P \to P_f$ factors uniquely through a morphism of monoids :

$$j_{f,g} : P_g \to P_f.$$

Likewise, if $D(g) \subset D(f)$ as well, the localization $j_g : P \to P_g$ factors through a unique map $j_{g,f} : P_f \to P_g$, whence the identities :

$$j_{f,g} \circ j_{g,f} \circ j_f = j_f \qquad j_{g,f} \circ j_{f,g} \circ j_g = j_g$$

and since j_f and j_g are epimorphisms, we see that $j_{f,g}$ and $j_{g,f}$ are mutually inverse isomorphisms.

6.5.3. Say that $D(f) \subset D(g) \subset D(h)$ for some $f, g, h \in P$; by direct inspection, it is clear that $j_{f,g} \circ j_{g,h} = j_{f,h}$, so the rule $D(f) \mapsto P_f$ yields a well defined presheaf of monoids on the site $\mathscr{C}_P$ of open subsets of $\operatorname{Spec} P$ of the form $D(f)$ for some $f \in P$. Then $\mathscr{O}_{\operatorname{Spec} P}$ is trivially a sheaf on $\mathscr{C}_P$ (notice that if $D(f) = \bigcup_{i \in I} D(g_i)$ is an open covering of $D(f)$, then $D(g_i) = D(f)$ for some $i \in I$). According to [59, Ch.0, §3.2.2] it follows that $\mathscr{O}_{\operatorname{Spec} P}$ extends uniquely to a well defined sheaf of monoids on $\operatorname{Spec} P$, whence a monoidal space $(\operatorname{Spec} P, \mathscr{O}_{\operatorname{Spec} P})$. By inspecting the construction, we find natural identifications :

$$(6.5.4) \qquad (\mathscr{O}_{\operatorname{Spec} P})_\mathfrak{p} \xrightarrow{\sim} P_\mathfrak{p} \qquad \text{for every } \mathfrak{p} \in \operatorname{Spec} P$$

and moreover :

$$P \xrightarrow{\sim} \Gamma(\operatorname{Spec} P, \mathscr{O}_{\operatorname{Spec} P}).$$

It is also clear that the rule

$$(6.5.5) \qquad P \mapsto (\operatorname{Spec} P, \mathscr{O}_{\operatorname{Spec} P})$$

defines a functor from the category $\mathbf{Mnd}^o$ to the category of monoidal spaces.

Proposition 6.5.6. *The functor* (6.5.5) *is right adjoint to the functor :*

$$(T, \mathscr{O}_T) \mapsto \Gamma(T, \mathscr{O}_T)$$

from the category of monoidal spaces, to the category $\mathbf{Mnd}^o$.

Proof. Let $f : P \to \Gamma(T, \mathscr{O}_T)$ be a map of monoids. We define a morphism

$$\varphi_f := (\varphi_f, \log \varphi_f) : (T, \mathscr{O}_T) \to (\operatorname{Spec} P, \mathscr{O}_{\operatorname{Spec} P})$$

as follows. Given $t \in T$, let $f_t : P \to \mathscr{O}_{T,t}$ be the morphism deduced from f, and denote by $\mathfrak{m}_t \subset \mathscr{O}_{T,t}$ the maximal ideal. We set $\varphi_f(t) := f_t^{-1}(\mathfrak{m}_t)$. In order to show that φ_f is continuous, it suffices to prove that $U_s := \varphi_f^{-1}(D(s))$ is open in M_T, for every $s \in P$. However, $U_s = \{t \in T \mid f_t(s) \in \mathscr{O}_{T,t}^\times\}$, and it is easily seen that this condition defines an open subset (details left to the reader). Next, we define $\log \varphi_f$ on the basic open subsets $D(s)$. Indeed, let $j_s : \Gamma(T, \mathscr{O}_T) \to \mathscr{O}_T(U_s)$ be the natural map; by construction, $j_s \circ f(s)$ is invertible in $\mathscr{O}_T(U_s)$, hence $j_s \circ f$ extends to a unique map of monoids :

$$P_s = \mathscr{O}_{\operatorname{Spec} P}(D(s)) \to \varphi_{f*}\mathscr{O}_T(D(s)).$$

By [59, Ch.0, §3.2.5], the above rule extends to a unique morphism $\mathscr{O}_{\operatorname{Spec} P} \to \varphi_{f*}\mathscr{O}_T$ of sheaves of monoids, whence – by adjunction – a well defined morphism $\log \varphi_f : \varphi_f^* \mathscr{O}_{\operatorname{Spec} P} \to \mathscr{O}_T$. In order to show that $(\varphi_f, \log \varphi_f)$ is the sought morphism of monoidal spaces, it remains to check that $(\log \varphi_f)_t : P_{\varphi_f(t)} \to \mathscr{O}_{T,t}$ is a local morphism, for every $t \in T$. However, let $i_t : P \to P_{\varphi_f(t)}$ be the localization map; by construction, we have $(\log \varphi_f)_t \circ i_t = f_t$, and the contention is a straightforward consequence.

Conversely, say that $(\varphi, \log \varphi) : (T, \mathscr{O}_T) \to (\operatorname{Spec} P, \mathscr{O}_{\operatorname{Spec} P})$ is a morphism of monoidal spaces; then $\log \varphi$ corresponds to a unique morphism $\psi : \mathscr{O}_{\operatorname{Spec} P} \to \varphi_* \mathscr{O}_T$, and we set

$$f_\varphi := \Gamma(\operatorname{Spec} P, \psi) : P \to \Gamma(T, \mathscr{O}_T).$$

By inspecting the definitions, it is easily seen that $f_{\varphi_f} = f$ for every morphism of monoids f as above. To conclude, it remains only to show that the rule $(\varphi, \log \varphi) \mapsto f_\varphi$ is injective. However, for a given morphism of monoidal spaces $(\varphi, \log \varphi)$ as above, and every point $t \in T$, we have a commutative diagram of monoids :

$$\begin{array}{ccc} P & \xrightarrow{f_\varphi} & \Gamma(T, \mathscr{O}_T) \\ \downarrow & & \downarrow{\scriptstyle j_t} \\ P_{\varphi(t)} & \xrightarrow{\log \varphi_t} & \mathscr{O}_{T,t} \end{array} .$$

Since $\log \varphi_t$ is local, it follows that $\varphi(t) = (f_\varphi \circ j_t)^{-1}\mathfrak{m}_t$, especially f_φ determines $\varphi : T \to \operatorname{Spec} P$. Finally, since the map $P \to P_{\varphi(t)}$ is an epimorphism, we see that $\log \varphi_t$ is determined by f_φ as well, and the proposition follows. □

Definition 6.5.7. Let $(T, \mathscr{O}_T)$ be a sharp monoidal space.

(i) We say that $(T, \mathscr{O}_T)$ is an *affine fan*, if there exist a monoid P and an isomorphism of sharp monoidal spaces $(\operatorname{Spec} P, \mathscr{O}_{\operatorname{Spec} P})^\sharp \xrightarrow{\sim} (T, \mathscr{O}_T)$.

(ii) In the situation of (i), if P can be chosen to be finitely generated (resp. fine), we say that $(T, \mathscr{O}_T)$ is a *finite* (resp. *fine*) *affine fan*.

(iii) We say that $(T, \mathscr{O}_T)$ is a *fan*, if there exists an open covering $T = \bigcup_{i \in I} U_i$, such that the induced sharp monoidal space $(U_i, \mathscr{O}_{T|U_i})$ is an affine fan, for every $i \in I$. We denote by $\mathbf{Fan}$ the full subcategory of the category of monoidal spaces, whose objects are the fans.

(iv) In the situation of (iii), if the covering $(U_i \mid i \in I)$ can be chosen, so that $(U_i, \mathscr{O}_{T|U_i})$ is a finite (resp. fine) affine fan for every $i \in I$, we say that $(T, \mathscr{O}_T)$ is *locally finite* (resp. *locally fine*).

(v) We say that the fan $(T, \mathscr{O}_T)$ is *finite* (resp. *fine*) if it is locally finite (resp. locally fine) and quasi-compact.

(vi) Let $(T, \mathscr{O}_T)$ be a fan. The *simplicial locus* $T_{\mathrm{sim}} \subset T$ is the subset of all $t \in T$ such that $\mathscr{O}_{T,t}$ is a free monoid of finite rank.

Remark 6.5.8. (i) For every monoid P, let T_P denote the affine fan $(\mathrm{Spec}\, P)^\sharp$. In light of proposition 6.5.6, it is easily seen that the functor $P \mapsto T_P$ is an equivalence from the opposite of the full subcategory of sharp monoids, to the category of affine fans.

(ii) Since the saturation functor commutes with localizations (lemma 6.2.9(i)), it is easily seen that the saturation of a fan is a fan, and more precisely, the saturation of an affine fan T_P, is naturally isomorphic to $T_{P^{\mathrm{sat}}}$.

(iii) Let Q_1 and Q_2 be two monoids; since the product $P \times Q$ is also the coproduct of P and Q in the category $\mathbf{Mnd}$ (see example 4.8.30(i)), we have a natural isomorphism in the category of fans :

$$T_{P\times Q} \xrightarrow{\sim} T_P \times T_Q.$$

More generally, suppose that $P \to Q_i$, for $i = 1, 2$, are two morphisms of monoids. Then we have a natural isomorphism of fans :

$$T_{Q_1 \otimes_P Q_2} \xrightarrow{\sim} T_{Q_1} \times_{T_P} T_{Q_2}.$$

From this, a standard argument shows that fibre products are representable in the category of fans.

(iv) Furthermore, lemma 6.1.17(ii) implies that the natural map :

$$\pi : \mathrm{Spec}\,(P \times Q) \to \mathrm{Spec}\, P \times \mathrm{Spec}\, Q$$

is a homeomorphism (where the product of $\mathrm{Spec}\, P$ and $\mathrm{Spec}\, Q$ is taken in the category of topological spaces and continuous maps).

(v) Moreover, we have natural isomorphisms of monoids :

$$\mathscr{O}_{T_{P\times Q},\pi^{-1}(s,t)} \xrightarrow{\sim} \mathscr{O}_{T_P,s} \times \mathscr{O}_{T_Q,t} \qquad \text{for every } s \in \mathrm{Spec}\, P \text{ and } t \in \mathrm{Spec}\, Q.$$

(vi) For any fan $T := (T, \mathscr{O}_T)$, and any monoid M, we shall use the standard notation :

$$T(M) := \mathrm{Hom}_{\mathbf{Fan}}((\mathrm{Spec}\, M)^\sharp, T).$$

Especially, if T is an affine fan, say $T = (\mathrm{Spec}\, P)^\sharp$, then $T(M) = \mathrm{Hom}_{\mathbf{Mnd}}(P, M^\sharp)$; for instance, if T is affine, $T(\mathbb{N})$ is a monoid, and $T(\mathbb{Q}_+)^{\mathrm{gp}}$ is a $\mathbb{Q}$-vector space. Furthermore, by standard general nonsense we have natural identifications of sets :

$$(T_1 \times_T T_2)(M) \xrightarrow{\sim} T_1(M) \times_{T(M)} T_2(M)$$

for any pair of T-fans T_1 and T_2, and every monoid M. If T, T_1 and T_2 are affine, this identification is also an isomorphism of monoids.

Example 6.5.9. (i) The topological space underlying the affine fan $(\mathrm{Spec}\, \mathbb{N}, \mathscr{O}_{\mathrm{Spec}\, \mathbb{N}})^\sharp$ consists of two points : $\mathrm{Spec}\, \mathbb{N} = \{\varnothing, \mathfrak{m}\}$, where $\mathfrak{m} := \mathbb{N} \setminus \{0\}$ is the closed point. The structure sheaf $\mathscr{O} := \mathscr{O}_{\mathrm{Spec}\, \mathbb{N}}$ is determined as follows. The two stalks are $\mathscr{O}_\varnothing = \{1\}$ (the trivial monoid) and $\mathscr{O}_\mathfrak{m} = \mathbb{N}$; the global sections are $\Gamma(\mathrm{Spec}\, \mathbb{N}, \mathscr{O}) = \mathbb{N}$.

(ii) Let $(T, \mathscr{O}_T)$ be any fan, P any monoid, with maximal ideal $\mathfrak{m}_P$, and $\varphi : T_P := (\mathrm{Spec}\, P, \mathscr{O}_{\mathrm{Spec}\, P})^\sharp \to (T, \mathscr{O}_T)$ a morphism of fans. Say that $\varphi(\mathfrak{m}_P) \in U$ for some affine open subset $U \subset F$; then $\varphi(\mathrm{Spec}\, P) \subset U$, hence φ factors through a morphism of fans $T_P \to (U, \mathscr{O}_{T|U})$. In view of proposition 6.5.6, such a morphism corresponds to a unique morphism of monoids $\varphi^\natural : \mathscr{O}_T(U) \to P^\sharp$, and then $\varphi(\mathfrak{m}_P) = \varphi^{\natural-1}(\mathfrak{m}_P) \in \operatorname{Spec} \mathscr{O}_T(U) = U$. The map on stalks determined by φ is the local morphism

$$\mathscr{O}_{T,\varphi(\mathfrak{m}_P)} \xrightarrow{\sim} \mathscr{O}_T(U)_{\varphi(\mathfrak{m}_P)}/\mathscr{O}_T(U)^\times_{\varphi(\mathfrak{m}_P)} \to P^\sharp$$

obtained from $\varphi^\natural$ after localization at the prime ideal $\varphi^{\natural-1}(\mathfrak{m}_P)$.

(iii) For any two monoids M and N, denote by $\mathrm{loc.Hom}_{\mathbf{Mnd}}(M, N)$ the set of local morphisms of monoids $M \to N$. The discussion in (ii) leads to a natural identification :

$$T(P) \xrightarrow{\sim} \coprod_{t\in T} \mathrm{loc.Hom}_{\mathbf{Mnd}}(\mathscr{O}_{T,t}, P^\sharp)$$

For any monoid P. The *support* of a P-point $\varphi \in T(P)$ is the unique point $t \in T$ such that φ corresponds to a local morphism $\mathscr{O}_{T,t} \to P^\sharp$.

Example 6.5.10. (i) Let P be any monoid, $k > 0$ any integer, set $T_P := (\operatorname{Spec} P)^\sharp$, and let $\boldsymbol{k}_P : P \to P$ be the k-Frobenius map of P (definition 4.8.40(ii)). Then

$$\boldsymbol{k}_{T_P} := \operatorname{Spec} \boldsymbol{k}_P : T_P \to T_P$$

is a well defined endomorphism inducing the identity on the underlying topological space.

(ii) More generally, let F be any fan; for every integer $k > 0$ we have the *k-Frobenius endomorphism*

$$\boldsymbol{k}_F : F \to F$$

which induces the identity on the underlying topological space, and whose restriction to any affine open subfan $U \subset F$ is the endomorphism $\boldsymbol{k}_U$ defined as in (i).

6.5.11. Let P be a monoid, M a P-module, and set $T_P := (\operatorname{Spec} P)^\sharp$. We define a presheaf $M^\sim$ on the site of basic affine open subsets $D(f) \subset \operatorname{Spec} P$ (for all $f \in P$), by the rule :

$$U \mapsto M^\sim(U) := M \otimes_P \mathscr{O}_{T_P}(U)$$

(and for an inclusion $U' \subset U$ of basic open subsets, the corresponding morphism $M^\sim(U) \to M^\sim(U')$ is deduced from the restriction map $\mathscr{O}_{T_P}(U) \to \mathscr{O}_{T_P}(U')$). It is easily seen that $M^\sim$ is a sheaf, hence it extends to a well defined sheaf of $\mathscr{O}_{T_P}$-modules on T_P ([59, Ch.0, §3.2.5]). Clearly $\Gamma(T_P, M^\sim) = M$, and the rule $M \mapsto M^\sim$ yields a well defined functor $P\text{-}\mathbf{Mod} \to \mathscr{O}_{T_P}\text{-}\mathbf{Mod}$, which is left adjoint to the global section functor on $\mathscr{O}_{T_P}$-modules : $\mathscr{M} \mapsto \Gamma(T_P, \mathscr{M})$ (verification left to the reader).

Definition 6.5.12. Let $(T, \mathscr{O}_T)$ be a fan, $\mathscr{M}$ a $\mathscr{O}_T$-module. We say that $\mathscr{M}$ is *quasi-coherent*, if there exist an open covering $T = \bigcup_{i\in I} U_i$ of T by affine open subsets, and for each $i \in I$ an $\mathscr{O}_T(U_i)$-module M_i with an isomorphism of $\mathscr{O}_{T|U_i}$-modules $\mathscr{M}_{|U_i} \xrightarrow{\sim} M_i^\sim$.

Remark 6.5.13. (i) Let $(T, \mathscr{O}_T)$ be a fan, $\mathscr{M}$ a quasi-coherent $\mathscr{O}_T$-module, and $U \subset T$ be any open subset, such that $(U, \mathscr{O}_{T|U})$ is an affine fan (briefly : an affine open subfan of T). Then, since U admits a unique closed point $t \in U$, it is easily seen that $\mathscr{M}_{|U}$ is naturally isomorphic to $\mathscr{M}_t^\sim$ as a $\mathscr{O}_{T|U}$-module.

(ii) In the same vein, if $\mathscr{M}$ is an invertible $\mathscr{O}_T$-module (see definition 4.8.6(iv)), then the restriction $\mathscr{M}_{|U}$ of $\mathscr{M}$ to any affine open subset, is isomorphic to $\mathscr{O}_{T|U}$.

(iii) For any fan $(T, \mathscr{O}_T)$, the sheaf of abelian groups $\mathscr{O}_T^{\mathrm{gp}}$ is quasi-coherent (exercise for the reader). Suppose that T is integral; then an $\mathscr{O}_T$-submodule $\mathscr{I} \subset \mathscr{O}_T^{\mathrm{gp}}$ is called a *fractional ideal* (resp. a *reflexive fractional ideal*) of $\mathscr{O}_T$ if $\mathscr{I}$ is quasi-coherent, and $\mathscr{I}(U)$ is a fractional ideal (resp. a reflexive fractional ideal) of $\mathscr{O}_T(U)$, for every affine open subset $U \subset T$.

(iv) Let P be any integral monoid, $I \subset P^{\mathrm{gp}}$ a fractional ideal of P, and set $T_P := (\operatorname{Spec} P)^\sharp$. It follows easily from lemma 6.4.39(i) that $I^\sim \subset \mathscr{O}_{T_P}^{\mathrm{gp}}$ is a fractional ideal of $\mathscr{O}_{T_P}$, and $I^\sim$ is reflexive if and only if I is a reflexive fractional ideal of P (lemma 6.4.39(ii)).

(v) Suppose that T is locally finite; in this case, it follows easily from proposition 6.1.9(ii) that every quasi-coherent ideal of $\mathscr{O}_T$ is coherent. Likewise, if T is also integral, and $\mathscr{I} \subset \mathscr{O}_T^{\mathrm{gp}}$ is a (quasi-coherent) fractional ideal of $\mathscr{O}_T$, then $\mathscr{I}$ is coherent, provided the stalks $\mathscr{I}_t$ are finitely generated $\mathscr{O}_{T,t}$-modules, for every $t \in T$. (Details left to the reader.)

Remark 6.5.14. (i) Let T be any integral fan. We define a sheaf $\mathscr{D}iv_T$ on T, by letting $\mathscr{D}iv_T(U)$ be the set of all reflexive fraction ideals of $\mathscr{O}_U$, for every open subset $U \subset T$.

(ii) Now, suppose that T is locally fine; in this case, we can endow $\mathscr{D}iv_T$ with a natural structure of T-monoid, as follows. First, we define a presheaf of monoids on the site $\mathscr{C}_T$ of affine open subsets of T by the rule :

$$U \mapsto \mathscr{D}iv_T(U) := (\mathrm{Div}(\mathscr{O}_T(U)), \odot)$$

(notation of (6.4.37)) and for an inclusion $U' \subset U$ of affine open subset, the corresponding morphism of monoids $\mathscr{D}iv_T(U) \to \mathscr{D}iv_T(U')$ is deduced from the flat map $\mathscr{O}_T(U) \to \mathscr{O}_T(U')$, by virtue of lemma 6.4.45(iv). Arguing as in (6.5.3), we see that $\mathscr{D}iv_T$ is a sheaf on $\mathscr{C}_T$, and then [59, Ch.0, §3.2.2] implies that $\mathscr{D}iv_T$ extends uniquely to a sheaf of monoids on T. It is then clear that the sheaf of sets underlying this T-monoid is (naturally isomorphic to) the sheaf defined in (i).

(iii) In the situation of (ii), we have likewise a T-submonoid $\mathscr{D}iv_T^+ \subset \mathscr{D}iv_T$ (remark 6.4.47), and we may also define a T-monoid $\overline{\mathscr{D}iv}_T$ (see (6.4.53)). Moreover, we have the global version of (6.4.54) : namely, the sequence of T-monoids

$$1 \to \mathscr{O}_T^{\mathrm{gp}} \xrightarrow{j_T} \mathscr{D}iv_T \to \overline{\mathscr{D}iv}_T \to 1$$

is exact (recall that $\mathscr{O}_T^\times = 1_T$, the initial T-monoid), and j_T restricts to a map of T-monoids

$$\mathscr{O}_T \to \mathscr{D}iv_T^+.$$

Indeed, the assertion can be checked on the stalks over each $t \in T$, where it reduces to the exact sequence (6.4.54) for $P := \mathscr{O}_{T,t}$. Lastly, we remark that if T is locally fine and saturated, then $\mathscr{D}iv_T$ and $\overline{\mathscr{D}iv}_T$ are abelian T-groups (proposition 6.4.42(i,ii)).

6.5.15. If $f : T' \to T$ is a morphism of fans, and $\mathscr{M}$ is any $\mathscr{O}_T$-module, then we define as usual the $\mathscr{O}_{T'}$-module :

$$f^*\mathscr{M} := f^{-1}\mathscr{M} \otimes_{f^{-1}\mathscr{O}_T} \mathscr{O}_{T'}$$

where $f^{-1}\mathscr{M}$ denotes the usual sheaf-theoretic inverse image of $\mathscr{M}$ (so $f^{-1}\mathscr{O}_T$ means here what was denoted $f^*\mathscr{O}_T$ in definition 6.5.1(ii)). The rule $\mathscr{M} \mapsto f^*\mathscr{M}$ yields a left adjoint to the functor

$$\mathscr{O}_{T'}\text{-}\mathbf{Mod} \to \mathscr{O}_T\text{-}\mathbf{Mod} \qquad \mathscr{N} \mapsto f_*\mathscr{N}$$

(verification left to the reader). Notice that if $\mathscr{M}$ is quasi-coherent, then $f^*\mathscr{M}$ is a quasi-coherent $\mathscr{O}_{T'}$-module. Indeed, the assertion is local on T', hence we are reduced to the case where $T' = (\mathrm{Spec}\, P')$ and $T = (\mathrm{Spec}\, P)$ for some monoids P and P'. In this case, the functor $M \mapsto f^*(M^\sim) : P\text{-}\mathbf{Mod} \to \mathscr{O}_{T'}\text{-}\mathbf{Mod}$ is left adjoint to the functor $\mathscr{M} \mapsto \Gamma(T', \mathscr{M})$ on $\mathscr{O}_{T'}$-modules. The latter functor also admits the left adjoint given by the rule : $M \mapsto (M \otimes_P P')^\sim$, whence a natural isomorphism of $\mathscr{O}_{T'}$-modules :

$$f^*(M^\sim) \xrightarrow{\sim} (M \otimes_P P')^\sim.$$

6.5.16. Let $T := (T, \mathscr{O}_T)$ be a fan, $t \in T$ any point. The *height* of t is :

$$\mathrm{ht}_T(t) := \dim \mathscr{O}_{T,t} \in \mathbb{N} \cup \{+\infty\}$$

(see definition 6.1.10(ii)) and the *dimension* of T is $\dim T := \sup(\mathrm{ht}_T(t) \mid t \in T)$. (If $T = \varnothing$ is the empty fan, we let $\dim T := -\infty$.)

Suppose that T is locally finite; then it follows from (6.5.4) and lemma 6.1.20(iii),(iv) that the height of any point of T is an integer. Moreover, let $U(t) \subset T$ denote the subset of all points $x \in T$ which specialize to t (*i.e.* such that the topological closure of $\{x\}$ in T contains t); clearly $U(t)$ is the intersection of all the open neighborhoods of t in T, and we have a natural homeomorphism :

$$\text{(6.5.17)} \qquad \operatorname{Spec} \mathscr{O}_{T,t} \xrightarrow{\sim} U(t).$$

Especially, if T is locally finite, $U(t)$ is a finite set, and moreover $U(t)$ is an open subset : indeed, if $U \subset T$ is any finite affine open neighborhood of t, we have $U(t) \subset U$, hence $U(t)$ can be realized as the intersection of the finitely many open neighborhoods of t in U. In this case, (6.5.17) induces an isomorphism of fans :

$$\text{(6.5.18)} \qquad (\operatorname{Spec} \mathscr{O}_{T,t})^\sharp \xrightarrow{\sim} (U(t), \mathscr{O}_{T|U(t)}).$$

Therefore, for every $h \in \mathbb{N}$, let $T_h \subset T$ be the subset of all points of T of height $\leq h$; clearly $U(t) \subset T_h$ whenever $t \in T_h$, hence the foregoing shows that – if T is locally finite – T_h is an open subset of T for every $h \in \mathbb{N}$, and $T = \bigcup_{h \in \mathbb{N}} T_h$.

Notice also that the simplicial locus of a fan T is closed under generizations. Therefore, T_{sim} is an open subset of T, whenever T is locally finite.

6.5.19. In the situation of remark 6.5.8(v), suppose additionally that P and Q are finitely generated. The natural projection $P \times Q \to P$ induces a morphism $j : T_P \to T_{P \times Q}$ of affine fans, and it is easily seen that $j(t) = \pi^{-1}(t, \varnothing)$ for every $t \in T_P$. It follows that the restriction of j is a homeomorphism $U(t) \xrightarrow{\sim} U(j(t))$ for every $t \in T_P$, and moreover $\log j : j^*\mathscr{O}_{T_{P\times Q}} \to \mathscr{O}_{T_P}$ is an isomorphism. We conclude that j is an open immersion.

6.5.20. Let P be a fine, sharp and saturated monoid, and set $Q := P^\vee$ (notation of (6.4.13)). By proposition 6.4.14(iv), we have a natural identification $P \xrightarrow{\sim} Q^\vee$. By proposition 6.4.9(ii) and corollary 6.3.12(ii), the rule

$$\text{(6.5.21)} \qquad F \mapsto F^* := F^*_{\mathbb{R}} \cap P$$

establishes a natural bijection from the faces of Q to those of P. For every face F of Q, set $\mathfrak{p}_F := P \setminus F^*$; there follows a natural bijection $F \mapsto \mathfrak{p}_F$ between the set of all faces of Q and $\operatorname{Spec} P$, such that

$$F \subset F' \Leftrightarrow \mathfrak{p}_F \subset \mathfrak{p}_{F'}.$$

Moreover, set $T_P := (\operatorname{Spec} P)^\sharp$; we have natural identifications :

$$F^\vee \xrightarrow{\sim} \mathscr{O}_{T_P, \mathfrak{p}_F} \qquad \text{for every face } F \text{ of } Q$$

under which, the specialization maps $\mathscr{O}_{T_P,\mathfrak{p}_{F'}} \to \mathscr{O}_{T_P,\mathfrak{p}_F}$ correspond to the restriction maps $(F')^\vee \to F^\vee : \varphi \mapsto \varphi_{|F}$.

Definition 6.5.22. Let $T := (T, \mathscr{O}_T)$ be a fan.

(i) An *integral* (resp. a *rational*) *partial subdivision* of T is a morphism $f : (T', \mathscr{O}_{T'}) \to T$ of fans such that, for every $t \in T'$, the group homomorphism

$$(\log f)^{\mathrm{gp}}_t : \mathscr{O}^{\mathrm{gp}}_{T,f(t)} \to \mathscr{O}^{\mathrm{gp}}_{T',t} \qquad \text{(resp. the } \mathbb{Q}\text{-linear map } (\log f)^{\mathrm{gp}}_t \otimes_{\mathbb{Z}} \mathbf{1}_{\mathbb{Q}})$$

is surjective.

(ii) If $f : T' \to T$ is an integral (resp. rational) partial subdivision, and the induced map

$$T'(\mathbb{N}) \to T(\mathbb{N}) \qquad (\text{resp. } T'(\mathbb{Q}_+) \to T(\mathbb{Q}_+)) \quad : \quad \varphi \mapsto f \circ \varphi$$

is bijective, we say that f is an integral (resp. a rational) *subdivision* of T.

(iii) A morphism of fans $f : T' \to T$ is *finite* (resp. *proper*), if the fibre $f^{-1}(t)$ is a finite (resp. and non-empty) set, for every $t \in T$.

(iv) A subdivision $T' \to T$ of T is *simplicial*, if $T'_{\rm sim} = T'$.

Remark 6.5.23. (i) Let T be any integral fan. Then the counit of adjunction

$$T^{\rm sat} \to T$$

is an integral subdivision. This morphism is also a homeomorphism on the underlying topological spaces, in light of 6.2.9(iv).

(ii) Let $f : T' \to T$ be any integral subdivision. Then f restricts to a bijection $T'_0 \xrightarrow{\sim} T_0$ on the sets of points of height zero. Indeed, notice that if $t' \in T'$ is a point of height zero, then $f(t') \in T_0$ since the map $(\log f)^{\rm gp}_{t'}$ must be local; moreover $\text{loc.Hom}_{\mathbf{Mnd}}(\mathscr{O}_{T,t'}, \mathbb{N})$ consists of precisely one element, namely the unique map $\sigma_t : \mathbb{N} \to \{1\}$, and if $t'_1, t'_2 \in T'_0$ have the same image in T_0, the sections $\sigma_{t'_1}$ and $\sigma_{t'_2}$ have the same image in $T(\mathbb{N})$, hence they must coincide, so that $t'_1 = t'_2$, as claimed.

(iii) Let $f : T' \to T$ be an integral subdivision of locally fine and saturated fans. In general, the image of a point $t' \in T'$ of height one may have height strictly greater than one. On the other hand, for any $t \in T$ of height one, and any $t' \in f^{-1}(t)$, the map $\mathbb{Z} \xrightarrow{\sim} \mathscr{O}^{\rm gp}_{T,t} \to \mathscr{O}^{\rm gp}_{T',t'}$ must be surjective (theorem 6.4.18(ii)), therefore $\mathscr{O}^{\rm gp}_{T',t'}$ is a cyclic group; however $\mathscr{O}_{T',t'}$ is also sharp and saturated, so it must be either the trivial monoid $\{1\}$ or $\mathbb{N}$. The first case is excluded by (ii), so $\text{ht}(t') = 1$, and moreover $(\log f)_{t'}$ is an isomorphism (and there exists a unique such isomorphism). Since the induced map $T'(\mathbb{N}) \to T(\mathbb{N})$ is bijective, it follows easily that $f^{-1}(t)$ consists of exactly one point, and therefore f restricts to an isomorphism $f^{-1}(T_1) \xrightarrow{\sim} T_1$.

Proposition 6.5.24. *Let $f : T' \to T$ be a morphism of fans, with T' locally finite, and consider the following conditions :*

(a) *The induced map $T'(\mathbb{N}) \to T(\mathbb{N})$ is injective.*
(b) *For every integral saturated monoid P, the induced map $T'(P) \to T(P)$ is injective.*
(c) *f is a partial rational subdivision.*

Then we have : (a) $\Leftrightarrow$ (b) $\Rightarrow$ (c).

Proof. Obviously (b) $\Rightarrow$ (a). Conversely, assume that (a) holds, let P be a saturated monoid, and suppose we have two sections in $T'(P)$ whose images in $T(P)$ agree. In light of example 6.5.9(iii), this means that we may find two points $t'_1, t'_2 \in T'$, such that $f(t'_1) = f(t'_2) = t$, and two local morphisms of monoids $\sigma_i : \mathscr{O}_{T',t'_1} \to P/P^\times$ whose compositions with $\log f_{t'_i}$ $(i = 1, 2)$ yield the same morphism $\mathscr{O}_{T,t} \to P/P^\times$, and we have to show that these maps are equal. In view of lemma 6.2.9(ii), we may then replace P by $P/P^\times$, and assume that P is sharp. Since the stalks of $\mathscr{O}_{T'}$ are finitely generated, the morphisms σ_i factor through a finitely generated submonoid $M \subset P$. We may then replace P by its submonoid $M^{\rm sat}$, which allows to assume additionally that P is finitely generated (corollary 6.4.1(ii)). In this case, we may find an injective map $j : P \to \mathbb{N}^{\oplus r}$ (corollary 6.4.12(iv); notice that j is trivially a local morphism), hence we may replace σ_i by $j \circ \sigma_i$ (for $i = 1, 2$), after which we may assume that $P = \mathbb{N}^{\oplus r}$ for some $r \in \mathbb{N}$. Let $\delta : P \to \mathbb{N}$ be the local morphism given by the rule : $(x_1, \dots, x_r) \mapsto x_1 + \cdots + x_r$ for every $x_1, \dots, x_r \in \mathbb{N}$; the compositions $\delta \circ \sigma_i$ (for $i = 1, 2$) are two elements of $T'(\mathbb{N})$ whose images agree in $T(\mathbb{N})$, hence they must coincide by assumption. This implies already that $t'_1 = t'_2$. Next, let $\pi_k : P \to \mathbb{N}$ (for $k = 1, \dots, r$) be the natural projections, and fix $k \leq r$; the morphisms $\pi_k \circ \sigma_i$ for $i = 1, 2$ are not necessarily local, but they determine elements of $T'(\mathbb{N})$ whose images agree again in $T(\mathbb{N})$, hence they must coincide. Since k is arbitrary, we deduce that $\sigma_1 = \sigma_2$, as stated.

Next, we suppose that (b) holds, and we wish to show assertion (c); the latter is local on F', hence we may assume that both F and F' are affine, say $F = (\text{Spec}\, Q)^\sharp$ and $F' = (\text{Spec}\, Q')^\sharp$, with Q' finitely generated and sharp, and then we are reduced to checking that the map $Q^{\rm gp} \otimes_{\mathbb{Z}}$

$\mathbb{Q} \to Q'^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$ induced by f is surjective, or equivalently, that the dual map :

$$\mathrm{Hom}_{\mathbf{Mnd}}(Q', \mathbb{Q}) \to \mathrm{Hom}_{\mathbf{Mnd}}(Q, \mathbb{Q})$$

is injective. To this aim, we may further assume that Q' is integral, in which case, by remark 6.4.16(i) we have $\mathrm{Hom}_{\mathbf{Mnd}}(Q', \mathbb{Q}) = \mathrm{Hom}_{\mathbf{Mnd}}(Q', \mathbb{Q}_+)^{\mathrm{gp}}$; the contention is an easy consequence. □

6.5.25. In light of proposition 6.5.24, we may ask whether the surjectivity of the map on P-points induced by a morphism f of fans can be similarly characterized. This turns out to be the case, but more assumptions must be made on the morphism f, and also some additional restrictions must be imposed on the type of monoid P. Namely, we shall consider monoids of the form Γ_+, where $(\Gamma, \leq)$ is any totally ordered abelian group, and $\Gamma_+ \subset \Gamma$ is the subgroup of all elements ≤ 1 (where $1 \in \Gamma$ denotes the neutral element). With this notation, we have the following :

Proposition 6.5.26. *Let $f : T' \to T$ be a finite partial integral subdivision, with T locally finite. The following conditions are equivalent :*

(a) *The induced map $T'(\mathbb{N}) \to T(\mathbb{N})$ is surjective.*
(b) *For every totally ordered abelian group $(\Gamma, \geq)$, the induced map $T'(\Gamma_+) \to T(\Gamma_+)$ is surjective.*

Proof. Obviously, we need only to show that (a) ⇒ (b). Thus suppose, by way of contradiction, that (a) holds, but nevertheless there exist a totally ordered abelian group $(\Gamma, \leq)$, and an element of $T(\Gamma_+)$ which is not in the image of $T'(\Gamma_+)$. Such element corresponds to a local morphism of monoids $\varphi : \mathscr{O}_{T,t} \to \Gamma_+$, for some $t \in T$, and the assumption means that φ does not factor through the monoid $\mathscr{O}_{T',s}$, for any $s \in f^{-1}(t)$. Set

$$P := \mathscr{O}_{T,t}^{\mathrm{int}} \qquad Q_s := P^{\mathrm{gp}} \times_{\mathscr{O}_{T',s}^{\mathrm{gp}}} \mathscr{O}_{T',s}^{\mathrm{int}} \quad \text{for every } s \in f^{-1}(t)$$

Notice that, since the map $P^{\mathrm{gp}} \to \mathscr{O}_{T',s}^{\mathrm{gp}}$ is surjective, we have

$$Q_s / Q_s^{\times} \simeq \mathscr{O}_{T',s}^{\mathrm{int}} / (\mathscr{O}_{T',s}^{\mathrm{int}})^{\times}$$

and there is a natural injective morphism of monoids $g_s : P \to Q_s$, determined by the pair $(i, (\log f)_s^{\mathrm{int}})$, where $i : P^{\mathrm{int}} \to P^{\mathrm{gp}}$ is the natural morphism; moreover, $P^{\mathrm{gp}} = Q_s^{\mathrm{gp}}$ for every $s \in f^{-1}(t)$. Clearly φ factors through a morphism $\overline{\varphi} : P \to \Gamma_+$; since the unit of adjunction $\mathscr{O}_{T,t} \to P$ is surjective, it follows that P is sharp and $\overline{\varphi}$ is local. Moreover, the group homomorphism $\overline{\varphi}^{\mathrm{gp}} : P^{\mathrm{gp}} \to \Gamma$ factors uniquely through each Q_s^{gp}. Our assumption then states that we may find, for each $s \in f^{-1}(t)$, an element $x_s \in Q_s$ whose image in Γ lies in the complement of Γ_+, *i.e.* the image of x_s^{-1} lies in the maximal ideal $\mathfrak{m} \subset \Gamma_+$. Let $P' \subset P^{\mathrm{gp}}$ be the submonoid generated by P and by $(x_s^{-1} \mid s \in f^{-1}(t))$. By construction, P' is finitely generated, and the morphism $\overline{\varphi}$ extends uniquely to a morphism $P' \to \Gamma_+$, which maps each x_s^{-1} into $\mathfrak{m}$. It follows that all the x_s^{-1} lie in the maximal ideal of P'. Let us now pick any local morphism $\psi' : P' \to \mathbb{N}$ (corollary 6.4.12(iii)); by restriction, ψ' induces a local morphism $\psi : P \to \mathbb{N}$, which – according to (a) – must factor through a local morphism $\psi_s : Q_s \to \mathbb{N}$, for at least one $s \in f^{-1}(t)$. However, on the one hand we have $\psi'^{\mathrm{gp}} = \psi^{\mathrm{gp}} = \psi_s^{\mathrm{gp}}$; on the other hand $\psi'(x_s^{-1}) \neq 0$, hence $\psi_s(x_s) = \psi'^{\mathrm{gp}}(x_s) \notin \mathbb{N}$, a contradiction. □

Example 6.5.27. Let T be a locally fine fan, $\varphi : F \to T$ an integral subdivision, with F locally fine and saturated, and $k > 0$ an integer. Suppose we have a commutative diagram of fans :

$$\begin{array}{ccc} F & \xrightarrow{\varphi} & T \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \boldsymbol{k}_T} \\ F & \xrightarrow{\varphi} & T. \end{array} \tag{6.5.28}$$

where $\boldsymbol{k}_T$ is the k-Frobenius endomorphism (example 6.5.10(ii)). Then we claim that necessarily $g = \boldsymbol{k}_F$. Indeed, suppose that this fails; then we may find a point $t \in F$ such that the composition of g and the open immersion $j_t : U(t) \to F$ is not equal to $j_t \circ \boldsymbol{k}_{U(t)}$. Set $P := \mathscr{O}_{F,t}$; then $g \circ j_t \neq j_t \circ \boldsymbol{k}_{U(t)}$ in $F(P)$. However, an easy computation shows that $\varphi(g \circ j_t) = \varphi(j_t \circ \boldsymbol{k}_{U(t)})$, which contradicts proposition 6.5.24.

6.5.29. Let $P := \coprod_{n\in\mathbb{N}} P_n$ be a $\mathbb{N}$-graded monoid (see definition 4.8.8); then P_0 is a submonoid of P, every P_n is a P_0-module, and $P_+ := \coprod_{n>0} P_n$ is an ideal of P. For every $a \in P$, the localization P_a is $\mathbb{Z}$-graded in an obvious way, and we denote by $P_{(a)} \subset P_a$ the submonoid of elements of degree 0. Notice that there is a natural identification of P_0-monoids

$$P_{(a^n)} \xrightarrow{\sim} P_{(a)} \qquad \text{for every integer } n > 0. \tag{6.5.30}$$

Set as well :

$$D_+(a) := (\operatorname{Spec} P_{(a)})^\sharp$$

and notice that the natural map $P \to P_{(a)}$ induces a morphism of fans $\pi_a : D_+(a) \to T_P := (\operatorname{Spec} P_0)^\sharp$. If $b \in P$ is any other element, in order to determine the fibre product $D_+(a) \times_{T_P} D_+(b)$ we may assume – in light of (6.5.30) – that $a, b \in P_n$ for the same integer n, in which case we have natural isomorphisms

$$P_{(a)} \otimes_{P_0} P_{(b)} \xrightarrow{\sim} P_{(ab)} \xleftarrow{\sim} P_{(a)}[b^{-1}a]$$

(see remark 6.1.25(i)) onto the localization of $P_{(a)}$ obtained by inverting its element $a^{-1}b$; this is of course the same as $P_{(b)}[a^{-1}b]$. In other words $D_+(a) \times_{T_P} D_+(b)$ is naturally isomorphic to $D_+(ab)$, identified to an open subfan in both $D_+(a)$ and $D_+(b)$. We may then glue the fans $D_+(a)$ for a ranging over all the elements of P, to obtain a new fan, denoted :

$$\operatorname{Proj} P$$

called the *projective fan* associated with P. By inspecting the construction, we see that the morphisms π_a assemble to a well defined morphism of fans $\pi_P : \operatorname{Proj} P \to T_P$. Each element $a \in P$ yields an open immersion $j_a : D_+(a) \to \operatorname{Proj} P$, and if $b \in P$ is any other element, j_{ab} factors through an open immersion $D_+(ab) \to D_+(a)$.

6.5.31. Let $\varphi : P \to P'$ be a morphism of $\mathbb{N}$-graded monoids (so $\varphi P_n \subset P'_n$ for every $n \in \mathbb{N}$). Set :

$$G(\varphi) := \bigcup_{a\in P} D_+(\varphi(a)) \subset \operatorname{Proj} P'.$$

Notice that, for every $a \in P$, φ induces a morphism $\varphi_{(a)} : P_{(a)} \to P'_{(a)}$, whence a morphism of affine fans $(\operatorname{Proj} \varphi)_a : D_+(\varphi(a)) \to D_+(a) \subset \operatorname{Proj} P$. Moreover, if $b \in P$ is any other element, it is easily seen that $(\operatorname{Proj} \varphi)_a$ and $(\operatorname{Proj} \varphi)_b$ agree on $D_+(\varphi(a)) \cap D_+(\varphi(b))$. Therefore, the morphisms $(\operatorname{Proj} \varphi)_a$ glue to a well defined morphism :

$$\operatorname{Proj} \varphi : G(\varphi) \to \operatorname{Proj} P.$$

Notice that $G(\varphi) = \operatorname{Proj} P'$, whenever φP generates the ideal P'_+. Moreover, we have

$$(\operatorname{Proj} \varphi)^{-1} D_+(a) = D_+(\varphi(a)) \qquad \text{for every } a \in P. \tag{6.5.32}$$

Indeed, say that $D_+(b) \subset G(\varphi)$ for some $b \in P'$, and $(\operatorname{Proj} \varphi) D_+(b) \subset D_+(a)$. In order to show that $D_+(b) \subset D(\varphi(a))$, it suffices to check that $D_+(b\varphi(c)) \subset D_+(\varphi(a))$ for every $c \in P$. However, the assumption means that the natural map

$$P_{(c)} \to P'_{(\varphi(c))} \to P'_{(b\varphi(c))} \to P'_{(b\varphi(c))}/P'^{\times}_{(b\varphi(c))}$$

factors through the localization $P_{(c)} \to P_{(ac)}$. This is equivalent to saying that $\varphi(c^{-1}a)$ is invertible in $P'_{(b\varphi(c))}$, in which case the localization $P'_{(\varphi(c))} \to P'_{(b\varphi(c))}$ factors through the localization

$P'_{(\varphi(c))} \to P'_{(\varphi(ac))}$. The latter means that the open immersion $D_+(b\varphi(c)) \subset D_+(\varphi(c))$ factors through the open immersion $D_+(\varphi(ac)) \subset D_+(\varphi(c))$, as claimed.

6.5.33. In the situation of (6.5.29), set $Y := \mathrm{Proj}\, P$ to ease notation. Let M be a $\mathbb{Z}$-graded P-module; for every $a \in P$, let $M_{(a)} \subset M_a := M \otimes_P P_a$ be the $P_{(a)}$-submodule of degree zero elements (for the natural grading on M_a). We deduce a quasi-coherent $\mathscr{O}_{D_+(a)}$-module $M^\sim_{(a)}$ (see definition 6.5.12). Moreover, if $b \in P$ is any other element, we have a natural identification

$$\widetilde{\omega}_{a,b} : M^\sim_{(a)|D_+(a)\cap D_+(b)} \xrightarrow{\sim} M^\sim_{(b)|D_+(a)\cap D_+(b)}.$$

This can be verified as follows. First, in view of (6.5.30), we may assume that $a, b \in P_n$, for some $n \in \mathbb{N}$, in which case we consider the P-linear morphism :

$$M_{(a)} \to M_{(b)} \otimes_{P_{(b)}} P_{(b)}[a^{-1}b] \quad : \quad \frac{x}{a^m} \mapsto \frac{x}{b^m} \otimes \frac{b^m}{a^m} \qquad \text{for every } x \in M_{nm}.$$

It is easily seen that this map is actually $P_{(a)}$-linear, hence it extends to a $\mathscr{O}_T(D_+(a) \cap D_+(b))$-linear morphism :

$$\omega_{a,b} : M_{(a)} \otimes_{P_{(a)}} P_{(a)}[b^{-1}a] \xrightarrow{\sim} M_{(b)} \otimes_{P_{(b)}} P_{(b)}[a^{-1}b].$$

Moreover, $\omega_{a,b} \circ \omega_{b,a}$ is the identity map, hence $\omega_{a,b}$ induces the sought isomorphism $\widetilde{\omega}_{a,b}$. Furthermore, for any $a, b, c \in P$, set $D_+(a, b, c) := D_+(a) \cap D_+(b) \cap D_+(c)$; we have the identity :

$$\widetilde{\omega}_{a,c|D_+(a,b,c)} = \widetilde{\omega}_{b,c|D_+(a,b,c)} \circ \widetilde{\omega}_{a,b|D_+(a,b,c)}$$

which shows that the locally defined sheaves $M^\sim_{(a)}$ glue to a well defined $\mathscr{O}_Y$-module, which we shall denote $M^\sim$. Especially, for every $n \in \mathbb{Z}$, let $P(n)$ be the $\mathbb{Z}$-graded P-module such that $P(n)_k := P_{n+k}$ for every $k \in \mathbb{Z}$ (with the convention that $P_n := \varnothing$ if $n < 0$); we set :

$$\mathscr{O}_Y(n) := P(n)^\sim.$$

Every element $a \in P_n$ induces a natural isomorphism :

$$\mathscr{O}_Y(n)_{|D_+(a)} \xrightarrow{\sim} \mathscr{O}_{D_+(a)} \quad : \quad x \mapsto f^{-k}x \quad \text{for every local section } x.$$

Hence on the open subset :

$$U_n(P) := \bigcup_{a \in P_n} D_+(a)$$

the sheaf $\mathscr{O}_Y(n)$ restricts to an invertible $\mathscr{O}_{U_n(P)}$-module (see definition 4.8.6(iv)). Especially, if P_1 generates P_+, the $\mathscr{O}_Y$-modules $\mathscr{O}_Y(n)$ are invertible, for every $n \in \mathbb{Z}$.

6.5.34. In the situation of (6.5.31), let M be a $\mathbb{Z}$-graded P-module. Then $M' := M \otimes_P P'$ is a $\mathbb{Z}$-graded P'-module, with the grading defined by the rule :

$$M'_n := \bigcup_{j+k=n} \mathrm{Im}(M_j \otimes_{P_0} P'_k \to M'). \tag{6.5.35}$$

There follows a $P_{(a)}$-linear morphism :

$$M_{(a)} \to M'_{(\varphi(a))} \quad : \quad \frac{x}{a^k} \mapsto \frac{x \otimes 1}{\varphi(a)^k} \qquad \text{for every } a \in P \tag{6.5.36}$$

and since both localization and tensor product commute with arbitrary colimits, it is easily seen that (6.5.36) extends an injective $P'_{(\varphi(a))}$-linear map

$$M_{(a)} \otimes_{P_{(a)}} P'_{(\varphi(a))} \to M'_{(\varphi(a))}$$

whence a map of $\mathscr{O}_{D_+(\varphi(a))}$-modules $(\mathrm{Proj}\,\varphi)^*M^\sim_{|D_+(\varphi(a))} \to (M')^\sim_{|D_+(\varphi(a))}$, and the system of such maps, for a ranging over the elements of P, is compatible with all open immersions $D_+(\varphi(ab)) \subset D_+(a)$, whence a well defined monomorphism of $\mathscr{O}_{G(\varphi)}$-modules

$$(6.5.37) \qquad (\mathrm{Proj}\,\varphi)^*M^\sim \to (M')^\sim_{|G(\varphi)}.$$

Moreover, if $a \in P_1$, then for every $m \in M_j$ and $x \in P'_k$, we may write

$$\frac{m \otimes x}{\varphi(a)^{j+k}} = \frac{m}{a^j} \otimes \frac{x}{\varphi(a)^j}$$

so the above map is an isomorphism on $D_+(a)$. Thus, (6.5.37) restricts to an isomorphism on the open subset

$$G_1(\varphi) := \bigcup_{a \in P_1} D_+(\varphi(a)).$$

Especially (6.5.37) is an isomorphism whenever P_1 generates P_+. Notice as well that $G_1(\varphi) \subset U_1(P') \cap G(\varphi)$, and actually $G_1(\varphi) = U_1(P')$ if $\varphi(P)$ generates P'_+.

6.5.38. Let P be as in (6.5.29), and $f : P_0 \to Q$ a given morphism of monoids. Then $P' := P \otimes_{P_0} Q$ is naturally $\mathbb{N}$-graded, so that the natural map $f_P : P \to P'$ is a morphism of graded monoids. Every element of P' is of the form $a \otimes b = (a \otimes 1) \cdot (1 \otimes b)$, where $a \in P$ and $b \in Q$. Then lemma 4.8.34 yields a natural isomorphism of Q-monoids :

$$P_{(a)} \otimes_{P_0} Q[b^{-1}] \stackrel{\sim}{\to} P'_{(a \otimes b)}$$

whence an isomorphism of affine fans :

$$\beta_{a \otimes b} : D_+(a \otimes b) \stackrel{\sim}{\to} D_+(a) \times_{T_P} D(b)$$

such that $(\pi_a \times_{T_P} j_b^*) \circ \beta_{a \otimes b} = \pi_{a \otimes b}$, where $j_b^* : D(b) \to T_Q := (\mathrm{Spec}\,Q)^\sharp$ is the natural open immersion. Especially, it is easily seen that the isomorphisms $\beta_{a \otimes 1}$ assemble to a well defined isomorphism of fans :

$$(\mathrm{Proj}\,f_P, \pi_{P'}) : \mathrm{Proj}\,P' \stackrel{\sim}{\to} \mathrm{Proj}\,P \times_{T_P} T_Q$$

such that $(\pi_P \times_{T_P} \mathbf{1}_{T_Q}) \circ (\mathrm{Proj}\,f_P, \pi_{P'}) = \pi_{P'}$. Lastly, if $g : Q \to R$ is another morphism of monoids, $T_R := (\mathrm{Spec}\,R)^\sharp$, and $P'' := P' \otimes_Q R$, then we have the identity :

$$(6.5.39) \qquad ((\mathrm{Proj}\,f_P, \pi_{P'}) \times_{T_Q} \mathbf{1}_{T_R}) \circ (\mathrm{Proj}\,g_{P'}, \pi_{P''}) = (\mathrm{Proj}\,(g \circ f)_P, \pi_{P''}).$$

Moreover, for every $\mathbb{Z}$-graded module M, the map $(\mathrm{Proj}\,f_P)^*M^\sim \to (M \otimes_{P_0} Q)^\sim_{|G(f_P)}$ of (6.5.37) is an isomorphism, regardless of whether or not P_1 generates P_+ (verification left to the reader). Especially, we get a natural identification :

$$(\mathrm{Proj}\,f_P)^*\mathscr{O}_Y(n) \stackrel{\sim}{\to} \mathscr{O}_{Y'}(n) \qquad \text{for every } n \in \mathbb{Z}$$

where $Y := \mathrm{Proj}\,P$ and $Y' := \mathrm{Proj}\,P'$.

6.5.40. In the situation of (6.5.38), let $\varphi : R \to P$ be a morphism of $\mathbb{N}$-graded monoids. There follow morphisms of fans :

$$\mathrm{Proj}\,\varphi : G(\varphi) \to \mathrm{Proj}\,R \qquad \mathrm{Proj}(f_P \circ \varphi) : G(f_P \circ \varphi) \to \mathrm{Proj}\,R$$

and in view of (6.5.32), it is easily seen that :

$$(6.5.41) \qquad G(f_P \circ \varphi) = (\mathrm{Proj}\,f_P)^{-1}(G(\varphi)).$$

6.5.42. Let now $(T, \mathscr{O}_T)$ be any fan. A $\mathbb{N}$-*graded* $\mathscr{O}_T$-*monoid* is a $\mathbb{N}$-graded T-monoid $\mathscr{P}$, with a morphism $\mathscr{O}_T \to \mathscr{P}$ of T-monoids. We say that such a $\mathscr{O}_T$-monoid is *quasi-coherent*, if it is such, when regarded as a $\mathscr{O}_T$-module. To a quasi-coherent $\mathbb{N}$-graded $\mathscr{O}_T$-monoid $\mathscr{P}$, we attach a morphism of fans :

$$\pi_{\mathscr{P}} : \mathrm{Proj}\,\mathscr{P} \to T$$

constructed as follows. First, for every affine open subfan $U \subset T$, the monoid $\mathscr{P}(U)$ is $\mathbb{N}$-graded, so we have the projective fan $\mathrm{Proj}\,\mathscr{P}(U)$, and the morphism of monoids $\mathscr{O}_T(U) \to \mathscr{P}(U)$ induces a morphism of fans $\mathrm{Proj}\,\mathscr{P}(U) \to U$. Next, say that $U_1, U_2 \subset T$ are two affine open subsets; for any affine open subset $V \subset U_1 \cap U_2$ we have restriction maps $\rho_{V,i} : \mathscr{P}(U_i) \to \mathscr{P}(V)$ inducing isomorphisms of graded $\mathscr{O}_T(V)$-monoids :

$$\mathscr{P}(U_i) \otimes_{\mathscr{O}_T(U_i)} \mathscr{O}_T(V) \xrightarrow{\sim} \mathscr{P}(V).$$

whence isomorphisms of V-fans :

$$\mathrm{Proj}\,\mathscr{P}(V) \xrightarrow{\sim} \mathrm{Proj}\,\mathscr{P}(U_i) \otimes_{\mathscr{O}_T(U_i)} \mathscr{O}_T(V) \xrightarrow{\mathrm{Proj}\,\rho_{V,i}} \mathrm{Proj}\,\mathscr{P}(U_i) \times_{U_i} V$$

which in turn yield natural identifications :

$$\vartheta_V : \mathrm{Proj}\,\mathscr{P}(U_1) \times_{U_1} V \xrightarrow{\sim} \mathrm{Proj}\,\mathscr{P}(U_2) \times_{U_2} V.$$

If $W \subset V$ is a smaller affine open subset, (6.5.39) implies that $\vartheta_V \times_V \mathbf{1}_W = \vartheta_W$, and therefore the isomorphisms ϑ_V glue to a single isomorphism of $U_1 \cap U_2$-fans :

$$\mathrm{Proj}\,\mathscr{P}(U_1) \times_{U_1} (U_1 \cap U_2) \xrightarrow{\sim} \mathrm{Proj}\,\mathscr{P}(U_2) \times_{U_2} (U_1 \cap U_2)$$

which is furthermore compatible with base change to any triple intersection $U_1 \cap U_2 \cap U_3$ of affine open subsets (details left to the reader). In such situation, we may glue the fans $\mathrm{Proj}\,\mathscr{P}(U)$ – with $U \subset T$ ranging over all the open affine subsets – along the above isomorphisms, to obtain the sought fan $\mathrm{Proj}\,\mathscr{P}$; the construction also comes with a well defined morphism to T, as required. Then, for every such open affine U, the induced morphism $\mathrm{Proj}\,\mathscr{P}(U) \to \mathrm{Proj}\,\mathscr{P}$ is an open immersion; finally a direct inspection shows that, for every smaller affine open subset $V \subset U$ we have :

$$U_n(\mathscr{P}(U)) \cap \mathrm{Proj}\,\mathscr{P}(V) = U_n(\mathscr{P}(V)) \qquad \text{for every } n \in \mathbb{N}$$

(where the intersection is taken in $\mathrm{Proj}\,\mathscr{P}$). Hence the union of all the open subsets $U_n(\mathscr{P}(U))$ is an open subset $U_n(\mathscr{P}) \subset \mathrm{Proj}\mathscr{P}$, intersecting each $\mathrm{Proj}\,\mathscr{P}(U)$ in its subset $U_n(\mathscr{P}(U))$.

6.5.43. To ease notation, set $Y := \mathrm{Proj}\,\mathscr{P}$, and let $\pi : Y \to T$ be the projection. Let $\mathscr{M}$ be a $\mathbb{Z}$-graded $\mathscr{P}$-module, quasi-coherent as a $\mathscr{O}_T$-module; for every affine open subset $U \subset T$, the graded $\mathscr{P}(U)$-module $\mathscr{M}(U)$ yields a quasi-coherent $\mathscr{O}_{\pi^{-1}U}$-module $\mathscr{M}_U^\sim$, and every inclusion of affine open subset $U' \subset U$ induces a natural isomorphism $\mathscr{M}^\sim_{U|U'} \xrightarrow{\sim} \mathscr{M}^\sim_{U'}$ of $\mathscr{O}_{\pi^{-1}U'}$-modules. Therefore the modules $\mathscr{M}_U^\sim$ glue to a well defined $\mathscr{O}_Y$-module $\mathscr{M}^\sim$.

For every $n \in \mathbb{Z}$, denote by $\mathscr{M}(n)$ the $\mathbb{Z}$-graded $\mathscr{P}$-module such that $\mathscr{M}(n)_k := \mathscr{M}_{n+k}$ for every $k \in \mathbb{Z}$ (especially, with the convention that $\mathscr{P}_k := 0$ whenever $k < 0$, we obtain in this way the $\mathscr{P}$-module $\mathscr{P}(n)$). We set :

$$\mathscr{O}_Y(n) := \mathscr{P}(n)^\sim \quad \text{and} \quad \mathscr{M}^\sim(n) := \mathscr{M}^\sim \otimes_{\mathscr{O}_Y} \mathscr{O}_Y(n).$$

Clearly the restriction of $\mathscr{O}_Y(n)$ to $U_n(\mathscr{P})$ is invertible, for every $n \in \mathbb{Z}$.

Moreover, for every $n \in \mathbb{Z}$, the scalar multiplication $\mathscr{P}(n) \otimes_{\mathscr{O}_T} \mathscr{M} \to \mathscr{M}(n)$ determines a well defined morphism of $\mathscr{O}_Y$-modules :

$$\mathscr{M}^\sim(n) \to \mathscr{M}(n)^\sim$$

and arguing as in (6.5.34) we see that the restriction of this map is an isomorphism on $U_1(\mathscr{P})$. Especially, we have natural morphisms of $\mathscr{O}_Y$-modules :

$$\mathscr{O}_Y(n) \otimes_{\mathscr{O}_Y} \mathscr{O}_Y(m) \to \mathscr{O}_Y(n+m) \qquad \text{for every } n, m \in \mathbb{Z}$$

whose restrictions to $U_1(\mathscr{P})$ are isomorphisms.

Example 6.5.44. Let T be a fan, $\mathscr{L}$ an invertible $\mathscr{O}_T$-module, and set $\mathscr{P}(\mathscr{L}) := \mathrm{Sym}^\bullet_{\mathscr{O}_T}\mathscr{L}$ (see example 4.8.10). Then the morphism

$$\pi_{\mathscr{P}(\mathscr{L})} : \mathbb{P}(\mathscr{L}) := \mathrm{Proj}\, \mathscr{P}(\mathscr{L}) \to T$$

is an isomorphism. Indeed, the assertion can be checked locally on every affine open subset $U \subset T$, hence say that $U = (\mathrm{Spec}\, P)^\sharp$ for some monoid P, and $\mathscr{L} \simeq \mathscr{O}_{T|U}$, in which case the P-monoid $\mathscr{P}(\mathscr{L})(U)$ is isomorphic to $P \times \mathbb{N}$ (with its natural morphism $P \to P \times \mathbb{N} : x \mapsto (x, 0)$ for every $x \in P$), and the sought isomorphism corresponds to the natural identification :

$$P = (P \times \mathbb{N})_{(1,1)} \tag{6.5.45}$$

where $(1, 1) \in P \times \{1\} = (P \times \mathbb{N})_1$. Likewise, $\mathscr{O}_{\mathbb{P}(\mathscr{L})}(n)$ is the $\mathscr{O}_{\mathbb{P}(\mathscr{L})}$-module associated with the graded $(P \times \mathbb{N})$-module $P \times \mathbb{N}(n) = \mathscr{L}^{\otimes n}(U) \otimes_P (P \times \mathbb{N})$, so (6.5.45) induces a natural isomorphism

$$\pi^*_{\mathscr{P}(\mathscr{L})}\mathscr{L}^{\otimes n} \xrightarrow{\sim} \mathscr{O}_{\mathbb{P}(\mathscr{L})}(n) \qquad \text{for every } n \in \mathbb{N}.$$

6.5.46. In the situation of (6.5.42), let $\varphi : \mathscr{P} \to \mathscr{P}'$ be a morphism of quasi-coherent $\mathbb{N}$-graded $\mathscr{O}_T$-monoids (defined in the obvious way). By the foregoing, for every affine open subset $U \subset T$, we have an induced morphism $\mathrm{Proj}\, \varphi(U) : G(\varphi(U)) \to \mathrm{Proj}\mathscr{P}(U)$ of U-fans, where $G(\varphi(U)) \subset \mathrm{Proj}\, \mathscr{P}(U)$ is an open subset of $\mathrm{Proj}\, \mathscr{P}'$. Let $V \subset U$ be a smaller affine open subset; in light of (6.5.41), we have

$$G(\varphi(V)) = G(\varphi(U)) \cap \mathrm{Proj}\, \mathscr{P}(V).$$

It follows that the union of all the open subsets $G(\varphi(U))$ is an open subset $G(\varphi)$ such that

$$G(\varphi) \cap \mathrm{Proj}\, \mathscr{P}(U) = G(\varphi(U)) \qquad \text{for every affine open subset } U \subset T$$

and the morphisms $\mathrm{Proj}\, \varphi(U)$ assemble to a well defined morphism

$$\mathrm{Proj}\, \varphi : G(\varphi) \to \mathrm{Proj}\, \mathscr{P}.$$

Moreover, if $\mathscr{M}$ is a $\mathbb{Z}$-graded quasi-coherent $\mathscr{P}$-module, the morphisms (6.5.37) assemble to a well defined morphism of $\mathscr{O}_{G(\varphi)}$-modules :

$$(\mathrm{Proj}\, \varphi)^*\mathscr{M}^\sim \to (\mathscr{M}')^\sim_{|G(\varphi)} \tag{6.5.47}$$

where the grading of $\mathscr{M}' := \mathscr{M} \otimes_{\mathscr{P}} \mathscr{P}'$ is defined as in (6.5.35). Likewise, the union of all open subsets $G_1(\varphi(U))$ is an open subset $G_1(\varphi) \subset U_1(\mathscr{P}) \cap G(\varphi)$, such that the restriction of (6.5.47) to $G_1(\varphi)$ is an isomorphism. Especially, set $Y := \mathrm{Proj}\, \mathscr{P}$ and $Y' := \mathrm{Proj}\, \mathscr{P}'$; we have a natural morphism :

$$(\mathrm{Proj}\, \varphi)^*\mathscr{O}_Y(n) \xrightarrow{\sim} \mathscr{O}_{Y'}(n)_{|G(\varphi)}$$

which is an isomorphism, if $\mathscr{P}_1$ generates $\mathscr{P}_+ := \coprod_{n>0} \mathscr{P}_n$ locally on T.

6.5.48. On the other hand, let $f : T' \to T$ be a morphism of fans. The discussion in (6.5.38) implies that f induces a natural isomorphism of T'-fans :

$$\mathrm{Proj}\, f^*\mathscr{P} \xrightarrow{\sim} \mathrm{Proj}\, \mathscr{P} \times_T T'. \tag{6.5.49}$$

Moreover, set $Y := \mathrm{Proj}\, \mathscr{P}$, $Y' := \mathrm{Proj}\, f^*\mathscr{P}$, and let $\pi_Y : Y' \to Y$ be the projection deduced from (6.5.49); then there follows a natural identification :

$$\mathscr{O}_{Y'}(n) \xrightarrow{\sim} \pi^*_Y\mathscr{O}_Y(n) \qquad \text{for every } n \in \mathbb{Z}.$$

6.5.50. Keep the notation of (6.5.42), and to ease notation, set $Y := \mathrm{Proj}\,\mathscr{P}$. Let $\mathscr{C}$ be the category whose objects are all the pairs $(\psi : X \to T, \mathscr{L})$, where ψ is a morphism of fans, and $\mathscr{L}$ is an invertible $\mathscr{O}_X$-module; the morphisms $(\psi : X \to T, \mathscr{L}) \to (\psi' : X' \to T, \mathscr{L}')$ are the pairs (β, h), where $\beta : X \to X'$ is a morphism of T-fans, and $h : \beta^*\mathscr{L}' \xrightarrow{\sim} \mathscr{L}$ is an isomorphism of $\mathscr{O}_{X'}$-modules (with composition of morphisms defined in the obvious way). Consider the functor

$$F_{\mathscr{P}} : \mathscr{C}^o \to \mathbf{Set}$$

which assigns to any object $(\psi, \mathscr{L})$ of $\mathscr{C}$, the set consisting of all morphisms of graded $\mathscr{O}_X$-monoids

$$g : \psi^*\mathscr{P} \to \mathrm{Sym}^\bullet_{\mathscr{O}_X}\mathscr{L}$$

which are epimorphisms on the underlying $\mathscr{O}_X$-modules (notation of example 4.8.10). On a morphism (β, h) as in the foregoing, and an element $g' \in F_{\mathscr{P}}(\psi', \mathscr{L}')$, the functor acts by the rule :

$$F_{\mathscr{P}}(\beta, h) := (\mathrm{Sym}^\bullet_{\mathscr{O}_X} h) \circ \beta^* g'.$$

Lemma 6.5.51. *In the situation of* (6.5.50), *the following holds :*

(i) *The object* $(\pi_{\mathscr{P}|U_1(\mathscr{P})} : U_1(\mathscr{P}) \to T, \mathscr{O}_Y(1)_{|U_1(\mathscr{P})})$ *represents the functor* $F_{\mathscr{P}}$.

(ii) *If* $\mathscr{P}$ *is an integral* T*-monoid, the* $\mathscr{O}_T$*-monoid* $\mathscr{P}^{\mathrm{sat}}$ *admits a unique grading such that the unit of adjunction* $\mathscr{P} \to \mathscr{P}^{\mathrm{sat}}$ *is a* $\mathbb{N}$*-graded morphism, and there is a natural isomorphism of* $\mathrm{Proj}\,\mathscr{P}$*-fans :*

$$\mathrm{Proj}\,(\mathscr{P}^{\mathrm{sat}}) \xrightarrow{\sim} (\mathrm{Proj}\,\mathscr{P})^{\mathrm{sat}}.$$

Proof. (i): The proof is *mutatis mutandis*, the same as that of lemma 10.6.33 (with some minor simplifications). We leave it as an exercise for the reader.

(ii): The first assertion shall be left to the reader. The second assertion is local on $\mathrm{Proj}\,\mathscr{P}$, hence we may assume that $T = (\mathrm{Spec}\,P_0)$, and $\mathscr{P} = P^\sim$ for some $\mathbb{N}$-graded integral P_0-monoid P. Let $a \in P^{\mathrm{sat}}$ be any element; by definition we have $a^n \in P$ for some $n > 0$, and we know that the open subsets $D_+(a)$ et $D_+(a^n)$ coincide in $\mathrm{Proj}(\mathscr{P}^{\mathrm{sat}})$; hence we come down to showing that $(P_{(a)})^{\mathrm{sat}} = (P^{\mathrm{sat}})_{(a)}$ for every $a \in P$, which can be left to reader. □

Definition 6.5.52. Let $(T, \mathscr{O}_T)$ be a fan (resp. an integral fan), $\mathscr{I} \subset \mathscr{O}_T$ an ideal (resp. a fractional ideal) of $\mathscr{O}_T$.

(i) Let $f : X \to T$ be a morphism of fans (resp. of integral fans); then $f^{-1}\mathscr{I}$ is an ideal (resp. a fractional ideal) of $f^{-1}\mathscr{O}_T$, and we let :

$$\mathscr{I}\mathscr{O}_X := \log f(f^{-1}\mathscr{I}) \cdot \mathscr{O}_X$$

which is the smallest ideal (res. fractional ideal) $\mathscr{O}_X$ containing the image of $f^{-1}\mathscr{I}$.

(ii) A *blow up* of the ideal $\mathscr{I}$ is a morphism of fans (resp. of integral fans) $\varphi : T' \to T$ which enjoys the following universal property. The ideal (resp. fractional ideal) $\mathscr{I}\mathscr{O}_{T'}$ is invertible, and every morphism of fans (resp. of integral fans) $X \to T$ such that $\mathscr{I}\mathscr{O}_X$ is invertible, factors uniquely through φ.

6.5.53. Let T be a fan (resp. an integral fan), $\mathscr{I} \subset \mathscr{O}_T$ a quasi-coherent ideal (resp. fractional ideal), and consider the $\mathbb{N}$-graded $\mathscr{O}_T$-monoid :

$$\mathscr{B}(\mathscr{I}) := \coprod_{n\in\mathbb{N}} \mathscr{I}^n$$

where $\mathscr{I}^n \subset \mathscr{O}_T$ is the ideal (resp. fractional ideal) associated with the presheaf $U \mapsto \mathscr{I}(U)^n$ for every open subset $U \subset T$ (notation of (6.1.1), with the convention that $\mathscr{I}^0 := \mathscr{O}_T$) and the multiplication law of $\mathscr{B}(\mathscr{I})$ is defined in the obvious way.

Proposition 6.5.54. *The natural projection*

$$\mathrm{Proj}\,\mathscr{B}(\mathscr{I}) \to T$$

is a blow up of the ideal $\mathscr{I}$.

Proof. We shall consider the case where T is not necessarily integral, and $\mathscr{I} \subset \mathscr{O}_T$; the case of a fractional ideal of an integral fan is proven in the same way. Set $Y := \mathrm{Proj}\,\mathscr{B}(\mathscr{I})$; to begin with, let us show that $\mathscr{I}\mathscr{O}_Y$ is invertible. The assertion is local on T, hence we may assume that $T = (\mathrm{Spec}\,P)^\sharp$, and $\mathscr{I} = I^\sim$ for some ideal $I \subset P$, so $Y = \mathrm{Proj}\,B(I)$, where $B(I) = \coprod_{n\in\mathbb{N}} I^n$. Let $a \in B(I)_1 = I$ be any element; then the restriction of $\mathscr{I}\mathscr{O}_Y$ to $D_+(a)$ is generated by $1 = a/a \in B(I)_{(a)}$, so clearly $\mathscr{I}_{|D_+(a)} \simeq \mathscr{O}_Y$; since $U_1(B(I)) = Y$ (notation of (6.5.33)), the contention follows.

Next, let $\varphi : X \to T$ be a morphism of fans, such that $\mathscr{I}\mathscr{O}_X$ is an invertible ideal. It follows easily that $\mathscr{I}^n\mathscr{O}_X$ is invertible for every $n \in \mathbb{N}$, so the natural map of $\mathbb{N}$-graded $\mathscr{O}_X$-monoids

$$\varphi^*\mathrm{Sym}^\bullet_{\mathscr{O}_T}(\mathscr{I}) \xrightarrow{\sim} \mathrm{Sym}^\bullet_{\mathscr{O}_X}(\mathscr{I}\mathscr{O}_X) \to \mathscr{B}(\mathscr{I}\mathscr{O}_X)$$

is an isomorphism. On the other hand, the projection $\mathrm{Proj}\,\mathscr{B}(\mathscr{I}\mathscr{O}_X) \to X$ is an isomorphism (example (6.5.44)), whence – in view of (6.5.49) – a natural morphism of T-fans :

$$X \to \mathrm{Proj}\,\mathscr{B}(\mathscr{I}). \tag{6.5.55}$$

To conclude, it remains to show that (6.5.55) is the only morphism of T-fans from X to $\mathrm{Proj}\,\mathscr{B}(\mathscr{I})$. The latter assertion can be checked again locally on T, so we are reduced as above to the case where T is the spectrum of P, and $\mathscr{I}$ is associated with I. We may also assume that $X = (\mathrm{Spec}\,Q)^\sharp$, and φ is given by a morphism of monoids $f : P \to Q$. Then the hypothesis means that the ideal $f(I)Q$ is isomorphic to Q (see remark 6.5.13(ii)), hence it is generated by an element of the form $f(a)$, for some $a \in I$, and the endomorphism $x \mapsto f(a)x$ of $f(I)Q$, is an isomorphism. In such situation, it is clear that f factors uniquely through a morphism of monoids $P \to B(I)_{(a)}$; namely, one defines $g : B(I)_{(a)} \to Q$ by the rule : $a^{-k}x \mapsto f(a)^{-k}f(x)$ (for every $x \in I^k$), which is well defined, by the foregoing observations. The morphism $(\mathrm{Spec}\,g)^\sharp : X \to D_+(a)$ must then agree with (6.5.55). □

Example 6.5.56. (i) Let P be a monoid, $I \subset P$ any finitely generated ideal, $\{a_1, \dots, a_n\}$ a finite system of generators of I; set $T := (\mathrm{Spec}\,P)^\sharp$, and let $\varphi : T' \to T$ be the blow up of the ideal $I^\sim \subset \mathscr{O}_T$. Then T' admits an open covering consisting of the affine fans $D_+(f_i)$. The latter are the spectra of the monoids Q_i consisting of all fractions of the form $a \cdot f_i^{-t}$, for every $a \in I^n$; we have $a \cdot f_i^{-t} = b \cdot f_i^{-s}$ in Q_i if and only if there exists $k \in \mathbb{N}$ such that $a \cdot f_i^{s+k} = b \cdot f_i^{t+k}$, if and only if the two fractions are equal in P_{f_i}, in other word, Q_i is the submonoid of P_{f_i} generated by P and $\{f_j \cdot f_i^{-1} \mid j \leq n\}$, for every $i = 1, \dots, n$.

(ii) Consider the special case where P is fine, and the ideal $I \subset P$ is generated by two elements $f, g \in P$. Let $t \in T'$ be any point; up to swapping f and g, we may assume that t corresponds to a prime ideal $\mathfrak{p} \subset P[f/g]$, hence $\varphi(t)$ corresponds to $\mathfrak{q} := j^{-1}\mathfrak{p} \subset P$, where $j : P \to P[f/g]$ is the natural map. We have the following two possibilities :

- Either $f/g \in \mathfrak{p}$, in which case let $y \in F' := P[f/g]\setminus\mathfrak{p}$ be any element; writing $x = y\cdot(f/g)^n$ for some $n \geq 0$ and $y \in P$, we deduce that $n = 0$, so $x = y \in F := P\setminus\mathfrak{q}$, therefore $F' = j(F)$. Notice as well that in this case f/g is not invertible in $P[f/g]$, hence $P[f/g]^\times = P^\times$, whence $\dim P[f/g] = \dim P$, and , by corollary 6.4.12(i).
- Or else $f/g \notin \mathfrak{p}$, in which case the same argument yields $F' = j(F)[f/g]$. In this case, f/g is invertible in $P[f/g]$ if and only if it is invertible in the face F', hence $\mathrm{ht}\,\mathfrak{p} = \dim P - \mathrm{rk}_\mathbb{Z} F' \geq \dim P - \dim F - 1$, by corollary 6.4.12(i),(ii).

In either event, corollary 6.4.12(i),(ii) implies the inequality :

$$1 \geq \mathrm{ht}(\varphi(t)) - \mathrm{ht}(t) \geq 0 \qquad \text{for every } t \in T'.$$

6.6. **Special subdivisions.** In this section we explain how to construct – either by geometrical or combinatorial means – useful subdivisions of given fans.

6.6.1. Let T be any locally fine and saturated fan, and $t \in T$ any point. By reflexivity (proposition 6.4.14(iv)), the elements $s \in \mathscr{O}^{\mathrm{gp}}_{T,t} \otimes_{\mathbb{Z}} \mathbb{Q}$ correspond bijectively to $\mathbb{Q}$-linear forms $\rho_s : U(t)(\mathbb{Q}_+)^{\mathrm{gp}} \to \mathbb{Q}$, and $s \in \mathscr{O}^{\mathrm{gp}}_{T,t}$ if and only if ρ_s restricts to a morphism of monoids $U(t)(\mathbb{N}) \to \mathbb{Z}$. Moreover, this bijection is compatible with specialization maps : if t' is a generization of t in T, then the form $U(t')(\mathbb{Q}_+)^{\mathrm{gp}} \to \mathbb{Q}$ induced by the image of s in $\mathscr{O}^{\mathrm{gp}}_{T,t'} \otimes_{\mathbb{Z}} \mathbb{Q}$ is the restriction of ρ_s (see (6.5.20)).

Hence, any global section $\lambda \in \Gamma(T, \mathscr{O}^{\mathrm{gp}}_T)$ yields a well defined function

$$\rho_\lambda : T(\mathbb{N}) \to \mathbb{Z}$$

whose restriction to $U(t)(\mathbb{N})$ is the restriction of a $\mathbb{Z}$-linear form on $U(t)(\mathbb{N})^{\mathrm{gp}}$, for every $t \in T$; conversely, any such function arises from a unique global section of $\mathscr{O}^{\mathrm{gp}}_T$. Likewise, we have a natural isomorphism between the $\mathbb{Q}$-vector space of global sections λ of $\mathscr{O}^{\mathrm{gp}}_T \otimes_{\mathbb{Z}} \mathbb{Q}$, and the space of functions $\rho_\lambda : T(\mathbb{Q}_+) \to \mathbb{Q}$ with a corresponding linearity property.

Let now $\rho : T(\mathbb{Q}_+) \to \mathbb{Q}$ be any function; we may attach to ρ a sheaf of fractional ideals of $\mathscr{O}_{T,\mathbb{Q}}$ (notation of (6.3.20)), by the rule :

$$\mathscr{I}_{\rho,\mathbb{Q}}(U) := \{s \in \mathscr{O}_{T,\mathbb{Q}}(U) \mid \rho_s \geq \rho_{|U}\} \qquad \text{for every open subset } U \subset T$$

In this generality, not much can be said concerning $\mathscr{I}_{\rho,\mathbb{Q}}$; to advance, we restrict our attention to a special class of functions, singled out by the following :

Definition 6.6.2. Let T be a locally fine and saturated fan.

(a) A *roof* on T is a function :

$$\rho : T(\mathbb{Q}_+) \to \mathbb{Q}$$

such that, for every $t \in T$, there exist $k := k(t) \in \mathbb{N} \setminus \{0\}$ and $\mathbb{Q}$-linear forms

$$\lambda_1, \dots, \lambda_k : U(t)(\mathbb{Q}_+)^{\mathrm{gp}} \to \mathbb{Q}$$

with $\rho(s) = \min(\lambda_i(s) \mid i = 1, \dots, k)$ for every $s \in U(t)(\mathbb{Q}_+)$.

(b) An *integral roof* on T is a roof ρ on T such that $\rho(s) \in \mathbb{Z}$ for every $s \in T(\mathbb{N})$.

6.6.3. The interest of the notion of roof on a fan T, is that it encodes in a geometrical way, an integral subdivision of T, together with a coherent sheaf of fractional ideals of $\mathscr{O}_T$ (see definition 4.8.6(iii)). This shall be seen in several steps. To begin with, let T and ρ be as in definition 6.6.2(a). For any $t \in T$, pick a system $\underline{\lambda} := \{\lambda_1, \dots, \lambda_k\}$ of $\mathbb{Q}$-linear forms fulfilling condition (b) of the definition; then for every $i = 1, \dots, k$ let us set :

$$U(t,i)(\mathbb{N}) := \{x \in U(t)(\mathbb{N}) \mid \rho(x) = \lambda_i(x)\}.$$

Notice that $U(t)(\mathbb{N}) = \mathscr{O}_{T,t}$, by proposition 6.4.14(iv). Moreover, say that $\underline{\lambda}$ is *irredundant for* t if no proper subsystem of $\underline{\lambda}$ fulfills condition (b) of definition 6.6.2 relative to $U(t)$.

Lemma 6.6.4. *With the notation of* (6.6.3)*, the following holds :*

(i) $U(t,i)(\mathbb{N})$ *is a saturated fine monoid for every* $i \leq k$.

(ii) *There is a unique system of* $\mathbb{Q}$*-linear forms which is irredundant for* t.

(iii) *If* $\underline{\lambda}$ *is irredundant for* t*, then* $\dim U(t,i)(\mathbb{N}) = \mathrm{ht}_T(t)$ *for every* $i = 1, \dots, k$.

Proof. (i): We leave to the reader the verification that $U(t,i)$ is a saturated monoid. Next, let $\sigma_i \subset U(t)(\mathbb{R}_+)^{\mathrm{gp}\vee}$ be the convex polyhedral cone spanned by the linear forms

$$((\lambda_j - \lambda_i) \otimes_{\mathbb{Q}} \mathbb{R} \mid j = 1, \dots, k).$$

Then σ_i is $\mathscr{O}^{\mathrm{gp}\vee}_{T,t}$-rational, so $\sigma_i^\vee$ is $\mathscr{O}^{\mathrm{gp}}_{T,t}$-rational, and $\sigma_i^\vee \cap \mathscr{O}^{\mathrm{gp}}_{T,t}$ is a fine monoid (propositions 6.3.21(i), and 6.3.22(i)), therefore the same holds for $U(t,i)(\mathbb{N}) = \sigma_i^\vee \cap \mathscr{O}_{T,t}$ (corollary 6.4.2).

(iii): Notice that $\mathrm{ht}_T(t) = \dim U(t)(\mathbb{N})$, by proposition 6.4.14(ii) and (6.5.18). In view of corollary 6.4.12(i), it follows already that $\dim U(t,i) \leq \mathrm{ht}_T(t)$. Now, let $\lambda_1, \dots, \lambda_k$ be an irredundant system, and suppose, by contradiction, that $\dim U(t,i)(\mathbb{N}) < \dim U(t)(\mathbb{N})$ for some $i \leq k$. Especially, $\sigma_i^\vee \cap U(t)(\mathbb{R}_+)$ does not span the $\mathbb{R}$-vector space $U(t)(\mathbb{R}_+)^{\mathrm{gp}}$, and therefore the dual $\sigma_i + U(t)(\mathbb{R})^\vee$ is not strictly convex (corollary 6.3.14). After relabeling, we may assume that $i = 1$. Hence there exist $a_j, b_j \in \mathbb{R}_+$ and $\varphi, \varphi' \in U(t)(\mathbb{R})^\vee$, and an identity :

$$\sum_{j=2}^{k} a_j(\lambda_j - \lambda_i) + \varphi = -\sum_{j=2}^{k} b_j(\lambda_j - \lambda_i) - \varphi'.$$

Moreover, $\sum_{j=2}^{k}(a_j + b_j) > 0$. It follows that there exist $\psi \in U(t)(\mathbb{R})^\vee$, and non-negative real numbers $(c_j \mid j = 2, \dots k)$ such that

$$\lambda_i = \sum_{j=2}^{k} c_j\lambda_j + \psi \quad \text{and} \quad \sum_{j=2}^{k} c_j = 1.$$

On the other hand, the irredundancy condition means that there exists $x \in U(t,1)(\mathbb{N})$ such that $\lambda_j(x) > \lambda_1(x)$ for every $j > 1$. Since $\psi(x) \geq 0$, we get a contradiction.

(ii): The assertion is clear, if $\mathrm{ht}_T(t) \leq 1$. Hence suppose that the height of t is ≥ 2, and let $\underline{\lambda} := \{\lambda_1, \dots, \lambda_k\}$ and $\underline{\mu} := \{\mu_1, \dots, \mu_r\}$ be two irredundant systems for t. Fix $i \leq k$, and pick $x \in U(t,i)(\mathbb{N})$ which does not lie on any proper face of $U(t,i)(\mathbb{N})$ (the existence of x is ensured by (iii) and proposition 6.4.9(i)); say that $\mu_1(x) = \rho(x)$. Since i is arbitrary, the assertion shall follow, once we have shown that μ_1 agrees with λ_i on the whole of $U(t,i)(\mathbb{N})$.

However, by definition we have $\mu_1(y) \geq \lambda_i(y)$ for every $y \in U(t,i)(\mathbb{N})$, and then it is easily seen that $\mathrm{Ker}(\mu_1 - \lambda_i) \cap U(t,i)(\mathbb{N})$ is a face of $U(t,i)(\mathbb{N})$; since $x \in \mathrm{Ker}(\mu_1 - \lambda_i)$, we deduce that $\mu_1 - \lambda_i$ vanishes identically on $U(t,i)(\mathbb{N})$. □

6.6.5. Henceforth, we denote by $\underline{\lambda}(t) := \{\lambda_1, \dots, \lambda_k\}$ the irredundant system of $\mathbb{Q}$-linear forms for t. Let $1 \leq i, j \leq k$; then we claim that $U(t,i,j)(\mathbb{N}) := U(t,i)(\mathbb{N}) \cap U(t,j)(\mathbb{N})$ is a face of both $U(t,i)(\mathbb{N})$ and $U(t,j)(\mathbb{N})$. Indeed say that $x, x' \in U(t,i)(\mathbb{N})$ and $x + x' \in U(t,i,j)(\mathbb{N})$; these conditions translate the identities :

$$\lambda_i(x) + \lambda_i(x') = \lambda_j(x) + \lambda_j(x') \qquad \lambda_i(x) \leq \lambda_j(x) \qquad \lambda_i(x') \leq \lambda_j(x')$$

whence $x, x' \in U(t,i,j)(\mathbb{N})$. Define :

$$U(t,i) := (\mathrm{Spec}\, U(t,i)(\mathbb{N})^\vee)^\sharp \qquad U(t,i,j) := (\mathrm{Spec}\, U(t,i,j)(\mathbb{N})^\vee)^\sharp \qquad \text{for every } i,j \leq k.$$

According to (6.5.20) and (6.5.18), the inclusion maps $U(t,i,j)(\mathbb{N}) \to U(t,l)(\mathbb{N})$ (for $l = i, j$) are dual to open immersions

$$U(t,i) \leftarrow U(t,i,j) \to U(t,j). \tag{6.6.6}$$

We may then attach to t and $\rho_{|U(t)}$ the fan $U(t,\rho)$ obtained by gluing the affine fans $U(t,i)$ along their common intersections $U(t,i,j)$. The duals of the inclusions $U(t,i)(\mathbb{N}) \to U(t)(\mathbb{N})$ determine a well defined morphism of locally fine and saturated fans :

$$U(t,\rho) \to U(t) \tag{6.6.7}$$

which, by construction, induces a bijection on $\mathbb{N}$-points : $U(t,\rho)(\mathbb{N}) \xrightarrow{\sim} U(t)(\mathbb{N})$, so it is a rational subdivision, according to proposition 6.5.24.

6.6.8. Let now $t' \in T$ be a generization of t; clearly the system $\underline{\lambda}' := \{\lambda'_1, \dots, \lambda'_k\}$ consisting of the restrictions λ'_i of the linear forms λ_i to the $\mathbb{Q}$-vector subspace $U(t')(\mathbb{Q}_+)^{\mathrm{gp}}$, fulfills condition (b) of definition 6.6.2, relative to t'. However, $\underline{\lambda}'$ may fail to be irredundant; after relabeling, we may assume that the subsystem $\{\lambda'_1, \dots, \lambda'_l\}$ is irredundant for t', for some $l \leq k$. With the foregoing notation, we have obvious identities :

$$U(t', i)(\mathbb{N}) = U(t, i)(\mathbb{N}) \cap U(t')(\mathbb{N}) \qquad U(t', i, j)(\mathbb{N}) = U(t, i, j)(\mathbb{N}) \cap U(t')(\mathbb{N})$$

for every $i, j \leq l$; whence, in light of remark 6.4.16(iii), a commutative diagram of fans :

$$\begin{array}{ccccc} U(t', i) & \longleftarrow & U(t', i, j) & \longrightarrow & U(t', j) \\ \downarrow & & \downarrow & & \downarrow \\ U(t, i) \times_{U(t)} U(t') & \longleftarrow & U(t, i, j) \times_{U(t)} U(t') & \longrightarrow & U(t, i) \times_{U(t)} U(t') \end{array}$$

whose top horizontal arrows are the open immersions (6.6.6) (with t replaced by t'), whose bottom horizontal arrows are the open immersions $(6.6.6) \times_{U(t)} U(t')$, and whose vertical arrows are natural isomorphisms. Since $U(t')$ is an open subset of $U(t)$, we deduce an open immersion

$$j_{t,t'} : U(t', \rho) \to U(t, \rho).$$

If t'' is a generization of t', it is clear that $j_{t',t''} \circ j_{t,t'} = j_{t,t''}$, hence we may glue the fans $U(t, \rho)$ along these open immersions, to obtain a locally fine and saturated fan $T(\rho)$. Furthermore, the morphisms (6.6.7) glue to a single rational subdivision :

$$T(\rho) \to T. \qquad (6.6.9)$$

Remark 6.6.10. (i) In the language of definition 6.3.25 the foregoing lengthy procedure translates as the following simple geometric operation. Given a fan Δ (consisting of a collection of convex polyhedral cones of a $\mathbb{R}$-vector space V), a roof on Δ is a piecewise linear function $F := \bigcup_{\sigma \in \Delta} \sigma \to \mathbb{R}$, which is concave on each $\sigma \in \Delta$ (and hence it is a roof on each such σ, in the sense of example 6.3.27). Then, such a roof determines a natural refinement Δ' of Δ; namely, Δ' is the coarsest refinement such that, for each $\sigma' \in \Delta'$, the function $\rho_{|\sigma'}$ is the restriction of a $\mathbb{R}$-linear form on V. This refinement Δ' corresponds to the present $T(\rho)$.

(ii) Moreover, let P be a fine, sharp and saturated monoid of dimension d, set $T_P := (\operatorname{Spec} P)^\sharp$, and suppose that $f : T \to T_P$ is any integral, fine, proper and saturated subdivision. Then f corresponds to a geometrical subdivision Δ of the strictly convex polyhedral cone $T_P(\mathbb{R}_+) = P^\vee_\mathbb{R}$, and we claim that Δ can be refined by the subdivision associated with a roof on T_P. Namely, let Δ_{d-1} be the subset of Δ consisting of all σ of dimension $d-1$; every $\sigma \in \Delta_{d-1}$ is the intersection of a d-dimensional element of Δ and a hyperplane $H_\sigma \subset P^{\mathrm{gp}\vee}_\mathbb{R}$; such hyperplane is the kernel of a linear form λ_σ on $P^{\mathrm{gp}\vee}_\mathbb{R}$. Let us define

$$\rho(x) := \sum_{\sigma \in \Delta_{d-1}} \min(0, \lambda_\sigma(x)) \qquad \text{for every } x \in T_P(\mathbb{R}_+).$$

Then it is easily seen that the subdivision of $T_P(\mathbb{R}_+)$ associated with the roof ρ as in (i), refines the subdivision Δ. In the language of fans, this construction translates as follows. For every point $\sigma \in T$ of height $d-1$, let $H_\sigma \subset P^{\mathrm{gp}}$ be the kernel of the surjection $P^{\mathrm{gp}} \to \mathscr{O}^{\mathrm{gp}}_{T,\sigma}$ induced by $\log f$; notice that H_σ is a free abelian group of rank one, and pick a generator s_σ of H_σ, which – as in (6.6.1) – corresponds to a function $\lambda_\sigma : T_P(\mathbb{N}) \to \mathbb{Z}$, so we may again consider the integral roof ρ on T_P defined as in the foregoing. Then it is easily seen that the rational subdivision $T(\rho) \to T_P$ associated with ρ, factors as the composition of f and a (necessarily unique) integral subdivision $g : T(\rho) \to T$. More precisely, for every mapping

$\varepsilon : \{\sigma \in T \mid \mathrm{ht}(\sigma) = d-1\} \to \{0,1\}$, let us set

$$\lambda_\varepsilon := \sum_{\mathrm{ht}(\sigma)=d-1} \varepsilon(\sigma)\cdot\lambda_\sigma \qquad \text{and} \qquad U(\varepsilon)(\mathbb{N}) := \{x \in T_P(\mathbb{N}) \mid \rho(x) = \lambda_\varepsilon(x)\}.$$

Whenever $U(\varepsilon)(\mathbb{N})$ has dimension d, let $t_\varepsilon \in T(\rho)$ be the unique point such that $U(\varepsilon)(\mathbb{N})^\vee = \mathscr{O}_{T(\rho),t_\varepsilon}$. As the reader may check, there exists a unique closed point $\tau \in T$, such that $\varepsilon(\sigma)\cdot\lambda_\sigma(x) \leq 0$ for every $x \in U(\tau)(\mathbb{N})$ and every $\sigma \in U(\tau)$ of height $d-1$. Then we have $g(t_\varepsilon) = \tau$, and the restriction $U(t_\varepsilon) \to U(\tau)$ of g is deduced from the inclusion $U(\varepsilon)(\mathbb{N}) \subset U(\tau)(\mathbb{N})$ of submonoids of $T_P(\mathbb{N})$.

(iii) Furthermore, in the situation of (ii), the roof ρ on T_P can also be viewed as a roof on T, and then it is clear from the construction that the morphism $g : T(\rho) \to T$ is also the subdivision of T attached to the roof ρ.

We wish now to establish some basic properties of the sheaf of fractional ideals

$$\mathscr{I}_\rho := \mathscr{I}_{\rho,\mathbb{Q}} \cap \mathscr{O}_T^{\mathrm{gp}}$$

attached to a given roof on T. First we remark :

Lemma 6.6.11. *Keep the notation of* (6.6.5)*, and let $s \in \mathscr{O}^{\mathrm{gp}}_{T,t} \otimes_\mathbb{Z} \mathbb{Q}$ be any element such that $\rho_s \geq \rho_{|U(t)}$. Then we have :*

(i) *There exist $\varphi \in (\mathscr{O}_{T,t})_\mathbb{Q}$ (notation of* (6.3.20)*) and $c_1,\dots,c_k \in \mathbb{R}_+$ such that :*

$$\rho_s = \rho_\varphi + \sum_{i=1}^k c_i\lambda_i \qquad \sum_{i=1}^k c_i = 1.$$

(ii) *The stalk $\mathscr{I}_{\rho,t}$ is a finitely generated $\mathscr{O}_{T,t}$-module.*

Proof. (i): Let $\sigma \subset U(t)(\mathbb{R}_+)^{\mathrm{gp}}$ be the convex polyhedral cone spanned by the linear forms $((\lambda_i - \rho_s)\otimes_\mathbb{Q}\mathbb{R} \mid i = 1,\dots,k)$. Then the assumption on s means that

$$U(t)(\mathbb{R}_+) \cap \sigma^\vee = U(t)(\mathbb{R}_+) \cap \mathrm{Ker}\,\rho_s \otimes_\mathbb{Q} \mathbb{R}.$$

Especially $\sigma^\vee \cap U(t)(\mathbb{R}_+)$ does not span $U(t)(\mathbb{R}_+)^{\mathrm{gp}}$. Then one can repeat the proof of lemma 6.6.4(iii) to derive the assertion.

(ii): By remark 6.4.16(i), we may write $\lambda_i = \rho_{s_i} - \rho_{s'_i}$, where $s_i, s'_i \in (\mathscr{O}_{T,t})_\mathbb{Q}$ for each $i \leq k$; pick $N \in \mathbb{N}$ large enough, so that $Ns'_i \in \mathscr{O}_{T,t}$ for every $i \leq k$, and set $\tau := N\sum_{i=1}^k s'_i$. Then $\tau + \mathscr{I}_{\rho,t} \subset \mathscr{O}^{\mathrm{gp}}_{T,t} \cap (\mathscr{O}_{T,t})_\mathbb{Q} = \mathscr{O}_{T,t}$. By proposition 6.1.9(ii), we deduce that $\tau + \mathscr{I}_{\rho,t}$ is a finitely generated ideal, whence the contention. ☐

Proposition 6.6.12. *Let T be a locally fine and saturated fan, ρ a roof on T. Then the associated fractional ideal $\mathscr{I}_\rho$ of $\mathscr{O}_T$ is coherent.*

Proof. In view of lemma 6.6.11(ii) and remark 6.5.13(v), it suffices to show that $\mathscr{I}_\rho$ is quasi-coherent, *i.e.* for every generization t' of t, the image of $\mathscr{I}_{\rho,t}$ in $\mathscr{O}_{T,t'}$ generates the $\mathscr{O}_{T,t'}$-module $\mathscr{I}_{\rho,t'}$. Fix such t'; by propositions 6.3.21(i) and 6.4.9(i), there exists $\lambda \in \mathscr{O}_{T,t} = U(t)(\mathbb{N})^\vee$ such that $U(t')(\mathbb{N}) = \mathrm{Ker}\,\lambda$; especially, we see that $U(t')(\mathbb{N})^{\mathrm{gp}}$ is a direct summand of $U(t)(\mathbb{N})^{\mathrm{gp}}$. Now, let $s' \in \mathscr{I}_{\rho,t'}$ be any local section; it follows that we may find $s \in \mathscr{O}^{\mathrm{gp}}_{T,t}$ such that $\rho_s : U(t)(\mathbb{N})^{\mathrm{gp}} \to \mathbb{Z}$ is a $\mathbb{Z}$-linear extension of the corresponding $\mathbb{Z}$-linear form $\rho_{s'} : U(t')(\mathbb{N})^{\mathrm{gp}} \to \mathbb{Z}$. Let also $\{\lambda_1,\dots,\lambda_k\}$ be the irredundant system of $\mathbb{Q}$-linear forms for t (relative to the roof ρ). For every $i \leq k$ we have the following situation :

$$\lambda_\mathbb{R} := \lambda \otimes_\mathbb{Q} \mathbb{R} \in U(t,i)(\mathbb{R}_+)^\vee \qquad (\rho_s - \lambda_i)\otimes_\mathbb{Q}\mathbb{R} \in U(t',i)(\mathbb{R}_+)^\vee.$$

However, $U(t',i)(\mathbb{R}_+) = U(t,i)(\mathbb{R}_+) \cap \mathrm{Ker}\,\lambda_\mathbb{R}$, hence $U(t',i)(\mathbb{R}_+)^\vee = U(t,i)(\mathbb{R}_+)^\vee + \mathbb{R}\lambda_\mathbb{R}$. Especially, there exist $r_i \in \mathbb{R}_+$ and $\varphi \in U(t)(\mathbb{R}_+)^\vee$ such that $(\rho_s - \lambda_i)\otimes_\mathbb{Q}\mathbb{R} = \varphi - r_i\lambda_\mathbb{R}$. Let

N be an integer greater than $\max(r_1, \dots, r_k)$; it follows that $s + N\lambda \in \mathscr{I}_{\rho,t}$ and its image in $\mathscr{I}_{\rho,t'}$ equals s'. ☐

Proposition 6.6.13. *In the situation of* (6.6.3) *suppose that ρ is an integral roof on T. Then the morphism* (6.6.9) *is the saturation of a blow up of the fractional ideal $\mathscr{I}_\rho$.*

Proof. We have to exhibit an isomorphism $f : T(\rho) \to X := \mathrm{Proj}\,\mathscr{B}(\mathscr{I})^{\mathrm{sat}}$ of T-fans. We begin with :

Claim 6.6.14. (i) The fractional ideal $\mathscr{I}_\rho \mathscr{O}_{T(\rho)}$ is invertible.

(ii) For every $n \in \mathbb{N}$, denote by $n\rho : T(\mathbb{Q}_+) \to \mathbb{Q}_+$ the function given by the rule $x \mapsto n \cdot \rho(x)$ for every $x \in T(\mathbb{Q}_+)$. Then :

$$\mathscr{B}(\mathscr{I})^{\mathrm{sat}} = \mathscr{B}' := \coprod_{n\in\mathbb{N}} \mathscr{I}_{n\rho}$$

Proof of the claim. (ii): The assertion is local on T, hence we may assume that $T = U(t)$ for some $t \in T$, in which case, denote by $\underline{\lambda} := \{\lambda_1, \dots, \lambda_k\}$ the irredundant system of $\mathbb{Q}$-linear forms for t. Let $n \in \mathbb{N}$, and $s \in \mathscr{I}_{n\rho}(U(t))$; it is easily seen that the T-monoid $\mathscr{B}'$ is saturated, hence it suffices to show that there exists an integer $k > 0$ such that $k\rho_s = \rho_{s'}$ for some $s' \in \mathscr{I}^k(U(t))$ (notation of (6.6.1)). However, lemma 6.6.11(ii) implies more precisely that we may find such k, so that the corresponding s' lies in the ideal generated by $\underline{\lambda}$.

(i): The assertion is local on $T(\rho)$, hence we consider again $t \in T$ and the corresponding $\underline{\lambda}$ as in the foregoing. It suffices to show that $\mathscr{J} := \mathscr{I}_\rho \mathscr{O}_{U(t,i)}$ is invertible for every $i = 1, \dots, k$ (notation of (6.6.5)). However, by inspecting the constructions it is easily seen that $\mathscr{J}(U(t,i))$ consists of all $s \in (U(t,i)(\mathbb{N})^\vee)^{\mathrm{gp}}$ such that $\rho_s(x) \geq \rho(x)$ for every $x \in U(t,i)(\mathbb{Q}_+)$, *i.e.* $\rho_s(x) \geq \lambda_i(x)$ for every $x \in U(t,i)(\mathbb{Q}_+)$. However, since ρ is integral, we have $\lambda_i \in \mathscr{O}^{\mathrm{gp}}_{T,t}$; if we apply lemma 6.6.11(i) with T replaced by $U(t,i)$, we conclude that $\mathscr{J}(U(t,i))$ is the fractional ideal generated by λ_i, whence the contention. ◊

In view of claim 6.6.14(i) we see that there exists a unique morphism f of T-fans from $T(\rho)$ to X. It remains to check that f is an isomorphism; the latter assertion is local on X, hence we may assume that $T = U(t)$ for some $t \in T$, and then we let again $\underline{\lambda}$ be the irredundant system for t. A direct inspection yields a natural identification of $\mathscr{O}_{T,t}$-monoids :

$$\mathscr{B}'(U(t,i))_{(\lambda_i)} \xrightarrow{\sim} U(t,i)(\mathbb{N})^\vee \qquad \text{for every } i = 1, \dots, k$$

whence an isomorphism $U(t,i) \xrightarrow{\sim} D_+(\lambda_i) \subset X$, which – by uniqueness – must coincide with the restriction of f. On the other hand, the proof of claim 6.6.14(ii) also shows that $X = D_+(\lambda_1) \cup \dots \cup D_+(\lambda_k)$, and the proposition follows. ☐

Example 6.6.15. Let P be a fine, sharp and saturated monoid.

(i) The simplest non-trivial roofs on $T_P := (\mathrm{Spec}\, P)^\sharp$ are the functions ρ_λ such that

$$\rho_\lambda(x) := \min(0, \lambda(x)) \qquad \text{for every } x \in T_P(\mathbb{Q}_+).$$

where $\lambda \neq 0$ is a given element of $\mathrm{Hom}_{\mathbb{Q}}(T_P(\mathbb{Q}_+)^{\mathrm{gp}}, \mathbb{Q}) \simeq P^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$. Such a ρ_λ is an integral roof, provided $\lambda \in P^{\mathrm{gp}}$. In the latter case, we may write $\lambda = \rho_{s_1} - \rho_{s_2}$, for some $s_1, s_2 \in P$. Set $\rho' := \rho_\lambda + \rho_{s_2}$, *i.e.* $\rho' = \min(\rho_{s_1}, \rho_{s_2})$; clearly $T(\rho_\lambda) = T(\rho')$, and on the other hand lemma 6.6.11(i) implies that the ideal $\mathscr{I}_{\rho'}$ is the saturation of the ideal generated by s_1 and s_2.

(ii) More generally, any system $\lambda_1, \dots, \lambda_n \in \mathrm{Hom}_{\mathbb{Q}}(T_P(\mathbb{Q}_+)^{\mathrm{gp}}, \mathbb{Q})$ of $\mathbb{Q}$-linear forms yields a roof ρ on T_P, such that $\rho(x) := \sum_{i=1}^n \min(0, \lambda_i(x))$ for every $x \in T_P(\mathbb{Q}_+)$. A simple inspection shows that the corresponding subdivision $T(\rho) \to T$ can be factored as the composition of n subdivisions $g_i : T_i \to T_{i-1}$, where $T_0 := T_P$, $T_n := T(\rho)$, and each g_i (for $i \leq n$) is the subdivision of T_{i-1} corresponding to the roof ρ_{λ_i} as defined in (i).

(iii) These subdivisions of T_P "by hyperplanes" are precisely the ones that occur in remark 6.6.10(ii),(iii). Summing up, we conclude that every proper integral and saturated subdivision $g : T \to T_P$ of T_P can be dominated by another subdivision $f : T(\rho) \to T_P$ of the type considered in (ii), so that f factors as the composition of g and a subdivision $h : T(\rho) \to T$ which is also of the type (ii). Especially, both f and h can be realized as the composition of finitely many saturated blow up of ideals generated by at most two elements of P.

6.6.16. Let P be a fine, sharp and saturated monoid. A proper, integral, fine and saturated subdivision of

$$T_P := (\operatorname{Spec} P)^\sharp$$

is essentially equivalent to a $(P^{\mathrm{gp}})^\vee$-rational subdivision of the polyhedral cone $\sigma := P^\vee_{\mathbb{R}}$ (see (6.4.8) and definition 6.3.25). A standard way to subdivide a polyhedron σ consists in choosing a point $x_0 \in \sigma \setminus \{0\}$, and forming all the polyhedra $x_0 * F$, where F is any proper face of σ, and $x_0 * F$ denotes the convex span of x_0 and F. We wish to describe the same operation in terms of the topological language of affine fans.

Namely, pick any non-zero $\varphi \in T_P(\mathbb{Q}_+)$ (φ corresponds to the point x_0 in the foregoing). Let $U(\varphi) \subset \operatorname{Spec} P$ be the set of all prime ideals $\mathfrak{p}$ such that $\varphi(P \setminus \mathfrak{p}) \neq \{0\}$; in other words, the complement of $U(\varphi)$ is the topological closure of the support of φ in T_P, especially, $U(\varphi)$ is an open subset of T_P. Denote by $j : P = \Gamma(T_P, \mathscr{O}_{T_P}) \to \Gamma(U(\varphi), \mathscr{O}_{T_P})$ the restriction map. The morphism of monoids :

$$P \to \Gamma(U(\varphi), \mathscr{O}_{T_P}) \times \mathbb{Q}_+ \qquad x \mapsto (j(x), \varphi(x))$$

determines a cocartesian diagram of fans

$$\begin{array}{ccc} U(\varphi) \times (\operatorname{Spec} \mathbb{Q}_+)^\sharp & \xrightarrow{\beta'} & U(\varphi) \times (\operatorname{Spec} \mathbb{N})^\sharp \\ \downarrow{\scriptstyle \psi'} & & \downarrow{\scriptstyle \psi} \\ T_P & \xrightarrow{\beta} & T_{\varphi^{-1}\mathbb{N}} := (\operatorname{Spec} \varphi^{-1}\mathbb{N})^\sharp \end{array}$$

Lemma 6.6.17. *With the notation of* (6.6.16)*, the morphisms ψ and ψ' are proper rational subdivisions, which we call the* subdivisions centered at φ.

Proof. Notice that both β and β' are homeomorphisms on the underlying topological spaces, and moreover both $\log \beta^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbf{1}_{\mathbb{Q}}$ and $\log \beta'^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbf{1}_{\mathbb{Q}}$ are isomorphisms. Thus, it suffices to show that ψ is a proper rational subdivision, hence we may replace P by $\varphi^{-1}\mathbb{N}$, which allows to assume that φ is a morphism of monoids $P \to \mathbb{N}$, and ψ is a morphism of fans

$$T' := U(\varphi) \times (\operatorname{Spec} \mathbb{N})^\sharp \to T_P.$$

In this situation, by inspecting the construction, we find that ψ restricts to an isomorphism :

$$\psi^{-1}U(\varphi) \xrightarrow{\sim} U(\varphi)$$

and the preimage of the closed subset $T_P \setminus U(\varphi)$ is the preimage of the closed point $\mathfrak{m} \in (\operatorname{Spec} \mathbb{N})^\sharp$ under the natural projection $T' \to (\operatorname{Spec} \mathbb{N})^\sharp$; in view of the discussion of (6.5.19), this is naturally identified with $U(\varphi) \times \{\mathfrak{m}\}$. Moreover, the restriction $U(\varphi) \times \{\mathfrak{m}\} \to T_P \setminus U(\varphi)$ of ψ is the map $(t, \mathfrak{m}) \mapsto t \cup \varphi^{-1}\mathfrak{m}$ (recall that $t \subset P$ is a prime ideal which does not contain $\varphi^{-1}\mathfrak{m}$). Thus, the assertion will follow from :

Claim 6.6.18. The $\mathbb{Q}$-linear map

$$\log \psi^{\mathrm{gp}}_{(t,\mathfrak{m})} \otimes_{\mathbb{Z}} \mathbf{1}_{\mathbb{Q}} : P^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q} \to (\mathscr{O}^{\mathrm{gp}}_{T_P,t} \times \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is surjective for every $t \in U(\varphi)$, and the induced map :

$$T'(\mathbb{Q}_+) \to T_{\varphi^{-1}\mathbb{N}}(\mathbb{Q}_+) = T_P(\mathbb{Q}_+) \qquad (6.6.19)$$

is bijective.

Proof of the claim. Indeed, let $F_t \subset P^\vee$ be the face of $P^\vee$ corresponding to the point $t \in U(\varphi)$, under the bijection (6.5.21); then $\varphi \notin F_t$, whence a natural isomorphism of monoids :

$$(F_t + \mathbb{N}\varphi)^\vee \xrightarrow{\sim} \mathscr{O}_{T_P,t} \times \mathbb{N} \quad : \quad \lambda \mapsto (\lambda_{|F}, \lambda(\varphi))$$

whose inverse, composed with $\log \psi_{(t,\mathfrak{m})}$, yields the restriction map $P \to (F_t + \mathbb{N}\varphi)^\vee$. This interpretation makes evident the surjectivity of $\log \psi^{\mathrm{gp}}_{(t,\mathfrak{m})} \otimes_{\mathbb{Z}} \mathbf{1}_{\mathbb{Q}}$. In view of example 6.5.9(iii), the bijectivity of (6.6.19) is also clear, if one remarks that :

$$P^\vee_{\mathbb{R}} = \bigcup_{t \in U(\varphi)} (F_t + \mathbb{N}\varphi)_{\mathbb{R}}.$$

The latter identity is obvious from the geometric interpretation in terms of polyhedral cones. A formal argument runs as follows. Let $\varphi' \in P^\vee_{\mathbb{R}}$; since P spans $P^{\mathrm{gp}}_{\mathbb{R}}$, the cone $(P_{\mathbb{R}})^\vee$ is strictly convex (corollary 6.3.14), hence the line $\varphi' + \mathbb{R}\varphi \subset P^{\mathrm{gp}}_{\mathbb{R}}$ is not contained in $P^\vee_{\mathbb{R}}$, therefore there exists a largest $r \in \mathbb{R}$ such that $\varphi' - r\varphi \in P^\vee_{\mathbb{R}}$, and necessarily $r \geq 0$. If $\varphi' - r\varphi = 0$, the assertion is clear; otherwise, let F be the minimal face of $P^\vee$ such that $\varphi' - r\varphi \in F_{\mathbb{R}}$, so that $\varphi' = (\varphi' - r\varphi) + r\varphi \in (F + \mathbb{N}\varphi)_{\mathbb{R}}$. Thus, we are reduced to showing that $\varphi \notin F$. But notice that $\varphi' - r\varphi$ lies in the relative interior of F; therefore, if $\varphi \in F$, we may find $\varepsilon > 0$ such that $\varphi' - (r + \varepsilon)\varphi$ still lies in $F_{\mathbb{R}}$, contradicting the definition of r. □

6.6.20. Let P be a fine, sharp and saturated monoid with $d := \dim P > 0$. Lemma 6.6.17 is frequently used to construct subdivisions centered at an *interior point* of T_P, *i.e.* a point $\varphi \in T_P(\mathbb{N})$ which does not lie on any proper face of $T_P(\mathbb{N})$ (equivalently, the support of φ is the closed point $\mathfrak{m}_P$ of T_P). In this case $U(\varphi) = T_P \setminus \{\mathfrak{m}_P\} = (T_P)_{d-1}$. By lemma 6.6.17, the $\mathbb{N}$-point φ lifts to a unique $\mathbb{Q}_+$-point $\widetilde{\varphi}$ of $(T_P)_{d-1} \times (\operatorname{Spec}\mathbb{N})^\sharp$, and by inspecting the definitions, it is easily seen that – under the identification of remark 6.5.8(iii) – the support of $\widetilde{\varphi}$ is the point $(\varnothing, \mathfrak{m}_{\mathbb{N}})$, where $\varnothing \in T_P$ is the generic point. More precisely, we may identify $(\operatorname{Spec}\mathbb{N})^\sharp(\mathbb{Q}_+)$ with $\mathbb{Q}_+$, and $(T_P)_{d-1}(\mathbb{Q}_+)$ with a cone in the $\mathbb{Q}$-vector space $T_P(\mathbb{Q}_+)^{\mathrm{gp}}$, and then $\widetilde{\varphi}$ corresponds to the point $(0,1) \in T_P(\mathbb{Q}_+)^{\mathrm{gp}} \times \mathbb{Q}_+$.

Suppose now that we have an integral roof

$$\rho : (T_P)_{d-1}(\mathbb{Q}_+) \to \mathbb{Q}$$

and let $\pi : T(\rho) \to (T_P)_{d-1}$ be the associated subdivision. Let also $\psi : (T_P)_{d-1} \times (\operatorname{Spec}\mathbb{N})^\sharp \to T_P$ be the subdivision centered at φ; we deduce a new subdivision :

$$T^* := T(\rho) \times (\operatorname{Spec}\mathbb{N})^\sharp \xrightarrow{\pi \times \mathbf{1}_{(\operatorname{Spec}\mathbb{N})^\sharp}} (T_P)_{d-1} \times (\operatorname{Spec}\mathbb{N})^\sharp \xrightarrow{\psi} T_P \qquad (6.6.21)$$

whose restriction to the preimage of $(T_P)_{d-1}$ is T_P-isomorphic to π.

Lemma 6.6.22. *In the situation of* (6.6.20)*, there exist an integral roof*

$$\widetilde{\rho} : T_P(\mathbb{Q}_+) \to \mathbb{Q}$$

whose restriction to $(T_P)_{d-1}(\mathbb{Q}_+)$ *agrees with* ρ*, and a morphism* $T^* \to T(\widetilde{\rho})$ *of* T_P*-fans, whose underlying continuous map is a homeomorphism.*

Proof. According to remark 6.5.8(vi), we have a natural identification :

$$T_P(\mathbb{Q}_+) = T(\rho)(\mathbb{Q}_+) \times (\operatorname{Spec}\mathbb{N})^\sharp(\mathbb{Q}_+) = T(\rho)(\mathbb{Q}_+) \times \mathbb{Q}_+ \subset T_P(\mathbb{Q}_+)^{\mathrm{gp}} \times \mathbb{Q}.$$

mapping the point φ to $(0,1)$. For a given $c \in \mathbb{R}$, denote by $\rho_c : T_P(\mathbb{Q}_+) \to \mathbb{Q}$ the function given by the rule : $(x,y) \mapsto \rho(x) + cy$ for every $x \in T(\rho)(\mathbb{Q}_+)$ and every $y \in \mathbb{Q}_+$.

Let $t \in T(\rho)$ be any point of height $d-1$; by assumption, there exists a $\mathbb{Q}$-linear form $\lambda_t : U(\pi(t))(\mathbb{Q}_+)^{\mathrm{gp}} \to \mathbb{Q}$ whose restriction to $U(t)(\mathbb{Q}_+)$ agrees with the restriction of ρ. Therefore, the restriction of ρ_c to $U(t,\mathfrak{m}_{\mathbb{N}})(\mathbb{Q}_+) = U(t)(\mathbb{Q}_+)\times\mathbb{Q}_+$ agrees with the restriction of the $\mathbb{Q}$-linear form

$$\lambda_{t,c} : U(\pi(t))(\mathbb{Q}_+)^{\mathrm{gp}}\times\mathbb{Q} = T_P(\mathbb{Q}_+)^{\mathrm{gp}}\to\mathbb{Q} \qquad (x,y)\mapsto\lambda_t(x)+cy.$$

For any two points $t,t'\in T(\rho)$ of height $d-1$, with $\pi(t)=\pi(t')$, pick a finite system of generators $\{x_1,\dots,x_n\}$ for $U(t')(\mathbb{N})$, and let $y_1,\dots,y_n\in U(t)(\mathbb{Q})^{\mathrm{gp}}$, $a_1,\dots,a_n\in\mathbb{Q}$ such that

$$x_i = y_i + a_i\varphi \qquad \text{in the vector space } U(\pi(t))(\mathbb{Q}_+)^{\mathrm{gp}}.$$

In case $x_i\in U(t)(\mathbb{N})$, we shall have $a_i=0$, and otherwise we remark that $a_i>0$. Indeed, if $a_i<0$ we would have $x_i-a_i\varphi\in U(t)(\mathbb{Q}_+)^{\mathrm{gp}}\cap T_P(\mathbb{Q}_+)\subset U(\pi(t))(\mathbb{Q}_+)$; however, $U(\pi(t))(\mathbb{Q}_+)$ is a proper face of $T_P(\mathbb{Q}_+)$, hence $\varphi\in U(\pi(t))(\mathbb{Q}_+)$, a contradiction.

We may then find $c>0$ large enough, so that $\lambda_{t'}(x_i)<\lambda_{t,c}(x_i)$ for every $x_i\notin U(t)(\mathbb{N})$. It follows easily that

(6.6.23) $$\lambda_{t',c}(x)<\lambda_{t,c}(x) \qquad \text{for all } x\in U(t',\mathfrak{m}_{\mathbb{N}})(\mathbb{Q}_+)\setminus U(t,\mathfrak{m}_{\mathbb{N}})(\mathbb{Q}_+).$$

Clearly we may choose c large enough, so that (6.6.23) holds for every pair t,t' as above, and then it is clear that ρ_c will be a roof on T_P. Notice that the points of height d of T^* are precisely those of the form $(t,\mathfrak{m}_{\mathbb{N}})$, for $t\in T_P$ of height $d-1$, so the points of $T(\rho_c)$ of height d are in natural bijection with those of T^*, and if $\tau\in T(\rho_c)$ corresponds to $\tau^*\in T^*$ under this bijection, we have an injective map $U(\tau^*)(\mathbb{N})\to U(\tau)(\mathbb{N})$, commuting with the induced projections to $T_P(\mathbb{N})$. There follows a morphism of T_P-fans $U(\tau^*)\to U(\tau)$ inducing a bijection $U(\tau^*)(\mathbb{Q}_+)\xrightarrow{\sim}U(\tau)(\mathbb{Q}_+)$. Since $T(\rho_c)$ (resp. T^*) is the union of all such $U(\tau)$ (resp. $U(\tau^*)$), we deduce a morphism $T^*\to T(\rho_c)$ inducing a homeomorphism on underlying topological spaces. Lastly, say that $\tau^*=(t,\mathfrak{m})$; then $U(\tau^*)(\mathbb{N})^{\mathrm{gp}}=U(t)(\mathbb{N})^{\mathrm{gp}}\oplus\mathbb{Z}\varphi$, and on the other hand $U(t)(\mathbb{N})^{\mathrm{gp}}$ is a direct factor of $U(\tau)(\mathbb{N})^{\mathrm{gp}}$ (since the specialization map $\mathscr{O}^{\mathrm{gp}}_{T(\rho_c),\tau}\to\mathscr{O}^{\mathrm{gp}}_{T,t}$ is surjective). It follows easily that we may choose for c a suitable positive integer, in such a way that the resulting roof $\widetilde{\rho}:=\rho_c$ will also be integral. □

6.6.24. Let P be a fine, sharp and saturated monoid, and set as usual $T_P:=(\operatorname{Spec}P)^\sharp$. There is a canonical choice of a point in $T_P(\mathbb{N})$ which does not lie on any proper face of $T_P(\mathbb{N})$; namely, one may take the $\mathbb{N}$-point φ_P defined as the sum of the generators of the one-dimensional faces of $T_P(\mathbb{N})$ (such faces are isomorphic to $\mathbb{N}$, by theorem 6.4.18(ii)).

Set $d:=\dim P$; if ρ is a given integral roof for $(T_P)_{d-1}$, and $c\in\mathbb{N}$ is a sufficiently large, we may then attach to the datum (φ_P,ρ,c) an integral roof $\widetilde{\rho}$ of T_P extending ρ as in lemma 6.6.22, and such that $\widetilde{\rho}(\varphi_P)=c$. More generally, let T be a fine and saturated fan of dimension d, and suppose we have a given integral roof ρ_{d-1} on T_{d-1}; for every point $t\in T$ of height d we have the corresponding canonical point φ_t in the "interior" of $U(t)(\mathbb{N})$, and we may then pick an integer c_d large enough, so that ρ_{d-1} extends to an integral roof $\rho_d : T(\mathbb{Q}_+)\to\mathbb{Q}$ with $\rho_d(\varphi_t)=c_d$ for every $t\in T$ of height d, and such that the associated subdivision is T_P-homeomorphic to $T(\rho_{d-1})\times(\operatorname{Spec}\mathbb{N})$.

This is the basis for the inductive construction of an integral roof on T_P which is canonical in a certain restricted sense. Indeed, fix an increasing sequence of positive integers $\underline{c}:=(c_2,\dots,c_d)$; first we define $\rho_1 : T_1(\mathbb{Q}_+)\to\mathbb{Q}_+$ to be the identically zero map. This roof is extended recursively to a function ρ_h on T_h, for each $h=2,\dots,d$, by the rule given above, in such a way that $\rho_h(\varphi_t)=c_i$ for every point t of height $i\le h$. By the foregoing we see that the sequence $\underline{c}$ can be chosen so that ρ_d shall again be an integral roof.

6.6.25. More generally, let $\mathscr{S}:=\{P_1,\dots,P_k\}$ be any finite set of fine, sharp and saturated monoids. We let $\mathscr{S}$-**Fan** be the full subcategory of **Fan** whose objects are the fans T such that, for every $t\in T$, there exist $P\in\mathscr{S}$ and an open immersion $U(t)\subset(\operatorname{Spec}P)^\sharp$. Then the foregoing shows that we may find a sequence of integers $\underline{c}(\mathscr{S}):=(c_2,\dots,c_d)$, with

$d := \max(\dim\, P_i \mid i = 1, \dots, k)$ such that the following holds. Every object T of $\mathscr{S}$-**Fan** is endowed with an integral roof $\rho_T : T(\mathbb{Q}_+) \to \mathbb{Q}_+$ such that :

- $\rho_T(\varphi_t) = c_i$ whenever $\mathrm{ht}(t) = i \geq 2$, and ρ_T vanishes on $T_1(\mathbb{Q}_+)$.
- Every open immersion $g : T' \to T$ in $\mathscr{S}$-**Fan** determines an open immersion $\widetilde{g} : T'(\rho_{T'}) \to T(\rho_T)$ such that the diagram

$$\begin{array}{ccc} T'(\rho_{T'}) & \xrightarrow{\widetilde{g}} & T(\rho_T) \\ {\scriptstyle \pi_{T'}}\downarrow & & \downarrow{\scriptstyle \pi_T} \\ T' & \xrightarrow{g} & T \end{array}$$

  commutes (where π_T and $\pi_{T'}$ are the subdivisions associated to ρ_T and $\rho_{T'}$).
- If $\dim\, T = d$, there exists a natural rational subdivision

$$T_{d-1}(\rho_{T_{d-1}}) \times (\mathrm{Spec}\, \mathbb{N})^\sharp \to T(\rho_T) \tag{6.6.26}$$

  which is a homeomorphism on the underlying topological spaces.

6.6.27. Notice as well that, by construction, $T_1(\rho_{T_1}) = T_1$; hence, by composing the morphisms (6.6.26), we obtain a rational subdivision of $T(\rho_T)$:

$$T_1 \times (\mathrm{Spec}\, \mathbb{N}^{\oplus d-1})^\sharp \to \cdots \to T_{d-2}(\rho_{T_{d-2}}) \times (\mathrm{Spec}\, \mathbb{N}^{\oplus 2})^\sharp \to T_{d-1}(\rho_{T_{d-1}}) \times (\mathrm{Spec}\, \mathbb{N})^\sharp \to T(\rho_T).$$

In view of theorem 6.4.18(ii), it is easily seen that every affine open subset of T_1 is isomorphic to $(\mathrm{Spec}\, \mathbb{N})^\sharp$, and any two such open subsets have either empty intersection, or else intersect in their generic points. In any case, we deduce a natural epimorphism

$$(\mathbb{Q}_+)_{T_1} \to \mathscr{O}_{T_1,\mathbb{Q}}$$

(notation of (6.3.20)) from the constant T_1-monoid arising from $\mathbb{Q}_+$; whence an epimorphism :

$$\vartheta_T : (\mathbb{Q}_+^{\oplus d})_{T(\rho_T)} \to \mathscr{O}_{T(\rho_T),\mathbb{Q}}$$

which is compatible with open immersions $g : T' \to T$ in $\mathscr{S}$-**Fan**, in the following sense. Set $d' := \dim\, T'$, and notice that $d' \leq d$; denote by $\pi_{dd'} : \mathbb{Q}_+^{\oplus d} \to \mathbb{Q}_+^{\oplus d'}$ the projection on the first d' direct summands; then the diagram of T'-monoids :

$$\begin{array}{ccc} (\mathbb{Q}_+^{\oplus d})_{T'(\rho_{T'})} & \xrightarrow{\widetilde{g}^*\vartheta_T} & \widetilde{g}^*\mathscr{O}_{T(\rho_T),\mathbb{Q}} \\ {\scriptstyle (\pi_{dd'})_{T'(\rho_{T'})}}\downarrow & & \downarrow{\scriptstyle (\log \widetilde{g})_{\mathbb{Q}}} \\ (\mathbb{Q}_+^{\oplus d'})_{T'(\rho_{T'})} & \xrightarrow{\vartheta_{T'}} & \mathscr{O}_{T'(\rho_{T'}),\mathbb{Q}} \end{array} \tag{6.6.28}$$

commutes.

6.6.29. Let $f : T' \to T$ be a proper morphism, with T' locally fine, such that the induced map $T'(\mathbb{Q}_+) \to T(\mathbb{Q}_+)$ is injective, and let $s \in T$ be any element. For every $t \in f^{-1}(s)$, we set

$$G_t := U(t)(\mathbb{Q}_+)^{\mathrm{gp}} \cap U(s)(\mathbb{N})^{\mathrm{gp}} \qquad H_t := U(t)(\mathbb{N})^{\mathrm{gp}} \qquad \delta_t := (G_t : H_t)$$

and define $\delta(f, s) := \max(\delta_t \mid t \in f^{-1}(s))$.

Lemma 6.6.30. *For every $s \in T$ we have :*

(i) $\delta(f, s) \in \mathbb{N} \setminus \{0\}$.
(ii) *If $t, t' \in f^{-1}(s)$, and t is a specialization of t' in T', then $\delta_{t'} \leq \delta_t$.*
(iii) *If f is a rational subdivision, the following conditions are equivalent :*
  (a) $\delta(f, s) = 1$.
  (b) $U(s)(\mathbb{N}) = \bigcup_{t \in f^{-1}(s)} U(t)(\mathbb{N})$.

Proof. (i): Since $f^{-1}(s)$ is a finite non-empty set, the assertion means that $\delta_t < +\infty$ for every $t \in f^{-1}(s)$. However, for such a t, let $P := \mathscr{O}_{T,s}$ and $Q := \mathscr{O}_{T',t}$; then

$$U(t)(\mathbb{Q}_+)^{\mathrm{gp}} = \mathrm{Hom}_{\mathbb{Z}}(Q^{\mathrm{gp}}, \mathbb{Q}) \qquad \text{and} \qquad U(t)(\mathbb{N})^{\mathrm{gp}} = \mathrm{Hom}_{\mathbb{Z}}(Q^{\mathrm{gp}}, \mathbb{Z})$$

(remark 6.4.16(i) and proposition 6.4.14(iii)). By proposition 6.5.24 we know that the map $(\log f)_t^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q} : P^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q} \to Q^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}$ is surjective. Since Q is finitely generated, it follows that the image Q' of P^{gp} in Q^{gp} is a subgroup of finite index, and then it is easily seen that $\delta_t = (Q^{\mathrm{gp}} : Q')$.

(ii): Under the stated assumptions we have :

$$G_{t'} = G_t \cap U(t')(\mathbb{Q}_+)^{\mathrm{gp}} \qquad H_{t'} = H_t \cap U(t')(\mathbb{Q}_+)^{\mathrm{gp}}$$

whence the contention.

(iii): Assume that (b) holds, and let $\varphi \in U(t)(\mathbb{Q}_+)^{\mathrm{gp}} \cap U(s)(\mathbb{N})^{\mathrm{gp}}$ for some $t \in f^{-1}(s)$. Pick any element $\varphi_0 \in U(t)(\mathbb{N})$ which does not lie on any proper face of $\mathscr{O}^{\vee}_{T',t}$; we may then find an integer $a > 0$ large enough, so that $\varphi + a\varphi_0 \in U(t)(\mathbb{Q}_+)$. Set $\varphi_1 := \varphi + a\varphi_0$ and $\varphi_2 := a\varphi_0$; then $\varphi_1, \varphi_2 \in U(t)(\mathbb{Q}_+) \cap U(s)(\mathbb{N}) = U(t)(\mathbb{N})$, hence $\varphi = \varphi_1 - \varphi_2 \in U(t)(\mathbb{N})^{\mathrm{gp}}$. Since φ is arbitrary, we see that $\delta_t = 1$, whence (a).

Conversely, suppose that (a) holds, and let $\varphi \in U(s)(\mathbb{N})$; then there exists $t \in f^{-1}(s)$ such that $\varphi \in U(t)(\mathbb{Q}_+)$. Thus, $\varphi \in U(t)(\mathbb{N})^{\mathrm{gp}} \cap U(t)(\mathbb{Q}_+) = U(t)(\mathbb{N})$, whence (b). ☐

Theorem 6.6.31. *Every locally fine and saturated fan T admits an integral, proper, simplicial subdivision $f : T' \to T$, whose restriction $f^{-1}T_{\mathrm{sim}} \to T_{\mathrm{sim}}$ is an isomorphism of fans.*

Proof. Let T be such a fan. By induction on $h \in \mathbb{N}$, we shall construct a system of integral, proper simplicial subdivisions $f_h : S(h) \to T_h$ of the open subsets T_h (notation of (6.5.16)), such that, for every $h \in \mathbb{N}$, the restriction $f_{h+1}^{-1}(T_h) \to T_h$ of f_{h+1} is isomorphic to f_h, and such that $f_h^{-1}T_{h,\mathrm{sim}} \to T_{h,\mathrm{sim}}$ is an isomorphism. Then, the colimit of the morphisms f_h will be the sought subdivision of T.

For $h \leq 1$, we may take $S(h) := T_h$.

Next, suppose that $h > 1$, and that $f_{h-1} : S(h-1) \to T_{h-1}$ has already been given; as a first step, we shall exhibit a rational, proper simplicial subdivision of T_h. Indeed, for every $t \in T_h \setminus T_{h-1}$, choose a $\mathbb{N}$-point $\varphi_t \in U(t)(\mathbb{N})$ in the following way. If $t \in T_{\mathrm{sim}}$, then let φ_t be the (unique) generator of an arbitrarily chosen one-dimensional face of $U(t)(\mathbb{N})$; and otherwise take any point φ_t which does not lie on any proper face of $U(t)(\mathbb{N})$.

With these choices, notice that $U(\varphi_t) \times (\mathrm{Spec}\,\mathbb{N})^\sharp = U(t)$ in case $t \in T_{\mathrm{sim}}$, and otherwise $U(\varphi_t) = U(t)_{h-1}$ (notation of (6.6.16)). By lemma 6.6.17, we obtain corresponding rational subdivisions $U(\varphi_t) \times (\mathrm{Spec}\,\mathbb{N})^\sharp \to U(t)$ of $U(t)$ centered at φ_t. Notice that if t lies in the simplicial locus, this subdivision is an isomorphism, and in any case, it restricts to an isomorphism on the preimage of $U(t)_{h-1}$.

By composing with the restriction of $f_{h-1} \times (\mathrm{Spec}\,\mathbb{N})^\sharp$, we get a rational subdivision:

$$g_t : T'_t := f_{h-1}^{-1} U(\varphi_t) \times (\mathrm{Spec}\,\mathbb{N})^\sharp \to U(t)$$

whose restriction to the preimage of $U(t)_{h-1}$ is an isomorphism. Moreover, g_t is an isomorphism if t lies in T_{sim}. Also notice that g_t is simplicial, since the same holds for f_{h-1}.

If t, t' are any two distinct points of T of height h, we deduce an isomorphism

$$g_t^{-1}(U(t) \cap U(t')) \xrightarrow{\sim} g_{t'}^{-1}(U(t) \cap U(t'))$$

hence we may glue the fans T'_t and the morphisms g_t along these isomorphism, to obtain the sought simplicial rational subdivision $g : T' \to T_h$.

For the next step, we shall refine g locally at every point s of height h; *i.e.* for such s, we shall find an integral, proper simplicial subdivision $f_s : T''_s \to U(s)$, whose restriction to $U(s)_{h-1}$

agrees with g, hence with f_{h-1}. Once this is accomplished, we shall be able to build the sought subdivision f_h by gluing the morphisms f_s and f_{h-1} along the open subsets $U(s)_{h-1}$.

Of course, if s lies in the simplicial locus of T, we will just take for f_s the restriction of g, which by construction is already an isomorphism.

Henceforth, we may assume that $T = U(s)$ is an affine fan of dimension h with $s \notin T_{\mathrm{sim}}$, and $g : T' \to T$ is a given proper rational simplicial subdivision, whose restriction to $g^{-1}T_{h-1}$ is an integral subdivision. We wish to apply the criterion of lemma 6.6.30(iii), which shows that g is an integral subdivision if and only if $\delta(g, s) = 1$. Thus, let $t_1, \dots, t_k$ be the points of T' such that $\delta_{t_i} = \delta(g, s)$ for every $i = 1, \dots, k$. Since $\delta(g, s)$ is anyway a strictly positive integer (lemma 6.6.30(i)), a simple descending induction reduces to the following :

Claim 6.6.32. Given g as above, we may find a proper rational simplicial subdivision $g' : T'' \to T$ such that the following holds :

(i) The restriction of g' to $g'^{-1}T_{h-1}$ is isomorphic to the restriction of g.
(ii) Let $t'_1, \dots, t'_{k'} \in T''$ be the points such that $\delta_{t'_i} = \delta(g', s)$ for every $i = 1, \dots, k'$. We have $\delta(g', s) \leq \delta(g, s)$, and if $\delta(g', s) = \delta(g, s)$ then $k' < k$.

Proof of the claim. Set $t := t_1$; by definition, there exists $\varphi' \in U(s)(\mathbb{N})$ which lies in $U(t)(\mathbb{Q}_+) \setminus U(t)(\mathbb{N})$; this means that there exists a morphism φ fitting into a commutative diagram :

$$\begin{array}{ccc} \mathscr{O}_{T,s} & \xrightarrow{(\log g)_t} & \mathscr{O}_{T',t} \\ \downarrow{\scriptstyle \varphi'} & & \downarrow{\scriptstyle \varphi} \\ \mathbb{N} & \longrightarrow & \mathbb{Q}_+. \end{array}$$

Say that $\mathscr{O}_{T',t} \simeq \mathbb{N}^{\oplus r}$, and let $(\pi_1, \dots, \pi_r)$ be the (essentially unique) basis of $\mathscr{O}^\vee_{T'(h),t}$; then $\varphi = a_1\pi_1 + \cdots a_r\pi_r$, for some $a_1, \dots, a_r \geq 0$, and after subtracting some positive integer multiple of π, we may assume that $0 \leq a_i < 1$ for every $i = 1, \dots, r$. Moreover, the coefficients are all strictly positive if and only if φ is a local morphism; more generally, we let t' be the unique generization of t such that φ factors through a local morphism $\mathscr{O}_{T',t'} \to \mathbb{N}$. Set $e := \mathrm{ht}(t')$, and denote by $\pi_1, \dots, \pi_e$ the basis of $\mathscr{O}^\vee_{T',t'}$, so that :

$$\varphi = b_1\pi_1 + \cdots + b_e\pi_e \qquad (6.6.33)$$

for unique rational coefficients $b_1, \dots, b_e$ such that $0 < b_i < 1$ for every $i = 1, \dots, e$.

Denote by $Z \subset T'$ the topological closure of $\{t'\}$; for every $u \in Z \cap T'$, the morphism φ factors through a morphism $\varphi_u : \mathscr{O}_{T,u} \to \mathbb{Q}_+$, and we may therefore consider the subdivision of $U(u)$ centered at φ_u as in (6.6.16), which fits into a commutative diagram of fans :

$$\begin{array}{ccc} (U(u)\backslash Z) \times (\mathrm{Spec}\,\mathbb{Q}_+)^\sharp & \longrightarrow & (U(u)\backslash Z) \times (\mathrm{Spec}\,\mathbb{N})^\sharp \\ \downarrow & & \downarrow{\scriptstyle \psi_u} \\ U(u) & \longrightarrow & U'(u) := (\mathrm{Spec}\,\varphi_u^{-1}\mathbb{N})^\sharp. \end{array}$$

We complete the family $(\psi_u \mid u \in Z)$, by letting $U'(u) := U(u)$ and $\psi_u := \mathbf{1}_{U(u)}$ for every $u \in T' \setminus Z$. Notice then, that the topological spaces underlying $U(u)$ and $U'(u)$ agree for every $u \in T'$, and for every $u_1, u_2 \in T'$, the restrictions of ψ_{u_1} and ψ_{u_2} :

$$\psi_{u_i}^{-1}(U(u_1) \cap U(u_2)) \to U(u_1) \cap U(u_2) \qquad (i = 1, 2)$$

are isomorphic. Furthermore, by construction, each restriction $U(u) \to T$ of g factors uniquely through a morphism $\beta_u : U'(u) \to T$, hence the family $(\beta_u \circ \psi_u \mid u \in T')$ glues to a well defined morphism of fans $g' : T'' \to T$. By a direct inspection, it is easily seen that g' is a proper rational simplicial subdivision which fulfills condition (i) of the claim.

Moreover, the map of topological spaces underlying g' factors naturally through a continuous map $p : T'' \to T'$, so that $g' = g \circ p$. The restriction $V := p^{-1}(T' \setminus Z) \to T$ of g' is isomorphic to the restriction $T' \setminus Z \to T$ of g, hence :

$$\delta_t = \delta_{p(t)} \qquad \text{for every } t \in p^{-1}(T' \setminus Z).$$

It follows that if $k > 1$ and $T' \setminus Z$ contains at least one of the points $t_2, \dots, t_k$, then $\delta(g', s) \geq \delta(g, s)$. On the other hand, $p^{-1}(T' \setminus Z)$ contains at most $k - 1$ points u of T'' such that $\delta_u = \delta(g, s)$, and for the remaining points $u' \in p^{-1}(T' \setminus Z)$ we have $\delta_{u'} < \delta(g, s)$. Since obviously

$$\delta(g', s) = \max(\delta(g'_{|p^{-1}(T'\setminus Z)}, s), \delta(g'_{|p^{-1}Z}, s))$$

we see that condition (ii) holds provided we show :

$$\delta(g'_{|p^{-1}Z}, s) = \max(b_1, \dots, b_e) \cdot \delta(g, s). \tag{6.6.34}$$

Hence, let us fix $u \in Z$, and let $(v, x) \in (U(u)\setminus Z) \times (\mathrm{Spec}\, \mathbb{N})^\sharp$ be any point (see (6.5.19)); if $x = \varnothing$, then $(v, x) \notin p^{-1}Z$, so it suffices to consider the points of the form $(v, \mathfrak{m})$ (where $\mathfrak{m} \subset \mathbb{N}$ is the maximal ideal). Moreover, say that $\mathrm{ht}(u) = d$; in view of lemma 6.6.30(ii), it suffices to consider the points $(v, \mathfrak{m})$ such that $\mathrm{ht}(v) = d - 1$. There are exactly e such points, namely the prime ideals $v_i := (\pi_i \circ \sigma)^{-1}\mathfrak{m}$, where $\pi_1, \dots, \pi_e$ are as in (6.6.33), and $\sigma : \mathscr{O}_{T',u} \to \mathscr{O}_{T',t'}$ is the specialization map.

In order to estimate $\delta_{(v,\mathfrak{m})}$ for some $v := v_i$, we look at the transpose of the map

$$(\log g')^{\mathrm{gp}}_{(v,\mathfrak{m})} : \mathscr{O}^{\mathrm{gp}}_{T,s} \to \mathscr{O}^{\mathrm{gp}}_{T',v} \times \mathbb{Z} \quad : \quad z \mapsto ((\log g)^{\mathrm{gp}}_v(z), \varphi^{\mathrm{gp}}(z)).$$

Let $(p_1, \dots, p_d)$ be the basis of $\mathscr{O}^\vee_{T,u}$, ordered in such a way that $p_i = \pi_i \circ \sigma$ for every $i = 1, \dots, e$. By a little abuse of notation, we may then denote $(p_j \mid j \neq i)$ the basis of $\mathscr{O}^\vee_{T',v}$, so that the dual group $(\mathscr{O}^{\mathrm{gp}}_{T',v} \times \mathbb{Z})^\vee$ admits the basis $\{p^{\mathrm{gp}}_j \mid j \neq i\} \cup \{q\}$, where q is the natural projection onto $\mathbb{Z}$. Set as well $p'_i := p_i \circ (\log g)^{\mathrm{gp}}_u$ for every $i = 1, \dots, d$. With this notation, the above transpose is the group homomorphism given by the rule :

$$p_j \mapsto p'_i \quad \text{for } j \neq i \qquad \text{and} \qquad q \mapsto b_1 p'_1 + \cdots + b_e p'_e$$

from which we deduce easily that $\delta_{(v,\mathfrak{m})} = b_i \cdot \delta_u$, whence (6.6.34). □

Proposition 6.6.35. *Let $(\Gamma, +, 0)$ be a fine monoid, M a fine Γ-graded monoid. Then :*

(i) *There exists a finite set of generators $C := \{\gamma_1, \dots, \gamma_k\}$ of Γ, with the following property. For every $\gamma \in \Gamma$, we may find $a_1, \dots, a_k \in \mathbb{N}$ such that:*

$$\gamma = a_1\gamma_1 + \cdots + a_k\gamma_k \quad \text{and} \quad M_\gamma = M^{a_1}_{\gamma_1} \cdots M^{a_k}_{\gamma_k}.$$

(ii) *There exists a subgroup $H \subset \Gamma^{\mathrm{gp}}$ of finite index, such that :*

$$M_{a\gamma} = M^a_\gamma \qquad \text{for every } \gamma \in H \cap \Gamma \text{ and every integer } a > 0.$$

(In (i) *and* (ii) *we use the multiplication law of $\mathscr{P}(M)$, as in* (6.1.1)*).*

Proof. Obviously we may assume that M maps surjectively onto Γ, in which case $G := \Gamma^{\mathrm{gp}}$ is finitely generated, and its image G' into $G_\mathbb{R} := G \otimes_\mathbb{Z} \mathbb{R}$ is a free abelian group of finite rank. The same holds as well for the image L of $\log M^{\mathrm{gp}}$ in $M^{\mathrm{gp}}_\mathbb{R} := \log M^{\mathrm{gp}} \otimes_\mathbb{Z} \mathbb{R}$. Let $p : G \to G'$ be the natural projection. According to proposition 6.3.22(iii), we have :

$$M_\mathbb{Q} = M_\mathbb{R} \cap M^{\mathrm{gp}}_\mathbb{Q} \tag{6.6.36}$$

(notation of (6.3.20)). Let $f^{\mathrm{gp}}_\mathbb{R} : M^{\mathrm{gp}}_\mathbb{R} \to G_\mathbb{R}$ be the induced $\mathbb{R}$-linear map, and denote by $f_\mathbb{R} : M_\mathbb{R} \to G_\mathbb{R}$ the restriction of $f^{\mathrm{gp}}_\mathbb{R}$. By proposition 6.3.28(ii), we may find a G'-rational subdivision Δ of $f_\mathbb{R}(M_\mathbb{R})$ such that for every $\sigma \in \Delta$ and every $a, b \in \sigma$ we have :

$$f^{-1}_\mathbb{R}(a + b) = f^{-1}_\mathbb{R}(a) + f^{-1}_\mathbb{R}(b) \tag{6.6.37}$$

After choosing a refinement, we may assume that Δ is a simplicial fan (theorem 6.6.31). Let $\tau \in \Delta$ be any cone; by proposition 6.3.22(i), the monoid $N := \tau \cap G'$ is finitely generated, and then the same holds for $M \times_{G'} N$, by corollary 6.4.2. However, the latter is just $M' := \bigoplus_{\gamma \in p^{-1}N} M_\gamma$ (lemma 4.8.29(iii)). Set $\Gamma' := \Gamma \cap p^{-1}N$.

Claim 6.6.38. $p(\Gamma')$ generates τ.

Proof of the claim. Clearly the Γ-grading of M induces a surjection $M_\mathbb{Q} \to \Gamma_\mathbb{Q}$ (notation of (6.3.20)). By proposition 6.3.22(iii), we have $\Gamma_\mathbb{Q} = f_\mathbb{R}(M_\mathbb{R}) \cap G_\mathbb{Q}$, hence $N \subset \tau \cap G_\mathbb{Q} \subset \Gamma_\mathbb{Q}$. Thus, for every $n \in N$ we may find $a \in \mathbb{N}$ such that $a \cdot n \in p(\Gamma)$, hence $a \cdot n \in p(\Gamma')$. On the other hand, N generates τ, since the latter is G'-rational. The claim follows. ◊

In view of claim 6.6.38, we may replace M by M', and Γ by Γ', which allows to assume that $f_\mathbb{R}(M_\mathbb{R})$ is a simplicial cone, and (6.6.37) holds for every $a, b \in f_\mathbb{R}(M_\mathbb{R})$. Next, let $S := \{e_1, \dots, e_n\} \subset p(\Gamma)$ be a set of generators of the cone $f_\mathbb{R}(M_\mathbb{R})$; from the discussion in (6.3.15) we see that, up to replacing S by a subset, the rays $\mathbb{R}_+ \cdot e_i$ (with $i = 1, \dots, n$) are precisely the extremal rays of $f_\mathbb{R}(M_\mathbb{R})$, especially, the vectors $e_1, \dots, e_n$ are $\mathbb{R}$-linearly independent. Choose $g_1, \dots, g_n \in \Gamma$ such that $p(g_i) = e_i$ for every $i = 1, \dots, n$. According to corollary 6.4.5, there exist finite subsets $\Sigma_1 \dots, \Sigma_n \subset M$, such that :

$$M_{g_i} = M_0 \cdot \Sigma_i \qquad \text{for every } i = 1, \dots, n.$$

Claim 6.6.39. $(M_0)_\mathbb{Q} = f_\mathbb{R}^{-1}(0) \cap M_\mathbb{Q}^{\mathrm{gp}}$.

Proof of the claim. To begin with, we may write $f_\mathbb{R}^{-1}(0) = (f_\mathbb{R}^{\mathrm{gp}})^{-1}(0) \cap M_\mathbb{R}$, hence $f_\mathbb{R}^{-1}(0) \cap M_\mathbb{Q}^{\mathrm{gp}} = (f_\mathbb{Q}^{\mathrm{gp}})^{-1}(0) \cap M_\mathbb{Q}$, by (6.6.36). Now, suppose $x \in (f_\mathbb{Q}^{\mathrm{gp}})^{-1}(0) \cap M_\mathbb{Q}$; then we may find an integer $a > 0$ such that $ax = m \otimes 1$ for some $m \in \log M$. Say that $m \in \log M_\gamma$; then $f_\mathbb{Q}^{\mathrm{gp}}(\gamma) = 0$, therefore γ is a torsion element of G', and consequently $bm \in \log M_0$ for an integer $b > 0$ large enough. We conclude that $x = (bm) \otimes (ba)^{-1} \in (M_0)_\mathbb{Q}$, as claimed. ◊

On the other hand, since $\mathbb{R}_+ e_i$ is a G'-rational polyhedral cone, $f_\mathbb{R}^{-1}(\mathbb{R}_+ e_i)$ is an L-rational polyhedral cone (proposition 6.3.21(ii,iii)), hence it admits a finite set of generators $S_i \subset M_\mathbb{Q}$. Up to replacing the elements of S_i by some positive rational multiples, we may assume that $f_\mathbb{R}(s)$ is either 0 or e_i, for every $s \in S_i$. Then, set $S'_i := \{s \in S_i \mid f_\mathbb{R}(s) = e_i\}$; we easily get :

$$f_\mathbb{R}^{-1}(e_i) = f_\mathbb{R}^{-1}(0) + T_i \qquad \text{where :} \qquad T_i := \Big\{ \sum_{s \in S'_i} t_s \cdot s \mid t_s \in \mathbb{R}_+, \sum_{s \in S'_i} t_s = 1 \Big\} \tag{6.6.40}$$

(so T_i is the convex hull of S'_i, and as usual, the addition of sets in (6.6.40) refers to the addition law of $\mathscr{P}(M_\mathbb{R}^{\mathrm{gp}})$, see (6.1.1)). Next, we may find an integer $a > 0$ such that $a \cdot s$ lies in the image of $\log M$, for every $s \in S_i$. After replacing e_i by $a \cdot e_i$ and S_i by $\{a \cdot s \mid s \in S_i\}$, we may then achieve that (6.6.40) holds, and furthermore S'_i lies in the image of $\log M$, therefore in the image of $\log M_{g_i}$. It follows easily that (6.6.40) still holds with S'_i replaced by the set $\Sigma_i \otimes 1 := \{m \otimes 1 \mid m \in \Sigma_i\}$. Let $\Sigma_0 \subset \log M$ be a finite set of generators for the monoid M_0; claim 6.6.39 implies that Σ_0 is also a set of generators for the L-rational polyhedral cone $f_\mathbb{R}^{-1}(0)$. Let $P \subset M$ be the submonoid generated by $\Sigma := \Sigma_0 \cup \Sigma_1 \cup \cdots \cup \Sigma_n$, and $\Gamma'' \subset \Gamma$ the submonoid generated by $g_1, \dots, g_n$; clearly the Γ-grading of M restricts to a Γ''-grading on P. Notice that $g_1 \otimes 1, \dots, g_n \otimes 1$ are linearly independent in $G_\mathbb{Q}$, since the same holds for $e_1, \dots, e_n$; especially, $\Gamma'' \simeq \mathbb{N}^{\oplus n}$.

Claim 6.6.41. (i) The set $\Sigma \otimes 1 := \{s \otimes 1 \mid s \in \Sigma\}$ generates the cone $M_\mathbb{R}$.

(ii) $P_{a+b} = P_a \cdot P_b$ for every $a, b \in \Gamma''$.

(iii) There exists a finite set $A \subset M$ such that $M = A \cdot P$.

Proof of the claim. By (6.6.40) and the foregoing discussion, we know that $(\Sigma_0 \cup \Sigma_i) \otimes 1$ generate $f_\mathbb{R}^{-1}(\mathbb{R}_+ e_i)$, for every $i = 1, \dots, n$. Since the additivity property (6.6.37) holds for every $a, b \in$

$f_{\mathbb{R}}(M_{\mathbb{R}})$, assertion (i) follows. (ii) is a straightforward consequence of the definitions. Next, from (i) and proposition 6.3.22(iii), we deduce that $M_{\mathbb{Q}} = P_{\mathbb{Q}}$. Thus, let $m_1, \dots, m_r$ be a system of generators for the monoid M; it follows that there are integers $k_1, \dots, k_r > 0$, such that $m_1^{k_1}, \dots, m_r^{k_r} \in P$, and therefore the subset $A := \{\prod_{i=1}^r m_i^{t_i} \mid 0 \le t_i < k_i \text{ for every } i \le r\}$ fulfills the condition of (iii). ◊

We introduce a partial ordering on G, by declaring that $a \le b$ for two elements $a, b \in G$, if and only if $b - a \in \Gamma''$. Now, let $a \in G$ be any element; we set

$$G(a) := \{g \in G \mid g \le a\}.$$

The subset $G(a)$ inherits a partial ordering from G, and a is the maximum of the elements of $G(a)$; moreover, notice that every non-empty finite subset $S \subset G(a)$ admits a supremum $\sup S \in G(a)$. Indeed, it suffices to show the assertion for a set of two elements $S = \{b_1, b_2\}$; then $a - b_i = \sum_{j=1}^n k_{ij} g_j$ for certain $k_{ij} \in \mathbb{N}$, so that $\sup(b_1, b_2) = a - \sum_{j=1}^n \min(k_{1j}, k_{2j}) \cdot g_j$.

Let A be as in claim 6.6.41(iii), and denote by $B \subset \Gamma$ the image of A; for every $a \in G$, let also $B(a) := B \cap G(a)$; when $B(a) \neq \varnothing$, invoking several times claim 6.6.41(ii,iii), we get :

$$M_a = \bigcup_{b \in B(a)} M_b \cdot P_{a-b} = \bigcup_{b \in B(a)} M_b \cdot P_{\sup B(a) - b} \cdot P_{a - \sup B(a)} \subset M_{\sup B(a)} \cdot P_{a - \sup B(a)}.$$

Finally, say that $a - \sup B(a) = \sum_{i=1}^n t_i g_i$ for certain $t_1, \dots, t_n \in \mathbb{N}$; applying once more claim 6.6.41(ii), we conclude that :

$$M_a \subset M_{\sup B(a)} \cdot \prod_{i=1}^n M_{g_i}^{t_i}.$$

The converse inclusion is clear, and therefore condition (i) of the proposition is fulfilled with :

$$C := \{g_1, \dots, g_n\} \cup \{\sup B(a) \mid a \in G, B(a) \neq \varnothing\}.$$

(ii): For $h \in \Gamma''^{\mathrm{gp}}$, say $h = \sum_{i=1}^n a_i g_i$, with integers $a_1, \dots, a_n$, we let $|h| := \sum_{i=1}^n |a_i| g_i \in \Gamma''$. Choose any positive integer α such that :

$$|b| \le \alpha \cdot \sum_{i=1}^n g_i \qquad \text{for every } b \in B \cap \Gamma''^{\mathrm{gp}} \tag{6.6.42}$$

and let $H \subset \Gamma''^{\mathrm{gp}}$ be the subgroup generated by $\alpha g_1, \dots, \alpha g_n$.

Claim 6.6.43. $B(h) = B(kh)$ for every $h \in H$ and every integer $k > 0$.

Proof of the claim. Let $h := \sum_{i=1}^n \alpha_i g_i \in H$, and suppose that $b \in B(kh)$ for some $k > 0$; therefore $kh - b \in \Gamma''$, hence $b \in \Gamma''^{\mathrm{gp}}$, and we can write $b = \sum_{i=1}^n \beta_i g_i$ for integers $\beta_1, \dots, \beta_n$, such that $k\alpha_i - \beta_i \ge 0$ for every $i = 1, \dots, n$. In this case, (6.6.42) implies that $\alpha_i \ge 0$ for every $i \le n$, and $\alpha_i \ge \alpha \ge \beta_i$ whenever $\beta_i > 0$. It follows easily that $k'h - b \in \Delta$ for every integer $k' > 0$, whence the claim. ◊

Using claims 6.6.41(ii) and 6.6.43, and arguing as in the foregoing, we may compute :

$$\begin{aligned} M_{ah} &= M_{\sup B(ah)} \cdot P_{ah - \sup B(ah)} \\ &= M_{\sup B(h)} \cdot P_{ah - \sup B(h)} \\ &= M_{\sup B(h)} \cdot P_{h - \sup B(h)} \cdot P_h^{a-1} \\ &\subset M_h^a. \end{aligned}$$

The converse inclusion is clear, so (ii) holds. □

6.6.44. Let M be an integral monoid, and $w \in \log M^{\mathrm{gp}}$ any element. For $\varepsilon \in \{1, -1\}$ we have a natural inclusion

$$j_\varepsilon : \log M \to M(\varepsilon) := \log M + \varepsilon \mathbb{N} w$$

(*i.e.* $M(\varepsilon)$ is the submonoid of M^{gp} generated by M and w^ε). Let us write $w := b^{-1}a$ for some $a, b \in M$; then the induced morphisms of affine schemes $\iota_\varepsilon := \mathrm{Spec}\, \mathbb{Z}[j_\varepsilon]$ have a natural geometric interpretation. Namely, let $f : X \to \mathrm{Spec}\, \mathbb{Z}[M]$ be the blow up of the ideal $I \subset \mathbb{Z}[M]$ generated by a and b; we have $X = U_1 \cup U_{-1}$, where U_ε, for $\varepsilon = \pm 1$, is the largest open subscheme of X such that $w^\varepsilon \in \mathscr{O}_X(U_\varepsilon)$. Then ι_ε is naturally identified with the restriction $U_\varepsilon \to \mathrm{Spec}\, \mathbb{Z}[M]$ of the blow up f. More generally, by adding to M any finite number of elements of M^{gp}, we may construct in a combinatorial fashion, the standard affine charts of a blow up of an ideal of $\mathbb{Z}[M]$ generated by finitely many elements of M. These considerations explain the significance of the following *flattening theorem* :

Theorem 6.6.45. *Let $j : M \to N$ be an inclusion of fine monoids. Then there exist a finite set $\Sigma \subset \log M^{\mathrm{gp}}$, and an integer $k > 0$ such that the following holds :*

(i) *For every mapping $\varepsilon : \Sigma \to \{\pm 1\}$, the induced inclusion :*

$$M(\varepsilon) := \log M + \sum_{\sigma \in \Sigma} \varepsilon(\sigma) \mathbb{N}\sigma \to N(\varepsilon) := \log N + \sum_{\sigma \in \Sigma} \varepsilon(\sigma) \mathbb{N}\sigma$$

is a flat morphism of fine monoids.

(ii) *Suppose that j is a flat morphism, and let $\iota : M \to M^{\mathrm{sat}}$ be the natural inclusion. Denote by Q the push-out of the diagram $N \xleftarrow{j} M \xrightarrow{\iota \circ \boldsymbol{k}_M} M^{\mathrm{sat}}$ (where $\boldsymbol{k}_M$ is the k-Frobenius). Then the natural map $M^{\mathrm{sat}} \to Q^{\mathrm{sat}}$ is flat and saturated.*

Proof. (i): (Notice that $\log M$, $\log N$ and $P(\varepsilon) := \sum_{\sigma \in \Sigma} \varepsilon(\sigma) \mathbb{N}\sigma$ may be regarded as submonoids of $\log N^{\mathrm{gp}}$, and then the above sum is taken in the monoid $(\mathscr{P}(\log N^{\mathrm{gp}}), +)$ defined as in (6.1.1).) Set $G := N^{\mathrm{gp}}/M^{\mathrm{gp}}$, and let $N^{\mathrm{gp}} = \bigoplus_{\gamma \in G} N^{\mathrm{gp}}_\gamma$ be the j-grading of N^{gp} (remark 6.2.5(iii)); notice that this grading restricts to j-gradings for N and $N(\varepsilon)$. Let also $\Gamma := \{\gamma \in G \mid N_\gamma \neq \varnothing\}$, and choose a finite generating set $\{\gamma_1, \dots, \gamma_r\}$ of Γ with the properties of proposition 6.6.35(i). According to corollary 6.4.5, for every $i \leq r$ there exists a finite subset $\Sigma_i := \{t_{i1}, \dots, t_{in_i}\} \subset M$ such that $N_{\gamma_i} = N_0 \cdot \Sigma_i$. Moreover, N_0 is a finitely generated monoid, by corollary 6.4.2. Let Σ_0 be a finite system of generators for N_0, and for every $i \leq r$ define $\Sigma'_i := \{t_{ij} - t_{il} \mid 1 \leq j < l \leq n_i\}$. We claim that the subset $\Sigma := \Sigma_0 \cup \Sigma'_1 \cup \dots \cup \Sigma'_r$ will do. Indeed, first of all notice that $\Sigma \subset \log M^{\mathrm{gp}}$. We shall apply the flatness criterion of remark 6.2.5(iv). Thus, we have to show that, for every $g \in \Gamma$, the $M(\varepsilon)$-module $N(\varepsilon)_g$ is a filtered union of cosets $\{m\} + M(\varepsilon)$ (for certain $m \in N_g$). Hence, let $x_1, x_2 \in N(\varepsilon)_g$; we may write $x_i = m_i + p_i$ for some $m_i \in \log N$ and $p_i \in P(\varepsilon)$ $(i = 1, 2)$; since $\{x_i\} + M(\varepsilon) \subset \{m_i\} + M(\varepsilon)$, we may then assume that $p_1 = p_2 = 0$, hence $x_1, x_2 \in N$. Thus, it suffices to show that N_g is contained in a filtered union of cosets as above. However, by assumption there exist $a_1, \dots, a_r \in \mathbb{N}$ such that $g = \sum_{i=1}^r a_i \gamma_i$ and $N_g = N^{a_1}_{\gamma_1} \cdots N^{a_r}_{\gamma_r}$; we are then easily reduced to the case where $g = \gamma_i$ for some $i \leq r$. Therefore, $x_j = \sigma_j + b_j$, where $\sigma_j \in \Sigma_i$ and $b_j \in N_0$, for $j = 1, 2$. Notice that $P(\varepsilon)$ contains either $\sigma_1 - \sigma_2$, or $\sigma_2 - \sigma_1$ (or both); in the first occurrence, set $\sigma := \sigma_2$, and otherwise, let $\sigma := \sigma_1$. Likewise, say that $\Sigma_0 = \{y_1, \dots, y_n\}$, so that $b_j = \sum_{s=1}^n a_{js} y_s$ for certain $a_{js} \in \mathbb{N}$ $(j = 1, 2)$; for every $s \leq n$, we set $a^*_s := \min(a_{1s}, a_{2s})$ if $y_s \in P(\varepsilon)$, and otherwise we set $a^*_s := \max(a_{1s}, a_{2s})$. One sees easily that $x_1, x_2 \in \{\sigma + \sum_{s=1}^n a^*_s y_s\} + M(\varepsilon)$, whence the contention.

(ii): By proposition 6.6.35(ii), there exists a subgroup $H \subset G$ of finite index, such that :

$$\pi^{-1}(h^n) = \pi^{-1}(h)^n$$

for every integer $n > 0$ and every $h \in H$. Let $k := (G : H)$, and define N' as the fibre product in the cartesian diagram :

$$\begin{array}{ccc} N' & \xrightarrow{\pi'} & G \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle \boldsymbol{k}_G} \\ N & \xrightarrow{\pi} & G \end{array} \tag{6.6.46}$$

The trivial morphism $\mathbf{0}_M : M \to G$ (*i.e.* the unique one that factors through $\{1\}$) and the inclusion j satisfy the identity : $\boldsymbol{k}_G \circ \mathbf{0}_M = \pi \circ j$, hence they determine a well-defined map $\varphi : M \to N'$.

Claim 6.6.47. φ is flat and saturated.

Proof of the claim. For the flatness, we shall apply the criterion of remark 6.2.5(iv). First, φ is injective, since $\mu \circ \varphi = j$. Next, notice that the sequence of abelian groups :

$$0 \to M^{\mathrm{gp}} \xrightarrow{\varphi^{\mathrm{gp}}} (N')^{\mathrm{gp}} \xrightarrow{(\pi')^{\mathrm{gp}}} G \to 0$$

is short exact; indeed, this is none else than the pullback $\mathscr{E} * \boldsymbol{k}_G^{\mathrm{gp}}$ along the morphism $\boldsymbol{k}_G$, of the short exact sequence

$$\mathscr{E} := (0 \to M^{\mathrm{gp}} \xrightarrow{j^{\mathrm{gp}}} N^{\mathrm{gp}} \xrightarrow{\pi^{\mathrm{gp}}} G \to 0). \tag{6.6.48}$$

It follows that φ is flat if and only if, for every $x \in \pi'(N')$, the preimage $(\pi')^{-1}(x)$ is a filtered union of cosets of the form $\{n\} \cdot \varphi(M')$. However, the induced map $(\pi')^{-1}(g) \to \pi^{-1}(g^k)$ is a bijection for every $g \in G$, and the flatness of j implies that $\pi^{-1}(x^k)$ is a filtered union of cosets, whence the contention. Notice also that $\mathrm{Im}\, \boldsymbol{k}_G \subset H$; hence, by the same token, we derive that $(\pi')^{-1}(g^n) = (\pi')^{-1}(g)^n$ for every $g \in G$, therefore φ is quasi-saturated, by proposition 6.2.31. Since we know already that j is integral (theorem 6.2.3), the claim follows. ◊

Next, we wish to consider the commutative diagram of monoids :

$$\begin{array}{ccc} M & \xrightarrow{j} & N \\ {\scriptstyle \boldsymbol{k}_M}\downarrow & & \downarrow{\scriptstyle\psi} \\ M & \xrightarrow{\varphi} & N' \end{array} \tag{6.6.49}$$

such that ψ is the map determined by the pair of morphisms $(f, \boldsymbol{k}_N)$. Let P be the push-out of the maps j and $\boldsymbol{k}_M$; the maps φ and ψ determine a morphism $\tau : P \to N'$.

Claim 6.6.50. (i) The diagram $(6.6.46)^{\mathrm{gp}}$ of associated abelian groups, is cartesian.

(ii) The diagram of abelian groups $(6.6.49)^{\mathrm{gp}}$ is cocartesian (*i.e.* τ^{gp} is an isomorphism).

(iii) There exists a morphism $\lambda : N' \to P$ such that $\lambda \circ \tau = \boldsymbol{k}_P$ and $\tau \circ \lambda = \boldsymbol{k}_{N'}$.

Proof of the claim. (i): Suppose that $x \in (N')^{\mathrm{gp}}$ is any element such that $(\pi')^{\mathrm{gp}}(x) = 1$ and $\mu^{\mathrm{gp}}(x) = 1$; we may write $x = b^{-1}a$ for some $a, b \in N'$, and it follows that $\pi'(a) = \pi'(b)$ and $\mu(a) = \mu(b)$, hence $a = b$ in N', since the forgetful functor $\mathbf{Mnd} \to \mathbf{Set}$ commutes with fibre products. Thus $x = 1$. On the other hand, suppose that $\pi^{\mathrm{gp}}(b^{-1}a) = x^k$ for some $a, b \in N$ and $x \in G$; therefore, $\pi(b^{k-1}a) = (bx)^k$, so there exists $c \in N'$ such that $\mu(c) = b^{k-1}a$ and $\pi'(c) = bx$. Likewise, there exists $d \in N'$ with $\mu(d) = b^k$ and $\pi'(d) = b$. Consequently, $(\pi')^{\mathrm{gp}}(d^{-1}c) = x$ and $\mu(d^{-1}c) = b^{-1}a$. The assertion follows.

(ii): Quite generally, let $E := (0 \to A \to B \to C \to 0)$ be a short exact sequence of objects in an abelian category $\mathscr{C}$, and for every object X of $\mathscr{C}$, let $\boldsymbol{k}_X := k \cdot \mathbf{1}_X : X \to X$. Then one has a natural map of complexes $\alpha_E : E * \boldsymbol{k}_C \to E$ (resp. $\beta_E : E \to \boldsymbol{k}_A * E$) from E to the pull-back of E along $\boldsymbol{k}_C$ (resp. from the push-out of E along $\boldsymbol{k}_A$ to E), and a natural isomorphism $\omega_E : \boldsymbol{k}_A * E \xrightarrow{\sim} E * \boldsymbol{k}_C$ in the category $\mathbf{Ext}_{\mathscr{C}}(C, A)$ of extensions of C by A (this is the category whose objects are all short exact sequences in $\mathscr{C}$ of the form $0 \to A \to X \to C \to 0$, and whose morphisms are the maps of complexes which are the identity on A and C). These maps are related by the identities :

$$\beta_E \circ \alpha_E \circ \omega_E = k \cdot \mathbf{1}_{\boldsymbol{k}_A * E} \qquad \omega_E \circ \beta_E \circ \alpha_E = k \cdot \mathbf{1}_{E * \boldsymbol{k}_C}. \tag{6.6.51}$$

We leave to the reader the construction of ω_E. In the case at hand, we obtain a natural isomorphism $\omega_{\mathscr{E}} : \boldsymbol{k}_M^{\mathrm{gp}} * \mathscr{E} \xrightarrow{\sim} \mathscr{E} * \boldsymbol{k}_G$, where $\mathscr{E}$ is the short exact sequence of (6.6.48). An inspection of the construction shows that the map τ^{gp} is precisely the isomorphism defined by $\omega_{\mathscr{E}}$.

(iii): Let $\mu' : N \to P$ be the natural map, and set $\lambda := \mu' \circ \mu$. By inspecting the constructions, one checks easily that λ^{gp} is the map defined by $\beta_{\mathscr{E}} \circ \alpha_{\mathscr{E}}$. Then the assertion follows from (6.6.51). ◇

Let P' be the push-out of the diagram $N' \xleftarrow{\varphi} M \xrightarrow{\iota} M^{\mathrm{sat}}$; from claim 6.6.47, lemma 6.2.2(i) and corollary 6.2.25(iii), we deduce that the natural map $M^{\mathrm{sat}} \to P'$ is flat and saturated, hence P' is saturated. On the other hand, directly from the definitions we get a cocartesian diagram

$$\begin{array}{ccc} P & \longrightarrow & Q \\ {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle\tau'} \\ N' & \longrightarrow & P'. \end{array} \tag{6.6.52}$$

The induced diagram $(6.6.52)^{\mathrm{sat}}$ of saturated monoids is still cocartesian (remark 4.8.41(v)); however, claim 6.6.50(iii) implies easily that τ^{sat} is an isomorphism, therefore the same holds for $(\tau')^{\mathrm{sat}}$, and assertion (ii) follows. □

7. HOMOLOGICAL ALGEBRA

This chapter lays the foundations of homological and homotopical algebra that shall be needed in the rest of the treatise. For some standard facts that we do not repeat here, we use the book [163] as a quick reference. Other two useful references are the books [110] and [162].

7.1. **Complexes in an additive category.** Let $\mathscr{A}$ be any additive category. We denote by $\mathsf{C}(\mathscr{A})$ the category of *(cochain) complexes* of objects of $\mathscr{A}$. Hence, an object of $\mathsf{C}(\mathscr{A})$ is a pair

$$K^\bullet := (K^\bullet, d_K^\bullet)$$

consisting of a system of objects $(K^n \mid n \in \mathbb{Z})$ and morphisms $(d_K^n : K^n \to K^{n+1} \mid n \in \mathbb{Z})$ of $\mathscr{A}$, called the *differentials* of the complex $K^\bullet$, such that

$$d_K^{n+1} \circ d_K^n = 0 \qquad \text{for every } n \in \mathbb{Z}.$$

The morphisms $\varphi^\bullet : K^\bullet \to L^\bullet$ in $\mathsf{C}(\mathscr{A})$ are the systems of morphisms $(\varphi^n : K^n \to L^n \mid n \in \mathbb{Z})$ of $\mathscr{A}$ such that

$$\varphi^{n+1} \circ d_K^n = d_L^{n+1} \circ \varphi^n \qquad \text{for every } n \in \mathbb{Z}.$$

We will usually omit the subscript when referring to the differentials of a complex, unless there is a danger of confusion. Also, let $I \subset \mathbb{Z}$ be any (bounded or unbounded) interval, *i.e.* I is either of the form $\mathbb{Z} \cap [a, +\infty[$, or $\mathbb{Z} \cap]-\infty, b]$ (for some $a, b \in \mathbb{Z}$), or the intersection of any of these two. We shall denote by

$$\mathsf{C}^I(\mathscr{A})$$

the full subcategory of $\mathsf{C}(\mathscr{A})$ whose objects are the complexes $K^\bullet$ such that $K^i = 0$ whenever $i \notin I$ (where we fix a zero object 0 of $\mathscr{A}$). For instance, if $I = \mathbb{Z} \cap [a, +\infty[$, then $\mathsf{C}^I(\mathscr{A})$ is also denoted $\mathsf{C}^{\geq a}(\mathscr{A})$, and likewise for the case of an upper bounded interval. Moreover, we set

$$\mathsf{C}^-(\mathscr{A}) := \bigcup_{n \in \mathbb{Z}} \mathsf{C}^{\leq n}(\mathscr{A}) \qquad \mathsf{C}^+(\mathscr{A}) := \bigcup_{n \in \mathbb{Z}} \mathsf{C}^{\geq n}(\mathscr{A}) \qquad \mathsf{C}^b(\mathscr{A}) := \bigcup_{n \in \mathbb{Z}} \mathsf{C}^{[-n,n]}(\mathscr{A})$$

so $\mathsf{C}^-(\mathscr{A})$ (resp. $\mathsf{C}^+(\mathscr{A})$, resp. $\mathsf{C}^b(\mathscr{A})$) is the full subcategory of $\mathsf{C}(\mathscr{A})$ whose objects are the *bounded above* (resp. *bounded below*, resp. *bounded*) complexes of $\mathscr{A}$.

7.1.1. For every $a \in \mathbb{Z}$, the inclusion functor

$$\mathsf{C}^{\geq a}(\mathscr{A}) \to \mathsf{C}(\mathscr{A}) \qquad (\text{resp. } \mathsf{C}^{\leq a}(\mathscr{A}) \to \mathsf{C}(\mathscr{A})\ ) \tag{7.1.2}$$

admits a right (resp. left) adjoint called the *brutal truncation functor*, and denoted

$$t^{\geq a} : \mathsf{C}(\mathscr{A}) \to \mathsf{C}^{\geq a}(\mathscr{A}) \qquad (\text{resp. } t^{\leq a} : \mathsf{C}(\mathscr{A}) \to \mathsf{C}^{\leq a}(\mathscr{A})\ ).$$

Namely, for any complex $K^\bullet$, we let $t^{\geq a}(K^\bullet)$ be the unique object of $\mathsf{C}^{\geq a}(\mathscr{A})$ that agrees with $K^\bullet$ in all degrees $\geq a$, and with the same differentials as $K^\bullet$, in this range of degrees (and likewise for $t^{\leq a}(K^\bullet)$). If $\varphi^\bullet : K^\bullet \to L^\bullet$ is any morphism in $\mathsf{C}(\mathscr{A})$, then $t^{\geq a}\varphi^\bullet$ is the unique morphism $t^{\geq a}K^\bullet \to t^{\geq a}L^\bullet$ which agrees with $\varphi^\bullet$ in all degrees $\geq a$ (and likewise for $t^{\leq a}\varphi^\bullet$).

Moreover, if $\mathscr{A}$ is an abelian category, (7.1.2) also admits a left (resp. a right) adjoint

$$\tau^{\geq a} : \mathsf{C}(\mathscr{A}) \to \mathsf{C}^{\geq a}(\mathscr{A}) \qquad (\text{resp. } \tau^{\leq a} : \mathsf{C}(\mathscr{A}) \to \mathsf{C}^{\leq a}(\mathscr{A})\ )$$

called the *normalized truncation functor* (or just the *truncation functor*). Namely, we fix representatives for all kernels and cokernels of $\mathscr{A}$, and for any complex $K^\bullet$, we let $\tau^{\geq a}(K^\bullet)$ be the unique object of $\mathsf{C}^{\geq a}(\mathscr{A})$ whose term in degree a equals $\mathrm{Coker}\, d_K^{a-1}$, and which agrees with $K^\bullet$ in all degrees $> a$, with the same differentials as $K^\bullet$, in this range of degrees; the differential in degree a is the unique morphism $\mathrm{Coker}\, d_K^{a-1} \to K^{a+1}$ whose composition with the natural map $K^a \to \mathrm{Coker}\, d_K^{a-1}$ equals d_K^{a-1}. If $\varphi^\bullet : K^\bullet \to L^\bullet$ is any morphism in $\mathsf{C}(\mathscr{A})$, then $\tau^{\geq a}\varphi^\bullet$ is the unique morphism $\tau^{\geq a}K^\bullet \to \tau^{\geq a}L^\bullet$ which agrees with $\varphi^\bullet$ in all degrees $> a$, and which, in degree a, is the morphism $\mathrm{Coker}\, d_K^a \to \mathrm{Coker}\, d_L^a$ induced by φ^a.

Likewise, $\tau^{\leq a}(K^\bullet)$ is the unique complex of $\mathsf{C}^{\leq a}(\mathscr{A})$ whose term in degree a equals $\mathrm{Ker}\, d_K^a$, and which agrees with $K^\bullet$ in all degrees $< a$, with the same differentials as $K^\bullet$, in all degrees $< a-1$; the differential in degree $a-1$ is the restriction of d_K^{a-1}. Again, for $\varphi^\bullet$ as above, $\tau^{\leq a}\varphi^\bullet$ agrees with $\varphi^\bullet$ in degrees $< a$, and is the restriction $\mathrm{Ker}\, d_K^a \to \mathrm{Ker}\, d_L^a$ of φ^a in degree a.

7.1.3. There is an obvious functor

$$\bullet[0] : \mathscr{A} \to \mathsf{C}(\mathscr{A}) \quad : \quad A \mapsto A[0]$$

that sends any object A of $\mathscr{A}$ to the complex with A *placed in degree zero*, *i.e.* such that $A[0]^i$ equals A if $i = 0$, and equals 0 otherwise (clearly, there is a unique such complex). On the other hand, the *shift operator* is the functor

$$\mathsf{C}(\mathscr{A}) \to \mathsf{C}(\mathscr{A}) \quad : \quad K^\bullet \to K^\bullet[1]$$

given by the rule :

$$K^\bullet[1]^n := K^{n+1} \qquad d^n_{K[1]} := -d_K^{n+1} \qquad \text{for every } n \in \mathbb{Z}.$$

Clearly the shift operator is an automorphism of $\mathsf{C}(\mathscr{A})$, and one defines the operator $K^\bullet \mapsto K^\bullet[n]$, for every $n \in \mathbb{Z}$, as the n-th power of the shift operator (in the automorphism group of $\mathsf{C}(\mathscr{A})$). Then, we can combine the two previous operators, to define the complex

$$A[n] := (A[0])[n] \qquad \text{for every } n \in \mathbb{Z} \text{ and every } A \in \mathrm{Ob}(\mathscr{A}).$$

Furthermore, notice that the shift operator restricts to functors

$$\mathsf{C}^+(\mathscr{A}) \to \mathsf{C}^+(\mathscr{A}) \qquad \mathsf{C}^-(\mathscr{A}) \to \mathsf{C}^-(\mathscr{A}) \qquad \mathsf{C}^b(\mathscr{A}) \to \mathsf{C}^b(\mathscr{A}).$$

Definition 7.1.4. Let $\mathscr{A}$ be an additive category, $\varphi^\bullet, \psi^\bullet : K^\bullet \to L^\bullet$ two morphisms in $\mathsf{C}(\mathscr{A})$.

(i) A *(chain) homotopy* from $\psi^\bullet$ to $\varphi^\bullet$ is the datum $s^\bullet := (s^n : K^n \to L^{n-1} \mid n \in \mathbb{Z})$ of a system of morphisms in $\mathscr{A}$ such that

$$\varphi^n - \psi^n = s^{n+1} \circ d_K^n + d_L^{n-1} \circ s^n \qquad \text{for every } n \in \mathbb{Z}.$$

(ii) We say that $\varphi^\bullet$ and $\psi^\bullet$ as in (i) are *chain homotopic* if there is a chain homotopy between them. It is easily seen that this defines an equivalence relation $\sim$ on the set $\mathrm{Hom}_{\mathsf{C}(\mathscr{A})}(K^\bullet, L^\bullet)$, which is preserved by composition of morphisms : if $\varphi^\bullet \sim \psi^\bullet$ and $\alpha^\bullet : K'^\bullet \to K^\bullet$, $\beta^\bullet : L^\bullet \to L'^\bullet$ are any two morphisms, then $\varphi^\bullet \circ \alpha^\bullet \sim \psi^\bullet \circ \alpha^\bullet$ and $\beta^\bullet \circ \varphi^\bullet \sim \beta^\bullet \circ \psi^\bullet$. It follows that, for every interval $I \subset \mathbb{Z}$ there exists a well defined *homotopy category*

$$\mathsf{Hot}^I(\mathscr{A})$$

whose objects are the same as those of $\mathsf{C}^I(\mathscr{A})$, and whose morphisms are the homotopy classes of morphisms of complexes, and a natural functor

$$\mathsf{C}^I(\mathscr{A}) \to \mathsf{Hot}^I(\mathscr{A}) \tag{7.1.5}$$

which is the identity on objects, and the quotient map on Hom-sets. If $I = \mathbb{Z}$, we write $\mathsf{Hot}(\mathscr{A})$ instead of $\mathsf{Hot}^{\mathbb{Z}}(\mathscr{A})$, and we denote

$$\mathrm{Hot}_{\mathscr{A}}(K^\bullet, L^\bullet)$$

the set of morphisms $K^\bullet \to L^\bullet$ in $\mathsf{Hot}(\mathscr{A})$. We may also define the subcategories $\mathsf{Hot}^+(\mathscr{A})$, $\mathsf{Hot}^-(\mathscr{A})$ and $\mathsf{Hot}^b(\mathscr{A})$ as in (7.1).

(iii) We also say that a morphism $\varphi : K^\bullet \to L^\bullet$ is a *homotopy equivalence*, if the class of $\varphi^\bullet$ is an isomorphism in $\mathsf{Hot}(\mathscr{A})$, *i.e.* if there exists a morphism $\psi^\bullet : L^\bullet \to K^\bullet$ such that $\psi^\bullet \circ \varphi^\bullet \sim \mathbf{1}_{K^\bullet}$ and $\varphi^\bullet \circ \psi^\bullet \sim \mathbf{1}_{L^\bullet}$. We say that $\varphi^\bullet$ is *null-homotopic* if it is chain homotopic to the zero morphism $K^\bullet \to L^\bullet$. We say that a complex $K^\bullet$ is *homotopically trivial*, if the zero endomorphism $0 \cdot \mathbf{1}_{K^\bullet}$ is a homotopy equivalence.

Remark 7.1.6. Let $\mathscr{A}$ and $\mathscr{A}'$ be any two additive categories.

(i) If $F : \mathscr{A} \to \mathscr{A}'$ is any additive functor, we get induced functors

$$\mathsf{C}(F) : \mathsf{C}(\mathscr{A}) \to \mathsf{C}(\mathscr{A}') \qquad \mathsf{Hot}(F) : \mathsf{Hot}(\mathscr{A}) \to \mathsf{Hot}(\mathscr{A}') \qquad K^\bullet \mapsto FK^\bullet$$

by the rule :

$$(FK^\bullet)^n := F(K^n) \quad \text{and} \quad d^n_{FK^\bullet} := F(d^n_K) \qquad \text{for every } n \in \mathbb{Z} \text{ and every } K^\bullet \in \mathrm{Ob}(\mathsf{C}(\mathscr{A})).$$

More generally, we get induced functors $\mathsf{C}^I(F)$, $\mathsf{Hot}^I(F)$ for any interval $I \subset \mathbb{Z}$, as well as $\mathsf{C}^+(F)$, $\mathsf{C}^-(F)$ and $\mathsf{C}^b(F)$ (and the corresponding homotopy category variants), in the obvious fashion. Also, notice that the shift operators descend to a functor on the homotopy category :

$$\mathsf{Hot}(\mathscr{A}) \to \mathsf{Hot}(\mathscr{A}) \qquad A^\bullet \mapsto A^\bullet[n] \qquad \text{for every } n \in \mathbb{Z}$$

and the latter restrict to endofunctors of the subcategories $\mathsf{Hot}^+(\mathscr{A})$, $\mathsf{Hot}^-(\mathscr{A})$ and $\mathsf{Hot}^b(\mathscr{A})$.

(ii) The category $\mathsf{C}(\mathscr{A})$ is additive. Namely, if $K^\bullet, L^\bullet$ are any two complexes of $\mathscr{A}$, and $\varphi^\bullet, \psi^\bullet : K^\bullet \to L^\bullet$ any two morphisms, we define

$$(\varphi + \psi)^i := \varphi^i + \psi^i \qquad \text{for every } i \in \mathbb{Z}$$

and it is clear that this rule yields a natural abelian group structure on $\mathrm{Hom}_{\mathscr{A}}(K^\bullet, L^\bullet)$, such that composition of morphisms is a bilinear operation. The zero object is the (unique) complex 0 which has the zero object of $\mathscr{A}$ in each degree, and biproducts in $\mathsf{C}(\mathscr{A})$ admit natural representatives, defined by the rule :

$$(A \oplus B)^i := A^i \oplus B^i \qquad \text{for every } A^\bullet, B^\bullet \in \mathrm{Ob}(\mathsf{C}(\mathscr{A})) \text{ and every } i \in \mathbb{Z}.$$

Moreover, it is clear that if $\varphi_1^\bullet, \varphi_2^\bullet : K^\bullet \to L^\bullet$ are any two morphisms in $\mathsf{C}(\mathscr{A})$ such that $\varphi_1^\bullet \sim \varphi_2^\bullet$, we have $\varphi_1^\bullet + \psi^\bullet \sim \varphi_2^\bullet + \psi^\bullet$ for every other morphism $\psi^\bullet : K^\bullet \to L^\bullet$; hence there is a unique abelian group structure on $\mathrm{Hot}_{\mathscr{A}}(K^\bullet, L^\bullet)$ for every $K^\bullet, L^\bullet \in \mathrm{Ob}(\mathsf{C}(\mathscr{A}))$, that makes $\mathsf{Hot}(\mathscr{A})$ into an additive category, such that (7.1.5) is an additive functor.

Furthermore, if $\mathscr{A}$ is abelian, the same holds for $\mathsf{C}(\mathscr{A})$. Indeed, we can take

$$(\mathrm{Ker}\,\varphi)^i := \mathrm{Ker}\,\varphi^i \qquad (\mathrm{Coker}\,\varphi)^i := \mathrm{Coker}\,\varphi^i \qquad \text{for every morphism } \varphi^\bullet \text{ of } \mathsf{C}(\mathscr{A})$$

and it is clear that the resulting morphism $\beta_{\varphi^\bullet}$ as in (3.7.33) is an isomorphism. It is then also clear that, for any additive functor F as in (i), the induced functors $\mathsf{C}(F)$ and $\mathsf{Hot}(F)$ are additive, and the same holds as well for the shift operators on $\mathsf{C}(\mathscr{A})$ and $\mathsf{Hot}(\mathscr{A})$.

All these considerations extend more generally to the full subcategories $\mathsf{C}^I(\mathscr{A})$ (for any interval $I \subset \mathbb{Z}$), $\mathsf{C}^-(\mathscr{A})$, $\mathsf{C}^+(\mathscr{A})$ and $\mathsf{C}^b(\mathscr{A})$, as well as their homotopy category variants : the detailed verifications shall be left to the reader.

(iii) Taking into account remark 3.7.37(i), we get a natural identification

$$\mathsf{C}(\mathscr{A})^o \stackrel{\sim}{\to} \mathsf{C}(\mathscr{A}^o) \qquad K^\bullet \mapsto (K^o)^\bullet$$

where

$$(K^o)^i := (K^{-i})^o \qquad \text{and} \qquad d^i_{K^o} := (-1)^{i+1} \cdot (d_K^{-i-1})^o$$

for every $i \in \mathbb{Z}$ and every $K^\bullet \in \mathrm{Ob}(\mathsf{C}(\mathscr{A}))$. If $\varphi^\bullet : K^\bullet \to L^\bullet$ is any morphism in $\mathsf{C}(\mathscr{A})$, then $(\varphi^o)^i := \varphi^{-i}$ for every $i \in \mathbb{Z}$. Also, if $s^\bullet$ is a homotopy between morphisms $f^\bullet, g^\bullet : K^\bullet \to L^\bullet$, then the system

$$(s^o)^\bullet := ((-1)^i \cdot (s^{1-i})^o \mid i \in \mathbb{Z})$$

is a homotopy between $(f^o)^\bullet, (g^o)^\bullet : (L^o)^\bullet \to (K^o)^\bullet$, whence an isomorphism of categories :

$$\mathsf{Hot}(\mathscr{A})^o \stackrel{\sim}{\to} \mathsf{Hot}(\mathscr{A}^o).$$

One may wonder about the reason for inserting apparently artificial signs in the above formulas; the point is that this sign convention makes the foregoing natural identifications behave slightly better in connection with additive contravariant functors (and their extensions to complexes : see *e.g.* the construction of the $\mathrm{Hom}^\bullet$ functor in example 7.1.7). On the other hand, there is a drawback as well : indeed, notice that by applying twice our natural isomorphism, we do *not* get the identity automorphism of $\mathsf{C}(\mathscr{A})^{oo} = \mathsf{C}(\mathscr{A})$; rather

$$d^\bullet_{K^{oo}} = -d^\bullet_K \qquad \text{and} \qquad (s^{oo})^\bullet = -s^\bullet$$

for every object $K^\bullet$ and every homotopy $s^\bullet$ of $\mathsf{C}(\mathscr{A})$ (details left to the reader).

(iv) The indexing notation that makes use of superscript to denote the degrees in a complex, is known traditionally as *cohomological degree notation*. Sometimes it is more natural to switch to the *homological degree notation*, that makes use of subscript indexing; namely, one associates with any cochain complex $K^\bullet$, the *chain complex* $K_\bullet$ given by the rule :

$$K_n := K^{-n} \qquad \text{and} \qquad d_n := d^{-n} : K_n \to K_{n-1} \qquad \text{for every } n \in \mathbb{Z}.$$

(v) If $\mathscr{A}$ is complete (resp. cocomplete) then the same holds for $\mathsf{C}^I(\mathscr{A})$, for any interval $I \subset \mathbb{Z}$. Indeed, we may regard these categories as full subcategories of the category $\mathrm{Fun}(\mathbb{Z}, \mathscr{A})$, where the ordered set $\mathbb{Z}$ (endowed with its standard ordering) is viewed as a category as explained in example 1.1.6(iii), and therefore the assertion follows from corollary 1.4.1(ii,iv), which shows more precisely that the limits and colimits in $\mathsf{C}^I(\mathscr{A})$ are formed *degree-wise*.

(vi) If $\mathscr{A}$ is complete (resp. cocomplete), then all products (resp. coproducts) of $\mathsf{Hot}^I(\mathscr{A})$ are representable, for every interval $I \subset \mathbb{Z}$. Indeed, let $K^\bullet_\bullet := (K^\bullet_j \mid j \in J)$ be any family of objects of $\mathsf{Hot}^I(\mathscr{A})$ indexed by a small set J; we claim that the product $K^\bullet$ in the category $\mathsf{C}^I(\mathscr{A})$ of the family $K^\bullet_\bullet$ also represents the product of the same family in $\mathsf{Hot}^I(\mathscr{A})$. Indeed, let $M^\bullet$ be any other object of $\mathsf{Hot}^I(\mathscr{A})$, and $\pi^\bullet_j : K^\bullet \to K^\bullet_j$ the canonical projection, for every $j \in J$; from any morphism $\varphi^\bullet : M^\bullet \to K^\bullet$ we obtain a system $(\varphi^\bullet_j := \pi^\bullet_j \circ \varphi^\bullet : M^\bullet \to K^\bullet_j \mid j \in J)$ of morphisms in $\mathsf{Hot}^I(\mathscr{A})$. Conversely, given such a system of morphisms, for every $j \in J$ pick arbitrarily a morphism of complexes $\widetilde{\varphi}^\bullet_j : M^\bullet \to K^\bullet_j$ in the class of $\varphi^\bullet_j$; the system $(\widetilde{\varphi}^\bullet_j \mid j \in J)$ yields by (v) a unique morphism of complexes $\widetilde{\varphi}^\bullet : M^\bullet \to K^\bullet$ and we let $\varphi^\bullet$ be the homotopy class of $\widetilde{\varphi}^\bullet$. We need to show that $\varphi^\bullet$ does not depend on the choices of representatives $\widetilde{\varphi}^\bullet_j$; to this aim, we come down to checking that if each $\widetilde{\varphi}^\bullet_j$ is null-homotopic, then the same holds for $\widetilde{\varphi}^\bullet$. Hence, pick for every $j \in J$ a homotopy $s^\bullet_j$ from $\widetilde{\varphi}^\bullet_j$ to the zero morphism. Since the products are computed degree-wise in $\mathsf{C}^I(\mathscr{A})$, the system $(s^n_j : M^n \to K^{n-1}_j \mid j \in J)$ corresponds, for every $n \in I$, to a unique morphism $s^n : M^n \to K^{n-1}$ of $\mathscr{A}$, and it is easily seen that the system $(s^n \mid n \in I)$ yields a well defined homotopy from $\widetilde{\varphi}^\bullet$ to the zero morphism $M^\bullet \to K^\bullet$. By applying the same argument to the opposite category $\mathscr{A}^o$ and invoking (iii) we get the assertion for coproducts.

Example 7.1.7. Denote by $\mathbb{Z}\text{-}\mathbf{Mod}_{\mathrm{fft}}$ the additive category of free abelian groups of finite rank. We have a natural isomorphism of categories :

$$\mathbb{Z}\text{-}\mathbf{Mod}^o_{\mathrm{fft}} \xrightarrow{\sim} \mathbb{Z}\text{-}\mathbf{Mod}_{\mathrm{fft}} \quad : \quad P \mapsto P^\vee := \mathrm{Hom}_{\mathbb{Z}}(P, \mathbb{Z})$$

that sends any $\mathbb{Z}$-linear map $f : P \to Q$ to its transpose $f^\vee : Q^\vee \to P^\vee$. In view of remark 7.1.6(iii) there follow natural isomorphisms of categories

$$\mathsf{C}(\mathbb{Z}\text{-}\mathbf{Mod}_{\mathrm{fft}})^o \xrightarrow{\sim} \mathsf{C}(\mathbb{Z}\text{-}\mathbf{Mod}_{\mathrm{fft}}) \qquad \mathsf{Hot}(\mathbb{Z}\text{-}\mathbf{Mod}_{\mathrm{fft}})^o \xrightarrow{\sim} \mathsf{Hot}(\mathbb{Z}\text{-}\mathbf{Mod}_{\mathrm{fft}}) \qquad K^\bullet \mapsto K^{\vee\bullet}.$$

Definition 7.1.8. Let $\mathscr{A}$ be an abelian category, $K^\bullet$ any object of $\mathsf{C}(\mathscr{A})$, and $i \in \mathbb{Z}$ any integer.

(i) The *cohomology of $K^\bullet$ in degree i* is the object of $\mathscr{A}$:

$$H^i K^\bullet := \mathrm{Ker}\, d^i / \mathrm{Im}\, d^{i-1}.$$

(ii) We say that $K^\bullet$ is *acyclic in degree i* (resp. *acyclic*) if $H^i K^\bullet = 0$ (resp. if $H^j K^\bullet = 0$ for every $j \in \mathbb{Z}$).

Remark 7.1.9. (i) Clearly, for every $i \in \mathbb{Z}$, the rule $K^\bullet \mapsto H^i K^\bullet$ extends to a functor

$$H^i : \mathsf{C}(\mathscr{A}) \to \mathscr{A}.$$

(ii) Moreover, if $\varphi^\bullet, \psi^\bullet : K^\bullet \to L^\bullet$ are chain homotopic morphisms in $\mathsf{C}(\mathscr{A})$, then it is easily seen that, for every $i \in \mathbb{Z}$, the induced morphisms in cohomology

$$H^i\varphi^\bullet, H^i\psi^\bullet : H^i K^\bullet \to H^i L^\bullet$$

coincide. Hence, the cohomology functor H^i factors (uniquely) through a functor

$$H^i : \mathsf{Hot}(\mathscr{A}) \to \mathscr{A} \qquad \text{for every } i \in \mathbb{Z}.$$

(iii) Notice as well that the cohomology is a *self-dual* operation : namely, we have an essentially commutative diagram of functors

$$\begin{array}{ccc} \mathsf{C}(\mathscr{A})^o & \xrightarrow{\sim} & \mathsf{C}(\mathscr{A}^o) \\ & {\scriptstyle (H^i)^o} \searrow \quad \swarrow {\scriptstyle H^{-i}} & \\ & \mathscr{A}^o & \end{array} \qquad \text{for every } i \in \mathbb{Z}$$

whose top horizontal arrow is the isomorphism of remark 7.1.6(iii). Indeed, if $K^\bullet$ is any complex of $\mathscr{A}$, the morphism d^{i-1}_K factors uniquely through a morphism $e^{i-1}_K : K^{i-1} \to \mathrm{Ker}\, d^i_K$, and we have a natural isomorphism $H^i K^\bullet \xrightarrow{\sim} \mathrm{Coker}\, e^{i-1}_K$. On the other hand, the morphism d^i_K factors uniquely through a morphism $f^i_K : \mathrm{Coker}\, d^{i-1}_K \to K^i$ and we have also the natural isomorphism $H^i K^\bullet \xrightarrow{\sim} \mathrm{Ker}\, f^i_K$, so the assertion follows from remark 3.7.37(i).

(iv) Just like in remark 7.1.6(iv), it is sometimes convenient to switch to a subscript notation; thus, for any chain complex $K_\bullet$ one sets

$$H_n K_\bullet := H^{-n} K_\bullet \qquad \text{for every } n \in \mathbb{Z}$$

and calls this object of $\mathscr{A}$ the *homology of $K_\bullet$* in degree n.

7.1.10. A *double complex* of $\mathscr{A}$ is an object of

$$\mathsf{C}_2(\mathscr{A}) := \mathsf{C}(\mathsf{C}(\mathscr{A}))$$

(and likewise for a morphism of double complexes); *i.e.* a double complex is a triple

$$K^{\bullet\bullet} := (K^{\bullet\bullet}, d_h^{\bullet\bullet}, d_v^{\bullet\bullet})$$

consisting of a system $(K^{pq} \mid p, q \in \mathbb{Z})$ of objects of $\mathscr{A}$, and morphisms d_h^{pq}, d_v^{pq} called respectively the *horizontal* and *vertical* differentials, fitting into a commutative diagram

$$\begin{array}{ccc} K^{pq} & \xrightarrow{d_h^{pq}} & K^{p+1,q} \\ {\scriptstyle d_v^{pq}}\downarrow & & \downarrow{\scriptstyle d_v^{p+1,q}} \\ K^{p,q+1} & \xrightarrow{d_h^{pq}} & K^{p+1,q+1} \end{array} \qquad \text{for every } p, q \in \mathbb{Z}$$

and such that

$$d_h^{p+1,q} \circ d_h^{pq} = 0 \qquad d_v^{p,q+1} \circ d_v^{pq} = 0 \qquad \text{for every } p, q \in \mathbb{Z}.$$

7.1.11. There are natural functors

$$\mathsf{C}_2(\mathscr{A}) \to \mathsf{C}_2(\mathscr{A}) \ : \ K^{\bullet\bullet} \mapsto \mathsf{fl}_{\mathscr{A}}(K^{\bullet\bullet}) \qquad \qquad \mathsf{C}_2(\mathscr{A}) \to \mathsf{C}(\mathscr{A}) \ : \ K^{\bullet\bullet} \mapsto (K^{\bullet\bullet})^{\Delta}$$

where :

- The *flip* $\mathsf{fl}_{\mathscr{A}}(K^{\bullet\bullet})$ of $K^{\bullet\bullet}$ is the double complex $F^{\bullet\bullet}$ such that $F^{pq} := K^{qp}$ for every $p, q \in \mathbb{Z}$, with differentials deduced from those of $K^{\bullet\bullet}$, in the obvious way.
- The *diagonal* $(K^{\bullet\bullet})^{\Delta}$ is the complex $D^{\bullet}$ such that $D^p := K^{pp}$ for every $p \in \mathbb{Z}$ and with differentials

$$d_v^{p+1,q} \circ d_h^{pq} : D^p \to D^{p+1} \qquad \text{for every } p \in \mathbb{Z}.$$

Suppose that all coproducts (resp. all products) are representable in $\mathscr{A}$. Then there are two other natural functors

$$\mathrm{Tot}^{\oplus}_{\mathscr{A}} : \mathsf{C}_2(\mathscr{A}) \to \mathsf{C}(\mathscr{A}) \qquad (\text{ resp. } \mathrm{Tot}^{\Pi}_{\mathscr{A}} : \mathsf{C}_2(\mathscr{A}) \to \mathsf{C}(\mathscr{A}) \ )$$

defined as follows. The *total complex* $\mathrm{Tot}^{\oplus}_{\mathscr{A}}(K^{\bullet\bullet})$ (resp. $\mathrm{Tot}^{\Pi}_{\mathscr{A}}(K^{\bullet\bullet})$) is the complex $T^{\bullet}$ with

$$T^n := \bigoplus_{p+q=n} K^{pq} \qquad (\text{ resp. } T^n := \prod_{p+q=n} K^{pq} \ )$$

and with differentials $T^n \to T^{n+1}$ given by the sum (resp. the product) of the morphisms

$$d_h^{pq} + (-1)^p \cdot d_v^{pq} : K^{pq} \to K^{p,q+1} \oplus K^{p+1,q} \qquad \text{for all } p, q \in \mathbb{Z} \text{ such that } p + q = n.$$

We usually drop the subscript $\mathscr{A}$, unless the omission would be a source of ambiguities, and we often omit as well the superscript $\oplus$ when dealing with the total complex functor; to avoid confusion, we stipulate that *the notation* Tot *shall always refer to the functor* $\mathrm{Tot}^{\oplus}$, so if we need to use the other total complex functor, we shall denote it explicitly by Tot^{Π}. Notice that we have natural isomorphisms

$$\mathrm{Tot}(K^{\bullet\bullet}) \xrightarrow{\sim} \mathrm{Tot}(\mathsf{fl}_{\mathscr{A}}(K^{\bullet\bullet})) \qquad (\text{resp. } \mathrm{Tot}^{\Pi}(K^{\bullet\bullet}) \xrightarrow{\sim} \mathrm{Tot}^{\Pi}(\mathsf{fl}_{\mathscr{A}}(K^{\bullet\bullet})) \ )$$

given, in each degree $n \in \mathbb{Z}$, by the direct sum (resp. the direct product) of the morphisms $(-1)^{pq} \cdot \mathbf{1}_{K^{pq}}$, for every $p, q \in \mathbb{Z}$ such that $p + q = n$.

7.1.12. The category of *triple complexes* of $\mathscr{A}$ can be realized in two equivalent ways : namely, we have natural identifications :

$$\mathsf{C}_2(\mathsf{C}(\mathscr{A})) \xrightarrow{\sim} \mathsf{C}(\mathsf{C}_2(\mathscr{A})) \tag{7.1.13}$$

and we denote either of these categories by $\mathsf{C}_3(\mathscr{A})$. In other words, a triple complex is a system

$$(K^{ijk}, d_1^{ijk}, d_2^{ijk}, d_3^{ijk} \mid i, j, k \in \mathbb{Z})$$

such that, for every fixed $p \in \mathbb{Z}$, the subsystems

$$(K^{p,\bullet\bullet}, d_2^{p,\bullet\bullet}, d_3^{p,\bullet\bullet}) \qquad (K^{\bullet,p,\bullet}, d_1^{\bullet,p,\bullet}, d_3^{\bullet,p,\bullet}) \qquad (K^{\bullet,\bullet,p}, d_1^{\bullet,\bullet,p}, d_2^{\bullet,\bullet,p})$$

are double complexes of $\mathscr{A}$. Corresponding to the two interpretations of $\mathsf{C}_3(\mathscr{A})$, we have two different way of forming total complexes out of triple complexes; namely, if $\mathscr{A}$ is cocomplete we have the functors

$$\mathrm{Tot}_{\mathscr{A}} \circ \mathrm{Tot}_{\mathsf{C}(\mathscr{A})}, \mathrm{Tot}_{\mathscr{A}} \circ \mathsf{C}(\mathrm{Tot}_{\mathscr{A}}) : \mathsf{C}_3(\mathscr{A}) \to \mathsf{C}(\mathscr{A})$$

and likewise for the Tot^Π variant, if $\mathscr{A}$ is complete. However, a simple computation shows that these two functors agree under the natural identification (7.1.13) : namely, both functors yield a complex whose differential is the direct sum (resp. direct product) of the morphisms

$$d_1^{ijk} + (-1)^i \cdot d_2^{ijk} + (-1)^{i+j} \cdot d_3^{ijk} : K^{ijk} \to K^{i+1,j,k} \oplus K^{i,j+1,k} \oplus K^{i,j,k+1}.$$

We shall therefore denote these functors indifferently by Tot (resp. by Tot^Π), just as for double complexes.

7.1.14. Let $\mathscr{A}, \mathscr{A}', \mathscr{A}''$ be three additive categories, and

$$B : \mathscr{A} \times \mathscr{A}' \to \mathscr{A}''$$

a *biadditive* functor, *i.e.* such that, for every $A \in \mathrm{Ob}(\mathscr{A})$ and every $A' \in \mathrm{Ob}(\mathscr{A}')$, both functors

$$B(A, -) : \mathscr{A}' \to \mathscr{A}'' \qquad B(-, A') : \mathscr{A}' \to \mathscr{A}''$$

are additive. The functor B induces a functor

$$B^{\bullet\bullet} : \mathsf{C}(\mathscr{A}) \times \mathsf{C}(\mathscr{A}') \to \mathsf{C}_2(\mathscr{A}'')$$

by the rule

$$B^{ij}(K^\bullet, L^\bullet) := B(K^i, L^j) \qquad \text{for every } K^\bullet \in \mathrm{Ob}(\mathsf{C}(\mathscr{A})) \text{ and } L^\bullet \in \mathrm{Ob}(\mathsf{C}(\mathscr{A}')).$$

The differentials of $B^{\bullet\bullet}(K^\bullet, L^\bullet)$ in degree (p, q) are respectively $B(d_K^p, \mathbf{1}_{L^q})$ and $B(\mathbf{1}_{K^p}, d_L^q)$. Suppose that $\mathscr{A}''$ is cocomplete (resp. complete); then, by composing with the functor $\mathrm{Tot}^\oplus$ (resp. with Tot^Π), we deduce a biadditive functor

$$B_\oplus^\bullet : \mathsf{C}(\mathscr{A}) \times \mathsf{C}(\mathscr{A}') \to \mathsf{C}(\mathscr{A}'') \qquad (\text{resp. } B_\Pi^\bullet : \mathsf{C}(\mathscr{A}) \times \mathsf{C}(\mathscr{A}') \to \mathsf{C}(\mathscr{A}'')).$$

For a general additive category $\mathscr{A}''$, the foregoing functors are still well defined at least on bounded above or bounded below complexes, *i.e.* we have functors

$$B_-^\bullet : \mathsf{C}^-(\mathscr{A}) \times \mathsf{C}^-(\mathscr{A}') \to \mathsf{C}^-(\mathscr{A}'') \qquad B_+^\bullet : \mathsf{C}^+(\mathscr{A}) \times \mathsf{C}^+(\mathscr{A}') \to \mathsf{C}^+(\mathscr{A}'').$$

Example 7.1.15. (i) If $\mathscr{A}$ is any additive category, the construction of (7.1.14) extends the biadditive functor

$$\mathrm{Hom}_{\mathscr{A}}(-, -) : \mathscr{A}^o \times \mathscr{A} \to \mathbb{Z}\text{-}\mathbf{Mod}$$

to a functor $\mathrm{Hom}_{\mathscr{A},\Pi}^\bullet : \mathsf{C}(\mathscr{A}) \times \mathsf{C}(\mathscr{A}^o) \to \mathsf{C}(\mathbb{Z}\text{-}\mathbf{Mod})$ (notice the swapping of the two factors, that is required in order to conform with established sign conventions) and combining with the natural identification of remark 7.1.6(iii), we obtain a biadditive functor

$$\mathrm{Hom}_{\mathscr{A}}^\bullet(-, -) : \mathsf{C}(\mathscr{A}) \times \mathsf{C}(\mathscr{A})^o \to \mathsf{C}(\mathbb{Z}\text{-}\mathbf{Mod}) \quad : \quad (L^\bullet, K^\bullet) \mapsto \mathrm{Hom}_{\mathscr{A},\Pi}^\bullet((K^o)^\bullet, L^\bullet).$$

We shall denote by $d^\bullet_{K,L}$ the differential of the complex $\mathrm{Hom}^\bullet_{\mathscr{A}}(K^\bullet, L^\bullet)$, for every $K^\bullet, L^\bullet \in \mathrm{Ob}(\mathsf{C}(\mathscr{A}))$. Let $n \in \mathbb{Z}$ be any integer; with this notation, a simple inspection shows that :

- $d^n_{K,L}$ is the product, for every $j \in \mathbb{Z}$, of the maps
$$\mathrm{Hom}_{\mathscr{A}}(K^{j-n}, L^j) \to \mathrm{Hom}_{\mathscr{A}}(K^{j-n}, L^{j+1}) \oplus \mathrm{Hom}_{\mathscr{A}}(K^{j-n-1}, L^j)$$
given by the rule : $(\varphi : K^{j-n} \to L^j) \mapsto d^j_L \circ \varphi - (-1)^n \cdot \varphi \circ d^{j-n-1}_K$.
- $\mathrm{Hom}_{\mathsf{C}(\mathscr{A})}(K^\bullet, L^\bullet[n]) = \mathrm{Ker}\, d^n_{K,L}$.
- Let $\varphi^\bullet, \psi^\bullet : K^\bullet \to L^\bullet[n]$ be any two morphisms; then the set of homotopies from $\varphi^\bullet$ to $\psi^\bullet$ is naturally identified with the subset
$$\{s^\bullet \in \mathrm{Hom}^{-1-n}_{\mathscr{A}}(K^\bullet, L^\bullet) \mid d^{-1-n}_{K,L}(s^\bullet) = \psi^\bullet - \varphi^\bullet\}.$$
- Consequently, we have a natural identification :
$$\mathrm{Hot}_{\mathscr{A}}(K^\bullet, L^\bullet[n]) \xrightarrow{\sim} H^n\mathrm{Hom}^\bullet_{\mathscr{A}}(K^\bullet, L^\bullet).$$

(ii) The constructions of (i) can be used to endow $\mathsf{C}(\mathscr{A})$ with a natural 2-category structure, whose 2-cells are given by homotopies of complexes. Indeed, let us write
$$s^\bullet : \varphi^\bullet \Rightarrow \psi^\bullet$$
if $s^\bullet := (s^n \mid n \in \mathbb{N})$ is a homotopy from $\varphi^\bullet$ to $\psi^\bullet$. Then, if $\lambda^\bullet : K^\bullet \to L^\bullet$ is another morphism, and $t^\bullet : \psi^\bullet \Rightarrow \lambda^\bullet$ another homotopy, we define a composition law by setting
$$s^\bullet \odot t^\bullet := (s^n + t^n \mid n \in \mathbb{N})$$
and it is immediate that $s^\bullet \odot t^\bullet$ is a homotopy $\varphi^\bullet \Rightarrow \lambda^\bullet$. Moreover, if $\beta^\bullet, \gamma^\bullet : L^\bullet \to P^\bullet$ are two other morphisms in $\mathsf{C}(\mathscr{A})$, and $u : \beta^\bullet \Rightarrow \gamma^\bullet$ another homotopy, we have a Godement composition law by the rule
$$u^\bullet * s^\bullet := (\beta^{n+1} \circ s^n + u^n \circ \psi^n \mid n \in \mathbb{N}) : \beta^\bullet \circ \varphi^\bullet \Rightarrow \gamma^\bullet \circ \psi^\bullet.$$
The associativity of the laws $*$ and $\odot$ thus defined are easily checked by direct computation.

Now, suppose that $\delta^\bullet : L^\bullet \to P^\bullet$ is yet another morphism, and $v^\bullet : \gamma^\bullet \Rightarrow \delta^\bullet$ another homotopy. A direct calculation yields the identity
$$(u^\bullet * s^\bullet) \odot (v^\bullet * t^\bullet) = (u^\bullet \odot v^\bullet) * (s^\bullet \odot t^\bullet) + c^\bullet$$
where $c^n := d^{n-2}_P \circ u^{n-1} \circ t^n - u^n \circ t^{n+1} \circ d^n_K$ for every $n \in \mathbb{Z}$, which can be rewritten as
$$c^\bullet = d^{-2}_{K,L}((u \circ t)^\bullet) \qquad \text{where } (u \circ t)^n := u^{n-1} \circ t^n \text{ for every } n \in \mathbb{Z}.$$
Summing up, we conclude that $\mathsf{C}(\mathscr{A})$ carries a 2-category structure, whose 2-cells $\varphi \Rightarrow \psi$ (for any two 1-cells $\varphi, \psi : K^\bullet \to L^\bullet$) are the classes
$$\overline{s} \in \mathrm{Hom}^{-1}_{\mathscr{A}}(K^\bullet, L^\bullet)/\mathrm{Im}(d^{-2}_{K,L}) \qquad \text{such that } d^{-1}_{K,L}(\overline{s}) = \psi^\bullet - \varphi^\bullet.$$

(iii) Moreover, if F is any additive functor as in remark 7.1.6(i), the induced functor $\mathsf{C}(F)$ extends to a pseudo-functor for the 2-category structures given by (ii); indeed, if $s^\bullet : K^\bullet \Rightarrow L^\bullet$ is a homotopy, obviously the system $(Fs^n \mid n \in \mathbb{N})$ is a homotopy $Fs^\bullet : F(K^\bullet) \Rightarrow F(L^\bullet)$.

Example 7.1.16. (i) Suppose that $(\mathscr{A}, \otimes, \Phi, \Psi)$ is a tensor abelian category, and set
$$B(A, A') := A \otimes A' \qquad \text{for every } A, A' \in \mathrm{Ob}(\mathscr{A}).$$
Following (7.1.14), we get a corresponding functor $B^{\bullet\bullet}_\oplus$, for which we use the notation
$$- \boxtimes - : \mathsf{C}(\mathscr{A}) \times \mathsf{C}(\mathscr{A}) \to \mathsf{C}_2(\mathscr{A}).$$
If all coproducts are representables in $\mathscr{A}$, we get as well the functor $B^\bullet_\oplus$, which we denote :
$$K^\bullet \otimes L^\bullet := \mathrm{Tot}(K^\bullet \boxtimes L^\bullet).$$

If $\mathscr{A}$ is not cocomplete, $K^\bullet \otimes L^\bullet$ is still well defined, provided $K^\bullet$ and $L^\bullet$ are both bounded below or if they are both bounded above. The commutativity constraints for $(\mathscr{A}, \otimes)$ yield natural isomorphisms $K^\bullet \boxtimes L^\bullet \xrightarrow{\sim} \mathsf{fl}_{\mathscr{A}}(L^\bullet \boxtimes K^\bullet)$, as well as

(7.1.17) $$\Psi^\bullet_{K,L} : K^\bullet \otimes L^\bullet \xrightarrow{\sim} L^\bullet \otimes K^\bullet.$$

Namely, one takes the direct sum of the maps $(-1)^{pq} \cdot \Psi_{K^p,L^q}$, for every $p, q \in \mathbb{Z}$.

(ii) Likewise, if $P^\bullet$ is another complex of $\mathscr{A}$, the associativity constraints of $\mathscr{A}$ assemble to an isomorphism of triple complexes

$$(\Phi_{K^i,L^j,P^k} \mid i,j,k) : (K^\bullet \boxtimes L^\bullet) \boxtimes P^\bullet \xrightarrow{\sim} K^\bullet \boxtimes (L^\bullet \boxtimes P^\bullet)$$

whence, taking into account the discussion of (7.1.12), a natural isomorphism in $\mathsf{C}(\mathscr{A})$:

$$\Phi^\bullet_{K,L,P} : K^\bullet \otimes (L^\bullet \otimes P^\bullet) \xrightarrow{\sim} (K^\bullet \otimes L^\bullet) \otimes P^\bullet.$$

With these natural isomorphisms, $\mathsf{C}(\mathscr{A})$ is then a tensor abelian category as well.

(iii) In the situation of (i), notice that the natural morphism $K^i \otimes L^j \to (K^\bullet \otimes L^\bullet)^{i+j}$ induces morphisms

$$\mathrm{Ker}\,(d^i_K) \otimes \mathrm{Ker}\,(d^j_L) \to \mathrm{Ker}\,(d^{i+j}_{K\otimes L})$$
$$(\mathrm{Ker}\,(d^i_K) \otimes \mathrm{Im}\,(d^{j-1}_L)) \oplus (\mathrm{Im}\,(d^{i-1}_K) \otimes \mathrm{Ker}\,(d^j_L)) \to \mathrm{Im}\,(d^{i+j}_{K\otimes L})$$

so, the induced map $\mathrm{Ker}\,(d^i_K) \otimes \mathrm{Ker}\,(d^j_L) \to H^{i+j}(K^\bullet \otimes L^\bullet)$ factors through a natural pairing :

$$H^i(K^\bullet) \otimes H^j(L^\bullet) \to H^{i+j}(K^\bullet \otimes L^\bullet) \qquad \text{for every } i, j \in \mathbb{Z}.$$

(iv) Moreover, if $P^\bullet$ is a third complex, in view of (ii) we get a commutative diagram

$$\begin{array}{ccccc} H^iK^\bullet \otimes (H^jL^\bullet \otimes H^kP^\bullet) & \xrightarrow{\alpha} & (H^iK^\bullet \otimes H^jL^\bullet) \otimes H^kP^\bullet & \xrightarrow{\beta} & H^{i+j}(K^\bullet \otimes L^\bullet) \otimes H^kP^\bullet \\ \downarrow{\scriptstyle\gamma} & & & & \downarrow{\scriptstyle\delta} \\ H^iK^\bullet \otimes H^{j+k}(L^\bullet \otimes P^\bullet) & \xrightarrow{\lambda} & H^{i+j+k}(K^\bullet \otimes (L^\bullet \otimes P^\bullet)) & \xrightarrow{\sigma} & H^{i+j+k}((K^\bullet \otimes L^\bullet) \otimes P^\bullet) \end{array}$$

where α is the associativity constraint, β, γ, δ and λ are given by the above pairing, and σ is deduced from $\Phi^\bullet_{K,L,P}$.

(v) If $\mathscr{A}$ also admits an internal Hom functor, we may define as well a functor

$$\mathscr{H}om^{\bullet\bullet} : \mathsf{C}(\mathscr{A}) \times \mathsf{C}(\mathscr{A})^o \to \mathsf{C}_2(\mathscr{A})$$

following the trace of example 7.1.15(i); namely, we set

$$\mathscr{H}om^{p,q}(K^\bullet, L^\bullet) := \mathscr{H}om(K^{-p}, L^q) \qquad \text{for every } p, q \in \mathbb{Z}$$

with differentials

$$d^{p,q}_h := \mathscr{H}om(\mathbf{1}_{K^{-p}}, d^q_L) \qquad d^{p,q}_v := (-1)^{p+1} \cdot \mathscr{H}om(d^{-p-1}_K \mathbf{1}_{L^q}) \qquad \text{for every } p, q \in \mathbb{Z}.$$

If all products are representable in $\mathscr{A}$, we may then define

$$\mathscr{H}om^\bullet(K^\bullet, L^\bullet) := \mathrm{Tot}^\Pi \mathscr{H}om^{\bullet\bullet}(K^\bullet, L^\bullet).$$

(vi) Suppose that $\mathscr{A}$ is both complete and cocomplete, and $P^\bullet$ is any other complex; to ease notation, set $H^\bullet := \mathscr{H}om^\bullet(K^\bullet, L^\bullet)$ and $N^\bullet := P^\bullet \otimes K^\bullet$. We have natural isomorphisms

$$\mathrm{Hom}_{\mathsf{C}(\mathscr{A})}(P^\bullet, H^\bullet) \xrightarrow{\sim} \mathrm{Equal}\Big(\mathrm{Hom}^0_{\mathscr{A}}(P^\bullet, H^\bullet) \overset{d_v}{\underset{d_h}{\rightrightarrows}} \mathrm{Hom}^1_{\mathscr{A}}(P^\bullet, H^\bullet)\Big)$$

where

$$d_h := \prod_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathscr{A}}(P^n, d^n_H) \qquad \text{and} \qquad d_v := \prod_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathscr{A}}(d^n_P, H^{n+1})$$

(see example 7.1.15(i)). However, notice that

$$\begin{aligned}\mathrm{Hom}^a_{\mathscr{A}}(P^\bullet, H^\bullet) &= \prod_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathscr{A}}\Big(P^n, \prod_{p+q=n+a} \mathscr{H}om(K^{-p}, L^q)\Big) \\ &= \prod_{n\in\mathbb{Z}} \prod_{p+q=n+a} \mathrm{Hom}_{\mathscr{A}}(P^n, \mathscr{H}om(K^{-p}, L^q)) \\ &= \prod_{n\in\mathbb{Z}} \prod_{p+q=n+a} \mathrm{Hom}_{\mathscr{A}}(P^n \otimes K^{-p}, L^q) \\ &= \prod_{q\in\mathbb{Z}} \mathrm{Hom}_{\mathscr{A}}\Big(\bigoplus_{n-p=q-a} P^n \otimes K^{-p}, L^q\Big)\end{aligned}$$

for every $n, a \in \mathbb{Z}$, whence

$$\mathrm{Hom}_{\mathsf{C}(\mathscr{A})}(P^\bullet, H^\bullet) \xrightarrow{\sim} \mathrm{Equal}\Big(\mathrm{Hom}^0_{\mathscr{A}}(N^\bullet, L^\bullet) \underset{d'_h}{\overset{d'_v}{\rightrightarrows}} \mathrm{Hom}^1_{\mathscr{A}}(N^\bullet, L^\bullet)\Big)$$

where

$$d'_h := \prod_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathscr{A}}(d^q_N, L^q) \qquad \text{and} \qquad d'_v := \prod_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathscr{A}}(N^q, d^q_L)$$

so finally :

$$\mathrm{Hom}_{\mathsf{C}(\mathscr{A})}(P^\bullet, \mathscr{H}om^\bullet(K^\bullet, L^\bullet)) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{C}(\mathscr{A})}(P^\bullet \otimes K^\bullet, L^\bullet)$$

which says that $\mathscr{H}om^\bullet$ is an internal Hom functor for $\mathsf{C}(\mathscr{A})$.

(vii) In the situation of (vi), set

$$\mathscr{H}om_{\mathsf{C}(\mathscr{A})}(K^\bullet, L^\bullet) := \mathrm{Ker}\,(d^0 : H^0 \to H^1)$$

and take $P^\bullet := Z[0]$, where Z is any object of $\mathscr{A}$; it is easily seen that the natural map

$$\mathrm{Hom}_{\mathscr{A}}(Z, \mathscr{H}om_{\mathsf{C}(\mathscr{A})}(K^\bullet, L^\bullet)) \to \mathrm{Hom}_{\mathsf{C}(\mathscr{A})}(P^\bullet, H^\bullet) \to \mathrm{Hom}_{\mathsf{C}(\mathscr{A})}(Z[0] \otimes K^\bullet, L^\bullet)$$

is an isomorphism : details left to the reader.

7.1.18. Suppose that $\mathscr{A}$ is an additive category with small Hom-sets, and let $(G^\bullet, d^\bullet_G)$ be any complex of free abelian groups of finite rank; with the notation of (3.7.47), we obtain an object $(G^\bullet_{\mathscr{A}}, d^\bullet_{\mathscr{A}})$ of $\mathsf{C}(\mathscr{A}^\dagger)$; on the other hand, if $(K^\bullet, d^\bullet_K)$ is any object of $\mathsf{C}(\mathscr{A})$, we may also consider the object $h^\dagger_K := (h^\dagger_{K^n}, h^\dagger_{d^n} \mid n \in \mathbb{N})$ of $\mathsf{C}(\mathscr{A}^\dagger)$, and since $\mathscr{A}^\dagger$ is an abelian tensor category, we may form the tensor product

$$G^\bullet \boxtimes_{\mathbb{Z}} K^\bullet := \mathscr{H}om^{\bullet\bullet}(G^\bullet_{\mathscr{A}}, h^\dagger_K)$$

according to example 7.1.16(v). Arguing as in (3.7.47), we see that this object of $\mathsf{C}_2(\mathscr{A}^\dagger)$ is isomorphic to an object of $\mathsf{C}_2(\mathscr{A})$, and after choosing representing objects, we get a functor

$$\mathsf{C}(\mathbb{Z}\text{-}\mathbf{Mod}_{\mathrm{fft}}) \times \mathsf{C}(\mathscr{A}) \to \mathsf{C}_2(\mathscr{A}) \qquad (G^\bullet, K^\bullet) \mapsto G^\bullet \boxtimes_{\mathbb{Z}} K^\bullet$$

which is additive in both arguments. If all direct sums are representable in $\mathscr{A}$, or else, if $G^\bullet$ is a bounded complex, it is then natural to define

$$G^\bullet \otimes_{\mathbb{Z}} K^\bullet := \mathrm{Tot}\, G^\bullet \boxtimes_{\mathbb{Z}} K^\bullet \qquad \text{for every } G^\bullet \text{ and } K^\bullet \text{ as above.}$$

Likewise, we set $K^\bullet \boxtimes_{\mathbb{Z}} G^\bullet := \mathsf{fl}_{\mathscr{A}}(G^\bullet \boxtimes_{\mathbb{Z}} K^\bullet)$ and $K^\bullet \otimes_{\mathbb{Z}} G^\bullet := \mathrm{Tot}(K^\bullet \boxtimes_{\mathbb{Z}} G^\bullet)$.

Remark 7.1.19. (i) With the notation of (7.1.18), notice the natural isomorphism

$$K^\bullet[1] \xrightarrow{\sim} \mathbb{Z}[1] \otimes_{\mathbb{Z}} K^\bullet \qquad \text{for every } K^\bullet \in \mathrm{Ob}(\mathsf{C}(\mathscr{A}))$$

which explains the sign convention in the definition of the shift operator in (7.1.3).

(ii) Moreover, denote by $\mathbb{K}\langle 1\rangle^\bullet \in \mathrm{Ob}(\mathsf{C}^{[-1,0]}(\mathbb{Z}\text{-}\mathbf{Mod}))$ the object such that

$$\mathbb{K}\langle 1\rangle^{-1} := \mathbb{Z} \qquad \mathbb{K}\langle 1\rangle^{0} := \mathbb{Z}\oplus\mathbb{Z}$$

and with differential d^{-1} given by the rule : $n \mapsto (n,-n)$ for every $n\in\mathbb{Z}$. Let $e_0 := (0,1)$ and $e_1 := (1,0)$ be the canonical basis of $\mathbb{K}\langle 1\rangle^0$; we have two morphisms

$$\iota_i^\bullet : \mathbb{Z}[0] \to \mathbb{K}\langle 1\rangle^\bullet \qquad \text{for } i=0,1$$

given, in degree zero, by the rule : $n \mapsto n\cdot e_i$ for every $n\in\mathbb{N}$. For any pair of morphisms $\varphi^\bullet,\psi^\bullet : L^\bullet \to M^\bullet$ in $\mathsf{C}(\mathscr{A})$, and any homotopy $s^\bullet := (s^n \mid n\in\mathbb{Z})$ from $\varphi^\bullet$ to $\psi^\bullet$, we obtain a morphism of complexes

$$\sigma^\bullet : \mathbb{K}\langle 1\rangle^\bullet \otimes_{\mathbb{Z}} L^\bullet \to M^\bullet$$

as follows. For every $n\in\mathbb{Z}$, the morphism $\sigma^n : L^n \oplus L^n \oplus L^{n+1} \to M^n$ restricts to φ^n (resp. ψ^n, resp. s^n) on the first (resp. second, resp. third) summand. Clearly

$$\varphi^\bullet = \sigma^\bullet\circ(\iota_0^\bullet \otimes_{\mathbb{Z}} L^\bullet) \qquad \psi^\bullet = \sigma^\bullet\circ(\iota_1^\bullet \otimes_{\mathbb{Z}} L^\bullet). \tag{7.1.20}$$

Conversely, the datum of a morphism $\sigma^\bullet : \mathbb{K}\langle 1\rangle^\bullet \otimes_{\mathbb{Z}} L^\bullet \to M^\bullet$ yields a homotopy from $\sigma^\bullet\circ(\iota_1^\bullet \otimes_{\mathbb{Z}} L^\bullet)$ to $\sigma^\bullet\circ(\iota_0^\bullet \otimes_{\mathbb{Z}} L^\bullet)$.

(iii) If $\mathscr{A}$ is abelian, the functor $-\boxtimes_{\mathbb{Z}}-$ extends to the category $\mathsf{C}(\mathbb{Z}\text{-}\mathbf{Mod}_{\mathrm{fg}})\times\mathsf{C}(\mathscr{A})$, and if $\mathscr{A}$ is also cocomplete, we can even extend it to the whole of $\mathsf{C}(\mathbb{Z}\text{-}\mathbf{Mod})\times\mathsf{C}(\mathscr{A})$ and we have as well a corresponding functor $-\otimes_{\mathbb{Z}}-$ on this category (see (3.7.47) and remark 3.7.49(ii)).

(iv) Let $M^\bullet$ and $N^\bullet$ be any two complexes of abelian groups, and $f_0^\bullet, f_1^\bullet : M^\bullet \to N^\bullet$ two homotopy equivalent morphisms. If $K^\bullet$ is any object of $\mathsf{C}(\mathscr{A})$ and the tensor product $M^\bullet\otimes_{\mathbb{Z}}K^\bullet$, $N^\bullet\otimes_{\mathbb{Z}}K^\bullet$ are both representable in $\mathsf{C}(\mathscr{A})$, then the induced morphisms $f_0^\bullet\otimes_{\mathbb{Z}}K^\bullet$ and $f_1^\bullet\otimes_{\mathbb{Z}}K^\bullet$ are homotopy equivalent. Indeed, by (ii), we have a morphism $\tau^\bullet : \mathbb{K}\langle 1\rangle^\bullet\otimes_{\mathbb{Z}} M^\bullet \to N^\bullet$ such that $\tau^\bullet\circ(\iota_i^\bullet\otimes_{\mathbb{Z}}\mathbf{1}_M) = f_i^\bullet$ for $i=0,1$; by example 7.1.16(ii) the tensor product $\tau^\bullet\otimes_{\mathbb{Z}}K^\bullet$ is naturally identified with a morphism $\tau_K^\bullet : \mathbb{K}\langle 1\rangle^\bullet\otimes_{\mathbb{Z}}(M^\bullet\otimes_{\mathbb{Z}}K^\bullet) \to N^\bullet\otimes_{\mathbb{Z}}K^\bullet$ such that $\tau_K^\bullet\circ(\iota_i^\bullet\otimes\mathbf{1}_{M\otimes_{\mathbb{Z}}K}) = f_i^\bullet\otimes_{\mathbb{Z}}K^\bullet$ for $i=0,1$, whence the claim, again by (ii).

(v) Let $P^\bullet$ be any bounded complex of free abelian groups of finite rank. Notice the natural identification

$$(P^\bullet\otimes_{\mathbb{Z}}K^\bullet)^o \xrightarrow{\sim} P^{\vee\bullet}\otimes_{\mathbb{Z}}(K^o)^\bullet \qquad \text{in } \mathsf{C}(\mathscr{A}^o)$$

deduced from example 7.1.7 and remark 7.1.6(iii).

Proposition 7.1.21. *Let $\mathscr{A}$ be any additive category, and $F : \mathsf{C}(\mathscr{A}) \to \mathscr{B}$ any functor. The following conditions are equivalent :*

(a) *F factors uniquely through the natural functor $\mathsf{C}(\mathscr{A}) \to \mathsf{Hot}(\mathscr{A})$.*
(b) *$F\varphi^\bullet$ is an isomorphism, for every homotopy equivalence $\varphi^\bullet$ in $\mathsf{C}(\mathscr{A})$.*

Proof. Clearly (a)⇒(b), hence, suppose that (b) holds, consider two chain homotopic morphisms $\varphi^\bullet,\psi^\bullet : L^\bullet \to M^\bullet$ in $\mathsf{C}(\mathscr{A})$, and define $\sigma^\bullet$ as in remark 7.1.19(ii); in view of (7.1.20), assertion (a) will follow, once we know that

$$F(\iota_0^\bullet\otimes_{\mathbb{Z}}L^\bullet) = F(\iota_1^\bullet\otimes_{\mathbb{Z}}L^\bullet).$$

To show the latter identity, consider the morphism

$$p^\bullet : \mathbb{K}\langle 1\rangle^\bullet \to \mathbb{Z}[0]$$

such that $p^0 : \mathbb{Z}\oplus\mathbb{Z}\to\mathbb{Z}$ is the addition map : $(a,b)\mapsto a+b$ for every $a,b\in\mathbb{Z}$. Clearly

$$p^\bullet\circ\iota_0^\bullet = \mathbf{1}_{\mathbb{Z}[0]} = p^\bullet\circ\iota_1^\bullet$$

so we are easily reduced to showing that $F(p^\bullet\otimes_{\mathbb{Z}}L^\bullet)$ is an isomorphism in $\mathscr{B}$. By assumption (b), the latter in turn will follow, once we have shown that $p^\bullet\otimes_{\mathbb{Z}}L^\bullet$ is a homotopy equivalence. Thus, it suffices to exhibit a morphism $j^\bullet : \mathbb{Z}[0]\to\mathbb{K}\langle 1\rangle^\bullet$ such that

$$p^\bullet\circ j^\bullet = \mathbf{1}_{\mathbb{Z}[0]} \tag{7.1.22}$$

and $(j^\bullet \circ p^\bullet) \otimes_\mathbb{Z} L^\bullet$ is homotopy equivalent to $\mathbf{1}_{\mathbb{K}\langle 1\rangle \otimes_\mathbb{Z} L}$. Set $\mathbb{K}\langle 1,1\rangle^\bullet := \mathbb{K}\langle 1\rangle^\bullet \otimes_\mathbb{Z} \mathbb{K}\langle 1\rangle^\bullet$; by remark 7.1.19(ii), the datum of a homotopy from $(j^\bullet \circ p^\bullet) \otimes_\mathbb{Z} L^\bullet$ to $\mathbf{1}_{\mathbb{K}\langle 1\rangle \otimes_\mathbb{Z} L}$ is equivalent to that of a morphism

$$\tau^\bullet : \mathbb{K}\langle 1,1\rangle^\bullet \otimes_\mathbb{Z} L^\bullet \to \mathbb{K}\langle 1\rangle^\bullet \otimes_\mathbb{Z} L^\bullet$$

such that

$$\tau^\bullet \circ (\iota_0^\bullet \otimes_\mathbb{Z} \mathbb{K}\langle 1\rangle^\bullet \otimes_\mathbb{Z} L^\bullet) = (j^\bullet \circ p^\bullet) \otimes_\mathbb{Z} L^\bullet \quad \text{and} \quad \tau^\bullet \circ (\iota_1^\bullet \otimes_\mathbb{Z} \mathbb{K}\langle 1\rangle^\bullet \otimes_\mathbb{Z} L^\bullet) = \mathbf{1}_{\mathbb{K}\langle 1\rangle \otimes_\mathbb{Z} L}.$$

Notice that (7.1.22) is fulfilled with $j^\bullet := \iota_0^\bullet$; we are then further reduced to showing the existence of a morphism

$$t^\bullet : \mathbb{K}\langle 1,1\rangle^\bullet \to \mathbb{K}\langle 1\rangle^\bullet$$

such that

$$t^\bullet \circ (\iota_0^\bullet \otimes_\mathbb{Z} \mathbb{K}\langle 1\rangle^\bullet) = \iota_0^\bullet \circ p^\bullet \quad \text{and} \quad t^\bullet \circ (\iota_1^\bullet \otimes_\mathbb{Z} \mathbb{K}\langle 1\rangle^\bullet) = \mathbf{1}_{\mathbb{K}\langle 1\rangle}.$$

To this aim, let e_0, e_1 be the canonical basis of $\mathbb{K}\langle 1\rangle^0$, so that $d^{-1}(1) = e_0 - e_1$. Then :

$$\mathbb{K}\langle 1,1\rangle^k \text{ is spanned by } \begin{cases} 1 \otimes 1 & \text{for } k = -2 \\ (e \otimes e_i, e_i \otimes e \mid i = 0,1) & \text{for } k = -1 \\ (e_i \otimes e_j \mid i,j = 0,1) & \text{for } k = 0 \end{cases}$$

with differentials given by the rules :

$$\begin{aligned} d^{-2}(1 \otimes 1) &:= (e_0 - e_1) \otimes 1 - 1 \otimes (e_0 - e_1) \\ d^{-1}(1 \otimes e_i) &:= (e_0 - e_1) \otimes e_i \\ d^{-1}(e_i \otimes 1) &:= e_i \otimes (e_0 - e_1). \end{aligned}$$

We define a morphism $t^\bullet$ as sought, by the rules :

$$\begin{aligned} t^{-2}(1 \otimes 1) &:= 0 \\ t^{-1}(1 \otimes e_0) = t^{-1}(e_0 \otimes 1) &:= 0 \\ t^{-1}(e_1 \otimes 1) = t^{-1}(1 \otimes e_1) &:= 1 \\ t^0(e_i \otimes e_0) &:= e_0 \quad (i = 0,1) \\ t^0(e_i \otimes e_1) &:= e_i \quad (i = 0,1). \end{aligned}$$

A direct inspection shows that $t^\bullet$ satisfies the required identities, and concludes the proof. □

7.1.23. In the situation of (7.1.14), let $A \in \mathrm{Ob}(\mathscr{A})$, $A' \in \mathrm{Ob}(\mathscr{A}')$ and let P be any free abelian group of finite rank. With the notation of (3.7.47), it is easily seen that there exist natural isomorphisms in $\mathscr{A}''$

$$B(P \otimes_\mathbb{Z} A, A') \xrightarrow{\sim} P \otimes_\mathbb{Z} B(A, A') \xrightarrow{\sim} B(A, P \otimes_\mathbb{Z} A'). \tag{7.1.24}$$

Now, let "?" be either $+$ or $-$, and $K^\bullet$ (resp. $L^\bullet$) any object of $\mathsf{C}^?(\mathscr{A})$ (resp. of $\mathsf{C}^?(\mathscr{A}')$); let also $P^\bullet$ be any bounded complex of free abelian groups of finite rank. The maps (7.1.24) assemble to a natural isomorphism of triple complexes

$$\omega^{ijk} : B(P^i \otimes_\mathbb{Z} K^j, L^k) \xrightarrow{\sim} P^i \otimes_\mathbb{Z} B(K^j, L^k) \qquad i,j,k \in \mathbb{Z}$$

whence, taking into account the discussion of (7.1.12), an isomorphism of total complexes

$$\omega^\bullet_{K,L,P} : B^\bullet_?(P^\bullet \otimes_\mathbb{Z} K^\bullet, L^\bullet) \xrightarrow{\sim} P^\bullet \otimes_\mathbb{Z} B^\bullet_?(K^\bullet, L^\bullet)$$

(notation of (7.1.18)). Likewise, we get a natural isomorphism

$$\sigma^\bullet_{P,K,L} : B^\bullet_?(K^\bullet, P^\bullet \otimes_\mathbb{Z} L^\bullet) \xrightarrow{\sim} P^\bullet \otimes_\mathbb{Z} B^\bullet_?(K^\bullet, L^\bullet).$$

(Notice that $\sigma^\bullet_{P,K,L}$ involves composition with the flip operator $\mathsf{C}(\mathsf{fl}_{\mathscr{A}}) : \mathsf{C}_3(\mathscr{A}) \stackrel{\sim}{\to} \mathsf{C}_3(\mathscr{A})$, hence, in each degree $p \in \mathbb{Z}$, it is the direct sum of morphisms $(-1)^{ij}\sigma^{ijk}$, for all $i, j, k \in \mathbb{Z}$ such that $i + j + k = p$, and where σ^{ijk} is a natural isomorphism as in (7.1.24)).

Especially, take $P^\bullet := \mathbb{K}\langle 1\rangle^\bullet$; it is easily seen that $\omega^\bullet_{K,L,\mathbb{K}\langle 1\rangle}$ identifies the morphism

$$B^\bullet_?(\iota^\bullet_i \otimes_{\mathbb{Z}} \mathbf{1}_K, L^\bullet) : B^\bullet_?(K^\bullet, L^\bullet) \to B^\bullet_?(\mathbb{K}\langle 1\rangle^\bullet \otimes_{\mathbb{Z}} K^\bullet, L^\bullet)$$

with the morphism

$$\iota^\bullet_i \otimes_{\mathbb{Z}} B^\bullet_?(K^\bullet, L^\bullet) : B^\bullet_?(K^\bullet, L^\bullet) \to \mathbb{K}\langle 1\rangle^\bullet \otimes_{\mathbb{Z}} B^\bullet_?(K^\bullet, L^\bullet).$$

Hence, suppose that $\varphi^\bullet_0, \varphi^\bullet_1 : L^\bullet \to M^\bullet$ are two homotopy equivalent morphisms in $\mathsf{C}^?(\mathscr{A}')$; taking into account remark 7.1.19(ii), we deduce that the induced morphisms $B^\bullet_?(K^\bullet, \varphi^\bullet_i)$ (for $i = 0, 1$) are also homotopy equivalent. A similar argument shows that the functor

$$\mathsf{C}^?(\mathscr{A}) \to \mathsf{C}^?(\mathscr{A}'') \quad : \quad K^\bullet \mapsto B^\bullet_?(K^\bullet, L^\bullet)$$

likewise preserves homotopy equivalences. We conclude that the functor $B^\bullet_?$ descends to the homotopy category, to give a functor

$$B^\bullet_? : \mathsf{Hot}^?(\mathscr{A}) \times \mathsf{Hot}^?(\mathscr{A}') \to \mathsf{Hot}^?(\mathscr{A}'')$$

which, in case $\mathscr{A}''$ is cocomplete (resp. complete), can even be extended to a functor

$$B^\bullet_\oplus : \mathsf{Hot}(\mathscr{A}) \times \mathsf{Hot}(\mathscr{A}') \to \mathsf{Hot}(\mathscr{A}'') \qquad (\text{resp. } B^\bullet_\Pi : \mathsf{Hot}(\mathscr{A}) \times \mathsf{Hot}(\mathscr{A}') \to \mathsf{Hot}(\mathscr{A}'')).$$

Example 7.1.25. (i) Consider any abelian tensor category $(\mathscr{A}, \otimes, \Phi, \Psi)$. Following (7.1.23), the tensor product of example 7.1.16(i) descends to a biadditive functor

$$\otimes : \mathsf{Hot}^?(\mathscr{A}) \times \mathsf{Hot}^?(\mathscr{A}) \to \mathsf{Hot}^?(\mathscr{A})$$

for "?" equal to either $+$ or $-$. If $\mathscr{A}$ is cocomplete, this functor is even defined on the whole of $\mathsf{Hot}(\mathscr{A})$.

(ii) The constructions of (7.1.23) can also be applied to the functor $\mathrm{Hom}^\bullet_{\mathscr{A}}$ of example 7.1.15(i) : taking into account remark 7.1.6(iii), we deduce that the functor $\mathrm{Hom}^\bullet_{\mathscr{A}}$ descends to a biadditive functor :

$$\mathrm{Hot}^\bullet_{\mathscr{A}} : \mathsf{Hot}(\mathscr{A}) \times \mathsf{Hot}(\mathscr{A})^o \to \mathsf{Hot}(\mathbb{Z}\text{-}\mathbf{Mod}) \quad : \quad (L^\bullet, K^\bullet) \mapsto \mathrm{Hom}^\bullet_{\mathscr{A}}(K^\bullet, L^\bullet).$$

7.2. **Filtered complexes and spectral sequences.** Let $\mathscr{A}$ be any abelian category; a *filtered object* of $\mathscr{A}$ is a datum

$$(A, \mathrm{Fil}^\bullet A)$$

consisting of an object A of $\mathscr{A}$ and a system of subobjects $(\mathrm{Fil}^p A \mid p \in \mathbb{Z})$ of A, with

$$\mathrm{Fil}^{p+1} A \subset \mathrm{Fil}^p A \qquad \text{for every } p \in \mathbb{Z}.$$

If $(B, \mathrm{Fil}^\bullet B)$ is any other filtered object of $\mathscr{A}$, a morphism of filtered objects

$$(A, \mathrm{Fil}^\bullet A) \to (B, \mathrm{Fil}^\bullet B) \tag{7.2.1}$$

is just a morphism $u : A \to B$ in $\mathscr{A}$ which restricts to a morphism

$$\mathrm{Fil}^p u : \mathrm{Fil}^p A \to \mathrm{Fil}^p B \qquad \text{for every } p \in \mathbb{Z}.$$

With these morphisms, clearly the filtered objects of $\mathscr{A}$ form a category

$$\mathsf{Fil}.\mathscr{A}.$$

Moreover, any functor $F : \mathscr{A} \to \mathscr{B}$ induces a functor

$$\mathsf{Fil}.F : \mathsf{Fil}.\mathscr{A} \to \mathsf{Fil}.\mathscr{B}$$

that assigns to any filtered object $(A, \mathrm{Fil}^\bullet A)$ of $\mathscr{A}$ the filtered object $(FA, \mathrm{Fil}^\bullet FA)$ such that $\mathrm{Fil}^k FA := \mathrm{Im}(F(\mathrm{Fil}^k A) \to FA)$ for every $k \in \mathbb{Z}$.

7.2.2. To any object $(A, \mathrm{Fil}^\bullet A)$ of $\mathsf{Fil}.\mathscr{A}$ we attach its *associated graded object*, which is the system of objects of $\mathscr{A}$:

$$(\mathrm{gr}^n A \mid n \in \mathbb{Z}) \qquad \text{where} \qquad \mathrm{gr}^n A := \mathrm{Fil}^n A/\mathrm{Fil}^{n+1} A \quad \text{for every } n \in \mathbb{Z}.$$

Notice that any morphism $u(A, \mathrm{Fil}^\bullet) \to (B, \mathrm{Fil}^\bullet)$ of filtered objects of $\mathscr{A}$ induces a system of morphisms in $\mathscr{A}$:

$$\mathrm{gr}^p u : \mathrm{gr}^p A \to \mathrm{gr}^p B \qquad \text{for every } p \in \mathbb{Z}.$$

We say that the filtration $\mathrm{Fil}^\bullet A$ of a filtered object $(A, \mathrm{Fil}^\bullet A)$ is *bounded above* (resp. *bounded below*) if there exists $N \in \mathbb{Z}$ such that $\mathrm{Fil}^N A = A$ (resp. such that $\mathrm{Fil}^N A = 0$). We say that the filtration $\mathrm{Fil}^\bullet$ is *finite* if it is bounded above and below. Clearly, if $\mathrm{Fil}^\bullet A$ is bounded above (resp. below), then $\mathrm{gr}^n A = 0$ whenever $-n$ (resp. n) is sufficiently large.

Definition 7.2.3. Let $\mathscr{A}$ be an abelian category, and $r_0 \in \mathbb{N}$ any integer.

(i) A *(cohomological)* r_0*-spectral sequence* in $\mathscr{A}$ is a datum

$$((E_r^{pq}, d_r^{pq}, \beta_r^{pq}) \mid p, q, r \in \mathbb{Z},\ r \geq r_0)$$

consisting of :

- objects E_r^{pq} of $\mathscr{A}$ and morphisms $d_r^{pq} : E_r^{pq} \to E_r^{p+r,q-r+1}$ in $\mathscr{A}$, such that :

(7.2.4) $$d_r^{p+r,q-r+1} \circ d_r^{pq} = 0 \qquad \text{for every } p, q, r \in \mathbb{Z} \text{ with } r \geq r_0.$$

- isomorphisms

$$\beta_r^{pq} : \mathrm{Ker}\, d_r^{pq}/\mathrm{Im}\, d_r^{p-r,q+r-1} \xrightarrow{\sim} E_{r+1}^{pq} \qquad \text{for every } p, q, r \in \mathbb{Z} \text{ with } r \geq r_0.$$

(ii) Let $(E_\bullet^{\bullet\bullet}, d_{E,\bullet}^{\bullet\bullet}, \beta_{E,\bullet}^{pq})$ and $(F_\bullet^{\bullet\bullet}, , d_{F,\bullet}^{\bullet\bullet}, \beta_{F,\bullet}^{pq})$ be two r_0-spectral sequences in $\mathscr{A}$. A *morphism of* r_0*-spectral sequences*

$$u_\bullet^{\bullet\bullet} : E_\bullet^{\bullet\bullet} \to F_\bullet^{\bullet\bullet}$$

is a system $(u_r^{pq} : E_r^{pq} \to F_r^{pq} \mid p, q, r \in \mathbb{Z},\ r \geq r_0)$ of morphisms of $\mathscr{A}$, such that the diagrams

$$\begin{array}{ccc} E_r^{pq} & \xrightarrow{d_{E,r}^{pq}} & E^{p+r,q-r+1} \\ {\scriptstyle u_r^{pq}}\downarrow & & \downarrow{\scriptstyle u_r^{p+r,q-r+1}} \\ F_r^{pq} & \xrightarrow{d_{F,r}^{pq}} & F^{p+r,q-r+1} \end{array} \qquad \begin{array}{ccccccc} E_r^{pq} & \longleftarrow & \mathrm{Ker}\, d_{E,r}^{pq} & \longrightarrow & \mathrm{Ker}\, d_{E,r}^{pq}/\mathrm{Im}\, d_{E,r}^{p-r,q+r-1} & \xrightarrow{\beta_{E,r}^{pq}} & E_{r+1}^{pq} \\ {\scriptstyle u_r^{pq}}\downarrow & & & & & & \downarrow{\scriptstyle u_{r+1}^{pq}} \\ F_r^{pq} & \longleftarrow & \mathrm{Ker}\, d_{F,r}^{pq} & \longrightarrow & \mathrm{Ker}\, d_{E,r}^{pq}/\mathrm{Im}\, d_{E,r}^{p-r,q+r-1} & \xrightarrow{\beta_{F,r}^{pq}} & F_{r+1}^{pq} \end{array}$$

commute for every $p, q, r \in \mathbb{Z}$ with $r \geq r_0$ (where the unmarked arrows are the natural monomorphisms and epimorphisms). Obviously, the r_0-spectral sequences of $\mathscr{A}$ and their morphisms form a category, which we denote by

$$\mathsf{Sp.Seq}_{r_0}(\mathscr{A})$$

7.2.5. Let $r \in \mathbb{N}$ be any integer. Clearly, for every integer $s \geq r$ there is a natural functor

$$\mathsf{Sp.Seq}_r(\mathscr{A}) \to \mathsf{Sp.Seq}_s(\mathscr{A})$$

that simply forgets the terms E_k^{pq} with $r \leq k < s$. One may also define a natural functor form s-spectral sequences to r-spectral sequences. Namely, let $E_\bullet^{\bullet\bullet}$ be any $(r+1)$-spectral sequence of $\mathscr{A}$. The *décalage* of $E^{\bullet\bullet}$ is the r-spectral sequence $(\mathrm{Déc}\,(E)^{\bullet\bullet}, d_{\mathrm{Déc}\,(E)}^{\bullet\bullet})$ given by the rule

$$\mathrm{Déc}\,(E)_s^{p,q-p} := E_{s+1}^{p+q,-p} \qquad d_{\mathrm{Déc}\,(E),s}^{p,q-p} := d_{E,s+1}^{p+q,-p} \qquad \text{for every } p, q, s \in \mathbb{Z} \text{ with } s \geq r$$

and whose isomorphisms $\beta_{\mathrm{Déc}\,(E)}^{\bullet\bullet}$ are deduced from the corresponding isomorphisms for $E^{\bullet\bullet}$, in the obvious way. Clearly, this rule extends to a natural functor

$$\mathrm{Déc}\ : \mathsf{Sp.Seq}_{r+1}(\mathscr{A}) \to \mathsf{Sp.Seq}_r(\mathscr{A}).$$

7.2.6. Let $E_\bullet^{\bullet\bullet}$ be any r_0-spectral sequence; with the notation of definition 7.2.3 we define, by induction on $k-r$, two systems of subobjects

$$(Z_k(E_r^{pq}), B_k(E_r^{pq}) \subset E_r^{pq} \mid p,q,r,k \in \mathbb{Z},\ k > r \geq r_0)$$

as follows. First, we set

$$Z_{r+1}(E_r^{pq}) := \mathrm{Ker}\, d_r^{pq} \qquad B_{r+1}(E_r^{pq}) := \mathrm{Im}\, d_r^{p-r,q+r-1} \qquad \text{for every } p,q,r \in \mathbb{Z} \text{ with } r \geq r_0.$$

Next, say that $i \geq 2$ is any integer, and suppose that both $Z_k(E_r^{pq})$ and $B_k(E_r^{pq})$ have already been defined, for every $p,q,r,k \in \mathbb{Z}$ with $k > r \geq r_0$ and $k - r < i$; we let

$$Z_{r+i}(E_r^{pq}) := \pi_{pqr}^{-1} Z_{r+i}(E_{r+1}^{pq}) \qquad B_{r+i}(E_r^{pq}) := \pi_{pqr}^{-1} B_{r+i}(E_{r+1}^{pq})$$

where π_{pqr} is the composition

$$Z_{r+1}(E_r^{pq}) \to Z_{r+1}(E_r^{pq})/B_{r+1}(E_r^{pq}) \xrightarrow{\beta_r^{pq}} E_{r+1}^{pq}.$$

Then it is clear that the system $(\beta_r^{pq} \mid p,q,r \in \mathbb{Z},\ r \geq r_0)$ induces isomorphisms

$$\beta_{rk}^{pq} : Z_k(E_r^{pq})/B_k(E_r^{pq}) \xrightarrow{\sim} E_k^{pq} \qquad \text{for every } p,q,r,k \in \mathbb{Z} \text{ with } k > r \geq r_0$$

and for every $p,q,r,k,n \in \mathbb{Z}$ with $k,n > r \geq r_0$ we have inclusions

$$B_k(E_r^{pq}) \subset B_{k+1}(E_r^{pq}) \subset Z_{n+1}(E_r^{pq}) \subset Z_n(E_r^{pq}) \subset E_r^{pq}.$$

Definition 7.2.7. In the situation of (7.2.6), we define :

(i) an *abutment* for the spectral sequence $E_\bullet^{\bullet\bullet}$ to be a datum consisting of

- a system of subobjects $(B_\infty(E_{r_0}^{pq}), Z_\infty(E_{r_0}^{pq}) \subset E_{r_0}^{pq} \mid p,q \in \mathbb{Z})$ such that

$$B_k(E_{r_0}^{pq}) \subset B_\infty(E_{r_0}^{pq}) \subset Z_\infty(E_{r_0}^{pq}) \subset Z_k(E_{r_0}^{pq}) \qquad \text{for every } p,q,k \in \mathbb{Z} \text{ with } k > r_0$$

- a system of filtered objects $((E_\infty^n, \mathrm{Fil}^\bullet E_\infty^n) \mid n \in \mathbb{Z})$ of $\mathscr{A}$
- a system of isomorphisms in $\mathscr{A}$

$$\beta_\infty^{pq} : E_\infty^{pq} := Z_\infty(E_{r_0}^{pq})/B_\infty(E_{r_0}^{pq}) \xrightarrow{\sim} \mathrm{gr}^p E_\infty^{p+q} \qquad \text{for every } p,q \in \mathbb{Z}.$$

We summarize these conditions via the traditional notation :

$$E_r^{pq} \Rightarrow E_\infty^{p+q}.$$

For any $n \in \mathbb{Z}$, we say that $E_\bullet^{\bullet\bullet}$ is *convergent in degree* n, if for every $p,q \in \mathbb{Z}$ with $p+q=n$ there exists $k \in \mathbb{N}$ such that $B_k(E_{r_0}^{pq}) = B_\infty(E_{r_0}^{pq})$ and $Z_k(E_{r_0}^{pq}) = Z_\infty(E_{r_0}^{pq})$. We say that $E_\bullet^{\bullet\bullet}$ is *convergent*, if it is convergent in all degrees.

(iii) Let $(E_\bullet^{\bullet\bullet}, B_\infty(E_{r_0}^{\bullet\bullet}), Z_\infty(E_{r_0}^{\bullet\bullet}), E_\infty^\bullet, \beta_{E,\infty}^{pq})$ and $(F_\bullet^{\bullet\bullet}, B_\infty(F_{r_0}^{\bullet\bullet}), Z_\infty(F_{r_0}^{\bullet\bullet}), F_\infty^\bullet, \beta_{F,\infty}^{pq})$ be two r_0-spectral sequences in $\mathscr{A}$ with abutments. A *morphism of r_0-spectral sequences with abutments* is a pair

$$u_\bullet^{\bullet\bullet} : E_\bullet^{\bullet\bullet} \to F_\bullet^{\bullet\bullet} \qquad (v^n : (E_\infty^n, \mathrm{Fil}^\bullet E_\infty^n) \to (F_\infty^n, \mathrm{Fil}^\bullet F_\infty^n) \mid n \in \mathbb{Z})$$

such that :

- $u_\bullet^{\bullet\bullet}$ is a morphism of r_0-spectral sequences.
- v^n is a morphism of filtered objects of $\mathscr{A}$, for every $n \in \mathbb{Z}$.
- $u_{r_0}^{pq}(B_\infty(E_{r_0}^{pq})) \subset B_\infty(F_{r_0}^{pq})$ and $u_{r_0}^{pq}(Z_\infty(E_{r_0}^{pq})) \subset Z_\infty(F_{r_0}^{pq})$ for every $p,q \in \mathbb{Z}$.
- The following diagrams commute, for every $p,q \in \mathbb{Z}$:

$$\begin{array}{ccccccc}
E_{r_0}^{pq} & \longleftarrow & Z_\infty(E_{r_0}^{pq}) & \longrightarrow & E_\infty^{pq} & \xrightarrow{\beta_{E,\infty}^{pq}} & \mathrm{gr}^p E_\infty^{p+q} \\
\downarrow{\scriptstyle u_{r_0}^{pq}} & & & & & & \downarrow{\scriptstyle \mathrm{gr}^p v^{p+q}} \\
F_{r_0}^{pq} & \longleftarrow & Z_\infty(F_{r_0}^{pq}) & \longrightarrow & E_\infty^{pq} & \xrightarrow{\beta_{F,\infty}^{pq}} & \mathrm{gr}^p F_\infty^{p+q}
\end{array}$$

(where the unmarked arrows are the natural epimorphisms and monomorphisms).

7.2.8. For every $r \in \mathbb{N}$, we denote by

$$\mathsf{Sp.Seq}_r^\infty(\mathscr{A})$$

the category of r-spectral sequences with abutments (and their morphisms, as in definition 7.2.7(ii)). For every integer $s > r$ there is an obvious functor

$$\mathsf{Sp.Seq}_r^\infty(\mathscr{A}) \to \mathsf{Sp.Seq}_s^\infty(\mathscr{A})$$

that simply forgets the terms E_k^{pq} with $r \leq k < s$, and replaces the terms $B_\infty(E_r^{pq}), Z_\infty(E_r^{pq})$ with their images $\beta_{rs}^{pq}(B_\infty(E_r^{pq})), \beta_{rs}^{pq}(Z_\infty(E_r^{pq})$ in E_s^{pq}.

Remark 7.2.9. Let $r_0 \in \mathbb{N}$, $n \in \mathbb{Z}$ and $E_\bullet^{\bullet\bullet}$ any r_0-spectral sequence of $\mathscr{A}$.

(i) Notice that if $E_r^{pq} = 0$ for some $p, q \in \mathbb{Z}$ and $r \geq r_0$, then $E_s^{pq} = 0$ for every $s \geq r$.

(ii) We say $E_\bullet^{\bullet\bullet}$ is *bounded above* (resp. *bounded below*) in degree n, if there exist $r, N \in \mathbb{N}$ with $r \geq r_0$, such that

$$E_r^{pq} = 0 \qquad \text{for all } p, q \in \mathbb{Z} \text{ such that } p + q = n \text{ and } q > N \text{ (resp. } p > N).$$

If $E_\bullet^{\bullet\bullet}$ is bounded above in degree $n+1$ and bounded below in degree $n-1$, notice that for every $p, q \in \mathbb{Z}$ with $p + q = n$ there exists an integer $r \geq r_0$ such that both d_s^{pq} and $d_s^{p-s,q+s-1}$ are zero morphisms for every $s \geq r$, in which case β_s^{pq} is an isomorphism $E_{s+1}^{pq} \stackrel{\sim}{\to} E_s^{pq}$, for every $s \geq r$, and for every $p, q, s, k \in \mathbb{Z}$ with $k > s \geq r$ and $p + q = n$ we have :

$$Z_k(E_s^{pq}) = Z_{s+1}(E_s^{pq}) \qquad B_k(E_s^{pq}) = B_{s+1}(E_s^{pq}).$$

Definition 7.2.10. Let $\mathscr{A}$ be any abelian category, and $n \in \mathbb{Z}$.

(i) A *filtered complex* $(K^\bullet, \mathrm{Fil}^\bullet K^\bullet)$ of $\mathscr{A}$ is just an object of the category $\mathsf{Fil.C}(\mathscr{A})$.

(ii) We say that $\mathrm{Fil}^\bullet K^\bullet$ is *bounded above* (resp. *bounded below*, resp. *finite*) in degree n, if the filtration $\mathrm{Fil}^\bullet K^n$ on the object K^n of $\mathscr{A}$ is bounded above (resp. bounded below, resp. finite). We say that $\mathrm{Fil}^\bullet K^\bullet$ is *bounded*, if it is finite in all degrees (see (7.2.2)).

Remark 7.2.11. Likewise, we may define a *filtered double complex* as an object of the category $\mathsf{Fil.C}_2(\mathscr{A})$. Then, the flip and diagonal functors of (7.1.11) can be upgraded to functors

$$\mathsf{Fil.C}_2(\mathscr{A}) \to \mathsf{Fil.C}_2(\mathscr{A}) \qquad \mathsf{Fil.C}_2(\mathscr{A}) \to \mathsf{Fil.C}(\mathscr{A}).$$

If all coproducts (resp. all products) are representable in $\mathscr{A}$, the total complex functor $\mathrm{Tot}^\oplus$ (resp. Tot^Π) induces a corresponding functor

$$\mathsf{Fil.Tot}^\oplus : \mathsf{Fil.C}_2(\mathscr{A}) \to \mathsf{Fil.C}(\mathscr{A}) \qquad (\text{resp. } \mathsf{Fil.Tot}^\Pi : \mathsf{Fil.C}_2(\mathscr{A}) \to \mathsf{Fil.C}(\mathscr{A}))$$

(see (7.2)) which we shall often denote just by Tot (resp. by Tot^Π).

7.2.12. Any filtered complex of $\mathscr{A}$ determines a spectral sequence of $\mathscr{A}$, whose terms are defined as follows. For every $p, q \in \mathbb{Z}$ and $r \in \mathbb{N}$, set

$$\begin{aligned}
Z(K)_r^{pq} &:= \mathrm{Ker}\,(\overline{d} : (\mathrm{Fil}^p K^\bullet)^{p+q} \to (\mathrm{Fil}^p K^\bullet)^{p+q+1}/(\mathrm{Fil}^{p+r} K^\bullet)^{p+q+1}) \\
D(K)_r^{pq} &:= (\mathrm{Fil}^{p+1} K^\bullet)^{p+q} + d_K^{p+q-1}((\mathrm{Fil}^{p-r+1} K^\bullet)^{p+q-1}) \subset (\mathrm{Fil}^{p-r+1} K^\bullet)^{p+q} \\
B(K)_r^{pq} &:= D(K)_r^{pq} \cap Z(K)_r^{pq} \\
E(K)_r^{pq} &:= Z(K)_r^{pq} / B(K)_r^{pq}
\end{aligned}$$

where $\overline{d}$ is the morphism in $\mathscr{A}$ induced by the differential

$$d_{\mathrm{Fil}^p K}^{p+q} : (\mathrm{Fil}^p K^\bullet)^{p+q} \to (\mathrm{Fil}^p K^\bullet)^{p+q+1}$$

of the complex $\mathrm{Fil}^p K^\bullet$. Notice that

$$(\mathrm{Fil}^{p+r} K^\bullet)^{p+q} \subset Z(K)_r^{pq} \subset (\mathrm{Fil}^p K^\bullet)^{p+q} \qquad \text{for every } p, q \in \mathbb{Z} \text{ and } r \in \mathbb{N}.$$

Lemma 7.2.13. *With the notation of* (7.2.12)*, we have :*

$$B(K)^{pq}_r = Z(K)^{p+1,q-1}_r + d^{p+q-1}_K(Z(K)^{p-r+1,q+r-2}_{r-1}) \qquad \textit{for every } p,q \in \mathbb{Z} \textit{ and } r \geq 1.$$

Proof. Notice first that

$$D(K)^{pq}_r \cap (\mathrm{Fil}^p K^\bullet)^{p+q} = (\mathrm{Fil}^{p+1} K^\bullet)^{p+q} + (\mathrm{Im}\, d^{p+q-1}_{\mathrm{Fil}^{p-r+1}K} \cap (\mathrm{Fil}^p K^\bullet)^{p+q})$$

and

$$\mathrm{Im}\, d^{p+q-1}_{\mathrm{Fil}^{p-r+1}K} \cap (\mathrm{Fil}^p K^\bullet)^{p+q} = d^{p+q-1}_K(Z(K)^{p-r+1,q+r-2}_{r-1}).$$

On the other hand, we have :

$$d^{p+q}_K(D(K)^{pq}_r) \cap (\mathrm{Fil}^{p+r} K^\bullet)^{p+q+1} = \mathrm{Im}\, d^{p+q}_{\mathrm{Fil}^{p+1}K} \cap (\mathrm{Fil}^{p+r} K^\bullet)^{p+q+1} = d^{p+q}_K(Z(K)^{p+1,q-1}_r).$$

The lemma follows easily, by comparing these identities. □

7.2.14. It is easily seen that $d^{p+q}_{\mathrm{Fil}^p K}$ restricts to a morphism

$$D(K)^{pq}_r + Z(K)^{pq}_r \to Z(K)^{p+r,q-r+1}_r \tag{7.2.15}$$

which sends $D(K)^{pq}_r$ into $B(K)^{p+r,q-r+1}_r$, and therefore induces a morphism

$$d(K)^{pq}_r : E(K)^{pq}_r \to E(K)^{p+r,q-r+1}_r \qquad \text{for every } p,q \in \mathbb{Z} \text{ and } r \in \mathbb{N}.$$

Clearly $d(K)^{p+r,q-r+1}_r \circ d(K)^{pq}_r = 0$. Furthermore, a simple inspection shows that

$$\mathrm{Ker}\, d(K)^{pq}_r = Z(K)^{pq}_{r+1}/(Z(K)^{pq}_{r+1} \cap D(K)^{pq}_r) \qquad \mathrm{Im}\, d(K)^{p-r,q+r-1}_r = D(K)^{pq}_{r+1}/D(K)^{pq}_r$$

whence a natural isomorphism

$$E(K)^{pq}_{r+1} \xrightarrow{\sim} \mathrm{Ker}\, d(K)^{pq}_r/\mathrm{Im}\, d(K)^{p-r,q+r-1}_r \qquad \text{for every } p,q \in \mathbb{Z} \text{ and } r \in \mathbb{N}$$

so the system $(E(K)^{\bullet\bullet}_\bullet, d(K)^{\bullet\bullet}_\bullet)$ is indeed a spectral sequence.

Remark 7.2.16. (i) A simple inspection shows that

$$E(K)^{pq}_0 = \mathrm{gr}^p K^{p+q} \qquad \text{and} \qquad d(K)^{pq}_0 = d^{p+q}_{\mathrm{gr}^p K} \qquad \text{for every } p,q \in \mathbb{Z}$$

so the resulting complex $(E(K)^{p,\bullet}_0, d(K)^{p,\bullet})$ is just $\mathrm{gr}^p K^\bullet[q]$, for every $p,q \in \mathbb{Z}$.

(ii) In view of (i), we have a natural isomorphism

$$E(K)^{pq}_1 \xrightarrow{\sim} H^{p+q}(\mathrm{gr}^p K^\bullet) \qquad \text{for every } p,q \in \mathbb{Z}$$

which identifies $d(K)^{pq}_1$ with a morphism

$$H^{p+q}(\mathrm{gr}^p K^\bullet) \to H^{p+q+1}(\mathrm{gr}^{p+1} K^\bullet).$$

A direct inspection shows that the latter is the boundary morphism in degree $p+q$ arising (by the snake lemma) from the short exact of complexes

$$0 \to \mathrm{gr}^{p+1} K^\bullet \to \mathrm{Fil}^p K^\bullet/\mathrm{Fil}^{p+2} K^\bullet \to \mathrm{gr}^p K^\bullet \to 0.$$

7.2.17. With the notation of (7.2.6) and (7.2.12), it is easily seen that

$$Z_k(E(K)^{pq}_r) = \mathrm{Im}\,(Z(K)^{pq}_k \to E(K)^{pq}_r)$$
$$B_k(E(K)^{pq}_r) = \mathrm{Im}\,(B(K)^{pq}_k \to E(K)^{pq}_r)$$

for every $p,q,r,k \in \mathbb{Z}$ with $k > r \geq 0$. We may then define a natural abutment for $E(K)^{\bullet\bullet}_\bullet$, as follows. For every $p,q \in \mathbb{Z}$ we set

$$Z_\infty(E(K)^{pq}_0) := \mathrm{Im}(\mathrm{Ker}\, d^{p+q}_{\mathrm{Fil}^p K} \to E(K)^{pq}_0)$$
$$B_\infty(E(K)^{pq}_0) := \mathrm{Im}((\mathrm{Fil}^p K^{p+q} \cap \mathrm{Im}\, d^{p+q-1}_K) \to E(K)^{pq}_0)$$

as well as

$$E(K)^n_\infty := H^n K^\bullet \qquad \mathrm{Fil}^p E(K)^n_\infty := \mathrm{Im}(H^n(\mathrm{Fil}^p K^\bullet) \to H^n(K^\bullet)) \qquad \text{for every } p,n \in \mathbb{Z}.$$

By inspecting the definitions we find natural isomorphisms

$$\beta(K)^{pq}_\infty : E(K)^{pq}_\infty := Z_\infty(E(K)^{pq}_0)/B_\infty(E(K)^{pq}_0) \xrightarrow{\sim} \mathrm{gr}^p E(K)^{p+q}_\infty$$

and the datum $A(K)^{\bullet\bullet}_\infty := (B_\infty(E(K)^{pq}_0), Z_\infty(E(K)^{pq}_0), \beta(K)^{pq}_\infty)$ is the sought abutment. Summing up, we get a well defined functor

$$\mathsf{Fil.C}(\mathscr{A}) \to \mathsf{Sp.Seq}^\infty_0(\mathscr{A}) \qquad (K^\bullet, \mathrm{Fil}^\bullet K^\bullet) \mapsto (E(K)^{\bullet\bullet}_\bullet, A(K)^{\bullet\bullet}_\infty).$$

Clearly, if $\mathrm{Fil}^\bullet K^\bullet$ is bounded in degree n, then $\mathrm{Fil}^\bullet E(K)^n_\infty$ is a finite filtration.

Proposition 7.2.18. (i) *Let $(K^\bullet, \mathrm{Fil}^\bullet K^\bullet)$ be a filtered complex in $\mathscr{A}$, and $N \in \mathbb{N}$ such that :*

$$H^{n-1}(K^\bullet/\mathrm{Fil}^{-p}K^\bullet) = 0 \qquad \textit{and} \qquad H^{n+1}(\mathrm{Fil}^p K^\bullet) = 0 \qquad \textit{for every } p \geq N.$$

Then the 1-spectral sequence with abutment $(E(K)^{\bullet\bullet}_\bullet, A(K)^{\bullet\bullet}_\infty)$ is convergent in degree n.

(ii) *Especially, the conclusion of* (i) *holds if $\mathrm{Fil}^\bullet K^\bullet$ is bounded above in degree $n-1$ and bounded below in degree $n+1$.*

Proof. Obviously (i)⇒(ii). Hence, let us assume that the condition of (i) holds, and consider, for every $r \in \mathbb{N}$ and every $p, q \in \mathbb{Z}$ with $p + q = n$, the composition

$$H^n(\mathrm{gr}^p K^\bullet) \xrightarrow{\delta} H^{n+1}(\mathrm{Fil}^{p+1}K^\bullet) \xrightarrow{H^{n+1}(\pi_r)} H^{n+1}(\mathrm{Fil}^{p+1}K^\bullet/\mathrm{Fil}^{p+r}K^\bullet)$$

where δ is the boundary morphism attached by the snake lemma to the short exact sequence of complexes $0 \to \mathrm{Fil}^{p+1}K^\bullet \to \mathrm{Fil}^p K^\bullet \to \mathrm{gr}^p K^\bullet \to 0$, and $\pi_r : \mathrm{Fil}^{p+1}K^\bullet \to \mathrm{Fil}^{p+1}K^\bullet/\mathrm{Fil}^{p+r}K^\bullet$ is the projection. By direct inspection, we see that :

$$\mathrm{Ker}\,(H^{n+1}(\pi_r) \circ \delta) = Z_r(E^{pq}_1) \qquad \text{and} \qquad \mathrm{Ker}\,\delta = Z_\infty(E^{pq}_1).$$

On the other hand, the long exact cohomology sequence attached to the short exact sequence of complexes $0 \to \mathrm{Fil}^{p+r}K^\bullet \to \mathrm{Fil}^{p+1}K^\bullet \to \mathrm{Fil}^{p+1}K^\bullet/\mathrm{Fil}^{p+r}K^\bullet \to 0$ shows that $\mathrm{Ker}\,H^{n+1}(\pi_r)$ is a quotient of $H^{n+1}(\mathrm{Fil}^{p+r}K^\bullet)$. Thus, $H^{n+1}(\pi_r)$ is a monomorphism for every $r \in \mathbb{N}$ such that $p + r \geq N$, and in that case we get $Z_r(E^{pq}_1) = Z_\infty(E^{pq}_1)$. Likewise, consider the composition :

$$H^{n-1}(\mathrm{Fil}^{p-r+1}K^\bullet/\mathrm{Fil}^p K^\bullet) \xrightarrow{H^{n-1}(j_r)} H^{n-1}(K^\bullet/\mathrm{Fil}^p K^\bullet) \xrightarrow{\delta'} H^n(\mathrm{gr}^p K^\bullet)$$

where δ' is the boundary morphism attached by the snake lemma to the short exact sequence $0 \to \mathrm{Fil}^p K^\bullet \to K^\bullet \to K^\bullet/\mathrm{Fil}^p K^\bullet \to 0$, and $j_r : \mathrm{Fil}^{p-r+1}K^\bullet/\mathrm{Fil}^p K^\bullet \to K^\bullet/\mathrm{Fil}^p K^\bullet$ is the natural inclusion. By direct inspection we see that

$$\mathrm{Im}(\delta' \circ H^{n-1}(j_r)) = B_r(E^{pq}_1) \qquad \text{and} \qquad \mathrm{Im}\,\delta' = B_\infty(E^{pq}_1).$$

On the other hand, the long exact cohomology sequence attached to the short exact sequence $0 \to \mathrm{Fil}^{p-r+1}K^\bullet/\mathrm{Fil}^p K^\bullet \to K^\bullet/\mathrm{Fil}^p K^\bullet \to K^\bullet/\mathrm{Fil}^{p-r+1}K^\bullet \to 0$ shows that $\mathrm{Coker}\,H^{n-1}(j_r)$ is a subobject of $H^{n-1}(K^\bullet/\mathrm{Fil}^{p-r+1}K^\bullet)$. Especially, $H^{n-1}(j_r)$ is an epimorphism whenever $p - r + 1 \leq -N$, and in that case we get $B_r(E^{pq}_1) = B_\infty(E^{pq}_1)$. □

7.2.19. The constructions of examples 7.1.15(i), 7.1.16(i) admit filtered counterparts. Namely, suppose that $(\mathscr{A}, \otimes, \Phi, \Psi)$ is a tensor abelian category, and let $(A, \mathrm{Fil}^\bullet A)$ and $(B, \mathrm{Fil}^\bullet B)$ be any two objects of $\mathsf{Fil}.\mathscr{A}$. Suppose that :

- at least one of the filtrations $\mathrm{Fil}^\bullet A$ and $\mathrm{Fil}^\bullet B$ is bounded
- or else, $\mathscr{A}$ is cocomplete.

In this situation we define a filtration $\mathrm{Fil}^\bullet(A \otimes B)$ on $A \otimes B$, by ruling that

$$\mathrm{Fil}^p(A \otimes B) := \sum_{k \in \mathbb{Z}} \mathrm{Im}\,(\mathrm{Fil}^k A \otimes \mathrm{Fil}^{p-k} B \to A \otimes B) \qquad \text{for every } p \in \mathbb{Z}.$$

In case $\mathscr{A}$ is cocomplete, clearly, this rule yields a functor

$$\mathsf{Fil}.\mathscr{A} \times \mathsf{Fil}.\mathscr{A} \to \mathsf{Fil}.\mathscr{A}.$$

Next, suppose that $\mathscr{A}$ is cocomplete, and let $(K^\bullet, \mathrm{Fil}^\bullet K^\bullet)$ and $(L^\bullet, \mathrm{Fil}^\bullet L^\bullet)$ be any two filtered complexes of $\mathscr{A}$; we get an induced filtration $\mathrm{Fil}^\bullet K^\bullet \boxtimes L^\bullet$ on the double complex $K^\bullet \boxtimes L^\bullet$, and applying the functor Tot of remark 7.2.11 we obtain a filtration on $K^\bullet \otimes L^\bullet$, whence a functor

$$\mathsf{Fil.C}(\mathscr{A}) \times \mathsf{Fil.C}(\mathscr{A}) \to \mathsf{Fil.C}(\mathscr{A}) \qquad (\mathrm{Fil}^\bullet K, \mathrm{Fil}^\bullet L) \mapsto \mathrm{Fil}^\bullet K^\bullet \otimes L^\bullet.$$

Notice that if $\mathscr{A}$ is not cocomplete, the filtered double complex $\mathrm{Fil}^\bullet K^\bullet \boxtimes L^\bullet$ is still well defined, provided at least one of the filtrations $\mathrm{Fil}^\bullet K^\bullet$ and $\mathrm{Fil}^\bullet L^\bullet$ is bounded. If moreover both $K^\bullet$ and $L^\bullet$ lie in $\mathsf{C}^-(\mathscr{A})$ or $\mathsf{C}^+(\mathscr{A})$, then also $\mathrm{Fil}^\bullet K^\bullet \otimes L^\bullet$ is well defined.

7.2.20. Let $(A, \mathrm{Fil}^\bullet A)$ and $(B, \mathrm{Fil}^\bullet B)$ be any two objects of $\mathsf{Fil}.\mathscr{A}$. We define a filtration $\mathrm{Fil}^\bullet \mathrm{Hom}_{\mathscr{A}}(A, B)$ on the abelian group $\mathrm{Hom}_{\mathscr{A}}(A, B)$, by ruling that

$$\mathrm{Fil}^p \mathrm{Hom}_{\mathscr{A}}(A, B) := \{f : A \to B \mid f(\mathrm{Fil}^k A) \subset \mathrm{Fil}^{p+k} B \text{ for every } k \in \mathbb{Z}\}$$

for every $p \in \mathbb{Z}$. Clearly this rule yields a functor

$$\mathsf{Fil}.\mathscr{A} \times (\mathsf{Fil}.\mathscr{A})^o \to \mathsf{Fil}.\mathbb{Z}\text{-}\mathbf{Mod}. \tag{7.2.21}$$

Let now $(K^\bullet, \mathrm{Fil}^\bullet K^\bullet)$ and $(L^\bullet, \mathrm{Fil}^\bullet L^\bullet)$ be any two filtered complexes of $\mathscr{A}$; we define a filtration $\mathrm{Fil}^\bullet \mathrm{Hom}^{\bullet\bullet}_{\mathscr{A}}(K^\bullet, L^\bullet)$ on the double complex of abelian groups $\mathrm{Hom}^{\bullet\bullet}_{\mathscr{A}}(K^\bullet, L^\bullet)$ (see example 7.1.15(i)), by ruling that

$$\mathrm{Fil}^k \mathrm{Hom}^{p,q}_{\mathscr{A}}(K^\bullet, L^\bullet) := \mathrm{Fil}^k \mathrm{Hom}_{\mathscr{A}}(K^{-p}, L^q) \qquad \text{for every } p, q, k \in \mathbb{Z}$$

(where the right-hand side is the filtered abelian group obtained by applying the functor (7.2.21) to the filtered objects $(K^{-p}, \mathrm{Fil}^\bullet K^{-p})$ and $(L^q, \mathrm{Fil}^\bullet L^q)$ of $\mathscr{A}$). After applying the functor Tot^{Π} of remark 7.2.11, we deduce a filtration on $\mathrm{Hom}^\bullet_{\mathscr{A}}(K^\bullet, L^\bullet)$, whence a functor

$$\mathsf{Fil.C}(\mathscr{A}) \times \mathsf{Fil.C}(\mathscr{A})^o \to \mathsf{Fil.C}(\mathbb{Z}\text{-}\mathbf{Mod}) \qquad (\mathrm{Fil}^\bullet L^\bullet, \mathrm{Fil}^\bullet K^\bullet) \mapsto \mathrm{Fil}^\bullet \mathrm{Hom}^\bullet_{\mathscr{A}}(K^\bullet, L^\bullet).$$

Taking into account remark 7.1.15(i), we also see that

$$\mathrm{Hom}_{\mathsf{Fil.C}(\mathscr{A})}(K^\bullet, L^\bullet) = \mathrm{Fil}^0 \mathrm{Hom}^\bullet(K^\bullet, L^\bullet) \cap \ker d^0_{K,L}.$$

Definition 7.2.22. Let $\mathscr{A}$ be any abelian category, $k \in \mathbb{Z}$ any integer, $\varphi^\bullet, \psi^\bullet : (K^\bullet, \mathrm{Fil}^\bullet K^\bullet) \to (L^\bullet, \mathrm{Fil}^\bullet L^\bullet)$ any two morphisms of filtered complexes of $\mathscr{A}$.

(i) A *homotopy of order* k from $\varphi^\bullet$ to $\psi^\bullet$ is an element $s^\bullet \in \mathrm{Fil}^k \mathrm{Hom}^{-1}_{\mathscr{A}}(K^\bullet, L^\bullet)$ such that $d^{-1}_{K,L}(s^\bullet) = \psi^\bullet - \varphi^\bullet$ (notation of example 7.1.15(i)).

(ii) We write $\varphi^\bullet \sim_k \psi^\bullet$ if there exists a homotopy of order k from $\psi^\bullet$ to $\varphi^\bullet$. It is easily seen that if this is the case, and $\alpha^\bullet : K'^\bullet \to K^\bullet$, $\beta^\bullet : L^\bullet \to L'^\bullet$ are any two morphisms, then $\varphi^\bullet \circ \alpha^\bullet \sim_k \psi^\bullet \circ \alpha^\bullet$ and $\beta^\bullet \circ \varphi^\bullet \sim_k \beta^\bullet \circ \psi^\bullet$. Moreover, $\sim_k$ is an equivalence relation on $\mathrm{Hom}_{\mathsf{Fil.C}(\mathscr{A})}(K^\bullet, L^\bullet)$. It follows that there exists a well defined *filtered homotopy category of order* k

$$\mathsf{Fil.Hot}(\mathscr{A}, k)$$

whose objects are the same as those of $\mathsf{Fil.C}(\mathscr{A})$, and whose morphisms are the order k homotopy classes of morphisms of complexes. Furthermore, we have a natural functor

$$\mathsf{Fil.C}(\mathscr{A}) \to \mathsf{Fil.Hot}(\mathscr{A}, k)$$

which is the identity on objects, and the quotient map on Hom-sets.

7.2.23. With the notation of definition 7.2.22, suppose that $k \geq 0$, and consider any element $s^\bullet \in \mathrm{Fil}^k\mathrm{Hom}^\bullet_{\mathscr{A}}(K^\bullet, L^\bullet)$; then $\sigma := d^{-1}_{K,L}(s^\bullet) : (K^\bullet, \mathrm{Fil}^\bullet K^\bullet) \to (L^\bullet, \mathrm{Fil}^\bullet L^\bullet)$ is a morphism of filtered complexes, and a simple inspection shows that

$$\sigma^{p+q}(Z(K)^{pq}_r) \subset \mathrm{Ker}\, d^{p+q}_{\mathrm{Fil}^p L} \qquad \text{for every } p, q, r \in \mathbb{Z} \text{ with } r \geq k$$

so the induced map $E(\sigma)^{pq}_r : E(K)^{pq}_r \to E(L)^{pq}_r$ vanishes for every $p, q, r \in \mathbb{Z}$ with $r \geq k$. Consequently, we obtain a commutative diagram of functors

$$\begin{array}{ccc} \mathsf{Fil.C}(\mathscr{A}) & \longrightarrow & \mathsf{Sp.Seq}^\infty_0(\mathscr{A}) \\ \downarrow & & \downarrow \\ \mathsf{Fil.Hot}(\mathscr{A}, k) & \longrightarrow & \mathsf{Sp.Seq}^\infty_k(\mathscr{A}) \end{array} \qquad \text{for every } k \geq 0$$

whose top horizontal arrow is the functor of (7.2.17), and whoses left (resp. right) vertical arrow is the functor of definition 7.2.22(ii) (resp. the forgetful functor of (7.2.8)).

7.2.24. We show next, how to lift the décalage functor of (7.2.5), to a functor on complexes

$$\mathrm{Déc} : \mathsf{Fil.C}(\mathscr{A}) \to \mathsf{Fil.C}(\mathscr{A}).$$

This construction is borrowed from [54]. Namely, given $(K^\bullet, \mathrm{Fil}^\bullet K^\bullet)$ as in (7.2.12), we set

$$D^\bullet := K^\bullet \qquad \text{and} \qquad \mathrm{Fil}^p D^n := Z(K)^{p+n,-p}_1 \quad \text{for every } p, n \in \mathbb{Z}.$$

It follows straightforwardly from (7.2.15), that the differential $d^\bullet_K$ of $K^\bullet$ restricts to a morphism $\mathrm{Fil}^p D^n \to \mathrm{Fil}^p D^{n+1}$ for every $p, n \in \mathbb{Z}$, so $(D^\bullet, \mathrm{Fil}^\bullet D^\bullet)$ is an object of $\mathsf{Fil.C}(\mathscr{A})$, and we let

$$\mathrm{Déc}\,(K^\bullet, \mathrm{Fil}^\bullet K^\bullet) := (D^\bullet, \mathrm{Fil}^\bullet D^\bullet).$$

Every morphism (7.2.1) induces in the obvious fashion a morphism

$$\mathrm{Déc}\,(K^\bullet, \mathrm{Fil}^\bullet K^\bullet) \to \mathrm{Déc}\,(L^\bullet, \mathrm{Fil}^\bullet L^\bullet)$$

so Déc is a functor as sought. Moreover, it is easily seen that if $\mathrm{Fil}^\bullet K^\bullet$ is a bounded filtration, the same holds for the filtration of $\mathrm{Déc}\,(K^\bullet, \mathrm{Fil}^\bullet K^\bullet)$. Notice that this definition is not self-dual, *i.e.* it is not preserved after switching to the opposite category $\mathscr{A}^o$.

7.2.25. We wish next to compare the spectral sequences

$$E(K)^{\bullet\bullet}_\bullet \qquad \text{and} \qquad E(\mathrm{Déc}\,K)^{\bullet\bullet}_\bullet$$

attached, as in (7.2.12), to $(K^\bullet, \mathrm{Fil}^\bullet)$ and respectively to $\mathrm{Déc}\,(K^\bullet, \mathrm{Fil}^\bullet K)$. To this aim, notice that

$$Z(\mathrm{Déc}\,K)^{p,n-p}_0 = Z(K)^{p+n,-p}_1 \qquad D(\mathrm{Déc}\,K)^{p,n-p}_0 = Z(K)^{p+1+n,-p-1}_1$$

and we have

$$Z(K)^{p+1+n,-p-1}_1 \subset (\mathrm{Fil}^{p+1+n}K^\bullet)^n \subset D(K)^{p+n,-p}_1 \subset Z(K)^{p+n,-p}_1$$

whence a natural epimorphism

$$u^{p,n-p} : E(\mathrm{Déc}\,K)^{p,n-p}_0 \to \mathrm{Déc}\,(E(K))^{p,n-p}_0 \qquad \text{for every } p, n \in \mathbb{Z}.$$

With this notation, we have :

Proposition 7.2.26. *With the notation of* (7.2.24)*, the following holds :*

(i) *The system* $(u^{p,n-p} \mid p, n \in \mathbb{Z})$ *extends to an epimorphism in* $\mathsf{Sp.Seq}_0(\mathscr{A})$

$$u^{\bullet\bullet}_\bullet : E(\mathrm{Déc}\,K)^{\bullet\bullet}_\bullet \to \mathrm{Déc}\,(E(K))^{\bullet\bullet}_\bullet.$$

(ii) *The morphism* $u^\bullet$ *induces isomorphisms in* $\mathsf{Sp.Seq}_1(\mathscr{A})$

$$(E(\mathrm{Déc}\,K)^{\bullet\bullet}_r \mid r \geq 1) \xrightarrow{\sim} (\mathrm{Déc}\,(E(K))^{\bullet\bullet}_r \mid r \geq 1).$$

Proof. For any filtered complex $(L^\bullet, \mathrm{Fil}^\bullet L^\bullet)$ of $\mathscr{A}$, let

$$L'^\bullet := L^\bullet \qquad (\mathrm{Fil}^p L'^\bullet)^n := (\mathrm{Fil}^{p-n} L^\bullet)^n \qquad \text{for every } p, n \in \mathbb{Z}.$$

It is easily seen that

$$(L^\bullet, \mathrm{Fil}^\bullet L^\bullet)' := (L'^\bullet, \mathrm{Fil}^\bullet L'^\bullet)$$

is a filtered complex of $\mathscr{A}$, and a direct inspection of the definitions yields the identities

$$E(L)_r^{p,n-p} = E(L')_{r+1}^{p+n,-p} \qquad d(L)_r^{p,n-p} = d(L')_{r+1}^{p+n,-p} \qquad \text{for every } p, n \in \mathbb{Z} \text{ and } r \in \mathbb{N}$$

which add up to an identity in $\mathsf{Sp.Seq}_0(\mathscr{A})$:

$$(E(L)_r^{\bullet\bullet} \mid r \geq 0) = \mathrm{Déc}\,(E(L')_r^{\bullet\bullet} \mid r \geq 1). \tag{7.2.27}$$

Especially, set $(M^\bullet, \mathrm{Fil}^\bullet M^\bullet) := (\mathrm{Déc}\,(K^\bullet, \mathrm{Fil}^\bullet K^\bullet))'$. Explicitly, we have

$$M^\bullet = K^\bullet \qquad (\mathrm{Fil}^p M^\bullet)^n = Z(K)_1^{p,n-p} \subset (\mathrm{Fil}^p K^\bullet)^n \qquad \text{for every } p, n \in \mathbb{Z}$$

so that the identity morphism $\mathbf{1}_K : M^\bullet \to K^\bullet$ is a morphism

$$(M^\bullet, \mathrm{Fil}^\bullet M^\bullet) \to (K^\bullet, \mathrm{Fil}^\bullet K^\bullet) \qquad \text{in } \mathsf{Fil.C}(\mathscr{A})$$

which in turns yields a morphism in $\mathsf{Sp.Seq}_0(\mathscr{A})$:

$$E(M)_\bullet^{\bullet\bullet} \to E(K)_\bullet^{\bullet\bullet}.$$

Combining with (7.2.27), we get a natural morphism

$$v_\bullet^{\bullet\bullet} : E(\mathrm{Déc}\,K)_\bullet^{\bullet\bullet} \to \mathrm{Déc}\,(E(K))_\bullet^{\bullet\bullet}$$

and a direct inspection shows that $u_\bullet^{\bullet\bullet} = v_\bullet^{\bullet\bullet}$, whence (i). To check (ii), we remark that

$$Z(M)_r^{pq} = \mathrm{Ker}\,(\bar{d} : Z(K)_1^{pq} \to Z(K)_1^{p,q+1}/Z(K)_1^{p+r,q-r+1}) = Z(K)_r^{pq} \qquad \text{for every } r \geq 1$$

whence :

$$B(M)_r^{pq} = B(K)_r^{pq} \qquad \text{for every } r \geq 1$$

due to lemma (7.2.13). □

7.3. **Derived categories and derived functors.** Let $\mathscr{A}$ be any additive category. With the notation of remark 7.1.19(ii), set

$$\mathbb{C}^\bullet := \mathrm{Coker}\,(\iota_1^\bullet : \mathbb{Z}[0] \to \mathbb{K}\langle 1\rangle^\bullet).$$

Explicitly, $\mathbb{C}^\bullet$ is the object of $\mathsf{C}^{[-1,0]}(\mathbb{Z}\text{-}\mathbf{Mod})$ such that $\mathbb{C}^{-1} = \mathbb{C}^0 = \mathbb{Z}$ and with $d_{\mathbb{C}}^{-1} = \mathbf{1}_{\mathbb{Z}}$. The morphism $\iota_0^\bullet$ induces a short exact sequence of complexes of abelian groups :

$$0 \to \mathbb{Z}[0] \xrightarrow{\iota^\bullet} \mathbb{C}^\bullet \xrightarrow{\pi^\bullet} \mathbb{Z}[1] \to 0.$$

Now, let $\varphi^\bullet : K^\bullet \to L^\bullet$ be any morphism in $\mathsf{C}(\mathscr{A})$. We define the *cone* of $\varphi^\bullet$ as the push-out in the cocartesian diagram of $\mathsf{C}(\mathscr{A})$:

$$\begin{array}{ccc} K^\bullet & \xrightarrow{\iota^\bullet \otimes_{\mathbb{Z}} \mathbf{1}_K} & \mathbb{C}^\bullet \otimes_{\mathbb{Z}} K^\bullet \\ {\scriptstyle\varphi^\bullet}\downarrow & & \downarrow{\scriptstyle\beta^\bullet} \\ L^\bullet & \xrightarrow{\psi^\bullet} & (\mathrm{Cone}\,\varphi)^\bullet. \end{array}$$

Let also $\gamma^\bullet : \mathbb{K}\langle 1\rangle^\bullet \to \mathbb{C}^\bullet$ be the natural projection; by remark 7.1.19(ii), the composition

$$\beta^\bullet \circ (\gamma^\bullet \otimes_{\mathbb{Z}} K^\bullet) : \mathbb{K}\langle 1\rangle^\bullet \otimes_{\mathbb{Z}} K^\bullet \to (\mathrm{Cone}\,\varphi)^\bullet$$

corresponds to a homotopy from $\psi^\bullet \circ \varphi^\bullet$ to $\beta^\bullet \circ (\gamma^\bullet \circ \iota_1^\bullet) \otimes_{\mathbb{Z}} K^\bullet$, and the latter is obviously the zero endomorphism of $K^\bullet$. Moreover, we get a natural identification

$$\mathrm{Coker}\,\psi^\bullet \xrightarrow{\sim} \mathrm{Coker}\,(\iota^\bullet \otimes_{\mathbb{Z}} K^\bullet) \xrightarrow{\sim} K^\bullet[1] \tag{7.3.1}$$

and we let $\partial^\bullet$ be the unique morphism of complexes which fits in the commutative diagram

$$\begin{array}{ccc} & (\mathrm{Cone}\,\varphi)^\bullet & \\ {\scriptstyle -\pi^\bullet}\swarrow & & \searrow{\scriptstyle \partial^\bullet} \\ \mathrm{Coker}\,\psi^\bullet & \longrightarrow & K^\bullet[1] \end{array}$$

whose horizontal arrow is the identification (7.3.1), and where $\pi^\bullet$ is the natural projection. Summing up, we have attached to $\varphi^\bullet$ a natural sequence of morphisms of complexes

$$\Theta(\varphi^\bullet) \quad : \quad K^\bullet \xrightarrow{\varphi^\bullet} L^\bullet \xrightarrow{\psi^\bullet} (\mathrm{Cone}\,\varphi)^\bullet \xrightarrow{\partial^\bullet} K^\bullet[1]$$

such that $\psi^\bullet \circ \varphi^\bullet$ is homotopically trivial, and the induced sequence of morphisms in $\mathscr{A}$

$$0 \to L^i \xrightarrow{\psi^i} (\mathrm{Cone}\,\varphi)^i \xrightarrow{\partial^i} K^{i+1} \to 0$$

is *split exact* for every $i \in \mathbb{Z}$, *i.e.* such that $(\mathrm{Cone}\,\varphi)^i = L^i \oplus K^{i+1}$, and ψ_i (resp. $-\partial^i$) is the natural monomorphism (resp. epimorphism) induced by this direct sum decomposition. A direct inspection shows that the differential

$$d^{i-1}_{\mathrm{Cone}\,\varphi} : L^{i-1} \oplus K^i \to L^i \oplus K^{i+1}$$

is given by the matrix

$$\begin{bmatrix} d^{i-1}_L & \varphi^i \\ 0 & -d^i_K \end{bmatrix}.$$

Moreover, it is easily seen that every morphism $\beta^\bullet : \varphi^\bullet_1 \to \varphi^\bullet_2$ in $\mathsf{Morph}(\mathsf{C}(\mathscr{A}))$ induces a natural commutative diagram in $\mathsf{C}(\mathscr{A})$:

$$\begin{array}{cccccccc} \Theta(\varphi^\bullet_1) & & K^\bullet_1 & \xrightarrow{\varphi^\bullet_1} & L^\bullet_1 & \xrightarrow{\psi^\bullet_1} & (\mathrm{Cone}\,\varphi_1)^\bullet & \xrightarrow{\partial^\bullet_1} & K^\bullet_1[1] \\ {\scriptstyle \Theta(\beta^\bullet)}\downarrow & & {\scriptstyle \beta^\bullet_1}\downarrow & & {\scriptstyle \beta^\bullet_2}\downarrow & & \downarrow{\scriptstyle \mathrm{Cone}(\beta_1,\beta_2)^\bullet} & & \downarrow{\scriptstyle \beta^\bullet_1[1]} \\ \Theta(\varphi^\bullet_2) & & K^\bullet_2 & \xrightarrow{\varphi^\bullet_2} & L^\bullet_2 & \xrightarrow{\psi^\bullet_2} & (\mathrm{Cone}\,\varphi_2)^\bullet & \xrightarrow{\partial^\bullet_2} & K^\bullet_2[1] \end{array} \qquad (7.3.2)$$

where $\psi^\bullet_1$ and $\psi^\bullet_2$ are the natural morphisms, as in the foregoing (details left to the reader).

Definition 7.3.3. Let $\mathscr{A}$ be any additive category, and denote by $\mathsf{T}(\mathscr{A})$ either of the categories $\mathsf{C}(\mathscr{A})$ or $\mathsf{Hot}(\mathscr{A})$. With the notation of (7.3) :

(i) We call $\Theta(\varphi^\bullet)$ the *true triangle* associated to the morphism $\varphi^\bullet$, and $\partial^\bullet$ is the *boundary morphism* of $\Theta(\varphi^\bullet)$.

(ii) A *triangle* of $\mathsf{T}(\mathscr{A})$ is any sequence of morphisms of $\mathsf{T}(\mathscr{A})$:

$$A^\bullet \xrightarrow{\alpha^\bullet} B^\bullet \xrightarrow{\beta^\bullet} C^\bullet \xrightarrow{\gamma^\bullet} A^\bullet[1]. \qquad (7.3.4)$$

(iii) Let $\Theta_i := (A^\bullet_i \xrightarrow{\alpha^\bullet_i} B^\bullet_i \xrightarrow{\beta^\bullet_i} C^\bullet_i \xrightarrow{\gamma^\bullet_i} A^\bullet_i[1])$ for $i = 1, 2$ be two triangles of $\mathsf{T}(\mathscr{A})$. A *morphism of triangles* $\Theta_1 \to \Theta_2$ is a commutative diagram of morphisms of $\mathsf{T}(\mathscr{A})$:

$$\begin{array}{ccccccc} A^\bullet_1 & \xrightarrow{\alpha^\bullet_1} & B^\bullet_1 & \xrightarrow{\beta^\bullet_1} & C^\bullet_1 & \xrightarrow{\gamma^\bullet_1} & A^\bullet_1[1] \\ {\scriptstyle \tau^\bullet}\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \tau^\bullet[1]} \\ A^\bullet_2 & \xrightarrow{\alpha^\bullet_2} & B^\bullet_2 & \xrightarrow{\beta^\bullet_2} & C^\bullet_2 & \xrightarrow{\gamma^\bullet_2} & A^\bullet_2[1]. \end{array}$$

Morphisms of triangles can be composed in the obvious fashion, and clearly the system of all triangles of $\mathsf{T}(\mathscr{A})$ and their morphisms, forms a category.

(iv) A *distinguished triangle* of $\mathsf{T}(\mathscr{A})$ is a triangle of $\mathsf{T}(\mathscr{A})$ that is isomorphic to a true triangle. We denote by
$$\Theta.\mathsf{T}(\mathscr{A})$$
the full subcategory of the category of triangles of $\mathsf{T}(\mathscr{A})$ whose objects are the distinguished triangles.

(v) Let $\mathscr{B}$ be any other additive category. A *triangulated functor* from $\mathsf{T}(\mathscr{A})$ to $\mathsf{T}(\mathscr{B})$ is a pair (F,τ) consisting of a functor $F:\mathsf{T}(\mathscr{A})\to\mathsf{T}(\mathscr{B})$ and a natural isomorphism
$$\tau^\bullet_K : F(K^\bullet[1]) \xrightarrow{\sim} (FK^\bullet)[1] \qquad \text{for every } K^\bullet\in\mathrm{Ob}(\mathsf{T}(\mathscr{A}))$$
such that, for every distinguished triangle (7.3.4) of $\mathsf{T}(\mathscr{A})$, the triangle
$$FA^\bullet \xrightarrow{F\alpha^\bullet} FB^\bullet \xrightarrow{F\beta^\bullet} FC^\bullet \xrightarrow{\tau^\bullet_A\circ F\gamma^\bullet} FA^\bullet[1]$$
is distinguished in $\mathsf{T}(\mathscr{B})$. We also say that F is *triangulated* if there exists a natural isomorphism τ as above, such that (F,τ) is a triangulated functor.

Remark 7.3.5. (i) With the terminology of definition 7.3.3, we may say that diagram (7.3.2) is a morphism of distinguished triangles, and clearly the rules : $\varphi^\bullet\mapsto\Theta(\varphi^\bullet)$ for any morphism $\varphi^\bullet$ in $\mathsf{C}(\mathscr{A})$ and $\beta^\bullet\mapsto\Theta(\beta^\bullet)$ for any morphism $\beta^\bullet$ in $\mathsf{Morph}(\mathsf{C}(\mathscr{A}))$ amount to a functor
$$\mathsf{Morph}(\mathsf{C}(\mathscr{A}))\to\Theta.\mathsf{C}(\mathscr{A}). \tag{7.3.6}$$
Moreover, if $\varphi^\bullet$ is a morphism of $\mathsf{C}^+(\mathscr{A})$ (resp. $\mathsf{C}^-(\mathscr{A})$, resp. $\mathsf{C}^b(\mathscr{A})$), then clearly $(\mathrm{Cone}\,\varphi)^\bullet$ lies in the same subcategory of $\mathsf{C}(\mathscr{A})$, so (7.3.6) restricts to functors
$$\mathsf{Morph}(\mathsf{C}^?(\mathscr{A}))\to\Theta.\mathsf{C}^?(\mathscr{A}) \qquad \text{with “?” equal to either } +, -, \text{ or } b$$
with obvious notation.

(ii) Let $F:\mathscr{A}\to\mathscr{B}$ be any additive functor between additive categories. Directly from the definitions, it is clear that the induced functors $\mathsf{C}(F)$ and $\mathsf{Hot}(F)$ are both triangulated.

(iii) Suppose now that $\mathscr{A}$ is an abelian category, so that the same holds for $\mathsf{C}(\mathscr{A})$, and the discussion of (7.3) yields a short exact sequence
$$0\to L^\bullet \xrightarrow{\psi^\bullet} (\mathrm{Cone}\,\varphi^\bullet) \xrightarrow{\partial^\bullet} K^\bullet[1]\to 0 \qquad \text{in } \mathsf{C}(\mathscr{A})$$
such that the boundary map in degree i of the induced long exact cohomology sequence is none else than φ^{i+1}, so we get a natural acyclic complex
$$\cdots\to H^{i-1}L^\bullet \xrightarrow{\psi^i} H^{i-1}(\mathrm{Cone}\,\varphi)^\bullet \xrightarrow{\partial^{i-1}} H^iK^\bullet \xrightarrow{\varphi^i} H^iL^\bullet\to\cdots$$
for every morphism $\varphi^\bullet$ in $\mathsf{C}(\mathscr{A})$. However, one of the subtleties of distinguished triangles, is that they provide a language for expressing certain exactness and cohomological assertions, that remains available even for general additive but not necessarily abelian categories. For instance, let us show the following :

Proposition 7.3.7. *Let $\mathscr{A}$ be any additive category. For any complex $L^\bullet$ of $\mathscr{A}$, the functors*
$$\mathsf{C}(\mathscr{A})\to\mathsf{C}(\mathbb{Z}\text{-}\mathbf{Mod}) \qquad K^\bullet\mapsto\mathrm{Hom}^\bullet_{\mathscr{A}}(L^\bullet,K^\bullet)$$
$$\mathsf{Hot}(\mathscr{A})\to\mathsf{Hot}(\mathbb{Z}\text{-}\mathbf{Mod}) \qquad K^\bullet\mapsto\mathrm{Hot}^\bullet_{\mathscr{A}}(L^\bullet,K^\bullet)$$
are triangulated (notation of example 7.1.25*).*

Proof. Let $\varphi^\bullet:K^\bullet_1\to K^\bullet_2$ be any morphism of complexes of $\mathscr{A}$. The universal properties of the push-out and of the cokernel yield a unique commutative diagram
$$\begin{array}{ccccccc}
\mathrm{Hom}^\bullet_{\mathscr{A}}(L^\bullet,\Theta(\varphi^\bullet)) & & \mathrm{Hom}^\bullet_{\mathscr{A}}(L^\bullet,K^\bullet_2) & \longrightarrow & \mathrm{Hom}^\bullet_{\mathscr{A}}(L^\bullet,(\mathrm{Cone}\,\varphi^\bullet)) & \longrightarrow & \mathrm{Hom}^\bullet_{\mathscr{A}}(L^\bullet,K^\bullet_1[1]) \\
\downarrow & & \| & & \downarrow{\scriptstyle\beta^\bullet} & & \downarrow{\scriptstyle\gamma^\bullet} \\
\Theta(\mathrm{Hom}^\bullet_{\mathscr{A}}(L^\bullet,\varphi^\bullet)) & & \mathrm{Hom}^\bullet_{\mathscr{A}}(L^\bullet,K^\bullet_2) & \longrightarrow & \mathrm{Cone}\,\mathrm{Hom}^\bullet_{\mathscr{A}}(L^\bullet,\varphi^\bullet) & \longrightarrow & \mathrm{Hom}^\bullet_{\mathscr{A}}(L^\bullet,K^\bullet_1)[1].
\end{array}$$

Now, since the sequence $0 \to K_2^i \to (\mathrm{Cone}\,\varphi)^i \to K_1^{i+1} \to 0$ is split exact in each degree $i \in \mathbb{Z}$, it is easily seen that the same holds for the induced sequence

$$0 \to \mathrm{Hom}^i_{\mathscr{A}}(L^\bullet, K_2^\bullet) \to \mathrm{Hom}^i_{\mathscr{A}}(L^\bullet, (\mathrm{Cone}\,\varphi)^\bullet) \to \mathrm{Hom}^i_{\mathscr{A}}(L^\bullet, K_1^\bullet[1]) \to 0.$$

It follows that $\gamma^\bullet$ is the obvious natural identification and $\beta^\bullet$ is the direct product of the natural identifications

$$\mathrm{Hom}_{\mathscr{A}}(L^i, \varphi^j \oplus \varphi^{j+1}) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{A}}(L^i, \varphi^j) \oplus \mathrm{Hom}_{\mathscr{A}}(L^i, \varphi^{j+1}) \qquad \text{for every } i, j \in \mathbb{Z}$$

whence the assertion for the functor $\mathrm{Hom}^\bullet_{\mathscr{A}}$. The assertion for $\mathrm{Hot}^\bullet_{\mathscr{A}}$ follows immediately. □

Remark 7.3.8. Resume the situation of (7.1.23). It is easily seen that, *mutatis mutandis*, the proof of proposition 7.3.7 shows also the following. For every $K^\bullet \in \mathrm{Ob}(\mathsf{Hot}(\mathscr{A}))$ the functor

$$\mathsf{Hot}^-(\mathscr{A}') \to \mathsf{Hot}^-(\mathscr{A}'') \qquad L^\bullet \mapsto B^\bullet_-(K^\bullet, L^\bullet)$$

is triangulated. Likewise, the same holds for the functor

$$\mathsf{Hot}^-(\mathscr{A}) \to \mathsf{Hot}^-(\mathscr{A}'') \qquad K^\bullet \mapsto B^\bullet_-(K^\bullet, L^\bullet)$$

for every $L^\bullet \in \mathrm{Ob}(\mathsf{Hot}(\mathscr{A}'))$: details left to the reader.

Lemma 7.3.9. *With the notation of* (7.3)*, let* $\varphi_1^\bullet, \varphi_2^\bullet : K^\bullet \to L^\bullet$ *be any two morphisms in* $\mathsf{C}(\mathscr{A})$*. The following holds :*

(i) *Any homotopy* $s^\bullet$ *from* $\varphi_1^\bullet$ *to* $\varphi_2^\bullet$ *induces a natural isomorphism*

$$\gamma_s^\bullet : (\mathrm{Cone}\,\varphi_1)^\bullet \xrightarrow{\sim} (\mathrm{Cone}\,\varphi_2)^\bullet \qquad \textit{in } \mathsf{C}(\mathscr{A})$$

fitting into a commutative diagram

$$\begin{array}{ccccccc}
K^\bullet & \xrightarrow{\varphi_1^\bullet} & L^\bullet & \xrightarrow{\psi_1^\bullet} & (\mathrm{Cone}\,\varphi_1)^\bullet & \xrightarrow{\partial^\bullet_{\varphi_1}} & K^\bullet[1] \\
 & & \| & & \downarrow{\scriptstyle \gamma_s^\bullet} & & \| \\
K^\bullet & \xrightarrow{\varphi_2^\bullet} & L^\bullet & \xrightarrow{\psi_2^\bullet} & (\mathrm{Cone}\,\varphi_2)^\bullet & \xrightarrow{\partial^\bullet_{\varphi_2}} & K^\bullet[1]
\end{array}$$

whose top (resp. bottom) row is $\Theta(\varphi_1^\bullet)$ *(resp.* $\Theta(\varphi_2^\bullet)$*).*

(ii) *There is a natural isomorphism*

$$\omega^\bullet : (\mathrm{Cone}\,\psi_1)^\bullet \xrightarrow{\sim} K^\bullet[1] \oplus (\mathbb{C}^\bullet \otimes_{\mathbb{Z}} L^\bullet) \qquad \textit{in } \mathsf{C}(\mathscr{A})$$

fitting into a commutative diagram

$$\begin{array}{ccccccc}
L^\bullet & \xrightarrow{\psi_1^\bullet} & (\mathrm{Cone}\,\varphi_1)^\bullet & \xrightarrow{\tau^\bullet} & (\mathrm{Cone}\,\psi_1)^\bullet & \xrightarrow{\partial^\bullet_{\psi_1}} & L^\bullet[1] \\
 & & \downarrow{\scriptstyle \partial^\bullet_{\varphi_1}} & & \downarrow{\scriptstyle \omega^\bullet} & & \| \\
 & & K^\bullet[1] & \xleftarrow{-p_1^\bullet} & K^\bullet[1] \oplus (\mathbb{C}^\bullet \otimes_{\mathbb{Z}} L^\bullet) & \xrightarrow{p_2^\bullet} & L^\bullet[1]
\end{array}$$

whose top row is $\Theta(\psi_1^\bullet)$*, and where* $p_1^\bullet$ *is the natural projection, and* $p_2^\bullet$ *is the morphism given by the matrix*

$$\begin{bmatrix} 0 \\ \varphi_1^{i+1} \\ -\mathbf{1}_{L^{i+1}} \end{bmatrix} : L^i \oplus K^{i+1} \oplus L^{i+1} \to L^{i+1} \qquad \textit{for every } i \in \mathbb{Z}.$$

Proof. (i): By assumption, $\varphi_1^i = \varphi_2^i + d^{i-1}s^i + s^{i+1}d^i$ for every $i \in \mathbb{Z}$. We let γ_s^i be the automorphism of $L^i \oplus K^{i+1}$ given by the matrix

$$\begin{bmatrix} \mathbf{1}_{L^i} & s^{i+1} \\ 0 & \mathbf{1}_{K^{i+1}} \end{bmatrix} \qquad \text{for every } i \in \mathbb{Z}.$$

A direct computation shows that the system $(\gamma_s^i \mid i \in \mathbb{Z})$ is the sought isomorphism, and the commutativity of the resulting diagram as in (i) follows by a simple inspection.

(ii): Set $\sigma^\bullet := (\iota^\bullet \otimes_\mathbb{Z} \mathbf{1}_L) \circ \varphi_1^\bullet$. By definition, we have a natural identification

$$(\mathrm{Cone}\,\sigma)^\bullet = (\mathrm{Cone}\,\psi_1)^\bullet$$

as well as a commutative diagram

$$\begin{array}{ccccc}
K^\bullet & \xrightarrow{\varphi_1^\bullet} & L^\bullet & \xrightarrow{\iota^\bullet \otimes_\mathbb{Z} \mathbf{1}_L} & \mathbb{C}^\bullet \otimes_\mathbb{Z} L^\bullet \\
{\scriptstyle \iota^\bullet \otimes_\mathbb{Z} \mathbf{1}_K}\Big\downarrow & & {\scriptstyle \psi_1^\bullet}\Big\downarrow & & \Big\downarrow \\
\mathbb{C}^\bullet \otimes_\mathbb{Z} K^\bullet & \longrightarrow & (\mathrm{Cone}\,\varphi_1)^\bullet & \xrightarrow{\tau^\bullet} & (\mathrm{Cone}\,\psi_1)^\bullet \\
\Big\downarrow & & {\scriptstyle \partial_{\varphi_1}^\bullet}\Big\downarrow & & {\scriptstyle \partial_\sigma^\bullet}\Big\downarrow \\
K^\bullet[1] & \xrightarrow{-\mathbf{1}_{K[1]}} & K^\bullet[1] & = & K^\bullet[1]
\end{array}$$

whose two square subdiagrams on the top ladder are cocartesian. Notice that $\mathbb{C}^\bullet$ is homotopically trivial (details left to the reader); by remark 7.1.19(iv), the same then holds for $\mathbb{C}^\bullet \otimes_\mathbb{Z} L^\bullet$. Thus, $\sigma^\bullet$ is null-homotopic. In light of (i) and (7.3.1), we deduce an isomorphism as sought, fitting into a commutative diagram

$$\begin{array}{ccccccc}
K^\bullet & \longrightarrow & \mathbb{C}^\bullet \otimes_\mathbb{Z} L^\bullet & \longrightarrow & (\mathrm{Cone}\,\psi_1)^\bullet & \xrightarrow{\partial_\sigma^\bullet} & K^\bullet[1] \\
 & & \Vert & & \Big\downarrow{\scriptstyle \omega^\bullet} & & \Vert \\
K^\bullet & \xrightarrow{0} & \mathbb{C}^\bullet \otimes_\mathbb{Z} L^\bullet & \longrightarrow & K^\bullet[1] \oplus (\mathbb{C}^\bullet \otimes_\mathbb{Z} L^\bullet) & \xrightarrow{\partial_0^\bullet} & K^\bullet[1]
\end{array}$$

whose bottom row is the true triangle associated to the zero morphism $K^\bullet \to \mathbb{C}^\bullet \otimes_\mathbb{Z} L^\bullet$. Explicitly, the differentials

$$d^i_{(\mathrm{Cone}\,\psi_1)}, d^i_{K[1]\oplus(\mathbb{C}\otimes_\mathbb{Z} L)} : L^i \oplus K^{i+1} \oplus L^{i+1} \to L^{i+1} \oplus K^{i+2} \oplus L^{i+2}$$

are given by the matrices

$$\begin{bmatrix} d_L^i & \varphi_1^{i+1} & \mathbf{1}_{L^{i+1}} \\ 0 & -d_K^{i+1} & 0 \\ 0 & 0 & -d_L^{i+1} \end{bmatrix} \qquad \begin{bmatrix} d_L^i & 0 & \mathbf{1}_{L^{i+1}} \\ 0 & -d_K^{i+1} & 0 \\ 0 & 0 & -d_L^{i+1} \end{bmatrix} \qquad \text{for every } i \in \mathbb{Z}$$

from which we see that we can take for ω^i the automorphism of $L^i \oplus K^{i+1} \oplus L^{i+1}$ given by the matrix

$$\begin{bmatrix} \mathbf{1}_{L^i} & 0 & 0 \\ 0 & \mathbf{1}_{K^{i+1}} & 0 \\ 0 & \varphi_1^{i+1} & \mathbf{1}_{L^{i+1}} \end{bmatrix} \qquad \text{for every } i \in \mathbb{Z}$$

and then the commutativity of the right square subdiagram of (ii) is immediate. Lastly, it is easily seen that $\partial_0^\bullet = -p_1^\bullet$, therefore

$$-p_1^\bullet \circ \omega^\bullet \circ \tau^\bullet = \partial_\sigma^\bullet \circ \tau^\bullet = \partial_{\varphi_1}^\bullet$$

which shows the commutativity of the left square subdiagram of (ii). ☐

Remark 7.3.10. (i) In the situation of lemma 7.3.9, notice that the diagram

$$\begin{array}{ccccccc}
K^\bullet & \xrightarrow{\varphi_1^\bullet} & L^\bullet & \xrightarrow{\psi_1^\bullet} & (\mathrm{Cone}\,\varphi_1)^\bullet & \xrightarrow{\partial_{\varphi_1}^\bullet} & K^\bullet[1] \\
\Vert & & \Vert & & {\scriptstyle \gamma_s^\bullet}\Big\downarrow & & \Vert \\
K^\bullet & \xrightarrow{\varphi_2^\bullet} & L^\bullet & \xrightarrow{\psi_2^\bullet} & (\mathrm{Cone}\,\varphi_2)^\bullet & \xrightarrow{\partial_{\varphi_2}^\bullet} & K^\bullet[1]
\end{array}$$

commutes in $\mathsf{Hot}(\mathscr{A})$, and amounts to a natural isomorphism

$$\Theta(\varphi_1^\bullet) \stackrel{\sim}{\to} \Theta(\varphi_2^\bullet) \qquad \text{in } \Theta.\mathsf{Hot}(\mathscr{A}).$$

(ii) Let $\Theta := (A^\bullet \xrightarrow{\alpha^\bullet} B^\bullet \xrightarrow{\beta^\bullet} C^\bullet \xrightarrow{\gamma^\bullet} A^\bullet[1])$ be any triangle of $\mathsf{C}(\mathscr{A})$ or $\mathsf{Hot}(\mathscr{A})$. We obtain a new triangle by setting

$$\Theta[1] := (B^\bullet \xrightarrow{\beta^\bullet} C^\bullet \xrightarrow{\gamma^\bullet} A^\bullet[1] \xrightarrow{-\alpha^\bullet[1]} B^\bullet[1])$$

and clearly, the rule $\Theta \mapsto \Theta[1]$ yields an endofunctor of the category of triangles. The change of sign on the last arrow is needed to ensure that this operator restricts to an endofunctor of the subcategory of distinguished triangles, at least up to homotopy. Indeed, we may state :

Proposition 7.3.11. *With the notation of remark* 7.3.10(ii)*, suppose that* Θ *is a distinguished triangle of* $\mathsf{Hot}(\mathscr{A})$*. Then the same holds for* $\Theta[1]$*, so we have an automorphism*

$$\Theta.\mathsf{Hot}(\mathscr{A}) \stackrel{\sim}{\to} \Theta.\mathsf{Hot}(\mathscr{A}) \qquad \Theta \mapsto \Theta[1].$$

Proof. We may assume that $\Theta = \Theta(\alpha^\bullet)$ for some morphism $\alpha^\bullet : A^\bullet \to B^\bullet$ of complexes of $\mathscr{A}$, in which case $C^\bullet = (\mathrm{Cone}\,\alpha)^\bullet$, and it suffices to show the more precise :

Claim 7.3.12. There is a natural isomorphism

$$\Theta(\beta^\bullet) \stackrel{\sim}{\to} \Theta(\alpha^\bullet)[1] \qquad \text{in } \Theta.\mathsf{Hot}(\mathscr{A}).$$

Proof of the claim. Indeed, notice that the natural projection $p_1^\bullet : A^\bullet[1] \oplus (\mathbb{C}^\bullet \otimes_{\mathbb{Z}} B^\bullet) \to A^\bullet[1]$ represents an isomorphism in $\mathsf{Hot}(\mathscr{A})$, whose inverse is the class of the natural monomorphism $e_1^\bullet : A^\bullet[1] \to A^\bullet[1] \oplus (\mathbb{C}^\bullet \otimes_{\mathbb{Z}} B^\bullet)$. Taking into account lemma 7.3.9(ii), we then get the commutative diagram in $\mathsf{Hot}(\mathscr{A})$:

$$\begin{array}{ccccccc} B^\bullet & \xrightarrow{\beta^\bullet} & (\mathrm{Cone}\,\alpha)^\bullet & \longrightarrow & (\mathrm{Cone}\,\beta)^\bullet & \xrightarrow{\partial_\beta^\bullet} & B^\bullet[1] \\ \| & & \| & & \downarrow{\scriptstyle p_1^\bullet \circ \omega^\bullet} & & \| \\ B^\bullet & \xrightarrow{\beta^\bullet} & (\mathrm{Cone}\,\alpha)^\bullet & \xrightarrow{\partial_\alpha^\bullet} & A^\bullet[1] & \xrightarrow{-\alpha^\bullet} & B^\bullet[1] \end{array}$$

where $\omega^\bullet : (\mathrm{Cone}\,\beta)^\bullet \stackrel{\sim}{\to} A^\bullet[1] \oplus (\mathbb{C}^\bullet \otimes_{\mathbb{Z}} B^\bullet)$ is the isomorphism provided by lemma 7.3.9(ii). The claim follows. ☐

Definition 7.3.13. Let $\mathscr{A}$ be any abelian category.

(i) A *quasi-isomorphism* is a morphism $\varphi^\bullet : K^\bullet \to L^\bullet$ in $\mathsf{C}(\mathscr{A})$ such that $H^i\varphi^\bullet$ is an isomorphism for every $i \in \mathbb{Z}$.

(ii) With the notation of (7.3), we let Σ be the subset of $\mathrm{Morph}(\mathsf{C}(\mathscr{A}))$ consisting of all quasi-isomorphisms. We set

$$\mathsf{D}(\mathscr{A}) := \mathsf{C}(\mathscr{A})[\Sigma^{-1}]$$

(see remark 1.6.10(i)) and we call this category the *derived category* of $\mathscr{A}$.

(iii) More generally, if $I \subset \mathbb{Z}$ is any interval as in (7.1), and $a \in \mathbb{Z}$ is any integer, we denote

$$\mathsf{D}^I(\mathscr{A}) \qquad \mathsf{D}^{\geq a}(\mathscr{A}) \qquad \mathsf{D}^{\leq a}(\mathscr{A}) \qquad \mathsf{D}^+(\mathscr{A}) \qquad \mathsf{D}^-(\mathscr{A}) \qquad \mathsf{D}^b(\mathscr{A})$$

the essential image in $\mathsf{D}(\mathscr{A})$ of the categories $\mathsf{C}^I(\mathscr{A})$, respectively $\mathsf{C}^{\geq a}(\mathscr{A})$, $\mathsf{C}^{\leq a}(\mathscr{A})$, $\mathsf{C}^+(\mathscr{A})$, $\mathsf{C}^-(\mathscr{A})$, $\mathsf{C}^b(\mathscr{A})$.

Remark 7.3.14. (i) Notice that, if $\varphi^\bullet$ is a quasi-isomorphism in $\mathsf{C}(\mathscr{A})$, and $\psi^\bullet$ is any other morphism that is homotopy equivalent to $\varphi^\bullet$, then $\psi^\bullet$ is a quasi-isomorphism as well (remark 7.1.9(ii)). We may then say that a morphism $\varphi^\bullet$ in $\mathsf{Hot}(\mathscr{A})$ is a *quasi-isomorphism*, if it admits a representative in $\mathsf{C}(\mathscr{A})$ which is a quasi-isomorphism; by the foregoing, this property can be checked on any representative for the homotopy class $\varphi^\bullet$.

(ii) In the same vein, notice that, by virtue of proposition 7.1.21, the localization functor $\mathsf{C}^?(\mathscr{A}) \to \mathsf{D}^?(\mathscr{A})$ factors uniquely through a functor

$$\mathsf{Hot}^?(\mathscr{A}) \to \mathsf{D}^?(\mathscr{A}) \tag{7.3.15}$$

whenever "?" equals either $+$, $-$, b, or any interval $I \subset \mathbb{Z}$.

(iii) In light of remark 7.1.9(iii), we see that the isomorphism of remark 7.1.6(iii) identifies the subset $\Sigma^o \subset \mathrm{Ob}(\mathsf{C}(\mathscr{A})^o)$ with the system of quasi-isomorphisms of $\mathsf{C}(\mathscr{A}^o)$. Combining with remark 1.6.10(ii) we get natural isomorphisms of categories :

$$\mathsf{D}^I(\mathscr{A})^o \xrightarrow{\sim} \mathsf{D}^{-I}(\mathscr{A}^o) \qquad \mathsf{D}^+(\mathscr{A})^o \xrightarrow{\sim} \mathsf{D}^-(\mathscr{A}^o) \qquad \mathsf{D}^b(\mathscr{A})^o \xrightarrow{\sim} \mathsf{D}^b(\mathscr{A}^o)$$

where $I \subset \mathbb{Z}$ is any interval, and we set $-I := \{-a \mid a \in I\}$.

(iv) Notice that the brutal truncation functors of (7.1.1) do not transform quasi-isomorphism into quasi-isomorphisms, hence they do not descend to the derived category. On the other hand, the normalized truncation functors do preserve quasi-isomorphisms, so they yield functors

$$\tau^{\geq a} : \mathsf{D}(\mathscr{A}) \to \mathsf{D}^{\geq a}(\mathscr{A}) \qquad \text{and} \qquad \tau^{\leq a} : \mathsf{D}(\mathscr{A}) \to \mathsf{D}^{\leq a}(\mathscr{A}). \qquad \text{for every } a \in \mathbb{Z}$$

Moreover, let $i^{\geq a} : \mathsf{C}^{\geq a}(\mathscr{A}) \to \mathsf{C}(\mathscr{A})$ and $j^{\geq a} : \mathsf{D}^{\geq a}(\mathscr{A}) \to \mathsf{D}(\mathscr{A})$ be the inclusion functors; so the counit of the adjoint pair $(\tau^{\geq a}, i^{\geq a})$ is just the identity endofunctor $\mathbf{1}_{\mathsf{C}^{\geq a}(\mathscr{A})}$, and it is easily seen that the unit of adjunction $\mathbf{1}_{\mathsf{C}(\mathscr{A})} \Rightarrow i^{\geq a} \circ \tau^{\geq a}$ induces a natural transformation $\eta : \mathbf{1}_{\mathsf{D}(\mathscr{A})} \Rightarrow j^{\geq a} \circ \tau^{\geq a}$. Then, the triangular identities for the adjunction $(\tau^{\geq a}, i^{\geq a})$ imply corresponding triangular identities for η and the identity endofunctor $\mathbf{1}_{\mathsf{D}^{\geq a}(\mathscr{A})}$; by proposition 1.1.15(i), we conclude that $\tau^{\geq a}$ is left adjoint to $j^{\geq a}$. Likewise, $\tau^{\leq a}$ is right adjoint to the inclusion functor $\mathsf{D}^{\leq a}(\mathscr{A}) \to \mathsf{D}(\mathscr{A})$.

(v) Directly from the definition (and the universal property of localization), we see that, for every $i \in \mathbb{Z}$, the cohomology functor in degree i factors uniquely through a functor

$$H^i : \mathsf{D}(\mathscr{A}) \to \mathscr{A}.$$

The set of such functors is *not* faithful : *i.e.* two morphisms $\varphi^\bullet, \psi^\bullet : K^\bullet \to L^\bullet$ such that $H^i(\varphi^\bullet) = H^i(\psi^\bullet)$ for every $i \in \mathbb{Z}$ do not necessarily coincide.

(vi) Moreover, the shift operators clearly descend to functors on the derived category :

$$\mathsf{D}(\mathscr{A}) \to \mathsf{D}(\mathscr{A}) \qquad A^\bullet \mapsto A^\bullet[n] \qquad \text{for every } n \in \mathbb{N}.$$

Hence, we have a category of distinguished triangles $\Theta.\mathsf{D}^?(\mathscr{A})$ as in definition 7.3.3(iv), and a notion of triangulated functor $\mathsf{D}^?(\mathscr{A}) \to \mathsf{D}^?(\mathscr{B})$ as in definition 7.3.3(v), for "?" equal to $+$, $-$, b or $\mathbb{Z}$, and for any pair of abelian categories $\mathscr{A}$, $\mathscr{B}$.

Theorem 7.3.16. *Let $\mathscr{A}$ be any abelian category, and denote by*

$$\Sigma_? \subset \mathrm{Morph}(\mathsf{Hot}^?(\mathscr{A})) \qquad \text{with "?" equal to either } +,\ -,\ b, \text{ or any interval } I \subset \mathbb{Z}$$

the subset of all quasi-isomorphisms. We have :

(i) *The functor* (7.3.15) *factors uniquely through an equivalence :*

$$\mathsf{Hot}^?(\mathscr{A})[\Sigma_?^{-1}] \xrightarrow{\sim} \mathsf{D}^?(\mathscr{A}).$$

(ii) *The set $\Sigma_?$ admits both a right and a left calculus of fractions.*

Proof. To begin with, we point out :

Claim 7.3.17. For any abelian category $\mathscr{A}$, the following holds :

(iii) Let $K^\bullet, L^\bullet$ be any two complexes of $\mathscr{A}$, with $L^\bullet \in \mathrm{Ob}(\mathsf{C}^{\leq a}(\mathscr{A}))$. If there exists a quasi-isomorphism $K^\bullet \to L^\bullet$, then the counit of adjunction $\tau^{\leq a}K^\bullet \to K^\bullet$ is a quasi-isomorphism.

(iv) In order to prove the theorem, it suffices to show that Σ admits a right calculus of fractions.

Proof of the claim. (iii) follows immediately from remark 7.3.14(iv) (details left to the reader).

(iv): Indeed, suppose that Σ admits a right calculus of fraction; in view of remark 7.3.14(iii) it follows that Σ admits as well a left calculus of fraction. Next, in case "?" equals $\mathbb{Z}$, assertion (i) says that the induced functor $\mathsf{Hot}(\mathscr{A})[\Sigma^{-1}] \xrightarrow{\sim} \mathsf{D}(\mathscr{A})$ is an equivalence; the latter follows immediately by comparing the respective universal properties : details left to the reader. So, the theorem is completely proven, for "?" equal to $\mathbb{Z}$.

Next, from (iii), the case "?"$= \mathbb{Z}$ of the theorem, and proposition 1.6.21 we get assertion (i) of the theorem for "?" equal to $]-\infty, a]$, and we also deduce that $\Sigma_{]-\infty,a]}$ admits a right calculus of fractions. We check directly that $\Sigma_{]-\infty,a]}$ admits also a left calculus of fractions. Indeed, (the duals of) axioms (CF1) and (CF2) are obviously fulfilled. For (CF3), let us consider any pair of morphisms $f^\bullet : B^\bullet \to A^\bullet$ and $s^\bullet : B^\bullet \to C^\bullet$ in $\mathsf{Hot}^{]-\infty,a]}(\mathscr{A})$, with $s^\bullet$ a quasi-isomorphism; since (the dual of) (CF3) is already known to hold for Σ, we may find morphisms $t : A \to D$ and $g : C \to D$ in $\mathsf{Hot}(\mathscr{A})$, such that $t \circ f = g \circ s$, and where t is a quasi-isomorphism. But then, by remark 7.3.14(iv), we may replace D (resp. t, resp. g) by $\tau^{\leq a} D$ (resp. by $\tau^{\leq a} t$, resp. by $\tau^{\leq a} g$), after which we may assume that $t \in \Sigma_{]-\infty,a]}$, whence (CF3). The proof of (CF4) is similar, and shall be left to the reader.

Dualizing the foregoing case, and taking into account remark 7.3.14(iii), we then get both assertions (i) and (ii) of the theorem also for "?" equal to $[a, +\infty[$, so the proof of theorem is complete for the case when "?" equals any unbounded interval $I \subset \mathbb{Z}$. One argues likewise to show the theorem in case "?" equals either $+$ or $-$ (details left to the reader). Lastly, for "?" equal to either b or an interval $[a, b]$, it suffices to apply proposition 1.6.21 and (iii) to the fully faithful inclusion functors

$$\mathsf{Hot}^{[a,b]}(\mathscr{A}) \to \mathsf{Hot}^{\geq a}(\mathscr{A}) \qquad \mathsf{Hot}^{b}(\mathscr{A}) \to \mathsf{Hot}^{+}(\mathscr{A})$$

and argue as in the foregoing cases to complete the proof of the claim. ◊

By claim 7.3.17(ii), it remains only to show that Σ admits a right calculus of fractions. To this aim, we check the conditions of definition 1.6.14. (CF1) and (CF2) are obvious. Next, consider two morphisms $\varphi^\bullet : A^\bullet \to B^\bullet$ and $\psi^\bullet : C^\bullet \to B^\bullet$ in $\mathsf{C}^?(\mathscr{A})$, with $\varphi^\bullet$ a quasi-isomorphism. Let $\beta^\bullet : B^\bullet \to (\mathrm{Cone}\,\psi)^\bullet$ be the natural morphism; with the notation of (7.3.2), we have an induced commutative diagram

$$\begin{array}{ccccccc} A^\bullet & \xrightarrow{\beta^\bullet\circ\varphi^\bullet} & (\mathrm{Cone}\,\psi)^\bullet & \longrightarrow & (\mathrm{Cone}\,\beta\circ\varphi)^\bullet & \longrightarrow & A^\bullet[1] \\ {\scriptstyle\varphi^\bullet}\downarrow & & \| & & \downarrow{\scriptstyle\gamma^\bullet} & & \downarrow{\scriptstyle\varphi^\bullet[1]} \\ B^\bullet & \xrightarrow{\beta^\bullet} & (\mathrm{Cone}\,\psi)^\bullet & \longrightarrow & (\mathrm{Cone}\,\beta)^\bullet & \longrightarrow & B^\bullet[1] \end{array}$$

where $\gamma^\bullet := \mathrm{Cone}\,(\varphi^\bullet, \mathbf{1})^\bullet$, and by the 5-lemma, it follows that $\gamma^\bullet$ is a quasi-isomorphism. However, lemma 7.3.9(ii) identifies $\gamma^\bullet$ with a morphism $(\mathrm{Cone}\,\beta\circ\varphi)^\bullet \to C^\bullet[1] \oplus (\mathbb{C}^\bullet \otimes_\mathbb{Z} B^\bullet)$, whence a commutative diagram

$$\begin{array}{ccc} (\mathrm{Cone}\,\beta\circ\varphi)^\bullet & \longrightarrow & A^\bullet[1] \\ {\scriptstyle\pi^\bullet\circ\gamma^\bullet}\downarrow & & \downarrow{\scriptstyle\varphi^\bullet[1]} \\ C^\bullet[1] & \xrightarrow{\psi^\bullet[1]} & B^\bullet[1] \end{array}$$

where $\pi^\bullet : C^\bullet[1] \oplus (\mathbb{C}^\bullet \otimes_\mathbb{Z} B^\bullet) \to C^\bullet[1]$ is the natural projection. Since $\mathbb{C}^\bullet \otimes_\mathbb{Z} B^\bullet$ is homotopically trivial (see the proof of lemma 7.3.9(ii)), we see that $\pi^\bullet$ is also a quasi-isomorphism, which shows (CF3). Lastly, let $f^\bullet, g^\bullet : X^\bullet \to Y^\bullet$ be any two morphisms in $\mathsf{Hot}^?(\mathscr{A})$, and $s^\bullet : Y^\bullet \to Z^\bullet$ a quasi-isomorphism such that $s^\bullet \circ f^\bullet = s^\bullet \circ g^\bullet$ in $\mathsf{Hot}^?(\mathscr{A})$. We rewrite the latter condition as $s^\bullet \circ (f^\bullet - g^\bullet) = 0$, and we notice the exact sequence

$$H^{-1}\mathrm{Hot}^\bullet_\mathscr{A}(X^\bullet, (\mathrm{Cone}\,s)^\bullet) \to H^0\mathrm{Hot}^\bullet_\mathscr{A}(X^\bullet, Y^\bullet) \to H^0\mathrm{Hot}^\bullet_\mathscr{A}(X^\bullet, Z^\bullet)$$

induced, via remark 7.3.5(i), by the distinguished triangle $\mathrm{Hot}^\bullet_{\mathscr{A}}(X^\bullet, \Theta(s^\bullet))$ provided by proposition 7.3.7. Taking into account example 7.1.15(i), we deduce that there exists a morphism

$$h^\bullet : X^\bullet \to (\mathrm{Cone}\, s^\bullet)[-1] \quad \text{such that} \quad \partial^\bullet_s[-1] \circ h^\bullet = f^\bullet - g^\bullet \qquad \text{in } \mathsf{Hot}^?(\mathscr{A})$$

where $\partial^\bullet_s : (\mathrm{Cone}\, s)^\bullet \to Y^\bullet[1]$ is the boundary morphism of the true triangle $\Theta(s^\bullet)$. Let now $\partial^\bullet_h : (\mathrm{Cone}\, h)^\bullet \to X^\bullet[1]$ be the boundary morphism of $\Theta(h^\bullet)$; from proposition 7.3.11 it follows easily that

$$h^\bullet \circ \partial^\bullet_h[-1] = 0 \qquad \text{in } \mathsf{Hot}^?(\mathscr{A})$$

whence $(f^\bullet - g^\bullet) \circ \partial^\bullet_h[-1] = 0$, and to deduce (CF4), it suffices to remark :

Claim 7.3.18. $\partial^\bullet_h$ is a quasi-isomorphism.

Proof of the claim. By assumption, $s^\bullet$ is a quasi-isomorphism; in light of the long exact sequence of remark 7.3.5(i), it follows easily that $H^i(\mathrm{Cone}\, s)^\bullet = 0$ for every $i \in \mathbb{Z}$. By the same token, we see that $H^i\partial^\bullet_h$ is an isomorphism for every $i \in \mathbb{Z}$, which is the claim. □

Corollary 7.3.19. *For every abelian category $\mathscr{A}$, and every interval $I \subset \mathbb{Z}$, we have :*

(i) $\mathsf{D}^?(\mathscr{A})$ *is an additive category for "?" equal to either $+, -, b$ or I.*

(ii) *If $\mathscr{A}$ is complete, and* its products are exact, i.e. *the functor* $\mathrm{Lim}_S : \mathsf{Fun}(S, \mathscr{A}) \to \mathscr{A}$ *is exact for every small discrete category S, then the products are representable in $\mathsf{D}^I(\mathscr{A})$, and the localization functor $j^I : \mathsf{C}^I(A) \to \mathsf{D}^I(A)$ commutes with direct products.*

(iii) *Dually, if $\mathscr{A}$ is cocomplete, and* its direct sums are exact, i.e. *the functor* Colim_S : $\mathsf{Fun}(S, \mathscr{A}) \to \mathscr{A}$ *is exact for every small discrete category S, then the direct sums are representable in $\mathsf{D}^I(\mathscr{A})$, and the localization functor j^I commutes with direct sums.*

Proof. (ii): In light of remark 7.1.6(vi), it suffices to check that the localization functor

$$\mathsf{Hot}^I(\mathscr{A}) \to \mathsf{D}^I(\mathscr{A})$$

commutes with direct products. Hence, consider any family $(K^\bullet_j \mid j \in J)$ of objects of $\mathsf{Hot}^I(\mathscr{A})$ indexed by a small set J, set $K^\bullet := \prod_{j\in J} K^\bullet_j$, the direct product in the category $\mathsf{C}^I(\mathscr{A})$, and denote by $\pi^\bullet_j : K^\bullet \to K^\bullet_j$ the canonical projection, for every $j \in J$; we know that $K^\bullet$ represents also the direct product of the family $K^\bullet_\bullet$ in $\mathsf{Hot}^I(\mathscr{A})$. Hence, let $L^\bullet$ be any other object of $\mathsf{Hot}^I(\mathscr{A})$, and $\varphi^\bullet, \varphi'^\bullet : L^\bullet \to K^\bullet$ any two morphisms in $\mathsf{D}^I(\mathscr{A})$ such that $\pi^\bullet_j \circ \varphi^\bullet = \pi^\bullet_j \circ \varphi'^\bullet$ for every $j \in J$; we need to check that $\varphi^\bullet = \varphi'^\bullet$. After composing with a quasi-isomorphism $L'^\bullet \to L^\bullet$ of $\mathsf{Hot}^I(\mathscr{A})$, we may assume that $\varphi^\bullet$ and $\varphi'^\bullet$ are two morphisms in $\mathsf{Hot}^I(\mathscr{A})$ (theorem 7.3.16(ii)), and we may find for every $j \in J$ a quasi-isomorphism $\psi^\bullet_j : K^\bullet_j \to K'^\bullet_j$ in $\mathsf{Hot}^I(\mathscr{A})$ such that $\psi^\bullet_j \circ \pi^\bullet_j \circ \varphi^\bullet = \psi^\bullet_j \circ \pi^\bullet_j \circ \varphi'^\bullet$. However, we have

$$H^r K^\bullet = \prod_{j\in J} H^r K^\bullet_j \qquad \text{for every } r \in \mathbb{Z} \tag{7.3.20}$$

and likewise for $K'^\bullet := \prod_{j\in J} K'^\bullet_j$, hence $\psi^\bullet := \prod_{j\in J} \psi^\bullet_j : K^\bullet \to K'^\bullet$ is a quasi-isomorphism with $\psi^\bullet \circ \varphi^\bullet = \psi^\bullet \circ \varphi'^\bullet$ in $\mathsf{Hot}^I(\mathscr{A})$, whence the assertion. Conversely, given a system of morphisms $(\varphi^\bullet_j : L^\bullet \to K^\bullet_j \mid j \in J)$ in $\mathsf{D}^I(\mathscr{A})$, we know by theorem 7.3.16(ii) that there exist for every $j \in J$ a quasi-isomorphism $\psi^\bullet_j : K^\bullet_j \to K'^\bullet_j$ and a morphism $\widetilde{\varphi}^\bullet_j : L^\bullet \to K'^\bullet$ in $\mathsf{C}^I(\mathscr{A})$ whose class in $\mathsf{Hot}^I(\mathscr{A})$ represents $\psi^\bullet_j \circ \varphi^\bullet_j$ in $\mathsf{D}^I(A)$. The systems $(\widetilde{\varphi}^\bullet_j \mid j \in J)$ and $(\psi^\bullet_j \mid j \in J)$ yield morphisms $\widetilde{\varphi}^\bullet : L^\bullet \to K'^\bullet := \prod_{j\in J} K'^\bullet_j$ and $\psi^\bullet : K^\bullet \to K'^\bullet$, and we come down to checking that $\psi^\bullet$ is a quasi-isomorphism. However, under the identification (7.3.20) and the corresponding one for $H^r K'^\bullet$, the map $H^r\psi^\bullet$ equals $\prod_{j\in J} H^r\psi^\bullet_j$, whence the contention.

(iii) follows from (ii), by duality.

(i): The proof of (ii) already shows that all finite products and coproducts are representable in $\mathsf{D}^?(\mathscr{A})$, and obviously the zero complex is a zero object for $\mathsf{D}^?(\mathscr{A})$. It remains to check that

$\mathsf{D}(\mathscr{A})$ is pre-additive. More precisely, there exists a unique pre-additive structure on $\mathsf{D}^?(\mathscr{A})$ such that the localization functor $\mathsf{Hot}^?(\mathscr{A}) \to \mathsf{D}^?(\mathscr{A})$ is additive. Indeed, given any pair of morphisms $\varphi^\bullet, \varphi'^\bullet : K^\bullet \to L^\bullet$ in $\mathsf{D}^?(\mathscr{A})$, we may find a quasi-isomorphism $\psi^\bullet : L^\bullet \to L'^\bullet$ in $\mathsf{Hot}^?(\mathscr{A})$ such that $\psi^\bullet \circ \varphi^\bullet$ and $\psi^\bullet \circ \varphi'^\bullet$ are represented by morphisms $\beta^\bullet, \beta'^\bullet : K^\bullet \to L'^\bullet$ of $\mathsf{Hot}^?(\mathscr{A})$ (theorem 7.3.16(ii)), and we set $\varphi^\bullet + \varphi'^\bullet := (\psi'^\bullet)^{-1} \circ (\beta^\bullet + \beta'^\bullet)$. It is easily seen that this definition does not depend on the choice of $\psi^\bullet$, and that it provides the required group law on $\mathrm{Hom}_{\mathsf{D}^?(\mathscr{A})}(K^\bullet, L^\bullet)$: the details are left to the reader. □

7.3.21. Let now $\mathscr{A}$ be an abelian category, $\mathscr{B}$ an additive subcategory of $\mathscr{A}$, and set

$$\Sigma_{\mathscr{B},+} := \Sigma \cap \mathrm{Morph}(\mathsf{Hot}^+(\mathscr{B})) \qquad \Sigma_{\mathscr{B},-} := \Sigma \cap \mathrm{Morph}(\mathsf{Hot}^-(\mathscr{B}))$$

where $\Sigma \subset \mathrm{Morph}(\mathsf{Hot}(\mathscr{A}))$ is the set of all quasi-isomorphisms. We may define the categories

$$\mathsf{D}^+_{\mathscr{B}}(\mathscr{A}) := \mathsf{Hot}^+(\mathscr{B})[\Sigma^{-1}_{\mathscr{B},+}] \qquad \mathsf{D}^-_{\mathscr{B}}(\mathscr{A}) := \mathsf{Hot}^-(\mathscr{B})[\Sigma^{-1}_{\mathscr{B},-}]$$

and clearly the inclusion functor $\mathscr{B} \to \mathscr{A}$ induces natural functors

$$i^+ : \mathsf{D}^+_{\mathscr{B}}(\mathscr{A}) \to \mathsf{D}^+(\mathscr{A}) \qquad i^- : \mathsf{D}^-_{\mathscr{B}}(\mathscr{A}) \to \mathsf{D}^-(\mathscr{A}).$$

Proposition 7.3.22. *In the situation of* (7.3.21)*, suppose additionally that the following holds :*

(a) *$\mathscr{B}$ is a full subcategory of $\mathscr{A}$.*

(b) *For every $A \in \mathrm{Ob}(\mathscr{A})$, there exists $B \in \mathrm{Ob}(\mathscr{B})$ with a monomorphism $A \to B$ in $\mathscr{A}$.*

Then the functor i^+ is an equivalence, and $\Sigma_{\mathscr{B},+}$ admits a left calculus of fractions.

Proof. Set $\Sigma_+ := \Sigma \cap \mathrm{Morph}(\mathsf{Hot}^+(\mathscr{A}))$. By theorem 7.3.16(i), it suffices to show that the induced functor

$$\mathsf{Hot}^+(\mathscr{B})[\Sigma^{-1}_{\mathscr{B},+}] \to \mathsf{Hot}^+(\mathscr{A})[\Sigma^{-1}_+]$$

is an equivalence. In light of remark 1.6.24 and theorem 7.3.16(ii), we are then reduced to checking :

Claim 7.3.23. Suppose that conditions (a) and (b) of the proposition hold. Then, for every $K^\bullet \in \mathsf{C}^+(\mathscr{A})$ there exists $L^\bullet \in \mathsf{C}^+(\mathscr{B})$ with a quasi-isomorphism $K^\bullet \to L^\bullet$.

Proof of the claim. Say that $K^\bullet \in \mathsf{C}^{\geq a}(\mathscr{A})$ for some $a \in \mathbb{Z}$; we shall construct first a double complex $A^{\bullet\bullet} \in \mathsf{C}^{\geq a}(\mathsf{C}^{\geq -1}(\mathscr{A}))$ such that

- $A^{p,-1} = K^p$ and $d_h^{p-1,-1} = d_K^{p-1}$ for every $p \in \mathbb{Z}$.
- $A^{pq} \in \mathrm{Ob}(\mathscr{B})$ for every $p \in \mathbb{Z}$ and every $q \geq 0$.
- For every $p \in \mathbb{Z}$, the cohomology of the complex $(A^{p\bullet}, d_v^{p\bullet})$ vanishes in every degree.

To this aim, we proceed by induction on p. Hence, we let $A^{pq} = 0$ for every $p < a$ and every $q \in \mathbb{Z}$. Suppose that $p \geq a$, and that A^{ij} is already defined for every $i < a$ and every $j \in \mathbb{Z}$, as well as all the differentials d_v^{ij} and $d_h^{i-1,j}$ for every $i < a$ and every $j \in \mathbb{Z}$. Then we construct A^{pj} and the differentials $d_v^{p,j-1}$, $d_h^{p-1,j}$ by induction on $j \in \mathbb{Z}$. Namely, for $j < -1$ we set $A^{pj} = 0$, and for $j = -1$ we set $A^{p,-1} = K^p$ and $d_h^{p-1,-1} = d_K^{p-1}$. Thus, suppose that $j \geq 0$, and $A^{p,j-1}$ is already given, as well as $d_h^{p-1,j-1}$ and $d_h^{p,j-2}$; we define the object B^{pj} of $\mathscr{A}$ as the coproduct in the push-out diagram

$$\begin{array}{ccc} A^{p-1,j} & \xrightarrow{\quad g \quad} & B^{pj} \\ {\scriptstyle \overline{d}_v}\uparrow & & \uparrow{\scriptstyle f} \\ \mathrm{Coker}\, d_v^{p-1,j-2} & \xrightarrow{\quad \overline{d}_h \quad} & \mathrm{Coker}\, d_v^{p,j-2} \end{array}$$

where $\overline{d}_v$ and $\overline{d}_h$ are induced by $d_v^{p-1,j-1}$ and respectively $d_h^{p-1,j-1}$. Notice that, by inductive assumption, $\overline{d}_v$ is a monomorphism, so the same holds for f (details left to the reader). By assumption (b), we may find a monomorphism $h : B^{pq} \to C$ for some $C \in \mathrm{Ob}(\mathscr{B})$, and we set $A^{pq} := C$, $d_h^{p-1,j} := h \circ g$ and $d_v^{p,j-1} := h \circ f \circ p$, where $p : A^{p,j-1} \to \mathrm{Coker}\, d_v^{p,j-2}$ is the natural projection. Then it is easily seen that $d_h^{p-1,j} \circ d_v^{p-1,j-1} = d_v^{p,j-1} \circ d_h^{p-1,j-1}$, and moreover

$$\mathrm{Ker}\, d_v^{p,j-1} / \mathrm{Im}\, d_v^{p,j-2} = 0$$

as required. Now, let $t^{\geq 0} : \mathsf{C}_2(\mathscr{A}) \to \mathsf{C}(\mathsf{C}^{\geq 0}(\mathscr{A}))$ be the brutal truncation functor, and set

$$B^{\bullet\bullet} := t^{\geq 0} A^{\bullet\bullet} \qquad C^{\bullet\bullet} := K^\bullet[0] \qquad L^\bullet := \mathrm{Tot}\, B^{\bullet\bullet}.$$

(Hence, $C^{\bullet\bullet}$ is the double complex whose rows $C^{\bullet q}$ are the zero complex, except for $q = 0$, where it agrees with $K^\bullet$). The differentials $d_v^{\bullet,-1}$ of $A^{\bullet\bullet}$ induce a morphism $f^\bullet : K^\bullet \to L^\bullet$, and it remains only to check that $f^\bullet$ is a quasi-isomorphism. To this aim, notice that the differentials $d_v^{\bullet,-1}$ also induce a morphism of double complexes

$$g^{\bullet\bullet} : C^{\bullet\bullet} \to B^{\bullet\bullet}$$

such that $\mathrm{Tot}\, g^{\bullet\bullet} = f^\bullet$. Moreover, we have standard convergent spectral sequences

$$E_2^{pq} := H^q(C^{p,\bullet}) \Rightarrow H^{p+q}\mathrm{Tot}\, C^{\bullet\bullet} \qquad F_2^{pq} := H^q(B^{p,\bullet}) \Rightarrow H^{p+q}\mathrm{Tot}\, B^{\bullet\bullet}$$

and by construction, $g^{\bullet\bullet}$ induces an isomorphism of spectral sequences $E_2^{\bullet\bullet} \stackrel{\sim}{\to} F_2^{\bullet\bullet}$, whence the claim. □

Remark 7.3.24. (i) In view of remark 7.3.14(iii), we see that the dual of proposition 7.3.22 also holds. Namely, in the situation of (7.3.21), suppose that

(a) $\mathscr{B}$ is a full subcategory of $\mathscr{A}$.
(b) For every $A \in \mathrm{Ob}(\mathscr{A})$, there exists $B \in \mathrm{Ob}(\mathscr{B})$ with an epimorphism $B \to A$ in $\mathscr{A}$.

Then the functor i^- is an equivalence, and $\Sigma_{\mathscr{B},-}$ admits a right calculus of fractions.

(ii) Two special cases of proposition 7.3.22 (and its dual) are especially important. Namely, let $\mathscr{I}$ (resp. $\mathscr{P}$) be the full subcategory of $\mathscr{A}$ whose objects are the injective (resp. projective) objects of $\mathscr{A}$. We say that $\mathscr{A}$ has *enough injectives* (resp. *enough projectives*) if $\mathscr{I}$ (resp. $\mathscr{P}$) satisfies condition (b) of proposition 7.3.22 (resp. condition (b) of (i)). When this holds, proposition 7.3.22 (and its dual) can be further sharpened; namely, we have :

Theorem 7.3.25. *Let $\mathscr{A}$ be any abelian category, and define the subcategories $\mathscr{I}$ and $\mathscr{P}$ of $\mathscr{A}$ as in remark* 7.3.24(ii)*. Then the following holds :*

(i) *Let $I^\bullet$ be any object of $\mathsf{C}^+(\mathscr{I})$. Let also $L^\bullet$ be any object of $\mathsf{C}(\mathscr{A})$, and $\alpha^\bullet : I^\bullet \to L^\bullet$ any quasi-isomorphism. Then there exists a morphism $\beta : L^\bullet \to I^\bullet$ in $\mathsf{C}(\mathscr{A})$ such that*

$$\beta^\bullet \circ \alpha^\bullet = \mathbf{1}_{I^\bullet} \qquad \textit{in } \mathsf{Hot}(\mathscr{A}).$$

(ii) *If $\mathscr{A}$ has enough injectives, the inclusion functor $\mathscr{I} \to \mathscr{A}$ induces an equivalence*

$$\mathsf{Hot}^+(\mathscr{I}) \stackrel{\sim}{\to} \mathsf{D}^+(\mathscr{A}).$$

(iii) *Dually, if $\mathscr{A}$ has enough projectives, the inclusion $\mathscr{P} \to \mathscr{A}$ induces an equivalence*

$$\mathsf{Hot}^-(\mathscr{P}) \stackrel{\sim}{\to} \mathsf{D}^-(\mathscr{A}).$$

Proof. (i): To begin with, we remark :

Claim 7.3.26. In the situation of (i), let also $K^\bullet$ be any acyclic complex of $\mathscr{A}$, and $\varphi^\bullet : K^\bullet \to I^\bullet$ any morphism in $\mathsf{C}(\mathscr{A})$. Then $\varphi^\bullet$ is null-homotopic.

Proof of the claim. We need to exhibit a system of morphisms $(s^n : K^n \to I^{n-1} \mid n \in \mathbb{Z})$ with

$$\varphi^n = d_I^{n-1} \circ s^n + s^{n+1} \circ d_K^n \qquad (7.3.27)$$

for every $n \in \mathbb{Z}$. To this aim, say that $I^\bullet \in \mathrm{Ob}(\mathsf{C}^{\geq a}(\mathscr{A}))$ for some $a \in \mathbb{Z}$; then we let s^j be the zero morphism for every $j \leq a$. Next, suppose that $j \geq a$, and that s^j has already been given, so that (7.3.27) holds with $n = j-1$. We compute :

$$
\begin{aligned}
(\varphi^j - d_I^{j-1} \circ s^j) \circ d_K^{j-1} &= d_I^{j-1} \circ \varphi^{j-1} - d_I^{j-1} \circ s^j \circ d_K^{j-1} \\
&= d_I^{j-1} \circ (\varphi^{j-1} - s^j \circ d_K^{j-1}) \\
&= d_I^{j-1} \circ d_I^{j-2} \circ s^{j-1} \\
&= 0.
\end{aligned}
$$

Since $\mathrm{Im}\, d_K^{j-1} = \mathrm{Ker}\, d_K^j$, it follows that $\varphi^j - d_I^{j-1} \circ s^j$ factors through a morphism $\mathrm{Im}\, d_K^j \to I^j$, and since I^j is injective, the latter can be extended to a morphism $s^{j+1} : K^{j+1} \to I^j$. By construction, (7.3.27) holds for $n = j$, with this choice of s^{j+1}, whence the claim. $\diamondsuit$

Now, under the condition of (i), the complex $\mathrm{Cone}\,\alpha^\bullet$ is acyclic (remark 7.3.5(iii)), so the boundary morphism $\partial^\bullet : \mathrm{Cone}\,\alpha^\bullet \to I^\bullet[1]$ is null-homotopic, by claim 7.3.26. Let us then fix a homotopy $s^\bullet$ from the zero morphism to $\partial^\bullet$; in each degree $j \in \mathbb{Z}$, the morphism s^j is the sum of a morphism $s_L^j : L^j \to I^j$ and a morphism $s_I^j : I^{j+1} \to I^j$, and the relation

$$
-\partial^j = d_{I[1]}^{j-1} \circ s^j + s^{j+1} \circ d_{\mathrm{Cone}\,\varphi}^j \qquad \text{for every } j \in \mathbb{Z}
$$

translates as the pair of identities :

$$
\begin{aligned}
-d_I^j \circ s_L^j + s_L^{j+1} \circ d_L^j &= 0 \\
-d_I^j \circ s_I^j + s_L^{j+1} \circ \alpha^{i+1} + s_I^{j+1} \circ d_I^{i+1} &= \mathbf{1}_{I^{j+1}}
\end{aligned}
$$

the first of which says that the system $(s_L^j \mid j \in \mathbb{Z})$ amounts to a morphism of complexes $s_L^\bullet : L^\bullet \to I^\bullet$, and the second says that the system $(s_I^j \mid j \in \mathbb{Z})$ yields a homotopy from $\mathbf{1}_I$ to $s_L^\bullet \circ \alpha^\bullet$. In other words, the morphism $\beta^\bullet := s_L^\bullet$ will do.

(ii): By proposition 7.3.22, it suffices to check that every quasi-isomorphism in $\mathsf{Hot}^+(\mathscr{I})$ is an isomorphism. But this follows immediately from (i).

(iii): Clearly the subcategory of injective objects of $\mathscr{A}^o$ is $\mathscr{P}^o$, so the assertion follows from (ii) and remark 7.3.14(iii). $\square$

Corollary 7.3.28. *With the notation of theorem* 7.3.25*, the natural map*

$$
\mathrm{Hot}_{\mathscr{A}}(K^\bullet, I^\bullet) \to \mathrm{Hom}_{\mathsf{D}(\mathscr{A})}(K^\bullet, I^\bullet)
$$

is an isomorphism, for every $K^\bullet \in \mathrm{Ob}(\mathsf{C}(\mathscr{A}))$ and every $I^\bullet \in \mathrm{Ob}(\mathsf{C}^+(\mathscr{I}))$.

Proof. By virtue of theorem 7.3.16, every morphism $g^\bullet : K^\bullet \to I^\bullet$ in $\mathsf{D}(\mathscr{A})$ is represented by a fraction $(f^\bullet, \alpha^\bullet)$, where $f^\bullet : K^\bullet \to L^\bullet$ is any morphism in $\mathsf{Hot}(\mathscr{A})$, and $\alpha^\bullet : I^\bullet \to L^\bullet$ is a quasi-isomorphism. Then, by theorem 7.3.25(i) we may find a morphism $\beta^\bullet : L^\bullet \to I^\bullet$ in $\mathsf{Hot}(\mathscr{A})$ such that $\beta^\bullet \circ \alpha^\bullet = \mathbf{1}_{I^\bullet}$; clearly $\beta^\bullet$ is also a quasi-isomorphism, and $g^\bullet = \beta^\bullet \circ f^\bullet$ in $\mathsf{D}(\mathscr{A})$, so the map of the corollary is surjective. For the injectivity, suppose that $f_1^\bullet, f_2^\bullet : K^\bullet \to I^\bullet$ are any two morphisms in $\mathsf{Hot}(\mathscr{A})$ whose images agree in $\mathsf{D}(\mathscr{A})$; invoking again theorem 7.3.16, we then find a quasi-isomorphism $\alpha^\bullet : I^\bullet \to L^\bullet$ in $\mathsf{Hot}(\mathscr{A})$ such that $\alpha^\bullet \circ f_1^\bullet = \alpha^\bullet \circ f_2^\bullet$ in $\mathsf{Hot}(\mathscr{A})$. Then pick again $\beta^\bullet$ as in the foregoing; we deduce that

$$
f_1^\bullet = \beta^\bullet \circ \alpha^\bullet \circ f_1^\bullet = \beta^\bullet \circ \alpha^\bullet \circ f_2^\bullet = f_2^\bullet \qquad \text{in } \mathsf{Hot}(\mathscr{A})
$$

whence the assertion. $\square$

Definition 7.3.29. Let $\mathscr{A}$ and $\mathscr{B}$ be two abelian categories, and $F : \mathsf{C}^+(\mathscr{A}) \to \mathsf{D}(\mathscr{B})$ a functor. Denote $\omega_{\mathscr{A}}^+ : \mathsf{C}^+(\mathscr{A}) \to \mathsf{D}^+(\mathscr{A})$ and $\omega_{\mathscr{A}}^- : \mathsf{C}^-(\mathscr{A}) \to \mathsf{D}^-(\mathscr{A})$ the localization functors.

(i) A *right derived functor* of F is a pair (RF, μ) consisting of a functor

$$RF : \mathsf{D}^+(\mathscr{A}) \to \mathsf{D}(\mathscr{B})$$

and a natural transformation

$$\mu : F \Rightarrow RF \circ \omega^+_{\mathscr{A}}$$

which satisfies the following universal property. For every other pair (G, ζ) consisting of a functor $G : \mathsf{D}^+(\mathscr{A}) \to \mathsf{D}(\mathscr{B})$ and a natural transformation $\zeta : F \Rightarrow G \circ \omega^+_{\mathscr{A}}$, there exists a unique natural transformation $\xi : RF \Rightarrow G$ such that

$$\zeta = (\xi * \omega^+_{\mathscr{A}}) \circ \mu. \tag{7.3.30}$$

(ii) Dually, if $F : \mathsf{C}^-(\mathscr{A}) \to \mathsf{D}(\mathscr{B})$ is any functor, a *left derived functor* of F is a pair (LF, μ) consisting of a functor

$$LF : \mathsf{D}^-(\mathscr{A}) \to \mathsf{D}(\mathscr{B})$$

and a natural transformation

$$\mu : F \Rightarrow LF \circ \omega^-_{\mathscr{A}}$$

such that $((LF)^o, \mu^o)$ is a right derived functor of F^o, under the natural identifications of remark 7.3.14(iii).

(iii) Especially, let $\varphi : \mathscr{A} \to \mathscr{B}$ be any additive functor, and for "?" equal to $+$ or $-$, let $\omega^?_{\mathscr{B}} : \mathsf{C}^?(\mathscr{B}) \to \mathsf{D}^?(\mathscr{B})$ be the localization functor. Then a right (resp. left) derived functor of $\omega^+_{\mathscr{B}} \circ \mathsf{C}^+(\varphi)$ (resp. of $\omega^-_{\mathscr{B}} \circ \mathsf{C}^-(\varphi)$) shall just be called a *right* (resp. *left*) *derived functor* of φ, and we shall just write $R\varphi$ (resp. $L\varphi$) to denote this functor.

Remark 7.3.31. Keep the situation of definition 7.3.29.

(i) As usual, if the right derived functor of F exists, it is unique up to isomorphism. More precisely, let (RF, μ) be any pair as in definition 7.3.29(i); if (G, ζ) is any other such pair fulfilling the same universal condition, there exists a unique isomorphism of functors $\xi : RF \xrightarrow{\sim} G$ such that (7.3.30) holds. Likewise one characterizes left functors up to isomorphism.

(ii) For F as in definition 7.3.29(i) (resp. definition 7.3.29(ii)), we shall use, for every $p \in \mathbb{Z}$ and every object $K^\bullet$ of $\mathsf{D}^+(\mathscr{A})$ (resp. of $\mathsf{D}^-(\mathscr{A})$), the standard notation :

$$R^pFK^\bullet := H^p(RFK^\bullet) \qquad (\text{resp. } L^pFK^\bullet := H^p(LFK^\bullet)).$$

Also, if φ is as in definition 7.3.29(iii), and the right (resp. left) derived functor of φ exists, then we have a natural transformation

$$\varphi A \to R^0\varphi A[0] \qquad (\text{resp. } L^0\varphi A[0] \to \varphi A) \qquad \text{for every } A \in \mathrm{Ob}(\mathscr{A})$$

and it is easily seen that this transformation is an isomorphism of functors if and only φ is left exact (resp. right exact : details left to the reader). We say that an object A of $\mathscr{A}$ is *φ-acyclic* if $R^p\varphi A[0] = 0$ (resp. $L^p\varphi A[0] = 0$) for every $p \neq 0$.

(iii) Suppose that $\mathscr{A}$ has enough injectives (see remark 7.3.24(ii)), and $F : \mathsf{C}^+(\mathscr{A}) \to \mathsf{D}(\mathscr{B})$ is a functor that factors through $\mathsf{Hot}^+(\mathscr{A})$ (via the natural functor (7.1.5)). Then the right derived functor of F exists, and can be constructed as follows. First, say that $f^\bullet : I^\bullet \to J^\bullet$ is a quasi-isomorphism, with $I^\bullet, J^\bullet \in \mathrm{Ob}(\mathsf{C}^+(\mathscr{I}))$ (where $\mathscr{I}$ is as in remark 7.3.24(ii)). By theorem 7.3.25(i), it follows that $f^\bullet$ is a homotopy equivalence, so $Ff^\bullet$ is an isomorphism by assumption, and therefore F induces a functor

$$RF_{\mathscr{I}} : \mathsf{D}^+_{\mathscr{I}}(\mathscr{A}) \to \mathsf{D}^+(\mathscr{B})$$

(notation of (7.3.21)). However, according to proposition 7.3.22, the natural functor $i^+ : \mathsf{D}^+_{\mathscr{I}}(\mathscr{A}) \to \mathsf{D}^+(\mathscr{A})$ admits a quasi-inverse $j^+ : \mathsf{D}^+(\mathscr{A}) \to \mathsf{D}^+_{\mathscr{I}}(\mathscr{A})$; in view of claim 7.3.23, the functor j^+ can be described by choosing, for every object $K^\bullet$ of $\mathsf{C}^+(\mathscr{A})$, a quasi-isomorphism

$$\tau^\bullet_K : K^\bullet \to I^\bullet_K \qquad \text{with } I^\bullet_K \in \mathrm{Ob}(\mathsf{C}^+(\mathscr{I})).$$

With this notation, we let

$$RF := RF_{\mathscr{I}} \circ j^+$$

and we define a natural transformation $\mu : F \Rightarrow RF \circ \omega^+_{\mathscr{A}}$, by the rule :

$$K^\bullet \mapsto F(\tau^\bullet_K) \qquad \text{for every } K^\bullet \in \mathrm{Ob}(\mathsf{C}^+(\mathscr{A})).$$

Now, suppose that $G : \mathsf{D}^+(\mathscr{A}) \to \mathsf{D}(\mathscr{B})$ is another functor, with a natural transformation $\zeta : F \Rightarrow G \circ \omega^+_{\mathscr{A}}$. There follows a commutative diagram in $\mathsf{D}(\mathscr{B})$

$$\begin{array}{ccc} FK^\bullet & \xrightarrow{\zeta^\bullet_K} & GK^\bullet \\ {\scriptstyle F\tau^\bullet_K}\downarrow & & \downarrow{\scriptstyle G\tau^\bullet_K} \\ FI^\bullet_K & \xrightarrow{\zeta^\bullet_{I_K}} & GI^\bullet_K \end{array}$$

in which $G\tau^\bullet_K$ is an isomorphism. It follows that (7.3.30) holds if and only if $\xi_K = (G\tau^\bullet_K)^{-1} \circ \zeta^\bullet_{I_K}$ for every $K^\bullet \in \mathrm{Ob}(\mathsf{C}^+(\mathscr{A}))$, and therefore (RF, μ) is a derived functor of F, as stated. This is essentially the original construction of the right derived functor proposed by Grothendieck in [84], which predates the invention of derived categories. Dually, one obtains likewise a left derived functor of F, in case $\mathscr{A}$ has enough projectives : in this case, one starts by fixing, for every bounded above complex $K^\bullet$ of $\mathscr{A}$, a quasi-isomorphism $P^\bullet_K \to K^\bullet$ with $P^\bullet$ in $\mathsf{C}^+(\mathscr{P})$.

(iv) A slight generalization of [84, Lemme 3.3.1] yields a construction which works only for additive functors $F : \mathscr{A} \to \mathscr{B}$, but which requires only that $\mathscr{A}$ has *enough F-injective objects*. Namely, let us suppose that $\mathscr{A}$ contains a full additive subcategory $\mathscr{M}$ such that :

(a) For every $A \in \mathrm{Ob}(\mathscr{A})$, there exists a monomorphism $A \to M$, with $M \in \mathrm{Ob}(\mathscr{M})$.
(b) For every acyclic bounded below complex $K^\bullet \in \mathsf{C}^+(\mathscr{M})$, the complex $FK^\bullet$ is acyclic.

Then we say that $\mathscr{M}$ is an *F-injective subcategory of $\mathscr{A}$*, and we obtain a right derived functor of F as follows. Let $f^\bullet : M^\bullet \to N^\bullet$ be any quasi-isomorphism in $\mathsf{C}^+(\mathscr{M})$; then $\mathrm{Cone}\, f^\bullet$ is acyclic, so the same holds for $F(\mathrm{Cone}\, f^\bullet)$, by (b), and since $\mathsf{C}^+(F)$ is a triangulated functor (remark 7.3.5(ii)), we see that $Ff^\bullet$ is still a quasi-isomorphism. Thus, $\mathsf{C}^+(F)$ induces a functor

$$RF_{\mathscr{M}} : \mathsf{D}^+_{\mathscr{M}}(\mathscr{A}) \to \mathsf{D}^+(\mathscr{B}).$$

On the other hand, proposition 7.3.22 says that the natural functor $i^+ : \mathsf{D}^+_{\mathscr{M}}(\mathscr{A}) \to \mathsf{D}^+(\mathscr{A})$ admits a quasi-inverse $j^+ : \mathsf{D}^+(\mathscr{A}) \to \mathsf{D}^+_{\mathscr{M}}(\mathscr{A})$, so we may set again

$$RF := RF_{\mathscr{M}} \circ j^+$$

and one obtains as in (iii) a natural transformation μ, such that the pair (RF, μ) fulfills the required universal property : details left to the reader. This derived functor RF is triangulated (see remark 7.3.14(vi)); moreover, the construction implies that every object of $\mathscr{M}$ is F-acyclic.

(v) Dually, we say that $\mathscr{A}$ has *enough F-projective objects* (for F as in (iv)) if the category $\mathscr{A}^o$ has enough F^o-objects, *i.e.* there exists a full additive subcategory $\mathscr{M}$ such that :

(a) For every $A \in \mathrm{Ob}(\mathscr{A})$, there exists an epimorphism $M \to A$, with $M \in \mathrm{Ob}(\mathscr{M})$.
(b) For every acyclic bounded above complex $K^\bullet \in \mathsf{C}^-(\mathscr{M})$, the complex $FK^\bullet$ is acyclic.

In this case we say that $\mathscr{M}$ is an *F-projective subcategory of $\mathscr{A}$*, and by dualizing the constructions of (iv) one obtains a triangulated left derived functor LF; also, every object of $\mathscr{M}$ is F-acyclic. This terminology agrees with that of [110, Def.13.3.4] (to check the equivalence of our definition with that of *loc.cit.* one can argue as in the proof of [110, Prop.13.3.5(ii)]).

(vi) The above quoted [84, Lemme 3.3.1] deals with the special case of (iv), where the additive full subcategory $\mathscr{M}$ fulfills condition (a), but (b) is replaced by the more restrictive :

(c) For every $M, M' \in \mathrm{Ob}(\mathscr{M})$ and every short exact sequence in $\mathscr{A}$

$$\Sigma \quad : \quad 0 \to M' \to M \to M'' \to 0$$

the object M'' is also in $\mathscr{M}$, and Σ induces a short exact sequence

$$F(\Sigma) \quad : \quad 0 \to F(M') \to F(M) \to F(M'') \to 0.$$

To see that (c)⇒(b), let $(M^\bullet, d^\bullet_M)$ be any acyclic object of $\mathsf{C}^+(\mathscr{M})$; an easy induction shows that the induced sequences

$$\Sigma^i \quad : \quad 0 \to \mathrm{Im}\, d^i \to M^{i+1} \to \mathrm{Im}\, d^{i+1} \to 0$$

are short exact and $\mathrm{Im}\, d^i \in \mathrm{Ob}(\mathscr{M})$ for every $i \in \mathbb{Z}$. By assumption, the sequences $F(\Sigma^i)$ are then also short exact, so $F(\mathrm{Im}\, d^i) = \mathrm{Im}(Fd^i)$ for every $i \in \mathbb{Z}$, and $FM^\bullet$ is acyclic. Notice also that any short exact sequence $0 \to I' \to I \to I'' \to 0$ in $\mathscr{A}$ with I' injective, is split, and I'' is injective if and only if the same holds for I. Hence, if $\mathscr{A}$ has enough injectives, the category $\mathscr{M} := \mathscr{I}$ fulfills conditions (a) and (c) for every additive functor F. Conversely, suppose that $\mathscr{M}$ fulfills condition (a), and moreover, that every $A \in \mathrm{Ob}(\mathscr{C})$ isomorphic to a direct factor of an object of $\mathscr{M}$, is an object of $\mathscr{M}$. Then $\mathscr{M}$ contains every injective object of $\mathscr{A}$. Indeed, if I is an injective object of $\mathscr{C}$, then by (a) we can find a monomorphism $I \to M$ with M in $\mathscr{M}$; hence I is a direct summand of M, so it is in $\mathscr{M}$, by our assumption.

Example 7.3.32. Let $\mathscr{A}$ be any abelian category with enough injectives; recall that the functor

$$\mathrm{Hom}^\bullet_{\mathscr{A}} : \mathsf{C}(\mathscr{A}) \times \mathsf{C}(\mathscr{A})^o \to \mathsf{C}(\mathbb{Z}\text{-}\mathbf{Mod})$$

factors through $\mathsf{Hot}(\mathscr{A}) \times \mathsf{Hot}(\mathscr{A})^o$ (see example 7.1.25); hence, after fixing an equivalence $j^+ : \mathsf{D}^+(\mathscr{A}) \to \mathsf{D}^+_{\mathscr{I}}(\mathscr{A})$ and arguing as in remark 7.3.31(iii), we deduce a functor

$$R\mathrm{Hom}^\bullet_{\mathscr{A}} : \mathsf{D}^+(\mathscr{A}) \times \mathsf{Hot}(\mathscr{A})^o \to \mathsf{D}(\mathbb{Z}\text{-}\mathbf{Mod})$$

such that, for every $K^\bullet \in \mathrm{Ob}(\mathsf{Hot}(\mathscr{A}))$, the restriction

$$\mathsf{D}^+(\mathscr{A}) \to \mathsf{D}(\mathbb{Z}\text{-}\mathbf{Mod}) \quad : \quad L^\bullet \mapsto R\mathrm{Hom}^\bullet_{\mathscr{A}}(K^\bullet, L^\bullet) \tag{7.3.33}$$

is the derived functor of the functor $\mathrm{Hom}^\bullet_{\mathscr{A}}(K^\bullet, -) : \mathsf{C}^+(\mathscr{A}) \to \mathsf{D}(\mathbb{Z}\text{-}\mathbf{Mod})$. Moreover, (7.3.33) is triangulated, by proposition 7.3.7. Furthermore, let $\varphi^\bullet : K^\bullet_1 \to K^\bullet_2$ be any quasi-isomorphism in $\mathsf{Hot}(\mathscr{A})$; we claim that $R\mathrm{Hom}^\bullet_{\mathscr{A}}(\varphi^\bullet, L^\bullet)$ is a quasi-isomorphism, for every $L^\bullet \in \mathrm{Ob}(\mathsf{D}^+(\mathscr{A}))$. Indeed, set $I^\bullet_L := j^+ L^\bullet$; by example 7.1.15(i) and corollary 7.3.28 we have natural isomorphisms of abelian groups

$$H^n R\mathrm{Hom}^\bullet_{\mathscr{A}}(\varphi^\bullet, L^\bullet) \xrightarrow{\sim} \mathrm{Hot}_{\mathscr{A}}(\varphi^\bullet, I^\bullet_L[n]) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{D}(\mathscr{A})}(\varphi^\bullet, I^\bullet_L[n])$$

whence the contention. It follows that $R\mathrm{Hom}^\bullet_{\mathscr{A}}$ descends to an additive functor

$$R\mathrm{Hom}^\bullet_{\mathscr{A}} : \mathsf{D}^+(\mathscr{A}) \times \mathsf{D}(\mathscr{A})^o \to \mathsf{D}(\mathbb{Z}\text{-}\mathbf{Mod})$$

with a natural isomorphism of abelian groups, for every $K^\bullet$ in $\mathsf{D}(\mathscr{A})$ and every $L^\bullet$ in $\mathsf{D}^+(\mathscr{A})$:

$$R^n\mathrm{Hom}^\bullet_{\mathscr{A}}(K^\bullet, L^\bullet) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{D}(\mathscr{A})}(K^\bullet, L^\bullet[n]). \tag{7.3.34}$$

Remark 7.3.35. In the situation of example 7.3.32, our methods do not allows us to extend the functor $R\mathrm{Hom}^\bullet_{\mathscr{A}}$ to unbounded complexes in both arguments, but still we can show that every $L^\bullet \in \mathsf{D}(\mathscr{A})$ and every distinguished triangle of $\mathsf{D}(\mathscr{A})$

$$\Theta \quad : \quad K^\bullet \to K'^\bullet \to K''^\bullet \to K^\bullet[1]$$

induce a long exact Ext-sequence :

$$\mathrm{Hom}_{\mathsf{D}(\mathscr{A})}(L^\bullet, K^\bullet) \to \mathrm{Hom}_{\mathsf{D}(\mathscr{A})}(L^\bullet, K'^\bullet) \to \mathrm{Hom}_{\mathsf{D}(\mathscr{A})}(L^\bullet, K''^\bullet) \to \mathrm{Hom}_{\mathsf{D}(\mathscr{A})}(L^\bullet, K^\bullet[1]) \to \cdots$$

Indeed, let $L'^\bullet \to L^\bullet$ be any quasi-isomorphism; we may assume that Θ is already a distinguished triangle of $\mathsf{Hot}(\mathscr{A})$, and due to proposition 7.3.7 and remark 7.3.5(iii), we get the long exact sequence

$$\mathrm{Hot}_{\mathscr{A}}(L'^\bullet, K^\bullet) \to \mathrm{Hot}_{\mathscr{A}}(L'^\bullet, K'^\bullet) \to \mathrm{Hot}_{\mathscr{A}}(L'^\bullet, K''^\bullet) \to \mathrm{Hot}_{\mathscr{A}}(L'^\bullet, K^\bullet[1]) \to \cdots$$

and on the other hand, theorem 7.3.16(ii) implies that $\mathrm{Hom}_{\mathsf{D}(\mathscr{A})}(L^\bullet, K^\bullet)$ is isomorphic to the colimit of the system $(\mathrm{Hot}_{\mathscr{A}}(L'^\bullet, K^\bullet) \mid L'^\bullet \to L^\bullet)$ indexed by the filtered set of all such quasi-isomorphisms (proposition 1.6.16(i)). Since all filtered colimits are exact in the category of abelian groups, the assertion follows. Likewise, from Θ and any $L^\bullet \in \mathrm{Ob}(\mathsf{D}(\mathscr{A}))$ we get as well the long exact Ext-sequence :

$$\mathrm{Hom}_{\mathsf{D}(\mathscr{A})}(K^\bullet[1], L^\bullet) \to \mathrm{Hom}_{\mathsf{D}(\mathscr{A})}(K''^\bullet, L^\bullet) \to \mathrm{Hom}_{\mathsf{D}(\mathscr{A})}(K'^\bullet, L^\bullet) \to \mathrm{Hom}_{\mathsf{D}(\mathscr{A})}(K^\bullet, L^\bullet) \to \cdots$$

Example 7.3.36. (i) Remark 7.3.31(iv) can be adapted to biadditive functors. Namely, in the situation of (7.1.23), let us say that an object A of $\mathscr{A}$ (resp. A' of $\mathscr{A}'$) is *left B-flat* (resp. *right B-flat*) if the additive functor $B(A, -)$ (resp. $B(-, A')$) is exact. We assume that :

- $\mathscr{A}$ has *enough left B-flat objects* and $\mathscr{A}'$ has *enough right B-flat objects*; *i.e.* for every $M \in \mathrm{Ob}(\mathscr{A})$ (resp. $M' \in \mathrm{Ob}(\mathscr{A}')$) there exists an epimorphism $A \to M$ (resp. $A' \to M'$), where A (resp. A') is a left (resp. right) B-flat object of $\mathscr{A}$ (resp. of $\mathscr{A}'$).

Let us denote by $\mathscr{F}$ (resp. $\mathscr{F}'$) the full subcategory of $\mathscr{A}$ (resp. of $\mathscr{A}'$) whose objects are the left B-flat (resp. right B-flat) objects of $\mathscr{A}$ (of $\mathscr{A}'$). By remark 7.3.24(i), the natural functors

$$\mathsf{D}^-_{\mathscr{F}}(\mathscr{A}) \to \mathsf{D}^-(\mathscr{A}) \qquad \mathsf{D}^-_{\mathscr{F}'}(\mathscr{A}') \to \mathsf{D}^-(\mathscr{A}') \tag{7.3.37}$$

are equivalences. On the other hand, let $P^\bullet$ be any object of $\mathsf{C}^-(\mathscr{F})$, and $f^\bullet : K^\bullet \to L^\bullet$ any quasi-isomorphism in $\mathsf{C}^-(\mathscr{A}')$; we wish to show that the induced morphism $B^\bullet_-(P^\bullet, f^\bullet)$ is a quasi-isomorphism as well. By remark 7.3.8, it suffices to check that $B^\bullet_-(P^\bullet, \mathrm{Cone}\, f^\bullet)$ is acyclic. However, the double complex $B^{\bullet\bullet}(P^\bullet, \mathrm{Cone}\, f^\bullet)$ yields a convergent spectral sequence

$$E_1^{ij} := H^i B(P^j, \mathrm{Cone}\, f^\bullet) \Rightarrow H^{i+j} B^\bullet_-(P^\bullet, \mathrm{Cone}\, f^\bullet)$$

and since P^j is left B-flat and $\mathrm{Cone}\, f^\bullet$ is acyclic, we get $E_1^{ij} = 0$ for every $i, j \in \mathbb{Z}$, whence the claim. Likewise, we see that $B^\bullet(g^\bullet, Q^\bullet)$ is a quasi-isomorphism, whenever $Q^\bullet \in \mathrm{Ob}(\mathsf{C}^-(\mathscr{F}'))$ and $g^\bullet$ is any quasi-isomorphism in $\mathsf{C}^-(\mathscr{A})$. Then, as in remark 7.3.31(iv), after fixing quasi-inverse functors for the equivalences (7.3.37), we obtain a functor

$$LB^\bullet_- : \mathsf{D}^-(\mathscr{A}) \times \mathsf{D}^-(\mathscr{A}') \to \mathsf{D}^-(\mathscr{A}'')$$

with a natural transformation

$$\mu : LB^\bullet_- \circ (\omega^-_{\mathscr{A}} \times \omega^-_{\mathscr{A}'}) \to \omega^-_{\mathscr{A}''} \circ B^\bullet_-$$

such that, for every $K^\bullet \in \mathrm{Ob}(\mathsf{C}^-(\mathscr{A}))$, the induced functor

$$\mathsf{D}^-(\mathscr{A}') \to \mathsf{D}^-(\mathscr{A}'') \quad : \quad X^\bullet \mapsto LB^\bullet(K^\bullet, X^\bullet)$$

together with the corresponding restriction of μ, is a left derived functor of $B^\bullet_-(K^\bullet, -)$. Symmetrically, for every $K^\bullet \in \mathrm{Ob}(\mathsf{C}^-(\mathscr{A}'))$, the functor $LB^\bullet_-(-, K^\bullet)$ provides a left derived functor of $B^\bullet_-(-, K^\bullet)$: details left to the reader.

(ii) By remark 7.3.31(ii), the following conditions are equivalent :

(a) The induced morphism

$$B^*(A, A') := L^0 B^\bullet(A[0], A'[0]) \to B(A, A')$$

is an isomorphism for every $A \in \mathrm{Ob}(\mathscr{A})$ and every $A' \in \mathrm{Ob}(\mathscr{A}')$.

(b) The functor $B(A, -) : \mathscr{A}' \to \mathscr{A}''$ is right exact for every $A \in \mathrm{Ob}(\mathscr{A})$.

(c) The functor $B(-, A') : \mathscr{A} \to \mathscr{A}''$ is right exact for every $A' \in \mathrm{Ob}(\mathscr{A})$.

Moreover, B^* is obviously another biadditive functor $\mathscr{A} \times \mathscr{A}' \to \mathscr{A}''$, and every right (resp. left) B-flat object of $\mathscr{A}$ (of $\mathscr{A}'$) is also a right (resp. left) B^*-flat object. Thus, $\mathscr{A}$ has enough right B^*-flat objects and $\mathscr{A}'$ has enough left B^*-flat objects, so we may define as well the left defived functor LB^* of B^*. Furthermore, a simple inspection shows that the functors $B^*(A, -)$ and $B^*(-, A')$ are right exact, for every $A \in \mathrm{Ob}(\mathscr{A})$ and $A' \in \mathrm{Ob}(\mathscr{A}')$, and the induced morphism

$$LB^* \to LB$$

is an isomorphism of functors. Lastly, an object P of $\mathscr{A}$ (resp. P' of $\mathscr{A}'$) is right (resp. left) B^*-flat if and only if $L^1B(P, -)$ (resp. $L^1B(-, P')$) is the (constant) zero functor. The detailed verifications of all these assertions shall be left as exercises for the reader.

7.3.38. Let $(\mathscr{A}, \otimes, \Phi, \Psi)$ be any abelian tensor category; example 7.3.36 applies especially to the functor $\otimes$. In this case, it is clear that an object of $\mathscr{A}$ is left $\otimes$-flat if and only if it is right $\otimes$-flat, and such objects shall therefore be called simply *$\otimes$-flat*. Thus, suppose that $\mathscr{A}$ has enough $\otimes$-flat objects; we conclude that the induced tensor functor for complexes of $\mathscr{A}$ (see example 7.1.16) admits a left derived functor

$$- \overset{\mathbf{L}}{\otimes} - : \mathsf{D}^-(\mathscr{A}) \times \mathsf{D}^-(\mathscr{A}) \to \mathsf{D}^-(\mathscr{A})$$

called the *derived tensor product*. Explicitly, this is defined as follows : for every bounded above complex $K^\bullet$ we fix a quasi-isomorphism $\rho^\bullet_K : P^\bullet_K \to K^\bullet$ with $P^\bullet_K$ a bounded above complex of $\otimes$-flat objects, and we set

$$K^\bullet \overset{\mathbf{L}}{\otimes} L^\bullet := P^\bullet_K \otimes P^\bullet_L.$$

By inspecting the construction, we also see that $K^\bullet \overset{\mathbf{L}}{\otimes} L^\bullet$ is naturally isomorphic – via $\rho^\bullet_K$ and $\rho^\bullet_L$ – to both $K^\bullet \otimes P^\bullet_L$ and $P^\bullet_K \otimes L^\bullet$. Moreover, if A is any object of $\mathscr{A}$, clearly the functor $A \otimes - : \mathscr{A} \to \mathscr{A}$ is right exact if and only if the same holds for the functor $- \otimes A : \mathscr{A} \to \mathscr{A}$, and remark 7.3.31(ii) implies that the latter condition holds if and only if the induced morphism

$$H^0(A[0] \overset{\mathbf{L}}{\otimes} B[0]) \to A \otimes B$$

is an isomorphism for every $B \in \mathrm{Ob}(\mathscr{A})$. We also let

$$\mathrm{Tor}^{\mathscr{A}}_i(K^\bullet, L^\bullet) := H_i(K^\bullet \overset{\mathbf{L}}{\otimes} L^\bullet) \qquad \text{for every } i \in \mathbb{Z}.$$

In case $\mathscr{A} = A\text{-}\mathbf{Mod}$ for some ring A, it is customary to denote this functor by $- \overset{\mathbf{L}}{\otimes}_A -$, and then one also writes Tor^A_i instead of $\mathrm{Tor}^{A\text{-}\mathbf{Mod}}_i$.

Remark 7.3.39. (i) Using the commutativity and associativity constraints for the tensor product in $\mathscr{A}$, we deduce – in light of example 7.1.16(i,ii) – natural *associativity isomorphisms*

$$K^\bullet \overset{\mathbf{L}}{\otimes} (L^\bullet \overset{\mathbf{L}}{\otimes} Q^\bullet) \xrightarrow{\sim} (K^\bullet \overset{\mathbf{L}}{\otimes} L^\bullet) \overset{\mathbf{L}}{\otimes} Q^\bullet \qquad \text{in } \mathsf{D}^-(\mathscr{A})$$

as well as *commutativity isomorphisms*

$$K^\bullet \overset{\mathbf{L}}{\otimes} L^\bullet \xrightarrow{\sim} L^\bullet \overset{\mathbf{L}}{\otimes} K^\bullet \qquad \text{in } \mathsf{D}^-(\mathscr{A})$$

for any bounded above complexes $K^\bullet, L^\bullet$, and $Q^\bullet$.

(ii) In the situation of (7.3.38), take $\mathscr{A} = A\text{-}\mathbf{Mod}$ for some ring A, and suppose furthermore that $\varphi : A \to B$ is a ring homomorphism, $K^\bullet$ a bounded above complex of A-modules, and $L^\bullet$ a complex of B-modules. Notice that $P^\bullet_K \otimes_A L^\bullet$ is naturally a complex of B-modules; also, if

$L^\bullet \to Q^\bullet$ is any morphism of complexes of B-modules, then the induced map $P^\bullet_K \otimes_A L^\bullet \to P^\bullet_K \otimes_A Q^\bullet$ is B-linear. It follows easily that the derived tensor product yields a functor

$$\mathsf{D}^-(A\text{-}\mathbf{Mod}) \times \mathsf{D}^-(B\text{-}\mathbf{Mod}) \to \mathsf{D}^-(B\text{-}\mathbf{Mod}) \qquad (K^\bullet, L^\bullet) \mapsto K^\bullet \overset{\mathbf{L}}{\otimes}_A L^\bullet$$

such that, denoting $\varphi^* : \mathsf{D}^-(B\text{-}\mathbf{Mod}) \to \mathsf{D}^-(A\text{-}\mathbf{Mod})$ the "forgetful" functor, we have a natural isomorphism

$$K^\bullet \overset{\mathbf{L}}{\otimes}_A \varphi^* L^\bullet \xrightarrow{\sim} \varphi^*(K^\bullet \overset{\mathbf{L}}{\otimes}_A L^\bullet) \qquad \text{in } \mathsf{D}^-(A\text{-}\mathbf{Mod}).$$

7.3.40. Let now $M^\bullet_1, M^\bullet_2, N^\bullet_1, N^\bullet_2$ be any four objects of $\mathsf{C}^-(\mathscr{A})$, and to ease notation, set

$$M^\bullet_{12} := M^\bullet_1 \otimes M^\bullet_2 \qquad N^\bullet_{12} := N^\bullet_1 \otimes N^\bullet_2 \qquad P^\bullet_{12} := P^\bullet_{M_{12}} \qquad \rho^\bullet_{12} := \rho^\bullet_{M_{12}}$$

as well as $P^\bullet_i := P^\bullet_{M_i}$ and $\rho^\bullet_i := \rho^\bullet_{M_i}$ for $i = 1, 2$. There is a commutative diagram in $\mathsf{D}^-(\mathscr{A})$

$$\begin{array}{ccc} P^\bullet_1 \otimes P^\bullet_2 & \xrightarrow{\varphi^\bullet_{12}} & P^\bullet_{12} \\ & {}_{\rho^\bullet_1 \otimes \rho^\bullet_2} \searrow \quad \swarrow {}_{\rho^\bullet_{12}} & \\ & M^\bullet_{12} & \end{array}$$

where $\rho^\bullet_{12}$ is an isomorphism, so $\varphi^\bullet_{12}$ is uniquely determined, whence a map

$$(M^\bullet_1 \overset{\mathbf{L}}{\otimes} N^\bullet_1) \otimes (M^\bullet_2 \overset{\mathbf{L}}{\otimes} N^\bullet_2) \xrightarrow{\sim} (P^\bullet_1 \otimes P^\bullet_2) \otimes N^\bullet_{12} \xrightarrow{\varphi^\bullet_{12} \otimes N^\bullet_{12}} P^\bullet_{12} \otimes N^\bullet_{12} \xrightarrow{\sim} M^\bullet_{12} \overset{\mathbf{L}}{\otimes} N^\bullet_{12}.$$

Taking into account example 7.1.16(iii) and remark 7.3.14(v), we deduce a bilinear pairing

$$\mathrm{Tor}^{\mathscr{A}}_i(M^\bullet_1, N^\bullet_1) \otimes \mathrm{Tor}^{\mathscr{A}}_j(M^\bullet_2, N^\bullet_2) \to \mathrm{Tor}^{\mathscr{A}}_{i+j}(M^\bullet_{12}, N^\bullet_{12}) \qquad \text{for every } i, j \in \mathbb{Z}.$$

Moreover, suppose that $M^\bullet_3$ and $N^\bullet_3$ are two other bounded above complexes of $\mathscr{A}$; by inspecting the constructions, we find a commutative diagram

$$\begin{array}{ccccc} P^\bullet_1 \otimes (P^\bullet_2 \otimes P^\bullet_3) & \xrightarrow{P^\bullet_1 \otimes \varphi^\bullet_{23}} & P^\bullet_1 \otimes P^\bullet_{23} & \xrightarrow{\varphi^\bullet_{1,23}} & P^\bullet_{1,23} \\ & {}_{\rho^\bullet_1 \otimes (\rho^\bullet_2 \otimes \rho^\bullet_3)} \searrow & \downarrow {\scriptstyle \rho^\bullet_1 \otimes \rho^\bullet_{23}} & \swarrow {}_{\rho^\bullet_{1,23}} & \\ \downarrow {\scriptstyle \Phi^\bullet_P} & & M^\bullet_{1,23} & & \downarrow {\scriptstyle P_{\Phi^\bullet_M}} \\ & & \downarrow {\scriptstyle \Phi^\bullet_M} & & \\ & & M^\bullet_{12,3} & & \\ & {}^{(\rho^\bullet_1 \otimes \rho^\bullet_2) \otimes \rho^\bullet_3} \nearrow & \uparrow {\scriptstyle \rho^\bullet_{12} \otimes \rho^\bullet_3} & \nwarrow {}^{\rho^\bullet_{12,3}} & \\ (P^\bullet_1 \otimes P^\bullet_2) \otimes P^\bullet_3 & \xrightarrow{\varphi^\bullet_{12} \otimes P^\bullet_3} & P^\bullet_{12} \otimes P^\bullet_3 & \xrightarrow{\varphi^\bullet_{12,3}} & P^\bullet_{12,3} \end{array}$$

where $M^\bullet_{1,23} := M^\bullet_{12} \otimes M^\bullet_3$, $M^\bullet_{23} := M^\bullet_2 \otimes M^\bullet_3$, $M^\bullet_{1,23} := M^\bullet_1 \otimes M^\bullet_{23}$, and likewise for $P^\bullet_{1,23}$, $P^\bullet_{23}$, and $P^\bullet_{12,3}$ and the morphism $\varphi^\bullet_{23}$, $\varphi^\bullet_{1,23}$, $\varphi^\bullet_{12,3}$, $\rho^\bullet_{23}$, $\rho^\bullet_{1,23}$, $\rho^\bullet_{12,3}$. Here $\Phi^\bullet_M$ and $\Phi^\bullet_P$ are the associativity constraints.

Therefore, set $T^j_i := \mathrm{Tor}^{\mathscr{A}}_i(M^\bullet_j, N^\bullet_j)$ for every $i \in \mathbb{Z}$ and $j = 1, 2, 3$, and also

$$T^{jk}_i := \mathrm{Tor}^{\mathscr{A}}_i(M^\bullet_{jk}, N^\bullet_{jk}) \qquad T^{1,23}_i := \mathrm{Tor}^{\mathscr{A}}_i(M^\bullet_{1,23}, N^\bullet_{1,23}) \qquad T^{12,3}_i := \mathrm{Tor}^{\mathscr{A}}_i(M^\bullet_{12,3}, N^\bullet_{12,3})$$

for every $i \in \mathbb{Z}$, with $j = 1, 2$ and $k = j + 1$; in light of example 7.1.16(iv), we deduce a commutative diagram in $\mathscr{A}$:

(7.3.41)
$$\begin{array}{ccccc} T^1_i \otimes (T^2_j \otimes T^3_k) & \longrightarrow & T^1_i \otimes T^{23}_{j+k} & \longrightarrow & T^{1,23}_{i+j+k} \\ \downarrow & & & & \downarrow \\ (T^1_i \otimes T^2_j) \otimes T^3_k & \longrightarrow & T^{12}_{i+j} \otimes T^3_k & \longrightarrow & T^{12,3}_{i+j+k} \end{array}$$

whose horizontal arrows are given by the above bilinear pairing, and whose left (resp. right) vertical arrow is the associativity constraint (resp. is induced by the associativity constraint $\Phi^\bullet_M$).

Lemma 7.3.42. *Let $\mathscr{A}$ be any abelian category and $a, b \in \mathbb{Z}$ any two integers. We have :*

(i) *Let $K^\bullet \in \mathrm{Ob}(\mathsf{D}^{\leq b}(\mathscr{A}))$ and $L^\bullet \in \mathrm{Ob}(\mathsf{D}^{\geq a}(\mathscr{A}))$ be any two complexes, and suppose that $\mathscr{A}$ has enough injectives. Then the complex $R\mathrm{Hom}^\bullet_{\mathscr{A}}(K^\bullet, L^\bullet)$ lies in $\mathsf{D}^{\geq a-b}(\mathscr{A})$.*

(ii) *Suppose that $(\mathscr{A}, \otimes, \Phi, \Psi)$ is an abelian tensor category with enough $\otimes$-flat objects, and let $K^\bullet \in \mathrm{Ob}(\mathsf{D}^{\leq a}(\mathscr{A}))$ and $L^\bullet \in \mathrm{Ob}(\mathsf{D}^{\leq b}(\mathscr{A}))$. then $K^\bullet \overset{\mathbf{L}}{\otimes} L^\bullet \in \mathrm{Ob}(\mathsf{D}^{\leq a+b}(\mathscr{A}))$.*

Proof. (i) follows easily from (7.3.34) and remark 7.3.14(iv) : details left to the reader. Assertion (ii) follows immediately from the construction of the derived tensor product. □

7.3.43. *Complexes of modules over a ring.* We conclude this section with a discussion of a few special features of the category of complexes of modules over a ring and of its derived category.

7.3.44. Let A be any ring; to ease notation we set

$$\mathsf{C}(A) := \mathsf{C}(A\text{-}\mathbf{Mod}) \qquad \mathsf{Hot}(A) := \mathsf{Hot}(A\text{-}\mathbf{Mod}) \qquad \mathsf{D}(A) := \mathsf{D}(A\text{-}\mathbf{Mod})$$

and likewise we define $\mathsf{C}^I(A)$, $\mathsf{Hot}^I(A)$ and $\mathsf{D}^I(A)$ for any interval $I \subset \mathbb{Z}$.

Proposition 7.3.45. *With the notation of* (7.3.44)*, the following holds :*

(i) *Let $(K^\bullet_n, \varphi^\bullet_n : K^\bullet_n \to K^\bullet_{n+1} \mid n \in \mathbb{N})$ be any direct system of objects of $\mathsf{C}^I(A)$, and $L^\bullet$ its colimit in $\mathsf{C}^I(A)$. Then $L^\bullet$ represents also the colimit of $(j^I(K^\bullet_n), j^I(\varphi^\bullet_n) \mid n \in \mathbb{N})$ in $\mathsf{D}^I(A)$.*

(ii) *Let $(K^\bullet_n, \varphi^\bullet_n : K^\bullet_{n+1} \to K^\bullet_n \mid n \in \mathbb{N})$ be any inverse system of complexes of A-modules, and $L^\bullet$ its limit in the category $\mathsf{C}(A)$. Suppose that for every $i \in \mathbb{Z}$ the inverse system $(K^i_n, \varphi^i_n \mid n \in \mathbb{N})$ satisfies the Mittag-Leffler condition. Then for every object $C^\bullet$ of $\mathsf{D}(A)$ we have a short exact sequence*

$$0 \to \lim_{n\in\mathbb{N}}{}^1 \mathrm{Hom}_{\mathsf{D}(A)}(C^\bullet, K^\bullet_n[-1]) \to \mathrm{Hom}_{\mathsf{D}(A)}(C^\bullet, L^\bullet) \to \lim_{n\in\mathbb{N}} \mathrm{Hom}_{\mathsf{D}(A)}(C^\bullet, K^\bullet_n) \to 0.$$

Proof. (i): We have commutative diagrams in $\mathsf{C}(A)$:

$$\begin{array}{ccccccccc} & & K^\bullet_i & \xrightarrow{\psi^\bullet_i} & K^\bullet_i \oplus K^\bullet_{i+1} & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & \bigoplus_{n\in\mathbb{N}} K^\bullet_n & \xrightarrow{\psi^\bullet} & \bigoplus_{n\in\mathbb{N}} K^\bullet_n & \longrightarrow & L^\bullet & \longrightarrow & 0 \end{array} \qquad \text{for every } i \in \mathbb{N}$$

whose bottom rows are the same short exact sequence for every $i \in \mathbb{N}$, and with $\psi^r_i(x) := (x, -\varphi^r_i(x))$ for every $i \in \mathbb{N}$, every $r \in \mathbb{Z}$ and every $x \in K^r_i$. For every object $C^\bullet$ of $\mathsf{D}(A)$ set

$$P(k) := \prod_{n\in\mathbb{N}} \mathrm{Hom}_{\mathsf{D}(A)}(K^\bullet_n, C^\bullet[k]) \qquad \widetilde{\psi}(k) := \mathrm{Hom}_{\mathsf{D}(A)}(\psi^\bullet, C^\bullet[k]) \qquad \text{for every } k \in \mathbb{Z}.$$

By corollary 7.3.19(ii) and remark 7.3.35, there follows a long exact sequence :

$$\mathrm{Hom}_{\mathsf{D}(A)}(L^\bullet, C^\bullet[k]) \to P(k) \xrightarrow{\widetilde{\psi}(k)} P(k) \to \mathrm{Hom}_{\mathsf{D}(A)}(L^\bullet, C^\bullet[k+1]) \to \cdots$$

and commutative diagrams for every $i \in \mathbb{N}$ and $k \in \mathbb{Z}$:

$$\begin{array}{ccc} P(k) & \xrightarrow{\widetilde{\psi}(k)} & P(k) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathsf{D}(A)}(K_i^\bullet \oplus K_{i+1}^\bullet, C^\bullet[k]) & \longrightarrow & \mathrm{Hom}_{\mathsf{D}(A)}(K_i^\bullet, C^\bullet[k]) \end{array} \tag{7.3.46}$$

whose bottom rows are induced by $\psi_i^\bullet$. Thus, $\mathrm{Ker}\,\widetilde{\psi}(0)$ consists of the systems of morphisms $(\beta_n^\bullet : K_n^\bullet \to C^\bullet \mid n \in \mathbb{N})$ such that $\beta_{n+1}^\bullet \circ \varphi_n^\bullet = \beta_n^\bullet$ for every $n \in \mathbb{N}$. In other words, $\mathrm{Ker}\,\widetilde{\psi}(0)$ is naturally identified with the set of cocones with basis $(K_n^\bullet \mid n \in \mathbb{N})$ and vertex $C^\bullet$ in the category $\mathsf{D}(A)$. Moreover, considering (7.3.46) with $k = -1$, a simple induction on i shows that $\widetilde{\psi}(-1)$ is surjective (details left to the reader); summing up, we have a natural identification

$$\mathrm{Hom}_{\mathsf{D}(A)}(L^\bullet, C^\bullet) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{D}(A)}(\mathop{\mathrm{colim}}_{n\in\mathbb{N}} j^I(K_n^\bullet), C^\bullet)$$

whence the assertion.

(ii): We argue similarly, considering the commutative diagram in $\mathsf{C}(A)$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L'^\bullet & \longrightarrow & \prod_{n\in\mathbb{N}} K_n^\bullet & \xrightarrow{\mu^\bullet} & \prod_{n\in\mathbb{N}} K_n^\bullet & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & K_i^\bullet \oplus K_{i+1}^\bullet & \xrightarrow{\mu_i^\bullet} & K_i^\bullet & & \end{array} \qquad \text{for every } i \in \mathbb{N}$$

whose top row is independent of i and with $\mu_i^r(x,y) := x - \varphi_i^r(y)$ for every $i \in \mathbb{N}$, every $r \in \mathbb{Z}$ and every $(x,y) \in K_i^r \oplus K_{i+1}^r$. Notice that the top row is a short exact sequence in $\mathsf{C}(A)$, since the system $(K_n^r \mid n \in \mathbb{N})$ satisfies the Mittag-Leffler condition for every $r \in \mathbb{Z}$. For every $k \in \mathbb{Z}$ and any $C^\bullet \in \mathrm{Ob}(\mathsf{D}(A))$ set

$$Q(k) := \prod_{n\in\mathbb{N}} \mathrm{Hom}_{\mathsf{D}(A)}(C^\bullet, K_n^\bullet[k]) \qquad \widetilde{\mu}(k) := \mathrm{Hom}_{\mathsf{D}(A)}(C^\bullet, \mu^\bullet[k]).$$

By corollary 7.3.19(ii) and remark 7.3.35, there follows a long exact sequence

$$\mathrm{Hom}_{\mathsf{D}(A)}(C^\bullet, L'^\bullet[k]) \to Q(k) \xrightarrow{\widetilde{\mu}(k)} Q(k) \to \mathrm{Hom}_{\mathsf{D}(A)}(C^\bullet, L'^\bullet[k+1]) \to$$

as well as commutative diagrams as in (7.3.46). Then, a simple inspection yields a natural identification of $\mathrm{Ker}\,\widetilde{\mu}(0)$ with the set of cones in $\mathsf{D}(A)$ with basis $(K_n^\bullet \mid n \in \mathbb{N})$ and vertex $C^\bullet$. Lastly, $\mathrm{Coker}\,\widetilde{\mu}(-1)$ is naturally identified with $\lim^1_{n\in\mathbb{N}} \mathrm{Hom}_{\mathsf{D}(A)}(C^\bullet, K_n^\bullet[-1])$, whence the contention. ☐

7.3.47. *Minimal resolutions.* Let A be a local ring, k its residue field, and :

$$\cdots \xrightarrow{d_3} L_2 \xrightarrow{d_2} L_1 \xrightarrow{d_1} L_0 \xrightarrow{\varepsilon} M$$

a resolution of an A-module M of finite type. We say that $(L_\bullet, d_\bullet, \varepsilon)$ is a *finite-free resolution* if each L_i is a free A-module of finite rank. We say that $(L_\bullet, d_\bullet, \varepsilon)$ is a *minimal free resolution* of M if it is a finite-free resolution, and moreover the induced maps $k \otimes_A L_i \to k \otimes_A \mathrm{Im}\, d_i$ are isomorphisms for all $i \in \mathbb{N}$ (where we let $d_0 := \varepsilon$). One verifies easily that if A is a coherent ring, then every finitely presented A-module admits a minimal resolution.

7.3.48. Let $\underline{L} := (L_\bullet, d_\bullet, \varepsilon)$ and $\underline{L}' := (L'_\bullet, d'_\bullet, \varepsilon')$ be two free resolutions of M. A *morphism of resolutions* $\underline{L} \to \underline{L}'$ is a map of complexes $\varphi_\bullet : (L_\bullet, d_\bullet) \to (L'_\bullet, d'_\bullet)$ that extends to a commutative diagram

$$\begin{array}{ccc} L_\bullet & \xrightarrow{\varepsilon} & M \\ {\scriptstyle\varphi_\bullet}\downarrow & & \| \\ L'_\bullet & \xrightarrow{\varepsilon'} & M \end{array}$$

Lemma 7.3.49. *Let $\underline{L} := (L_\bullet, d_\bullet, \varepsilon)$ be a minimal free resolution of an A-module M of finite type, $\underline{L}' := (L'_\bullet, d'_\bullet, \varepsilon')$ any other finite-free resolution, $\varphi_\bullet : \underline{L}' \to \underline{L}$ a morphism of resolutions. Then $\varphi_\bullet$ is an epimorphism (in the category of complexes of A-modules),* $\mathrm{Ker}\,\varphi_\bullet$ *is a null homotopic complex of free A-modules, and there is an isomorphism of complexes :*

$$L'_\bullet \xrightarrow{\sim} L_\bullet \oplus \mathrm{Ker}\,\varphi_\bullet.$$

Proof. Suppose first that $\underline{L} = \underline{L}'$. We set $d_0 := \varepsilon$, $L_{-1} := M$, $\varphi_{-1} := \mathbf{1}_M$ and we show by induction on n that φ_n is an isomorphism. Indeed, this holds for $n = -1$ by definition. Suppose that $n \geq 0$ and that the assertion is known for all $j < n$; by a little diagram chasing (or the five lemma) we deduce that φ_{n-1} induces an automorphism $\mathrm{Im}\, d_n \xrightarrow{\sim} \mathrm{Im}\, d_n$, therefore $\varphi_n \otimes_A \mathbf{1}_k : k \otimes_A L_n \to k \otimes_A L_n$ is an automorphism (by minimality of $\underline{L}$), so the same holds for φ_n (*e.g.* by looking at the determinant of φ_n).

For the general case, by standard arguments we construct a morphism of resolutions: $\psi_\bullet : \underline{L} \to \underline{L}'$. By the foregoing case, $\varphi_\bullet \circ \psi_\bullet$ is an automorphism of $\underline{L}$, so $\varphi_\bullet$ is necessarily an epimorphism, and $\underline{L}'$ decomposes as claimed. Finally, it is also clear that $\mathrm{Ker}\,\varphi_\bullet$ is an acyclic bounded above complex of free A-modules, hence it is null homotopic. □

Remark 7.3.50. (i) Suppose that A is a coherent local ring, and let $\underline{L} := (L_\bullet, d_\bullet, \varepsilon)$ and $\underline{L}' := (L'_\bullet, d'_\bullet, \varepsilon')$ be two minimal resolutions of the finitely presented A-module M. It follows easily from lemma 7.3.49 that $\underline{L}$ and $\underline{L}'$ are isomorphic as resolutions of M.

(ii) Moreover, any two isomorphisms $\underline{L} \to \underline{L}'$ are homotopic, hence the rule: $M \mapsto L_\bullet$ extends to a functor

$$A\text{-}\mathbf{Mod}_{\mathrm{coh}} \to \mathsf{Hot}(A\text{-}\mathbf{Mod})$$

from the category of finitely presented A-modules to the homotopy category of complexes of A-modules.

(iii) The sequence of A-modules $(\mathrm{Syz}^i_A M := \mathrm{Im}\, d_i \mid i > 0)$ is determined uniquely by M (up to non-unique isomorphism). The graded module $\mathrm{Syz}^\bullet_A M$ is sometimes called the *syzygy* of the module M. Moreover, if $\underline{L}''$ is any other finite free resolution of M, then we can choose a morphism of resolutions $\underline{L}'' \to \underline{L}$, which will be a split epimorphism by lemma 7.3.49, and the submodule $d_\bullet(L''_\bullet) \subset L_\bullet$ decomposes as a direct sum of $\mathrm{Syz}^\bullet_A M$ and a free A-module of finite rank.

Lemma 7.3.51. *Let $A \to B$ a faithfully flat homomorphism of coherent local rings, M a finitely presented A-module. Then there exists an isomorphism of graded B-modules :*

$$B \otimes_A \mathrm{Syz}^\bullet_A M \to \mathrm{Syz}^\bullet_B(B \otimes_A M).$$

Proof. Left to the reader. □

Definition 7.3.52. A ring homomorphism $A \to B$ is said to be *essentially of finite type* (resp. *essentially of finite presentation*) if B is a localization of a finite type (resp. finitely presented) A-algebra; in this case one also says that B is an A-algebra *of essentially finite type* (resp. an *essentially finitely presented* A-algebra).

Proposition 7.3.53. *Let $A \to B$ be a flat and essentially finitely presented local ring homomorphism of local rings, M an A-flat finitely presented B-module. Then the B-module M admits a minimal free resolution*

$$\Sigma_\bullet \quad : \quad \cdots \xrightarrow{d_3} L_2 \xrightarrow{d_2} L_1 \xrightarrow{d_1} L_0 \xrightarrow{d_{-1}} L_{-1} := M.$$

Moreover, Σ is universally A-exact, i.e. *for every A-module N, the complex $\Sigma_\bullet \otimes_A N$ is still exact.*

Proof. To start out, let us notice :

Claim 7.3.54. In order to prove the proposition, it suffices to show that, for every A-flat finitely presented B-module N, and every B-linear surjective map $d : L \to N$, from a free B-module L of finite rank, $\mathrm{Ker}\, d$ is also an A-flat finitely presented B-module.

Proof of the claim. Indeed, in that case, we can build inductively a minimal resolution $\Sigma_\bullet$ of M, such that $N_i := \mathrm{Ker}(d_i : L_i \to L_{i-1})$ is an A-flat finitely presented B-module for every $i \in \mathbb{N}$. Namely, suppose that a complex $\Sigma_\bullet^{(i)}$ with these properties has already been constructed, up to degree i, and let κ_B be the residue field of B; by Nakayama's lemma, we may find a surjection $d_{i+1} : L_{i+1} \to N_i$, where L_{i+1} is a free B-module of rank $\dim_{\kappa_B}(N_i \otimes_B \kappa_B)$. Under the assumption of the claim, the resulting complex $\Sigma_\bullet^{(i+1)} : (L_{i+1} \to L_i \to L_{i-1} \to \cdots \to M)$ fulfills the sought conditions, up to degree $i+1$.

It is easily seen that the complex $\Sigma_\bullet$ thus obtained shall be universally A-exact. ◊

Let us write A as the union of the filtered family $(A_\lambda \mid \lambda \in \Lambda)$ of its noetherian subalgebras. Say that $B = C_\mathfrak{p}$, for some finitely presented A-algebra C, and a prime ideal $\mathfrak{p} \subset C$, and $M = N_\mathfrak{p}$ for some finitely presented C-module N. We may find $\lambda \in \Lambda$, a finitely generated A_λ-algebra C_λ and a C_λ-module N_λ such that $C = C_\lambda \otimes_{A_\lambda} A$, and $N = N_\lambda \otimes_{A_\lambda} A$; for every $\mu \geq \lambda$, let $C_\mu := A_\mu \otimes_{A_\lambda} C_\lambda$ and $N_\mu := A_\mu \otimes_{A_\lambda} N_\lambda$; also, denote by $\mathfrak{p}_\mu$ the preimage of $\mathfrak{p}$ in C_μ, and set $B_\mu := (C_\mu)_{\mathfrak{p}_\mu}$, $M_\mu := (N_\mu)_{\mathfrak{p}_\mu}$. According to [65, Ch.IV, Cor.11.2.6.1(ii)], we may assume that M_μ is a flat A_μ-module, for every $\mu \geq \lambda$. Moreover, suppose that $d : L \to M$ is a B-linear surjection from a free B-module L of rank r; then we may find $\mu \in \Lambda$ such that d descends to a B_μ-linear surjection $d_\mu : L_\mu \to M_\mu$ from a free B_μ-module L_μ of rank r. It follows easily that $K_\mu := \mathrm{Ker}\, d_\mu$ is a flat A_μ-module, for every $\mu \geq \lambda$, and the induced map $K_\mu \otimes_{B_\mu} B \to K := \mathrm{Ker}\, d$ is a surjection, whose kernel is a quotient of $\mathrm{Tor}_1^{A_\mu}(A, M_\mu)_\mathfrak{p}$; the latter vanishes, since M_μ is A_μ-flat. Hence, K is A-flat; furthermore, K_μ is clearly a finitely generated B_μ-module, hence K is a finitely presented B-module. Then the proposition follows from claim 7.3.54. □

7.4. **Simplicial objects.**

In this section, we introduce the simplicial formalism, which provides the language for the homotopical algebra of section 7.10.

Definition 7.4.1. Let $\mathscr{C}$ be any category, and $k \in \mathbb{N}$ any integer.

(i) We denote by Δ the *simplicial category*, whose objects are the finite ordered sets :

$$[n] := \{0 < 1 < \cdots < n\} \qquad \text{for every } n \in \mathbb{N}$$

and whose morphisms are the non-decreasing functions.

(ii) Δ is a full subcategory of the *augmented simplicial category* $\Delta^\wedge$, whose set of objects is $\mathrm{Ob}(\Delta) \cup \{\varnothing\}$, with $\mathrm{Hom}_{\Delta^\wedge}(\varnothing, [n])$ consisting of the unique mapping of sets $\varnothing \to [n]$, for every $n \in \mathbb{N}$. It is convenient to set $[-1] := \varnothing$.

(iii) The *augmented k-truncated simplicial category* $\Delta_k^\wedge$, is the full subcategory of $\Delta^\wedge$ whose objects are the elements of $\mathrm{Ob}(\Delta^\wedge)$ of cardinality $\leq k+1$. The *k-truncated simplicial category* is the full subcategory Δ_k of $\Delta_k^\wedge$ whose set of objects is $\mathrm{Ob}(\Delta_k^\wedge) \setminus \{\varnothing\}$.

(iv) A *simplicial object* (resp. an *augmented simplicial object*, resp. a *k-truncated simplicial object*, resp. a *k-truncated augmented simplicial object*) of $\mathscr{C}$ is a functor $\Delta^o \to \mathscr{C}$ (resp. $(\Delta^\wedge)^o \to \mathscr{C}$, resp. $\Delta_k^o \to \mathscr{C}$, resp. $(\Delta_k^\wedge)^o$). The morphisms of simplicial objects of $\mathscr{C}$ are just the natural transformations (and likewise for the truncated or augmented variants). Clearly, these objects form a category, and we use the notation

$$s.\mathscr{C} := \mathsf{Fun}(\Delta^o, \mathscr{C}) \qquad \widehat{s}.\mathscr{C} := \mathsf{Fun}((\Delta^\wedge)^o, \mathscr{C})$$
$$s_k.\mathscr{C} := \mathsf{Fun}(\Delta_k^o, \mathscr{C}) \qquad \widehat{s}_k.\mathscr{C} := \mathsf{Fun}((\Delta_k^\wedge)^o, \mathscr{C}).$$

(v) Dually, a *cosimplicial object* $F^\bullet$ of $\mathscr{C}$ is a functor $F : \Delta \to \mathscr{C}$, or – which is the same – a simplicial object in $\mathscr{C}^o$. Likewise one defines the truncated or augmented cosimplicial variants, and we set

$$c.\mathscr{C} := \mathsf{Fun}(\Delta, \mathscr{C}) \qquad \widehat{c}.\mathscr{C} := \mathsf{Fun}(\Delta^\wedge, \mathscr{C})$$
$$c_k.\mathscr{C} := \mathsf{Fun}(\Delta_k, \mathscr{C}) \qquad \widehat{c}_k.\mathscr{C} := \mathsf{Fun}(\Delta_k^\wedge, \mathscr{C}).$$

7.4.2. Notice the *front-to-back* involution :

$$\Delta \to \Delta \quad : \quad (\alpha : [n] \to [m]) \mapsto (\alpha^\vee : [n] \to [m]) \quad \text{for every } n, m \in \mathbb{N} \tag{7.4.3}$$

defined as the endofunctor which induces the identity on $\mathrm{Ob}(\Delta)$, and such that :

$$\alpha^\vee(i) := m - \alpha(n-i) \qquad \text{for every } \alpha \in \mathrm{Hom}_\Delta([n],[m]) \text{ and every } i \in [n].$$

Another construction of interest is the endofunctor

$$\gamma : \Delta \to \Delta$$

given by the rule : $[n] \mapsto [n+1]$ for every $n \in \mathbb{N}$, and which takes any morphism $\alpha : [n] \to [m]$ of Δ, to the morphism $\gamma(\alpha) : [n+1] \to [m+1]$ which is the unique extension of α such that $\gamma(\alpha)(n+1) := m+1$. Notice that γ restricts to functors $\gamma_k : \Delta_{k+1} \to \Delta_k$ for every $k \in \mathbb{N}$.

• Given a simplicial object F of $\mathscr{C}$, one gets a cosimplicial object F^o of $\mathscr{C}$, by the (obvious) rule : $(F^o)[n] := F[n]$ for every $n \in \mathbb{N}$, and $F^o(\alpha) := F(\alpha)^o$ for every morphism α in Δ.

• Moreover, by composing a simplicial (resp. cosimplicial) object F (resp. G) with the involution (7.4.3), one obtains a simplicial (resp. cosimplicial) object $F^\vee$ (resp. $G^\vee$). Likewise, given a morphism $\alpha : F_1 \to F_2$, the Godement product $\alpha^\vee := \alpha * (7.4.3)$ is a morphism $F_1^\vee \to F_2^\vee$.

• For F and G as above, we may also consider the simplicial (resp. cosimplicial) object $\gamma F := F \circ \gamma^o$ (resp. $\gamma G := G \circ \gamma$), and this definition extends again to morphisms, by taking Godement products. The object γF (resp. γG) is called the *path space* of F (resp. of G). If F is a $(k+1)$-truncated simplicial object, then we can consider $\gamma_k F := F \circ \gamma_k^o$, which is a k-truncated simplicial object (and likewise for truncated cosimplicial objects).

7.4.4. There is an obvious fully faithful functor :

$$\mathscr{C} \to s.\mathscr{C} \quad : \quad A \mapsto s.A \qquad (\text{ resp. } \quad \mathscr{C} \to s_k.\mathscr{C} \quad : \quad A \mapsto s_k.A \ )$$

that assigns to each object A of $\mathscr{C}$ the *constant simplicial object* $s.A$ (resp. *constant truncated simplicial object* $s_k.A$) such that $s.A[n] := A$ for every $n \in \mathbb{N}$ (resp. for every $n \leq k$), and $s.A(\alpha) := \mathbf{1}_A$ for every morphism α of Δ (resp. of Δ_k). Of course, we have as well augmented variants $\widehat{s}.A$ and $\widehat{s}_k.A$, and cosimplicial versions $c.A$, $c_k.A$, $\widehat{c}.A$, $\widehat{c}_k.A$.

Moreover, we have, for every integer $k \in \mathbb{N}$, the *k-truncation functor*

$$s.\mathrm{trunc}_k : s.\mathscr{C} \to s_k.\mathscr{C} \qquad (\text{ resp. } \quad \widehat{s}.\mathrm{trunc}_k : \widehat{s}.\mathscr{C} \to \widehat{s}_k.\mathscr{C} \ )$$

that assigns to any simplicial (resp. augmented simplicial) object $F : \Delta^o \to \mathscr{C}$ (resp. $F : (\Delta^\wedge)^o \to \mathscr{C}$) its composition with the inclusion functor $\Delta_k^o \to \Delta^o$ (resp. $(\Delta_k^\wedge)^o \to (\Delta^\wedge)^o$).

Again, we have as well the corresponding cosimplicial versions $c.\mathrm{trunc}_k$ and $\widehat{c}.\mathrm{trunc}_k$. Also, for every $n \in \mathbb{N}$, we have the functor

$$\bullet[n] : s.\mathscr{A} \to \mathscr{A} \qquad A \mapsto A[n].$$

Lastly, any functor $\varphi : \mathscr{B} \to \mathscr{C}$ induces functors

$$s.\varphi : s.\mathscr{B} \to s.\mathscr{C} \qquad s_k.\varphi : s_k.\mathscr{B} \to s_k.\mathscr{C} \quad : \quad F \mapsto \varphi \circ F$$

and there are of course augmented variants $\widehat{s}.\varphi$ and $\widehat{s}_k.\varphi$, as well as the corresponding cosimplicial versions.

Example 7.4.5. (i) For every integer $k \geq -1$, we denote by $\mathbf{\Delta}_k$ the simplicial set represented by $[k]$. Explicitly, $\mathbf{\Delta}_k$ is given by the rule

$$\mathbf{\Delta}_k[n] := \mathrm{Hom}_\Delta([n], [k]) \quad \text{and} \quad \mathbf{\Delta}_k[\varphi] := \mathrm{Hom}_\Delta(\varphi, [k])$$

for every $n \in \mathbb{N}$ and every morphism φ in Δ. For instance, $\mathbf{\Delta}_{-1}$ (resp. $\mathbf{\Delta}_0$) is the constant simplicial set associated to the empty set (resp. to the set with one element); this is also the initial (resp. final) object for $s.\mathbf{Set}$.

(ii) Any morphism $\varphi : [k] \to [n]$ in $\Delta^\wedge$ induces a morphism which we shall denote

$$\mathbf{\Delta}_\varphi := \mathrm{Hom}_{\Delta^\wedge}(-, \varphi) : \mathbf{\Delta}_k \to \mathbf{\Delta}_n$$

given by left composition with φ on $\mathbf{\Delta}_k[i]$, for every $i \in \mathbb{N}$.

7.4.6. For given $n \in \mathbb{N}$, and every $i = 0, \dots, n$, let

$$\varepsilon_i : [n-1] \to [n] \qquad (\text{resp. } \eta_i : [n+1] \to [n])$$

be the unique injective map in $\Delta^\wedge$ whose image misses i (resp. the unique surjective map in Δ with two elements mapping to i). The morphisms ε_i (resp. η_i) are called *face maps* (resp. *degeneracy maps*). By direct inspection, one checks that they fulfill the identities :

$$\begin{aligned} \varepsilon_j \circ \varepsilon_i &= \varepsilon_i \circ \varepsilon_{j-1} && \text{if } i < j \\ \eta_j \circ \eta_i &= \eta_i \circ \eta_{j+1} && \text{if } i \leq j \\ \eta_j \circ \varepsilon_i &= \begin{cases} \varepsilon_i \circ \eta_{j-1} & \text{if } i < j \\ \mathbf{1} & \text{if } i = j \text{ or } i = j+1 \\ \varepsilon_{i-1} \circ \eta_j & \text{if } i > j+1. \end{cases} \end{aligned}$$

Example 7.4.7. (i) For instance, notice the identities :

$$\varepsilon_i^\vee = \varepsilon_{n-i} \qquad \eta_i^\vee = \eta_{n-i} \qquad \text{for every } n \in \mathbb{N} \text{ and every } i = 0, \dots, n.$$

(ii) For every $r, s, i \in \mathbb{N}$ with $i \leq r$, we set

$$\varepsilon_{r,i}^s := \varepsilon_i \circ \cdots \circ \varepsilon_i : [r] \to [r+s].$$

This is the injective mapping whose image is $\{0, \dots, i-1, s+i, \dots, r+s\}$; for instance, the front-to-back dual

$$\varepsilon_{r,0}^{s\vee} := \varepsilon_{r+s} \circ \cdots \circ \varepsilon_{r+1} : [r] \to [r+s]$$

is just the natural inclusion map. We shall also use the notation

$$\varepsilon_{-1,0}^s : [-1] \to [s] \qquad \text{for every } s \in \mathbb{N}$$

for the unique morphism $\varnothing \to [s]$ in $\Delta^\wedge$; of course, we have $\varepsilon_{-1,0}^{s\vee} = \varepsilon_{-1,0}^s$ for every $s \in \mathbb{N}$.

7.4.8. It is easily seen that every morphism $\alpha : [n] \to [m]$ in Δ admits a unique factorization $\alpha = \varepsilon \circ \eta$, where the monomorphism ε is uniquely a composition of faces :

$$\varepsilon = \varepsilon_{i_1} \circ \cdots \circ \varepsilon_{i_s} \qquad \text{with } 0 \leq i_s \leq \cdots \leq i_1 \leq m$$

and the epimorphism η is uniquely a composition of degeneracy maps :

$$\eta = \eta_{j_1} \circ \cdots \circ \eta_{j_t} \qquad \text{with } 0 \leq j_1 < \cdots < j_t \leq m$$

(see [163, Lemma 8.1.2]). It follows that, to give a simplicial object $A[\bullet]$ of a category $\mathscr{C}$, it suffices to give a sequence of objects $(A[n] \mid n \in \mathbb{N})$ of $\mathscr{C}$, together with *face operators*

$$\partial_i := A[\varepsilon_i] : A[n] \to A[n-1] \qquad i = 0, \dots, n$$

for every integer $n > 0$ and *degeneracy operators*

$$\sigma_i := A[\eta_i] : A[n] \to A[n+1] \qquad i = 0, \dots, n$$

for every $n \in \mathbb{N}$, satisfying the following *simplicial identities* :

$$\begin{aligned} &\partial_i \circ \partial_j = \partial_{j-1} \circ \partial_i && \text{if } i < j \\ &\sigma_i \circ \sigma_j = \sigma_{j+1} \circ \sigma_i && \text{if } i \leq j \\ &\partial_i \circ \sigma_j = \begin{cases} \sigma_{j-1} \circ \partial_i & \text{if } i < j \\ \mathbf{1} & \text{if } i = j \text{ or } i = j+1 \\ \sigma_j \circ \partial_{i-1} & \text{if } i > j+1. \end{cases} \end{aligned} \tag{7.4.9}$$

Under this correspondence we have $\partial_i = A(\varepsilon_i)$ and $\sigma_i = A(\eta_i)$ ([163, Prop.8.1.3]). Likewise, a k-truncated simplicial object of $\mathscr{C}$ is the same as the datum of a sequence $(A[n] \mid n = 0, \dots, k)$ of objects of $\mathscr{C}$, and of a system of face and degeneracy operators restricted to this sequence of objects, and fulfilling the same identities (7.4.9).

Dually, a cosimplicial object $A[\bullet]$ of $\mathscr{C}$ is the same as the datum of a sequence $(A[n] \mid n \in \mathbb{N})$ of objects of $\mathscr{C}$, together with *coface operators*

$$\partial^i : A[n-1] \to A[n] \qquad i = 0, \dots, n$$

and *codegeneracy operators*

$$\sigma^i : A[n+1] \to A[n] \qquad i = 0, \dots, n$$

which satisfy the *cosimplicial identities* :

$$\begin{aligned} &\partial^j \circ \partial^i = \partial^i \circ \partial^{j-1} && \text{if } i < j \\ &\sigma^j \circ \sigma^i = \sigma^i \circ \sigma^{j+1} && \text{if } i \leq j \\ &\sigma^j \circ \partial^i = \begin{cases} \partial^i \circ \sigma^{j-1} & \text{if } i < j \\ \mathbf{1} & \text{if } i = j \text{ or } i = j+1 \\ \partial^{i-1} \circ \sigma^j & \text{if } i > j+1 \end{cases} \end{aligned} \tag{7.4.10}$$

and likewise for k-truncated cosimplicial objects.

7.4.11. An augmented simplicial object of a category $\mathscr{C}$ can be viewed as the datum of a simplicial object $A[\bullet]$ of $\mathscr{C}$, together with an object $A[-1] \in \mathrm{Ob}(\mathscr{C})$, and a morphism $\varepsilon : A[0] \to A[-1]$, which is an *augmentation*, *i.e.* such that :

$$\varepsilon \circ \partial_0 = \varepsilon \circ \partial_1.$$

Dually, an augmented cosimplicial object of $\mathscr{C}$ can be viewed as a cosimplicial object $A[\bullet]$, together with a morphism $\eta : A[-1] \to A[0]$ in $\mathscr{C}$, such that η^o is an augmentation for $A^o[\bullet]$. We say that η is an *augmentation* for $A[\bullet]$.

Remark 7.4.12. Let A be a simplicial object of the category $\mathscr{C}$.

(i) For every $n \in \mathbb{N}$ we have $\gamma A[n] := A[n+1]$ (notation of (7.4.2)), and the face operators $\gamma A[\varepsilon_i] : \gamma A[n+1] \to \gamma A[n]$ for $i \leq n+1$ (resp. degeneracy operators $\gamma A[\eta_i] : \gamma A[n] \to \gamma A[n+1]$ for $i \leq n$) of γA are $\partial_i : A[n+2] \to A[n+1]$ (resp. $\sigma_i : A[n+1] \to A[n+2]$); *i.e.* we drop ∂_{n+2} and σ_{n+1}. Likewise for the truncated variants.

(ii) The discarded faces ∂_{n+2} and degeneracies σ_{n+1} can be used to produce natural morphisms

$$s.A[0] \xrightarrow{f_A} \gamma A \xrightarrow{g_A} A.$$

Namely, we set

$$f_A[n] := \sigma_n \circ \cdots \circ \sigma_1 \qquad g_A[n] := \partial_{n+1} \qquad \text{for every } n \in \mathbb{N}.$$

For every $k \in \mathbb{N}$, the same operation on an object $A \in \mathrm{Ob}(s_{k+1}.\mathscr{C})$ yields natural morphisms

$$s.\mathrm{trunc}_k A \xrightarrow{f_A} \gamma_k A \xrightarrow{g_A} s.\mathrm{trunc}_k A.$$

(iii) Since there is a unique morphism $\sigma_{n,0} : [n] \to [0]$ in Δ for every $n \in \mathbb{N}$, it is easily seen that the system $(A[\sigma_{n,0}] : A[0] \to A[n])$ defines a natural morphism

$$s.A[0] \to A \qquad \text{in } s.\mathscr{C}.$$

(iv) Likewise, suppose $\varepsilon : A[0] \to A[-1]$ is an augmentation for A; since there is exactly one morphism $\varepsilon^{n+1}_{-1,0} : \varnothing \to [n]$ in $\Delta^\wedge$ for every $n \in \mathbb{N}$, we see that the system $(A[\varepsilon^{n+1}_{-1,0}] \mid n \in \mathbb{N})$ defines a natural morphism

$$A \to s.A[-1] \qquad \text{in } s.\mathscr{C}.$$

Definition 7.4.13. Let $\mathscr{C}$ be any category. Denote by

$$e_i : \Delta^o \to [1]/\Delta^o \qquad i = 0, 1$$

the functor that assigns to each $[n] \in \mathrm{Ob}(\Delta)$ the unique morphism $[n] \to [1]$ of Δ whose image is $\{i\}$. Let also $\mathsf{t} : [1]/\Delta^o \to \Delta^o$ be the target functor (see (1.1.24)). Let A and B be two simplicial objects of $\mathscr{C}$, and $f, g : A \to B$ two morphisms.

(i) A *homotopy* from f to g is the datum of a natural transformation

$$u : A \circ \mathsf{t} \Rightarrow B \circ \mathsf{t} \qquad \text{such that } u * e_0 = f \text{ and } u * e_1 = g.$$

(ii) Dually, if C and D are cosimplicial objects of $\mathscr{C}$, and $p, q : C \to D$ any two morphisms in $c.\mathscr{C}$, then a *homotopy* from p to q is a homotopy from p^o to q^o in $s.\mathscr{C}^o$.

Remark 7.4.14. (i) In the situation of definition 7.4.13, suppose that $p : A' \to A$ and $q : B \to B'$ are any two morphisms in $s.\mathscr{C}$; then the natural transformation

$$(q * \mathsf{t}) \circ u \circ (p * \mathsf{t}) : A' \circ \mathsf{t} \Rightarrow B' \circ \mathsf{t}$$

is a homotopy from $q \circ f \circ p$ to $q \circ g \circ p$. Moreover, if $F : \mathscr{C} \to \mathscr{D}$ is any functor, then

$$F * u : FA * \mathsf{t} \Rightarrow FB * \mathsf{t}$$

is a homotopy from Ff to Fg.

(ii) However, unlike the case for chain homotopies, simplicial homotopies cannot be composed in this generality; hence, the simplicial category $s.\mathscr{C}$ cannot be made into a 2-category, by taking the homotopies as 2-cells.

(iii) In the same vein, for a general category $\mathscr{C}$, the relation :

$$R(f, g) \Leftrightarrow \text{there exists a homotopy from } f \text{ to } g$$

on the set $\mathrm{Morph}(s.\mathscr{C})$ of morphisms of $s.\mathscr{C}$ is neither symmetric nor transitive (though it will follow from theorem 7.4.60 that this is an equivalence relation, in case $\mathscr{C}$ is abelian). Since $\mathsf{t} \circ e_i = \mathbf{1}_{\Delta^o}$ for $i = 0, 1$, the natural transformation $f * \mathsf{t} : A \circ \mathsf{t} \Rightarrow B \circ \mathsf{t}$ is a homotopy from

f to f, for every morphism $f : A \to B$ of $s.\mathsf{C}$, so the relation R is at least reflexive. Let R^* be the minimal equivalence relation on $\mathrm{Morph}(s.\mathscr{C})$ such that $R(f,g) \Rightarrow R^*(f,g)$ for every $f, g \in \mathrm{Morph}(f,g)$. By virtue of (i), we have $R^*(f,g) \Rightarrow R^*(q \circ f \circ p, q \circ g \circ p)$ for every pair of morphisms $f, g : A \to B$ and every pair of morphisms $p : A' \to A$ and $q : B \to B'$ in $s.\mathscr{C}$. Hence, there exists a well defined *simplicial homotopy category*

$$s.\mathsf{Hot}(\mathscr{C})$$

with $\mathrm{Ob}(s.\mathsf{Hot}(\mathscr{C})) = \mathrm{Ob}(s.\mathscr{C})$, and with morphisms given by the sets :

$$s.\mathrm{Hot}_{\mathscr{C}}(A,B) := \mathrm{Hom}_{s.\mathscr{C}}(A,B)/R^* \qquad \text{for every } A, B \in \mathrm{Ob}(s.\mathscr{C})$$

of R^*-equivalence classes of morphisms $A \to B$ in $s.\mathscr{C}$. The composition law for $s.\mathsf{Hot}(\mathscr{C})$ is given by the unique system of maps $s.\mathrm{Hot}_{\mathscr{C}}(A,B) \times s.\mathrm{Hot}_{\mathscr{C}}(B,C) \to s.\mathrm{Hot}_{\mathscr{C}}(A,C)$ such that we get a projection functor :

$$s.\mathscr{C} \to s.\mathsf{Hot}(\mathscr{C}) \qquad A \mapsto A$$

by the rule : $f \mapsto [f]$, the class of f in $s.\mathrm{Hot}_{\mathscr{C}}(A,B)$, for every $A, B \in \mathrm{Ob}(s.\mathscr{C})$ and every morphism $f : A \to B$. Likewise, by virtue of (i), every functor $F : \mathscr{C} \to \mathscr{D}$ induces a functor

$$s.\mathsf{Hot}(F) : s.\mathsf{Hot}(\mathscr{C}) \to s.\mathsf{Hot}(\mathscr{D}) \qquad A \mapsto FA \qquad [f] \mapsto [Ff].$$

Definition 7.4.15. (i) Let $\mathscr{C}$ be any category, and $A[\bullet]$ any augmented simplicial object of $\mathscr{C}$, with augmentation given by a morphism $\varphi : A \to s.A[-1]$ in $\mathscr{C}$. Then we say that A is *homotopically trivial* if the class of φ is an isomorphism in $s.\mathsf{Hot}(\mathscr{C})$.

(ii) Dually, we say that an augmented cosimplicial object $B[\bullet]$ of $\mathscr{C}$ is homotopically trivial, if $B[\bullet]^o$ is a homotopically trivial object of $s.\mathscr{C}^o$.

7.4.16. Notice that there are exactly $n+1$ morphisms $\varphi : [n] \to [1]$ for every $[n] \in \mathrm{Ob}(\Delta)$, and they can be labeled by the cardinality of $\varphi^{-1}(0)$: for every $n \in \mathbb{N}$ and every $k \leq n+1$, we shall write $\varphi_{n,k} : [n] \to [1]$ for the unique morphism such that $\varphi_{n,k}^{-1}(0)$ has cardinality k. With this notation, notice that

$$\varphi_{n,k} \circ \varepsilon_i = \begin{cases} \varphi_{n-1,k} & \text{if } i \geq k \\ \varphi_{n-1,k-1} & \text{if } i < k \end{cases} \qquad \text{and} \qquad \varphi_{n,k} \circ \eta_i = \begin{cases} \varphi_{n+1,k} & \text{if } i \geq k \\ \varphi_{n+1,k+1} & \text{if } i < k. \end{cases}$$

Hence, a homotopy u from f to g as in definition 7.4.13, is the same as a system of morphisms

$$u_{n,k} : A[n] \to B[n] \qquad \text{for every } n \in \mathbb{N} \text{ and every } k \leq n+1$$

such that $u_{n,n+1} = f[n]$ and $u_{n,0} = g[n]$ for every $n \in \mathbb{N}$, and the diagrams

$$\begin{array}{ccc} A[n] & \xrightarrow{u_{n,k}} & B[n] \\ {\scriptstyle \partial_i}\downarrow & & \downarrow{\scriptstyle \partial_i} \\ A[n-1] & \xrightarrow{u_{n-1,k-a}} & B[n-1] \end{array} \qquad \begin{array}{ccc} A[n] & \xrightarrow{u_{n,k}} & B[n] \\ {\scriptstyle \sigma_i}\downarrow & & \downarrow{\scriptstyle \sigma_i} \\ A[n+1] & \xrightarrow{u_{n+1,k+a}} & B[n+1] \end{array}$$

commute for every $n \in \mathbb{N}$ and every $k \leq n+1$, where $a := 0$ if $i \geq k$, and $a := 1$ if $i < k$.

7.4.17. A *bisimplicial object* in a category $\mathscr{C}$ is an object of the category $s.(s.\mathscr{C})$. The latter can also be regarded as the category of all functors $\Delta^o \times \Delta^o \to \mathscr{C}$; it follows that a bisimplicial object of $\mathscr{C}$ is the same as a system $(A[p,q] \mid (p,q) \in \mathbb{N} \times \mathbb{N})$ of objects of $\mathscr{C}$, together with morphisms

$$A[\alpha,\beta] : A[p,q] \to A[p',q'] \qquad \text{for all morphisms } \alpha : [p'] \to [p], \beta : [q'] \to [q] \text{ of } \Delta$$

compatible with compositions of morphisms in Δ, in the obvious way. More generally, we may define inductively the category of n-simplicial objects $s^n.\mathscr{A}$, for every $n \in \mathbb{N}$, by letting $s^0.\mathscr{A} := \mathscr{A}$, and $s^n.\mathscr{A} := s.(s^{n-1}.\mathscr{A})$, for every $n > 0$. The *diagonal functor*

$$\Delta \to \Delta \times \Delta \qquad [n] \mapsto ([n],[n]) \qquad \alpha \mapsto (\alpha,\alpha) \qquad \text{for all } n \in \mathbb{N} \text{ and all morphisms } \alpha \text{ of } \Delta$$

induces a functor

$$\Delta_{\mathscr{C}} : s^2.\mathscr{C} \to s.\mathscr{C} \qquad A \mapsto A^\Delta.$$

Especially, we have $A^\Delta[n] := A[n,n]$ for every $n \in \mathbb{N}$, and the face operators ∂_i on $A^\Delta[n]$ are of the form $A[\varepsilon_i, \varepsilon_i]$, for every $i = 0, \dots, n$ (and likewise for the degeneracies). Also, the *flip functor*

$$\Delta \times \Delta \to \Delta \times \Delta \qquad ([m],[n]) \mapsto ([n],[m])$$

induces an endofunctor

$$\mathsf{fl} : s^2.\mathscr{A} \to s^2.\mathscr{A}$$

in the obvious way. Furthermore, the endofunctors

$$\Delta \times \Delta \xrightarrow{\gamma \times \mathbf{1}_\Delta} \Delta \times \Delta \xleftarrow{\mathbf{1}_\Delta \times \gamma} \Delta \times \Delta$$

induce functors

$$s^2.\mathscr{A} \xleftarrow{\gamma_1} s^2.\mathscr{A} \xrightarrow{\gamma_2} s^2.\mathscr{A}$$

that admit descriptions as in remark 7.4.12(i). Correspondingly, we get natural morphisms

$$g_A^{(i)} : \gamma_i A \to A \qquad \text{for } i = 1,2 \text{ and every } A \in \mathrm{Ob}(s^2.\mathscr{A})$$

as in remark 7.4.12(ii).

Remark 7.4.18. (i) Let $\mathscr{C}$ be a category whose finite coproducts are representable. Let also $\mathbf{f}.\mathbf{Set}$ be the category of finite sets. To every object S of $s.\mathbf{f}.\mathbf{Set}$ and every $X \in \mathrm{Ob}(s.\mathscr{C})$, we attach a bisimplicial object $S \boxtimes X$ of $\mathscr{C}$ as follows. For every $n, m \in \mathbb{N}$, we let $S \boxtimes X[n,m]$ be the coproduct of finitely many copies of $X[m]$, indexed by the elements of $S[n]$; hence, for every $a \in S[n]$ we have a natural morphism $i_a : X[m] \to S \boxtimes X[n,m]$. If $\varphi : [n] \to [n']$ and $\psi : [m] \to [m']$ are any two morphisms in Δ^o, we let $S \boxtimes X[\varphi,\psi] : S \boxtimes X[n,m] \to S \boxtimes X[n',m']$ be the unique morphism such that $S \boxtimes X[\varphi,\psi] \circ i_a = i_{S[\varphi](a)} \circ X[\psi]$ for every $a \in S[n]$. Clearly, this rules extends to a well defined functor

$$s.\mathbf{f}.\mathbf{Set} \times s.\mathscr{C} \to s^2.\mathscr{C} \qquad (S,X) \mapsto S \boxtimes X.$$

Likewise, we define $X \boxtimes S := \mathsf{fl}(S \boxtimes X)$ (notation of (7.4.17)). If all coproducts of $\mathscr{C}$ are representable, we may even extend the above construction to arbitrary simplicial sets.

(ii) In the same vein, let $\mathscr{A}$ be any abelian category, and M any object of $s.\mathbb{Z}\text{-}\mathbf{Mod}_{\mathrm{fg}}$ (notation of (3.7.47)). For any $A \in \mathrm{Ob}(s.\mathscr{A})$, we may define a bisimplicial object $M \boxtimes_{\mathbb{Z}} A$ of $\mathscr{A}$, by the rule : $[n,m] \mapsto M[n] \otimes_{\mathbb{Z}} A[n]$ for every $n, m \in \mathbb{N}$ and $[\varphi,\psi] \mapsto M[\varphi] \otimes_{\mathbb{Z}} A[\psi]$ for all morphisms φ, ψ of Δ (where these mixed tensor products are as defined in (3.7.47)). Clearly these rules yield a well defined functor

$$s.\mathbb{Z}\text{-}\mathbf{Mod}_{\mathrm{fg}} \times s.\mathscr{A} \to s^2.\mathscr{A} \qquad (M,A) \mapsto M \boxtimes_{\mathbb{Z}} A.$$

Likewise, we set $A \boxtimes_{\mathbb{Z}} M := \mathsf{fl}(M \boxtimes_{\mathbb{Z}} A)$.

(iii) Furthermore, if $(\mathscr{C}, \otimes)$ is any tensor category, and X, Y any two simplicial objects of $\mathscr{C}$, we may define a bisimplicial object $X \boxtimes Y$, by the same rule as in (ii). This yields a functor

$$s.\mathscr{C} \times s.\mathscr{C} \to s^2.\mathscr{C} \qquad (X,Y) \mapsto X \boxtimes Y.$$

In this situation (resp. in the situation of (i), resp. of (ii)), we shall let also

$$X \otimes Y := (X \boxtimes Y)^\Delta \qquad (\text{resp. } S \otimes X := (S \boxtimes X)^\Delta, \text{ resp. } M \otimes_{\mathbb{Z}} A := (M \boxtimes_{\mathbb{Z}} A)^\Delta\ )$$

which we shall call the *tensor product* of X and Y (resp. of S and X, resp. of M and A). Notice the natural identifications

$$S \boxtimes A \stackrel{\sim}{\to} (S \otimes s.\mathbb{Z}) \boxtimes_{\mathbb{Z}} A \qquad \text{for every } S \in \mathrm{Ob}(s.\mathbf{f}.\mathbf{Set}) \text{ and every } A \in \mathrm{Ob}(s.\mathscr{A}).$$

Likewise, if U is any unit object for $(\mathscr{C}, \otimes)$, we get natural identifications

$$S \boxtimes X \stackrel{\sim}{\to} (S \otimes s.U) \boxtimes X \qquad \text{for every } S \in \mathrm{Ob}(s.\mathbf{f}.\mathbf{Set}) \text{ and every } X \in \mathrm{Ob}(s.\mathscr{C})$$

via the isomorphisms $u_X : X \stackrel{\sim}{\to} U \otimes X$ provided by proposition 3.7.6. In the same vein, we have a natural identification $\mathbf{\Delta}_0 \otimes X \stackrel{\sim}{\to} X$ for every $X \in \mathrm{Ob}(\mathscr{C})$ (notation of example 7.4.5(i)).

(iv) Notice also that if $f, g : S \to T$ are any two morphisms of simplicial finite sets, then – in light of remark 7.4.14(i) – any homotopy u from f to g induces a homotopy $u \otimes X$ from $f \otimes X$ to $g \otimes X$.

(v) Let $f, g : A \to B$ be as in definition 7.4.13. Notice that the datum of a homotopy u from f to g is the same as that of a morphism

$$\widetilde{u} : \mathbf{\Delta}_1 \otimes A \to B \qquad \text{such that } \widetilde{u} \circ (\mathbf{\Delta}_{\varepsilon_0} \otimes \mathbf{1}_A) = f \text{ and } \widetilde{u} \circ (\mathbf{\Delta}_{\varepsilon_1} \otimes \mathbf{1}_A) = g$$

(where $\varepsilon_0, \varepsilon_1 : [0] \to [1]$ are the face maps : notation of example 7.4.5(ii)). Indeed, given u, we construct $\widetilde{u}$ as follows. For every $n \in \mathbb{N}$ and every $\varphi \in \mathbf{\Delta}_1[n]$, let $\widetilde{u}[n]$ be the unique morphism such that $\widetilde{u}[n] \circ i_\varphi = u_\varphi$. The naturality of u easily implies that this rule amounts to a morphism $\widetilde{u}$ as sought. Conversely, given $\widetilde{u}$, we can construct a natural transformation u, by reversing the foregoing rule.

Remark 7.4.19. (i) Let $(\mathscr{C}, \otimes)$ be a tensor category with internal Hom functor $\mathscr{H}om$, and unit object U. To every two objects X, Y of $s.\mathscr{C}$, we may attach the object of $\mathscr{C}$

$$\mathscr{H}om_{s.\mathscr{C}}(X, Y) := \mathrm{Equal}(\textstyle\prod_{n \in \mathbb{N}} \mathscr{H}om(X[n], Y[n]) \overset{d_1}{\underset{d_0}{\rightrightarrows}} \textstyle\prod_{\varphi:[n]\to[m]} \mathscr{H}om(X[m], Y[n]))$$

where the second product ranges over the morphisms φ of Δ, and where

$$d_0 := \prod_{\varphi:[n]\to[m]} \mathscr{H}om(X[\varphi], Y[n]) \qquad \text{and} \qquad d_1 := \prod_{\varphi:[n]\to[m]} \mathscr{H}om(X[m], Y[\varphi]).$$

Arguing as in example 7.1.16(vi), it is easily seen that there are natural isomorphisms

$$\mathrm{Hom}_{\mathscr{C}}(Z, \mathscr{H}om_{s.\mathscr{C}}(X, Y)) \stackrel{\sim}{\to} \mathrm{Hom}_{s.\mathscr{C}}(s.Z \otimes X, Y)$$

for every $Z \in \mathrm{Ob}(\mathscr{C})$ and every $X, Y \in \mathrm{Ob}(s.\mathscr{C})$.

(ii) Suppose additionally, that all finite coproducts of $\mathscr{C}$ are representable. For every $Z \in \mathrm{Ob}(\mathscr{C})$, and every $X \in \mathrm{Ob}(s.\mathscr{C})$, consider the simplicial set

$$\mathrm{Hom}_{\mathscr{C}}(Z, X)$$

such that $\mathrm{Hom}_{\mathscr{C}}(Z, X)[n] := \mathrm{Hom}_{\mathscr{C}}(Z, X[n])$ for every $n \in \mathbb{N}$, with face and degeneracies deduced from those of X, in the obvious way. For any $k \in \mathbb{N}$, let also $\mathbf{\Delta}_k$ be the simplicial set defined in example 7.4.5(i); we have natural isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\mathscr{C}}(Z, \mathscr{H}om_{s.\mathscr{C}}(\mathbf{\Delta}_k \otimes s.U, X)) &\stackrel{\sim}{\to} \mathrm{Hom}_{s.\mathscr{C}}(\mathbf{\Delta}_k \otimes s.Z, X) \\ &\stackrel{\sim}{\to} \mathrm{Hom}_{s.\mathbf{Set}}(\mathbf{\Delta}_k, \mathrm{Hom}_{\mathscr{C}}(Z, X)) \\ &\stackrel{\sim}{\to} \mathrm{Hom}_{\mathscr{C}}(Z, X)[k]\end{aligned}$$

where the last isomorphism follows from Yoneda's lemma (proposition 1.2.6(ii) : details left to the reader). Applying again Yoneda's lemma, we deduce a natural isomorphism

$$\mathscr{H}om_{s.\mathscr{C}}(\mathbf{\Delta}_k \otimes s.U, X) \stackrel{\sim}{\to} X[k] \qquad \text{for every } k \in \mathbb{N}.$$

(iii) Notice that $\mathbf{\Delta}_\psi \circ \mathbf{\Delta}_\varphi = \mathbf{\Delta}_{\psi\circ\varphi}$ for every two morphisms $\varphi : [k] \to [k']$ and $\psi : [k'] \to [k'']$ of $\Delta^\wedge$ (notation of example 7.4.5(ii)). Hence, the system $(\mathbf{\Delta}_i \mid i \in \mathbb{N})$ amounts to an object of $c.s.\mathbf{Set}$. Moreover, since the Yoneda isomorphisms are natural in both X and $\mathbf{\Delta}_k$, we get a commutative diagram

$$\begin{array}{ccc} \mathscr{H}om_{s.\mathscr{C}}(\mathbf{\Delta}_{k'} \otimes s.U, X) & \xrightarrow{\sim} & X[k'] \\ {\scriptstyle \mathscr{H}om_{s.\mathscr{C}}(\mathbf{\Delta}_\varphi \otimes s.U, X)} \downarrow & & \downarrow {\scriptstyle X[\varphi]} \\ \mathscr{H}om_{s.\mathscr{C}}(\mathbf{\Delta}_k \otimes s.U, X) & \xrightarrow{\sim} & X[k] \end{array}$$

for every morphism φ as above.

(iv) Notice as well that the considerations of (ii) and (iii) can be repeated, *mutatis mutandis*, for truncated simplicial objects : if X and Y are objects of $s_n.\mathscr{C}$ and Z is an object of $\mathscr{C}$, then $\mathscr{H}om_{s_n.\mathscr{C}}(X, Y)$ (resp. $\mathrm{Hom}_{\mathscr{C}}(Z, X)$) shall be an object of $\mathscr{C}$ (resp. of $s_k.\mathbf{Set}$), and we shall have natural isomorphisms

$$\mathscr{H}om_{s_n.\mathscr{C}}(s.\mathrm{trunc}_n(\mathbf{\Delta}_k \otimes s.U), X) \xrightarrow{\sim} X[k] \qquad \text{for every } k \le n$$

and similarly for the commutative diagrams of (iii) (details left to the reader).

7.4.20. Let $\mathscr{C}$ be a finitely cocomplete category. Theorem 1.3.4 say that, for every integer $k \in \mathbb{N}$, the k-truncation functor on $s.\mathscr{C}$ admits a left adjoint

$$\mathrm{sk}_k : s_k.\mathscr{C} \to s.\mathscr{C}$$

which is called the *k-th skeleton functor*. By inspecting the proof of *loc.cit.* we see that, for every k-truncated simplicial object F, this adjoint is calculated by the rule :

$$\mathrm{sk}_k A[n] := \operatorname*{colim}_{\varphi:[i]\to[n]} F[i] \tag{7.4.21}$$

where i ranges over all the integers $\le k$, and φ over all the morphisms $[i] \to [n]$ in Δ^o, and the transition maps $F[i] \to F[j]$ in the colimit are the morphisms $F[\psi]$ given by all commutative triangles

$$\begin{array}{ccc} [i] & \xrightarrow{\psi} & [j] \\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle\varphi} & \\ & [n] & \end{array} \qquad \text{in } \Delta^o. \tag{7.4.22}$$

A morphism $\alpha : [n] \to [m]$ in Δ^o induces a morphism $\mathrm{sk}_k F[\alpha] : \mathrm{sk}_k A[n] \to \mathrm{sk}_k A[m]$; namely, for every $\varphi : [i] \to [m]$ one has a natural morphism $j_\varphi : F[i] \to \mathrm{sk}_k F[m]$, and $\mathrm{sk}_k F[\alpha]$ is the colimit of the system of morphisms

$$j_{\alpha\circ\psi} : F[i] \to \mathrm{sk}_k A[m] \qquad \text{for all } \psi : [i] \to [n].$$

It is clear that, for every $n, m \le k$ and every $\alpha : [n] \to [m]$, the colimit (7.4.21) is realized by $F[n]$, and under this identification, $\mathrm{sk}_k A[\alpha]$ agrees with $F[\alpha]$, so the unit of adjunction

$$F \to s.\mathrm{trunc}_k \circ \mathrm{sk}_k F$$

is an isomorphism. Dually, if all finite limits are representable in $\mathscr{C}$, the truncation functor admits a right adjoint

$$\mathrm{cosk}_k : s_k.\mathscr{C} \to s.\mathscr{C}$$

called the *k-th coskeleton functor*, and a simple inspection of the proof of *loc.cit.* yields the rule:

$$\mathrm{cosk}_k F[n] := \lim_{\varphi:[n]\to[i]} F[i]$$

where i ranges over the integers $\leq k$, and $\varphi : [n] \to [i]$ over the morphisms in Δ^o, and the transition maps are as in the foregoing (except that the downwards arrows in the commutative triangles (7.4.22) are reversed). Especially, we easily deduce that the counit of adjunction

$$s.\mathrm{trunc}_k \circ \mathrm{cosk}_k F \to F$$

is an isomorphism. Moreover, if $\mathscr{B}$ is another category with small Hom-sets, whose finite colimits (resp. finite limits) are all representable, and $\varphi : \mathscr{B} \to \mathscr{C}$ is any functor, then for any $F \in \mathrm{Ob}(s_k.\mathscr{B})$ there is a natural transformation

$$\mathrm{sk}_k(s_k.\varphi F) \to s.\varphi(\mathrm{sk}_k F) \qquad (\text{ resp. } s.\varphi(\mathrm{cosk}_k F) \to \mathrm{cosk}_k(s_k.\varphi F)\ )$$

which is an isomorphism, if φ is right exact (resp. if φ is left exact).

7.4.23. Let $\mathscr{A}$ be an abelian category, and A any object of $s.\mathscr{A}$. For every $n > 0$, set

$$d_n := \sum_{i=0}^{n} (-1)^i \cdot \partial_i \qquad A[n] \to A[n-1].$$

Directly from the simplicial identities (7.4.9) we may compute

$$\begin{aligned} d_n \circ d_{n+1} &= \sum_{i=0}^{n} \sum_{j=0}^{n+1} (-1)^{i+j} \cdot \partial_i \circ \partial_j \\ &= \sum_{i=0}^{n} \sum_{j>i}^{n+1} (-1)^{i+j} \cdot \partial_{j-1} \circ \partial_i + \sum_{i=0}^{n} \partial_i \circ \partial_i + \sum_{i=0}^{n} \sum_{j<i}^{n} (-1)^{i+j} \cdot \partial_i \circ \partial_j \\ &= \sum_{i=0}^{n} \sum_{j-1>i}^{n+1} (-1)^{i+j} \cdot \partial_{j-1} \circ \partial_i + \sum_{i=0}^{n} \sum_{j<i}^{n} (-1)^{i+j} \cdot \partial_i \circ \partial_j \\ &= 0 \end{aligned}$$

for every $n \in \mathbb{N}$, so we are led to the following :

Definition 7.4.24. Let $\mathscr{A}$ be an abelian category, and $k \in \mathbb{N}$ any integer.

(i) If $A[\bullet]$ is any simplicial object of $\mathscr{A}$, with face operators ∂_i, the *unnormalized complex associated to* $A[\bullet]$ is the complex $(A_\bullet, d_\bullet) \in \mathrm{Ob}(\mathsf{C}^{\leq 0}(\mathscr{A}))$ such that $A_n := A[n]$ for every $n \in \mathbb{N}$, and d_n is defined as in (7.4.23), for every $n > 0$.

(ii) If $A[\bullet]$ is a k-truncated simplicial object of $\mathscr{A}$, the *unnormalized complex associated to* $A[\bullet]$ as the complex $(A_\bullet, d_\bullet)$ where A_n and d_n are defined as in (i) for every $n \leq k$, and $A_n := 0$, $d_n := 0$ for every $n > k$.

(iii) If $A[\bullet]$ is as in (i) (resp. as in (ii)), the *normalized complex associated to* $A[\bullet]$ is the subcomplex $N_\bullet A$ of $A_\bullet$ such that

$$N_0 A := A[0] \quad \text{and} \quad N_n A := \bigcap_{i=1}^{n} \mathrm{Ker}\, \partial_i \qquad \text{for every } n > 0 \qquad (\text{resp. for } 0 < n \leq k).$$

So, the differential $N_n A \to N_{n-1} A$ equals ∂_0, for every $n \in \mathbb{N}$ (resp. for every $n \leq k$).

(iv) If $A[\bullet]$ is as in (i) or (ii), the *homology of* A in degree n is

$$H_n A := H_n A_\bullet \qquad \text{for every } n \in \mathbb{N}.$$

(v) Let $\varepsilon : A \to A_{-1}$ be an augmentation for $A[\bullet]$. One says that the augmented simplicial object (A, ε) is *aspherical*, if $H_n A = 0$ for every $n > 0$, and ε induces an isomorphism $H_0 A \xrightarrow{\sim} A_{-1}$.

7.4.25. Let $\mathscr{A}$ be an abelian category, and recall that $s^n.\mathscr{A}$ and $s_k.\mathscr{A}$ are both abelian categories as well, for every $n, k \in \mathbb{N}$ (remark 3.7.37(ii)); also, clearly the rule of definition 7.4.24(iii) yields natural additive functors

$$\mathsf{N}_{\mathscr{A}} : s.\mathscr{A} \to \mathsf{C}^{\leq 0}(\mathscr{A}) \qquad \mathsf{N}_{\mathscr{A},k} : s_k.\mathscr{A} \to \mathsf{C}^{[-k,0]}(\mathscr{A}) \qquad A[\bullet] \mapsto N_\bullet A \qquad \text{for every } k \in \mathbb{N}$$

and the rules of definition 7.4.24(i,ii) yield additive functors

$$\mathsf{U}_{\mathscr{A}} : s.\mathscr{A} \to \mathsf{C}^{\leq 0}(\mathscr{A}) \qquad \mathsf{U}_{\mathscr{A},k} : s_k.\mathscr{A} \to \mathsf{C}^{[-k,0]}(\mathscr{A}) \qquad A[\bullet] \mapsto A_\bullet \qquad \text{for every } k \in \mathbb{N}.$$

Remark 7.4.26. Let $\mathscr{A}$ and A be as in (7.4.23); directly from the simplicial identities (7.4.9) we may compute

$$d_n \circ \sigma_k = \sum_{i=0}^{k-1} (-1)^i \cdot \sigma_{k-1} \circ \partial_i + \sum_{i=k+2}^{n} (-1)^i \cdot \sigma_k \circ \partial_{i-1}.$$

Especially, if we let

$$D_0 := 0 \qquad \text{and} \qquad D_n A := \sum_{i=0}^{n-1} \mathrm{Im}\,(\sigma_i : A[n-1] \to A[n]) \qquad \text{for every } n > 0$$

we see that d_n restricts to a morphism $d'_n : D_n A \to D_{n-1} A$ for every $n > 0$, hence $(D_\bullet A, d'_\bullet)$ is a subcomplex of $A_\bullet$, called the *degenerate subcomplex*, and clearly we obtain an additive functor

$$\mathsf{D}_{\mathscr{A}} : s.\mathscr{A} \to \mathsf{C}(\mathscr{A}) \qquad A \mapsto D_\bullet A.$$

Proposition 7.4.27. *The natural injections induce a decomposition*

$$A_\bullet = N_\bullet A \oplus D_\bullet A \qquad \textit{in } \mathsf{C}^{\leq 0}(\mathscr{A}).$$

Proof. First, we notice that $N_n A \cap D_n A = 0$ for every $n \in \mathbb{N}$. Indeed, it suffices to check that $N_n A \cap \mathrm{Im}(\sigma_i) = 0$ for every $i = 0, \dots, n-1$, but the latter follows easily from the identity $\partial_{i+1} \circ \sigma_i = \mathbf{1}_{A[n-1]}$. To conclude the proof, it suffices to show that $N_n A + D_n A = A_n$ for every $n \in \mathbb{N}$. Indeed, set $K_0 := A_n$ and define inductively $K_i := K_{i-1} \cap \mathrm{Ker}\,\partial_i$ for every $i = 1, \dots, n$; to prove the latter assertion, it suffices to check that

(7.4.28) $$(\mathbf{1}_{A_n} - \sigma_i \cdot \partial_i)(K_i) \subset K_{i+1} \qquad \text{for every } i = 0, \dots, n.$$

However, we have :

$$\partial_j \circ (\mathbf{1}_{A_n} - \sigma_i \cdot \partial_i) = \partial_j - \sigma_{i-1} \circ \partial_{i-1} \circ \partial_j \qquad \text{whenever } j < i$$

and $\partial_i \circ (\mathbf{1}_{A_n} - \sigma_i \cdot \partial_i) = \partial_i - \partial_i = 0$, whence (7.4.28). □

Example 7.4.29. (i) Let S be any simplicial set, and set $\mathbb{Z}^S := S \otimes s.\mathbb{Z}$ (notation of remark 7.4.18(iii)); we wish to give an explicit description of the complex $N_\bullet \mathbb{Z}^S$. To begin with, notice that $\mathbb{Z}^S[n]$ is the free abelian group with basis indexed by $S[n]$, for every $n \in \mathbb{N}$, and for every $x \in S[n]$ denote by $e_x \in \mathbb{Z}^S[n]$ the corresponding basis element. Next, set

$$D_n S := \bigcup_{i=0}^{n-1} \mathrm{Im}\,(\sigma_i : S[n-1] \to S[n]) \quad \text{and} \quad N_n S := S[n] \setminus D_n S \qquad \text{for every } n \in \mathbb{N}.$$

From proposition 7.4.27, we see that $N_n \mathbb{Z}^S$ can be naturally identified with the direct summand of $\mathbb{Z}^S[n]$ generated by the system $(e_x \mid x \in N_n S)$, for every $n \in \mathbb{N}$. In order to describe the differential d_n, let us define $\overline{e}_x := e_x$ if $x \in N_n S$, and $\overline{e}_x := 0$ if $x \in D_n S$. Then we may write

$$d_n(\overline{e}_x) = \sum_{i=0}^{n} (-1)^i \cdot \overline{e}_{\partial_i x} \qquad \text{for every } n \in \mathbb{N} \text{ and every } x \in N_n S.$$

The verifications are straightforward, and shall be left to the reader.

(ii) For instance, for any $i \in \mathbb{N}$ consider the simplicial set $\mathbf{\Delta}_i$ of example 7.4.5(i), and set

$$\mathbb{K}\langle i\rangle_\bullet := N_\bullet \mathbb{Z}^{\mathbf{\Delta}_i}$$

where $\mathbb{Z}^{\mathbf{\Delta}_i}$ is defined as in (i). Notice that this notation agrees with that of remark 7.1.19(ii). It is easily seen that a morphism $[n] \to [i]$ of Δ lies in $D_n\mathbf{\Delta}_i$ if and only if it is not injective; hence $N_n\mathbf{\Delta}_i$ is the set of all injective maps $\varphi : [n] \to [i]$ of Δ. We deduce a natural isomorphism

$$\mathbb{K}\langle i\rangle_n \xrightarrow{\sim} \Lambda^{n+1}_{\mathbb{Z}}\mathbb{Z}^{\oplus i+1} \qquad \text{for every } i, n \in \mathbb{N}.$$

Namely, to any map φ as above, we assign the exterior product $e_{\varphi(0)} \wedge \cdots \wedge e_{\varphi(n)}$, where $e_0, \dots, e_i$ denotes the canonical basis of $\mathbb{Z}^{\oplus i+1}$. Under this isomorphism, the differential of $\mathbb{K}\langle i\rangle_\bullet$ gets identified with the differential of the Koszul complex attached to the sequence $\mathbf{1}_{i+1} := (1, \dots, 1) \in \mathbb{Z}^{\oplus i+1}$, that will be introduced in (7.8). Summing up, we obtain natural short exact sequences

$$0 \to \mathbb{K}\langle i\rangle_\bullet[1] \to \mathbf{K}_\bullet(\mathbf{1}_{i+1}) \to \mathbb{Z}[0] \to 0 \qquad \text{for every } i \in \mathbb{N}$$

where $[1]$ denotes the shift operator, and $\mathbb{Z}[0]$ is the complex with $\mathbb{Z}$ in degree zero : see (7.1.3).

(iii) Let $\mathscr{A}$ be any abelian category, $A \in \mathrm{Ob}(s.\mathscr{A})$ and $Z \in \mathrm{Ob}(\mathscr{A})$. Notice that the simplicial set $\mathrm{Hom}_{\mathscr{A}}(Z, A)$ of remark 7.4.19(ii) is actually a simplicial abelian group, and a direct inspection of the arguments of *loc.cit.* yields a natural isomorphism of abelian groups

$$\mathrm{Hom}_{s.\mathbb{Z}\text{-}\mathbf{Mod}}(\mathbb{Z}^{\mathbf{\Delta}_i}, \mathrm{Hom}_{\mathscr{A}}(Z, A)) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{A}}(Z, A[i]) \qquad \text{for every } i \in \mathbb{N}.$$

7.4.30. Keep the notation of remark 7.4.26, and let

$$\mathsf{j}^A_\bullet : N_\bullet A \to A_\bullet \qquad \mathsf{q}^A_\bullet : A_\bullet \to N_\bullet A$$

be respectively the injection and the projection with kernel $D_\bullet A$. Then the rules : $A \mapsto \mathsf{j}^A_\bullet$ and $A \mapsto \mathsf{q}^A_\bullet$ define natural transformations

$$\mathsf{j}_\bullet : \mathsf{N}_{\mathscr{A}} \Rightarrow \mathsf{U}_{\mathscr{A}} \qquad \mathsf{q}_\bullet : \mathsf{U}_{\mathscr{A}} \Rightarrow \mathsf{N}_{\mathscr{A}}.$$

With this notation, we may state :

Theorem 7.4.31. *With the notation of remark* (7.4.30)*, we have :*

(i) *The injection $\mathsf{j}^A_\bullet$ is a homotopy equivalence, and $D_\bullet A$ is homotopically trivial.*

(ii) *More precisely, there exist natural modifications*

$$\mathsf{j}_\bullet \circ \mathsf{q}_\bullet \rightsquigarrow \mathbf{1}_{\mathsf{U}_{\mathscr{A}}} \qquad \mathsf{q}_\bullet \circ \mathsf{j}_\bullet \rightsquigarrow \mathbf{1}_{\mathsf{N}_{\mathscr{A}}}$$

on the 2*-category $\mathsf{C}(\mathscr{A})$ as in example* 7.1.15(ii) *(see definition* 2.2.9*).*

(iii) *Especially, the natural map*

$$H_i N_\bullet A \to H_i A$$

is an isomorphism, for every $i \in \mathbb{N}$.

Proof. Clearly (ii)⇒(i), and (i)⇒(iii), by virtue of remark 7.1.9(ii). To show (ii), set $\varphi^A_0 := \mathbf{1}_{A_0}$ and $\varphi^A_n := \mathbf{1}_{A[n]} - \sigma_{n-1} \circ \partial_n : A_n \to A_n$ for every $n > 0$. We notice :

Claim 7.4.32. The system $(\varphi^A_n \mid n \in \mathbb{N})$ defines an endomorphism of $A_\bullet$, which is homotopically equivalent to $\mathbf{1}_{A_\bullet}$.

Proof of the claim. Indeed, let $s_n := (-1)^n \cdot \sigma_n : A_n \to A_{n+1}$ for every $n \in \mathbb{N}$, and define $s_n := 0$ for every $n < 0$. Using the simplicial identities (7.4.9) we compute :

$$\begin{aligned} s_{n-1} \circ d_n + d_{n+1} \circ s_n &= (-1)^{n-1} \cdot \sum_{j=0}^{n} (-1)^j \cdot (\sigma_{n-1} \circ \partial_j - \partial_j \circ \sigma_n) \\ &= \varphi_n^A - \mathbf{1}_{A[n]} + (-1)^{n-1} \cdot \sum_{j=0}^{n-1} (-1)^j \cdot (\sigma_{n-1} \circ \partial_j - \sigma_{n-1} \circ \partial_j) \\ &= \varphi_n^A - \mathbf{1}_{A[n]} \end{aligned}$$

for every $n > 0$, and for $n = 0$ we have as well $s_{-1} \circ d_0 + d_1 \circ s_0 = 0$, whence the claim. ◊

Notice that the homotopy exhibited in the proof of claim 7.4.32 is natural in A, so it already yields the first sought modification. Next, for every simplicial object B of $\mathscr{A}$, let γB and $g_B : \gamma B \to B$ be as in remark 7.4.12(i,ii), and define inductively

$$\beta^0 A := A \qquad \text{and} \qquad \beta^{n+1} A := \mathrm{Ker}\, g_{\beta^n A} \qquad \text{for every } n \in \mathbb{N}.$$

A simple inspection shows that

$$N_{n+i} A \subset \beta^n A[i] \qquad \text{and} \qquad \beta^n A[0] = N_n A \qquad \text{for every } i, n \in \mathbb{N}.$$

Hence, we may define a subcomplex $B_\bullet^{(n)}$ of $A_\bullet$ for every $n \in \mathbb{N}$, as follows. For $i = 0, \dots, n$ we let $B_i^{(n)} := N_i A$, and for $i > n$ we let $B_i^{(n)} := (\beta^n A)[n+i]$ (notation of (7.1.3)). The differential of $B_\bullet^{(n)}$ is of course just the restriction of that of $A_\bullet$. By construction, $N_\bullet A$ is a subcomplex of $B_\bullet^{(n)}$, for all $n \in \mathbb{N}$. Next, we define an endomorphism $f_\bullet^{(n)}$ of $B_\bullet^{(n)}$, by setting

$$f_i^{(n)} := \mathbf{1}_{B_i^{(n)}} \qquad \text{for every } i \leq n, \text{ and} \qquad f_i^{(n)} := \varphi_i^{\beta^n A} \qquad \text{for } i > n.$$

Notice that the restriction of $f_\bullet^{(n)}$ to the subcomplex $N_\bullet A$ is just the inclusion map $N_\bullet A \to B_\bullet^{(n)}$. Clearly $B^{(n+1)} \subset B^{(n)}$ for every $n \in \mathbb{N}$, and we remark that $f_\bullet^{(n)}$ factors through the inclusion map $B^{(n+1)} \subset B^{(n)}$. Indeed, since $B_i^{(n)} = B_i^{(n+1)}$ for every $i \leq n$, the assertion is obvious for this range of degrees; so we have only to check that $\varphi_\bullet^{\beta^n A}$ factors through $\beta^{n+1} A_\bullet$ for every $n \in \mathbb{N}$, and an easy induction reduces to checking that $\varphi_\bullet^A$ factors through $B_\bullet^{(1)}$. But the latter assertion comes down to the identity $\partial_i \circ \varphi_i^A = 0$ for every $i > 0$, which follows easily from the simplicial identities (7.4.9).

If $(s_i \mid i \in \mathbb{N})$ is the homotopy between $\mathbf{1}_{\beta^n A_\bullet}$ and $\varphi_\bullet^{\beta^n A}$ supplied by claim 7.4.32, then we obtain a homotopy $(t_i^{(n)} \mid i \in \mathbb{N})$ between $\mathbf{1}_{B_\bullet^{(n)}}$ and $f_\bullet^{(n)}$, by setting

$$t_i^{(n)} := 0 \qquad \text{for every } i < n, \text{ and} \qquad t_i^{(n)} := s_{i-n} \qquad \text{for } i \geq n.$$

Notice that $\mathrm{Im}\, t_i^{(n)} \subset D_{i+1} A$ for every $i, n \in \mathbb{N}$. Thus, for every $n \in \mathbb{N}$, let $h_\bullet^{(n)} : B_\bullet^{(n)} \to B_\bullet^{(n+1)}$ be the morphism of complexes deduced from $f_\bullet^{(n)}$ and $j_\bullet^{(n)} : B^{(n)} \to A_\bullet$ the inclusion map, and set $q_\bullet^{(n)} := h_\bullet^{(n)} \circ \cdots \circ h_\bullet^{(0)}$; it follows easily that the composition

$$p_\bullet^{(n)} := j_\bullet^{(n+1)} \circ q_\bullet^{(n)} : A_\bullet \to A_\bullet$$

is homotopically equivalent to $\mathbf{1}_{A_\bullet}$, for every $n \in \mathbb{N}$. More precisely, a direct inspection shows that the system of morphisms

$$\tau_i^{(n)} := \sum_{k=0}^{n-1} j_\bullet^{(k+1)} \circ t_\bullet^{(k+1)} \circ q_\bullet^{(k)} \qquad \text{for every } i \in \mathbb{N}$$

provides a homotopy between $\mathbf{1}_{A_\bullet}$ and $p_\bullet^{(n)}$. Furthermore, it is clear that

$$p_i^{(n)} = p_i^{(m)} \qquad \text{and} \qquad \tau_i^{(n)} = \tau_i^{(m)} \qquad \text{for every } m \geq n \text{ and every } i \leq n$$

so, we finally get an endomorphism $p_\bullet : A_\bullet \to A_\bullet$ by setting $p_i := p_i^{(i)}$ for every $i \in \mathbb{N}$, and a homotopy $\tau_\bullet$ between $p_\bullet$ and $\mathbf{1}_{A_\bullet}$, with $\tau_i := \tau_i^{(i)}$ for every $i \in \mathbb{N}$. By construction, $p_\bullet$ factors through $\mathsf{j}_\bullet^A$, and moreover, the restriction of $p_\bullet$ to the subcomplex $N_\bullet A$ is just $\mathsf{j}_\bullet^A$, so $\mathsf{j}_\bullet^A$ is a homotopy equivalence, via natural homotopies, as stated.

Lastly, notice that $\mathrm{Im}\,\tau_i \subset D_{i+1}A$ for every $i \in \mathbb{N}$, from which it follows easily that $p_i(D_iA) \subset D_iA$ for every $i \in \mathbb{N}$, and then the foregoing implies that $\mathrm{Ker}\,p_\bullet = D_\bullet A$. We conclude that $D_\bullet A$ is homotopically trivial, as stated. □

7.4.33. By iterating U, we get a functor from bisimplicial objects to double complexes

$$\mathsf{U}^2_{\mathscr{A}} : s^2.\mathscr{A} \xrightarrow{s.\mathsf{U}_{\mathscr{A}}} s.\mathsf{C}(\mathscr{A}) \xrightarrow{\mathsf{U}_{\mathsf{C}(\mathscr{A})}} \mathsf{C}(\mathsf{C}(\mathscr{A})) \qquad A[\bullet,\bullet] \mapsto A_{\bullet\bullet}$$

and notice that this functor is naturally isomorphic to the functor

$$s^2.\mathscr{A} \xrightarrow{\mathsf{U}_{s.\mathscr{A}}} \mathsf{C}(s.\mathscr{A}) \xrightarrow{\mathsf{C}(\mathsf{U}_{\mathscr{A}})} \mathsf{C}(\mathsf{C}(\mathscr{A})).$$

In the same vein, theorem 7.4.31 yields natural decompositions of additive functors :

$$\mathsf{U}^2_{\mathscr{A}} = (\mathsf{C}(\mathsf{N}_{\mathscr{A}}) \oplus \mathsf{C}(\mathsf{D}_{\mathscr{A}})) \circ (\mathsf{N}_{s.\mathscr{A}} \oplus \mathsf{D}_{s.\mathscr{A}}) \tag{7.4.34}$$

and if we let

$$\mathsf{N}^2_{\mathscr{A}} := \mathsf{C}(\mathsf{N}_{\mathscr{A}}) \circ \mathsf{N}_{s.\mathscr{A}} : s^2.\mathscr{A} \to \mathsf{C}(\mathsf{C}(\mathscr{A})) \qquad A \mapsto N_{\bullet\bullet}A$$

the natural morphism

$$\mathsf{i}^A_\bullet : \mathrm{Tot}\, N_{\bullet\bullet}A \to \mathrm{Tot}\, A_{\bullet\bullet}$$

is a homotopy equivalence.

7.4.35. We wish next to exhibit two natural transformations

$$A^\Delta_\bullet \xrightarrow{\mathsf{AW}^A_\bullet} \mathrm{Tot} A_{\bullet\bullet} \xrightarrow{\mathsf{Sh}^A_\bullet} A^\Delta_\bullet \qquad \text{for every } A \in \mathrm{Ob}(s^2.\mathscr{A}).$$

Namely, for every $n \in \mathbb{N}$, define :

- Sh^A_n as the sum, for all $p, q \in \mathbb{N}$ such that $p + q = n$, of the *shuffle maps*

$$\mathsf{Sh}^A_{p,q} := \sum_{\mu\nu} \varepsilon_{\mu\nu} \cdot A[\eta_{\nu_1} \circ \cdots \circ \eta_{\nu_q}, \eta_{\mu_1} \circ \cdots \circ \eta_{\mu_p}] : A_{p,q} \to A_{n,n}$$

  where the sum ranges over the *shuffle permutations* (μ, ν) of type (p, q) of the set $\{0, \dots, p + q - 1\}$ (these are the permutations described in [75, §4.3.15]), and $\varepsilon_{\mu\nu}$ is the sign of the permutation (μ, ν)
- AW^A_n as the sum, for all $p, q \in \mathbb{N}$ such that $p + q = n$, of the *Alexander-Whitney maps*

$$\mathsf{AW}^A_{p,q} := A[\varepsilon^{q\vee}_{p,0}, \varepsilon^p_{q,0}] : A_{n,n} \to A_{p,q}$$

(notation of example 7.4.7(ii)). Clearly, for every $p, q \in \mathbb{N}$ the rule $A \mapsto A_{p,q}$ defines a functor

$$\bullet[p, q] : s^2.\mathscr{A} \to \mathscr{A}$$

and the maps $\mathsf{Sh}^A_{p,q}$ and $\mathsf{AW}^A_{p,q}$ yield natural transformations

$$\mathsf{Sh}_{p,q} : \bullet[p, q] \Rightarrow \bullet[p + q, p + q] \qquad \mathsf{AW}_{p,q} : \bullet[p + q, p + q] \Rightarrow \bullet[p, q].$$

Proposition 7.4.36. *With the notation of* (7.4.25)*, the sequence* $(\mathsf{Sh}^A_n \mid n \in \mathbb{N})$ *defines a morphism of chain complexes*

$$\mathsf{Sh}^A_\bullet : \mathrm{Tot} A_{\bullet\bullet} \to A^\Delta_\bullet \qquad \textit{for every } A \in \mathrm{Ob}(s^2.\mathscr{A}).$$

Proof. For any bisimplicial object A of $\mathscr{A}$, set $A_{-1,q} = A_{p,-1} := 0$ for every $p, q \in \mathbb{Z}$, and let

$$A[\varepsilon_0, \mathbf{1}_{[q]}] : A_{0,q} \to A_{-1,q} \qquad A[\mathbf{1}_{[p]}, \varepsilon_0] : A_{p,0} \to A_{p,-1} \qquad A[\varepsilon_0, \varepsilon_0] : A_{0,0} \to A_{-1,-1}$$

be the zero maps; likewise, let $\mathsf{Sh}^A_{-1,q} : A_{-1,q} \to A_{q-1,q-1}$ and $\mathsf{Sh}^A_{p,-1} : A_{p,-1} \to A_{p-1,p-1}$ be the zero maps, and notice that also $\mathsf{Sh}^A_{0,0}$ is the zero map. We define

$$d^{A,h}_{p,q} := \sum_{i=0}^{p} (-1)^i \cdot A[\varepsilon_i, \mathbf{1}_{[q]}] : A_{p,q} \to A_{p-1,q} \qquad d^{A,v}_{p,q} := \sum_{i=0}^{q} (-1)^j \cdot A[\mathbf{1}_{[p]}, \varepsilon_i] : A_{p,q} \to A_{p,q-1}$$

so, for every $n \in \mathbb{N}$ the differential d_n in degree n of $\mathrm{Tot} A_{\bullet\bullet}$ is the sum of the maps

$$d^{A,h}_{p,q} + (-1)^p \cdot d^{A,v}_{p,q} : A_{p,q} \to A_{p-1,q} \oplus A_{p,q-1} \qquad \text{for all } p, q \in \mathbb{N} \text{ such that } p + q = n$$

whereas the differential of $A^\Delta_\bullet$ is the morphism

$$d^A_n := \sum_{i=1}^{n} A[\varepsilon_i, \varepsilon_i] : A_{n,n} \to A_{n-1,n-1}$$

and we have to check the identity

$$(7.4.37) \qquad d^A_{p+q} \circ \mathsf{Sh}^A_{p,q} = \mathsf{Sh}^A_{p-1,q} \circ d^{A,h}_{p,q} + (-1)^p \cdot \mathsf{Sh}^A_{p,q-1} \circ d^{A,v}_{p,q} \qquad \text{for all } p, q \in \mathbb{N}.$$

Now, we set $B := \gamma_1 A$, $C := \gamma_2 A$, $D := \gamma_2 B$ (notation of (7.4.17)), but for the purpose of this proof, we shall modify the differentials of the double complexes $B_{\bullet\bullet}$ and $C_{\bullet\bullet}$ in certain low degrees : namely, we define

$$d^{B,h}_{0,q} := A[\varepsilon_0, \mathbf{1}_{[q]}] \qquad d^{C,v}_{p,0} := A[\mathbf{1}_{[p]}, \varepsilon_0] \qquad \text{for every } p, q \in \mathbb{N}.$$

Claim 7.4.38. With the foregoing notation, the following holds :

(i) $\mathsf{Sh}^A_{p,q} = (-1)^q \cdot \mathsf{Sh}^D_{p-1,q} \circ A[\mathbf{1}_{[p]}, \eta_q] + \mathsf{Sh}^D_{p,q-1} \circ A[\eta_p, \mathbf{1}_{[q]}]$ for every $p, q \in \mathbb{N}$.

(ii) $A[\varepsilon_{p+q-1}, \varepsilon_{p+q-1}] \circ \mathsf{Sh}^D_{p-1,q-1} = \mathsf{Sh}^A_{p-1,q-1} \circ A[\varepsilon_p, \varepsilon_q]$ for every $p, q > 0$.

(iii) $A[\varepsilon_{p+q}, \varepsilon_{p+q}] \circ \mathsf{Sh}^A_{p,q} = (-1)^q \cdot \mathsf{Sh}^A_{p-1,q} \circ A[\varepsilon_p, \mathbf{1}_{[q]}] + \mathsf{Sh}^A_{p,q-1} \circ A[\mathbf{1}_{[p]}, \varepsilon_q]$ for every $p, q \in \mathbb{N}$.

Proof of the claim. (i): First, we notice the identities :

$$(7.4.39) \qquad \begin{aligned} A[\mathbf{1}_{[p+q]}, \eta_{p+q}] \circ \mathsf{Sh}^A_{p,q} &= \mathsf{Sh}^C_{p,q} \circ A[\mathbf{1}_{[p]}, \eta_q] \\ A[\eta_{p+q}, \mathbf{1}_{[p+q]}] \circ \mathsf{Sh}^A_{p,q} &= \mathsf{Sh}^B_{p,q} \circ A[\eta_p, \mathbf{1}_{[q]}]. \end{aligned} \qquad \text{for every } p, q \in \mathbb{N}$$

that are deduced from the identities

$$\eta_{\mu_1} \circ \cdots \circ \eta_{\mu_p} \circ \eta_{p+q} = \eta_q \circ \eta_{\mu_1} \circ \cdots \circ \eta_{\mu_p}$$
$$\eta_{\nu_1} \circ \cdots \circ \eta_{\nu_q} \circ \eta_{p+q} = \eta_p \circ \eta_{\nu_1} \circ \cdots \circ \eta_{\nu_q}$$

which in turn follow from the simplicial identities for degeneracy maps. By applying the first (resp. second) identity (7.4.39) with A replaced by B (resp. by C) and p replaced by $p-1$ (resp. with q replaced by $q-1$) we get :

$$(7.4.40) \qquad \begin{aligned} A[\mathbf{1}_{[p+q]}, \eta_{p+q-1}] \circ \mathsf{Sh}^B_{p-1,q} &= \mathsf{Sh}^D_{p-1,q} \circ A[\mathbf{1}_{[p]}, \eta_q] \\ A[\eta_{p+q-1}, \mathbf{1}_{[p+q]}] \circ \mathsf{Sh}^C_{p,q-1} &= \mathsf{Sh}^D_{p,q-1} \circ A[\eta_p, \mathbf{1}_{[q]}]. \end{aligned} \qquad \text{for every } p, q \in \mathbb{N}.$$

Now, suppose first that $p, q > 0$, and let (μ, ν) be any (p, q)-shuffle of $\{0, \dots, p+q-1\}$; then either $\mu_p = p+q-1$ or $\nu_q = p+q-1$. In the first (resp. second) case, after removing μ_p (resp. ν_q) we get a $(p-1, q)$-shuffle (resp. a $(p, q-1)$-shuffle) $(\overline{\mu}, \overline{\nu})$ with

$$\varepsilon_{\overline{\mu\nu}} = (-1)^q \cdot \varepsilon_{\mu\nu} \qquad (\text{resp. } \varepsilon_{\overline{\mu\nu}} = \varepsilon_{\mu\nu}).$$

However, the first (resp. second) left-hand side of (7.4.40) contains precisely all the terms of the first (resp. second) type occurring in the definition of $\mathsf{Sh}^A_{p,q}$, so we get (i) in this case. The cases where either $p = 0$ or $q = 0$ can be dealt with by a similar, but simpler, argument.

(ii) follows by naturality of $\mathsf{Sh}_{p-1,q-1}$, applied to the morphism $g^{(1)}_A \circ g^{(2)}_B : D \to A$ from (7.4.17).

(iii): The case where $p = q = 0$ is obvious, and the other cases follow by composing both sides of (i) with $A[\varepsilon_{p+q}, \varepsilon_{p+q}]$, applying (ii), and recalling that $\eta_q \circ \varepsilon_q = \mathbf{1}_{[q]}$: details left to the reader. ◊

Now, a simple inspection shows that (7.4.37) follows from claim 7.4.38(iii) and the following

Claim 7.4.41. $d^D_{p+q-1} \circ \mathsf{Sh}^A_{p,q} = \mathsf{Sh}^A_{p-1,q} \circ d^{B,h}_{p-1,q} + (-1)^p \cdot \mathsf{Sh}^A_{p,q-1} \circ d^{C,v}_{p,q-1}$ for every $p, q \in \mathbb{N}$.

Proof of the claim. Consider first the case where $p = 0$. If $q \leq 1$, it is easily seen that both sides of the stated identity vanish; if $q \geq 2$, the left-hand side is

$$d^D_{q-1} \circ A[\eta_0 \circ \cdots \circ \eta_{q-1}, \mathbf{1}_{[q]}] = \sum_{i=0}^{q-1} (-1)^i \cdot A[\eta_0 \circ \cdots \circ \eta_{q-1} \circ \varepsilon_i, \varepsilon_i]$$

and the right-hand side is

$$A[\eta_0 \circ \cdots \circ \eta_{q-2}, \mathbf{1}_{[q]}] \circ d^{C,v}_{0,q-1} = \sum_{i=0}^{q-1} (-1)^i \cdot A[\eta_0 \circ \cdots \circ \eta_{q-2}, \varepsilon_i]$$

so in this case the assertion comes down to the obvious identity

$$\eta_0 \circ \cdots \circ \eta_{q-1} \circ \varepsilon_i = \eta_0 \circ \cdots \circ \eta_{q-2} : [q-2] \to [0] \qquad \text{for every } i = 0, \dots, q-1.$$

Likewise we deal with the case where $q = 0$. It follows already that (7.4.37) holds for these values of (p, q), and for every bisimplicial object A of $\mathscr{A}$; especially, we get

$$d^D_q \circ \mathsf{Sh}^D_{0,q} = \mathsf{Sh}^D_{0,q-1} \circ d^{D,v}_{0,q} \qquad \text{for every } q \in \mathbb{N}. \tag{7.4.42}$$

Next, for $p = q = 1$, a direct computation shows that both sides equal

$$A[\varepsilon_0 \circ \eta_0, \mathbf{1}_{[1]}] - A[\mathbf{1}_{[1]}, \varepsilon_0 \circ \eta_0] : A_{1,1} \to A_{1,1}.$$

We prove now, by induction on q, that the assertion holds for $p = 1$. This is already known for $q \leq 1$, so suppose that $r > 1$, and that the assertion holds for $p = 1$ and every $q < r$. The latter implies that also (7.4.37) holds for these values of (p, q), and for every bisimplicial object A of $\mathscr{A}$; especially, we get

$$d^D_r \circ \mathsf{Sh}^D_{1,r-1} = \mathsf{Sh}^D_{0,r-1} \circ d^{D,h}_{1,r-1} - \mathsf{Sh}^D_{1,r-2} \circ d^{D,v}_{1,r-1}. \tag{7.4.43}$$

On the other hand, claim 7.4.38(i) says that

$$d^D_r \circ \mathsf{Sh}^A_{1,r} = (-1)^r \cdot d^D_r \circ \mathsf{Sh}^D_{0,r} \circ A[\mathbf{1}_{[1]}, \eta_r] + d^D_r \circ \mathsf{Sh}^D_{1,r-1} \circ A[\eta_1, \mathbf{1}_{[r]}]$$

Combining with (7.4.42) and (7.4.43) we obtain

$$d^D_r \circ \mathsf{Sh}^A_{1,r} = (-1)^r \cdot \mathsf{Sh}^D_{0,r-1} \circ d^{D,v}_{0,r} \circ A[\mathbf{1}_{[1]}, \eta_r] + (\mathsf{Sh}^D_{0,r-1} \circ d^{D,h}_{1,r-1} - \mathsf{Sh}^D_{1,r-2} \circ d^{D,v}_{1,r-1}) \circ A[\eta_1, \mathbf{1}_{[r]}].$$

However, we have

$$d^{D,h}_{1,r-1} \circ A[\eta_1, \mathbf{1}_{[r]}] = A[\eta_1 \circ \varepsilon_0, \mathbf{1}_{[r]}] - A[\mathbf{1}_{[1]}, \mathbf{1}_{[r]}] = A[\varepsilon_0 \circ \eta_0, \mathbf{1}_{[r]}] - \mathbf{1}_{A[1,r]}$$

so we can rewrite $d^D_r \circ \mathsf{Sh}^A_{1,r} = \varphi_1 + \varphi_2 - \varphi_3$, where

$$\begin{aligned}
\varphi_1 &:= \mathsf{Sh}^D_{0,r-1} \circ ((-1)^r \cdot d^{D,v}_{0,r} \circ A[\mathbf{1}_{[1]}, \eta_r] - \mathbf{1}_{A[1,r]}) = (-1)^r \cdot \mathsf{Sh}^D_{0,r-1} \circ A[\mathbf{1}_{[1]}, \eta_{r-1}] \circ d^{C,v}_{1,r-1} \\
\varphi_2 &:= \mathsf{Sh}^D_{0,r-1} \circ A[\varepsilon_0 \circ \eta_0, \mathbf{1}_{[r]}] \\
\varphi_3 &:= \mathsf{Sh}^D_{1,r-2} \circ d^{D,v}_{1,r-1} \circ A[\eta_1, \mathbf{1}_{[r]}].
\end{aligned}$$

On the other hand, using claim 7.4.38(i), we can compute

$$\begin{aligned}\mathsf{Sh}^A_{0,r}\circ d^{B,h}_{0,r-1} &= \mathsf{Sh}^D_{0,r-1}\circ A[\eta_0,\mathbf{1}_{[r]}]\circ d^{B,h}_{0,r-1} = \varphi_2 \\ \mathsf{Sh}^A_{1,r-1}\circ d^{C,v}_{1,r-1} &= ((-1)^{r-1}\cdot\mathsf{Sh}^D_{0,r-1}\circ A[\mathbf{1}_{[1]},\eta_{r-1}] + \mathsf{Sh}^D_{1,r-2}\circ A[\eta_1,\mathbf{1}_{[r-1]}])\circ d^{C,v}_{1,r-1} \\ &= \varphi_3-\varphi_1\end{aligned}$$

which shows that the claim holds for $p=1$ and $q=r$, and concludes the induction.

Lastly, we prove the claim for every $p,q\in\mathbb{N}$, by induction on $p+q$; notice that the foregoing already shows that the assertion holds whenever $p+q\leq 2$. Thus, let $r>2$, and suppose that the claim is already known for every pair (p,q) such that $p+q<r$; then also (7.4.37) holds for such values of p and q, and especially we get

$$d^D_{p+q}\circ\mathsf{Sh}^D_{p,q} = \mathsf{Sh}^D_{p-1,q}\circ d^{D,h}_{p,q} + (-1)^p\cdot\mathsf{Sh}^D_{p,q-1}\circ d^{D,v}_{p,q} \qquad \text{whenever } p+q<r. \tag{7.4.44}$$

Let (p',q') be a pair such that $p'+q'=r$; combining (7.4.44) with claim 7.4.38(i), we get

$$\begin{aligned}d^D_{p'+q'-1}\circ\mathsf{Sh}^A_{p',q'} &= (-1)^{q'}\cdot(\mathsf{Sh}^D_{p'-2,q'}\circ d^{D,h}_{p'-1,q'} - (-1)^{p'}\cdot\mathsf{Sh}^D_{p'-1,q'-1}\circ d^{D,v}_{p'-1,q'})\circ A[\mathbf{1}_{[p']},\eta_{q'}] \\ &\quad + (\mathsf{Sh}^D_{p'-1,q'-1}\circ d^{D,h}_{p',q'-1} + (-1)^{p'}\cdot\mathsf{Sh}^D_{p',q'-2}\circ d^{D,v}_{p',q'-1})\circ A[\eta_{p'},\mathbf{1}_{[q']}].\end{aligned}$$

On the other hand, after noticing that

$$\begin{aligned}d^{D,h}_{p',q'-1}\circ A[\eta_{p'},\mathbf{1}_{[q']}] - (-1)^{p'}\cdot\mathbf{1}_{A[p',q']} &= A[\eta_{p'-1},\mathbf{1}_{[q']}]\circ d^{B,h}_{p'-1,q'} \\ d^{D,v}_{p'-1,q'}\circ A[\mathbf{1}_{[p']},\eta_{q'}] - (-1)^{q'}\cdot\mathbf{1}_{A[p',q']} &= A[\mathbf{1}_{[p']},\eta_{q'-1}]\circ d^{C,v}_{p',q'-1}\end{aligned}$$

we may apply claim 7.4.38(i), to compute

$$\begin{aligned}\mathsf{Sh}^A_{p'-1,q'}\circ d^{B,h}_{p'-1,q'} &= ((-1)^{q'}\cdot\mathsf{Sh}^D_{p'-2,q'}\circ A[\mathbf{1}_{[p'-1]},\eta_{q'}] + \mathsf{Sh}^D_{p'-1,q'-1}\circ A[\eta_{p'-1},\mathbf{1}_{[q']}])\circ d^{B,h}_{p'-1,q'} \\ &= (-1)^{q'}\cdot\mathsf{Sh}^D_{p'-2,q'}\circ d^{D,h}_{p'-1,q'}\circ A[\mathbf{1}_{[p']},\eta_{q'}] - (-1)^{p'}\cdot\mathsf{Sh}^D_{p'-1,q'-1} \\ &\quad + \mathsf{Sh}^D_{p'-1,q'-1}\circ d^{D,h}_{p',q'-1}\circ A[\eta_{p'},\mathbf{1}_{[q']}]\end{aligned}$$

and

$$\begin{aligned}\mathsf{Sh}^A_{p',q'-1}\circ d^{C,v}_{p',q'-1} &= ((-1)^{q'-1}\cdot\mathsf{Sh}^D_{p'-1,q'-1}\circ A[\mathbf{1}_{[p']},\eta_{q'-1}] \\ &\quad + \mathsf{Sh}^D_{p',q'-2}\circ A[\eta_{p'},\mathbf{1}_{[q'-1]}])\circ d^{C,v}_{p',q'-1} \\ &= (-1)^{q'-1}\cdot\mathsf{Sh}^D_{p'-1,q'-1}\circ d^{D,v}_{p'-1,q'}\circ A[\mathbf{1}_{[p']},\eta_{q'}] + \mathsf{Sh}^D_{p'-1,q'-1} \\ &\quad + \mathsf{Sh}^D_{p',q'-2}\circ d^{D,v}_{p',q'-1}\circ A[\eta_{p'},\mathbf{1}_{[q']}].\end{aligned}$$

Finally, the sought identity for the pair (p',q') follows by comparing the last two identities with the previous one for $d^D_{p'+q'-1}\circ\mathsf{Sh}^A_{p',q'}$, and this concludes the proof of the inductive step. □

Proposition 7.4.45. *With the notation of* (7.4.25)*, the sequence* $(\mathsf{AW}^A_n \mid n\in\mathbb{N})$ *defines a morphism of chain complexes*

$$\mathsf{AW}^A_\bullet : A^\Delta_\bullet \to \mathrm{Tot}A_{\bullet\bullet} \qquad \textit{for every } A\in\mathrm{Ob}(s^2.\mathscr{A}).$$

Proof. Denote by $d^T_\bullet$ the differential of $\mathrm{Tot}A_{\bullet\bullet}$, and keep the notation of the proof of proposition 7.4.36; we have to check the identity

$$\mathsf{AW}_{n-1}\circ d^A_n = d^T_n\circ\mathsf{AW}_n \qquad \text{for every } n\in\mathbb{N}. \tag{7.4.46}$$

However, say that $p,q\in\mathbb{N}$ and $p+q=n$; the projection of the right-hand side of (7.4.46) on the direct summand $A_{p-1,q}$ of $(\mathrm{Tot}A_{\bullet\bullet})_{n-1}$ equals

$$d^{A,h}_{p,q}\circ\mathsf{AW}_{p,q} + (-1)^{p-1}\cdot d^{A,v}_{p-1,q+1}\circ\mathsf{AW}_{p-1,q+1}. \tag{7.4.47}$$

Unwinding the definition, (7.4.47) is found to be

$$\sum_{i=0}^{p}(-1)^i \cdot A[\varepsilon^{q\vee}_{p,0} \circ \varepsilon_i, \varepsilon^p_{q,0}] - (-1)^p \cdot \sum_{i=0}^{q+1}(-1)^i \cdot A[\varepsilon^{q+1\vee}_{p-1,0}, \varepsilon^{p-1}_{q+1,0} \circ \varepsilon_i]$$

which we may rewrite as

$$(7.4.48) \qquad \sum_{i=0}^{p-1}(-1)^i \cdot A[\varepsilon^{q\vee}_{p,0} \circ \varepsilon_i, \varepsilon^p_{q,0}] + \sum_{i=p}^{p+q}(-1)^i \cdot A[\varepsilon^{q+1\vee}_{p-1,0}, \varepsilon^{p-1}_{q+1,0} \circ \varepsilon_{i-p+1}].$$

On the other hand, the projection of the left-hand side of (7.4.46) onto $A_{p-1,q}$ equals

$$\sum_{i=0}^{n}(-1)^i \cdot A[\varepsilon_i \circ \varepsilon^{q\vee}_{p-1,0}, \varepsilon_i \circ \varepsilon^{p-1}_{q,0}].$$

To compare the latter with (7.4.48), it suffices to remark that

$$\varepsilon_i \circ \varepsilon^{q\vee}_{p-1,0} = \begin{cases} \varepsilon^{q\vee}_{p,0} \circ \varepsilon_i & \text{if } i < p \\ \varepsilon^{q+1\vee}_{p-1,0} & \text{if } i \geq p \end{cases} \qquad \text{and} \qquad \varepsilon_i \circ \varepsilon^{p-1}_{q,0} = \begin{cases} \varepsilon^{p}_{q,0} & \text{if } i < p \\ \varepsilon^{p-1}_{q+1,0} \circ \varepsilon_i & \text{if } i \geq p \end{cases}$$

which are all deduced from the simplicial identities for the ε_i. The proposition follows. □

7.4.49. Propositions 7.4.36 and 7.4.45 yield natural transformations of functors

$$\mathrm{Tot}\, \mathsf{U}^2_{\mathscr{A}} \underset{\mathsf{AW}_\bullet}{\overset{\mathsf{Sh}_\bullet}{\rightleftarrows}} \mathsf{U}_{\mathscr{A}} \circ \Delta_{\mathscr{A}}.$$

Now, denote by $\mathsf{h}_{\mathscr{A}} : \mathsf{C}(\mathscr{A}) \to \mathsf{Hot}(\mathscr{A})$ the natural functor (this induces the identity on the objects, and the projection on the group of morphisms); there follow natural transformations

$$\mathsf{h}_{\mathscr{A}} \circ \mathrm{Tot}\, \mathsf{U}^2_{\mathscr{A}} \underset{\mathsf{h}_{\mathscr{A}} * \mathsf{AW}_\bullet}{\overset{\mathsf{h}_{\mathscr{A}} * \mathsf{Sh}_\bullet}{\rightleftarrows}} \mathsf{h}_{\mathscr{A}} \circ \mathsf{U}_{\mathscr{A}} \circ \Delta_{\mathscr{A}}. \qquad \text{between functors } s^2.\mathscr{A} \to \mathsf{Hot}(\mathscr{A}).$$

Theorem 7.4.50 (Eilenberg-Zilber-Cartier)**.** *With the notation of* (7.4.49)*, we have :*

(i) $\mathsf{h}_{\mathscr{A}} * \mathsf{AW}_\bullet$ *and* $\mathsf{h}_{\mathscr{A}} * \mathsf{Sh}_\bullet$ *are mutually inverse isomorphisms of functors.*

(ii) *More precisely, there exist natural modifications (see definition* 2.2.9*)*

$$\mathsf{AW}_\bullet \circ \mathsf{Sh}_\bullet \rightsquigarrow \mathbf{1}_{\mathrm{Tot}\,\mathsf{U}^2_{\mathscr{A}}} \qquad \mathsf{Sh}_\bullet \circ \mathsf{AW}_\bullet \rightsquigarrow \mathbf{1}_{\mathsf{U}_{\mathscr{A}} \circ \Delta_{\mathscr{A}}}.$$

Proof. Let $\mathsf{p}^A_\bullet : \mathrm{Tot}\, A_{\bullet\bullet} \to \mathrm{Tot}\, N_{\bullet\bullet}A$ be the projection whose kernel is the sum of the remaining three direct summands in the decomposition Tot (7.4.34); explicitly, in each degree $n \in \mathbb{N}$, this kernel is the sum of the subobjects $\mathrm{Im}\, A[\eta_i, \mathbf{1}_{[q]}]$ and $\mathrm{Im}\, A[\mathbf{1}_{[p]}, \eta_j]$ of $A_{p,q}$, for every $i = 0, \dots, p-1$, $j = 0, \dots, q-1$, and every $p, q \in \mathbb{N}$ such that $p + q = n$. Likewise, let $\mathsf{j}^A_\bullet : N_\bullet A^\Delta \to A^\Delta_\bullet$ and $\mathsf{q}^A_\bullet : A^\Delta_\bullet \to N_\bullet A^\Delta$ be as in (7.4.30); theorem 7.4.31 and the discussion of (7.4.33) show that the pairs $(\mathsf{p}^A_\bullet, \mathsf{i}^A_\bullet)$ and $(\mathsf{j}^A_\bullet, \mathsf{q}^A_\bullet)$ induce mutually inverse isomorphisms in $\mathsf{Hot}(\mathscr{A})$, so it suffices to check that the same holds for the compositions

$$\overline{\mathsf{Sh}}^A_\bullet := \mathsf{q}^A_\bullet \circ \mathsf{Sh}^A_\bullet \circ \mathsf{i}^A_\bullet : \mathrm{Tot}\, N_{\bullet\bullet}A \to N_\bullet A^\Delta \qquad \overline{\mathsf{AW}}^A_\bullet := \mathsf{p}^A_\bullet \circ \mathsf{AW}^A_\bullet \circ \mathsf{j}^A_\bullet : N_\bullet A^\Delta \to \mathrm{Tot}\, N_{\bullet\bullet}A$$

for every bisimplicial object A of $\mathscr{A}$. However, we have

Claim 7.4.51. $\mathsf{p}^A_\bullet \circ \mathsf{AW}^A_\bullet \circ \mathsf{Sh}^A_\bullet \circ \mathsf{i}^A_\bullet = \mathbf{1}_{\mathrm{Tot}\, N_{\bullet\bullet}A}$.

Proof of the claim. More precisely, say that $p + q = n$, consider the composition

$$f_{p,q} : A_{p,q} \xrightarrow{\mathsf{Sh}^A_{p,q}} A_{p+q,p+q} \xrightarrow{\mathsf{AW}^A_n} (\mathrm{Tot}\, A_{\bullet\bullet})_n$$

and let $g_{p,q} : A_{p,q} \to (\mathrm{Tot}\, A_{\bullet\bullet})_n$ be the inclusion map; we shall show that $f_{p,q} - g_{p,q} \subset \mathrm{Ker}\, \mathsf{p}^A_n$. Indeed, we have

$$f_{p,q} = \sum_{i=0}^{n} \sum_{(\mu,\nu)} \varepsilon_{\mu\nu} \cdot A[\eta_{\nu_1} \circ \cdots \circ \eta_{\nu_q} \circ \varepsilon^{n-i\vee}_{i,0}, \eta_{\mu_1} \circ \cdots \circ \eta_{\mu_p} \circ \varepsilon^{i}_{n-i,0}]$$

(notation of (7.4.35), and $\varepsilon^i_{n-i,0}$ is defined as in the proof of proposition 7.4.45). However, if $i < p$, then $\eta_{\mu_1} \circ \cdots \circ \eta_{\mu_p} \circ \varepsilon^i_{n-i,0}$ is of the form $\tau \circ \eta_{\mu_p}$ for some $\tau : [p+q-i-1] \to [q]$, so the corresponding term does lie in $\mathrm{Ker}\, \mathsf{p}^A_n$. If $i > p$, then $\eta_{\nu_1} \circ \cdots \circ \eta_{\nu_q} \circ \varepsilon^{n-i\vee}_{i,0}$ is of the form $\tau \circ \eta_k$ for some $k \leq \nu_q$ and some $\tau : [i-1] \to [p]$, so this term likewise lies in $\mathrm{Ker}\, \mathsf{p}^A_n$. It remains to consider the terms with $i = p$; however, a simple inspection shows that $\eta_{\nu_1} \circ \cdots \circ \eta_{\nu_q} \circ \varepsilon^{n-p\vee}_{p,0}$ is of the same form as above, unless ν_1 is either $p-1$ or p (details left to the reader); furthermore, if $\nu_1 = p-1$, then $\mu_p \geq p$, in which case $\eta_{\mu_1} \circ \cdots \circ \eta_{\mu_p} \circ \varepsilon^i_{n-i,0}$ is of the form described above. In all these cases, the corresponding term again lies in $\mathrm{Ker}\, \mathsf{p}^A_n$. So, it remains only to consider the single case where (μ, ν) is the identity permutation, whose sign equals 1; in this case, the corresponding term is none else than $A[\mathbf{1}_{[p]}, \mathbf{1}_{[q]}]$, whence the claim. ◊

Claim 7.4.51 and theorem 7.4.31(ii) already yield the existence of the sought modification $\mathsf{AW}_\bullet \circ \mathsf{Sh}_\bullet \rightsquigarrow \mathbf{1}_{\mathrm{Tot}\, \mathsf{U}^2_{\mathscr{A}}}$. Next we define, for every $n \in \mathbb{N}$, a natural transformation

$$s_n : \bullet[n] \circ \Delta_{\mathscr{A}} \Rightarrow \bullet[n+1] \circ \Delta_{\mathscr{A}}$$

(notation of (7.4.4); so s^A_n is a morphism $A[n,n] \to A[n+1,n+1]$ for every object A of $s^2.\mathscr{A}$). The construction is by induction on n : for $n = 0$ we let $s^A_0 : A[0,0] \to A[1,1]$ be the zero morphism; for $n > 0$ we set

$$s^A_n := \mathsf{Sh}^{A'}_n \circ \mathsf{AW}^{A'}_n \circ A[\eta_0, \eta_0] - s^{A'}_{n-1} \qquad \text{where } A' := (\gamma_2 \circ \gamma_1(A^\vee))^\vee$$

(notation of (7.4.17) and (7.4.2) : explicitly, we have $A'[p,q] := A[p+1, q+1]$ for every $p, q \in \mathbb{N}$, and $A'[\varepsilon_i, \varepsilon_j] := A[\varepsilon_{i+1}, \varepsilon_{j+1}]$, and likewise for $A'[\varepsilon_i, \eta_j]$, $A'[\eta_i, \varepsilon_j]$ and $A'[\eta_i, \eta_j]$, for every face and degeneracy map of Δ). We have

Claim 7.4.52. The system $(\mathsf{q}^A_{n+1} \circ s^A_n \circ \mathsf{j}^A_n \mid n \in \mathbb{N})$ is a homotopy $\mathsf{q}^A_\bullet \circ \mathsf{Sh}^A_\bullet \circ \mathsf{AW}^A_\bullet \circ \mathsf{j}^A_\bullet \Rightarrow \mathbf{1}_{N_\bullet A^\Delta}$.

Proof of the claim. Denote by $d^A_\bullet$ the differential of $A^\Delta_\bullet$, and let $d^A_0 : A^\Delta_0 \to 0$ and $s^A_{-1} : 0 \to A^\Delta_0$ be the zero morphisms. We check, more precisely, that

$$\mathsf{q}^A_n \circ (d^A_{n+1} \circ s^A_n + s^A_{n-1} \circ d^A_n) = \mathsf{q}^A_n \circ \mathsf{Sh}^A_n \circ \mathsf{AW}^A_n - \mathsf{q}^A_n \qquad \text{for every } n \in \mathbb{N}.$$

We argue by induction on n, and the assertion is clear for $n = 0$. For $n = 1$, notice that

$$\mathsf{Sh}^A_1 \circ \mathsf{AW}^A_1 = A[\varepsilon_1 \circ \eta_0, \mathbf{1}_{[1]}] + A[\mathbf{1}_{[1]}, \varepsilon_0 \circ \eta_0] : A_{1,1} \to A_{1,1}.$$

It follows that

$$s^A_1 = \mathsf{Sh}^{A'}_1 \circ \mathsf{AW}^{A'}_1 \circ A[\eta_0, \eta_0] = (A[\varepsilon_1 \circ \eta_0 \circ \eta_0, \eta_0] + A[\eta_0, \eta_1])$$

whence

$$d^A_2 \circ s^A_1 = \mathsf{Sh}^A_1 \circ \mathsf{AW}^A_1 - \mathbf{1}_{A[1,1]} + A[\varepsilon_1 \circ \varepsilon_1 \circ \eta_0, \varepsilon_1 \circ \eta_0]$$

(details left to the reader). Since $\mathrm{Im}\, A[\eta_0, \eta_0] \subset \mathrm{Ker}\, \mathsf{q}^A_2$, the assertion follows in this case. Next, suppose that $r > 1$, and that the sought identity is already known for every $n < r$, and every bisimplicial object A; especially, it holds for A', so if we let $d^{A'}_\bullet$ be the differential of $A'_\bullet$, we get

$$\mathsf{q}^{A'}_{r-1} \circ (d^{A'}_r \circ s^{A'}_{r-1} + s^{A'}_{r-2} \circ d^{A'}_{r-1}) = \mathsf{q}^{A'}_{r-1} \circ \mathsf{Sh}^{A'}_{r-1} \circ \mathsf{AW}^{A'}_{r-1} - \mathsf{q}^{A'}_{r-1}. \tag{7.4.53}$$

On the other hand, after noticing that

$$d^{A'}_r \circ A[\eta_0, \eta_0] = \mathbf{1}_{A[r,r]} - A[\eta_0, \eta_0] \circ d^{A'}_{r-1}$$

we may compute

$$\begin{aligned} d^{A'}_r \circ s^A_r &= d^{A'}_r \circ \mathsf{Sh}^{A'}_r \circ \mathsf{AW}^{A'}_r \circ A[\eta_0,\eta_0] - d^{A'}_r \circ s^{A'}_{r-1} \\ &= \mathsf{Sh}^{A'}_r \circ \mathsf{AW}^{A'}_r \circ d^{A'}_r \circ A[\eta_0,\eta_0] - d^{A'}_r \circ s^{A'}_{r-1} \\ &= \mathsf{Sh}^{A'}_r \circ \mathsf{AW}^{A'}_r - \mathsf{Sh}^{A'}_r \circ \mathsf{AW}^{A'}_r \circ A[\eta_0,\eta_0] \circ d^{A'}_{r-1} - d^{A'}_r \circ s^{A'}_{r-1} \end{aligned}$$

which, combined with (7.4.53), implies

$$\mathsf{q}^{A'}_{r-1} \circ (d^{A'}_r \circ s^A_r + s^A_{r-1} \circ d^{A'}_{r-1}) = -\mathsf{q}^{A'}_{r-1}. \tag{7.4.54}$$

Furthermore, notice the natural morphism in $s^2.\mathscr{A}$

$$(g^{(1)}_{A^\vee} \circ g^{(2)}_{\gamma_1 A^\vee})^\vee : A' \to A$$

supplied by (7.4.17); explicitly, for every $p, q \in \mathbb{N}$, this morphism is given by the discarded face operator $A[\varepsilon_0,\varepsilon_0] : A[p+1,q+1] \to A[p,q]$. Then, the naturality of $s_\bullet$, $\mathsf{Sh}_\bullet$ and $\mathsf{AW}_\bullet$ implies

$$\begin{aligned} A[\varepsilon_0,\varepsilon_0] \circ s^A_r &= A[\varepsilon_0,\varepsilon_0] \circ \mathsf{Sh}^{A'}_r \circ \mathsf{AW}^{A'}_r \circ A[\eta_0,\eta_0] - A[\varepsilon_0,\varepsilon_0] \circ s^{A'}_{r-1} \\ &= \mathsf{Sh}^A_r \circ \mathsf{AW}^A_r \circ A[\eta_0 \circ \varepsilon_0, \eta_0 \circ \varepsilon_0] - s^A_{r-1} \circ A[\varepsilon_0,\varepsilon_0] \\ &= \mathsf{Sh}^A_r \circ \mathsf{AW}^A_r - s^A_{r-1} \circ A[\varepsilon_0,\varepsilon_0]. \end{aligned}$$

Lastly, recalling that $d^A_r = A[\varepsilon_0,\varepsilon_0] - d^{A'}_{r-1}$, the latter identity can be added to (7.4.54), to deduce

$$\mathsf{q}^{A'}_{r-1} \circ (d^A_{r+1} \circ s^A_r + s^A_{r-1} \circ d^A_r) = \mathsf{q}^{A'}_{r-1} \circ \mathsf{Sh}^A_r \circ \mathsf{AW}^A_r - \mathsf{q}^{A'}_{r-1}.$$

Now, to prove the assertion in degree r, it suffices to observe that q^A_r factors through $\mathsf{q}^{A'}_{r-1}$. ◊

Claim 7.4.52 and theorem 7.4.31(ii) supply the second sought modification, and conclude the proof of the theorem. □

7.4.55. Let $(\mathscr{A}, \otimes)$ be an abelian tensor category, $A[\bullet]$ and $B[\bullet]$ two objects of $s.\mathscr{A}$, and define the bisimplicial objects $A \boxtimes B$ and $B \boxtimes A$, as well as the simplicial objects $A \otimes B$ and $B \otimes A$ of $\mathscr{A}$ as in remark 7.4.18(iii). Notice that the system of commutativity constraints $(\Psi_{A[n],B[n]} \mid n \in \mathbb{N})$ amounts to an isomorphism

$$\Psi_{A\otimes B} : A \otimes B \xrightarrow{\sim} B \otimes A \qquad \text{in } s.\mathscr{A}$$

whence an isomorphism $\Psi_{(A\otimes B)_\bullet} : (A \otimes B)_\bullet \xrightarrow{\sim} (B \otimes A)_\bullet$ on the respective unnormalized complexes.

Proposition 7.4.56. *With the notation of* (7.4.55)*, the diagram of chain complexes*

$$\begin{array}{ccc} \mathrm{Tot}(A \boxtimes B)_{\bullet\bullet} & \xrightarrow{\Psi^\bullet_{A_\bullet,B_\bullet}} & \mathrm{Tot}(B \boxtimes A)_{\bullet\bullet} \\ {\scriptstyle \mathsf{Sh}^{A\boxtimes B}_\bullet}\downarrow & & \downarrow{\scriptstyle \mathsf{Sh}^{B\boxtimes A}_\bullet} \\ (A \otimes B)_\bullet & \xrightarrow{\Psi_{(A\otimes B)_\bullet}} & (B \otimes A)_\bullet \end{array}$$

commutes, where $\Psi^\bullet_{A_\bullet,B_\bullet}$ is the commutativity constraint for the unnormalized chain complexes $A_\bullet$ and $B_\bullet$, as in (7.1.17)*.*

Proof. The assertion boils down to the identity

$$(-1)^{pq} \cdot \mathsf{Sh}^{B\boxtimes A}_{q,p} \circ \Psi_{A[p],B[q]} = \Psi_{A[n],B[n]} \circ \mathsf{Sh}^{A\boxtimes B}_{p,q} \tag{7.4.57}$$

for every $p, q \in \mathbb{N}$ with $p + q = n$. For the latter, suppose first that $p = 0$, in which case

$$\mathsf{Sh}^{A\boxtimes B}_{p,q} = A[\eta_0 \circ \cdots \circ \eta_{q-1}] \otimes \mathbf{1}_{B[q]} \qquad \text{and} \qquad \mathsf{Sh}^{B\boxtimes A}_{q,p} = \mathbf{1}_{B[q]} \otimes A[\eta_0 \circ \cdots \circ \eta_{q-1}]$$

from which we derive (7.4.57), using the naturality of Ψ (details left to the reader). Likewise we argue for the case where $q = 0$. For the general case, we proceed by induction on n. The cases $n = 0, 1$ have already been dealt with, so suppose $r \geq 2$, and that the sought identity is already known for every pair of integers whose sum is $< n$, and every objects A, B of $s.\mathscr{A}$. By the foregoing, we may also assume that both $p, q > 0$, and then claim 7.4.38(i) implies that

$$\mathsf{Sh}^{A\boxtimes B}_{p,q} = (-1)^q \cdot \mathsf{Sh}^{\gamma A\boxtimes\gamma B}_{p-1,q} \circ (\mathbf{1}_{A[p]} \otimes B[\eta_q]) + \mathsf{Sh}^{\gamma A\boxtimes\gamma B}_{p,q-1} \circ (A[\eta_p] \otimes \mathbf{1}_{B[q]}).$$

However, it follows easily from remark 3.7.34(iii) that Ψ is an additive functor in both of its arguments; combining with the inductive assumption, we deduce that

$$\begin{aligned}\Psi_{A[n],B[n]} \circ \mathsf{Sh}^{A\boxtimes B}_{p,q} &= (-1)^{pq} \cdot \mathsf{Sh}^{\gamma B\boxtimes\gamma A}_{q,p-1} \circ \Psi_{A[p],B[q+1]} \circ (\mathbf{1}_{A[p]} \otimes B[\eta_q]) \\ &\quad + (-1)^{p(q-1)} \cdot \mathsf{Sh}^{\gamma B\boxtimes\gamma A}_{q-1,p} \circ \Psi_{A[p+1],B[q]} \circ (A[\eta_p] \otimes \mathbf{1}_{B[q]}) \\ &= (-1)^{pq} \cdot \mathsf{Sh}^{\gamma B\boxtimes\gamma A}_{q,p-1} \circ (B[\eta_q] \otimes \mathbf{1}_{A[p]}) \circ \Psi_{A[p],B[q]} \\ &\quad + (-1)^{p(q-1)} \cdot \mathsf{Sh}^{\gamma B\boxtimes\gamma A}_{q-1,p} \circ (\mathbf{1}_{B[q]} \otimes A[\eta_p]) \circ \Psi_{A[p],B[q]} \\ &= (-1)^{pq} \cdot \mathsf{Sh}^{B\boxtimes A}_{q,p} \circ \Psi_{A[p],B[q]}\end{aligned}$$

where the last equality follows again from claim 7.4.38(i) and the additivity of Ψ. $\square$

7.4.58. The shuffle map is also *associative*, in the following sense. Let

$$A := (A[p, q, r] \mid p, q, r \in \mathbb{N})$$

be any triple simplicial object of the abelian category $\mathscr{A}$, and denote by $A^{(1,2)}$ (resp. $A^{(2,3)}$) the diagonal bisimplicial object of $\mathscr{A}$ extracted from A by the rule

$$A^{(1,2)}[p, q] := A[p, p, q] \qquad (\text{ resp. } A^{(2,3)}[p, q] := A[p, q, q]\ ) \qquad \text{for every } p, q \in \mathbb{N}.$$

Let also $A_{\bullet\bullet\bullet}$ be the triple chain complex associated to A, and $A^{\Delta}_{\bullet\bullet\bullet}$ the diagonal chain complex extracted from $A_{\bullet\bullet\bullet}$. Moreover, denote by A' (resp. A'') the bisimplicial object of $s.\mathscr{A}$ given by the rule :

$$[p, q] \mapsto A[p, q, \bullet] \quad (\text{ resp. } [p, q] \mapsto A[\bullet, p, q]\ ) \qquad \text{for every } p, q \in \mathbb{N}.$$

Proposition 7.4.59. *With the notation of* (7.4.58)*, we have a commutative diagram in* $\mathsf{C}(\mathscr{A})$

$$\begin{array}{ccc}\mathrm{Tot}(A_{\bullet\bullet\bullet}) & \longrightarrow & \mathrm{Tot}(A^{(1,2)}_{\bullet\bullet}) \\ \downarrow & & \downarrow {\scriptstyle \mathsf{Sh}^{A^{(1,2)}}_{\bullet}} \\ \mathrm{Tot}(A^{(2,3)}_{\bullet\bullet}) & \xrightarrow{\mathsf{Sh}^{A^{(2,3)}}_{\bullet}} & A^{\Delta}_{\bullet\bullet\bullet}\end{array}$$

whose top horizontal (resp. left vertical) arrow is obtained as the composition of $\mathsf{Sh}^{A'}_{\bullet}$ *(resp.* $\mathsf{Sh}^{A''}_{\bullet}$*) with the functor* $\mathrm{Tot} \circ \mathsf{C}(\mathsf{U}_{\mathscr{A}}) : \mathsf{C}(s.\mathscr{A}) \to \mathsf{C}(\mathsf{C}(\mathscr{A})) \to \mathsf{C}(\mathscr{A})$.

Proof. For every $p, q, r \in \mathbb{N}$, denote by

$$\begin{aligned}\mathsf{Sh}^{A'}_{p,q}[r] &: A[p, q, r] \to A[p + q, p + q, r] = A^{(1,2)}[p + q, r] \\ \mathsf{Sh}^{A''}_{q,r}[p] &: A[p, q, r] \to A^{(2,3)}[p, q + r]\end{aligned}$$

respectively the $[r]$-component of $\mathsf{Sh}^{A'}_{p,q}$ and the $[p]$-component of $\mathsf{Sh}^{A''}_{q,r}$. The assertion boils down to the identity :

$$\mathsf{Sh}^{A^{(1,2)}}_{p+q,r} \circ \mathsf{Sh}^{A'}_{p,q}[r] = \mathsf{Sh}^{A^{(2,3)}}_{p,q+r} \circ \mathsf{Sh}^{A''}_{q,r}[p] \qquad \text{for every } p, q, r \in \mathbb{N}.$$

To check the latter, set

$$D' := \gamma_2 \circ \gamma_1 A' \qquad D^{(1,2)} := \gamma_2 \circ \gamma_1 (A^{(1,2)})$$

and define likewise D'' and $D^{(2,3)}$ (notation of (7.4.17)). Also, let $A[p, q, r] := 0$ whenever one of the indices p, q, r is < 0, and define $\mathsf{Sh}_{p,q}$ to be the zero map, when either p or q is strictly negative (cp. the proof of proposition 7.4.36). We argue by induction on $n := p+q+r$; the case $n=0$ is trivial, so suppose that $n>0$, and that the assertion is already known for all indices whose sum is $<n$, and all triple simplicial objects of $\mathscr{A}$. Notice as well that the assertion trivially holds as well if any of the indices p,q,r is strictly negative, since in this case both sides are the zero map. Hence, we may assume that $p,q,r\in\mathbb{N}$; in this case, by applying claim 7.4.38(i), first to $A^{(1,2)}$ and then to A', we compute

$$\begin{aligned}
\mathsf{Sh}^{A^{(1,2)}}_{p+q,r}\circ\mathsf{Sh}^{A'}_{p,q}[r] &= (-1)^r\cdot\mathsf{Sh}^{D^{(1,2)}}_{p+q-1,r}\circ A[\mathbf{1}_{[p+q]},\mathbf{1}_{[p+q]},\eta_r]\circ\mathsf{Sh}^{A'}_{p,q}[r] \\
&\quad+\mathsf{Sh}^{D^{(1,2)}}_{p+q,r-1}\circ A[\eta_{p+q},\eta_{p+q},\mathbf{1}_{[r]}]\circ\mathsf{Sh}^{A'}_{p,q}[r] \\
&= (-1)^r\cdot\mathsf{Sh}^{D^{(1,2)}}_{p+q-1,r}\circ\mathsf{Sh}^{A'}_{p,q}[r+1]\circ A[\mathbf{1}_{[p]},\mathbf{1}_{[q]},\eta_r] \\
&\quad+\mathsf{Sh}^{D^{(1,2)}}_{p+q,r-1}\circ\mathsf{Sh}^{D'}_{p,q}[r]\circ A[\eta_p,\eta_q,\mathbf{1}_{[r]}] \qquad\text{(by (7.4.39))} \\
&= (-1)^{q+r}\cdot\mathsf{Sh}^{D^{(1,2)}}_{p+q-1,r}\circ\mathsf{Sh}^{D'}_{p-1,q}[r+1]\circ A[\mathbf{1}_{[p]},\eta_q,\eta_r] \\
&\quad+(-1)^r\cdot\mathsf{Sh}^{D^{(1,2)}}_{p+q-1,r}\circ\mathsf{Sh}^{D'}_{p,q-1}[r+1]\circ A[\eta_p,\mathbf{1}_{[q]},\eta_r] \\
&\quad+\mathsf{Sh}^{D^{(1,2)}}_{p+q,r-1}\circ\mathsf{Sh}^{D'}_{p,q}[r]\circ A[\eta_p,\eta_q,\mathbf{1}_{[r]}] \\
&= (-1)^{q+r}\cdot\mathsf{Sh}^{D^{(2,3)}}_{p-1,q+r}\circ\mathsf{Sh}^{D''}_{q,r}[p]\circ A[\mathbf{1}_{[p]},\eta_q,\eta_r] \\
&\quad+(-1)^r\cdot\mathsf{Sh}^{D^{(2,3)}}_{p,q+r-1}\circ\mathsf{Sh}^{D''}_{q-1,r}[p+1]\circ A[\eta_p,\mathbf{1}_{[q]},\eta_r] \\
&\quad+\mathsf{Sh}^{D^{(2,3)}}_{p,q+r-1}\circ\mathsf{Sh}^{D''}_{q,r-1}[p+1]\circ A[\eta_p,\eta_q,\mathbf{1}_{[r]}]
\end{aligned}$$

where the last identity holds by inductive assumption. On the other hand, by applying claim 7.4.38(i) to $A^{(2,3)}$ we get

$$\begin{aligned}
\mathsf{Sh}^{A^{(2,3)}}_{p,q+r}\circ\mathsf{Sh}^{A''}_{q,r}[p] &= (-1)^{q+r}\cdot\mathsf{Sh}^{D^{(2,3)}}_{p-1,q+r}\circ A[\mathbf{1}_{[p]},\eta_{q+r},\eta_{q+r}]\circ\mathsf{Sh}^{A''}_{q,r}[p] \\
&\quad+\mathsf{Sh}^{D^{(2,3)}}_{p,q+r-1}\circ A[\eta_p,\mathbf{1}_{[q+r]},\mathbf{1}_{[q+r]}]\circ\mathsf{Sh}^{A''}_{q,r}[p] \\
&= (-1)^{q+r}\cdot\mathsf{Sh}^{D^{(2,3)}}_{p-1,q+r}\circ\mathsf{Sh}^{D''}_{q,r}[p]\circ A[\mathbf{1}_{[p]},\eta_q,\eta_r] \\
&\quad+\mathsf{Sh}^{D^{(2,3)}}_{p,q+r-1}\circ\mathsf{Sh}^{A''}_{q,r}[p+1]\circ A[\eta_p,\mathbf{1}_{[q]},\mathbf{1}_{[r]}] \qquad\text{(by (7.4.39))}
\end{aligned}$$

and after applying again claim 7.4.38(i) to A'' and comparing with the foregoing expression for $\mathsf{Sh}^{A^{(1,2)}}_{p+q,r}\circ\mathsf{Sh}^{A'}_{p,q}[r]$, we obtain the sought identity. □

Theorem 7.4.60 (Dold-Puppe-Kan)**.** *For any abelian category $\mathscr{A}$, and any $k\in\mathbb{N}$, we have :*

(i) *The functors $\mathsf{N}_{\mathscr{A}}$ and $\mathsf{N}_{\mathscr{A},k}$ are equivalences.*

(ii) *If f,g are any two morphisms in $s.\mathscr{A}$, then there exists a simplicial homotopy from f to g if and only if there exists a chain homotopy from $N_\bullet f$ to $N_\bullet g$.*

Proof. We easily reduce to the case where $\mathscr{A}$ is small, and then there exists a fully faithful embedding $\mathscr{A}\to\mathscr{B}$, where $\mathscr{B}$ is a complete and cocomplete abelian tensor category, with internal Hom functor (lemma 3.7.46).

(i): We first construct an explicit quasi-inverse for the functors $\mathsf{N}_{\mathscr{B}}$ and $\mathsf{N}_{\mathscr{B},k}$, as follows. For every $i\in\mathbb{N}$, consider the cochain complex $\mathbb{K}\langle i\rangle_\bullet$ defined in example 7.4.29(ii); notice that every morphism $\varphi:[i]\to[i']$ in Δ induces a morphism

$$\mathbb{K}\langle\varphi\rangle_\bullet := N_\bullet\mathbb{Z}^{\mathbf{\Delta}_\varphi}:\mathbb{K}\langle i\rangle_\bullet\to\mathbb{K}\langle i'\rangle_\bullet$$

(notation of remark 7.4.19(iii)). Hence, the system $(\mathbb{K}\langle i\rangle_\bullet\mid i\in\mathbb{N})$ amounts to a cosimplicial object of $\mathsf{C}^{\leq 0}(\mathbb{Z}\text{-}\mathbf{Mod})$. Now, let U be a unit of the tensor category $\mathscr{B}$; for any object $C_\bullet$ of $\mathsf{C}^{\leq 0}(\mathscr{B})$ (resp. of $\mathsf{C}^{[-k,0]}(\mathscr{B})$), we set

$$K_C[i] := \mathscr{H}\!om_{\mathsf{C}(\mathscr{B})}(\mathbb{K}\langle i\rangle_\bullet\otimes_{\mathbb{Z}}U[0],C_\bullet)\qquad\text{for every } i\in\mathbb{N}\text{ (resp. for every } i\leq k)$$

where $\mathscr{H}om_{\mathsf{C}(\mathscr{B})}$ is the functor constructed in example 7.1.16(vii), and the mixed tensor product is defined as in (7.1.18). By the foregoing, it is clear that the system $(K_C[i] \mid i \in \mathbb{N})$ amounts to an object of $s.\mathscr{B}$ (resp. of $s_k.\mathscr{B}$). In order to compute $N_\bullet K_C$, let us set

$$\mathbb{Z}_+^{\mathbf{\Delta}_i} := \sum_{n=1}^{i} \mathrm{Im}\,(\mathbb{Z}^{\mathbf{\Delta}_{\varepsilon_n}} : \mathbb{Z}^{\mathbf{\Delta}_{i-1}} \to \mathbb{Z}^{\mathbf{\Delta}_i}) \qquad \mathbb{Z}_0^{\mathbf{\Delta}_i} := \mathbb{Z}^{\mathbf{\Delta}_i}/\mathbb{Z}_+^{\mathbf{\Delta}_i} \qquad \overline{\mathbb{K}}\langle i\rangle_\bullet := N_\bullet \mathbb{Z}_0^{\mathbf{\Delta}_i}$$

for every $i > 0$, as well as $\mathbb{Z}_0^{\mathbf{\Delta}_0} := \mathbb{Z}^{\mathbf{\Delta}_0}$ and $\overline{\mathbb{K}}\langle 0\rangle_\bullet := \mathbb{K}\langle 0\rangle_\bullet$; thus, $\overline{\mathbb{K}}\langle i\rangle_\bullet$ is also the quotient of $\mathbb{K}\langle i\rangle_\bullet$ by the sum of the images of the morphisms $\mathbb{K}\langle\varepsilon_n\rangle$, for $n = 1, \dots, i$. With this notation, a simple inspection of the definition shows that

$$N_i K_C = \mathscr{H}om_{\mathsf{C}(\mathscr{B})}(\overline{\mathbb{K}}\langle i\rangle_\bullet \otimes_{\mathbb{Z}} U[0], C_\bullet) \qquad \text{for every } i \in \mathbb{N} \text{ (resp. for every } i \leq k\text{)}.$$

On the other hand, using the explicit description of example 7.4.29(ii), it is easily seen that

$$\overline{\mathbb{K}}\langle i\rangle_n \xrightarrow{\sim} \begin{cases} \mathbb{Z} & \text{if } n = i \text{ or } n = i-1 \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

More precisely, $\overline{\mathbb{K}}\langle i\rangle_i$ (resp. $\overline{\mathbb{K}}\langle i\rangle_{i-1}$) is generated by the basis element $e_{\mathbf{1}}$ of $\mathbb{K}\langle i\rangle_i$ (resp. e_{ε_0} of $\mathbb{K}\langle i\rangle_{i-1}$) corresponding to the identity map $\mathbf{1}_{[i]}$ (resp. corresponding to $\varepsilon_0 : [i-1] \to [i]$). Furthermore, example 7.4.29(i) shows that the differential $\mathbb{K}\langle i\rangle_i \to \mathbb{K}\langle i\rangle_{i-1}$ maps $e_{\mathbf{1}}$ to e_{ε_0}, so it corresponds to the identity map $\mathbb{Z} \to \mathbb{Z}$, under the foregoing identification. Taking into account remark 3.7.12(iv), we deduce that $N_i K_C$ is the kernel of the morphism

$$C_i \oplus C_{i-1} \to C_{i-1} \oplus C_{i-2} \tag{7.4.61}$$

given by the matrix

$$\begin{pmatrix} (-1)^i \cdot d_i^C & \mathbf{1}_{C_{i-1}} \\ 0 & (-1)^{i+1} \cdot d_{i-1}^C \end{pmatrix}$$

and since $d_{i-1}^C \circ d_i^C = 0$, the latter is just C_i; more precisely, C_i is identified with this kernel, via the monomorphism

$$(\mathbf{1}_{C_i}, (-1)^{i+1} \cdot d_i^C) : C_i \to C_i \oplus C_{i-1}. \tag{7.4.62}$$

This identification $N_i K_C \xrightarrow{\sim} C_i$ can also be described as follows. For every $Z \in \mathrm{Ob}(\mathscr{B})$, let

$$\mathrm{Hom}_{\mathscr{B}}(Z, C_\bullet)$$

be the cochain complex such that $\mathrm{Hom}_{\mathscr{B}}(Z, C_\bullet)_n := \mathrm{Hom}_{\mathscr{B}}(Z, C_n)$ for every $n \in \mathbb{Z}$, with differentials induced by those of $C_\bullet$, in the obvious way; then we have natural identifications

$$\mathrm{Hom}_{\mathsf{C}(\mathbb{Z}\text{-}\mathbf{Mod})}(\overline{\mathbb{K}}\langle i\rangle_\bullet, \mathrm{Hom}_{\mathscr{B}}(Z, C_\bullet)) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{C}(\mathscr{B})}(\overline{\mathbb{K}}\langle i\rangle_\bullet \otimes_{\mathbb{Z}} Z[0], C_\bullet) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{B}}(Z, N_i K_C)$$

(see example 7.1.16(vii)) whose composition with the induced isomorphism

$$\mathrm{Hom}_{\mathscr{B}}(Z, N_i K_C) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{B}}(Z, C_i)$$

is given by the rule :

$$(\varphi_\bullet : \overline{\mathbb{K}}\langle i\rangle_\bullet \to \mathrm{Hom}_{\mathscr{B}}(Z, C_\bullet)) \mapsto (\varphi_i(\varepsilon_{\mathbf{1}}) : Z \to C_i). \tag{7.4.63}$$

It remains to determine the differential $d_{i+1}^N : N_{i+1} K_C \to N_i K_C$; by definition, the latter is induced by the morphism $\overline{\mathbb{K}}\langle\varepsilon_0\rangle_\bullet : \overline{\mathbb{K}}\langle i\rangle_\bullet \to \overline{\mathbb{K}}\langle i+1\rangle_\bullet$. In turn, the foregoing description says that $\overline{\mathbb{K}}\langle\varepsilon_0\rangle_\bullet$ is naturally identified with the morphism of $\mathsf{C}(\mathbb{Z}\text{-}\mathbf{Mod})$

$$\begin{array}{ccccccccccc} \overline{\mathbb{K}}\langle i\rangle_\bullet & & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\ \downarrow & & & & \downarrow & & \downarrow{\scriptstyle \mathbf{1}_{\mathbb{Z}}} & & \downarrow & & \\ \overline{\mathbb{K}}\langle i+1\rangle_\bullet & & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

(where the two copies of $\mathbb{Z}$ on the top horizontal row are placed in homological degrees i and $i-1$), so d^N_{i+1} is deduced from the morphism in $\mathsf{C}(\mathscr{B})$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_{i+1}\oplus C_i & \longrightarrow & C_i\oplus C_{i-1} & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \mathbf{1}} & & \downarrow & & \\ 0 & \longrightarrow & 0 & \longrightarrow & C_i\oplus C_{i-1} & \longrightarrow & C_{i-1}\oplus C_{i-2} & \longrightarrow & 0 \end{array}$$

whose rows are given by the morphisms (7.4.61). Therefore, via the identification (7.4.62), the morphism d^N_{i+1} becomes none else than $(-1)^{i+1}\cdot d^C_{i+1}$, *i.e.* we have obtained a natural isomorphism

$$\omega^C_\bullet : N_\bullet K_C \xrightarrow{\sim} C_\bullet$$

for every complex in $\mathsf{C}^{\leq 0}(\mathscr{B})$ (resp. in $\mathsf{C}^{[-k,0]}(\mathscr{B})$).

Conversely, let $B[\bullet]$ be any object of $s.\mathscr{B}$, and Z any object of $\mathscr{B}$; clearly

$$N_\bullet(\mathbf{\Delta}_k\otimes s.Z) = \mathbb{K}\langle i\rangle_\bullet\otimes_{\mathbb{Z}} Z[0] \qquad \text{for every } i\in\mathbb{N}.$$

In view of example 7.1.16(vii) and remark 7.4.19(ii), we deduce a natural transformation

$$\text{(7.4.64)}\qquad \begin{array}{ccc} \mathrm{Hom}_{\mathscr{B}}(Z,B[i]) & \xrightarrow{\ a\ } & \mathrm{Hom}_{s.\mathscr{B}}(\mathbf{\Delta}_i\otimes s.Z,B) \\ & & \downarrow{\scriptstyle b} \\ \mathrm{Hom}_{\mathscr{B}}(Z,K_{N_\bullet B}[i]) & \xleftarrow{\ c\ } & \mathrm{Hom}_{\mathsf{C}(\mathscr{B})}(\mathbb{K}\langle i\rangle_\bullet\otimes_{\mathbb{Z}} Z[0],N_\bullet B) \end{array}$$

which, by Yoneda's lemma, comes from a unique morphism in $\mathscr{B}$

$$\psi^B_i : B[i]\to K_{N_\bullet B}[i] \qquad \text{for every } i\in\mathbb{N}.$$

The same construction applies, in case B is an object of $s_k.\mathscr{B}$: we need only replace the group $\mathrm{Hom}_{s.\mathscr{B}}(\mathbf{\Delta}_i\otimes s.Z,B)$ by $\mathrm{Hom}_{s_k.\mathscr{B}}(s.\mathrm{trunc}_k(\mathbf{\Delta}_i\otimes s.Z),B)$ in (7.4.64) : see remark 7.4.19(iv). Moreover, remark 7.4.19(iii) implies that the system $\psi^B := (\psi^B_i \mid i\in\mathbb{N})$ amounts to a morphism $B\to K_{N_\bullet B}$ in $s.\mathscr{B}$ (resp. in $s_k.\mathscr{B}$). By the same token, the $\mathbb{Z}$-linear isomorphism a maps the abelian subgroup $\mathrm{Hom}_{\mathscr{B}}(Z,N_iB)$ isomorphically onto the subgroup

$$\mathrm{Hom}_{s.\mathscr{B}}(\mathbb{Z}_0^{\mathbf{\Delta}_i}\otimes_{\mathbb{Z}} s.Z,B)\xrightarrow{\sim}\mathrm{Hom}_{s.\mathbb{Z}\text{-}\mathbf{Mod}}(\mathbb{Z}_0^{\mathbf{\Delta}_i},\mathrm{Hom}_{\mathscr{B}}(Z,B))$$

for every $Z\in\mathrm{Ob}(\mathscr{B})$ and every $i\in\mathbb{N}$, where $\mathrm{Hom}_{\mathscr{B}}(Z,B)$ is the simplicial abelian group as in example 7.4.29(iii). Explicitly, if $\beta : Z\to N_iB$ is any morphism, then $a(\beta)$ is the unique morphism $\mathbb{Z}_0^{\mathbf{\Delta}_i}\to\mathrm{Hom}_{\mathscr{B}}(Z,B)$ of simplicial abelian groups such that $a(\beta)(e_{\mathbf{1}})=\beta$, where $e_{\mathbf{1}}\in\overline{\mathbb{K}}\langle i\rangle_i\subset\mathbb{Z}_0^{\mathbf{\Delta}_i}[i]$ is the basis element described in the foregoing. Likewise, we have natural identifications

$$\mathrm{Hom}_{\mathsf{C}(\mathbb{Z}\text{-}\mathbf{Mod})}(\mathbb{K}\langle i\rangle_\bullet,\mathrm{Hom}_{\mathscr{B}}(Z,N_\bullet B))\xrightarrow{\sim}\mathrm{Hom}_{\mathsf{C}(\mathscr{B})}(\mathbb{K}\langle i\rangle_\bullet\otimes_{\mathbb{Z}} Z[0],N_\bullet B)$$

and b restricts to a map

$$\mathrm{Hom}_{s.\mathbb{Z}\text{-}\mathbf{Mod}}(\mathbb{Z}_0^{\mathbf{\Delta}_i},\mathrm{Hom}_{\mathscr{B}}(Z,B))\to\mathrm{Hom}_{\mathsf{C}(\mathbb{Z}\text{-}\mathbf{Mod})}(\overline{\mathbb{K}}\langle i\rangle_\bullet,\mathrm{Hom}_{\mathscr{B}}(Z,N_\bullet B)) \qquad \varphi\mapsto N_\bullet\varphi.$$

Again, the same applies to an object of $s_k.\mathscr{B}$, by taking suitable truncated variants of the above constructions. Taking into account (7.4.63), we conclude that ψ^B induces an isomorphism

$$N_\bullet\psi^B : N_\bullet B\xrightarrow{\sim} N_\bullet K_{N_\bullet B}$$

which is inverse to $\omega^{N_\bullet B}$. To finish the proof of assertion (i) for the category $\mathscr{B}$, it then suffices to remark :

Claim 7.4.65. The functors $\mathsf{N}_{\mathscr{B}}$ and $\mathsf{N}_{\mathscr{B},k}$ are conservative.

Proof of the claim. We show, by induction on n, that if a morphism h in $s.\mathscr{B}$ or $s_k.\mathscr{B}$ (for any $k \in \mathbb{N}$) induces an isomorphism $N_\bullet h$, then $h[n]$ is an isomorphism for every $n \in \mathbb{N}$ (resp. for every $n \leq k$). The assertion is obvious for $n = 0$, hence suppose that $n > 0$, and that $h[n-1]$ is known to be an isomorphism whenever h is a morphism in $s.\mathscr{B}$ or in $s_k.\mathscr{B}$ (for arbitrary $k \in \mathbb{N}$) such that $N_\bullet h$ is an isomorphism. Let $h : A \to B$ be any such morphism in $s.\mathscr{B}$ or in $s_k.\mathscr{B}$. If h is a morphism in $s_0.\mathscr{B}$, we are done, hence we may suppose that either h is a morphism in $s.\mathscr{B}$ or $k > 0$. Set

$$A' := \mathrm{Ker}(g_A : \gamma A \to A) \qquad (\text{ resp. } A' := \mathrm{Ker}(g_A : \gamma_{k-1} A \to s.\mathrm{trunc}_{k-1} A)\ )$$

as well as $B' := \mathrm{Ker}(g_B)$ (notation of remark 7.4.12(ii)). Notice that g_A is an epimorphism, since ∂_{n+2} admits the section σ_{n+1}, for every $n \in \mathbb{N}$ (resp. for every $n \leq k$). Therefore, we have a commutative diagram in $s.\mathscr{B}$ with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \longrightarrow & \gamma A & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle \gamma h} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & A' & \longrightarrow & \gamma A & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

(resp. a corresponding diagram in $s_{k-1}.\mathscr{B}$). By inspecting the definitions, it is easily seen that $N_\bullet A' = (N_\bullet A)[-1]$, $N_\bullet A' = (N_\bullet A)[-1]$, and $N_\bullet h' = (N_\bullet h)[-1]$; especially, $N_\bullet h'$ is an isomorphism, so $h'[n-1]$ is an isomorphism, by inductive assumption. The same holds also for $h[n-1]$, and we conclude that $\gamma h[n]$ is an isomorphism. But $\gamma h[n] = h[n+1]$, so we are done. ♢

Lastly, in order to prove assertion (i) for the original category $\mathscr{A}$, it suffices to notice :

Claim 7.4.66. Let $C_\bullet$ be any object of $\mathsf{C}(\mathscr{A})$ (resp. of $\mathsf{C}^{[-k,0]}(\mathscr{A})$), and regard $C_\bullet$ as an object of $\mathsf{C}(\mathscr{B})$ (resp. of $\mathsf{C}^{[-k,0]}(\mathscr{B})$), via the fully faithful embedding $\mathscr{A} \to \mathscr{B}$. Then K_C is isomorphic to an object of $s.\mathscr{A}$ (resp. $s_k.\mathscr{A}$), regarded as a full subcategory of $s.\mathscr{B}$ (resp. of $s_k.\mathscr{B}$), via the same embedding.

Proof of the claim. This follows easily, by remarking that $\mathbb{K}\langle i\rangle_\bullet$ lies in $\mathsf{C}^{[-i,0]}(\mathbb{Z}\text{-}\mathbf{Mod})$ for every $i \in \mathbb{N}$, and $\mathbb{K}\langle i\rangle_j$ is a finitely generated abelian group for every $i, j \in \mathbb{N}$: details left to the reader. ♢

(ii): First, let $f, g : A \to B$ be two morphisms in $s.\mathscr{A}$, and $u : \mathbf{\Delta}_1 \otimes A \to B$ a homotopy from f to g (see remark 7.4.18(v)); especially,

$$u \circ (\mathbf{\Delta}_{\varepsilon_1} \otimes A) = f \qquad \text{and} \qquad u \circ (\mathbf{\Delta}_{\varepsilon_0} \otimes A) = g.$$

Notice that

$$\mathbb{Z}_\bullet^{\mathbf{\Delta}_1} \otimes_{\mathbb{Z}} A_\bullet = \mathrm{Tot}(\mathbf{\Delta}_1 \boxtimes A)_{\bullet\bullet}$$

(notation of remark 7.4.18(iii) and example 7.4.29(i)); there follows a morphism in $\mathsf{C}(\mathscr{A})$

$$\widetilde{u}_\bullet : \mathbb{Z}_\bullet^{\mathbf{\Delta}_1} \otimes_{\mathbb{Z}} A_\bullet \xrightarrow{\mathsf{Sh}_\bullet} (\mathbf{\Delta}_1 \otimes A)_\bullet \xrightarrow{\mathsf{q}_\bullet} N_\bullet(\mathbf{\Delta}_1 \otimes A) \xrightarrow{N_\bullet u} N_\bullet B$$

where $\mathsf{Sh}_\bullet$ denotes the shuffle map for the bisimplicial object $A \boxtimes \mathbf{\Delta}_1$, and $\mathsf{q}_\bullet$ is the projection defined in (7.4.30). Moreover, the maps $\mathbf{\Delta}_{\varepsilon_i}[0] : \mathbf{\Delta}_0[0] \to \mathbf{\Delta}_1[0]$ ($i = 0, 1$) induce morphisms

$$\widetilde{e}_{i,n} := \mathbb{Z}^{\mathbf{\Delta}_{\varepsilon_i}[0]} \otimes_{\mathbb{Z}} A_n : A_n \to \mathbb{Z}^{\mathbf{\Delta}_1[0]} \otimes_{\mathbb{Z}} A_n \subset (\mathbb{Z}_\bullet^{\mathbf{\Delta}_1} \otimes_{\mathbb{Z}} A_\bullet)_n$$

that amount to morphisms of cochain complexes $\widetilde{e}_{i,\bullet} : A_\bullet \to \mathbb{Z}_\bullet^{\mathbf{\Delta}_1} \otimes_{\mathbb{Z}} A_\bullet$ ($i = 0, 1$), and a simple inspection of the definitions shows that

$$\mathsf{Sh}_\bullet \circ \widetilde{e}_{i,\bullet} = (\Delta_{\varepsilon_i} \otimes A)_\bullet \qquad \text{for } i = 0, 1$$

whence

$$\widetilde{u}_\bullet \circ \widetilde{e}_{1,\bullet} = f_\bullet \qquad \text{and} \qquad \widetilde{u}_\bullet \circ \widetilde{e}_{0,\bullet} = g_\bullet.$$

The construction makes it clear that $\widetilde{e}_i$ restricts to a morphism $N_\bullet A \to \mathbb{K}\langle 1\rangle_\bullet \otimes_\mathbb{Z} N_\bullet A$, and the latter is none else than the map $\iota_i \otimes_\mathbb{Z} N_\bullet A$, with the notation of remark 7.1.19(ii). We conclude that the morphism

$$\overline{u}_\bullet : \mathbb{K}\langle 1\rangle_\bullet \otimes_\mathbb{Z} N_\bullet A \hookrightarrow \mathbb{Z}_\bullet^{\boldsymbol{\Delta}_1} \otimes_\mathbb{Z} A_\bullet \xrightarrow{\widetilde{u}_\bullet} N_\bullet B$$

is a homotopy from $N_\bullet f$ to $N_\bullet g$ (notation of (7.4.30)).

Conversely, suppose that $\overline{v}_\bullet : \mathbb{K}\langle 1\rangle_\bullet \otimes_\mathbb{Z} N_\bullet A \to N_\bullet B$ is a homotopy from $N_\bullet f$ to $N_\bullet g$. Since $\mathbb{K}\langle 1\rangle_\bullet$ is a direct summand of $\mathbb{Z}_\bullet^{\boldsymbol{\Delta}_1}$ and $N_\bullet A$ is a direct summand of $A_\bullet$, we may extend $\overline{v}_\bullet$ to a morphism

$$\widetilde{v}_\bullet : \mathbb{Z}_\bullet^{\boldsymbol{\Delta}_1} \otimes_\mathbb{Z} A_\bullet \to N_\bullet B$$

such that $\widetilde{v}_\bullet$ is the zero morphism on the direct summands other than $\mathbb{K}\langle 1\rangle_\bullet \otimes_\mathbb{Z} N_\bullet A$. Next, consider the composition

$$v_\bullet : N_\bullet(\boldsymbol{\Delta}_1 \otimes A) \xrightarrow{\mathsf{j}_\bullet} (\boldsymbol{\Delta}_1 \otimes A)_\bullet \xrightarrow{\mathsf{AW}_\bullet} A_\bullet \otimes_\mathbb{Z} \mathbb{Z}_\bullet^{\boldsymbol{\Delta}_1} \xrightarrow{\Psi_\bullet} \mathbb{Z}_\bullet^{\boldsymbol{\Delta}_1} \otimes_\mathbb{Z} A_\bullet \xrightarrow{\widetilde{v}_\bullet} N_\bullet B$$

where $\mathsf{j}_\bullet$ is the natural injection (see (7.4.30)), $\mathsf{AW}_\bullet$ is the Alexander-Whitney map for the bisimplicial object $A \boxtimes \boldsymbol{\Delta}_1$ (notice that $\boldsymbol{\Delta}_1 \otimes A = (A \boxtimes \boldsymbol{\Delta}_1)^\Delta$), and $\Psi_\bullet$ is the commutativity constraint (see example 7.1.16(i)). By (i), the morphism $v_\bullet$ comes from a unique morphism

$$v : \boldsymbol{\Delta}_1 \otimes A \to B \qquad \text{in } s.\mathscr{A}.$$

On the other hand, since $N_{p+q}A$ is contained in the kernel of $A[\varepsilon_{p,0}^{q\vee}]$ for every $p, q \in \mathbb{N}$, it is easily seen that the diagram

$$\begin{array}{ccc} N_\bullet A & \xrightarrow{\iota_i \otimes_\mathbb{Z} N_\bullet A} & \mathbb{K}\langle 1\rangle_\bullet \otimes_\mathbb{Z} N_\bullet A \\ {\scriptstyle N_\bullet(\boldsymbol{\Delta}_{\varepsilon_i} \otimes A)}\downarrow & & \downarrow \\ N_\bullet(\boldsymbol{\Delta}_1 \otimes A) & \xrightarrow{\Psi_\bullet \circ \mathsf{AW}_\bullet \circ \mathsf{j}_\bullet} & \mathbb{Z}_\bullet^{\boldsymbol{\Delta}_1} \otimes_\mathbb{Z} A_\bullet \end{array}$$

commutes for $i = 0, 1$. We conclude that v is a homotopy from f to g. □

Corollary 7.4.67. *Let $\mathscr{A}$ be any abelian category. We have :*

(i) *If $k \in \mathbb{N}$ is any integer, and A any k-truncated simplicial object of $\mathscr{A}$, then*

$$H_i(\mathrm{cosk}_k A) = 0 \qquad \textit{for every } i \geq k.$$

(ii) *Every homotopically trivial augmented simplicial object of $\mathscr{A}$ is aspherical.*

Proof. (i): Denote by $t_{\leq k} : \mathsf{C}^{\leq 0}(\mathscr{A}) \to \mathsf{C}^{[-k,0]}(\mathscr{A})$ the brutal truncation functor (see (7.1.1)). In light of theorem 7.4.60, we see that $t_{\leq k}$ admits a right adjoint $v_k : \mathsf{C}^{[-k,0]}(\mathscr{A}) \to \mathsf{C}^{\leq 0}(\mathscr{A})$, and clearly there are natural isomorphisms

$$N_\bullet \mathrm{cosk}_k A \xrightarrow{\sim} v_k N_\bullet A \qquad \text{for every } A \in \mathrm{Ob}(s_k.\mathscr{A}).$$

Taking into account theorem 7.4.31(iii), we are then reduced to showing

Claim 7.4.68. $H_i(v_k K_\bullet) = 0$ for every $(K_\bullet, d_\bullet) \in \mathrm{Ob}(\mathsf{C}^{[-k,0]})$ and every $i \geq k$.

Proof of the claim. Indeed it is easily seen that :

$$(v_k K_\bullet)_i = \begin{cases} K_i & \text{for } \leq k \\ \mathrm{Ker}\, d_k & \text{for } i = k+1 \\ 0 & \text{for } i > k+1 \end{cases}$$

and the differential of $v_k K_\bullet$ in degree $\leq k$ agrees with that of $K_\bullet$, whereas in degree $k+1$ it is the natural inclusion map (details left to the reader). The claim follows immediately. ◊

(ii) follows directly from theorems 7.4.60(ii) and 7.4.31(iii), and remark 7.1.9(ii). □

7.5. **Simplicial sets.** This section, which complements the previous one, collects some classical material pertaining to the homotopy theory of the category of simplicial sets. The presentation is borrowed from [80] and [108], where much more may be found.

7.5.1. To begin with, notice that the category $s.\mathbf{Set}$ of *simplicial sets* is none else than the category of presheaves on the category Δ, so $s.\mathbf{Set}$ is complete and cocomplete, and all limits and colimits in $s.\mathbf{Set}$ are computed argumentwise (see (4.1)); also, all colimits and monomorphisms are universal, and all epimorphisms are universal effective.

Example 7.5.2. (i) We have already introduced in example 7.4.5 the simplicial set $\mathbf{\Delta}_k$ (for any $k \in \mathbb{N}$), which is just the image of $[k]$ under the Yoneda embedding

$$h : \Delta \to s.\mathbf{Set}.$$

To ease notation, for any morphism $\varphi : [n] \to [m]$ in Δ, we shall usually write $\varphi : \mathbf{\Delta}_n \to \mathbf{\Delta}_m$ instead of $\mathbf{\Delta}_\varphi$. By Yoneda's lemma (proposition 1.2.6), we have natural identifications

$$(7.5.3) \qquad A[n] \xrightarrow{\sim} \mathrm{Hom}_{s.\mathbf{Set}}(\mathbf{\Delta}_n, A) \qquad \text{for every } A \in \mathrm{Ob}(s.\mathbf{Set}) \text{ and every } n \in \mathbb{N}.$$

(ii) More generally, the category of simplicial sets contains all the products $\mathbf{\Delta}_n \times \mathbf{\Delta}_m$ (for every $m, n \in \mathbb{N}$). Furthermore, if S is any set and A any simplicial set, we may form the product $s.S \times A$ (notation of (7.4.4)), which we shall denote simply by $S \times A$. Explicitly, we have

$$(S \times A)[n] := S \times A[n] \qquad \text{for every } n \in \mathbb{N}$$

and the faces and degeneracies of $S \times A$ are derived from those of A, in the obvious fashion.

(iii) Moreover, lemma 1.4.8 yields a natural presentation

$$A \xrightarrow{\sim} \operatorname*{colim}_{h\Delta/A} h_{s.\mathbf{Set}} \circ \iota_A$$

for every simplicial set A. By inspecting the definitions, we see that the latter amounts to the following description of A. Notice the natural identification $\mathbf{\Delta}_j[i] \xrightarrow{\sim} \mathrm{Hom}_{s.\mathbf{Set}}(\mathbf{\Delta}_i, \mathbf{\Delta}_j)$ provided by (7.5.3); we have a natural diagram in $s.\mathbf{Set}$

$$(7.5.4) \qquad \coprod_{i,j\in\mathbb{N}} (A[i] \times \mathbf{\Delta}_i[j]) \times \mathbf{\Delta}_j \overset{p_\bullet}{\underset{c_\bullet}{\rightrightarrows}} \coprod_{n\in\mathbb{N}} A[n] \times \mathbf{\Delta}_n \xrightarrow{t_\bullet} A$$

where $p_\bullet$ are $c_\bullet$ are the (unique) morphisms which restrict respectively to morphisms

$$\{a\} \times \mathbf{\Delta}_j \xleftarrow{\varphi} \{(a,\varphi)\} \times \mathbf{\Delta}_i \xrightarrow{\mathbf{1}_{\mathbf{\Delta}_i}} \{A[\varphi](a)\} \times \mathbf{\Delta}_i$$

for every $i, j \in \mathbb{N}$ and every $(a, \varphi) \in A[i] \times \mathbf{\Delta}_j[i]$, and where $t_\bullet$ is the (unique) morphism whose restriction to $\{b\} \times \mathbf{\Delta}_n \to A$ is the morphism corresponding to $b \in A[n]$ under the identification (7.5.3), for every $b \in A[n]$. Then $t_\bullet$ is an epimorphism, and (7.5.4) naturally identifies A with the coequalizer of $p_\bullet$ and $c_\bullet$.

(iv) Another useful simplicial set is the *boundary* of $\mathbf{\Delta}_k$, denoted

$$\partial\mathbf{\Delta}_k \qquad \text{for every integer } k > 0$$

which is defined as the smallest subobject of $\mathbf{\Delta}_k$ containing the images of all the face morphisms $\varepsilon_i : \mathbf{\Delta}_{k-1} \to \mathbf{\Delta}_k$, for $i = 0, \dots, k$ (see (7.4.6)). Thus, we have a natural epimorphism in $s.\mathbf{Set}$

$$(7.5.5) \qquad [k] \times \mathbf{\Delta}_{k-1} \to \partial\mathbf{\Delta}_k \qquad \text{for every } k > 0$$

whose restriction to the subobject $\{i\} \times \mathbf{\Delta}_{k-1}$ agrees with ε_i, for every $i = 0, \dots, k$.

Notice that the simplicial identities yield a cartesian diagram in $\Delta^\wedge$

$$\begin{array}{ccc} [k-2] & \xrightarrow{\varepsilon_{j-1}} & [k-1] \\ {\scriptstyle\varepsilon_i}\downarrow & & \downarrow{\scriptstyle\varepsilon_i} \\ [k-1] & \xrightarrow{\varepsilon_j} & [k] \end{array} \qquad \text{whenever } 0 \leq i < j \leq k.$$

Since the Yoneda embedding commutes with representable limits (corollary 1.4.3(vi)), there follows a cartesian diagram in $s.\mathbf{Set}$

$$\begin{array}{ccc} \mathbf{\Delta}_{k-2} & \xrightarrow{\varepsilon_{j-1}} & \mathbf{\Delta}_{k-1} \\ {\scriptstyle\varepsilon_i}\downarrow & & \downarrow{\scriptstyle\varepsilon_i} \\ \mathbf{\Delta}_{k-1} & \xrightarrow{\varepsilon_j} & \mathbf{\Delta}_k \end{array} \qquad \text{for } 0 \leq i < j \leq k$$

(details left to the reader). Now, denote by $S_k \subset [k] \times [k]$ the subset of all pairs (i,j) with $i < j$; since (7.5.5) is effective (by (7.5.1)), we deduce a commutative diagram

$$S_k \times \mathbf{\Delta}_{k-2} \underset{\varepsilon''_\bullet}{\overset{\varepsilon'_\bullet}{\rightrightarrows}} [k] \times \mathbf{\Delta}_{k-1} \longrightarrow \partial\mathbf{\Delta}_k \qquad \text{in } s.\mathbf{Set}$$

which presents $\partial\mathbf{\Delta}_k$ as the coequalizer of $\varepsilon'_\bullet$ and $\varepsilon''_\bullet$, where $\varepsilon'_\bullet$ (resp. $\varepsilon''_\bullet$) is the morphism whose restriction to $\{(i,j)\} \times \mathbf{\Delta}_{k-2}$ agrees with $\varepsilon_i : \{(i,j)\} \times \mathbf{\Delta}_{k-2} \to \{j\} \times \mathbf{\Delta}_{k-1}$ (resp. with $\varepsilon_{j-1} : \{(i,j)\} \times \mathbf{\Delta}_{k-2} \to \{i\} \times \mathbf{\Delta}_{k-1}$) for every $(i,j) \in S_k$. As an immediate corollary, we see that if A is any simplicial set, a morphism $\partial\mathbf{\Delta}_k \to A$ is the same as the datum of an ordered sequence $x_0, \dots, x_k$ of $k+1$ elements of $A[k-1]$, such that

$$\partial_i x_j = \partial_{j-1} x_i \qquad \text{whenever } 0 \leq i < j \leq k.$$

Moreover, if $x \in A[n]$ is any element, we may regard x as a morphism $\overline{x} : \mathbf{\Delta}_n \to A$ via the natural identification (7.5.3), and then we shall often denote by

$$\partial x : \partial\mathbf{\Delta}_n \to A$$

the restriction of $\overline{x}$ to the suboboject $\partial\mathbf{\Delta}_n$. Lastly, we set $\partial\mathbf{\Delta}_0 := s.\varnothing$.

(v) Another important simplicial set is $\mathbf{\Lambda}^n_k$, which is defined for every $k, n \in \mathbb{N}$ such that $k > 0$ and $n = 0, \dots, k$; namely, it is the smallest subobject of $\mathbf{\Delta}_k$ that contains the images of all the face morphisms $\varepsilon_i : \Delta_{k-1} \to \Delta_k$, except for the face ε_n. Thus, for every such k and n we have a natural monomorphism

$$\iota^n_k : \mathbf{\Lambda}^n_k \to \mathbf{\Delta}_k.$$

The same argument as in (iv) yields a presentation of $\mathbf{\Lambda}^n_k$ as the coequalizer of two morphisms :

$$S^n_k \times \mathbf{\Delta}_{k-2} \underset{\varepsilon''_\bullet}{\overset{\varepsilon'_\bullet}{\rightrightarrows}} ([k] \setminus \{n\}) \times \mathbf{\Delta}_{k-1} \longrightarrow \mathbf{\Lambda}^n_k$$

where $S^n_k \subset S_k$ is the subset of all pairs (i,j) with $i, j \neq n$, and $\varepsilon'_\bullet$, $\varepsilon''_\bullet$ are the restrictions of the morphisms with the same name appearing in (iii).

(vi) Let $\psi : X \to Y$ be any morphism of simplicial sets, $y \in Y[0]$ any element, and $j_y : \Delta_0 \to Y$ the corresponding morphism of simplicial sets; the resulting fibre product

$$\psi^{-1}(y) := \Delta_0 \times_Y X$$

is called the *fibre of ψ over y*. Explicitly, set $\{y_n\} := \mathrm{Im}\, j_y[n]$ for every $n \in \mathbb{N}$. Then

$$\psi^{-1}(y)[n] = \psi[n]^{-1}(y_n) \qquad \text{for every } n \in \mathbb{N}$$

and the faces and degeneracies of $\psi^{-1}(y)$ are the restrictions of the corresponding maps for X.

Remark 7.5.6. (i) From the simplicial identities of (7.4.6) we also get a commutative diagram

$$\begin{array}{ccc} \mathbf{\Delta}_{n+1} & \xrightarrow{\eta_i} & \mathbf{\Delta}_n \\ {\scriptstyle \eta_{j+1}}\downarrow & & \downarrow{\scriptstyle \eta_j} \\ \mathbf{\Delta}_n & \xrightarrow{\eta_i} & \mathbf{\Delta}_{n-1} \end{array}$$

for every integer $n > 0$ and every $i, j \leq n-1$ with $i \leq j$. We claim that this diagram is cocartesian in $s.\mathbf{Set}$. Indeed, let X be any simplicial set, and $f, g : \mathbf{\Delta}_n \to X$ any two morphisms such that $f \circ \eta_i = g \circ \eta_{j+1}$. We set $h := f \circ \varepsilon_{j+1} : \mathbf{\Delta}_{n-1} \to X$, and we notice that

$$h = f \circ \eta_i \circ \varepsilon_i \circ \varepsilon_{j+1} = g \circ \eta_{j+1} \circ \varepsilon_i \circ \varepsilon_{j+1} = g \circ \eta_{j+1} \circ \varepsilon_{j+2} \circ \varepsilon_i = g \circ \varepsilon_i.$$

Therefore :

$$\begin{aligned} h \circ \eta_j &= g \circ \varepsilon_i \circ \eta_j = g \circ \eta_{j+1} \circ \varepsilon_i = f \circ \eta_i \circ \varepsilon_i = f \\ h \circ \eta_i &= f \circ \varepsilon_{j+1} \circ \eta_i = f \circ \eta_i \circ \varepsilon_{j+2} = g \circ \eta_{j+1} \circ \varepsilon_{j+2} = g. \end{aligned}$$

Lastly, since both η_i and η_j are epimorphisms, h is uniquely determined by either of the identities $h \circ \eta_j = f$ and $h \circ \eta_i = g$, whence the assertion.

(ii) We point out that also the diagram

$$\begin{array}{ccc} \mathbf{\Delta}_{n+1} & \xrightarrow{\eta_i} & \mathbf{\Delta}_n \\ {\scriptstyle \eta_i}\downarrow & & \downarrow{\scriptstyle \mathbf{1}_{\mathbf{\Delta}_n}} \\ \mathbf{\Delta}_n & \xrightarrow{\mathbf{1}_{\mathbf{\Delta}_n}} & \mathbf{\Delta}_n \end{array}$$

is trivially cocartesian for every $n \in \mathbb{N}$ and every $i \leq n$, since η_i is an epimorphism.

(iii) Let $k \in \mathbb{N}$ be any integer. The categories $\mathbf{Set}$ of all small sets and $\mathbf{f}.\mathbf{Set}$ of finite sets, both satisfy the conditions of (7.4.20), so we get left and right adjoints

$$\mathrm{sk}_k, \mathrm{cosk}_k : s_k.\mathbf{Set} \to s.\mathbf{Set} \qquad \mathrm{sk}_k, \mathrm{cosk}_k : s_k.\mathbf{f}.\mathbf{Set} \to s.\mathbf{f}.\mathbf{Set}$$

for the respective k-truncation functors.

(iv) For any simplicial set X and any integer $r > 0$, let us say that an element $x \in X[r]$ is a *degenerate simplex*, if $x = \sigma_i y$ for some $i = 0, \dots, r-1$ and some $y \in X[r-1]$. We claim that, for any k-truncated simplicial set Y, and every $r > k$, every element of $\mathrm{sk}_k Y[r]$ is a degenerate simplex. Indeed, let $\alpha : [k] \to [r]$ be any morphism in Δ^o; notice first that, under the identification $Y[k] \xrightarrow{\sim} \mathrm{sk}_k Y[k]$ given by the unit of adjunction, the map $\mathrm{sk}_k Y[\alpha] : \mathrm{sk}_k Y[k] \to \mathrm{sk}_k Y[r]$ corresponds to the natural map $j_\alpha : F[k] \to \mathrm{sk}_k Y[r]$. But $\mathrm{sk}_k Y[r]$ is the union of the images of all such maps (see example 1.2.23(i)), and on the other hand, any such map factors through some degeneracy map σ_i, whence the assertion.

(v) For any simplicial set X and every $k, n \in \mathbb{N}$ with $n \geq k$, the counits of adjunction give a natural commutative diagram

$$\begin{array}{ccc} \mathrm{sk}_k(s.\mathrm{trunc}_k X) & \xrightarrow{\varepsilon_X^{(k,n)}} & \mathrm{sk}_n(s.\mathrm{trunc}_n X) \\ & {\scriptstyle \varepsilon_X^{(k)}}\searrow \quad \swarrow{\scriptstyle \varepsilon_X^{(n)}} & \\ & X & \end{array}$$

amounting to a cocone with vertex X, and it follows easily from the discussion of (7.4.20) that $s.\mathrm{trunc}_k(\varepsilon_X^{(k,n)})$ is an isomorphism for every such $k, n \in \mathbb{N}$, and the resulting morphism

$$\operatorname*{colim}_{n \in \mathbb{N}} \mathrm{sk}_n(s.\mathrm{trunc}_n X) \to X$$

is an isomorphism. We define the *n-th skeleton of X* as the simplicial subset

$$\mathrm{Sk}_n X := \mathrm{Im}\, \varepsilon_X^{(n)} \qquad \text{for every } n \in \mathbb{N}.$$

Lemma 7.5.7. *Let X be any simplicial set, $r > 0$ any integer, and $x, y \in X[r]$ be any two degenerate simplices (see remark* 7.5.6(iii)*). The following holds :*

(i) *If $\partial x = \partial y$, then $x = y$.*
(ii) *There exists a unique pair (p, z) where $p : [r] \to [n]$ is an epimorphism and $z \in X[n]$ is a non-degenerate simplex such that $x = X[p](z)$.*
(iii) *The natural morphism $\mathrm{sk}_n(s.\mathrm{trunc}_n X) \to \mathrm{Sk}_n X$ is an isomorphism for every $n \in \mathbb{N}$.*

Proof. (i): Say that $x = \sigma_m z$ and $y = \sigma_n w$ for some $z, w \in X[r-1]$. If $m = n$, we have

$$z = \partial_m \sigma_m z = \partial_m x = \partial_m y = \partial_m \sigma_m w = w$$

whence the assertion. Hence, we may assume that $m < n$, in which case we get :

$$z = \partial_m x = \partial_m \sigma_n w = \sigma_{n-1} \partial_m w$$

so that $x = \sigma_m \sigma_{n-1} \partial_m w = \sigma_n \sigma_m \partial_m w$. Thus, we may replace z by $\sigma_m \partial_m w$ and m by n, and reduce to the foregoing case, so the proof is complete.

(ii): The existence of such a pair (p, z) is obvious. Next, suppose that $(p' : [r] \to [n'], z')$ is another such pair; from remark 7.5.6, it follows easily that there exists a cocartesian diagram of the form

$$\begin{array}{ccc} \mathbf{\Delta}_r & \xrightarrow{\mathbf{\Delta}_p} & \mathbf{\Delta}_n \\ {\scriptstyle \mathbf{\Delta}_{p'}}\downarrow & & \downarrow{\scriptstyle \mathbf{\Delta}_{q'}} \\ \mathbf{\Delta}_{n'} & \xrightarrow{\mathbf{\Delta}_q} & \mathbf{\Delta}_k \end{array}$$

for a suitable $k \leq n, n'$ and epimorphisms $q : [n] \to [k]$ and $q' : [n'] \to [k]$ (details left to the reader). Regarding z and z' as morphisms $x : \mathbf{\Delta}_n \to X$ and respectively $x' : [n'] \to X$, our assumption gives the identity :

$$z \circ \mathbf{\Delta}_p = z' \circ \mathbf{\Delta}_{p'}$$

whence a unique $w \in X[k]$ such that $X[q'](w) = z$ and $X[q](w) = z'$. Since z and z' are non-degenerate, it follows that $n = k = n'$ and $q = q' = \mathbf{1}_{\mathbf{\Delta}_k}$, whence $z = z'$, as stated.

(iii): Set $Y := \mathrm{sk}_n(s.\mathrm{trunc}_n X)$ for every $n \in \mathbb{N}$; the natural map $f[k] : Y[k] \to \mathrm{Sk}_n[k]$ is surjective for every $k \in \mathbb{N}$, and is bijective for every $k \leq n$, so it suffices to check that $f[k]$ is injective for every $k > n$. Thus, fix $k > N$, and let $y, y' \in Y[k]$ be any two simplices such that $f[k](y) = f[k](y')$. Remark 7.5.6(iv) tells us that y and y' are degenerate, say $y = Y[p](z)$ and $y' = Y[p'](z')$ for epimorphisms $p : [k] \to [m]$ and $p : [k] \to [m']$, and non-degenerate simplices $z \in Y[m]$, $z' \in Y[m']$. We must then have $m, m' \leq n$, and therefore $x := f[m](z)$ and $x' := f[m'](z')$ are two non-degenerate simplices of $\mathrm{Sk}_n X$, such that

$$\mathrm{Sk}_n[p](x) = f[k](y) = f[k](y') = \mathrm{Sk}_n[p'](x').$$

By (ii), we deduce that $p = p'$ and $x = x'$, whence $y = y'$, as required. □

Remark 7.5.8. Let X be any simplicial set and $n \in \mathbb{N}$ any integer.

(i) Clearly, $\mathrm{Sk}_n X \subset \mathrm{Sk}_{n+1} X$, and we have

$$X = \bigcup_{r \in \mathbb{N}} \mathrm{Sk}_r X. \tag{7.5.9}$$

We say that X has dimension $\leq n$, if $X = \mathrm{Sk}_n X$. Let $s.\mathbf{Set}_{\leq n}$ be the full subcategory of $s.\mathbf{Set}$ whose objects are the simplicial set of dimension $\leq n$; it follows easily from lemma 7.5.7(iii) that the adjunction $(\mathrm{sk}_n, \mathrm{trunc}_n)$ establishes an equivalence

$$s.\mathbf{Set}_{\leq n} \xrightarrow{\sim} s_k.\mathbf{Set}.$$

(ii) Moreover, $\mathrm{Sk}_n X$ can be obtained by "attaching n-cells" to $\mathrm{Sk}_{n-1}X$, if $n > 0$. Namely, let $E^n_X \subset X[n]$ denote the set of non-degenerate n-simplices of X; we get a commutative diagram

$$\mathscr{D} \quad : \quad \begin{array}{ccc} E^n_X \times \partial\mathbf{\Delta}_n & \xrightarrow{E^n_X \times i_n} & E^n_X \times \mathbf{\Delta}_n \\ \downarrow & & \downarrow{\scriptstyle c} \\ \mathrm{Sk}_{n-1}X & \longrightarrow & \mathrm{Sk}_n X \end{array}$$

where $i_n : \partial\mathbf{\Delta}_n \to \mathbf{\Delta}_n$ and the bottom horizontal arrow are the natural inclusion maps, and c is the unique morphism whose restriction to the factor $\{x\} \times \mathbf{\Delta}_n$ is the morphism $\overline{x} : \mathbf{\Delta}_n \to X$ corresponding to x, for every $x \in E^n_X$. We claim that $\mathscr{D}$ is a cocartesian diagram. For the proof, notice that all the simplicial sets in $\mathscr{D}$ have dimension $\leq n$; by (i), it then suffices to check that $\mathrm{trunc}_n\mathscr{D}$ is cocartesian, *i.e.* that the diagram of sets $\mathscr{D}[k]$ is cocartesian, for every $k \leq n$ (see (7.5.1)). The latter assertion is clear for $k < n$, since in this case both horizontal arrows of $\mathscr{D}[k]$ are isomorphisms. For $k = n$, notice that the complement S of $E^n_X \times \partial\mathbf{\Delta}_n[n]$ in $E^n_X \times \mathbf{\Delta}_n[n]$ is naturally identified with E^n_X, which is also the complement of $\mathrm{Sk}_{n-1}X[n]$ in $\mathrm{Sk}_n X[n]$; it then suffices to remark that, under these natural bijections, the restriction of $c[n]$ to S is identified with the identity map $E^n_X \xrightarrow{\sim} E^n_X$ (details left to the reader).

Definition 7.5.10. Let $\varphi : A \to B$ and $\psi : X \to Y$ be two morphisms of simplicial sets.

(i) We say that ψ *has the right lifting property with respect to* φ, if for every commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & X \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle \psi} \\ B & \xrightarrow{\beta} & Y \end{array} \qquad \text{in } s.\mathbf{Set}$$

there exists a morphism $\gamma : B \to X$ such that $\gamma \circ \varphi = \alpha$ and $\psi \circ \gamma = \beta$.

(ii) We say that $\psi : X \to Y$ is a *Kan fibration* (or briefly, that ψ is a *fibration*), if ψ has the right lifting property with respect to all the monomorphisms $\iota^n_k : \mathbf{\Lambda}^n_k \to \mathbf{\Delta}_k$, for every $k, n \in \mathbb{N}$ with $k > 0$ and $n \leq k$ (notation of example 7.5.2(v)).

(iii) We say that φ is a *retract of* ψ if there exists a commutative diagram of simplicial sets

$$\begin{array}{ccccc} & & \overset{\mathbf{1}_A}{\frown} & & \\ A & \longrightarrow & X & \longrightarrow & A \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \varphi} \\ B & \longrightarrow & Y & \longrightarrow & B. \\ & & \underset{\mathbf{1}_B}{\smile} & & \end{array}$$

(iv) We say that a simplicial set A is *fibrant* if the (unique) morphism $A \to \mathbf{\Delta}_0$ is a fibration.

Remark 7.5.11. In light of example 7.5.2(iv,v), it is clear that a simplicial set A is fibrant if and only if the following *Kan extension condition* holds. For every $n, k \in \mathbb{N}$ with $0 \leq k \leq n+1$, and every sequence $x_0, \dots, x_{k-1}, x_{k+1}, \dots, x_{n+1}$ of elements of $A[n]$ such that

$$\partial_i x_j = \partial_{j-1} x_i \qquad \text{whenever } 0 \leq i < j \leq n+1 \text{ and } i, j \neq k$$

there exists an element $x \in A[n+1]$ such that

$$\partial_i x = x_i \qquad \text{for every } i = 0, \dots, k-1, k+1, \dots, n+1.$$

Example 7.5.12. Let A be any *simplicial group*, *i.e.* an object of $s.\mathbf{Grp}$, where $\mathbf{Grp}$ denotes the category of groups. Then the simplicial set underlying A is fibrant. Indeed, suppose that n, k and $x_0, \dots, x_{n+1}$ are as in remark 7.5.11. We construct by induction on $i = -1, \dots, n+1$

an element $y_i \in A[n+1]$ such that $\partial_j y_i = x_j$ whenever $0 \leq j \leq i$, $j \neq k$. Indeed, the condition is fulfilled trivially for $i = -1$, by setting $y_{-1} := 1$, the neutral element of the group $A[n+1]$. Suppose that $i \geq 0$, and y_{i-1} has already been exhibited as required; if $i = k$, we set $y_i := y_{i-1}$, which again trivially fulfills the stated condition. Otherwise, consider the element $u := x_i^{-1} \cdot (\partial_i y_{i-1})$, and notice that

$$\partial_j u = (\partial_j x_i^{-1}) \cdot (\partial_j \partial_i y_{i-1}) = (\partial_j x_i^{-1}) \cdot (\partial_{i-1} \partial_j y_{i-1}) = (\partial_j x_i^{-1}) \cdot (\partial_{i-1} x_j) = 1$$

for every $j = 0, \dots, i-1$ with $j \neq k$. Hence, set $y_i := y_{i-1} \cdot (\sigma_i u)^{-1}$; we have

$$\partial_j y_i = (\partial_j y_{i-1}) \cdot (\partial_j \sigma_i u)^{-1} = x_i \cdot (\sigma_{i-1} \partial_j u) = x_i \qquad \text{whenever } 0 \leq j < i.$$

Likewise :

$$\partial_i y_i = (\partial_i y_{i-1}) \cdot (\partial_i \sigma_i u)^{-1} = (\partial_i y_{i-1}) \cdot (\partial_i \sigma_i \partial_i y_{i-1})^{-1} \cdot (\partial_i \sigma_i x_i) = x_i$$

as needed. Thus, the element $x := y_{n+1}$ fulfills the condition of remark 7.5.11.

For any given morphism ψ of $s.\mathbf{Set}$, the set of all morphisms φ such that ψ has the right lifting property with respect to φ is closed under certain elementary operations, that are singled out in the following :

Definition 7.5.13. Let Σ be a set of monomorphisms of $s.\mathbf{Set}$.

(i) We say that Σ is *saturated* if the following conditions hold :

(a) All isomorphisms of $s.\mathbf{Set}$ lie in Σ.

(b) For every countable system of morphisms in $s.\mathbf{Set}$

$$A_0 \xrightarrow{\varphi_0} A_1 \xrightarrow{\varphi_1} A_2 \to \cdots$$

such that $\varphi_n \in \Sigma$ for every $n \in \mathbb{N}$, also the induced morphism

$$A_0 \to \operatorname*{colim}_{n \in \mathbb{N}} A_n$$

lies in Σ.

(c) For any cocartesian diagram of simplicial sets :

$$\begin{array}{ccc} A & \longrightarrow & A' \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi'} \\ B & \longrightarrow & B \amalg_A A' \end{array}$$

such that $\varphi \in \Sigma$, we have $\varphi' \in \Sigma$.

(d) If $(\varphi_i : A_i \to B_i \mid i \in I)$ is an arbitrary family of elements of Σ, then also

$$\coprod_{i \in I} \varphi_i : \coprod_{i \in I} A_i \to \coprod_{i \in I} B_i$$

lies in Σ.

(e) Any retract of any element of Σ lies also in Σ (see definition 7.5.10(iii)).

(ii) The *saturation* of Σ is the intersection of all saturated sets of monomorphisms of $s.\mathbf{Set}$ containing Σ.

7.5.14. With this terminology, for any morphism ψ of $s.\mathbf{Set}$, let $\Sigma(\psi)$ be the set of monomorphisms φ of $s.\mathbf{Set}$ such that ψ has the right lifting property with respect to φ. We point out the following simple observation :

Lemma 7.5.15. *With the notation of* (7.5.14)*, the following holds :*

(i) $\Sigma(\psi)$ *is saturated, for any morphism* ψ *of* $s.\mathbf{Set}$.

(ii) *For any cartesian square of simplicial sets*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ {\scriptstyle\psi'}\downarrow & & \downarrow{\scriptstyle\psi} \\ Y' & \longrightarrow & Y \end{array}$$

we have $\Sigma(\psi) \subset \Sigma(\psi')$.

(iii) *If* ψ' *is any retract of* ψ*, then* $\Sigma(\psi) \subset \Sigma(\psi')$ *(see definition* 7.5.10(iii)*).*

(iv) *Especially, in both* (ii) *and* (iii)*, if* ψ *is a fibration, the same holds for* ψ'*, and for any* $y \in Y[0]$*, the fibre* $\psi^{-1}(y)$ *is a fibrant simplicial set (see example* 7.5.2(vi)*).*

Proof. Left to the reader. □

7.5.16. The Kan extension condition isolates a class of simplicial sets on which the standard constructions of homotopy theory can be carried out. Many of these constructions come down to exhibiting certain morphisms from a product of the type $\mathbf{\Delta}_k \times \mathbf{\Delta}_1$ (for variable $k \in \mathbb{N}$) to given simplicial sets. To manipulate with ease such basic products, it will be useful to have at our disposal presentations for them, in the vein of example 7.5.2(iv,v).

To this aim, notice that Δ is a full subcategory of the category $\mathbf{POSet}$ of partially ordered sets (notation of example 1.1.6(iii)). We may then extend the Yoneda embedding of Δ into $s.\mathbf{Set}$ to a well defined functor

$$\mathbf{POSet} \to s.\mathbf{Set} \qquad (P, \leq) \mapsto \mathbf{\Delta}_P. \tag{7.5.17}$$

Namely, for every partially ordered set $(P, \leq)$, let $h_P : \mathbf{POSet}^o \to \mathbf{Set}$ be the image of P under the Yoneda embedding of the category $\mathbf{POSet}$; then $\mathbf{\Delta}_P$ is the restriction of h_P to the subcategory Δ^o. We notice :

Lemma 7.5.18. *The functor* (7.5.17) *is fully faithful and commutes with all limits.*

Proof. The Yoneda embedding commutes with representable limits (corollary 1.4.3(vi)), and the same holds for the restriction functor $\mathbf{POSet}^\wedge \to s.\mathbf{Set}$, since limits are computed argumentwise in both of these categories (corollary 1.4.3(ii)). To see that (7.5.17) is faithful, it suffices to remark that there is a natural isomorphism

$$\mathbf{\Delta}_P[0] \xrightarrow{\sim} P \qquad \text{for every partially ordered set } (P, \leq) \tag{7.5.19}$$

which induces a natural identification : $\mathbf{\Delta}_\varphi = \varphi$, for every morphism $\varphi : (P, \leq) \to (Q, \leq)$ of partially ordered sets. To check that (7.5.17) is full, let $(P, \leq)$ and $(Q, \leq)$ be any two objects of $\mathbf{POSet}$, and $f : \mathbf{\Delta}_P \to \mathbf{\Delta}_Q$ a morphism of simplicial sets. The natural identification (7.5.19) yields a map $\varphi := f[0] : P \to Q$, and it suffices to check that φ is a map of ordered sets such that $\mathbf{\Delta}_\varphi = f$. Thus, let $x_0, x_1 \in P$ be any two elements with $x_0 < x_1$; there follows a unique morphism $\psi : \mathbf{\Delta}_1 \to P$ of partially ordered sets such that $\psi(i) = x_i$ for $i = 0, 1$. Then $\psi \in \mathbf{\Delta}_P[1]$, and we may set

$$\psi' := f[1](\psi) \in \mathbf{\Delta}_Q[1] \qquad y_i := \mathbf{\Delta}_Q[\varepsilon_{i-1}](\psi') \qquad \text{for } i = 0, 1$$

(notation of (7.4.8)). Tracing back the definitions, we see that $\varphi(x_i) = y_i$ for $i = 0, 1$, and the existence of ψ' tells us that $y_0 \leq y_1$, *i.e.* φ is a morphism in $\mathbf{POSet}$, as required. It remains only to show that $\mathbf{\Delta}_\varphi[n] = f[n]$ for every $n \in \mathbb{N}$. However, let $x : [n] \to P$ be any element of $\mathbf{\Delta}_P[n]$ and set $y := f[n](x)$, so the image of x (resp. of y) is an increasing sequence $(x_0, \ldots, x_n)$ of elements of P (resp. $(y_0, \ldots, y_n)$ of Q); we have to check that $\varphi(x_k) = y_k$ for every $k = 0, \ldots, n$. Now, for any such $k \leq n$, let $\beta_k : [0] \to [n]$ be the unique map such that $\beta_k(0) = k$; the correspondence (7.5.19) identifies x_k with $\mathbf{\Delta}_P[\beta_k](x)$ and y_k with $\mathbf{\Delta}_Q[\beta_k](y)$, whence the contention (details left to the reader). □

7.5.20. The category of partially ordered sets is complete; especially, all finite products

$$[k_0] \times \cdots \times [k_n]$$

are representable in $\mathbf{POSet}$: explicitly, the partial ordering on such a product is defined by declaring that

$$\underline{a} := (a_0, \dots, a_n) \leq \underline{b} := (b_0, \dots, b_n) \qquad \text{if and only if } a_i \leq b_i \text{ for every } i = 0, \dots, n$$

for every pair of elements $\underline{a}, \underline{b} \in [k_0] \times \cdots \times [k_n]$. Now, for any $k, r \in \mathbb{N}$ with $r \leq k$, consider the morphism in $\mathbf{POSet}$

$$\varphi_{k,r} : [k+1] \to [k] \times [1] \quad \text{such that} \quad \varphi_{k,r}(i) = \begin{cases} (0, i) & \text{for } i = 0, \dots, r \\ (1, i-1) & \text{for } i = r+1, \dots, k+1. \end{cases}$$

It is easily seen that every morphism $[t] \to [k] \times [1]$ in $\mathbf{POSet}$ (for any $t \in \mathbb{N}$) factors through some $\varphi_{k,r}$, so we get an epimorphism of simplicial sets

$$\varphi_k : [k] \times \mathbf{\Delta}_{k+1} \to \mathbf{\Delta}_k \times \mathbf{\Delta}_1$$

whose restriction to each subobject $\{r\} \times \mathbf{\Delta}_{k+1}$ is the morphism $\mathbf{\Delta}_{\varphi_{k,r}}$ (notation of (7.5.17)). Now, let r, s be any two integers such that $0 \leq r < s \leq k$; obviously, the intersection of the images of $\varphi_{k,r}$ and $\varphi_{k,s}$ is the subset

$$\{(0,0), \dots, (0,r), (1,s), \dots, (1,k)\}$$

so we get a cartesian diagram in $\mathbf{POSet}$

$$\begin{array}{ccc} [t] & \xrightarrow{\varepsilon^{s-r}_{t,r+1}} & [k+1] \\ {\scriptstyle \varepsilon^{s-r}_{t,r+1}} \downarrow & & \downarrow {\scriptstyle \varphi_{k,r}} \\ [k+1] & \xrightarrow{\varphi_{k,s}} & [k] \times [1] \end{array} \qquad \text{with } t := k+1-s+r$$

(notation of example (7.4.7)(ii)). Since (7.5.17) commutes with fibre products, and φ_k is an effective epimorphism (by (7.5.1)), we conclude that the diagram

$$\text{(7.5.21)} \qquad \bigcup_{0 \leq r < s \leq k} \{(r,s)\} \times \mathbf{\Delta}_{k+1-s+r} \underset{\varepsilon''_\bullet}{\overset{\varepsilon'_\bullet}{\rightrightarrows}} [k] \times \mathbf{\Delta}_{k+1} \xrightarrow{\varphi_k} \mathbf{\Delta}_k \times \mathbf{\Delta}_1$$

identifies $\mathbf{\Delta}_k \times \mathbf{\Delta}_1$ with the coequalizer of $\varepsilon'_\bullet$ and $\varepsilon''_\bullet$, where $\varepsilon'_\bullet$ (resp. $\varepsilon''_\bullet$) is the morphism whose restrictions to each subobject $\{(r,s)\} \times \mathbf{\Delta}_t$ agrees with $\varepsilon^{s-r}_{t,r+1} : \{(r,s)\} \times \mathbf{\Delta}_t \to \{r\} \times \mathbf{\Delta}_{k+1}$ (resp. with $\varepsilon^{s-r}_{t,r+1} : \{(r,s)\} \times \mathbf{\Delta}_t \to \{s\} \times \mathbf{\Delta}_{k+1}$).

However, (7.5.21) can be simplified as follows. Notice that if $i_1, \dots, i_r$ is any sequence of non-negative integers such that $1 \geq i_{n+1} - i_n \geq 0$ for every $n = 0, \dots, r-1$, then

$$\varepsilon_{i_r} \circ \cdots \circ \varepsilon_{i_1} = \varepsilon^r_{t,i_1} \qquad \text{for every } t \in \mathbb{N}.$$

From this observation, it is easily seen that the terms $\{(r,s)\} \times \Delta_{k+1-s-r}$ in (7.5.21) with $s > r+1$ are redundant; so we arrive at the presentation

$$\text{(7.5.22)} \qquad [k-1] \times \mathbf{\Delta}_k \underset{\varepsilon''_\bullet}{\overset{\varepsilon'_\bullet}{\rightrightarrows}} [k] \times \mathbf{\Delta}_{k+1} \xrightarrow{\varphi_k} \mathbf{\Delta}_k \times \mathbf{\Delta}_1$$

where each term $\{r\} \times \mathbf{\Delta}_k$ corresponds to the term $\{(r, r+1)\} \times \mathbf{\Delta}_k$ of (7.5.21), and the morphisms $\varepsilon'_\bullet, \varepsilon''_\bullet$ are redefined accordingly.

7.5.23. We consider now the following sets of monomorphisms of $s.\mathbf{Set}$:

- Σ_1 is the set of all morphisms $\iota_k^n : \mathbf{\Lambda}_k^n \to \mathbf{\Delta}_k$, for every $k, n \in \mathbb{N}$ with $k \geq n$.
- Σ_2 is the set of all morphisms
$$i_k^j : (\partial\mathbf{\Delta}_k \times \mathbf{\Delta}_1) \cup (\mathbf{\Delta}_k \times \mathbf{\Lambda}_1^j) \to \mathbf{\Delta}_k \times \mathbf{\Delta}_1 \qquad \text{for every } k \in \mathbb{N} \text{ and } j = 0, 1.$$
- Σ_3 is the set of all monomorphisms of the form
$$i_{K,L}^j : (K \times \mathbf{\Delta}_1) \cup (L \times \mathbf{\Lambda}_1^j) \to L \times \mathbf{\Delta}_1$$
where $K \to L$ is an arbitrary monomorphism of $s.\mathbf{Set}$, and $j = 0, 1$.

With this notation, we may now state :

Proposition 7.5.24. *The sets Σ_1, Σ_2 and Σ_3 have the same saturation.*

Proof. In order to check that the saturation of Σ_1 contains Σ_2, fix $k \in \mathbb{N}$; if $j = 0$ (resp. if $j = 1$), for every $r = 0, \dots, k + 1$, denote by $A_r \subset \mathbf{\Delta}_k \times \mathbf{\Delta}_1$ the image of the restriction of φ_k to the subobject $\{0, \dots, r\} \times \mathbf{\Delta}_{k+1}$ (resp. the subobject $\{k - r + 1, \dots, k + 1\} \times \mathbf{\Delta}_{k+1}$), and define B_r as the fibre product in the cartesian diagram
$$\begin{array}{ccc} B_r & \longrightarrow & (\partial\mathbf{\Delta}_k \times \mathbf{\Delta}_1) \cup (\mathbf{\Delta}_k \times \mathbf{\Lambda}_1^j) \\ \downarrow & & \downarrow{\scriptstyle i_k^j} \\ A_r & \longrightarrow & \mathbf{\Delta}_k \times \mathbf{\Delta}_1. \end{array}$$
Set also $A_{-1} := s.\varnothing$ (the initial object of $s.\mathbf{Set}$). Now, a straightforward induction reduces to checking that the natural morphisms
$$B_k \amalg_{B_r} A_r \to B_k \amalg_{B_{r+1}} A_{r+1} \qquad \text{for } r = -1, \dots, k - 1$$
lie in the saturation of Σ_1. To this aim, it suffices to show that the same holds for the natural morphisms
$$\tau_r : B_{r+1} \amalg_{B_r} A_r \to A_{r+1} \qquad \text{for } r = -1, \dots, k - 1.$$
This, in turns, follows immediately from :

Claim 7.5.25. (i) The natural isomorphism $\{0\} \times \mathbf{\Delta}_{k+1} \xrightarrow{\sim} A_0$ identifies B_0 with $\{0\} \times \mathbf{\Lambda}_{k+1}^1$.
(ii) For every $r = 0, \dots, k - 1$ there is a commutative diagram in $s.\mathbf{Set}$
$$\begin{array}{ccccc} \mathbf{\Delta}_k & \xrightarrow{\varepsilon_{r+1}} & \mathbf{\Lambda}_{k+1}^{r+2} & \xrightarrow{\iota_{k+1}^{r+2}} & \mathbf{\Delta}_{k+1} \\ {\scriptstyle \varepsilon_{r+1}}\downarrow & & \downarrow & & \downarrow \\ A_r & \longrightarrow & B_{r+1} \amalg_{B_r} A_r & \xrightarrow{\tau_r} & A_{r+1} \end{array}$$
whose two square subdiagrams are cocartesian.

Proof of the claim. To ease notation, we shall identify $\{r\} \times \mathbf{\Delta}_k$ with its image under φ_k, which is a subobject of $\mathbf{\Delta}_k \times \mathbf{\Delta}_1$, for every $r = 0, \dots, k$. Notice that $B_{r+1} \amalg_{B_r} A_r$ is the smallest subobject of $\mathbf{\Delta}_k \times \mathbf{\Delta}_1$ that contains A_r and $Z_{r+1} := (\{r + 1\} \times \mathbf{\Delta}_{k+1}) \cap B_k$. However, a morphism $[t] \to B_k$ is the same as an increasing sequence of elements of $[k] \times [1]$
$$(a_n, b_n) \qquad n = 0, \dots, t \tag{7.5.26}$$
such that :

(a) either the cardinality of the set $\{a_0, \dots, a_t\}$ is strictly less than $k + 1$
(b) or else $b_n \neq j$ for every $n = 0, \dots, t$.

On the other hand, recall that the inclusion $\{r+1\} \times \mathbf{\Delta}_{k+1} \subset \mathbf{\Delta}_k \times \mathbf{\Delta}_1$ corresponds to the inclusion map of partially ordered sets $\varphi_{k,r+1}$ (notation of (7.5.20)). It follows that an injective morphism $\mathbf{\Delta}_t \to Z_{r+1}$ corresponds to a strictly increasing sequence (7.5.26) contained in

$$\{(0,0),\dots,(0,r+1),(1,r+1),\dots,(1,k)\}$$

and fulfilling either of the foregoing conditions (a) or (b). Taking $r = -1$, we already see that (i) holds. If $r \geq 0$, take $t := k$, and notice that no such sequence can have $a_r \neq r+1$, so we see that of all the $k+2$ monomorphisms

$$\mathbf{1}_{\{r+1\}} \times \varepsilon_j : \{r+1\} \times \mathbf{\Delta}_k \to \{r+1\} \times \mathbf{\Delta}_{k+1}$$

only those with $j \neq r+1, r+2$ factor through Z_{r+1}. On the other hand, from the presentation (7.5.22) we see that the intersection of A_r and $\{r+1\} \times \mathbf{\Delta}_{k+1}$ is the image of $\{r\} \times \mathbf{\Delta}_k$, which maps to both of them via the morphism ε_{r+1}. From this, we conclude already that the left square subdiagram of (ii) is indeed cocartesian, if we let the central vertical arrow to be the morphism that naturally identifies $\mathbf{\Lambda}^{r+2}_{k+1}$ with the subobject $\{r+1\} \times \mathbf{\Lambda}^{r+2}_{k+1}$ of $\{r+1\} \times \mathbf{\Delta}_{k+1}$. By the same token, it is clear the right square subdiagram is likewise cocartesian, provided we let the rightmost vertical arrow to be the morphism that naturally identifies $\mathbf{\Delta}_{k+1}$ with $\{r+1\} \times \mathbf{\Delta}_{k+1}$. $\Diamond$

Next, let us check that the saturation of Σ_2 contains Σ_3. Indeed, let $\mu : K \to L$ be any monomorphism of $s.\mathbf{Set}$, and fix $j \in \{0,1\}$; according to example 7.5.2(iii), there exist :

- A countable system of monomorphisms

$$K_{-1} := K \xrightarrow{\mu_{-1}} K_0 \to \cdots \to K_n \xrightarrow{\mu_n} K_{n+1} \to \cdots$$

  such that μ is isomorphic to the induced morphism

$$K \to \operatorname*{colim}_{n\in\mathbb{N}} K_n.$$

- For every $n \in \mathbb{N}$, a set I_n and an epimorphism $K_{n-1} \amalg (I_n \times \mathbf{\Delta}_n) \to K_n$ whose restriction to K_{n-1} agrees with μ_{n-1}.

There follows, for every integer $n \in \mathbb{N}$, a cocartesian diagram

$$\begin{array}{ccc} (K_{n-1} \times \mathbf{\Delta}_1) \cup (K_n \times \mathbf{\Lambda}^j_1) & \longrightarrow & (K_{n-1} \times \mathbf{\Delta}_1) \cup (L \times \mathbf{\Lambda}^j_1) \\ {\scriptstyle i^j_{K_{n-1},K_n}} \downarrow & & \downarrow {\scriptstyle \varphi_n} \\ K_n \times \mathbf{\Delta}_1 & \longrightarrow & (K_n \times \mathbf{\Delta}_1) \cup (L \times \mathbf{\Lambda}^j_1) \end{array}$$

and the system $(\varphi_n \mid n \in \mathbb{N})$ induces a morphism

$$(K \times \mathbf{\Delta}_1) \cup (L \times \mathbf{\Lambda}^j_1) \to \operatorname*{colim}_{n\in\mathbb{N}} (K_n \times \mathbf{\Delta}_1) \cup (L \times \mathbf{\Lambda}^j_1) = L \times \mathbf{\Delta}_1$$

which is isomorphic to $i^j_{K,L}$. Thus, we may assume that there exist a set I and an epimorphism $K \amalg (I \times \mathbf{\Delta}_n) \to L$ for some $n \in \mathbb{N}$, and we shall argue by induction on n. Set $Q := (I \times \mathbf{\Delta}_n) \times_L K$ and notice that Q is a subobject of $I \times \mathbf{\Delta}_n$. Also, since all epimorphisms are effective, the induced commutative diagram

$$\begin{array}{ccc} Q & \longrightarrow & K \\ \downarrow & & \downarrow \\ I \times \mathbf{\Delta}_n & \longrightarrow & L \end{array}$$

is cocartesian. Furthermore, since all colimits are universal in $s.\mathbf{Set}$, we deduce that

$$Q = \coprod_{i \in I_n} Q_i \qquad \text{where} \qquad Q_i := (\{i\} \times \mathbf{\Delta}_n) \cap Q \quad \text{for every } i \in I.$$

We are then reduced to the case where $L = \mathbf{\Delta}_n$. If $n = 0$, then clearly K is either $\mathbf{\Delta}_0$ or $\mathbf{\Delta}_{-1}$, and in either case we have $i^j_{K,L} \in \Sigma_2$.

Next, suppose that $n > 0$, and that the assertion is already known for every integer $k < n$. If $K = \mathbf{\Delta}_n$, we are done; otherwise, it is easily seen that the inclusion $K \to \mathbf{\Delta}_n$ factors through $\partial\mathbf{\Delta}_n$, and arguing as in the foregoing, we are reduced to considering the morphisms $i^j_{K,\partial\mathbf{\Delta}_n}$ and $i^j_{\partial\mathbf{\Delta}_n,\mathbf{\Delta}_n}$. The latter lies in Σ_2, and from example 7.5.2(iv) it is clear that there is an epimorphism $K \amalg ([n] \times \mathbf{\Delta}_{n-1}) \to \partial\mathbf{\Delta}_n$, so $i^j_{K,\partial\mathbf{\Delta}_n}$ lies in the saturation of Σ_2, by inductive assumption.

Lastly, we show that the saturation of Σ_3 contains Σ_1. To this aim, it suffices to construct a commutative diagram

$$\begin{array}{ccccc} \mathbf{\Lambda}^r_k & \longrightarrow & (\mathbf{\Lambda}^r_k \times \mathbf{\Delta}_1) \cup (\mathbf{\Delta}_k \times \mathbf{\Lambda}^j_1) & \longrightarrow & \mathbf{\Lambda}^r_k \\ \downarrow{\scriptstyle \iota^r_k} & & \downarrow{\scriptstyle i^j_{\mathbf{\Lambda}^r_k,\mathbf{\Delta}_k}} & & \downarrow{\scriptstyle \iota^r_k} \\ \mathbf{\Delta}_k & \xrightarrow{\ i\ } & \mathbf{\Delta}_k \times \mathbf{\Delta}_1 & \xrightarrow{\ p\ } & \mathbf{\Delta}_k \end{array}$$

for every $n, r \in \mathbb{N}$ with $r \leq k$, such that $p \circ i = \mathbf{1}_{\mathbf{\Delta}_k}$. However, notice that a morphism $[t] \to (\mathbf{\Lambda}^r_k \times \mathbf{\Delta}_1) \cup (\mathbf{\Delta}_k \times \mathbf{\Lambda}^j_1)$ (for any $t \in \mathbb{N}$) is the same as an increasing sequence (7.5.26) of elements of $[k] \times [1]$ such that

(a) either $b_n \neq j$ for every $n = 0, \dots, t$

(b) or else, the cardinality of of the set $\{a_0, \dots, a_t\} \cup \{r\}$ is strictly less than $k + 1$.

Now, any such p and i shall be the images, under the functor (7.5.17), of corresponding maps of ordered sets

$$[k] \xrightarrow{i^*} [k] \times [1] \xrightarrow{p^*} [k] \qquad \text{such that } p^* \circ i^* = \mathbf{1}_{[k]}.$$

We obtain a suitable $j \in \{0, 1\}$ and suitable p^*, i^* by the following rule. If $r < k$, we let $j := 1$, and define p^* and i^* as the maps such that

$$i^*(a) = (a, 1) \quad \text{for every } a = 0, \dots, k \qquad p^*(a, b) = \begin{cases} a & \text{if } b = 1 \text{ or } a \leq r \\ r & \text{otherwise.} \end{cases}$$

If $r = k$, we let $j := 0$, and set

$$i^*(a) := (a, 0) \quad \text{for every } a = 0, \dots, k \qquad p^*(a, b) = \begin{cases} a & \text{if } b = 0 \\ r & \text{otherwise.} \end{cases}$$

The reader may easily check that the resulting maps i and p will do. □

Definition 7.5.27. A monomorphism in $s.\mathbf{Set}$ is an *anodyne extension*, if it lies in the saturation of the set Σ_1.

In view of lemma 7.5.15(i), we see that a fibration has the right lifting property with respect to every anodyne extension. We also notice :

Corollary 7.5.28. *Let $K \to L$ be any anodyne extension, and $A \to B$ any monomorphism. Then the induced monomorphism*

$$(K \times B) \cup (L \times A) \to L \times B$$

is an anodyne extension.

Proof. Let Σ be the set of monomorphisms $K' \to L'$ such that the induced morphism

$$(K' \times B) \cup (L' \times A) \to L' \times B$$

is an anodyne extension. It is easily seen that Σ is saturated (details left to the reader; cp. the proof of proposition 7.5.24); taking into account proposition 7.5.24, it then suffices to check that Σ contains the morphisms $i^j_{X,Y}$, where $f : X \to Y$ is an arbitrary monomorphism (notation of (7.5.23)). However, for any such f we have a commutative diagram

$$\begin{array}{ccc} (((X \times \mathbf{\Delta}_1) \cup (Y \times \mathbf{\Lambda}^j_1)) \times B) \cup ((Y \times \mathbf{\Delta}_1) \times A) & \longrightarrow & ((Y \times \mathbf{\Delta}_1) \times B \\ \downarrow & & \downarrow \\ (((X \times B) \cup (Y \times A)) \times \mathbf{\Delta}_1) \cup ((Y \times B) \times \mathbf{\Lambda}^j_1) & \xrightarrow{i^j_{(X\times B)\cup(Y\times A), Y\times B}} & (Y \times B) \times \mathbf{\Delta}_1 \end{array}$$

whose vertical arrows are isomorphisms. Then the contention follows by appealing again to proposition 7.5.24. □

Theorem 7.5.29. *For every morphism $f : X \to Y$ of simplicial sets there exists a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{i_f} & E_f \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle p_f} & \\ & Y & \end{array} \qquad \text{in } s.\mathbf{Set}$$

such that i_f is an anodyne extension, and p_f is a fibration.

Proof. For every $k, n \in \mathbb{N}$ with $k \geq \min(1, n)$, consider the set $\mathscr{L}^n_k$ of all commutative diagrams

$$\begin{array}{ccc} \mathbf{\Lambda}^n_k & \longrightarrow & X \\ {\scriptstyle \iota^n_k}\downarrow & & \downarrow{\scriptstyle f} \\ \mathbf{\Delta}_k & \longrightarrow & Y \end{array}$$

and set

$$\mathbf{\Lambda}_f := \coprod_{n\in\mathbb{N}} \coprod_{k\geq\min(1,n)} \mathscr{L}^n_k \times \mathbf{\Lambda}^n_k \qquad \mathbf{L}_f := \coprod_{n\in\mathbb{N}} \coprod_{k\geq\min(1,n)} \mathscr{L}^n_k \times \mathbf{\Delta}_k.$$

There follows a natural morphism

$$\begin{array}{ccc} \mathbf{\Lambda}_f & \xrightarrow{g} & X \\ {\scriptstyle \iota}\downarrow & & \downarrow{\scriptstyle f} \\ \mathbf{L}_f & \longrightarrow & Y \end{array}$$

with ι an anodyne extension. Define E^0_f as the push-out in the induced cocartesian diagram

$$\begin{array}{ccc} \mathbf{\Lambda}_f & \xrightarrow{g} & X \\ {\scriptstyle \iota}\downarrow & & \downarrow{\scriptstyle i^0_f} \\ \mathbf{L}_f & \longrightarrow & E^0_f \end{array}$$

so that i^0_f is anodyne as well, and we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{i^0_{(X,f)}} & E^0_f \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle p^0_f} & \\ & Y. & \end{array}$$

Clearly, the rule $(X, f) \mapsto (E^0_f, p^0_f)$ extends to a well defined functor

$$\underline{E}^0 : s.\mathbf{Set}/Y \to s.\mathbf{Set}/Y$$

and the rule $(X, f) \mapsto i^0_{(X,f)}$ yields a natural transformation

$$i^0 : \mathbf{1}_{s.\mathbf{Set}/Y} \Rightarrow \underline{E}^0.$$

Thus, we may define inductively :

$$\underline{E}^n(X, f) := \underline{E}^0(\underline{E}^{n-1}(X, f)) \qquad \text{for every integer } n \geq 1$$

so $\underline{E}^n(X, f)$ is a pair (E^n_f, p^n_f), and we get as well as a natural transformation

$$i^n_{(X,f)} := i^0_{\underline{E}^{n-1}(X,f)} : E^{n-1}_f \to E^n_f \qquad \text{for every integer } n \geq 1$$

which is again an anodyne extension. We set

$$E_f := \operatorname*{colim}_{n\in\mathbb{N}} E^n_f$$

where the transition maps in the colimit are given by the system of morphisms $(i^n_f \mid n \in \mathbb{N})$. The colimit of the system of morphism $(p^n_f \mid n \in \mathbb{N})$ is then a morphism $p_f : E_f \to Y$; moreover, the induced morphism $i_f : X \to E_f$ is anodyne, and obviously $p_f \circ i_f = f$. To conclude the proof, it suffices to show that p_f is a fibration. Thus, consider any commutative diagram

$$\begin{array}{ccc} \mathbf{\Lambda}^n_k & \xrightarrow{g} & E_f \\ {\scriptstyle \iota^n_k}\downarrow & & \downarrow{\scriptstyle p_f} \\ \mathbf{\Delta}_k & \xrightarrow{h} & Y. \end{array}$$

It follows easily from example 7.5.2(v) that – for $r \in \mathbb{N}$ large enough – g factors through a morphism $g_r : \mathbf{\Lambda}^n_k \to E^r_f$ and the natural morphism $E^r_f \to E_f$, whence a commutative diagram

$$\begin{array}{ccccc} \mathbf{\Lambda}^n_k & \xrightarrow{g_r} & E^r_f & \xrightarrow{i^{r+1}_f} & E^{r+1}_f \\ {\scriptstyle \iota^n_k}\downarrow & & \downarrow{\scriptstyle p^r_f} & & \downarrow{\scriptstyle p^{r+1}_f} \\ \mathbf{\Delta}_k & \xrightarrow{h} & Y & = & Y. \end{array}$$

But then, by construction of E^{n+1}_f, we see that h lifts to a morphism $h' : \mathbf{\Delta}_k \to E^{n+1}_f$ such that $p^{r+1}_f \circ h' = h$ and $h' \circ \iota^n_k = i^{r+1}_f \circ g_r$. The composition of h' with the natural morphism $E^{r+1}_f \to E_f$ is a morphism $h'' : \mathbf{\Delta}_k \to E_f$ such that $p_f \circ h'' = h$ and $h'' \circ \iota^n_k = g$, as required. □

A useful way to produce new fibrant simplicial sets out of old ones is provided by the general construction introduced in the following :

Definition 7.5.30. Let A and B be any two simplicial sets; the *function complex*

$$s.\mathscr{H}om(A, B)$$

associated with A and B is the simplicial set given by the rule :

$$n \mapsto \mathrm{Hom}_{s.\mathbf{Set}}(\mathbf{\Delta}_n \times A, B) \qquad \varphi \mapsto \mathrm{Hom}_{s.\mathbf{Set}}(\mathbf{\Delta}_\varphi \times \mathbf{1}_A, B)$$

for every $n \in \mathbb{N}$ and every morphism φ of Δ.

Remark 7.5.31. (i) Notice that the system of isomorphisms $A[n] \xrightarrow{\sim} \mathrm{Hom}_{s.\mathbf{Set}}(\mathbf{\Delta}_n, A)$ given by Yoneda's lemma for every $n \in \mathbb{N}$, amounts to a natural identification

$$\iota_A : A \xrightarrow{\sim} s.\mathscr{H}om(\mathbf{\Delta}_0, A) \qquad \text{for every } A \in \mathrm{Ob}(s.\mathbf{Set}).$$

(ii) Also, if A, B and C are any three simplicial sets, there is a natural transformation

$$\tau_{A,B,C} : s.\mathscr{H}om(A, B) \to s.\mathscr{H}om(A \times C, B \times C)$$

which, to any $n \in \mathbb{N}$, assigns the map

$$\mathrm{Hom}_{s.\mathbf{Set}}(\mathbf{\Delta}_n \times A, B) \to \mathrm{Hom}_{s.\mathbf{Set}}(\mathbf{\Delta}_n \times A \times C, B \times C) \qquad \varphi \mapsto \varphi \times \mathbf{1}_C.$$

(iii) There is a natural *evaluation morphism*

$$\mathrm{ev}_{A,B} : A \times s.\mathscr{H}om(A, B) \to B \qquad \text{for every } A, B \in \mathrm{Ob}(s.\mathbf{Set})$$

which, to every $n \in \mathbb{N}$, assigns the map

$$A[n] \times \mathrm{Hom}_{s.\mathbf{Set}}(\mathbf{\Delta}_n \times A, B) \to B[n] \qquad (a, f) \mapsto f[n](\mathbf{1}_{\mathbf{\Delta}_n}, a).$$

Indeed, for any morphism $\varphi : [k] \to [n]$ in $\mathbf{\Delta}$, and (a, f) any element of $(A \times \mathscr{H}om(A, B))[n]$ we may compute

$$\begin{aligned} B[\varphi] \circ \mathrm{ev}_{A,B}[n](a, f) &= B[\varphi] \circ f[n](\mathbf{1}_{\mathbf{\Delta}_n}, a) \\ &= f[k] \circ (\mathbf{\Delta}_n[\varphi] \times A[\varphi])(\mathbf{1}_{\mathbf{\Delta}_n}, a) \\ &= f[k](\varphi, A[\varphi](a)) \\ &= f[k] \circ (\mathbf{\Delta}_\varphi \times \mathbf{1}_A)[k](\mathbf{1}_{\mathbf{\Delta}_k}, A[\varphi](a)) \\ &= (f \circ (\mathbf{\Delta}_\varphi \times \mathbf{1}_A))[k](\mathbf{1}_{\mathbf{\Delta}_k}, A[\varphi](a)) \\ &= \mathrm{ev}_{A,B}[k](A[\varphi](a), f \circ (\mathbf{\Delta}_\varphi \times \mathbf{1}_A)) \\ &= \mathrm{ev}_{A,B}[k] \circ (A \times \mathscr{H}om(A, B))[\varphi](a, f) \end{aligned}$$

which shows that $\mathrm{ev}_{A,B}$ is a morphism of simplicial sets.

(iv) Clearly $s.\mathbf{Set}$ is a tensor category, with tensor product given by the product of simplicial sets, and with $\mathbf{\Delta}_0$ as unit object. We claim that the functor

$$s.\mathscr{H}om : (s.\mathbf{Set})^o \times s.\mathbf{Set} \to s.\mathbf{Set}$$

is an internal Hom functor for $(s.\mathbf{Set}, \times, \mathbf{\Delta}_0)$. Indeed, given $A, B, C \in \mathrm{Ob}(s.\mathbf{Set})$ and any morphism $f : A \times B \to C$, we obtain a morphism

$$A \xrightarrow{\iota_A} s.\mathscr{H}om(\mathbf{\Delta}_0, A) \xrightarrow{\tau_{A,B,C}} s.\mathscr{H}om(B, A \times B) \xrightarrow{\mathscr{H}om(B,f)} s.\mathscr{H}om(B, C).$$

Conversely, given a morphism $g : A \to \mathscr{H}om(B, C)$, we get a morphism

$$A \times B \xrightarrow{g \times \mathbf{1}_B} s.\mathscr{H}om(B, C) \times B \overset{\sim}{\to} B \times s.\mathscr{H}om(B, C) \xrightarrow{\mathrm{ev}_{B,C}} C.$$

It is easily seen that these two rules yield mutually inverse natural bijections, as required (details left to the reader).

7.5.32. Let $f : A \to B$ and $g : C \to D$ be morphisms of simplicial sets. We have a commutative diagram

$$\begin{array}{ccc} s.\mathscr{H}om(D, A) & \xrightarrow{s.\mathscr{H}om(D,f)} & s.\mathscr{H}om(D, B) \\ {\scriptstyle s.\mathscr{H}om(g,A)} \downarrow & & \downarrow {\scriptstyle s.\mathscr{H}om(g,B)} \\ s.\mathscr{H}om(C, A) & \xrightarrow{s.\mathscr{H}om(C,f)} & s.\mathscr{H}om(C, B) \end{array}$$

whence an induced morphism in $s.\mathbf{Set}$

(7.5.33) $$s.\mathscr{H}om(D, A) \to s.\mathscr{H}om(D, B) \times_{s.\mathscr{H}om(C,B)} s.\mathscr{H}om(C, A).$$

Proposition 7.5.34. *In the situation of* (7.5.32)*, suppose that f is a fibration and g a monomorphism. Then* (7.5.33) *is a fibration as well.*

Proof. The datum of a commutative diagram

$$\begin{array}{ccc} \boldsymbol{\Lambda}^i_k & \longrightarrow & s.\mathscr{H}om(D,A) \\ {\scriptstyle \iota^i_k}\downarrow & & \downarrow \\ \boldsymbol{\Delta}_k & \longrightarrow & s.\mathscr{H}om(D,B) \times_{s.\mathscr{H}om(C,B)} s.\mathscr{H}om(C,A) \end{array}$$

is equivalent, by remark 7.5.31(iv), to that of a commutative diagram

$$\begin{array}{ccc} (\boldsymbol{\Lambda}^i_k \times D) \cup (\boldsymbol{\Delta}_k \times C) & \longrightarrow & A \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ \boldsymbol{\Delta}_k \times D & \longrightarrow & B. \end{array}$$

On the other hand, f has the right lifting property with respect to g (corollary 7.5.28); it follows easily that (7.5.33) has the right lifting property with respect to ι^i_k, as stated. □

Corollary 7.5.35. *Let A be any fibrant simplicial set. We have :*

(i) *$s.\mathscr{H}om(D,A)$ is fibrant, for any simplicial set D.*

(ii) *Any monomorphism $D \to D'$ of simplicial sets induces a fibrant morphism*

$$s\mathscr{H}om(D',A) \to s.\mathscr{H}om(D,A).$$

Proof. (i): Take $B := \boldsymbol{\Delta}_0$, $C := \boldsymbol{\Delta}_{-1}$, and let $f : A \to B$, $g : C \to D$ be the unique morphisms of simplicial sets. In view of proposition 7.5.34, it suffices to notice the natural identifications

$$s.\mathscr{H}om(D,\boldsymbol{\Delta}_0) \stackrel{\sim}{\to} \boldsymbol{\Delta}_0 \stackrel{\sim}{\to} s.\mathscr{H}om(\boldsymbol{\Delta}_{-1},D)$$

for every simplicial set A. The proof of (ii) is similar : details left to the reader. □

7.5.36. Let $f,g : X \to Y$ be two morphisms of simplicial sets. Recall (see remark 7.4.18(v)) that a *homotopy u from f to g* is the datum of a commutative diagram of simplicial sets :

$$\begin{array}{ccccc} X \times \boldsymbol{\Delta}_0 & \xrightarrow{\mathbf{1}_X \times \varepsilon_0} & X \times \boldsymbol{\Delta}_1 & \xleftarrow{\mathbf{1}_X \times \varepsilon_1} & X \times \boldsymbol{\Delta}_0 \\ \| & & \downarrow{\scriptstyle u} & & \| \\ X & \xrightarrow{f} & Y & \xleftarrow{g} & X. \end{array}$$

Definition 7.5.37. In the situation of (7.5.36), suppose that $i : X' \to X$ is a monomorphism of simplicial sets, such that $f \circ i = g \circ i$. We say that u is a *homotopy from f to g relative to X'*, if u is a homotopy from f to g, and the following diagram commutes :

$$\begin{array}{ccc} X' \times \boldsymbol{\Delta}_1 & \longrightarrow & X' \\ {\scriptstyle i \times \mathbf{1}_{\boldsymbol{\Delta}_1}}\downarrow & & \downarrow{\scriptstyle f \circ i} \\ X \times \boldsymbol{\Delta}_1 & \xrightarrow{u} & Y \end{array}$$

where the top horizontal arrow is the natural projection.

Remark 7.5.38. Let $f,g : X \to Y$ be any two morphisms of simplicial sets.

(i) Of course, a homotopy from f to g can be viewed as a homotopy relative to the sub-object $\boldsymbol{\Delta}_{-1} \subset X$, so definition 7.5.37 generalizes the previous notion. Notice also that to f one may attach a *constant homotopy*

$$X \times \boldsymbol{\Delta}_1 \xrightarrow{\mathbf{1}_X \times \eta_0} X \xrightarrow{f} Y.$$

(ii) Let u be any homotopy from f to g relative to a subobject X' of X, and $h : Y \to Z$ any morphism of simplicial sets. Then clearly $h \circ u$ is a homotopy from $h \circ f$ to $h \circ g$ relative to X'. Likewise, if $t : Z \to X$ is any other morphism, then $u \circ (t \times \boldsymbol{\Delta}_1)$ is a homotopy from $f \circ t$ to $g \circ t$ relative to $Z \times_X X'$.

7.5.39. Let A be any simplicial set; on the set $\mathrm{Hom}_{s.\mathbf{Set}}(\mathbf{\Delta}_0, A) \simeq A[0]$ we consider the binary relation $\sim$ such that

$$a \sim b \qquad \Leftrightarrow \qquad \text{there exists a homotopy from } a \text{ to } b.$$

Lemma 7.5.40. *With the notation of* (7.5.39)*, suppose that A is fibrant. Then $\sim$ is an equivalence relation on $A[0]$.*

Proof. Clearly $a \sim b$ if and only if there exists a morphism $u : \Delta_1 \to X$ such that $u \circ \varepsilon_0 = a$ and $u \circ \varepsilon_1 = b$. Taking $u := a \circ \sigma_0$, where $\sigma_0 : \mathbf{\Delta}_1 \to \mathbf{\Delta}_0$ is the unique morphism, and $a \in A[0]$ any element, we obtain the reflexivity of $\sim$. Next, suppose that $a \sim b$ and $b \sim c$ for some $a, b, c \in A[0]$, and pick $u, v : \mathbf{\Delta}_1 \to A$ that give a homotopy from a to b, and respectively from b to c. Then it is easily seen that the datum of u and v determines a unique morphism $t : \mathbf{\Lambda}^1_2 \to A$ such that $t \circ \varepsilon_0 = u$ and $t \circ \varepsilon_2 = v$. By assumption, t extends to a morphism $s : \mathbf{\Delta}_2 \to A$, and then $s \circ \varepsilon_1$ is a homotopy from a to c, whence the transitivity of $\sim$.

Lastly, let $u : \mathbf{\Delta}_1 \to A$ be a homotopy from a to b, for some $a, b \in A[0]$; we let $t : \mathbf{\Lambda}^0_2 \to A$ be the unique morphism such that $t \circ \varepsilon_2 = u$ and $t \circ \varepsilon_1 = b \circ \sigma_0$. By assumption t extends to a morphism $s : \mathbf{\Delta}_2 \to A$, and it is easily seen that $s \circ \varepsilon_0$ is a homotopy from b to a, which shows that $\sim$ is also symmetric. □

7.5.41. More generally, if $i : K \to X$ is any monomorphism in $s.\mathbf{Set}$, and Y is any simplicial set, we consider on $\mathrm{Hom}_{s.\mathbf{Set}}(X, Y)$ the binary relation

$$f \sim_K g \qquad \Leftrightarrow \qquad \text{there exists a homotopy from } f \text{ to } g \text{ relative to } K.$$

In case $K = s.\varnothing$, we simply write $f \sim g$ instead of $f \sim_{s.\varnothing} g$.

Theorem 7.5.42. *With the notation of* (7.5.41)*, suppose that Y is fibrant. Then $\sim_K$ is an equivalence relation.*

Proof. The fibres of the map of sets

$$\mathrm{Hom}_{s.\mathbf{Set}}(X, Y) \to \mathrm{Hom}_{s.\mathbf{Set}}(K, Y) \qquad f \mapsto f \circ i \tag{7.5.43}$$

give a partition of $\mathrm{Hom}_{s.\mathbf{Set}}(X, Y)$, and if $f \sim_K g$, then f and g lie in the same fibre of (7.5.43). Hence, we may restrict attention to a single fibre of (7.5.43), say the fibre over the morphism $h : K \to Y$. Now, h corresponds to a morphism $h^* : \mathbf{\Delta}_0 \to s.\mathscr{H}om(K, Y)$ in $s.\mathbf{Set}$, and likewise, the datum of a pair of morphisms $f, g : X \to Y$ such that $f \circ i = h = g \circ i$ corresponds to that of a pair of morphisms $f^*, g^* : \mathbf{\Delta}_0 \to s.\mathscr{H}om(X, Y)$ with $i^* \circ f^* = h^* = i^* \circ g^*$, where

$$i^* : s.\mathscr{H}om(X, Y) \to s.\mathscr{H}om(K, Y)$$

is the morphism deduced from i. Furthermore, a homotopy from f to g relative to K is the same as a morphism $t : \mathbf{\Delta}_1 \to s.\mathscr{H}om(X, Y)$ such that $i^* \circ t$ factors through h^* and such that

$$t \circ \varepsilon_0 = f^* \qquad t \circ \varepsilon_1 = g^*.$$

Set $A := i^{*-1}(h^*)$ (notation of example 7.5.2(vi)). We conclude that t corresponds to a homotopy between the elements of $A[0]$ induced by f^* and g^*. However, A is fibrant by corollary 7.5.35(ii) and lemma 7.5.15(iv), so the assertion follows from lemma 7.5.40. □

7.5.44. Let A be any simplicial set, $\xi \in A[0]$ any element, and denote also by $\xi : \mathbf{\Delta}_0 \to A$ the corresponding morphism of simplicial sets. We shall frequently abuse notation, and denote indifferently by ξ the unique element of $A[n]$ that lies in the image of the map $\xi[n]$. We call the pair (A, ξ) a *pointed simplicial set*. With this terminology, we have :

Definition 7.5.45. Let (A, ξ) be a pointed simplicial set, such that A is fibrant.

(i) For every integer $n \in \mathbb{N}$, let $A(\xi, n)$ be the set of all $a \in A[n]$ such that ∂a factors through ξ (notation of example 7.5.2(iv)). We set

$$\pi_n(A, \xi) := A(\xi, n)/\sim_{\partial \mathbf{\Delta}_n}$$

(where $\sim_{\partial \mathbf{\Delta}_n}$ is the homotopy equivalence relation defined in (7.5.41)), and for every $n > 0$ we call $\pi_n(A, \xi)$ the *n-th homotopy group* of A.

(ii) Notice that $\pi_0(A, \xi)$ is actually independent of ξ, so we shall usually just denote it $\pi_0(A)$. We say that A is *connected* if the cardinality of $\pi_0(A)$ equals one.

Lemma 7.5.46. *In the situation of definition* 7.5.45*, let* $n \in \mathbb{N}$ *be any integer and* $a \in A(\xi, n)$ *any simplex. The following conditions are equivalent :*

(a) $a \sim_{\partial \mathbf{\Delta}_n} \xi$.
(b) *There exists* $b \in A[n+1]$ *such that* $\partial_i b = \xi$ *for every* $i = 1, \dots, n+1$ *and* $\partial_0 b = a$.

Proof. (a)⇒(b): Let $h : \mathbf{\Delta}_n \times \mathbf{\Delta}_1 \to A$ be any morphism in $s.\mathbf{Set}$. From the presentation (7.5.22) we see that h is the same as a system $x_0, \dots, x_n$ of elements of $A[n+1]$ such that

$$\partial_i x_i = \partial_i x_{i-1} \qquad \text{for } i = 1, \dots, n.$$

Namely, $x_i := h \circ \mathbf{\Delta}_{\varphi_{n,i}}$ for every $i = 0, \dots, n$, where $\varphi_{n,i} : [n+1] \to [n] \times [1]$ is defined as in (7.5.20). Moreover, recall that the fully faithful functor (7.5.17) associates with any map of partially ordered sets $f : [n] \to [n] \times [1]$ a morphism $\mathbf{\Delta}_f : \mathbf{\Delta}_n \to \mathbf{\Delta}_n \times \mathbf{\Delta}_1$, and it is easily seen that $\mathbf{\Delta}_f$ factors through the subobject $(\partial \mathbf{\Delta}_n \times \mathbf{\Delta}_1) \cup (\mathbf{\Delta}_n \times \partial \mathbf{\Delta}_1)$ if and only if f fulfills the following condition. Set $f(j) := (r_j, s_j)$ for every $j \in [n]$; then, for every $j = 0, \dots, n-1$ we have either $r_j = r_{j+1}$ or $s_j = s_{j+1}$ (details left to the reader). It follows easily that h is a homotopy from ξ to a relative to $\partial \mathbf{\Delta}_n$ if and only if we have

$$\partial_0 x_0 = a \qquad \partial_{n+1} x_n = \xi \qquad \text{and} \qquad \partial_j x_i = \xi \quad \text{for } 0 < i \leq n \text{ and } j \neq i, i+1.$$

Consider then the system $y_\bullet := (y_1, \dots, y_{n+2})$ of elements of $A[n+1]$ such that $y_{n+2} := \xi$ and $y_i = x_{i-1}$ for every $i = 1, \dots, n+1$. A direct calculation shows that

$$\partial_i y_j = \partial_{j-1} y_i \qquad \text{whenever } 1 \leq i < j \leq n+2$$

so $y_\bullet$ corresponds to a unique morphism $\varphi : \mathbf{\Lambda}^0_{n+2} \to A$ whose restriction with the i-th face of $\mathbf{\Lambda}^{n+2}_{n+2}$ agrees with y_i, for every $i = 1, \dots, n+2$. Since A is fibrant, we may then find $c \in A[n+2]$ such that $\partial_i c = y_i$ for every $i = 1, \dots, n+2$. Set $b := \partial_0 c$; then

$$\partial_i b = \partial_i \partial_0 c = \partial_0 \partial_{i+1} c = \partial_0 x_i \qquad \text{for every } i = 0, \dots, n+1$$

whence the assertion.

(b)⇒(a): Consider the map $f : [n] \times [1] \to [n+1]$ in the category $\mathbf{POSet}$ given by the rule :

$$(i, 0) \mapsto i \quad \text{and} \quad (i, 1) \mapsto n+1 \quad \text{for every } i \in [n]$$

and let $\overline{b} : \mathbf{\Delta}_{n+1} \to A$ be the morphism in $s.\mathbf{Set}$ corresponding to b; then it is easily seen that $\overline{b} \circ \mathbf{\Delta}_f : \mathbf{\Delta}_n \times \mathbf{\Delta}_1 \to A$ is a homotopy from a to ξ relative to $\partial \mathbf{\Delta}_n$ (details left to the reader). □

7.5.47. Let (A, ξ) be any pointed simplicial set, such that A is fibrant. To justify the name of $\pi_n(A, \xi)$, we shall endow it with a natural group structure, for every $n > 0$. To this aim, say that $\alpha, \beta \in \pi_n(A, \xi)$ are any two classes, and pick representatives $x, y \in A(\xi, n)$ for α and respectively β. Then, obviously the sequence

$$\xi, \dots, \xi, x, y$$

of $n+1$ elements of $A[n]$ satisfies the compatibility condition of remark 7.5.11, with $k := n$. We may therefore find $z \in A[n+1]$ such that $\partial_{n-1} z = x$, $\partial_{n+1} z = y$, and $\partial_i z = \xi$ for $i = 0, \dots, n-2$. The first observation is :

Lemma 7.5.48. *With the notation of* (7.5.47), *the class of* $\partial_n z$ *in* $\pi_n(A, \xi)$ *depends only on* α *and* β *(and not on* z*, nor the choice of representatives* x *and* y*).*

Proof. Let x' and y' be two other elements of $A(\xi, n)$, and h (resp. h') a homotopy from x to x' (resp. from y to y') relative to $\partial\mathbf{\Delta}_n$. Let also z (resp. z') be a morphism $\mathbf{\Delta}_{n+1} \to A$ such that

$$z \circ \varepsilon_i = \xi = z' \circ \varepsilon_i \qquad \text{for } i = 0, \dots, n-2$$

and

$$z \circ \varepsilon_{n-1} = x \qquad z' \circ \varepsilon_{n-1} = x' \qquad z \circ \varepsilon_{n+1} = y \qquad z' \circ \varepsilon_{n+1} = y'.$$

Then there exists a unique morphism $\psi : (\mathbf{\Delta}_{n+1} \times \partial\mathbf{\Delta}_1) \cup (\mathbf{\Lambda}^n_{n+1} \times \mathbf{\Delta}_1) \to A$ such that

- $\psi \circ (\mathbf{1}_{\mathbf{\Delta}_{n+1}} \times \varepsilon_0) = z$ and $\psi \circ (\mathbf{1}_{\mathbf{\Delta}_{n+1}} \times \varepsilon_1) = z'$.
- $\psi \circ (\varepsilon_i \times \mathbf{1}_{\mathbf{\Delta}_1})$ factors through ξ for $i = 0, \dots, n-2$.
- $\psi \circ (\varepsilon_{n-1} \times \mathbf{1}_{\mathbf{\Delta}_1}) = h$ and $\psi \circ (\varepsilon_{n+1} \times \mathbf{1}_{\mathbf{\Delta}_1}) = h'$

and it is easily seen that $\psi \circ (\varepsilon_n \times \mathbf{1}_{\mathbf{\Delta}_1})$ is a homotopy from $\partial_n z$ to $\partial_n z'$. □

7.5.49. Keep the notation of (7.5.47); lemma 7.5.48 says that the rule : $(\alpha, \beta) \mapsto \partial_n z$ yields a well defined pairing

(7.5.50) $$\pi_n(A, \xi) \times \pi_n(A, \xi) \to \pi_n(A, \xi) \quad (\alpha, \beta) \mapsto \alpha \cdot \beta \qquad \text{for every integer } n > 0.$$

For any $x \in A(\xi, n)$, we shall write $[x]$ for the class of x in $\pi_n(A, \xi)$.

Theorem 7.5.51. *For every integer* $n > 0$*, the pairing* (7.5.50) *defines a group law on* $\pi_n(A, \xi)$*, whose neutral element is the class* $[\xi]$.

Proof. Fix $n > 0$ and any $\alpha \in \pi_n(A, \xi)$; to see that $[\xi] \cdot \alpha = \alpha \cdot [\xi] = \alpha$, pick any representative $x : \mathbf{\Delta}_n \to A$ for α, and set $z := x \circ \eta_n$, $z' := x \circ \eta_{n+1}$. From the simplicial identities (7.4.6) it is easily seen that $\partial_i z = \partial_i z' = \xi$ for $i = 0, \dots, n-2$, and $\partial_{n-1} z = \partial_{n+1} z' = x$, so $\partial_n z$ and $\partial_n z'$ represent respectively $[\xi] \cdot \alpha$ and $\alpha \cdot [\xi]$. But we have $\partial_n z = \partial_n z' = x$, whence the claim.

Next, we check that the map

$$\pi_n(A, \xi) \to \pi_n(A, \xi) \quad : \quad \beta \mapsto \alpha \cdot \beta$$

is surjective. Indeed, given any $y \in A(\xi, n)$, there exists a morphism $\varphi : \mathbf{\Lambda}^{n+1}_{n+1} \to A$ such that $\varphi \circ \varepsilon_i = \xi$ for $i = 0, \dots, n-2$, $\varphi \circ \varepsilon_{n-1} = x$ and $\varphi \circ \varepsilon_n = y$. We can then extend φ to a morphism $z : \mathbf{\Delta}_n \to A$, and if we denote by β (resp. γ) the class of y (resp. of $\partial_{n+1} z$) in $\pi_n(A, \xi)$, we see that $\alpha \cdot \gamma = \beta$.

Likewise, one shows that right multiplication by α defines a surjection on $\pi_n(A, \xi)$ (details left to the reader). It remains only to check the associativity of (7.5.50), and to this aim, let $x, y, z \in A(\xi, n)$ be any three elements, representing respectively $\alpha, \beta, \gamma \in \pi_n(A, \xi)$. Then there exist $u_{n-1}, u_{n+1}, u_{n+2} \in A[n+1]$ such that

$$\partial_i u_{n-1} = \partial_i u_{n+1} = \partial_i u_{n+2} = \xi \qquad \text{for every } i = 0, \dots, n-2$$

and

$$\begin{aligned} \partial_{n-1} u_{n-1} &= x & \partial_{n+1} u_{n-1} &= y \\ \partial_{n-1} u_{n+1} &= \partial_n u_{n-1} & \partial_{n+1} u_{n+1} &= z \\ \partial_{n-1} u_{n+2} &= y & \partial_{n+1} u_{n+2} &= z. \end{aligned}$$

By remark 7.5.11, we may then find some $v \in A[n+2]$ such that $\partial_i v$ equals ξ for $i = 0, \dots, n-2$, and equals u_i for $i = n-1, n+1, n+2$. It follows that

$$\partial_{n-1}\partial_n v = x \qquad \partial_{n+1}\partial_n v = \partial_n u_{n+2} \quad \text{and} \quad \partial_i \partial_n v = \xi \quad \text{for } i = 0, \dots, n-2$$

We may then compute :

$$\begin{aligned}(\alpha\cdot\beta)\cdot\gamma &= [\partial_n u_{n-1}]\cdot\gamma \\ &= [\partial_n u_{n+1}] \\ &= [\partial_n\partial_{n+1}v] \\ &= [\partial_n\partial_n v] \\ &= \alpha\cdot(\beta\cdot\gamma)\end{aligned}$$

as required. □

Remark 7.5.52. (i) Let us denote

$$s.\mathbf{Set}^{\mathrm{f}}_{\circ}$$

the category of *fibrant pointed simplicial sets*, whose objects are all the pairs (A,ξ) consisting of a simplicial set A and an element $\xi\in A[0]$. A morphism $f:(A,\xi)\to(A',\xi')$ in $s.\mathbf{Set}^{\mathrm{f}}_{\circ}$ is a morphism $f:A\to A'$ of simplicial sets such that $f[0](\xi)=\xi'$. It follows easily from remark 7.5.38(ii) that, for every integer $n>0$ (resp. for $n=0$), the rule $(A,\xi)\mapsto\pi_n(A,\xi)$ (resp. $(A,\xi)\mapsto\pi_0(A)$) yields a functor

$$\pi_n : s.\mathbf{Set}^{\mathrm{f}}_{\circ}\to\mathbf{Grp} \qquad (\text{resp. } \pi_0 : s.\mathbf{Set}^{\mathrm{f}}_{\circ}\to\mathbf{Set}_{\circ})$$

with values in the category of groups (resp. of pointed sets). Namely, if f is a morphism in $s.\mathbf{Set}^{\mathrm{f}}_{\circ}$ as in the foregoing, and $x\in A(\xi,n)$ (resp. $x\in A[0]$) is any element, we let $\pi_n(f,\xi)[x]$ (resp. $\pi_0(f)[x]$) be the class of $f[n](x)$ in $\pi_n(A',\xi')$ (resp. in $\pi_0(A')$). A simple inspection shows that this rule is well defined and does yield a group homomorphism $\pi_n(A,\xi)\to\pi_n(A',\xi')$, whenever $n>0$.

(ii) Let X and Y be two fibrant simplicial sets, $\xi\in X[0]$ any element, and denote also by ξ the image of the corresponding morphism $\mathbf{\Delta}_0\to X$, which is therefore a simplicial subset of X. Let $f,g:X\to Y$ be two morphisms of simplicial sets such that $f[0](\xi)=g[0](\xi)$ and $f\sim_\xi g$. Then we claim that

$$\pi_0(f)=\pi_0(g) \quad\text{and}\quad \pi_n(f,\xi)=\pi_n(g,\xi) \qquad \text{for every } n>0.$$

Indeed, let u be a homotopy from f to g relative to ξ; if $x\in X(\xi,n)$, denote also by $x:\mathbf{\Delta}_n\to X$ the corresponding morphism, and notice that $u\circ(x\times\mathbf{1}_{\mathbf{\Delta}_1}):\mathbf{\Delta}_n\times\mathbf{\Delta}_1\to Y$ is a homotopy from $f[n](x)$ to $g[n](x)$ relative to $\partial\mathbf{\Delta}_n$, whence the contention.

7.5.53. Let $p:X\to Y$ be a fibration between fibrant simplicial sets X and Y. Let also $\xi\in X[0]$ be any vertex, and set $\xi':=p[0](\xi)$ and $F:=p^{-1}(\xi')$ (notation of example 7.5.2(vi)), so that F is fibrant as well, by lemma 7.5.15(iv). For any $n\in\mathbb{N}$ and any $a\in Y(\xi',n)$, we get a commutative diagram

$$\begin{array}{ccc} \mathbf{\Lambda}^0_n & \xrightarrow{\ c\ } & X \\ \downarrow & & \downarrow{\scriptstyle p} \\ \mathbf{\Delta}_n & \xrightarrow{\ a\ } & Y \end{array}$$

where c is the unique morphism in $s.\mathbf{Set}$ that factors through $\xi:\mathbf{\Delta}_0\to X$. We may then find $b\in X[n]$ such that $p[n](b)=a$ and $\partial_i b=\xi$ for every $i=1,\dots,n$.

Lemma 7.5.54. *In the situation of* (7.5.53)*, we have $\partial_0 b\in F(\xi,n-1)$, and the class $[\partial_0 b]\in\pi_{n-1}(F,\xi)$ depends only on $[a]\in\pi_n(Y,\xi')$ (and is independent of the representative a for $[a]$ and of the simplex b).*

Proof. We have $\partial_i\partial_0 b=\partial_0\partial_{i+1}b=\xi$ for every $i=0,\dots,n-1$, whence the first assertion. Next, suppose that $a'\in Y(\xi',n)$ is another simplex such that $a\sim_{\partial\mathbf{\Delta}_n}a'$, and pick any homotopy

$h : \mathbf{\Delta}_n \times \mathbf{\Delta}_1 \to Y$ from a to a' relative to $\partial\mathbf{\Delta}_n$. Let also $b' \in X[n]$ be any simplex such that $p(b') = a'$ and $\partial_i b' = \xi$ for $i = 1, \dots, n$. We obtain a commutative diagram

$$\begin{array}{ccc} (\mathbf{\Delta}_n \times \partial\mathbf{\Delta}_1) \cup (\mathbf{\Lambda}^0_n \times \mathbf{\Delta}_1) & \xrightarrow{\ c'\ } & X \\ \downarrow & & \downarrow{\scriptstyle p} \\ \mathbf{\Delta}_n \times \mathbf{\Delta}_1 & \xrightarrow{\ h\ } & Y \end{array}$$

where c' is the unique morphism whose restriction to $\mathbf{\Delta}_n \times \mathbf{\Lambda}^0_1$ (resp. to $\mathbf{\Delta}_n \times \mathbf{\Lambda}^1_1$) agrees with b (resp. with b') and whose restriction to $\mathbf{\Lambda}^0_n \times \mathbf{\Delta}_1$ is the unique morphism that factors through $\xi : \mathbf{\Delta}_0 \to X$. Since p is a fibration, we may then find a morphism $h' : \mathbf{\Delta}_n \times \mathbf{\Delta}_1 \to X$ which extends c', and such that $p \circ h' = h$ (corollary 7.5.28). Set $h'' := h' \circ (\varepsilon_0 \times \mathbf{1}_{\mathbf{\Delta}_1}) : \mathbf{\Delta}_{n-1} \times \mathbf{\Delta}_1 \to X$. It is easily seen that h'' is a homotopy from $\partial_0 b$ to $\partial_0 b'$ relative $\partial\mathbf{\Delta}_{n-1}$, whence the lemma. □

7.5.55. Keep the situation of (7.5.53); it follows from lemma 7.5.54 that the rule $[a] \mapsto [\partial_0 b]$ yields a well defined map

$$\delta_n : \pi_n(Y, \xi') \to \pi_{n-1}(F, \xi) \qquad \text{for every } n > 0$$

and it is easily seen that $\delta_n[\xi] = [\xi']$, *i.e.* δ_n is a map of pointed sets. Let also $i : F \to X$ be the natural monomorphism; there follows, for every $n \in \mathbb{N}$, a natural sequence of pointed sets

$$\pi_{n+1}(X, \xi) \xrightarrow{\pi_{n+1}(p,\xi)} \pi_{n+1}(Y, \xi') \xrightarrow{\delta_{n+1}} \pi_n(F, \xi) \xrightarrow{\pi_n(i,\xi)} \pi_n(X, \xi) \xrightarrow{\pi_n(p,\xi)} \pi_n(Y, \xi')$$

called the *homotopy sequence in degree* n attached to the fibration p (and the vertex ξ). Let $\underline{1}$ denote the monoid with one element (the initial object in the category of monoids); noticed that the category $\mathbf{Set}_\circ$ can also be regarded naturally as the category of pointed (left) $\underline{1}$-modules (see (4.8.13)), and therefore the homotopy sequence in every degree is naturally a sequence of pointed $\underline{1}$-modules.

Theorem 7.5.56. *With the notation of* (7.5.55)*, the following holds :*

(i) *The homotopy sequence attached to* p *is exact in every degree* $n \in \mathbb{N}$, i.e. *we have*

$$\operatorname{Im} \pi_{n+1}(p, \xi) = \operatorname{Ker} \delta_{n+1} \qquad \operatorname{Im} \delta_{n+1} = \operatorname{Ker} \pi_n(i, \xi) \qquad \operatorname{Im} \pi_n(i, \xi) = \operatorname{Ker} \pi_n(p, \xi).$$

(ii) δ_n *is a group homomorphism for every* $n > 1$.

Proof. (See remark 4.8.17(iii) for the notion of kernel of a morphism of pointed $\underline{1}$-modules.)

(ii): Let $a_{n-1}, a_n, a_{n+1} \in Y(\xi', n)$ be three elements, $b_{n-1}, b_n, b_{n+1} \in X[n]$ and $\varphi \in Y[n+1]$ four simplices such that

- $p(b_i) = a_i$ and $\partial_j b_i = \xi$ for every $i = n-1, n, n+1$ and $j = 1, \dots, n$
- $\partial_i \varphi = \xi'$ for $i = 0, \dots, n-2$ and $\partial_i \varphi = a_i$ for $i = n-1, n, n+1$.

We deduce a commutative diagram

$$\begin{array}{ccc} \mathbf{\Lambda}^0_{n+1} & \xrightarrow{\ c\ } & X \\ \downarrow & & \downarrow{\scriptstyle p} \\ \mathbf{\Delta}_{n+1} & \xrightarrow{\ \varphi\ } & Y \end{array}$$

where c is the unique morphism whose restriction to the i-th face of $\mathbf{\Lambda}^{n+1}_0$ factors through ξ if $i = 1, \dots, n-2$, and equals b_i for $i = n-1, n, n+1$. Since p is a fibration, we may then find $\psi : \mathbf{\Delta}_{n+1} \to X$ whose restriction to $\mathbf{\Lambda}^0_{n+1}$ agrees with c, and such that $p \circ \psi = \varphi$. Set $t := \partial_0 \psi \in X[n]$; then $p[n](t) = \partial_0(p[n+1](\psi)) = \partial_0 \varphi = \xi$, so that $t \in F[n]$. Moreover,

$\partial_i t = \partial_0 \partial_{i+1} \psi$ for every $i = 0, \dots, n$, hence $\partial_i t = \xi$ for $i < n-2$ and $\partial_i t = \partial_0 b_{i+1}$ for $i = n-2, n-1, n$. This means that

$$[\partial_0 b_n] = [\partial_0 b_{n-1}] \cdot [\partial_0 b_{n+1}] \qquad \text{in } \pi_{n-1}(F, \xi)$$

and on the other hand we have $[a_n] = [a_{n-1}] \cdot [a_{n+1}]$ in $\pi_n(Y, \xi')$, whence the assertion.

(i): The inclusion $\operatorname{Im} \pi_n(i, \xi) \subset \operatorname{Ker} \pi_n(p, \xi)$ is immediate, and the identity $\operatorname{Im} \delta_{n+1} = \operatorname{Ker} \pi_n(i, \xi)$ follows easily from lemma 7.5.46, by inspection of the definition of δ_{n+1}. The inclusion $\operatorname{Im} \pi_{n+1}(p, \xi) \subset \operatorname{Ker} \delta_{n+1}$ is likewise immediate, from the definition of δ_{n+1} (details left to the reader).

• Next, let us show that $\operatorname{Ker} \delta_{n+1} \subset \operatorname{Im} \pi_{n+1}(p, \xi)$. Indeed, let $a \in Y(\xi', n+1)$, pick $b \in X[n+1]$ such that $p[n+1](b) = a$ and $\partial_i b = \xi$ for every $i = 1, \dots, n+1$, and suppose that $\partial_0 b \sim_{\partial \mathbf{\Delta}_n} \xi$. Let $h : \mathbf{\Delta}_n \times \mathbf{\Delta}_1 \to F$ be a homotopy from $\partial_0 b$ to ξ relative to $\partial \mathbf{\Delta}_n$. Then we get a unique morphism

$$t : (\mathbf{\Delta}_{n+1} \times \mathbf{\Lambda}^1_1) \cup (\partial \mathbf{\Delta}_{n+1} \times \mathbf{\Delta}_1) \to X$$

whose restriction to $\mathbf{\Delta}_{n+1} \times \mathbf{\Lambda}^1_1$ agrees with b, whose restriction to $\mathbf{\Lambda}^0_{n+1} \times \mathbf{\Delta}_1$ factors through ξ, and whose restriction to $(\operatorname{Im} \varepsilon_0) \times \mathbf{\Delta}_1 \subset \partial \mathbf{\Delta}_{n+1} \times \mathbf{\Delta}_1$ agrees with h. Since X is fibrant, t extends to a morphism $t' : \mathbf{\Delta}_{n+1} \times \mathbf{\Delta}_1 \to X$, and we denote by $x \in X[n+1]$ the restriction of t' to $\mathbf{\Delta}_{n+1} \times \mathbf{\Lambda}^0_1$. Then it is easily seen that $x \in X(\xi, n+1)$, and $p \circ t'$ is a homotopy relative to $\partial \mathbf{\Delta}_{n+1}$ from a to $p(x)$, whence the assertion.

• Lastly, let $b \in X(\xi, n)$ be any simplex such that $[p(b)] = [\xi']$ in $\pi_n(Y, \xi')$, and pick any homotopy $h : \mathbf{\Delta}_n \times \mathbf{\Delta}_1 \to Y$ from $p(b)$ to ξ' relative to $\partial \mathbf{\Delta}_n$. We get a commutative diagram

$$\begin{array}{ccc} (\mathbf{\Delta}_n \times \mathbf{\Lambda}^0_1) \cup (\partial \mathbf{\Delta}_n \times \mathbf{\Delta}_1) & \xrightarrow{\ \varphi\ } & X \\ \big\downarrow & & \big\downarrow{\scriptstyle p} \\ \mathbf{\Delta}_n \times \mathbf{\Delta}_1 & \xrightarrow{\ h\ } & Y \end{array}$$

where φ is the unique morphism whose restriction to $\partial \mathbf{\Delta}_n \times \mathbf{\Delta}_1$ factors through ξ, and whose restriction to $\mathbf{\Delta}_n \times \mathbf{\Lambda}^0_1$ agrees with b. Since p is a fibration, we may then find a morphism $h' : \mathbf{\Delta}_n \times \mathbf{\Delta}_1 \to X$ extending φ and such that $p \circ h' = h$. Set $b' := h' \circ (\mathbf{1}_{\mathbf{\Delta}_n} \times \varepsilon_1)$, so that h' is a homotopy from b to b' relative to $\partial \mathbf{\Delta}_n$; however, notice that $p(b') = \xi'$, so $b' \in F(\xi, n)$, therefore $[b] \in \pi_n(i, \xi)$, which concludes the proof of the theorem. ☐

7.5.57. The following device shall be useful for explaining the functorial behaviour of the homotopy groups under change of base points. Let A be any fibrant simplicial set; we set

$$\pi(A) := A[1]/\sim_{\partial \mathbf{\Delta}_1} .$$

Given $a \in A[1]$, we denote by $[a] \in \pi(A)$ the class of a. Evidently, the vertices $\partial_i a \in A[0]$ $(i = 0, 1)$ depend only on $[a]$, so we shall also denote them by $\partial_i[a]$. We use the notation

$$[a] : x \to y$$

to signal that $x = \partial_0[a]$ and $y = \partial_1[a]$. If $[b] : y \to z$ is another class, we define a composition $[b] \circ [a] : x \to z$ as follows. Let $\varphi : \mathbf{\Lambda}^1_2 \to A$ be the unique morphism such that $\varphi \circ \varepsilon_0 = a$ and $\varphi \circ \varepsilon_2 = b$; since A is fibrant, φ extends to a morphism $c : \mathbf{\Delta}_2 \to A$, and we let

$$[b] \circ [a] := [\partial_1 c].$$

In order to show that this rule is well defined, suppose that $a', b' \in A[1]$ and $c' \in A[2]$ are any three other simplices such that $[a] = [a']$, $[b] = [b']$, $\partial_0 c' = a'$ and $\partial_2 c' = b'$. Pick a homotopy h (resp. h') from a to a' (resp. from b to b') relative to $\partial \mathbf{\Delta}_1$; there follows a morphism

$$t : (\mathbf{\Delta}_2 \times \partial \mathbf{\Delta}_1) \cup (\mathbf{\Lambda}^1_2 \times \mathbf{\Delta}_1) \to A$$

whose restriction to $\mathbf{\Delta}_2 \times \mathbf{\Lambda}^0_1$ (resp. to $\mathbf{\Delta}_2 \times \mathbf{\Lambda}^1_1$) agrees with c (resp. with c') and whose restriction to $(\mathrm{Im}\,\varepsilon_0) \times \mathbf{\Delta}_1$ (resp. to $(\mathrm{Im}\,\varepsilon_2) \times \mathbf{\Delta}_1$) agrees with h (resp. with h'). Since A is fibrant, t extends to a morphism $\mathbf{\Delta}_2 \times \mathbf{\Delta}_1 \to A$, whose restriction to $(\mathrm{Im}\,\varepsilon_1) \times \mathbf{\Delta}_1$ is a homotopy from $\partial_1 c$ to $\partial_1 c'$ relative to $\partial\mathbf{\Delta}_1$, whence the claim. Moreover, we claim that this composition law is associative : indeed, the proof is the same as that of the associativity of the group law on $\pi_1(A, \xi)$ (details left to the reader). Furthermore, for every vertex $x \in A[0]$, let $\mathbf{1}_x : \mathbf{\Delta}_1 \to A$ be the unique morphism which factors through x; then $[a] \circ [\mathbf{1}_x] = [a]$ for every $y \in A[0]$ and every $[a] : x \to y$, since

$$\partial_0(a \circ \eta_1) = \mathbf{1}_x \qquad \partial_1(a \circ \eta_1) = a \qquad \partial_2(a \circ \eta_1) = a.$$

Likewise, we see that $[\mathbf{1}_x] \circ [b] = [b]$ for every $[b] : y \to x$, by considering $\partial(b \circ \eta_0) = b$ (details left to the reader). Lastly, for any $[a] : x \to y$ consider the unique morphism $t : \mathbf{\Lambda}^0_2 \to A$ such that $t \circ \varepsilon_1 = \mathbf{1}_x$ and $t \circ \varepsilon_0 = a$; since A is fibrant, t extends to a morphism $c : \mathbf{\Delta}_2 \to A$, and it follows that $[\partial_2 c] \circ [a] = \mathbf{1}_x$. Similarly, we see easily that $[a]$ admits a right inverse. We conclude that the datum

$$\underline{\pi}(A) := (A[0], \pi(A), \circ)$$

is a *groupoid*, *i.e.* a category all whose morphisms are isomorphisms. It shall be called the *fundamental groupoid of* A.

Remark 7.5.58. Let $f : X \to Y$ be any morphism of fibrant simplicial sets. A simple inspection of the definitions shows that f induces a functor

$$\underline{\pi}(f) : \underline{\pi}(X) \to \underline{\pi}(Y).$$

Explicitly, $\underline{\pi}(f)$ is the functor whose map on objects $X[0] \to Y[0]$ is given by $f[0]$, and whose map on morphisms is induced by $f[1]$. It is then clear that the rule $f \mapsto \underline{\pi}(f)$ defines a functor

$$\underline{\pi} : s.\mathbf{Set} \to \mathsf{Gpd}$$

with values in the full subcategory of $\mathbf{Cat}$ whose objects are all (small) groupoids.

7.5.59. Now, let $n \in \mathbb{N}$ be any integer, $x \in A[0]$ any vertex, $b : \mathbf{\Delta}_n \to A$ any element of $A(x, n)$, and $[\alpha] : x \to y$ any element of $\pi(A)$. We have a unique morphism

$$t : (\partial\mathbf{\Delta}_n \times \mathbf{\Delta}_1) \cup (\mathbf{\Delta}_n \times \mathbf{\Lambda}^1_1) \to A$$

such that :

- the restriction of t to $\partial\mathbf{\Delta}_n \times \mathbf{\Delta}_1$ equals $\alpha \circ p$, where $p : \partial\mathbf{\Delta}_n \times \mathbf{\Delta}_1 \to \mathbf{\Delta}_1$ is the natural projection
- the restriction of t to $\mathbf{\Delta}_n \times \mathbf{\Lambda}^1_1$ agrees with b.

Since A is fibrant, t extends to a morphism $h : \mathbf{\Delta}_n \times \mathbf{\Delta}_1 \to A$, and we denote by $\partial_1 h \in A[n]$ the restriction of h to $\mathbf{\Delta}_n \times \mathbf{\Lambda}^0_1$. Notice that $\partial_1 h \in A(y, n)$.

Lemma 7.5.60. *With the notation of* (7.5.59)*, the class* $[\partial_1 h] \in \pi_n(A, y)$ *depends only on* $[\alpha]$ *and* $[b] \in \pi_n(A, x)$ *(and is independent of* h *and of the representatives* α *and* b *for these classes).*

Proof. Say that $[\beta] = [\alpha]$ in $\pi(A)$, so we have a homotopy $\gamma : \mathbf{\Delta}_1 \times \mathbf{\Delta}_1 \to A$ from α to β relative to $\partial\mathbf{\Delta}_1$. Suppose also that $[b] = [b']$ in $\pi_n(A, x)$, and let $H : \mathbf{\Delta}_n \times \mathbf{\Delta}_1 \to A$ be a given homotopy from b to b' relative to $\partial\mathbf{\Delta}_n$. Pick also a morphism $h' : \mathbf{\Delta}_n \times \mathbf{\Delta}_1 \to A$ whose restriction to $\partial\mathbf{\Delta}_n \times \mathbf{\Delta}_1$ (resp. to $\mathbf{\Delta}_n \times \mathbf{\Lambda}^1_1$) equals $\beta \circ p$ (resp. b'). To ease notation, set

$$K := (\mathbf{\Lambda}^1_1 \times \mathbf{\Delta}_1) \cup (\mathbf{\Delta}_1 \times \partial\mathbf{\Delta}_1) \qquad \text{and} \qquad L := \mathbf{\Delta}_1 \times \mathbf{\Delta}_1$$

and notice that the monomorphism $K \to L$ is anodyne. We then have a unique morphism

$$u : (\partial\mathbf{\Delta}_n \times L) \cup (\mathbf{\Delta}_n \times K) \to A$$

such that

- the restriction of u to $\partial\mathbf{\Delta}_n \times L$ equals $q \circ \gamma$, where $q : \partial\mathbf{\Delta}_n \times L \to L$ is the natural projection
- the restriction of u to $\mathbf{\Delta}_n \times (\mathbf{\Lambda}^1_1 \times \mathbf{\Delta}_1)$ equals H
- the restriction of u to $\mathbf{\Delta}_n \times (\mathbf{\Lambda}^1_1 \times \mathbf{\Delta}_1)$ (resp. to $\mathbf{\Delta}_n \times (\mathbf{\Lambda}^0_1 \times \mathbf{\Delta}_1)$) equals h (resp. h').

By corollary 7.5.28, the morphism u extends to a morphism $h'' : \mathbf{\Delta}_n \times L \to A$, and it is easily seen that the restriction of h'' to $\mathbf{\Delta}_n \times (\mathbf{\Lambda}^0_1 \times \mathbf{\Delta}_1)$ is a homotopy from $\partial_1 h$ to $\partial_1 h'$ relative to $\partial\mathbf{\Delta}_n$, whence the assertion. □

7.5.61. In light of lemma 7.5.60, the rule $(\alpha, b) \mapsto [\partial_1 h]$ yields a well defined mapping

$$\pi_n(A, [\alpha]) : \pi_n(A, x) \to \pi_n(A, y)$$

for every $n \in \mathbb{N}$, every $x, y \in A[0]$ and every $[\alpha] : x \to y$. For $n = 0$, a simple inspection shows that the resulting map $\pi_0(A, [\alpha]) : \pi_0(A) \to \pi_0(A)$ is the identity. Thus, in the following we assume that $n > 0$, in which case we have :

Proposition 7.5.62. *With the notation of* (7.5.61)*, the following holds for every* $n > 0$ *:*

(i) $\pi_n(A, [\alpha])$ *is a group homomorphism, and* $\pi_n(A, [\mathbf{1}_x]) = \mathbf{1}_{\pi_n(A,x)}$ *for every* $x \in A[0]$.

(ii) *If* $[\beta] : y \to z$ *is any other element of* $\pi(A)$*, we have*

$$\pi_n(A, [\beta]) \circ \pi_n(A, [\alpha]) = \pi_n(A, [\beta] \circ [\alpha]).$$

(iii) *Therefore, the rule*

$$x \mapsto \pi_n(A, x) \qquad [\alpha] \mapsto \pi_n(A, [\alpha]) \qquad \textit{for every } x \in A[0] \textit{ and every } [\alpha] \in \pi(A)$$

yields a well defined functor

$$\pi_n : \underline{\pi}(A) \to \mathbf{Grp}.$$

Proof. (i): Let $b, b' \in A(x, n)$ be any two elements, and pick a morphism $c : \mathbf{\Delta}_{n+1} \to A$ such that $\partial_i c = x$ for $i = 0, \dots, n-2$, $\partial_{n-1} c = b$ and $\partial_{n+1} c = b'$. Let also $[\alpha] : x \to y$ be any element of $\pi(A)$, and choose morphisms $h, h' : \mathbf{\Delta}_n \times \mathbf{\Delta}_1 \to A$ whose restrictions to $\partial\mathbf{\Delta}_n \times \mathbf{\Delta}_1$ equal $p \circ \alpha$ (where p is as in (7.5.59)) and whose restrictions to $\mathbf{\Delta}_n \times \mathbf{\Lambda}^1_1$ agree with b and respectively b'. Then there exists a unique morphism $T : (\partial\mathbf{\Delta}_{n+1} \times \mathbf{\Delta}_1) \cup (\mathbf{\Delta}_{n+1} \times \mathbf{\Lambda}^1_1) \to A$ such that

- for $i = 0, \dots, n-2$, the restriction of T to $(\mathrm{Im}\, \varepsilon_i) \times \mathbf{\Delta}_1$ equals $p' \circ \alpha$, where $p' : (\mathrm{Im}\, \varepsilon_i) \times \mathbf{\Delta}_1 \to \mathbf{\Delta}_1$ is the projection
- the restriction of T to $(\mathrm{Im}\, \varepsilon_{n-1}) \times \mathbf{\Delta}_1$ (resp. to $(\mathrm{Im}\, \varepsilon_{n+1}) \times \mathbf{\Delta}_1$) agrees with h (resp. with h')
- the restriction of T to $\mathbf{\Delta}_{n+1} \times \mathbf{\Lambda}^1_1$ agrees with c.

Since A is fibrant, T extends to a morphism $H : \mathbf{\Delta}_{n+1} \times \mathbf{\Delta}_1 \to A$, and we denote by $U : \mathbf{\Delta}_n \times \mathbf{\Delta}_1 \to A$ the restriction of H to $(\mathrm{Im}\, \varepsilon_n) \times \mathbf{\Delta}_1$. By inspecting the definition, it is easily seen that $\partial_1 U$ is a representative for both $\pi_n(A, [\alpha])([b] \cdot [b'])$ and $\pi_n(A, [\alpha])([b]) \cdot \pi_n(A, [\alpha])([b'])$, so $\pi_n(A, [\alpha])$ is a group homomorphism. A simple inspection shows that $\pi_n(A, [\mathbf{1}_x])$ is the identity map of $\pi_n(A, x)$.

(ii): Pick any morphism $\varphi : \mathbf{\Delta}_2 \to A$ such that $\partial_0 \varphi = \alpha$ and $\partial_2 \varphi = \beta$. Let also b and h as in (7.5.59), and choose similarly a morphism $h' : \mathbf{\Delta}_n \times \mathbf{\Delta}_1 \to A$ whose restriction to $\mathbf{\Delta}_n \times \mathbf{\Lambda}^1_1$ equals $\partial_1 h$, and whose restriction to $\partial\mathbf{\Delta}_n \times \mathbf{\Delta}_1$ equals $p \circ \beta$. Then there exists a unique morphism $u : (\partial\mathbf{\Delta}_n \times \mathbf{\Delta}_2) \cup (\mathbf{\Delta}_n \times \mathbf{\Lambda}^1_2) \to A$ such that

- the restriction of u to $\partial\mathbf{\Delta}_n \times \mathbf{\Delta}_2$ equals $q \circ \varphi$, where $q : \partial\mathbf{\Delta}_n \times \mathbf{\Delta}_2 \to \mathbf{\Delta}_2$ is the projection
- the restriction of u to $\mathbf{\Delta}_n \times (\mathrm{Im}\, \varepsilon_0)$ (resp. to $\mathbf{\Delta}_n \times (\mathrm{Im}\, \varepsilon_2)$) equals h (resp. h').

Since A is fibrant, u extends to a morphism $w : \mathbf{\Delta}_n \times \mathbf{\Delta}_2 \to A$, and we denote by $w_1 : \mathbf{\Delta}_n \times \mathbf{\Delta}_1 \to A$ the restriction of w to $\mathbf{\Delta}_n \times (\mathrm{Im}\, \varepsilon_1)$. Hence, the restriction of w_1 to $\partial\mathbf{\Delta}_n \times \mathbf{\Delta}_1$ equals $p \circ (\partial_1 \varphi)$, and its restriction $\partial_1 w_1$ to $\mathbf{\Delta}_n \times \mathbf{\Lambda}^0_1$ (resp. to $\mathbf{\Delta}_n \times \mathbf{\Lambda}^1_1$) equals b (resp. $\partial_1 h'$). Unwinding the definitions, we find that

$$\begin{aligned}\pi_n(A, [\beta]) \circ \pi_n(A, [\alpha])([b]) &= \pi_n(A, [\beta])([\partial_1 h]) \\ &= [\partial_1 w_1] \\ &= \pi_n(A, [\partial_1 \varphi])([b]) \\ &= \pi_n(A, [\beta] \circ [\alpha])([b])\end{aligned}$$

whence the contention. Of course, (iii) results by combining (i) and (ii). □

Remark 7.5.63. Let $f : X \to Y$ be any morphism of fibrant simplicial sets, $[\alpha] : x \to x'$ a morphism in $\underline{\pi}(X)$, and set $y := f[0](x)$, $y' := f[0](y')$ and $\beta := f[1](\alpha)$, so that $[\beta] = \underline{\pi}(f)[\alpha]$ in $\underline{\pi}(Y)$ (notation of remark 7.5.58). A direct inspection shows that the resulting diagram

$$\begin{array}{ccc} \pi_n(X, x) & \xrightarrow{\pi_n(X,[\alpha])} & \pi_n(X, x') \\ {\scriptstyle \pi_n(f,x)}\downarrow & & \downarrow{\scriptstyle \pi_n(f,x')} \\ \pi_n(Y, y) & \xrightarrow{\pi_n(Y,[\beta])} & \pi_n(Y, y') \end{array}$$

commutes (details left to the reader).

Definition 7.5.64. Consider any commutative diagram of simplicial sets :

$$\begin{array}{ccc} X' & \underset{g}{\overset{f}{\rightrightarrows}} & X \\ & {\scriptstyle p'}\searrow \quad \swarrow{\scriptstyle p} & \\ & Y. & \end{array}$$

(i) A *p-fibrewise homotopy from f to g* is a homotopy h from f to g such that $p \circ h$ is a constant homotopy (see remark 7.5.38(i)).

(ii) Let also $K \subset X'$ be any simplicial subset. A *p-fibrewise homotopy from f to g relative to K* is a p-fibrewise homotopy from f to g which is also a homotopy relative to K (see definition 7.5.37). We write

$$f \overset{p}{\sim} g \qquad (\text{resp. } f \overset{p}{\sim}_K g)$$

if there exists a p-fibrewise homotopy from f to g (resp. relative to K).

(iii) Recall that, for every $n \in \mathbb{N}$, we may regard any $x \in X[n]$ as a morphism $x : \mathbf{\Delta}_n \to X$. Then, we say that p is a *minimal fibration*, if p is a fibration, and

$$a = b \quad \Leftrightarrow \quad a \overset{p}{\sim}_{\partial\mathbf{\Delta}_n} b$$

for every $n \in \mathbb{N}$ and every $a, b \in X[n]$ such that $p[n](a) = p[n](b)$.

(iv) A *strong deformation retract* of p is a datum (i, r, h) consisting of a monomorphism $i : X' \to X$, an epimorphism $r : X \to X'$ such that

$$p \circ i \circ r = p \qquad \text{and} \qquad r \circ i = \mathbf{1}_{X'}$$

and a p-fibrewise homotopy $h : X \times \mathbf{\Delta}_1 \to X$ from $\mathbf{1}_X$ to $i \circ r$ relative to X'.

(v) We say that f is a *fibrewise homotopy equivalence from p' to p*, if there exists a morphism of simplicial sets $f' : X \to X'$ with

$$p' \circ f' = p \qquad f' \circ f \overset{p'}{\sim} \mathbf{1}_{X'} \qquad f \circ f' \overset{p}{\sim} \mathbf{1}_X.$$

(vi) For any simplicial set A, let $p_A : A \to \mathbf{\Delta}_0$ be the unique morphism of simplicial sets. If A is fibrant, we say that A is *minimal*, if p_A is a minimal fibration. We say that a morphism of simplicial sets $f : A \to A'$ is a *homotopy equivalence*, if f is a fibrewise homotopy equivalence from p_A to $p_{A'}$. We say that a monomorphism $i : A' \to A$ is a *strong deformation retract of* A, if there exist a morphism $r : A \to A'$ and a homotopy h from $\mathbf{1}_A$ to $i \circ r$, such that (i, r, h) is a strong deformation retract of p_A.

Remark 7.5.65. (i) By inspecting the definitions, it is easily seen that a composition of fibrewise homotopy equivalences is a fibrewise homotopy equivalence. Especially, a composition of homotopy equivalences is a homotopy equivalence.

(ii) Let (i, r, h) be any strong deformation retract; then, clearly, both i and r are fibrewise homotopy equivalences.

Lemma 7.5.66. *Let $p : X \to Y$ be any fibration, $p' : X' \to Y$ any object of $s.\mathbf{Set}/Y$, and $i : K \to X'$ any monomorphism. Then :*

(i) *$\overset{p}{\sim}_K$ is an equivalence relation on $\mathrm{Hom}_{s.\mathbf{Set}/Y}(p', p)$.*

(ii) *More precisely, let $f, f', f'' \in \mathrm{Hom}_{s.\mathbf{Set}/Y}(p', p)$ be any three elements such that*

$$f \overset{p}{\sim} f' \qquad f' \overset{p}{\sim}_K f''$$

and h a p-fibrewise homotopy from f to f'. Then there exists a p-fibrewise homotopy h' from f to f'' such that

$$h' \circ (i \times \mathbf{1}_{\mathbf{\Delta}_1}) = h \circ (i \times \mathbf{1}_{\mathbf{\Delta}_1}).$$

(iii) *Let f, f' be as in* (ii)*, and suppose that f is a fibrewise homotopy equivalence. Then the same holds for f'.*

Proof. (i): It is a relative variant of the proof of theorem 7.5.42. Namely, consider first the case of $\overset{p}{\sim}_K$; on the one hand, by proposition 7.5.34, the morphism

$$t : s.\mathscr{H}om(X', X) \to P := s.\mathscr{H}om(X', Y) \times_{s.\mathscr{H}om(K,Y)} s.\mathscr{H}om(K, X)$$

induced by p and i is a fibration. On the other hand, let $f, f' \in \mathrm{Hom}_{s.\mathbf{Set}/Y}(p', p)$ be any two elements; the datum of a homotopy h from f to f' relative to K is the same as that of a morphism

$$h^* : \mathbf{\Delta}_1 \to s.\mathscr{H}om(X', X)$$

such that $s.\mathscr{H}om(i, X) \circ h^*$ factors through a morphism $a : \mathbf{\Delta}_0 \to s.\mathscr{H}om(K, X)$ and the unique morphism $\pi : \mathbf{\Delta}_1 \to \mathbf{\Delta}_0$. Then, h is a p-fibrewise homotopy if and only if

$$s.\mathscr{H}om(K, p) \circ a = s.\mathscr{H}om(i, Y) \circ b$$
$$s.\mathscr{H}om(X', p) \circ h^* = b \circ \pi$$

for the unique morphism $b : \mathbf{\Delta}_0 \to s.\mathscr{H}om(X', Y)$ corresponding to p'. Now, the pair (a, b) determines a unique morphism $(a, b) : \mathbf{\Delta}_0 \to P$, and the foregoing conditions tell us that $t \circ h^* = (a, b) \circ \pi$. Thus, set $Q := t^{-1}(a, b)$ (notation of example 7.5.2(vi)); by lemma 7.5.15(iv), the simplicial set Q is fibrant, and we have a natural bijection

$$Q[0] \xrightarrow{\sim} \{f \in \mathrm{Hom}_{s.\mathbf{Set}/Y}(p', p) \mid f \circ i = a\}$$

which identifies the restriction of the relation $\overset{p}{\sim}_K$ with the relation $\sim$ on $Q[0]$. The latter is an equivalence relation, by lemma 7.5.40, whence the contention.

(ii): Let again $b : \mathbf{\Delta}_0 \to s.\mathscr{H}om(X', Y)$ be the morphism corresponding to p', and set

$$P_b := q^{-1}(b) \qquad Q_b := P_b \times_P s.\mathscr{H}om(X', X)$$

where $q : P \to s.\mathscr{H}om(X', Y)$ is the natural projection. So, the morphism t restricts to a fibration $t_b : Q_b \to P_b$. By assumption, f' and f'' can be regarded as morphisms $\mathbf{\Delta}_0 \to$

$s.\mathscr{H}om(X', X)$ such that $a := s.\mathscr{H}om(i, X) \circ f' = s.\mathscr{H}om(i, X) \circ f''$, and there exists a morphism $g : \mathbf{\Delta}_1 \to s.\mathscr{H}om(X', X)$ such that $s.\mathscr{H}om(i, X) \circ g$ factors through a and $s.\mathscr{H}om(X', p) \circ g$ factors through b, and moreover $g \circ \varepsilon_1 = f'$, $g \circ \varepsilon_0 = f''$. Likewise, the homotopy h can be regarded as a morphism $\mathbf{\Delta}_1 \to s.\mathscr{H}om(X', X)$ such that $s.\mathscr{H}om(X', p) \circ h$ factors through b, and moreover $h \circ \varepsilon_1 = f$, $h \circ \varepsilon_0 = f'$. The pair (h, g) then amounts to a morphism $(h, g) : \mathbf{\Lambda}^1_2 \to Q_b$ fitting into a commutative diagram

$$\begin{array}{ccccc} \mathbf{\Lambda}^1_2 & & \xrightarrow{(h,g)} & & Q_b \\ {\scriptstyle \iota^1_2}\downarrow & & & & \downarrow{\scriptstyle t_b} \\ \mathbf{\Delta}_2 & \xrightarrow{\eta_0} & \mathbf{\Delta}_1 & \longrightarrow & P_b \end{array}$$

(namely, (h, g) is the unique morphism such that $(h, g) \circ \varepsilon_0 = h$ and $(h, g) \circ \varepsilon_2 = g$). Thus, (h, g) extends to a morphism $u : \mathbf{\Delta}_2 \to Q_b$ such that $t_b \circ u$ factors as well through η_0; it is easily seen that the morphism $u \circ \varepsilon_1$ yields the sought homotopy h'.

(iii): By assumption, there exists a morphism $g : X \to X'$ of $s.\mathbf{Set}/Y$ such that $g \circ f \overset{p}{\sim} \mathbf{1}_{X'}$ and $f \circ g \overset{p'}{\sim} \mathbf{1}_X$. On the other hand, since $f \overset{p}{\sim} f'$, remark 7.5.38(ii) implies that $g \circ f \overset{p'}{\sim} g \circ f'$ and $f \circ g \overset{p}{\sim} f' \circ g$. Then the assertion follows from (i). ☐

7.5.67. Consider a fibration $p : X \to Y$ of simplicial sets, and two morphisms $f_0, f_1 : A \to Y$, and for $i = 0, 1$ define the simplicial set X_i as the fibre product in the cartesian diagram

$$\begin{array}{ccc} X_i & \longrightarrow & X \\ {\scriptstyle p_i}\downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow{f_i} & Y. \end{array}$$

Proposition 7.5.68. *In the situation of* (7.5.67)*, suppose that* $f_0 \sim f_1$ *(notation of* (7.5.41)*). Then there exists a fibrewise homotopy equivalence from* p_0 *to* p_1*.*

Proof. Let $h : A \times \mathbf{\Delta}_1 \to Y$ be a homotopy from f_0 to f_1, and define the simplicial set H as the fibre product in the resulting diagram whose two square subdiagrams are cartesian

$$\begin{array}{ccccc} X_i & \xrightarrow{j_i} & H & \longrightarrow & X \\ {\scriptstyle p_i}\downarrow & & \downarrow{\scriptstyle p_H} & & \downarrow{\scriptstyle p} \\ A & \xrightarrow{A\times\varepsilon_i} & A \times \mathbf{\Delta}_1 & \xrightarrow{h} & Y \end{array} \qquad \text{for } i = 0, 1.$$

Then, p_H is a fibration (lemma 7.5.15(iv)), hence there exist commutative diagrams

$$\begin{array}{ccc} X_i & \xrightarrow{j_i} & H \\ {\scriptstyle X_i\times\varepsilon_i}\downarrow & \nearrow{\scriptstyle \vartheta_i} & \downarrow{\scriptstyle p_H} \\ X_i \times \mathbf{\Delta}_1 & \xrightarrow{p_i\times\mathbf{\Delta}_1} & A \times \mathbf{\Delta}_1 \end{array} \qquad \text{for } i = 0, 1$$

(proposition 7.5.24) whence two commutative diagrams with cartesian square subdiagrams

$$\begin{array}{ccccc} & & \overset{p_i}{\frown} & & \\ X_i & \xrightarrow{\omega_i} & X_{1-i} & \xrightarrow{p_{1-i}} & A \\ {\scriptstyle X_i\times\varepsilon_{1-i}}\downarrow & & \downarrow{\scriptstyle j_{1-i}} & & \downarrow{\scriptstyle A\times\varepsilon_{1-i}} \\ X_i \times \mathbf{\Delta}_1 & \xrightarrow{\vartheta_i} & H & \xrightarrow{p_H} & A \times \mathbf{\Delta}_1 \\ & & \underset{p_i\times\mathbf{\Delta}_1}{\smile} & & \end{array} \qquad \text{for } i = 0, 1.$$

Notice now that

$$\vartheta_0 \circ (X_0 \times \varepsilon_1) = j_1 \circ \omega_0 = \vartheta_1 \circ (X_1 \times \varepsilon_1) \circ \omega_0 = \vartheta_1 \circ (\omega_0 \times \mathbf{\Delta}_1) \circ (X_0 \times \varepsilon_1)$$

from which we deduce that there exists a commutative diagram

$$\begin{array}{ccc} X_0 \times \mathbf{\Lambda}_2^0 & \xrightarrow{\ \beta\ } & H \\ {\scriptstyle X_0 \times \iota_2^0}\downarrow & \overset{\gamma}{\nearrow} & \downarrow{\scriptstyle p_H} \\ X_0 \times \mathbf{\Delta}_2 & \xrightarrow{\ p_0 \times \eta_1\ } & A \times \mathbf{\Delta}_1 \end{array}$$

$$\gamma : X_0 \times \mathbf{\Delta}_2 \to H$$

where β is the unique morphism such that

$$\beta \circ (X_0 \times \varepsilon_1) = \vartheta_0 \qquad \text{and} \qquad \beta \circ (X_0 \times \varepsilon_2) = \vartheta_1 \circ (\omega_0 \times \mathbf{\Delta}_1).$$

Set $\gamma_0 := \gamma \circ (X_0 \times \varepsilon_0) : X_0 \times \mathbf{\Delta}_1 \to H$, and notice that $\eta_1 \circ \varepsilon_0 : \mathbf{\Delta}_1 \to \mathbf{\Delta}_1$ factors through $\varepsilon_0 : \mathbf{\Delta}_0 \to \mathbf{\Delta}_1$; it follows that γ_0 factors through $j_0 : X_0 \to H$ and a morphism $\overline{\gamma}_0 : X_0 \times \mathbf{\Delta}_1 \to X_0$. Lastly, notice that

$$j_0 \circ \overline{\gamma}_0 \circ (X_0 \times \varepsilon_0) = \vartheta_0 \circ (X_0 \times \varepsilon_0) = j_0$$

$$j_0 \circ \overline{\gamma}_0 \circ (X_0 \times \varepsilon_1) = \vartheta_1 \circ (\omega_0 \times \mathbf{\Delta}_1) \circ (X_0 \times \mathbf{\Delta}_0) = \vartheta_1 \circ (X_1 \times \varepsilon_0) \circ \omega_0 = j_0 \circ \omega_1 \circ \omega_0$$

whence

$$\overline{\gamma}_0 \circ (X_0 \times \varepsilon_0) = \mathbf{1}_{X_0} \qquad \text{and} \qquad \overline{\gamma}_0 \circ (X_0 \times \varepsilon_1) = \omega_1 \circ \omega_0$$

so $\overline{\gamma}_0$ is a homotopy from $\mathbf{1}_{X_0}$ to $\omega_1 \circ \omega_0$. A similar construction yields likewise a homotopy from $\mathbf{1}_{X_1}$ to $\omega_0 \circ \omega_1$, whence the proposition (details left to the reader). □

Proposition 7.5.69. *Every fibration $p : X \to Y$ admits a strong deformation retract (j, r, h) such that $p \circ j$ is a minimal fibration.*

Proof. For every $k, n \in \mathbb{N}$ with $n \geq k$, denote by

$$\mathrm{Sk}_k X \xrightarrow{\varepsilon_X^{(k,n)}} \mathrm{Sk}_n X \xrightarrow{\eta_X^{(n)}} X$$

the inclusion maps (notation of remark 7.5.6(v)) and set $p^{(n)} := p \circ \eta_X^{(n)}$ for every $n \in \mathbb{N}$.

Claim 7.5.70. For every $n \in \mathbb{N}$ there exist morphisms

$$j^{(n)} : Z^{(n)} \to \mathrm{Sk}_n X \qquad r^{(n)} : \mathrm{Sk}_n X \to Z^{(n)} \qquad h^{(n)} : \mathrm{Sk}_n X \times \mathbf{\Delta}_1 \to X$$

such that :

(a) $r^{(n)} \circ j^{(n)} = \mathbf{1}_{Z^{(n)}}$ and $p^{(n)} \circ j^{(n)} \circ r^{(n)} = p^{(n)}$.
(b) $h^{(n)}$ is a homotopy from $\eta_X^{(n)}$ to $\eta_X^{(n)} \circ j^{(n)} \circ r^{(n)}$.
(c) Both $p \circ h^{(n)}$ and the restriction $h_Z^{(n)} : Z^{(n)} \times \mathbf{\Delta}_1 \to X$ of $h^{(n)}$ are constant homotopies.
(d) $\varepsilon_X^{(k,n)}$ restricts to a morphism $e^{(k,n)} : Z^{(k)} \to Z^{(n)}$ for every $k \leq n$.
(e) $s.\mathrm{trunc}_k e^{(k,n)} : s.\mathrm{trunc}_k Z^{(n)} \to s.\mathrm{trunc}_k Z^{(k)}$ is an isomorphism, and $s.\mathrm{trunc}_k h^{(n)} = s.\mathrm{trunc}_k h^{(k)}$ for every $k \leq n$.
(f) The induced map $Z^{(n)}[n] \to X[n]/\overset{p}{\sim}_{\partial\mathbf{\Delta}_n}$ is injective.

Proof of the claim. We construct inductively the sought simplicial sets and morphisms, as follows. First, set $Z^{(-1)} := s.\varnothing$. Next, let $n \in \mathbb{N}$ be any integer such that either $n = 0$ or else $Z^{(n-1)} \subset \mathrm{Sk}_{n-1} X$, $r^{(n-1)}$ and $h^{(n-1)}$ have already been defined. Since $Z^{(n-1)}[n]$ consists entirely of degenerate simplices (see remark 7.5.6(iv)), lemma 7.5.7(i) implies that the natural projection $Z^{(n-1)}[n] \to X[n]/\overset{p}{\sim}_{\partial\mathbf{\Delta}_n}$ is injective.

Now, for $n = 0$, set $C_0 := X[0]/\overset{p}{\sim}_{\partial\mathbf{\Delta}_n}$, and if $n > 0$, let $C_n \subset X[n]/\overset{p}{\sim}_{\partial\mathbf{\Delta}_n}$ be the subset of all equivalence classes of elements $x \in X[n]$ such that $\partial_i x \in Z^{(n-1)}[n-1]$ for every $i = 0, \dots, n$ (notice that $\partial_i x$ depends only on the class of x in $X[n]/\overset{p}{\sim}_{\partial\mathbf{\Delta}_n}$). Pick a representative $x \in X[n]$ for every equivalence class $\overline{x} \in C_n$ that does not lie in the image of $Z^{(n-1)}[n]$, and let $R_n \subset X[n]$ be the subset of all such chosen representatives; we denote by $T^{(n)} \subset s.\mathrm{trunc}_n X$ the n-truncated simplicial set such that

$$T^{(n)}[n] = Z^{(n-1)}[n] \cup R_n \qquad \text{and} \qquad T^{(n)}[k] = Z^{(n-1)}[k] \qquad \text{for every } k < n.$$

The inclusion map $T^{(n)} \to s.\mathrm{trunc}_n X$ induces a morphism $\mathrm{sk}_n T^{(n)} \to \mathrm{Sk}_n X$, so we may set

$$Z^{(n)} := \mathrm{Im}\,(\mathrm{sk}_n T^{(n)} \to \mathrm{Sk}_n X)$$

and we let $j^{(n)} : Z^{(n)} \to \mathrm{Sk}_n X$ be the inclusion map. With this definition, we already see that conditions (d) and (f) are fulfilled, as well the part of condition (e) concerning $Z^{(n)}$. Next, we define the morphism $r^{(n)} : \mathrm{Sk}_n X \to Z^{(n)}$. Namely, we set $r^{(n)}[k] := r^{(n-1)}[k]$ for every $k < n$, and the map $r^{(n)}[n]$ is constructed as follows. If $n = 0$, we let $r^{(0)}[0] : X[0] \to Z^{(0)}[0] = R_0$ be the map that sends any $x \in X[0]$ to the unique representative of the equivalence class of x in $X[0]/\overset{p}{\sim}_{\partial\mathbf{\Delta}_n}$ that lies in R_0. In case $n > 0$, let $x \in \mathrm{Sk}_n X[n] = X[n]$ be any element; if $x \in Z^{(n)}[n]$, we let $r^{(n)}[n](x) := x$, and if $x \notin Z^{(n)}[n]$ we let $h_{\partial x} : \partial\mathbf{\Delta}_n \times \mathbf{\Delta}_1 \to X$ be the composition

$$\partial\mathbf{\Delta}_n \times \mathbf{\Delta}_1 \xrightarrow{(\partial x)\times \mathbf{1}_{\mathbf{\Delta}_1}} \mathrm{Sk}_{n-1}X \times \mathbf{\Delta}_1 \xrightarrow{h^{(n-1)}} X.$$

The pair $(h_{\partial x}, x)$ amounts to a morphism γ_x fitting into a commutative diagram

$$\begin{array}{ccccc}
(\mathbf{\Delta}_n \times \mathbf{\Lambda}^1_j) \cup ((\partial\mathbf{\Delta}_n) \times \mathbf{\Delta}_1) & & \xrightarrow{\quad\gamma_x\quad} & & X \\
\big\downarrow & & & & \big\downarrow{\scriptstyle p} \\
\mathbf{\Delta}_n \times \mathbf{\Delta}_1 & \longrightarrow & \mathbf{\Delta}_n & \xrightarrow{p \circ x} & Y
\end{array}$$

whose horizontal (resp. vertical) unmarked arrow is the natural projection (resp. the natural inclusion map). Since p is a fibration, proposition 7.5.24 and lemma 7.5.15(i) imply that γ_x extends to a morphism $h_x : \mathbf{\Delta}_n \times \mathbf{\Delta}_1 \to X$ which, by inspection, is a p-fibrewise homotopy from x to

$$x' := h_x \circ (\mathbf{1}_{\mathbf{\Delta}_n} \times \varepsilon_1) : \mathbf{\Delta}_n \to X$$

(notation of example 7.5.2(iv)). Moreover, $\partial_i x' \in Z^{(n-1)}[n-1]$ for every $i = 0, \dots, n$, so there exists a unique $x'' \in T^{(n)}[n]$ such that $x' \overset{p}{\sim}_{\partial\mathbf{\Delta}_n} x''$, and we set

$$r^{(n)}[n](x) := x''.$$

We have to check that the resulting system $r^{(n)}[\bullet] := (r^{(n)}[k] \mid k = 0, \dots, n)$ is a morphism

$$s.\mathrm{trunc}_n X \to T^{(n)}$$

of n-truncated simplicial sets. If $n = 0$, there is nothing to show. If $n > 0$, since the sub-system $(r^{(n)}[k] \mid k = 0, \dots, n-1)$ is already, by assumption, a morphism $s.\mathrm{trunc}_{n-1} X \to s.\mathrm{trunc}_{n-1} Z^{(n-1)}$, we have only to show that

$$\partial(r^{(n)}[n](x)) = r^{(n)}[n-1] \circ \partial x \qquad \text{for every } x \in X[n].$$

If $x \in T^{(n)}[n]$, the identity is obvious, since the face and degeneracies of $T^{(n)}$ are the restrictions of those of X. Hence, suppose $x \notin T^{(n)}[n]$; then, by construction, $\partial(r^{(n)}[n](x))$ is the morphism $h_{\partial x} \circ (\mathbf{1}_{\partial\mathbf{\Delta}_n} \times \varepsilon_1) : \partial\mathbf{\Delta}_n \to X$. The latter is the same as

$$(\eta_X^{(n-1)} \circ j^{(n-1)} \circ r^{(n-1)})[n-1] \circ \partial x$$

(because $h^{(n-1)}$ is a homotopy from $\eta_X^{(n-1)}$ to $\eta_X^{(n-1)} \circ j^{(n-1)} \circ r^{(n-1)}$) whence the contention.

By adjunction, the morphism $r^{(n)}[\bullet]$ yields the required morphism

$$r^{(n)} : \mathrm{Sk}_n X \to Z^{(n)}.$$

Also, again by adjunction, the identities of (a) can be checked on the respective n-truncations, where they are clear (details left to the reader).

It remains to exhibit the homotopy $h^{(n)} : \mathrm{Sk}_n X \times \mathbf{\Delta}_1 \to X$. However, to any given $x \in \mathrm{Sk}_n X[n] \setminus Z^{(n)}[n]$, we have already attached a p-fibrewise homotopy h_x from x to x', and we have $x' \overset{p}{\sim}_{\partial \mathbf{\Delta}_n} r^{(n)}[n](x)$; by lemma 7.5.66(ii), it follows that there exists a p-fibrewise homotopy $h'_x : \mathbf{\Delta}_n \times \mathbf{\Delta}_1 \to X$ from x to $r^{(n)}[n](x)$ such that

$$h'_x \circ (i_n \times \mathbf{1}_{\mathbf{\Delta}_1}) = h_x \circ (i_n \times \mathbf{1}_{\mathbf{\Delta}_1}) = h_{\partial x} \tag{7.5.71}$$

where $i_n : \partial \mathbf{\Delta}_n \to \mathbf{\Delta}_n$ is the inclusion map. If $x \in Z^{(n)}[n]$, we have $x = r^{(n)}[n](x)$, so we may take h'_x to be a constant homotopy. There results a well defined map

$$\mu : X[n] \to s.\mathscr{H}\!om(\mathbf{\Delta}_1, X)[n] \qquad x \mapsto h'_x.$$

On the other hand, by remark 7.5.31(iv), the homotopy $h^{(n-1)}$ corresponds to a morphism of simplicial sets

$$\nu^{(n-1)} : \mathrm{Sk}_{n-1} X \to s.\mathscr{H}\!om(\mathbf{\Delta}_1, X).$$

We claim that the datum of μ and the maps $(\nu^{(n-1)}[k] \mid k = 0, \dots, n-1)$ amounts to a morphism of n-truncated simplicial sets

$$s.\mathrm{trunc}_n X \to s.\mathrm{trunc}_n(s.\mathscr{H}\!om(\mathbf{\Delta}_1, X)). \tag{7.5.72}$$

The assertion comes down to checking the identity

$$\partial_i \circ \mu(x) = \nu^{(n-1)}[n-1] \circ \partial_i(x) \qquad \text{for every } x \in X[n] \text{ and every } i = 0, \dots, n \tag{7.5.73}$$

and recall that the i-th face operator on $s.\mathscr{H}\!om(\mathbf{\Delta}_1, X)[n]$

$$\partial_i : \mathrm{Hom}_{s.\mathbf{Set}}(\mathbf{\Delta}_n \times \mathbf{\Delta}_1, X) \to \mathrm{Hom}_{s.\mathbf{Set}}(\mathbf{\Delta}_{n-1} \times \mathbf{\Delta}_1, X)$$

is given by the rule : $\varphi \mapsto \varphi \circ (\varepsilon_i \times \mathbf{1}_{\mathbf{\Delta}_1})$ for every morphism $\varphi : \mathbf{\Delta}_n \times \mathbf{\Delta}_1 \to X$.

Hence, if $x \in Z^{(n)}[n]$, the morphism $\partial_i(h'_x)$ is the constant homotopy from $\partial_i x$ to itself; since $h^{(n-1)}$ fulfills condition (c), this is the same as $\nu^{(n-1)}[n-1](\partial_i x)$, so in this case (7.5.73) holds. In case $x \notin Z^{(n)}[n]$, identity (7.5.71) says that $\partial(h'_x) = h_{\partial x}$. But a simple inspection shows that $h_{\partial x} = \nu^{(n-1)}[n-1] \circ \partial x$, so (7.5.73) holds also in this case, and we get the sought morphism (7.5.72); by adjunction, the latter corresponds to a morphism

$$h^{(n)} : \mathrm{Sk}_n X \times \mathbf{\Delta}_1 \to X$$

as required, and clearly condition (e) for $h^{(n)}$ is fulfilled by construction. Moreover, a simple inspection shows that (7.5.72) fits into a commutative diagram

$$\begin{array}{ccc}
 & & s.\mathrm{trunc}_n X \\
 & \nearrow^{\mathbf{1}_{s.\mathrm{trunc}_n X}} & \uparrow{\scriptstyle s.\mathrm{trunc}_n(s.\mathscr{H}\!om(\varepsilon_0, X))} \\
s.\mathrm{trunc}_n X & \longrightarrow & s.\mathrm{trunc}_n(s.\mathscr{H}\!om(\mathbf{\Delta}_1, X)) \\
 & \searrow_{s.\mathrm{trunc}_n(j^{(n)} \circ r^{(n)})} & \downarrow{\scriptstyle s.\mathrm{trunc}_n(s.\mathscr{H}\!om(\varepsilon_1, X))} \\
 & & s.\mathrm{trunc}_n X
\end{array}$$

where we have used the natural identification $X \xrightarrow{\sim} s.\mathscr{H}\!om(\mathbf{\Delta}_0, X)$ for the target of the morphisms $s.\mathscr{H}\!om(\varepsilon_i, X)$ $(i = 0, 1)$. By adjunction, this diagram translates as our condition (b) for $h^{(n)}$. To conclude the proof of the claim, it remains to check condition (c). However, in order to prove that $p \circ h^{(n)}$ is a constant homotopy it suffices – by adjunction – to show that the morphism $\nu^{(n)} : \mathrm{Sk}_n X \to s.\mathscr{H}om(\mathbf{\Delta}_1, X)$ corresponding to $h^{(n)}$ fits into a commutative diagram

$$\mathscr{D} \quad : \quad \begin{array}{ccc} \mathrm{Sk}_n X & \xrightarrow{\nu^{(n)}} & s.\mathscr{H}om(\mathbf{\Delta}_1, X) \\ {\scriptstyle p^{(n)}}\downarrow & & \downarrow{\scriptstyle s.\mathscr{H}om(\mathbf{\Delta}_1, p)} \\ Y = s.\mathscr{H}om(\mathbf{\Delta}_0, Y) & \xrightarrow{s.\mathscr{H}om(\pi, Y)} & s.\mathscr{H}om(\mathbf{\Delta}_1, Y) \end{array}$$

where π is the unique morphism $\mathbf{\Delta}_1 \to \mathbf{\Delta}_0$. Again by adjunction, it then suffices to check that $s.\mathrm{trunc}_n\mathscr{D}$ commutes; the latter assertion follows easily by inspecting the construction of the morphism (7.5.72). We argue similarly to show that $h_Z^{(n)}$ is a constant homotopy : the assertion comes down to checking the commutativity of the diagram

$$\mathscr{D}_Z \quad : \quad \begin{array}{ccc} Z^{(n)} & \xrightarrow{\nu^{(n)} \circ j^{(n)}} & s.\mathscr{H}om(\mathbf{\Delta}_1, X) \\ \| & & \uparrow{\scriptstyle s.\mathscr{H}om(\mathbf{\Delta}_1, \eta_X^{(n)} \circ j^{(n)})} \\ Z^{(n)} = s.\mathscr{H}om(\mathbf{\Delta}_0, Z^{(n)}) & \xrightarrow{s.\mathscr{H}om(\pi, Z^{(n)})} & s.\mathscr{H}om(\mathbf{\Delta}_1, Z^{(n)}) \end{array}$$

and since the counit of adjunction $\mathrm{sk}_n T^{(n)} \to Z^{(n)}$ is an epimorphism, we are reduced to showing that $s.\mathrm{trunc}_n\mathscr{D}_Z$ commutes, which again follows by inspecting the construction of (7.5.72) : details left to the reader. ♢

Now, if the system $(Z^{(n)}, j^{(n)}, r^{(n)}, h^{(n)} \mid n \in \mathbb{N})$ fulfills conditions (a)–(f) of claim 7.5.70, we let

$$Z := \operatorname*{colim}_{n \in \mathbb{N}} Z^{(n)}$$

where the transition maps $Z^{(k)} \to Z^{(n)}$ are the morphisms $e^{(k,n)}$, for every $k, n \in \mathbb{N}$ with $k \leq n$. In view of (7.5.9), the colimit of the system of monomorphisms $(j^{(n)} \mid n \in \mathbb{N})$ (resp. epimorphisms $(r^{(n)} \mid n \in \mathbb{N})$) is therefore a monomorphism

$$j : Z \to X \qquad \text{(resp. an epimorphism } r : X \to Z\text{)}$$

such that $r \circ j = \mathbf{1}_Z$ and $p \circ j \circ r = p$. It is also clear that there exists a unique homotopy $h : X \times \mathbf{\Delta}_1 \to X$ from $\mathbf{1}_X$ to $j \circ r$, such that

$$s.\mathrm{trunc}_n(h) = s.\mathrm{trunc}_n(h^{(n)}) \qquad \text{for every } n \in \mathbb{N}$$

and then the resulting datum (j, r, h) is the sought strong deformation retract. It remains to check that $p \circ j$ is a minimal fibration. First, it follows from lemma 7.5.15(iv) that $p \circ j$ is a fibration. Lastly, fix $n \in \mathbb{N}$, and suppose that $z, z' \in Z[n]$ are any two elements such that $z \overset{p \circ j}{\sim}_{\partial \mathbf{\Delta}_n} z'$; then $j(z) \overset{p}{\sim}_{\partial \mathbf{\Delta}_n} j(z')$, so $z = z'$, by condition (f) of claim 7.5.70. □

Proposition 7.5.74. *Consider a commutative diagram of simplicial sets*

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ & {\scriptstyle p'}\searrow \quad \swarrow{\scriptstyle p} & \\ & Y & \end{array}$$

such that p and p' are minimal fibrations, and f is a fibrewise homotopy equivalence. Then f is an isomorphism of simplicial sets.

Proof. The proposition is an immediate consequence of the following more precise :

Claim 7.5.75. In the situation of the proposition, suppose that :

(a) $f' : X' \xrightarrow{\sim} X$ is an isomorphism such that $p \circ f' = p'$ and $f \overset{p}{\sim} f'$.

(b) p is a minimal fibration.

Then f is an isomorphism as well.

Proof of the claim. We show first that $f[n]$ is injective for every $n \in \mathbb{N}$. We argue by induction on $n \in \mathbb{N}$. For $n = 0$, notice that $\partial\mathbf{\Delta}_0 = s.\varnothing$; hence, if $a, b \in X'[0]$ and $f[0](a) = f[0](b)$, assumption (a) yields :

$$f'[0](a) \overset{p}{\sim}_{\partial\mathbf{\Delta}_0} f[0](a) \qquad f'[0](b) \overset{p}{\sim}_{\partial\mathbf{\Delta}_0} f[0](b)$$

whence $f'[0](a) \overset{p}{\sim}_{\partial\mathbf{\Delta}_0} f'[0](b)$ by virtue of lemma 7.5.66(i); then assumption (b) implies that $f'[0](a) = f'[0](b)$, and finally $a = b$, as stated. Next, suppose that $n > 0$ and $f[n-1]$ is already known to be injective, and let $a, b \in X'[n]$ be any two elements with $f[n](a) = f[n](b)$; condition (a) says that there exists a p-fibrewise homotopy h_a (resp. h_b) : $\mathbf{\Delta}_n \times \mathbf{\Delta}_1 \to X$ from $f[n](a)$ to $f'[n](a)$ (resp. from $f[n](b)$ to $f'[n](b)$), and on the other hand, the inductive assumption implies that $\partial_i a = \partial_i b$, for every $i = 0, \dots, n$, so the restrictions of h_a and h_b are both equal to the same morphism $w : \partial\mathbf{\Delta}_n \times \mathbf{\Delta}_1 \to X$. There follows a commutative diagram

$$\begin{array}{ccc} (\mathbf{\Delta}_n \times \mathbf{\Lambda}^2_2) \cup (\partial\mathbf{\Delta}_n \times \mathbf{\Delta}_2) & \xrightarrow{u} & X \\ \downarrow & & \downarrow p \\ \mathbf{\Delta}_n \times \mathbf{\Delta}_2 & \xrightarrow{v} & Y \end{array}$$

such that

$$u \circ (\mathbf{1}_{\mathbf{\Delta}_n} \times \varepsilon_1) = h_a \qquad u \circ (\mathbf{1}_{\mathbf{\Delta}_n} \times \varepsilon_0) = h_b$$

and the restriction of u to $\partial\mathbf{\Delta}_n \times \mathbf{\Delta}_2$ equals $w \circ (\mathbf{1}_{\partial\mathbf{\Delta}_n} \times \eta_0)$. Also, v is the composition of the projection $\mathbf{\Delta}_n \times \mathbf{\Delta}_2 \to \mathbf{\Delta}_n$ and the morphism $(p \circ f)[n](a) : \mathbf{\Delta}_n \to Y$. Then there exists a morphism $t : \mathbf{\Delta}_n \times \mathbf{\Delta}_2 \to X$ that extends u and lifts v; it is easily seen that $t \circ (\mathbf{\Delta}_n \times \varepsilon_2) : \mathbf{\Delta}_n \times \mathbf{\Delta}_1 \to X$ is a p-fibrewise homotopy from $f'[n](a)$ to $f'[n](b)$ relative to $\partial\mathbf{\Delta}_n$, and since p is minimal, we conclude that $f'[n](a) = f'[n](b)$, so finally $a = b$, as required.

Next, we show by induction on $n \in \mathbb{N}$ that $f[k]$ is surjective, for every $k < n$. If $n = 0$, there is nothing to show, hence suppose that $n > 0$, and let $x \in X[n]$ be any element; by inductive assumption, for every $i = 0, \dots, n$ there exists $a_i \in X'[n-1]$ such that $f[n-1](a_i) = \partial_i x$. Since the injectivity of f has already been established, we see easily that $\partial_i a_j = \partial_{j-1} a_i$ whenever $0 \leq i < j \leq n$, so the system $(a_0, \dots, a_n)$ determines a unique morphism

$$w' : \partial\mathbf{\Delta}_n \to X'$$

such that $f \circ w'$ is the restriction of $x : \mathbf{\Delta}_n \to X$ to $\partial\mathbf{\Delta}_n$ (see example 7.5.2(iv)). Now, let $h : X' \times \mathbf{\Delta}_1 \to X$ be a p-fibrewise homotopy from f to f'; we deduce a commutative diagram

$$\begin{array}{ccc} (\partial\mathbf{\Delta}_n \times \mathbf{\Delta}_1) \cup (\mathbf{\Delta}_n \times \mathbf{\Lambda}^0_1) & \xrightarrow{u'} & X \\ \downarrow & & \downarrow p \\ \mathbf{\Delta}_n \times \mathbf{\Delta}_1 & \xrightarrow{v'} & Y \end{array}$$

such that the restriction of u' to $\partial\mathbf{\Delta}_n \times \mathbf{\Delta}_1$ (resp. to $\mathbf{\Delta}_n \times \mathbf{\Lambda}^0_1$) agrees with $h \circ (w' \times \mathbf{1}_{\mathbf{\Delta}_1})$ (resp. with the composition of the projection $\mathbf{\Delta}_n \times \mathbf{\Lambda}^0_1 \xrightarrow{\sim} \mathbf{\Delta}_n$ and the morphism x). Also, v' is a constant homotopy. There follows a morphism $t' : \mathbf{\Delta}_n \times \mathbf{\Delta}_1 \to X$ that extends u' and lifts v'. Let

$$x' : \mathbf{\Delta}_n \to X'$$

be the unique morphism such that $f' \circ x' = t' \circ (\mathbf{1}_{\mathbf{\Delta}_n} \times \varepsilon_1)$. By construction, we have

$$f' \circ x' \circ i_n = f' \circ w'$$

(where $i_n : \partial\mathbf{\Delta}_n \to \mathbf{\Delta}_n$ is the natural inclusion); whence $x' \circ i_n = w'$, and we obtain yet another commutative diagram

$$\begin{array}{ccc} (\mathbf{\Delta}_n \times \mathbf{\Lambda}^0_2) \cup (\partial\mathbf{\Delta}_n \times \mathbf{\Delta}_2) & \xrightarrow{u''} & X \\ \downarrow & & \downarrow{\scriptstyle p} \\ \mathbf{\Delta}_n \times \mathbf{\Delta}_2 & \xrightarrow{v''} & Y \end{array}$$

such that

$$u'' \circ (\mathbf{1}_{\mathbf{\Delta}_n} \times \varepsilon_1) = t' \qquad u'' \circ (\mathbf{1}_{\mathbf{\Delta}_n} \times \varepsilon_2) = h \circ (x' \times \mathbf{1}_{\mathbf{\Delta}_1})$$

and where v'' factors through the projection $\mathbf{\Delta}_n \times \mathbf{\Delta}_2 \to \mathbf{\Delta}_n$. As usual, we find a morphism $t'' : \mathbf{\Delta}_n \times \mathbf{\Delta}_2 \to X$ that lifts v'' and extends u'', and it is easily seen that $t'' \circ (\mathbf{\Delta}_n \times \varepsilon_0)$ is a p-fibrewise homotopy from x to $f[n](x')$ relative to $\partial\mathbf{\Delta}_n$. By the minimality of p, it follows that $x = f[n](x')$, whence the sought surjectivity of $f[n]$. □

Definition 7.5.76. Let $f : X \to Y$ be any morphism of fibrant simplicial sets. We say that f is a *weak equivalence*, if the induced map $\pi_n(f, \xi)$ is an isomorphism, for every $\xi \in X[0]$ and every $n \in \mathbb{N}$.

Proposition 7.5.77. *Let $f : X \to Y$ and $g : Y \to Z$ be any two morphisms of fibrant simplicial sets. The following holds :*

(i) *If f has the right lifting property with respect to the natural monomorphism $i_n : \partial\mathbf{\Delta}_n \to \mathbf{\Delta}_n$ for every $n \in \mathbb{N}$, then f is both a fibration and a weak equivalence.*
(ii) *If f is a homotopy equivalence, then it is a weak equivalence.*
(iii) *If any two of the three morphisms $f, g, g \circ f$ is a weak equivalence, then so is the third.*

Proof. (i): It follows easily from lemma 7.5.15(i) and remark 7.5.8 that f has the right lifting property with respect to every monomorphism of simplicial sets (details left to the reader). Especially, f is a fibration. It also follows easily that $f[0]$ is surjective. Moreover, suppose that $a, b \in X[0]$ and $[f(a)] = [f(b)]$ in $\pi_0(Y)$, and pick a homotopy $h : \mathbf{\Delta}_1 \to Y$ from $f(a)$ to $f(b)$; there follows a commutative diagram

$$\begin{array}{ccc} \partial\mathbf{\Delta}_1 & \xrightarrow{c} & X \\ {\scriptstyle i_1}\downarrow & & \downarrow{\scriptstyle f} \\ \mathbf{\Delta}_1 & \xrightarrow{h} & Y \end{array}$$

such that the restriction of c to $\mathbf{\Lambda}^0_1$ (resp. to $\mathbf{\Lambda}^1_1$) agrees with a (resp. with b). By assumption, c extends to a morphism $h' : \mathbf{\Delta}_1 \to X$ such that $f \circ h' = h$. We conclude that $\pi_0(f)$ is bijective. Next, let $\xi \in X[0]$ be any vertex, and set $F := f^{-1}(f(\xi))$. By lemma 7.5.15(ii), the unique morphism $F \to \mathbf{\Delta}_0$ has the right lifting properties with respect to i_n, for every $n \in \mathbb{N}$. By the foregoing, we know already that $\pi_0(F)$ is a set of cardinality one; moreover, we have :

Claim 7.5.78. $\pi_n(F, \xi) = 0$ for every $n \geq 1$.

Proof of the claim. Let $a \in F(\xi, n)$ be any element; then there exists a unique morphism $b : \partial\mathbf{\Delta}_{n+1} \to F$ whose restriction to $\mathbf{\Lambda}^0_{n+1}$ factors through ξ, and whose composition with $\varepsilon_0 : \mathbf{\Delta}_n \to \partial\mathbf{\Delta}_{n+1}$ agrees with a. By assumption, b extends to a morphism $c : \mathbf{\Delta}_{n+1} \to X$, and then the claim follows from lemma 7.5.46. ◊

The proposition now follows easily from theorem 7.5.56 and claim 7.5.78.

(ii): Let $f' : Y \to X$ be a morphism and $h : X \times \mathbf{\Delta}_1 \to X$ a homotopy from $\gamma_0 := \mathbf{1}_X$ to $\gamma_1 := f' \circ f$. By adjunction, h corresponds to a morphism $\beta : X \to s.\mathscr{H}om(\mathbf{\Delta}_1, X)$ fitting into the commutative diagrams

$$\begin{array}{ccc} X & \xrightarrow{\beta} & s.\mathscr{H}om(\mathbf{\Delta}_1, X) \\ & \underset{\gamma_i}{\searrow} & \Big\downarrow{\scriptstyle s.\mathscr{H}om(\varepsilon_i, X)} \\ & & s.\mathscr{H}om(\mathbf{\Lambda}^i_1, X) = X \end{array} \qquad i = 0, 1.$$

Recall that $s.\mathscr{H}om(\mathbf{\Delta}_1, X)$ is fibrant (corollary 7.5.35(i)).

Claim 7.5.79. β and γ_1 are weak equivalences.

Proof of the claim. As for β, since γ_0 is an isomorphism, it suffices to show that $p_0 := s.\mathscr{H}om(\varepsilon_0, X)$ is a weak equivalence. To this aim, by (i), we are reduced to checking that p_0 has the right lifting property with respect to the monomorphisms i_n, for every $n \in \mathbb{N}$. Thus, let $a : \partial\mathbf{\Delta}_n \to s.\mathscr{H}om(\mathbf{\Delta}_1, X)$ be a morphism such that $p_0 \circ a$ extends to a morphism $b : \mathbf{\Delta}_n \to X$; by adjunction, a and b determine a unique morphism $(\mathbf{\Delta}_n \times \mathbf{\Lambda}^0_1) \cup (\partial\mathbf{\Delta}_n \times \mathbf{\Delta}_1) \to X$ which, since X is fibrant, extends to a morphism $\mathbf{\Delta}_n \times \mathbf{\Delta}_1 \to X$. The latter corresponds, by adjunction, to a unique morphism $\mathbf{\Delta}_n \to s.\mathscr{H}om(\mathbf{\Delta}_1, X)$ which is the sought extension of a.

The same argument shows as well that $s.\mathscr{H}om(\varepsilon_1, X)$ is a weak equivalence, and so the same follows for γ_1, as stated. $\diamondsuit$

Now, if f is a homotopy equivalence, we have also a homotopy $h' : Y \times \mathbf{\Delta}_1 \to Y$ from $\mathbf{1}_Y$ to $f \circ f'$. Taking into account claim 7.5.79, we deduce that both

$$\pi_n(g, f(\xi)) \circ \pi_n(f, \xi) \quad \text{and} \quad \pi_n(f, f'(\xi')) \circ \pi_n(f', \xi')$$

are bijections, for every $n \in \mathbb{N}$, every $\xi \in X[0]$ and every $\xi' \in Y[0]$. The assertion follows.

(iii): The only non-trivial assertion is that g is a weak equivalence if both f and $g \circ f$ are. However, in this case it is clear that $\pi_0(g)$ is an isomorphism, so it remains only to check that $\pi_n(g, y)$ is an isomorphism for every $y \in Y[0]$. Thus, fix such vertex y; since $\pi_0(f)$ is a bijection, there exist $x \in X[0]$ and a morphism $[\alpha] : y \to y' := f[0](x)$ in $\underline{\pi}(Y)$ (notation of (7.5.57)). Set $z := g[0](y)$, $z' := g[0](z')$ and $\beta := g[1](\alpha)$; there follows a commutative diagram (see remark 7.5.63)

$$\begin{array}{ccccc} \pi_n(Y, y) & \xrightarrow{\pi_n(Y, [\alpha])} & \pi_n(Y, y') & \xleftarrow{\pi_n(f, x)} & \pi_n(X, x) \\ \Big\downarrow{\scriptstyle \pi_n(g, y)} & & \Big\downarrow{\scriptstyle \pi_n(g, y')} & \underset{\pi_n(g \circ f, x)}{\swarrow} & \\ \pi_n(Z, z) & \xrightarrow{\pi_n(Z, [\beta])} & \pi_n(Z, z') & & \end{array}$$

from which we see first that $\pi_n(g, y')$ is an isomorphism, and then the assertion follows. $\square$

The following is the main result of this section :

Theorem 7.5.80 (Whitehead)**.** *Every weak equivalence of fibrant simplicial sets is a homotopy equivalence.*

Proof. We begin with the following special case :

Claim 7.5.81. Let $f : X \to Y$ be any minimal fibration of fibrant simplicial sets, and suppose that f is a weak equivalence. Then f is an isomorphism.

Proof of the claim. We consider first the case where $Y = \mathbf{\Delta}_0$ (notice that $\mathbf{\Delta}_0$ is trivially a fibrant simplicial set), so that f is the unique morphism $X \to \mathbf{\Delta}_0$. We show by induction that $X[n]$ contains exactly one element, for every $n \in \mathbb{N}$. Say first that $n = 0$; since $\pi_0(\mathbf{\Delta}_0)$ is non-empty, we may find a vertex $\xi \in X[0]$, and we set $\xi' := f[0](\xi)$. Let $a \in X[0]$ be any other vertex; since $f[0](a) = \xi'$, by assumption we may find a homotopy from a to ξ, and therefore $a = \xi$, since X is minimal. Next, suppose that $n > 0$, and the assertion is already known for every integer $< n$; let $b \in X[n]$ be any simplex, and notice that ∂b must factor through ξ, by inductive assumption, so $b \in X(\xi, n)$. However, obviously $\pi_n(\mathbf{\Delta}_0, \xi') = 0$, hence $b \sim_{\partial\mathbf{\Delta}_n} \xi$, and finally $b = \xi$, again by the minimality of X.

Next, consider an arbitrary minimal fibration f, fix $\xi' \in Y[0]$ and pick any $\xi \in f[0]^{-1}(\xi')$ (notice that $Y[0]$ and $f[0]^{-1}(\xi')$ are non-empty, since Y is fibrant and f is a fibration). Set $F := f^{-1}(\xi')$; theorem 7.5.56(i) implies that $\pi_n(F, \xi)$ has cardinality equal to one, for every $n \in \mathbb{N}$. On the other hand, since f is minimal, it is easily seen that the same holds for F (details left to the reader) therefore $F \simeq \mathbf{\Delta}_0$, by the foregoing case. Since ξ' is arbitrary, the claim follows. ◊

Now, let $f : X \to Y$ be any weak equivalence, with X and Y fibrant simplicial sets. By theorem 7.5.29, there exist an anodyne extension $i : X \to E$ and a fibration $p : E \to Y$ such that $p \circ i = f$. Since Y is fibrant, the same holds for E.

Claim 7.5.82. The morphism i is a strong deformation retract of E.

Proof of the claim. Since i is anodyne and X is fibrant, the identity $\mathbf{1}_X$ extends to a morphism $r : E \to X$. Next, consider the unique morphism

$$t : (X \times \mathbf{\Delta}_1) \cup (E \times \partial\mathbf{\Delta}_1) \to E$$

whose restriction to $X \times \mathbf{\Delta}_1$ is the composition of i with the projection onto X, and whose restriction with $E \times \mathbf{\Lambda}_1^0$ (resp. with $E \times \mathbf{\Lambda}_1^1$) agrees with $\mathbf{1}_E$ (resp. with $i \circ r$). Since E is fibrant, t extends to a morphism $h : E \times \mathbf{\Delta}_1 \to E$ (corollary 7.5.28), and it is easily seen that (i, r, h) is a strong deformation retract for the unique morphism $E \to \mathbf{\Delta}_0$. ◊

By claim 7.5.82 and remark 7.5.65(ii) we see that i is a homotopy equivalence, hence it suffices to check that the same holds for p. Moreover, i is a weak equivalence (proposition 7.5.77(ii)), hence the same holds for p (proposition 7.5.77(iii)). Hence, we may replace X by E and f by p, and assume from start that f is also a fibration. We may then find a strong deformation retract $(j : F \to X, r, h)$ of f such that $q := f \circ j$ is a minimal fibration. Arguing as in the foregoing, we see that j is a weak equivalence, and therefore the same holds for q (again, by proposition 7.5.77(iii)); then claim 7.5.81 says that q is an isomorphism. Lastly, r is a homotopy equivalence (remark 7.5.65(ii)), so the same holds for $f = q \circ r$. □

7.6. **Graded rings.**

Definition 7.6.1. Let $(\Gamma, +, 0)$ be a commutative monoid, R a ring.

(i) A Γ-*graded (associative, unital) R-algebra* is a pair $\underline{B} := (B, \mathrm{gr}_\bullet B)$ consisting of an associative unital R-algebra B and a Γ-grading $B = \bigoplus_{\gamma\in\Gamma} \mathrm{gr}_\gamma B$ of the R-module B, such that

$$1 \in \mathrm{gr}_0 B \qquad \text{and} \qquad \mathrm{gr}_\gamma B \cdot \mathrm{gr}_{\gamma'} B \subset \mathrm{gr}_{\gamma+\gamma'} B \qquad \text{for every } \gamma, \gamma' \in \Gamma.$$

A morphism of Γ-graded R-algebras is a map of R-algebras which is compatible with the gradings, in the obvious way.

(ii) Let $\underline{B} := (B, \mathrm{gr}_\bullet B)$ be a Γ-graded R-algebra. A Γ-*graded (left) $\underline{B}$-module* is a datum $\underline{M} := (M, \mathrm{gr}_\bullet M)$ consisting of a B-module M and a Γ-grading $M = \bigoplus_{\gamma\in\Gamma} \mathrm{gr}_\gamma M$ of the R-module underlying M, such that

$$\mathrm{gr}_\gamma B \cdot \mathrm{gr}_{\gamma'} M \subset \mathrm{gr}_{\gamma+\gamma'} M \qquad \text{for every } \gamma, \gamma' \in \Gamma.$$

A morphism of Γ-graded $\underline{B}$-module is a map $f : M \to N$ of B-modules, such that $f(\mathrm{gr}_\gamma M) \subset \mathrm{gr}_\gamma N$, for every $\gamma \in \Gamma$. Likewise we define Γ-*graded right $\underline{B}$-modules* and Γ-*graded $\underline{B}$-bimodules*, and their respective morphisms.

(iii) A *graded ideal* of $\underline{B}$ is a Γ-graded sub-$\underline{B}$-bimodule $(I, \mathrm{gr}_\bullet I)$ of $\underline{B}$. Obviously, in this case we must have $\mathrm{gr}_\gamma I = I \cap \mathrm{gr}_\gamma B$ for every $\gamma \in \Gamma$.

(iv) If $f : \Gamma' \to \Gamma$ is any morphism of commutative monoids, and $\underline{M}$ is any Γ-graded R-module, we define the Γ'-graded R-module $\Gamma' \times_\Gamma \underline{M}$ by setting

$$\mathrm{gr}_\gamma(\Gamma' \times_\Gamma M) := \mathrm{gr}_{f(\gamma)} M \qquad \text{for every } \gamma \in \Gamma'.$$

Notice that if $\underline{B} := (B, \mathrm{gr}_\bullet B)$ is a Γ-graded R-algebra, then $\Gamma' \times_\Gamma \underline{B}$ is a Γ'-graded R-algebra, with multiplication and addition laws induced by those of B, in the obvious way. Likewise, if $(M, \mathrm{gr}_\bullet M)$ is a Γ-graded $\underline{B}$-module, then $\Gamma' \times_\Gamma M$ is naturally a Γ'-graded $\Gamma' \times_\Gamma \underline{B}$-module.

(v) Furthermore, if N is any Γ'-graded R-module, we define the Γ-graded R-module $N_{/\Gamma}$ whose underlying R-module is the same as N, and whose grading is given by the rule

$$\mathrm{gr}_\gamma(N_{/\Gamma}) := \bigoplus_{\gamma' \in f^{-1}(\gamma)} \mathrm{gr}_\gamma N.$$

Just as in (iii), if $\underline{C} := (C, \mathrm{gr}_\bullet C)$ is a Γ'-graded R-algebra, then we get a Γ-graded R-algebra $\underline{C}_{/\Gamma}$, whose underlying R-algebra is the same as C. Lastly, if $(N, \mathrm{gr}_\bullet N)$ is a Γ'-graded $\underline{C}$-module, then $N_{/\Gamma}$ is a Γ-graded $\underline{C}_{/\Gamma}$-module.

In this section we will consider only *commutative* graded algebras, but in the next section (7.7) we will encounter certain special classes of non-commutative graded algebras as well.

Remark 7.6.2. (i) For instance, the R-algebra $R[\Gamma]$ is naturally a Γ-graded R-algebra, when endowed with the Γ-grading such that $\mathrm{gr}_\gamma R[\Gamma] := \gamma R$ for every $\gamma \in \Gamma$.

(ii) Let R be a ring; consider a cartesian diagram of monoids

$$\begin{array}{ccc} \Gamma_3 & \longrightarrow & \Gamma_1 \\ \downarrow & & \downarrow \\ \Gamma_2 & \longrightarrow & \Gamma_0 \end{array}$$

and let B be any Γ_1-graded R-algebra. A simple inspection of the definitions yields an identity of Γ_2-graded R-algebras :

$$\Gamma_2 \times_{\Gamma_0} B_{/\Gamma_0} = (\Gamma_3 \times_{\Gamma_1} B)_{/\Gamma_2}.$$

Definition 7.6.3. Let A be a ring, $S \subset A$ the multiplicative subset of regular elements of A.

(i) For every ring homomorphism $A \to B$, we denote the integral closure of A in B by :

$$\mathrm{i.c.}(A, B).$$

(ii) The *total ring of fractions* of A is the ring :

$$\mathrm{Frac}(A) := S^{-1}A.$$

(iii) The *normalization* A is $A^\nu := \mathrm{i.c.}(A, \mathrm{Frac}(A))$. We say that A is *normal* if $A = A^\nu$.

(iv) We say that a A is *nice* if either $A = 0$ or $\mathrm{Frac}(A)$ has Krull dimension zero.

Remark 7.6.4. Let A be a ring.

(i) Notice that the localization map $A \to \mathrm{Frac}(A)$ is injective, and every regular element of $\mathrm{Frac}(A)$ is invertible : the details are left to the reader.

(ii) If A has Krull dimension zero, then every regular element of A is invertible : indeed, if $a \in A$ is regular and $\mathfrak{p} \in \mathrm{Spec}\, A$, the image of a in $A_\mathfrak{p}$ is still regular, hence $a \notin \mathfrak{p}A_\mathfrak{p}$, because $\mathfrak{p}A_\mathfrak{p}$ is the nilradical of $A_\mathfrak{p}$. Thus, a doesn't lie in any prime ideal of A, whence the contention.

(iii) If A is reduced and has finitely many minimal prime ideals, then A is nice. Indeed, in this case $\mathrm{Frac}(A)$ has also finitely many minimal prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_k$; now, if $x \in \mathrm{Frac}(A) \setminus (\mathfrak{p}_1 \cup \cdots \mathfrak{p}_k)$, and x is not invertible, then x is not regular, by (i). Say that $xy = 0$ for some $y \in \mathrm{Frac}(A) \setminus \{0\}$; it follows that $y \in \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_k = 0$, a contradiction. So every non-invertible element of $\mathrm{Frac}(A)$ lies in $\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_k$, whence the contention.

(iv) If A is noetherian and has no embedded prime ideals, then A is nice. Indeed, in this case $\mathrm{Frac}(A)$ fulfills the same conditions; then, take again $x \in \mathrm{Frac}(A)$ in the complement of the finitely many minimal prime ideals. If x is not invertible, it is not regular according to (i), hence it lies in some associated prime ideal of A ([126, Th.6.1(ii)]), a contradiction.

(v) Every flat ring homomorphism $f : A \to B$ maps the regular elements of A to the regular elements of B, hence it extends uniquely to a flat ring homomorphism

$$\mathrm{Frac}(f) : \mathrm{Frac}(A) \to \mathrm{Frac}(B).$$

Moreover, if f is injective, the same holds for $\mathrm{Frac}(f)$, since the localization map $B \to \mathrm{Frac}(B)$ is injective by (i) : details left to the reader.

(vi) Let $f : A \to B$ be a flat ring homomorphism, such that A is nice and $\mathrm{Spec}\,\kappa(\mathfrak{p}) \otimes_A B$ is either empty or has Krull dimension zero for every prime ideal $\mathfrak{p}$ of A (where $\kappa(\mathfrak{p})$ denotes the residue field of $\mathfrak{p}$). Then B is nice and $\mathrm{Frac}(f)$ induces an isomorphism of B-algebras :

$$f' : \mathrm{Frac}(A) \otimes_A B \xrightarrow{\sim} \mathrm{Frac}(B).$$

Indeed, let $b \in B$ be any regular element; since the induced map $g : B \to \mathrm{Frac}(A) \otimes_A B$ is flat, $g(b)$ is regular in $\mathrm{Frac}(A) \otimes_A B$. On the other hand, by assumption $\mathrm{Spec}\,\mathrm{Frac}(A)$ and the fibres of $\mathrm{Spec}(f)$ are either empty or have Krull dimension zero, hence $\mathrm{Frac}(A) \otimes_A B$ is either 0 or has Krull dimension zero, so $g(b)$ is invertible, by (ii). This shows that g extends to a ring homomorphism $\mathrm{Frac}(B) \to \mathrm{Frac}(A) \otimes_A B$ which is the inverse of f', whence the contention.

(vii) If A is nice, $(A_\mathfrak{p})^\nu = (A^\nu)_\mathfrak{p}$ for every $\mathfrak{p} \in \mathrm{Spec}\,A$. Indeed, in light of (vi) we have :

$$(A_\mathfrak{p})^\nu = \mathrm{i.c.}(A_\mathfrak{p}, \mathrm{Frac}(A_\mathfrak{p})) = \mathrm{i.c.}(A_\mathfrak{p}, \mathrm{Frac}(A)_\mathfrak{p}) = (A^\nu)_\mathfrak{p}.$$

(viii) Especially, if A is a nice and reduced ring, then A is normal if and only if A is *locally normal*, *i.e.* if and only if $A_\mathfrak{p}$ is a reduced local normal domain for every $\mathfrak{p} \in \mathrm{Spec}\,A$. Indeed, the condition is sufficent, by virtue of (vii). Conversely, suppose that A is nice, reduced and normal. In view of (vii), it suffices to check that $A_\mathfrak{p}$ is a domain for every $\mathfrak{p} \in \mathrm{Spec}\,A$, and since $A_\mathfrak{p}$ is reduced, this amounts to showing that $A_\mathfrak{p}$ has a unique minimal prime ideal. However, $A_\mathfrak{p}$ is nice due to (vi), hence $X := \mathrm{Spec}\,\mathrm{Frac}(A_\mathfrak{p})$ has Krull dimension zero, and especially it is a totally disconnected topological space. Moreover, it follows from (i) that the inclusion map $X \to \mathrm{Spec}\,A_\mathfrak{p}$ has dense image, so it must contain every minimal prime ideal of $\mathrm{Spec}\,A_\mathfrak{p}$, and then X is precisely the set of minimal prime ideals of $A_\mathfrak{p}$. Now, if X contains two distinct points, we may find non-empty open and closed subsets $U, U' \subset X$ with $U \cap U' = \varnothing$ and $U \cup U' = X$; then there is an idempotent $e \in \mathrm{Frac}(A_\mathfrak{p})$ which vanishes exactly on U. Since $A_\mathfrak{p}$ is normal by (vii), it follows that $e \in A_\mathfrak{p}$, but then $\mathrm{Spec}\,A_\mathfrak{p}$ is the disjoint union of two non-empty open and closed subsets, which is absurd, since it has a unique closed point.

Definition 7.6.5. Let $(\Gamma, +, 0)$ be an integral monoid, $\underline{A} := (A, \mathrm{gr}_\bullet A)$ a Γ-graded $\mathbb{Z}$-algebra. Set $\mathrm{gr}^*_\gamma A := \mathrm{gr}_\gamma A \setminus \{0\}$ for every $\gamma \in \Gamma$, and $S^* := \bigcup_{\gamma\in\Gamma} \mathrm{gr}^*_\gamma A$.

(i) We say that $\underline{A}$ is a $*$-*domain* if we have $\mathrm{gr}^*_\gamma A \cdot \mathrm{gr}^*_{\gamma'} A \subset \mathrm{gr}^*_{\gamma+\gamma'} A$ for every $\gamma, \gamma' \in \Gamma$.

(ii) We say that $\underline{A}$ is a $*$-*field* if $A \neq 0$ and every element of S^* is invertible in A.

(iii) Notice that if $\underline{A}$ is a $*$-domain, then every element of S^* is regular in A, and obviously S^* is a multiplicative subset of A, so we have a unique Γ^{gp}-grading on

$$\mathrm{Frac}^*(\underline{A}) := S^{*-1} A$$

such that the localization map $A \to \mathrm{Frac}^*(\underline{A})$ is a morphism of Γ^{gp}-graded $\mathbb{Z}$-algebras.

Lemma 7.6.6. *Let $\underline{A}$ be a Γ-graded $*$-field, and $\underline{M} := (M, \mathrm{gr}_\bullet M)$ a Γ-graded $\underline{A}$-module. Then M is a free A-module, and it admits a basis consisting of homogeneous elements of M.*

Proof. Indeed, notice first that $\mathrm{gr}_0 A$ is a field, and $\Gamma_0 := \{\gamma \in \Gamma \mid \mathrm{gr}^* A \neq \varnothing\}$ is an abelian group (with the addition law of Γ). Now, let $\Lambda \subset \Gamma$ be a system of representatives for Γ/Γ^0 (say, with $0 \in \Lambda$), and for every $\lambda \in \Lambda$, pick a basis $(m_{\lambda,i} \mid i \in I_\lambda)$ for the $\mathrm{gr}_0 A$-vector space $\mathrm{gr}_\lambda M$. Let us show that $\Sigma := (m_{\lambda,i} \mid \lambda \in \Lambda, i \in I_\lambda)$ is a basis for the A-module M. To see that Σ is a generating family for M, consider any $\gamma \in \Gamma$ and any $m \in \mathrm{gr}_\gamma M$; then there exist $\lambda \in \Lambda$ and $\gamma' \in \Gamma_0$ such that $\gamma = \lambda + \gamma'$, and for any $a \in \mathrm{gr}_{\gamma'} A$ we have a $\mathrm{gr}_0 A$-linear combination $a^{-1}m = \sum_{i\in J} a_i m_{\lambda,i}$ for a finite subset $J \subset I_\lambda$. Thus $m = \sum_{i\in J} aa_i m_{\lambda,i}$, whence the contention. Next, say that we have an A-linear relation of the form : $\sum_{\lambda\in\Lambda}\sum_{i\in I_\lambda} a_{\lambda,i} m_{\lambda,i} = 0$ (where $a_{\lambda,i} = 0$ for all but finitely many indices (λ, i)); then it is easily seen that we have already $\sum_{i\in I_\lambda} a_{\lambda,i} m_{\lambda,i} = 0$ for every $\lambda \in \Lambda$. Then it also follows straightforwardly that $\sum_{i\in I_\lambda} \mathrm{gr}_\gamma(a_{\lambda,i}) m_{\lambda,i} = 0$ for every $\gamma \in \Gamma_0$, where $\mathrm{gr}_\gamma(a_{\lambda,i})$ denotes the image of $a_{\lambda,i}$ under the projection $A \mapsto \mathrm{gr}_\gamma A$. For every such γ, pick $b_\gamma \in \mathrm{gr}^*_\gamma A$; then we get the $\mathrm{gr}_0 A$-linear relation $\sum_{i\in I_\lambda} b_\gamma^{-1}\mathrm{gr}_\gamma(a_{\lambda,i}) m_{\lambda,i} = 0$ in $\mathrm{gr}_\lambda M$, and therefore $\mathrm{gr}_\gamma(a_{\lambda,i}) = 0$ for every $\lambda \in \Lambda$, every $\gamma \in \Gamma_0$ and every $i \in I_\lambda$, *i.e.* $a_{\lambda,i} = 0$ for all such λ and i, as required. □

Remark 7.6.7. (i) For every Γ-graded $*$-domain $\underline{A}$, the Γ^{gp}-graded $\mathbb{Z}$-algebra $\mathrm{Frac}^*(\underline{A})$ is a $*$-field, and the natural maps $A \to \mathrm{Frac}^*(\underline{A}) \to \mathrm{Frac}(A)$ are injective (details left to the reader).

(ii) Let $\underline{A}$ and $\underline{B}$ be two Γ-graded $*$-domains, and $f : \underline{A} \to \underline{B}$ an injective morphism of Γ-graded $\mathbb{Z}$-algebras. Then f extends to a unique morphism of Γ^{gp}-graded $\mathbb{Z}$-algebras $\mathrm{Frac}(f)^* : \mathrm{Frac}^*(\underline{A}) \to \mathrm{Frac}^*(\underline{B})$, and taking into account (i) and lemma 7.6.6, we see that $\mathrm{Frac}^*(f)$ is a flat and injective ring homomorphism, hence it extends in turn uniquely to a flat and injective homomorphism of total rings of fractions $\mathrm{Frac}(f) : \mathrm{Frac}(A) \to \mathrm{Frac}(B)$, by remark 7.6.4(v).

(iii) Especially, let $(\underline{A}_\lambda := (A_\lambda, \mathrm{gr}^\lambda_\bullet A) \mid \lambda \in \Lambda)$ be a filtered system of Γ-graded $\mathbb{Z}$-algebras, with injective transition maps, such that $\underline{A}_\lambda$ is a $*$-domain for every $\lambda \in \Lambda$. It follows easily that the colimit A of the filtered system of underlying rings $(A_\lambda \mid \lambda \in \Lambda)$ carries a unique Γ-grading $\mathrm{gr}_\bullet$ such that the natural ring homomorphism $j_\lambda : A_\lambda \to A$ is a morphism of Γ-graded $\mathbb{Z}$-algebras $\underline{A}_\lambda \to \underline{A} := (A, \mathrm{gr}_\bullet)$; moreover, $\underline{A}$ is also a $*$-domain, and the resulting cocone $(\underline{A}_\lambda \to \underline{A} \mid \lambda \in \Lambda)$ is universal in the category of Γ-graded $\mathbb{Z}$-algebras (details left to the reader). Furthermore, according to (ii), for every $\lambda, \mu \in \Lambda$ with $\mu \geq \lambda$, the transition map $A_\lambda \to A_\mu$ extends uniquely to a flat and injective ring homomorphism $\mathrm{Frac}(A_\lambda) \to \mathrm{Frac}(A_\mu)$; similarly we get a unique flat and injective extension $\mathrm{Frac}(j_\lambda) : \mathrm{Frac}(A_\lambda) \to \mathrm{Frac}(A)$ of j_λ, for every $\lambda \in \Lambda$. Then it is easily seen that the resulting cocone $(\mathrm{Frac}(j_\lambda) \mid \lambda \in \Lambda)$ is universal.

(iv) In the situation of (iii), suppose moreover that A_λ is nice for every $\lambda \in \Lambda$. Then the same holds for A. Indeed, from (iii) we deduce a natural identification of $\mathrm{Spec}\,\mathrm{Frac}(A)$ with the inverse limit of the cofiltered system of topological spaces $(\mathrm{Spec}\,\mathrm{Frac}(A_\lambda) \mid \lambda \in \Lambda)$. But it is easily seen that the limit of a cofiltered system of spectral spaces of dimension zero has dimension zero (details left to the reader), whence the contention.

Proposition 7.6.8. *Let* $(\Gamma, +, 0)$ *be an abelian group,* $\underline{A} := (A, \mathrm{gr}_\bullet A)$ *a* Γ*-graded* $*$*-field,* $p \geq 1$ *the characteristic exponent of the field* $\mathrm{gr}_0 A$*, and suppose that* $\Gamma \setminus \{0\}$ *has no* p*-torsion elements. Then* A *is a nice reduced normal ring.*

Proof. Clearly $\underline{A}$ is the filtered colimit of its Γ-graded $\mathbb{Z}$-subalgebras $\Delta \times_\Gamma \underline{A}$, where Δ ranges over the finitely generated subgroups of Γ. Moreover, A^ν is the filtered union of its subrings $(\Delta \times_\Gamma \underline{A})^\nu$. Taking into account remark 7.6.7(iv), we are thus easily reduced to the case where Γ is finitely generated, say $\Gamma = \Delta_1 \oplus \cdots \oplus \Delta_k$, for cyclic abelian groups $\Delta_1, \dots, \Delta_k$. Furthermore, after replacing Γ by a subgroup, we may assume that $\mathrm{gr}^*_\gamma \neq \varnothing$ for every $\gamma \in \Gamma$, in which case it is easily seen that $\mathrm{gr}_\gamma A$ is a one-dimensional $\mathrm{gr}_0 A$-vector space, for every $\gamma \in \Gamma$ (details left to the reader), and we deduce an isomorphism of $\mathrm{gr}_0 A$-algebras :

$$(\Delta_1 \times_\Gamma \underline{A}) \otimes_{\mathrm{gr}_0 A} \cdots \otimes_{\mathrm{gr}_0 A} (\Delta_k \times_\Gamma \underline{A}) \xrightarrow{\sim} A.$$

In view of remark 7.6.4(iii) we are then reduced to checking :

Claim 7.6.9. If Γ is a cyclic abelian group, A is a smooth $\mathrm{gr}_0 A$-algebra.

Proof of the claim. Let γ be a generator of Γ; recall that $\mathrm{gr}^*_\gamma A \neq \varnothing$, and pick any $a \in \mathrm{gr}^*_\gamma A$. If $\Gamma \stackrel{\sim}{\to} \mathbb{Z}$, then it is easily seen that the map : $P(T) \mapsto P(a)$ yields an isomorphism of $\mathrm{gr}_0 A$-algebras $\mathrm{gr}_0 A[T, T^{-1}] \stackrel{\sim}{\to} A$. Lastly, say that $\Gamma = \mathbb{Z}/n\mathbb{Z}$ for some $n \in \mathbb{N} \setminus \{0\}$; by assumption n is invertible in $\mathrm{gr}_0 A$, and $b := a^n \in \mathrm{gr}^*_0 A$. Then the same rule yields an isomorphism of $\mathrm{gr}_0 A$-algebras $\mathrm{gr}_0 A[T]/(T^n - b) \stackrel{\sim}{\to} A$. □

Corollary 7.6.10. *Let Γ be a monoid, $\underline{A} := (A, \mathrm{gr}_\bullet A)$ a Γ-graded $*$-domain and suppose that Γ^{gp} is a torsion-free abelian group. Then A is a domain.*

Proof. In view of remark 7.6.7(i) we may replace $\underline{A}$ by the Γ^{gp}-graded $\mathbb{Z}$-algebra $\mathrm{Frac}^*(A)$, and assume from start that Γ is a torsion-free abelian group, and $\underline{A}$ is a $*$-field. Then $\underline{A}$ is the filtered colimit of the system of Γ-graded $\mathbb{Z}$-subalgebras $\Delta \times_\Gamma \underline{A}$, where Δ ranges over the finitely generated subgroups of Γ, and it suffices to prove the assertion for such subalgebras. Thus, we may assume $\Gamma = \mathbb{Z}^{\oplus r}$ for some $r \in \mathbb{N}$, and after replacing Γ by a subgroup, we may even assume that $\mathrm{gr}^*_\gamma A \neq \varnothing$ for every $\gamma \in \Gamma$. In this case, the proof of proposition 7.6.8 shows that A is isomorphic to $\mathrm{gr}_0 A[T_1^{\pm 1}, \dots, T_r^{\pm 1}]$, whence the contention. □

Proposition 7.6.11. *Suppose that $(\Gamma', +) \to (\Gamma, +)$ is a morphism of fine monoids, B a finitely generated (resp. finitely presented) Γ-graded R-algebra, and M a finitely generated (resp. finitely presented) Γ-graded B-module. We have :*

(i) *$\Gamma' \times_\Gamma B$ is a finitely generated (resp. finitely presented) R-algebra.*
(ii) *$\Gamma' \times_\Gamma M$ is a finitely generated (resp. finitely presented) $\Gamma' \times_\Gamma B$-module.*
(iii) *$\mathrm{gr}_\gamma M$ is a finitely generated (resp. finitely presented) $\mathrm{gr}_0 B$-module, for every $\gamma \in \Gamma$.*

Proof. Let B_M denote the direct sum $B \oplus M$, endowed with the R-algebra structure given by the rule :

$$(b_1, m_1) \cdot (b_2, m_2) = (b_1 b_2, b_1 m_2 + b_2 m_1) \qquad \text{for every } b_1, b_2 \in B \text{ and } m_1, m_2 \in M.$$

Notice that B_M is characterized as the unique R-algebra structure for which M is an ideal with $M^2 = 0$, the natural projection $B_M \to B$ is a map of R-algebras, and the B-module structure on M induced via π agrees with the given B-module structure on M.

Claim 7.6.12. The following conditions are equivalent :

(a) The R-algebra B_M is finitely generated (resp. finitely presented).
(b) B is a finitely generated (resp. finitely presented) R-algebra and M is a finitely generated (resp. finitely presented) B-module.

Proof of the claim. (b)⇒(a): Suppose first that B is a finitely generated R-algebra, and M is a finitely generated B-module. Pick a system of generators $\Sigma_B := \{b_1, \dots, b_s\}$ for B and $\Sigma_M := \{m_1, \dots, m_k\}$ for M. Then it is easily seen that $\Sigma_B \cup \Sigma_M$ generates the R-algebra B_M.

For the finitely presented case, pick a surjection of R-algebras $\varphi : R[T_1, \dots, T_s] \to B$ and of B-module $\psi : B^{\oplus k} \to M$. Let Σ'_B be a finite system of generators of the ideal $\mathrm{Ker}\,\varphi$. Pick also a finite system $b_1, \dots, b_r$ of generators of the B-module $\mathrm{Ker}\,\psi$; we may write $b_i = \sum_{j=1}^k b_{ij} e_j$ for certain $b_{ij} \in B$ (where $e_1, \dots, e_k$ is the standard basis of $B^{\oplus k}$). For every $i \leq r$ and $j \leq k$, pick $P_{ij} \in \varphi^{-1}(b_{ij})$. It is easily seen that B_M is isomorphic to the R-algebra $R[T_1, \dots, T_{s+k}]/I$, where I is generated by $\Sigma'_B \cup \{\sum_{j=1}^k P_{ij} T_{j+s} \mid i = 1, \dots, r\} \cup \{T_{i+s} T_{j+s} \mid 0 \leq i, j \leq k\}$.

(a)⇒(b): Suppose that B_M is a finitely generated R-algebra, and let $c_1, \dots, c_n$ be a system of generators. For every $i = 1, \dots, n$, we may write $c_i = b_i + m_i$ for unique $b_i \in B$ and $m_i \in M$. Since $M^2 = 0$, it is easily seen that $m_1, \dots, m_n$ is a system of generators for the B-module M, and clearly $b_1, \dots, b_n$ is a system of generators for the R-algebra B.

Next, suppose that B_M is finitely presented over R. We may find a system of generators $b_1, \dots, b_s, m_1, \dots, m_k$ of B_M with $b_i \in B$ and $m_j \in M$ for every $i \leq s$ and $j \leq k$. We deduce a surjection of R-algebras

$$\varphi : R[T_1, \dots, T_{s+k}]/(T_{s+i}T_{s+j} \mid 0 \leq i, j \leq k) \to B_M$$

such that $T_i \mapsto b_i$ for every $i \leq s$ and $T_{j+s} \mapsto m_j$ for every $j \leq k$. It is easily seen that $\mathrm{Ker}\,\varphi$ is generated by the classes of finitely many polynomials $P_1, \dots, P_r$, where

$$P_i = Q_i(T_1, \dots, T_s) + \sum_{j=1}^{k} T_{s+j} Q_{ij}(T_1, \dots, T_s) \qquad i = 1, \dots, r$$

for certain polynomials $Q_i, Q_{ij} \in R[T_1, \dots, T_s]$. It follows easily that $B = R[T_1, \dots, T_s]/I$, where I is the ideal generated by $Q_1, \dots, Q_r$, and M is isomorphic to the B-module $B^{\oplus k}/N$, where N is the submodule generated by the system $\{\sum_{j=1}^{k} Q_{ij}(b_1, \dots, b_s)e_j \mid i = 1, \dots, r\}$ $\diamondsuit$

Suppose now that B is a finitely generated R-algebra; then B is generated by finitely many homogeneous elements, say $b_1, \dots, b_s$ of degrees respectively $\gamma_1, \dots, \gamma_s$. Thus, we may define surjections of monoids

(7.6.13) $$\mathbb{N}^{\oplus s} \to \Gamma \quad : \quad e_i \mapsto \gamma_i \qquad \text{for } i = 1, \dots, s$$

(where $e_1, \dots, e_s$ is the standard basis of $\mathbb{N}^{\oplus s}$) and of R-algebras $\varphi : C \to B$, where $C := R[\mathbb{N}^{\oplus s}]$ is a free polynomial R-algebra. Notice that C is a $\mathbb{N}^{\oplus s}$-graded R-algebra, and via (7.6.13) we may regard φ as a morphism of Γ-graded R-algebras $C_{/\Gamma} \to B$. Then $I := \ker \varphi$ is a Γ-graded ideal of C, and if we set $P := \mathbb{N}^{\oplus s} \times_\Gamma \Gamma'$ we deduce an isomorphism of Γ'-graded R-algebras

$$B' := \Gamma' \times_\Gamma B \xrightarrow{\sim} (P \times_{\mathbb{N}^{\oplus s}} C)_{/\Gamma'}/(\Gamma' \times_\Gamma I)$$

(see remark 7.6.2(ii)).

Claim 7.6.14. $P \times_{\mathbb{N}^{\oplus s}} C$ is a finitely presented R-algebra.

Proof of the claim. Indeed, this R-algebra is none else than $R[P]$, hence the assertion follows from corollary 6.4.2 and lemma 6.1.7(i). $\diamondsuit$

From claim 7.6.14 it follows already that B' is a finitely generated R-algebra. Now, suppose that M is a finitely generated B-module, and set $M' := \Gamma' \times_\Gamma M$; notice that

(7.6.15) $$\Gamma' \times_\Gamma (B_M) = B'_{M'}.$$

In view of claim 7.6.12, we deduce that M' is a finitely generated B'-module. Next, in case B is a finitely presented R-algebra, I is a finitely generated ideal of $C_{/\Gamma}$; as we have just seen, this implies that $\Gamma' \times_\Gamma I$ is a finitely generated $(\Gamma' \times_\Gamma C_{/\Gamma})$-module, and then claim 7.6.14 shows that B' is a finitely presented R-algebra. This concludes the proof of (i).

Lastly, if moreover M is a finitely presented B-module, assertion (i), together with (7.6.15) and claim 7.6.12 say that M' is a finitely presented B'-module; thus, also assertion (ii) is proven.

(iii): For any given $\gamma \in \Gamma$, let us consider the morphism $f : \mathbb{N} \to \Gamma$ such that $1 \mapsto \gamma$, and set $B' := \mathbb{N} \times_\Gamma B$. By (i), the R-algebra B' is finitely generated (resp. finitely presented), and the B'-module $M' := \mathbb{N} \times_\Gamma M$ is finitely generated (resp. finitely presented). After replacing B by B' and M by M', we may then assume from start that $\Gamma = \mathbb{N}$, and we are reduced to showing that $\mathrm{gr}_1 M$ is a finitely generated (resp. finitely presented) $\mathrm{gr}_0 B$-module.

Let $m_1, \dots, m_t$ be a system of generators of M consisting of homogeneous elements of degrees respectively $j_1, \dots, j_t$. We endow $B^{\oplus t}$ with the $\mathbb{N}$-grading such that

$$\mathrm{gr}_k B^{\oplus t} := \bigoplus_{i=1}^{t} \mathrm{gr}_{k-j_i} B e_i$$

(where $e_1, \dots, e_t$ is the standard basis of $B^{\oplus t}$); then the B-linear map $B^{\oplus t} \to M$ given by the rule $e_i \mapsto m_i$ for every $i = 1, \dots, t$ is a morphism of $\mathbb{N}$-graded B-modules, and if M is finitely presented, its kernel is generated by finitely many homogeneous elements $b_1, \dots, b_s$. In the latter case, endow again $B^{\oplus s}$ with the unique $\mathbb{N}$-grading such that the B-linear map $\varphi : B^{\oplus s} \to B^{\oplus t}$ given by the rule $e_i \mapsto b_i$ for every $i = 1, \dots, s$ is a morphism of $\mathbb{N}$-graded B-modules. Now, in order to check that $\mathrm{gr}_1 M$ is a finitely generated $\mathrm{gr}_0 B$-module, it suffices to show that the same holds for $\mathrm{gr}_1 B^{\oplus t}$. The latter is a direct sum of $\mathrm{gr}_0 B$-modules isomorphic to either $\mathrm{gr}_0 B$ or $\mathrm{gr}_1 B$. Likewise, if M is finitely presented, $\mathrm{gr}_1 M = \mathrm{Coker}\, \mathrm{gr}_1 \varphi$, and again, $\mathrm{gr}_1 B^{\oplus s}$ is a direct sum of $\mathrm{gr}_0 B$-modules isomorphic to either $\mathrm{gr}_0 B$ or $\mathrm{gr}_1 B$; hence in order to check that $\mathrm{gr}_1 M$ is a finitely presented $\mathrm{gr}_0 B$-module, it suffices to show that $\mathrm{gr}_1 B$ is a finitely presented $\mathrm{gr}_0 B$-module. In either event, we are reduced to the case where $\Gamma = \mathbb{N}$ and $M = B$.

However, from (ii) we deduce especially that $\mathrm{gr}_0 B = \{0\} \times_{\mathbb{N}} B$ is a finitely generated (resp. finitely presented) R-algebra, hence B is a finitely generated (resp. finitely presented) B_0-algebra as well; we may then assume that $R = \mathrm{gr}_0 B$. Let Σ be a system of homogeneous generators for the R-algebra B; we may then also assume that

$$\Sigma \cap \mathrm{gr}_0 B = \varnothing. \tag{7.6.16}$$

Then it is easily seen that the R-module $\mathrm{gr}_1 B$ is generated by $\Sigma \cap \mathrm{gr}_1 B$. Lastly, if B is a finitely presented R-algebra, we consider the natural surjection $\psi : R[\Sigma] \to B$ from the free polynomial R-algebra generated by the set Σ, and endow $R[\Sigma]$ with the unique grading for which ψ is a map of $\mathbb{N}$-graded R-algebras; then $I := \mathrm{Ker}\, \psi$ is a finitely generated $\mathbb{N}$-graded ideal with $\mathrm{gr}_0 I = 0$. As usual, we pick a finite system Σ' of homogeneous generators for I; clearly B_1 is isomorphic to $\mathrm{gr}_1 B_0[\Sigma]/\mathrm{gr}_1 I$. On the other hand, (7.6.16) easily implies that $\mathrm{gr}_1 R[\Sigma]$ is a free R-module of finite rank, and moreover $\mathrm{gr}_1 I$ is generated by $\Sigma' \cap \mathrm{gr}_1 I$; especially, $\mathrm{gr}_1 B$ is a finitely presented R-module in this case, and the proof is complete. $\square$

7.6.17. Let $(\Gamma, +)$ be a monoid, R a ring, $\underline{B} := (B, \mathrm{gr}_\bullet B)$ a Γ-graded R-algebra, and M a Γ-graded $\underline{B}$-module. We denote by $M[\gamma]$ the Γ-graded $\underline{B}$-module whose underlying B-module is M, and whose grading is given by the rule :

$$\mathrm{gr}_\beta M[-\gamma] := \bigoplus_{\delta+\gamma=\beta} \mathrm{gr}_\delta M \qquad \text{for every } \gamma \in \Gamma.$$

Remark 7.6.18. (i) In the situation of (7.6.17), pick any system $\mathbf{x} := (x_i \mid i \in I)$ of homogeneous generators of M, and say that $x_i \in \mathrm{gr}_{\gamma_i} M$ for every $i \in I$. Then we may define a surjective map of Γ-graded $\underline{B}$-modules

$$L := \bigoplus_{i \in I} B[-\gamma_i] \to M \quad : \quad e_i \mapsto x_i \qquad \text{for every } i \in I$$

where $(e_i \mid i \in I)$ denotes the canonical basis of the free B-module L (notice that $e_i \in \mathrm{gr}_{\gamma_i} L$ for every $i \in I$).

(ii) In case M is a finitely generated B-module, we may pick a finite system $\mathbf{x}$ as above, and then L shall be a free B-module of finite rank.

(iii) Especially, suppose that B is a coherent ring and M is finitely presented as a B-module; then, in the situation of (ii), the kernel of the surjection $L \to M$ shall be again a finitely presented Γ-graded $\underline{B}$-module, so we can repeat the above construction, and find inductively a resolution

$$\Sigma \quad : \quad \cdots \to L_n \xrightarrow{d_n} L_{n-1} \to \cdots \to L_0 \xrightarrow{d_0} M$$

such that L_n is a free B-module of finite rank, and the map d_n is a morphism of Γ-graded $\underline{B}$-modules, for every $n \in \mathbb{N}$.

(iv) In the situation of (iii), suppose furthermore that B is a flat R-algebra, in which case $\mathrm{gr}_\gamma B$ is a flat R-module, for every $\gamma \in \Gamma$. Then it is clear that the resolution Σ yields, in each degree $\gamma \in \Gamma$ a flat resolution Σ_γ of the R-module $\mathrm{gr}_\gamma M$.

7.6.19. *Graded rings and diagonalizable groups.* We conclude this section we a brief review of the well known correspondance between Γ-gradings on a module M (for a given commutative group Γ), and actions of the diagonalizable group $D(\Gamma)$ on M. Let S be a scheme; on the category Sch/S of S-schemes we have the presheaf of rings :

$$\mathscr{O}_{\mathsf{Sch}/S} : \mathsf{Sch}/S \to \mathbb{Z}\text{-}\mathsf{Alg} \qquad (X \to S) \mapsto \Gamma(X, \mathscr{O}_X).$$

Also, let M be an $\mathscr{O}_S$-module; following [55, Exp.I, Déf.4.6.1], we attach to M the $\mathscr{O}_{\mathsf{Sch}/S}$-module $\mathscr{W}_M$ given by the rule : $(f : X \to S) \mapsto \Gamma(X, f^*M)$. If M and N are two quasi-coherent $\mathscr{O}_S$-modules, and $f : S' \to S$ an affine morphism, it is easily seen that

$$\Gamma(S', \mathscr{H}om_{\mathscr{O}_{\mathsf{Sch}/S}}(\mathscr{W}_M, \mathscr{W}_N)) = \mathrm{Hom}_{\mathscr{O}_S}(M, f_*\mathscr{O}_{S'} \otimes_{\mathscr{O}_S} N) \tag{7.6.20}$$

(see [55, Exp.I, Prop.4.6.4] for the details).

Let $f : G \to S$ be a group S-scheme, *i.e.* a group object in the category Sch/S. If f is affine, we say that G is an *affine group S-scheme*; in that case, the mutiplication law $G \times_S G \to G$ and the unit section $S \to G$ correspond respectively to morphisms of $\mathscr{O}_S$-algebras

$$\Delta_G : f_*\mathscr{O}_G \to f_*\mathscr{O}_G \otimes_{\mathscr{O}_S} f_*\mathscr{O}_G \qquad \varepsilon_G : \mathscr{O}_S \to f_*\mathscr{O}_G$$

which make commute the diagram :

$$\begin{array}{ccc} f_*\mathscr{O}_G & \xrightarrow{\Delta_G} & f_*\mathscr{O}_G \otimes_{\mathscr{O}_S} f_*\mathscr{O}_G \\ {\scriptstyle \Delta_G}\downarrow & & \downarrow{\scriptstyle \mathbf{1}_{f_*\mathscr{O}_S} \otimes \Delta_G} \\ f_*\mathscr{O}_G \otimes_{\mathscr{O}_S} f_*\mathscr{O}_G & \xrightarrow{\Delta_G \otimes \mathbf{1}_{f_*\mathscr{O}_S}} & f_*\mathscr{O}_G \otimes_{\mathscr{O}_S} f_*\mathscr{O}_G \otimes_{\mathscr{O}_S} f_*\mathscr{O}_G \end{array}$$

as well as a similar diagram, which expresses the unit property of ε_G : see [55, Exp.I, §4.2].

Example 7.6.21. Let G be any commutative group. The presheaf of groups

$$D_S(G) : \mathsf{Sch}/S \to \mathbb{Z}\text{-}\mathbf{Mod} \qquad (X \to S) \mapsto \mathrm{Hom}_{\mathbb{Z}\text{-}\mathbf{Mod}}(G, \mathscr{O}_X^\times(X))$$

is representable by an affine group S-scheme $D_S(G)$, called the *diagonalizable group scheme* attached to G. Explicitly, if $S = \operatorname{Spec} R$ is an affine scheme, the underlying S-scheme of $D_S(G)$ is $\operatorname{Spec} R[G]$, and the group law is given by the map of R-algebras

$$\Delta_G : R[G] \to R[G] \otimes_R R[G] \xrightarrow{\sim} R[G \times G] \qquad g \mapsto (g, g) \quad \text{for every } g \in G$$

with unit $\varepsilon_G : R[G] \to R$ given by the standard augmentation (see [55, Exp.I, §4.4]). For a general scheme S, we have $D_S(G) = D_{\operatorname{Spec}\mathbb{Z}}(G) \times_{\operatorname{Spec}\mathbb{Z}} S$ (with the induced group law and unit section).

Definition 7.6.22. Let M be an $\mathscr{O}_S$-module, G a group S-scheme. A *G-module structure* on M is the datum of a morphism of presheaves of groups on Sch/S :

$$h_G \to \mathscr{A}ut_{\mathscr{O}_{\mathsf{Sch}/S}}(\mathscr{W}_M)$$

(where h_G denotes the Yoneda embedding : see (1.2.4)).

7.6.23. Suppose now that $f : G \to S$ is an affine group S-scheme, and M a quasi-coherent $\mathscr{O}_S$-module; in view of (7.6.20), a G-module structure on M is then the same as a map of $\mathscr{O}_S$-modules

$$\mu_M : M \to f_*\mathscr{O}_G \otimes_{\mathscr{O}_S} M$$

which makes commute the diagrams :

$$\begin{array}{ccc} M & \xrightarrow{\mu_M} & f_*\mathscr{O}_G \otimes_{\mathscr{O}_S} M \\ {\scriptstyle \mu_M}\downarrow & & \downarrow{\scriptstyle \Delta_G \otimes \mathbf{1}_M} \\ f_*\mathscr{O}_G \otimes_{\mathscr{O}_S} M & \xrightarrow{\mathbf{1}_{f_*\mathscr{O}_G} \otimes \mu_M} & f_*\mathscr{O}_G \otimes_{\mathscr{O}_S} f_*\mathscr{O}_G \otimes_{\mathscr{O}_S} M \end{array} \qquad \begin{array}{ccc} M & \xrightarrow{\mu_M} & f_*\mathscr{O}_G \otimes_{\mathscr{O}_S} M \\ & {\scriptstyle \mathbf{1}_M}\searrow & \downarrow{\scriptstyle \varepsilon_G \otimes \mathbf{1}_M} \\ & & M. \end{array}$$

Example 7.6.24. Let Γ be a commutative group; a $D_S(\Gamma)$-module structure on a quasi-coherent $\mathscr{O}_S$-module M is the datum of a morphism of $\mathscr{O}_S$-modules

$$\mu_M : M \to \mathscr{O}_S[\Gamma] \otimes_{\mathscr{O}_S} M = \mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}} M$$

which makes commute the diagrams of (7.6.23). If $S = \operatorname{Spec} R$ is affine, M is associated with an R-module which we denote also by M; in this case, μ_M is the same as a system $(\mu_M^{(\gamma)} \mid \gamma \in \Gamma)$ of $\mathscr{O}_S$-linear endomorphisms of M, such that :

- for every $x \in M$, the subset $\{\gamma \in \Gamma \mid \mu_M^{(\gamma)}(x) \neq 0\}$ is finite.
- $\mu_M^{(\gamma)} \circ \mu_M^{(\tau)} = \delta_{\gamma,\tau} \cdot \mathbf{1}_M$ and $\sum_{\gamma \in \Gamma} \mu_M = \mathbf{1}_M$.

In other words, the $\mu_M^{(\gamma)}$ form an orthogonal system of projectors of M, summing up to the identity $\mathbf{1}_M$. This is the same as the datum of a Γ-grading on M : namely, for a given $D_S(\Gamma)$-module structure μ_M, one lets

$$\mathrm{gr}_\gamma M := \mu_M^{(\gamma)}(M) \qquad \text{for every } \gamma \in \Gamma$$

and conversely, given a Γ-grading $\mathrm{gr}_\bullet M$ on M, one defines μ_M as the R-linear map given by the rule : $x \mapsto \gamma \otimes x$ for every $\gamma \in \Gamma$ and every $x \in \mathrm{gr}_\gamma M$.

7.6.25. Suppose now that $g : X \to S$ is an affine S-scheme, and $f : G \to S$ an affine group S-scheme. A *G-action* on X is a morphism of presheaves of groups :

$$h_G \to \mathscr{A}ut_{\mathsf{Sch}/S^\wedge}(h_X).$$

(notation of (1.2.4)); the latter is the same as a morphism of S-schemes

$$G \times_S X \to X \tag{7.6.26}$$

inducing a G-module structure on $g_*\mathscr{O}_X$:

$$g_*\mathscr{O}_X \to f_*\mathscr{O}_G \otimes_{\mathscr{O}_S} g_*\mathscr{O}_X$$

which is also a morphism of $\mathscr{O}_S$-algebras. For instance, if $S = \operatorname{Spec} R$ is affine, and $G = D_S(\Gamma)$ for an abelian group Γ, we may write $X = \operatorname{Spec} A$ for some R-algebra B, and in view of example 7.6.24, the G-action on X is the same as the datum of a Γ-graded R-algebra structure on B, in the sense of definition 7.6.1(i).

Remark 7.6.27. (i) Suppose that Γ is an integral monoid. Then, to a Γ-graded R-algebra B, the correspondance described in (7.6.25) attaches a $D_S(\Gamma^{\mathrm{gp}})$-action on $\operatorname{Spec} B$ (where $S := \operatorname{Spec} R$), given by the map of R-algebras

$$\vartheta_B : B \to B[\Gamma] \subset B[\Gamma^{\mathrm{gp}}] \quad : \quad b \mapsto b \cdot \gamma \qquad \text{for every } \gamma \in \Gamma, \text{ and every } b \in \mathrm{gr}_\gamma B.$$

(ii) If Γ is a finite abelian group whose order is invertible in $\mathscr{O}_S$, then $D_S(\Gamma)$ is an étale S-scheme. Indeed, in light of (4.8.52), the assertion is reduced to the case where $\Gamma = \mathbb{Z}/n\mathbb{Z}$ for some integer $n > 0$ which is invertible in $\mathscr{O}_S$. However, $R[\mathbb{Z}/n\mathbb{Z}] \simeq R[T]/(T^n - 1)$, which is an étale R-algebra, if $n \in R^\times$.

(iii) More generally, suppose that Γ is a finitely generated abelian group, such that the order of its torsion subgroup is invertible in $\mathscr{O}_S$. Then we may write $\Gamma = L \oplus \Gamma_{\mathrm{tor}}$, where L is a free abelian group of finite rank, and Γ_{tor} is a finite abelian group as in (ii). In view of (4.8.52) and (ii), we conclude that $D_S(\Gamma)$ is a smooth S-scheme in this case.

(iv) Let Γ be as in (iii), and suppose that X is an S-scheme with an action of $G := D_S(\Gamma)$. Then the corresponding morphism (7.6.26) and the projection $p_G : G \times_S X \to G$ induce an automorphism of the G-scheme $G \times_S X$, whose composition with the projection $p_X : G \times_S X \to X$ equals (7.6.26). We then deduce that both (7.6.26) and p_X are smooth morphisms. This observation, together with the above correspondance between Γ-graded algebras and $D_S(\Gamma)$-actions, is the basis for a general method that allows to prove properties of graded rings, provided they can be translated as properties of the corresponding schemes *which are well behaved under smooth base changes*. We shall present hereafter a few applications of this method.

Proposition 7.6.28. *Let Γ be an integral monoid, B a Γ-graded (commutative, unital) ring, and suppose that the order of any torsion element of Γ^{gp} is invertible in B. Then :*

(i) $\mathrm{nil}(B[\Gamma]) = \mathrm{nil}(B) \cdot B[\Gamma]$.

(ii) $\mathrm{nil}(B)$ *is a Γ-graded ideal of B.*

Proof. (i): Clearly $\mathrm{nil}(B) \cdot B[\Gamma] \subset \mathrm{nil}(B[\Gamma])$. To show the converse inclusion, it suffices to prove that $\mathrm{nil}(B[\Gamma]) \subset \mathfrak{p}B[\Gamma]$ for every prime ideal $\mathfrak{p} \subset B$, or equivalently that $B/\mathfrak{p}[\Gamma]$ is a reduced ring for every such $\mathfrak{p}$. Since the natural map $\Gamma \to \Gamma^{\mathrm{gp}}$ is injective, we may further replace Γ by Γ^{gp}, and assume that Γ is an abelian group. In this case, $B/\mathfrak{p}[\Gamma]$ is the filtered union of the rings $B/\mathfrak{p}[H]$, where H runs over the finitely generated subgroups of Γ; it suffices therefore to prove that each $B/\mathfrak{p}[H]$ is reduced, so we may assume that Γ is finitely generated, and the order of its torsion subgroup is invertible in B. In this case, $B/\mathfrak{p}[\Gamma]$ is a smooth $B/\mathfrak{p}$-algebra (remark 7.6.27(iii)), and the assertion follows from [66, Ch.IV, Prop.17.5.7].

(ii): The assertion to prove is that $\mathrm{nil}(B) = \bigoplus_{\gamma\in\Gamma} \mathrm{nil}(B) \cap \mathrm{gr}_\gamma B$. However, let $\vartheta_B : B \to B[\Gamma]$ be the map defined as in remark 7.6.27(i); now, (i) implies that ϑ_B restricts to a map $\mathrm{nil}(B) \to \mathrm{nil}(B) \cdot B[\Gamma]$, whence the contention. □

Proposition 7.6.29. *Let $(\Gamma, +, 0)$ be an integral monoid, $f : A \to B$ a morphism of Γ-graded rings, and suppose that the order of any torsion element of Γ^{gp} is invertible in A. We have :*

(i) $(\mathrm{i.c.}(A, B))[\Gamma^{\mathrm{gp}}] = \mathrm{i.c.}(A[\Gamma^{\mathrm{gp}}], B[\Gamma^{\mathrm{gp}}])$.

(ii) *The grading of B restricts to a Γ-grading on the subring* $\mathrm{i.c.}(A, B)$.

(iii) *Suppose moreover that B is reduced. Then for every $\gamma \in \Gamma$ such that* $\mathrm{gr}_\gamma \mathrm{i.c.}(A, B) \neq 0$ *there exists an integer $m > 0$ with* $\mathrm{gr}_{m\gamma} A \neq 0$.

Proof. (i): We easily reduce to the case where Γ is finitely generated, in which case $A[\Gamma^{\mathrm{gp}}]$ is a smooth A-algebra (remark 7.6.27(iii)), and the assertion follows from [64, Ch.IV, Prop.6.14.4].

(ii): Define ϑ_A and ϑ_B as in remark 7.6.27(i), and consider the commutative diagram :

$$\begin{array}{ccc} A & \xrightarrow{\vartheta_A} & A[\Gamma] \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f[\Gamma]} \\ B & \xrightarrow{\vartheta_B} & B[\Gamma]. \end{array}$$

Say that $b \in \mathrm{i.c.}(A, B)$; then $\vartheta_B(b) \in \mathrm{i.c.}(A[\Gamma], B[\Gamma])$. In light of (i), we deduce that $\vartheta_B(b) \in (\mathrm{i.c.}(A, B))[\Gamma^{\mathrm{gp}}] \cap B[\Gamma] = (\mathrm{i.c.}(A, B))[\Gamma]$. The claim follows easily.

(iii): Let x be any non-zero element of $\mathrm{gr}_\gamma \mathrm{i.c.}(A, B)$. By definition there exist an integer $n > 0$ and $a_1, \dots, a_n \in A$ such that $x^n + a_1 x^{n-1} + \cdots + a_n = 0$ in B. For $i = 1, \dots, n$ let a'_i be the image of a_i under the projection $A \to \mathrm{gr}_{i\gamma} A$; it is easily seen that $x^n + a'_1 x^{n-1} + \cdots + a'_n = 0$. If $a'_i = 0$ for every $i = 1, \dots, n$, we deduce that $x^n = 0$, which is absurd, since B is reduced. Thus, we have $a'_m \neq 0$ for at least one index $m \leq n$, so $\mathrm{gr}_{m\gamma} A \neq 0$. □

Corollary 7.6.30. *Let Γ be an integral monoid, $\underline{A} := (A, \mathrm{gr}_\bullet A)$ a Γ-graded $*$-domain, $\Gamma^\nu \subset \Gamma^{\mathrm{gp}}$ the saturation of Γ, and suppose that the order of any torsion element of Γ^{gp} is invertible in A. Then there exists a unique Γ^ν-grading $\mathrm{gr}_\bullet A^\nu$ on A^ν (notation of definition* 7.6.3(iii)*) such that $(A^\nu, \mathrm{gr}_\bullet A^\nu)$ is a Γ^ν-graded nice reduced $*$-domain, and the inclusion map $A \to A^\nu$ is a morphism of Γ^ν-graded rings.*

Proof. From proposition 7.6.8 and remark 7.6.7(i) we see that $A^\nu = \mathrm{i.c.}(A, \mathrm{Frac}^*(A))$. The corollary then follows straightforwardly from proposition 7.6.29. □

7.7. **Differential graded algebras.** The material of this paragraph shall be applied in section 7.10, in order to study certain strictly anti-commutative graded algebras constructed via homotopical algebra. Especially, the graded algebras appearing in this paragraph are usually *not* commutative, unless it is explicitly said otherwise.

Definition 7.7.1. (i) Let A be any ring, and $\underline{B} := (B, \mathrm{gr}_\bullet B)$ a $\mathbb{Z}$-graded (associative, unital) A-algebra. We say that $\underline{B}$ is *strictly anti-commutative* (or *alternating*), if we have

(a) $a \cdot b = (-1)^{pq} \cdot b \cdot a$ for every $p, q \in \mathbb{Z}$, every $a \in \mathrm{gr}_p B$, and every $b \in \mathrm{gr}_q B$

(b) $a \cdot a = 0$ for every $p \in \mathbb{Z}$ and every $a \in \mathrm{gr}_{2p+1} B$.

If only condition (a) holds, we say that $\underline{B}$ is *anti-commutative*.

(ii) Let $\underline{M} := (M, \mathrm{gr}_\bullet M)$ be any $\mathbb{Z}$-graded $\underline{B}$-bimodule. An *A-linear graded derivation* from $\underline{B}$ to $\underline{M}$ is a morphism of graded A-modules $\partial : \underline{B} \to \underline{M}$ such that

$$\partial(xy) = \partial(x) \cdot y + x \cdot \partial(y) \qquad \text{for every } x, y \in B.$$

Remark 7.7.2. (i) Let $A\text{-}\mathsf{Alt}$ be the full subcategory of the category of $\mathbb{Z}$-graded A-algebras, whose objects are the strictly anti-commutative graded A-algebras. Define also the category $A\text{-}\mathsf{Alg.Mod}$ as in [75, Def.2.5.22]. We have an obvious functor

$$\mathrm{gr}_{0,1} : A\text{-}\mathsf{Alt} \to A\text{-}\mathsf{Alg.Mod} \qquad (B, \mathrm{gr}_\bullet B) \mapsto (\mathrm{gr}_0 B, \mathrm{gr}_1 B)$$

and it is easily seen that the functor $\mathrm{gr}_{0,1}$ is right adjoint to the functor

$$\Lambda^\bullet_\bullet : A\text{-}\mathsf{Alg.Mod} \to A\text{-}\mathsf{Alt} \qquad (B, M) \mapsto \Lambda^\bullet_B M$$

that assigns to every object (B, M) the graded A-algebra of the B-linear exterior powers of M; the details shall be left to the reader.

(ii) For every pair of graded A-algebras $(B, \mathrm{gr}_\bullet B), (B', \mathrm{gr}_\bullet B')$ we define a *tensor product*

$$(B'', \mathrm{gr}_\bullet B'') := (B, \mathrm{gr}_\bullet B) \otimes_A (B', \mathrm{gr}_\bullet B')$$

which is the graded A-algebra with $B'' := B \otimes_A B'$, and $\mathrm{gr}_n B'' := \bigoplus_{i+j=n} \mathrm{gr}_i B \otimes_A \mathrm{gr}_j B'$ for every $n \in \mathbb{Z}$. The multiplication law of B'' is the direct sum of the maps

$$(\mathrm{gr}_i B \otimes_A \mathrm{gr}_j B') \times (\mathrm{gr}_k B \otimes_A \mathrm{gr}_l B') \to \mathrm{gr}_{i+k} B \otimes_A \mathrm{gr}_{j+l} B' \qquad (b \otimes b', c \otimes c') \mapsto (-1)^{jk} bc \otimes b'c'.$$

It is easily seen that if $(B, \mathrm{gr}_\bullet B)$ and $(B', \mathrm{gr}_\bullet B')$ are anti-commutative (resp. alternating), the same holds for their tensor product. Moreover, if $(B, \mathrm{gr}_\bullet B)$ and $(B', \mathrm{gr}_\bullet B')$ are alternating, then $(B'', \mathrm{gr}_\bullet B'')$ represents the coproduct of $(B, \mathrm{gr}_\bullet B)$ and $(B', \mathrm{gr}_\bullet B')$ in the category $A\text{-}\mathsf{Alt}$. Again, we leave the details to the reader.

(iii) Likewise, all finite coproducts are representable in $A\text{-}\mathsf{Alg.Mod}$: namely, the coproduct of any two objects $(B, M), (B', M')$ is represented by :

$$(B'', M'') := (B \otimes_A B', (M \otimes_A B') \oplus (B \otimes_A M'))$$

with the obvious universal cocone $(B, M) \to (B'', M'') \leftarrow (B', M')$. By (i), (ii), and proposition 1.3.25(iv), we deduce a natural isomorphism of alternating graded A-algebras :

$$\Lambda^\bullet_{B''} M'' \xrightarrow{\sim} \Lambda^\bullet_B M \otimes_A \Lambda^\bullet_{B'} M'.$$

Definition 7.7.3. Let A be any ring.

(i) A *differential graded A-algebra* is the datum of

- a complex $(B^\bullet, d^\bullet_B)$ of A-modules
- an A-linear map $\mu^{pq} : B^p \otimes_A B^q \to B^{p+q}$ for every $p, q \in \mathbb{Z}$

such that the following holds :

(a) Set $B := \bigoplus_{p \in \mathbb{Z}} B^p$; then the system of maps $\mu^{\bullet\bullet}$ adds up to a map $\mu : B \otimes_A B \to B$, and one requires that the resulting pair (B, μ) is an associative unital $\mathbb{Z}$-graded A-algebra. Then, one sets $a \cdot b := \mu(a \otimes b)$, for every $a, b \in B$.

(b) We have the identities

$$d_B^{p+q}(a \cdot b) = d_B^p(a) \cdot b + (-1)^p \cdot a \cdot d_B^q(b) \qquad \text{for every } p, q \in \mathbb{Z} \text{ and every } a \in B^p, b \in B^q.$$

We call B the *graded A-algebra associated to $B^\bullet$*. Then we also say that $B^\bullet$ is *anti-commutative* (resp. *strictly anti-commutative*) if the same holds for B (definition 7.7.1).

(ii) A *morphism $B^\bullet \to C^\bullet$ of differential graded A-algebras* is a map of complexes of A-modules such that the induced map of associated graded A-modules is a map of A-algebras. We denote the resulting category of differential graded A-algebras by :

$$A\text{-}\mathsf{dgAlg}.$$

We shall also consider the full subcategory of A-dgAlg denoted

$$A\text{-}\mathsf{dgAlt}$$

whose objects are the strictly anti-commutative differential graded A-algebras.

(iii) Let $(B^\bullet, d_B^\bullet)$ be a differential graded A-algebra, B its associated graded A-algebra, and $(M^\bullet, d_M^\bullet)$ a complex of A-modules.

(a) We say that $M^\bullet$ is a *left $B^\bullet$-module* if the A-module $M := \bigoplus_{p \in \mathbb{Z}} M^p$ is a graded left B-module (for the natural $\mathbb{Z}$-grading on M), and we have

$$d_M^{p+q}(b \cdot m) = (d_B^p b) \cdot m + (-1)^p \cdot b \cdot d_M^q(m)$$

for every $p, q \in \mathbb{Z}$, every $b \in B^p$, and every $m \in M^q$.

(b) We say that $M^\bullet$ is a *right $B^\bullet$-module* if M is a graded right B-module, and we have

$$d_M^{p+q}(m \cdot a) = d_M^q(m) \cdot a + (-1)^q \cdot m \cdot d_M^p(a)$$

for every $p, q \in \mathbb{Z}$, every $b \in B^p$, and every $m \in M^q$.

(c) We say that $M^\bullet$ is a *B-bimodule* if it is both a left and right $B^\bullet$-module, and with these $B^\bullet$-modules structures, the A-module M becomes a B-bimodule (*i.e.* the left multiplication commutes with the right multiplication).

We call M the *graded B-module associated to $M^\bullet$*. A morphism $M^\bullet \to N^\bullet$ of left (resp. right, resp. bi-) $B^\bullet$-modules is a map of complexes of A-modules, such that the induced map of associated graded A-modules is a map of left (resp. right, resp. bi-) B-modules.

Remark 7.7.4. Let $B^\bullet$ be any differential graded A-algebra.

(i) Notice that condition (b) of definition 7.7.3(i) is the same as saying that $\mu^{\bullet\bullet}$ induces a map of complexes

$$\mu^\bullet : B^\bullet \otimes_A B^\bullet \to B^\bullet.$$

Moreover, in light of example 7.1.16(i), we see that $B^\bullet$ is anti-commutative if and only if the diagram

$$\begin{array}{ccc} B^\bullet \otimes_A B^\bullet & \xrightarrow{\ \sim\ } & B^\bullet \otimes_A B^\bullet \\ & {\scriptstyle \mu^\bullet} \searrow \quad \swarrow {\scriptstyle \mu^\bullet} & \\ & B^\bullet & \end{array}$$

commutes, where the horizontal arrow is the isomorphism (7.1.17) that swaps the two tensor factors. Hence, in some sense this is actually a commutativity condition.

(ii) Likewise, if $M^\bullet$ is a complex of A-modules with a graded left (right) B-module structure on the associated graded A-module M, then $M^\bullet$ is a left (resp. right) $B^\bullet$-module if and only if the scalar multiplication of the B-module M induces a map of complexes

$$B^\bullet \otimes_A M^\bullet \to M^\bullet \qquad (\text{ resp. } M^\bullet \otimes_A B^\bullet \to M^\bullet \text{ }).$$

(iii) It is easily seen that the multiplication maps μ^{pq} induce A-linear maps

$$(H^p B^\bullet) \otimes_A (H^q B^\bullet) \to H^{p+q} B^\bullet \qquad \text{for every } p, q \in \mathbb{Z}$$

anf if we let $H^\bullet B^\bullet := \bigoplus_{p\in\mathbb{Z}} H^p B^\bullet$, then the resulting map

$$(H^\bullet B^\bullet) \otimes_A (H^\bullet B^\bullet) \to H^\bullet B^\bullet$$

endows $H^\bullet B^\bullet$ with a structure of $\mathbb{Z}$-graded associative unital A-algebra, which shall be strictly anti-commutative whenever the same holds for $B^\bullet$. Likewise, if $M^\bullet$ is any left (resp. right, resp. bi) $B^\bullet$-module, then $H^\bullet M^\bullet$ is naturally a $\mathbb{Z}$-graded left (resp. right, resp. bi) $H^\bullet B^\bullet$-module.

(iv) Let $M^\bullet$ be a left $B^\bullet$-module, and denote by $\mu_M^{pq} : B^p \otimes_A M^q \to M^{p+q}$ the (p,q)-graded component of the scalar multiplication of $M^\bullet$, for every $p, q \in \mathbb{Z}$. Then $M^\bullet[1]$ is also naturally a left $B^\bullet$-module, with scalar multiplication $\mu_{M[1]}^{\bullet\bullet}$ given by the rule :

$$\mu_{M[1]}^{pq} := (-1)^p \cdot \mu_M^{p,q+1} \qquad \text{for every } p, q \in \mathbb{Z}.$$

Likewise, if $N^\bullet$ is a right $B^\bullet$-module, with scalar multiplication $\mu_N^{\bullet\bullet}$, then $N^\bullet[1]$ is naturally a right $B^\bullet$-module, with multiplication $\mu_{N[1]}^{\bullet\bullet}$ given by the rule :

$$\mu_{N[1]}^{pq} := \mu_N^{p,q+1} \qquad \text{for every } p, q \in \mathbb{Z}.$$

Lastly, if $P^\bullet$ is a $B^\bullet$-bimodule, then the left and right $B^\bullet$-module structure defined above on $P^\bullet[1]$, amount to a natural B-bimodule structure on $P^\bullet[1]$.

(v) Let $C^\bullet$ be any $\mathbb{Z}$-graded A-algebra, $N^\bullet$ any $\mathbb{Z}$-graded left (resp. right, resp. bi-) B-module. We let $N[1]^\bullet$ be the graded A-module given by the rule $N[1]^p := N^{p+1}$ for every $p \in \mathbb{Z}$. We shall always view $N[1]^\bullet$ as a left (resp. right, resp. bi-) B-module, via the scalar multiplications obtained from those of $N^\bullet$, following the rules spelled out in (iv). This ensures that the functor from $B^\bullet$-modules to $H^\bullet B^\bullet$-modules that assigns to any $B^\bullet$-module $M^\bullet$ its homology $H^\bullet M^\bullet$, is compatible with shift operators.

7.7.5. Let A be ring, $(B^\bullet, d_B^\bullet)$ any differential graded A-algebra, and $I^\bullet \subset B^\bullet$ a (graded) two-sided ideal of $B^\bullet$ (*i.e.* a bi-submodule of the $B^\bullet$-bimodule $B^\bullet$). Let

$$\partial : H^\bullet(B^\bullet/I^\bullet) \to H^\bullet I^\bullet[1]$$

denote the natural map induced by the short exact sequence of complexes

$$0 \to I^\bullet \to B^\bullet \to B^\bullet/I^\bullet \to 0.$$

We have :

Lemma 7.7.6. *In the situation of* (7.7.5)*, the map ∂ is an A-linear graded derivation of the graded A-algebra $H^\bullet(B^\bullet/I^\bullet)$.*

Proof. Indeed, let a and b be any two cycles of the complex $B^\bullet/I^\bullet$ in degree p and q, and $\overline{a}$, $\overline{b}$ the respective classes; lift a and b to some elements $\widetilde{a} \in B^p$ and $\widetilde{b} \in B^q$, so that $\partial(\overline{a}\cdot\overline{b})$ is the class in $H^{p+q+1}I^\bullet$ of $d_B(\widetilde{a}\cdot\widetilde{b}) = d_B(\widetilde{a})\cdot\widetilde{b} + (-1)^p\cdot\widetilde{a}\cdot d_B(\widetilde{b})$. Since $d_B(\widetilde{a})$ (resp. $d_B(\widetilde{b})$) represents the class of $\partial(\overline{a})$ (resp. of $\partial(\overline{b})$), the assertion follows from the explicit description of the bimodule structure on $I^\bullet[1]$ provided by remark 7.7.4(iv). □

Remark 7.7.7. Let $B^\bullet$ and $C^\bullet$ be two differential graded A-algebras, and denote by B and C the respective associated graded A-algebras. As in remark 7.7.2(ii), we may endow $B\otimes_A C$ with a natural structure of graded A-algebra. In terms of morphisms of complexes, the multiplication law of $B \otimes_A C$ then corresponds to the composition

$$(B^\bullet \otimes_A C^\bullet) \otimes_A (B^\bullet \otimes_A C^\bullet) \xrightarrow{\sim} (B^\bullet \otimes_A B^\bullet) \otimes_A (C^\bullet \otimes_A C^\bullet) \xrightarrow{\mu_B^\bullet \otimes_A \mu_C^\bullet} B^\bullet \otimes_A C^\bullet$$

where the first isomorphism is obtained by composing the associativity isomorphisms of example 7.1.16(ii) and the isomorphisms (7.1.17) that swap the tensor factors. Here $\mu_B^\bullet$ and $\mu_C^\bullet$ are the multiplication maps of $B^\bullet$ and $C^\bullet$. Thus, we obtain on the complex $B^\bullet \otimes_A C^\bullet$ a natural structure of differential graded A-algebra, and taking into account remark 7.7.2(ii) it is easily seen that if $B^\bullet$ and $C^\bullet$ are strictly anti-commutative, then $B^\bullet \otimes_A C^\bullet$ represents the coproduct of $B^\bullet$ and $C^\bullet$ in A-dgAlt.

7.7.8. Suppose now that both $B^\bullet$ and $C^\bullet$ are bounded above complexes; presumably, in this case there is a canonical way to define a differential graded algebra structure as well on (suitable representatives for)

$$D^\bullet := B^\bullet \overset{\mathbf{L}}{\otimes}_A C^\bullet$$

in such a way that this structure is well defined as an object of the derived category of A-dgAlg (the latter should be, as usual, the localization of A-dgAlg, by the multiplicative system of quasi-isomorphisms). More modestly, we shall endow the graded A-module $H_\bullet D^\bullet$ with a natural structure of associative graded A-algebra, in such a way that the natural map

$$(7.7.9) \qquad H_\bullet D^\bullet := \bigoplus_{i \in \mathbb{Z}} \mathrm{Tor}_i^A(B^\bullet, C^\bullet) \to H_\bullet(B^\bullet \otimes_A C^\bullet)$$

is a morphism of graded A-algebras. The multiplication of $H^\bullet D^\bullet$ is defined as the composition

$$H_i D^\bullet \otimes_A H_j D^\bullet \xrightarrow{\alpha} \mathrm{Tor}_{i+j}^A(B^\bullet \otimes_A B^\bullet, C^\bullet \otimes_A C^\bullet) \xrightarrow{\mu} H_{i+j} D^\bullet \qquad \text{for every } i, j \in \mathbb{Z}$$

where α is the bilinear pairing provided by (7.3.40), and with $\mu := \mathrm{Tor}_{i+j}^A(\mu_B^\bullet, \mu_C^\bullet)$.

Let us check first that the foregoing rule does define an associative multiplication on $H_\bullet D^\bullet$. To this aim, set $B_2^\bullet := B^\bullet \otimes_A B^\bullet$, $B_3^\bullet := B^\bullet \otimes_A B_2^\bullet$ and define likewise $C_2^\bullet$ and $C_3^\bullet$; a little diagram chase, together with (7.3.41), reduces to verifying the commutativity of the diagram

$$\begin{array}{ccccc}
\mathrm{Tor}_i^A(B_2^\bullet, C_2^\bullet) \otimes_A H_j D^\bullet & \xrightarrow{\mu \otimes_A \mathbf{1}_{H_i D^\bullet}} & H_i D^\bullet \otimes_A H_j D^\bullet & \xleftarrow{\mathbf{1}_{H_j D^\bullet} \otimes_A \mu} & H_i D^\bullet \otimes_A \mathrm{Tor}_j^A(B_2^\bullet, C_2^\bullet) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Tor}_{i+j}^A(B_3^\bullet, C_3^\bullet) & \xrightarrow{\gamma} & \mathrm{Tor}_{i+j}^A(B_2^\bullet, C_2^\bullet) & \xleftarrow{\delta} & \mathrm{Tor}_{i+j}^A(B_3^\bullet, C_3^\bullet)
\end{array}$$

whose vertical arrows are the bilinear pairings of (7.3.40), and with

$$\gamma := \mathrm{Tor}_{i+j}^A(\mu_B^\bullet \otimes_A \mathbf{1}_{B^\bullet}, \mu_C^\bullet \otimes_A \mathbf{1}_{C^\bullet}) \qquad \delta := \mathrm{Tor}_{i+j}^A(\mathbf{1}_{B^\bullet} \otimes_A \mu_B^\bullet, \mathbf{1}_{C^\bullet} \otimes_A \mu_C^\bullet).$$

We show the commutativity of the left subdiagram; the same argument applies to the right one. Unwinding the definitions, we come down to checking the commutativity, in $\mathsf{D}(A\text{-}\mathbf{Mod})$, of the diagram of complexes

$$\begin{array}{ccccc}
(P_{B_2}^\bullet \otimes_A C_2^\bullet) \otimes_A (P_B^\bullet \otimes_A C^\bullet) & \xrightarrow{\sim} & (P_{B_2}^\bullet \otimes_A P_B^\bullet) \otimes_A C_3^\bullet & \xrightarrow{\varphi_{12,3}^\bullet \otimes_A \mathbf{1}_{C_3^\bullet}} & P_{B_3}^\bullet \otimes_A C_3^\bullet \\
{\scriptstyle \vartheta^\bullet}\downarrow & & & & \downarrow{\scriptstyle \eta^\bullet} \\
(P_B^\bullet \otimes_A C^\bullet) \otimes_A (P_B^\bullet \otimes_A C^\bullet) & \xrightarrow{\sim} & (P_B^\bullet \otimes_A P_B^\bullet) \otimes_A C_2^\bullet & \xrightarrow{\varphi_{1,23}^\bullet \otimes_A \mathbf{1}_{C_3^\bullet}} & P_{B_2}^\bullet \otimes_A C_2^\bullet
\end{array}$$

(notation of (7.3.38)), with $\vartheta^\bullet := (P_{\mu_B}^\bullet \otimes_A \mu_C^\bullet) \otimes_A \mathbf{1}_{P_B^\bullet \otimes_A C^\bullet}$ and $\eta^\bullet := P_{\mu_B \otimes_A \mathbf{1}_B}^\bullet \otimes_A (\mu_C^\bullet \otimes_A \mathbf{1}_{C^\bullet})$, and where $\varphi_{12,3}^\bullet$ and $\varphi_{1,23}^\bullet$ are as in (7.3.40) (and with the isomorphisms given by compositions of associativity and swapping isomorphisms). A direct inspection shows that this diagram commutes up to homotopy, as required.

Next, we check that (7.7.9) is a map of graded A-algebras. Unwinding the definitions, we see that – with the notation of (7.3.40) – the multiplication of $H_\bullet D^\bullet$ is the map on homology induced by the composition

$$(P_B^\bullet \otimes_A C^\bullet) \otimes_A (P_B^\bullet \otimes_A C^\bullet) \xrightarrow{\sim} (P_B^\bullet \otimes_A P_B^\bullet) \otimes_A C_2^\bullet \to P_{B_2}^\bullet \otimes_A C^\bullet \to P_B^\bullet \otimes_A C$$

where the first isomorphism is again a composition of associativity isomorphisms and isomorphisms that swap the factors; the second map is $\varphi_{12}^\bullet \otimes_A \mu_C^\bullet$, where $\varphi_{12}^\bullet : P_B^\bullet \otimes_A P_B^\bullet \to P_{B_2}^\bullet$ is defined as in (7.3.40). The last map is $P_{\mu_B}^\bullet \otimes_A \mathbf{1}_{C^\bullet}$, where $P_{\mu_B}^\bullet$ is given by (7.3.38). Since the associativity and swapping isomorphisms are obviously natural in all their arguments, we come down to checking the commutativity, in $\mathsf{D}(A\text{-}\mathbf{Mod})$, of the diagram of complexes

$$\begin{array}{ccccc}
(P^\bullet_B \otimes_A P^\bullet_B) \otimes_A C^\bullet_2 & & \xrightarrow{\varphi^\bullet_{12}\otimes_A \mu^\bullet_C} & & P^\bullet_{B_2} \otimes_A C^\bullet \\
\Big\downarrow {\scriptstyle (\rho^\bullet_B\otimes_A\rho^\bullet_B)\otimes_A \mathbf{1}_{C^\bullet_2}} & & & & \Big\downarrow {\scriptstyle P^\bullet_{\mu_B}\otimes_A \mathbf{1}_{C^\bullet}} \\
B^\bullet_2 \otimes_A C^\bullet_2 & \xrightarrow{\mu^\bullet_B\otimes_A\mu^\bullet_C} & B^\bullet \otimes_A C^\bullet & \xleftarrow{\rho^\bullet_B\otimes_A \mathbf{1}_{C^\bullet}} & P^\bullet_B \otimes_A C^\bullet.
\end{array}$$

The latter is further reduced to the commutativity of

$$\begin{array}{ccccc}
P^\bullet_B \otimes_A P^\bullet_B & & \xrightarrow{\varphi^\bullet_{12}} & & P^\bullet_{B_2} \\
\Big\downarrow {\scriptstyle \rho^\bullet_B\otimes_A\rho^\bullet_B} & & & & \Big\downarrow {\scriptstyle P^\bullet_{\mu_B}} \\
B^\bullet_2 & \xrightarrow{\mu^\bullet_B} & B^\bullet & \xleftarrow{\rho^\bullet_B} & P^\bullet_B.
\end{array}$$

But a simple inspection shows that this diagram indeed commutes up to homotopy.

Lastly, we claim that if $B^\bullet$ and $C^\bullet$ are both anti-commutative, then the same holds for $H_\bullet D^\bullet$. Indeed, consider the diagram

$$\begin{tikzcd}
(P^\bullet_B \otimes_A P^\bullet_B) \otimes_A C^\bullet_2 \arrow[rrrr, "\sim"] \arrow[dr, "(\rho^\bullet_B\otimes_A\rho^\bullet_B)\otimes_A\mathbf{1}_{C^\bullet_2}"] \arrow[ddd, "\varphi^\bullet_{12}\otimes_A\mu^\bullet_C"'] & & & & (P^\bullet_B \otimes_A P^\bullet_B) \otimes_A C^\bullet_2 \arrow[dl, "(\rho^\bullet_B\otimes_A\rho^\bullet_B)\otimes_A\mathbf{1}_{C^\bullet_2}"'] \arrow[ddd, "\varphi^\bullet_{12}\otimes_A\mu^\bullet_C"] \\
& B^\bullet_2 \otimes_A C^\bullet_2 \arrow[rr, "\sim"] \arrow[dr, "\mu^\bullet_B\otimes_A\mu^\bullet_C"'] & & B^\bullet_2 \otimes_A C^\bullet_2 \arrow[dl, "\mu^\bullet_B\otimes_A\mu^\bullet_C"] & \\
& & B^\bullet \otimes_A C^\bullet & & \\
P^\bullet_{B_2} \otimes_A C^\bullet \arrow[rr, "P^\bullet_{\mu_B}\otimes_A\mathbf{1}_{C^\bullet}"] & & P^\bullet_B \otimes_A C^\bullet \arrow[u, "\rho^\bullet_B\otimes_A\mathbf{1}_{C^\bullet}"'] & & P^\bullet_{B_2} \otimes_A C^\bullet. \arrow[ll, "P^\bullet_{\mu_B}\otimes_A\mathbf{1}_{C^\bullet}"']
\end{tikzcd}$$

(The upper isomorphism is obtained from the automorphism of $P^\bullet_B \otimes_A P^\bullet_B$ that swaps the two factors, and from the automorphism of $C^\bullet_2$ of the same type; likewise for the lower isomorphism.) The assumption on $B^\bullet$ and $C^\bullet$ says that the inner triangular subdiagram commutes, and the same holds – up to homotopy – for the upper and the left and right subdiagrams, by a simple inspection. Then also the outer rectangular subdiagram commutes up to homotopy, and the contention follows easily.

Remark 7.7.10. (i) Simplicial A-algebras are an important source of differential graded algebras, thanks to the following construction. Let R be any simplicial A-algebra, $R_\bullet$ the associated chain complex of A-modules, and denote by $\mu_R : R \otimes_A R \to R$ the multiplication map of R. By considering the shuffle map for the bisimplicial A-module $R \boxtimes_A R$ (notation as in (7.4.55)), we deduce a natural map of complexes

$$\mu_{R_\bullet} : R_\bullet \otimes_A R_\bullet \xrightarrow{\mathsf{Sh}^{R\boxtimes_A R}_\bullet} (R \otimes_A R)_\bullet \xrightarrow{(\mu_R)_\bullet} R_\bullet$$

and taking into account propositions 7.4.56 and 7.4.59, it is easily seen that $(R_\bullet, \mu_{R_\bullet})$ is a strictly anti-commutative differential graded algebra. By remark 7.7.4(iii), we deduce that the graded A-module $H_\bullet R := \bigoplus_{p\in\mathbb{N}} H_p R$ is naturally an $\mathbb{N}$-graded associative unital and strictly anti-commutative A-algebra.

(ii) Likewise, if M is any R-module, then we obtain on the associated chain complex $M_\bullet$ a natural structure of $R_\bullet$-bimodule, so that $H_\bullet M$ is naturally a graded $H_\bullet R$-bimodule.

(iii) Clearly, a morphism $\varphi : R \to S$ of simplicial A-algebras induces a morphism $\varphi_\bullet : R_\bullet \to S_\bullet$ of differential graded A-algebras, and a morphism $f : M \to N$ of R-modules induces a morphism $\varphi_\bullet : M_\bullet \to N_\bullet$ of $R_\bullet$-bimodules.

7.8. **Koszul algebras and regular sequences.** For any ring A, let us consider the category

$$A\text{-}\mathsf{dAlg.Mod}$$

whose objects are the triples (B, M, ∂), where B is an A-algebra, M is a B-module, and $\partial : M \to B$ is an A-linear map. The morphisms $(B, M, \partial) \to (B', M', \partial')$ in A-$\mathsf{dAlg.Mod}$ are the pairs (f, g) where $f : B \to B'$ is a map of A-algebras, and $g : B' \otimes_B M \to M'$ is a B'-linear map, such that

$$f \circ \partial(m) = \partial' \circ g(1 \otimes m) \qquad \text{for every } m \in M$$

with the obvious composition law for such morphisms. We have a natural functor :

$$A\text{-}\mathsf{dgAlt} \to A\text{-}\mathsf{dAlg.Mod} \qquad (B^\bullet, d^\bullet_B) \mapsto (\mathrm{Ker}\, d^0_B, B^{-1}, d^{-1}_B)$$

which admits a left adjoint

$$\mathbf{K}_\bullet : A\text{-}\mathsf{dAlg.Mod} \to A\text{-}\mathsf{dgAlt} \qquad (B, M, \partial) \mapsto \mathbf{K}_\bullet(B, M, \partial)$$

that associates to every object (B, M, ∂) its *Koszul algebra* $\mathbf{K}_\bullet(B, M, \partial)$, whose underlying strictly anti-commutative A-algebra is the exterior algebra $\Lambda^\bullet_B M$, as in remark 7.7.2(i). The differentials of the complex $\mathbf{K}_\bullet(B, M, \partial)$ are given by the maps

$$d_{n+1} : \Lambda^{n+1}_B M \to \Lambda^n_B M \qquad x_0 \wedge \cdots \wedge x_n \mapsto \sum_{i=0}^{n} (-1)^i \partial(x_i) \cdot x_0 \wedge \cdots \wedge x_{i-1} \wedge x_{i+1} \wedge \cdots \wedge x_n$$

for every $n \in \mathbb{N}$. The detailed verifications shall be left to the reader.

Remark 7.8.1. (i) In light of remark 7.7.2(iii), it is easily seen that the coproduct of any two objects (B, M, ∂) and (B', M', ∂') of A-$\mathsf{dAlg.Mod}$ is representable by the object

$$(B'' := B \otimes_A B', M'' := (M \otimes_A B') \oplus (B \otimes_A M'), \partial'')$$

where $\partial'' : M'' \to B''$ is the A-linear map such that

$$\partial''(m \otimes b', b \otimes m') := \partial(m) \otimes b' + b \otimes \partial'(m') \qquad \text{for every } m \in M, m' \in M', b \in B, b' \in B'.$$

Combining with remark 7.7.7 and proposition 1.3.25(iv), we deduce a natural isomorphism :

$$\mathbf{K}_\bullet(B'', M'', \partial'') \xrightarrow{\sim} \mathbf{K}_\bullet(B, M, \partial) \otimes_A \mathbf{K}_\bullet(B', M', \partial') \qquad \text{in } A\text{-}\mathsf{dgAlt}.$$

(ii) Let $\mathbf{f} := (f_1, \dots, f_r)$ be a finite sequence of elements of A, and $I \subset A$ the ideal generated by $\mathbf{f}$; let also $\partial_\mathbf{f} : A^{\oplus r} \to A$ be the A-linear form such that $\partial_\mathbf{f}(a_1, \dots, a_r) := \sum_{i=1}^r f_i a_i$ for every $a_1, \dots, a_r \in A$. The *Koszul complex of the sequence* $\mathbf{f}$ is the complex :

$$\mathbf{K}_\bullet(\mathbf{f}) := \mathbf{K}_\bullet(A, A^{\oplus r}, \partial_\mathbf{f})$$

(see [61, Ch.III, §1.1]). Thus, for $r = 1$, so that $\mathbf{f} = (f)$ for a single element $f \in A$, we have

$$\mathbf{K}_\bullet(f) = (0 \to A \xrightarrow{f} A \to 0)$$

concentrated in homological degrees 0 and 1. In the general case, combining with (i) we get a natural isomorphism :

$$\mathbf{K}_\bullet(\mathbf{f}) \xrightarrow{\sim} \mathbf{K}_\bullet(f_1) \otimes_A \cdots \otimes_A \mathbf{K}_\bullet(f_r) \qquad \text{in } A\text{-}\mathsf{dgAlt}.$$

(iii) For every complex of A-modules $M^\bullet$, we also use the customary notation :

$$\mathbf{K}_\bullet(\mathbf{f}, M^\bullet) := M^\bullet \otimes_A \mathbf{K}_\bullet(\mathbf{f}) \qquad \mathbf{K}^\bullet(\mathbf{f}, M^\bullet) := \mathrm{Tot}^\bullet(\mathrm{Hom}^\bullet_A(\mathbf{K}_\bullet(\mathbf{f}), M^\bullet))$$

and denote by $H_\bullet(\mathbf{f}, M^\bullet)$ (resp. $H^\bullet(\mathbf{f}, M^\bullet)$) the homology of $\mathbf{K}_\bullet(\mathbf{f}, M^\bullet)$ (resp. the cohomology of $\mathbf{K}^\bullet(\mathbf{f}, M^\bullet)$). Especially, if M is any A-module :

$$H_0(\mathbf{f}, M) = M/IM \qquad H^0(\mathbf{f}, M) = \mathrm{Hom}_A(A/I, M) = \mathrm{Ann}_M(I)$$

(where, as usual, we regard M as a complex placed in degree 0).

Lemma 7.8.2. *With the notation of remark* 7.8.1(iii), *let $g : A \to B$ be a ring homomorphism, and $M^\bullet$ (resp. $N^\bullet$) a complex of A-modules (resp. B-modules). The following holds :*

(i) *For every $x \in I$, scalar multiplication by x induces the zero endomorphism on the objects $\mathbf{K}_\bullet(\mathbf{f}, M^\bullet)$ and $\mathbf{K}^\bullet(\mathbf{f}, M^\bullet)$ of* $\mathsf{Hot}(A\text{-}\mathbf{Mod})$.

(ii) *Especially, $H_i(\mathbf{f}, M^\bullet)$ and $H^i(\mathbf{f}, M^\bullet)$ are A/I-modules, for every $i \in \mathbb{Z}$.*

(iii) *If $I = A$, the complexes $\mathbf{K}_\bullet(\mathbf{f}, M^\bullet)$ and $\mathbf{K}^\bullet(\mathbf{f}, M^\bullet)$ are homotopically trivial.*

(iv) *Denote by $g(\mathbf{f})$ the image in B of the sequence* $\mathbf{f}$. *Then we have natural identifications*

$$\mathbf{K}_\bullet(\mathbf{f}, N^\bullet) \xrightarrow{\sim} \mathbf{K}_\bullet(g(\mathbf{f}), N^\bullet) \qquad \mathbf{K}^\bullet(\mathbf{f}, N^\bullet) \xrightarrow{\sim} \mathbf{K}^\bullet(g(\mathbf{f}), N^\bullet) \qquad \textit{in } \mathsf{C}(B\text{-}\mathbf{Mod}).$$

(v) *We have a natural isomorphism*

$$\mathbf{K}^\bullet(\mathbf{f}, M^\bullet) \xrightarrow{\sim} \mathbf{K}_\bullet(\mathbf{f}, M^\bullet)[-r] \qquad \textit{in } \mathsf{C}(A\text{-}\mathbf{Mod}).$$

Proof. (i): It suffices to notice that scalar multiplication by f induces the zero endomorphism of $\mathbf{K}_\bullet(f)$, for every $f \in A$: indeed, a homotopy from $f \cdot \mathbf{1}_{\mathbf{K}_\bullet(f)}$ to the zero endomorphism is given by the system of maps $(s_n \mid n \in \mathbb{N})$ with $s_n := 0$ for $n \neq 0$, and $s_0 := \mathbf{1}_A$.

(ii) and (iii) are immediate consequences of (i), and (iv) follows directly from the definitions.

(v): Recall that for every free A-module L of finite rank and every A-module M we have a natural isomorphism of A-modules

$$\mathrm{Hom}_A(L, M) \xrightarrow{\sim} L^\vee \otimes_A M \qquad \text{where } L^\vee := \mathrm{Hom}_A(L, A).$$

There follows, for every complex $L_\bullet$ of free A-modules of finite rank, a natural isomorphism

$$\mathrm{Hom}^\bullet_A(L_\bullet, M^\bullet) \xrightarrow{\sim} L^\vee_\bullet \boxtimes_A M^\bullet \qquad \text{in } \mathsf{C}_2(A\text{-}\mathbf{Mod})$$

(notation of example 7.1.16(i)). Moreover, for any two free A-modules of finite rank L_1 and L_2 we have as well a natural isomorphism

$$(L_1 \otimes_A L_2)^\vee \xrightarrow{\sim} L_1^\vee \otimes_A L_2^\vee$$

whence, for any complexes $L_{1,\bullet}$ and $L_{2,\bullet}$ of free A-modules of finite rank, a natural isomorphism

$$(L_{1,\bullet} \boxtimes_A L_{2,\bullet})^\vee \xrightarrow{\sim} L^\vee_{1,\bullet} \boxtimes_A L^\vee_{2,\bullet} \qquad \text{in } \mathsf{C}_2(A\text{-}\mathbf{Mod}).$$

Summing up, it then suffices to observe that there is a natural isomorphism

$$\mathbf{K}_\bullet(f) \xrightarrow{\sim} \mathbf{K}_\bullet(f)^\vee[1] \qquad \text{in } \mathsf{C}(A\text{-}\mathbf{Mod})$$

for every $f \in A$. $\square$

7.8.3. Let A be a ring, $\mathbf{f} := (f_1, \dots, f_r)$ a sequence of elements of A, and $\mathbf{f}' := (f_1, \dots, f_{r-1})$; let also $I \subset A$ (resp. $I' \subset A$) be the ideal generated by the sequence $\mathbf{f}$ (resp. by $\mathbf{f}'$). We have a short exact sequence of complexes : $0 \to A[0] \to \mathbf{K}_\bullet(f_r) \to A[1] \to 0$, and after tensoring with $\mathbf{K}_\bullet(\mathbf{f}')$, in view of remark 7.8.1(ii) we get a distinguished triangle :

$$\mathbf{K}_\bullet(\mathbf{f}') \to \mathbf{K}_\bullet(\mathbf{f}) \to \mathbf{K}_\bullet(\mathbf{f}')[1] \xrightarrow{\partial} \mathbf{K}_\bullet(\mathbf{f}')[1].$$

By inspecting the definitions one checks easily that the boundary map ∂ is induced by multiplication by f_r. There follow exact sequences :

$$(7.8.4) \qquad 0 \to H_0(f_r, H_p(\mathbf{f}', M)) \to H_p(\mathbf{f}, M) \to H^0(f_r, H_{p-1}(\mathbf{f}', M)) \to 0$$

for every A-module M and for every $p \in \mathbb{N}$, and together with remark 7.8.1(iii), we deduce :

Lemma 7.8.5. *With the notation of* (7.8.3)*, the following conditions are equivalent :*

(a) $H_i(\mathbf{f}, M) = 0$ *for every* $i > 0$.

(b) *The scalar multiplication by* f_r *is a bijection on* $H_i(\mathbf{f}', M)$ *for every* $i > 0$*, and is an injection on* $M/I'M$. □

Definition 7.8.6. We say that the sequence $\mathbf{f} := (f_1, \dots, f_r)$ of elements of A is *completely secant* on the A-module M, if we have $H_i(\mathbf{f}, M) = 0$ for every $i > 0$.

The interest of definition 7.8.6 is due to its relation to the notion of *regular sequence* of elements of A (see *e.g.* [37, §9, n.6]). Namely, we have the following criterion :

Proposition 7.8.7. *With the notation of* (7.8.3)*, the following conditions are equivalent :*

(a) *The sequence* $\mathbf{f}$ *is* M*-regular.*

(b) *For every* $j \leq r$*, the sequence* $(f_1, \dots, f_j)$ *is completely secant on* M.

Proof. Lemma 7.8.5 shows that (b) implies (a). Conversely, suppose that (a) holds; we show that (b) holds, by induction on r. If $r = 0$, there is nothing to prove. Assume that the assertion is already known for all $j < r$. Since $\mathbf{f}$ is M-regular by assumption, the same holds for the subsequence $\mathbf{f}' := (f_1, \dots, f_{r-1})$, and f_r is regular on $M/(\mathbf{f}')M$. Hence $H_p(\mathbf{f}', M) = 0$ for every $p > 0$, by inductive assumption. Then lemma 7.8.5 shows that $H_p(\mathbf{f}, M) = 0$ for every $p > 0$, as claimed. □

Notice that any permutation of a completely secant sequence is again completely secant, whereas a permutation of a regular sequence is not always regular. As an application of the foregoing, we point out the following :

Corollary 7.8.8. *With the notation of* (7.8.3)*, the following holds :*

(i) *If a sequence* (f, g) *of elements of* A *is* M*-regular, and* M *is* f*-adically separated, then* (g, f) *is* M*-regular.*

(ii) *If* $(n_1, \dots, n_r)$ *is any sequence of strictly positive integers, then* $\mathbf{f}$ *is completely secant on* M *(resp.* M*-regular) if and only if the same holds for the sequence* $(f_1^{n_1}, \dots, f_r^{n_r})$.

Proof. (i): According to proposition 7.8.7, we only need to show that the sequence (g) is completely secant, *i.e.* that g is regular on M. Hence, suppose that $gm = 0$ for some $m \in M$; it suffices to show that $m \in f^n M$ for every $n \in \mathbb{N}$. We argue by induction on n. By assumption g is regular on M/fM, hence $m \in fM$, which shows the claim for $n = 1$. Let $n > 1$, and suppose we already know that $m = f^{n-1}m'$ for some $m' \in M$. Hence $0 = gf^{n-1}m'$, so $gm' = 0$ and the foregoing case shows that $m' = fm''$ for some $m'' \in M$, thus $m = f^n m''$, as required.

(ii): We deal first with the assertion for completely secant sequences. First, notice that, since every permutation of a completely secant sequence is still completely secant, it suffices to show the assertion for the sequence of integers $(1, 1, \dots, 1, n_r)$. However, set $\mathbf{f}' := (f_1, \dots, f_{r-1})$; by lemma 7.8.5, $\mathbf{f}$ is completely secant if and only if scalar multiplication by f_r is a bijection on $H_i(\mathbf{f}', M)$ for every $i > 0$, and an injection on $M/(\mathbf{f}')M$. But f_r fulfills the latter conditions if and only if $f_r^{n_r}$ does, whence the contention. The assertion for M-regular sequences follows from the foregoing, in view of proposition 7.8.7. □

7.8.9. Let A be a ring, $\mathbf{f} := (f_1, \dots, f_r)$ a completely secant sequence of elements of A (see definition 7.8.6); denote by J the ideal generated by $\mathbf{f}$, and set $A_0 := A/J$. We may regard the complex $A_0[0]$ as an (especially simple) differential graded A-algebra, with multiplication $\overline{\mu}^\bullet$ deduced from that of A_0, in the obvious way. We may then state :

Proposition 7.8.10. *In the situation of* (7.8.9)*, the* A_0*-module* J/J^2 *is free of rank* r*, and there exists a unique isomorphism of strictly anti-commutative graded* A*-algebras*

$$\Lambda^\bullet_{A_0}(J/J^2) \xrightarrow{\sim} H_\bullet(A_0[0] \overset{\mathbf{L}}{\otimes}_A A_0[0]) \tag{7.8.11}$$

which restricts, in degree 1*, to the natural identification*

$$\Lambda^1_{A_0}(J/J^2) = J/J^2 \xrightarrow{\sim} \mathrm{Tor}^A_1(A_0, A_0).$$

Proof. By assumption, the Koszul complex, with its natural augmentation, yields a resolution

$$\varepsilon^\bullet : \mathbf{K}_\bullet(\mathbf{f}) \to A_0[0]$$

by free A-modules. Hence, there is a unique isomorphism $\omega^\bullet : P^\bullet_{A_0} \xrightarrow{\sim} \mathbf{K}_\bullet(\mathbf{f})$ in $\mathsf{D}(A\text{-}\mathbf{Mod})$, whose composition with $\varepsilon^\bullet$ agrees with $\rho^\bullet_{A_0}$ (notation of (7.3.38)). Then $\varepsilon^\bullet$ is a map of differential graded A-algebras, and we easily deduce a commutative diagram in $\mathsf{D}(A\text{-}\mathbf{Mod})$

$$\begin{array}{ccc} P^\bullet_{A_0} \otimes_A P^\bullet_{A_0} & \xrightarrow{\ P^\bullet_{\overline{\mu}}\ } & P^\bullet_{A_0} \\ {\scriptstyle \omega^\bullet \otimes_A \omega^\bullet}\downarrow & & \downarrow{\scriptstyle \omega^\bullet} \\ \mathbf{K}_\bullet(\mathbf{f}) \otimes_A \mathbf{K}_\bullet(\mathbf{f}) & \longrightarrow & \mathbf{K}_\bullet(\mathbf{f}) \end{array}$$

whose bottom horizontal arrow is the multiplication map of $\mathbf{K}_\bullet(\mathbf{f})$, and where $P^\bullet_{\overline{\mu}}$ is defined as in (7.3.38). We conclude that $\omega^\bullet$ induces an isomorphism of anti-commutative graded A-algebras

$$H_\bullet(A_0[0] \overset{\mathbf{L}}{\otimes}_A A_0[0]) \xrightarrow{\sim} \mathbf{K}_\bullet(\mathbf{f}, A_0[0]) \xrightarrow{\sim} H_\bullet((\Lambda^\bullet_A(A^{\oplus r}), d_{\mathbf{f},\bullet}) \otimes_A A_0). \tag{7.8.12}$$

By simple inspection, we see that $d_{\mathbf{f},i} \otimes_A \mathbf{1}_{A_0} = 0$ for every $i \in \mathbb{Z}$, whence an isomorphism of strictly anti-commutative A-graded algebras :

$$H_\bullet((\Lambda^\bullet_A(A^{\oplus r}), d_{\mathbf{f},\bullet}) \otimes_A A_0) \xrightarrow{\sim} \Lambda^\bullet_{A_0}(A_0^{\oplus r}).$$

Combining with (7.8.12), we obtain in degree one a natural isomorphism

$$\Lambda^1_{A_0}(A_0^{\oplus r}) = A_0^{\oplus r} \xrightarrow{\sim} \mathrm{Tor}^A_1(A_0, A_0) \xrightarrow{\sim} J/J^2$$

which, finally, delivers the sought isomorphism of differential graded A-algebras. The uniqueness of (7.8.11) is clear, since the exterior algebra is generated by its degree one summand. □

Definition 7.8.13. Let A be a ring, $\mathbf{f} := (f_1, \dots, f_r)$ a finite sequence of elements of A, and M an A-module. Set also $R := \mathbb{Z}[T_1, \dots, T_r]$, let $I \subset R$ (resp. $J \subset A$) be the ideal generated by $\mathbf{T} := (T_1, \dots, T_r)$ (resp. by $\mathbf{f}$), and denote by $\mathrm{gr}_\bullet R$ the graded ring associated with the I-adic filtration of R; we associate with $\mathbf{f}$ the R-module structure on M such that $T_i x := f_i x$ for every $x \in M$ and every $i = 1, \dots, r$, and denote likewise by $\mathrm{gr}_\bullet M$ the graded $\mathrm{gr}_\bullet R$-module associated with the J-adic filtration on M. We say that $\mathbf{f}$ is *M-quasi-regular*, if the natural map

$$\mathrm{gr}_\bullet R \otimes_{\mathrm{gr}_0 R} \mathrm{gr}_0 M \to \mathrm{gr}_\bullet M$$

is an isomorphism.

Lemma 7.8.14. (i) *With the notation of definition* 7.8.13*, we have a natural isomorphism :*

$$R/I \overset{\mathbf{L}}{\otimes}_R M \xrightarrow{\sim} \mathbf{K}_\bullet(\mathbf{f}, M) \qquad \textit{in } \mathsf{D}(R\text{-}\mathbf{Mod}).$$

(ii) *If moreover* $\mathbf{f}$ *is completely secant on* A*, we have a natural isomorphism :*

$$A/J \overset{\mathbf{L}}{\otimes}_A M \xrightarrow{\sim} \mathbf{K}_\bullet(\mathbf{f}, M) \qquad \textit{in } \mathsf{D}(A\text{-}\mathbf{Mod}).$$

Proof. (i): Clearly $\mathbf{T}$ is a regular sequence on R, hence $H_i(\mathbf{T}, R) = 0$ for every $i > 0$ (proposition 7.8.7), *i.e.* $\mathbf{K}_\bullet(\mathbf{T}, R)$ is a resolution of R/I by free R-modules (remark 7.8.1(iii)). On the other hand, let $g : R \to A$ be the unique ring homomorphism such that $g(T_i) := f_i$ for $i = 1, \dots, r$; clearly the R-module structure of M is obtained by restriction of scalars along g from its A-module structure, so we have a natural identification $\mathbf{K}_\bullet(\mathbf{T}, M) \xrightarrow{\sim} \mathbf{K}_\bullet(\mathbf{f}, M)$ (lemma 7.8.2(iv)). The assertion follows immediately.

(ii): If $\mathbf{f}$ is completely secant on A, then $\mathbf{K}_\bullet(\mathbf{f})$ is a resolution of A/J by free A-modules (remark 7.8.1(iii)), whence the assertion. □

The following result summarizes and completes the list of interdependencies found thus far between the properties of finite sequences of elements in a ring that have been introduced at various stages in the text.

Proposition 7.8.15. *In the situation of definition* 7.8.13*, consider the following conditions :*

(a) *The sequence* $\mathbf{f}$ *is* M*-regular.*
(b) *The sequence* $\mathbf{f}$ *is completely secant on* M*.*
(c) $\mathrm{Tor}_1^R(R/I, M) = 0$*, or equivalently,* $H_1(\mathbf{f}, M) = 0$*.*
(d) *The sequence* $\mathbf{f}$ *is* M*-quasi-regular.*
(e) T_1 *is* $\mathrm{gr}_\bullet M$*-regular, and* $f_1 M \cap J^{n+1}M = f_1 J^n M$ *for every* $n \in \mathbb{N}$*.*
(f) M *is* J*-adically complete and separated.*
(g) *For every* $k = 0, \dots, r$*, the* A*-module* $M/\sum_{i=1}^k f_i M$ *is* J*-adically separated.*

Then (a)⇒(b)⇒(c)⇒(d)⇒(e)*, and if* (d) *holds, then* (f)⇒(g)⇒(a)*.*

Proof. (a)⇒(b) is already known, by proposition 7.8.7, and (b)⇒(c), by lemma 7.8.14(i).

(c)⇒(d): The assumption means that the natural map $I \otimes_R M \to IM$ is bijective. On the other hand, notice that $\mathrm{gr}_0 := R/I$ is isomorphic to $\mathbb{Z}$, and $\mathrm{gr}_n R := I^n/I^{n+1}$ is a free $\mathbb{Z}$-module for every $n \in \mathbb{N}$; consider then for every $n \in \mathbb{N}$ the commutative diagram :

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I^{n+1} \otimes_R M & \longrightarrow & I^n \otimes_R M & \longrightarrow & \mathrm{gr}_n R \otimes_R M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I^{n+1}M & \longrightarrow & I^n M & \longrightarrow & \mathrm{gr}_n M & \longrightarrow & 0 \end{array}$$

all of whose vertical arrows are surjections, and whose horizontal rows are short exact sequences, as $\mathrm{Tor}_1^R(\mathrm{gr}_n R, M) = 0$. Then, a simple induction shows that all three vertical arrows are isomorphisms for every $n \in \mathbb{N}$, whence the assertion.

(d)⇒(e): It is easily seen that $I^{n+1}/(T_1 I^n + I^{n+2})$ is a free $\mathbb{Z}$-module for every $n \in \mathbb{N}$, hence we get a short exact sequence

$$0 \to \mathrm{gr}_n R \otimes_{\mathrm{gr}_0 R} \mathrm{gr}_0 M \xrightarrow{\varphi} \mathrm{gr}_{n+1} R \otimes_{\mathrm{gr}_0 R} \mathrm{gr}_0 M \to (I^{n+1}/(T_1 I^n + I^{n+2})) \otimes_{\mathrm{gr}_0 R} \mathrm{gr}_0 M \to 0$$

for every $n \in \mathbb{N}$ where φ is induced by scalar multiplication by T_1 on R, which maps I^n into I^{n+1} for every such n. Then (d) implies that T_1 is regular on the $\mathrm{gr}_\bullet R$-module $\mathrm{gr}_\bullet M$; moreover

$$\{x \in I^n M \mid f_1 x \in I^{n+2}M\} = I^{n+1}M \qquad \text{for every } n \in \mathbb{N}$$

whence $f_1 I^n M \cap I^{n+2}M = f_1 I^{n+1}M$ for every $n \in \mathbb{N}$. By a simple induction on k, we deduce that $f_1 I^{n-k} M \cap I^{n+2}M = f_1 I^{n+1}M$ for $k = 0, \dots, n$, whence the sought identity.

(d) and (f)⇒(g): We shall argue by induction on the length r of $\mathbf{f}$. For $r = 0$, there is nothing to prove. Next, suppose that $r \geq 1$, and the assertion is already known for every M-quasi-regular sequence of length $r-1$; by the foregoing, (e) holds for M, so that scalar multiplication by f_1 on the A-module M induces an injective map $\mathrm{gr}_n M \to \mathrm{gr}_{n+1} M$ for every $n \in \mathbb{N}$. Since the J-adic topology is separated on M, we deduce already that f_1 is M-regular and the J-adic topology on $f_1 M$ agrees with the topology induced by the J-adic topology of M; since $f_1 M$ is isomorphic to M, we conclude that $f_1 M$ is complete for the topology induced by M, and by virtue of proposition 8.2.13(i,v), condition (f) holds therefore as well for $\overline{M} := M/f_1 M$.

Now, let $R' := \mathbb{Z}[T_2, \dots, T_r] \subset R$, set $I' := I \cap R'$, $J' := I'A$, and denote by $\mathrm{gr}_\bullet R'$ the graded ring associated with the I'-adic filtration of R'. The projection $R \to R/(T_1) \xleftarrow{\sim} R'$ induces isomorphisms of rings and respectively graded rings :

$$\mathrm{gr}_0 R \xrightarrow{\sim} \mathrm{gr}_0 R' \qquad (\mathrm{gr}_\bullet R)/(\overline{T}_1) \xrightarrow{\sim} \mathrm{gr}_\bullet R'$$

where $\overline{T}_1 \in \mathrm{gr}_1 R$ denotes the class of T_1. Let us endow $\overline{M}$ with the R'-module structure obtained by restriction of scalars along the inclusion map $R' \to R$, and let $\mathrm{gr}_\bullet \overline{M}$ be the graded $\mathrm{gr}_\bullet R'$-module associated with the J'-adic filtration on $\overline{M}$. Then clearly the J'-adic topology of $\overline{M}$ coincides with its J-adic topology; moreover, notice that

$$\mathrm{gr}_\bullet R' \otimes_{\mathrm{gr}_\bullet R} \mathrm{gr}_\bullet M = \bigoplus_{n\in\mathbb{N}} \frac{I^n M}{f_1 I^{n-1} M + I^{n+1} M} \qquad \mathrm{gr}_\bullet \overline{M} = \bigoplus_{n\in\mathbb{N}} \frac{I^n M}{(f_1 M \cap I^n M) + I^{n+1} M}$$

(here we set $I^{-1}M := M$). Hence, (e) also implies that the projection $\mathrm{gr}_\bullet M \to \mathrm{gr}_\bullet \overline{M}$ induces isomorphisms of $\mathrm{gr}_0 R$-modules and respectively graded $\mathrm{gr}_\bullet R'$-modules :

$$\mathrm{gr}_0 M \xrightarrow{\sim} \mathrm{gr}_0 \overline{M} \qquad \mathrm{gr}_\bullet R' \otimes_{\mathrm{gr}_\bullet R} \mathrm{gr}_\bullet M \xrightarrow{\sim} \mathrm{gr}_\bullet \overline{M}$$

Therefore, the natural map

$$\mathrm{gr}_\bullet R' \otimes_{\mathrm{gr}_0 R'} \mathrm{gr}_0 \overline{M} \to \mathrm{gr}_\bullet \overline{M}$$

is bijective, *i.e.* the sequence $\mathbf{f}' := (f_2, \dots, f_n)$ is $\overline{M}$-quasi-regular; by inductive assumption, $\overline{M}/\sum_{i=2}^k f_i \overline{M} = M/\sum_{i=1}^k f_i M$ is then J-adically separated for $k = 1, \dots, n$, as required.

(d) and (g)⇒(a): Again, we argue by induction on r. If $r = 0$, there is nothing to show; suppose then that $r \geq 1$, and the assertion is already known for every A-module M fulfilling condition (g), relative to any M-quasi-regular sequence of length $r-1$. As in the foregoing, we see that f_1 is M-regular, and $\mathbf{f}'$ is $\overline{M}$-quasi-regular; then by inductive assumption, $\mathbf{f}'$ is $\overline{M}$-regular, so $\mathbf{f}$ is M-regular. □

Remark 7.8.16. (i) Condition (g) of proposition 7.8.15 holds in either of the following cases :

(a) A is noetherian, M is an A-module of finite type and I lies in the Jacobson radical of A ([126, Th.8.1(i)]).

(b) For a given fine monoid P, the ring A is P-graded, M is a P-graded A-module, and each of $f_1, \dots, f_r$ are homogeneous elements whose degrees lie in the maximal ideal $\mathfrak{m}$ of P : indeed, recall that the $\mathfrak{m}$-adic filtration is separated on P (lemma 6.4.21(i)).

(ii) In the situation of definition 7.8.13, notice that the sequence $\mathbf{f}$ is quasi-regular on A if and only if the following two conditions hold :

(a) J/J^2 is a free A/J-module of rank r

(b) the natural map of graded A/J-algebras

$$\mathrm{Sym}^\bullet_{A/J}(J/J^2) \to \bigoplus_{n\in\mathbb{N}} J^n/J^{n+1}$$

is an isomorphism : the details shall be left to the reader. See also corollary 7.9.22(iv).

For future reference, we point out :

Lemma 7.8.17. (Cp. [20, Exp.VII, Prop.1.8(ii)]]) *Let A be a ring, $n \geq 1$ an integer, and $\mathbf{f} := (f_0, \dots, f_n)$ a completely secant sequence of elements of A. The following holds :*

(i) *The sequences $\mathbf{g} := (f_0, f_1 - f_0T_1, \dots, f_n - f_0T_n)$ and $\mathbf{g}' := (f_1 - f_0T_1, \dots, f_n - f_0T_n)$ are completely secant in the polynomial A-algebra $A[T_1, \dots, T_n]$.*

(ii) *Let $A[f_1/f_0, \dots, f_n/f_0] \subset A[1/f_0]$ be the A-subalgebra generated by $f_1/f_0, \dots, f_n/f_0$. Then the kernel of the surjective map of A-algebras*

$$A[T_1, \dots, T_n] \to A[f_1/f_0, \dots, f_n/f_0] \qquad T_i \mapsto f_i/f_0 \qquad \textit{for } i = 1, \dots, n$$

is the ideal $I \subset A[T_1, \dots, T_n]$ generated by the sequence $\mathbf{g}'$.

Proof. (i): Since the inclusion map $A \to B := A[T_1, \dots, T_n]$ is flat, it is clear that the sequence $\mathbf{f}$ is completely secant in B. Define the B-linear forms $\partial_{\mathbf{f}}, \partial_{\mathbf{g}} : B^{\oplus n+1} \to B$ as in remark 7.8.1(ii), so that $\mathbf{K}_\bullet(\mathbf{f}) = \mathbf{K}_\bullet(B, B^{\oplus n+1}, \partial_{\mathbf{f}})$, and likewise for $\mathbf{K}_\bullet(\mathbf{g})$. We have an isomorphism

$$(\mathbf{1}_B, \varphi) : (B, B^{\oplus n+1}, \partial_{\mathbf{g}}) \xrightarrow{\sim} (B, B^{\oplus n+1}, \partial_{\mathbf{f}}) \qquad \text{in } A\text{-}\mathsf{dAlg.Mod}$$

with $\varphi : B^{\oplus n+1} \xrightarrow{\sim} B^{\oplus n+1}$ the B-linear automorphism such that $\varphi(e_0) := e_0$ and $\varphi(e_i) := e_i - T_i e_0$ for $i = 1, \dots, n$. The assertion for $\mathbf{g}$ is an immediate consequence.

Combining with lemma 7.8.5, we see that the scalar multiplication by f_0 is injective on $H_i(\mathbf{g}', B)$, for every $i \in \mathbb{N}$. Hence, the natural map

$$H_i(\mathbf{g}', B) \to H_i(\mathbf{g}', B[f_0^{-1}])$$

is injective for every $i \in \mathbb{N}$, and in order to show that $\mathbf{g}'$ is completely secant in B, we may then assume that $f_0 \in A^\times$. To this aim, we consider also the sequence $\mathbf{T} := (T_1, \dots, T_n)$; we have an isomorphism

$$(\psi, \psi^{\oplus n}) : (B, B^{\oplus n}, \partial_{\mathbf{T}}) \xrightarrow{\sim} (B, B^{\oplus n}, \partial_{\mathbf{g}'}) \qquad \text{in } A\text{-}\mathsf{dAlg.Mod}$$

where $\psi : B \xrightarrow{\sim} B$ is the ring isomorphism such that $\psi(T_i) := f_i - f_0 T_i$ for $i = 1, \dots, n$. Since the sequence $\mathbf{T}$ is obviously B-regular, the assertion then follows from proposition 7.8.7.

(ii) We have a commutative diagram of rings :

$$\begin{array}{ccc} B/IB & \longrightarrow & A[f_1/f_0, \dots, f_n/f_0] \\ \downarrow & & \downarrow \\ B/IB[1/f_0] & \longrightarrow & A[1/f_0] \end{array}$$

whose vertical arrows are the localizations, and it is easily seen that the bottom horizontal arrow is an isomorphism. On the other hand, from (i) and lemma 7.8.5 we see that the scalar multiplication by f_0 is injective on B/I; hence the top horizontal arrow is injective, whence the contention. □

Definition 7.8.18. Let $\mathscr{A}$ be any additive category, and $A_\bullet$ any object of $\mathsf{Fun}(\mathbb{N}^o, \mathscr{A})$ (where the partially ordered set $(\mathbb{N}, \leq)$ is regarded as category, as in example 1.1.6(iii)). In other words, $A_\bullet$ is a datum $(A_\bullet, \varphi_{\bullet\bullet})$ consisting of a family $(A_p \mid p \in \mathbb{N})$ of objects of $\mathscr{A}$ and a system of morphisms $\varphi_{q,p} : A_q \to A_p$ of $\mathscr{A}$ for every integers $q \geq p \geq 0$, such that

$$\varphi_{q,p} \circ \varphi_{r,q} = \varphi_{r,p} \qquad \text{for every } r \geq q \geq p \geq 0.$$

The objects of $\mathsf{Fun}(\mathbb{N}^o, \mathscr{A})$ are called also *inverse systems* of $\mathscr{A}$ indexed by $\mathbb{N}$. Then :

(i) We say that $A_\bullet$ is *essentially zero*, if for every $p \in \mathbb{N}$ there exists $q \geq p$ such that $\varphi_{q,p}$ is the zero morphism.

(ii) We say that $A_\bullet$ is *uniformly essentially zero*, if there exists $c \in \mathbb{N}$ such that $\varphi_{p+c,p}$ is the zero morphism for every $p \in \mathbb{N}$. In this case, the smallest of such c is called the *step* of $A_\bullet$.

Recall that if $\mathsf{Fun}(\mathbb{N}^o, \mathscr{A})$ is an additive category, and it is even abelian, if the same holds for $\mathscr{A}$; moreover, the kernels and cokernels in $\mathsf{Fun}(\mathbb{N}^o, \mathscr{A})$ are computed argumentwise : see remark 3.7.37(ii). We notice :

Lemma 7.8.19. *Let $\mathscr{A}$ be any abelian category, and consider a short exact sequence*

$$0 \to A'_\bullet \to A_\bullet \to A''_\bullet \to 0 \qquad \textit{in } \mathsf{Fun}(\mathbb{N}^o, \mathscr{A}).$$

Then, $A_\bullet$ is essentially zero (resp. uniformly essentially zero) if and only if the same holds for both $A'_\bullet$ and $A''_\bullet$.

Proof. Indeed, it is clear that if $A_n \to A_m$ is the zero morphism for some $n \geq m$, then the same holds for the morphisms $A'_n \to A'_m$ and $A''_n \to A''_m$. Conversely, suppose that $A'_n \to A'_m$ and $A''_p \to A''_n$ are the zero morphisms, for some $p \geq n \geq m$; then it follows easily that the same holds for the morphism $A_p \to A_m$. □

7.8.20. Let $\mathbf{f} := (f_1, \dots, f_r)$ and $\mathbf{g} := (g_i \mid i = 1, \dots, r)$ be two finite sequences of elements of a ring A; we set $\mathbf{fg} := (f_i g_i \mid i = 1, \dots, r)$ and define as follows a map of Koszul complexes

$$\boldsymbol{\varphi}_{\mathbf{g}} : \mathbf{K}_\bullet(\mathbf{fg}) \to \mathbf{K}_\bullet(\mathbf{f})$$

(see remark 7.8.1(ii)). First, suppose that $r = 1$; then $\mathbf{f} = (f)$, $\mathbf{g} = (g)$ and the sought map $\boldsymbol{\varphi}_g$ is the commutative diagram :

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \xrightarrow{fg} & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle g} & & \| & & \\ 0 & \longrightarrow & A & \xrightarrow{f} & A & \longrightarrow & 0. \end{array}$$

For the general case we let :

$$\boldsymbol{\varphi}_{\mathbf{g}} := \boldsymbol{\varphi}_{g_1} \otimes_A \cdots \otimes_A \boldsymbol{\varphi}_{g_r}.$$

Especially, for every $m, n \geq 0$ we have maps $\boldsymbol{\varphi}_{\mathbf{f}^n} : \mathbf{K}_\bullet(\mathbf{f}^{n+m}) \to \mathbf{K}_\bullet(\mathbf{f}^m)$, whence maps

$$\boldsymbol{\varphi}^\bullet_{\mathbf{f}^n} : \mathbf{K}^\bullet(\mathbf{f}^m, M) \to \mathbf{K}^\bullet(\mathbf{f}^{m+n}, M)$$

and clearly $\boldsymbol{\varphi}^\bullet_{\mathbf{f}^{p+q}} = \boldsymbol{\varphi}^\bullet_{\mathbf{f}^p} \circ \boldsymbol{\varphi}^\bullet_{\mathbf{f}^q}$ for every $m, p, q \geq 0$.

7.8.21. In the situation of (7.8.20), set $A_r := \mathbb{Z}[T_1, \dots, T_r]$, and denote by $\beta_{\mathbf{f}} : A_r \to A$ the unique ring homomorphism such that $\beta_{\mathbf{f}}(T_i) := f_i$ for every $i = 1, \dots, r$. Let also $I_r \subset A_r$ be the ideal generated by $(T_1, \dots, T_r)$, and set $I := I_r A$. We consider the following conditions :

- $(\mathrm{a})_{\mathbf{f}}$ The inverse system $(\mathrm{Tor}_i^{A_r}(A_r/I_r^n, A) \mid n \in \mathbb{N})$ is essentially zero for every $i > 0$.
- $(\mathrm{b})_{\mathbf{f}}$ The inverse system $(\mathrm{Tor}_i^A(A/I^n, A/I) \mid n \in \mathbb{N})$ is essentially zero for every $i > 0$.
- $(\mathrm{c})_{\mathbf{f}}$ The inverse system $(H_i(\mathbf{f}, I^n) \mid n \in \mathbb{N})$ is essentially zero for every $i > 0$.
- $(\mathrm{d})_{\mathbf{f}}$ The inverse system $(H_i \mathbf{K}_\bullet(\mathbf{f}^n) \mid n \in \mathbb{N})$ is essentially zero for every $i > 0$.
- $(\mathrm{e})_{\mathbf{f}}$ The inverse system $(\mathrm{Tor}_{i+1}^A(A/I^n, M) \mid n \in \mathbb{N})$ is essentially zero, for every $i, k \in \mathbb{N}$ and every A/I^k-module M.
- $(\mathrm{f})_{\mathbf{f}}$ For every $i \in \mathbb{Z}$ and every bounded above complex $C_\bullet$ of flat A-modules such that for every $j \in \mathbb{Z}$ the annihilator ideal of $H_j C_\bullet = 0$ contains a power of I, the inverse system $(H_i(I^n C_\bullet) \mid n \in \mathbb{N})$ is essentially zero.

Moreover, we shall also consider the conditions $(\mathrm{a})^{\mathrm{un}}_{\mathbf{f}}, \dots, (\mathrm{f})^{\mathrm{un}}_{\mathbf{f}}$ obtained from $(\mathrm{a})_{\mathbf{f}}, \dots, (\mathrm{f})_{\mathbf{f}}$ after replacing "essentially zero" by "uniformly essentially zero".

Remark 7.8.22. Condition $(\mathrm{d})_{\mathbf{f}}$ seems to have been first considered in [83, Lemma 2.4], and in particular, our remark 7.8.45 was already observed in [83, Lemma 2.5 and th.2.8]. The same condition reappears in [85, Exp.II, Lemme 9], as well as in in [1, Lemma 3.1.1], and it is baptised "weak proregularity" in an *errata*[1] for that paper. A non-commutative extension is proposed in [160]. For motivation, we point out the following lemma, which says that completely secant sequences $\mathbf{f}$ trivially satisfy condition $(\mathrm{a})^{\mathrm{un}}_{\mathbf{f}}$:

Lemma 7.8.23. *With the notation of* (7.8.21)*, let* $k \in \mathbb{N}$ *such that* $H_k(\mathbf{f}, A) = 0$*. Then we have*

$$\mathrm{Tor}_k^{A_r}(A_r/I_r^n, A) = 0 \qquad \textit{for every } n \in \mathbb{N}.$$

[1] see `http://www.math.purdue.edu/~lipman/papers/homologyfix.pdf`

Proof. Notice first that $A_r/I_r = \mathbb{Z}$, and I_r^n/I_r^{n+1} is a free $\mathbb{Z}$-module, for every $n \in \mathbb{N}$; using the long Tor-exact sequences arising from the short exact sequences

$$0 \to I_r^n/I_r^{n+1} \to A_r/I_r^{n+1} \to A_r/I_r^n \to 0$$

a simple induction reduces to the case where $n = 1$ (details left to the reader). In this case, the assertion follows from lemma 7.8.14(i). □

Lemma 7.8.24. *With the notation of* (7.8.21)*, for every $i, p \in \mathbb{N}$ with $i > 0$, and every A_r/I_r^p-module N, the natural morphism $\mathrm{Tor}_i^{A_r}(A_r/I_r^{n+p}, N) \to \mathrm{Tor}_i^{A_r}(A_r/I_r^n, N)$ is the zero map.*

Proof. Let $\mathrm{Fil}^\bullet N$ be the I_r-adic filtration of N, and $\mathrm{gr}^\bullet N$ the associated graded A_r/I_r-module; the long exact Tor-sequences arising from the short exact sequences $0 \to \mathrm{Fil}^{q+1}N \to \mathrm{Fil}^q N \to \mathrm{gr}^q N \to 0$ (for every $q \in \mathbb{N}$), and an easy induction argument, reduce to the case where $p = 1$ (details left to the reader), so N is an A_r/I_r-module, *i.e.* a $\mathbb{Z}$-module; we may moreover reduce to the case where N is a finitely generated $\mathbb{Z}$-module, and then we may even assume that $N = \mathbb{Z}/a\mathbb{Z}$ for some $a \in \mathbb{N}$ (details left to the reader). In this situation, set $B := A_r/aA_r$, $J := I_r B_r$, consider the standard 2-spectral sequence ([163, Th.5.6.6])

$$E^2_{pq} := \mathrm{Tor}_p^B(\mathrm{Tor}_q^{A_r}(A_r/I_r^n, B), \mathbb{Z}/a\mathbb{Z}) \Rightarrow \mathrm{Tor}_{p+q}^{A_r}(A_r/I_r^n, \mathbb{Z}/a\mathbb{Z})$$

and notice that, on the one hand, the scalar multiplication by a is injective on A_r/I_r^n, and on the other hand, the complex $0 \to A_r \to A_r \to 0$ with differential given by $a \cdot \mathbf{1}_{A_r}$ is a free resolution of B; we deduce that $E^2_{pq} = 0$ whenever $q > 0$, whence natural isomorphisms :

$$\mathrm{Tor}_i^B(B/J^n, \mathbb{Z}/a\mathbb{Z}) \xrightarrow{\sim} \mathrm{Tor}_i^{A_r}(A_r/I_r^n, \mathbb{Z}/a\mathbb{Z}) \qquad \text{for every } i, n \in \mathbb{N}$$

and we are reduced to checking that the natural map

$$\mathrm{Tor}_i^B(B/J^{n+1}, \mathbb{Z}/a\mathbb{Z}) \to \mathrm{Tor}_i^B(B/J^n, \mathbb{Z}/a\mathbb{Z})$$

vanishes for every $i > 0$. To this aim, notice that the Koszul complex $\mathbf{K}_\bullet := \mathbf{K}_\bullet(T_\bullet, B)$ associated with the sequence $T_\bullet := (T_1, \dots, T_k)$ of elements of B, is a resolution of $Z/a\mathbb{Z}$ consisting of free B-modules (proposition 7.8.7), so we have a natural isomorphism

$$B/J^n \otimes_B \mathbf{K}_\bullet \xrightarrow{\sim} B/J^n \overset{\mathbf{L}}{\otimes}_B \mathbb{Z}/a\mathbb{Z} \qquad \text{in } \mathsf{D}(B\text{-}\mathbf{Mod}).$$

Now, denote by $\mathrm{gr}_\bullet B$ the standard graded $\mathbb{Z}/a\mathbb{Z}$-algebra structure on B such that $\mathrm{gr}_1 B$ is generated by $T_1, \dots, T_k$. For every $i \in \mathbb{N}$ we define a grading on $\mathbf{K}_i$ as follows. Let $(e_1, \dots, e_k)$ be the canonical basis of $B^{\oplus k}$, and $\mathscr{F}_i$ the set of all strictly increasing maps $\{1, \dots, i\} \to \{1, \dots, r\}$; for every $\varphi \in \mathscr{F}_i$ and every $r \in \mathbb{N}$ set

$$e_\varphi := e_{\varphi(1)} \wedge \cdots \wedge e_{\varphi(i)} \qquad \text{and} \qquad \mathrm{gr}_r \mathbf{K}_i := \sum_{\varphi \in \mathscr{F}_i} e_\varphi \cdot \mathrm{gr}_{r-i} B.$$

A simple inspection then shows that the differential of the Koszul complex is, in each degree, a map of graded $\mathbb{Z}/a\mathbb{Z}$-modules, therefore we get a natural decomposition

$$\mathbf{K}_\bullet \xrightarrow{\sim} \bigoplus_{r \in \mathbb{N}} \mathrm{gr}_r \mathbf{K}_\bullet \qquad \text{in } \mathsf{C}(\mathbb{Z}/a\mathbb{Z}\text{-}\mathbf{Mod})$$

and since $J^n = \bigoplus_{r \geq n} \mathrm{gr}_r B$, the complex $\mathbf{K}_\bullet \otimes_B B/J^n$ admits a corresponding decomposition, and we have

$$(\mathrm{gr}_r \mathbf{K}_\bullet \otimes_B B/J^n)_i = \begin{cases} \mathrm{gr}_r \mathbf{K}_i & \text{if } r < i + n \\ 0 & \text{otherwise.} \end{cases}$$

Since $H_i \mathbf{K}_\bullet = 0$ for every $i > 0$, it follows easily that

$$\mathrm{gr}_r H_i(\mathbf{K}_\bullet \otimes_B B/J^n) = 0 \qquad \text{for every } i, r, n \in \mathbb{N} \text{ such that } r \neq i - 1 + n \text{ and } i > 0.$$

Thus, fix $i, n \in \mathbb{N}$ with $i > 0$, and consider any $x \in \mathrm{Tor}_i^B(B/J^{n+1}, \mathbb{Z}/a\mathbb{Z})$; by the foregoing, we have $x \in H_i(\mathrm{gr}_{i+n}\mathbf{K}_\bullet \otimes_B B/J^{n+1})$, and $H_i(\mathrm{gr}_{i+n}\mathbf{K}_\bullet \otimes_B B/J^n) = 0$, so the image of x vanishes in $\mathrm{Tor}_i^B(B/J^n, \mathbb{Z}/a\mathbb{Z})$, as required. □

Proposition 7.8.25. *With the notation of* (7.8.21)*, fix* $i \in \mathbb{N}$*, and let also* $\mathbf{g} := (g_i \mid i = 1, \dots, s)$ *be another sequence of elements of* A *that generates an ideal* J*. We have :*

(i) $(\mathrm{a})_\mathbf{f} \Leftrightarrow (\mathrm{b})_\mathbf{f} \Leftrightarrow (\mathrm{c})_\mathbf{f} \Leftrightarrow (\mathrm{d})_\mathbf{f} \Leftrightarrow (\mathrm{e})_\mathbf{f} \Leftrightarrow (\mathrm{f})_\mathbf{f}$.

(ii) $(\mathrm{a})_\mathbf{f}^{\mathrm{un}} \Rightarrow (\mathrm{b})_\mathbf{f}^{\mathrm{un}} \Leftrightarrow (\mathrm{c})_\mathbf{f}^{\mathrm{un}} \Leftrightarrow (\mathrm{e})_\mathbf{f}^{\mathrm{un}} \Leftrightarrow (\mathrm{f})_\mathbf{f}^{\mathrm{un}}$.

(iii) *If the radical of* I *equals the radical of* J*, then* $(\mathrm{a})_\mathbf{f} \Leftrightarrow (\mathrm{a})_\mathbf{g}$.

(iv) *If* $I = J$*, then* $(\mathrm{a})_\mathbf{f}^{\mathrm{un}} \Leftrightarrow (\mathrm{a})_\mathbf{g}^{\mathrm{un}}$.

Proof. Let us check first that $(\mathrm{a})_\mathbf{f} \Rightarrow (\mathrm{b})_\mathbf{f}$ and $(\mathrm{a})_\mathbf{f}^{\mathrm{un}} \Rightarrow (\mathrm{b})_\mathbf{f}^{\mathrm{un}}$. To this aim, we consider the change of rings spectral sequence

$$E(n)_{ij}^2 := \mathrm{Tor}_i^A(\mathrm{Tor}_j^{A_r}(A_r/I_r^n, A), A/I) \Rightarrow \mathrm{Tor}_{i+j}^{A_r}(A_r/I_r^n, A/I).$$

Notice that the projection $A_r/I_r^m \to A_r/I_r^n$ induces a morphism of spectral sequences

(7.8.26) $$E(m)_{\bullet\bullet}^\bullet \to E(n)_{\bullet\bullet}^\bullet \qquad \text{for every } m \geq n.$$

Then the assertion is the case $p = 0$ of the following :

Claim 7.8.27. If condition $(\mathrm{a})_\mathbf{f}$ (resp. $(\mathrm{a})_\mathbf{f}^{\mathrm{un}}$) holds, then the inverse system $(E(n)_{i+1,0}^{p+2} \mid n \in \mathbb{N})$ is essentially zero (resp. uniformly essentially zero) for every $i, p \in \mathbb{N}$.

Proof of the claim. We fix $i \in \mathbb{N}$, and we argue by descending induction on p. Notice first that

$$E(n)_{i,0}^{i+1} = E(n)_{i,0}^\infty \qquad \text{for every } i, n \in \mathbb{N}.$$

Then, lemma 7.8.24 trivially implies that the assertion holds for every $i, p \in \mathbb{N}$ with $p \geq i$. Now, let $0 \leq q \leq i - 1$, and suppose that the assertion is already known for every $p > q$ (with our fixed i). We get an exact sequence of inverse systems

$$0 \to (E(n)_{i+1,0}^{q+3} \mid n \in \mathbb{N}) \to (E(n)_{i+1,0}^{q+2} \mid n \in \mathbb{N}) \to (E(n)_{i-q-1,q+1}^{q+2} \mid n \in \mathbb{N})$$

and by inductive assumption the system $(E(n)_{i+1,0}^{q+3} \mid n \in \mathbb{N})$ is essentially zero (resp. uniformly essentially zero). Moreover, $(\mathrm{a})_\mathbf{f}$ (resp. $(\mathrm{a})_\mathbf{f}^{\mathrm{un}}$) implies that the same holds for the system $(E(n)_{i-q-1,q+1}^{q+2} \mid n \in \mathbb{N})$. Then the contention follows from lemma 7.8.19. ◊

- In order to show that $(\mathrm{b})_\mathbf{f} \Rightarrow (\mathrm{a})_\mathbf{f}$ we remark :

Claim 7.8.28. Let $j \in \mathbb{N}$ be any integer. Then :

(i) If $(E(n)_{0,j}^2 \mid n \in \mathbb{N})$ is an essentially zero inverse system, the same holds for the inverse system $(\mathrm{Tor}_j^{A_r}(A_r/I_r^n, A) \mid n \in \mathbb{N})$.

(ii) If $(\mathrm{Tor}_j^{A_r}(A_r/I_r^n, A) \mid n \in \mathbb{N})$ is an essentially zero inverse system, the same holds for the inverse system $(E_{i,j}^{p+2} \mid n \in \mathbb{N})$, for every $i, p \in \mathbb{N}$.

(iii) The inverse system $(E(n)_{0,j}^{j+2} \mid n \in \mathbb{N})$ is essentially zero.

(iv) $(\mathrm{b})_\mathbf{f}$ implies that $(E(n)_{i,0}^{j+2} \mid n \in \mathbb{N})$ is an essentially zero system for every $i \in \mathbb{N}$.

Proof of the claim. (i): Denote by $\varphi_{n,c} : \mathrm{Tor}_j^{A_r}(A_r/I_r^{n+c}, A) \to \mathrm{Tor}_j^{A_r}(A_r/I_r^n, A)$ the natural map. The hypothesis means that for every $n \in \mathbb{N}$ there exists $c \in \mathbb{N}$ such that $\varphi_{n,c} \otimes_A A/I = 0$, *i.e.* $\mathrm{Im}\,\varphi_{n,c} \subset I \cdot \mathrm{Tor}_j^{A_r}(A_r/I_r^n, A)$. By a simple induction we deduce that for every $n, k \in \mathbb{N}$ there exists $d \in \mathbb{N}$ such that $\mathrm{Im}\,\varphi_{n,c} \subset I^k \cdot \mathrm{Tor}_j^{A_r}(A_r/I_r^n, A)$, and letting $k = n$, the assertion follows.

(ii): Obviously, the assumption implies that $(E_{i,j}^2 \mid n \in \mathbb{N})$ is an essentially zero inverse system; the assertion is an immediate consequence.

(iv) is clear, and notice that $E(n)^{j+2}_{0,j} = E(n)^\infty_{0,j}$ for every $n \in \mathbb{N}$. In light of lemma 7.8.24, assertion (iii) follows easily as well. ◊

In light of claim 7.8.28(i) it suffices to show that $(E(n)^2_{0,j})$ is essentially zero for every $j > 0$. We argue by induction on j; for $j = 1$ we consider the exact sequence of inverse systems :

$$(E(n)^2_{2,0} \mid n \in \mathbb{N}) \to (E(n)^2_{0,1} \mid n \in \mathbb{N}) \to (E(n)^3_{0,1}) \to 0.$$

By claim 7.8.28(iii,iv), the first and third terms are essentially zero, and therefore the same holds for the middle term, by lemma 7.8.42.

Next, let $j > 1$, and suppose that the inverse systems $(E(n)^2_{0,k} \mid n \in \mathbb{N})$ are essentially zero for $0 < k < j$. By claim 7.8.28(i,ii) we deduce that $(E(n)^{p+2}_{i,k} \mid n \in \mathbb{N})$ is essentially zero for every $p, i \in \mathbb{N}$ and whenever $0 < k < j$. We show, by descending induction on p, that $(E(n)^{p+2}_{0,j} \mid n \in \mathbb{N})$ is essentially zero for every $p \in \mathbb{N}$. First, the assertion holds for $p \geq j$, by virtue of claim 7.8.28(iii). Suppose then that $0 \leq k < j$, and that the assertion is already known for every $p > k$; we consider the exact sequence of inverse systems

$$(E(n)^{k+2}_{k,j-k-1} \mid n \in \mathbb{N}) \to (E(n)^{k+2}_{0,j} \mid n \in \mathbb{N}) \to (E(n)^{k+3}_{0,j}) \to 0.$$

Our inductive assumtions imply that the first and third terms are essentially zero; then the same holds for the middle term (lemma 7.8.42), and the proof is concluded.

• Next, we show that $(\mathrm{b})_\mathbf{f} \Rightarrow (\mathrm{e})_\mathbf{f}$ and $(\mathrm{b})^{\mathrm{un}}_\mathbf{f} \Rightarrow (\mathrm{e})^{\mathrm{un}}_\mathbf{f}$. For any A/I^k-module M, we need to check that the inverse system $(\mathrm{Tor}^A_{i+1}(A/I^n, M) \mid n \in \mathbb{N})$ is essentially zero (resp. uniformly essentially zero); to this aim, we consider the I-adic filtration $\mathrm{Fil}^\bullet M$ on M, and the associated graded A/I-module $\mathrm{gr}_\bullet M$. Applying the long exact Tor-sequences arising from the short exact sequences $0 \to \mathrm{Fil}^{q+1} M \to \mathrm{Fil}^q M \to \mathrm{gr}^q M \to 0$, together with lemma 7.8.19, we reduce to checking that $(\mathrm{Tor}^A_{i+1}(A/I^n, \mathrm{gr}^\bullet M) \mid n \in \mathbb{N})$ is an essentially zero (resp. uniformly essentially zero) inverse system. Thus, we may suppose that $IM = 0$, in which case we show more precisely :

Claim 7.8.29. Let $p > 0$ be any integer, and suppose that $(\mathrm{Tor}^A_i(A/I^n, A/I) \mid n \in \mathbb{N})$ is an essentially zero (resp. uniformly essentially zero) inverse system for every $i = 1, \dots, p$. Then, for every A/I-module M the inverse system $(\mathrm{Tor}^A_p(A/I^n, M) \mid n \in \mathbb{N})$ is essentially zero (resp. uniformly essentially zero).

Proof of the claim. We consider the spectral sequence

$$E(n)^2_{i,j} := \mathrm{Tor}^{A/I}_i(\mathrm{Tor}^A_j(A/I^n, A/I), M) \Rightarrow \mathrm{Tor}^A_{i+j}(A/I^n, M) \qquad \text{for every } n \in \mathbb{N}$$

and the corresponding morphisms of spectral sequences as in (7.8.26), induced by the projections $A/I^m \to A/I^n$. Notice that $E(n)^2_{i,0} = 0$, and therefore $E(n)^\infty_{i,0} = 0$ for every $i > 0$. On the other hand, our assumption implies that the inverse system $(E(n)^2_{i,j} \mid n \in \mathbb{N})$ is essentially zero (resp. uniformly essentially zero) for every $i, j \in \mathbb{N}$ such that $j > 0$ and $i + j = q$. Summing up, we deduce that the inverse system $(E(n)^\infty_{i,j} \mid n \in \mathbb{N})$ is essentially zero (resp. uniformly essentially zero) for every $i, j \in \mathbb{N}$ such that $i + j = p$. Taking into account lemma 7.8.19, the contention now follows by a simple induction. ◊

• We check next that $(\mathrm{e})_\mathbf{f} \Rightarrow (\mathrm{f})_\mathbf{f}$ and $(\mathrm{e})^{un}_\mathbf{f} \Rightarrow (\mathrm{f})^{un}_\mathbf{f}$. To this aim, notice that $H_i(I^n C_\bullet) = H_i(I^n[0] \overset{\mathbf{L}}{\otimes}_A C_\bullet)$ for every $i, n \in \mathbb{Z}$, since $C_\bullet$ is a complex of flat A-modules. On the other hand, we have a standard spectral sequence

$$E(n)^2_{p,q} := \mathrm{Tor}^A_p(I^n, H_q C_\bullet) \Rightarrow H_{p+q}(I^n[0] \overset{\mathbf{L}}{\otimes}_A C_\bullet).$$

Since $\mathrm{Tor}^A_p(I^n, H_q C_\bullet) \xrightarrow{\sim} \mathrm{Tor}^A_{p+1}(A/I^n, H_q C_\bullet)$ for every $p, n \in \mathbb{N}$, condition $(\mathrm{e})_\mathbf{f}$ (resp. $(\mathrm{e})^{\mathrm{un}}_\mathbf{f}$) implies that $(E(n)^2_{p,q} \mid n \in \mathbb{N})$ is an essentially zero (resp. uniformly essentially zero) inverse system for every $p \in \mathbb{N}$ and every $q \in \mathbb{Z}$, hence the same holds for the system $(E(n)^\infty_{p,q} \mid n \in \mathbb{N})$.

Since $C_\bullet$ is bounded above, lemma 7.8.19 easily implies that $H_i(I^n[0] \overset{\mathbf{L}}{\otimes}_A C_\bullet)$ is an essentially zero (resp. uniformly essentially zero) system for every $i \in \mathbb{Z}$ (details left to the reader), whence the contention.

Clearly $(\mathrm{f})_{\mathbf{f}} \Rightarrow (\mathrm{c})_{\mathbf{f}}$ and $(\mathrm{f})^{\mathrm{un}}_{\mathbf{f}} \Rightarrow (\mathrm{c})^{\mathrm{un}}_{\mathbf{f}}$, since the Koszul complex is a bounded complex of flat A-modules whose cohomology is an A/I-module in each degree (lemma 7.8.2(ii)).

- To show that $(\mathrm{c})_{\mathbf{f}} \Rightarrow (\mathrm{b})_{\mathbf{f}}$ and $(\mathrm{c})^{\mathrm{un}}_{\mathbf{f}} \Rightarrow (\mathrm{b})^{\mathrm{un}}_{\mathbf{f}}$ we consider the spectral sequence

$$E(n)^2_{i,j} := \mathrm{Tor}^A_i(I^n, H_j\mathbf{K}_\bullet(\mathbf{f})) \Rightarrow H_{i+j}(\mathbf{f}, I^n) \qquad \text{for every } n \in \mathbb{N}$$

and the corresponding system of morphisms of spectral sequences as in (7.8.26). Recalling that $H_0\mathbf{K}_\bullet(\mathbf{f}_\bullet) = A/I$, as well as the natural isomorphism

$$\text{(7.8.30)} \qquad \mathrm{Tor}^A_i(I^n, A/I) \xrightarrow{\sim} \mathrm{Tor}^A_{i+1}(A/I^n, A/I) \qquad \text{for every } i, n \in \mathbb{N}$$

the assertion will follow from the case $j = p = 0$ of the following more general :

Claim 7.8.31. If $(\mathrm{c})_{\mathbf{f}}$ (resp. $(\mathrm{c})^{\mathrm{un}}_{\mathbf{f}}$) holds, the inverse system $(E(n)^{p+2}_{i,j} \mid n \in \mathbb{N})$ is essentially zero (resp. uniformly essentially zero) for every $i, j, p \in \mathbb{N}$.

Proof of the claim. We argue by induction on i. For $i = 0$, notice that $E(n)^2_{0,0} = E(n)^\infty_{0,0}$ for every $n \in \mathbb{N}$; condition $(\mathrm{c})_{\mathbf{f}}$ (resp. $(\mathrm{c})^{\mathrm{un}}_{\mathbf{f}}$) then yields the claim for the system $(E(n)^2_{0,0} \mid n \in \mathbb{N})$. In light of (7.8.30) and claim 7.8.29, we deduce that $(E(n)^2_{0,j} \mid n \in \mathbb{N})$ is essentially zero (resp. uniformly essentially zero) for every $j \in \mathbb{N}$, and then the same follows for $(E(n)^{p+2}_{0,j} \mid n \in \mathbb{N})$, for every $j, p \in \mathbb{N}$.

Next, suppose that $k > 0$, and the claim is already known for every $i < k$ and every $j, p \in \mathbb{N}$. We show, by descending induction on q, that $(E(n)^q_{k,0} \mid n \in \mathbb{N})$ is an essentially zero (resp. uniformy essentially zero) system for every $q \geq 2$. Indeed, notice first that $E(n)^{k+1}_{k,0} = E(n)^\infty_{k,0}$; in light of $(\mathrm{c})_{\mathbf{f}}$ (resp. $(\mathrm{c})^{\mathrm{un}}_{\mathbf{f}}$), the assertion follows already for every $q \geq k+1$. Next, let $p \leq k$ and suppose that the assertion is already known for $q = p + 1$. Notice the exact sequence of inverse systems :

$$0 \to (E(n)^{p+1}_{k,0} \mid n \in \mathbb{N}) \to (E(n)^p_{k,0} \mid n \in \mathbb{N}) \to (E(n)^p_{k-p,p-1} \mid n \in \mathbb{N}).$$

By inductive assumption, both the first and third terms are essentially zero (resp. uniformly essentially zero), so the same holds for the middle term. We know now that $(E(n)^2_{k,0} \mid n \in \mathbb{N})$ is essentially zero (resp. uniformly essentially zero); from (7.8.30) and claim 7.8.29 we deduce that the same holds for $(E(n)^2_{k,j} \mid n \in \mathbb{N})$, for every $j \in \mathbb{N}$, and finally the same follows for $(E(n)^{p+2}_{k,j} \mid n \in \mathbb{N})$, for every $p, j \in \mathbb{N}$, whence the claim. ◊

- Lastly, let us check that $(\mathrm{a})_{\mathbf{f}} \Leftrightarrow (\mathrm{d})_{\mathbf{f}}$. To this aim, set $T_\bullet := (T_1, \dots, T_r)$, and notice that $H_i(T^n_\bullet, A_r) = 0$ for every $i, n > 0$ (proposition 7.8.7), whence a natural isomorphism

$$\mathrm{Tor}^{A_r}_i(A_r/T^n_\bullet A_r, A) \xrightarrow{\sim} H_i(\mathbf{K}_\bullet(T^n_\bullet) \otimes_{A_r} A) \xrightarrow{\sim} H_i(\mathbf{f}^n, A) \qquad \text{for every } i, n \in \mathbb{N}.$$

On the other hand, it is clear that for every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that

$$I^m_r \subset T^n_\bullet A_r \subset I^n_r$$

whence the contention.

(iv): Set $B_r := A_r \otimes_{\mathbb{Z}} A$, and notice that $\beta_{\mathbf{f}}$ factors as a composition

$$A_r \xrightarrow{\beta'_{\mathbf{f}}} B_r \xrightarrow{\gamma_{\mathbf{f}}} A_r$$

where $\beta'_{\mathbf{f}} : A_r \to B_r$ is the ring homomorphism such that $\beta'_{\mathbf{f}}(T_i) := 1 \otimes f_i$ for $i = 1, \dots, r$, and $\gamma_{\mathbf{f}}$ is the map of $\mathbb{Z}$-algebras induced by the pair $(\beta_{\mathbf{f}}, \mathbf{1}_A)$. Let us endow B_r with the A_r-module structure given by $\beta'_{\mathbf{f}}$ and A with the B_r-module structure given by $\gamma_{\mathbf{f}}$; there follows, for every $n \in \mathbb{N}$ a change of rings spectral sequence

$$E(n)^2_{i,j} := \mathrm{Tor}_i^{B_r}(\mathrm{Tor}_j^{A_r}(A_r/I_r^n, B_r), A) \Rightarrow \mathrm{Tor}_{i+j}^{A_r}(A_r/I_r^n, A).$$

Claim 7.8.32. $\mathrm{Tor}_j^{A_r}(A_r/I_r^n, B_r) = 0$ for every $j > 0$.

Proof of the claim. We have a change of rings spectral sequence

$$E^2_{p,q} := \mathrm{Tor}_p^{A_r}(\mathrm{Tor}_q^{\mathbb{Z}}(A, A_r), A_r/I_r^n) \Rightarrow \mathrm{Tor}_{p+q}^{\mathbb{Z}}(A, A_r/I_r^n)$$

and clearly $E^2_{p,q} = 0$ for every $q > 0$, so there follows a natural isomorphism

$$\mathrm{Tor}_p^{A_r}(B_r, A_r/I_r^n) \xrightarrow{\sim} \mathrm{Tor}_p^{\mathbb{Z}}(A, A_r/I_r^n) \qquad \text{for every } p \in \mathbb{N}.$$

However, A_r/I_r^n is a free $\mathbb{Z}$-module, whence the claim. ◊

In view of claim 7.8.32 we have $E^2_{i,j} = 0$ for every $j > 0$, whence a natural isomorphism

$$\mathrm{Tor}_i^{B_r}(B_r/I_r^n B_r, A) \xrightarrow{\sim} \mathrm{Tor}_i^{A_r}(A_r/I_r^n, A) \qquad \text{for every } i \in \mathbb{N}.$$

Especially, $(\mathrm{a})_{\mathbf{f}}$ is equivalent to :

$(\mathrm{g})_{\mathbf{f}}$ The inverse system $(\mathrm{Tor}_i^{B_r}(B_r/I_r^n B_r, A) \mid n \in \mathbb{N})$ is essentially zero for every $i > 0$

and $(\mathrm{a})^{\mathrm{un}}_{\mathbf{f}}$ is equivalent to the corresponding condition $(\mathrm{g})^{\mathrm{un}}_{\mathbf{f}}$.

Now, set $(\mathbf{f}, \mathbf{g}) := (f_1, \dots, f_r, g_1, \dots, g_s)$. In light of the foregoing we see that in order to prove the equivalence $(\mathrm{a})_{\mathbf{f}} \Leftrightarrow (\mathrm{a})_{\mathbf{g}}$ it suffices to check that

$$(\mathrm{g})_{\mathbf{f}} \Leftrightarrow (\mathrm{g})_{(\mathbf{f},\mathbf{g})} \qquad \text{and} \qquad (\mathrm{g})_{\mathbf{g}} \Leftrightarrow (\mathrm{g})_{(\mathbf{f},\mathbf{g})}$$

and likewise for the uniformly essentially zero variants, so we may assume from start that $s > r$ and $f_i = g_i$ for $i = 1, \dots, r$. In this case, an easy induction on $s - r$ further reduces to the case where $s = r + 1$. Now, let us say that $g_s = \sum_{i=1}^r a_i f_i$ for some $a_1, \dots, a_r \in A$, and consider the automorphism of A-algebras $\omega : B_{r+1} \xrightarrow{\sim} B_{r+1}$ such that $\omega(T_i) := T_i$ for $i = 1, \dots, r$ and $\omega(T_{r+1}) := T_{r+1} - \sum_{i=1}^r a_i T_i$. Clearly $\omega(I_{r+1}^n B_{r+1}) = I_{r+1}^n B_{r+1}$ for every $n \in \mathbb{N}$, and $\gamma_{\mathbf{g}} \circ \omega(\mathbf{g}) = (\mathbf{f}, 0) := (f_1, \dots, f_r, 0)$. It follows easily that $(\mathrm{g})_{\mathbf{g}} \Leftrightarrow (\mathrm{g})_{(\mathbf{f},0)}$ and $(\mathrm{g})^{\mathrm{un}}_{\mathbf{g}} \Leftrightarrow (\mathrm{g})^{\mathrm{un}}_{(\mathbf{f},0)}$, so we may further assume that $\mathbf{g} = (\mathbf{f}, 0)$.

Let $\pi : B_{r+1} \to B_r$ be the map of A-algebras such that $\pi(T_i) := T_i$ for $i = 1, \dots, r$ and $\pi(T_{r+1}) := 0$, and notice that $\gamma_{(\mathbf{f},0)} = \gamma_{\mathbf{f}} \circ \pi$; if we endow B_r with the B_{r+1}-module structure given by π, there follows, for every $n \in \mathbb{N}$, a change of rings spectral sequence

$$E(n)^2_{i,j} := \mathrm{Tor}_i^{B_r}(\mathrm{Tor}_j^{B_{r+1}}(B_{r+1}/I_{r+1}^n B_{r+1}, B_r), A) \Rightarrow \mathrm{Tor}_{i+j}^{B_{r+1}}(B_{r+1}/I_{r+1}^n B_{r+1}, A)$$

as well as a system of morphisms of spectral sequences as in (7.8.26).

Claim 7.8.33. (i) $\mathrm{Tor}_j^{B_{r+1}}(B_{r+1}/I_{r+1}^{n+1} B_{r+1}, B_r) = 0$ for every $j > 1$.

(ii) The projection $B_{r+1}/I_{r+1}^{n+1} \to B_{r+1}/I_{r+1}^n$ induces the zero map

$$\mathrm{Tor}_1^{B_{r+1}}(B_{r+1}/I_{r+1}^{n+1} B_{r+1}, B_r) \to \mathrm{Tor}_1^{B_{r+1}}(B_{r+1}/I_{r+1}^n B_{r+1}, B_r) \qquad \text{for every } n \in \mathbb{N}.$$

Proof of the claim. The B_{r+1}-module B_r admits the free resolution $B_{r+1} \to B_{r+1} \xrightarrow{\pi} B_r$ given by scalar multiplication by T_{r+1}. Assertion (i) follows immediately, and we also deduce that

$$\mathrm{Tor}_1^{B_{r+1}}(B_{r+1}/I_{r+1}^{n+1} B_{r+1}, B_r) \xrightarrow{\sim} \mathrm{Ker}(B_{r+1}/I_{r+1}^{n+1} B_{r+1} \xrightarrow{T_{r+1}} B_{r+1}/I_{r+1}^{n+1} B_{r+1}) \subset I_{r+1}^n/I_{r+1}^{n+1}.$$

Moreover, the transition maps in (ii) are induced by the inclusions $I_{r+1}^n \subset I_{r+1}^{n-1}$ for every $n > 0$, whence (ii). ◊

From claim 7.8.33 we see that $E(n)^2_{i,j} = 0$ for every $j > 1$, and $(E(n)^2_{i,1} \mid n \in \mathbb{N})$ is a uniformly essentially zero inverse system. Hence, $E(n)^\infty_{i,j}$ is uniformly essentially zero for every $j > 0$, and $E(n)^\infty_{i,0} = E(n)^2_{i,0}$ for every $n, i \in \mathbb{N}$. Taking into account lemma 7.8.19 we conclude that $(\mathrm{g})_{(\mathbf{f},0)}$ (resp. $(\mathrm{g})^{\mathrm{un}}_{(\mathbf{f},0)}$) holds if and only if $(E(n)^2_{i,0} \mid n \in \mathbb{N})$ is essentially zero (resp. uniformly essentially zero) for every $i > 0$. But $E(n)^2_{i,0} = \mathrm{Tor}^{B_r}_i(B_r/I^n_r B_r, A)$, so the latter condition is precisely $(\mathrm{g})_{\mathbf{f}}$ (resp. $(\mathrm{g})^{\mathrm{un}}_{\mathbf{f}}$).

(iii): By assumption, there exists an integer $k > 0$ such that $f^k_i \in J$ and $g^k_j \in I$ for every $i = 1, \dots, r$ and every $j = 1, \dots, s$. By the foregoing proof of (iv), we deduce that

$$(\mathrm{a})_{\mathbf{f}} \Leftrightarrow (\mathrm{a})_{(\mathbf{f},\mathbf{g}^k)} \qquad \text{and} \qquad (\mathrm{a})_{\mathbf{g}} \Leftrightarrow (\mathrm{a})_{(\mathbf{f}^k,\mathbf{g})}.$$

Hence, we are reduced to checking that $(\mathrm{a})_{(\mathbf{f}^k,\mathbf{g})} \Leftrightarrow (\mathrm{a})_{(\mathbf{f},\mathbf{g})} \Leftrightarrow (\mathrm{a})_{(\mathbf{f},\mathbf{g}^k)}$. We show the first equivalence; obviously the same argument will yield the second one. To this aim, for every integers $n \geq m \geq 0$ let $\varphi^{(n,m)}_\bullet : \mathbf{K}_\bullet(\mathbf{f}^n, \mathbf{g}^n) \to \mathbf{K}_\bullet(\mathbf{f}^m, \mathbf{g}^m)$ and $\psi^{(n,m)}_\bullet : \mathbf{K}_\bullet(\mathbf{f}^n, \mathbf{g}^{kn}) \to \mathbf{K}_\bullet(\mathbf{f}^m, \mathbf{g}^{km})$ be the transition morphisms of the inverse systems $(\mathbf{K}_\bullet(\mathbf{f}^n, \mathbf{g}^n) \mid n \in \mathbb{N})$ and $(\mathbf{K}_\bullet(\mathbf{f}^n, \mathbf{g}^{kn}) \mid n \in \mathbb{N})$. Then, for every $n, k \in \mathbb{N}$ we have morphisms

$$\mathbf{K}_\bullet(\mathbf{f}^{kn}, \mathbf{g}^{k^2 n}) \xrightarrow{\beta_\bullet} \mathbf{K}_\bullet(\mathbf{f}^{kn}, \mathbf{g}^{kn}) \xrightarrow{\beta'_\bullet} \mathbf{K}_\bullet(\mathbf{f}^{n}, \mathbf{g}^{kn}) \xrightarrow{\beta''_\bullet} \mathbf{K}_\bullet(\mathbf{f}^{n}, \mathbf{g}^{n})$$

such that $\beta'_\bullet \circ \beta_\bullet = \psi^{(kn,n)}_\bullet$ and $\beta''_\bullet \circ \beta'_\bullet = \varphi^{(kn,n)}_\bullet$ (see (7.8.20)), whence the contention. □

Remark 7.8.34. (i) By carefully tracking the estimates arising in the proof, it is possible to refine proposition 7.8.25(ii) as follows. For every $r \in \mathbb{N}$ and every sequence of integers $(c(i) \mid i \in \mathbb{N})$ there exists a sequence of integers $(d(i) \mid i \in \mathbb{N})$, such that the following holds : for A and $\mathbf{f}$ of length r as in (7.8.21), suppose that the inverse system $(\mathrm{Tor}^{A_r}_{i+1}(A_r/I^n_r, A) \mid n \in \mathbb{N})$ is uniformly essentially zero with step $\leq c(i)$, for every $i \in \mathbb{N}$. Then the inverse system $(H_{i+1}(\mathbf{f}, I^n) \mid n \in \mathbb{N})$ is uniformly essentially zero of step $\leq d(i)$, for every $i \in \mathbb{N}$.

(ii) A shorter alternative proof for the refinement of (i) can be proven as follows. Fix such a sequence $c_\bullet$, and suppose that the assertion fails; then we may find a system of pairs $((R_\lambda, \mathbf{f}_\lambda) \mid \lambda \in \mathbb{N})$ consisting of a ring R_λ and a sequence $\mathbf{f}_\lambda$ of r elements of R_λ, for every λ, such that the following holds. For every $\lambda \in \mathbb{N}$, denote by $\beta_\lambda : A_r \to R_\lambda$ the ring homomorphism associated with the sequence $\mathbf{f}_\lambda$, as in (7.8.21); then :

(a) the inverse system $(\mathrm{Tor}^{A_r}_{i+1}(A_r/I^n_r, R_\lambda) \mid n \in \mathbb{N})$ is uniformly essentially zero with step $\leq c(i)$, for every $i, \lambda \in \mathbb{N}$
(b) the inverse system $(H_{i+1}(\mathbf{f}, I^n) \mid n \in \mathbb{N})$ is uniformly essentially zero of step $d(i, \lambda)$, for every $i \in \mathbb{N}$
(c) the sequence of integers $(d(i_0, \lambda) \mid \lambda \in \mathbb{N})$ is strictly increasing, for some $i_0 \in \mathbb{N}$.

Now, set $J := \oplus_{\lambda \in \mathbb{N}} R_\lambda$, $B := A_r \oplus J$, and endow B with the unique ring structure such that $(a, b) \cdot (a', b') := (aa', ba' + a'b)$ for every $a, a' \in A_r$ and $b, b' \in J$. Then J is an ideal of B with $J^2 = 0$, the natural projection induces a ring isomorphism $\omega : B/J \xrightarrow{\sim} A_r$, and the inclusion map $\beta : A_r \to B$ is a ring homomorphism; moreover, the A_r-module structure induced via ω on J agrees with the A_r-module structure induced by the system of maps $(\beta_\lambda \mid \lambda \in \mathbb{N})$. We consider the sequence $\mathbf{t} := (T_1, \dots, T_r)$ of A_r, and its image $\mathbf{f} := \beta(\mathbf{t})$ in B, and we notice the natural isomorphisms of A_r-modules

$$\mathrm{Tor}^{A_r}_{i+1}(A_r/I^n_r, B) \xrightarrow{\sim} \bigoplus_{\lambda \in \mathbb{N}} \mathrm{Tor}^{A_r}_{i+1}(A_r/I^n_r, A_r) \qquad \text{for every } i, n \in \mathbb{N}.$$

In light of condition (a), we deduce that B satisfies condition $(\mathrm{a})^{\mathrm{un}}_{\mathbf{f}}$, so it also satisfies $(\mathrm{c})^{\mathrm{un}}_{\mathbf{f}}$, by virtue of proposition 7.8.25(ii); on the other hand, clearly we have as well natural isomorphisms

$$H_i(\mathbf{f}, I^n_r B) \xrightarrow{\sim} H_i(\mathbf{t}, I^n_r) \oplus \bigoplus_{\lambda \in \mathbb{N}} H_i(\mathbf{f}_\lambda, I^n_r R_\lambda) \qquad \text{for every } i, n \in \mathbb{N}$$

which, in view of our assumption (c), imply that the inverse system $(H_{i_0}(\mathbf{f}, I^n_r B) \mid n \in \mathbb{N})$ cannot be uniformly essentially zero, a contradiction.

7.8.35. Let $\mathbf{f} := (f_1, \dots, f_r)$ be a finite sequence of elements of a ring A, and set

$$I := f_1A + \cdots + f_rA \qquad \text{and} \qquad J := \bigcup_{n\in\mathbb{N}} \mathrm{Ann}_A(I^n).$$

Denote by $\overline{\mathbf{f}} := (\overline{f}_1, \dots, \overline{f}_r)$ the image of $\mathbf{f}$ in $\overline{A} := A/J$.

Proposition 7.8.36. *With the notation of* (7.8.35)*, the following conditions are equivalent :*

(a) *The ring A satisfies condition* $(\mathrm{a})_{\mathbf{f}}$ *(resp.* $(\mathrm{a})^{\mathrm{un}}_{\mathbf{f}}$*, resp.* $(\mathrm{c})^{\mathrm{un}}_{\mathbf{f}}$*) of* (7.8.21)*.*

(b) $\overline{A}$ *satisfies condition* $(\mathrm{a})_{\overline{\mathbf{f}}}$ *(resp.* $(\mathrm{a})^{\mathrm{un}}_{\overline{\mathbf{f}}}$*, resp* $(\mathrm{c})^{\mathrm{un}}_{\overline{\mathbf{f}}}$*) and there exists $k \in \mathbb{N}$ with $I^kJ = 0$.*

Proof. (a)⇒(b): by proposition 7.8.25(i) the ring A satisfies condition $(\mathrm{c})_{\mathbf{f}}$, so there exists $k \in \mathbb{N}$ such that the natural map $H_r(\mathbf{f}, I^k) \to H_r(\mathbf{f}, A)$ is the zero morphism. In other words

$$I^k \cap \mathrm{Ann}_A(I) = 0.$$

Therefore we have as well $I^k \cap J = 0$, and it follows easily that $J = \mathrm{Ann}_A(I^k)$. If A satisfies $(\mathrm{c})^{\mathrm{un}}_{\overline{\mathbf{f}}}$, notice that for every $n \geq k$ the natural map $I^n \to I^n\overline{A}$ is then bijective, and thus the same holds for the induced map $H_i(\mathbf{f}, I^n) \to H_i(\mathbf{f}, I^n\overline{A})$, for every $i \in \mathbb{N}$, so $\overline{A}$ fulfills $(\mathrm{c})^{\mathrm{un}}_{\overline{\mathbf{f}}}$.

If A satisfies $(\mathrm{a})_{\overline{\mathbf{f}}}$ (resp. $(\mathrm{a})^{\mathrm{un}}_{\overline{\mathbf{f}}}$), define A_r and $\beta_{\mathbf{f}}$ as in (7.8.21), and endow A and J with the A_r-module structure induced by $\beta_{\mathbf{f}}$; from lemma 7.8.24 we deduce that the inverse system $T(J, i)_\bullet := (\mathrm{Tor}_i^{A_r}(A_r/I_r^n, J) \mid n \in \mathbb{N})$ is uniformly essentially zero for every $i > 0$. By assumption, the inverse system $T(A, i) := (\mathrm{Tor}_i^{A_r}(A_r/I_r^n, A) \mid n \in \mathbb{N})$ is essentially zero (resp. uniformly essentially zero); moreover, the short exact sequence of A_r-modules $0 \to J \to A \to \overline{A} \to 0$ yields exact sequences of inverse systems :

$$T(J, i)_\bullet \to T(A, i)_\bullet \to T(\overline{A}, i)_\bullet := (\mathrm{Tor}_i^{A_r}(A_r/I_r^n, \overline{A}) \mid n \in \mathbb{N}) \to T(J, i-1)_\bullet \tag{7.8.37}$$

for every $i > 0$. By virtue of lemma 7.8.19, it follows already that $T(\overline{A}, i)_\bullet$ is essentially zero (resp. uniformly essentially zero) for every $i > 1$. By the same token, the case $i = 1$ yields the exact sequence of inverse systems

$$T(A, 1)_\bullet \to T(\overline{A}, 1) \to (J/I^nJ \mid n \in \mathbb{N}) \xrightarrow{\varphi_\bullet} (A/I^n \mid n \in \mathbb{N}).$$

But the foregoing shows that φ_n is injective for every $n \geq k$, hence $\mathrm{Ker}\,\varphi_n$ is uniformly essentially zero, and finally $T(\overline{A}, 1)_\bullet$ is essentially zero (resp. uniformly essentially zero), again by lemma 7.8.19. Summing up, this shows that $\overline{A}$ fulfills condition $(\mathrm{a})_{\overline{\mathbf{f}}}$ (resp. $(\mathrm{a})^{\mathrm{un}}_{\overline{\mathbf{f}}}$), as stated.

(b)⇒(a): Suppose first that $\overline{A}$ satisfies $(\mathrm{a})_{\mathbf{f}}$ (resp. $(\mathrm{a})^{\mathrm{un}}_{\mathbf{f}}$); by assumption, J is an A_r/I_r^k-module, so the inverse system $T(J, i)_\bullet$ is uniformly essentially zero for every $i > 0$, by lemma 7.8.24; the inverse system $T(\overline{A}, i)_\bullet$ is essentially zero (resp. uniformly essentially zero), and thus the same holds also for $T(A, i)_\bullet$, in view of (7.8.37) and lemma 7.8.19. Lastly, if $\overline{A}$ satisfies $(\mathrm{c})^{\mathrm{un}}_{\overline{\mathbf{f}}}$, then it also satisfies condition $(\mathrm{a})_{\mathbf{f}}$, by proposition 7.8.25(i), hence A satisfies $(\mathrm{a})_{\mathbf{f}}$, by the foregoing case; but as we have seen, this implies that the natural map $H_i(\mathbf{f}, I^n) \to H_i(\mathbf{f}, I^n\overline{A})$ is bijective for every $i \in \mathbb{N}$ and sufficiently large n, so A satisfies $(\mathrm{c})^{\mathrm{un}}_{\overline{\mathbf{f}}}$. □

7.8.38. Let A, I and $\mathbf{f}$ be as in (7.8.35), and for every $n > 0$ denote by $I^{(n)} \subset A$ the ideal generated by $\mathbf{f}^n := (f_i^n \mid i = 1, \dots, r)$. For all $m \geq n > 0$ we deduce natural commutative diagrams of complexes :

$$\begin{array}{ccc} \mathbf{K}_\bullet(\mathbf{f}^m) & \longrightarrow & A/I^{(m)}[0] \\ {\scriptstyle \varphi_{\mathbf{f}^{m-n}}} \downarrow & & \downarrow {\scriptstyle \pi_{mn}} \\ \mathbf{K}_\bullet(\mathbf{f}^n) & \longrightarrow & A/I^{(n)}[0] \end{array}$$

(notation of (7.8.20)) where π_{mn} is the natural surjection, whence a compatible system of maps:

$$\mathrm{Hom}_{\mathsf{D}(A\text{-}\mathbf{Mod})}(A/I^{(n)}[0], M^\bullet) \to \mathrm{Hom}_{\mathsf{D}(A\text{-}\mathbf{Mod})}(\mathbf{K}_\bullet(\mathbf{f}^n), M^\bullet). \tag{7.8.39}$$

for every $n \geq 0$ and every complex $M^\bullet$ in $\mathsf{D}^+(A\text{-}\mathbf{Mod})$. Since $\mathbf{K}_\bullet(\mathbf{f}^n)$ is a complex of free A-modules, (7.8.39) translates as a direct system of maps :

$$R^i\mathrm{Hom}_A(A/I^{(n)}, M^\bullet) \to H^i(\mathbf{f}^n, M^\bullet) \qquad \text{for all } n \in \mathbb{N} \text{ and every } i \in \mathbb{Z}. \tag{7.8.40}$$

Notice that $I^{nr-r+1} \subset I^{(n)} \subset I^n$ for every $n > 0$, hence the colimit of the system (7.8.40) is equivalent to a natural map :

$$\operatorname*{colim}_{n\in\mathbb{N}} R^i\mathrm{Hom}^\bullet_A(A/I^n, M^\bullet) \to \operatorname*{colim}_{n\in\mathbb{N}} H^i(\mathbf{f}^n, M^\bullet) \qquad \text{for every } i \in \mathbb{Z}. \tag{7.8.41}$$

Lemma 7.8.42. *With the notation of* (7.8.38)*, the following conditions are equivalent :*

(a) *The map* (7.8.41) *is an isomorphism for every* $M^\bullet \in \mathrm{Ob}(\mathsf{D}^+(A\text{-}\mathbf{Mod}))$ *and all* $i \in \mathbb{Z}$.
(b) $\mathrm{colim}_{n\in\mathbb{N}} H^i(\mathbf{f}^n, J) = 0$ *for every* $i > 0$ *and every injective A-module J.*
(c) *The ring A satisfies condition* $(\mathrm{d})_\mathbf{f}$ *of* (7.8.21).

Proof. (a) ⇒ (b) is obvious. Next, if J is an injective A-module, we have natural isomorphisms

$$H^i(\mathbf{f}^n, J) \simeq \mathrm{Hom}_A(H_i\mathbf{K}_\bullet(\mathbf{f}^n), J) \qquad \text{for all } n \in \mathbb{N}. \tag{7.8.43}$$

which easily implies that (c) ⇒ (b).

(b) ⇒ (c) : Indeed, for any $p \in \mathbb{N}$ let us choose an injection $\varphi : H_i\mathbf{K}_\bullet(\mathbf{f}^n) \to J$ into an injective A-module J. By (7.8.43) we can regard φ as an element of $H^i(\mathbf{f}^n, J)$; by (b) the image of φ in $\mathrm{Hom}_A(H_i\mathbf{K}_\bullet(\mathbf{f}^q), J)$ must vanish if $q > p$ is large enough. This can happen only if $H_i\mathbf{K}_\bullet(\mathbf{f}^q) \to H_i\mathbf{K}_\bullet(\mathbf{f}^p)$ is the zero map.

(b) ⇒ (a) : Let $M^\bullet \to J^\bullet$ be an injective resolution of the complex $M^\bullet$. The double complex $\mathrm{colim}_{n\in\mathbb{N}} \mathrm{Hom}^\bullet_A(\mathbf{K}_\bullet(\mathbf{f}^n), J^\bullet)$ determines two spectral sequences :

$$E_1^{pq} := \operatorname*{colim}_{n\in\mathbb{N}} \mathrm{Hom}_A(\mathbf{K}_p(\mathbf{f}^n), H^qJ^\bullet) \Rightarrow \operatorname*{colim}_{n\in\mathbb{N}} R^{p+q}\mathrm{Hom}^\bullet_A(\mathbf{K}_\bullet(\mathbf{f}^n), M^\bullet)$$
$$F_1^{pq} := \operatorname*{colim}_{n\in\mathbb{N}} H^p(\mathbf{f}^n, J^q) \simeq \operatorname*{colim}_{n\in\mathbb{N}} \mathrm{Hom}_A(H_p\mathbf{K}_\bullet(\mathbf{f}^n), J^q) \Rightarrow \operatorname*{colim}_{n\in\mathbb{N}} R^{p+q}\mathrm{Hom}^\bullet_A(\mathbf{K}_\bullet(\mathbf{f}^n), M^\bullet).$$

Clearly $E_1^{pq} = 0$ whenever $q > 0$, and (b) says that $F_1^{pq} = 0$ for $p > 0$. Hence these two spectral sequences degenerate and we deduce natural isomorphisms :

$$\operatorname*{colim}_{n\in\mathbb{N}} R^q\mathrm{Hom}^\bullet_A(A/I^{(n)}, M^\bullet) \simeq F_2^{0q} \xrightarrow{\sim} E_2^{q0} \simeq \operatorname*{colim}_{n\in\mathbb{N}} H^q(\mathbf{f}^n, M^\bullet).$$

By inspection, one sees easily that these isomorphisms are the same as the maps (7.8.41). □

Lemma 7.8.44. *In the situation of* (7.8.38)*, suppose that the following holds. For every finitely presented quotient B of A and every $i = 1, \dots, r$, there exists $p \in \mathbb{N}$ such that*

$$\mathrm{Ann}_B(f_i^q) = \mathrm{Ann}_B(f_i^p) \qquad \textit{for every } q \geq p.$$

Then the ring A satisfies condition $(\mathrm{d})_\mathbf{f}$ *of* (7.8.21).

Proof. We shall argue by induction on r. If $r = 1$, then $\mathbf{f} = (f)$ for a single element $f \in A$. In this case, our assumption ensures that there exists $p \in \mathbb{N}$ such that $\mathrm{Ann}_A(f^q) = \mathrm{Ann}_A(f^p)$ for every $q \geq p$. It follows easily that $H_1(\boldsymbol{\varphi}_{f^p}) : H_1\mathbf{K}_\bullet(f^{p+k}) \to H_1\mathbf{K}_\bullet(f^k)$ is the zero map for every $k \geq 0$ (notation of (7.8.20)), whence the claim.

Next, suppose that $r > 1$ and that the claim is known for all sequences of less than r elements. Set $\mathbf{g} := (f_1, \dots, f_{r-1})$ and $f := f_r$. Specializing (7.8.4) to our current situation, we derive short exact sequences :

$$0 \to H_0(f^n, H_p\mathbf{K}_\bullet(\mathbf{g}^n)) \to H_p\mathbf{K}_\bullet(\mathbf{f}^n) \to H^0(f^n, H_{p-1}\mathbf{K}_\bullet(\mathbf{g}^n)) \to 0$$

for every $p > 0$ and $n \geq 0$; for a fixed p, this is an inverse system of exact sequences. By induction, the inverse system $(H_i\mathbf{K}_\bullet(\mathbf{g}^n) \mid n \in \mathbb{N})$ is essentially zero for $i > 0$; in light of lemma 7.8.19, we deduce already that the inverse system $(H_i\mathbf{K}_\bullet(\mathbf{f}^n) \mid n \in \mathbb{N})$ is essentially zero for all $i > 1$. To conclude, we are thus reduced to showing that the inverse system

$(T_n := H^0(f^n, H_0\mathbf{K}_\bullet(\mathbf{g}^n)) \mid n \in \mathbb{N})$ is essentially zero. However $A_n := H_0\mathbf{K}_\bullet(\mathbf{g}^n) = A/(g_1^n, \dots, g_{r-1}^n)$ is a finitely presented quotient of A for any fixed $n \in \mathbb{N}$, hence the foregoing case $r = 1$ shows that the inverse system $(T_{mn} := \mathrm{Ann}_{A_n}(f^m) \mid m \in \mathbb{N})$ is essentially zero. Let $m \geq n$ be chosen so that $T_{mn} \to T_{nn}$ is the zero map; then the composition $T_m = T_{mm} \to T_{mn} \to T_{nn} = T_n$ is zero as well. □

Remark 7.8.45. Notice that the condition of lemma 7.8.44 holds when A is noetherian. Combining with lemma 7.8.42, we conclude that if A is noetherian, (7.8.41) is an isomorphism, for every finite system $\mathbf{f}$ of elements of A, and every $i \in \mathbb{N}$. See also corollary 7.9.22(iii).

7.9. **Filtered rings and Rees algebras.** Some of the following material is borrowed from [27, Appendix III], where much more can be found.

Definition 7.9.1. Let R be a ring, A an R-algebra.

(i) An *R-algebra filtration on A* is an increasing exhaustive filtration $\mathrm{Fil}_\bullet A$ indexed by $\mathbb{Z}$ and consisting of R-submodules of A, such that :
$$1 \in \mathrm{Fil}_0 A \qquad \text{and} \qquad \mathrm{Fil}_i A \cdot \mathrm{Fil}_j A \subset \mathrm{Fil}_{i+j} A \qquad \text{for every } i, j \in \mathbb{Z}.$$
The pair $\underline{A} := (A, \mathrm{Fil}_\bullet A)$ is called a *filtered R-algebra*.

(ii) Let M be an A-module. An *$\underline{A}$-filtration* on M is an increasing exhaustive filtration $\mathrm{Fil}_\bullet M$ consisting of R-submodules, and such that :
$$\mathrm{Fil}_i A \cdot \mathrm{Fil}_j M \subset \mathrm{Fil}_{i+j} M \qquad \text{for every } i, j \in \mathbb{Z}.$$
The pair $\underline{M} := (M, \mathrm{Fil}_\bullet M)$ is called a *filtered $\underline{A}$-module*.

(iii) Let U be an indeterminate. The *Rees algebra* of $\underline{A}$ is the $\mathbb{Z}$-graded subring of $A[U, U^{-1}]$
$$\mathsf{R}(\underline{A})_\bullet := \bigoplus_{i \in \mathbb{Z}} U^i \cdot \mathrm{Fil}_i A.$$
We occasionally consider also the two $\mathbb{N}$-graded subrings of $\mathsf{R}(\underline{A})_\bullet$ denoted :
$$\mathsf{R}^-(\underline{A})_\bullet := A[U] \cap \mathsf{R}(\underline{A})_\bullet \qquad \text{and} \qquad \mathsf{R}^+(\underline{A})_\bullet := A[U^{-1}] \cap \mathsf{R}(\underline{A})_\bullet$$

(iv) Let $\underline{M} := (M, \mathrm{Fil}_\bullet M)$ be a filtered $\underline{A}$-module. The *Rees module* of $\underline{M}$ is the graded $\mathsf{R}(\underline{A})_\bullet$-module :
$$\mathsf{R}(\underline{M})_\bullet := \bigoplus_{i \in \mathbb{Z}} U^i \cdot \mathrm{Fil}_i M.$$
We have similarly the $\mathsf{R}^-(\underline{A})$-submodule $\mathsf{R}^-(\underline{M})_\bullet := \bigoplus_{i \in \mathbb{N}} U^i \cdot \mathrm{Fil}_i M$ and the $\mathsf{R}^+(\underline{A})$-submodule $\mathsf{R}^+(\underline{M})_\bullet := \bigoplus_{i \in \mathbb{N}} U^{-i} \cdot \mathrm{Fil}_{-i} M$

Lemma 7.9.2. *Let R be a ring, $\underline{A} := (A, \mathrm{Fil}_\bullet A)$ a filtered R-algebra, $\mathrm{gr}_\bullet \underline{A}$ the associated graded R-algebra, $\mathsf{R}(\underline{A})_\bullet \subset A[U, U^{-1}]$ the Rees algebra of $\underline{A}$. Then there are natural isomorphisms of graded R-algebras :*
$$\mathsf{R}(\underline{A})_\bullet / U\mathsf{R}(\underline{A})_\bullet \simeq \mathrm{gr}_\bullet \underline{A} \qquad \mathsf{R}(\underline{A})_\bullet[U^{-1}] \simeq A[U, U^{-1}]$$
and of R-algebras :
$$\mathsf{R}(\underline{A})_\bullet / (1 - U)\mathsf{R}(\underline{A})_\bullet \simeq A.$$

Proof. The isomorphisms with $\mathrm{gr}_\bullet \underline{A}$ and with $A[U, U^{-1}]$ follow directly from the definitions. For the third isomorphism, it suffices to remark that $A[U, U^{-1}]/(1 - U) \simeq A$. □

Example 7.9.3. Let A be any ring, $I \subset A$ any ideal, and endow A with its I-adic filtration $\mathrm{Fil}_\bullet A$, so that $\mathrm{Fil}_{-i} A := I^i$ for every $i \in \mathbb{N}$ and $\mathrm{Fil}_i A = A$ for every $i > 0$. The pair $\underline{A} := (A, \mathrm{Fil}_\bullet)$ is obviously a filtered A-algebra, and we denote by $\mathsf{R}(A, I)_\bullet$ its Rees algebra.

Definition 7.9.4. Let R be a ring, $\underline{A} := (A, \mathrm{Fil}_\bullet A)$ a filtered R-algebra.

(i) Suppose that A is of finite type over R, let $\mathbf{x} := (x_1, \dots, x_n)$ be a finite set of generators for A as an R-algebra, and $\mathbf{k} := (k_1, \dots, k_n)$ a sequence of n integers; the *good filtration* $\mathrm{Fil}_\bullet A$ *attached to the pair* $(\mathbf{x}, \mathbf{k})$ is the R-algebra filtration such that $\mathrm{Fil}_i A$ is the R-submodule generated by all the elements of the form

$$\prod_{j=1}^n x_j^{a_j} \qquad \text{where :} \qquad \sum_{j=1}^n a_j k_j \leq i \quad \text{and} \quad a_1, \dots, a_n \geq 0$$

for every $i \in \mathbb{Z}$. A filtration $\mathrm{Fil}_\bullet A$ on A is said to be *good* if it is the good filtration attached to some system of generators $\mathbf{x}$ and some sequence of integers $\mathbf{k}$.

(ii) The filtration $\mathrm{Fil}_\bullet A$ is said to be *positive* if it is the good filtration associated with a pair $(\mathbf{x}, \mathbf{k})$ as in (i), such that moreover $k_i > 0$ for every $i = 1, \dots, n$.

(iii) Let M be a finitely generated A-module. An $\underline{A}$-filtration $\mathrm{Fil}_\bullet M$ is called a *good filtration* if $\mathsf{R}(M, \mathrm{Fil}_\bullet M)_\bullet$ is a finitely generated $\mathsf{R}(\underline{A})_\bullet$-module.

Example 7.9.5. Let $A := R[t_1, \dots, t_n]$ be the free R-algebra in n indeterminates. Choose any sequence $\mathbf{k} := (k_1, \dots, k_n)$ of integers, and denote by $\mathrm{Fil}_\bullet A$ the good filtration associated with $\mathbf{t} := (t_1, \dots, t_n)$ and $\mathbf{k}$. Then $\mathsf{R}(A, \mathrm{Fil}_\bullet A)_\bullet$ is isomorphic, as a graded R-algebra, to the free polynomial algebra $A[U] = R[U, t_1, \dots, t_n]$, endowed with the grading such that $U \in \mathrm{gr}_1 A[U]$ and $t_j \in \mathrm{gr}_{k_j} A[U]$ for every $j \leq n$. Indeed, a graded isomorphism can be defined by the rule : $U \mapsto U$ and $t_j \mapsto U^{k_j} \cdot t_j$ for every $j \leq n$. The easy verification shall be left to the reader.

7.9.6. Let $\underline{A}$ and M be as in definition 7.9.4(iii); suppose that $\mathbf{m} := (m_1, \dots, m_n)$ is a finite system of generators of M, and let $\mathbf{k} := (k_1, \dots, k_n)$ be an arbitrary sequence of n integers. To the pair $(\mathbf{m}, \mathbf{k})$ we attach a filtered $\underline{A}$-module $\underline{M} := (M, \mathrm{Fil}_\bullet M)$, by declaring that :

$$\mathrm{Fil}_i M := m_1 \cdot \mathrm{Fil}_{i-k_1} A + \cdots + m_n \cdot \mathrm{Fil}_{i-k_n} A \qquad \text{for every } i \in \mathbb{Z}. \tag{7.9.7}$$

Notice that the homogeneous elements $m_1 \cdot U^{k_1}, \dots, m_n \cdot U^{k_n}$ generate the graded Rees module $\mathsf{R}(\underline{M})_\bullet$, hence $\mathrm{Fil}_\bullet M$ is a good $\underline{A}$-filtration. Conversely :

Lemma 7.9.8. *For every good filtration* $\mathrm{Fil}_\bullet M$ *on* M*, there exist a sequence* $\mathbf{m}$ *of generators of* M *and a sequence of integers* $\mathbf{k}$*, such that* $\mathrm{Fil}_\bullet M$ *is of the form* (7.9.7).

Proof. Suppose that $\mathrm{Fil}_\bullet M$ is a good filtration; then $\mathsf{R}(\underline{M})_\bullet$ is generated by finitely many homogeneous elements $m_1 \cdot U^{k_1}, \dots, m_n \cdot U^{k_n}$. Thus,

$$\mathsf{R}(\underline{M})_i := U^i \cdot \mathrm{Fil}_i M = m_1 \cdot U^{k_1} \cdot A_{i-k_1} + \cdots + m_n \cdot U^{k_n} \cdot A_{i-k_n} \qquad \text{for every } i \in \mathbb{Z}$$

which means that the sequences $\mathbf{m} := (m_1, \dots, m_n)$ and $\mathbf{k} := (k_1, \dots, k_n)$ will do. $\square$

7.9.9. Let $A \to B$ and $A \to C$ be maps of noetherian rings, and suppose that either B or C is an A-algebra of finite type; let also M be a finitely generated B-module, N a finitely generated C-module, and $I \subset B$ any ideal. Then the A-modules

$$T_i := \mathrm{Tor}_i^A(M, N)$$

inherit natural $B \otimes_A C$-module structures, and we remark :

Lemma 7.9.10. *In the situation of* (7.9.9)*, the* $B \otimes_A C$*-module* T_i *is finitely generated for every* $i \in \mathbb{N}$.

Proof. Say that B is an A-algebra of finite type; the same argument will apply also to the case where C is an A-algebra of finite type. Then, we have $B = D/J$ for some free polynomial A-algebra D of finite type, and some ideal $J \subset D$. Hence, M is also a D-module of finite type; then D is a noetherian ring, so we may find a resolution $L_\bullet \to M$ consisting of free D-modules of finite type, and since D is a free A-module, we get natural A-linear isomorphisms

$\omega_i : T_i \xrightarrow{\sim} H_i(L_\bullet \otimes_A N)$ for every $i \in \mathbb{N}$. Moreover, $L_i \otimes_A N$ is naturally a complex of $D \otimes_A C$-modules of finite type, so T_i inherits via ω_i a structure of $D \otimes_A C$-module of finite type. Lastly, a simple inspection shows that the latter $D \otimes_A C$-module structure on T_i agrees with the one induced from the natural $B \otimes_A C$-module structure, via the surjection $D \otimes_A C \to B \otimes_A C$, whence the contention. □

We endow B (resp. $B \otimes_A C$) with its I-adic (resp. $I \cdot (B \otimes_A C)$-adic) filtration as in example 7.9.3, and denote by $\underline{B}$ (resp. $\underline{B} \otimes_A C$) the resulting filtered ring. Also, set $\mathrm{Fil}_n M := \mathrm{Fil}_n B \cdot M$ for every $n \in \mathbb{Z}$; we define a $\underline{B} \otimes_A C$-filtration on T_i, by the rule :

$$\mathrm{Fil}_n T_i := \mathrm{Im}\,(\mathrm{Tor}_i^A(\mathrm{Fil}_n M, N) \to T_i) \qquad \text{for every } i \in \mathbb{N} \text{ and } n \in \mathbb{Z}.$$

Proposition 7.9.11. *The $\underline{B} \otimes_A C$-filtration $\mathrm{Fil}_\bullet T_i$ is good, for every $i \in \mathbb{N}$.*

Proof. Pick a finite set of generators $f_1, \dots, f_r$ for I, and consider the surjective map of graded B-algebras

$$(7.9.12) \qquad B[U, t_1, \dots, t_r] \to \mathsf{R}(\underline{B})_\bullet \quad : \quad U \mapsto U \in \mathsf{R}(\underline{B})_1 \quad t_i \mapsto f_i \qquad \text{for } i = 1, \dots, r$$

(for the grading of $B[U, t_1, \dots, t_r]$ that places the indeterminates $t_1, \dots, t_r$ in degree -1, and U in degree 1). The $B \otimes_A C$-module

$$(7.9.13) \qquad P_{i,\bullet} := \bigoplus_{n \in \mathbb{Z}} \mathrm{Tor}_i^A(\mathrm{Fil}_n M, N)$$

carries a natural structure of graded $\mathsf{R}(\underline{B})_\bullet \otimes_A C$-module, whence a graded $B \otimes_A C[U, t_1, \dots, t_r]$-module structure as well, via (7.9.12), and it suffices to show :

Claim 7.9.14. $P_{i,\bullet}$ is a finitely generated $B \otimes_A C[U, t_1, \dots, t_r]$-module, for every $i \in \mathbb{N}$.

Proof of the claim. Notice that $\mathsf{R}(M, \mathrm{Fil}_\bullet M)_\bullet$ is a finitely generated $\mathsf{R}(\underline{B})_\bullet$-module; *a fortiori*, it is a finitely generated graded $B[U, t_1, \dots, t_r]$-module. By remark 7.6.18(iv), we may then find a resolution

$$\cdots \to L_n \xrightarrow{d_n} L_{n-1} \to \cdots \to L_0 \xrightarrow{d_0} \mathsf{R}(M, \mathrm{Fil}_\bullet M)_\bullet$$

where each L_n is a free $B[U, t_1, \dots, t_r]$-module of finite rank, each d_n is a morphism of graded $B[U, t_1, \dots, t_r]$-modules, and the restriction of the resolution to the summands of degree i is a flat resolution of the B-module $I^i M$, for every $i \in \mathbb{N}$. There follows a natural isomorphism of graded $B \otimes_A C$-modules

$$(7.9.15) \qquad P_{i,\bullet} \xrightarrow{\sim} H_i(L_\bullet \otimes_A N) \qquad \text{for every } i \in \mathbb{N}$$

and a simple inspection shows that the $B \otimes_A C[U, t_1, \dots, t_r]$-module structure on $P_{i,\bullet}$ deduced via (7.9.15) agrees with the foregoing one, so the assertion follows. □

7.9.16. In the situation of (7.9.9), set $B_n := B/I^n$ for every $n \in \mathbb{N}$. We deduce, for every $i \in \mathbb{N}$, a morphism of projective systems of $B \otimes_A C$-modules

$$X^i_\bullet := (B_n \otimes_B \mathrm{Tor}_i^A(M, N) \mid n \in \mathbb{N}) \xrightarrow{\varphi_{i,\bullet}} Y^i_\bullet := (\mathrm{Tor}_i^A(B_n \otimes_B M, N) \mid n \in \mathbb{N})$$

where the transition maps of $X^i_\bullet$ and $Y^i_\bullet$ are induced by the projections $B_{n+1} \to B_n$, for every $n \in \mathbb{N}$, and the morphisms $\varphi_{i,n}$ are induced by the projections $M \to B_n \otimes_B M$.

Corollary 7.9.17. *With the notation of* (7.9.16)*, we have :*

(i) *The morphism $\varphi_{i,\bullet}$ is an isomorphism of pro-$B \otimes_A C$-modules, for every $i \in \mathbb{N}$.*
(ii) *The morphism $\varphi_{i,\bullet}$ induces an isomorphism of $B \otimes_A C$-modules :*

$$\lim_{n \in \mathbb{N}} B_n \otimes_B \mathrm{Tor}_i^A(M, N) \xrightarrow{\sim} \lim_{n \in \mathbb{N}} \mathrm{Tor}_i^A(B_n \otimes_B M, N) \qquad \textit{for every } i \in \mathbb{N}.$$

Proof. (i): The assertion means that the systems $(\mathrm{Ker}\,\varphi_{i,n} \mid n \in \mathbb{N})$ and $(\mathrm{Coker}\,\varphi_{i,n} \mid n \in \mathbb{N})$ are essentially zero, for every $i \in \mathbb{N}$. We shall prove the more precise :

Claim 7.9.18. For every $i \in \mathbb{N}$ the systems $(\mathrm{Ker}\,\varphi_{i,n} \mid n \in \mathbb{N})$ and $(\mathrm{Coker}\,\varphi_{i,n} \mid n \in \mathbb{N})$ are uniformly essentially zero (see definition 7.8.18(ii)).

Proof of the claim. The long exact $\mathrm{Tor}^A_\bullet(-, N)$-sequence arising from the short exact sequence

$$0 \to I^n M \to M \to M/I^n M \to 0$$

yields a natural identification

$$\mathrm{Coker}\,\varphi_{i,n} = \mathrm{Ker}\,(U^n : P_{i-1,-n} \to P_{i-1,0})$$

where $P_{i-1,\bullet}$ is defined as in (7.9.13), and U^n denotes the scalar multiplication by the same element of $\mathsf{R}(\underline{B})_\bullet$, for the natural $\mathsf{R}(\underline{B})_\bullet$-module structure of $P_{i-1,\bullet}$. Under this identification, the system $(\mathrm{Coker}\,\varphi_{i,n} \mid n \in \mathbb{N})$ becomes a direct summand of the system of $B \otimes_A C$-modules

$$(\mathrm{Ker}\,(U^n : P_{i-1,\bullet} \to P_{i-1,\bullet}) \mid n \in \mathbb{N}) \qquad (7.9.19)$$

whose transition maps are given by multiplication by U. By the same token, the inverse system $(\mathrm{Ker}\,\varphi_{i,n} \mid n \in \mathbb{N})$ is naturally identified with the inverse system $(\mathrm{Fil}_{-n}T_i/I^nT_i \mid n \in \mathbb{N})$, for every $i \in \mathbb{N}$. Now, it follows easily from proposition 7.9.11 and lemma 7.9.8 that, for every $i \in \mathbb{N}$, there exists $c \in \mathbb{N}$ such that

$$\mathrm{Fil}_{-n-c}T_i \subset I^nT_i \qquad \text{for every } n \in \mathbb{N}$$

whence the sought vanishing for the maps between kernels. Lastly, since $B \otimes_A C[U, t_1, \dots, t_r]$ is noetherian, claim 7.9.14 implies that there exists $c \in \mathbb{N}$ such that $\mathrm{Ker}\,U^{c+n} = \mathrm{Ker}\,U^c$ for every $n \in \mathbb{N}$. Therefore the transition map $\mathrm{Ker}\,U^{c+n} \to \mathrm{Ker}\,U^n$ of (7.9.19) vanishes for every $n \in \mathbb{N}$, and the proof of the claim is complete. ◊

(ii) is a standard consequence of (i) : see [163, Prop.3.5.7]. □

7.9.20. Let $\mathbf{f} := (f_1, \dots, f_r)$ be a finite sequence of elements of a ring A, and $I \subset A$ the ideal generated by $\mathbf{f}$. Denote by $\mathrm{Fil}_\bullet A$ the I-adic filtration of A, and set $B_\bullet := \mathsf{R}^+(A, \mathrm{Fil}_\bullet A)$ (notation of definition 7.9.1(iii)); let moreover $\mathbf{f}_0$ be the image in B_0 of the sequence $\mathbf{f}$. The inclusion map $I = B_1 \subset B_\bullet$ extends uniquely to a map of A-algebras

$$\mathrm{Sym}^\bullet_A(I) \to B_\bullet. \qquad (7.9.21)$$

Part (iv) of the following corollary is included in [37, §9, Th.1]; an earlier reference is [128, Ch.I, Th.1] where the same result is proved under the assumption that $\mathbf{f}$ is a regular sequence.

Corollary 7.9.22. *In the situation of* (7.9.20)*, the following holds :*

(i) $B_\bullet$ *satisfies condition* $(\mathrm{c})_{\mathbf{f}_0}$ *if and only if* $B_\bullet$ *satisfies condition* $(\mathrm{c})^{\mathrm{un}}_{\mathbf{f}_0}$*, if and only if* A *satisfies condition* $(\mathrm{c})^{\mathrm{un}}_{\mathbf{f}}$ *(notation of* (7.8.21)*).*
(ii) $B_\bullet$ *satisfies condition* $(\mathrm{a})^{\mathrm{un}}_{\mathbf{f}_0}$ *if and only if* A *satisfies condition* $(\mathrm{a})^{\mathrm{un}}_{\mathbf{f}}$.
(iii) *If* A *is noetherian, then* A *satisfies condition* $(\mathrm{a})^{\mathrm{un}}_{\mathbf{f}}$.
(iv) *If* $H_1(\mathbf{f}, A) = 0$*, then* (7.9.21) *is an isomorphism.*

Proof. (iii): It suffices to apply claim 7.9.18 with A (resp. I, resp. B, resp. C) replaced by A_r (resp. by I_r, resp. by A_r, resp. by A).

(i): Indeed, if condition $(\mathrm{c})_{\mathbf{f}_0}$ holds, for every $i \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that the natural map $H_i(\mathbf{f}_0, I^kB_\bullet) \to H_i(\mathbf{f}_0, B)$ is the zero map. However, clearly the latter is the direct sum of the natural maps

$$H_i(\mathbf{f}, I^{n+k}) \to H_i(\mathbf{f}, I^n) \qquad \text{for every } n \in \mathbb{N}$$

whence $(\mathrm{c})^{\mathrm{un}}_{\mathbf{f}}$. By the same token, we immediately see that $(\mathrm{c})^{\mathrm{un}}_{\mathbf{f}} \Rightarrow (\mathrm{c})^{\mathrm{un}}_{\mathbf{f}_0}$.

(ii): Define A_r, I_r, $\beta_{\mathbf{f}} : A_r \to A$ and $\beta_{\mathbf{f}_0} : A_r \to B_\bullet$ as in (7.8.21), endow A_r with its I_r-adic filtration $\mathrm{Fil}_\bullet A_r$, and set

$$B_{r,\bullet} := \mathsf{R}(A_r, \mathrm{Fil}_\bullet A_r) \qquad C_\bullet := B_{r,\bullet} \otimes_{A_r} A.$$

Let also $\mathbf{t}$ be the sequence $(T_1, \dots, T_r)$ of homogenous elements of degree zero in $B_{r,\bullet}$, and $\mathbf{g}$ the image of $\mathbf{t}$ under the natural map of graded rings $B_{r,\bullet} \to C_\bullet$; since the ring $B_{r,\bullet}$ is noetherian, (iii) tells us that it satisfies condition $(\mathrm{a})^{\mathrm{un}}_{\mathbf{t}}$, which means that we have uniformly essentially zero inverse systems

$$K_{r,\bullet} := (\mathrm{Ker}\,(I_r^n \otimes_{A_r} B_{r,\bullet} \to I_r^n B_{r,\bullet}) \mid n \in \mathbb{N}) \qquad \text{and} \qquad T^i_{r,\bullet} := (\mathrm{Tor}_i^{A_r}(I_r^n, B_{r,\bullet}) \mid n \in \mathbb{N})$$

for every $i > 0$. Suppose now that $(\mathrm{a})^{\mathrm{un}}_{\mathbf{f}}$ holds; this means that also the inverse systems

$$K_\bullet := (\mathrm{Ker}\,(I_r^n \otimes_{A_r} A \to I^n) \mid n \in \mathbb{N}) \quad \text{and} \quad T^i_\bullet := (\mathrm{Tor}_i^{A_r}(I_r^n, A) \mid n \in \mathbb{N}) \quad \text{for every } i > 0$$

are uniformly essentially zero; in this case, we notice :

Claim 7.9.23. For every $i > 0$ there exists $c \in \mathbb{N}$ such that $I^c \cdot T^i_n = 0$ for every $n \in \mathbb{N}$.

Proof of the claim. By assumption, for every $i \in \mathbb{N}$ there exists $c \in \mathbb{N}$ such that the transition map $\varphi_{n+c,n} : T^i_{n+c} \to T^i_n$ is the zero morphism for every $n \in \mathbb{N}$. Now, if $a \in I^c$, the scalar multiplication by a on I^n factors as the composition of an A-linear map $I^n \to I^{n+c}$ and the inclusion map $I^{n+c} \to I^n$. It follows that scalar multiplication by a on T^i_n factors as the composition of an A-linear map $T^i_n \to T^i_{n+c}$ and $\varphi_{n+c,n}$, whence the claim. ◊

Claim 7.9.24. If $C_\bullet$ satisfies condition $(\mathrm{a})^{\mathrm{un}}_{\mathbf{g}}$, then $B_\bullet$ satisfies condition $(\mathrm{a})^{\mathrm{un}}_{\mathbf{f}_0}$.

Proof of the claim. We have a natural surjective homomorphism $f_\bullet : C_\bullet \to B_\bullet$ of graded rings, such that $\mathrm{Ker}\, f_\bullet = \oplus_{n \in \mathbb{N}} K_n$, and arguing as in the proof of claim 7.9.23, we see that there exists $c \in \mathbb{N}$ such that $I^c \cdot K_n = 0$ for every $n \in \mathbb{N}$. Set

$$J_C := \bigcup_{n \in \mathbb{N}} \mathrm{Ann}_{C_\bullet}(I^n) \qquad J_B := \bigcup_{n \in \mathbb{N}} \mathrm{Ann}_{B_\bullet}(I^n)$$

and $\overline{C} := C_\bullet / J_C$; it follows that the projection $C_\bullet \to \overline{C}_\bullet$ factors through a surjective ring homomorphism $B_\bullet \to \overline{C}$ whose kernel equals J_B, so that $f_\bullet(J_C) = J_B$. Suppose now that $C_\bullet$ satisfies condition $(\mathrm{a})^{\mathrm{un}}_{\mathbf{g}}$, and let $\overline{\mathbf{g}}$ be the image of $\mathbf{g}$ in $\overline{C}$; then there exists $k \in \mathbb{N}$ such that $I^k J_C = 0$, and $\overline{C}$ satisfies condition $(\mathrm{a})^{\mathrm{un}}_{\overline{\mathbf{g}}}$ (proposition 7.8.36). Hence $I^k J_B = 0$, and the assertion follows, again by proposition 7.8.36. ◊

In view of claim 7.9.24, we are reduced to checking that for every $i > 0$ the inverse systems

$$K'_\bullet := (\mathrm{Ker}\,(I_r^n \otimes_{A_r} C_\bullet \to I_r^n C_\bullet) \mid n \in \mathbb{N}) \qquad \text{and} \qquad T'^i_\bullet := (\mathrm{Tor}_i^{A_r}(I_r^n, C_\bullet) \mid n \in \mathbb{N})$$

are uniformly essentially zero. However, notice that the natural map

$$K_{r,\bullet} \otimes_{A_r} A \to K'_\bullet$$

is an epimorphism of inverse systems, so we get already the assertion for $K'_\bullet$. Next, we remark:

Claim 7.9.25. The inverse system $(H_i(A \overset{\mathbf{L}}{\otimes}_{A_r} B_{r,\bullet} \overset{\mathbf{L}}{\otimes}_{A_r} I_r^n) \mid n \in \mathbb{N})$ is uniformly essentially zero for every $i > 0$.

Proof of the claim. We have a spectral sequence :

$$E(n)^2_{pq} := \mathrm{Tor}_p^{A_r}(A, \mathrm{Tor}_q^{A_r}(B_{r,\bullet}, I_r^n)) \Rightarrow H_{p+q}(A \overset{\mathbf{L}}{\otimes}_{A_r} B_{r,\bullet} \overset{\mathbf{L}}{\otimes}_{A_r} I_r^n) \qquad \text{for every } n \in \mathbb{N}$$

hence it suffices to check that the inverse system $(E(n)^2_{pq} \mid n \in \mathbb{N})$ is uniformly essentially zero for every $p, q \in \mathbb{N}$ such that $p + q > 0$. The latter assertion is already known for every

$q > 0$, since the system $T^q_{r,\bullet}$ is uniformly essentially zero. For $q = 0$, we consider the short exact sequence of inverse systems

$$0 \to K_{r,\bullet} \to (B_{r,\bullet} \otimes_{A_r} I^n_r \mid n \in \mathbb{N}) \to (I^n_r B_{r,\bullet} \mid n \in \mathbb{N}) \to 0$$

whence, for every $p \in \mathbb{N}$ an exact sequence of inverse systems

$$X^p_\bullet := \mathrm{Tor}^{A_r}_p(A, K_{r,n} \mid n \in \mathbb{N}) \to (E(n)^{p0}_2 \mid n \in \mathbb{N}) \to Y^p_\bullet := (\mathrm{Tor}^{A_r}_p(A, I^n_r B_{r,\bullet}) \mid n \in \mathbb{N}).$$

Now, the system $X^p_\bullet$ is uniformly essentially zero for every $p \in \mathbb{N}$, since the same holds for $K_{r,\bullet}$; moreover, $Y^p_\bullet$ is uniformly essentially zero for $p > 0$, since the same holds for $T^p_\bullet$. Hence, $(E(n)^{p0}_2 \mid n \in \mathbb{N})$ is uniformly essentially zero for $p > 0$, and the proof is concluded. ◊

Lastly, for every $n \in \mathbb{N}$ let us consider the spectral sequence

$$E(n)^2_{pq} := \mathrm{Tor}^{A_r}_p(\mathrm{Tor}^{A_r}_q(A, B_{r,\bullet}), I^n_r) \Rightarrow H_{p+q}(A \overset{\mathbf{L}}{\otimes}_{A_r} B_{r,\bullet} \overset{\mathbf{L}}{\otimes}_{A_r} I^n_r)$$

and notice that $(E(n)^2_{p0} \mid n \in \mathbb{N}) = T'^p_\bullet$ for every $p \in \mathbb{N}$. To conclude, it suffices therefore to show :

Claim 7.9.26. The inverse system $(E(n)^k_{p0} \mid n \in \mathbb{N})$ is uniformly essentially zero, for every $k \geq 2$ and every $p > 0$.

Proof of the claim. We fix $p > 0$ and argue by descending induction on k. Notice that $E(n)^{p+1}_{p0} = E(n)^\infty_{p0}$ for every $n \in \mathbb{N}$, so claim 7.9.25 implies that the assertion holds whenever $k \geq p + 1$. Next, let $2 \leq i \leq p$ and suppose that the assertion is known for every $k > i$; we have an exact sequence of inverse systems

$$(E(n)^{i+1}_{p0} \mid n \in \mathbb{N}) \to (E(n)^i_{p0} \mid n \in \mathbb{N}) \to (E(n)^i_{p-i,i-1} \mid n \in \mathbb{N}).$$

In view of our inductive assumption (and of lemma 7.8.19), we are therefore reduced to showing that the inverse system $(E(n)^i_{p-i,i-1} \mid n \in \mathbb{N})$ is uniformly essentially zero. However, by claim 7.9.23 there exists $c \in \mathbb{N}$ such that $I^c_r \cdot \mathrm{Tor}^{A_r}_{i-1}(A, B_{r,\bullet}) = 0$, and then the contention follows from lemma 7.8.24. ◊

Conversely, if $(\mathrm{a})^{\mathrm{un}}_{\mathbf{f}_0}$ holds for $B_\bullet$, then we get immediately condition $(\mathrm{a})^{\mathrm{un}}_{\mathbf{f}}$ for A, since the latter is a direct summand of the A_r-module $B_\bullet$.

(iv): We consider first the case where $A = A_r$ and $\mathbf{f} = (T_1, \dots, T_r)$, so $I = I_r$ (notation of (7.8.21)). We have natural A_r-linear surjections :

$$\otimes^n_{A_r} I_r \to \mathrm{Sym}^n_{A_r} I_r \xrightarrow{\beta_n} I^n_r \qquad \text{for every } n \in \mathbb{N}$$

and we need to check that β_n is an isomorphism for every $n \in \mathbb{N}$. The assertion is clear for $n = 0, 1$. Let then $n \geq 1$, and notice that $\otimes^n_{A_r} I^n_r$ can be presented as the quotient L_n/R_n where:

- L_n is the free $\mathbb{Z}$-module with basis $\mathscr{B}_n$ given by all sequences $(m_1, \dots, m_n)$ consisting of unital monomials $m_1, \dots, m_n$ of total degree > 0 in the variables $T_1, \dots, T_r$
- $R_n \subset L_n$ is the subgroup generated by all elements of the form :

  $$(m_1, \dots, m_{i-1}, m'm_i, m_{i+1}, \dots, m_n) - (m_1, \dots, m_{j-1}, m'm_j, m_{j+1}, \dots, m_n)$$

  for every $(m_1, \dots, m_n) \in \mathscr{B}_n$, every $m' \in \mathscr{B}_1$, and every $1 \leq i < j \leq n$

(details left to the reader). Likewise, $\mathrm{Sym}^n_{A_r} I_r$ can be presented as the quotient L_n/R'_n, where $R'_n \subset L_n$ is the subgroup generated by R_n and the additional system of elements of the form :

$$(m_1, \dots, m_n) - (m_{\sigma(1)}, \dots, m_{\sigma(n)})$$

for every $(m_1, \dots, m_n) \in \mathscr{B}_n$ and every permutation σ of $\{1, \dots, n\}$. Moreover, for every $n \in \mathbb{N}$ and every $\lambda \in \{1, \dots, r\}$ we have a $\mathbb{Z}$-linear map

$$\varphi_{\lambda,n} : L_n \to L_{n+1} \qquad m_\bullet := (m_1, \dots, m_n) \mapsto (T_\lambda, m_1, \dots, m_n) \qquad \text{for every } m_\bullet \in \mathscr{B}_n$$

and clearly $\varphi_{\lambda,n}(R_n) \subset R_{n+1}$ and $\varphi_{\lambda,n}(R'_n) \subset R'_{n+1}$ for every such n and λ. With this notation, we come down to checking that for every pair $(m_1, \dots, m_n), (m'_1, \dots, m'_n) \in \mathscr{B}_n$ such that $d(m_\bullet) := \prod_{i=1}^n m_n = \prod_{i=1}^n m'_n$, we have $\delta := (m_1, \dots, m_n) - (m'_1, \dots, m'_n) \in R'_n$. We argue by induction on n. The case $n = 1$ is trivial. If $n = d(m_\bullet)$, then $m_i, m'_i \in \{T_1, \dots, T_r\}$ for $i = 1, \dots, n$, so clearly there exists a permutation σ with $m'_i = m_{\sigma(i)}$ for every $i = 1, \dots, n$. Let then $d(m_\bullet) > n \geq 2$, and suppose that the assertion is known for every $p_\bullet, p'_\bullet \in \mathscr{B}_{n-1}$. Pick $\lambda \in \{1, \dots, r\}$ such that T_λ divides some m_i (and hence, some m'_j); it suffices to prove that there exists $p_\bullet, p'_\bullet \in \mathscr{B}_{n-1}$ such that

$$m_\bullet - \varphi_{\lambda,n-1}(p_\bullet), m'_\bullet - \varphi_{\lambda,n-1}(p'_\bullet) \in R'_n \qquad \text{and} \qquad \prod_{i=1}^n m_i = T_\lambda \cdot \prod_{i=2}^n p_i = T_\lambda \cdot \prod_{i=2}^n p'_i.$$

Indeed, in that case, by inductive assumption we have $p_\bullet - p'_\bullet \in R'_{n-1}$, so that $\varphi_{\lambda,n-1}(p_\bullet - p'_\bullet) \in R'_n$, whence the assertion. It suffices to exhibit $p_\bullet$, as the same argument will apply to give $p'_\bullet$. Now, say that T_λ divides m_i; if $m_i = T_\lambda$, there exists $j \in \{1, \dots, n\}$ such that m_j has degree > 1, so $m_j = ab$ for two unital monomials a, b of degrees > 0. Then let $q_\bullet \in \mathscr{B}_n$ be the basis element such that $q_i := am_i$, $q_j := b$, and $q_l := m_l$ for every $l \neq i, j$. Hence $m_\bullet - q_\bullet \in R_n$, and after replacing $m_\bullet$ by $q_\bullet$ we may assume that the degree of m_i is > 1. In that case, $m_i = cT_\lambda$ for a monomial c of degree > 0, and we define $q'_\bullet \in \mathscr{B}_n$ by setting $q'_1 := T_\lambda m_1$, $q'_i := c$, and $q'_l := m_l$ for every $l \neq i, 1$. Again, $m_\bullet - q'_\bullet \in R_n$, so after replacing $m_\bullet$ by $q'_\bullet$ we may moreover assume that $i = 1$. Then, say that $m_1 = dT_\lambda$ for some monomial d of degree > 0; the sought $p_\bullet \in \mathscr{B}_{n-1}$ is obtained by setting $p_1 := m_2 d$ and $p_i := m_{i+1}$ for every $i = 2, \dots n-1$.

Next, let A be a general ring; since $H_1(\mathbf{f}, A) = 0$, we have $\mathrm{Tor}_1^{A_r}(A_r/I_r^n, A) = 0$ for every $n \in \mathbb{N}$ (lemma 7.8.23); *i.e.* the natural $A \otimes_{A_r} I_r^n \to I^n$ is bijective for every $n \in \mathbb{N}$. Composing with the natural isomorphism $\mathrm{Sym}^\bullet_A(A \otimes_{A_r} I_r) \xrightarrow{\sim} A \otimes_{A_r} \mathrm{Sym}^\bullet_{A_r}(I_r)$, the assertion for A and $\mathbf{f}$ follows from that for A_r and $(T_1, \dots, T_r)$. □

Remark 7.9.27. By carefully tracking the estimates in the proof, it is easily seen that for every sequence of integers $(c(i) \mid i \in \mathbb{N})$ there exists a sequence of integers $(d(i) \mid i \in \mathbb{N})$ such that the following refinement of corollary 7.9.22(ii) holds. Suppose that for every $i \in \mathbb{N}$ the inverse system $(\mathrm{Tor}_i^{A_r}(A_r/I_r^n, A) \mid n \in \mathbb{N})$ is uniformly essentially zero with step $\leq c(i)$; then the inverse system $(\mathrm{Tor}_i^{A_r}(A_r/I_r^n, B_\bullet) \mid n \in \mathbb{N})$ is uniformly essentially zero with step $\leq d(i)$.

7.9.28. Let $(A, \mathrm{Fil}_\bullet A)$ be a filtered ring, *i.e.* a filtered $\mathbb{Z}$-algebra, in the sense of definition 7.9.1(i), and denote by $\mathrm{gr}_\bullet A$ the associated graded ring. Suppose that the filtration $\mathrm{Fil}_\bullet A$ is *exhaustive* and *separated*, that is

$$\bigcup_{n \in \mathbb{Z}} \mathrm{Fil}_n A = A \qquad \text{and} \qquad \bigcap_{n \in \mathbb{Z}} \mathrm{Fil}_n A = 0.$$

The filtration $\mathrm{Fil}_\bullet A$ defines a linear topology $\mathscr{T}$ on the $\mathbb{Z}$-module A, and we suppose moreover that $\mathscr{T}$ is complete (and notice that $\mathscr{T}$ is obviously separated). Furthermore, let $\mathbf{f} := (f_1, \dots, f_r)$ be a finite sequence of elements of A; for every $i = 1, \dots, r$, pick $n_i \in \mathbb{Z}$ such that $f_i \in \mathrm{Fil}_{n_i} A$, denote by $\overline{f}_i \in \mathrm{gr}_{n_i} A$ the image of f_i, and let $\overline{\mathbf{f}} := (\overline{f}_1, \dots, \overline{f}_r)$ be the resulting sequence of elements of $\mathrm{gr}_\bullet A$.

Proposition 7.9.29. *In the situation of* (7.9.28)*, suppose that the sequence* $\overline{\mathbf{f}}$ *is completely secant (see definition* 7.8.6*). Then the same holds for the sequence* $\mathbf{f}$.

Proof. We define as follows a filtration on the Koszul complex $\mathbf{K}_\bullet(\mathbf{f}) = \mathbf{K}_\bullet(\mathbf{f}, A)$, *i.e.* a compatible system of filtrations on the exterior powers $\Lambda^i_A(A^{\oplus r})$ for $i = 1, \dots, r$. To this aim, for every $i \leq r$ denote by S_i the set of all strictly increasing maps $\{1, \dots, i\} \to \{1, \dots, r\}$; for every $\varphi \in S_i$ set

$$e_\varphi := e_{\varphi(1)} \wedge \cdots \wedge e_{\varphi(i)} \qquad \text{and} \qquad n_\varphi := n_{\varphi(1)} + \cdots + n_{\varphi(i)}.$$

We let $\mathrm{Fil}_\bullet\Lambda^i_A(A^{\oplus r})$ be the good filtration associated, as in (7.9.6), with the basis $(e_\varphi \mid \varphi \in S_i)$ and the sequence of integers $(n_\varphi \mid \varphi \in S_i)$. The explicit description in (7.8) shows that the differentials of the Koszul complex are morphisms of filtered abelian groups, for the resulting filtrations on the terms $\mathbf{K}_i(\mathbf{f})$. Thus, we have the sought filtration $\mathrm{Fil}_\bullet\mathbf{K}_\bullet(\mathbf{f}, A)$, and we let $\mathrm{gr}_\bullet\mathbf{K}_\bullet(\mathbf{f}, A)$ be the associated complex of graded abelian groups; with this notation, a simple inspection shows that we have a natural isomorphism of complexes of abelian groups

$$\mathrm{gr}_\bullet\mathbf{K}_\bullet(\mathbf{f}, A) \xrightarrow{\sim} \mathbf{K}_\bullet(\overline{\mathbf{f}}, \mathrm{gr}_\bullet A). \qquad (7.9.30)$$

For every $j, k \in \mathbb{Z}$ with $j \geq k$ we set

$$\mathbf{K}^{(j,k)}_\bullet(\mathbf{f}, A) := \mathrm{Fil}_j\mathbf{K}_\bullet(\mathbf{f}, A)/\mathrm{Fil}_{k-1}\mathbf{K}_\bullet(\mathbf{f}, A).$$

We endow $\mathbf{K}^{(j,k)}_\bullet(\mathbf{f}, A)$ with the filtration induced by $\mathrm{Fil}_\bullet\mathbf{K}_\bullet(\mathbf{f}, A)$, and we let $\mathrm{gr}_\bullet\mathbf{K}^{(j,k)}_\bullet(\mathbf{f}, A)$ be the complex of graded abelian groups associated with this filtered subquotient of $\mathbf{K}_\bullet(\mathbf{f}, A)$. According to remark 7.2.16(i), we have a 1-spectral sequence

$$E_1^{pq} := H^{p+q}(\mathrm{gr}_{-p}\mathbf{K}^{(j,k)}_\bullet(\mathbf{f}, A)) \Rightarrow H^{p+q}\mathbf{K}^{(j,k)}_\bullet(\mathbf{f}, A).$$

However, (7.9.30) and the assumption on $\overline{\mathbf{f}}$ imply that $E_1^{pq} = 0$ whenever $p + q < 0$, and then proposition 7.2.18 implies that $H^n(\mathbf{K}^{(j,k)}_\bullet(\mathbf{f}, A)) = 0$ for every j, k, n with $j \geq k$ and $n < 0$. Set as well $\mathbf{K}^{(k)}_\bullet(\mathbf{f}, A) := \mathbf{K}_\bullet(\mathbf{f}, A)/\mathrm{Fil}_{k-1}\mathbf{K}_\bullet(\mathbf{f}, A)$ for every $k \in \mathbb{Z}$; it follows easily that

$$H^n\mathbf{K}^{(k)}_\bullet(\mathbf{f}, A) = 0 \qquad \text{for every } n < 0 \text{ and every } k \in \mathbb{Z}.$$

Next, since $\mathbf{K}_n(\mathbf{f}, A)$ is a free A-module of finite rank for every $n \in \mathbb{Z}$, and since the filtration $\mathrm{Fil}_\bullet A$ is separated and defines a complete topology, we get a natural isomorphism of abelian groups

$$\mathbf{K}_n(\mathbf{f}, A) \xrightarrow{\sim} \lim_{k\in\mathbb{Z}} \mathbf{K}^{(k)}_n(\mathbf{f}, A) \qquad \text{for every } n \in \mathbb{Z}.$$

Lastly, in view of [163, Prop.3.5.7, Th.3.5.8] we conclude that $H^n\mathbf{K}_\bullet(\mathbf{f}, A) = 0$ for every $n < 0$, *i.e.* the sequence $\mathbf{f}$ is completely secant, as stated. □

7.10. **Some homotopical algebra.** The methods of homotopical algebra allow to construct derived functors of non-additive functors, in a variety of situations. In this section, we present the basics of this theory, beginning with its cornerstone, the *standard resolution associated with a triple*, as explained in the following paragraph.

7.10.1. A *triple* $(\top, \boldsymbol{\eta}, \boldsymbol{\mu})$ on a category $\mathscr{C}$ is the datum of a functor $\top : \mathscr{C} \to \mathscr{C}$ together with natural transformations :

$$\boldsymbol{\eta} : \mathbf{1}_\mathscr{C} \Rightarrow \top \qquad \boldsymbol{\mu} : \top \circ \top \Rightarrow \top$$

such that the following diagrams commute :

$$\begin{array}{ccccc} (\top\circ\top)\circ\top & = & \top\circ(\top\circ\top) & \xRightarrow{\top*\boldsymbol{\mu}} & \top\circ\top \\ {\scriptstyle \boldsymbol{\mu}*\top}\Downarrow & & & & \Downarrow{\scriptstyle \boldsymbol{\mu}} \\ \top\circ\top & & \xRightarrow{\quad\boldsymbol{\mu}\quad} & & \top \end{array} \qquad \begin{array}{ccccc} \top & \xRightarrow{\top*\boldsymbol{\eta}} & \top\circ\top & \xLeftarrow{\boldsymbol{\eta}*\top} & \top \\ & {\scriptstyle \mathbf{1}_\top}\searrow & \Downarrow{\scriptstyle \boldsymbol{\mu}} & \swarrow{\scriptstyle \mathbf{1}_\top} & \\ & & \top. & & \end{array}$$

Dually, a cotriple $(\perp, \boldsymbol{\varepsilon}, \boldsymbol{\delta})$ on a category $\mathscr{C}$ is a functor $\perp : \mathscr{C} \to \mathscr{C}$ together with natural transformations :

$$\boldsymbol{\varepsilon} : \perp \Rightarrow \mathbf{1}_\mathscr{C} \qquad \boldsymbol{\delta} : \perp \Rightarrow \perp \circ \perp$$

such that the following diagrams commute :

$$\begin{array}{ccccc} \perp & \xRightarrow{\quad\boldsymbol{\delta}\quad} & & & \perp\circ\perp \\ {\scriptstyle\boldsymbol{\delta}}\Downarrow & & & & \Downarrow{\scriptstyle\boldsymbol{\delta}*\perp} \\ \perp\circ\perp & \xRightarrow{\perp*\boldsymbol{\delta}} & \perp\circ(\perp\circ\perp) & = & (\perp\circ\perp)\circ\perp \end{array} \qquad \begin{array}{ccccc} & & \perp & & \\ & {\scriptstyle\mathbf{1}_\perp}\swarrow & \Downarrow{\scriptstyle\boldsymbol{\delta}} & \searrow{\scriptstyle\mathbf{1}_\perp} & \\ \perp & \xLeftarrow{\perp*\boldsymbol{\varepsilon}} & \perp\circ\perp & \xRightarrow{\boldsymbol{\varepsilon}*\perp} & \perp . \end{array}$$

7.10.2. A cotriple $(\perp, \boldsymbol{\varepsilon}, \boldsymbol{\delta})$ on $\mathscr{C}$ and an object A of $\mathscr{C}$ yield a simplicial object $\perp A[\bullet]$ in $\mathscr{C}$; namely, for every $n \in \mathbb{N}$ set $\perp A[n] :=\perp^{n+1} A$, and define face and degeneracy operators :

$$\partial_i := (\perp^i *\boldsymbol{\varepsilon}* \perp^{n-i})_A \quad : \quad \perp A[n] \to\perp A[n-1]$$
$$\sigma_i := (\perp^i *\boldsymbol{\delta}* \perp^{n-i})_A \quad : \quad \perp A[n] \to\perp A[n+1].$$

Using the foregoing commutative diagrams, one verifies easily that the simplicial identities (7.4.9) hold. Moreover, the morphism $\boldsymbol{\varepsilon}_A :\perp A \to A$ defines an augmentation $\perp A[\bullet] \to A$.

Dually, a triple $(\top, \boldsymbol{\eta}, \boldsymbol{\mu})$ and an object A of $\mathscr{C}$ determine a cosimplicial object $\top A[\bullet] := \top^{\bullet+1} A$, such that

$$\partial^i := (\top^i * \boldsymbol{\eta} * \top^{n-i})_A \quad : \quad \top^n A \to \top^{n+1} A$$
$$\sigma^i := (\top^i * \boldsymbol{\mu} * \top^{n-i})_A \quad : \quad \top^{n+2} A \to \top^{n+1} A$$

which is augmented by the morphism $\boldsymbol{\eta}_A : A \to \top A$. Clearly, the rule : $A \mapsto\perp A[\bullet]$ (resp. $A \mapsto \top A[\bullet]$) defines a functor

$$\mathscr{C} \to \widehat{s.\mathscr{C}} \qquad (\text{resp. } \mathscr{C} \to \widehat{c.\mathscr{C}}).$$

7.10.3. An adjoint pair of functors $(G : \mathscr{B} \to \mathscr{A}, F : \mathscr{A} \to \mathscr{B})$ with its unit $\boldsymbol{\eta}$ and counit $\boldsymbol{\varepsilon}$ (see (1.1.13)), determines a triple $(\top, \boldsymbol{\eta}, \boldsymbol{\mu})$ on $\mathscr{B}$, where :

$$\top := F \circ G : \mathscr{B} \to \mathscr{B} \qquad \boldsymbol{\mu} := F * \boldsymbol{\varepsilon} * G : \top \circ \top \Rightarrow \top$$

as well as a cotriple $(\perp, \boldsymbol{\varepsilon}, \boldsymbol{\delta})$ on $\mathscr{A}$, where :

$$\perp := G \circ F : \mathscr{A} \to \mathscr{A} \qquad \boldsymbol{\delta} := G * \boldsymbol{\eta} * F : \perp \Rightarrow \perp \circ \perp .$$

Indeed, the naturality of $\boldsymbol{\varepsilon}$ and $\boldsymbol{\eta}$ easily implies the commutativity of the diagrams of (7.10.1).

Remark 7.10.4. (i) Notice that a cotriple on $\mathscr{C}$ is the same as a triple on $\mathscr{C}^o$: more precisely, to every triple $(\top, \boldsymbol{\eta}, \boldsymbol{\mu})$ on $\mathscr{C}$ there corresponds a dual cotriple on $\mathscr{C}^o$, given by the datum

$$(\top, \boldsymbol{\eta}, \boldsymbol{\mu})^o := (\top^o, \boldsymbol{\eta}^o, \boldsymbol{\mu}^o).$$

(ii) Also, clearly the dual of the cosimplicial set $\top A[\bullet]$ is the simplicial set in $\mathscr{C}^o$ attached to the dual cotriple $(\top, \boldsymbol{\eta}, \boldsymbol{\mu})^o$ and the object A (seen as an object of $\mathscr{C}^o$).

(iii) Recall that every adjoint pair $(G : \mathscr{B} \to \mathscr{A}, F : \mathscr{A} \to \mathscr{B})$ yields by duality an adjoint pair $(F^o : \mathscr{A}^o \to \mathscr{B}^o, G^o : \mathscr{B}^o \to \mathscr{A}^o)$, and if $\boldsymbol{\eta}$ (resp. $\boldsymbol{\varepsilon}$) is the unit (resp. the counit) of a given adjunction for the pair (G, F), then $\boldsymbol{\varepsilon}^o$ (resp. $\boldsymbol{\eta}^o$) is a unit (resp. a counit) of an adjunction for the dual pair (F^o, G^o) (see remark 1.1.19(iv)). Then, by direct inspection we easily see that the cotriple $(\perp, \boldsymbol{\varepsilon}, \boldsymbol{\delta})$ attached to the adjoint pair (G, F) and to $(\boldsymbol{\eta}, \boldsymbol{\varepsilon})$ is the dual of the triple attached to the adjoint pair (F^o, G^o) and to $(\boldsymbol{\varepsilon}^o, \boldsymbol{\eta}^o)$.

The following proposition explains how to use triples and cotriples to construct functorial resolutions.

Proposition 7.10.5. *In the situation of* (7.10.3)*, the following holds :*

(i) *For every* $A \in \mathrm{Ob}(\mathscr{A})$ *and* $B \in \mathrm{Ob}(\mathscr{B})$*, the augmented simplicial objects :*

$$\perp GB[\bullet] \xrightarrow{\boldsymbol{\varepsilon}_{GB}} GB \qquad F\perp A[\bullet] \xrightarrow{F*\boldsymbol{\varepsilon}_A} FA$$

are homotopically trivial (notation of (7.10.2)*).*

(ii) *Dually, the same holds for the augmented cosimplicial objects :*

$$GB \xrightarrow{G*\boldsymbol{\eta}_B} G\top B[\bullet] \qquad FA \xrightarrow{\boldsymbol{\eta}_{FA}} \top FA[\bullet].$$

Proof. Notice that

$$F\bot A[n] = \top FA[n] \qquad \text{for every } [n] \in \mathrm{Ob}(\Delta^\wedge).$$

Therefore, for every morphism $\varphi : [n] \to [m]$ in $\Delta^\wedge$, we have two morphisms

$$F\bot A[\varphi] : F\bot A[m] \to F\bot A[n] \qquad \top FA[\varphi] : F\bot A[n] \to F\bot A[m].$$

Now, recall that the augmentation ε_A defines a natural morphism

$$\bot A[\varepsilon^{\bullet+1}_{-1,0}] : \bot A[\bullet] \to s.A \qquad [n] \mapsto \bot A[\varepsilon^{n+1}_{-1,0}]$$

(notation of remark 7.4.12(iv)), and one sees that the system $(\top FA[\varepsilon^{n+1}_{-1,0}] \mid [n] \in \mathrm{Ob}(\Delta))$ defines a morphism

$$\top FA[\varepsilon^{\bullet+1}_{-1,0}] : s.FA \to F\bot A[\bullet]$$

which is right inverse to $F\bot A[\varepsilon^{\bullet+1}_{-1,0}]$. For every $n, k \in \mathbb{N}$ such that $k \leq n+1$, set

$$u_{n,k} := (\top FA[\varepsilon^k_{n-k,0}]) \circ (F\bot A[\varepsilon^k_{n-k,0}])$$

(notation of example 7.4.7(ii)).

Claim 7.10.6. The system $u_{\bullet\bullet}$ defines a homotopy from $\mathbf{1}_{F\bot A[\bullet]}$ to $(\top FA[\varepsilon^{\bullet+1}_{-1,0}]) \circ (F\bot A[\varepsilon^{\bullet+1}_{-1,0}])$ (see (7.4.16)).

Proof of the claim. By unwinding the definitions, we see that $F\bot A[\varepsilon^k_{n-k,0}]$ is the composition

$$F(\boldsymbol{\varepsilon}_{\bot^{n+1-k}A} \circ \cdots \circ \boldsymbol{\varepsilon}_{\bot^{n-1}A} \circ \boldsymbol{\varepsilon}_{\bot^n A}) : F\bot^{n+1}A \to F\bot^{n+1-k}A \qquad \text{for } k > 0$$

and $\top FA[\varepsilon^k_{n-k,0}]$ is the composition

$$\boldsymbol{\eta}_{F\bot^n A} \circ \boldsymbol{\eta}_{F\bot^{n-1}A} \circ \cdots \circ \boldsymbol{\eta}_{F\bot^{n+1-k}A} : F\bot^{n+1-k}A \to F\bot^{n+1}A \qquad \text{for } k > 0$$

and both equal $\mathbf{1}_{F\bot^{n+1}A}$ for $k = 0$. Now, since the face operator ∂_i of $F\bot A[n]$ is $F\bot^i\,(\boldsymbol{\varepsilon}_{\bot^{n-i}A})$ for every $i \leq n$, the naturality of $\boldsymbol{\eta}$ yields a commutative diagram

$$\begin{array}{ccc} F\bot^n A & \xrightarrow{\partial_{i-1}} & F\bot^{n-1}A \\ \Big\downarrow{\scriptstyle \boldsymbol{\eta}_{F\bot^n A}} & & \Big\downarrow{\scriptstyle \boldsymbol{\eta}_{F\bot^{n-1}A}} \\ F\bot^{n+1}A & \xrightarrow{\partial_i} & F\bot^n A \end{array} \qquad \text{for every } i > 0 \text{ and every } n \in \mathbb{N}$$

whereas $\partial_0 \circ \boldsymbol{\eta}_{F\bot^n A} = \mathbf{1}_{F\bot^n A}$ for every $n \in \mathbb{N}$; whence the identities :

$$\partial_i \circ \top FA[\varepsilon^k_{n-k,0}] = \begin{cases} \top FA[\varepsilon^{k-1}_{n-k,0}] & \text{for } i < k \\ \top FA[\varepsilon^k_{n-k-1,0}] \circ \partial_{i-k} & \text{for } i \geq k. \end{cases}$$

On the other hand, we have the simplicial identities :

$$\varepsilon_i \circ \varepsilon^k_{n-k,0} = \begin{cases} \varepsilon^{k+1}_{n-k} & \text{for } i < k \\ \varepsilon^k_{n+1-k} \circ \varepsilon_{i-k} & \text{for } i \geq k. \end{cases}$$

Summing up, we conclude that

$$\partial_i \circ u_{n,k} = \begin{cases} u_{n-1,k-1} \circ \partial_i & \text{for } i < k \\ u_{n-1,k} \circ \partial_i & \text{for } i \geq k. \end{cases}$$

Arguing likewise, one deduces as well the corresponding commutation rule – spelled out in (7.4.16) – for the degeneracies σ_i, and the claim follows. ◊

Likewise, notice that

$$G\top B[n] = \bot GB[n] \qquad \text{for every } [n] \in \Delta^\wedge.$$

Therefore, for every morphism $\varphi : [n] \to [m]$ in $\Delta^\wedge$, we have as well two morphisms

$$\perp GB[\varphi] :\perp GB[m] \to\perp GB[n] \qquad G\top B[\varphi] :\perp GB[n] \to\perp GB[m]$$

and the system $(G\top B[\varepsilon^{n+1}_{-1,0}] \mid [n] \in \mathrm{Ob}(\Delta^\wedge))$ defines a morphism

$$G\top B[\varepsilon^{\bullet+1}_{-1,0}] : s.GB \to\perp GB[\bullet]$$

which is right inverse to the morphism $\perp GB[\varepsilon^{\bullet+1}_{-1,0}]$ deduced from the augmentation. Arguing as in the proof of claim 7.10.6, one checks that the rule

$$v_{n,k} := (G\top B[\varepsilon^{k\vee}_{n-k,0}]) \circ (\perp GB[\varepsilon^{k\vee}_{n-k,0}]) \qquad \text{for every } n, k \in \mathbb{N} \text{ such that } k \leq n+1$$

yields a homotopy v from $\mathbf{1}_{\perp GB[\bullet]}$ to $(G\top B[\varepsilon^{\bullet+1}_{-1,0}]) \circ (\perp GB[\varepsilon^{\bullet+1}_{-1,0}])$.

The dual statements admit the dual proof. □

7.10.7. We explain first an application of the foregoing constructions to the construction of the derived functors of certain additive functors. Indeed, let now $\mathscr{A}$ be an abelian category, and $(\perp, \boldsymbol{\varepsilon}, \boldsymbol{\delta})$ a cotriple on $\mathscr{A}$. In view of (7.10.2) and (7.4.25) we get a functor

$$\triangle_0 := \mathsf{U}\circ\perp^\bullet : \mathscr{A} \to \mathsf{C}^{\leq 0}(\mathscr{A})$$

that assigns to any object A of $\mathscr{A}$ the unnormalized chain complex associated to the simplicial object $\perp A[\bullet]$. Also, the augmentation $\boldsymbol{\varepsilon}_A :\perp A[\bullet] \to A$ induces a natural transformation $\mathsf{U} * \boldsymbol{\varepsilon} : \triangle_0 \Rightarrow \bullet[0]$ (notation of (7.1.3)). Then we can consider the composition :

$$\triangle : \mathsf{C}^-(\mathscr{A}) \xrightarrow{\mathsf{C}^-(\triangle_0)} \mathsf{C}^-(\mathsf{C}^{\leq 0}(\mathscr{A})) \xrightarrow{\mathrm{Tot}} \mathsf{C}^-(\mathscr{A})$$

and notice that $\mathrm{Tot}\circ\mathsf{C}^-(\bullet[0])$ is the identity automorphism of $\mathsf{C}^-(\mathscr{A})$, so $\mathsf{U}*\boldsymbol{\varepsilon}$ induces a natural transformation :

$$\omega_{K^\bullet} : \triangle K^\bullet \to K^\bullet \qquad \text{for every } K^\bullet \in \mathsf{C}^-(\mathscr{A}).$$

Proposition 7.10.8. *Let $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{A}'$ be three abelian categories, $\varphi : \mathscr{A} \to \mathscr{A}'$ an additive functor, and $(G : \mathscr{B} \to \mathscr{A}, F : \mathscr{A} \to \mathscr{B})$ an adjoint pair of functors. Denote by $G\mathscr{B}$ the full subcategory of $\mathscr{A}$ with $\mathrm{Ob}(G\mathscr{B}) := \{GB \mid B \in \mathrm{Ob}(\mathscr{B})\}$, and suppose that :*

(a) *G and $\varphi \circ G$ are both exact.*
(b) *F is exact and faithful.*

Then we have :

(i) *$\omega_{K^\bullet}$ is a quasi-isomorphism for every $K^\bullet \in \mathrm{Ob}(\mathsf{C}^-(\mathscr{A}))$.*
(ii) *$G\mathscr{B}$ is a φ-projective subcategory of $\mathscr{A}$ (see remark* 7.3.31(v)*).*

Proof. Let $(\perp, \boldsymbol{\varepsilon}, \boldsymbol{\delta})$ be the cotriple associated with the adjoint pair (G, F), as in in (7.10.3). By proposition 7.10.5(i), the counit $\boldsymbol{\varepsilon}_A$ induces a quasi-isomorphism $F(\triangle_0 A) \xrightarrow{\sim} FA[0]$ for every $A \in \mathrm{Ob}(\mathscr{A})$, where $\triangle_0$ is defined as in (7.10.7); then the natural transformation $\triangle_0 A \to A[0]$ is already a quasi-isomorphism for every such A, since F is exact and faithful. The assertion then follows from [163, Lemma 2.7.3].

(ii): For every $i \in \mathbb{N}$, consider the functor

$$\varphi_i := H_i \circ \mathsf{C}(\varphi) \circ \triangle_0 : \mathscr{A} \to \mathscr{B} \qquad A \mapsto H_i(\varphi(\triangle_0 A)).$$

Again by proposition 7.10.5(i), $\boldsymbol{\varepsilon}_{GB}$ induces a quasi-isomorphism $\triangle_0(GB) \xrightarrow{\sim} GB[0]$ for every $B \in \mathrm{Ob}(\mathscr{B})$, so $\varphi_i(GB) = 0$ for every $i > 0$, and we get a natural transformation $\varphi_0 \Rightarrow \varphi$ whose restriction to $G\mathscr{B}$ is an isomorphism of functors. Since G, F and φ are exact, the same holds for $\perp$ and $\varphi\circ\perp$; hence, every short exact sequence $\Sigma := (0 \to A' \to A \to A'' \to 0)$ of $\mathscr{A}$ induces a short exact sequence

$$\varphi(\triangle_0\Sigma) \quad : \quad 0 \to \varphi(\triangle_0 A') \to \varphi(\triangle_0 A) \to \varphi(\triangle_0 A'') \to 0 \qquad \text{in } \mathsf{C}(\mathscr{A}').$$

Especially, if $\varphi_i(A) = \varphi_i(A'') = 0$ for every $i > 0$, we have as well $\varphi_i(A') = 0$ for every $i > 0$, by the long exact cohomology sequence associated with $\varphi(T\Sigma)$. Let then $X_\bullet := (X_n, d_n \mid n \in \mathbb{N})$ be any acyclic object of $\mathsf{C}^-(G\mathscr{B})$; by considering the short exact sequences $\Sigma_n := (0 \to \mathrm{Im}(d_{n+1}) \to X_n \to \mathrm{Im}(d_i) \to 0)$, we deduce inductively that $\varphi_i(\mathrm{Im}\, d_n) = 0$ for every $i > 0$ and every $n \in \mathbb{N}$, so the sequence $\varphi_0(\Sigma_n)$ is still short exact for every such n. Hence, the complex $\varphi_0(X_\bullet)$ is still acyclic, and then the same holds for the complex $\varphi(X_\bullet)$. Lastly, since F is faithful, the counit $\varepsilon_A : GFA \to A$ is an epimorphism (proposition 1.1.20(ii)), whence the assertion. □

Remark 7.10.9. (i) Proposition 7.10.8(i) implies that the functor $\triangle$ of (7.10.7) takes quasi-isomorphisms to quasi-isomorphisms, so it descends to a well defined functor

$$\triangle : \mathsf{D}^-(\mathscr{A}) \to \mathsf{D}^-_{G\mathscr{B}}(\mathscr{A})$$

(notation of (7.3.21)) and taking into account remark 7.3.24(i) we see that $\triangle$ is a quasi-inverse for the functor $i^- : \mathsf{D}^-_{G\mathscr{B}}(\mathscr{A}) \to \mathsf{D}^-(\mathscr{A})$ induced by the inclusion functor $i : G\mathscr{B} \to \mathscr{A}$. By virtue of proposition 7.10.8(ii) and remark 7.3.31(v), we then conclude that the left derived functor of φ is given by :

$$L\varphi : \mathsf{D}^-(\mathscr{A}) \to \mathsf{D}^-(\mathscr{A}') \qquad K^\bullet \mapsto \varphi(\triangle K^\bullet).$$

(ii) In view of remark 7.10.4, the constructions of (7.10.7) can be dualized : every triple $(\top, \boldsymbol{\eta}, \boldsymbol{\mu})$ on $\mathscr{A}$ induces first a functor $\nabla_0 : \mathscr{A} \to \mathsf{C}^{\geq 0}(\mathscr{A})$ that attaches to every $A \in \mathrm{Ob}(\mathscr{A})$ the unnormalized cochain complex of the cosimplicial object $\top A[\bullet]$ (the latter is obtained by regarding $\top A[\bullet]$ as a simplicial object of $\mathscr{A}^o$, and applying the corresponding functor $\mathsf{U} : s.\mathscr{A}^o \to \mathsf{C}^{\leq 0}(\mathscr{A}^o) = \mathsf{C}^{\geq 0}(\mathscr{A})^o$: see remark 7.1.6(iii)), which in turns induces a functor

$$\nabla : \mathsf{C}^{\geq 0}(\mathscr{A}) \to \mathsf{C}^{\geq 0}(\mathscr{A})$$

together with a natural transformation $\tau_{K^\bullet} : K^\bullet \to \nabla K^\bullet$ for every $K^\bullet \in \mathrm{Ob}(\mathsf{C}^{\geq 0}(\mathscr{A}))$.

(iii) Likewise, proposition 7.10.8 dualizes as follows. Let $\mathscr{A}, \mathscr{B}$ and $\mathscr{A}'$ be three abelian categories, $\varphi : \mathscr{A} \to \mathscr{A}'$ an additive functor, and $(F : \mathscr{A} \to \mathscr{B}, G : \mathscr{B} \to \mathscr{A})$ an adjoint pair of functors (so, now F is left adjoint to G). Define the subcategory $G\mathscr{B}$ as in proposition 7.10.8, and suppose that conditions (a) and (b) of *loc.cit.* hold. Then the morphism $\tau_{K^\bullet}$ of (ii) is a quasi-isomorphism for every $K^\bullet \in \mathsf{D}^+(\mathscr{A})$, and $G\mathscr{B}$ is a φ-injective subcategory of $\mathscr{A}$. Invoking remark 7.3.31(iv), we deduce that, in this situation, the right derived functor of φ is given by :

$$R\varphi : \mathsf{D}^+(\mathscr{A}) \to \mathsf{D}^+(\mathscr{A}') \qquad K^\bullet \mapsto \varphi(\nabla K^\bullet).$$

Example 7.10.10. (i) The following application is already suggested in [77, App.2, Rem.3.7]. Let I, J be two small categories, $\mathscr{A}$ a complete abelian category, and $f : I \to J$ a functor. By theorem 1.3.4, the right Kan extension $\int_f$ along f is right adjoint to $\mathsf{Fun}(f, \mathscr{A}) : \mathsf{Fun}(J, \mathscr{A}) \to \mathsf{Fun}(I, \mathscr{A})$. Moreover, the limit functor $\mathrm{Lim}_J : \mathsf{Fun}(J, \mathscr{A}) \to \mathscr{A}$ is right adjoint to the constant functor $c_J : \mathscr{A} \to \mathsf{Fun}(J, \mathscr{A})$. It is easily seen that the functor $\mathsf{Fun}(f, \mathscr{A})$ is exact; suppose then that :

(a) f is essentially surjective
(b) Both Lim_I and $\mathrm{Lim}_{j/fI}$ for every $j \in \mathrm{Ob}(J)$ are exact.

It is easily seen that (a) implies that $\mathsf{Fun}(f, \mathscr{A})$ is faithful, and (b) implies that $\int_f$ is exact; taking into account remark 7.10.9(iii), we deduce that the right derived functor of Lim_J is given by :

$$R\lim_J : \mathsf{D}^+(\mathsf{Fun}(J, \mathscr{A})) \to \mathsf{D}^+(\mathscr{A}) \qquad K^\bullet \mapsto \lim_J(\nabla K^\bullet)$$

where $\nabla : \mathsf{C}^+(\mathsf{Fun}(J, \mathscr{A})) \to \mathsf{C}^+(\mathsf{Fun}(J, \mathscr{A}))$ is induced by the triple associated with the adjoint pair $(\mathsf{Fun}(f, \mathscr{A}), \int_f)$.

(ii) With the notation of (i), take for I a discrete subcategory of J, such that every object of J is isomorphic to some element of $\mathrm{Ob}(I)$, and let $f : I \to J$ be the inclusion functor. Then f is essentially surjective, and the category j/fI is discrete for every $j \in \mathrm{Ob}(J)$. Suppose then that *the products are exact* in the complete abelian category $\mathscr{A}$, *i.e.* that the functor Lim_S : $\mathsf{Fun}(S, \mathscr{A}) \to \mathscr{A}$ is exact for every small discrete category S. In this situation, conditions (a) and (b) of (i) are verified, so we get a construction of $R\,\mathrm{Lim}_J$ via the standard resolutions of proposition 7.10.5(ii). Let us explicit this construction, for any given functor $F : J \to \mathscr{A}$: $j \mapsto F_j$. First, for every $n \in \mathbb{N}$ and $j \in \mathrm{Ob}(J)$, let $S(n, j)$ be the set of all functors $\beta : [n] \to J$ with $\beta(0) = j$ and $\beta(i) \in \mathrm{Ob}(I)$ for every $i = 1, \dots, n$ (recall that $[n]$ is the set of integers $\{0, \dots, n\}$, with its standard ordering). Then $\nabla_0 F$ is the complex of $\mathsf{Fun}(J, \mathscr{A})$:

$$j \mapsto (\nabla_0 F)^\bullet_j := \Big(0 \to \prod_{\beta \in S(1,j)} F_{\beta(1)} \xrightarrow{d^0_j} \prod_{\beta \in S(2,j)} F_{\beta(2)} \xrightarrow{d^1_j} \cdots\Big)$$

where, in view of the sign conventions of remark 7.1.6(iii), we set $d^n_j = \sum_{i=0}^{n+1} (-1)^{i+n+1} \cdot \partial^{i+1}_j$, and ∂^i_j is the unique morphism that makes commute the diagram :

$$\begin{array}{ccc} \prod_{\beta \in S(n+1,j)} F_{\beta(n+1)} & \xrightarrow{\partial^i_j} & \prod_{\beta \in S(n+2,j)} F_{\beta(n+2)} \\ \downarrow{\scriptstyle p_{\gamma \circ \varepsilon_i}} & & \downarrow{\scriptstyle p_\gamma} \\ F_{\gamma \circ \varepsilon_i(n+1)} & \xrightarrow{F_{\gamma(\varepsilon_i(n+1) \to n+2)}} & F_{\gamma(n+2)} \end{array} \qquad \text{for every } \gamma \in S(n+2, j)$$

where $\varepsilon_i : [n+1] \to [n+2]$ is the i-th face map, and p_γ and $p_{\gamma \circ \varepsilon_i}$ are the natural projections. To every morphism $\varphi : j \to k$ of J, there corresponds the morphism $(\nabla_0 F)^\bullet_\varphi : (\nabla_0 F)^\bullet_j \to (\nabla_0 F)^\bullet_k$ of $\mathsf{C}^{\geq 0}(\mathscr{A})$ that makes commute the diagram :

$$\begin{array}{ccc} \prod_{\beta \in S(n,j)} F_{\beta(n)} & \xrightarrow{(\nabla_0 F)^{n-1}_\varphi} & \prod_{\beta \in S(n,k)} F_{\beta(n)} \\ \downarrow{\scriptstyle p_{\varphi^*(\gamma)}} & & \downarrow{\scriptstyle p_\gamma} \\ F_{\gamma(n)} & = & F_{\gamma(n)} \end{array} \qquad \text{for every } n \in \mathbb{N} \setminus \{0\} \text{ and every } \gamma \in S(n, k)$$

where $\varphi^* : S(n, k) \to S(n, j)$ is the map given by the rule :

$$(k \xrightarrow{\psi_0} \beta(1) \xrightarrow{\psi_1} \cdots \xrightarrow{\psi_{n-1}} \beta(n)) \mapsto (j \xrightarrow{\psi_0 \circ \varphi} \beta(1) \xrightarrow{\psi_1} \cdots \xrightarrow{\psi_{n-1}} \beta(n)).$$

Lastly, the augmentation $\tau_F : F \to \nabla_0 F$ is the unique map that makes commute the diagram :

$$\begin{array}{ccc} F_j & \xrightarrow{\tau_{F,j}} & \prod_{\beta \in S(1,j)} F_{\beta(1)} \\ & \searrow^{F_{\gamma(0 \to 1)}} & \downarrow{\scriptstyle p_\gamma} \\ & & F_{\gamma(1)} \end{array} \qquad \text{for every } \gamma \in S(1, j) \text{ and every } j \in \mathrm{Ob}(J).$$

(iii) We deduce from (ii) the following explicit expression for the complex $\lim_J(\nabla_0 F)$. Notice that for every $X \in \mathrm{Ob}(\mathscr{A})$ and every $n \in \mathbb{N}$, the datum of a morphism $X \to \lim_J(\nabla_0 F)^n_\bullet$ is equivalent to that of a system $f_\bullet := (f_\beta \mid \beta \in \mathrm{Fun}([n+1], J))$, where $f_\beta : X \to F_{\beta(n+1)}$ is a morphism of $\mathscr{A}$ for every $\beta \in \mathrm{Fun}([n+1], J)$, fulfilling the compatibility condition :

$$f_{\varphi^*(\beta)} = f_\beta \qquad \text{for every } \varphi \in \mathrm{Morph}(J) \text{ and every } \beta \in \mathrm{Fun}([n+1, J]).$$

In turn, such a system $f_\bullet$ is equivalent to that of a system $g_\bullet := (g_\gamma \mid \gamma \in \mathrm{Fun}([n], J))$, where $g_\gamma : X \to F_{\gamma(n)}$ is also a morphism of $\mathscr{A}$ (and without any compatibility condition); namely, to the system $g_\bullet$ there corresponds the system $f_\bullet$ with $f_\beta := g_{\beta \circ \varepsilon_0}$, where $\varepsilon_0 : [n] \to [n+1]$ is the 0-th face map, so that $\varepsilon_0(i) := i + 1$ for $i = 0, \dots, n$. Hence,

$\lim_J(\nabla_0 F)^n_\bullet$ is represented by $\prod_{\gamma\in\mathrm{Fun}([n],J)} F_{\gamma(n)}$, for every $n \in \mathbb{N}$. Under these identifications, it is easily seen that $d^n := \lim_J d^n_\bullet : \lim_J(\nabla_0 F)^n_\bullet \to \lim_J(\nabla_0 F)^{n+1}_\bullet$ corresponds to the map $\sum_{i=0}^{n+1}(-1)^{i+n+1}\partial^i : \prod_{\gamma\in\mathrm{Fun}([n],J)} F_{\gamma(n)} \to \prod_{\gamma\in\mathrm{Fun}([n+1],J)} F_{\gamma(n)}$, where ∂^i is the unique morphism of $\mathscr{A}$ that makes commute the diagram :

$$\begin{array}{ccc} \prod_{\gamma\in\mathrm{Fun}([n],J)} F_{\gamma(n)} & \xrightarrow{\partial^i} & \prod_{\gamma\in\mathrm{Fun}([n+1],J)} F_{\gamma(n+1)} \\ {\scriptstyle p_{\beta\circ\varepsilon_i}}\downarrow & & \downarrow{\scriptstyle p_\beta} \\ F_{\beta\circ\varepsilon_i(n)} & \xrightarrow{F_{\gamma(\varepsilon_i(n)\to n+1)}} & F_{\beta(n+1)} \end{array} \qquad \text{for every } \beta \in \mathrm{Fun}([n+1],J)$$

where again, $\varepsilon_i : [n] \to [n+1]$ is the i-th face map, and p_β and $p_{\beta\circ\varepsilon_i}$ are the natural projections. Summing up, $\lim_J(\nabla_0 F)$ is (up to natural isomorphisms of $\mathsf{C}(\mathscr{A})$) the complex :

$$0 \to \prod_{\beta\in\mathrm{Fun}([0],J)} F_{\beta(0)} \xrightarrow{d^0} \prod_{\beta\in\mathrm{Fun}([1],J)} F_{\beta(1)} \xrightarrow{d^1} \cdots$$

with the differentials d^n given above.

(iv) Let $G : J' \to J$ be a functor; by remark 1.2.11(i), we have a natural transformation

$$(\lim_G \mathbf{1}_{\mathscr{A}})_F : \mathop{\mathrm{Lim}}_J F \to \mathop{\mathrm{Lim}}_{J'}(F\circ G).$$

Since the functor $\mathsf{Fun}(G,\mathbf{1}_{\mathscr{A}}) : \mathsf{Fun}(J,\mathscr{A}) \to \mathsf{Fun}(J',\mathscr{A})$ is exact, we deduce a natural transformation

$$(R\lim_G \mathbf{1}_{\mathscr{A}})_{F^\bullet} : R\lim_J F^\bullet \to R\lim_{J'}(F^\bullet\circ G) \qquad \text{for every } F^\bullet \in \mathrm{Ob}(\mathsf{D}^+(\mathsf{Fun}(J,\mathscr{A}))).$$

We may upgrade this transformation to a natural map of cochain complexes

$$\lambda_F : \lim_J \nabla_0 F \to \lim_{J'} \nabla_0(F\circ G)$$

as follows. It suffices to exhibit a natural map μ_F that makes commute the diagram

$$\begin{array}{ccc} & F\circ G & \\ {\scriptstyle \tau_F * G}\swarrow & & \searrow{\scriptstyle \tau_{F\circ G}} \\ (\nabla_0 F)\circ G & \xrightarrow{\mu_F} & \nabla_0(F\circ G) \end{array}$$

where τ_F and $\tau_{F\circ G}$ are the respective augmentations, as in (ii). For every $n\in\mathbb{N}$, we let μ^n_F be the unique morphism that makes commute the diagram :

$$\begin{array}{ccc} \prod_{\beta\in S(n,Gj')} F_{\beta(n)} & \xrightarrow{(\mu^n_F)_{j'}} & \prod_{\beta\in S(n,j')} F_{G\beta(n)} \\ & {\scriptstyle p_{G_*(\gamma)}}\searrow & \downarrow{\scriptstyle p_\gamma} \\ & & F_{G\gamma(n)} \end{array} \qquad \text{for every } j' \in \mathrm{Ob}(J') \text{ and every } \gamma \in S(n,j')$$

where $G_* : S(n,j') \to S(n,Gj')$ is the map such that $(\beta : [n] \to J') \mapsto (G\circ\beta : [n] \to J)$.

In terms of the natural identifications of (iii), it follows that λ_F corresponds to the unique morphism of complexes that makes commute the diagram :

$$\begin{array}{ccc} \prod_{\gamma\in\mathrm{Fun}([n],J)} F_{\gamma(n)} & \xrightarrow{\lambda^n_F} & \prod_{\gamma'\in\mathrm{Fun}([n],J')} F_{G\gamma'(n)} \\ & {\scriptstyle p_{G_*(\beta)}}\searrow & \downarrow{\scriptstyle p_\beta} \\ & & F_{G\beta(n)} \end{array} \qquad \text{for every } n\in\mathbb{N} \text{ and every } \beta\in\mathrm{Fun}([n],J').$$

(v) Let I, J and the inclusion functor $f : I \to J$ be as in (ii), and suppose now that $\mathscr{A}$ is cocomplete and that *the direct sums are exact* in $\mathscr{A}$, *i.e.* that the functor $\mathrm{Colim}_S : \mathsf{Fun}(S,\mathscr{A}) \to$

$\mathscr{A}$ is exact for every small discrete category S. Then we can follow remark 7.10.9(i) to obtain a construction of the left derived functor $L\,\mathrm{Colim}_J$ of the functor Colim_J, which shall be dual to (ii). Let us explicit the resulting complex $\mathrm{colim}_J(\triangle_0 F)$ for any given functor $F : J \to \mathscr{A}$. Proceeding as in (iii), we find that it is the complex of $\mathsf{C}^-(\mathscr{A})$:

$$\cdots \xrightarrow{d_2} \bigoplus_{\beta\in\mathrm{Fun}([1],J)} F_{\beta(0)} \xrightarrow{d_1} \bigoplus_{\beta\in\mathrm{Fun}([0],J)} F_{\beta(0)} \to 0$$

with $d_n = \sum_{i=0}^n (-1)^i \cdot \partial_{n-i}$, and ∂_i is the morphism that makes commute the diagram :

$$\begin{array}{ccc} F_{\gamma(0)} & \xrightarrow{F_{\gamma(0\to\varepsilon_i(0))}} & F_{\gamma\circ\varepsilon_i(0)} \\ {\scriptstyle e_\gamma}\downarrow & & \downarrow{\scriptstyle e_{\gamma\circ\varepsilon_i}} \\ \bigoplus_{\beta\in\mathrm{Fun}([n],J)} F_{\beta(0)} & \xrightarrow{\partial_i} & \bigoplus_{\beta\in\mathrm{Fun}([n-1],J)} F_{\beta(0)} \end{array} \qquad \text{for every } \gamma\in\mathrm{Fun}([n],J)$$

where e_γ and $e_{\gamma\circ\varepsilon_i}$ are the natural monomorphisms.

7.10.11. Let $\mathscr{A}$ be a complete abelian category with exact products, J a small category, $I \subset J$ a discrete subcategory such that the inclusion functor $f : I \to J$ is essentially surjective. Then the functor $\triangledown : \mathsf{C}^+(\mathsf{Fun}(J,\mathscr{A})) \to \mathsf{C}^+(\mathsf{Fun}(J,\mathscr{A}))$ is well defined, as in example 7.10.10(ii).

We consider also the special case where we take $\mathscr{A} = \mathbb{Z}$, and we denote the corresponding functor by $\triangledown' : \mathsf{C}^+(\mathsf{Fun}(J,\mathbb{Z}\text{-}\mathbf{Mod})) \to \mathsf{C}^+(\mathsf{Fun}(J,\mathbb{Z}\text{-}\mathbf{Mod}))$.

Proposition 7.10.12. *In the situation of* (7.10.11)*, we have a natural isomorphism in* $\mathsf{C}(\mathscr{A})$ *:*

$$\mathrm{Hom}^\bullet_{\mathscr{A}}(C^\bullet, \lim_J(\triangledown D^\bullet)) \xrightarrow{\sim} \lim_J(\triangledown'\mathrm{Hom}^\bullet_{\mathscr{A}}(C^\bullet, D^\bullet))$$

for every $C^\bullet \in \mathrm{Ob}(\mathsf{C}^-(\mathscr{A}))$ *and every* $D^\bullet \in \mathrm{Ob}(\mathsf{C}^+(\mathsf{Fun}(J,\mathscr{A})))$.

Proof. (Here the functor $\mathrm{Hom}^\bullet_{\mathscr{A}}(-,-)$ is defined as in example 7.1.15(i).) We have two functors

$$\mathrm{Hom}_{\mathscr{A}}(-, \mathrm{Lim}_J), \mathrm{Lim}_J\, \mathrm{Hom}_{\mathscr{A}}(-,-) : \mathscr{A}^o \times \mathsf{Fun}(J,\mathscr{A}) \to \mathbb{Z}\text{-}\mathbf{Mod}$$

and an isomorphism $\omega : \mathrm{Hom}_{\mathscr{A}}(-,\mathrm{Lim}_J) \xrightarrow{\sim} \mathrm{Lim}_J\,\mathrm{Hom}_{\mathscr{A}}(-,-)$. Then, for every $K^\bullet \in \mathrm{Ob}(\mathsf{C}(\mathscr{A}))$ and every $L^\bullet \in \mathrm{Ob}(\mathsf{C}(\mathsf{Fun}(J,\mathscr{A})))$, the system $(\omega_{K^q,L^p} \mid p,q\in\mathbb{Z})$ yields an isomorphism : $\mathrm{Hom}^\bullet_{\mathscr{A}}(K^\bullet, \lim_J L^\bullet) \xrightarrow{\sim} \lim_J \mathrm{Hom}^\bullet_{\mathscr{A}}(K^\bullet, L^\bullet)$ in $\mathsf{C}(\mathbb{Z}\text{-}\mathbf{Mod})$. Especially, we get a natural isomorphism:

$$\mathrm{Hom}^\bullet_{\mathscr{A}}(C^\bullet, \mathrm{Lim}_J(\triangledown D^\bullet)) \xrightarrow{\sim} \mathrm{Lim}_J\, \mathrm{Hom}^\bullet_{\mathscr{A}}(C^\bullet, \triangledown D^\bullet). \qquad \text{in } \mathsf{C}(\mathbb{Z}\text{-}\mathbf{Mod}) \tag{7.10.13}$$

and recall that $\triangledown D^\bullet = \mathrm{Tot}(\triangledown_0 D^\bullet)$ (see remark 7.10.9(ii)); also, $\mathrm{Hom}^\bullet_{\mathscr{A}}$ is the composition

$$\mathsf{C}(\mathscr{A}) \times \mathsf{C}(\mathscr{A})^o \xrightarrow{\mathbf{1}_{\mathsf{C}(\mathscr{A})}\times\Phi} \mathsf{C}(\mathscr{A}) \times \mathsf{C}(\mathscr{A}^o) \xrightarrow{\mathrm{Hom}^{\bullet\bullet}_{\mathscr{A}}} \mathsf{C}_2(\mathscr{A}) \xrightarrow{\mathrm{Tot}^\Pi_{\mathscr{A}}} \mathsf{C}(\mathscr{A})$$

where $\Phi : \mathsf{C}(\mathscr{A})^o \xrightarrow{\sim} \mathsf{C}(\mathscr{A}^o)$ is the isomorphism of categories of remark 7.1.6(iii), and $\mathrm{Hom}^{\bullet\bullet}_{\mathscr{A}}$ is deduced from the biadditive functor $\mathrm{Hom}_{\mathscr{A}} : \mathscr{A} \times \mathscr{A}^o \to \mathscr{A}$, as prescribed in (7.1.14). Consider the functor $H : \mathsf{C}(\mathscr{A}) \times \mathscr{A}^o \to \mathsf{C}(\mathscr{A})$ such that $H(L^\bullet, K)^p := \mathrm{Hom}_{\mathscr{A}}(K, L^p)$ for every $((L^\bullet, d^\bullet_L), K) \in \mathrm{Ob}(\mathsf{C}(\mathscr{A}) \times \mathscr{A}^o)$ and every $p\in\mathbb{Z}$, with differential $d^p := \mathrm{Hom}_{\mathscr{A}}(K, d^p_L)$ for every such p. The functor H is biadditive, so by applying to H the construction of (7.1.14) we get another functor :

$$\mathrm{Hom}^{\bullet\bullet\bullet}_{\mathscr{A}} := H^{\bullet\bullet} : \mathsf{C}_2(\mathscr{A}) \times \mathsf{C}(\mathscr{A}^o) \to \mathsf{C}_3(\mathscr{A}) \qquad (L^{\bullet\bullet}, K^\bullet) \mapsto \mathrm{Hom}^{\bullet\bullet\bullet}_{\mathscr{A}}(K^\bullet, L^{\bullet\bullet}).$$

With this notation, for every $K^\bullet \in \mathrm{Ob}(\mathsf{C}(\mathscr{A}))$ and every $L^{\bullet\bullet} \in \mathrm{Ob}(\mathsf{C}_2(\mathscr{A}))$ we have :

$$\mathrm{Hom}^{\bullet\bullet}_{\mathscr{A}}(K^\bullet, \mathrm{Tot}^\Pi(L^{\bullet\bullet})) = \mathsf{C}(\mathrm{Tot}^\Pi_{\mathscr{A}})(\mathrm{Hom}^{\bullet\bullet\bullet}_{\mathscr{A}}(K^\bullet, L^{\bullet\bullet}))$$

(recall that $\mathsf{C}(\mathrm{Tot}^{\Pi}_{\mathscr{A}})(M^{\bullet\bullet\bullet})^{nr} = \Pi_{p+q=n} M^{pqr}$, and $\mathrm{Hom}^{\bullet\bullet\bullet}_{\mathscr{A}}(K^\bullet, L^{\bullet\bullet})^{pqr} = \mathrm{Hom}_{\mathscr{A}}(K^r, L^{pq})$ for every $p, q, r \in \mathbb{Z}$). Summing up, we get a natural isomorphism :

$$\mathrm{Hom}^\bullet_{\mathscr{A}}(C^\bullet, \mathrm{Tot}(\nabla_0 D^\bullet)) \xrightarrow{\sim} \mathrm{Tot}^{\Pi}_{\mathscr{A}} \circ \mathsf{C}(\mathrm{Tot}^{\Pi}_{\mathscr{A}})(\mathrm{Hom}^{\bullet\bullet\bullet}_{\mathscr{A}}(\Phi(C^\bullet), \nabla_0 D^\bullet)) \tag{7.10.14}$$

and recall that $(\nabla_0 D^\bullet)^{pq} = \top^{p+1}(D^q)$ for every $p, q \in \mathbb{N}$. Next, by a direct inspection of the constructions in (ii), we get an isomorphism of functors $\mathscr{A}^o \times \mathscr{A} \to \mathsf{C}^{\geq 0}(\mathbb{Z}\text{-}\mathbf{Mod})$:

$$\mathrm{Hom}_{\mathscr{A}}(-, \nabla_0(-)) \xrightarrow{\sim} \nabla'_0 \mathrm{Hom}_{\mathscr{A}}(-, -)$$

whence a natural isomorphism :

$$\mathrm{Hom}^{\bullet\bullet\bullet}_{\mathscr{A}}(\Phi(C^\bullet), \nabla_0 D^\bullet) \xrightarrow{\sim} \nabla'_0 \mathrm{Hom}^{\bullet\bullet}_{\mathscr{A}}(\Phi(C^\bullet), D^\bullet) \qquad \text{in } \mathsf{C}_3(\mathbb{Z}\text{-}\mathbf{Mod}).$$

Lastly, recall that $\mathrm{Tot}^{\Pi}_{\mathscr{A}} \circ \mathsf{C}(\mathrm{Tot}^{\Pi}_{\mathscr{A}}) = \mathrm{Tot}^{\Pi}_{\mathscr{A}} \circ \mathrm{Tot}^{\Pi}_{\mathsf{C}(\mathscr{A})}$, and we have $\mathrm{Tot}^{\Pi}_{\mathsf{C}(\mathscr{A})}(M^{\bullet\bullet\bullet})^{pn} = \prod_{q+r=n} M^{pqr}$ for every $p, n \in \mathbb{Z}$ (see (7.1.12)), and notice that

$$\mathrm{Tot}^{\Pi}_{\mathsf{C}(\mathscr{A})}(\nabla'_0 \mathrm{Hom}^{\bullet\bullet}_{\mathscr{A}}(\Phi(C^\bullet), D^\bullet)) = \nabla'_0(\mathrm{Tot}^{\Pi}_{\mathscr{A}} \mathrm{Hom}^{\bullet\bullet}_{\mathscr{A}}(\Phi(C^\bullet), D^\bullet)) = \nabla'_0(\mathrm{Hom}^\bullet_{\mathscr{A}}(C^\bullet, D^\bullet)).$$

Combining with (7.10.14) and (7.10.13), the assertion follows. □

Remark 7.10.15. (i) In the situation of proposition 7.10.12, suppose moreover that $\mathscr{A}$ has enough projective objets. Then we may choose a quasi-isomorphism $P^\bullet \to C^\bullet$ in $\mathsf{C}^-(\mathscr{A})$, with $P^\bullet$ a complex of projectives; from the double complexes

$$\mathrm{Hom}_{\mathscr{A}}(P^\bullet, \lim_J(\nabla D^\bullet)) \qquad \text{and} \qquad \lim_J(\nabla' \mathrm{Hom}_{\mathscr{A}}(P^\bullet, D^\bullet))$$

and taking into account example 7.10.10(i), we then obtain two spectral sequences

$$E_2^{pq} := \mathrm{Ext}^p_{\mathscr{A}}(C^\bullet, R^q \lim_J D^\bullet) \qquad F_2^{pq} := R^p \lim_J \mathrm{Ext}^q_{\mathscr{A}}(C^\bullet, D^\bullet)$$

that converge both to $\mathrm{Hom}_{\mathsf{D}(\mathscr{A})}(C^\bullet, R\lim_J D^\bullet[p+q])$ (and with finite induced filtrations on the abutment : see [163, §5.6]). These spectral sequences already appear in [106, Th.4.2, Th.4.3] and in [144, Th.1].

(ii) On the other hand, by applying proposition 7.10.12 to the opposite category $\mathscr{A}^o$, and taking into account example 7.10.10(v), we deduce a natural isomorphism in $\mathsf{C}(\mathscr{A})$:

$$\mathrm{Hom}^\bullet_{\mathscr{A}}(\operatorname*{colim}_J(\triangle D^\bullet), C^\bullet) \xrightarrow{\sim} \lim_J(\nabla' \mathrm{Hom}^\bullet_{\mathscr{A}}(D^\bullet, C^\bullet))$$

for every $C^\bullet \in \mathrm{Ob}(\mathsf{C}^+(\mathscr{A}))$ and every $D^\bullet \in \mathsf{C}^-(\mathsf{Fun}(J, \mathscr{A}))$, where $\triangle : \mathsf{C}^-(\mathsf{Fun}(J, \mathscr{A})) \to \mathsf{C}^-(\mathsf{Fun}(J, \mathscr{A}))$ is the functor associated with the adjoint pair $(\int^f, \mathsf{Fun}(f, \mathscr{A}))$. If moreover, $\mathscr{A}$ has enough injective objects, we may choose a quasi-isomorphism $C^\bullet \to Q^\bullet$ in $\mathsf{C}^+(\mathscr{A})$ with a complex $Q^\bullet$ of injectives, and arguing as in (i) we deduce two spectral sequences :

$$E_2^{pq} := \mathrm{Ext}^p_{\mathscr{A}}(L^q \operatorname*{colim}_J D^\bullet, C^\bullet) \qquad F_2^{pq} := R^p \lim_J \mathrm{Ext}^q_{\mathscr{A}}(D^\bullet, C^\bullet)$$

that converge both to $\mathrm{Hom}_{\mathsf{D}(\mathscr{A})}(L\operatorname{colim}_J D^\bullet, C^\bullet[p+q])$.

(iii) Even in case $\mathscr{A}$ does not have enough projective objects, we can still deduce from proposition 7.10.12 a spectral sequence

$$E_2^{pq} := R^p \lim_J \mathrm{Hom}_{\mathsf{D}(\mathscr{A})}(C^\bullet, D^\bullet[q]) \Rightarrow \mathrm{Hom}_{\mathsf{D}(\mathscr{A})}(C^\bullet, R\lim_J D^\bullet[p+q]).$$

To this aim, we notice first that proposition 7.10.12 yields a spectral sequence

$$F(C^\bullet)_2^{pq} := R^p \lim_J \mathsf{Hot}_{\mathscr{A}}(C^\bullet, D^\bullet[q]) \Rightarrow \mathsf{Hot}_{\mathscr{A}}(C^\bullet, R\lim_J D^\bullet[p+q]).$$

Next, let Σ be the full subcategory of $\mathsf{Hot}^-(\mathscr{A})/C^\bullet$ whose objects are the quasi-isomorphisms $C'^\bullet \to C^\bullet$. By virtue of theorem 7.3.16, we deduce a spectral sequence :

$$\operatorname*{colim}_{C'^\bullet \in \mathrm{Ob}(\Sigma)} F(C'^\bullet)_2^{pq} \Rightarrow \mathrm{Hom}_{\mathsf{D}(\mathscr{A})}(C^\bullet, R\lim_J D^\bullet[p+q]).$$

On the other hand, set $G(C'^\bullet)^{pq}_j := \prod_{\beta\in S(p,j)} \mathsf{Hot}_{\mathscr{A}}(C'^\bullet, D^\bullet_{\beta(p)})$ for every $p,q\in\mathbb{N}$ every $j\in \mathrm{Ob}(J)$, and every object $C'^\bullet \to C^\bullet$ of Σ (notation of example 7.10.10(ii)). Then $F(C'^\bullet)^{pq}_2$ is the kernel of a map $\prod_{\varphi\in\mathrm{Morph}(J)} G(C'^\bullet)^{pq}_{\mathsf{t}(\varphi)} \to \prod_{j\in\mathrm{Ob}(J)} G(C'^\bullet)^{pq}_j$. Since Σ is filtered, we are then reduced to checking that for any family $(L^\bullet_\lambda \mid \lambda\in\Lambda)$ of objects of $\mathsf{C}^+(\mathscr{A})$ we have

$$\operatorname*{colim}_{C'^\bullet\in\mathrm{Ob}(\Sigma)} \prod_{\lambda\in\Lambda} \mathsf{Hot}_{\mathscr{A}}(C'^\bullet, D^\bullet_\lambda) = \prod_{\lambda\in\Lambda} \mathrm{Hom}_{\mathsf{D}(\mathscr{A})}(C^\bullet, D^\bullet_\lambda).$$

The latter follows easily from corollary 7.3.19(ii), remark 7.1.6(vi), and theorem 7.3.16.

7.10.16. Now, denote by $\mathbf{Set}_\circ$ the category of *pointed sets*, whose objects are all the pairs (S,s) where S is a small set, and $s\in S$ is any element; the morphisms $f:(S,s)\to(S',s')$ are the mappings $f:S\to S'$ such that $f(s)=s'$. Next, let A be any ring, and consider the *pointed forgetful functor*

$$\mathsf{F}: A\text{-}\mathbf{Mod}\to\mathbf{Set}_\circ \qquad M\mapsto (M,0_M)$$

(where $0_M\in M$ is the zero element of M). The functor F admits the left adjoint

$$\mathsf{L}:\mathbf{Set}_\circ\to A\text{-}\mathbf{Mod} \qquad (S,s)\mapsto A^{(S)}/As$$

(where $A^{(S)}$ denotes the free A-module with basis given by S, so $As\subset A^{(S)}$ is the direct summand generated by the basis element $s\in S$). According to proposition 7.10.5, the adjoint pair (L,F) yields a functor

$$\perp^A_\bullet: A\text{-}\mathbf{Mod}\to\widehat{s}.A\text{-}\mathbf{Mod}$$

into the category of augmented simplicial A-modules (notation of (7.4.11)), such that

$$\mathsf{F}\perp^A_\bullet M\to \mathsf{F}M$$

is a homotopically trivial augmented pointed simplicial set, for every A-module M. This functor is obtained by iterating the functor $\perp^A:=\mathsf{L}\circ\mathsf{F}$, so in each degree it consists of a free A-module. We call $\perp^A_\bullet M$ the *standard free resolution of* M.

Remark 7.10.17. Notice that $\perp^A_\bullet 0 = s.0$, the constant simplicial A-module associated with the trivial A-module (see (7.4.4)). This is the reason why we prefer the adjoint pair (L,F), rather than the analogous pair considered in [103, I.1.5.5], arising from the forgetful functor from A-modules to non-pointed sets.

7.10.18. Let R be a simplicial A-algebra, and define the category R-$\mathbf{Mod}$ of R-modules, as in [75, §8.1]. Notice that any morphism $S\to R$ of simplicial rings induces a base change functor

$$S\text{-}\mathbf{Mod}\to R\text{-}\mathbf{Mod} \qquad (M[n]\mid n\in\mathbb{N})\mapsto (R[n]\otimes_{S[n]} M[n]\mid n\in\mathbb{N})$$

which is left adjoint to the forgetful functor (details left to the reader).

Now, by applying the functors $\perp^{R[n]}_\bullet$ to the terms $M[n]$ of a given R-module M, we obtain an augmented simplicial R-module

$$\perp^R_\bullet M\to M \tag{7.10.19}$$

which amounts to a bisimplicial A-module, whose columns $\perp^{R[n]}_\bullet M[n]\to M[n]$ are augmented simplicial $R[n]$-modules, for every $n\in\mathbb{N}$. Likewise, the row $\perp^R_n M$ is an R-module, for every $n\in\mathbb{N}$.

Lemma 7.10.20. *In the situation of* (7.10.18)*, let $\varphi: M\to N$ be any quasi-isomorphism of R-modules. Then the induced morphism*

$$\perp^R_n\varphi:\perp^R_n M\to\perp^R_n N$$

is a homotopy equivalence, for every $n\in\mathbb{N}$.

Proof. Set $\perp^R_{-1} M := M$, and notice the natural isomorphisms

$$\perp^R_n M \xrightarrow{\sim} R\otimes_{s.\mathbb{Z}} \perp^{s.\mathbb{Z}} \circ \perp^R_{n-1} M \qquad \text{for every } n \in \mathbb{N} \tag{7.10.21}$$

(where $s.\mathbb{Z}$ is the constant simplicial ring arising from $\mathbb{Z}$ (see (7.4.4)), which is the initial object in the category of simplicial rings). The assumption means that $\perp^R_{-1} \varphi$ is a quasi-isomorphism. However, (7.10.21) and Whitehead's theorem ([103, I.2.2.3]) imply that if – for a given $n \in \mathbb{N}$ – the map $\perp^R_{n-1} \varphi$ is a quasi-isomorphism, then $\perp^R_n \varphi$ is a homotopy equivalence. We may thus conclude by a simple induction on n. ◻

Example 7.10.22. Let M and N two R-modules. We may define two functors

$$M \overset{\ell}{\otimes}_R - \quad (\text{ resp. } - \overset{\ell}{\otimes}_R N\ ) \quad : \quad R\text{-}\mathbf{Mod} \to R\text{-}\mathbf{Mod}$$

by the rules :

$$P \mapsto M \otimes_R (\perp^R_\bullet P)^\Delta \quad (\text{ resp. } P \mapsto (\perp^R_\bullet P)^\Delta \otimes_R N\ ) \qquad \text{for every } R\text{-module } P$$

where Δ is the diagonal functor, from $\mathscr{C}$-bisimplicial to $\mathscr{C}$-simplicial objects (see (7.4.17)).

(i) We claim that these functors descend to the derived category. That is, suppose that $\varphi : P \to P'$ is a quasi-isomorphism; then $M \overset{\ell}{\otimes}_R \varphi$ and $\varphi \overset{\ell}{\otimes}_R N$ are both quasi-isomorphisms. Indeed, lemma 7.10.20 implies that the induced morphisms

$$M\otimes_R \perp^R_n \varphi : M\otimes_R \perp^R_n P \to M\otimes_R \perp^R_n P'$$

are quasi-isomorphisms, for every $n \in \mathbb{N}$, and likewise for $\varphi\otimes_R \perp^R_n N$, so the assertion follows from Eilenberg-Zilber's theorem 7.4.50(i).

(ii) Also, we claim that the notation $M \overset{\ell}{\otimes}_R N$ is unambiguous. That is, we may compute this object by applying the functor $M \overset{\ell}{\otimes}_R -$ to N, or by applying the functor $- \overset{\ell}{\otimes}_R N$ to M, and the resulting two R-modules are naturally isomorphic in $\mathsf{D}(R\text{-}\mathbf{Mod})$. Indeed, it suffices to check that the natural morphisms

$$M \otimes_R (\perp^R_\bullet N)^\Delta \xleftarrow{\alpha} (\perp^R_\bullet M)^\Delta \otimes_R (\perp^R_\bullet N)^\Delta \xrightarrow{\beta} (\perp^R_\bullet M)^\Delta \otimes_R N$$

induced by the augmentation (7.10.19), are both quasi-isomorphisms. We check this for α; the same argument shall apply to β. To ease notation, set $L := (\perp^R_\bullet N)^\Delta$. Then α is deduced by extracting the diagonal from the augmented simplicial R-module

$$\perp^R_\bullet M \otimes_R L \to M \otimes_R L \tag{7.10.23}$$

(so, the n-th column of $s.L$ is a constant and flat simplicial $R[n]$-module, for every $n \in \mathbb{N}$). Thus, we are reduced to checking that the columns of (7.10.23) are aspherical. However, for every $n \in \mathbb{N}$, the n-th column is the augmented simplicial $R[n]$-module $(\perp^{R[n]}_\bullet M[n])\otimes_{R[n]} L[n]$; since $L[n]$ is flat, it then suffices to recall that the standard free resolution is aspherical.

(iii) The foregoing discussion yields a well defined functor

$$- \overset{\ell}{\otimes}_R - : \mathsf{D}(R\text{-}\mathbf{Mod}) \times \mathsf{D}(R\text{-}\mathbf{Mod}) \to \mathsf{D}(R\text{-}\mathbf{Mod})$$

called the *left derived tensor product*. The same method shall be applied hereafter to construct derived functors of certain non-additive functors.

Remark 7.10.24. Let M and N be as in example 7.10.22 and set $P := (\perp^R_\bullet M)\otimes_R s.N$ (this is a simplicial R-module; especially, it is a bisimplicial A-module); by Eilenberg-Zilber's theorem 7.4.50, the cochain complex associated with $M \overset{\ell}{\otimes}_R N$ is naturally isomorphic, in $\mathsf{D}(A\text{-}\mathbf{Mod})$

to the complex $\mathrm{Tot}(P^{\bullet\bullet})$, where $P^{\bullet\bullet}$ denotes the double complex associated with P. There follows a spectral sequence :

$$E^1_{pq} := \mathrm{Tor}^{R[p]}_q(M[p], N[p]) \Rightarrow H_{p+q}(M \overset{\ell}{\otimes}_R N)$$

whose differentials $d^1_{pq} : E^1_{pq} \to E^1_{p-1,q}$ are obtained as follows. For every integer $q \in \mathbb{N}$, let T_q be the R-module given by the rule : $T_q[p] := E^1_{pq}$, with face and degeneracy maps deduced from those of M, N and R, in the obvious way. Then d^1_{pq} is the differential in degree p of the chain complex $T_{q\bullet}$ associated with T_q (details left to the reader).

Example 7.10.25. (i) Another useful cotriple arises from the forgetful functor $A\text{-}\mathsf{Alg} \to \mathbf{Set}$ and its left adjoint, that attaches to any set S the free A-algebra $A[S]$. There results, for every A-algebra B, a standard simplicial resolution by free A-algebras

$$F_A(B) \to B.$$

Now, if R is again a simplicial A-algebra, and S any R-algebra, we can proceed as in the foregoing, to obtain a bisimplicial resolution $F_R(S) \to S$ whose columns $F_{R[n]}(S[n]) \to S[n]$ are augmented simplicial $R[n]$-algebras, for every $n \in \mathbb{N}$. A simple inspection shows that the proof of lemma 7.10.20 carries over – *mutatis mutandis* – to the present situation, hence a quasi-isomorphism $S \to S'$ of R-algebra induces a morphism $F_R(S)[n] \to F_R(S')[n]$ of R-algebras, which is a homotopy equivalence on the underlying R-modules, for every $n \in \mathbb{N}$.

(ii) We may then define a *derived tensor product* for R-algebras, proceeding as in example 7.10.22. Namely, if S and S' are any two R-algebras, we define two functors

$$S \overset{\ell}{\otimes}_R - \quad (\text{ resp. } - \overset{\ell}{\otimes}_R S' \,) \quad : \quad R\text{-}\mathsf{Alg} \to R\text{-}\mathsf{Alg}$$

by the rules :

$$T \mapsto S \otimes_R F_R(T)^\Delta \quad (\text{ resp. } T \mapsto F_R(T)^\Delta \otimes_R N\,) \qquad \text{for every } R\text{-algebra } T.$$

Arguing like in *loc.cit.* we see that both functors transform quasi-isomorphisms into quasi-isomorphisms, hence they descend to the derived category $\mathsf{D}(R\text{-}\mathsf{Alg})$, and moreover, the two functors are naturally isomorphic, so the notation $S \overset{\ell}{\otimes}_R S'$ is unambiguous : the detailed verification is left as an exercise for the reader.

7.10.26. Let $A\text{-}\mathsf{Alg.Mod}$ be the category of all pairs (B, M), where B is any A-algebra, and M any B-module. The morphisms $(B, M) \to (B', M')$ are the pairs (f, φ), where $f : B \to B'$ is a morphism of A-algebras, and $\varphi : M \to f^*M'$ a B-linear map (here f^*M' denotes the B-module obtained from the B'-module M', by restriction of scalars along the map f). Suppose now that we have a commutative diagram of categories :

$$\begin{array}{ccc} A\text{-}\mathsf{Alg.Mod} & \xrightarrow{T} & \mathscr{C} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle f} \\ A\text{-}\mathsf{Alg} & \xrightarrow{g} & \mathscr{B} \end{array}$$

such that :

- For every $X \in \mathrm{Ob}(B)$, the fibre $f^{-1}X$ is an abelian category whose filtered colimits are representable and exact.
- F is given by the rule $(B, M) \mapsto B$ for every $(B, M) \in \mathrm{Ob}(A\text{-}\mathsf{Alg.Mod})$.
- For every A-algebra B, the restriction

$$T_B : F^{-1}B \to f^{-1}(gB)$$

of T commutes with filtered colimits, *i.e.*, if $((B, M_i) \mid i \in I)$ is any filtered system of objects of A-Alg.Mod (over the same A-algebra B), then the induced morphism

$$\operatorname*{colim}_{i\in I} T_B M_i \to T_B(\operatorname*{colim}_{i\in I} M_i)$$

is an isomorphism.

The functors f and g extend to functors $s.f : s.\mathscr{C} \to s.\mathscr{B}$, respectively $s.g : s.A\text{-Alg} \to s.\mathscr{B}$, and notice that – for R any simplicial A-algebra – T induces a functor

$$T_R : R\text{-}\mathbf{Mod} \to s.\mathscr{C}_{/R} := s.f^{-1}(s.gR) \qquad (M[n] \mid n \in \mathbb{N}) \mapsto (T_{R[n]}M[n] \mid n \in \mathbb{N}).$$

For every R-module M, we set

$$LT_R(M) := T_R(\perp_\bullet^R M)^\Delta.$$

The augmentation $T_R(\perp_\bullet^R M) \to T_R M$ can be regarded as a morphism of bisimplicial objects

$$T_R(\perp_\bullet^R M) \to s.T_R M$$

(the columns of $s.T_R M$ are constant $\mathscr{C}$-simplicial objects), whence a morphism in $s.\mathscr{C}_{/R}$:

(7.10.27) $$LT_R(M) \to s.T_R M^\Delta = T_R M \qquad \text{for every } R\text{-module } M.$$

In many applications, $s.\mathscr{C}_{/R}$ will be also an abelian category, but anyhow we can state the following :

Proposition 7.10.28. *With the notation of* (7.10.26)*, suppose that $\mathscr{C}$ is an abelian category. Then the following holds :*

(i) *If M is a flat R-module, then* (7.10.27) *is a quasi-isomorphism.*
(ii) *If $\varphi : M \to N$ is a quasi-isomorphism of R-modules, then the induced map*

$$LT_R(M) \to LT_R(N)$$

is a quasi-isomorphism.

Proof. (i): The assumption means that $M[n]$ is a flat $R[n]$-module for every $n \in \mathbb{N}$. Denote by C the unnormalized double complex associated with $T_R(\perp_\bullet^R M)$, and by C^Δ (resp. by D) the unnormalized complex associated with $LT_R(M)$ (resp. to $T_R M$). By Eilenberg-Zilber's theorem 7.4.50, we have a natural quasi-isomorphism

(7.10.29) $$C^\Delta \to \mathrm{Tot}(C)$$

given by the Alexander-Whitney map of (7.4.35), and a simple inspection shows that the map $C^\Delta \to D$ induced by (7.10.27) factors through (7.10.29). Hence, it suffices to show that the map

$$\mathrm{Tot}(C) \to D$$

induced by the augmentation of $T_R(\perp_\bullet^R M)$, is a quasi-isomorphism, under the stated condition.

To this aim, it suffices to check that the augmented $\mathscr{C}$-simplicial object

(7.10.30) $$T_R(\perp_\bullet^{R[n]} M[n]) \to T_R M[n]$$

is aspherical for every $n \in \mathbb{N}$. However, since $M[n]$ is a flat $R[n]$-module, it can be written as a filtered colimit of a system of free $R[n]$-modules ([120, Ch.I, Th.1.2]); on the other hand, the functor $\perp_\bullet^{R[n]}$ commutes with all filtered colimits, and the same holds for T_R, by assumption. Since the filtered colimits of the fibres of the functor f are exact, we are then reduced to checking the assertion in case $M[n]$ is a free $R[n]$-module; but in this case, (7.10.30) is even homotopically trivial, by virtue of proposition 7.10.5.

(ii): In light of Eilenberg-Zilber's theorem 7.4.50, it suffices to show that the induced map

$$T_R(\perp_n^R M) \to T_R(\perp_n^R N)$$

is a quasi-isomorphism for every $n \in \mathbb{N}$. In turns, this follows readily from lemma 7.10.20. $\Box$

Remark 7.10.31. (i) The proof of proposition 7.10.28(i) applies as well to the derived tensor product; namely, for any two R-modules M and N there is a natural morphism of R-modules

$$M \overset{\ell}{\otimes}_R N \to M \otimes_R N$$

which is a quasi-isomorphism, if either M or N is a flat R-module (details left to the reader).

(ii) Especially, let S and S' be any two R-algebras. Then, since the standard free resolution $F_R(S)$ of example 7.10.25 is a flat simplicial R-module, (i) implies that the R-module underlying the R-algebra $S \overset{\ell}{\otimes}_R S'$, is naturally isomorphic, in $\mathsf{D}(R\text{-}\mathbf{Mod})$, to the derived tensor product of the R-modules underlying S and S'. In other words, the notation $- \overset{\ell}{\otimes}_R -$ is unambiguous, whether one refers to derived tensor products of algebras, or of their underlying modules.

(iii) Denote by $\sigma, \omega : R\text{-}\mathbf{Mod} \to R\text{-}\mathbf{Mod}$ respectively the suspension and loop functors ([103, I.3.2.1]), and recall that σ is left adjoint to ω. A simple inspection of the definitions yields natural identifications

$$(\sigma M) \otimes_R N \xrightarrow{\sim} \sigma(M \otimes_R N) \qquad (\omega M) \otimes_R N \xrightarrow{\sim} \omega(M \otimes_R N)$$

for every R-modules M and N. By the same token, it is clear that σ and ω transform flat R-modules into flat R-modules. In view of (i), we deduce natural isomorphisms

$$(\sigma M) \overset{\ell}{\otimes}_R N \xrightarrow{\sim} \sigma(M \overset{\ell}{\otimes}_R N) \qquad (\omega M) \overset{\ell}{\otimes}_R N \xrightarrow{\sim} \omega(M \overset{\ell}{\otimes}_R N) \qquad \text{in } \mathsf{D}(R\text{-}\mathbf{Mod})$$

for every R-modules M and N.

(iv) Let $f : N \to N'$ be any morphism of R-modules; in the same vein, we get a natural identification :

$$\mathrm{Cone}(M \otimes_R f) \xrightarrow{\sim} M \otimes_R \mathrm{Cone}\, f \qquad \text{for every } R\text{-module } M$$

(see [103, I.3.2.2] for the definition of the cone of a morphism of R-modules). It follows immediately that the functor $M \overset{\ell}{\otimes}_R - : \mathsf{D}(R\text{-}\mathbf{Mod}) \to \mathsf{D}(R\text{-}\mathbf{Mod})$ transforms distinguished triangles into distinguished triangles (see [103, I.3.2.2.4] for the definition of distinguished triangle in $\mathsf{D}(R\text{-}\mathbf{Mod})$). For future reference, let us also point out :

Lemma 7.10.32. *Let R be a simplicial A-algebra, X and Y two R-modules, $n, m \in \mathbb{N}$ two integers, and suppose that $H_i X = 0 = H_j Y$ for every $i < n$ and $j < m$. Then*

$$H_i(X \overset{\ell}{\otimes}_R Y) = 0 \qquad \textit{for every } i < n + m.$$

Proof. Set $\sigma^0 := \mathbf{1}_{R\text{-}\mathbf{Mod}}$, and define inductively $\sigma^k : R\text{-}\mathbf{Mod} \to R\text{-}\mathbf{Mod}$ by the rule : $\sigma^k := \sigma \circ \sigma^{k-1}$ for every $k > 0$. Define likewise ω^k, and notice that σ^k is left adjoint to ω^k, for every $k \in \mathbb{N}$ (remark 7.10.31(iii)). Let $f : \sigma^n \circ \omega^n X \to X$ denote the counit of adjunction, and notice that $\mathrm{Cone}\, f = 0$ in $\mathsf{D}(R\text{-}\mathbf{Mod})$. In view of remark 7.10.31(iv,v), we deduce that the natural morphisms

$$\sigma^n(\omega^n X \overset{\ell}{\otimes}_R Y) \to (\sigma^n \circ \omega^n X) \overset{\ell}{\otimes}_R Y \to X \overset{\ell}{\otimes}_R Y$$

are isomorphisms in $\mathsf{D}(R\text{-}\mathbf{Mod})$. Likewise, the natural morphism

$$\sigma^m(\omega^n X \overset{\ell}{\otimes}_R \omega^m Y) \to \omega^n X \overset{\ell}{\otimes}_R Y$$

is an isomorphism in $\mathsf{D}(R\text{-}\mathbf{Mod})$, so finally the same holds for the morphism

$$\sigma^{n+m}(\omega^n X \overset{\ell}{\otimes}_R \omega^m Y) \to X \overset{\ell}{\otimes}_R Y$$

whence the claim. $\Box$

7.10.33. In view of proposition 7.10.28, it is clear that if $\mathscr{C}$ is an abelian category, LT_R yields a well defined functor on derived categories

$$LT_R : \mathsf{D}(R\text{-}\mathbf{Mod}) \to \mathsf{D}(s.\mathscr{C}_{/R}) \qquad M \mapsto LT_R(M)$$

(where $\mathsf{D}(s.\mathscr{C}_{/R})$ is the localization of $s.\mathscr{C}_{/R}$ with respect to the class of quasi-isomorphisms, and likewise for $\mathsf{D}(R\text{-}\mathbf{Mod})$: see [75, Def.8.1.3]). More generally, suppose that $\mathscr{C}$ is endowed with a functor

$$\Phi : \mathscr{C} \to \mathscr{A}$$

to an abelian category $\mathscr{A}$ (usually, this will be a forgetful functor of some sort), and denote by $\mathsf{D}_\Phi(s.\mathscr{C}_{/R})$ the localization of $s.\mathscr{C}_{/R}$ with respect to the system of its morphisms whose image under $s.\Phi$ are quasi-isomorphisms in $s.\mathscr{A}$. Then, since clearly

$$s.\Phi \circ LT_R = L(s.\Phi \circ T_R)$$

we see that LT_R descends to a well defined functor

$$LT_R : \mathsf{D}(R\text{-}\mathbf{Mod}) \to \mathsf{D}_\Phi(s.\mathscr{C}_{/R}).$$

We will need also the following slight refinement :

Corollary 7.10.34. *In the situation of proposition* 7.10.28*, the following holds :*

(i) *Let $\varphi : M \to N$ be a morphism of R-modules, and $n \in \mathbb{N}$ an integer such that*

$$H_i\varphi : H_iM \to H_iN$$

is an isomorphism, for every $i < n$. Then the same holds for the induced map

$$H_i(LT_R\varphi) : H_i(LT_RM) \to H_i(LT_RN).$$

(ii) *Suppose that $T(B, 0) = 0_{gB}$ (the initial and final object of $f^{-1}(gB)$) for every A-algebra B. Let $n \in \mathbb{N}$ be an integer, and M an R-module such that $H_iM = 0$ for every $i < n$. Then $H_i(LT_RM) = 0_{gR_i}$ for every $i < n$.*

Proof. (i): Denote by

$$\varphi_X : X \to \mathrm{cosk}_n X \qquad \text{for every } X \in \mathrm{Ob}(R\text{-}\mathbf{Mod})$$

the unit of adjunction (see (7.4.20)). Taking into account corollary 7.4.67(i), we have the following properties :

- $X[i] = \mathrm{cosk}_n X[i]$ and $\varphi[i]$ is the identity map of $X[i]$, for every $i \leq n$.
- $H_i(\mathrm{cosk}_n X) = 0$ for every $i \geq n$.

In view of proposition 7.10.28, we are then easily reduced to checking the assertion for the special where $N := \mathrm{cosk}_n M$, and $\varphi := \varphi_M$. However, since in this case $N[i] = M[i]$ for every $i \leq n$, it is clear that $\perp_\bullet^{R[i]} \varphi[i]$ is the identity automorphism of $\perp_\bullet^{R[i]} M[i]$, for every $i \leq n$. After applying the functor T_R, and extracting the diagonal, we see that the induced morphism $LT_RM \to LT_RN$ in $s.\mathscr{C}$ is given by the identity automorphism of $(LT_RM)[i]$, in every degree $i \leq n$, whence the contention.

(ii): In light of (i), we may assume that $M = s.0$ (the trivial R-module). In this case, the assertion follows easily from remark 7.10.17. □

7.10.35. In the situation of (7.10.26), take $\mathscr{C} := A\text{-}\mathsf{Alg.Mod}$, $f := F$, and g the identity automorphism of A-Alg. If $B \to B'$ is a map of A-algebras, and $(B, M) \in \mathrm{Ob}(\mathscr{C})$, we define $B' \otimes_B (B, M) := (B', B' \otimes_B M)$.

Corollary 7.10.36. *In the situation of* (7.10.35)*, suppose moreover that, for every flat morphism* $B \to B'$ *of* A*-algebras, the natural map*

$$B' \otimes_B T(B, M) \to T(B' \otimes_B (B, M))$$

is an isomorphism, for every $(B, M) \in \mathrm{Ob}(\mathscr{C})$*. Then, for every flat morphism* $\varphi : R \to S$ *of simplicial* V*-algebras, the natural morphism*

$$S \otimes_R LT_R M \to LT_S(S \otimes_R M) \qquad (7.10.37)$$

is an isomorphism in $\mathsf{D}(S\text{-}\mathbf{Mod})$*, for every* R*-module* M*.*

Proof. (Recall that φ is flat if and only if $S[n]$ is a flat $R[n]$-algebra, for every $n \in \mathbb{N}$. The map of the proposition is deduced from the natural morphism $\perp_\bullet^R M \to \perp_\bullet^S (S \otimes_R M)$, given by functoriality of the standard free resolution.)

Let $\pi : \perp_\bullet^R M \to M$ be the standard free resolution of M; by the flatness assumption on S, the morphism $S \otimes_R \pi : S \otimes_R \perp_\bullet^R M \to S \otimes_R M$ is a free resolution of the S-module $S \otimes_R M$, whence a natural isomorphism

$$LT_S(S \otimes_R M) \xrightarrow{\sim} T_S((S \otimes_R \perp_\bullet^R M)^\Delta) \qquad \text{in } \mathsf{D}(S\text{-}\mathbf{Mod})$$

by proposition 7.10.28(i) and Eilenberg-Zilber's theorem 7.4.50. On the other hand, the assumptions on T yield a natural isomorphism

$$S \otimes_R LT_R(M) = S \otimes_R T_R((\perp_\bullet^R M)^\Delta) \xrightarrow{\sim} T_S((S \otimes_R \perp_\bullet^R M)^\Delta) \qquad \text{in } \mathsf{D}(S\text{-}\mathbf{Mod}).$$

By combining these isomorphisms, we get an isomorphism $S \otimes_R LT_R M \to LT_S(S \otimes_R M)$, and a simple inspection shows that the latter is realized by the natural map (7.10.37). □

7.10.38. Let now R be any simplicial A-algebra, and $I \subset R$ any ideal, and set $R_0 := R/I$. The *positive Rees algebra of* I is the graded R-algebra

$$\mathsf{R}^+(R, I)^\bullet := \bigoplus_{p \in \mathbb{N}} I^p$$

(cp. definition 7.9.1(iii) and example 7.9.3). Notice the natural isomorphism of graded R_0-algebras

$$\mathsf{R}^+(R, I) \otimes_R R_0 \xrightarrow{\sim} \mathrm{gr}_I^\bullet R := \bigoplus_{p \in \mathbb{N}} I^p / I^{p+1}$$

as well as the exact sequence of $\mathsf{R}^+(R, I)$-modules

$$0 \to \mathrm{gr}_I^{\bullet+1} R \to \mathsf{R}^+(R, I) \otimes_R R/I^2 \to \mathrm{gr}_I^\bullet R \to 0$$

deduced from the natural projection $R/I^2 \to R_0$. Next, pick any quasi-isomorphism of R-algebras $P \to R_0$, with P a flat R-algebra; there follows an exact sequence of $P \otimes_R \mathsf{R}^+(R, I)$-modules

$$0 \to P \otimes_R \mathrm{gr}_I^{\bullet+1} R \to P \otimes_R \mathsf{R}^+(R, I) \otimes_R R/I^2 \xrightarrow{\beta} P \otimes_R \mathrm{gr}_I^\bullet R \to 0$$

and notice that β is actually a morphism of $P \otimes_R \mathsf{R}^+(R, I)$-algebras. According to remark 7.7.10(i), after forming the associated cochain complexes, we obtain an epimorphism $\beta^\bullet$ of differential graded $(P \otimes_R \mathsf{R}^+(R, I))^\bullet$-algebras, whose kernel $(P \otimes_R \mathrm{gr}_I^{\bullet+1} R)^\bullet$ is a two-sided ideal of $(P \otimes_R \mathsf{R}^+(R, I) \otimes_R R/I^2)^\bullet$. In this situation, we deduce a map of $H_\bullet(R_0 \overset{\ell}{\otimes}_R \mathsf{R}^+(P, I))$-modules

$$\delta : H_\bullet(R_0 \overset{\ell}{\otimes}_R \mathrm{gr}_I^\bullet R) \to H_\bullet(R_0 \overset{\ell}{\otimes}_R \mathrm{gr}_I^{\bullet+1} R)[1] \qquad (7.10.39)$$

which is a graded derivation, according to lemma 7.7.6 (recall that here the shift $[1]$ refers to the homological grading; the notation $\mathrm{gr}^{\bullet+1}$ denotes also a shift in degrees, which however *does not* alter the signs of the scalar multiplication in the way prescribed by remark 7.7.4(v)).

7.10.40. Keep the situation of (7.10.38), and suppose furthermore that R is a constant A-algebra and I a constant ideal, *i.e.* $R = s.B$, and $I = s.J$ for some A-algebra B and some ideal $J \subset B$. We set $B_0 := B/J$ and

$$G^\bullet := R_0 \overset{\ell}{\otimes}_R \mathrm{gr}^\bullet_I R \qquad \Lambda_\bullet := \Lambda^\bullet_B(J/J^2) \qquad S^\bullet := \mathrm{Sym}^\bullet_B(J/J^2)$$

where the derived tensor product is formed in $\mathsf{D}(R\text{-}\mathsf{Alg})$, as in example 7.10.25, so $G^\bullet$ is represented by $P \otimes_R \mathrm{gr}^\bullet_I R$, and $H_\bullet G^\bullet$ is a bigraded A-algebra, strictly anti-commutative for the (homological) grading $H_\bullet$, and commutative for the (cohomological) grading deduced from the grading of $G^\bullet$ (see remark 7.7.10(i)). Also, $\Lambda_\bullet$ (resp. $S^\bullet$) is a strictly anti-commutative (resp. a commutative) graded B_0-algebra, and we use the homological (resp. cohomological) conventions for grading, so Λ_p (resp. S^p) is placed in degree $-p$ (resp. p). Moreover, a standard calculation yields a natural isomorphism of B_0-modules

$$H_0 G^0 \xrightarrow{\sim} B_0 \qquad H_1 G^0 \xrightarrow{\sim} \mathrm{Tor}^B_1(B_0, B_0) \xrightarrow{\sim} J/J^2$$

whence a natural map $\Lambda_\bullet \to H_\bullet G^0$ of strictly anti-commutative graded B_0-algebras, restricting to an isomorphism in degrees ≤ 1. On the other hand, we have a surjective map of commutative graded B_0-algebras

$$S^\bullet \to \mathrm{gr}^\bullet_J B := \bigoplus_{p\in\mathbb{N}} J^p/J^{p+1} \tag{7.10.41}$$

restricting as well to an isomorphism in degrees ≤ 1; since $P \otimes_R \mathrm{gr}^\bullet_I R$ is naturally a simplicial $\mathrm{gr}^\bullet_J B$-algebra, we obtain a natural map of bigraded $S^\bullet$-algebras

$$\omega^\bullet_\bullet : \Lambda_\bullet \otimes_B S^\bullet \to H_\bullet G^\bullet.$$

Now, $H_\bullet G^\bullet$ has been endowed with a derivation $\delta : H_\bullet G^\bullet \to H_\bullet G^{\bullet+1}[1]$ in (7.10.39); on the other hand, there exists on $\Lambda_\bullet \otimes_B S^\bullet$ a natural $S^\bullet$-linear graded derivation

$$\partial : \Lambda_\bullet \otimes_B S^\bullet \to \Lambda_\bullet \otimes_B S^{\bullet+1}[1]$$

as explained in [103, I.4.3.1.2]. We may then consider the diagram of $S^\bullet$-linear maps

$$\begin{array}{ccc} \Lambda_\bullet \otimes_B S^\bullet & \xrightarrow{\ \omega^\bullet_\bullet\ } & H_\bullet G^\bullet \\ {\scriptstyle\partial}\downarrow & & \downarrow{\scriptstyle\delta} \\ \Lambda_\bullet \otimes_B S^{\bullet+1}[1] & \xrightarrow{\ \omega^{\bullet+1}_\bullet[1]\ } & H_\bullet G^{\bullet+1}[1]. \end{array} \tag{7.10.42}$$

Definition 7.10.43. (i) In the situation of (7.10.40), we say that the ideal J is *quasi-regular*, if the following conditions hold :

- The map $\omega^\bullet_\bullet$ restricts to an isomorphism $\omega^0_\bullet : \Lambda_\bullet \xrightarrow{\sim} H_\bullet G^0$.
- The B_0-module J/J^2 is flat.

(ii) Let R be any simplicial A-algebra, and $I \subset R$ any ideal. We say that I is *quasi-regular*, if $I[n]$ is a quasi-regular ideal of $R[n]$, for every $n \in \mathbb{N}$.

Remark 7.10.44. (i) Definition 7.10.43 is due to Quillen ([139, Def.8.4]). Notice that a sequence of elements $(f_1, \dots, f_r)$ in a ring B is B-quasi-regular, in the sense of definition 7.8.13, if and only if it generates an ideal $J \subset B$ such that J/J^2 is a free B_0-module of rank r, and (7.10.41) is an isomorphism (notation of (7.10.40)). The relationship between these two quasi-regularity notions is partially elucidated by the following proposition 7.10.45(ii).

(ii) Suppose that $\underline{B} := (B_i \mid i \in I)$ is any filtered system of rings, and $\underline{J} := (J_i \subset B_i \mid i \in I)$ a filtered system of ideals, such that J_i is quasi-regular in B_i, for every $i \in I$. Denote by B (resp. J) the colimit of $\underline{B}$ (resp. of $\underline{J}$); then it is easily seen that J is quasi-regular in B.

(iii) Let B be any ring, and $C := B[X_i \mid i \in I]$ a free B-algebra (for any set I). Then the ideal $J := (X_i \mid i \in I) \subset C$ is quasi-regular. Indeed, (ii) reduces to the case where I is a finite set, for which the assertion is a special case of proposition 7.10.47 below.

Proposition 7.10.45. *In the situation of* (7.10.40)*, we have :*

(i) (7.10.42) *is a commutative diagram.*

(ii) *Suppose that J is a quasi-regular ideal. Then*

(a) $\omega^\bullet_\bullet$ *is an isomorphism.*

(b) *There exists a natural isomorphism of B_0-modules :*

$$\mathrm{Tor}_i^B(B_0, B/J^n) \stackrel{\sim}{\to} \mathrm{Coker}(\partial^{n-2}_{i+1} : \Lambda_{i+1} \otimes_B S^{n-2} \to \Lambda_i \otimes_B S^{n-1}) \qquad \textit{for every } n, i > 0.$$

Proof. (i): Since both derivations are $S^\bullet$-linear, it suffices to check that (7.10.42) commutes in (upper) degree 0. Moreover, since the B-algebra $\Lambda_\bullet$ is generated by Λ_1, it suffices to check the commutativity of the diagram

(7.10.46)
$$\begin{array}{ccc} \Lambda_1 & \xrightarrow{\omega_1^0} & H_1G^0 \\ {\scriptstyle \partial_1^0}\downarrow & & \downarrow{\scriptstyle \delta_1^0} \\ S^1 & \xrightarrow{\omega_0^1} & H_0G^1. \end{array}$$

However, according to [103, I.4.3.1.2], the map ∂_1^0 is just the identity of J/J^2. The map δ_1^0 is the boundary map arising – by the snake lemma – from the exact sequence of complexes

$$0 \to JP^\bullet/J^2P^\bullet \to P^\bullet/J^2P^\bullet \to P^\bullet/JP^\bullet \to 0$$

(where $P^\bullet$ denotes the cochain complex associated with the flat resolution $P \to s.B_0$). The same boundary map is used to define the isomorphism ω_1^0, and ω_0^1 is the natural isomorphism $J/J^2 \to H_0(P^\bullet/JP^\bullet)$. Summing up, the commutativity of (7.10.46) follows by simple inspection.

(ii.a): We prove, by induction on $n \in \mathbb{N}$, that every $\omega^n_\bullet$ is an isomorphism. For $n = 0, 1$ there is nothing to show, so suppose that $n > 1$, and the assertion is already known for every degree $< n$. In order to prove the assertion in degree n, it suffices to check that $\omega_0^n : S^n \to \mathrm{gr}^n_J B$ is an isomorphism; indeed, in this case $\mathrm{gr}^n_J B$ is a flat B_0-module, and then a standard spectral sequence argument allows to conclude that ω_j^n is also an isomorphism, for every $j \in \mathbb{N}$.

Let $P \to R_0$ be a resolution as in (7.10.38). The I-adic filtration has finite length on P/I^kP, for every $k \in \mathbb{N}$, and therefore yields a convergent spectral sequence

$$E^1_{pq} \Rightarrow \mathrm{Tor}^B_{p+q}(B_0, B/J^k)$$

with

$$E^1_{pq} := \begin{cases} \mathrm{Tor}^B_{p+q}(B_0, \mathrm{gr}_J B^{-p}) & \text{for } p > -k \\ 0 & \text{otherwise} \end{cases}$$

whose differentials $E^1_{pq} \to E^1_{p-1,q}$ agree – by direct inspection – with the derivation δ^{-p}_{p+q}, whenever $p > 1 - k$. Especially, for $k = n + 1$, we get a complex

$$\Sigma \quad : \quad H_2G^{n-2} \xrightarrow{\delta_2^{n-2}} H_1G^{n-1} \xrightarrow{\delta_1^{n-1}} \mathrm{gr}^n_J B \to 0.$$

On the other hand, since J/J^2 is a flat B_0-module, the corresponding sequence

$$\Sigma' \quad : \quad \Lambda_2 \otimes_B S^{n-2} \xrightarrow{\partial} \Lambda_1 \otimes_B S^{n-1} \xrightarrow{\partial} S^n \to 0$$

is an exact complex (this is a segment of the Koszul complex of [103, I.4.3.1.7]). In light of (i), the maps ω_2^{n-2} and ω_1^{n-1} yield a morphism of complexes $\Sigma' \to \Sigma$, so we are reduced to checking that Σ is exact, and the inductive assumption already says that Σ is exact at the middle term $H_1 G^{n-1}$. Now notice that, since the Koszul complex is exact, the inductive assumption implies that $E^2_{pq} = 0$ for all (p, q) such that $p > 1 - n$ and $q > 0$; moreover, obviously we have $E^1_{pq} = 0$ for $p + q < 0$ or $p > 0$, therefore

$$E^\infty_{00} = E^1_{00} \qquad \text{and} \qquad E^\infty_{-n,n} = E^2_{-n,n} = \mathrm{Coker}\, \delta_1^{n-1}.$$

But a simple inspection shows that the natural map $H_0 P \to E^\infty_{00}$ is an isomorphism, so $E^\infty_{-n,n} = 0$, therefore δ_1^{n-1} is surjective, and Σ is exact, as required.

(ii.b): We use the previous spectral sequence, for $k = n$. Taking into account (ii.a), and the exactness of the Koszul complex ([103, I.4.3.1.7]) we get

$$\begin{aligned} &E^1_{pq} = 0 && \text{for } p + q < 0 \text{ or } p > 0 \text{ or } p \le -n \\ &E^2_{p,q} = 0 && \text{for every } (p, q) \text{ with } p > 1 - n \text{ and } q > 0 \\ &E^2_{1-n,i+n-1} \xrightarrow{\sim} \mathrm{Coker}\, \partial^{n-2}_{i+1} && \text{for every } i \in \mathbb{N}. \end{aligned}$$

It follows that $E^r_{1-n+r,i+n-r} = 0$ for every $r \ge 2$ and every $i \in \mathbb{N}$; indeed, this is clear if $1 - n + r > 0$, and if the latter fails, we get $1 - n + r > 1 - n$ and $i + n - r > 0$, so the stated vanishing holds nevertheless. We deduce that $E^\infty_{1-n,i+n-1} = E^2_{1-n,i+n-1}$ for every $i \in \mathbb{N}$. Next, suppose that $i > 0$. Clearly we have $E^\infty_{p,i-p} = 0$ both for $p > 0$ and $p \le -n$; and if $0 \ge p > 1 - n$, we get $i - p \ge i > 0$, so again $E^\infty_{p,i-p} = 0$. In conclusion, $E^\infty_{p,i-p} = 0$ except possibly for $p = 1 - n$, and the contention follows. ☐

Proposition 7.10.47. *Let B be a ring, $f_\bullet := (f_1, \dots, f_r)$ a finite sequence of elements of B. Denote by J the ideal generated by $f_\bullet$, set $B_0 := B/J$, and consider the following conditions :*

(a) *The sequence $f_\bullet$ is B-regular.*
(b) *The sequence $f_\bullet$ is completely secant in B.*
(c) *The ideal J is quasi-regular and J/J^2 is a free B_0-module of rank r.*
(d) *The sequence $f_\bullet$ is B-quasi-regular.*

Then (a)⇒(b)⇒(c)⇒(d)*, and if B is J-adically complete and separated, then* (d)⇒(a).

Proof. (a)⇒(b) and (c)⇒(d) have already been pointed out respectively in proposition 7.8.7 and proposition 7.10.45(ii.a), and the same for the implication (d)⇒(a), when B is J-adically complete and separated.

(b)⇒(c): This becomes a paraphrase of proposition 7.8.10, once we have identified the graded B-algebra $H_\bullet G^\bullet$ with the graded B-algebra $H_\bullet(B_0[0] \overset{\mathbf{L}}{\otimes}_B B_0[0])$. However, denote by $P^\bullet$ the cochain complex associated with the flat resolution $P \to s.B_0$; according to remark 7.7.10(i), $P^\bullet$ is naturally a differential graded B-algebra, and the induced map $\varepsilon^\bullet : P^\bullet \to B_0[0]$ is a morphism of differential graded B-algebras (for the multiplication law $\overline{\mu}^\bullet$ on $B_0[0]$ inherited from that of B_0). Then there exists in $\mathsf{D}(B\text{-}\mathbf{Mod})$ a unique isomorphism $\omega^\bullet : P^\bullet_{B_0} \to P^\bullet$ whose composition with $\varepsilon^\bullet$ agrees with $\rho^\bullet_{B_0}$ (notation of (7.3.38)), and moreover we get a commutative diagram in $\mathsf{D}(B\text{-}\mathbf{Mod})$:

$$\begin{array}{ccc} P^\bullet_{B_0} \otimes_B P^\bullet_{B_0} & \xrightarrow{P^\bullet_{\overline{\mu}}} & P^\bullet_{B_0} \\ {\scriptstyle \omega^\bullet \otimes_B \omega^\bullet} \downarrow & & \downarrow {\scriptstyle \omega^\bullet} \\ P^\bullet \otimes_B P^\bullet & \longrightarrow & P^\bullet \end{array}$$

whose bottom horizontal arrow is the multiplication map of $P^\bullet$, and where $P^\bullet_{\overline{\mu}}$ is defined as in (7.3.38). The assertion follows immediately. ☐

The following is a well known theorem of Quillen, which is found in his typewritten notes [139, Th.8.8] that have been circulated since the late 60s, but have never been published. A version of this theorem, under more restrictive assumptions, is also found in [3, Ch.XIII, Prop.3].

Theorem 7.10.48. *Let R be any simplicial A-algebra, and $I \subset R$ a quasi-regular ideal such that $H_0 I = 0$. Then we have :*

$$H_i I^n = 0 \qquad \text{for every } n \in \mathbb{N} \text{ and every } i < n.$$

Proof. We argue by induction on $n \in \mathbb{N}$. For $n \leq 1$ there is nothing to prove, so suppose that $n > 1$, and that the assertion is already known for I^{n-1}. Set $R_k := R/I^{k+1}$ for every $k \in \mathbb{N}$, and let

$$\Lambda_\bullet \otimes_R S^\bullet := \Lambda^\bullet_R(I/I^2) \otimes_R \mathrm{Sym}^\bullet_R(I/I^2)$$

which is a bigraded R_0-module, endowed with the bigraded derivation $\partial^\bullet_\bullet$ such that $\partial^\bullet_\bullet[i]$ is the Koszul derivation of $\Lambda_\bullet \otimes_R S^\bullet[i]$, defined as in [103, I.4.3.1.2]. Set as well

$$C_i := \mathrm{Coker}\,(\partial^{n-3}_i : \Lambda_i \otimes_R S^{n-3} \to \Lambda_{i-1} \otimes_R S^{n-2}) \qquad \text{for every } i \in \mathbb{N}.$$

From the inductive assumption and lemma 7.10.32, we already know that

$$H_i(I \overset{\ell}{\otimes}_R I^{n-1}) = 0 \qquad \text{for every } i < n. \tag{7.10.49}$$

However, according to remark 7.10.24, there is a spectral sequence

$$E^1_{pq} := \mathrm{Tor}^{R[p]}_q(I[p], I[p]^{n-1}) \Rightarrow H_{p+q}(I \overset{\ell}{\otimes}_R I^{n-1})$$

and proposition 7.10.45(ii.b) yields natural isomorphisms

$$E^1_{pq} \xrightarrow{\sim} \mathrm{Tor}^{R[p]}_{q+1}(R_0[p], I[p]^{n-1}) \xrightarrow{\sim} \mathrm{Tor}^{R[p]}_{q+2}(R_0[p], R_{n-2}[p]) \xrightarrow{\sim} C_{q+3}[p] \qquad \text{for every } q > 0$$

under which, the differentials $E^1_{pq} \to E^1_{p-1,q}$ are identified with those of the chain complex defining the R_0-module C_{q+3}. Now, since I/I^2 is a flat R_0-module, [103, I.4.3.1.7] says that the sequence of R_0-modules

$$\Sigma^i \quad : \quad 0 \to \Lambda_{n+i-3} \xrightarrow{\partial} \Lambda_{n+i-4} \otimes_R S^1 \to \cdots \to \Lambda_i \otimes_R S^{n-3} \xrightarrow{\partial} \Lambda_{i-1} \otimes_R S^{n-2} \to C_i \to 0$$

is an exact complex, for every $i > 0$. On the other hand, since $H_0 I/I^2 = 0$, (and again, since I/I^2 is a flat R_0-module), combining [103, I.4.3.2.1] and lemma 7.10.32, we see that

$$H_i(\Lambda_j \otimes_R S^{q-j}) = 0 \qquad \text{for every } q \in \mathbb{N}, \text{ every } j \leq q, \text{ and every } i < q.$$

We recall the following general

Claim 7.10.50. Let $\mathscr{A}$ be any abelian category, $d \in \mathbb{N}$ any integer, and consider an exact sequence

$$0 \to K_d \xrightarrow{f_d} K_{d-1} \to \cdots \to K_1 \xrightarrow{f_1} K_0 \to 0$$

of objects of $\mathsf{C}(\mathscr{A})$, such that $K_i = 0$ in $\mathsf{D}(\mathscr{A})$, for every $i = 1, \dots, d-1$. Then there is a natural isomorphism

$$K_0 \xrightarrow{\sim} K_d[d-1] \qquad \text{in } \mathsf{D}(\mathscr{A}).$$

Proof of the claim. For every $i = 0, \dots, d$, set $Z_i := \mathrm{Ker}\, f_i$. We have short exact sequences

$$0 \to Z_i \to K_i \to Z_{i-1} \to 0 \qquad \text{for } i = 1, \dots, d.$$

Since $K_i = 0$ in $\mathsf{D}(\mathscr{A})$, there follows natural isomorphisms $Z_{i-1} \xrightarrow{\sim} Z_i[1]$ for every $i = 1, \dots, d-1$. However, $Z_0 = K_0$ and $Z_{d-1} = K_d$, whence the claim. ◊

By applying claim 7.10.50 to the exact sequence $\mathrm{cosk}_{n+i-3}\Sigma^i$, we easily deduce that

$$H_j C_i = 0 \qquad \text{for every } i > 0 \text{ and every } j < n+i-3$$

so finally $E^2_{pq} = 0$ for every $q > 0$ and every $p < n + q$, and clearly $E^1_{pq} = 0$ for every $q < 0$. Consequently, for every $i < n$ the term $E^\infty_{i-q,q} = 0$ vanishes for $q > 0$, and equals $E^2_{i,0}$ for $q = 0$. Lastly, a simple calculation shows that

$$E^2_{i,0} = H_i(I \otimes_R I^{n-1}) \qquad \text{for every } i \in \mathbb{N}.$$

Summing up, and taking into account 7.10.49, we conclude that $H_i(I \otimes_R I^{n-1}) = 0$ for every $i < n$, so we are reduced to checking :

Claim 7.10.51. The natural map $H_i(I \otimes_R I^{n-1}) \to H_i(I^n)$ is surjective, for every $i \leq n$.

Proof of the claim. Indeed, denote by K the kernel of the epimorphism $I \otimes_R I^{n-1} \to I^n$; as in the foregoing, we get natural isomorphisms

$$K[i] \xrightarrow{\sim} \mathrm{Tor}_1^{R[i]}(R_0[i], I[i]^{n-1}) \xrightarrow{\sim} \mathrm{Tor}_2^{R[i]}(R_0[i], R[i]/I[i]^{n-1}) \xrightarrow{\sim} C_3[i] \qquad \text{for every } i \in \mathbb{N}$$

that amount to an isomorphism $K \xrightarrow{\sim} C_3$ of R_0-modules. We have already seen that $H_i C_3 = 0$ for every $i < n$, whence the claim. □

7.11. **Injective modules, flat modules and indecomposable modules.**

7.11.1. *Indecomposable modules.* Recall that a unital (not necessarily commutative) ring R is said to be *local* if $R \neq 0$ and, for every $x \in R$ either x or $1-x$ is invertible. If R is commutative, this definition is equivalent to the usual one.

7.11.2. Let $\mathscr{C}$ be any abelian category and M an object of $\mathscr{C}$. One says that M is *indecomposable* if it is non-zero and cannot be presented in the form $M = N_1 \oplus N_2$ with non-zero objects N_1 and N_2. If such a decomposition exists, then $\mathrm{End}_{\mathscr{C}}(N_1) \times \mathrm{End}_{\mathscr{C}}(N_2) \subset \mathrm{End}_{\mathscr{C}}(M)$, especially the unital ring $\mathrm{End}_{\mathscr{C}}(M)$ contains an idempotent element $e \neq 1, 0$ and therefore it is not a local ring.

However, if M is indecomposable, it does not necessarily follow that $\mathrm{End}_{\mathscr{C}}(M)$ is a local ring. Nevertheless, one has the following:

Theorem 7.11.3 (Krull-Remak-Schmidt)**.** *Let $(A_i)_{i\in I}$ and $(B_j)_{j\in J}$ be two finite families of objects of $\mathscr{C}$, such that:*

(a) $\oplus_{i\in I} A_i \simeq \oplus_{j\in J} B_j$.
(b) $\mathrm{End}_{\mathscr{C}}(A_i)$ *is a local ring for every $i \in I$, and $B_j \neq 0$ for every $j \in J$.*

Then we have :

(i) *There is a surjection $\varphi : I \to J$, and isomorphisms $B_j \xrightarrow{\sim} \oplus_{i\in\varphi^{-1}(j)} A_i$, for every $j \in J$.*
(ii) *Especially, if B_j is indecomposable for every $j \in J$, then I and J have the same cardinality, and φ is a bijection.*

Proof. Clearly, (i)⇒(ii). To show (i), let us begin with the following :

Claim 7.11.4. Let M_1, M_2 be two objects of $\mathscr{C}$, and set $M := M_1 \oplus M_2$. Denote by $e_i : M_i \to M$ (resp. $p_i : M \to M_i$) the natural injection (resp. projection) for $i = 1, 2$. Suppose that $\alpha : P \to M$ is a subobject of M, such that $p_1\alpha : P \to M_1$ is an isomorphism. Then the natural morphism $\beta : P \oplus M_2 \to M$ is an isomorphism.

Proof of the claim. Denote by $\pi_P : P \oplus M_2 \to P$ and $\pi_2 : P \oplus M_2 \to M_2$ the natural projections; then $\beta := \alpha\pi_P + e_2\pi_2$. Of course, the assertion follows easily by applying the 5-lemma (which holds in any abelian category) to the commutative ladder with exact rows :

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_2 & \longrightarrow & P \oplus M_2 & \xrightarrow{\pi_P} & P & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle p_1\alpha} & & \\ 0 & \longrightarrow & M_2 & \xrightarrow{e_2} & M & \xrightarrow{p_1} & M_1 & \longrightarrow & 0. \end{array}$$

Equivalently, one can argue directly as follows. By definition, $\mathrm{Coker}\,\beta$ represents the functor

$$\mathscr{C} \to \mathbb{Z}\text{-}\mathbf{Mod} \quad : \quad X \mapsto \mathrm{Ker}\,\mathrm{Hom}_{\mathscr{C}}(\beta, X)$$

and likewise for $\mathrm{Coker}\,p_1\alpha$; however, it is easily seen that these functors are naturally isomorphic, hence the natural morphism $\mathrm{Coker}\,\beta \to \mathrm{Coker}\,p_1\alpha$ is an isomorphism, so β is an epimorphism. Next, let $t : \mathrm{Ker}\,\beta \to P \oplus M_2$ be the natural morphism; since $p_1\alpha$ is an isomorphism, $\pi_P \circ t = 0$, so t factors through a morphism $\bar{t} : \mathrm{Ker}\,\beta \to M_2$. It follows that $e_2 \circ \bar{t} = \beta \circ t = 0$, therefore $\bar{t} = 0$, and finally $\mathrm{Ker}\,\beta = 0$, since t is a monomorphism. $\diamondsuit$

Set $A := \oplus_{i \in I} A_i$, and denote $e_i : A_i \to A$ (resp. $p_i : A \to A_i$) the natural injection (resp. projection), for every $i \in I$. Also, endow $I' := I \cup \{\infty\}$ with any total ordering $(I', <)$ for which ∞ is the maximal element.

Claim 7.11.5. Let $(a_j : A \to A)_{j \in J}$ be a finite system of endomorphisms such that $\sum_{j \in J} a_j = \mathbf{1}_A$. Then there exists a mapping $\varphi : I \to J$ such that the following holds :

(i) $a_{\varphi(i)} e_i : A_i \to A$ is a monomorphism, for every $i \in I$, and let $P_i := \mathrm{Im}(a_{\varphi(i)} e_i)$.
(ii) For every $l \in I'$, the natural morphism $\beta_l : \bigoplus_{i<l} P_i \oplus \bigoplus_{i \geq l} A_i \to A$ is an isomorphism.

Proof of the claim. We both define $\varphi(l)$ and prove (ii), by induction on l. If l is the smallest element of I, then $\beta_l = \mathbf{1}_A$, so (ii) is obvious. Hence, let $l \in I$ be any element, $l' \in I'$ the successor of l, and assume that β_l is an isomorphism. Set $a'_j := \beta_l^{-1} a_j \beta_l$ for every $j \in J$, and $M_l := \bigoplus_{i<l} P_i \oplus \bigoplus_{i \geq l} A_i$; also, let $e'_l : A_l \to M_l$ and $p'_l : M_l \to A_l$ be the natural morphisms. Clearly $\sum_{j \in J} a'_j = \mathbf{1}_{M_l}$, so $\sum_{j \in J} p'_l a'_j e'_l = \mathbf{1}_{A_l}$; since $\mathrm{End}_{\mathscr{C}}(A_l)$ is a local ring, it follows that there exists $j_l \in J$ such that $p'_l a'_{j_l} e'_l$ is an automorphism of A_l. However,

$$a'_{j_l} e'_l = \beta_l^{-1} a_{j_l} \beta_l e'_l = \beta_l^{-1} a_{j_l} e_l \qquad (7.11.6)$$

so assertion (i) follows for the index l, by setting $\varphi(l) := j_l$.

Next, set $M_{l,1} := A_l$ and $M_{l,2} := \bigoplus_{i<l} P_i \oplus \bigoplus_{i>l} A_i$. The foregoing argument provides a subobject $\alpha : P'_l := \mathrm{Im}(a'_{\varphi(l)} e'_l) \to M_l = M_{l,1} \oplus M_{l,2}$ such that $p'_l \alpha : P'_l \to M_{l,1}$ is an isomorphism. By claim 7.11.4, it follows that the natural map $\beta'_{l'} : P'_l \oplus M_{l,2} \to M_l$ is an isomorphism. On the other hand, (7.11.6) implies that β_l restricts to an isomorphism $\gamma_l : P'_l \xrightarrow{\sim} P_l$. We deduce a commutative diagram

$$\begin{array}{ccc} P'_l \oplus M_{l,2} & \xrightarrow{\beta'_{l'}} & M_l \\ {\scriptstyle \gamma_l \oplus \mathbf{1}_{M_{l,2}}} \downarrow & & \downarrow {\scriptstyle \beta_l} \\ M_{l'} & \xrightarrow{\beta_{l'}} & A \end{array}$$

whose vertical arrows and top arrow are isomorphisms. Thus, the bottom arrow is an isomorphism as well, as required. $\diamondsuit$

For every $j \in J$, let $e'_j : B_j \to A$ (resp. $p'_j : A \to B_j$) be the injection (resp. projection) deduced from a given isomorphism as in (a), and set $a_j := e'_j p'_j$ for every $j \in J$. Then $\sum_{j \in J} a_j = \mathbf{1}_A$, so claim 7.11.5 yields a mapping $\varphi : I \to J$ such that the following holds. Set $P_j := \oplus_{i \in \varphi^{-1}(j)} \mathrm{Im}(a_j e_i)$ for every $j \in J$; then the natural morphism $\beta : \oplus_{j \in J} P_j \xrightarrow{\sim} A$ is an isomorphism. However, clearly $\beta(P_j)$ is a subobject of $\mathrm{Im}(e'_j)$, for every $j \in J$. Hence, φ must be a surjection, and β restricts to isomorphisms $P_j \xrightarrow{\sim} \mathrm{Im}(e'_j)$ for every $j \in J$, whence isomorphisms $P_j \xrightarrow{\sim} B_j$, from which (ii) follows immediately. $\square$

Remark 7.11.7. There are variants of theorem 7.11.3, that hold under different sets of assumptions. For instance, in [76, Ch.I, §6, Th.1] it is stated that part (ii) of the theorem still holds when I and J are small (not necessarily finite) sets, provided that $\mathscr{C}$ admits a small set of generators and that all small filtered colimits are representable and exact in $\mathscr{C}$. (One can moreover show that the latter result still holds even in the absence of a small set of generators for $\mathscr{C}$.)

7.11.8. Let us now specialize to the case of the category A-$\mathbf{Mod}$, where A is a (commutative) local ring, say with residue field k. Let M be any finitely generated A-module; arguing by induction on the dimension of $M \otimes_A k$, one shows easily that M admits a finite decomposition $M = \oplus_{i=1}^r M_i$, where M_i is indecomposable for every $i \leq r$.

Lemma 7.11.9. *Let A be a henselian local ring, and M an A-module of finite type. Then M is indecomposable if and only if $\mathrm{End}_A(M)$ is a local ring.*

Proof. In view of (7.11.2), we can assume that M is indecomposable, and we wish to prove that $\mathrm{End}_A(M)$ is local. Thus, let $\varphi \in \mathrm{End}_A(M)$.

Claim 7.11.10. The subalgebra $A[\varphi] \subset \mathrm{End}_A(M)$ is integral over A.

Proof of the claim. This is standard: choose a finite system of generators $(f_i)_{1\leq i\leq r}$ for M, then we can find a matrix $\mathbf{a} := (a_{ij})_{1\leq i,j\leq r}$ of elements of A, such that $\varphi(f_i) = \sum_{j=1}^r a_{ij} f_j$ for every $i \leq r$. The matrix $\mathbf{a}$ yields an endomorphism ψ of the free A-module $A^{\oplus r}$; on the other hand, let $e_1, \dots, e_r$ be the standard basis of $A^{\oplus r}$ and define an A-linear surjection $\pi : A^{\oplus r} \to M$ by the rule: $e_i \mapsto f_i$ for every $i \leq r$. Then $\varphi \circ \pi = \pi \circ \psi$; by Cayley-Hamilton, ψ is annihilated by the characteristic polynomial $\chi(T)$ of the matrix $\mathbf{a}$, whence $\chi(\varphi) = 0$. ◊

Since A is henselian, claim 7.11.10 implies that $A[\varphi]$ decomposes as a finite product of local rings. If there were more than one non-zero factor in this decomposition, the ring $A[\varphi]$ would contain an idempotent $e \neq 1, 0$, whence the decomposition $M = eM \oplus (1-e)M$, where both summands would be non-zero, which contradicts the assumption. Thus, $A[\varphi]$ is a local ring, so that either φ or $\mathbf{1}_M - \varphi$ is invertible. Since φ was chosen arbitrarily in $\mathrm{End}_A(M)$, the claim follows. □

Corollary 7.11.11. *Let A be a henselian local ring. Then:*

(i) *If $(M_i)_{i\in I}$ and $(N_j)_{j\in J}$ are two finite families of indecomposable A-modules of finite type such that $\oplus_{i=1}^r M_i \simeq \oplus_{j=1}^s N_j$, then there is a bijection $\beta : I \xrightarrow{\sim} J$ such that $M_i \simeq N_{\beta(i)}$ for every $i \in I$.*

(ii) *If M and N are two finitely generated A-modules such that $M^{\oplus k} \simeq N^{\oplus k}$ for some integer $k > 0$, then $M \simeq N$.*

(iii) *If M, N and X are three finitely generated A-modules such that $X \oplus M \simeq X \oplus N$, then $M \simeq N$.*

Proof. It follows easily from theorem 7.11.3 and lemma 7.11.9; the details are left to the reader. □

Proposition 7.11.12. *Let $A \to B$ be a faithfully flat map of local rings, M and N two finitely presented A-modules with a B-linear isomorphism $\omega : B \otimes_A M \xrightarrow{\sim} B \otimes_A N$. Then $M \simeq N$.*

Proof. Under the standing assumptions, the natural B-linear map

$$B \otimes_A \mathrm{Hom}_A(M, N) \to \mathrm{Hom}_B(B \otimes_A M, B \otimes_A N)$$

is an isomorphism ([75, Lemma 2.4.29(i.a)]). Hence we can write $\omega = \sum_{i=1}^r b_i \otimes \varphi_i$ for some $b_i \in B$ and $\varphi_i : M \to N$ $(1 \leq i \leq r)$. Denote by k and K the residue fields of A and B respectively; we set $\overline{\varphi}_i := \mathbf{1}_k \otimes_A \varphi_i : k \otimes_A M \to k \otimes_A N$. From the existence of ω we deduce easily that $n := \dim_k k \otimes_A M = \dim_k k \otimes_A N$. Hence, after choosing bases, we can view $\overline{\varphi}_1, \dots, \overline{\varphi}_r$ as endomorphisms of the k-vector space $k^{\oplus n}$. We consider the polynomial $p(T_1, \dots, T_r) := \det(\sum_{i=1}^r T_i \cdot \overline{\varphi}_i) \in k[T_1, \dots, T_r]$. Let $\overline{b}_1, \dots, \overline{b}_r$ be the images of $b_1, \dots, b_r$ in K; it follows that $p(\overline{b}_1, \dots, \overline{b}_r) \neq 0$, especially $p(T_1, \dots, T_r) \neq 0$.

Claim 7.11.13. The proposition holds if k is an infinite field or if $k = K$.

Proof of the claim. Indeed, in either of these cases we can find $\overline{a}_1, \dots, \overline{a}_r \in k$ such that $p(\overline{a}_1, \dots, \overline{a}_r) \neq 0$. For every $i \leq r$ choose an arbitrary representative $a_i \in A$ of $\overline{a}_i$, and set $\varphi := \sum_{i=1}^r a_i \varphi_i$. By construction, $\mathbf{1}_k \otimes_A \varphi : k \otimes_A M \to k \otimes_A N$ is an isomorphism. By Nakayama's lemma we deduce that φ is surjective. Exchanging the roles of M and N, the same argument yields an A-linear surjection $\psi : N \to M$. Finally, [126, Ch.1, Th.2.4] shows that both $\psi \circ \varphi$ and $\varphi \circ \psi$ are isomorphisms, whence the claim. ◊

Let A^{h} be the henselization of A, and A^{sh}, B^{sh} the strict henselizations of A and B respectively. Now, the induced map $A^{\mathrm{sh}} \to B^{\mathrm{sh}}$ is faithfully flat, the residue field of A^{sh} is infinite, and ω induces a B^{sh}-linear isomorphism $B^{\mathrm{sh}} \otimes_A M \xrightarrow{\sim} B^{\mathrm{sh}} \otimes_A N$. Hence, claim 7.11.13 yields an A^{sh}-linear isomorphism $\beta : A^{\mathrm{sh}} \otimes_A M \xrightarrow{\sim} A^{\mathrm{sh}} \otimes_A N$. However, A^{sh} is the colimit of a filtered family $(A_\lambda \mid \lambda \in \Lambda)$ of finite étale A^{h}-algebras, so β descends to an A_λ-linear isomorphism $\beta_\lambda : A_\lambda \otimes_A M \simeq A_\lambda \otimes_A N$ for some $\lambda \in \Lambda$. The A^{h}-module A_λ is free, say of finite rank n, hence β_λ can be regarded as an A^{h}-linear isomorphism $(A^{\mathrm{h}} \otimes_A M)^{\oplus n} \xrightarrow{\sim} (A^{\mathrm{h}} \otimes_A N)^{\oplus n}$. Then $A^{\mathrm{h}} \otimes_A M \simeq A^{\mathrm{h}} \otimes_A N$, by corollary 7.11.11(ii). Since the residue field of A^{h} is k, we conclude by another application of claim 7.11.13. □

7.11.14. *Injective hulls.* The notion of injective hull plays a central role in the theory of local duality : for a noetherian local ring, one constructs a dualizing module as the injective hull of the residue field (see [85, Exp.IV, Th.4.7]), and in section 11.6, injective hulls of the residue fields of a monoid algebra will also enable us to perform a certain computation of local cohomology, which is a crucial step in the proof of Hochster's theorem. We present here the basic results on injective hulls, in the context of arbitrary abelian categories.

Definition 7.11.15. Let $\mathscr{A}$ be any abelian category, and $f : N \to M$ a monomorphism in $\mathscr{A}$.

(i) We say that M is an *essential extension of* N if the following holds. For any subobject $P \subset M$ we have either $P = 0$ or $P \cap \mathrm{Im}\, f \neq 0$ (here 0 denotes the zero object of $\mathscr{A}$: see remark 3.7.29(i)).

(ii) We say that M is a *proper essential extension of* N if it is an essential extension of N, and f is not an isomorphism.

(iii) We say that M is an *injective hull* of N, if M is both an essential extension of N, and an injective object of $\mathscr{A}$.

Lemma 7.11.16. *Let $\mathscr{A}$ be an abelian category, and I an object of $\mathscr{A}$. Suppose that :*

(a) *$\mathscr{A}$ is cocomplete.*

(b) *All colimits of $\mathscr{A}$ are universal (see example* 1.4.17*).*

Then we have :

(i) *I is injective if and only if it does not admit any proper essential extensions.*

(ii) *If $N \to M$ is any monomorphism, the set of all essential extensions of N contained in M admits maximal elements.*

Proof. (i): Suppose that I is injective, and let $f : I \to M$ be any monomorphism which is not an isomorphism. Then f admits a left inverse, so I is a direct summand of M, hence M is not an essential extension of I. Conversely, suppose that I does not admit proper essential extensions; let $f : N \to M$ be a monomorphism in $\mathscr{A}$ and $g : N \to I$ any morphism in $\mathscr{A}$. We consider the cocartesian diagram in $\mathscr{A}$

$$\begin{array}{ccc} N & \xrightarrow{f} & M \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ I & \xrightarrow{f'} & P \end{array}$$

and notice that f' is a monomorphism, since the same holds for f. By Zorn's lemma – and due to conditions (a) and (b) – we may find a maximal subobject $Q \subset P$ such that $Q \cap I = 0$, and clearly P/Q is an essential extension of I (via h); therefore $P/Q = I$, so $P = I \oplus Q$, and if we let $p : P \to I$ be the resulting projection, we see that $p \circ g' : M \to I$ is an extension of f. This shows that I is injective.

(ii): By Zorn's lemma, it suffices to check the following assertion. Suppose that $(P_j \mid j \in J)$ is a non-empty totally ordered family of essential extensions of N contained in M (for some totally ordered small indexing set J). Then $P := \bigcup_{j\in J} P_j$ is again an essential extension of N (notice that this union – *i.e.* colimit – is a subobject of M, due to condition (b)). However, say that $Q \subset P$ and $Q \cap N = 0$; due to condition (b), we have $Q = \bigcup_{j\in J}(Q \cap P_j)$, and clearly $(Q \cap P_j) \cap N = 0$, so $Q \cap P_j = 0$ for every $j \in J$, and finally $Q = 0$. □

Proposition 7.11.17. *Let $\mathscr{A}$ be an abelian category fulfilling condition* (a) *and* (b) *of lemma* 7.11.16*, and M any object of $\mathscr{A}$. We suppose moreover that :*

(c) *$\mathscr{A}$ has enough injective objects.*

Then we have :

(i) *M admits an injective hull. More precisely, if $M \to I$ is any monomorphism into an injective object, then a maximal essential extension of M in I is an injective hull of M.*

(ii) *Let $f : M \to E$ be an injective hull of M, and $g : M \to I$ any monomorphism into an injective object. Then g factors through f and a monomorphism $h : E \to I$.*

(iii) *If $f : M \to E$ and $g : M \to E'$ are two injective hulls of M, there exists an isomorphism $h : E \xrightarrow{\sim} E'$ such that $h \circ f = f'$.*

Proof. (i): Let $E \subset I$ be such a maximal essential extension of M (which exists by lemma 7.11.16(ii)); by virtue of lemma 7.11.16(i), it suffices to check that E does not admit proper essential extensions. However, suppose that $E \to E'$ is a proper essential extension; since I is injective, the inclusion morphism $E \to I$ extends to a morphism $f : E' \to I$. By maximality of E, we must then have $\mathrm{Ker}\, f \neq 0$; on the other hand, obviously $E \cap \mathrm{Ker}\, f = 0$, which is absurd, since E' is an essential extension of E.

(ii): Since I is injective, g extends to a morphism $h : E \to I$, and clearly $M \cap \mathrm{Ker}\, h = 0$. Since E is an essential extension of M, it follows that $\mathrm{Ker}\, h = 0$.

(iii): By (ii) there exists a monomorphism $h : E \to E'$ such that $h \circ f = f'$. Since E is injective, it follows that $\mathrm{Im}\, h$ is a direct summand of E'; but E' is an essential extension of M, so $\mathrm{Im}\, h = E'$, *i.e.* h is also an epimorphism, hence an isomorphism. □

7.11.18. We specialize now to the case where $\mathscr{A}$ is the category A-$\mathbf{Mod}$ of A-modules, with A an arbitrary noetherian ring. Recall that if M is any A-module, then the set $\mathrm{Ass}\, M$ of all *associated primes* of M consists of the prime ideals $\mathfrak{p} \subset A$ such that there exists $m \in M$ with $\mathrm{Ann}_A(m) = \mathfrak{p}$. (This is the correct definition only for noetherian rings : we shall see in definition 10.5.1 a more general notion that is well behaved for arbitrary rings.) By proposition 7.11.17(i,iii) the injective hull of M exists and is well defined up to (in general, non-unique) isomorphism, and we shall denote by $E_A(M)$ a choice of such hull.

Lemma 7.11.19. *Let A be a noetherian ring. The following holds :*

(i) *If $(I_\lambda \mid \lambda \in \Lambda)$ is a (small) family of injective A-modules, then $I := \bigoplus_{\lambda\in\Lambda} I_\lambda$ is an injective A-module.*

(ii) *If $S \subset A$ is any multiplicative subset, and M any injective A-module, then M_S is an injective A_S-module.*

Proof. (i): Let us first recall :

Claim 7.11.20. Let R be any ring, M an R-module. The following conditions are equivalent :

(a) M is an injective R-module.
(b) For every ideal $J \subset R$, every R-linear map $J \to M$ extends to an R-linear map $R \to M$.
(c) $\mathrm{Ext}^1_R(R/J, M) = 0$ for every ideal $J \subset R$.

Proof of the claim. Of course, (a)⇒(b)⇒(c). To check that (c)⇒(a), let $N \subset P$ be an inclusion of R-modules, and $f : N \to M$ an A-linear map. We let S be the set of all pairs (N', f'), where $N' \subset P$ is an R-submodule containing N, and $f' : N' \to M$ an R-linear map extending f. The set S is partially ordered, by declaring that $(N', f') \geq (N'', f'')$ if $N'' \subset N'$ and $f'_{|N''} = f''$, for any two pairs $(N', f'), (N'', f'') \in S$. By Zorn's lemma, S admits a maximal element (Q, g). Suppose $Q \neq P$, and let $x \in P \setminus Q$. Set $Q' := Q + Rx$ and $J := \mathrm{Ann}_R(Q'/Q)$; there follows an exact sequence of R-modules

$$0 \to Q \to Q' \to R/J \to 0$$

and then (c) implies that the restriction map $\mathrm{Hom}_R(Q', M) \to \mathrm{Hom}_R(Q, M)$ is surjective; especially, g extetnds to an R-linear map $Q' \to M$, contradicting the maximality of Q. Hence $Q = P$, which shows (a). ♢

In view of claim 7.11.20, it suffices to show that every A-linear map $f : J \to I$ from any ideal $J \subset A$, extends to an A-linear map $A \to I$. However, since J is finitely generated, there exists a finite subset $\Lambda' \subset \Lambda$ such that the image of f is contained in $I' := \bigoplus_{\lambda \in \Lambda'} I_\lambda$. Clearly I' is an injective A-module, so f extends to an A-linear map $A \to I'$, and the assertion follows.

(ii): Let $J \subset A_S$ be any ideal, and write $J = I_S$ for some ideal $I \subset A$; since A is noetherian, we have

$$\mathrm{Ext}^1_{A_S}(A_S/J, M_S) = A_S \otimes_A \mathrm{Ext}^1_A(A/I, M)$$

so the assertion follows from claim 7.11.20. □

We may now state :

Proposition 7.11.21. *Let A be any noetherian ring, M any A-module. We have :*

(i) *For every $\mathfrak{p} \in \mathrm{Spec}\, A$, the A-module $E_A(A/\mathfrak{p})$ is indecomposable (see (7.11.2)).*
(ii) *Let I be any non-zero injective A-module, and $\mathfrak{p} \in \mathrm{Ass}\, I$ any associated prime. Then $E_A(A/\mathfrak{p})$ is a direct summand of I. Especially, if I is indecomposable, then I is isomorphic to $E_A(A/\mathfrak{p})$.*
(iii) $\mathrm{Ass}_A M = \mathrm{Ass}_A E_A(M)$.
(iv) *If $\mathfrak{p}, \mathfrak{q} \in \mathrm{Spec}\, A$ are any two prime ideals, then the A-modules $E_A(A/\mathfrak{p})$ and $E_A(A/\mathfrak{q})$ are isomorphic if and only if $\mathfrak{p} = \mathfrak{q}$.*
(v) *$E_{A_S}(M_S) \simeq A_S \otimes_A E_A(M)$ for any multiplicative subset $S \subset A$.*

Proof. (i): Set $B := A/\mathfrak{p}$, and suppose that $E_A(B)$ is decomposable; especially, there exist non-zero submodules $M_1, M_2 \subset E_A(B)$ with $M_1 \cap M_2 = 0$. Thus, $(M_1 \cap B) \cap (M_2 \cap B) = 0$, and since B is a domain, we deduce that $M_i \cap B = 0$ for $i = 1, 2$. But since $E_A(B)$ is an essential extension of B, this is absurd.

(ii): By assumption, there exists $x \in I$ such that $\mathrm{Ann}_A(x) = \mathfrak{p}$, so the submodule $Ax \subset I$ is isomorphic to $A/\mathfrak{p}$; then, by proposition 7.11.17(ii) there exists a monomorphism $f : E_A(A/\mathfrak{p}) \to I$, and since $E_A(A/\mathfrak{p})$ in injective, the image of f is a direct summand of I.

(iii): Clearly $\mathrm{Ass}_A(M) \subset \mathrm{Ass}_A E_A(M)$. Conversely, suppose $\mathfrak{p} \in \mathrm{Ass}_A E_A(M)$; then there exists an A-submodule $N \subset E_A(M)$ isomorphic to $A/\mathfrak{p}$. Since $E_A(M)$ is an essential extension of M, we have $N \cap M \neq 0$, so $\mathfrak{p} \in \mathrm{Ass}_A(M)$.

(iv) follows directly from (iii).

(v): We know already that $E' := A_S \otimes_A E_A(M)$ contains M_S and is injective (lemma 7.11.19(iii)), so it remains only to check that E' is an essential extension of M. Thus, let $x \in E' \setminus M_S$; we have to check that $N := A_S x \cap M_S \neq 0$, and clearly we may assume that

$x \in E_A(M)$. Set $\mathscr{F} := \{\mathrm{Ann}_A(tx) \mid t \in S\}$; since A is noetherian, $\mathscr{F}$ admits maximal elements, and notice that $A_S x = A_S tx$ for any $t \in S$. Hence, we may replace x by tx for some $t \in S$, after which we may assume that $\mathrm{Ann}_A(x)$ is maximal in $\mathscr{F}$. We may write $Ax \cap M = Ix$ for some ideal $I \subset A$, so $N = I_S x$; let $a_1, \dots, a_k \in A$ be a system of generators for I, and notice that $Ix \neq 0$, since $E_A(M)$ is an essential extension of M. Suppose that $N = 0$; then there exists $t \in S$ such that the identity $ta_i x = 0$ holds in A for $i = 1, \dots, k$. However, $\mathrm{Ann}_A(x) = \mathrm{Ann}_A(tx)$ by construction, so $Ix = 0$, a contradiction. □

Theorem 7.11.22. *Let A be a noetherian ring, I an injective A-module. We have :*

(i) *I decomposes as a direct sum of the form*

$$I \simeq \bigoplus_{\mathfrak{p} \in \mathrm{Spec}\, A} E_A(A/\mathfrak{p})^{(R_\mathfrak{p})} \qquad (7.11.23)$$

for a system of (small) sets $(R_\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Spec}\, A)$.

(ii) *Moreover, the cardinality of $R_\mathfrak{p}$ equals $\dim_{\kappa(\mathfrak{p})} \mathrm{Hom}_{A_\mathfrak{p}}(\kappa(\mathfrak{p}), I_\mathfrak{p})$, for every $\mathfrak{p} \in \mathrm{Spec}\, A$ (where $\kappa(\mathfrak{p}) := A_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}$). Especially, this cardinality is independent of the decomposition* (7.11.23).

Proof. (i): Denote by $\mathscr{F}$ the set of all indecomposable injective submodules of I, and by $\mathscr{S}$ the set of all subsets $\mathscr{G} \subset \mathscr{F}$ such that the natural map $I_\mathscr{G} := \bigoplus_{G \in \mathscr{G}} G \to I$ is injective. The set $\mathscr{S}$ is partially ordered by inclusion, and by Zorn's lemma, $\mathscr{S}$ admits a maximal element $\mathscr{M}$; in light of proposition 7.11.21(ii), it suffices to check that $I_\mathscr{M} = I$. However, $I_\mathscr{M}$ is injective (lemma 7.11.19(i)), hence $I = I_\mathscr{M} \oplus J$ for some A-module J, and it is easily seen that J is injective as well. We are thus reduced to showing that $J = 0$. Now, if $J \neq 0$, let $\mathfrak{p} \in \mathrm{Ass}_A J$ be any associated prime ([126, Th.6.1(i)]); then $E_A(A/\mathfrak{p})$ is an indecomposable injective direct summand of J (proposition 7.11.21(i,ii)), and therefore $I_\mathscr{M} \oplus E_A(A/\mathfrak{p})$ is a submodule of I, contradicting the maximality of $\mathscr{M}$.

(ii): In light of (i) and proposition 7.11.21(iii,v) we have

$$\mathrm{Hom}_{A_\mathfrak{p}}(\kappa(\mathfrak{p}), I_\mathfrak{p}) = \mathrm{Hom}_{A_\mathfrak{p}}(\kappa(\mathfrak{p}), E_A(A/\mathfrak{p})_\mathfrak{p}^{(R_\mathfrak{p})}) = \kappa(\mathfrak{p})^{(R_\mathfrak{p})} \otimes_{\kappa(\mathfrak{p})} \mathrm{Hom}_{A_\mathfrak{p}}(\kappa(\mathfrak{p}), E_{A_\mathfrak{p}}(\kappa(\mathfrak{p})))$$

so we are reduced to showing the following :

Claim 7.11.24. Let $\mathfrak{m} \subset A$ be any maximal ideal, and set $\kappa := A/\mathfrak{m}$; then

$$d := \dim_\kappa \mathrm{Hom}_A(\kappa, E_A(\kappa)) = 1.$$

Proof of the claim. Since $\kappa \subset E_A(\kappa)$, obviously $d \geq 1$. On the other hand, we have a natural identification

$$\mathrm{Hom}_A(\kappa, E_A(\kappa)) = F := \{x \in E_A(\kappa) \mid \mathfrak{m}x = 0\}.$$

If $d > 1$, we may find $x \in F$ such that $Ax \cap \kappa = 0$; but this is absurd, since $E_A(\kappa)$ is an essential extension of κ. □

7.11.25. When dealing with the more general coherent rings that appear in sections 11.4 and 11.5, the injective hull is no longer suitable for the study of local cohomology, but it turns out that one can use instead a "coh-injective hull", which works just as well.

Definition 7.11.26. Let A be a ring; we denote by A-$\mathbf{Mod}_{\mathrm{coh}}$ the full subcategory of A-$\mathbf{Mod}$ consisting of all coherent A-modules.

(i) An A-module J is said to be *coh-injective* if the functor

$$A\text{-}\mathbf{Mod}_{\mathrm{coh}} \to A\text{-}\mathbf{Mod}^o \quad : \quad M \mapsto \mathrm{Hom}_A(M, J)$$

is exact.

(ii) An A-module M is said to be *ω-coherent* if it is countably generated, and every finitely generated submodule of M is finitely presented.

Lemma 7.11.27. *Let A be a ring.*

(i) *Let M, J be two A-modules, $N \subset M$ a submodule. Suppose that J is coh-injective and both M and M/N are ω-coherent. Then the natural map:*

$$\mathrm{Hom}_A(M, J) \to \mathrm{Hom}_A(N, J)$$

is surjective.

(ii) *Let $(J_\lambda \mid \lambda \in \Lambda)$ be a filtered family of coh-injective A-modules. Then $\mathrm{colim}_{\lambda\in\Lambda} J_\lambda$ is coh-injective.*

(iii) *Assume that A is coherent, let $M_\bullet$ be an object of $\mathsf{D}^-(A\text{-}\mathbf{Mod}_{\mathrm{coh}})$ and $I^\bullet$ a bounded below complex of coh-injective A-modules. Then the natural map*

$$\mathrm{Hom}^\bullet_A(M_\bullet, I^\bullet) \to R\mathrm{Hom}^\bullet_A(M_\bullet, I^\bullet)$$

is an isomorphism in $\mathsf{D}(A\text{-}\mathbf{Mod})$.

Proof. (i): Since M is ω-coherent, we may write it as an increasing union $\bigcup_{n\in\mathbb{N}} M_n$ of finitely generated, hence finitely presented submodules. For each $n \in \mathbb{N}$, the image of M_n in M/N is a finitely generated, hence finitely presented submodule; therefore $N_n := N \cap M_n$ is finitely generated, hence it is a coherent A-module, and clearly $N = \bigcup_{n\in\mathbb{N}} N_n$. Suppose $\varphi : N \to J$ is any A-linear map; for every $n \in \mathbb{N}$, set $P_{n+1} := M_n + N_{n+1} \subset M$, and denote by $\varphi_n : N_n \to J$ the restriction of φ. We wish to extend φ to a linear map $\varphi' : M \to J$, and to this aim we construct inductively a compatible system of extensions $\varphi'_n : P_n \to J$, for every $n > 0$. For $n = 1$, one can choose arbitrarily an extension φ'_1 of φ_1 to P_1. Next, suppose $n > 0$, and that φ'_n is already given; since $P_n \subset M_n$, we may extend φ'_n to a map $\varphi''_n : M_n \to J$. Since $M_n \cap N_{n+1} = N_n$, and since the restrictions of φ''_n and φ_{n+1} agree on N_n, there exists a unique A-linear map $\varphi'_{n+1} : P_{n+1} \to J$ that extends both φ''_n and φ_{n+1}.

(ii): Recall that a coherent A-module is finitely presented. However, it is well known that an A-module M is finitely presented if and only if the functor $Q \mapsto \mathrm{Hom}_A(M, Q)$ on A-modules, commutes with filtered colimits (see *e.g.* [75, Prop.2.3.16(ii)]). The assertion is an easy consequence.

(iii): Since A is coherent, one can find a resolution $\varphi : P_\bullet \to M_\bullet$ consisting of free A-modules of finite rank (cp. [75, §7.1.20]). One looks at the spectral sequences

$$E_1^{pq} : \mathrm{Hom}_A(P_p, I^q) \Rightarrow R^{p+q}\mathrm{Hom}^\bullet_A(M_\bullet, I^\bullet)$$
$$F_1^{pq} : \mathrm{Hom}_A(M_p, I^q) \Rightarrow H^{p+q}\mathrm{Hom}^\bullet_A(M_\bullet, I^\bullet).$$

The resolution φ induces a morphism of spectral sequences $F_1^{\bullet\bullet} \to E_1^{\bullet\bullet}$; on the other hand, since I^q is coh-injective, we have : $E_2^{pq} \simeq \mathrm{Hom}_A(H_pF_\bullet, I^q) \simeq \mathrm{Hom}_A(H_pM_\bullet, I^q) \simeq F_2^{pq}$, so the induced map $F_2^{pq} \to E_2^{pq}$ is an isomorphism for every $p, q \in \mathbb{N}$, whence the claim. $\square$

7.11.28. Let A be a coherent ring, $Z \subset \mathrm{Spec}\, A$ a constructible closed subset; we denote by

$$A\text{-}\mathbf{Mod}_{\mathrm{coh},Z}$$

the full subcategory of $A\text{-}\mathbf{Mod}_{\mathrm{coh}}$ whose objects are the (coherent) A-modules with support contained in Z. Let $I \subset A$ be any finitely generated ideal such that $Z = \mathrm{Spec}\, A/I$; it is easily seen that

$$\mathrm{Ob}(A\text{-}\mathbf{Mod}_{\mathrm{coh},Z}) = \bigcup_{n\in\mathbb{N}} \mathrm{Ob}(A/I^n\text{-}\mathbf{Mod}_{\mathrm{coh}}).$$

Now, consider a functor

$$T : A\text{-}\mathbf{Mod}^o_{\mathrm{coh},Z} \to \mathbb{Z}\text{-}\mathbf{Mod}.$$

Notice that TM is naturally an A-module, for every coherent A-module M : indeed, if $a \in A$ is any element, we may define the scalar multiplication by a on TM as the $\mathbb{Z}$-linear endomorphism $T(a \cdot \mathbf{1}_M)$. In other words, T factors through the forgetful functor $A\text{-}\mathbf{Mod} \to \mathbb{Z}\text{-}\mathbf{Mod}$; especially, if we set

$$H_T := \operatorname*{colim}_{n\in\mathbb{N}} T(A/I^n)$$

we get a natural A-linear map

$$M \xrightarrow{\sim} \operatorname*{colim}_{n\in\mathbb{N}} \mathrm{Hom}_A(A/I^n, M) \to \operatorname*{colim}_{n\in\mathbb{N}} \mathrm{Hom}_A(TM, T(A/I^n)) \to \mathrm{Hom}_A(TM, H_T)$$

whence a bilinear pairing

$$M \times TM \to H_T$$

which in turns yields a natural transformation

$$\omega_M : TM \to \mathrm{Hom}_A(M, H_T) \qquad \text{for every } M \in \mathrm{Ob}(A\text{-}\mathbf{Mod}_{\mathrm{coh},Z}).$$

Lemma 7.11.29. *In the situation of* (7.11.28)*, the following conditions are equivalent :*

(a) *ω is an isomorphism of functors $T \xrightarrow{\sim} \mathrm{Hom}_A(-, H_T)$.*
(b) *T is left exact.*

Proof. Clearly (a)⇒(b). For the converse, suppose first that $Z = \mathrm{Spec}\, A$, in which case, notice that $H_T = TA$ and ω_A is the natural isomorphism $TA \xrightarrow{\sim} \mathrm{Hom}_A(A, TA)$; then, if M is any coherent A-module, pick a finite presentation

$$\Sigma \quad : \quad A^{\oplus n} \to A^{\oplus m} \to M \to 0$$

and apply the 5-lemma to the resulting ladder of A-modules ω_Σ with left exact rows, to deduce that ω_M is an isomorphism. Next, for a general ideal I and M an A-module with $\mathrm{Supp}\, M \subset Z$, we may find $n \in \mathbb{N}$ such that M is an A/I^n-module; since A/I^k is coherent, the foregoing case then shows that the induced map

$$TM \to \mathrm{Hom}_A(M, T(A/I^k))$$

is an isomorphism, for every $k \geq n$. To conclude, it suffices to remark that the natural map

$$\operatorname*{colim}_{k\in\mathbb{N}} \mathrm{Hom}_A(M, T(A/I^k)) \to \mathrm{Hom}_A(M, H_T)$$

is an isomorphism, since M is finitely presented ([75, Prop.2.3.16(ii)]). □

We wish next to present a criterion that allows to detect, among the functors T as in (7.11.28), those that are exact. A complete characterization shall be given only for a restricted class of coherent ring; namely, we make the following :

Definition 7.11.30. Let A be a coherent ring. We say that A is an *Artin-Rees* ring, if the following holds. For every finitely generated ideal $I \subset A$, every coherent A-module M, and every finitely generated A-submodule $N \subset M$, the I-adic topology of M induces the I-adic topology on N.

We can then state :

Proposition 7.11.31. *In the situation of* (7.11.28) *suppose furthermore that A is an Artin-Rees ring. Then the following conditions are equivalent :*

(a) *H_T is a coh-injective A-module.*
(b) *T is exact.*

Proof. Clearly (a)⇒(b). For the converse, let M be any coherent A-module, and $N \subset M$ any finitely generated A-submodule; it suffices to check that the induced map

$$\mathrm{Hom}_A(M, H_T) \to \mathrm{Hom}_A(N, H_T)$$

is surjective. However, let $f : N \to H_T$ be any A-linear map; since N is finitely presented, there exists $n \in \mathbb{N}$ such that f factors through an A-linear map $f_n : N \to T(A/I^n)$ ([75, Prop.2.3.16(ii)]), and clearly $I^n N \subset \mathrm{Ker}\, f_n$, so $I^n N \subset \mathrm{Ker}\, f$ as well. Since the I-adic topology of N agrees with the topology induced by the I-adic topology of M, there exists $k \in \mathbb{N}$ such that $I^k M \cap N \subset I^n N$. Set $\overline{N} := N/(I^k M \cap N)$; then $\overline{N}$ is a finitely generated A-submodule of $M/I^k M$, and especially it is a coherent A-module. By construction, f factors through an A-linear map $\overline{f} : \overline{N} \to H_T$; assumption (b) and lemma 7.11.29 imply that $\overline{f}$ extends to an A-linear map $M/I^k M \to H_T$, and the resulting map $M \to H_T$ extends f, whence (a). □

Remark 7.11.32. (i) In the terminology of definition 7.11.30, the standard Artin-Rees lemma implies that every noetherian ring is an Artin-Rees ring. We shall see later that if V is any valuation ring, then every essentially finitely presented V-algebra is an Artin-Rees ring (corollary 9.1.28 and theorem 11.4.46).

(ii) On the other hand, if A is noetherian, claim 7.11.20 easily implies that an A-module is coh-injective if and only if it is injective.

(iii) Combining (i) and (ii) with proposition 7.11.31, we recover [85, Exp.IV, Prop.2.1].

Example 7.11.33. Let κ_0 be a field, A a noetherian κ_0-algebra, $\mathfrak{m} \subset A$ a maximal ideal such that $\kappa := A/\mathfrak{m}$ is a finite extension of κ_0, and set $Z := \{\mathfrak{m}\} \subset \mathrm{Spec}\, A$. Then every object of $A\text{-}\mathbf{Mod}_{\mathrm{coh},Z}$ is a finite dimensional κ_0-vector space, and therefore the functor

$$T : A\text{-}\mathbf{Mod}_{\mathrm{coh},Z} \to \kappa_0\text{-}\mathbf{Mod} \qquad M \mapsto \mathrm{Hom}_{\kappa_0}(M, \kappa_0)$$

is exact. Proposition 7.11.31 and remark 7.11.32(ii) then say that

$$H_T := \operatorname*{colim}_{n\in\mathbb{N}} \mathrm{Hom}_{\kappa_0}(A/\mathfrak{m}^n, \kappa_0)$$

is an injective A-module. More precisely, notice that $\mathrm{Hom}_A(\kappa, H_T) = \mathrm{Ann}_{H_T}(\mathfrak{m}) = T(\kappa) \simeq \kappa$, therefore H_T is the injective hull of the residue field κ.

7.11.34. *Flatness criteria.* The following generalization of the local flatness criterion answers affirmatively a question raised in [65, Ch.IV, Rem.11.3.12].

Lemma 7.11.35. *Let A be a ring, $I \subset A$ an ideal, B a finitely presented A-algebra, $\mathfrak{p} \subset B$ a prime ideal containing IB, and M a finitely presented B-module. Then the following two conditions are equivalent:*

(a) *$M_\mathfrak{p}$ is a flat A-module.*

(b) *$M_\mathfrak{p}/IM_\mathfrak{p}$ is a flat A/I-module and* $\mathrm{Tor}_1^A(M_\mathfrak{p}, A/I) = 0$.

Proof. Clearly, it suffices to show that (b)⇒(a), hence we assume that (b) holds. We write A as the union of a filtered family $(A_\lambda \mid \lambda \in \Lambda)$ of its $\mathbb{Z}$-subalgebras of finite type, and set $A' := A/I$, $I_\lambda := I \cap A_\lambda$, $A'_\lambda := A_\lambda/I_\lambda$ for every $\lambda \in \Lambda$. Then, for some $\lambda \in \Lambda$, the A-algebra B descends to an A_λ-algebra B_λ of finite type, and M descends to a finitely presented B_λ-module M_λ. For every $\mu \geq \lambda$ we set $B_\mu := A_\mu \otimes_{A_\lambda} B_\lambda$ and $M_\mu := B_\mu \otimes_{B_\lambda} M_\lambda$. Up to replacing Λ by a cofinal family, we can assume that B_μ and M_μ are defined for every $\mu \in \Lambda$. Then, for every $\lambda \in \Lambda$ let $g_\lambda : B_\lambda \to B$ be the natural map, and set $\mathfrak{p}_\lambda := g_\lambda^{-1}\mathfrak{p}$.

Claim 7.11.36. There exists $\lambda \in \Lambda$ such that $M_{\lambda,\mathfrak{p}_\lambda}/I_\lambda M_{\lambda,\mathfrak{p}_\lambda}$ is a flat A'_λ-module.

Proof of the claim. Set $B'_\lambda := B_\lambda \otimes_{A_\lambda} A'_\lambda$ and $M'_\lambda := M_\lambda \otimes_{A_\lambda} A'_\lambda$ for every $\lambda \in \Lambda$; clearly the natural maps

$$\operatorname*{colim}_{\lambda\in\Lambda} A'_\lambda \to A/IA \qquad \operatorname*{colim}_{\lambda\in\Lambda} B'_\lambda \to B/IB \qquad \operatorname*{colim}_{\lambda\in\Lambda} M'_\lambda \to M/IM$$

are isomorphisms. Then the claim follows from (b) and [65, Ch.IV, Cor.11.2.6.1(i)]. ♢

In view of claim 7.11.36, we can replace Λ by a cofinal subset, and thereby assume that $M_{\lambda,\mathfrak{p}_\lambda}/I_\lambda M_{\lambda,\mathfrak{p}_\lambda}$ is a flat A'_λ-module for every $\lambda \in \Lambda$.

Claim 7.11.37. (i) More generally, let $R_\bullet := (R_\lambda \mid \lambda \in \Lambda)$ be any system of rings indexed by any filtered set Λ, and $N_\bullet := (N_\lambda \mid \lambda \in \Lambda)$, $N'_\bullet := (N'_\lambda \mid \lambda \in \Lambda)$ two systems consisting of R_λ-modules N_λ and N'_λ for every $\lambda \in \Lambda$, and R_λ-linear transition maps $N_\lambda \to N_\mu$, $N'_\lambda \to N'_\mu$ for every $\lambda, \mu \in \Lambda$ with $\lambda \leq \mu$. Denote by R, N, N' the colimits of $R_\bullet$, $N_\bullet$ and respectively $N'_\bullet$; then the natural maps $R_\lambda \to R$, $N_\lambda \to N$, $N'_\lambda \to N$ induce isomorphisms :

$$\operatorname*{colim}_{\lambda\in\Lambda} \mathrm{Tor}_i^{R_\lambda}(N_\lambda, N'_\lambda) \to \mathrm{Tor}_i^R(N, N') \qquad \text{for every } i \in \mathbb{N}.$$

(ii) For every $\lambda, \mu \in \Lambda$ with $\mu \geq \lambda$, the natural map:

$$f_{\lambda\mu} : B_{\mu,\mathfrak{p}_\mu} \otimes_{B_\lambda} \mathrm{Tor}_1^{A_\lambda}(M_{\lambda,\mathfrak{p}_\lambda}, A'_\lambda) \to \mathrm{Tor}_1^{A_\mu}(M_{\mu,\mathfrak{p}_\mu}, A'_\mu)$$

is surjective.

Proof of the claim. (i): For every $\lambda \in \Lambda$, let $L_\bullet(N'_\lambda)$ be the canonical free resolution of the R_λ-module N'_λ ([37, §3, n.3]). Similarly, denote by $L_\bullet(N')$ the canonical free resolution of the R-module N'. It follows from [38, Ch.II, §6, n.6, Cor.] and the exactness properties of filtered colimits, that the natural map: $\mathrm{colim}_{\lambda\in\Lambda} L_\bullet(N'_\lambda) \to L_\bullet(N')$ is an isomorphism. Hence :

$$\begin{aligned}
\operatorname*{colim}_{\lambda\in\Lambda} H_\bullet(N_\lambda \overset{\mathbf{L}}{\otimes}_{R_\lambda} N'_\lambda) &\simeq \operatorname*{colim}_{\lambda\in\Lambda} H_\bullet(N_\lambda \otimes_{R_\lambda} L_\bullet(N'_\lambda)) \\
&\simeq H_\bullet(N \otimes_R \operatorname*{colim}_{\lambda\in\Lambda} L_\bullet(N'_\lambda)) \\
&\simeq H_\bullet(N \otimes_R L_\bullet(N')) \\
&\simeq H_\bullet(N \overset{\mathbf{L}}{\otimes}_R N').
\end{aligned}$$

(ii): We use the change of rings spectral sequences for Tor ([163, Th.5.6.6])

$$E^2_{pq} : \mathrm{Tor}_p^{A_\mu}(\mathrm{Tor}_q^{A_\lambda}(M_{\lambda,\mathfrak{p}_\lambda}, A_\mu), A'_\mu) \Rightarrow \mathrm{Tor}_{p+q}^{A_\lambda}(M_{\lambda,\mathfrak{p}_\lambda}, A'_\mu)$$
$$F^2_{pq} : \mathrm{Tor}_p^{A'_\lambda}(\mathrm{Tor}_q^{A_\lambda}(M_{\lambda,\mathfrak{p}_\lambda}, A'_\lambda), A'_\mu) \Rightarrow \mathrm{Tor}_{p+q}^{A_\lambda}(M_{\lambda,\mathfrak{p}_\lambda}, A'_\mu).$$

Since $F^2_{10} = F^2_{20} = 0$, the natural map

$$\alpha : F^2_{01} := A'_\mu \otimes_{A'_\lambda} \mathrm{Tor}_1^{A_\lambda}(M_{\lambda,\mathfrak{p}_\lambda}, A'_\lambda) \to \mathrm{Tor}_1^{A_\lambda}(M_{\lambda,\mathfrak{p}_\lambda}, A'_\mu)$$

is an isomorphism. On the other hand, we have a surjection:

$$\beta : \mathrm{Tor}_1^{A_\lambda}(M_{\lambda,\mathfrak{p}_\lambda}, A'_\mu) \to E^2_{10} := \mathrm{Tor}_1^{A_\mu}(M_{\mu,\mathfrak{p}_\lambda}, A'_\mu)$$

and $f_{\lambda\mu} = (\beta \circ \alpha)_{\mathfrak{p}_\mu}$, whence the claim. ♢

We deduce from claim 7.11.37(i) that the natural map

$$\operatorname*{colim}_{\lambda\in\Lambda} \mathrm{Tor}_1^{A_\lambda}(M_{\lambda,\mathfrak{p}_\lambda}, A'_\lambda) \to \mathrm{Tor}_1^{A}(M_{\mathfrak{p}}, A') = 0$$

is an isomorphism. However, $\mathrm{Tor}_1^{A_\lambda}(M_{\lambda,\mathfrak{p}_\lambda}, A'_\lambda)$ is a finitely generated $B_{\lambda,\mathfrak{p}_\lambda}$-module by [37, §6, n.4, Cor.]. We deduce that $f_{\lambda\mu,\mathfrak{p}_\lambda} = 0$ for some $\mu \geq \lambda$; therefore $\mathrm{Tor}_1^{A_\mu}(M_{\mu,\mathfrak{p}_\mu}, A'_\mu) = 0$, in view of claim 7.11.37(ii), and then the local flatness criterion of [61, Ch.0, §10.2.2] says that $M_{\mu,\mathfrak{p}_\mu}$ is a flat A_μ-module, so finally $M_{\mathfrak{p}}$ is a flat A-module, as stated. □

8. Complements of topology and topological algebra

This chapter is a miscellanea of results of topology and topological algebra that shall be needed in the rest of the treatise.

8.1. **Spectral spaces and constructible subsets.** The class of spectral topological spaces was first introduced in Hochster's paper [90], where it was shown to coincide precisely with the class of those spaces that are homeomorphic to (Zariski) spectra of commutative rings. Since then, spectral spaces have reappeared in a variety of unrelated contexts; their ubiquity has all to do with the very agreeable properties of their boolean algebras of constructible subsets; especially, the constructible subsets of a spectral space generate a quasi-compact and separated topology. The many corollaries springing from this basic observation add up to a handy toolkit for dealing in a uniform way with the kind of non-separated spaces that most frequently occur in geometric applications of an algebraic bent.

In this section we introduce spectral spaces and prove their basic properties. We also collect a few results showing the interplay between algebraic and topological properties of schemes and their constructible subsets.

Definition 8.1.1. Let X be a topological space.

- (i) We say that X is *quasi-separated* if the intersection of any two quasi-compact open subsets of X is still quasi-compact.
- (ii) We say that X is *coherent* if it is quasi-compact and it admits a basis consisting of quasi-compact open subsets, and closed under finite intersections.
- (iii) We say that X is *reducible* if there exist non-empty closed subsets $Z, Z' \subset X$ such that $Z \cup Z' = X$ and $Z, Z' \neq X$. We say that X is *irreducible* if it is non-empty and not reducible.
- (iv) We say that X is *sober* if for every irreducible closed subset $Z \subset X$ (with the topology induced from X) there exists a unique point $\eta_Z \in Z$ such that Z is the topological closure of $\{\eta_Z\}$ in X. In this case, we say that η_Z is the *generic point* of Z.
- (v) We say that X is *spectral* if it is sober and coherent.
- (vi) We say that X is *noetherian*, if for every descending chain
$$\cdots \subset Z_2 \subset Z_1 \subset Z_0$$
of closed subsets of X, there exists $n \in \mathbb{N}$ such that $Z_m = Z_n$ for every $m \geq n$.
- (vii) We say that X is *locally coherent* (resp. *locally spectral*, resp. *locally noetherian*) if every point of X admits an open neighborhood which is coherent (resp. spectral, resp. noetherian).
- (viii) We say that X is T_0 if, for every pair of points $x, y \in X$, there exists an open subset $U \subset X$ such that $U \cap \{x, y\}$ contains exactly one point.
- (ix) We say that X is *compact* if it is quasi-compact and separated.

Remark 8.1.2. (i) Let X be a topological space, $(Z_i \mid i \in I)$ a family of irreducible closed subsets of X, totally ordered by inclusion, and denote by Z the topological closure in X of $\bigcup_{i \in I} Z_i$. Then Z is also irreducible. Indeed, say that $Z = Y_1 \cup Y_2$ for two closed subsets Y_1, Y_2 of X. If $Z_i \subset Y_1 \cap Y_2$ for every $i \in I$, then $Z \subset Y_1$; otherwise, say that $Z_i \not\subset Y_1$ for some $i \in I$; then $Z_i \subset Y_2$ for every $i \in I$, whence $Z \subset Z_2$.

(ii) By Zorn's lemma, it follows that every irreducible closed subset of X is contained in a maximal one. A maximal irreducible closed subset of X is called an *irreducible component* of X, and clearly X is the union of its irreducible components.

(iii) Let $Z_1, Z_2 \subset X$ be any two closed subsets, and endow Z_1, Z_2 and $Z_3 := Z_1 \cup Z_2$ with the topology induced by X. For $j \leq 3$, denote by I_j the set of all irreducible components of Z_i; then $I_3 \subset I_1 \cup I_2$. Indeed, let $Y \in I_3$; then $Y = (Y \cap Z_1) \cup (Y \cap Z_2)$, so we must have $Y = Y \cap Z_i$ for some $i \in \{1, 2\}$.

(iv) Let $k \in \mathbb{N}$ be any integer, and $Z_\bullet := (Z_i \mid i = 0, \dots, k)$ a strictly descending finite chain of irreducible closed subsets of X. The integer k is the *length* of $Z_\bullet$. The supremum of the lengths of all such chains is an invariant called the *(Krull) dimension* of X, and denoted

$$\dim X$$

(especially, $\dim \varnothing = -\infty$). We say that X has *finite Krull dimension* if $\dim X \in \mathbb{N} \cup \{-\infty\}$.

(v) Suppose that X is the union of finitely many closed subsets $Z_1, \dots, Z_k$. Then it follows directly from (iii) that $\dim X = \sup(\dim Z_1, \dots, \dim Z_k)$.

(vi) If X is sober, then X is also T_0. Indeed, let $x, y \in X$ be any two distinct points, and denote by Z_y the topological closure of $\{y\}$ in X; if X is sober, we may assume – up to swapping the roles of x and y – that $x \notin Z_y$. In this case, the set $X \setminus Z_y$ is an open neighborhood of x that does not contain y, whence the claim.

Lemma 8.1.3. *Let X be a topological space, $T \subset X$ any subset, $\overline{T}$ the topological closure of T in X, and endow T and $\overline{T}$ with the topologies induced from X. The following holds :*

(i) *T is irreducible if and only if the same holds for $\overline{T}$.*
(ii) *If T is non-empty and open in X, and X is irreducible, then T is irreducible.*
(iii) *If X is sober, and T is locally closed in X, then T is sober as well.*
(iv) *Suppose that $T = T_1 \cup T_2$ for two subsets T_1 and T_2, and endow also T_1 and T_2 with the topologies induced from X. If T_1, T_2 and X are sober, the same holds for T.*
(v) *Let $f : X \to Y$ be any continuous map, and endow $f(X)$ with the topology induced by Y. If X is irreducible, the same holds for $f(X)$.*

Proof. For any subset $Y \subset X$, let $\overline{Y}$ denote the topological closure of Y in X.

(i): Suppose that T is irreducible, and say that $\overline{T} = V \cup V'$ for two closed subsets V, V' of $\overline{T}$; then $T = (T \cap V) \cup (T \cap V')$, so we may assume that $T = T \cap V$, in which case $\overline{T} = V$. Conversely, suppose that $\overline{T}$ is irreducible, and $T = Z_1 \cup Z_2$ for some closed subsets Z_1, Z_2 of T; then $\overline{T} = \overline{Z}_1 \cup \overline{Z}_2$, so we may assume that $\overline{T} = \overline{Z}_1$, in which case $T = T \cap \overline{Z}_1 = Z_1$.

(ii): Indeed, say that $T = Z_1 \cup Z_2$ for two non-empty closed subsets of T; then $X = \overline{Z}_1 \cup \overline{Z}_2 \cup (X \setminus T)$. By assumption, $X \not\subset X \setminus T$; as X is irreducible, we may then assume that $X = \overline{Z}_1$, whence $T = Z_1$.

(iii): This is clear if T is closed in X, so we may assume that T is open in X. However, say that $Z \subset T$ is an irreducible closed subset; by (i), the topological closure $\overline{Z}$ of Z in X is irreducible, and clearly its generic point $\eta_{\overline{Z}}$ lies in Z, so $\eta_{\overline{Z}}$ is the unique point of Z such that the closure of $\{\eta_{\overline{Z}}\}$ in T equals Z.

(iv): Let Z be any irreducible closed subset of T, and set $Z_i := Z \cap T_i$ for $i = 1, 2$. According to (i), $\overline{Z}$ is irreducible; since $\overline{Z} = \overline{Z}_1 \cup \overline{Z}_2$, we may then assume that $\overline{Z} = \overline{Z}_1$, especially $\overline{Z}_1$ is irreducible, so Z_1 is irreducible, by (i). Therefore Z_1 admits a generic point η, since T_1 is sober. By construction, the topological closure of $\{\eta\}$ in T equals $T \cap \overline{Z} = Z$, and it remains to see that η is the unique point of Z with this property. However, if η' is any other such point, then clearly the topological closures in X of both $\{\eta\}$ and $\{\eta'\}$ equal $\overline{Z}$, so $\eta = \eta'$, since X is sober.

(v): Indeed, suppose that $f(X) = Z_1 \cup Z_2$ for two closed subsets Z_1, Z_2 of $f(X)$; then $X = f^{-1}(Z_1) \cup f^{-1}(Z_2)$, so we may assume that $X = f^{-1}Z_1$, in which case $f(X) = Z_1$. □

Proposition 8.1.4. *Let X be any topological space. Then X is noetherian if and only if the following conditions hold :*

(a) *Every closed subset of X has a finite number of irreducible components.*
(b) *For every descending chain $(Z_n \mid n \in \mathbb{N})$ of irreducible closed subsets of X, there exists $N \in \mathbb{N}$ such that $Z_n = Z_N$ for every $n \geq N$.*

Proof. (This was first shown in in [136, Prop.1.1], with a different proof.) Suppose first that X is noetherian. Then (b) clearly holds. To show (a), let S denote the set of all closed subsets of X that have infinitely many irreducible components; since X is noetherian, if S is not empty, it admits minimal elements, so say that Z is such a minimal element. Then clearly Z cannot be irreducible, so $Z = Z_1 \cup Z_2$ for some non-empty closed subsets of X strictly contained in Z. By minimality of Z, we have $Z_1, Z_2 \notin S$; but then remark 8.1.2(iii) implies that $Z_1 \cup Z_2 \notin S$ as well, a contradiction. For the converse, we show more precisely the following :

Claim 8.1.5. Let X be any topological space, $Z_\bullet := (Z_n \mid n \in \mathbb{N})$ a descending non-stationary chain of closed subsets of X, such that Z_n has a finite number of irreducible components, for every $n \in \mathbb{N}$. Then there exists a descending, non-stationary chain $(Z'_n \mid n \in \mathbb{N})$ of closed subsets of X, such that Z'_n is an irreducible component of Z_n, for every $n \in \mathbb{N}$.

Proof of the claim. For every $n \in \mathbb{N}$, let I_n be the (finite) set of irreducible components of Z_n, and denote by $I'_n \subset I_n$ the subset of all $T \in I_n$ such that $T \in I_m$ for every $m \geq n$. Notice that, since $Z_\bullet$ is descending and non-stationary, the subset $I''_n := I_n \setminus I'_n$ is non-empty, for every $n \in \mathbb{N}$. Moreover, for every $n > 0$, every $T \in I''_n$ and every $T' \in I_{n-1}$ such that $T \subset T'$, we must have $T' \in I''_{n-1}$; indeed, otherwise we would have $T' \in I_n$, hence $T = T'$, hence $T \in I'_n$, a contradiction. Hence, for every $n \in \mathbb{N}$, denote by S_n the set of all chains $(T_k \mid k = 0, \dots, n)$ such that

- $T_k \in I''_k$ for every $k = 0, \dots, n$.
- $T_k \subset T_{k-1}$ for every $k = 1, \dots, n$.

The foregoing observations easily imply that S_n is a finite non-empty set for every $n \in \mathbb{N}$. Moreover, we have an obvious map $S_m \to S_n$ whenever $m \geq n$, that assigns to any $T_\bullet \in S_m$ the truncated chain $(T_k \mid k = 0, \dots, n)$. The limit S of the system $(S_n \mid n \in \mathbb{N})$ is then non-empty; say that $(T^k_\bullet \mid k \in \mathbb{N})$ is any element of S, and set $Z'_n := T^n_n$ for every $n \in \mathbb{N}$. We claim that the chain $(Z'_n \mid n \in \mathbb{N})$ will do. Indeed, suppose by way of contradiction, that the chain $Z'_\bullet$ is stationary, so there exists $N \in \mathbb{N}$ such that $Z'_n = Z'_N$ for every $n \geq N$; but then we would have $Z_N \in I_n$ for every $n \geq N$, hence $Z_N \in I'_N$, which is absurd. □

Remark 8.1.6. Let X be a topological space, $U \subset X$ an open subset, and endow U with the topology induced from X.

(i) If X is quasi-separated, then the same holds for U. If X is coherent, then X is quasi-separated. Indeed, let $(U_i \mid i \in I)$ be a basis for X consisting of quasi-compact open subsets and closed under finite intersections, and U_1, U_2 two quasi-compact open subsets. Then there exist finite subsets $J_i \subset I$ such that $U_i = \bigcup_{i \in J_i} U_i$ for $i = 1, 2$; consequently $U_1 \cap U_2 = \bigcup_{(j,j') \in J_1 \times J_2} (U_j \cap U_{j'})$, and by assumption each $U_j \cap U_{j'}$ is quasi-compact, whence the claim.

(ii) If X is coherent (resp. spectral) and U is quasi-compact, then U is coherent (resp. spectral) for the topology induced from X. Indeed, lemma 8.1.3(iii) says that U is sober whenever the same holds for X, and if $(U_i \mid i \in I)$ is a basis of quasi-compact open subsets of X, then $(U \cap U_i \mid i \in I)$ is a basis of quasi-compact open subsets of U.

(iii) It follows easily from (ii) that if X is locally coherent (resp. locally spectral) then the same holds for U.

(iv) If X is noetherian, every open subset of X is quasi-compact, so X is coherent. Also, every locally closed subset of X is noetherian as well.

(v) Furthermore, suppose that $T_1, \dots, T_k$ is any finite family of subsets of X, such that $X = \bigcup_{i=1}^k T_i$, and endow each T_i with the topology induced from X; then, if each T_i is noetherian, the same holds for X.

8.1.7. Let us denote by $\mathbf{Top}$ the category of topological spaces (whose morphisms are the continuous maps), and by $\mathbf{Sober}$ the full subcategory of $\mathbf{Top}$ whose objects are the sober topological spaces.

Proposition 8.1.8. *The inclusion functor*

$$\mathbf{Sober} \to \mathbf{Top}$$

admits a left adjoint.

Proof. For any topological space X, denote by S_X the set of all irreducible closed subsets of X. If Z is any irreducible closed subset of X, we shall use the notation η_Z to refer to Z, when we wish to regard the latter as an element of S_X. From lemma 8.1.3(i,ii) we see that, for every open subset $U \subset X$, we have a natural bijection

$$S_U \stackrel{\sim}{\to} \{\eta_Z \in S_X \mid Z \cap U \neq \varnothing\} \qquad T \mapsto \eta_{\overline{T}}$$

(where, for any subset T of X, we let $\overline{T}$ be the topological closure of T in X). We may then identify S_U with its image in S_X, and we notice

Claim 8.1.9. (i) $S_U \cap S_V = S_{U \cap V}$ for every open subsets $U, V \subset X$.

(ii) For any family $(U_i \mid i \in I)$ of open subsets of X, we have $S_{\bigcup_{i\in I} U_i} = \bigcup_{i\in I} S_{U_i}$.

Proof of the claim. (i): Clearly $S_{U\cap V} \subset S_U \cap S_V$. For the converse inclusion, suppose that Z is closed subset of X, such that $Z \cap U, Z \cap V \neq \varnothing$ and $Z \cap U \cap V = \varnothing$. Then $Z = (Z \backslash U) \cup (Z \backslash V)$, so Z is not irreducible in X. The assertion is an immediate consequence.

(ii) is trivial. ◊

From claim 8.1.9 we see that the family $(S_U \mid U \text{ is open in } X)$ is a topology on S_X, and we endow S_X with this topology. Next, there is a natural map

$$f_X : X \to S_X \qquad x \mapsto \eta_{\overline{\{x\}}}$$

and it is easily seen that $f_X^{-1}(S_U) = U$ for every open subset $U \subset X$; especially f_X is continuous. It also follows that the rule $Z \mapsto f_X^{-1} Z$ establishes a bijection from the closed subsets of S_X to those of X; since this bijection respects inclusion of closed subsets, we deduce that it restricts to a bijection from the irreducible closed subsets of S_X to those of X. Moreover, in light of lemma 8.1.3(v), it is easily seen that the inverse of this bijection assigns to any irreducible closed subset Z of X, the topological closure of $f_X(Z)$ in S_X.

Claim 8.1.10. S_X is a sober topological space.

Proof of the claim. Let Z be any irreducible closed subset of X, and denote by S_Z the topological closure of $\{\eta_Z\}$ in S_X; then $S_Z = S_X \backslash S_U$, where S_U is the largest open subset of S_X that does not contain η_Z, *i.e.* U is the largest open subset of X that does not intersect Z. So $U = X \backslash Z$, and therefore $f_X^{-1}(S_Z) = Z$. We conclude that the topological closure of $f_X(Z)$ in S_X equals S_Z. Summing up, we see that the rule $\eta_Z \mapsto S_Z$ establishes a bijection from S_X to the set of irreducible closed subsets of S_X, whence the claim. ◊

Now, let Y be any sober space, and $g : X \to Y$ any continuous map; in light of claim 8.1.10, it suffices to show that there exists a unique continuous map $h : S_X \to Y$ such that $g = h \circ f_X$. However, it is easily seen that any continuous map h with this property must assign to every $\eta_Z \in S_X$ the generic point $\eta_{g(Z)}$ of the topological closure of $g(Z)$ in S (lemma 8.1.3(v)); conversely, one checks easily that the rule

$$\eta_Z \mapsto \eta_{g(Z)} \qquad \text{for every irreducible closed subset } Z \text{ of } X$$

defines a map $h : S_X \to Y$ such that $h^{-1}U = S_{g^{-1}U}$ for every open subset $U \subset Y$; especially, h is continuous : the details shall be left to the reader. □

Remark 8.1.11. Keep the notation of the proof of proposition 8.1.8, and let $U \subset X$ be any open subset; it is easily seen that $U = \varnothing$ if and only if $S_U = \varnothing$. It follows that the rule $V \mapsto f_X^{-1} V$ defines a bijection from the set of open subsets of S_X to the set of open subsets of X. Consequently, X is quasi-compact (resp. quasi-separated) if and only if the same holds for S_X, and X is coherent if and only if S_X is spectral.

Remark 8.1.12. It is clear that the topological space underlying every affine scheme is spectral, and that of any scheme is locally spectral. Conversely, it is shown in [90] that to every spectral space X one can attach naturally a commutative ring A_X whose spectrum is isomorphic to X. This rule $X \mapsto A_X$ is functorial with respect to continuous maps of a certain type, which are introduced in the next definition.

Definition 8.1.13. Let $f : X \to Y$ be a continuous map of topological spaces, $T \subset X$ any subset, and endow T with the topology induced from X.

(i) We say that f is *quasi-compact* (resp. *quasi-separated*) if, for every quasi-compact (resp. quasi-separated) open subset $U \subset X$, the preimage $f^{-1}U$ is quasi-compact (resp. quasi-separated).

(ii) We say that f is *spectral* if, for every pair of quasi-compact quasi-separated open subsets $U \subset X$, $V \subset Y$ with $f(U) \subset V$, the restriction $f_{|U} : U \to V$ is quasi-compact.

(iii) We say that T is *retro-compact in* X if the inclusion map $T \to X$ is quasi-compact.

(iv) We say that T is *globally constructible* if T lies in the boolean algebra of subsets of X generated by all retro-compact open subsets of X.

(v) We say that T is *constructible* if every point of X admits an open neighborhood U such that $T \cap U$ is globally constructible in U.

(vi) We say that T is *pro-constructible* (resp. *ind-constructible*) if every point of X admits an open neighborhood U such that $T \cap U$ is the intersection (resp. the union) of constructible subsets of U.

Remark 8.1.14. Let $f : X \to Y$ and $g : Y \to Z$ be two continuous maps of topological spaces.

(i) If f and g are quasi-compact (resp. quasi-separated), then the same holds for $g \circ f$.

(ii) If f and g are spectral, and both X and Y are locally coherent, then $g \circ f$ is spectral. Indeed, say that $U \subset X$ and $V \subset Z$ are quasi-compact and quasi-separated open subsets such that $g \circ f(U) \subset V$; for every $x \in U$ we may find a quasi-compact and quasi-separated open subset $W_x \subset Y$ such that $g(W_x) \subset V$ and $f(x) \in W_x$. Since X is locally coherent, we may find a quasi-compact and quasi-separated open neighborhood U_x of x in X, such that $f(U_x) \subset W_x$. Then both $f_{|U_x} : U_x \to W_x$ and $g_{|W_x} : W_x \to V$ are quasi-compact, so (i) says that the same holds for their composition $(g \circ f)_{|U_x} : U_x \to V$. However, finitely many such U_x suffice to cover U, whence the assertion.

(iii) Let us say that f is *strongly spectral* if the following holds. For every quasi-compact inclusion map $U \to U'$ between open subsets of Y, the induced open immersion $f^{-1}U \to f^{-1}U'$ is also quasi-compact. We claim that if f is strongly spectral, then f is spectral. Indeed, let $U \subset X$, $V \subset Y$ be two quasi-compact and quasi-separated open subsets, and $V' \subset V$ another quasi-compact open subset; by assumption, the induced inclusion map $i : f^{-1}V' \to f^{-1}V$ is quasi-compact, so $f^{-1}_{|U}V' = i^{-1}U$ is quasi-compact, whence the claim.

(iv) Obviously, a composition of strongly spectral maps is strongly spectral. Moreover, it is easily seen that any open immersion $i : U \to X$ is strongly spectral. In view of (iii), it follows that every open immersion is spectral.

Lemma 8.1.15. *Let X be any topological space. We have :*

(i) *X is quasi-separated if and only if it admits a covering consisting of retro-compact open quasi-separated subsets.*

(ii) *X is sober if and only if it admits a covering consisting of sober open subsets.*

(iii) *The following conditions are equivalent :*

   (a) *X is locally coherent and quasi-separated.*

   (b) *X admits a basis consisting of quasi-compact open subsets, which is closed under finite intersections.*

(iv) *The following conditions are equivalent :*

(a) *X is coherent (resp. spectral).*
(b) *X is quasi-compact and admits a covering consisting of coherent (resp. spectral) retro-compact open subsets.*
(c) *X is quasi-compact, quasi-separated and locally coherent (resp. locally spectral).*

(v) *X is noetherian if and only if it is locally noetherian and quasi-compact.*
(vi) *If X is locally noetherian, then X is locally coherent and quasi-separated.*

Proof. (i): The condition is trivially necessary. Suppose then that $(U_i \mid i \in I)$ is a covering of X such that U_i is quasi-separated, retro-compact and open in X for every $i \in I$; pick any two quasi-compact open subsets U, U'. We may find a finite subset $J \subset I$ such that $U \cup U' \subset \bigcup_{j \in J} U_j$, and our assumptions imply that, for every $j \in J$, the subset $V_j := U \cap U' \cap U_j$ is quasi-compact, so the same holds for $\bigcup_{j \in J} V_j = U \cap U'$.

(ii): Again, necessity is trivial. For the converse, let $(U_i \mid i \in I)$ be an open covering of X such that U_i is sober for every $i \in I$. Say that $Z \subset X$ is any irreducible closed subset, and that $Z_i := U_i \cap Z \neq \varnothing$ for some $i \in I$; then Z_i is an irreducible subset of U_i and the topological closure of Z_i in X equals Z (lemma 8.1.3(ii)). Let η be the generic point of Z_i; then clearly the topological closure of $\{\eta\}$ in X equals Z. It remains to check that η is the unique element of Z with this property. However, say that η' is another such element; then it is easily seen that $\eta' \in U_i$ as well, and η' is then also the generic point of Z_i, whence the contention.

(iii): It is easily seen that (b)⇒(a). Conversely, suppose that (a) holds, and choose a covering $(U_i \mid i \in I)$ of X consisting of coherent open subsets; for each $i \in I$ we then have a basis $(U_{ij} \mid j \in J_i)$ of the topology of U_i, consisting of quasi-compact open subsets. Set $\mathscr{F} := (U_{ij} \mid i \in I,\ j \in I_j)$, and let $\mathscr{F}'$ be the family of all finite non-empty intersections of elements of $\mathscr{F}$; then $\mathscr{F}'$ is a basis of the topology of X, and since X is quasi-separated, any element of $\mathscr{F}'$ is still a quasi-compact open subset of X, whence (b).

(iv): Obviously (a)⇒(b), and (b)⇒(c) by virtue of (i) and remark 8.1.6(i). Lastly, (c)⇒(a), by (iii) and (ii).

(v): Suppose that X is locally noetherian and quasi-compact, in which case it admits a finite covering $(U_i \mid i \in I)$ consisting of open noetherian subsets. Now, let $Z_\bullet := (Z_n \mid n \in \mathbb{N})$ be a descending chain of closed subsets of X; by assumption, every resulting chain $(Z_n \cap U_i \mid n \in \mathbb{N})$ is stationary, and since I is finite, it follows that $Z_\bullet$ is stationary as well.

(vi): X is locally coherent, by virtue of remark 8.1.6(iv); next, if $U, V \subset X$ are quasi-compact open subsets, then they are noetherian, by (v), so their intersection is quasi-compact. □

Proposition 8.1.16. *Let $f : X \to Y$ be a continuous map of topological spaces, $(Y_i \mid i \in I)$ an open covering of Y, and $f_i : f^{-1}Y_i \to Y_i$ the restriction of f, for every $i \in I$. We have :*

(i) *If Y is locally coherent, then f is quasi-compact if and only if f_i is quasi-compact for every $i \in I$.*
(ii) *If both X and Y are locally coherent, then f is quasi-separated and spectral if and only if f_i is quasi-separated and spectral for every $i \in I$.*
(iii) *Suppose that both X and Y are locally coherent, and let $(X_i \mid i \in I)$ be an open covering of X such that $X_i \subset f^{-1}Y_i$ for every $i \in I$. Then f is spectral if and only if the restriction $X_i \to Y_i$ of f is spectral for every $i \in I$.*

Proof. (i): Clearly, if f is quasi-compact, the same holds for every f_i. Hence we may assume that f_i is quasi-compact for every $i \in I$, and we have to check that f is quasi-compact. To this aim, let $U \subset Y$ be any quasi-compact open subset, and set $U_i := Y_i \cap U$ for every $i \in I$; since Y is locally coherent, we may find, for every $i \in I$, a covering $(U_{i,\lambda} \mid \lambda \in \Lambda_i)$ of U_i consisting of quasi-compact open subsets. Since U is quasi-compact, finitely many of these $U_{i,\lambda}$ suffice to cover U, hence $f^{-1}U$ is covered by finitely many subsets of the type $f_i^{-1}U_{i,\lambda}$, and the latter are quasi-compact by assumption, so the same holds for U.

(iii): To begin with, we remark

Claim 8.1.17. Suppose that both X and Y are locally coherent. Then f is spectral if and only if it is strongly spectral (see remark 8.1.14(iii)).

Proof of the claim. In view of remark 8.1.14(iii), we may assume that f is spectral, and we show that f is strongly spectral. Indeed, let $U \subset U'$ be any inclusion of open subset of Y, with U retro-compact in U'. We need to check that $f^{-1}U$ is retro-compact in $f^{-1}U'$, and by (i), the assertion can be checked locally on $f^{-1}U'$. However, since X and Y are locally coherent, for every $x \in f^{-1}U'$ we may find quasi-compact and quasi-separated open neighborhoods V' of x in $f^{-1}U'$ and W' of $f(x)$ in U' such that $f(V') \subset W'$. By assumption, $W' \cap U$ is quasi-compact, so the same holds for $V := V' \cap f^{-1}U$ (because f is spectral); since V' is quasi-separated, the inclusion map $V \to V'$ is then quasi-compact, as required. ◊

Now, in light of claim 8.1.17, for the proof of (iii) we may assume that each restriction $X_i \to Y_i$ is strongly spectral, and it suffices to check that the same holds for f. However, let $U_1 \subset U_2$ be any quasi-compact inclusion map of open subsets of Y; by (i), the induced inclusion map $U_1 \cap Y_i \to U_2 \cap Y_i$ is also quasi-compact, so by assumption the same holds for the inclusion

$$X_i \cap f^{-1}U_1 = X_i \cap f^{-1}(U_1 \cap Y_i) \to X_i \cap f^{-1}(U_2 \cap Y_i) = X_i \cap f^{-1}U_2 \qquad \text{for every } i \in I.$$

Again (i) then implies that the inclusion $f^{-1}U_1 \to f^{-1}U_2$ is quasi-compact, as required.

(ii): Again, we may assume that f_i is quasi-separated and spectral for every $i \in I$, and we show that the same follows for f. By (iii), we know already that f is spectral, so we let $U \subset Y$ be any quasi-separated open subset, and show that $f^{-1}U$ is quasi-separated in X. To this aim, choose a family of quasi-compact open subsets $(U_{i,\lambda} \mid \lambda \in \Lambda_i)$ which gives a covering of $U \cap Y_i$, for every $i \in I$; since U is quasi-separated, $U_{i,\lambda}$ is retro-compact in U for every $i \in I$ and every $\lambda \in \Lambda_i$, hence $f^{-1}U_{i,\lambda}$ is retro-compact in $f^{-1}U$, by claim 8.1.17. Now, let $V_1, V_2 \subset f^{-1}U$ be any two quasi-compact open subsets; it follows that $V_1 \cap f^{-1}U_{i,\lambda}$ and $V_2 \cap f^{-1}U_{i,\lambda}$ are quasi-compact for every $i \in I$ and every $\lambda \in \Lambda_i$. Since f_i is quasi-separated, we deduce that $V_{i,\lambda} := V_1 \cap V_2 \cap f^{-1}U_{i,\lambda}$ is also quasi-compact, for every such i and λ. Since V_1 is quasi-compact, finitely many of these $f^{-1}U_{i,\lambda}$ suffice to cover V_1, and therefore finitely many $V_{i,\lambda}$ cover $V_1 \cap V_2$, so the latter is quasi-compact, as required. □

Example 8.1.18. We return to remark 8.1.12 : with the terminology of definition 8.1.13(ii), we may now say that the continuous map $\mathrm{Spec}\,\varphi : \mathrm{Spec}\,B \to \mathrm{Spec}\,A$ attached to any ring homomorphism $\varphi : A \to B$ is spectral (verification left to the reader). In light of proposition 8.1.16(iii) it follows that the continuous map underlying any morphism of schemes $f : X \to Y$ is spectral as well. Moreover, let $(Y_i \mid i \in I)$ be any covering of Y consisting of open subschemes; proposition 8.1.16(i,ii) implies that f is quasi-compact (resp. quasi-separated) if and only the same holds for each restriction $f_{|Y_i} : f^{-1}Y_i \to Y_i$.

Lemma 8.1.19. *Let X be a topological space, $U \subset X$ an open subset, $T \subset X$ any subset, and endow U and T with the topology induced from X. The following holds :*

(i) *Every closed subset of X is retro-compact. The union of any finite number of retro-compact subsets of X is still retro-compact. The intersection of any open retro-compact subset of X with an arbitrary retro-compact subset of X is retro-compact.*

(ii) *Every globally constructible subset of X is retro-compact, and it is a finite union of subsets of the form $V \setminus V'$, where V and V' are retro-compact open subsets of X.*

(iii) *Suppose that T is retro-compact, and let $S \subset T$ be any subset. If S is retro-compact in T, then it is retro-compact in X as well. If T is also open and S is globally constructible in T, then S is globally constructible in X as well.*

(iv) *T is pro-constructible if and only if $X \setminus T$ is ind-constructible.*

(v) *If T is retro-compact (resp. globally constructible, resp. pro-constructible, resp. ind-constructible) in X, then $T \cap U$ is retro-compact (resp. globally constructible, resp. constructible, resp. pro-constructible, resp. ind-constructible) in U.*

(vi) *The constructible subsets of X form a boolean algebra.*

(vii) *If X is quasi-separated and U is quasi-compact, then U is retro-compact in X.*

(viii) *Suppose that X is quasi-compact; then we have :*

- (a) *If T is retro-compact in X, then T is quasi-compact.*
- (b) *Every globally constructible subset of X is quasi-compact.*

(ix) *Suppose that X is quasi-compact and quasi-separated; then we have :*

- (a) *U is retro-compact in X if and only if it is quasi-compact.*
- (b) *The globally constructible subsets of X are precisely the finite unions of subsets of the form $V \setminus V'$, where V and V' are arbitrary quasi-compact open subsets of X.*

(x) *Suppose that X is coherent; then we have :*

- (a) *T is constructible in X if and only if it is globally constructible in X (in which case T is also retro-compact, by* (ii)*).*
- (b) *T is pro-constructible (resp. ind-constructible) if and only if it is the intersection (resp. the union) of a family of constructible subsets of X.*
- (c) *The constructible open (resp. closed) subsets of X are precisely the quasi-compact open subsets (resp. the complements of quasi-compact open subsets) of X.*
- (d) *If T is retro-compact in X and S is any constructible (resp. pro-constructible, resp. ind-constructible) subset of X, then $T \cap S$ is constructible (resp. pro-constructible, resp. ind-constructible) in T.*
- (e) *If T is retro-compact in X, then it is a coherent topological space, and the inclusion map $i_T : T \to X$ is quasi-separated.*
- (f) *If T is constructible in X, and S is any constructible subset of T, then S is constructible in X.*

(xi) *If X is spectral and T is constructible in X, then T is spectral as well.*

Proof. (i) shall be left to the reader, and (ii) follows easily from (i).

(iii): The first assertion is obvious, and the second follows from (ii) and the first assertion.

(v): The proof for the case where T is retro-compact or globally constructible is left to the reader. Suppose that T is constructible in X; then, for any given point $x \in U$, we may find an open neighborhood U' of x in X, such that $T \cap U'$ is globally constructible in U'; then by the foregoing case, we know that $U' \cap U$ is an open neighborhood of x in U such that $T \cap U' \cap U$ is globally constructible in $U' \cap U$, so $T \cap U$ is constructible in U. The assertion for the case where T is pro-constructible or ind-constructible, is an immediate consequence.

(vi): It is easily seen that the complement of a constructible subset is constructible. Next, suppose that T and T' are two constructible subsets of X, and $x \in X$ is any point; then we may find open neighborhoods U and U' of x such that $T \cap U$ (resp. $T' \cap U'$) is globally constructible in U (resp. in U'). By (v), it follows that $T \cap U \cap U'$ and $T \cap U \cap U'$ are globally constructible in $U \cap U'$, and therefore the same holds for both $(T \cap T') \cap U \cap U'$ and $(T \cup T') \cap U \cap U'$. We conclude that both $T \cap T'$ and $T \cup T'$ are constructible, whence the assertion.

(iv) follows easily from from (vi).

(vii) and (viii.a) are obvious, and (viii.b) follows from (viii.a) and (ii).

(ix.a) follows from (vii) and (viii.a), and (ix.b) follows from (ix.a) and (ii).

(x.a): Indeed, by assumption we may find for every $x \in X$ an open neighborhood U_x of x in X such that $T \cap U_x$ is globally constructible in U_x; by (v) we may then assume that U_x is quasi-compact, so U_x is retro-compact, by (ix.a) and remark 8.1.6(i), and consequently $T \cap U_x$

is globally constructible in X, by (iii). Finitely many of such U_x suffice to cover X, whence the assertion.

(x.b): Indeed, say that T is ind-constructible; arguing as in the proof of (x.a) we may find, for every $x \in X$, a quasi-compact open subset U_x such that $T \cap U_x$ is the union of a family $(T_{i,x} \mid i \in I_x)$ of constructible subsets of U_x. By (x.a) and remark 8.1.6(ii), each $T_{i,x}$ is globally constructible in U_x, hence also in X, by (viii.a), (iii) and remark 8.1.6(i); since $T = \bigcup_{x\in X} \bigcup_{i\in I_x} T_{i,x}$, the claim follows. The assertion for pro-constructible subsets is reduced to this case, by taking complements and using (iv).

(x.c): Indeed, every quasi-compact open subset is constructible, by (vii); conversely, (x.a) and (ix.b) show that every constructible subset of X is quasi-compact. The assertion about constructible closed subsets of X then follows immediately, taking into account (vi).

(x.d): In view of (x.b), it suffices to consider the case where S is constructible, and so S is even globally constructible in X, by (x.a). Then, we are further reduced to the case where S is an open retro-compact subset of X, *i.e.* S is open and quasi-compact in X, by (x.c), and it suffices to check that $T \cap S$ is retro-compact in T. Thus, let V be any quasi-compact open subset of T; since X is coherent, we may write $V = V' \cap T$, for some quasi-compact open subset V' of X (details left to the reader), and then $V \cap (T \cap S) = (V' \cap S) \cap T$. Now, $V' \cap S$ is open and quasi-compact in X (remark 8.1.6(i)) and therefore $V \cap (T \cap S)$ is quasi-compact in T, and therefore also in $T \cap S$, since the latter is open in T.

(x.e): T is quasi-compact by (viii.a), and if $(U_i \mid i \in I)$ is a basis of quasi-compact open subsets of X which is closed under finite intersections, then $(T \cap U_i \mid i \in I)$ is a basis of quasi-compact open subsets for T which is also closed under finite intersections, so T is coherent. Next, let $U \subset X$ be any open subset (then U is quasi-separated, by remark 8.1.6(i)), and V_1, V_2 any two quasi-compact open subsets of $U \cap T$; then V_1 and V_2 are also quasi-compact in T, so the same follows for $V_1 \cap V_2$ (again, by remark 8.1.6(i)), which shows that i_T is quasi-separated.

(x.f): In view of (x.e), the topological space T is coherent, so S is globally constructible in T by (x.a), and in light of (ii) and (vi) we may assume that $S = V \setminus V'$, for two retro-compact open subsets V, V' of T. Then V and V' are quasi-compact in T, by (viii.a), and since X is coherent, we may find quasi-compact open subsets W, W' of X such that $V = W \cap T$ and $V' = W' \cap T$, so that $S = (W \setminus W') \cap T$, and the latter is constructible in X, again by (vi) and (x.c).

(xi): In light of (x.e) and (x.a), it remains to show that T is sober, and taking into account (ii) and lemma 8.1.3(iv), we may assume that T is locally closed in X, in which case it suffices to invoke lemma 8.1.3(iii). □

Remark 8.1.20. Let $f : X \to Y$ be any quasi-compact continuous map of topological spaces.

(i) If $T \subset X$ is a retro-compact subset of X, then $f(T)$ is a retro-compact subset of Y. Indeed, if $U \subset Y$ is any quasi-compact open subset, we have $f(T) \cap U = f(T \cap f^{-1}U)$, and $f^{-1}U$ is a quasi-compact open subset of X, by assumption.

(ii) If f is quasi-separated, Y is locally coherent, and $T \subset Y$ is any constructible (resp. pro-constructible, resp. ind-constructible) subset in Y, then $f^{-1}T$ enjoys the same property in X. To see this, we may assume that Y is coherent, in which case X is quasi-separated and T is constructible (resp. an intersection, resp. a union of constructible subsets, by lemma 8.1.19(x.b)); then it suffices to consider the case where T is constructible, and we are further reduced to the case where T is open and quasi-compact (lemma 8.1.19(ix.b,x.a)), for which the assertion follows from lemma 8.1.19(vii).

(iii) If f is quasi-separated, then f is spectral. Indeed, say that $U \subset X$ and $V \subset U$ are quasi-compact and quasi-separated open subsets such that $f(U) \subset V$, and let $V' \subset V$ be any quasi-compact open subset of V; then $f^{-1}V$ is quasi-compact and quasi-separated, and $f^{-1}_{|U}V' = U \cap f^{-1}V'$, which is quasi-compact, under the current assumptions, because U and $f^{-1}V'$ are quasi-compact open subsets of $f^{-1}V$.

(iv) Suppose that X is locally coherent, $T \subset X$ is any retro-compact subset, and endow T with the topology induced from X. Then the inclusion map $i_T : T \to X$ is spectral. To see this, first we invoke proposition 8.1.16(i,iii) to reduce to the case where X is coherent. Then the assertion follows from (iii) and lemma 8.1.19(x.e). Together with lemma 8.1.19(i), we see especially that any closed immersion $Z \to X$ is quasi-compact, quasi-separated and spectral.

Proposition 8.1.21. *Let X be a coherent topological space. Then X is noetherian if and only if every irreducible closed subset is constructible.*

Proof. If X is noetherian, then every open subset of X is constructible, by virtue of lemma 8.1.19(x.c), so the condition is necessary. Conversely, suppose that X is coherent, and that every irreducible closed subset of X is constructible. Denote by $\mathscr{F}$ the set of closed non-constructible subsets of X, partially ordered by inclusion. If $(Z_\lambda \mid \lambda \in \Lambda)$ is a non-empty totally ordered subset of $\mathscr{F}$, let us show that $Z := \bigcap_{\lambda\in\Lambda} Z_\lambda$ lies in $\mathscr{F}$. Indeed, if $Z \notin \mathscr{F}$, then $X \setminus Z$ would be a quasi-compact open subset of X (lemma 8.1.19(x.c)), and since $X \setminus Z = \bigcup_{\lambda\in\Lambda} X \setminus Z_\lambda$, we would have $X \setminus Z = X \setminus Z_\lambda$ for some $\lambda \in \Lambda$, *i.e.* $Z = Z_\lambda$, which is absurd, since $Z_\lambda \in \mathscr{F}$. By Zorn's lemma, $\mathscr{F}$ admits therefore a minimal element Z_0, if it is non-empty. Let us check that Z_0 is irreducible : indeed, otherwise we have $Z_0 = Z_0' \cup Z_0''$ for two closed subsets Z_0', Z_0'' strictly contained in Z_0; by minimality of Z_0, the subsets Z_0' and Z_0'' are therefore constructible, and then the same holds for Z_0, a contradiction. Now, by assumption every irreducible subset of X is constructible; hence we must have $\mathscr{F} = \varnothing$, and this easily implies that X is noetherian, again by virtue of lemma 8.1.19(x.c). □

8.1.22. Let $X_\bullet := (X_i \mid i \in \mathrm{Ob}(I))$ be a system of sober and quasi-compact topological spaces, indexed by a category I, with continuous transition maps $p_u : X_i \to X_j$ for every morphism $u : i \to j$ in I, and set

$$X := \lim_{i\in I} X_i$$

where the limit is taken in the category of topological spaces. For every $i \in \mathrm{Ob}(I)$, denote by $p_i : X \to X_i$ the induced projection, and recall that the system of all subsets of the form $\bigcap_{j\in J} p_j^{-1} U_j$, where J ranges over all finite subsets of $\mathrm{Ob}(I)$, and U_j ranges over all open subsets of X_j, is a basis of the topology of X.

Proposition 8.1.23. *In the situation of* (8.1.22)*, the following holds :*

(i) *X is a sober topological space.*
(ii) *Suppose furthermore, that the system $X_\bullet$ is cofiltered. We have :*
  (a) *If $X_i \neq \varnothing$ for every $i \in \mathrm{Ob}(I)$, then X is quasi-compact and non-empty.*
  (b) *If p_u is quasi-compact for every morphism u of I, then p_i is quasi-compact, for every $i \in \mathrm{Ob}(I)$.*

Proof. (i) follows immediately from corollary 1.3.26(i,ii) and proposition 8.1.8.

(ii.a): By proposition 1.5.21(ii), we may assume that I is a partially ordered set, and to ease notation, for every $u : i \to j$ in I (*i.e.* for every $i, j \in I$ such that $i \leq j$), we set $p_{ij} := p_u$.

Now, consider the family $\mathscr{F}$ consisting of all compatible systems $Z_\bullet := (Z_i \mid i \in I)$, where :

- Z_i is a non-empty closed subset of X_i, for every $i \in I$
- $p_{ij}(Z_i) \subset Z_j$ for every $i, j \in I$ with $i \leq j$

and endow $\mathscr{F}$ with a partial ordering, by declaring that, for any $Z_\bullet, Z'_\bullet \in \mathscr{F}$, we have $Z_\bullet \leq Z'_\bullet$ if and only if $Z_i \subset Z'_i$ for every $i \in I$. Suppose now that $(Z^\lambda_\bullet \mid \lambda \in \Lambda)$ is a totally ordered subset of $\mathscr{F}$, and set $C_i := \bigcap_{\lambda\in\Lambda} Z^\lambda_i$ for every $i \in I$.

Claim 8.1.24. The resulting system $C_\bullet$ lies in $\mathscr{F}$.

Proof of the claim. Indeed, the assertion comes down to saying that $C_i \neq \varnothing$ for every $i \in I$. But notice that, for every finite subset $\Lambda' \in \Lambda$ and every $i \in I$, the intersection $\bigcap_{\lambda \in \Lambda'} Z_i^\lambda$ is obviously closed and non-empty in X_i. Since X_i is quasi-compact for every $i \in I$, the claim follows. ◊

Since $(X_i \mid i \in I) \in \mathscr{F}$, we have $\mathscr{F} \neq \varnothing$. From claim 8.1.24 and Zorn's lemma, we deduce that $\mathscr{F}$ admits minimal elements. We notice:

Claim 8.1.25. Let $Z_\bullet$ be a minimal element of $\mathscr{F}$. We have :

(i) Z_i is irreducible in X_i, for every $i \in I$.
(ii) $p_{ij}(Z_i)$ is dense in Z_j, for every $i, j \in I$ with $i \leq j$.

Proof of the claim. (i): Suppose, by way of contradiction, that there exist $i \in I$ and closed non-empty proper subsets $Z^{(1)}, Z^{(2)}$ of Z_i, such that $Z_i = Z^{(1)} \cup Z^{(2)}$. For $t = 1, 2$, we consider the compatible system $Z_\bullet^{(t)}$ such that

$$Z_j^{(t)} := \begin{cases} Z_j \cap p_{ji}^{-1} Z^{(t)} & \text{if } j \leq i \\ Z_j & \text{otherwise.} \end{cases}$$

By the minimality of $Z_\bullet$, neither $Z_\bullet^{(1)}$ nor $Z_\bullet^{(2)}$ lies in $\mathscr{F}$, hence there must exist $j, k \in I$ such that $Z_j^{(1)} = Z_k^{(2)} = \varnothing$. Since I is cofiltered, we may then assume that $j = k$, in which case $Z_k = Z_k \cap p_{ki}^{-1} Z_i = Z_k^{(1)} \cup Z_k^{(2)} = \varnothing$, which is absurd.

(ii): Fix any $i, j \in I$ with $i \leq j$, and denote by T the topological closure of $p_{ij}(Z_i)$ in Z_j. We consider the compatible system $T_\bullet$ such that

$$T_k := \begin{cases} p_{kj}^{-1} T & \text{if } k \leq j \\ Z_k & \text{otherwise.} \end{cases}$$

Notice that $T_k = Z_k$ for every $k \leq i$; especially $T_k \neq \varnothing$ for every $k \in I$, and therefore $T_\bullet \in \mathscr{F}$. By the minimality of $Z_\bullet$, we must have $T_\bullet = Z_\bullet$, and therefore $T = Z_j$, as stated. ◊

Let now $Z_\bullet$ be any minimal element of $\mathscr{F}$; according to claim 8.1.25, every Z_i is irreducible, say with generic point η_i, and p_{ij} maps η_i to η_j, whenever $i, j \in I$ with $i \leq j$. The compatible system of points $(\eta_i \mid i \in I)$ is then a point of X, so the latter is not empty.

Next, notice that every open covering of X can be refined to a covering of the form :

$$X = \bigcup_{i \in I} p_i^{-1} U_i \qquad \text{for a family of open subsets } U_i \in X_i \tag{8.1.26}$$

hence, in order to prove that X is quasi-compact, it suffices to check that, for a covering as (8.1.26), there exists a finite subset $J \subset I$ such that the family $(p_j^{-1} U_j \mid j \in J)$ already covers X. To this aim, set

$$Z_i := X_i \backslash U_i \qquad \text{and} \qquad Z'_i := \bigcap_{j \geq i} p_{ij}^{-1} Z_j \qquad \text{for every } i \in I.$$

Notice that (8.1.26) means that $\bigcap_{i \in I} p_i^{-1} Z_i = \varnothing$, and *a fortiory* we have

$$\bigcap_{i \in I} p_i^{-1} Z'_i = \varnothing \tag{8.1.27}$$

as well. However, by construction we have $p_{ij}(Z'_i) \subset Z'_j$ for every $i, j \in I$ with $i \leq j$, so $(Z'_i \mid i \in I)$ is a cofiltered system of sober and quasi-compact topological spaces (lemma 8.1.3(iii)), and (8.1.27) shows that the limit of this system is the empty topological space; in light of the foregoing, we conclude that there exists $i \in I$ such that $Z'_i = \varnothing$. Since X_i is quasi-compact, this in turns means that there exists a finite subset $J \subset \{j \in I \mid j \geq i\}$ such that $\bigcap_{j \in J} p_{ij}^{-1} Z_j = \varnothing$. Lastly, this means precisely that the family $(p_j^{-1} U_j \mid j \in J)$ covers X.

(ii.b): Fix $i_0 \in I$, and let U_{i_0} be any quasi-compact open subset of X_{i_0}; the subset $I_0 := \{j \in I \mid j \leq i_0\}$ is cofinal in I, so we may replace I by I_0 and assume that i_0 is the largest element of I. In this case, for every $j \leq i_0$ we may set $U_j := p_{ji_0}^{-1}U$, and $(U_i \mid i \in I)$ is a cofiltered system of quasi-compact and sober topological spaces (lemma 8.1.3(iii)), hence its limit U is quasi-compact (and sober), by (i) and (ii.a); on the other hand, the induced map $U \to X$ is an open immersion, whence the contention. □

Remark 8.1.28. Proposition 8.1.23 fails in general for non-cofiltered systems of quasi-compact and sober spaces. For instance, a fibre product of such spaces shall not be quasi-compact, in general.

Definition 8.1.29. Let $(X, \mathscr{T})$ be a topological space. The set of ind-constructible subsets of X is closed under finite unions and finite intersections, and forms a basis for a topology $\mathscr{C}$ on X called the *constructible topology*. We shall denote by X^c the topological space $(X, \mathscr{C})$.

Remark 8.1.30. Let $f : X \to Y$ be a continuous map of topological spaces.

(i) The map f is not necessarily continuous with respect to the constructible topologies on X and Y, but remark 8.1.20(ii) says that if f is quasi-separated and quasi-compact, and Y is locally coherent, then the induced map

$$f^c : X^c \to Y^c$$

(whose underlying map of sets is the same as f) is continuous.

(ii) Let U be any open subset of a topological space X, and endow U with the topology induced from X. Then the topology of U^c is finer than the topology on U induced from the topology of X^c (lemma 8.1.19(v)). The same holds if X is coherent and U is retro-compact (lemma 8.1.19(x.d)).

(iii) Suppose that X is locally coherent. By lemma 8.1.19(iv, x.b), it follows that the open (resp. closed) subsets of X^c are the ind-constructible (resp. pro-constructible) subsets of X.

Proposition 8.1.31. *Let $f : X \to Y$ be any spectral map between locally coherent topological spaces, $T \subset Y$ any subset. The following holds :*

(i) *f^c is continuous.*
(ii) *If T is constructible (resp. ind-constructible, resp. pro-constructible) in Y, then $f^{-1}T$ is constructible (resp. ind-constructible, resp. pro-constructible) in X.*

Proof. Clearly, it suffices to check (ii). However, let $T \subset Y$ be any subset as in (ii), and pick a covering $(U_i \mid i \in I)$ of Y consisting of open coherent subsets; it suffices to check that $f^{-1}(T \cap U_i)$ is constructible (resp. ind-constructible, resp. pro-constructible) in $f^{-1}U_i$ for every $i \in I$, so we may replace Y by any U_i, and assume that Y is coherent, in which case, lemma 8.1.19(x.b) implies that it suffices to consider the case where T is constructible in Y. Then, T shall be a finite union of subsets of the form $T' \setminus T''$, where $T'' \subset T'$ are any two quasi-compact open subsets of Y (lemma 8.1.19(ix.b,x.a)), and we may further reduce to the case where T is an open quasi-compact subset of Y. Now, let U be any coherent open subset of X; it suffices to check that $W := U \cap f^{-1}V$ is quasi-compact in U (lemma 8.1.19(x.c)), and the latter holds, since f is spectral. □

Lemma 8.1.32. *Let X be a locally coherent topological space, $T \subset X$ any subset, and endow T with the topology induced from X. We have :*

(i) *If T is either constructible or open in X, the topology on T induced from X^c agrees with the topology of T^c.*
(ii) *The topology of X^c is finer than that of X.*

Proof. (i): It suffices to show that :

(a) if S is any ind-constructible subset of X, then $S \cap T$ is ind-constructible in T
(b) if S is any ind-constructible subset of T, then S is ind-constructible in X.

However, fix a basis $(U_i \mid i \in I)$ of X consisting of coherent open subsets, and set $T_i := T \cap U_i$ for every $i \in I$. If S is as in (a), and if $S \cap T_i$ is ind-constructible in T_i for every $i \in I$, then clearly $S \cap T$ is ind-constructible in T. If S is as in (b), and $S \cap U_i$ is ind-constructible in U_i for every $i \in I$, then S is ind-constructible in X. Hence, it suffices to prove (a) and (b) with X and T replaced by respectively U_i and T_i, for every $i \in I$. We may then assume from start that X is coherent. In this case, T is either open or retro-compact in X (lemma 8.1.19(x.a)), hence the topology of T^c is finer than the topology $\mathscr{T}$ on T induced from X^c (remark 8.1.30(ii)), whence (a). To show (b), suppose first that T is constructible in X; then, any ind-constructible subset of T is a union of constructible subsets in T (lemma 8.1.19(x.b,e)), and therefore it is also a union of constructible subsets in X (lemma 8.1.19(x.f)), whence (b), in this case. Lastly, if T is open in X, and S is ind-constructible in T, pick a covering $(T_i \mid i \in I)$ of T consisting of open coherent subsets; then $S_i := S \cap T_i$ is ind-constructible in T_i for every $i \in I$, so for every such i we may find a family $(S_{i,\lambda} \mid \lambda \in \Lambda_i)$ of globally constructible subsets of T_i, whose union is S_i (lemma 8.1.19(x.a,b)). However, T_i is retro-compact in X, so $S_{i,\lambda}$ is constructible in X, for every $i \in I$ and every $\lambda \in \Lambda_i$ (lemma 8.1.19(iii)); finally, $S = \bigcup_{i \in I} \bigcup_{\lambda \in \Lambda_i} S_{i,\lambda}$ is ind-constructible in X, so (b) holds also in this case.

(ii): We have to check that every open subset U of X is ind-constructible in X; however, U is ind-constructible in U, so the assertion follows from the foregoing. $\square$

8.1.33. We come now to the main theorem of this section. To prepare the proof, we introduce the following terminology. Let $\mathscr{F}$ be any family of subsets of a topological space T; we say that $\mathscr{F}$ is a *closed filter* (resp. a *constructible filter*) if :

- every $Z \in \mathscr{F}$ is closed (resp. constructible) in T
- $\varnothing \notin \mathscr{F}$ and $\mathscr{F} \neq \varnothing$
- if $Z, Z' \in \mathscr{F}$, then $Z \cap Z' \in \mathscr{F}$
- if $Z \in \mathscr{F}$ and Z' is any closed (resp. constructible) subset of T containing Z, then $Z' \in \mathscr{F}$.

The *center* of a filter $\mathscr{F}$ is the subset $\bigcap_{Z \in \mathscr{F}} Z$. It is easily seen that a topological space T is quasi-compact if and only if every closed filter of T has non-empty center.

Theorem 8.1.34. *Let X be a topological space. We have :*

(i) *If X is spectral, X^c is compact.*
(ii) *If X is locally spectral and quasi-separated, then X^c is locally compact and separated.*

Proof. (We say that a topological space is *locally compact* if it admits a covering consisting of compact open subsets.)

(i): We check first that X^c is separated. To this aim, let x, y be any two points of X, and denote by $\overline{\{x\}}$ and $\overline{\{y\}}$ the topological closures of $\{x\}$ and $\{y\}$ with respect to $\mathscr{T}$. If $x \notin \overline{\{y\}}$, it follows from lemma 8.1.32(ii) that we may find a constructible subset $T \subset X$ such that $x \in T$ and $y \notin T$; then the complement S of T in X is also constructible, and T and S separate the points x and y. Likewise one argues in the symmetric case where $y \notin \overline{\{x\}}$. Lastly, if neither of these conditions hold, we must have $x = y$, since X is sober and both x and y are generic points of the same closed subset of X.

To show that X^c is quasi-compact, we check that its closed filters have non-empty center. However, let $\mathscr{F}$ be any such filter, and denote by $\mathscr{F}'$ the set of all constructible subsets T of X such that there exists $F \in \mathscr{F}$ with $F \subset T$. Clearly $\mathscr{F}'$ is a constructible filter of X, thus, we come down to :

Claim 8.1.35. (i) $\mathscr{F}$ and $\mathscr{F}'$ have the same center.

(ii) Every constructible filter of X has non-empty center.

Proof of the claim. (i): Clearly the center of $\mathscr{F}'$ contains the center of $\mathscr{F}$. For the converse, it suffices to remark that every closed subset of X is pro-constructible in X (lemma 8.1.32(ii) and remark 8.1.30(iii)), and therefore it is the intersection of the constructible subsets that contain it (lemma 8.1.19(x.b)).

(ii): In view of lemma 8.1.19(xi), every such filter can be regarded as a cofiltered system of spectral spaces, with continuous transition maps. Then the assertion is a special case of proposition 8.1.23(ii.a). ◊

(ii): Lemma 8.1.32 and (i) imply that X^c is locally compact. To see that X is separated, let $x, y \in X$ be any two distinct points. We may then find a quasi-compact open subset U of X containing both x and y, and the induced map $U^c \to X^c$ is an open immersion (lemma 8.1.32), so we may replace X by U, and assume from start that X is is spectral (lemmata 8.1.15(iv) and 8.1.19(vii)), which is the case covered by (i). □

Corollary 8.1.36. *Let X be any spectral topological space, $U \subset X$ a subset. We have :*

(i) *U is open and quasi-compact in X^c if and only if it is constructible in X.*
(ii) *X^c is a spectral topological space.*
(iii) *If Y is any locally spectral topological space, the same holds for Y^c.*

Proof. (i): Suppose first that U is open and quasi-compact in X^c; then U is a union of a family $(U_i \mid i \in I)$ constructible subsets of X (lemma 8.1.19(x.b)), and there exists a finite subset $J \subset I$ such that $(U_i \mid i \in J)$ already covers U, so U is constructible.

Conversely, suppose that U is constructible in X, in which case it is both open and closed in X^c; since the latter is quasi-compact (theorem 8.1.34(i)), we see that U is quasi-compact in X^c.

(ii): Indeed, (i) and lemma 8.1.19(vi) easily imply that X^c is coherent, and X^c is trivially sober, since in any separated space the irreducible closed subsets are exactly those that contain a unique point.

(iii) follows from (ii) and lemma 8.1.32 : details left to the reader. □

Corollary 8.1.37. *Let $f : X \to Y$ be a continuous map of locally spectral topological spaces. The following holds :*

(i) *f is spectral if and only if the same holds for f^c.*
(ii) *Suppose furthermore that f is spectral. Then :*
   (a) *f is quasi-compact if and only if the same holds for $f^c : X^c \to Y^c$.*
   (b) *f is quasi-separated if and only if the same holds for f^c.*

Proof. We notice first :

Claim 8.1.38. In order to prove the corollary, we may assume that Y is spectral.

Proof of the claim. Let $(Y_i \mid i \in I)$ be a covering of Y consisting of spectral open subsets; then each Y_i^c is open in Y^c, and its topology agrees with the topology induced from Y^c (lemma 8.1.32(i,ii)), so $(Y_i^c \mid i \in I)$ is an open covering of Y^c. Then the claim follows immediately from proposition 8.1.16(iii) and corollary 8.1.36(iii). ◊

Thus, we assume henceforth that Y is spectral. Next, we remark, quite generally :

Claim 8.1.39. Let $g : T \to S$ be any continuous map between locally spectral topological spaces, with S locally compact. Then g is spectral.

Proof of the claim. By lemma 8.1.16(iii), we may assume that S is compact and that T is quasi-compact and quasi-separated. Now, let $V \subset S$ be any quasi-compact open subset; since S is separated, V is also closed in S, hence $g^{-1}V$ is closed in T, so it is quasi-compact, since the same holds for T. Thus, g is quasi-compact, and it is also obviously quasi-separated (due to remark 8.1.6(i)), so the claim follows from remark 8.1.20(iii). ◊

(i): In light of claim 8.1.39, proposition 8.1.31(i) and theorem 8.1.34(i), we see that if f is spectral, the same holds for f^c, hence we may assume that f^c is spectral, and we show that the same holds for f. Thus, let $U \subset X$ and $V \subset Y$ be quasi-compact and quasi-separated open subsets such that $f(U) \subset V$; it suffices to check that $f_{|U} : U \to V$ is quasi-compact, and since U^c (resp. V^c) is open in X^c (resp. in Y^c, by lemma 8.1.32(ii)), the restriction $f^c_{|U} : U^c \to V^c$ is still spectral, so we may replace X by U and Y by V, and assume that X is quasi-compact and quasi-separated, in which case it is spectral (lemma 8.1.15(iv)), and we need to show that f is quasi-compact. Now, let $V \subset Y$ be any quasi-compact open subset; then V is spectral (remark 8.1.6(ii)), and V^c is open and quasi-compact in Y^c (lemma 8.1.32(ii) and theorem 8.1.34(i)). Hence, V^c is closed in Y^c, and therefore $f^{-1}V^c$ is closed, and therefore quasi-compact, in X^c. Since the topology of the latter is finer than that of X, we conclude that $f^{-1}V$ is quasi-compact in X.

(ii.a): If f is quasi-compact, it follows that X admits a finite covering consisting of spectral open subsets, and if U is any such subset, the restriction $f_{|U}$ is quasi-compact, since f is spectral. Clearly it suffices to check that $f^c_{|U}$ is quasi-compact for every such U, so we may replace X by U and assume that X is spectral as well, in which case, notice that f is obviously quasi-separated. Now, say that $U \subset Y^c$ is open and quasi-compact; so, U is constructible in Y (corollary 8.1.36(i)), and then $V := f^{-1}U$ is constructible in X, by remark 8.1.20(ii), so it is quasi-compact in X^c (corollary 8.1.36(i)).

Conversely, say that f^c is quasi-compact, and U is a quasi-compact open subset of Y. Then U is constructible in Y, so it is open and quasi-compact in Y^c (corollary 8.1.36(i)), therefore $f^{-1}U$ is quasi-compact in X^c, and since the topology of X^c is finer than that of X (lemma 8.1.32(ii)), we conclude that $f^{-1}U$ is quasi-compact in X.

(ii.b): Suppose that f is quasi-separated; then X is quasi-separated, so X^c is separated (theorem 8.1.34(ii)), and it follows easily that f^c is quasi-separated (details left to the reader). Conversely, if f^c is quasi-separated, then X^c is quasi-separated, since Y^c is separated (by theorem 8.1.34(i)). Then, say that U, V are two quasi-compact open subsets of X; we may cover U (resp. V) by a finite family $(U_i \mid i \in I)$ (resp. $(V_j \mid j \in J)$) of spectral open subsets, and the resulting maps $U_i^c \to X^c$, $V_j^c \to X^c$ are open immersions (lemma 8.1.32). Moreover, each U_i^c and V_j^c is compact, by theorem 8.1.34(i), so $U_i^c \cap V_j^c$ is quasi-compact in X^c, for every $i \in I$ and every $j \in J$; consequently $U \cap V$ is quasi-compact in X^c, and *a fortiori* also in X, since the topology of the latter is coarser than that of X^c (lemma 8.1.32(ii)). □

Corollary 8.1.40. (i) *In the situation of* (8.1.22)*, suppose moreover that :*

(a) *The topological space X_i is spectral for every $i \in \mathrm{Ob}(I)$.*
(b) *The transition map p_u is quasi-compact, for every morphism u of I.*

Then X is spectral.

(ii) *Suppose furthermore, that the system $X_\bullet$ is cofiltered. Then :*

(a) *For every constructible (resp. open and quasi-compact) subset $U \subset X$ there exist $i \in \mathrm{Ob}(I)$ and a constructible (resp. open and quasi-compact) subset $U_i \subset X_i$ such that $U = p_i^{-1}U_i$.*
(b) *For any $i \in \mathrm{Ob}(I)$ and any two constructible subsets U, U' of X_i such that $p_i^{-1}U \subset p_i^{-1}U'$, there exists a morphism $u : j \to i$ in I, such that $p_u^{-1}U \subset p_u^{-1}U'$.*

Proof. (i): By proposition 8.1.23(i) we know already that X is sober, so it remains only to check that X is coherent. However, under the stated conditions, the datum $(X_i^c \mid i \in \mathrm{Ob}(I))$ is a system of quasi-compact and separated topological spaces, with continuous transition maps (theorem 8.1.34 and remark 8.1.30(i)); denote by X^c the limit of this system (in the category of topological spaces). Now, the product $Y := \prod_{i\in\mathrm{Ob}(I)} X_i^c$ is likewise quasi-compact and separated, and we remark :

Claim 8.1.41. The natural map $X^c \to Y$ is a closed immersion.

Proof of the claim. For every $i \in \mathrm{Ob}(I)$, denote by $q_i : Y \to X_i^c$ the natural projection; the image of X^c in Y is the intersection of the subsets $Y_u := \{y_\bullet \in Y \mid p_u \circ q_i(y_\bullet) = q_j(y_\bullet)\}$, for every morphism $u : i \to j$ in I, so it suffices to check that Y_u is closed in Y, for every such u. This is a special case of the following more general assertion. Let $f, g : T \to T'$ be two continuous maps of topological spaces, with T' separated; then the subset

$$S := \{t \in T \mid f(t) = g(t)\}$$

is closed in T. To show the latter, denote $F : T \to T' \times T'$ the map such that $F(t) = (f(t), g(t))$ for every $t \in T$; then $S = F^{-1}\Delta_{T'}$, where $\Delta_{T'} := \{(t', t') \mid t' \in T'\}$ is the diagonal of $T' \times T'$, and F is continuous for the product topology on $T' \times T'$. We then come down to showing that $\Delta_{T'}$ is closed in $T' \times T'$, which holds if (and only if) T' is separated. ♢

It follows from claim 8.1.41 that X^c is quasi-compact and separated; the topology of X^c is obviously finer than that of X, so the latter is quasi-compact as well. It remains to check that X admits a basis of quasi-compact open subsets closed under finite intersections. To this aim, let $J \subset \mathrm{Ob}(I)$ be any finite subset, pick a quasi-compact open subset U_j of X_j for each $j \in J$, and set

$$V := \prod_{i\in I\setminus J} X_j \times \prod_{j\in J} U_j \qquad U := X \cap V.$$

It suffices to show that U is quasi-compact in X. However, notice that U_j is constructible in X_j, so it is both open and closed in X_j^c, and therefore V is closed in Y; since X^c is closed in Y by claim 8.1.41, it follows that U is quasi-compact in X^c, and *a fortiori*, also in X.

(ii.a): Suppose first that U is open and quasi-compact; then U is a finite union of open subsets of the form $U_J := \bigcap_{j\in J} p_j^{-1}U_j$, where $J \subset \mathrm{Ob}(I)$ is a finite subset, and U_j is an open quasi-compact subset of X_j, for every $j \in J$. For such a given U_J, since I is cofiltered, there exists $i \in \mathrm{Ob}(I)$ with morphisms $u_j : i \to j$ for every $j \in J$, and therefore $U_J = p_i^{-1}V_i$, where $V_i := \bigcap_{j\in J} u_j^{-1}U_j$ is quasi-compact in X_i, since each u_j is quasi-compact and X_i is quasi-separated (remark 8.1.6(ii)). Thus, $U = \bigcup_{j\in J'} V_j$ for a finite subset $J' \subset \mathrm{Ob}(I)$ and quasi-compact open subsets $V_j \subset X_j$; again, we may find $i \in \mathrm{Ob}(I)$ and a morphism $v_j : i \to j$ for every $j \in J'$, so that $U = p_i^{-1}U_i$, where $U_i := \bigcup_{j\in J'} v_j^{-1}V_j$ is open and quasi-compact in X_i.

The case where U is constructible is reduced easily to the foregoing case, taking into account (i) and lemma 8.1.19(ix.b,x.a), and arguing as above, using the assumption that $X_\bullet$ is cofiltered (details left to the reader).

(ii.b) Let $Z := U\setminus U'$, set $Z_u := p_u^{-1}Z$ for every morphism $u : j \to i$ in I, and endow Z_u with the topology induced from X_j^c. From theorem 8.1.34(i) and lemma 8.1.19(xi) we see that $Z_\bullet := (Z_u \mid u \in \mathrm{Ob}(I/i))$ is a cofiltered system of compact topological spaces (see example 1.5.9(i)), and the assumption means that the limit of $Z_\bullet$ is empty, so there exists $u \in \mathrm{Ob}(I/i)$ such that $Z_u = \varnothing$, whence the assertion. □

Corollary 8.1.42. *Let X be any spectral topological space, $T \subset X$ a pro-constructible subset, and endow T with the topology induced from X. Then T is spectral and retro-compact in X.*

Proof. By lemma 8.1.19(x.b), we may write T as the intersection of a system $(T_i \mid i \in I)$ of constructible subsets of X. Let J be the set of all finite subsets of I, and for every $S \in J$, endow $T_S := \bigcap_{i\in S} T_i$ with the topology induced from X. Then T_S is a constructible subset of X for every $S \in J$, and clearly the topological space T is the limit of the (cofiltered) inverse system $(T_S \mid S \in J)$. By lemma 8.1.19(xi) each such T_S is spectral and retro-compact in X (lemma

8.1.19(x.a)); therefore, if $S \subset S'$ are any two elements of J, then T_S is constructible in $T_{S'}$ (lemma 8.1.19(x.d)), hence also retro-compact in $T_{S'}$ (lemma 8.1.19(x.a)), *i.e.* the induced map $T_S \to T_{S'}$ is quasi-compact, and the assertion follows from corollary 8.1.40(i) and proposition 8.1.23(ii.b). □

Proposition 8.1.43. *Let $(X, \mathscr{T})$ be a compact topological space, $\mathscr{U}$ a family of open and closed subsets of X, and endow X with the coarsest topology $\mathscr{T}_{\mathscr{U}}$ containing $\mathscr{U}$. Then the following conditions are equivalent :*

(a) *$(X, \mathscr{T}_{\mathscr{U}})$ is a T_0 topological space.*
(b) *$(X, \mathscr{T}_{\mathscr{U}})$ is a spectral space.*

Moreover, if these conditions hold, then $(X, \mathscr{T}_{\mathscr{U}})^c = (X, \mathscr{T})$.

Proof. From remark 8.1.2(vi) we get (b)⇒(a).

(a)⇒(b): Since $\mathscr{T}$ is compact, every $U \in \mathscr{U}$ is compact as well in $\mathscr{T}$, and therefore it is quasi-compact in $\mathscr{T}_{\mathscr{U}}$. Thus, $(X, \mathscr{T}_{\mathscr{U}})$ is coherent, and it remains only to check that X is sober. To this aim, let Z be any irreducible closed subset of $(X, \mathscr{T}_{\mathscr{U}})$, and set

$$\mathscr{U}_Z := \{U \in \mathscr{U} \mid U \cap Z \neq \varnothing\} \qquad \text{and} \qquad Z' := Z \cap \bigcap_{U \in \mathscr{U}_Z} U.$$

Since $\mathscr{T}_{\mathscr{U}}$ is T_0, the subset Z' contains at most one point of X, and to conclude, it suffices to check that $Z' \neq \varnothing$. However, notice that Z is closed also in $\mathscr{T}$, hence it is compact in this latter topology, so the same holds for every finite intersection $U_1 \cap \cdots \cap U_k \cap Z$ with $U_1, \dots, U_k \in \mathscr{U}_Z$. We are then reduced to showing that every such finite intersection is non-empty. Suppose that the latter fails, and set $V_i := Z \setminus U_i$ for every $i = 1, \dots, k$; then $Z = V_1 \cup \cdots \cup V_k$, and since Z is irreducible in $\mathscr{T}_{\mathscr{U}}$, it follows that $Z = V_i$ for some $i \leq k$, which is absurd.

Lastly, suppose that $(X, \mathscr{T}_{\mathscr{U}})$ is spectral; we claim that in this case the identity map of X yields a continuous map $i : (X, \mathscr{T}) \to (X, \mathscr{T}_{\mathscr{U}})^c$. Indeed, let $T \subset X$ be a subset that is globally constructible for the topology $\mathscr{T}_{\mathscr{U}}$; in view of lemma 8.1.19(x.a,x.b), it suffices to show that T is open in the topology $\mathscr{T}$. Then lemma 8.1.19(ix.b) further reduces to the case where $T = V \setminus V'$ for two quasi-compact open subsets V, V' of $(X, \mathscr{T}_{\mathscr{U}})$. However, V and V' are finite unions of subsets of the form $U_1 \cap \cdots \cap U_k$, where $k \in \mathbb{N}$ is arbitrary, and $U_1, \dots, U_k \in \mathscr{U}$; it follows that V and V' are open and closed in $(X, \mathscr{T})$, and therefore the same holds for their difference. Lastly, since $(X, \mathscr{T})$ is separated and $(X, \mathscr{T}_{\mathscr{U}})$ is quasi-compact (theorem 8.1.34(i)), it follows that i is a homeomorphism, whence the proposition. □

Definition 8.1.44. Let X be a topological space, $x, x' \in X$ any two points.

(i) We say that x *is a specialization of* x' *in* X, or equivalently, that x' *is a generization of* x *in* X, if x lies in the topological closure of $\{x'\}$ in X.
(ii) We say that x *is an immediate specialization of* x' *in* X, or equivalently, that x' *is an immediate generization of* x *in* X, if $x \neq x'$, x is a specialization of x' in X, and any other point of X that is both a specialization of x' and a generization of x in X is equal to either x or x'.
(iii) We denote by $X(x)$ the set of all generizations of x in X, and we endow $X(x)$ with the topology induced by X.
(iv) Let $f : X \to Y$ be any continuous map of topological spaces. We say that f is *specializing* (resp. *generizing*) if the following holds. For every $x \in X$ and every specialization (resp. generization) y of $f(x)$ in Y, there exists a specialization (resp. generization) x' of x in X, such that $f(x') = y$.

Remark 8.1.45. Let X be a topological space, and x any point of X.

(i) Clearly $X(x)$ is the intersection of all open neighborhoods of x in X. Especially, if X is coherent, $X(x)$ is a pro-constructible subset of X. If X is T_0 (*i.e.* for any two distinct points of X, at least one of them admits an open neighborhood not containing the other), x is the unique closed point of $X(x)$. If X is spectral, the same holds for $X(x)$, and the inclusion map $j_x : X(x) \to X$ is quasi-compact (corollary 8.1.42); since $X(x)$ is quasi-separated, j_x is also quasi-separated, and even spectral, by remark 8.1.20(iii).

(ii) Clearly, the relation

$$x \leq x' \quad \Leftrightarrow \quad x \text{ is a specialization of } x' \text{ in } X$$

defines a preordering on X (see example 1.1.6(iii)), and every continuous map $f : X \to Y$ is also a morphism of preordered sets $(X, \leq) \to (Y, \leq)$. If X is T_0, then $(X, \leq)$ is even a partially ordered set. Suppose that X is T_0 and non-empty; in general, $(X, \leq)$ admits neither maximal nor minimal points, but if X is sober, it follows easily from remark 8.1.2(ii) that $(X, \leq)$ admits maximal points. If $X \neq \varnothing$ is quasi-compact and T_0, then $(X, \leq)$ also admits minimal elements: indeed, if $(x_i \mid i \in I)$ is any (non-empty) totally ordered subset of X, let $\overline{\{x_i\}}$ be the topological closure of $\{x_i\}$ in X, for every $i \in I$; then $Z := \bigcap_{i\in I} \overline{\{x_i\}} \neq \varnothing$, so any point of Z is smaller than every x_i, and then the assertion follows as usual, by Zorn's lemma.

(iii) Suppose that X is sober. Then, X is noetherian if and only if the following holds :

(a) Every closed subset of X has finitely many maximal points.
(b) Every descending sequence $(x_n \mid n \in \mathbb{N})$ in $(X, \leq)$ is stationary, *i.e.* it admits $N \in \mathbb{N}$ such that $x_n = x_N$ for every $n \geq N$.

Indeed, for a sober space, (a) and (b) are rephrasing of the corresponding conditions of proposition 8.1.4. It is also clear that, in this situation, the dimension of X is the supremum of the lengths of all finite strictly descending sequences in $(X, \leq)$ (see remark 8.1.2(iv)).

(iv) If $f : X \to Y$ is a continuous map of T_0 topological spaces, it is easily seen that f is also a map of partially ordered sets $(X, \leq) \to (Y, \leq)$ for the partial orderings induced by specializations, as in (ii).

Definition 8.1.46. (i) Let X be a T_0 topological space. We denote by

$$\mathrm{Max}\, X \qquad \text{and} \qquad \mathrm{Min}\, X$$

the sets of maximal and respecively minimal points of the partially ordered set $(X, \leq)$, where $\leq$ is the partial ordering induced by specialization, as in remark 8.1.45(ii).

(ii) Let $f : X \to Y$ be a continuous map of T_0 topological spaces. We say that f is *maximizing* (resp. *minimizing*) if it restricts to a map

$$\mathrm{Max}(f) : \mathrm{Max}\, X \to \mathrm{Max}\, Y \qquad (\text{resp. } \mathrm{Min}(f) : \mathrm{Min}\, X \to \mathrm{Min}\, Y).$$

Proposition 8.1.47. *Let X be any locally spectral topological space, $T \subset X$ a pro-constructible subset, and $U \subset X$ an ind-constructible subset. The following holds :*

(i) *The topological closure of T in X is the set of all specializations of all points of T.*
(ii) *U is open in X if and only if it contains the generizations of all its points.*
(iii) *T is dense in X if and only if $\mathrm{Max}\, X \subset T$ (notation of definition 8.1.46(i)).*

Proof. (i): Denote by $\overline{T}$ the topological closure of T in X, and by T^s the set of all specializations in X of points of T. Clearly $T^s \subset \overline{T}$, so we need only prove the converse inclusion. However, let V be any open subset of X; on the one hand, the topological closure of $T \cap V$ in V equals $V \cap \overline{T}$. On the other hand, say that $x \in V$ is a specialization in X of a point $y \in T$; then clearly $y \in T \cap V$, *i.e.* x is a specialization in V of y, so $V \cap T^s$ is the set of specializations in V of the points of $V \cap T$. Since X admits a covering consisting of spectral open subsets, we may then replace X by any of these subsets, and assume from start that X is spectral.

Now, suppose that there exists $x \in \overline{T}\backslash T^s$, denote by $\mathscr{F}$ the family of all quasi-compact open neighborhoods of x in X, and notice that $\mathscr{F}$ is cofiltered, since X is quasi-separated, and not empty, since X is quasi-compact. Since $x \in \overline{T}$, we must have $W \cap T \neq \varnothing$ for every $W \in \mathscr{F}$; on the other hand, we have

$$\bigcap_{W \in \mathscr{F}} (W \cap T) = \varnothing$$

since $x \notin T^s$. But both T and the elements of $\mathscr{F}$ are closed in X^c, so $(W \cap T \mid W \in \mathscr{F})$ is a closed filter of X^c, therefore its center cannot be empty (theorem 8.1.34(i) and (8.1.33)), a contradiction.

(ii) follows easily from (i), by considering the pro-constructible subset $X \backslash U$.

(iii) is an immediate consequence of (i). □

Example 8.1.48. Let X be a spectral space of dimension ≤ 0. It follows easily from proposition 8.1.47(i) that every quasi-compact open subset of X is closed. Then X is separated : indeed, let $x, x' \in X$ be any two distinct points; since X is sober, we may assume that x does not lie in the topological closure of $\{x'\}$, in which case there exists a quasi-compact open neighborhood U of x that does not contain x', and the foregoing implies that $X \setminus U$ is an open neighborhood of x', whence the contention. Conversely, every separated and compact topological space whose quasi-compact open subsets form a basis of the topology is a spectral space of dimension ≤ 0; indeed, such a space is coherent, and it is obviously sober, since all its points are closed. A topological space with this properties is sometimes called a *compact boolean space* : see [115, p.168,169], where it is also explained that, by virtue of the Stone representation theorem, such spaces are characterized as those topological spaces which are homeomorphic to the maximal spectra of (unital) boolean algebras. More precisely, to such any compact boolean space X, one attaches its *characteristic ring* R_X, which is the boolean algebra of all continuous functions $X \to \mathbb{Z}/2\mathbb{Z}$, and then X is naturally homeomorphic to $\mathrm{Max}\, R_X$.

Corollary 8.1.49. *Let X be a spectral space, and $T \subset X$ a subset that contains the generizations of all its points, and endow T with the topology induced by X. The following conditions are equivalent :*

(a) *T is quasi-compact.*
(b) *T is a pro-constructible subset of X.*
(c) *T is spectral.*

Proof. We know already that (b)⇒(c), by corollary 8.1.42, and clearly (c)⇒(a).

(a)⇒(b): For every $x \in X \backslash T$, let $\overline{\{x\}}$ be the topological closure of $\{x\}$ in X; by assumption, $\overline{\{x\}} \cap T = \varnothing$, so for every $t \in T$ there exists a quasi-compact open subset $U_{x,t}$ such that $t \in U_{t,x}$ and $\overline{\{x\}} \cap T = \varnothing$. Since T is quasi-compact, we then have a finite subset $S_x \subset T$, such that $T \subset U_x := \bigcup_{t \in S_x} U_{t,x}$, and clearly U_x is a constructible open subset of X. We have $T = \bigcap_{x \in X \backslash T} U_x$, whence (b). □

Corollary 8.1.50. *Let $f : X \to Y$ be a continuous map of locally spectral topological spaces, and $T \subset X$ (resp. $S \subset Y$) any pro-constructible subset. Then :*

(i) *If f is quasi-compact, $f(T)$ is a pro-constructible subset of Y.*
(ii) *If f is spectral and generizing, the topological closure in X of $f^{-1}S$ is the preimage of the topological closure of S in Y.*

Proof. (i): The assertion is local on Y, hence we may assume that Y is spectral; then X admits a finite covering consisting of open spectral subsets, and we are easily reduced to the case where X is spectral as well. Now, by theorem 8.1.34(i,ii), the subset T is closed, and therefore quasi-compact, in X^c. Then $f(T)$ is quasi-compact in Y^c; hence it is closed in Y^c, and therefore pro-constructible in Y.

(ii): Let $\overline{S}$ denote the topological closure of S in Y; clearly the topological closure of $f^{-1}S$ lies in $f^{-1}\overline{S}$, so we have only to check the converse inclusion. Thus, say that $x \in f^{-1}\overline{S}$, so

$s := f(x) \in \overline{S}$, and therefore s is a specialization of a point $s' \in S$ (proposition 8.1.47(i)); since f is generizing, it follows that there exists a generization x' of x in X, such that $f(x') = s'$. This shows that x lies in the set of specializations of the points of $f^{-1}S$, and the contention follows. □

Proposition 8.1.51. *Let $f : Y \to X$ be a surjective, spectral map of locally spectral topological spaces, and suppose that f is either specializing or generizing. Then X is the (topological) quotient of Y, under the equivalence relation induced by f.*

Proof. The assertion means that a subset $Z \subset X$ is open (resp. closed) in X if and only if $Z' := f^{-1}Z$ is closed in Y. For any subset Z of X (resp. of Y), we denote by $\overline{Z}$ the topological closure of Z in X (resp. in Y). Of course, we may assume that Z' is closed in Y, and we check that Z is closed in X. Now, Z' is pro-constructible in Y (lemma 8.1.32(ii)), so $f(Z') = Z$ is pro-constructible in X (corollary 8.1.50(i)). If f is generizing, it follows that $Z' = \overline{f^{-1}Z} = f^{-1}\overline{Z}$ (corollary 8.1.50(ii)); since f is surjective, we deduce that $Z = \overline{Z}$, *i.e.* Z is closed, as required. Lastly, suppose that f is specializing, and notice that $\overline{Z}$ is the set of all specializations of all points of Z (proposition 8.1.47(i)); thus, let $x \in Z$, $x' \in X$ a specialization of x and $y \in Z'$ such that $f(y) = x$; since f is specializing, there exists $y' \in Y$ with $f(y') = x'$. But since Z' is closed, we have $y' \in Z'$, so $x' \in Z$, as required. □

Corollary 8.1.52. *Let X be a spectral topological space such that $\mathrm{Min}\, X$ (resp. $\mathrm{Max}\, X$) is finite. If $Z \subset X$ is a subset closed under specializations and generizations, then Z is open and closed.*

Proof. When $\mathrm{Max}\, X$ is finite, any such Z is a finite union of irreducible components, hence it is closed. But the same applies to the complement of Z, whence the contention. When $\mathrm{Min}\, X$ is finite, consider the continuous map

$$g : Y := \coprod_{x \in \mathrm{Min}\, X} X(x) \to X$$

whose restriction to each $X(x)$ is the inclusion map. Now, Y is spectral, and g is surjective, continuous and spectral (remark 8.1.45(i)), and it is obviously also generizing; on the other hand, clearly the assertion holds for Y, hence for X, by proposition 8.1.51. □

Remark 8.1.53. Hochster defines in [90] an involution

$$X \mapsto X^*$$

of the category of spectral topological spaces and spectral maps, with the following properties. For every spectral space X, the set underlying X^* is the same as that underlying X, and the constructible closed subsets of X form a basis of the topology of X^*. He shows that the partially ordered set $(X^*, \leq)$ is the opposite of $(X, \leq)$ ([90, Prop.8]). Using this construction, it is possible to give an alternative proof of corollary 8.1.52.

8.1.54. In the study of the valuation spectrum to be carried out in section 9.2 we shall encounter certain subsets of a spectral space that are not pro-constructible, but nevertheless are spectral spaces, when endowed with the subspace topology. In the following proposition we describe the general principle underlying these constructions.

Let X be any topological space, $S \subset X \times X$ a subset of *specializations of* X : *i.e.* for every $(x, y) \in S$ the point y is a specialization in X of the point x, in which case we shall also say that y is an S*-admissible specialization* of x, and x is an S*-admissible generization* of y. We say that a subset $T \subset X$ is S*-closed* if every S-admissible specialization of every point of T lies also in T. Furthermore, we shall say that S is a *transitive set of specializations*, if the following holds : if $x, y, z \in X$ are any three points, and $(x, y), (y, z) \in S$, then $(x, z) \in S$ as well. We may then state :

Proposition 8.1.55. *Let X be any spectral space, S a given set of specializations of X, set*

$$Y := \{x \in X \mid x \text{ has no proper } S\text{-admissible specializations}\}$$

and endow Y with the topology induced by the inclusion map $Y \to X$. Suppose that :

(S1) *Every $y \in Y$ has a fundamental system of open neighborhoods in X consisting of constructible S-closed open subsets.*

(S2) *For every $x \in X$ there exists an S-admissible specialization y of x that lies in Y.*

Then the following holds :

(i) *For every $x \in X$ there exists a unique point $r(x) \in Y$ that is an S-admissible specialization of x in X.*

(ii) *The topological space Y is spectral, and the resulting retraction*

$$r : X \to Y$$

is spectral.

(iii) *More precisely, a subset $T \subset Y$ is constructible in Y if and only if $r^{-1}T$ is constructible in X.*

Proof. (i): Suppose that a given $x \in X$ has two S-admissible specialization $y, y \in Y$; by remark 8.1.2(vi) we know that X is T_0, so we may assume that there exists an open neighborhood U of y that does not contain y'. If $V \subset U$ is any open neighborhood of y, then $x \in V$, but V does not contain the S-admissible specialization y' of x, contradicting (S1).

(ii): Let $\mathscr{P}$ (resp. $\mathscr{Q}$) be the set of all open subsets $T \subset Y$ such that $r^{-1}T$ is constructible (resp. open and constructible) in X.

Claim 8.1.56. $\mathscr{P}$ is a basis of the topology of Y, and Y is a T_0 topological space.

Proof of the claim. Since X is T_0, condition (S1) implies easily that the same holds for Y. Next, notice that we have $U = r^{-1}r(U)$ for every open and S-closed subset $U \subset X$, and therefore $U = r^{-1}(U \cap Y)$ for every such U. Thus, $U \cap Y \in \mathscr{P}$ for every open, constructible and S-closed subset $U \subset X$, and the claim follows from (S1). ◊

Let now $\mathscr{T}$ be the topology whose basis is $\mathscr{Q}$; clearly every element of $\mathscr{Q}$ is open and closed in the topology $\mathscr{T}$. Moreover, since X is quasi-compact, it is easily seen that the same holds for $(Y, \mathscr{T})$. Then proposition 8.1.43 and claim 8.1.56 show that Y is spectral. It is also clear that r is continuous and quasi-compact, so it is spectral, by remark 8.1.20(iii).

(iii): Proposition 8.1.43 also shows that $\mathscr{Q}$ is the set of constructible subsets of Y, whence the assertion. □

Corollary 8.1.57. *In the situation of proposition* 8.1.55*, suppose moreover that the following condition holds :*

(S3) *For every $x \in X$, any two S-admissible specializations are* comparable*,* i.e. *if $y, z \in X$ are any two S-admissible specializations of x, then either y is an S-admissible specialization of z or else z is an S-admissible specialization of y.*

Then, for every subset $T \subset X$ the following conditions are equivalent :

(a) *$T = r^{-1}r(T)$.*

(b) *T is S-closed, and every S-admissible generization of every point of T lies in T.*

Proof. It is easily seen that (b)⇒(a). Hence, suppose that (a) holds, let $x \in T$ be any point, and $y \in X$ any S-admissible generization of x. By (S3), either x is an S-admissible specialization of $r(y)$, or $r(y)$ is an S-admissible specialization of x. If $r(y)$ is an S-admissible specialization of x, then $r(x) = r(y)$, by proposition 8.1.55(i); therefore $y \in T$, by (a). If x is an S-admissible specialization of $r(y)$, we must have $x = r(y)$, since $r(y) \in Y$; then (a) yields again $y \in T$.

Lastly, let y be an S-admissible specialization of x. By (S3), either $r(x)$ is an S-admissible specialization of y, or y is an S-admissible specialization of $r(x)$. If $r(x)$ is an S-admissible specialization of y, then $r(x) = r(y)$, by proposition 8.1.55(i); then (a) yields $y \in T$. If y is an S-admissible specialization of $r(x)$, then $y = r(x)$, since $r(x) \in Y$; then again $y \in T$, by (a). □

The foregoing notions shall be applied also to the study of schemes and their underlying topological spaces, which are always locally spectral (see remark 8.1.12). For future reference, we make the following :

Definition 8.1.58. Let X be any scheme.

(i) We say that X is *locally coherent*, if $\mathscr{O}_X$ is a coherent sheaf of rings.
(ii) We say that X is *coherent*, if it is locally coherent, quasi-compact and quasi-separated.

Notice that the topological space underlying a coherent scheme is always spectral, by lemma 8.1.15(iv). We conclude this section with a few miscellaneous results concerning the topology of schemes.

Proposition 8.1.59. *Let X be any reduced and coherent scheme, and endow the set $\mathrm{Max}\, X$ of maximal points of X with the topology induced from X. Then $\mathrm{Max}\, X$ is quasi-compact.*

Proof. Clearly we have $\mathrm{Max}\, U = U \cap \mathrm{Max}\, X$ for any open subset $U \subset X$; then, if $(U_i \mid i \in I)$ is any finite covering of X consisting of affine open subsets, it suffices to show that $\mathrm{Max}\, U_i$ is quasi-compact for every $i \in I$. We may therefore assume from start that X is affine, say $X = \mathrm{Spec}\, A$ for some reduced coherent ring A. We notice :

Claim 8.1.60. Let R be any reduced ring and $f \in R$ any element; we have :

$$\mathrm{Max}\,(\mathrm{Spec}\, R_f) = \mathrm{Max}\,(\mathrm{Spec}\, R) \cap \mathrm{Spec}\, R/\mathrm{Ann}_R(f).$$

Proof of the claim. Set $I := \mathrm{Ann}_R(f)$ and let $\mathfrak{p}$ be any minimal prime ideal of R; then $\mathfrak{p} \in \mathrm{Max}\,(\mathrm{Spec}\, R_f)$ if and only if $\mathfrak{p} \in \mathrm{Max}\,(\mathrm{Spec}\, R)$ and $f \notin \mathfrak{p}$. Thus, it suffices to check that $f \notin \mathfrak{p}$ if and only if $I \subset \mathfrak{p}$. However, if $f \notin \mathfrak{p}$, clearly $I \subset \mathfrak{p}$. For the converse, notice that $I_\mathfrak{p} = \mathrm{Ann}_{R_\mathfrak{p}}(f)$, and $R_\mathfrak{p}$ is a field, since R is reduced; hence $I_\mathfrak{p}$ is either $R_\mathfrak{p}$ or 0, depending on whether the image of f is invertible or not in $R_\mathfrak{p}$, and the assertion follows easily. ◊

Now, pick any covering $(U_i \mid i \in I)$ of $\mathrm{Max}\, X$ consisting of open subsets; we have to show that there exists a finite subset $J \subset I$ such that $(U_i \mid i \in J)$ already covers $\mathrm{Max}\, X$. To this aim, we may assume without loss of generality that every U_i is of the form $\mathrm{Max}\,(\mathrm{Spec}\, A_{f_i})$ for some $f_i \in A$. Notice that $\mathrm{Ann}_A(f_i)$ is an ideal of finite type, since A is coherent; in view of claim 8.1.60, we deduce that for every $i \in I$ there exists a constructible open subset V_i of X such that $\mathrm{Max}\, X \backslash U_i = V_i \cap \mathrm{Max}\, X$. By assumption, $\mathrm{Max}\, X \cap \bigcap_{i\in I} V_i = \varnothing$; since every V_i contains the generizations of all its points, it follows that $\bigcap_{i\in I} V_i = \varnothing$. By corollary 8.1.36(i) and theorem 8.1.34, we deduce that there exists a finite subset $J \subset I$ such that $\bigcap_{i\in J} V_i = \varnothing$. Then clearly this subset J will do. □

Lemma 8.1.61. *Let A be a ring, $I \subset A$ an ideal. Then :*

(i) *The following are equivalent :*
  (a) *The map $A \to A/I$ is flat.*
  (b) *The map $A \to A/I$ is a localization.*
  (c) *For every prime ideal $\mathfrak{p} \subset A$ containing I, we have $IA_\mathfrak{p} = 0$, especially, $V(I)$ is closed under generizations in* $\mathrm{Spec}\, A$.
(ii) *Suppose I fulfills the conditions* (a)-(c) *of* (i). *Then the following are equivalent :*
  (a) *I is finitely generated.*
  (b) *I is generated by an idempotent.*

(c) $V(I) \subset \operatorname{Spec} A$ *is open.*

Proof. (i): Clearly (b)⇒(a). Also (a)⇒(c), since every flat local homomorphism is faithfully flat. Suppose that (c) holds; we show that the natural map $B := (1 + I)^{-1}A \to A/I$ is an isomorphism. Indeed, notice that IB is contained in the Jacobson radical of B, hence it vanishes, since it vanishes locally at every maximal ideal of B.

(ii): Clearly (ii.c)⇒(ii.b)⇒(ii.a). If (ii.a) holds, then $V(I)$ is constructible and closed under generizations, by (ii.c), so it is open (proposition 8.1.47(ii)), *i.e.* (ii.a)⇒(ii.c). ☐

8.1.62. For every ring A, we let $\mathscr{I}(A)$ be the set of all ideals $I \subset A$ fulfilling the equivalent conditions (i.a)-(i.c) of lemma 8.1.61, and $\mathscr{Z}(A)$ the set of all closed subset $Z \subset \operatorname{Spec} A$ that are closed under generizations in $\operatorname{Spec} A$. In light of lemma 8.1.61(i.c), we have a natural mapping:

$$\mathscr{I}(A) \to \mathscr{Z}(A) \quad : \quad I \mapsto V(I). \tag{8.1.63}$$

Lemma 8.1.64. *The mapping* (8.1.63) *is a bijection, whose inverse assigns to any $Z \in \mathscr{Z}(A)$ the ideal $I[Z]$ consisting of all the elements $f \in A$ such that $fA_\mathfrak{p} = 0$ for every $\mathfrak{p} \in Z$.*

Proof. Notice first that $I[V(I)] = I$ for every $I \in \mathscr{I}(A)$; indeed, clearly $I \subset I[V(I)]$, and if $f \in I[V(I)]$, then $fA_\mathfrak{p} = 0$ for every prime ideal $\mathfrak{p}$ containing I, hence the image of f in A/I vanishes, so $f \in I$. To conclude the proof, it remains to show that if Z is closed and closed under generizations, then $V(I[Z]) = Z$. However, say that $Z = V(J)$, let $\mathfrak{p} \in Z$ be any prime ideal, and suppose that $f \in J$; then $\operatorname{Spec} A_\mathfrak{p} \subset Z$, hence f is nilpotent in $A_\mathfrak{p}$, so there exist $n \in \mathbb{N}$ and an open neighborhood $U \subset \operatorname{Spec} A$ of $\mathfrak{p}$ such that $f^n = 0$ in U. Since Z is quasi-compact, finitely many such U suffice to cover Z, hence we may find $n \in \mathbb{N}$ large enough such that $f^n \in I[Z]$, whence the contention. ☐

The following result improves upon [136, Cor.2.6].

Proposition 8.1.65. *Let R be a ring, R' any finitely generated R-algebra, $\varphi : \operatorname{Spec} R' \to \operatorname{Spec} R$ the induced morphism of schemes, $\Sigma \subset \operatorname{Spec} R$ any subset, and endow Σ (resp. $\varphi^{-1}\Sigma$) with the topology induced from $\operatorname{Spec} R$ (resp. from $\operatorname{Spec} R'$). If Σ is a noetherian topological space, then the same holds for $\varphi^{-1}\Sigma$.*

Proof. Set $X := \operatorname{Spec} R$ and $X' := \operatorname{Spec} R'$; we may find an integer $n \in \mathbb{N}$ and a closed immersion $X' \to \operatorname{Spec} R[T_1, \dots, T_n]$, so it suffices to prove the assertion for $R' = R[T_1, \dots, T_n]$, and by factoring the map $R \to R'$ as a composition $R \to R[T_1, \dots, T_{n-1}] \to R'$, an easy induction reduces to the case where $R' = R[T]$.

Now suppose, by way of contradiction, that the proposition fails, and let $\mathscr{F}$ be the family of all closed subsets Z of Σ such that $\varphi^{-1}Z$ is not noetherian. Since Σ is noetherian, and $\mathscr{F}$ is not empty, $\mathscr{F}$ admits minimal elements. Let Z be a minimal element of $\mathscr{F}$, and endow Z with the topology induced from Σ; then Z is also noetherian (remark 8.1.6(iv)), so we may replace Σ by Z, and assume that no proper closed subset of Σ lies in $\mathscr{F}$. We claim that in this case, Σ is irreducible. Indeed, let $\Sigma_1, \dots, \Sigma_k$ be the finitely many irreducible components of Σ (proposition 8.1.4); if $k > 1$, we have $\Sigma_i \notin \mathscr{F}$ for $i = 1, \dots, k$, but then remark 8.1.6(v) easily implies that $\Sigma \notin \mathscr{F}$ as well, a contradiction. Hence, the topological closure $\overline{\Sigma}$ of Σ in X is irreducible in X (lemma 8.1.3(i)); we endow $\overline{\Sigma}$ with the reduced closed subscheme structure induced by the closed immersion $\overline{\Sigma} \to X$, so $\overline{\Sigma}$ is isomorphic to the spectrum of some quotient of R. We may then replace X by $\overline{\Sigma}$, after which we may assume that R is a domain, and Σ is dense in X. In this case, denote by K the field of fractions of R. We notice :

Claim 8.1.66. Let A be a domain, K the field of fractions of A, $I \subset A[T]$ any ideal, and $p(T) \in I$ a monic polynomial that generates $I \cdot K[T]$. Then $p(T)$ generates I.

Proof of the claim. Consider any $g(T) \in I$, and let $g(T) = q(T) \cdot p(T) + r(T)$ be the euclidean division of g by p in $K[T]$. Since p generates $I \cdot K[T]$, we have $r = 0$, and since p is monic, it is easily seen that $q \in A[T]$, whence the claim. ◊

Now, suppose that $(Z_n \mid n \in \mathbb{N})$ is a strictly descending chain of closed subsets of $\varphi^{-1}\Sigma$, and for each $n \in \mathbb{N}$, let $\overline{Z}_n$ be the topological closure of Z_n in $\operatorname{Spec} R'$, and $I_n \subset R'$ the largest ideal such $\operatorname{Spec} R'/I_n = \overline{Z}_n$. There follows an increasing chain of ideals $(I_n \mid n \in \mathbb{N})$. The corresponding sequence $(I_n \cdot K[T] \mid n \in \mathbb{N})$ is stationary and consists of principal ideals of $K[T]$, so we may find $N \in \mathbb{N}$ and a monic polynomial $p(T)$ such that $p(T)$ generates $I_n \cdot K[T]$ for every $n \geq N$. Moreover, we may find some $f \in R\backslash\{0\}$ such that $p(T) \in R_f[T]$ (where $R_f := R[f^{-1}]$), and we may even assume that $p(T) \in I_n \cdot R_f[T]$ for every $n \geq N$, in which case claim 8.1.66 shows that $p(T)$ generates $I_n \cdot R_f[T]$ for every $n \geq N$. Set $U := \operatorname{Spec} R_f$; we conclude that the chain of subsets $(\overline{Z}_n \cap \varphi^{-1}U \mid n \in \mathbb{N})$ is stationary. However, $Z'_n := Z_n \cap \varphi^{-1}U = \overline{Z}_n \cap \varphi^{-1}(U \cap \Sigma)$ for every $n \in \mathbb{N}$, so the chain $(Z'_n \mid n \in \mathbb{N})$ is stationary as well. On the other hand, by construction $U \neq \varnothing$, so $\Sigma' := \Sigma\backslash U$ is a proper closed subset of Σ, therefore $\varphi^{-1}\Sigma'$ is noetherian, and especially, the chain $(Z_n \cap \varphi^{-1}\Sigma' \mid n \in \mathbb{N})$ is stationary. Summing up, we deduce that the chain $(Z_n \mid n \in \mathbb{N})$ is stationary, which is absurd. □

8.2. **Topological groups.** This section collects a few generalities on topological groups that shall be used in later sections.

Definition 8.2.1. A *topological group* is a pair $(G, \mathscr{T}_G)$ where $(G, \cdot, e_G)$ is a group and $\mathscr{T}_G$ a topology on the set underlying G, such that :

- the composition law of G is a continuous map $(G, \mathscr{T}_G) \times (G, \mathscr{T}_G) \to (G, \mathscr{T}_G)$
- the rule $g \mapsto g^{-1}$ for every $g \in G$ yields a continuous map $(G, \mathscr{T}_G) \to (G, \mathscr{T}_G)$.

A *morphism of topological groups* $\varphi : (G, \mathscr{T}_G) \to (H, \mathscr{T}_H)$ is a group homomorphism $\varphi : G \to H$ that is continuous for the topologies $\mathscr{T}_G$ and $\mathscr{T}_H$.

Remark 8.2.2. Let $(G, \mathscr{T})$ be any topological abelian group.

(i) If $U \subset G$ is any open neighborhood of the neutral element 0, set $-U := \{-g \mid g \in U\}$. Clearly $-U$ is still an open neighborhood of 0 in G, and thus the same holds for $U \cap -U$. Especially, G admits a fundamental system of open neighborhoods of 0 consisting of subsets U such that $U = -U$.

(ii) Let $\mathscr{U}$ be any fundamental system of open neighborhoods of 0 in G. It is easily seen that a subset $V \subset G$ is open if and only if for every $g \in V$ there exists $U \in \mathscr{U}$ such that $g + U := \{g + h \mid h \in U\} \subset V$. Especially, $\mathscr{U}$ determines completely the topology of G. Notice also that, due to the continuity of the composition law of G, for every $U \in \mathscr{U}$ there exists $U' \in \mathscr{U}$ such that $U' + U' := \{g + g' \mid g, g' \in U'\} \subset U$.

(iii) Conversely, let $\mathscr{U}$ be any family of subsets of G such that :

(a) $0 \in U$ for every $U \in \mathscr{U}$
(b) $U = -U$ for every $U \in \mathscr{U}$
(c) for every $U \in \mathscr{U}$ there exists $U' \in \mathscr{U}$ such that $U' + U' \subset U$
(d) for every $U, U' \in \mathscr{U}$ there exists $U'' \in \mathscr{U}$ such that $U'' \subset U \cap U'$.
(e) for every $U \in \mathscr{U}$ and every $g \in U$ there exists $U' \in \mathscr{U}$ such that $g + U' \subset U$.

Then there exists a unique topology $\mathscr{T}$ on G such that $\mathscr{U}$ is a fundamental system of neighborhoods of 0 in G relative to the topology $\mathscr{T}$, and $(G, \mathscr{T})$ is a topological group. Indeed, the uniqueness is clear from (ii), which also describes $\mathscr{T}$ explicitly, and it is easily seen that the composition law $G \times G \to G$ and the map $G \to G : g \mapsto -g$ are both continuous for this topology : details left to the reader.

(iv) Let $\varphi : G' \to G$ be a group homomorphism with G' also abelian, and endow G' with the topology $\mathscr{T}'$ induced by $\mathscr{T}$. Then $(G', \mathscr{T}')$ is a topological group. Indeed, pick any fundamental system $\mathscr{U}$ of open neighborhoods of 0 in G fulfilling conditions (a)-(e) of (iii); it is easily seen that the system $\varphi^{-1}\mathscr{U} := (\varphi^{-1}U \mid U \in \mathscr{U})$ is a fundamental system of open neighborhoods of the neutral element $0'$ of G' relative to the topology $\mathscr{T}'$, and clearly $\varphi^{-1}\mathscr{U}$ fulfills the same conditions (a)-(e), whence the assertion.

(v) Likewise, let $\psi : G \to G''$ be a group homomorphism with G'' also abelian; for any system $\mathscr{U}$ of open neighborhoods of 0 as in (iv), set $\psi(\mathscr{U}) := (\psi(U) \mid U \in \mathscr{U})$; it is easily seen that the system $\psi(\mathscr{U})$ fulfills again conditions (a)-(e) of (iii), hence there exists a unique topology $\mathscr{T}''$ on G'' for which $\psi(\mathscr{U})$ is a fundamental system of open neighborhoods of the neutral element $0'' \in G''$, and such that $(G'', \mathscr{T}'')$ is a topological abelian group. If ψ is surjective, it is easily seen that $\mathscr{T}''$ agrees with the topology induced by $\mathscr{T}$ via ψ. More generally, $\psi(G)$ is an open subgroup of G'' for the topology $\mathscr{T}''$, and the topology on $\psi(G'')$ induced by $\mathscr{T}''$ agrees with the quotient topology induced by $\mathscr{T}$ via the surjection $\psi : G \to \psi(G)$: details left to the reader. Especially, $\mathscr{T}''$ is independent of the choice of $\mathscr{U}$, and is the finest topology on G'' such that $\psi : (G, \mathscr{T}) \to (G'', \mathscr{T}'')$ is a morphism of topological abelian groups.

(vi) Let $(G', \mathscr{T}')$ be another topological abelian group, and $\varphi : G \to G'$ a group homomorphism. Then φ is continuous if and only if it is *continuous at the point* $0 \in G$, *i.e.* if and only if for every open neighborhood $U' \subset G'$ of the neutral element $0' \in G'$ there exists an open neighborhood $U \subset G$ of 0 in G with $\varphi(U) \subset U'$. Indeed, the condition is obviously necessary. Conversely, suppose this condition holds, and let $V' \subset G'$ be any open subset; for every $g \in \varphi^{-1}V$ we may pick an open neighborhood U' of $0'$ in G' such that $U' \subset -\varphi(g) + V$, and by assumption there exists an open neighborhood $U \subset G$ of 0 such that $U \subset \varphi^{-1}(U')$, whence $g + U \subset \varphi^{-1}V$, so $\varphi^{-1}V$ is open in G, whence the contention.

(vii) Notice that G is separated if and only if the subset $\{0\}$ is closed in G. Indeed, the condition is obviously necessary. For the converse, suppose that $\{0\}$ is closed, and let $g \in G \setminus \{0\}$ be any other point; it suffices to show that there exist open neighborhoods U and U' respectively of 0 and of g, such that $U \cap U' = \varnothing$. However, by assumption there exists an open neighborhood V of 0 that does not contain g; then let U be any open neighborhood of 0 such that $U = -U$ and $U + U \subset V$, and set $U' := g + U$. Suppose $u \in U \cap U'$; then there exists $u' \in U$ such that $u = g + u'$, whence $g = u - u' \in U + U \subset V$, a contradiction.

Lemma 8.2.3. *Let G be a topological abelian group, $I_0 \subset G_0 \subset G$ two subgroups, that we endow with the topology induced from G. Also, let $\mathscr{T}_0$ and $\mathscr{T}$ be the quotient topologies of $\overline{G}_0 := G_0/I_0$ and respectively $\overline{G} := G/I_0$. The following holds :*

(i) *$\mathscr{T}_0$ agrees with the topology induced by $\mathscr{T}$ via the inclusion map $j : \overline{G}_0 \to \overline{G}$.*

(ii) *Suppose that I_0 is closed in G_0, denote by $I \subset G$ the topological closure of I_0 in G, and endow $\overline{G}' := G/I$ with the quotient topology $\mathscr{T}'$. Then the induced map*

$$j' : \overline{G}_0 \to \overline{G}'$$

is injective, and $\mathscr{T}_0$ agrees with the topology induced by $\mathscr{T}'$ via j'.

Proof. (i): We remark, quite generally :

Claim 8.2.4. Let

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

be a cartesian diagram of topological spaces. If f is an open map, the same holds for f'.

Proof of the claim. Every open subset of X' is a union of subsets of the form $U \times_Y V = g'^{-1}U \cap f'^{-1}V$, for arbitrary open subsets $U \subset X$ and $V \subset Y'$. Then $f'(U \times_Y V) = V \cap g^{-1}f(U)$ is open in Y', whence the claim. $\Diamond$

We consider the diagram

$$\begin{array}{ccc} G_0 & \longrightarrow & G \\ {\scriptstyle\pi_0}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ \overline{G}_0 & \xrightarrow{\ j\ } & \overline{G} \end{array}$$

whose vertical arrows are the projections, and whose top horizontal arrow is the inclusion map. It is easily seen that this diagram is cartesian, provided we endow G_0 and $\overline{G}_0$ with the topologies induced by the inclusions into G and respectively $\overline{G}$. Then π is an open map, so the same holds for π_0, by claim 8.2.4, whence the assertion.

(ii): Since I_0 is closed in G_0, we have $I_0 = G_0 \cap I$, so j' is clearly injective. Let $\mathscr{T}_0'$ be the topology on $\overline{G}_O$ induced by $\mathscr{T}'$ via j'; the identity map is a continuous bijection

$$(\overline{G}_0, \mathscr{T}_0) \to (\overline{G}_0, \mathscr{T}_0')$$

so it suffices to check that it is also a closed map. Now, let $\pi' : G \to \overline{G}'$ and $\pi_0' : G_0 \to \overline{G}_0$ be the projections, and let $Z \subset \overline{G}_0$ be a subset that is closed relative to the topology $\mathscr{T}_0$; by definition, this means that $\pi_0'^{-1}Z = Z + I_0$ is closed in G_0. Let $(Z + I_0)^c$ be the topological closure of $Z + I_0$ in G; it is easily seen that $(Z + I_0)^c = \pi'^{-1}\pi'((Z + I_0)^c)$, so $\pi'((Z + I_0)^c)$ is closed in $\overline{G}'$, and hence $\overline{G}_0 \cap \pi'((Z + I_0)^c)$ is closed in $\overline{G}_0$, relative to the topology $\mathscr{T}_0'$; lastly

$$\overline{G}_0 \cap \pi'((Z + I_0)^c) = (G_0 \cap (Z + I_0)^c)/I_0 = (Z + I_0)/I_0 = Z$$

since $Z + I_0$ is closed in G_0. The assertion follows. □

Lemma 8.2.5. *Let $f : G \to H$ and $g : H \to K$ be two continuous homomorphisms of topological groups. If $g \circ f$ is an open map, the same holds for g.*

Proof. Set $h := g \circ f$, and let U be any open neighborhood of the neutral element e_H in H; since $h(f^{-1}U) \subset g(U)$, we see that $g(U)$ contains an open neighborhood of the neutral element e_K in K. Now, let $x \in g(U)$ be any element, and pick $y \in U$ such that $h(y) = x$; let also V be an open neighborhood of e_H in H such that $y \cdot V \subset U$; it follows that $g(y \cdot V) \subset g(U)$. On the other hand, $g(V)$ contains an open neighborhood W of e_K, so $x \cdot W$ is an open neighborhood of x contained in $g(U)$. Since x is arbitrary, the assertion follows. □

Definition 8.2.6. Let G be any abelian topological group.

(i) A *Cauchy net* on G is a family $(g_\lambda \mid \lambda \in \Lambda)$ of elements of G indexed by a filtered partially ordered set Λ, such that the following holds. For every open neighborhood U of the neutral element $0 \in G$ there exists $\lambda \in \Lambda$ such that $g_\mu - g_\nu \in U$ for every $\mu, \nu \geq \lambda$.

(ii) We say that two Cauchy nets $g_\bullet := (g_\lambda \mid \lambda \in \Lambda)$ and $g'_\bullet := (g'_{\lambda'} \mid \lambda' \in \Lambda')$ of G are *equivalent*, and we write $g_\bullet \sim g'_\bullet$, if the following holds. For every open neighborhood U of $0 \in G$ there exist $\lambda \in \Lambda$ and $\lambda' \in \Lambda'$ such that $g_\mu - g'_{\mu'} \in U$ for every $\mu \geq \lambda$ and every $\mu' \geq \lambda'$.

(iii) We say that an element $h \in G$ is a *limit point* for the Cauchy net $g_\bullet := (g_\lambda \mid \lambda \in \Lambda)$ if $g_\bullet$ is equivalent to the trivial Cauchy net $(h_{\lambda'} := h \mid \lambda' \in \Lambda')$ indexed by a partially ordered set Λ' that has exactly one element. In this case, we also say that $g_\bullet$ *converges* to h in G.

(iv) We say that G is *complete* if every Cauchy net of G admits a limit point. We denote by

$$\mathsf{AbTopGrp} \qquad \text{and} \qquad \mathsf{AbTopGrp}_{\text{cp.sep.}}$$

respectively the category of abelian topological groups (with morphisms given by the continuous group homomorphisms) and its full subcategory whose objects are the complete and separated topological groups.

Remark 8.2.7. (i) In view of remark 8.2.2(vii), it is easily seen that an abelian topological group G is separated if and only if every Cauchy net of G admits at most one limit point. In this case, the unique limit point of a Cauchy net $(g_\lambda \mid \lambda \in \Lambda)$ shall be often denoted

$$\lim_{\lambda\in\Lambda} g_\lambda.$$

(ii) Moreover, it is easily seen that the relation $\sim$ introduced by definition 8.2.6 is indeed an equivalence relation on the set of all Cauchy nets on G. Clearly, if $g_\bullet \sim g'_\bullet$, then an element $h \in G$ is a limit point for $g_\bullet$ if and only if it is a limit point for $g'_\bullet$.

(iii) Furthermore, if $\varphi : G \to G'$ is a continuous group homomorphism of topological abelian groups, and $(g_\lambda \mid \lambda \in \Lambda)$ a Cauchy net of G, then $\varphi(g_\bullet) := (\varphi(g_\lambda) \mid \lambda \in \Lambda)$ is a Cauchy net of G', and if $g'_\bullet$ is another Cauchy net of G with $g_\bullet \sim g'_\bullet$, then $\varphi(g_\bullet) \sim \varphi(g'_\bullet)$.

(iv) Suppose that G is complete and separated, and let $H \subset G$ be any subgroup, that we endow with the topology induced by G. Then it is easily seen that H is complete if and only if it is topologically closed in G.

Theorem 8.2.8. (i) *The inclusion functor* $\mathsf{AbTopGrp}_{\mathrm{cp.sep}} \to \mathsf{AbTopGrp}$ *admits a left adjoint*

$$\mathsf{AbTopGrp} \to \mathsf{AbTopGrp}_{\mathrm{cp.sep.}} \qquad (G, \mathscr{T}) \mapsto (G^\wedge, \mathscr{T}^\wedge)$$

called the completion functor.

(ii) *For every topological abelian group* $(G, \mathscr{T})$*, the unit of adjunction* $i_G : G \to G^\wedge$ *is called the* completion map *of* G*. It has dense image, and* $\mathscr{T}$ *agrees with the topology induced by* $\mathscr{T}^\wedge$ *via* i_G.

(iii) *Let* $\varphi : (G, \mathscr{T}) \to (G', \mathscr{T}')$ *be a morphism of topological abelian groups with dense image and with* G' *complete and separated, and such that* $\mathscr{T}$ *agrees with the topology induced by* $\mathscr{T}'$ *via* φ*. By the adjunction of* (i)*, the map* φ *factors through* i_G *and a unique continuous group homomorphism* $\varphi^\wedge : G^\wedge \to G'$*, and* $\varphi^\wedge$ *is an isomorphism of topological groups.*

Proof. Let $(G, +, 0)$ be any topological abelian group. We let $G^\wedge$ be the set of equivalence classes of Cauchy nets of G. For any such Cauchy net $g_\bullet$, we denote by $[g_\bullet]$ the class of $g_\bullet$ in $G^\wedge$. We define a composition law on $G^\wedge$ as follows. Given two Cauchy nets $g_\bullet := (g_\lambda \mid \lambda \in \Lambda)$ and $g'_\bullet := (g'_{\lambda'} \mid \lambda' \in \Lambda')$, let $\Lambda \times \Lambda'$ be the product of Λ and Λ' in the category of partially ordered sets (whose underlying set is the cartesian product of Λ and Λ'); it is easily seen that the family $g''_{(\bullet,\bullet)} := (g_\lambda + g'_{\lambda'} \mid (\lambda, \lambda') \in \Lambda \times \Lambda')$ is also a Cauchy net of G, and we set

$$[g_\bullet] + [g'_\bullet] := [g''_{(\bullet,\bullet)}].$$

One checks easily that $[g''_{(\bullet,\bullet)}]$ depends only on the classes of $g_\bullet$ and $g'_\bullet$, so we get by this rule a well defined mapping $G^\wedge \times G^\wedge \to G^\wedge$. Clearly the neutral element for this addition law is the class $[0]$ of any constant Cauchy net $(0_\lambda \mid \lambda \in \Lambda)$ with $0_\lambda := 0$ for every $\lambda \in \Lambda$. Also, set $-g_\bullet := (-g_\lambda \mid \lambda \in \Lambda)$; then it is easily seen that $[g_\bullet] + [-g_\bullet] = [0]$. The associativity and commutativity of the addition law are likewise immediate, so $G^\wedge$ is indeed an abelian group.

Claim 8.2.9. The set $G^\wedge$ is essentially small.

Proof of the claim. Let $g_\bullet := (g_\lambda \mid \lambda \in \Lambda)$ be any Cauchy net of G. Fix any fundamental system $\mathscr{U}$ of open neighborhoods of 0 in G, and endow it with the partial ordering such that $U \leq U'$ if and only $U' \subset U$, for every $U, U' \in \mathscr{U}$; for every $U \in \mathscr{U}$ there exists $\lambda(U) \in \Lambda$ such that $g_\lambda - g_\mu \in U$ for every $\lambda, \mu \geq \lambda(U)$. It is easily seen that the family $(g_{\lambda(U)} \mid U \in \mathscr{U})$ is a Cauchy net of G equivalent to $g_\bullet$. Thus, $G^\wedge$ has the cardinality of the set of equivalence classes $G^\wedge_{\mathscr{U}}$ of all Cauchy nets indexed by $\mathscr{U}$; clearly $\mathscr{U}$ is small, so the same holds for $G^\wedge_{\mathscr{U}}$, whence the claim. $\diamondsuit$

In light of claim 8.2.9, we may replace $G^\wedge$ by an isomorphic small group, that we shall still denote by $G^\wedge$. We also have a natural group homomorphism

$$i_G : G \to G^\wedge \qquad g \mapsto [g]$$

that assigns to every $g \in G$ the class of any constant Cauchy net with value g. Next, by remark 8.2.2(i), we may find a fundamental system $\mathscr{U}$ of open neighborhoods of 0 in G such that $-U = U$ for every $U \in \mathscr{U}$; for every $U \in \mathscr{U}$, we denote by $U^\wedge \subset G^\wedge$ the subset consisting of the classes of all Cauchy nets $(g_\lambda \mid \lambda \in \Lambda)$ fulfilling the following condition. There exists $U' \in \mathscr{U}$ such that $g_\lambda + U' \subset U$ for every $\lambda \in \Lambda$.

Claim 8.2.10. The system $\mathscr{U}^\wedge := (U^\wedge \mid U \in \mathscr{U})$ fulfills conditions (a)-(e) of remark 8.2.2(iii).

Proof of the claim. Indeed, (a) and (d) are obvious. To see that (b) holds, let $(g_\lambda \mid \lambda \in \Lambda)$ be any Cauchy net, and $U', U \in \mathscr{U}$ such that $g_\lambda + U' \subset U$ for every $\lambda \in \Lambda$; then $-g_\lambda + (-U') \subset -U$, *i.e.* $-g_\lambda + U' \subset U$ for every $\lambda \in \Lambda$, whence the contention. Next, for any $U \in \mathscr{U}$, pick $U' \in \mathscr{U}$ such that $U' + U' \subset U$; to check (c) it suffices to show that $U'^\wedge + U'^\wedge \subset U^\wedge$. Thus, say that $(g_\lambda \mid \lambda \in \Lambda)$ and $(g'_{\lambda'} \mid \lambda' \in \Lambda')$ are two Cauchy nets whose classes lie in $U'^\wedge$, and choose $U'' \in \mathscr{U}$ such that $g_\lambda + U'', g'_{\lambda'} + U'' \subset U'$ for every $\lambda \in \Lambda$ and every $\lambda' \in \Lambda'$; it follows that $g_\lambda + g'_{\lambda'} + U'' \subset U' + U' \subset U$ for every such λ and λ', as arrested. To check (e), let $(g_\lambda \mid \lambda \in \Lambda)$ be a Cauchy net, $U, U' \in \mathscr{U}$ such that $g_\lambda + U' \subset U$ for every $\lambda \in \Lambda$, and $(g'_{\lambda'} \mid \lambda' \in \Lambda)$ another Cauchy net such that there exists $U'' \in \mathscr{U}$ with $g'_{\lambda'} + U'' \subset U'$ for every $\lambda' \in \Lambda'$; it follows that $g_\lambda + g'_{\lambda'} + U'' \subset U$ for every such λ and λ', whence the contention. ◊

From claim 8.2.10 and remark 8.2.2(iii) we deduce that there exists a unique topology $\mathscr{T}^\wedge$ on $G^\wedge$ for which $\mathscr{U}^\wedge$ is a fundamental system of open neighborhoods of 0, and $(G^\wedge, \mathscr{T}^\wedge)$ is a topological group. Moreover, we claim that

$$i_G^{-1}(U^\wedge) = U \qquad \text{for every } U \in \mathscr{U}.$$

Indeed, the inclusion $U \subset i_G^{-1}(U^\wedge)$ is obvious. For the converse, suppose that the class of the Cauchy net $g_\bullet := (g_\lambda \mid \lambda \in \Lambda)$ lies in $U^\wedge$, and there exists $h \in G$ such that $i_G(h) = [g_\bullet]$. This means that there exists $U' \in \mathscr{U}$ such that $g_\lambda + U' \subset U$ for every $\lambda \in \Lambda$ and $\lambda_0 \in \Lambda$ such that $h - g_\lambda \in U'$ for every $\lambda \geq \lambda_0$; thus, $h \in g_\lambda + U' \subset U$, whence the contention. Taking into account remark 8.2.2(vi), it follows that i_G is a continuous group homomorphism, and more precisely, the topology of G agrees with the topology induced by $\mathscr{T}^\wedge$ via i_G. Furthermore, it is clear that the image of i_G is dense in $G^\wedge$. Let us remark, more precisely :

Claim 8.2.11. (i) For every $U \in \mathscr{U}$ and every open subset $V \subset G^\wedge$, we have :

$$V \subset U^\wedge \qquad \Leftrightarrow \qquad i_G^{-1}V \subset U.$$

(ii) For every $U \in \mathscr{U}$, the subset $U^\wedge$ lies in the topological closure $i_G(U)^c$ of $i_G(U)$ in $G^\wedge$.

Proof of the claim. (i): By the foregoing, we know already that if $V \subset U^\wedge$, then $i_G^{-1}V \subset U$. Conversely, suppose that $i_G^{-1}V \subset U$, and let $h \in V \setminus U^\wedge$; then h is the class of a Cauchy net $h_\bullet := (h_\lambda \mid \lambda \in \Lambda)$ with the following property. For every $U' \in \mathscr{U}$ there exist $x_{U'} \in U'$ and $\lambda(U') \in \Lambda$ such that $g_{U'} := h_{\lambda(U')} + x_{U'} \notin U$. It is easily seen that the family $g_\bullet := (g_{U'} \mid U' \in \mathscr{U})$ is a Cauchy net of G equivalent to $h_\bullet$. We may thus replace $h_\bullet$ by $g_\bullet$, and assume from start that $h_\lambda \in G \setminus U$ for every $\lambda \in \Lambda$. Next, since V is open in $G^\wedge$, and since we know already that i_G is continuous, there exists $\lambda \in \Lambda$ such that $i_G(h_\lambda) \in V$; by assumption, this implies that $h_\lambda \in U$, a contradiction.

(ii): Explicitly, $i_G(U)^c$ is the set of equivalence classes of all Cauchy nets $g_\bullet := (g_\lambda \mid \lambda \in \Lambda)$ such that $([g_\bullet] + V^\wedge) \cap i_G(U) \neq \varnothing$ for every $V \in \mathscr{U}$. Now, suppose that $[g_\bullet] \in U^\wedge$, so we may assume that $g_\lambda \in U$ for every $\lambda \in U$, and for any given $V \in \mathscr{U}$ pick $V' \in \mathscr{U}$ such that $V' + V' \subset V$; we may then find $\lambda(V') \in \Lambda$ such that $g_\lambda - g_{\lambda'} \in V'$ for every $\lambda, \lambda' \geq \lambda(V')$. Set $\Lambda' := \{\lambda \in \Lambda \mid \lambda \geq \lambda(V')\}$, $h := g_{\lambda(V')}$ and $k_\lambda := h - g_\lambda$ for every $\lambda \in \Lambda'$. Clearly $k_\bullet := (k_\lambda \mid \lambda \in \Lambda')$ is a Cauchy net, $h \in U$ and $[g_\bullet] + [k_\bullet] = i_G(h)$; lastly, we have $k_\lambda + V' \subset V' + V' \subset V$ for every $\lambda \in \Lambda'$, whence $[k_\bullet] \in V^\wedge$. ◊

Claim 8.2.12. $(G^\wedge, \mathscr{T}^\wedge)$ is complete and separated.

Proof of the claim. Let $(g_{\bullet,i} \mid i \in I)$ a family of Cauchy nets of G indexed by a partially ordered set I, such that $([g_{\bullet,i}] \mid i \in I)$ is a Cauchy net in $G^\wedge$. Hence, for every $i \in I$ we have a partially ordered set Λ_i such that $g_{\bullet,i} = (g_\lambda \mid \lambda \in \Lambda_i)$. For every $U \in \mathscr{U}$ we may find $i(U) \in I$ such that $[g_{\bullet,i(U)}] - [g_{\bullet,j}] \in U^\wedge$ for every $j \geq i(U)$. The latter means that there exist a Cauchy net $(h_k \mid k \in K)$ of G and $U' \in \mathscr{U}$ such that $h_k + U' \subset U$ for every $k \in K$, as well as $\lambda(U) \in \Lambda_{i(U)}$, $\mu(U,j) \in \Lambda_j$ and $k_0 \in K$ such that $g_{\mu,j} - g_{\lambda,i(U)} - h_k \in U'$ for every $k \geq k_0$, every $\mu \geq \mu(U,j)$ and every $\lambda \geq \lambda(U)$. Therefore $g_{\mu,j} \in g_{\lambda,i(U)} + h_k + U' \subset g_{\lambda,i(U)} + U$ for every such j, μ and λ. Lastly, we may also assume that $g_{\lambda,i(U)} - g_{\lambda(U),i(U)} \in U$ for every $\lambda \geq \lambda(U)$, whence $g_{\mu,j} \in g_{\lambda(U),i(U)} + U + U$ for every $j \geq i(U)$ and every $\mu \geq \mu(U,j)$. Hence, the system $l_\bullet := (l_U := g_{\lambda(U),i(U)} \mid U \in \mathscr{U})$ is a Cauchy net in G (for the partial ordering on $\mathscr{U}$ such that $U \leq U'$ if and only if $U' \subset U$, for every $U, U' \in \mathscr{U}$), and it is easily seen that $[l_\bullet]$ is a limit point of $([g_{\bullet,i}] \mid i \in I)$, which shows that $G^\wedge$ is complete.

Lastly, suppose that $g_\bullet := (g_\lambda \mid \lambda \in \Lambda)$ is a Cauchy net such that $[g_\bullet]$ lies in $U^\wedge$ for every $U \in \mathscr{U}$. This means that for every such U there exists a Cauchy net $h^U_\bullet := (h^U_{\lambda'} \mid \lambda' \in \Lambda_U)$ such that $h^U_{\lambda'} \in U$ for every $\lambda' \in \Lambda_U$ and such that $g_\bullet \sim h^U_\bullet$. The latter condition implies that there exist $\lambda(U) \in \Lambda$ and $\lambda'(U) \in \Lambda_U$ such that $g_\lambda - h^U_{\lambda'} \in U$ for every $\lambda \geq \lambda(U)$ and every $\lambda' \geq \lambda'(U)$. We conclude that $g_\lambda \in U + U$ for every $\lambda \geq \lambda(U)$, so g_λ is equivalent to $[0]$, which shows that $G^\wedge$ is separated. ◊

Next, consider a continuous group homomorphism $\varphi : G \to H$ to a complete and separated topological abelian group H. In view of remark 8.2.7(i,iii), it follows easily that φ factors through i_G and the group homomorphism $\varphi^\wedge : G^\wedge \to H$ that assigns to every equivalence class $[g_\bullet]$ the unique limit point of the Cauchy net $\varphi(g_\bullet)$ in H. In view of remark 8.2.2(vi), we need to check that $\varphi^\wedge$ is continuous at the point 0. Thus, let $V \subset H$ be any open neighborhood of the neutral point 0_H of H, and let $g_\bullet := (g_\lambda \mid \lambda \in \Lambda)$ be a Cauchy net in G with $h := \varphi^\wedge([g_\bullet]) \in V$; we may find an open neighborhood V' of 0_H in H such that $V' = -V'$ and $h+V'+V'+V' \subset V$, and set $U := \varphi^{-1}V'$, so that $U = -U$. Now, let $g'_\bullet := (g'_{\lambda'} \mid \lambda' \in \Lambda')$ be any Cauchy net in G such that $[g'_\bullet] \in [g_\bullet] + U^\wedge$. The condition means that there exist a Cauchy net $(h_i \mid i \in I)$ in G and an open neighborhood U' of 0 in G such that $h_i + U' \subset U$ for every $i \in I$, and moreover there exist $\lambda_0 \in \Lambda$, $\lambda'_0 \in \Lambda'$, and $i_0 \in I$ such that $g'_{\lambda'} - g_\lambda - h_i \in U'$ for every $\lambda \geq \lambda_0$, every $\lambda' \geq \lambda'_0$ and every $i \geq i_0$. Thus, $g'_{\lambda'} \in g_\lambda + U$, and consequently $\varphi(g'_{\lambda'}) \in \varphi(g_\lambda) + V'$ for every $\lambda \geq \lambda_0$ and $\lambda' \geq \lambda'_0$. Set $h' := \varphi^\wedge([g'_\bullet])$; we may also assume that $\varphi(g_\lambda) - h, h' - \varphi(g'_{\lambda'}) \in V'$ for every $\lambda \geq \lambda_0$ and every $\lambda' \geq \lambda'_0$. We deduce that

$$h' \in \varphi(g'_{\lambda'}) + V' \subset \varphi(g_\lambda) + V' + V' \subset h + V' + V' + V' \subset V$$

i.e. $\varphi^\wedge([g_\bullet] + U^\wedge) \subset V$, whence the assertion. Lastly, since the image of i_G is dense in $G^\wedge$, the uniqueness of $\varphi^\wedge$ is clear.

(iii): The existence of the continuous group homomorphism $\varphi^\wedge : G^\wedge \to G'$ has just been established. Next, since φ has dense image, every $g' \in G'$ is the unique limit point of a Cauchy net of G' of the form $(\varphi(g_\lambda) \mid \lambda \in \Lambda)$ for a family $g_\bullet := (g_\lambda \mid \lambda \in \Lambda)$ of elements of G, and since $\mathscr{T}$ agrees with the topology induced by $\mathscr{T}'$, it is easily seen that $g_\bullet$ is a Cauchy net in G. It then follows that $\varphi^\wedge([g_\bullet]) = g'$, which shows that $\varphi^\wedge$ is surjective. Let $g_\bullet := (g_\lambda \mid \lambda \in \Lambda)$ be a Cauchy net in G whose class $[g_\bullet]$ lies in the kernel of $\varphi^\wedge$; this means that $(\varphi(g_\lambda) \mid \lambda \in \Lambda)$ converges to 0 in G', and since $\mathscr{T}$ is induced by $\mathscr{T}'$, it follows easily that $g_\bullet$ converges to 0 in G, which shows that $\varphi^\wedge$ is also injective. It remains therefore only to check that the topology on $G^\wedge$ induced by $\mathscr{T}'$ via $\varphi^\wedge$ is finer than $\mathscr{T}^\wedge$. To this aim, consider any $U \in \mathscr{U}$; by assumption, there exists an open neighborhood W of 0 in G' such that $\varphi^{-1}W \subset U$. Set $V := \varphi^{\wedge\,-1}W$; then $i_G^{-1}V \subset U$, and therefore $V \subset U^\wedge$, by virtue of claim 8.2.11(i). Since $\mathscr{U}^\wedge$ is a fundamental system of open neighborhoods of 0 in $G^\wedge$, the assertion follows. □

Proposition 8.2.13. *Let $(G, \mathscr{T}_G)$ be a topological abelian group, $H \subset G$ a subgroup, and endow H (resp. G/H) with the topology $\mathscr{T}_H$ (resp. $\mathscr{T}_{G/H}$) induced by $\mathscr{T}_G$ via the inclusion map $j : H \to G$ (resp. via the projection $\pi : G \to G/H$). Let also $i_G : G \to G^\wedge$ be the completion map. Then we have :*

(i) *The resulting sequence of separated completions is exact :*

$$0 \to H^\wedge \xrightarrow{j^\wedge} G^\wedge \xrightarrow{\pi^\wedge} (G/H)^\wedge.$$

(ii) *The topology $\mathscr{T}_H^\wedge$ of $H^\wedge$ agrees with the one induced by the topology $\mathscr{T}_G^\wedge$ of $G^\wedge$ via $j^\wedge$.*
(iii) *The image of $j^\wedge$ is the topological closure in $G^\wedge$ of $i_G(H)$.*
(iv) *Endow $L := G^\wedge/H^\wedge$ with the quotient topology $\mathscr{T}_L$ induced by $G^\wedge$ via the projection $G^\wedge \to L$. Also, let $k : L \to (G/H)^\wedge$ be the injective map deduced from $\pi^\wedge$. Then $\mathscr{T}_L$ agrees with the subspace topology induced by $(G/H)^\wedge$ via k.*
(v) *Suppose that G admits a countable fundamental system of open neighborhoods of 0. Then $\pi^\wedge$ is surjective.*

Proof. (i): Let $g_\bullet := (g_\lambda \mid \lambda \in \Lambda)$ be a Cauchy net of H; a direct inspection of the definitions shows that 0 is a limit point of $g_\bullet$ (relative to the topology $\mathscr{T}_H$) if and only if it is a limit point of $j(g_\bullet)$, relative to the topology $\mathscr{T}_G$, whence the injectivity of $j^\wedge$. Next, by functoriality we have $\pi^\wedge \circ j^\wedge = (\pi \circ j)^\wedge = 0 : H^\wedge \to (G/H)^\wedge$. To conclude, suppose then that $g_\bullet := (g_\lambda \mid \lambda \in \Lambda)$ is a Cauchy net of G such that $\pi(g_\bullet) := (\pi(g_\lambda) \mid \lambda \in \Lambda)$ admits 0 as limit point, and pick a fundamental system $\mathscr{U}$ of open neighborhoods of 0 in G; the assertion means that for every $U \in \mathscr{U}$ there exists $\lambda(U) \in \Lambda$ such that $\pi(g_\lambda) \in \pi(U)$ for every $\lambda \geq \lambda(U)$, *i.e.* $g_\lambda \in H + U := \{h + u \mid h \in H,\ u \in U\}$ for every such λ. Thus, for every $U \in \mathscr{U}$ pick $h_U \in H$ such that $g_{\lambda(U)} - h_U \in U$; it follows that the families $(g_{\lambda(U)} \mid U \in \mathscr{U})$ and $h_\bullet := (h_U \mid U \in \mathscr{U})$ are also Cauchy nets in G, and

$$g_\bullet \sim (g_{\lambda(U)} \mid U \in \mathscr{U}) \sim h_\bullet$$

which shows that the class of $g_\bullet$ in $G^\wedge$ lies in the image of $j^\wedge$, as stated.

(ii): Let $h_\bullet := (h_\lambda \mid \lambda \in \Lambda)$ be a Cauchy net of H whose class $[h_\bullet] \in H^\wedge$ lies in $j^{\wedge\,-1}(U^\wedge)$, for a given $U \in \mathscr{U}$ (notation of the proof of theorem 8.2.8); this means that there exist a Cauchy net $g_\bullet := (g_{\lambda'} \mid \lambda' \in \Lambda')$ of G, and $U' \in \mathscr{U}$, such that $g_{\lambda'} + U' \subset U$ for every $\lambda' \in \Lambda'$, and such that $h_\bullet \sim g_\bullet$. Pick any $U'' \in \mathscr{U}$ such that $U'' + U'' \subset U'$; after replacing Λ and Λ' by some cofinal subsets, we may assume that $h_\lambda - g_{\lambda'} \in U''$ for every $\lambda \in \Lambda$ and every $\lambda' \in \Lambda'$. Whence, $h_\lambda + U'' \in g_{\lambda'} + U'' + U'' \subset U$ for every $\lambda \in \Lambda$, which shows that $H^\wedge \cap U^\wedge \subset (H \cap U)^\wedge$. The converse inclusion is clear, and the assertion follows.

(iii): On the one hand, the image of $j^\wedge$ is a complete subgroup of $G^\wedge$, so it is topologically closed (remark 8.2.7(iv)); on the other hand, $i_G(H)$ is dense in in $H^\wedge$, whence the contention.

(iv): Denote by $(\overline{G}, \overline{\mathscr{T}}_G)$ the maximal separated quotient of G (*i.e.* the quotient $G/\{0\}^c$, where $\{0\}^c$ denotes the topological closure of $\{0\}$ in G), endowed with the quotient topology induced by G; let also $\overline{H}$ be the topological closure of the image of H in $\overline{G}$, and endow $\overline{H}$ with the topology $\overline{\mathscr{T}}_H$ induced by $\overline{\mathscr{T}}_G$ via the inclusion map $\overline{H} \to \overline{G}$. Endow as well $\overline{G}/\overline{H}$ with the quotient topology $\overline{\mathscr{T}}_{G/H}$ induced by $\overline{G}$; we get a sequence of separated abelian topological groups

$$0 \to \overline{H} \to \overline{G} \to \overline{G}/\overline{H} \to 0$$

and after taking separated completions, a commutative ladder of continuous group homomorphisms :

$$\begin{array}{ccccccc}
0 & \longrightarrow & H^\wedge & \longrightarrow & G^\wedge & \longrightarrow & (G/H)^\wedge \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \overline{H}^\wedge & \longrightarrow & \overline{G}^\wedge & \longrightarrow & (\overline{G}/\overline{H})^\wedge.
\end{array}$$

On the other hand, it is easily seen that $\mathscr{T}_H$ (resp. $\mathscr{T}_{G/H}$) agrees with the topology induced by $\overline{\mathscr{T}}_H$ (resp. by $\overline{\mathscr{T}}_{G/H}$) via the natural map $H \to \overline{H}$ (resp. $G/H \to \overline{G}/\overline{H}$) : details left to the reader. It follows immediately that the central and right-most vertical arrows are isomorphisms of topological groups. By the 5-lemma, we deduce that the left-most vertical arrow is a group isomorphism; but then it is also a homeomorphism, by virtue of (ii). Summing up, we may replace G by $\overline{G}$ and H by $\overline{H}$, and assume from start that G is separated and H is topologically closed in G; especially, the completion map $G \to G^\wedge$ is also injective. We now consider the commutative diagram of continuous group homomorphisms

$$\begin{array}{ccccc} G/H & \xrightarrow{f} & L & \xrightarrow{i_L} & L^\wedge \\ & {\scriptstyle i_{G/H}} \searrow & \downarrow {\scriptstyle k} & \swarrow {\scriptstyle g} & \\ & & (G/H)^\wedge & & \end{array}$$

where $i_{G/H}$ and i_L are the completion maps. By (iii) and lemma 8.2.3(ii), the map f is injective, and $\mathscr{T}_{G/H}$ agrees with the topology induced by $\mathscr{T}_L$ via f. By theorem 8.2.8(ii), the topology $\mathscr{T}_L$ in turn is induced via i_L by the topology $\mathscr{T}_L^\wedge$ of its completion $L^\wedge$. Thus, $\mathscr{T}_{G/H}$ is induced by $\mathscr{T}_L^\wedge$ via $i_L \circ f$; lastly, $i_L \circ f$ has dense image. Taking into account theorem 8.2.8(iii), we conclude that g is an isomorphism of topological groups, whence the assertion.

(v): Let $\overline{g}_\bullet := (\overline{g}_\lambda \mid \lambda \in \Lambda)$ be a Cauchy net in G/H, and pick a countable fundamental system $\mathscr{U} := (U_n \mid n \in \mathbb{N})$ of open neighborhoods of 0 in G. A simple induction shows that we may assume :

$$U_{n+1} + U_{n+1} \subset U_n \qquad \text{for every } n \in \mathbb{N}. \tag{8.2.14}$$

Now, for every $n \in \mathbb{N}$ there exists $\lambda(n) \in \Lambda$ such that $\overline{g}_\lambda - \overline{g}_\mu \in \pi(U_n)$ for every $\lambda, \mu \geq \lambda(n)$. It follows that the family $(\overline{g}_{\lambda(n)} \mid n \in \mathbb{N})$ is a Cauchy net in G/H equivalent to $g_\bullet$ (for the standard ordering on $\mathbb{N}$); thus, we may assume from start that the class $[\overline{g}_\bullet]$ in $(G/H)^\wedge$ is represented by a Cauchy *sequence* $(\overline{g}_n \mid n \in \mathbb{N})$ of G/H, such that $\overline{g}_{n+1} - \overline{g}_n \in \pi(U_n)$ for every $n \in \mathbb{N}$. Pick an arbitrary sequence $(g_n \mid n \in \mathbb{N})$ in G with $\pi(g_n) = \overline{g}_n$ for every $n \in \mathbb{N}$; by construction, for every $n \in \mathbb{N}$ there exists $h_n \in H$ such that $g_{n+1} + h_n - g_n \in U_n$. Set $g'_n := g_n + \sum_{i=0}^{n-1} h_i$ for every $n \in \mathbb{N}$; we then see that $g'_{n+1} - g'_n \in U_n$ for every $n \in \mathbb{N}$. Lastly, using (8.2.14), a simple induction shows more generally that $g'_i - g'_j \in U_{n-1}$ whenever $i \geq j > 0$. Therefore $g'_\bullet := (g'_n \mid n \in \mathbb{N})$ is a Cauchy sequence in G, and clearly $\pi^\wedge([g'_\bullet]) = [g_\bullet]$, whence the contention. □

Corollary 8.2.15. *Suppose that the topological group G is the product of a (small) family $(G_i \mid i \in I)$ of topological abelian groups, and for every $i \in I$ denote by $G_i^\wedge$ the separated completion of G_i. Then the natural map $j : G \to \prod_{i\in I} G_i^\wedge$ factors uniquely through the completion map $G \to G^\wedge$ and an isomorphism of topological groups :*

$$G^\wedge \xrightarrow{\sim} \prod_{i\in I} G_i^\wedge.$$

Proof. It is easily seen that the product $P := \prod_{i\in I} G_i^\wedge$ is complete and separated; also, j has dense image, and the topology of G is induced by that of P via j. Then the assertion follows from theorem 8.2.8(iii). □

Corollary 8.2.16. *Let $G_\bullet := (G_i \mid i \in \mathrm{Ob}(I))$ be any system of topological abelian groups, indexed by any small category I. We have :*

(i) *If G_i is complete and separated for every $i \in \mathrm{Ob}(I)$, then $\lim_I G_\bullet$ is a complete and separated topological abelian group.*

(ii) *Suppose that I is cofiltered and countable (see example* 1.5.26*), and that the system $G_\bullet$ satisfies the following* Mittag-Leffler condition *:*

(ML) *For every* $i \in \mathrm{Ob}(I)$ *there exists a morphism* $\varphi : j \to i$ *in* I *such that for every morphism* $\psi : k \to j$ *in* I *we have*

$$\mathrm{Im}(G(\varphi \circ \psi) : G_k \to G_i) = \mathrm{Im}(G(\varphi) : G_j \to Gi).$$

Then the natural map

$$(\lim_I G_\bullet)^\wedge \to \lim_I G_\bullet^\wedge$$

is an isomorphism of topological groups.

Proof. (i): Indeed, the limit L of the system $G_\bullet$ is a closed subgroup of the product $P := \prod_{i \in \mathrm{Ob}(I)} G_i$, and the topology of L is induced by that of P via this closed inclusion map; then the assertion follows from corollary 8.2.15 and remark 8.2.7(iv).

(ii): In light of theorem 8.2.8(ii), it is easily seen that the topology of L is induced by that of $L' := \lim_I G_\bullet^\wedge$, via the natural map $\beta : L \to L'$. Taking into account (i) and theorem 8.2.8(iii), we are then reduced to checking that β has dense image. Moreover, by virtue of example 1.5.26, we may assume that I is the totally ordered set $\mathbb{N}^o$. Now, for every $i, j \in \mathbb{N}$ with $j \geq i$ let $\varphi_{j,i} : G_j \to G_i$ be the transition map; for every $i \in \mathbb{N}$ we set

$$H_i := \bigcap_{j \geq i} \mathrm{Im}\, \varphi_{j,i}$$

and we endow H_i with the topology induced by G_i. Condition (ML) easily implies that $\varphi_{j,i}$ restricts to a surjective continuous group homomorphism $H_j \to H_i$ for every $i, j \in \mathbb{N}$ with $j \geq i$, and the natural morphisms

$$\lim_{\mathbb{N}^o} H_\bullet \to L \qquad \lim_{\mathbb{N}^o} H_\bullet^\wedge \to L'$$

are isomorphisms of topological groups. We may then replace $G_\bullet$ by $H_\bullet$, and assume from start that the map $\varphi_{j,i}$ is a surjection, for every $j \geq i$. In this case we come down to checking that the inclusion map $H_i \to H_i^\wedge$ is dense for every $i \in \mathrm{Ob}(I)$, which we know by theorem 8.2.8(ii). □

Corollary 8.2.17. *Let* G *be any topological abelian group,* $H \subset G$ *an open subgroup,* $H^\wedge$ *and* $G^\wedge$ *the separated completions of* H *and* G*, and* $i_G : G \to G^\wedge$ *the completion map. Then :*

(i) $H^\wedge$ *is an open subgroup of* $G^\wedge$*, and the natural map* $G/H \to G^\wedge/H^\wedge$ *is a group isomorphism.*

(ii) $H = i_G^{-1} H^\wedge$.

(iii) *Suppose that* G *admits a fundamental system* $(H_\lambda \mid \lambda \in \Lambda)$ *of open neighborhoods of* 0*, consisting of open subgroups. Then* $(H_\lambda^\wedge \mid \lambda \in \Lambda)$ *is a fundamental system of open neighboroods of* 0 *for* $G^\wedge$.

Proof. Under these assumptions, G induces the discrete topology on the quotient G/H, hence the latter is complete and separated; taking into account proposition 8.2.13(i,ii), we get an exact sequence

$$0 \to H^\wedge \to G^\wedge \to G/H$$

from which both (i) and (ii) follow easily. Assertion (iii) follows by inspecting the proof of theorem 8.2.8(i). □

Proposition 8.2.18. *Let* $\Lambda \neq \varnothing$ *be a finite set,* $((H_\lambda, +, 0_\lambda) \mid \lambda \in \Lambda)$ *a family of topological abelian groups,* $\varphi : \prod_{\lambda \in \Lambda} H_\lambda \to G$ *a continuous* $\mathbb{Z}$*-multilinear map into a complete and separated topological abelian group* $(G, +, 0_G)$*, and for every* $\lambda \in \Lambda$ *denote by* $i_\lambda : H_\lambda \to H_\lambda^\wedge$ *the completion map. Then there exists a unique continuous* $\mathbb{Z}$*-multilinear map*

$$\varphi^\wedge : \textstyle\prod_{\lambda \in \Lambda} H_\lambda^\wedge \to G \qquad \text{such that} \qquad \varphi^\wedge \circ (\prod_{\lambda \in \Lambda} i_\lambda) = \varphi.$$

Proof. For every subset $\Delta \subsetneq \Lambda$ and every $h_\bullet := (h_\lambda \mid \lambda \in \Delta) \in \prod_{\lambda\in\Delta} H_\lambda$, let

$$j_{\Delta,h_\bullet} : \prod_{\lambda\in\Lambda\setminus\Delta} H_\lambda \to \prod_{\lambda\in\Lambda} H_\lambda$$

be the map such that $j_{\Delta,h_\bullet}(k_\bullet) := (l_\lambda \mid \lambda \in \Lambda)$ for every $k_\bullet := (k_\lambda \mid \lambda \in \Lambda\setminus\Delta) \in \prod_{\lambda\in\Lambda\setminus\Delta} H_\lambda$, where $l_\lambda := h_\lambda$ for every $\lambda \in \Delta$, and $l_\lambda := k_\lambda$ for every $\lambda \in \Lambda\setminus\Delta$. Let us first show :

Claim 8.2.19. Let $((H_\lambda, +, 0_\lambda) \mid \lambda \in \Lambda)$ and G be as in the proposition, and $\psi : \prod_{\lambda\in\Lambda} H_\lambda \to G$ a $\mathbb{Z}$-multilinear map. Then the following conditions are equivalent :

(a) ψ is continuous.
(b) $\psi \circ j_{\Delta,h_\bullet}$ is continuous at $(0_\lambda \mid \lambda \in \Lambda\setminus\Delta)$ for every $\Delta \subsetneq \Lambda$ and every $h_\bullet \in \prod_{\lambda\in\Delta} H_\lambda$.

Proof of the claim. Clearly every such map $j_{\Delta,h_\bullet}$ is continuous, hence (a)⇒(b). Conversely, let $h_\bullet \in \prod_{\lambda\in\Lambda} H_\lambda$ be any element, and for every subset $\Delta \subset \Lambda$ set as well $j_\Delta := j_{\Delta,k_\bullet}$, where $k_\bullet := (h_\lambda \mid \lambda \in \Delta)$; we have

$$\psi(h_\bullet + u_\bullet) = \psi(h_\bullet) + \sum_{\Delta\subsetneq\Lambda} \psi \circ j_\Delta(u_\lambda \mid \lambda \in \Lambda\setminus\Delta) \qquad \text{for every } u_\bullet := (u_\lambda \mid \lambda \in \Lambda) \in \prod_{\lambda\in\Lambda} H_\lambda$$

which easily implies that (b)⇒(a) : the details are left to the reader. ◊

Now, the uniqueness of $\varphi^\wedge$ is clear, since the map $\prod_{\lambda_i n\Lambda} i_\lambda$ has dense image. For the existence of $\varphi^\wedge$, we are easily reduced to checking that for every $\mu \in \Lambda$ the map φ extends to a continuous $\mathbb{Z}$-multilinear map $\varphi_\mu : H^\wedge_\mu \times \prod_{\lambda\in\Lambda\setminus\{\mu\}} H_\lambda \to G$, so that $\varphi_\mu \circ (i_\mu \times \prod_{\lambda\in\Lambda\setminus\{\mu\}} \mathbf{1}_{H_\lambda}) = \varphi$.

Thus, let $h_\bullet \in \prod_{\lambda\in\Lambda\setminus\{\mu\}} H_\lambda$ be any element; the composition $\varphi_{h_\bullet} := \varphi \circ j_{\Lambda\setminus\{\mu\},h_\bullet} : H_\mu \to G$ is a continuous group homomorphism, which by adjunction factors through i_μ and a unique continuous group homomorphism $\varphi^\wedge_{h_\bullet} : H^\wedge_\mu \to G$, and we set

$$\varphi_\mu(x, h_\bullet) := \varphi^\wedge_{h_\bullet}(x) \qquad \text{for every } x \in H^\wedge_\mu \text{ and every } h_\bullet \in \prod_{\lambda\in\Lambda\setminus\{\mu\}} H_\lambda.$$

To check that φ_μ is $\mathbb{Z}$-multilinear, say that $h_\bullet \in \prod_{\lambda\in\Lambda\setminus\{\mu\}} H_\lambda$, $\nu \in \Lambda\setminus\{\mu\}$, and $k \in H_\nu$; set $h'_\bullet := (h'_\lambda \mid \lambda \in \Lambda\setminus\{\mu\})$ and $h''_\bullet := (h''_\lambda \mid \lambda \in \Lambda\setminus\{\mu\})$ with $h''_\lambda = h'_\lambda := h_\lambda$ if $\lambda \neq \nu$, and $h'_\nu := k$, $h''_\nu := h_\nu + h'_\nu$. Since φ is $\mathbb{Z}$-multilinear, $\varphi_{h''_\bullet} = \varphi_{h_\bullet} + \varphi_{h'_\bullet}$, whence $\varphi^\wedge_{h''_\bullet} = \varphi^\wedge_{h_\bullet} + \varphi^\wedge_{h'_\bullet}$, so that $\varphi_\mu(x, h''_\bullet) = \varphi_\mu(x, h_\bullet) + \varphi_\mu(x, h'_\bullet)$ for every $x \in H^\wedge_\mu$, whence the assertion.

Next, for every $x \in H^\wedge_\mu$, let also $\varphi_{\mu,x} : \prod_{\lambda\in\Lambda\setminus\{\mu\}} H_\lambda \to G$ be the map such that $\varphi_{\mu,x}(h_\bullet) := \varphi_{h_\bullet}(x)$ for every $h_\bullet \in \prod_{\lambda\in\Lambda\setminus\{\mu\}} H_\lambda$. To show the continuity of φ_μ, by claim 8.2.19 and a simple induction on the cardinality c of Λ, it suffices to check that :

(a) φ_μ is continuous at $(0_\lambda \mid \lambda \in \Lambda) \in H^\wedge_\mu \times \prod_{\lambda\in\Lambda\setminus\{\mu\}} H_\lambda$
(b) if $c \geq 2$, then for every $x \in H^\wedge_\mu$ the map $\varphi_{\mu,x}$ is continuous at $(0_\lambda \mid \lambda \in \Lambda\setminus\{\mu\})$.

To this aim, for every open neighborhood U of 0_μ in H_μ, define the open neighborhood $U^\wedge$ of 0_μ in $H^\wedge_\mu$, as in the proof of theorem 8.2.8, so that $U^\wedge$ lies in the topological closure of $i_\mu(U)$ in $H^\wedge_\mu$ (claim 8.2.11(ii)). Let $V \subset G$ be any open neighborhood of 0_G, and $W \subset G$ another open neighborhood of 0_G with $W + W + W \subset V$, and denote by W^c (resp. $(W + W)^c$) the topological closure of W (resp. of $W + W$) in G; since φ is continuous, there exist a family $U_\bullet := (U_\lambda \mid \lambda \in \Lambda)$ such that U_λ is an open neighborhood of 0_λ in H_λ for every $\lambda \in \Lambda$, and $\varphi(\prod_{\lambda\in\Lambda} U_\lambda) \subset W$. It follows that $\varphi_\mu(U^\wedge_\mu \times \prod_{\lambda\in\Lambda\setminus\{\mu\}} U_\lambda) \subset W^c \subset W + W \subset V$, which shows (a). Lastly, say that x is the limit of a Cauchy net $(x_i \mid i \in I)$ in H_μ, and keep $U_\bullet$ as in the foregoing; hence, there exists $i \in I$ such that $x_j - x_k \in U_\mu$ for every $j, k \in I$ with $j, k \geq i$. Since φ is continuous, there exist moreover for every $\lambda \in \Lambda\setminus\{\mu\}$ an open neighborhood $U'_\lambda \subset U_\lambda$ of 0_λ such that $\varphi(\{x_i\} \times \prod_{\lambda\in\Lambda\setminus\{\mu\}} U'_\lambda) \subset W$. Thus, $\varphi(\{x_j\} \times \prod_{\lambda\in\Lambda\setminus\{\mu\}} U'_\lambda) \subset W + W$ for every $j \geq i$, so $\varphi_\mu(\{x\} \times \prod_{\lambda\in\Lambda\setminus\{\mu\}} U'_\lambda) \subset (W + W)^c \subset W + W + W \subset V$, whence (b). □

8.3. **Topological rings and topological modules.** If T is any topological space and $S \subset T$ any subset, we shall denote by S^c the topological closure of S in T.

Definition 8.3.1. (i) A *topological ring* is a pair $(A, \mathscr{T}_A)$ where A is a (commutative, unital) ring and $\mathscr{T}_A$ a topology on the set underlying A, such that :

- the multiplication law of A is a continuous map $(A, \mathscr{T}_A) \times (A, \mathscr{T}_A) \to (A, \mathscr{T}_A)$
- the pair $((A, +, 0), \mathscr{T}_A)$ is a topological group.

(ii) Let $(A, \mathscr{T}_A)$ be a topological ring, M an A-module, $\mathscr{T}_M$ a topology on the set underlying M. We say that $(M, \mathscr{T}_M)$ is a *topological* $(A, \mathscr{T}_A)$*-module* (briefly : a *topological* A*-module*) if

- the scalar multiplication is a continuous map $(A, \mathscr{T}_A) \times (M, \mathscr{T}_M) \to (M, \mathscr{T}_M)$
- the pair $((M, +, 0), \mathscr{T}_M)$ is a topological abelian group.

(iii) Let $(A, \mathscr{T}_A), (B, \mathscr{T}_B)$ be two topological rings. A *morphism of topological rings* is a ring homomorphism $A \to B$ that is continuous for the topologies $\mathscr{T}_A$ and $\mathscr{T}_B$. Such a morphism $(A, \mathscr{T}_A) \to (B, \mathscr{T}_B)$ is also called a *topological* $(A, \mathscr{T}_A)$*-algebra*. The topological A-algebras form a category

$$A\text{-}\mathsf{TopAlg}$$

whose morphisms are the continuous A-algebra homomorphisms. Especially, the ring $\mathbb{Z}$ with its discrete topology is a topological ring, and $\mathbb{Z}$-TopAlg is the category of topological rings and continuous ring homomorphisms. Likewise, the topological $(A, \mathscr{T}_A)$-modules form a category

$$A\text{-}\mathsf{TopMod}$$

whose morphisms are the continuous A-linear maps.

(iv) Let $(A, \mathscr{T}_A)$ be a topological ring, $B \to A$ a ring homomorphism, $(M, \mathscr{T}_M)$ a topological A-module. We say that the topology $\mathscr{T}_M$ is B*-linear* if there exists a fundamental system of open neighborhoods of $0 \in M$ for the topology $\mathscr{T}_M$, consisting of B-submodules of M.

(v) Especially, in the situation of (iv) we say that the topology $\mathscr{T}_A$ is B*-linear*, if the same holds for the topological A-module $(A, \mathscr{T}_A)$. We say also simply that $\mathscr{T}_A$ is *linear* if it is A-linear. We denote by

$$A\text{-}\mathsf{TopAlg}_{B\text{-lin}}$$

the full subcategory of A-TopAlg whose objects are the B-linear topological A-algebras.

Remark 8.3.2. (i) Let A be a ring, and $\mathscr{T}_A$ a topology on A such that $(A, \mathscr{T}_A)$ is a topological group, for the addition law of A. Let $\mathscr{U}$ be a fundamental system of open neighborhoods of 0 in A; it is easily seen that $(A, \mathscr{T}_A)$ is a topological ring if and only if the system $\mathscr{U}$ fulfills the following conditions :

(a) For every $U \in \mathscr{U}$ there exists $U' \in \mathscr{U}$ such that $U' \cdot U' := \{a \cdot b \mid a, b \in U'\} \subset U$.
(b) For every $a \in A$ and every $U \in \mathscr{U}$ there exists $U' \in \mathscr{U}$ such that $a \cdot U' \subset U$.

(ii) Especially, if $I_\bullet := (I_\lambda \mid \lambda \in \Lambda)$ is any system of ideals of A that is cofiltered by inclusion, then there is a unique group topology $\mathscr{T}_{I_\bullet}$ on A for which $I_\bullet$ is a fundamental system of open neighborhoods of 0, and $(A, \mathscr{T}_{I_\bullet})$ is a topological ring. We call $\mathscr{T}_{I_\bullet}$ the $I_\bullet$*-adic topology* on A. Likewise, if M is any A-module, the $I_\bullet$*-adic topology* on M is the unique group topology $\mathscr{T}_M$ for which $(I_\lambda M \mid \lambda \in \Lambda)$ is a fundamental system of open neighborhoods of 0 in M, and $(M, \mathscr{T}_M)$ is a topological $(A, \mathscr{T}_{I_\bullet})$-module.

Remark 8.3.3. Let $(A, \mathscr{T}_A)$ be any topological ring and $(M, \mathscr{T}_M)$ any topological A-module.

(i) Let $(A^\wedge, \mathscr{T}_A^\wedge)$ be the separated completion of the topological group underlying A; from proposition 8.2.18 we see that the multiplication law of A induces a continuous $\mathbb{Z}$-bilinear map $A^\wedge \times A^\wedge \to A^\wedge$, which furnishes $A^\wedge$ with a natural structure of topological ring, such that the completion map $A \to A^\wedge$ is a morphism of topological rings. Likewise, the separated completion $(M^\wedge, \mathscr{T}_M^\wedge)$ of $(M, \mathscr{T}_M)$ is naturally a topological $A^\wedge$-module.

(ii) Suppose that $\mathscr{T}_A$ is the linear topology defined by a cofiltered system $I_\bullet := (I_\lambda \mid \lambda \in \Lambda)$ of ideals of A, and let $I \subset A$ be any open ideal. Also, suppose that $\mathscr{T}_M$ is the $I_\bullet$-adic topology on M. Then it follows easily from (i) and corollary 8.2.17(iii), that $\mathscr{T}_M^\wedge$ is the $A^\wedge$-linear topology defined by the cofiltered system $((I_\lambda M)^\wedge \mid \lambda \in \Lambda)$ of submodules of $M^\wedge$. Moreover, $(IM)^\wedge$ is an open $A^\wedge$-submodule in $M^\wedge$, the natural map $M/IM \to M^\wedge/(IM)^\wedge$ is an isomorphism, and the completion map $M \to M^\wedge$ identifies $(IM)^\wedge$ with the topological closure of the image of IM in $M^\wedge$ (corollary 8.2.17(i)).

(iii) In the situation of (ii), suppose furthermore that for every $\lambda \in \Lambda$ there exists $\mu \in \Lambda$ with $I_\mu \subset I_\lambda^2$ (where I_λ^2 is the usual square of the ideal I_λ). Then it is easily seen that the $I_\bullet$-adic topology on IM agrees with the topology induced by $\mathscr{T}_M$ on IM.

(iv) In the situation of (iii), suppose additionally that Λ is countable and I is finitely generated. Then the image of $(IM)^\wedge$ in $M^\wedge$ equals $IM^\wedge$: indeed, by assumption we may find $k \in \mathbb{N}$ and a surjective A-linear map $\varphi : M^{\oplus k} \to IM$, and it is easily seen that φ is an open map; especially, the quotient topology induced on IM via the map φ agrees with $\mathscr{T}_{IM}$. By proposition 8.2.13(v), the separated completion of φ is a surjective map

$$\varphi^\wedge : (M^{\oplus k})^\wedge \to (IM)^\wedge$$

and on the other hand we have a natural identification $(M^{\oplus k})^\wedge \xrightarrow{\sim} M^{\wedge\oplus k}$, whence the contention (details left to the reader).

Example 8.3.4. (i) Let $A \neq 0$ be a ring, Σ a set, and $\mathscr{F}$ a *filter* of Σ, *i.e.* a family of subsets of Σ fulfilling the following conditions : (a) for every $F, F' \in \mathscr{F}$ we have $F \cap F' \in \mathscr{F}$; (b) for every $F \subset F' \subset \Sigma$ with $F \in \mathscr{F}$, we have $F' \in \mathscr{F}$. Let us endow A with the discrete topology, and $M := A^{(\Sigma)}$ with the linear topology $\mathscr{T}_M$ defined by the system of submodules $(A^{(F)} \mid F \in \mathscr{F})$. Notice that $\mathscr{T}_M$ is separated if and only if $\bigcap_{F\in\mathscr{F}} F = \varnothing$. Notice as well the natural isomorphism of A-modules

$$M/A^{(F)} \xrightarrow{\sim} A^{(\Sigma\setminus F)} \qquad \text{for every subset } F \subset \Sigma.$$

For $F \subset F' \subset \Sigma$, these isomorphisms identify the natural projection $M/A^{(F)} \to M/A^{(F')}$ with the A-linear map $\pi_{F,F'} : A^{(\Sigma\setminus F)} \to A^{(\Sigma\setminus F')} : (a_\sigma \mid \sigma \in \Sigma \setminus F) \mapsto (a_\sigma \mid \sigma \in \Sigma \setminus F')$. The completion of M is then isomorphic to the inverse limit of the cofiltered system $((A^{\Sigma\setminus F} \mid F \in \mathscr{F}), \pi_{\bullet\bullet})$. If $\mathscr{T}_M$ is separated, a direct inspection shows that this limit is represented by the submodule L of A^Σ consisting of all sequences $(a_\sigma \mid \sigma \in \Sigma)$ with support $\Lambda := \{\sigma \in \Sigma \mid a_\sigma \neq 0\}$ satisfying the following condition : $\Lambda \cap (\Sigma \setminus F)$ is a finite set for every $F \in \mathscr{F}$.

(ii) In the situation of (i), suppose moreover that for every finite or countable subset $\Lambda \subset \Sigma$ there exists $F \in \mathscr{F}$ with $\Lambda \cap F = \varnothing$. Then M is complete and separated. Indeed, the condition implies that $\bigcap_{F\in\mathscr{F}} F = \varnothing$, so M is separated. To check the completeness of M, say that $a_\bullet := (a_\sigma \mid \sigma \in \Sigma) \in L$; if the support Λ of $a_\bullet$ is not finite, we may find a countable subset $\Lambda' \subset \Lambda$, and by assumption there exists $F \in \mathscr{F}$ such that $\Lambda' \cap F = \varnothing$. But then $\Lambda' \subset \Lambda \cap (\Sigma \setminus \Lambda')$, so $\Lambda \cap (\Sigma \setminus \Lambda')$ is not a finite set, a contradiction. Thus $a_\bullet \in M$, whence the assertion.

Example 8.3.5. Let A be any topological ring whose topology is linear, complete and separated, $I \subset A$ a closed ideal, and endow A/I with the quotient topology. In case A admits a fundamental system of open neighborhoods of 0 consisting of a *countable* family of ideals, then A/I is still complete and separated (proposition 8.2.13(v)). But if this latter condition is omitted, A/I will still be separated, but it will not necessarily be complete. For a counterexample, take $A := K[T_\lambda \mid \lambda \in \Lambda]$, where K is a non-zero ring, and Λ is the (small) set of all countable ordinal numbers, and let $I \subset A$ be the ideal generated by $(T_\lambda - T_\mu \mid \lambda, \mu \in \Lambda)$. We endow A with the linear topology defined by the cofiltered system of ideals $(J_\lambda^n \mid (\lambda, n) \in \Lambda \times \mathbb{N})$, where $J_\lambda \subset A$ is the ideal generated by $(T_\mu \mid \mu \in \Lambda,\ \mu > \lambda)$, for every $\lambda \in \Lambda$. Let $\Sigma \subset A$ be the subset of monomials, which is in natural bijection with the set of all maps $\varphi : \Lambda \to \mathbb{N}$ such that $\varphi^{-1}(\mathbb{N} \setminus \{0\})$ is a finite set. For every $(\lambda, n) \in \Lambda \times \mathbb{N}$, set $\Sigma_{\lambda,n} := J^n_\lambda \cap \Sigma$, and let $\mathscr{F}$ be the set of all subsets $F \subset \Sigma$ such that $\Sigma_{\lambda,n} \subset F$ for some $(\lambda, n) \in \Lambda \times \mathbb{N}$. Then we have a natural identification $A \xrightarrow{\sim} K^{(\Sigma)}$, and the topology of A is defined by the cofiltered system of K-submodules $(K^{(F)} \mid F \in \mathscr{F})$. It is easily seen that $\mathscr{F}$ fulfills the condition of example 8.3.4, hence A is complete and separated. Now, the quotient A/I is isomorphic to $K[T]$, endowed with its T-adic topology, which is separated but not complete.

Lemma 8.3.6. *Let A be a topological ring, $B \subset A$ an open subring, C the integral closure of B in A; endow B and C with the topologies induced by A, and denote by $A^\wedge$, $B^\wedge$ and $C^\wedge$ the respective separated completions. Then $C^\wedge$ is the integral closure of $B^\wedge$ in $A^\wedge$.*

Proof. Notice that $C^\wedge = B^\wedge \cdot C$, hence $C^\wedge$ lies in the integral closure of $B^\wedge$ in $A^\wedge$, and we are therefore reduced to checking that B is integrally closed in A if and only if $B^\wedge$ is integrally closed in $A^\wedge$. Let $j : A \to A^\wedge$ be the completion map. Suppose first that B is integrally closed in A, and let $a \in A^\wedge$ be any element that is integral over $B^\wedge$; pick any monic polynomial $P(T) \in B^\wedge[T]$ with $P(a) = 0$. Since B is open in A, we may then find $a' \in A$ and a monic polynomial $Q(T) \in B[T]$ such that

$$j(a') - a, P(a'), j(Q(a')) - P(j(a')) \in B^\wedge.$$

It follows that $Q(a') \in j^{-1}B^\wedge = B$ (corollary 8.2.17(ii)). Set $R(T) := Q(T) - Q(a')$; then $R(T) \in B[T]$ and $R(a') = 0$, so $a' \in B$, by assumption, and finally $a \in B^\wedge$, as required. Conversely, if $a \in A$ is integral over B, it follows easily that $j(a)$ is integral over $B^\wedge$, and hence $j(a) \in B^\wedge$, if $B^\wedge$ is integrally closed in $A^\wedge$; in this case, $a \in j^{-1}B^\wedge = B$, which shows that B is integrally closed in A. $\square$

8.3.7. For any topological ring A, let $\mathscr{C}_A$ denote the full subcategory of A-TopAlg whose objects are the complete and separated topological A-algebras whose topology is linear. Then the finite coproducts of $\mathscr{C}_A$ are representable. Indeed, consider more generally any two topological A-modules M, N whose topologies are A-linear; we denote by $\mathscr{T}^{\otimes}_{M,N}$ the A-linear topology on $M \otimes_A N$ defined by the system of submodules

$$\mathrm{Im}(M' \otimes_A N + M \otimes_A N' \to M \otimes_A N)$$

where M' (resp. N') ranges over the system of all open A-submodules of M (resp. of N), and following [59, Ch.0,§7.7.2] we denote by

$$M \widehat{\otimes}_A N$$

the separated completion of $(M \otimes_A N, \mathscr{T}^{\otimes}_{M,N})$. Notice that if $(M_\lambda \mid \lambda \in \Lambda)$ and $(N_{\lambda'} \mid \lambda' \in \Lambda')$ are any two fundamental systems of open submodules of M and respectively N, we have a natural isomorphism of topological A-modules

$$M \widehat{\otimes}_A N \xrightarrow{\sim} \lim_{(\lambda,\lambda')\in\Lambda\times\Lambda'} M/M_\lambda \otimes_A N/N_{\lambda'}$$

where the tensor products $M/M_\lambda \otimes_A N/N_{\lambda'}$ are endowed with their discrete topologies.

By the same token, if M' and N' are two other topological A-modules whose topologies are A-linear, and $f : M \to M'$ and $g : N \to N'$ are any two continuous A-linear maps, then the map $f \otimes_A g : M \otimes_A N \to M' \otimes_A N'$ is continuous for the topologies $\mathscr{T}^{\otimes}_{M,N}$ and $\mathscr{T}^{\otimes}_{M',N'}$, and therefore its separated completion is a well defined continuous A-linear map

$$f \widehat{\otimes}_A g : M \widehat{\otimes}_A N \to M' \widehat{\otimes}_A N'.$$

Now, if B and C are any two topological A-algebras whose topologies are linear, notice that the topology $\mathscr{T}^{\otimes}_{B,C}$ is defined by a cofiltered system of ideals of $B \otimes_A C$, hence $(B \otimes_A C, \mathscr{T}^{\otimes}_{B,C})$ is a topological A-algebra. We claim that if B and C are complete and separated, then $B \widehat{\otimes}_A C$ represents the coproduct of B and C in the category $\mathscr{C}_A$. For the proof, consider any two morphisms $f : B \to D$ and $g : C \to D$ in the category $\mathscr{C}_A$; there exists a unique morphism of A-algebras $h : B \otimes_A C \to D$ such that $h(b \otimes c) := f(b) \cdot g(c)$ for every $b \in B$ and $c \in C$, and since D is complete and separated, it remains only to check that h is continuous for the topology $\mathscr{T}^{\otimes}_{B,C}$. However, let $I \subset D$ be any open ideal; then $J := f^{-1}I$ (resp. $K := g^{-1}I$) is an open ideal of B (resp. of C) and clearly $h(J \otimes_A C + B \otimes_A K) \subset I$, whence the contention.

Definition 8.3.8. Let $(A, \mathscr{T})$ be any topological ring, and $S \subset A$ any subset.

(i) $S \subset A$ is *bounded* in A, if for every open neighborhood U of $0 \in A$ there exists an open neighborhood V of $0 \in A$ such that $v \cdot s \in U$ for every $v \in V$ and $s \in S$.

(ii) For every $n \in \mathbb{N}$, let $S(n) := \{a_1 \cdots a_n \mid a_1, \dots, a_n \in S\}$. We say that S is *power bounded* in A if the subset $\bigcup_{n\in\mathbb{N}} S(n)$ is bounded in A. An element $a \in A$ is *power bounded* (resp. *topologically nilpotent*) in A, if the subset $\{a\}$ is power bounded in A (resp. if the sequence $(a^n \mid n \in \mathbb{N})$ converges to 0 in the topology $\mathscr{T}$). We denote by

$$A^\circ \qquad \text{and} \qquad A^{\circ\circ}$$

respectively the subset of all power-bounded elements and the subset of all topologically nilpotent elements of A. An ideal $I \subset A$ is *topologically nilpotent*, if for every open neighborhood U of 0 in A there exists $n \in \mathbb{N}$ such that $I^n \subset U$.

(iii) We say that $(A, \mathscr{T})$ is *adic* (resp. *c-adic*) if there exists an ideal I of A such that

$$(I^n \mid n \in \mathbb{N}) \qquad (\text{resp. } ((I^n)^c \mid n \in \mathbb{N})\ )$$

is a fundamental system of open neighborhood of zero in A. Then, we call any such I an *ideal of adic definition* (resp. *an ideal of c-adic definition*) of $(A, \mathscr{T})$, and we also say that $\mathscr{T}$ is *I-adic* (resp. *I-c-adic*). If M is any A-module, the *I-adic topology* on M is the (unique) A-linear topology such that $(I^n M \mid n \in \mathbb{N})$ is a fundamental system of open neighborhoods of 0.

(iv) We say that $(A, \mathscr{T})$ is *f-adic* if there exists an open subring $A_0 \subset A$ that is adic for the topology induced by $\mathscr{T}$, and such that A_0 admits a finitely generated ideal of adic definition.

(v) Let $(A, \mathscr{T})$ be any f-adic ring. An open subring A_0 of A is called a *ring of definition of* A, if $\mathscr{T}$ induces a linear topology on A_0.

(vi) We say that $(A, \mathscr{T})$ is a *Tate ring* if it is f-adic and $A^\times \cap A^{\circ\circ} \neq \varnothing$.

Remark 8.3.9. Let $(A, \mathscr{T})$ be any topological ring.

(i) Let $S_1, S_2 \subset A$ be two bounded subsets. Clearly $S_1 \cup S_2$ is bounded. Moreover, also $S_1 S_2 := \{s_1 s_2 \mid s_1 \in S_1,\ s_2 \in S_2\}$ is bounded. Indeed, for any given open neighborhood U of 0 in A, let us pick an open neighborhood V of 0 in A such that $S_1 V \subset U$, and an open neighborhood V' of 0 in A such that $S_2 V' \subset V$; then $S_1 S_2 V' \subset U$, whence the contention.

(ii) Let $T_1, \dots, T_k \subset A$ be any finite family of subsets. Then $T := T_1 \cup \cdots \cup T_k$ is power bounded in A if and only if every $T_1, \dots, T_k$ is power bounded in A. Indeed, set $S_i := \bigcup_{n\in\mathbb{N}} T_i(n)$ for $i = 1, \dots, k$ (notation of definition 8.3.8(ii)); by definition, T is power bounded if and only if the product $S_1 \cdots S_k$ is bounded, so the assertion follows from (i).

(iii) Let $T \subset A$ be any subset. Then T is bounded (resp. power bounded) in A if and only if the same holds for the topological closure T^c of T in A. Indeed, suppose that T is bounded, and let $U \subset A$ be any open neighborhood of $0 \in A$; pick an open neighborhood U' of 0 in A such that $U' + U' \subset U$. By assumption, there exists an open neighborhood V of 0 in A such that $T \cdot V \subset U'$, whence $T^c \cdot V \subset U'^c$; but notice that $U'^c \subset U' + U'$: indeed, if $x \in U'^c$, we have $(x - U') \cap U' \neq \varnothing$, whence $x \in U' + U'$. Summing up, we get $T^c \cdot V \subset U$, which shows that T^c is bounded. Next, suppose that T is power bounded, *i.e.* $T' := \bigcup_{n\in\mathbb{N}} T(n)$ is bounded, and notice that $\bigcup_{n\in\mathbb{N}} T^c(n) \subset T'^c$; by the foregoing T'^c is bounded, so T^c is power bounded.

(iv) Let $a \in A$ be any element, and $n \geq 1$ any integer. Then a is topologically nilpotent if and only if the same holds for a^n. Indeed, obviously if a is topologically nilpotent, the same holds for a^n. Conversely, suppose that a^n is topologically nipotent; since the map $A \to A$:

$b \mapsto a^r b$ is continuous for every $r \in \mathbb{N}$, it follows that the sequences $(a^{kn+r} \mid k \in \mathbb{N})$ converge to 0 in A for every $r = 0, \dots, n-1$; the claim is an immediate consequence.

(v) Let $a, b \in A$ be any two elements. If a is power bounded and b is topologically nilpotent, ab is topologically nilpotent. Indeed, for any open neighborhood U of 0 in A there exists an open neighborhood V of 0 in A such that $\{b^k \mid k \in \mathbb{N}\} \cdot V \subset U$, and on the other hand, there exists $n \in \mathbb{N}$ such that $a^k \in V$ for every integer $k \geq n$; thus, $(ab)^k \in U$ for every $k \geq n$, whence the claim.

(vi) The class of f-adic rings was introduced by R.Huber in [98]. Such topological rings are now often called *Huber rings*.

Remark 8.3.10. Let $(A, \mathscr{T})$ be a topological ring whose topology $\mathscr{T}$ is $\mathbb{Z}$-linear.

(i) The subset $A^{\circ\circ}$ is an additive subgroup of A. Indeed, let $a, b \in A^{\circ\circ}$ be any two elements, U any open additive subgroup in A, and pick an open neighborhood V of 0 in A such that $V \cdot V \subset U$; we may find $r \in \mathbb{N}$ such that $a^i, b^i \in V$ for every $i \geq r$. Then, we may also find an integer $s \geq r$ such that $a^i b^j \in U$ for every $i, j \in \mathbb{N}$ such that $i + j \geq s$ and $\min(i, j) < r$. It follows easily that $(a+b)^n \in U$ for every $n \geq s$, and since U is arbitrary, we conclude that $a + b$ is topologically nilpotent.

(ii) For any subset $T \subset A$, denote by $\langle T \rangle \subset A$ the additive subgroup generated by T in A. Then T is bounded (resp. power bounded) if and only if the same holds for $\langle T \rangle$. Indeed, suppose that T is bounded, and let $U \subset A$ be any open additive subgroup; by assumption there exists an open neighborhood V of 0 in A such that $T \cdot V \subset U$. But then clearly $\langle T \rangle \cdot V \subset U$, so $\langle T \rangle$ is bounded. Likewise, suppose that T is power bounded, and set $S := \bigcup_{n \in \mathbb{N}} T(n)$ (notation of definition 8.3.8(ii)); since S is bounded, we know already that the same holds for $\langle S \rangle$. However, clearly $\langle S \rangle = \bigcup_{n \in \mathbb{N}} \langle T \rangle(n)$, so $\langle T \rangle$ is power bounded.

(iii) Let $T \subset A$ be any power bounded subset; in light of (ii), it is easily seen that the $\mathbb{Z}$-subalgebra $\mathbb{Z}[T] \subset A$ generated by T is bounded in A, and especially, every element of $\mathbb{Z}[T]$ is power bounded. Combining with remark 8.3.9(ii), we deduce that A° is the filtered union of all the bounded subrings of A. Especially, A° is a subring of A.

(iv) Moreover, A° is integrally closed in A, and $A^{\circ\circ}$ is a radical ideal of A°. Indeed, say that $x \in A$ is integral over A°, so that there exist $a_1, \dots, a_n \in A^\circ$ (for some $n \in \mathbb{N}$) such that $x^n + a_1 x^{n-1} + \cdots + a_n = 0$, and set $B := \mathbb{Z}[a_1, \dots, a_n] \subset A$; by (iii), the subring B is bounded in A, and it is easily seen that $B[x] = \sum_{i=0}^{n-1} Bx^i$, so $B[x]$ is bounded as well, therefore $B[x] \subset A^\circ$, which shows the first assertion. Lastly, (i) and remark 8.3.9(v) imply easily that $A^{\circ\circ}$ is an ideal of A°, and remark 8.3.9(iv) shows that $A^{\circ\circ}$ is a radical ideal.

(v) Lastly, if the topology $\mathscr{T}$ is also complete and separated, $A^{\circ\circ}$ lies in the Jacobson radical of A°. Indeed, in this case, for every $a \in A^{\circ\circ}$ the series $1 - a + a^2 - a^3 + \cdots$ converges in A to a unique element b such that $b \cdot (1 + a) = 1$. Now, b lies in the topological closure $\mathbb{Z}[a]^c$ of $\mathbb{Z}[a] \subset A$, and notice that $\mathbb{Z}[a]$ is the additive subgroup generated by the subset $T := \{a^n \mid n \in \mathbb{N}\}$; since a is power bounded, obviously T is a power bounded subset, and then the same holds for $\mathbb{Z}[a]^c$, by (ii) and remark 8.3.9(iii). Thus, $b \in A^\circ$, whence the contention.

Lemma 8.3.11. *Let A be a ring, and $I \subset A$ an ideal; consider a cartesian diagram of topological A-modules*

$$\begin{array}{ccc} M_0 & \longrightarrow & M_1 \\ \downarrow & & \downarrow {\scriptstyle f_1} \\ M_2 & \xrightarrow{f_2} & M_3 \end{array}$$

and suppose that the topology of M_3 is discrete, and the topologies of M_1 and M_2 are I-adic. Then the topology of M_0 is I-adic as well.

Proof. By assumption, we may find $c \in \mathbb{N}$ such that $f_1(I^c M_1) = f_2(I^c M_2) = 0$, and it follows easily that $I^n M_1 \times I^n M_2 \subset M_0$ for every $n \geq c$. Especially, $(I^n M_1 \times I^n M_2 \mid n \geq c)$ is a fundamental system of open neighborhoods of 0 in M_0. We also see that

$$I^{n+c} M_1 \times I^{n+c} M_2 = I^n(I^c M_1 \times I^c M_2) \subset I^n M_0 \subset I^n M_1 \times I^n M_2 \qquad \text{for every } n \in \mathbb{N}$$

whence the claim. □

Lemma 8.3.12. *Let A be a topological ring, $(M, \mathscr{T}_M)$ a topological A-module whose topology is A-linear, complete and separated, and $I \subset A$ an ideal such that the I-adic topology on M is finer than $\mathscr{T}_M$. Suppose moreover that either one of the following conditions holds :*

(a) *$I^n M$ is closed in the topology $\mathscr{T}_M$, for every $n \in \mathbb{N}$.*
(b) *I is finitely generated.*

Then M is separated and complete for the I-adic topology.

Proof. Let $(J_\lambda \mid \lambda \in \Lambda)$ be a fundamental system of open submodules in M; by assumption, for every $\lambda \in \Lambda$ there exists $n(\lambda) \in \mathbb{N}$ such that $I^{n(\lambda)} M \subset J_\lambda$. Suppose first that (a) holds; it follows that the natural map

$$M/I^n M \to \lim_{\lambda \in \Lambda} M/(I^n M + J_\lambda)$$

is injective for every $n \in \mathbb{N}$. Taking into account example 1.5.15(ii), we deduce that both of the induced maps

$$M \to \lim_{n \in \mathbb{N}} M/I^n M \qquad \text{and} \qquad \lim_{n \in \mathbb{N}} M/I^n M \to \lim_{(\lambda,n) \in \Lambda \times \mathbb{N}} M/(I^n M + J_\lambda)$$

are injective. However, notice that the subset $\Sigma \subset \Lambda \times \mathbb{N}$ of all (n, λ) such that $n \geq n(\lambda)$, is cofinal, hence the composition of these two maps is an isomorphism. We conclude that both these maps are bijective, and the assertion follows.

Next, suppose that (b) holds, and let $a_1, \dots, a_r$ be a finite system of generators for I. Clearly M is separated for the I-adic topology. Now, suppose $x_\bullet := (x_n \mid n \in \mathbb{N})$ is a sequence of elements of M such that $x_n - x_m \in I^n M$ for every $n, m \in \mathbb{N}$ with $m \geq n$. The sequence $x_\bullet$ converges for the topology $\mathscr{T}_M$ to an element $x \in M$, and it remains to check that x is also the limit of $x_\bullet$ for the I-adic topology of M. To this aim, for every $n \in \mathbb{N}$, let

$$S_n := \{(j_1, \dots, j_r) \in \mathbb{N}^r \mid j_1 + \cdots + j_r = n\}$$

and for every $\sigma := (j_1, \dots, j_r) \in \mathbb{N}^r$ set $a^\sigma := a_1^{j_1} \cdots a_r^{j_r}$. For every $k, l \in \mathbb{N}$ we may find inductively a system of elements $(y_{l,\sigma} \mid \sigma \in S_k)$ of M such that

- $x_{k+l} - x_k = \sum_{\sigma \in S_k} y_{l,\sigma} a^\sigma$
- $y_{l',\sigma} - y_{l,\sigma} \in I^l M$ whenever $l' \geq l$ and for every $\sigma \in \mathbb{N}^r$.

Thus, the sequence $(y_{l,\sigma} \mid l \in \mathbb{N})$ converges in the topology $\mathscr{T}_M$ to some element $y_\sigma \in M$ for every $\sigma \in \mathbb{N}^r$, and we get $x - x_k = \sum_{\sigma \in S_k} y_\sigma a^\sigma$; so this difference lies in $I^k M$ for every $k \in \mathbb{N}$, whence the contention. □

Proposition 8.3.13. *Let A be a ring, $I \subset A$ an ideal, and M a faithful A-module of finite type. The following holds :*

(i) *If M is I-adically complete and separated, then I lies in the Jacobson radical of A.*
(ii) *If moreover A is noetherian, then M is I-adically complete and separated if and only if the same holds for A.*

Proof. (i): Let $A^\wedge$ be the I-adic completion of A, so that M is naturally an $A^\wedge$-module. Suppose by contradiction that there exists a maximal ideal $\mathfrak{m} \subset A$ that does not contain I. Then there exist $a \in I$ and $b \in \mathfrak{m}$ with $a + b = 1$, whence $b = 1 - a \in (A^\wedge)^\times$ (with $b^{-1} = \sum_{n \in \mathbb{N}} a^n$), so that $M = \mathfrak{m}M$, and therefore $M_\mathfrak{m} = 0$, by Nakayama's lemma. On the other hand, since by assumption we have $\mathrm{Ann}_A(M) = 0$, we get $\mathrm{Ann}_{A_\mathfrak{m}}(M_\mathfrak{m}) = 0$, so $A_\mathfrak{m} = 0$, which is absurd.

(ii): If A is I-adically complete and separated, the same holds for M, by [126, Th.8.7]. Conversely, if M is I-adically complete and separated, then I lies in the Jacobson radical of A, by (i). Moreover, since M is faithful and of finite type, there exists $k \in \mathbb{N}$ and an injective map $f : A \to M^{\oplus k}$ (details left to the reader). Clearly $M^{\oplus k}$ is I-adically complete and separated, and $f(A)$ is a closed subset for the I-adic topology of $M^{\oplus k}$ ([126, Th.8.10]), hence A is complete and separated for the topology $\mathscr{T}$ induced by $M^{\oplus k}$; but the latter agrees with the I-adic topology ([126, Th.8.6]). □

Proposition 8.3.14. *Let A be a noetherian ring, $I \subset J \subset A$ two ideals, and M an A-module; set $\overline{A} := A/I$, $\overline{J} := J/I$ and $\overline{M} := M/IM$. Suppose that M is I-adically separated and complete, and that either one of the following conditions holds :*

(a) *M is of finite type.*
(b) *M is flat.*

Then M is J-adically separated and complete if and only if the same holds for $\overline{M}$.

Proof. For every A-module N, let $N^\wedge$ be the J-adic completion of N. We notice :

Claim 8.3.15. For every flat A-module P, the following holds :

(i) Every A-module of finite type N induces an isomorphism $f_N : N \otimes_A P^\wedge \xrightarrow{\sim} (N \otimes_A P)^\wedge$.
(ii) Moreover, $P^\wedge$ is a flat $A^\wedge$-module.

Proof of the claim. (i): We consider a short exact sequence of A-modules of finite type

$$\Sigma \qquad : \qquad 0 \to N_1 \to N_2 \to N_3 \to 0$$

and we first show that the induced sequence of J-adic completions :

$$0 \to (N_1 \otimes_A P)^\wedge \to (N_2 \otimes_A P)^\wedge \to (N_3 \otimes_A P)^\wedge \to 0$$

is still short exact. To this aim, since $\Sigma \otimes_A P$ is still short exact, it suffices to check that the J-adic topology of $N_1 \otimes_A P$ is induced by that of $N_2 \otimes_A P$ (proposition 8.2.13(i,v)). But by the Artin-Rees lemma there exists $c \in \mathbb{N}$ such that $J^{n+c}N_2 \cap N_1 \subset J^n N_1 \subset J^n N_2 \cap N_1$ for every $n \in \mathbb{N}$. Since P is a flat A-module, it follows that

$$J^{n+c}(N_2 \otimes_A P) \cap (N_1 \otimes_A P) \subset J^n(N_1 \otimes_A P) \subset J^n(N_2 \otimes_A P) \cap (N_1 \otimes_A P)$$

for every $n \in \mathbb{N}$, whence the assertion. Now, let $\varphi : A^{\oplus n} \to N$ be an A-linear surjection, and $N' := \mathrm{Ker}\,\varphi$. By the foregoing, we have a commutative ladder :

$$\begin{array}{ccccccccc}
 & & N' \otimes_A P^\wedge & \longrightarrow & A^{\oplus n} \otimes_A P^\wedge & \longrightarrow & N \otimes_A P^\wedge & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_{N'}} & & \downarrow{\scriptstyle f_{A^{\oplus n}}} & & \downarrow{\scriptstyle f_N} & & \\
0 & \longrightarrow & (N' \otimes_A P)^\wedge & \longrightarrow & (A^{\oplus n} \otimes_A P)^\wedge & \longrightarrow & (N \otimes_A P)^\wedge & \longrightarrow & 0
\end{array}$$

whose bottom (resp. top) horizontal row is short exact (resp. right exact). Clearly $f_{A^{\oplus n}}$ is an isomorphism, hence f_N is surjective for every A-module N of finite type. Especially, $f_{N'}$ is surjective; but the snake lemma yields an exact sequence $\mathrm{Ker}\, f_{A^{\oplus n}} \to \mathrm{Ker}\, f_N \to \mathrm{Coker}\, f_{N'}$, so f_N is an isomorphism.

(ii): It is easily seen that the natural map $P \to A^\wedge \otimes_A P$ induces an isomorphism on J-adic completions; after replacing A by $A^\wedge$, and P by $A^\wedge \otimes_A P$, we may thus assume from start that A is J-adically complete and separated, and then we need to check that P is a flat A-module. Now, let $i : L \to A$ be the inclusion map of an ideal; by (i), the map $i \otimes_A P^\wedge : L \otimes_A P^\wedge \to P^\wedge$ is naturally identified with the J-adic completion $(i \otimes_A P)^\wedge$ of $i \otimes_A P : L \otimes_A P \to P$. Since

P is a flat A-module, $i \otimes_A P$ is injective; then the foregoing shows that the same holds for $(i \otimes_A P)^\wedge$, hence also for $i \otimes_A P^\wedge$, so $P^\wedge$ is a flat A-module ([126, Th.7.2]). $\Diamond$

Now, if (b) holds, claim 8.3.15(ii) implies that the natural map $M^\wedge/IM^\wedge \to \overline{M}^\wedge$ is an isomorphism; the latter holds also under condition (a), by [126, Th.8.11]. In either case, we already deduce that if M is J-adically complete and separated, then the same holds for $\overline{M}$.

Conversely, suppose that $\overline{M}$ is J-adically complete and separated. If (a) holds, let $\overline{A}_0 := \overline{A}/\mathrm{Ann}_{\overline{A}}(\overline{M})$; then $\overline{M}$ is a faithful $\overline{A}_0$-module of finite type, and therefore $\overline{A}_0$ is J-adically complete and separated (proposition 8.3.13(ii)), so that every $\overline{A}_0$-module of finite type is J-adically complete and separated ([126, Th.8.7]). Now, notice that, for every $n \in \mathbb{N}$, the $\overline{A}$-module $I^nM/I^{n+1}M$ is a quotient of $I^n/I^{n+1} \otimes_{\overline{A}} \overline{M}$, hence it is an $\overline{A}_0$-module of finite type, so it is J-adically complete and separated. By [126, Th.8.7], this means that the natural map $I^nM/I^{n+1}M \to (I^nM/I^{n+1}M) \otimes_A A^\wedge$ is an isomorphism for every $n \in \mathbb{N}$. Since $A^\wedge$ is a flat A-algebra, it follows easily that, for every $n \in \mathbb{N}$, the natural map

$$f_n : M/I^nM \to (M/I^nM) \otimes_A A^\wedge \xrightarrow{\sim} M^\wedge/I^nM^\wedge$$

is an isomorphism. Since M is I-adically complete and separated, the limit of the system $(f_n \mid n \in \mathbb{N})$ is an isomorphism from M to the I-adic completion of $M^\wedge$. But $M^\wedge$ is I-adically complete and separated (lemma 8.3.12), so M is J-adically complete and separated, as stated.

Lastly, if (b) holds, then $M^\wedge$ is a flat A-module (claim 8.3.15(ii)), and therefore, for every $n \in \mathbb{N}$, both vertical arrows of the commutative diagram :

$$\begin{array}{ccc} (I^n/I^{n+1}) \otimes_{\overline{A}} \overline{M} & \longrightarrow & (I^n/I^{n+1}) \otimes_{\overline{A}} M^\wedge/IM^\wedge \\ \downarrow & & \downarrow \\ I^nM/I^{n+1}M & \longrightarrow & I^nM^\wedge/I^{n+1}M^\wedge \end{array}$$

are isomorphisms. The same holds for the top horizontal arrow, by claim 8.3.15(i), so also for the bottom horizontal one. Then we argue as in the foregoing case, to conclude. $\square$

Remark 8.3.16. Claim 8.3.15 is a special case of proposition 14.2.43, and for local rings it can already be found in [15, Cor.3.15]. For a non-noetherian variant, see [75, Lemma 7.1.6].

Corollary 8.3.17. *Let A be a noetherian ring, $I \subset J \subset A$ two ideals, M an A-module that is either flat or of finite type, and such that M/IM is J-adically complete and separated. Denote by $M_I^\wedge$ (resp. $M_J^\wedge$) the I-adic (resp. J-adic) completion of M. Then the natural map*

$$\alpha : M_I^\wedge \to M_J^\wedge$$

is an isomorphism.

Proof. Let also $A_I^\wedge$ be the I-adic completion of A; if M is a flat (resp. finitely generated) A-module, then $M_I^\wedge$ is a flat (resp. finitely generated) $A_I^\wedge$-module (claim 8.3.15(ii)); moreover, M/IM is trivially I-adically complete and separated. Hence, the natural map $M/IM \to M_I^\wedge/IM_I^\wedge$ is bijective, by claim 8.3.15(i) (resp. by [126, Th.8.11]). Especially, $M_I^\wedge/IM_I^\wedge$ is J-adically complete and separated, so the same holds for $M_I^\wedge$, by proposition 8.3.14. Thus, there exists a unique A-linear map $\beta : M_J^\wedge \to M_I^\wedge$ whose composition with the J-adic completion map $i_{M,J} : M \to M_J^\wedge$ agrees with the I-adic completion map $i_{M,I} : M \to M_I^\wedge$. Likewise, by definition we have $\alpha \circ i_{M,I} = i_{M,J}$, so that

$$\alpha \circ \beta \circ i_{M,J} = i_{M,J} \qquad \text{and} \qquad \beta \circ \alpha \circ i_{M,I} = i_{M,I}.$$

By the universal properties of $i_{M,I}$ and $i_{M,J}$, it follows that $\alpha \circ \beta = \mathbf{1}_{M_J^\wedge}$ and $\beta \circ \alpha = \mathbf{1}_{M_I^\wedge}$, whence the assertion. $\square$

Proposition 8.3.18. *Let $(A, \mathscr{T})$ be any f-adic ring. We have :*

(i) *If A_0 is any ring of definition of A, then the topology of A_0 is adic and admits a finitely generated ideal of adic definition.*
(ii) *A subring of A is a ring of definition of A if and only if it is open and bounded in A.*
(iii) *The following conditions are equivalent :*
  (a) *The topology $\mathscr{T}$ is linear.*
  (b) *The topology $\mathscr{T}$ is adic and admits a finitely generated ideal of adic definition.*

Proof. Fix an open subring A_1 and a finitely generated ideal I of A_1, such that $\mathscr{T}$ induces the I-adic topology on A_1.

(i): Since A_0 is open in A, it follows that there exists an integer $n > 0$ with $I^n \subset A_0$; set $J := I^n A_0$. If K is any open ideal of A_0, we may likewise find $m \in \mathbb{N}$ such that $I^{mn} \subset K$, so that $J^m \subset K$ and J^m is open in A_0. Hence, $\mathscr{T}$ induces the J-adic topology on A_0; however, J is not necessarily finitely generated (unless $A_1 \subset A_0$). Hence, let $T \subset A_0$ be a finite system of generators of the ideal I^n of A_1, and set $J' := A_0 T$; then $J' \subset J$, and on the other hand $J^2 = J' I_0^n \subset J'$, so the J-adic topology on A_0 coincides with the J'-adic topology.

(ii): Clearly, every ring of definition of A is open and bounded. Conversely, suppose that A_0 is open and bounded. As in the foregoing, we find an integer $n > 0$ such that $I^n \subset A_0$ and we set $J := I^n A_0$. Let now U be any open neighborhood of zero in A_0; since A_1 admits a fundamental system of open neighborhoods of zero consisisting of additive subgroups, we may assume that U is an additive subgroup of A_0. Since A_0 is bounded, there exists $m \in \mathbb{N}$ such that $x \cdot a \in U$ for every $x \in I^m$ and every $a \in A_0$; then clearly $J^m \subset U$. Lastly, it is easily seen that J^k is open in A_0 for every $k \in \mathbb{N}$, whence (ii).

(iii): If $\mathscr{T}$ is linear, then A is bounded in itself, so the assertion follows directly from (ii). □

Corollary 8.3.19. *Let $(A, \mathscr{T})$ be any f-adic ring. We have :*

(i) *If A_0 and A_1 are rings of definition of A, then the same holds for $A_0 \cap A_1$ and $A_0 \cdot A_1$.*
(ii) *Let B (resp. C) be a bounded (resp. open) subring of A, with $B \subset C$. Then there exists a ring of definition A_0 of A with $B \subset A_0 \subset C$.*
(iii) *$(A, \mathscr{T})$ is adic if and only if A is bounded in the topology $\mathscr{T}$.*
(iv) *A° is the filtered union of all the subrings of definition of A.*

Proof. (i): The assertion for $A_0 \cap A_1$ is immediate from proposition 8.3.18(ii). The assertion for $A_0 \cdot A_1$ follows likewise, taking into account that A has a fundamental system of open neighborhoods of zero consisting of additive subgroups : details left to the reader.

(ii): Let A_1 be any ring of definition of A; by proposition 8.3.18(ii), the subring $C_1 := A_1 \cap C$ is a ring of definition of A, and we may take $A_0 := B \cdot C_1$.

(iii): If $(A, \mathscr{T})$ is adic, then clearly A is bounded. Conversely, if A is bounded, (ii) implies that A is a subring of definition of $(A, \mathscr{T})$, and then $(A, \mathscr{T})$ is adic, by proposition 8.3.18(i).

(iv): By remark 8.3.10(iii), we know that A° is the filtered union of the family $\mathscr{F}$ of bounded subrings of A; notice that any subring containing an open subring of A is also open in A. Since the family of bounded and open subring of A is not empty (proposition 8.3.18(ii)), we conclude that this family is cofinal in $\mathscr{F}$, whence the contention. □

Corollary 8.3.20. *Let $(A, \mathscr{T}_A)$ be any topological ring, $B \subset A$ an open subring, and endow B with the topology $\mathscr{T}_B$ induced from $\mathscr{T}_A$. The following holds :*

(i) *$(A, \mathscr{T}_A)$ is f-adic if and only if the same holds for $(B, \mathscr{T}_B)$.*
(ii) *Suppose that $(A, \mathscr{T}_A)$ is a Tate ring, and B is a ring of definition of A. Then we have :*
  (a) *B contains an element of $A^\times \cap A^{\circ\circ}$.*
  (b) *Let $s \in B \cap A^\times \cap A^{\circ\circ}$ be any element. Then sB is an ideal of adic definition for B, and $B_s = A$.*

(iii) *Conversely, let C be any ring, $f \in C$ any element, D the image of C in the localization C_f, and endow C_f with the unique D-linear topology $\mathscr{T}_f$ such that $\{f^n D \mid n \in \mathbb{N}\}$ is a fundamental system of neighborhoods of 0 in C_f. Then $(C_f, \mathscr{T}_f)$ is a Tate ring.*

Proof. (i): Directly from the definition we see that if $(B, \mathscr{T}_B)$ is f-adic, the same holds for $(A, \mathscr{T}_A)$. Conversely, if $(A, \mathscr{T}_A)$ is f-adic, let $A_0 \subset A$ be any subring of definition; by corollary 8.3.19(ii) we may find a subring of definition A_1 of A contained in $A_0 \cap B$; especially, A_1 is an open subring of B whose topology is adic with a finitely generated ideal of adic definition (proposition 8.3.18(i)), whence the assertion.

(ii.a) is clear. Next, for every $a \in A$ there exists $n \in \mathbb{N}$ such that $s^n a \in B$, so $B_s = A$. Also, the map $a \mapsto s^n a$ is an automorphism of $(A, \mathscr{T}_A)$, so $s^n B$ is open for every $n \in \mathbb{N}$. Lastly, since B is bounded in A, for every open neighborhood U of $0 \in A$ there exists $n \in \mathbb{N}$ such that $s^n B \subset U$, whence (ii.b).

(iii): Explicitly, the open subsets of $(C_f, \mathscr{T}_f)$ are the arbitrary unions of subsets of the form $a + f^m D$, where $a \in C_f$ is arbitrary and $m \in \mathbb{N}$ is any integer; indeed, it is easily seen that the intersection of any two subsets of this type is either empty, or equal to one of them, so such arbitrary unions define a topology $\mathscr{T}_f$, and it is easily seen that $(C_f, \mathscr{T}_f)$ is a topological ring (details left to the reader). A simple inspection then shows that $(C_f, \mathscr{T}_f)$ is a Tate ring. □

Lemma 8.3.21. *Let A be any c-adic, complete and separated topological ring, $I, J \subset A$ two ideals, with I finitely generated. We have :*

(i) *I is open in A if and only if the same holds for I^c.*
(ii) *If I is an ideal of c-adic definition for A, the following holds :*
  (a) *A is adic and I is an ideal of adic definition for A.*
  (b) *J is open in A if and only if the same holds for J^c.*

Proof. Assertion (i) is a special case of [75, Lemma 5.3.5(ii) and 5.3.8(i)], and (ii.a) follows directly from (i). Lastly, suppose that J^c is open in A; by (ii.a) there exists $n \in \mathbb{N}$ such that $I^n \subset J^c \subset J + I^{n+1}$. Letting $M := (I^n + J)/J$, it follows that $IM = M$, and since A is complete and separated for its I-adic topology, I lies in the Jacobson radical of A (remark 8.3.10(v)); therefore $M = 0$ by Nakayama's lemma, *i.e.* $I^n \subset J$, so J is open in A. □

Lemma 8.3.22. *Let A be a complete and separated adic topological ring, $I \subset A$ an ideal of adic definition, M an A-module whose I-adic topology $\mathscr{T}_M$ is separated, $N \subset M$ a finitely generated submodule, and N^c the topological closure of N in $(M, \mathscr{T}_M)$. We have :*

(i) *If the I-adic topology $\mathscr{T}_N$ of N^c agrees with the one induced by $\mathscr{T}_M$, then $N = N^c$.*
(ii) *N is open in $(M, \mathscr{T}_M)$ if and only if the same holds for N^c.*

Proof. (i): By assumption, IN^c is an open submodule of N^c for the topology induced by $(M, \mathscr{T}_M)$, hence $N^c = N + IN^c$, and the assertion follows from [126, Th.8.4].

(ii): If N is open in M, then $N = N^c$ and the assertion is clear. Conversely, if N^c is open in M, then $I^n M \subset N^c$ for some $n \in \mathbb{N}$, hence $\mathscr{T}_N$ agrees with the topology induced from $\mathscr{T}_M$; then we get again $N = N^c$, by (i). □

Definition 8.3.23. Let $(A, \mathscr{T})$ and $(A', \mathscr{T}')$ be two topological rings, and $f : A \to A'$ a ring homomorphism.

(i) We say that f is *c-adic* (resp. *adic*) if $\mathscr{T}$ and $\mathscr{T}'$ are linear, and the family of ideals

$$((f(I) \cdot A')^c \mid I \text{ open ideal in } A) \qquad (\text{resp. } (f(I) \cdot A' \mid I \text{ open ideal in } A)\ )$$

is a fundamental system of open neighborhoods of zero in A'.

(ii) Suppose that A and A' are f-adic. Then we say that f is *f-adic* if there are rings of definition A_0 of A and A'_0 of A' such that $f(A_0) \subset A'_0$ and the restriction $f_{|A_0}$ is adic.

(iii) Let $\mathbf{Q}$ be any property of ring homomorphisms (*e.g.* “of finite type”, “flat”, “étale” and so on). We say that f is an *adically* $\mathbf{Q}$ *morphism* (resp. a *c-adically* $\mathbf{Q}$ *morphism*) if f is adic (resp. c-adic) and for every open ideal $I \subset A$ the induced map $A/I \to A'/(IA')^c$ is a $\mathbf{Q}$ ring homomorphism.

Lemma 8.3.24. *Let $f : (A, \mathscr{T}) \to (A', \mathscr{T}')$ be any ring homomorphism of topological rings.*

(i) *If f is c-adic, the following holds :*

(a) *f is a continuous ring homomorphism.*

(b) *If the topology $\mathscr{T}$ is c-adic, the same holds for $\mathscr{T}'$, and if I is any ideal of c-adic definition for A, then $f(I)A'$ is an ideal of c-adic definition for A'.*

(c) *If the topologies $\mathscr{T}$ and $\mathscr{T}'$ are f-adic, linear and complete, then f is adic.*

(ii) *If the topologies $\mathscr{T}$ and $\mathscr{T}'$ are adic, the following conditions are equivalent :*

(a) *f is adic.*

(b) *There exists an ideal of adic definition I of A such that $f(I) \cdot A'$ is an ideal of adic definition of A'.*

(c) *For every ideal I of adic definition of A, the ideal $f(I) \cdot A'$ is of adic definition for A'.*

(iii) *Suppose that f is an f-adic map of f-adic topological rings. Then the following holds :*

(a) *f is continuous and bounded (*i.e.* if S is any bounded subset of A, the subset $f(S)$ is bounded in A'). Especially, $f(A^\circ) \subset A'^\circ$.*

(b) *For every ring of definition A_0 of A and A'_0 of A' such that $f(A_0) \subset A'_0$, and every ideal of adic definition I of A_0, the ideal $f(I) \cdot A'_0$ is of adic definition for A'_0.*

(c) *For every ring of definition A_0 of A and every open subring $B \subset A'$ such that $f(A_0) \subset B$, there exists a ring of definition A'_0 of A' such that $f(A_0) \subset A'_0 \subset B$.*

(d) *If $B \subset A$, $B' \subset A'$ are open subrings with $f(B) \subset B'$, then the restriction $f_{|B} : B \to B'$ is f-adic, for the topologies of B and B' induced by $\mathscr{T}$ and $\mathscr{T}'$.*

(iv) *If f is a continuous open map, and the topologies $\mathscr{T}$ and $\mathscr{T}'$ are f-adic, then f is f-adic.*

(v) *Suppose that $f(A)$ is open in A', that $\mathscr{T}$ is a linear topology and that $\mathscr{T}'$ is adic. Then f is adic if and only if it is a continuous open map.*

Proof. (i.a) is obvious, and (i.c) follows easily from proposition 8.3.18(iii) and lemma 8.3.21(i). To show (i.b), let us first remark, quite generally :

Claim 8.3.25. Let $f : X \to Y$ be any continuous map between arbitrary topological spaces, and $S, T \subset X$ two subsets such that $S^c = T^c$. Then $(fS)^c = (fT)^c$.

Proof of the claim. It suffices to show that $fS \subset (fT)^c$. However, clearly we have $T \subset f^{-1}((fT)^c)$, therefore $S \subset T^c \subset f^{-1}((fT)^c)$, whence the contention. ◊

Claim 8.3.26. Let B be any topological ring, $J_1, J_2 \subset B$ two additive subgroups. Denote J_1J_2 the additive subgroup generated by $(j_1j_2 \mid j_1 \in J_1;\ j_2 \in J_2)$, and define likewise $J_1^cJ_2^c$. Then

$$(J_1J_2)^c = (J_1^cJ_2^c)^c \qquad \text{for every } n \in \mathbb{N}.$$

Proof of the claim. Since $J_1J_2 \subset J_1^cJ_2^c$, we have $(J_1J_2)^c \subset (J_1^cJ_2^c)^c$. For the converse, it suffices to show that $J_1^cJ_2^c \subset (J_1J_2)^c$. However, for every $k \in \mathbb{N}$, consider the map

$$\mu_k : (B^{\oplus 2})^{\oplus k} \to B \qquad (b_{ij} \mid i = 1, 2;\ j = 1, \dots, k) \mapsto \sum_{t=1}^{k} b_{1t}b_{2t}.$$

Then $J_1^cJ_2^c = \bigcup_{k\in\mathbb{N}} \mu_k((J_1^c \oplus J_2^c)^{\oplus k})$. Now, μ_k is clearly continuous for the product topology on $(B^{\oplus 2})^{\oplus k}$, and $(J_1^c \oplus J_2^c)^{\oplus k}$ is the topological closure of $(J_1 \oplus J_2)^{\oplus k}$ in this topology, so $\mu_k((J_1^c \oplus J_2^c)^{\oplus k})$ is contained in the topological closure in B of $\mu_k((J_1 \oplus J_2)^{\oplus k})$ (claim 8.3.25), which in turns is contained in $(J_1J_2)^c$, whence the contention. ◊

Now, let I be any ideal of c-adic definition of A; by assumption, for every open ideal I' of A' there exists $n \in \mathbb{N}$ such that $(f((I^n)^c)A')^c \subset I$. Set $J := f(I)A'$; then J is open in B, and by claims 8.3.26 and 8.3.25 we deduce that

$$(f((I^n)^c)A')^c = ((f((I^n)^c))^c A')^c = ((f(I^n))^c A')^c = (f(I^n)A')^c = (J^n)^c$$

whence the contention.

(ii): If I_1, I_2 are any two ideals of adic definition of A, then there exist $n, m \in \mathbb{N}$ such that $I_1^n \subset I_2^m \subset I_1$. The assertion is an easy consequence : details left to the reader.

(iii.a): The continuity of f is obvious. Suppose that $S \subset A$ is a bounded subset, and let $U' \subset A'$ be any open neighborhood of zero; we have to exhibit an open neighborhood of zero V' in A' such that $f(S) \cdot V' \subset U'$. To this aim, let A_0 and A'_0 be rings of definition of A and A', such that $f(A_0) \subset A'_0$, and $I \subset A_0$ an ideal of adic definition such that IA'_0 is an ideal of adic definition of A'_0; we may assume that U is an open ideal of A'_0, and we may find $n, m \in \mathbb{N}$ such that $f(I^n) \subset U'$ and $S \cdot I^m \subset I^n$. Thus, $f(S) \cdot f(I^m) \subset U'$, so $V' := I^m A'_0$ will do.

(iii.b): Let A_1 and A'_1 be rings of definitions of A and A' such that $f(A_1) \subset A'_1$, and $J \subset A_1$ an ideal of adic definition such that $K := f(J) \cdot A'_1$ is an ideal of adic definition of A'_1. Then there exist $n, m \in \mathbb{N}$ such that $J^n \subset I$ and $K^m \subset A'_0$, so that $K^{n+m} = f(J^n) \cdot K^m \subset f(I) \cdot A'_0$; especially, $f(I) \cdot A'_0$ is open in A'_0. Moreover, since I is topologically nilpotent and finitely generated in A_0, it is easily seen that $f(I)A'_0$ is topologically nilpotent in A'_0; thus $f(I)A'_0$ is an ideal of adic definition of A'_0, as stated.

(iii.c) follows easily from (iii.a) and corollary 8.3.19(ii).

(iii.d): By corollary 8.3.20(i), both B and B' are f-adic rings. Let $A_0 \subset A$ and $A'_0 \subset A'$ be subrings of definition with $f(A_0) \subset A'_0$; hence $B_0 := A_0 \cap B$ and $B'_0 := A'_0 \cap B'$ are subrings of definition of B and B' respectively, with $f(B_0) \subset B'_0$ (proposition 8.3.18(ii)); then the assertion follows from (iii.b).

(iv): Let A_0 and A'_0 be rings of definition of A and A'; after replacing A_0 by $A_0 \cap f^{-1}A'_0$ we may assume that $f(A_0) \subset A'_0$ (proposition 8.3.18(ii)); by assumption, $f(A_0)$ is open in A', so we may further replace A'_0 by $f(A'_0)$, and assume from start that f restricts to an open surjective map $A_0 \to A'_0$. In this case, let $I \subset A_0$ be any ideal of adic definition; it follows easily that $f(I)$ is an ideal of adic definition for A'_0, so f is adic.

(v) If f is a continuous open map, then for every pair of open ideals $J \subset A'$ and $I \subset A$ with $f(I) \subset J$, we have $f(I) \cdot A' \subset J$, and clearly $f(I) \cdot A'$ is open in A'. This shows that f is adic.

Conversely, if f is adic and $f(A)$ is open in A', let $J \subset f(A')$ be an open ideal of A'. For every open ideal $I \subset A$, the ideal $f(I) \cdot A'$ is open in A', hence the same holds for $f(I) \cdot J$; but $f(I) \cdot J \subset f(I)$, so $f(I)$ is open for every such I, and thus f is open. □

Example 8.3.27. Let $(A, \mathscr{T})$ be a topological ring, $f : A \to A'$ a surjective ring homomorphism, and endow A' with the topology $\mathscr{T}'$ induced by A via f. We have :

(i) It is easily seen that a subset $U' \subset A'$ is open if and only if $U' = f(U)$ for some open subset U of A. It follows that $(A', \mathscr{T}')$ is a topological ring.

(ii) Moreover, if $(A, \mathscr{T})$ is c-adic, the same holds for $(A', \mathscr{T}')$, and $f : (A, \mathscr{T}) \to (A', \mathscr{T}')$ is c-adic. Indeed, let $I \subset A$ be an ideal of c-adic definition, and set $J := f(I)$; by (i) we know that $f((I^n)^c)$ is open in A' for every $n \in \mathbb{N}$, and then claim 8.3.25 implies that $f((I^n)^c) = (J^n)^c$, so the latter is open in A' for every $n \in \mathbb{N}$, again by (i), and the system of such ideals is a fundamental system of open neighborhoods of 0 in A'.

(iii) Likewise, if $(A, \mathscr{T})$ is adic (resp. f-adic), the same holds for $(A', \mathscr{T}')$, and f is adic (resp f-adic).

Definition 8.3.28. For any f-adic ring A, set

$$X_A := \operatorname{Spec} A \qquad X_A^{\circ\circ} := \operatorname{Spec} A/A^{\circ\circ}A.$$

We call $X_A^{\circ\circ}$ the *non-analytic locus* of X_A and its complement $X_A \setminus X_A^{\circ\circ}$ the *analytic locus*.

Lemma 8.3.29. *Let A and B be two f-adic rings, $f : A \to B$ a continuous ring homomorphism, and set $\varphi := \mathrm{Spec}\, f$. We have :*

(i) $X_A^{\circ\circ} = \{\mathfrak{p} \in X_A \mid \mathfrak{p}$ *is open in* $A\}$.
(ii) φ *restricts to a map* $X_B^{\circ\circ} \to X_A^{\circ\circ}$.
(iii) *If f is an injective and open map, φ restricts to an isomorphism of schemes :*

$$X_B \setminus X_B^{\circ\circ} \xrightarrow{\sim} X_A \setminus X_A^{\circ\circ}.$$

(iv) *The homomorphism f is f-adic if and only if* $\varphi^{-1}(X_A^{\circ\circ}) = X_B^{\circ\circ}$.
(v) *Let $J \subset A$ be any ideal. Then J is open if and only if* $\mathrm{Spec}\, A/J \subset X_A^{\circ\circ}$.

Proof. (i) and (ii) are clear.

(v): If J is open, then the same holds for every prime ideal $\mathfrak{p}$ containing J; in light of (i) it then follows that $A^{\circ\circ} \subset \mathfrak{p}$ for every such $\mathfrak{p}$, so the radical of J also contains $A^{\circ\circ}$, *i.e.* $\mathrm{Spec}\, A/J \subset X_A^{\circ\circ}$. Conversely, suppose that the radical of J contains $A^{\circ\circ}$, and pick any subring of definition $A_0 \subset A$, and a finitely generated ideal $I \subset A_0$ of adic definition; it follows easily that J contains I^n for every sufficiently large $n \in \mathbb{N}$, so J is open.

Next, let us check that if f is adic, then $\varphi^{-1}(X_A^{\circ\circ}) = X_B^{\circ\circ}$. Indeed, pick subrings of definition $A_0 \subset A$, $B_0 \subset B$ such that $f(A_0) \subset B_0$, and an ideal of adic definition I of A_0; then $J := f(I) \cdot B_0$ is an ideal of adic definition of B_0 (lemma 8.3.24(iii.b)). It is easily seen that $X_A^{\circ\circ} = \mathrm{Spec}\, A/IA$ and $X_B^{\circ\circ} = \mathrm{Spec}\, B/JB$, whence the stated condition.

(iii): Notice first that f is f-adic in the current situation, so the foregoing already shows that $\varphi^{-1}(X_A^{\circ\circ}) = X_B^{\circ\circ}$. The latter means that $X_B \setminus X_B^{\circ\circ} = \bigcup_{s \in A^{\circ\circ}} \mathrm{Spec}\, B_s$. Now, let $\mathfrak{p} \in X_A \setminus X_A^{\circ\circ}$ be any non-open prime ideal of A; then there exists $s \in A^{\circ\circ}$ such that $s \notin \mathfrak{p}$, and it suffices to check that the localization $f_s : A_s \to B_s$ is an isomorphism. However, f_s is obviously injective; let $b \in B$ be any element; since A is open in B, there exists $n \in \mathbb{N}$ such that $s^n b \in A$, so f_s is surjective as well.

(iv): By the foregoing, we may assume that $\varphi^{-1}(X_A^{\circ\circ}) = X_B^{\circ\circ}$, and we check that f is adic. Indeed, pick a subring of definition $B_0 \subset B$; since f is continuous, $f^{-1}B_0$ is open in A_0, so we may find a subring of definition A_0 of A such that $A_0 \subset f^{-1}B_0$ (corollary 8.3.19(ii)). Let $f_0 : A_0 \to B_0$ be the restriction of f, and set $\varphi_0 := \mathrm{Spec}\, f_0$; we get a commutative diagram

$$\begin{array}{ccc} X_B & \xrightarrow{\varphi} & X_A \\ \downarrow & & \downarrow \\ X_{B_0} & \xrightarrow{\varphi_0} & X_{A_0} \end{array}$$

whose vertical arrows are induced by the corresponding inclusion maps. The assumption implies that $\varphi^{-1}(X_A \setminus X_A^{\circ\circ}) = X_B \setminus X_B^{\circ\circ}$; together with (iii), we deduce that $\varphi_0^{-1}(X_{A_0} \setminus X_{A_0}^{\circ\circ}) = X_{B_0} \setminus X_{B_0}^{\circ\circ}$, which is equivalent to $\varphi_0^{-1}(X_{A_0}^{\circ\circ}) = X_{B_0}^{\circ\circ}$. On the other hand, it suffices to show that f_0 is an adic ring homomorphism, so we may replace A and B by A_0 and B_0, and assume from start that A and B are adic topological rings. However, our assumption on $\varphi^{-1}(X_A^{\circ\circ})$ means that the radical of $f(A^{\circ\circ}) \cdot B$ equals $B^{\circ\circ}$; now, pick any finitely generated ideal of adic definition I of A (resp. J of B); then the radical of $f(I) \cdot B$ also equals $B^{\circ\circ}$, and according to (v), the latter condition implies that $f(I) \cdot B$ is open in B, so f is adic. □

Proposition 8.3.30. *With the notation of definition* 8.3.28, *let $U \subset X_A$ be a quasi-compact open subset containing the analytic locus. We have :*

(i) *There exists a unique ring topology $\mathscr{T}_U$ on $A_U := \mathscr{O}_{X_A}(U)$ such that $(A_U, \mathscr{T}_U)$ is f-adic and the restriction map $\rho_U : A \to A_U$ is open.*
(ii) *Suppose moreover that A admits a complete, separated and noetherian ring of definition. Then $(A_U, \mathscr{T}_U)$ is complete and separated.*

(iii) *Let $\varphi : A \to B$ be any morphism of f-adic topological rings, such that the image of* $\mathrm{Spec}\,\varphi$ *lies in U. Then the resulting map $\varphi_U : (A_U, \mathscr{T}_U) \to B$ is continuous.*

Proof. (i): Clearly, there exists a unique topology $\mathscr{T}_U$ on A_U such that ρ_U is open and $(A_U, \mathscr{T}_U)$ is a topological group for its additive group structure, and we need to check that $(A_U, \mathscr{T}_U)$ is a topological ring. Let I be a finitely generated ideal of adic definition for a ring of definition A_0 of A; the assertion comes down to the following. For every $a \in A_U$ and every $n \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that

$$a \cdot \rho_U(I^k) \subset \rho_U(I^n).$$

Now, pick a finite system of generators $(f_1, \dots, f_r)$ of I, and notice that $A^{\circ\circ}$ is the radical of the ideal IA° in the subring A° of A (remark 8.3.10(iv)), hence

$$X_A^{\circ\circ} = \mathrm{Spec}\,A/IA. \tag{8.3.31}$$

Let $j : U \to X_A$ be the inclusion map; then $j_*\mathscr{O}_U$ is a quasi-coherent $\mathscr{O}_{X_A}$-module ([59, Ch.I, Cor.9.2.2]), and on the other hand, from (8.3.31) and our assumption on U we see that $\mathrm{Spec}\,A_{f_i} \subset U$. Hence

$$A_{f_i} = \mathscr{O}_{X_A}(\mathrm{Spec}\,A_{f_i}) = j_*\mathscr{O}_U(\mathrm{Spec}\,A_{f_i}) = j_*\mathscr{O}_U(X_A)_{f_i} = A_U[f_i^{-1}] \qquad \text{for } i = 1, \dots, r.$$

Thus, there exists $m \in \mathbb{N}$ such that $f_i^m a \in \rho_U(A)$ for every $i = 1, \dots, r$. Then there also exists $t \in \mathbb{N}$ such that $\rho_U(I^t) \cdot f_i^m a \subset \rho_U(I^n)$ for $i = 1, \dots, r$, and the assertion follows easily. It is then also clear that the topology $\mathscr{T}_U$ is f-adic, and $\rho_U(A_0)$ is a ring of definition of A_U, with $\rho_U(I)$ as ideal of adic definition.

(ii): Let $A_0 \subset A$ be a noetherian, complete and separated subring of definition; it suffices to check that $A_0 \cap \mathrm{Ker}\,\rho_U$ is a closed ideal, but this holds, by [126, Th.8.10(i)].

(iii): Let $H \subset B$ be any open additive subgroup; then $\varphi_U^{-1}H$ contains the open subgroup $\rho_U(\varphi^{-1}H)$, so it is open, whence the contention. □

Lemma 8.3.32. *Let $f : (A, \mathscr{T}_A) \to (B, \mathscr{T}_B)$ be a c-adic morphism of topological rings. Then:*

(i) *The completion map $(A, \mathscr{T}_A) \to (A^\wedge, \mathscr{T}_A^\wedge)$ is c-adic, and A is c-adic if and only if the same holds for $A^\wedge$.*

(ii) *The separated completion $f^\wedge : (A^\wedge, \mathscr{T}_A^\wedge) \to (B^\wedge, \mathscr{T}_B^\wedge)$ of f is c-adic.*

(iii) *Let* **Q** *be any property of ring homomorphisms. If f is a c-adically* **Q** *morphism, the same holds for $f^\wedge$ (see definition* 8.3.23(iii)*).*

(iv) *If the topology $\mathscr{T}_A$ is I-adic for some finitely generated ideal $I \subset A$, the topology $\mathscr{T}_A^\wedge$ is $IA^\wedge$-adic.*

Proof. (i): Let $(I_\lambda \mid \lambda \in \Lambda)$ be any fundamental system of open ideals of A, and set $I_\lambda^\wedge := (I_\lambda A^\wedge)^c$ for every $\lambda \in \Lambda$. From remark 8.3.3(ii) we see that $(I_\lambda^\wedge \mid \lambda \in \Lambda)$ is a fundamental system of open neighborhoods of zero in $A^\wedge$, so $\mathscr{T}_A^\wedge$ is a linear topology, and the completion map $A \to A^\wedge$ is c-adic. In view of lemma 8.3.24(i.b), it follows that if A is c-adic, the same holds for $A^\wedge$. For the converse, suppose that $A^\wedge$ is c-adic, and pick an ideal of c-adic definition of the form $(IA^\wedge)^c$, where I is an open ideal of A; for every $n \in \mathbb{N}$ we let $(I^n)^c$ (resp. $(I^n)^\wedge$) be the topological closure of I^n in A (resp. in $A^\wedge$); then

$$(I^n)^c = A \cap (I^n)^\wedge \qquad \text{for every } n \in \mathbb{N}.$$

Now, let $J \subset A$ be any open ideal; the topological closure $J^\wedge$ of J in $A^\wedge$ is open in $A^\wedge$, so there exists $n \in \mathbb{N}$ such that $(I^n)^\wedge \subset J^\wedge$, and therefore $(I^n)^c \subset A \cap J^\wedge = J$, which says that A is c-adic with I as ideal of c-adic definition.

(ii): We know already that $\mathscr{T}_A^\wedge$ and $\mathscr{T}_B^\wedge$ are both linear topologies. Next, by assumption, for every open ideal $J \subset B$ there exists $\lambda \in \Lambda$ such that $(I_\lambda B)^c \subset J$; it follows that $(I_\lambda^\wedge B^\wedge)^c = (I_\lambda B^\wedge)^c \subset (JB^\wedge)^c$. Likewise, for every $\lambda \in \Lambda$ there exists an open ideal J of B such that $J \subset (I_\lambda B)^c$, and therefore $(JB^\wedge)^c \subset (I_\lambda^\wedge B^\wedge)^c$. Since the family $((JB^\wedge)^c \mid J \text{ open in } B)$

is a fundamental system of open ideals of B, it follows that the same holds for the family $((I^\wedge_\lambda B^\wedge)^c \mid \lambda \in \Lambda)$, whence the contention.

(iii): This follows easily from (ii), taking into account the natural isomorphisms

$$A^\wedge/(IA^\wedge)^c \xrightarrow{\sim} A/I \qquad \text{and} \quad B^\wedge/(JB^\wedge)^c \xrightarrow{\sim} B/J$$

for every open ideals I of A and J of B, as well as the equality : $((IA^\wedge)^c B^\wedge)^c = (IB^\wedge)^c$, which identify the map $A^\wedge/(IA^\wedge)^c \to B^\wedge/(IB^\wedge)^c$ induced by $f^\wedge$ with the map $A/I \to B/(IB)^c$ induced by f.

(iv) follows easily from remark 8.3.3(ii,iv). □

Proposition 8.3.33. *Let A be an f-adic topological ring, A_0 an open subring of A, and $A_0^\wedge$, $A^\wedge$ the separated completions of A_0 and respectively A. We have :*

(i) *$A^\wedge$ is an f-adic topological ring.*
(ii) *If A_0 is a ring of definition of A, then $A_0^\wedge$ is a ring of definition for $A^\wedge$.*
(iii) *The natural map $j : A \otimes_{A_0} A_0^\wedge \to A^\wedge$ is a ring isomorphism.*

Proof. By corollary 8.2.17(i), the natural map $A_0^\wedge \to A^\wedge$ is an open immersion, so (i) and (ii) follow from lemma 8.3.32(iv).

(iii): Suppose first that A_0 is a ring of definition of A. Since A_0 is open in A, the topology induced by A on $B := A/A_0$ via the projection $A \to B$ is discrete, so the completion map $B \to B^\wedge$ is an isomorphism of topological groups, and there follows a commutative ladder :

$$\begin{array}{ccccccccc} 0 & \to & A_0^\wedge & \to & A \otimes_{A_0} A_0^\wedge & \to & B \otimes_{A_0} A_0^\wedge & \to & 0 \\ & & \| & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j_B} & & \\ 0 & \to & A_0^\wedge & \to & A^\wedge & \to & B & \to & 0 \end{array}$$

whose bottom row is short exact, by proposition 8.2.13(v). It follows that the top row is exact as well, and to conclude, it suffices to check that j_B is an isomorphism. To this aim, let us write B as the colimit of the filtered system $(B_\lambda \mid \lambda \in \Lambda)$ of its finitely generated A_0-submodules; clearly it suffices to show that the restriction $j_\lambda : B_\lambda \otimes_{A_0} A_0^\wedge \to B_\lambda$ of j_B is an isomorphism for every such λ. However, for every $\lambda \in \Lambda$ there exists an open ideal $I_\lambda \subset A_0$ such that $I_\lambda B_\lambda = 0$, so j_λ factors as the composition of the isomorphism $B_\lambda \otimes_{A_0} A_0^\wedge \xrightarrow{\sim} B_\lambda \otimes_{A_0} A_0^\wedge/I_\lambda A_0^\wedge$ and the natural identifications $B_\lambda \otimes_{A_0} A_0^\wedge/I_\lambda A_0^\wedge \xrightarrow{\sim} B_\lambda \otimes_{A_0} A_0/I_\lambda \xrightarrow{\sim} B_\lambda$, whence the contention, in this case. Next, if A_0 is an arbitrary open subring of A, then A_0 is an f-adic ring (corollary 8.3.20(i)); we let A_1 be a ring of definition of A_0, and we notice that A_1 is also a ring of definition for A (proposition 8.3.18(ii)). By the foregoing, both natural maps $A_0 \otimes_{A_1} A_1^\wedge \to A_0^\wedge$ and $A \otimes_{A_1} A_1^\wedge \to A^\wedge$ are isomorphisms, so the same follows for j. □

Proposition 8.3.34. *Let A, B, B' be three f-adic topological rings and $f : B \to A$, $g : B \to B'$ two f-adic ring homomorphisms. Then we have :*

(i) *There exists a unique f-adic ring topology $\mathscr{T}_{A'}$ on $A' := B' \otimes_B A$ such that the resulting diagram (with $f' := \mathbf{1}_{B'} \otimes_B f$ and $g_A := g \otimes_B \mathbf{1}_A$)*

(8.3.35)
$$\begin{array}{ccc} B & \xrightarrow{f} & A \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g_A} \\ B' & \xrightarrow{f'} & (A', \mathscr{T}_{A'}) \end{array}$$

is cocartesian in the category $\mathbb{Z}\text{-}\mathsf{TopAlg}_{\mathbb{Z}\text{-lin}}$ (notation of definition 8.3.1(v)). Moreover, both f' and g_A are f-adic ring homomorphisms.

(ii) *Suppose additionally that f is open. Then the same holds for f', and (8.3.35) is cocartesian in the category $\mathbb{Z}\text{-}\mathsf{TopAlg}$.*

(iii) *In the situation of* (ii)*, suppose moreover that B is a ring of definition of A, and g induces an isomorphism $g^\wedge : B^\wedge \xrightarrow{\sim} B'^\wedge$ on separated completions; then*
  (a) *g_A induces an isomorphism $g_A^\wedge : A^\wedge \xrightarrow{\sim} A'^\wedge$ on separated completions*
  (b) *A'° (resp. $A'^{\circ\circ}$) is the image of $B' \otimes_B A^\circ$ (resp. of $B' \otimes_B A^{\circ\circ}$) in A'.*

Proof. (i): In light of corollary 8.3.19(i) and lemma 8.3.24(iii.b) we may find subrings of definitions $B_0 \subset B$, $A_0 \subset A$ and $B'_0 \subset B'$ such that f and g restrict to adic ring homomorphisms $f_0 : B_0 \to A_0$ and $g_0 : B_0 \to A'_0$. Let also $I_0 \subset B_0$ be an ideal of adic definition; denote by $\overline{A}'_0 \subset A'$ the image of the induced map $A'_0 := B'_0 \otimes_{B_0} A'_0 \to A'$, and endow $\overline{A}'_0$ with its $I_0\overline{A}'_0$-adic topology. There exists a unique group topology $\mathscr{T}'$ on A' such that $\overline{A}'_0$ is an open subgroup, and we claim that $\mathscr{T}'$ is a ring topology. This comes down to the following assertion. For every $a' \in A'$ and every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that

$$a' \cdot I_0^m \overline{A}'_0 \subset I_0^n \overline{A}'_0.$$

However, we may write a' as a finite sum of terms of the form $b' \otimes a$, with $a \in A$ and $b' \in B'$, so we are easily reduced to the case where $a' = b' \otimes a$ for such a and b'. But since f and g are f-adic, we may find, for every $n \in \mathbb{N}$, integers $m_1, m_2 \in \mathbb{N}$ such that $a \cdot I_0^{m_1} A_0 \subset I^n A_0$, and $b' \cdot I_0^{m_2} B'_0 \subset B'_0$; obviously $m := m_1 + m_2$ will do. It is now clear that f' and g_A are both f-adic ring homomorphisms for this topology $\mathscr{T}_{A'}$ on A'. Lastly, let $(C, \mathscr{T}_C)$ be any topological ring whose topology is $\mathbb{Z}$-linear, $h : B' \to C$ and $k : A \to C$ two continuous ring homomorphisms such that $k \circ f = h \circ g$; there follows a unique ring homomorphism $l : A' \to C$ such that $l \circ i = h$ and $l \circ g_A = k$. To conclude, we have to check that l is continuous, and it suffices to show that the restriction $\bar{l}_0 : \overline{A}'_0 \to C$ is continuous. But the topology of $\overline{A}'_0$ agrees with the quotient topology induced from the $I_0 A'_0$-adic topology $\mathscr{T}_{A'_0}$ on A'_0, so we reduced to checking that the induced map $l_0 : (A'_0, \mathscr{T}_{A'_0}) \to C$ is continuous in a neighborhood of 0. Thus, let U be a neighborhood of 0 in C; since $\mathscr{T}_C$ is $\mathbb{Z}$-linear, we may assume that U is a subgroup of C. Moreover, there exists an open neighborhood U' of 0 in C such that $U' \cdot U' \subset U$. Then we may find $n \in \mathbb{N}$ such that $k(I_0^n A_0), h(I_0^n B'_0) \subset U'$, so that $l_0(I_0^{2n} A'_0) \subset U$, whence the contention.

(ii): Indeed, if f is open, we may replace A_0 by $f(B_0)$ in the foregoing, in which case the induced map $B'_0 \to A'_0$ is surjective and open; since the same holds for the projection $A'_0 \to \overline{A}'_0$, we see that f' is open. Next, let C be any topological ring, and h, k two continuous ring homomorphisms as in the foregoing. We have to show that the induced map $l : A' \to C$ is continuous, and we are again reduced to showing that the same holds for its restriction $\bar{l}_0 : \overline{A}'_0 \to C$. But we have just seen that the topology of $\overline{A}'_0$ agrees with the quotient topology induced by the surjective map $f'_0 : B'_0 \to \overline{A}'$ (the restriction of f'); thus $\bar{l}_0$ is continuous if and only if the same holds for $\bar{l}_0 \circ f'_0$, and the latter is none else than the restriction of h.

(iii): Indeed, under these assumptions, we get a cocartesian diagram of rings

$$\begin{array}{ccccccccc} B' & \xrightarrow{\quad f' \quad} & A' & & & & & & \\ {\scriptstyle j}\downarrow & & \downarrow & & & & & & \\ B'^\wedge & \xrightarrow{g^{\wedge -1}} & B^\wedge & \longrightarrow & B^\wedge \otimes_{B'} A' & \xrightarrow{\sim} & B^\wedge \otimes_B A & \xrightarrow{\sim} & A^\wedge \end{array}$$

where j is the completion map, and the righmost isomorphism on the bottom row is provided by proposition 8.3.33(iii). Since the natural map $B^\wedge \to A^\wedge$ is injective, it follows especially that $\mathrm{Ker}\, f' \subset \mathrm{Ker}\, j$, and therefore f' induce an isomorphism $B'^\wedge \xrightarrow{\sim} \overline{A}'^\wedge_0$ on separated completions.

Invoking again proposition 8.3.33(iii) we deduce a commutative diagram

$$\begin{array}{ccccc} B^\wedge \otimes_B A & \longrightarrow & B'^\wedge \otimes_{B'} A' & \longrightarrow & \overline{A}'^\wedge_0 \otimes_{\overline{A}'_0} A' \\ \downarrow & & & & \downarrow \\ A^\wedge & & \xrightarrow{\quad g_A^\wedge \quad} & & A'^\wedge \end{array}$$

whose top horizontal and vertical arrows are isomorphisms, so the same holds for the bottom horizontal arrow. Set $\overline{A} := \mathrm{Im}\, g_A$, and endow $\overline{A}$ with the quotient topology induced by A via the surjection $A \to \overline{A}$. Recall that the topology of A (resp. of A') agrees with the topology induced by $A^\wedge$ (resp. by $A'^\wedge$) via the completion map $A \to A^\wedge$ (resp. $A' \to A'^\wedge$); since we already know that $g_A^\wedge$ is an isomorphism of topological rings, it follows easily that the topology of $\overline{A}$ agrees with the topology induced by A' via the inclusion map $\overline{A} \to A'$ (details left to the reader). Moreover, recall that g_A is f-adic, so lemma 8.3.24(iii) implies that the image of $B' \otimes_B A^\circ$ lies in A'°; for the converse, say that $x \in A'^\circ$, and write $x = \sum_{i=1}^n b_i \otimes a_i$ for some $n \in \mathbb{N}$ and elements $a_i \in A$, $b_i \in B'$, for $i = 1, \dots, n$. Let $I \subset B$ be any ideal of definition; we may find $m \in \mathbb{N}$ such that $a_i I^m \subset A^{\circ\circ}$ for every $i = 1, \dots, n$, and since $g^\wedge$ is an isomorphism, we may also find $b'_i \in B$, $b''_i \in I^m B'$ such that $b_i = g(b'_i) + b''_i$ for $i = 1, \dots, n$. Clearly $b''_i \otimes a_i$ lies in the image of $B' \otimes_B A^{\circ\circ}$ for every $i = 1, \dots, n$, so $y := 1 \otimes (\sum_{i=1}^n a_i b'_i) \in \overline{A} \cap A'^\circ \subset \overline{A}^\circ$, and it remains only to observe that $\overline{A}^\circ = g_A(A^\circ)$, because the kernel of the surjection $A \to \overline{A}$ is contained in the kernel of the completion map $A \to A^\wedge$ (details left to the reader). Likewise, if $x \in A'^{\circ\circ}$, then $y \in \overline{A} \cap A'^{\circ\circ} = \overline{A}^{\circ\circ} = g(A^{\circ\circ})$, which shows that $A'^{\circ\circ}$ lies in the image of $B' \otimes_B A^{\circ\circ}$, and the converse inclusion is obvious. $\square$

8.3.36. Let A be any topological ring; for any two subsets $S, T \subset A$ we denote by $S \cdot T$ the additive subgroup of A generated by $\{st \mid s \in S,\ t \in T\}$. We define inductively

$$T^0 := \{1\} \qquad \text{and} \qquad T^{n+1} := T \cdot T^n \qquad \text{for every } n \in \mathbb{N}.$$

Now, let $(T_\lambda \mid \lambda \in \Lambda)$ be any (small) family of subsets of A; then, for every $\nu \in \mathbb{N}^{(\Lambda)}$ the additive subgoup

$$T^\nu := \prod_{\lambda \in \Lambda} T_\lambda^{\nu(\lambda)}$$

is well defined. We consider the polynomial A-algebra

$$A[X_\bullet] := A[X_\lambda \mid \lambda \in \Lambda]$$

and we also set $X^\nu := \prod_{\lambda \in \Lambda} X_\lambda^{\nu(\lambda)}$ for every such ν. Furthermore, let $\mathscr{U}_A$ be the set of all open neighborhood U of 0 in A, and set

$$U[T_\bullet X_\bullet] := \Big\{ \sum_{\nu \in \mathbb{N}^{(\Lambda)}} a_\nu X^\nu \in A[X_\bullet] \mid a_\nu \in T^\nu \cdot U \text{ for every } \nu \in \mathbb{N}^{(\Lambda)} \Big\} \qquad \text{for every } U \in \mathscr{U}_A.$$

Proposition 8.3.37. *In the situation of* (8.3.36), *suppose that A is an f-adic ring, and $T_\lambda \cdot A$ is an open ideal of A for every $\lambda \in \Lambda$, and let $A_0 \subset A$ be a subring of definition. Then we have :*

(i) *There exists a unique ring topology $\mathscr{T}_{A,T_\bullet}$ on $A[X_\bullet]$ for which the family*

$$\mathscr{U}_{A[X_\bullet]} := (U[T_\bullet X_\bullet] \mid U \in \mathscr{U}_A)$$

is a fundamental system of open neighborhoods of 0.

(ii) *The topological ring*

$$A[X_\bullet]_T := (A[X_\bullet], \mathscr{T}_{A,T_\bullet})$$

is f-adic, and $A_0[T_\bullet X_\bullet]$ is a subring of definition of $A[X_\bullet]$. Moreover, if $I \subset A_0$ is an ideal of adic definition, then $I[T_\bullet X_\bullet]$ is an ideal of adic definition for $A_0[T_\bullet X_\bullet]$.

(iii) *The structure map* $h : A \to A[X_\bullet]$ *is f-adic for the topology* $\mathscr{T}_{A,T_\bullet}$*, and the subset* $\{tX_\lambda \mid \lambda \in \Lambda,\ t \in T_\lambda\}$ *is power bounded in* $A[X_\bullet]$.

(iv) *Moreover, for every f-adic ring* B*, every family* $(b_\lambda \mid \lambda \in \Lambda)$ *of elements of* B*, and every continuous ring homomorphism* $f : A \to B$ *such that* $F := \{f(t) \cdot b_\lambda \mid \lambda \in \Lambda,\ t \in T_\lambda\}$ *is power bounded in* B*, there exists a unique continuous ring homomorphism* $g : (A[X_\bullet], \mathscr{T}_{A,T_\bullet}) \to B$ *with* $g \circ h = f$ *and* $g(X_\lambda) = b_\lambda$ *for every* $\lambda \in \Lambda$.

Proof. (i): Notice that every element of $\mathscr{U}_{A[X_\bullet]}$ is an additive subgroup of $A[X_\bullet]$; it follows easily that there exists a unique group topology on $A[X_\bullet]$ that admits $\mathscr{U}_{A[X_\bullet]}$ as a fundamental system of neighborhoods of 0. Moreover, for $U \in \mathscr{U}_A$ we may find $V \in \mathscr{U}_A$ with $V \cdot V \subset U$, whence $V[T_\bullet X_\bullet] \cdot V[T_\bullet X_\bullet] \subset U[T_\bullet X_\bullet]$. Therefore, it remains only to check that for every $P \in A[X_\bullet]$ and every $U \in \mathscr{U}_A$ there exists $V \in \mathscr{U}_A$ such that

$$P \cdot V[T_\bullet X_\bullet] \subset U[T_\bullet X_\bullet].$$

To this aim, notice first that, since A is f-adic and $T_\lambda \cdot A$ is open in A for every $\lambda \in \Lambda$, then $T^\nu \cdot U \in \mathscr{U}_A$ for every $\nu \in \mathbb{N}^{(\Lambda)}$ and every $U \in \mathscr{U}_A$ (details left to the reader). Say that $P = \sum_{\nu \in \mathbb{N}^{(\Lambda)}} a_\nu X^\nu$; since $a_\nu = 0$ except for finitely many $\nu \in \mathbb{N}^{(\Lambda)}$, we may find $V \in \mathscr{U}_A$ small enough that $a_\nu \cdot V \in T^\nu \cdot U$ for every $\nu \in \mathbb{N}^{(\Lambda)}$. It is easily seen that such V will do.

(ii): From the foregoing, it is clear that $A_0[T_\bullet X_\bullet]$ is an open subring of $A[X_\bullet]$ and $I[T_\bullet X_\bullet]$ is an ideal of $A_0[T_\bullet X_\bullet]$. Moreover, $I^n[T_\bullet X_\bullet] = I[T_\bullet X_\bullet]^n$ for every $n \in \mathbb{N}$, so the topology $\mathscr{T}_{A,T_\bullet}$ induces the $I[T_\bullet X_\bullet]$-adic topology on $A_0[T_\bullet X_\bullet]$.

(iii): The continuity of h is obvious; moreover, h restricts to a map $A_0 \to A_0[T_\bullet X_\bullet]$, and $I \cdot A_0[T_\bullet X_\bullet] = I[T_\bullet X_\bullet]$, so h is f-adic. Lastly, we have $tX_\lambda \in A_0[T_\bullet X_\bullet]$ for every $\lambda \in \Lambda$ and $t \in T_\lambda$, whence the assertion.

(iv): Obviously there exists a unique map of A-algebras $g : A[X_\bullet] \to B$ such that $g(X_\lambda) = b_\lambda$ for every $\lambda \in \Lambda$, and it suffices to check that g is continuous. To this aim, for any open additive subgroup U of B pick $V \in \mathscr{U}_B$ such that $F^n \cdot V \subset U$ for every $n \in \mathbb{N}$. Set also $W := f^{-1}V$; then $g(W[T_\bullet X_\bullet]) \subset U$, whence the contention. □

8.3.38. *Modules of finite type over topological rings.* Let $(A, \mathscr{T}_A)$ be a topological ring, and M any A-module of finite type. We choose any surjective A-linear map $\pi : A^{\oplus n} \to M$ (for some $n \in \mathbb{N}$) and we endow $A^{\oplus n} := A \times \cdots \times A$ with the product of the topologies of its direct factors, and M with the topology $\mathscr{T}_M^A$ induced by $A^{\oplus n}$ via the map π.

Lemma 8.3.39. *With the notation of* (8.3.38)*, the following holds :*

(i) *The topology* $\mathscr{T}_M^A$ *is independent of the choice of* π.

(ii) $\pi : (A, \mathscr{T}_A)^n \to (M, \mathscr{T}_M^A)$ *is an open map.*

Proof. (i): Let $\pi' : A^{\oplus m} \to M$ be any other A-linear surjection; we define $\pi'' : A^{\oplus n+m} = A^{\oplus n} \oplus A^{\oplus m} \to M$ as the unique A-linear map whose restriction to $A^{\oplus n} \oplus 0$ agrees with π and whose restriction to $0 \oplus A^{\oplus m}$ agrees with π'. It suffices to check that the topologies on M induced by $A^{\oplus n}$ and $A^{\oplus m}$ via π and respectively π' agree with the topology $\mathscr{T}''$ induced by $A^{\oplus n+m}$ via π''. Thus, we are reduced to comparing $\mathscr{T}_M^A$ and $\mathscr{T}''$. Now, let $e_1, \dots, e_{n+m}$ be the standard basis of $A^{\oplus n+m}$, and for each $i = 1, \dots, m$ choose $f_i \in A^{\oplus n} \oplus 0$ such that $\pi''(f_i) = \pi''(e_{i+n})$; we consider the A-linear automorphism

$$\omega : A^{\oplus n+m} \xrightarrow{\sim} A^{\oplus n+m} \qquad e_j \mapsto \begin{cases} e_j & \text{for } j = 1, \dots, n \\ e_j - f_{j-n} & \text{for } j = n+1, \dots, n+m. \end{cases}$$

Clearly $\omega : (A, \mathscr{T}_A)^{n+m} \xrightarrow{\sim} (A, \mathscr{T}_A)^{n+m}$ is a homeomorphism, and the restriction of $\pi'' \circ \omega$ to $A^{\oplus n} \oplus 0$ still agrees with π. We may then replace π'' by $\pi'' \circ \omega$ and assume from start that $0 \oplus A^{\oplus m} \subset \operatorname{Ker} \pi''$. Now, let $U \subset M$ be any subset; we have $U \in \mathscr{T}_M^A$ if and only if $\pi^{-1}U$

is open in $(A, \mathscr{T}_A)^n$, and the latter holds if and only if $(\pi^{-1}U) \times A^{\oplus m}$ is open in $(A, \mathscr{T}_A)^{n+m}$, which in turn is equivalent to $U \in \mathscr{T}''$, whence the contention.

(ii): Let $U \subset A^{\oplus n}$ be any open subset and set $U' := U + \mathrm{Ker}\,\pi := \{u+v \mid u \in U,\ v \in \mathrm{Ker}\,\pi\}$; then U' is open in $A^{\oplus n}$ and $\pi^{-1}(\pi U) = U'$, whence the contention. □

Definition 8.3.40. In the situation of (8.3.38), we call $\mathscr{T}^A_M$ the *canonical topology* of the A-module M of finite type.

Proposition 8.3.41. *Let $(A, \mathscr{T}_A)$ be a topological ring, $M_1, \dots, M_k$ a finite sequence of A-modules of finite type, $(N, \mathscr{T}_N)$ a topological A-module and $\beta : M_1 \times \cdots \times M_k \to N$ any A-multilinear map. Then we have :*

(i) *The canonical topology of the A-module $M_1 \times \cdots \times M_k$ is the product of the canonical topologies $\mathscr{T}^A_{M_1}, \dots, \mathscr{T}^A_{M_k}$.*
(ii) *β is a continuous map $(M_1, \mathscr{T}^A_{M_1}) \times \cdots \times (M_k, \mathscr{T}^A_{M_K}) \to (N, \mathscr{T}_N)$.*

Proof. (i): By a simple induction on k, we are easily reduced to the case where $k = 2$. Let then $\mathscr{T}'$ be the product of the topologies $\mathscr{T}^A_{M_1}$ and $\mathscr{T}_{M_2}$; pick A-linear surjections $\pi_i : A^{\oplus n_i} \to M_i$ for $i = 1, 2$. It is easily seen that if $U \in \mathscr{T}'$, then $(\pi_1 \times \pi_2)^{-1}U$ is open in $(A, \mathscr{T}_A)^{n_1+n_2}$, hence $U \in \mathscr{T}^A_{M_1 \times M_2}$ (details left to the reader). Conversely, if $U \in \mathscr{T}^A_{M_1 \times M_2}$, then $(\pi_1 \times \pi_2)^{-1}U$ is a union of subsets of the form $V_1 \times V_2$, where $V_i \subset A^{\oplus n_1}$ is an arbitrary open subset, for $i = 1, 2$; therefore $(\pi_1 \times \pi_2)(V_1 \times V_2) = (\pi_1 V_1) \times (\pi_2 V_2) \in \mathscr{T}''$ since π_1 and π_2 are open maps (lemma 8.3.39(ii)), and so $U \in \mathscr{T}''$.

(ii): Pick A-linear surjections $\pi_i : A^{\oplus n_i} \to M_i$ for $i = 1, \dots, k$, and set $n := \sum_{i=1}^k n_i$ and $\beta' := \beta \circ (\pi_1 \times \cdots \times \pi_k)$. It is easily seen that $\beta' : (A, \mathscr{T}_A)^n \to (N, \mathscr{T}_N)$ is a continuous map (details left to the reader). Now, let $U \subset N$ be any open subset; we have $\beta^{-1}(U) = (\pi_1 \times \cdots \times \pi_k)(\beta'^{-1}U)$, which is open in $M_1 \times \cdots \times M_k$, since $\pi_1 \times \cdots \times \pi_k$ is an open map (lemma 8.3.39(ii)). □

Corollary 8.3.42. *Let $(A, \mathscr{T}_A)$ be any topological ring, $B \to A$ a ring homomorphism, $I \subset A$ an ideal, M and N two A-modules of finite type. The following holds :*

(i) *$(M, \mathscr{T}^A_M)$ is a topological A-module.*
(ii) *Every A-linear surjection $M \to N$ is a continuous open map $(M, \mathscr{T}^A_M) \to (N, \mathscr{T}^A_N)$.*
(iii) *If the topology $\mathscr{T}_A$ is B-linear (resp. I-adic), the same holds for $\mathscr{T}^A_M$.*

Proof. Pick an A-linear surjection $\pi : A^{\oplus n} \to M$.

(i): Let $\sigma_M : M \oplus M \to M$ and $\mu_M : A \times M \to M$ be respectively the addition law and the scalar multiplication law of M; we need to check that these maps are continuous for the product topologies $(M, \mathscr{T}^A_M) \times (M, \mathscr{T}^A_M)$ and $(A, \mathscr{T}_A) \times (M, \mathscr{T}^A_M)$. However, we have the commutative diagrams

$$\begin{array}{ccc} A^{\oplus n} \oplus A^{\oplus n} & \xrightarrow{\sigma_{A^n}} & A^{\oplus n} \\ \downarrow{\scriptstyle \pi\oplus\pi} & & \downarrow{\scriptstyle \pi} \\ M \oplus M & \xrightarrow{\sigma_M} & M \end{array} \qquad \begin{array}{ccc} A \times A^{\oplus n} & \xrightarrow{\mu_{A^n}} & A^{\oplus n} \\ \downarrow{\scriptstyle \mathbf{1}_A\times\pi} & & \downarrow{\scriptstyle \pi} \\ A \times M & \xrightarrow{\mu_M} & M \end{array}$$

where σ_{A^n} and μ_{A^n} are the addition law and the scalar multiplication law of $A^{\oplus n}$, which are clearly continuous for the product topologies. Now, if $U \in \mathscr{T}^A_M$, then $V := \pi^{-1}U$ is open in $(A, \mathscr{T}_A)^n$ and $W := \sigma^{-1}_{A^n}(V)$ is open in $A^{\oplus n} \oplus A^{\oplus n}$; hence $\sigma^{-1}_M(U) = (\pi \oplus \pi)(W)$ is open in $M \oplus M$, since $\pi \oplus \pi$ is open (lemma 8.3.39(ii)). Likewise one checks the continuity of μ_M.

(ii): The continuity of $f : M \to N$ follows from (i) and proposition 8.3.41(ii). Next, we easily reduce to checking that the map $f \circ \pi : (A, \mathscr{T}_A)^n \to (N, \mathscr{T}^A_N)$ is open; but this map is surjective, so the assertion follows from lemma 8.3.39(i).

(iii): Suppose that $(J_\lambda \mid \lambda \in \Lambda)$ is a fundamental system of open neighborhoods of $0 \in A$ consisting of B-submodules; since π is open, the family $(\pi(J_\lambda^{\oplus n}) \mid \lambda \in \Lambda)$ is a fundamental system of open neighborhoods of $0 \in M$ (details left to the reader); in view of (i), the assertion follows. Likewise, if $\mathscr{T}_A$ is the I-adic topology, then $(I^k A^{\oplus n} \mid k \in \mathbb{N})$ is a fundamental system of open neighborhoods of $0 \in A^{\oplus n}$, so $(I^k M \mid k \in \mathbb{N})$ is a fundamental system of open neighborhoods of $0 \in M$, *i.e.* $\mathscr{T}_M^A$ is the I-adic topology of M. □

Remark 8.3.43. (i) Let $(A, \mathscr{T}_A)$ be a topological ring and B any finite A-algebra. In light of proposition 8.3.41(ii) and corollary 8.3.42(i), it is easily seen that $(B, \mathscr{T}_B^A)$ is a topological A-algebra; moreover, if $\mathscr{T}_A$ is the I-adic topology (for some ideal $I \subset A$), then $\mathscr{T}_B^A$ is the IB-adic topology (corollary 8.3.42(ii)).

(ii) Let N be any finite B-module. On the one hand, we may endow N with its canonical topology $\mathscr{T}_N^B$ as B-module of finite type (that is, relative to the topological ring $(B, \mathscr{T}_B^A)$); on the other hand, by restriction of scalars, N is also an A-module of finite type, so we have as well the canonical topology $\mathscr{T}_N^A$. We claim that these topologies coincide. Indeed, pick an A-linear surjection $\pi : A^{\oplus n} \to B$ and a B-linear surjection $\pi' : B^{\oplus m} \to N$; there follows an A-linear surjection $\pi^{\oplus m} : A^{\oplus nm} \to B^{\oplus m}$ and by virtue of proposition 8.3.41(i) the product topology of $B^{\oplus m} = (B, \mathscr{T}_B^A) \times \cdots \times (B, \mathscr{T}_B^A)$ agrees with the topology induced by $A^{\oplus nm}$ via $\pi^{\oplus m}$. Then the topology $\mathscr{T}_N^B$ is also induced by $\pi' \circ \pi^{\oplus m}$, whence the contention.

(iii) In the situation of (i), suppose that the topology $\mathscr{T}_A$ is f-adic; let $A_0 \subset A$ be a subring of definition, $I_0 \subset A_0$ a finitely generated ideal of adic definition and $(x_\lambda \mid \lambda \in \Lambda)$ a finite system of generators of I_0. Say that $B = \sum_{i=1}^n Ab_i$ for some elements $b_1, \dots, b_n \in B$ with $b_1 := 1$, and let $\pi : A^{\oplus n} \to B$ be the A-linear surjection such that $\pi(a_1, \dots, a_n) = \sum_{i=1}^n a_i b_i$ for every $a_1, \dots, a_n \in A$. We have a system $(c_{ijk} \mid i, j, k = 1, \dots, n)$ of elements of A such that

$$b_i b_j = \sum_{k=1}^n c_{ijk} b_k \qquad \text{for every } i, j = 1, \dots, n.$$

After replacing I_0 by I_0^k for $k \in \mathbb{N}$ large enough, we may assume that $I_0 c_{ijk} \in A_0$ for every $i, j, k = 1, \dots, n$. Let $B_0 \subset B$ be the A_0-submodule generated by $\{1\} \cup \{b_i x_\lambda \mid \lambda \in \Lambda, i = 2, \dots, n\}$. It is easily seen that B_0 is a subring of B, and moreover $J := \pi(I_0^{\oplus n}) \subset B_0$, so B_0 is open in B (lemma 8.3.39(ii)). Furthermore, a simple calculation shows that J is an ideal of B_0, and clearly $I_0 B_0 \subset J$. Lastly, the family $(\pi((I_0^{k+1})^{\oplus n}) \mid k \in \mathbb{N})$ is a fundamental system of neighborhoods of $0 \in B$, and we have $\pi((I_0^{k+1})^{\oplus n}) = I_0^k J$ for every $k \in \mathbb{N}$, so the topology of B_0 induced by the inclusion into $(B, \mathscr{T}_B^A)$ is $I_0 B_0$-adic. We conclude that the topology $\mathscr{T}_B^A$ is f-adic, and the structure map $A \to B$ is f-adic and restricts to an adic finite ring homomorphism $A_0 \to B_0$ between subrings of definition.

8.3.44. *Linearly compact modules.* Let A be any ring, and M any A-module; recall that a *coset* of M is a subset of the form $x + N$, where $N \subset M$ is any submodule, and $x \in M$ any element. If A is a topological ring, and if M is a topological A-module, we say that a coset $x + N$ is *open* (resp *closed*) if it is an open (resp. closed) subset of M, equivalently, if N is an open (resp. closed) submodule. We say that a collection $(x_i + N_i \mid i \in I)$ of cosets of M *has the finite intersection property*, if $\bigcap_{j \in J}(x_j + N_j) \neq \varnothing$ for every finite subset $J \subset I$.

Definition 8.3.45. Let A be a topological ring, M a topological A-module whose topology is A-linear and separated. We say that M is *linearly compact* if we have $\bigcap_{i \in I}(x_i + N_i) \neq \varnothing$ for every system $(x_i + N_i \mid i \in I)$ of open cosets of M with the finite intersection property.

Remark 8.3.46. Let A be a topological ring, M a topological A-module whose topology is A-linear, and $N \subset M$ a submodule that is closed for the topology of M.

(i) Then N is the intersection of the open submodules that contain it. Indeed, for every $x \in M \setminus N$ there exists an open submodule $N' \subset M$ with $x + N' \cap N = \varnothing$, so that x does not lie in the open submodule $N + N'$, whence the assertion.

(ii) Suppose that M is linearly compact, and let $(x_i + N_i \mid i \in I)$ be a family of closed cosets with the finite intersection property. Then it follows easily from (i) that $\bigcap_{i\in I}(x_i + N_i) \neq \varnothing$: the details are left to the reader.

(iii) Endow N with the topology induced by M. Then, if M is linearly compact, the same holds for N. Indeed, if $(x_i + N_i \mid i \in I)$ is a family of open cosets of N with the finite intersection property, then each $x_i + N_i$ is also a closed coset in N, therefore in M as well, and one concludes with (ii).

(iv) Moreover, endow M/N with the topology induced by the projection $p : M \to M/N$; then if M is linearly compact, the same holds for M/N. Indeed, clearly the topology of M/N is A-linear and separated, since N is closed in M, and if $(x_i + Q_i \mid i \in I)$ is any family of open cosets of M/N with the finite intersection property, then each $p^{-1}(x_i + Q_i)$ is an open coset of M, so that $\bigcap_{i\in I}(x_i + Q_i) = p(\bigcap_{i\in I} p^{-1}(x_i + Q_i)) \neq \varnothing$.

(v) Let $f : M \to M'$ be a continuous homomorphism of A-modules, and suppose that M is linearly compact, and that the topology of M' is A-linear and separated. Then $f(M)$ is a closed subset of M'. Indeed, let x be any element of the topological closure of $f(M)$ in M', and denote by $\mathscr{U}$ the system of open submodules of M'; then $f(M) \cap (x + N') \neq \varnothing$ for every $N' \in \mathscr{U}$, so that the family of cosets $(f^{-1}(x + N') \mid N' \in \mathscr{U})$ of M has the finite intersection property. By assumption, there exists $y \in \bigcap_{N'\in\mathscr{U}} f^{-1}(x + N')$, and then $x - f(y) \in \bigcap_{N'\in\mathscr{U}} N' = \{0\}$, whence the assertion.

(vi) If M is linearly compact, the completion map $j : M \to M^\wedge$ is an isomorphism of topological modules. Indeed, it follows easily from corollary 8.2.17(iii) that the topology of $M^\wedge$ is A-linear, hence the image of j is a closed submodule, by (v); but then j is bijective, since its image is dense in $M^\wedge$, and since M is separated. Then we conclude with theorem 8.2.8(ii).

(vii) If M is an artinian A-module, then M is linearly compact for its discrete topology. Indeed, let $(x_i + N_i \mid i \in I)$ be any family of cosets of M with the finite intersection property; since M is artinian, there exists a finite subset $J \subset I$ such that $N_J := \bigcap_{j\in J} N_j = \bigcap_{i\in I} N_i$, and notice that $Q_J := \bigcap_{j\in J}(x_i + N_i) = y + N_J$ for any $y \in Q$. Now, for every $i \in I$ we have $N_J \subset N_i$, and since $Q_J \cap (x_i + N_i) \neq \varnothing$, we must have $Q_J \subset x_i + N_i$, so that $\bigcap_{i\in I}(x_i + N_i) = Q_J \neq \varnothing$.

Proposition 8.3.47. *Let A be a topological ring, and $(M_\lambda \mid \lambda \in \Lambda)$ a family of linearly compact topological A-modules. Then the topological product $\prod_{\lambda\in\Lambda} M_\lambda$ is linearly compact.*

Proof. It is easily seen that the topology of $M := \prod_{\lambda\in\Lambda} M_\lambda$ is A-linear and separated. Hence, let $\mathscr{U}$ be any set of open cosets of M with the finite intersection property; it remains to check that $\bigcap_{U\in\mathscr{U}} U \neq \varnothing$. To this aim, consider the set $\mathscr{F}$ of all such sets; we endow $\mathscr{F}$ with the partial ordering such that $\mathscr{U} \leq \mathscr{U}'$ if and only if $\mathscr{U} \subset \mathscr{U}'$. Then, let also $\mathscr{F}/\mathscr{U}$ be the subset of all elements of $\mathscr{F}$ larger than $\mathscr{U}$. If $(\mathscr{U}_i \mid i \in I)$ is any totally ordered subset of $\mathscr{F}/\mathscr{U}$, it is easily seen that $\bigcup_{i\in I} \mathscr{U}_i$ lies also in $\mathscr{F}/\mathscr{U}$; by Zorn's lemma, $\mathscr{F}/\mathscr{U}$ then admits a maximal element $\mathscr{V}$. Clearly it suffices to check that $\bigcap_{V\in\mathscr{V}} V \neq \varnothing$, hence we may assume from start that $\mathscr{U}$ is maximal in $\mathscr{F}$. Next, since the projection $p_\lambda : M \to M_\lambda$ is an open map, $(p_\lambda(U) \mid U \in \mathscr{U})$ is a system of open cosets of M_λ, for every $\lambda \in \Lambda$, and obviously it has the finite intersection property, since the same holds for $\mathscr{U}$. Hence, for every $\lambda \in \Lambda$ pick $x_\lambda \in \bigcap_{U\in\mathscr{U}} p_\lambda(U)$. Notice that $p_\lambda(p_\lambda^{-1}(x_\lambda + N) \cap U) = (x_\lambda + N) \cap p_\lambda(U)$ for every $\lambda \in \Lambda$, every $U \in \mathscr{U}$, and every open submodule $N \subset M_\lambda$. Especially, $p_\lambda^{-1}(x_\lambda + N) \cap U \neq \varnothing$ for every such λ, U and N. Moreover, for every finite subset $\mathscr{U}' \subset \mathscr{U}$ we have $\bigcap_{U\in\mathscr{U}'} U \in \mathscr{U}$, by maximality of $\mathscr{U}$. Hence, for every λ, U and N as in the foregoing, the family $\mathscr{U} \cup \{p_\lambda^{-1}(x_\lambda + N)\}$ has the finite intersection property, so that it must coincide with $\mathscr{U}$, by maximality of $\mathscr{U}$. Especially, for every finite subset $\Lambda' \subset \Lambda$, and every system $(N_\lambda \mid \lambda \in \Lambda')$ such that N_λ is an open submodule of M_λ for every $\lambda \in \Lambda'$, we have $U \cap \bigcap_{\lambda\in\Lambda'} p_\lambda^{-1}(x_\lambda + N_\lambda) \neq \varnothing$ for every $U \in \mathscr{U}$. Since U is a closed subset of M, it follows that $(x_\lambda \mid \lambda \in \Lambda) \in U$ for every $U \in \mathscr{U}$, as required. □

Part (ii) of the following corollary is already found in [106, Th.7.1].

Corollary 8.3.48. *Let A be a topological ring, J any category, and $F : J \to A\text{-}\mathsf{TopMod}$ any functor such that Fj is a linearly compact A-module for every $j \in \mathrm{Ob}(J)$. We have :*

(i) $\lim_J F$ *is a linearly compact A-module.*

(ii) *Let also $\varphi : A\text{-}\mathsf{TopMod} \to A\text{-}\mathbf{Mod}$ be the forgetful functor, and suppose moreover that J is cofiltered. Then $R^p \lim_J(\varphi \circ F) = 0$ for every $p > 0$.*

Proof. (i): Set $P := \prod_{j\in\mathrm{Ob}(J)} Fj$ and $Q := \prod_{\varphi\in\mathrm{Morph}(J)} F \circ \mathsf{t}(\varphi)$, where $\mathsf{t} : \mathsf{Morph}(J) \to J$ is the target functor of (1.1.30). According to the proof of proposition 1.2.22(i), the limit L of F is represented by the kernel of a continuous map $P \to Q$; since P and Q are both linearly compact (proposition 8.3.47), it follows that L is a closed submodule of P, and we conclude with remark 8.3.46(iii).

(ii): Set $G := \varphi \circ F$. Let $\mathscr{F}$ be the set of finite subsets $S \subset \mathrm{Morph}(J)$, and endow $\mathscr{F}$ with the partial ordering induced by inclusion of finite subsets. For every $S \in \mathscr{F}$, let J_S be the smallest subcategory of J with $J_S \subset \mathrm{Morph}(J_S)$, and denote by $G_S : J_S \to A\text{-}\mathbf{Mod}$ the restriction of F; we compute $R\lim_J G$ and $R\lim_{J_S} G_S$ by the canonical resolutions $\lim_J \nabla G$ and $\lim_{J_S} \nabla G_S$ exhibited in example 7.10.10(ii,iii) (with the notation of *loc.cit.*, we take $f : I \to J$ to be the inclusion functor of the discrete subcategory I with $\mathrm{Ob}(I) = \mathrm{Ob}(J)$). Recall that $\lim_J \nabla^p G = \prod_{\beta\in\mathrm{Fun}([p],J)} G_{\beta(p)}$; especially, $\lim_J \nabla^p G$ is linearly compact for the product topology (proposition 8.3.47). Moreover, a simple inspection shows that the differential $d^p : \lim_J \nabla^p G \to \lim_J \nabla^{p+1} G$ is a continuous map, for every $p \in \mathbb{N}$. From these explicit descriptions, we also see that $\lim_{J_S} \nabla G_S$ is a quotient of the complex $\lim_J \nabla G$, for every $S \in \mathscr{F}$, and we have :

$$(8.3.49) \qquad \lim_J \nabla G = \lim_{S\in\mathscr{F}} \lim_{J_S} \nabla G_S \qquad \text{in } \mathsf{C}(A\text{-}\mathbf{Mod}).$$

For any finite subset $T \subset \mathrm{Ob}(J)$, any $p > 0$, and every $j \in T$ let $U_j \subset S(p, j)$ be a finite subset. Clearly there exists $S \in \mathscr{F}$ such that every element of $\bigcup_{j\in T} U_j$ is a functor $[p] \to J_S$.

Claim 8.3.50. There exists a category J^*_S with an initial object e, and functors $J_S \xrightarrow{t} J^*_S \xrightarrow{u} J$ whose composition is the inclusion functor $J_S \to J$.

Proof of the claim. Clearly $\mathrm{Ob}(J_S)$ is a finite set; then, since J is cofiltered, we can find $j_0 \in \mathrm{Ob}(J)$ and a system of morphisms $(f_j : j_0 \to j \mid j \in \mathrm{Ob}(J_S))$ of J such that $g \circ f_j = h \circ f_j$ for every $j, j' \in \mathrm{Ob}(J_S)$ and every $g, h \in \mathrm{Hom}_J(j, j') \cap S$. Let then J^*_S be the unique category whose set of objects is the disjoint union $\mathrm{Ob}(J_S) \sqcup \{e\}$, such that e is the unique initial object of J^*_S, and such that J_S is a fully faithful subcategory of J^*_S. We let t be the inclusion functor, and we take u to be the unique functor such that $u \circ t$ is the inclusion functor, and with $u(e) := j_0$, and $u(e \to j) := f_j$ for every $j \in \mathrm{Ob}(J_S)$. ◊

With the notation of claim 8.3.50, let $G^*_S := G \circ u : J^*_S \to A\text{-}\mathbf{Mod}$, and denote also by ∇G^*_S the canonical resolution provided by example 7.10.10(ii). Then u and t induce maps of complexes

$$(8.3.51) \qquad \lim_J \nabla G \to \lim_{J^*_S} \nabla G^*_S \to \lim_{J_S} \nabla G_S$$

whose composition is the natural projection. Moreover, $\lim_{J^*_S}$ is isomorphic to the functor such that $X \mapsto X(e)$ for every $X \in \mathrm{Ob}(\mathsf{Fun}(J^*_S, A\text{-}\mathbf{Mod}))$, and in particular $\lim_{J^*_S}$ is exact, so that $R^p \lim_{J^*_S} = 0$ for every $p > 0$. It follows that the projection $\lim_J \nabla G \to \lim_{J_S} \nabla G_S$ induces the zero map $H^p(\lim_J \nabla G) \to H^p(\lim_{J_S} \nabla G_S)$ for every $p > 0$. Thus, for every $p > 0$ and every $x \in \mathrm{Ker}\, d^p$, the image of x in $\lim_{J_S} \nabla^p G_S$ lies in the image of the corresponding differential $d^{p-1}_S : \lim_{J_S} \nabla^{p-1} G_S \to \lim_{J_S} \nabla^p G_S$. In view of (8.3.49) and the surjectivity of the composed map (8.3.51), this implies that x lies in the topological closure of the image of d^{p-1}; but the image of d^{p-1} is closed (remark 8.3.46(v)), whence the assertion. □

Example 8.3.52. Let $(A, \mathfrak{m})$ be a local noetherian ring, complete and separated for its $\mathfrak{m}$-adic topology. Then every A-module M of finite type is linearly compact for its $\mathfrak{m}$-adic topology. Indeed, M is $\mathfrak{m}$-adically complete and separated ([126, Th.8.7]), so corollary 8.3.48(i) reduces to checking that $M_n := M/\mathfrak{m}^n N$ is linearly compact for every $n \in \mathbb{N}$. But M_n is an $A/\mathfrak{m}^n$-module of finite type, and $A/\mathfrak{m}^n$ is artinian, so the assertion follows from remark 8.3.46(vii) and [126, Th.3.1(iii)].

Corollary 8.3.53. *Let $(A, \mathfrak{m})$ be a local noetherian ring, complete and separated for its $\mathfrak{m}$-adic topology, J a filtered category, $M^\bullet \in \mathrm{Ob}(\mathsf{C}^-(\mathsf{Fun}(J, A\text{-}\mathbf{Mod})))$, and $N^\bullet \in \mathsf{C}^+(A\text{-}\mathbf{Mod})$. Suppose that $M(j)^p$ is an A-module of finite type for every $p \in \mathbb{Z}$ and every $j \in \mathrm{Ob}(J)$, and that $H^q(N)$ is an A-module of finite type for every $q \in \mathbb{Z}$. Then the natural map*

$$\mathrm{Ext}^n_A(\operatorname*{colim}_J M^\bullet, N^\bullet) \to \lim_J \mathrm{Ext}^n_A(M^\bullet, N^\bullet)$$

is an isomorphism.

Proof. Since J is filtered, the functor $\mathrm{Colim}_J : \mathsf{Fun}(J, A\text{-}\mathbf{Mod}) \to A\text{-}\mathbf{Mod}$ is exact, so the natural transformation $L\,\mathrm{Colim}_J M^\bullet \to \mathrm{colim}_J M^\bullet$ is an isomorphism in $\mathsf{D}(A\text{-}\mathbf{Mod})$. By example 7.10.15(ii), we have therefore a spectral sequence

$$E_2^{pq} := R^p \lim_J \mathrm{Ext}^q_A(M^\bullet, N^\bullet) \Rightarrow \mathrm{Ext}^{p+q}_A(\operatorname*{colim}_J M^\bullet, N^\bullet)$$

and we are reduced to checking that $E_2^{pq} = 0$ for every $p > 0$. In light of example 8.3.52 and corollary 8.3.48(ii), it then suffices to check that $\mathrm{Ext}^q_A(M(j)^\bullet, N^\bullet)$ is an A-module of finite type, for every $q \in \mathbb{N}$ and every $j \in \mathrm{Ob}(J)$. Now, since A is noetherian, we may find a quasi-isomorphism $P^\bullet \to M(j)^\bullet$ in $\mathsf{C}^-(A\text{-}\mathbf{Mod})$, such that P^n is a free A-module of finite rank for every $n \in \mathbb{Z}$; the double complex $\mathrm{Hom}_A(P^\bullet, N^\bullet)$ yields a spectral sequence

$$F_2^{pq} := \mathrm{Ext}^p_A(P^\bullet, H^q N^\bullet) \Rightarrow \mathrm{Ext}^{p+q}_A(M(j)^\bullet, N^\bullet).$$

Since the induced filtration on the abutment is finite, we are further reduced to checking that $\mathrm{Ext}^p_A(P^\bullet, H^q N^\bullet)$ is an A-module of finite type for every $p, q \in \mathbb{Z}$, which is clear, since A is noetherian. □

Remark 8.3.54. With more work, one can see that corollary 8.3.53 still holds for every $M^\bullet \in \mathrm{Ob}(\mathsf{C}^-(\mathsf{Fun}(J, A\text{-}\mathbf{Mod})))$ and every $N^\bullet \in \mathsf{C}^+(A\text{-}\mathbf{Mod})$ (without finiteness assumptions on the terms $M(j)^p$). Indeed, for every $j \in \mathrm{Ob}(J)$ let $\mathscr{F}_j$ be the set of all subcomplexes $M'^\bullet \subset M(j)^\bullet$ such that M'^p is an A-module of finite type for every $p \in \mathbb{Z}$. We let I be the category with $\mathrm{Ob}(I) = \bigcup_{j \in \mathrm{Ob}(J)} \mathscr{F}_j \times \{j\}$, and such that the morphisms $(M'^\bullet, j) \to (M''^\bullet, k)$ are the commutative diagrams of $\mathsf{C}^-(A\text{-}\mathbf{Mod})$:

$$\mathscr{D}(f^\bullet, \varphi) \qquad : \qquad \begin{array}{ccc} M'^\bullet & \longrightarrow & M(j)^\bullet \\ {\scriptstyle f^\bullet}\downarrow & & \downarrow{\scriptstyle M(\varphi)^\bullet} \\ M''^\bullet & \longrightarrow & M(k)^\bullet \end{array}$$

whose horizontal arrows are the inclusion maps, for every $(M'^\bullet, j), (M''^\bullet, k) \in \mathrm{Ob}(I)$ and every morphism $\varphi : j \to k$ of J, and with composition law such that $\mathscr{D}(f^\bullet, \varphi) \circ \mathscr{D}(g^\bullet, \psi) := \mathscr{D}(f^\bullet \circ g^\bullet, \varphi \circ \psi)$ for any composable pair of morphisms of I. We have a functor $\pi : I \to J$ such that $(M'^\bullet, j) \mapsto j$ for every $(M'^\bullet, j) \in \mathrm{Ob}(I)$, and $\mathscr{D}(f^\bullet, \varphi) \mapsto \varphi$ for every morphism $\mathscr{D}(f^\bullet, \varphi)$ of I. Moreover, we have a functor $F : I \to \mathsf{C}^-(A\text{-}\mathbf{Mod})$ such that $F(M'^\bullet, j) := M'^\bullet$ for every $(M'^\bullet, j) \in \mathrm{Ob}(I)$ and $F(\mathscr{D}(f^\bullet, \varphi)) := f^\bullet$ for every morphism $\mathscr{D}(f^\bullet, \varphi)$ of I. It is easily seen that I is filtered, and $\pi^o : I^o \to J^o$ is a fibration (details left to the reader).

Furthermore, for every $j \in \mathrm{Ob}(J)$ let $\iota_j : \pi^{-1}(j) \to I$ be the inclusion functor; it is easily seen that $\mathrm{colim}_{\pi^{-1}(j)} F \circ \iota_j = M(j)^\bullet$. In view of example 3.1.14(iv), we get a natural identification : $\int^\pi F = M^\bullet$, so that $\mathrm{colim}_I F = \mathrm{colim}_J M^\bullet$. By corollary 8.3.53, we have natural isomorphisms for every $n \in \mathbb{N}$:

$$\mathrm{Ext}^n_A(\operatorname*{colim}_J M^\bullet, N^\bullet) \xrightarrow{\sim} \mathrm{Ext}^n_A(\operatorname*{colim}_I F, N^\bullet) \xrightarrow{\sim} \lim_I \mathrm{Ext}^n_A(F, N^\bullet) \xrightarrow{\sim} \lim_J \int_\pi \mathrm{Ext}^n_A(F, N^\bullet).$$

Lastly, notice that the category $\pi^{-1}(j)$ is also filtered, for every $j \in \mathrm{Ob}(J)$; invoking again corollary 8.3.53 together with example 3.1.14(iii), we deduce a natural isomorphism

$$\int_\pi \mathrm{Ext}^n_A(F, N^\bullet) \xrightarrow{\sim} \mathrm{Ext}^n_A(M^\bullet, N) \qquad \text{for every } n \in \mathbb{N}$$

whence the assertion.

8.4. **Topologically local and topologically henselian rings.** Localization and henselization are basic techniques in the study of commutative algebra; in this section we introduce suitable variants of these constructions for f-adic topological rings.

Definition 8.4.1. (i) Let $\underline{A} := A$ be any f-adic ring.

(i) We say that A is *topologically local* if $A^{\circ\circ}$ lies in the Jacobson radical of A°.
(ii) We say that A is *topologically henselian* if $(A^\circ, A^{\circ\circ})$ is a henselian pair.

Proposition 8.4.2. *Let A be any f-adic ring, and B any open subring of A. We have :*

(i) *A is topologically local if and only if the same holds for B.*
(ii) *If A is topologically henselian, then A is also topologically local.*
(iii) *A is topologically henselian if and only if the same holds for B.*
(iv) *Suppose that A is topologically henselian, let C be any finite A-algebra, and endow C with its canonical A-module topology $\mathscr{T}^A_C$ (see remark* 8.3.43(i)*). Then $(C, \mathscr{T}^A_C)$ is a topologically henselian f-adic ring.*

Proof. (i): Indeed, suppose first that B is topologically local, and let $a \in A^{\circ\circ}$ be any element; we may find $n \in \mathbb{N}$ such that $a^{n+1} \in B^{\circ\circ}$, therefore $1 - a^{n+1} \in B^{\circ\times}$ and

$$(1-a)^{-1} = (1 + a + \cdots + a^n)/(1 - a^{n+1}) \in A^\circ$$

so a lies in the Jacobson radical of A°. Conversely, suppose that A is topologically local, and let $b \in B^{\circ\circ}$ be any element; then $1 - b \in A^{\circ\times}$, and we may find $n \in \mathbb{N}$ such that $b^{n+1}/(1-b) \in B^\circ$. Therefore

$$(1-b)^{-1} = b^{n+1}/(1-b) + 1 + b + \cdots + b^n \in B^\circ$$

so that b lies in the Jacobson radical of B°.

(ii): Quite generally, if (R, I) is a henselian pair, then I lies in the Jacobson radical of R (details left to the reader).

(iii): Indeed, suppose first that B is topologically henselian; then B is topologically local, by (ii), and therefore the same holds for A, by (i); in view of [75, Rem.5.1.10(ii)] we then see that A is topologically henselian if and only if the same holds for the open subring $C := B[A^{\circ\circ}]$ (which is f-adic, by corollary 8.3.20(i)). However, it is easily seen that the inclusion map $B^\circ \to C^\circ$ is integral, and moreover the ideal $A^{\circ\circ}$ of C° is the radical of $B^{\circ\circ} \cdot C^\circ$ (details left to the reader). Then the assertion follows from [75, Rem.5.1.10(i,v)].

In order to prove the converse, consider – for every scheme X – the set $\mathrm{oc}(X)$ of all open and closed subsets of X; clearly any morphism of schemes $\varphi : Y \to X$ induces a mapping

$$\mathrm{oc}(\varphi) : \mathrm{oc}(X) \to \mathrm{oc}(Y) \qquad Z \mapsto \varphi^{-1}Z.$$

With this notation we have, quite generally :

Claim 8.4.3. Let $\varphi : Y \to X$ be any closed (resp. open) and surjective morphism of schemes, and $p_1, p_2 : Y \times_X Y \to Y$ the induced projections. Then the diagram of sets

$$\mathrm{oc}(X) \xrightarrow{\mathrm{oc}(\varphi)} \mathrm{oc}(Y) \overset{\mathrm{oc}(p_1)}{\underset{\mathrm{oc}(p_2)}{\rightrightarrows}} \mathrm{oc}(Y \times_X Y)$$

identifies $\mathrm{oc}(X)$ with the equalizer of the mappings $\mathrm{oc}(p_1)$ and $\mathrm{oc}(p_2)$.

Proof of the claim. Since φ is surjective, $\mathrm{oc}(\varphi)$ is injective, and obviously its image lies in the equalizer of $\mathrm{oc}(p_1)$ and $\mathrm{oc}(p_2)$. Conversely, let Z be an open and closed subset of Y such that $p_1^{-1}Z = p_2^{-1}Z$. Since φ is surjective, we have $Z \subset \varphi^{-1}\varphi(Z)$, and we claim that in fact $Z = \varphi^{-1}\varphi(Z)$. Indeed, let $y \in \varphi^{-1}\varphi(Z)$ be any point; then there exists $z \in Z$ such that $\varphi(y) = \varphi(z)$, and we may find $w \in Y \times_X Y$ such that $p_1(w) = z$ and $p_2(w) = y$. The condition on Z then implies that $y \in Z$ as well, as required. Next, set $Z' := Y \setminus Z$, and notice that Z' lies as well in the equalizer of $\mathrm{oc}(p_1)$ and $\mathrm{oc}(p_2)$, so we have also $Z' = \varphi^{-1}\varphi(Z')$; it follows that $\varphi(Z) \cap \varphi(Z') = \varnothing$, and since φ is a closed (resp. open) mapping, we conclude that $\varphi(Z)$ is an open and closed subset of X whose preimage in Y equals Z, whence the claim. $\Diamond$

Now, if A is topologically henselian, the foregoing shows that the same holds for C, and taking into account lemma 8.3.29(iii), to conclude the proof of (iii) it suffices to remark :

Claim 8.4.4. Let $f : R \to S$ be an integral ring homomorphism, $I \subset R$ any ideal. Suppose that the image of the induced morphism of schemes $\varphi : \mathrm{Spec}\, S \to \mathrm{Spec}\, R$ contains $\mathrm{Spec}\, R \setminus \mathrm{Spec}\, R/I$. Then we have :

(i) I lies in the Jacobson radical of R if and only if IS lies in the Jacobson radical of S.
(ii) The pair (R, I) is henselian if and only if the same holds for the pair (S, IS).

Proof of the claim. Set $S' := S \times (R/I)$ and let $\pi : R \to R/I$ be the projection; the pair (f, π) determines a unique integral ring homomorphism $f' : R \to S'$, and it is easily seen that the pair (S, IS) is henselian if and only if the same holds for the pair (S', IS'). Likewise, obviously IS lies in the Jacobson radical of S if and only if IS' lies in the Jacobson radical of S'. Moreover, $\mathrm{Spec}\, f'$ is surjective; thus, we may replace S by S', and assume from start that φ is surjective.

(i): Suppose first that I lies in the Jacobson radical of R, and let $\mathfrak{m} \subset S$ be any maximal ideal; then $\mathfrak{m}' := \varphi(\mathfrak{m})$ is a maximal ideal of R ([126, Th.9.4(i)]), therefore $I \subset \mathfrak{m}'$, and hence $IS \subset \mathfrak{m}$, so IS lies in the Jacobson radical of S. Conversely, suppose that the latter condition holds, and let $\mathfrak{m}'$ be any maximal ideal of R; by assumption, we may find a prime ideal $\mathfrak{m}$ of S with $\varphi(\mathfrak{m}) = \mathfrak{m}'$, and then $\mathfrak{m}$ must be a maximal ideal, so $IS \subset \mathfrak{m}$, and finally $I \subset \mathfrak{m}'$.

(ii): If (R, I) is henselian, the same holds for (S, IS) ([75, Rem.5.1.10(v)]), so we may assume that (S, IS) is henselian, and we show that the same holds for (R, I). Indeed, let $R \to R'$ be any finite ring homomorphism; we have to check that the induced map $R' \to R'/IR'$ restricts to a bijection $\mathrm{oc}(\mathrm{Spec}\, R') \xrightarrow{\sim} \mathrm{oc}(\mathrm{Spec}\, R'/IR')$. However, set $S' := R' \otimes_R S$, and notice that the pair (S', IS') is henselian (again, by [75, Rem.5.1.10(v)]), and moreover the induced map $\mathrm{Spec}\, S' \to \mathrm{Spec}\, R'$ is surjective; thus, we may replace R by R', and we reduce to showing that the projection $R \to R/I$ induces a bijection $\mathrm{oc}(\mathrm{Spec}\, R) \to \mathrm{oc}(\mathrm{Spec}\, R/I)$. However, φ is a universally closed morphism ([60, Ch.II, Prop.6.1.10]), so claim 8.4.3 further reduces to checking that the induced projections $S \to S_0 := S/IS$ and $S \otimes_R S \to S_0 \otimes_R S_0$ induce bijections

$$\mathrm{oc}(\mathrm{Spec}\, S) \to \mathrm{oc}(\mathrm{Spec}\, S_0) \qquad \mathrm{oc}(\mathrm{Spec}\, S \otimes_R S) \to \mathrm{oc}(\mathrm{Spec}\, S_0 \otimes_R S_0).$$

In turns, this is clear, since (S, IS) and $(S \otimes_R S, IS \otimes_R S)$ are both henselian pairs (again, by [75, Rem.5.1.10(v)]). $\Diamond$

(iv): Let $A_0 \subset A$ be a subring of definition, and $I_0 \subset A_0$ an ideal of adic definition; by remark 8.3.43(iii), we know that the topology $\mathscr{T}_C^A$ is f-adic, and C admits a subring C_0 of definition such that the structure map $A \to C$ restricts to an adic finite ring homomorphism $A_0 \to C_0$. Then the pair (C_0, I_0C_0) is henselian (([75, Rem.5.1.10(v)]), and therefore $(C, \mathscr{T}_C^A)$ is topologically henselian, by (iii). □

Remark 8.4.5. If A is a topologically local f-adic ring, we have a homeomorphism

$$1 + A^{\circ\circ} \xrightarrow{\sim} 1 + A^{\circ\circ} \quad : \quad x \mapsto x^{-1}. \tag{8.4.6}$$

Indeed, say that $a \in A^{\circ\circ}$; then there exists $n \in \mathbb{N}$ such that

$$1/(1-a) - (1 + a + \cdots + a^n) = a^{n+1}/(1-a) \in A^{\circ\circ} \tag{8.4.7}$$

and since $A^{\circ\circ}$ is an ideal of A°, we deduce that $1/(1-a) \in 1 + A^{\circ\circ}$. It remains only to check that (8.4.6) is continuous on $A^{\circ\circ}$. However, if $a, b \in A^{\circ\circ}$, we may write

$$(1-a)^{-1} - (1-a-b)^{-1} = (1-a)^{-1} \cdot (1 - 1/(1-(1-a)^{-1}b))$$

so we are reduced to checking the continuity of (8.4.6) at the point $x = 1$, and the latter follows easily from (8.4.7). Notice that the same argument proves more precisely that

$$(1+U)^{-1} = 1 + U$$

for every open additive subgroup $U \subset A^{\circ\circ}$ such that $U \cdot U \subset U$: details left to the reader.

8.4.8. Let A be any f-adic ring, B a ring of definition of A, and I a finitely generated ideal of adic definition of B. We let

$$B_{\mathrm{loc}} := (1+I)^{-1}B \qquad \text{and} \qquad A_{\mathrm{loc}} := A \otimes_B B_{\mathrm{loc}}$$

and endow B_{loc} with the unique topology such that the localization map $B \to B_{\mathrm{loc}}$ is adic, and A_{loc} with the unique f-adic ring topology $\mathscr{T}_{\mathrm{loc}}$ such that the inclusion map $B_{\mathrm{loc}} \to A_{\mathrm{loc}}$ is open and the induced map $A \to A_{\mathrm{loc}}$ is f-adic (proposition 8.3.34(ii)). We claim that A_{loc} is topologically local. Indeed, suppose that $a \in A_{\mathrm{loc}}^{\circ\circ}$; we may find $n \in \mathbb{N}$ such that $a^n \in IB_{\mathrm{loc}}$, so $a^n = b/(1+t)$ for some $b, t \in I$. Therefore, $1 - a^n = (1+t-b)/(1+t) \in B_{\mathrm{loc}}^\times$, and arguing as in the proof of proposition 8.4.2(i) we deduce that $1 - a \in A_{\mathrm{loc}}^\times$, whence the claim. It follows especially that A_{loc} is independent of the choice of I : indeed, suppose that $J \subset I$ is another ideal of adic definition for B, and let $A' := (1+J)^{-1}A$; then there exist a unique map of A-algebras $\rho' : A' \to A_{\mathrm{loc}}$, and a natural isomorphism $A_{\mathrm{loc}} \xrightarrow{\sim} (1+I)^{-1}A'$ of A-algebras that identifies ρ' with the localization map; however, the foregoing shows that the image of $1 + I$ lies in $A'^\times$, so ρ' is bijective, and a simple inspection then shows that ρ'' is also an isomorphism of topological rings. Lastly, let $f : A \to C$ be any continuous ring homomorphism of f-adic rings, with C topologically local, and let C_0 be a ring of definition of C, and J an ideal of adic definition for C_0; by the foregoing, we may replace I by a smaller open ideal, and assume that $f(I) \subset J$. Moreover C_0 is topologically local, by proposition 8.4.2(i), whence $f(1+I) \subset 1 + J \subset C_0^\times$, so that f factors uniquely through the localization map $A \to A_{\mathrm{loc}}$ and a ring homomorphism $g : A_{\mathrm{loc}} \to C$. Furthermore, g is a continuous map : indeed, $g(IB_{\mathrm{loc}}) = g((1+I)^{-1}I) \subset (1+J)^{-1}J \subset J$, whence the assertion. Thus, the localization map $\rho : A \to A_{\mathrm{loc}}$ is initial in the category of topologically local topological A-algebras with f-adic topologies; especially, the pair (A_{loc}, ρ) is determined up to unique isomorphism, and we call it the *topological localization* of the f-adic ring A.

8.4.9. For any topological ring R, denote by

$$R\text{-}\mathsf{TopAlg}_{\mathrm{f-adic}} \qquad \text{and} \qquad R\text{-}\mathsf{TopHens}_{\mathrm{f-adic}}$$

the full subcategories of R-TopAlg whose objects are respectively the f-adic topological R-algebras, and the topologically henselian f-adic topological R-algebras (definition 8.3.1(iii)).

Now, let again A, B and I as in (8.4.8), denote by $(B^{\mathrm{h}}_I, I^{\mathrm{h}})$ the henselization of the pair (B, I), endow B^{h}_I with its I^{h}-adic topology, and set

$$A^{\mathrm{h}} := A \otimes_B B^{\mathrm{h}}_I.$$

We endow A^{h} with the unique topology $\mathscr{T}^{\mathrm{h}}_A$ such that $(A^{\mathrm{h}}, \mathscr{T}^{\mathrm{h}}_A)$ is an f-adic topological ring, the natural map $B^{\mathrm{h}}_I \to A^{\mathrm{h}}$ is open and the natural map $A \to A^{\mathrm{h}}$ is f-adic (proposition 8.3.34(ii)). We may then state :

Theorem 8.4.10. *With the notation of* (8.4.9)*, the following holds :*

(i) $(A^{\mathrm{h}}, \mathscr{T}^{\mathrm{h}}_A)$ *is a topologically henselian f-adic ring, independent of the choice of* B *and* I*, up to unique isomorphism of topological* A*-algebras.*

(ii) *The rule :* $A \mapsto (A^{\mathrm{h}}, \mathscr{T}^{\mathrm{h}}_A)$ *extends to a functor*

$$\mathbb{Z}\text{-}\mathsf{TopAlg}_{\mathrm{f-adic}} \to \mathbb{Z}\text{-}\mathsf{TopHens}_{\mathrm{f-adic}}$$

that is left adjoint to the forgetful functor (here $\mathbb{Z}$ *is endowed with its discrete topology).*

Proof. Let us consider as well the full subcategories of $\mathbb{Z}$-TopAlg denoted

$$\mathbb{Z}\text{-}\mathsf{TopAlg}_{\mathrm{adic}} \qquad \text{and} \qquad \mathbb{Z}\text{-}\mathsf{TopHens}_{\mathrm{adic}}$$

whose objects are respectively the adic topological rings, and the adic topological rings R such that the pair $(R, R^{\circ\circ})$ is henselian (notice that $R^{\circ\circ}$ is an ideal of R). For every adic ring R, we let $(R^{\mathrm{h}}, (R^{\circ\circ})^{\mathrm{h}})$ the henselization of the pair $(R, R^{\circ\circ})$, and we endow R^{h} with the unique ring topology $\mathscr{T}^{\mathrm{h}}_R$ such that the natural map $R \to R^{\mathrm{h}}$ is adic. We notice :

Claim 8.4.11. The rule $R \mapsto (R^{\mathrm{h}}, \mathscr{T}^{\mathrm{h}}_R)$ extends to a functor

$$\mathbb{Z}\text{-}\mathsf{TopAlg}_{\mathrm{adic}} \to \mathbb{Z}\text{-}\mathsf{TopHens}_{\mathrm{adic}}$$

that is left adjoint to the forgetful functor.

Proof of the claim. More generally, let $f : R \to S$ and $g : R \to R'$ be two ring homomorphisms, and $I \subset R$ and $J \subset S$ two ideals such that :

- $f(I) \subset J$
- J lies in the Jacobson radical ideal of S
- g is unramified, and $g \otimes_R R/I$ is an isomorphism.

Then we claim that there exists at most one ring homomorphism $h : R' \to S$ such that $h \circ g = f$. Indeed, let $h, h' : R' \to S$ be any two such maps; according to [66, Ch.IV, Prop.17.4.6], the maximal subscheme $U \subset \operatorname{Spec} S$ such that $(\operatorname{Spec} h)_{|U} = (\operatorname{Spec} h')_{|U}$ is open and closed in $\operatorname{Spec} S$, and on the other hand, U contains $\operatorname{Spec} S/J$, since $g \otimes_R R/I$ is an isomorphism. But $\operatorname{Spec} S/J$ meets every open and closed subset of $\operatorname{Spec} S$, since J lies in the Jacobson radical of S, whence the assertion. Next, suppose that the pair (S, J) is henselian, and g is étale; in this case, the map $g \otimes_R S : S \to R' \otimes_R S$ admits a section $s : R' \otimes_R S \to S$ ([66, Ch.IV, Prop.18.5.4]), and we set $h := s \circ (R' \otimes_R f) : R' \to S$. By the foregoing, h is the unique ring homomorphism such that $h \circ g = f$. Now, let $(R^{\mathrm{h}}, I^{\mathrm{h}})$ be the henselization of the pair (R, I), and recall that R^{h} is the colimit of a filtered system $(g_\lambda : R \to R'_\lambda \mid \lambda \in \Lambda)$ of étale ring homomorphisms such that $g_\lambda \otimes_R R/I$ is an isomorphism for every $\lambda \in \Lambda$. Summing up, we conclude that f factors uniquely through a morphism of R-algebras $R^{\mathrm{h}} \to S$. Lastly, if R and S are adic topological rings, then every continuous ring homomorphism $g : R \to S$ maps $I := R^{\circ\circ}$ into $J := S^{\circ\circ}$, and if $(S, S^{\circ\circ})$ is henselian, the foregoing yields a unique ring homomorphism $f^{\mathrm{h}} : R^{\mathrm{h}} \to S^{\mathrm{h}}$ extending f; to conclude, it suffices to check that f^{h} is continuous. However, if J' is any ideal of adic definition for S, we may find an ideal I' of adic definition for R such that $f(I') \subset J'$; by construction, $I'R^{\mathrm{h}}$ is an ideal of adic definition of R^{h}, and $f^{\mathrm{h}}(I'R^{\mathrm{h}}) \subset J'$, whence the contention. ◊

(i): Let us check first that $A^{\rm h}$ is independent of the choice of B and I. Indeed, since $B^{\circ\circ}$ is the radical of I, there exists a unique isomorphism $\varphi : B^{\rm h}_I \stackrel{\sim}{\to} B^{\rm h}$ of B-algebras, where $B^{\rm h}$ is as in the foregoing; it is then clear that the $I^{\rm h}$-adic topology on $B^{\rm h}$ agrees with the topology $\mathscr{T}^{\rm h}_B$, under the isomorphism φ. Let us endow $A \otimes_B B^{\rm h}$ with the unique ring topology such that the natural map $B^{\rm h} \to A \otimes_B B^{\rm h}$ is open (proposition 8.3.34(ii)); it follows that $\varphi \otimes_B A : A^{\rm h} \stackrel{\sim}{\to} A \otimes_B B^{\rm h}$ is an isomorphism of topological A-algebras, which shows that $A^{\rm h}$ does not depend on I. By the same token, we also deduce that $A^{\rm h}$ is topologically henselian (proposition 8.4.2(iii)). Next, let $C \subset A$ be another ring of definition, $I_C \subset C$ an ideal of definition of C, and $(C^{\rm h}, I^{\rm h}_C)$ the henselization of the pair (C, I_C); endow $C^{\rm h}$ with its $I^{\rm h}_C$-adic topology, set $A' := A \otimes_C C^{\rm h}$, and endow again A' with the unique ring topology such that the natural map $C^{\rm h} \to A'$ is open and the induced map $A \to A'$ is f-adic. We need to show :

Claim 8.4.12. There exists a unique isomorphism of topological A-algebras $A^{\rm h} \stackrel{\sim}{\to} A'$.

Proof of the claim. In light of corollary 8.3.19(i) we are easily reduced to the case where $B \subset C$, and since we have already established that the construction of $C^{\rm h}$ and $B^{\rm h}$ is independent of the ideals of definition, we may moreover assume that $I_C = I$ (details left to the reader). In this situation, set $D := B^{\rm h} \otimes_B C$; the induced ring homomorphism $B^{\rm h} \to D$ identifies $I^{\rm h} = B^{\rm h} \otimes_B I$ with the ideal ID of D, and therefore the pair (D, ID) is henselian ([75, Rem.5.1.10(ii)]). Let us endow D with its ID-adic topology; then $D^{\circ\circ}$ is the radical of ID, and therefore D is topologically henselian ([75, Rem.5.1.10(i)]); moreover, the natural map $C \to D$ is continuous, so by claim 8.4.11 it factors uniquely through a continuous map $C^{\rm h} \to D$ of C-algebras. We claim that the latter is an isomorphism of topological rings. Indeed, let E be any other object of $\mathbb{Z}$-$\mathsf{TopHens}_{\rm adic}$; any continuous ring homomorphism $B \to E$ admits a unique continuous extension $B^{\rm h} \to E$, so the set of continuous ring homomorphisms $D \to E$ is in natural bijection with the set of continuous ring homomorphisms $C \to E$, and the latter are as well in natural bijection with the set of continuous ring homomorphisms $C^{\rm h} \to E$, whence the claim (details left to the reader). There follows an isomorphism of A-algebras

$$\psi : A^{\rm h} \stackrel{\sim}{\to} A \otimes_C D \stackrel{\sim}{\to} A'$$

and since the structure maps of these A-algebras are f-adic, it is clear that ψ is an isomorphism of topological rings. Furthermore, suppose that $\psi' : A^{\rm h} \stackrel{\sim}{\to} A'$ is another isomorphism of topological A-algebras, and set $R := \psi(B^{\rm h}) \cdot \psi'(B^{\rm h})$; then R is a ring of definition of A' (corollary 8.3.19(i)), so it is an object of $\mathbb{Z}$-$\mathsf{TopHens}_{\rm adic}$ (proposition 8.4.2(iii)), and the restrictions $B^{\rm h} \to R$ of ψ and ψ' agree on B. By claim 8.4.11, it follows that ψ and ψ' agree on $B^{\rm h}$, and therefore they coincide, whence the contention. ◊

Lastly, let $f : A \to R$ be any continuous ring homomorphism from A to a topologically henselian f-adic topological ring, and pick any ring of definition R_0 of R; after replacing B by $B \cap f^{-1}R_0$ we may assume that f restricts to a map $f_0 : B \to R_0$ (proposition 8.3.18(ii)), and since R_0 is topologically henselian (proposition 8.4.2(iii)), the map f_0 factors uniquely through a continuous ring homomorphism $f^{\rm h}_0 : B^{\rm h} \to E$ (claim 8.4.11). The datum of f and $f^{\rm h}_0$ determines a unique continuous map of A-algebras $A^{\rm h} \to E$ (proposition 8.3.34(ii)), so the proof is concluded. □

Definition 8.4.13. Let A be any f-adic topological ring. The topological A-algebra $(A^{\rm h}, \mathscr{T}^{\rm h}_A)$ provided by theorem 8.4.10 is called the *topological henselization* of A.

Remark 8.4.14. Our notions of topological henselian f-adic ring and of topological henselization already appear in [101, §3.1], where they are called respectively henselian f-adic ring and henselization of an f-adic ring.

Corollary 8.4.15. *Let A be any f-adic topological ring, and denote by A_{loc}, A^{h}, $A^{\wedge}$ respectively the topologically localization, the topological henselization, and the separated completion of A. The following holds :*

(i) *The localization map $A \to A_{\mathrm{loc}}$ and the henselization map $A \to A^{\mathrm{h}}$ induce isomorphisms of topological rings on separated completions :*

$$A^{\mathrm{h}\wedge} \xleftarrow{\sim} A^{\wedge} \xrightarrow{\sim} A^{\wedge}_{\mathrm{loc}}.$$

(ii) *The inclusion maps $A^{\circ} \to A$ and $A^{\circ\circ} \to A$ induce natural identifications :*

$$(A^{\circ})_{\mathrm{loc}} \xrightarrow{\sim} (A_{\mathrm{loc}})^{\circ} \qquad (A^{\circ})^{\mathrm{h}} \xrightarrow{\sim} (A^{\mathrm{h}})^{\circ} \qquad (A^{\circ})^{\wedge} \xrightarrow{\sim} (A^{\wedge})^{\circ}$$

$$A^{\circ\circ} \otimes_{A^{\circ}} A^{\circ}_{\mathrm{loc}} \xrightarrow{\sim} (A_{\mathrm{loc}})^{\circ\circ} \qquad A^{\circ\circ} \otimes_{A^{\circ}} A^{\circ\mathrm{h}} \xrightarrow{\sim} (A^{\mathrm{h}})^{\circ\circ} \qquad (A^{\circ\circ})^{\wedge} \xrightarrow{\sim} (A^{\wedge})^{\circ\circ}.$$

(iii) *Let R be any open subring of A°, and denote by $(R^{\mathrm{h}}, R^{\circ\circ\mathrm{h}})$ the henselization of the pair $(R, R^{\circ\circ})$. Then we have :*
  (a) *There exists a unique ring topology $\mathscr{T}^{\mathrm{h}}_R$ on R^{h} such that the natural map $R \to R^{\mathrm{h}}$ is f-adic and $(R^{\mathrm{h}}, \mathscr{T}^{\mathrm{h}}_R)$ is isomorphic to the topological henselization of R.*
  (b) *Endow $A \otimes_R R^{\mathrm{h}}$ with the unique f-adic topology such that the natural map $R^{\mathrm{h}} \to A \otimes_R R^{\mathrm{h}}$ is open (see proposition* 8.3.34(ii)*). There exists a unique isomorphism of topological A-algebras $A^{\mathrm{h}} \xrightarrow{\sim} A \otimes_R R^{\mathrm{h}}$.*

Proof. (i): Let B be any ring of definition of A; a simple inspection of the construction in (8.4.8) shows that the natural map

$$A' := A \otimes_B B_{\mathrm{loc}} \to A_{\mathrm{loc}}$$

is an isomorphism of topological rings, for the f-adic topology on A' described in proposition 8.3.34(ii). Moreover, the localization map $B \to B_{\mathrm{loc}}$ clearly induces an isomorphism $B^{\wedge} \xrightarrow{\sim} (B_{\mathrm{loc}})^{\wedge}$. Then the assertion for $A^{\wedge}_{\mathrm{loc}}$ follows from proposition 8.3.34(iii.a). Likewise, the natural map $B \to B^{\mathrm{h}}$ induces an isomorphism on separated completions, so the same argument yields the assertion for $A^{\mathrm{h}\wedge}$.

(ii): The assertions for $(A_{\mathrm{loc}})^{\circ}$ and $(A_{\mathrm{loc}})^{\circ\circ}$ follow by the same token, from proposition 8.3.34(iii.b). Likewise, we get the assertions for $(A^{\mathrm{h}})^{\circ}$ and $(A^{\mathrm{h}})^{\circ\circ}$. In the case of the completion functor, we know that the natural map $A \otimes_B B^{\wedge} \to A^{\wedge}$ is an isomorphism of topological rings (proposition 8.3.33(iii)), so we may still apply proposition 8.3.34(iii) to deduce that $(A^{\wedge})^{\circ}$ (resp. $(A^{\wedge})^{\circ\circ}$) is the topological closure in $A^{\wedge}$ of the image of A° (resp. of $A^{\circ\circ}$).

(iii): Suppose first that $R = A$; especially, $A^{\circ} = A$. Then, according to corollary 8.3.19(iv), the ring A is the filtered union of the system $(B_\lambda \mid \lambda \in \Lambda)$ of its subrings of definitions containing B, whence an induced isomorphism of A-algebras

$$A^{\mathrm{h}} \xrightarrow{\sim} \operatorname*{colim}_{\lambda \in \Lambda} B_\lambda \otimes_B B^{\mathrm{h}}$$

where B^{h} denotes the topological henselization of B. On the other hand, B is also a subring of definition of B_λ, and $B^{\circ\circ}_\lambda = B_\lambda \cap B^{\circ\circ}$ for every $\lambda \in \Lambda$. Let $(B^{\mathrm{h}}_\lambda, J_\lambda)$ be the henselization of the pair $(B_\lambda, B^{\circ\circ}_\lambda)$; by theorem 8.4.10(i) we deduce a unique isomorphism

$$B^{\mathrm{h}}_\lambda \xrightarrow{\sim} B_\lambda \otimes_B B^{\mathrm{h}} \qquad \text{for every } \lambda \in \Lambda$$

of topological B_λ-algebras. Lastly, we have a unique isomorphism of A-algebras

$$\operatorname*{colim}_{\lambda \in \Lambda} B^{\mathrm{h}}_\lambda \xrightarrow{\sim} R^{\mathrm{h}}$$

whence the sought isomorphism $\varphi : A^{\mathrm{h}} \xrightarrow{\sim} R^{\mathrm{h}}$ of A-algebras. The uniqueness of φ follows from the universal property of the henselization functor. If we may endow R^{h} with the topology induced from A^{h} via φ, clearly both (iii.a) and (iii.b) hold in this case.

Next, if $R \subset A^\circ$ is an arbitrary open subring, choose a ring of definition B of A contained in R; by the foregoing case, R^{h} is naturally isomorphic to the topological henselization of R, so that we have unique isomorphisms

$$R^{\mathrm{h}} \xrightarrow{\sim} R \otimes_{A_0} A_0^{\mathrm{h}} \qquad A^{\mathrm{h}} \xrightarrow{\sim} A \otimes_{A_0} A_0^{\mathrm{h}}$$

respectively of topological R-algebras and topological A-algebras, such that both natural maps $A_0^{\mathrm{h}} \to R^{\mathrm{h}}$ and $A_0^{\mathrm{h}} \to A^{\mathrm{h}}$ are open. There follows an isomorphism of topological A-algebras $\varphi : A^{\mathrm{h}} \xrightarrow{\sim} A \otimes_R R^{\mathrm{h}}$ as sought. The uniqueness of φ follows again from the universal property of the topological henselization A^{h} of A. □

8.5. **Graded structures on topological rings.** Let Γ be a monoid, $A := \bigoplus_{\gamma\in\Gamma} A_\gamma$ a Γ-graded ring, and $\mathscr{T}$ a topology on A. The Γ-graded structure of A is not usually inherited by the completion $(A, \mathscr{T})^\wedge$ of $(A, \mathscr{T})$. This observation motivates the following definition, that introduces a more flexible notion of graded structure on topological rings, preserved under completions and related operations.

Definition 8.5.1. Let $(A, \mathscr{T})$ be a separated topological ring, Γ a monoid.

(i) A Γ-*pre-graded structure* on $(A, \mathscr{T})$ is a datum $(A, \underline{B}, \Gamma)$ consisting of a subring $B \subset A$ with a Γ-graded $\mathbb{Z}$-algebra structure $\underline{B}$ on B (definition 7.6.1(i)) such that the following holds :

- (a) The topology induced on B by $\mathscr{T}$ agrees with the linear topology defined by a cofiltered system of graded ideals of B (see definition 7.6.1(iii)).
- (b) B is a dense subset of A.

(ii) A Γ-*graded structure* on $(A, \mathscr{T})$ is a Γ-pre-graded structure $(A, \underline{B})$ such that $\mathrm{gr}_\gamma B$ is a closed subset of A, for every $\gamma \in \Gamma$.

(iii) Let $(A, \underline{B}, \Gamma)$ and $(A', \underline{B}', \Gamma')$ be two topological rings with pre-graded structures. A *morphism of topological rings with pre-graded structures* $(A, \underline{B}) \to (A', \underline{B}')$ is a pair (f, φ) consisting of a continuous ring homomorphism $f : A \to A'$ and a morphism of monoids $\varphi : \Gamma \to \Gamma'$ such that $f(B) \subset B'$, and such that the restriction $f_{|B} : B \to B'$ induces a morphism of Γ-graded $\mathbb{Z}$-algebras (notation of definition 7.6.1(iv))

$$\underline{B} \to \Gamma \times_{\Gamma'} \underline{B}'.$$

Clearly, we get therefore a category of topological rings with pre-graded structures :

$$\mathsf{pre\text{-}gr.TopAlg}.$$

We also have the full subcategory of $\mathsf{pre\text{-}gr.TopAlg}$ denoted

$$\mathsf{gr.TopAlg}$$

whose objects are the topological rings with graded structures.

Remark 8.5.2. Let $(\Gamma, +, 0)$ be a monoid, $(A, \underline{B})$ a Γ-pre-graded structure on the topological ring $(A, \mathscr{T})$. Pick a fundamental system $(J_\lambda \mid \lambda \in \Lambda)$ of graded open ideals of B, and endow $\mathrm{gr}_\gamma B$ with the topology $\mathscr{T}_\gamma$ induced by $\mathscr{T}$, for every $\gamma \in \Gamma$.

(i) Since $J_\lambda = \bigoplus_{\gamma\in\Gamma} \mathrm{gr}_\gamma J_\lambda$ for every $\lambda \in \Lambda$, the topology $\mathscr{T}_\gamma$ agrees with the linear topology defined by the cofiltered system of $\mathrm{gr}_0 B$-submodules $(\mathrm{gr}_\gamma J_\lambda \mid \lambda \in \Lambda)$, for every $\gamma \in \Gamma$. Moreover, a graded ideal I of B is closed in the topology of B if and only if $\mathrm{gr}_\gamma I$ is closed in $\mathrm{gr}_\gamma B$ for every $\gamma \in \Gamma$. Indeed, since $I \cap \mathrm{gr}_\gamma B = \mathrm{gr}_\gamma I$ for every such γ, the condition is clearly necessary. Conversely, notice that the natural projection $\pi_\gamma : B \to \mathrm{gr}_\gamma B$ is continuous for every $\gamma \in \Gamma$, hence the same holds for the composition $\rho_\gamma : B \to \mathrm{gr}_\gamma B/I$ of π_γ with the quotient map $\mathrm{gr}_\gamma B \to \mathrm{gr}_\gamma B/I$; now, if $\mathrm{gr}_\gamma I$ is closed in $\mathrm{gr}_\gamma B$, the topology of $\mathrm{gr}_\gamma B/I$ is separated, hence $\mathrm{Ker}\,\rho_\gamma$ is closed in B, and to conclude it suffices to remark that I is the intersection of such kernels, for γ ranging over all elements of Γ.

(ii) For every $\gamma \in \Gamma$, denote by $(\mathrm{gr}_\gamma B^\wedge, \mathscr{T}_\gamma^\wedge)$ the completion of $(\mathrm{gr}_\gamma B, \mathscr{T}_\gamma)$. Also, let $\mathrm{gr}_\gamma J_\lambda^\wedge$ be the topological closure of $\mathrm{gr}_\gamma J_\lambda$ in $\mathrm{gr}_\gamma B^\wedge$, for every $\lambda \in \Lambda$ and $\gamma \in \Gamma$, and set

$$J'_\lambda := \prod_{\gamma\in\Gamma} \mathrm{gr}_\gamma J_\lambda^\wedge \qquad B' := \prod_{\gamma\in\Gamma} \mathrm{gr}_\gamma B^\wedge \qquad J''_\lambda := \bigoplus_{\gamma\in\Gamma} \mathrm{gr}_\gamma J_\lambda^\wedge \qquad B'' := \bigoplus_{\gamma\in\Gamma} \mathrm{gr}_\gamma B^\wedge.$$

Notice that B'' (resp. B') is a ring (resp. a $\mathrm{gr}_0 B$-module), and J''_λ (resp. J'_λ) is an ideal of B'' (resp. a $\mathrm{gr}_0 B$-submodule of B'), for every $\lambda \in \Lambda$. We endow B' with the linear topology defined by the cofiltered system of submodules $(J'_\lambda \mid \lambda \in \Lambda)$. Notice that B' is complete and separated, and the induced map

$$j_B : B \to B'$$

is continuous; more precisely, the topology of B agrees with the topology induced by B' via j_B. Moreover, for every $\gamma \in \Gamma$, let $\pi'_\gamma : B' \to \mathrm{gr}_\gamma B^\wedge$ be the projection; since $J'_\lambda \subset \pi'^{-1}_\gamma(\mathrm{gr}_\gamma J_\lambda)$ for every $\lambda \in \Lambda$, we see that π'_γ is continuous for the topology $\mathscr{T}_\gamma$, for every $\gamma \in \Gamma$.

(iii) Next, since B is dense in A, the inclusion map $B \to A$ extends uniquely to an isomorphism between the completion $(A^\wedge, \mathscr{T}^\wedge)$ of $(A, \mathscr{T})$ and the completion of B (theorem 8.2.8(iii)); it follows that j_B factors uniquely through a continuous map of topological $\mathrm{gr}_0 B$-modules

$$j_A : A^\wedge \to B' \qquad a \mapsto (a_\gamma \mid \gamma \in \Gamma)$$

and j_A is injective, since the topology of B is induced by that of B' (proposition 8.2.13(i)). For every $\gamma \in \Gamma$ we call $\pi^\wedge_\gamma := \pi'_\gamma \circ j_A : A^\wedge \to \mathrm{gr}_\gamma B^\wedge : a \mapsto a_\gamma$ the *canonical γ-projection* and set

$$a_S := \sum_{\gamma\in S} a_\gamma \qquad \text{for every } a \in A^\wedge \text{ and every finite subset } S \subset \Gamma.$$

(iv) For every finite subset $S \subset \Gamma$, let $i_S : \mathrm{gr}_S B := \bigoplus_{\gamma\in S} \mathrm{gr}_\gamma B \to A$ be the inclusion map; since J_λ is a graded ideal for every $\lambda \in \Lambda$, it is easily seen that the topology on $\mathrm{gr}_S B$ induced by A via i_S is the product topology $\prod_{\gamma\in S} \mathscr{T}_\gamma$. Hence, i_S extends uniquely to a continuous map

$$\prod_{\gamma\in S} (\mathrm{gr}_\gamma B^\wedge, \mathscr{T}_\gamma^\wedge) = (\mathrm{gr}_S B)^\wedge \to (A^\wedge, \mathscr{T}^\wedge)$$

which is still injective (proposition 8.2.13(i)), whence an injective map $B'' \to A^\wedge$. We endow the Γ-graded ring $\underline{B}'' := (B'', \mathrm{gr}_\bullet B'')$ with the topology induced from $A^\wedge$ via this map, and set

$$(A, \underline{B})^\wedge := (A^\wedge, \underline{B}'').$$

(v) Notice that, in case $\Gamma \neq 0$, the separation condition on $\mathscr{T}$ can be deduced from condition (b) in definition 8.5.1 : indeed, if γ, γ' are two distinct elements of Γ, we have

$$\{0\}^c \subset \mathrm{gr}_\gamma B \cap \mathrm{gr}_{\gamma'} B = 0$$

whence the contention. Thus, the separation condition in definition 8.5.1 only serves to rule out the somewhat trivial case of a non-separated topological space A endowed with the $\{0\}$-graded $\mathbb{Z}$-algebra structure.

Proposition 8.5.3. *Let $(\Gamma, +, 0)$ be a monoid, $(A, \underline{B})$ a Γ-pre-graded structure on the topological ring $(A, \mathscr{T})$. We have :*

(i) *The topology $\mathscr{T}$ is linear.*
(ii) *In the situation of remark* 8.5.2(iii)*, the system $(a_S \mid S \subset \Gamma)$, with S ranging over the filtered set of finite subsets of Γ, is a Cauchy net in B'', whose unique limit point is a.*
(iii) *In the situation of remark* 8.5.2(iv)*, the datum $(A, \underline{B})^\wedge$ is a Γ-graded structure on $(A^\wedge, \mathscr{T}^\wedge)$.*

(iv) *The inclusion functor* $\mathsf{gr.TopAlg} \to \mathsf{pre\text{-}gr.TopAlg}$ *admits a left adjoint*

$$\mathsf{pre\text{-}gr.TopAlg} \to \mathsf{gr.TopAlg} \qquad (A, \underline{B}, \Gamma) \mapsto (A, \underline{B}, \Gamma)^c$$

that assigns to each topological ring with pre-graded structure its associated graded structure.

Proof. Fix a fundamental system $(J_\lambda \mid \lambda \in \Lambda)$ of graded open ideals of B.

(i): Since A is separated, the completion map $(A, \mathscr{T}) \to (A^\wedge, \mathscr{T}^\wedge)$ is injective, B is naturally identified with a dense subring of $A^\wedge$, and the inclusion map $B \to A^\wedge$ extends to an isomorphism between $(A^\wedge, \mathscr{T}^\wedge)$ and the completion $B^\wedge$ of B ([41, Ch.II, §3, n.7, Prop.13]). However, for every $\lambda \in \Lambda$, let J^c_λ be the topological closure of J_λ in $B^\wedge$; then $(J^c_\lambda \mid \lambda \in \Lambda)$ is a fundamental system of open ideals in $B^\wedge$ (remark 8.3.3(ii)), and lastly, $(J^c_\lambda \cap A \mid \lambda \in \Lambda)$ is a fundamental system of open ideals in A, whence the contention.

(iii): By remark 8.2.7(iv), the subset $\mathrm{gr}_\gamma B''$ is closed in the topology of $A^\wedge$, for every $\gamma \in \Gamma$. Moreover, $B \subset B''$, so B'' is dense in $A^\wedge$. In light of the proof of (i), it remains only to check that $J^c_\lambda \cap B'' = J''_\lambda$, for every $\lambda \in \Lambda$. To this aim, say that $x \in B'' \cap J^c_\lambda$ for some such λ; then there exists a finite subset $S \subset \Gamma$ such that $x = \sum_{\gamma \in S} x_\gamma$ with $x_\gamma \in \mathrm{gr}_\gamma B''$ for every $\gamma \in S$. It follows that $\pi^\wedge_\gamma(x) = x_\gamma$ for $\gamma \in S$ and $\pi^\wedge_\gamma(x) = 0$ otherwise. On the other hand, from claim 8.3.25 we get

$$\pi^\wedge_\gamma(x) \in \pi^\wedge_\gamma(J^c_\lambda) \subset (\pi_\gamma(J_\lambda))^c = \mathrm{gr}_\gamma J^\wedge_\lambda$$

and the contention follows.

(ii): In the light of the proof of (iii), we have to check that for every $\lambda \in \Lambda$ there exists a finite subset $S_J \subset \Gamma$ such that $a_S - a_{S'} \in J''_\lambda$ for every finite subsets $S, S' \subset \Gamma$ containing S_J. However, the proof of (i) shows that the topological closure J^c_λ of J_λ in A is an open ideal of A; since B is dense in A, it follows that there exists $b \in B$ such that $a - b \in J^c_\lambda$. Write $b = \sum_{\gamma \in T} b_\gamma$ for some finite subset $T \subset \Gamma$; in view of claim 8.3.25, we deduce that

$$\pi^\wedge_\gamma(b - a) = b_\gamma - a_\gamma \in \pi^\wedge_\gamma(J^c_\lambda) \subset \pi_\gamma(J)^c = \mathrm{gr}_\gamma J^\wedge_\lambda \qquad \text{for every } \gamma \in \Gamma$$

(notation of remark 8.5.2(ii)). Especially, $a_\gamma \in \mathrm{gr}_\gamma J^\wedge_\lambda$ for every $\gamma \in \Gamma \setminus T$, and consequently $\pi^\wedge_\gamma(a_S - a_{S'}) \in \mathrm{gr}_\gamma J^\wedge_\lambda$ for every $\gamma \in \Gamma$ and every $S, S' \subset \Gamma$ containing T; *i.e.* $a_S - a_{S'} \in J''_\lambda$ for every such S, S', so that $S_J := T$ will do. By the same token, $a - a_S = (a - b) + (b - a_S) \in J^c_\lambda$ for every finite subset $S \subset \Gamma$ containing T, so a is the unique limit of this Cauchy net.

(iv): For every $\gamma \in \Gamma$, let $\mathrm{gr}_\gamma B^c$ be the topological closure of $\mathrm{gr}_\gamma B$ in A, and set $\underline{B}^c := \oplus_{\gamma \in \Gamma} \mathrm{gr}_\gamma B^c$; clearly $\underline{B}^c \subset \underline{B}''$, and in light of (iii), it is clear that the pair $(A, \underline{B}^c)$ is a topological ring with graded ring structure. Moreover, if $(f, \varphi) : (A, \underline{B}, \Gamma) \to (X, \underline{Y}, \Gamma')$ is any morphism to a topological ring with graded structure, then claim 8.3.25 implies easily that f maps $\underline{B}^c$ into $\underline{Y}$, so we get the sought left adjoint by setting $(A, \underline{B}, \Gamma)^c := (A, \underline{B}^c, \Gamma)$. □

Example 8.5.4. Let Γ be a monoid and $(A, \underline{B})$ a topological ring with Γ-graded structure. We have the following elementary operations :

(i) If $I \subset B$ is any graded ideal, let I^c (resp. I_A) be the topological closure of I in B (resp. in A), and endow A/I_A with the quotient topology arising from the projection $A \to A/I_A$. Remark 8.5.2(i) implies that I^c is a graded ideal of B : namely $\mathrm{gr}_\gamma I^c$ is the topological closure of $\mathrm{gr}_\gamma I$ in B, for every $\gamma \in \Gamma$. It follows that B/I^c is a Γ-graded subring of A/I_A, and the topology induced by A/I_A on B/I^c agrees with the quotient topology induced by B (lemma 8.2.3(ii)), so the former is linear, and we get therefore a topological ring with pre-graded structure $(A/I_A, \underline{B}/I^c)$. We then may form the associated graded structure

$$(A, \underline{B})/I := (A/I_A, \underline{B}/I^c)^c$$

(notation of proposition 8.5.3(iv)). Explicitly, this is the pair $(A/I_A, \underline{C})$, where

$$\mathrm{gr}_\gamma C := (\mathrm{Im}(\mathrm{gr}_\gamma B \to A/I_A))^c = (\mathrm{gr}_\gamma B/I^c)^c \qquad \text{for every } \gamma \in \Gamma.$$

(ii) Keep the situation of (i), and suppose additionally that A is complete and separated. Then we claim that $\mathrm{Im}(\mathrm{gr}_\gamma B \to A/I_A)$ is already closed in A/I_A, so the graded subring of A/I_A is just B/I^c in this case. Indeed, under these assumptions $\mathrm{gr}_\gamma B$ is also complete and separated, hence the canonical γ-projection $\pi^\wedge_\gamma : A/I_A \to \mathrm{gr}_\gamma C^\wedge$ factors through the completion map $\mathrm{gr}_\gamma(B/I^c) \to \mathrm{gr}_\gamma C^\wedge$; on the other hand, $\pi^\wedge_\gamma$ restricts to an injective map $\mathrm{gr}_\gamma C \to \mathrm{gr}_\gamma C^\wedge$, so there results an injective map $\mathrm{gr}_\gamma C \to \mathrm{gr}_\gamma(B/I^c)$, whose restriction to $\mathrm{gr}_\gamma(B/I^c)$ is the identity map. Thus, $\mathrm{gr}_\gamma C = \mathrm{gr}_\gamma(B/I^c)$, as stated.

(iii) Lastly, let $\underline{C} := (C, \mathrm{gr}_\bullet C) \subset \underline{B}$ be any closed Γ-graded subring, and we endow C and the topological closure C^c of C in A with the topologies induced from A; then $\mathrm{gr}_\gamma C = C \cap \mathrm{gr}_\gamma B$ is closed in $\mathrm{gr}_\gamma B$, and hence also in A, for every $\gamma \in \Gamma$, so the pair

$$(A, \underline{B}) \cap C := (C^c, \underline{C})$$

is again a topological ring with Γ-graded structure.

8.5.5. Let $p \geq 2$ be a prime integer, $(A, \underline{B})$ any Γ-graded topological ring, $a \in A^\wedge$ any element, and $(a_\gamma \mid \gamma \in \Gamma)$ the sequence attached to a, as in remark 8.5.2(iii). For every finite subset $S \subset \Gamma$, let S^* be the set of non-constant mappings $\{1, \dots, p\} \to S$; choose also a cyclic subgroup G of order p of the group of permutations of $\{1, \dots, p\}$, and endow S^* with the right G-action given by the rule : $g(\varphi) := \varphi \circ g$ for every $g \in G$ and $\varphi \in S^*$. Clearly, for every $\varphi \in S^*$, the product $a_{\varphi(i)} \cdots a_{\varphi(p)}$ depends only on the class $\overline{\varphi}$ of φ in S^*/G, so we denote it $a_{\overline{\varphi}}$. Also, notice that every such class $\overline{\varphi}$ has cardinality p, since p is a prime; with this notation, we may then write

$$(a^p)_S = c_S + p \cdot d_S \qquad \text{where } c_S := \sum_{\gamma \in S} (a_\gamma)^p \quad \text{and} \quad d_S := \sum_{\overline{\varphi} \in S^*/G} a_{\overline{\varphi}}.$$

Lemma 8.5.6. *With the notation of* (8.5.5)*, we have :*

(i) *The systems* $(c_S \mid S \subset \Gamma)$ *and* $(d_S \mid S \subset \Gamma)$ *are Cauchy nets in* B''.

(ii) *Suppose moreover that the topology of A is coarser than the p-adic topology, and the p-Frobenius $\boldsymbol{p}_\Gamma$ of Γ is injective. Then the following conditions are equivalent :*

(a) *a is topologically nilpotent in $A^\wedge$.*

(b) *a_γ is topologically nilpotent in B'' for every $\gamma \in \Gamma$.*

Proof. (i): According to proposition 8.5.3(ii), for every open graded ideal $J \subset B''$ there exists a finite subset $S_J \subset \Gamma$ such that $a_\gamma \in \mathrm{gr}_\gamma J$ for every $\gamma \in \Gamma \setminus S_J$. Now, if $S \subset \Gamma$ is any finite subset containing S_J, let S^*_1 be the subset of S^* consisting of all mappings $\{1, \dots, p\} \to S$ whose image is contained in S_J, and set $S^*_2 := S^* \setminus S^*_1$. Clearly $a_{\overline{\varphi}} \in J$ for every $\overline{\varphi} \in S^*_2/G$ (notation of remark 8.5.2(ii)), and S^*_1 is (finite and) independent of S, whence $d_S - d_{S'} \in J$ for every finite subsets $S, S' \subset \Gamma$ containing S_J. The assertion for d_S follows. Likewise, it is clear that $c_S - c_{S'} \in J^p$ for every S, S' as in the foregoing, so the assertion for c_S follows immediately.

(ii): Let $I \subset B$ be any graded open ideal; by assumption there exists $n \in \mathbb{N}$ such that $p^n \in B$. We consider the image $\overline{a}$ of a in the quotient topological ring with Γ-graded structure $(A, \underline{B})/(I + pB)$, as in example 8.5.4, whose underlying topological ring is A/J, where J is the topological closure of $I + pB$ in A. Let also $I^\wedge$ (resp. $J^\wedge$) be the topological closure of I (resp. of J) in $A^\wedge$. Then the system of γ-projections $(\overline{a}_\gamma \mid \gamma \in \Gamma)$ of $\overline{a}$ is the image of $(a_\gamma \mid \gamma \in \Gamma)$. Now, if a is topologically nilpotent, we may find $k \in \mathbb{N}$ such that $\overline{a}^{p^k} = 0$; since $p \in J$ and since $\boldsymbol{p}_\Gamma$ is injective, assertion (i) implies that $(\overline{a}_\gamma)^{p^k} = 0$ for every $\gamma \in \Gamma$, *i.e.* $(a_\gamma)^{p^k} \in J^\wedge$, and therefore $(a_\gamma)^{p^{kn}} \in (J^\wedge)^n \subset I^\wedge$ (claim 8.3.26). Since I is arbitrary, this shows that a_γ is topologically nilpotent for every $\gamma \in \Gamma$. Conversely, if the latter condition holds, for every $\gamma \in \Gamma$ there exists $i \geq n$ such that $\overline{a}^{p^i}_\gamma = 0$. On the other hand, notice that the topology of A/J

is discrete, so $A/J = B/I^c$, where I^c denotes the topological closure of I in B; especially, there exists a finite set $S \subset \Gamma$ such that $\overline{a}_\gamma = 0$ for every $\gamma \in \Gamma \setminus S$. Thus, we may find $j \geq n$ large enough, such that $\overline{a}_\gamma^{p^j} = 0$ for every $\gamma \in \Gamma$; but then (i) implies that $\overline{a}^{p^j} = 0$, and again, since I is arbitrary, this implies that a is topologically nilpotent. □

Example 8.5.7. (i) Let Γ, Δ be two monoids, and $(A, \underline{B})$ a topological ring with Δ-graded structure. Endow the ring $A[\Gamma]$ with the unique topology such that the inclusion map $A \to A[\Gamma]$ is adic, and $B[\Gamma]$ with its natural $\Delta \oplus \Gamma$-graded structure. Then we claim that the pair

$$(A, \underline{B})[\Gamma] := (A[\Gamma], B[\Gamma])$$

is a topological ring with $\Delta \oplus \Gamma$-graded structure. Indeed, let $(I_\lambda \mid \lambda \in \Lambda)$ be a fundamental system of Δ-graded open ideals for B; then the family $(I^c_\lambda[\Gamma] \mid \lambda \in \Lambda)$ is a fundamental system of open ideals for $A[\Gamma]$, and $I^c_\lambda[\Gamma] \cap B[\Gamma] = I_\lambda[\Gamma]$ for every $\lambda \in \Lambda$. Moreover, it is easily seen that the projection $p_\gamma : A[\Gamma] \to A : \sum_{\gamma' \in \Gamma} \gamma \cdot a_\gamma \mapsto a_\gamma$ is a continuous map, for every $\gamma \in \Gamma$; now, each direct summand $\mathrm{gr}_{(\delta,\gamma)} B[\Gamma]$ equals $p_\gamma^{-1}(\mathrm{gr}_\delta B) \bigcap_{\gamma' \neq \gamma} (\mathrm{Ker}\, p_{\gamma'})$, so it is a closed subset, whence the contention.

(ii) Let $\varphi : \Gamma \to \Gamma'$ be any morphism of monoids, $(A, \underline{B})$ any topological ring with Γ-graded structure. We obtain a topological ring with Γ'-pre-graded structure $(A, \underline{B}_{/\Gamma'})$, and by virtue of proposition 8.5.3(iv), we may then form the associated Γ'-graded structure

$$(A, \underline{B})_{/\Gamma'} := (A, \underline{B}_{/\Gamma'})^c.$$

(iii) In the situation of (ii), notice also that the pair $(\mathbf{1}_A, \varphi)$ yields a morphism of topological rings with graded structures :

$$(A, \underline{B}, \Gamma) \to (A, \underline{B}, \Gamma)_{/\Gamma'}.$$

Moreover, suppose that $(A, \underline{D}, \Gamma')$ is another Γ'-graded structure on A, such that $(\mathbf{1}_A, \varphi)$ induces a morphism of topological rings with graded structures

$$(A, \underline{B}, \Gamma)_{/\Gamma'} \to (A, \underline{D}, \Gamma')$$

and set $(A, \underline{C}) := (A, \underline{B}, \Gamma)_{/\Gamma'}$. Then we claim that $\underline{D} = \underline{C}$. Indeed, by assumption $\mathrm{gr}_{\gamma'} D$ is closed in A, hence also in $\mathrm{gr}_{\gamma'} C$, for every $\gamma' \in \Gamma'$; arguing as in remark 8.5.2(i) we easily see that D is a closed subset of C, whence the contention, as D is dense in A, hence also in B.

We deduce that the morphism $(\mathbf{1}_A, \varphi)$ enjoys the following universal property. For every morphism $(f, \psi) : (A, \underline{B}, \Gamma) \to (A', \underline{B}', \Gamma')$ such that ψ factors through φ, there exists a unique morphism of topological rings with Γ'-graded structures $(f, \mathbf{1}_{\Gamma'}) : (A, \underline{B}, \Gamma)_{/\Gamma'} \to (A', \underline{B}', \Gamma')$ such that $(f, \psi) = (f, \mathbf{1}_{\Gamma'}) \circ (\mathbf{1}_A, \psi)$.

8.5.8. To state the following proposition 8.5.11, it is convenient to introduce a special class of monoids; we shall say that a monoid $(\Gamma, +, 0)$ is *weakly integral* if we have the following partial cancellation property :

$$2 \cdot \gamma + \delta = 2 \cdot \gamma + \delta' \quad \Rightarrow \quad \gamma + \delta = \gamma + \delta' \qquad \text{for every } \gamma, \delta, \delta' \in \Gamma.$$

Lemma 8.5.9. *Let $(\Gamma, +, 0)$ be any monoid, $n \geq 2$ any integer, and suppose that the n-Frobenius endomorphism of Γ is injective. Then Γ is weakly integral.*

Proof. Indeed, let $\gamma, \delta, \delta' \in \Gamma$ be three elements such that $2 \cdot \gamma + \delta = 2 \cdot \gamma + \delta'$. Then clearly $n \cdot \gamma + \delta = n \cdot \gamma + \delta'$. A simple induction on k then shows that $n \cdot \gamma + k \cdot \delta = n \cdot \gamma + k \cdot \delta'$ for every $k \in \mathbb{N}$. Especially $n \cdot (\gamma + \delta) = n \cdot (\gamma + \delta')$, whence $\gamma + \delta = \gamma + \delta'$, as required. □

Lemma 8.5.10. *Let $(\Gamma, +, 0)$ be a weakly integral monoid, $(A, \underline{B})$ a topological ring with Γ-graded structure, $a \in A$, $\alpha \in \Gamma$ and $f \in \mathrm{gr}_\alpha B$ any three elements. Then we have :*

(i) $\pi^\wedge_{2\alpha+\delta}(f^2 a) = f \cdot \pi^\wedge_{\alpha+\delta}(fa)$ *for every $\delta \in \Gamma$.*

(ii) $\pi^\wedge_\gamma(fa) = 0$ *whenever $\gamma \notin \alpha + \Gamma$.*

Proof. Since π_γ is a continuous map for every $\gamma \in \Gamma$, it suffices to check these identities for $a \in B$, and then there exists a finite subset $S \subset \Gamma$ such that $a = a_S$.

(ii): We have $fa = \sum_{\delta\in S} f \cdot \pi^\wedge_\delta(a)$, and $f \cdot \pi_\delta(a) \in \mathrm{gr}_{\alpha+\delta}B$ for every $\delta \in S$, whence the assertion.

(i): Clearly $\pi^\wedge_{2\alpha+\delta}(f^2a) = \sum_{\gamma\in T} a \cdot \pi^\wedge_\gamma(fa)$, where $T := \{\gamma \in S \mid \alpha + \gamma = 2\alpha + \delta\}$. But due to (ii), we have $\pi^\wedge_\gamma(fa) = 0$ unless $\gamma \in \alpha + \Gamma$, so we may replace T by the subset of all elements of the form $\alpha+\delta'$ such that $2\alpha+\delta' = 2\alpha+\delta$. Since Γ is weakly integral, $T = \{\alpha+\delta\}$, whence the assertion. □

Proposition 8.5.11. *Let $(\Gamma, +, 0)$ be a monoid, $(A, \underline{B})$ a Γ-graded structure on the topological ring $(A, \mathscr{T})$, and $i_0 : \mathrm{gr}_0B \to A$ the inclusion map. We have :*

- (i) *Suppose that the topology of A is f-adic, complete and separated. Then :*
  - (a) *There exists a finitely generated graded ideal J of B such that JA is an ideal of adic definition for A.*
  - (b) *If moreover, $J = (\mathrm{gr}_0J) \cdot B$ or else Γ is weakly integral, $\mathscr{T}$ induces on B the J-adic topology.*
- (ii) *Suppose that i_0 is c-adic. Then A is c-adic if and only if the same holds for gr_0B.*
- (iii) *Suppose that A is complete and separated, and i_0 is c-adic. Then A is f-adic if and only if the same holds for gr_0B.*

Proof. (i.a): Let I be any finitely generated ideal of adic definition of A; by assumption, we may find a graded ideal I_B of B and an integer $n \in \mathbb{N}$ with $I^n \cap B \subset I_B \subset I$. For every subset S of A we denote as usual by S^c the topological closure of S in A; since I^n is an open and closed subset of A, we have

$$I^n = I^n \cap B^c = (I^n \cap B)^c \subset (I^n \cap I_B)^c$$

so that $I^n = (I^n \cap I_B) + I^{n+1}$. We may then find a finitely generated graded subideal J of I_B such that $I^n = (I^n \cap J) + I^{n+1}$. On the other hand, since A is I-adically complete and separated, the ideal I is contained in the Jacobson radical of A (remark 8.3.10(v)), so that $I^n = (I^n \cap J)A$, by Nakayama's lemma, and therefore $I^n \subset JA$; by construction, we have as well $JA \subset I$, so J is an ideal with the sought properties.

(i.b): We prove more precisely :

Claim 8.5.12. In the situation of (i), let $J \subset B$ be any graded ideal, and suppose that either one of the following conditions holds :

- (a) Γ is integral.
- (b) Γ is weakly integral and JA is open in A.
- (c) $J = (\mathrm{gr}_0J) \cdot B$.

Then $J = B \cap JA$ for every $n \in \mathbb{N}$.

Proof of the claim. Say that $x \in B \cap JA$, so we may find a mapping $\varphi : \{1, \dots, k\} \to \Gamma$ and elements $a_i \in \mathrm{gr}_{\varphi(i)}J$, $y_i \in A$ for $i = 1, \dots, k$, such that $x = \sum_{i=1}^k a_iy_i$, as well as a finite subset $S \subset \Gamma$ such that $x = x_S$. Suppose first that $J = (\mathrm{gr}_0J) \cdot B$, in which case we may assume that $a_i \in \mathrm{gr}_0J$ for $i = 1, \dots, k$; we compute

$$x = \sum_{\gamma\in S} \sum_{i=1}^k a_i \cdot \pi^\wedge_\gamma(y_i)$$

whence the claim, in this case. Next, suppose that Γ is integral, so that the natural map $\Gamma \to \Gamma^{\mathrm{gp}}$ is injective; we may then replace Γ by Γ^{gp}, and assume from start that Γ is an abelian group.

We compute

$$x = \sum_{\gamma \in S} \sum_{i=1}^{k} a_i \cdot \pi^{\wedge}_{\gamma - \varphi(i)}(y_i)$$

so the claim follows also in this case. Lastly, suppose that condition (b) holds; then we may find homogeneous elements $f_1, \dots, f_r \in J$ such that $\sum_{i=1}^r f_i A$ is an open ideal of A, and the same holds therefore for the ideal $I := \sum_{i=1}^r f_i^2 A$. Next, we may find $y'_i \in B$ such that $y_i - y'_i \in I$ for $i = 1, \dots, k$, and clearly it suffices to check that $x' := x - \sum_{i=1}^k y'_i a_i \in J$. We may then replace x by x', and assume as well that $x \in I$; in this case, lemma 8.5.10 says that $\pi_\gamma(x) \in \sum_{i=1}^r f_i B$ for every $\gamma \in S$, whence the contention. ◊

(ii): Due to lemma 8.3.32(i) and proposition 8.5.3(iii), we may replace $(A, \underline{B})$ by $(A, \underline{B})^\wedge$, after which we may assume that A is complete and separated. We know already that if the topology of $\mathrm{gr}_0 B$ is c-adic, the same holds for $\mathscr{T}$ (lemma 8.3.24(i.b)). Thus, suppose that $\mathscr{T}$ is c-adic, and let I be any ideal of c-adic definition of A; by assumption, there exists an open ideal $I_0 \subset \mathrm{gr}_0 B$ with $(I_0 A)^c \subset I$. Now, let $J \subset \mathrm{gr}_0 B$ be any other open ideal; we know that there exists $m \in \mathbb{N}$ such that $(I^m)^c \subset (JA)^c$, and taking into account claim 8.3.26 we deduce that $(I_0^m A)^c \subset (JA)^c$. Combining with claim 8.3.25 we get

$$\pi_0^\wedge(I_0^m A)^c \subset \pi_0^\wedge(JA)^c.$$

But notice that, since A is complete, $\pi_0^\wedge(I_0^m A) = I_0^m$ and $\pi_0^\wedge(JA) = J$, so finally $(I_0^m)^c \subset J$, whence the assertion.

(iii): Suppose first that $\mathrm{gr}_0 B$ is f-adic (in which case, it is also adic, by proposition 8.3.18(iii)). Then lemmata 8.3.24(i.b) and 8.3.21(ii.a) imply that the same holds for A. Conversely, suppose that A is f-adic (and adic), and let $I \subset A$ be any finitely generated ideal of adic definition. By assumption, there exists an open ideal $J \subset \mathrm{gr}_0 B$ such that $I^n \subset (JA)^c \subset I$ for some $n \in \mathbb{N}$. Since I^n is both open and closed in A, we deduce

$$I^n = I^n \cap (JA)^c = (I^n \cap JA)^c = I^{n+1} + (JA \cap I^n).$$

It follows that there exists a finitely generated graded subideal $K \subset J$ such that $I^n \subset KA + I^{n+1}$, and so $I^n \subset KA \subset I$, by virtue of Nakayama's lemma and remark 8.3.10(v). Then $(K^r A \mid r \in \mathbb{N})$ is a fundamental system of open ideals in A, and therefore $(K^r A \cap \mathrm{gr}_0 B \mid r \in \mathbb{N})$ is a fundamental system of open ideals in $\mathrm{gr}_0 B$. But from claims 8.3.26 and 8.3.25 we get

$$K^r \subset K^r A \cap \mathrm{gr}_0 B \subset \pi_0^\wedge(K^r A) \subset \pi_0^\wedge(K^r B)^c = K^r$$

so $K^r A \cap \mathrm{gr}_0 B = K^r$ for every $r \in \mathbb{N}$, and finally $\mathrm{gr}_0 B$ is f-adic. □

8.6. **Homological algebra for topological modules.** Let A be any topological ring. The category A-TopMod of topological A-modules and continuous A-linear maps is additive and with representable kernels and cokernels but, generally, not abelian; indeed, if $f : M \to N$ is a continuous map of topological A-modules, $\mathrm{Coker}(\mathrm{Ker}\, f \to M)$ is not necessarily isomorphic to $\mathrm{Ker}(N \to \mathrm{Coker}\, f)$, since the quotient topology (induced from M) on $\mathrm{Im}\, f$ may be finer than the subspace topology (induced from N). It is therefore useful to introduce the following :

Definition 8.6.1. Let A be a topological ring, $f : M \to N$ a morphism of topological A-modules. We say that f is *strict*, if the natural map

$$\mathrm{Coker}(\mathrm{Ker}\, f \to M) \to \mathrm{Ker}(N \to \mathrm{Coker}\, f)$$

is an isomorphism of topological A-modules (where these kernels and cokernels are formed in the additive category A-TopMod : see remark 3.7.29(v)).

The category A-TopMod is *exact* in the sense of [140]; namely, the admissible monomorphisms (resp. epimorphisms) are the continuous injections (resp. surjections) $f : M \to N$ that are strict, in the sense of definition 8.6.1, *i.e.* that induce homeomorphisms $M \stackrel{\sim}{\to} f(M)$ (resp. $M/\mathrm{Ker}\, f \stackrel{\sim}{\to} N$), where $f(M)$ (resp. $M/\mathrm{Ker}\, f$) is endowed with the subspace (resp. quotient) topology induced from N (resp. from M). An admissible epimorphism is also called a *quotient map* of topological A-modules. Correspondingly there is a well defined class of *admissible short exact sequences* of topological A-modules. The following lemma exhibits a useful class of admissible short exact sequences.

Lemma 8.6.2. *Consider an inverse system, with surjective transition maps*

$$(\underline{E}_n \mid n \in \mathbb{N}) \quad : \quad 0 \to (M'_n \mid n \in \mathbb{N}) \to (M_n \mid n \in \mathbb{N}) \to (M''_n \mid n \in \mathbb{N}) \to 0$$

of short exact sequences of discrete A-modules. Then the induced complex of inverse limits :

$$\lim_{n\in\mathbb{N}} \underline{E}_n \quad : \quad 0 \to M' := \lim_{n\in\mathbb{N}} M'_n \to M := \lim_{n\in\mathbb{N}} M_n \to M'' := \lim_{n\in\mathbb{N}} M''_n \to 0$$

is an admissible short exact sequence of topological A-modules.

Proof. For every pair of integers $i, j \in \mathbb{N}$ with $i \leq j$, let $\varphi_{ji} : M_j \to M_i$ be the transition map in the inverse system $(M_n \mid n \in \mathbb{N})$, and define likewise φ'_{ji} and φ''_{ji}; the assumption means that all these maps are onto. Set $K_{ji} := \mathrm{Ker}\, \varphi_{ji}$ and define likewise K'_{ji}, K''_{ji}. By the snake lemma we deduce, for every $i \in \mathbb{N}$, an inverse system of short exact sequences :

$$0 \to (K'_{ji} \mid j \geq i) \to (K_{ji} \mid j \geq i) \to (K''_{ji} \mid j \geq j) \to 0$$

where again, all the transition maps are surjective. However, by definition, the decreasing family of submodules $(K_i := \lim_{j\geq i} K_{ji} \mid i \in \mathbb{N})$ is a fundamental system of open neighborhoods of $0 \in M$ (and likewise one defines the topology on the other two inverse limits). It follows already that the surjection $M \to M''$ is a quotient map of topological A-modules. To conclude it suffices to remark :

Claim 8.6.3. The natural map $K'_i := \lim_{j\geq i} K'_{ji} \to K_i$ induces an identification :

$$K'_i = K_i \cap M' \qquad \text{for every } i \in \mathbb{N}.$$

Proof of the claim. Indeed, for every $j \geq i$ we have a left exact sequence :

$$\underline{F}_j \quad : \quad (0 \to K'_{ji} \to M'_j \oplus K_{ji} \xrightarrow{\alpha} M_j)$$

where $\alpha(m', m) = m' - m$ for every $m' \in M'_j$ and $m \in K_{ji}$. Therefore $\lim_{j\geq i} \underline{F}_j$ is the analogous left exact sequence $0 \to K'_i \to M' \oplus K_i \to M$, whence the claim. □

Remark 8.6.4. Let $(C^\bullet, d^\bullet)$ be a complex of topological A-modules with continuous differentials. Then, for every $i \in \mathbb{Z}$, the cohomology $H^iC^\bullet$ inherits a well defined topology : namely, we can take either

$$\mathrm{Coker}(d^{i-1} : C^{i-1} \to \mathrm{Ker}\, d^i) \qquad \text{or} \qquad \mathrm{Ker}(d^i : \mathrm{Coker}\, d^{i-1} \to C^{i+1})$$

(where these kernels and cokernels are formed in the additive category A-TopMod) and by lemma 8.2.3(i), these two topologies coincide (regardless of whether the differentials are strict or otherwise). Notice also that for every $i \in \mathbb{Z}$ the following conditions are equivalent :

(a) The differential d^i factors through an open map $C^i \to \mathrm{Ker}\, d^{i+1}$.
(b) The differential d^i is strict and the topology of $H^{i+1}C^\bullet$ is discrete.

8.6.5. Consider an *admissible short exact sequence* of complexes of topological A-modules

$$0 \to (C_1^\bullet, d_1^\bullet) \xrightarrow{f^\bullet} (C_2^\bullet, d_2^\bullet) \xrightarrow{g^\bullet} (C_3^\bullet, d_3^\bullet) \to 0$$

i.e. a double complex of A-TopMod whose rows $0 \to C_1^k \to C_2^k \to C_3^k \to 0$ are admissible short exact sequences for every $k \in \mathbb{Z}$. For every $i \in \mathbb{Z}$ and for $j = 1, 2, 3$, denote by $\delta_j^i : C^i \to \mathrm{Ker}\, d_j^{i+1}$ the continuous map induced by the differential $d_j^i : C_j^i \to C_j^{i+1}$ of $C_j^\bullet$.

Proposition 8.6.6. *In the situation of* (8.6.5), *fix also* $i \in \mathbb{Z}$. *We have :*

(i) *If* δ_2^i *and* δ_3^{i-1} *are open maps, the same holds for* δ_1^i.
(ii) *If* δ_1^i *and* δ_3^i *are open maps, the same holds for* δ_2^i.
(iii) *If* δ_1^i *and* δ_2^{i-1} *are open maps, the same holds for* δ_3^{i-1}.

Proof. (i): Endow $D := g^{i+1}(\mathrm{Ker}\, d_2^{i+1})$ with the quotient topology induced by the restriction $h : \mathrm{Ker}\, d_2^{i+1} \to D$ of g^{i+1}, and notice that $\mathrm{Im}\, d_3^i \subset D$; we claim that d_3^i induces a continuous and open map $C_3^i \to D$. Indeed, since g^i is an admissible epimorphism, it suffices to check that the composition $d_3^i \circ g^i : C_2^i \to D$ is continuous and open; however, the latter equals $h \circ \delta_2^i$, whence the assertion. Moreover, f^{i+1} restricts to an admissible monomorphism $\mathrm{Ker}\, d_1^{i+1} \to \mathrm{Ker}\, d_2^{i+1}$. Summing up, we may replace C_j^{i+1} with $\mathrm{Ker}\, d_j^{i+1}$ for $j = 1, 2$, and C_3^{i+1} with D, and assume from start that $C_j^{i+2} = 0$ for $j = 1, 2, 3$. Then $d_j^i : C_j^i \to C_j^{i+1}$ is open for $j = 2, 3$, and it remains to check that d_1^i is open as well.

Next, set $\overline{C}_1^i := \mathrm{Coker}\, d_1^{i-1}$ and $\overline{C}_2^i := \mathrm{Coker}\, d_2^{i-1} \circ f^{i-1}$ (endowed with the topologies induced by C_1^i and respectively C_2^i). Clearly d_j^i factors through a continuous map $\overline{d}_j^i : \overline{C}_j^i \to C_j^{i+1}$ for $j = 1, 2$, and f^i (resp. g^i) induces an injective (resp. surjective) continuous map $\overline{f}^i : \overline{C}_1^i \to \overline{C}_2^i$ (resp. $\overline{g}^i : \overline{C}_2^i \to C_3^i$). Moreover, by lemma 8.2.3(i) the topology of $\overline{C}_1^i$ is induced by the topology of $\overline{C}_2^i$ via $\overline{f}^i$, *i.e.* $\overline{f}^i$ is an admissible monomorphism. Likewise, $\overline{g}^i$ is an admissible epimorphism. We may then replace C_j^i by $\overline{C}_j^i$ for $j = 1, 2$, and assume from start $C_1^{i-1} = 0$, in which case we obtain a commutative diagram of continuous maps

(8.6.7)
$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & C_2^{i-1} & \xrightarrow{g^{i-1}} & C_3^{i-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle d_2^{i-1}} & & \downarrow{\scriptstyle d_3^{i-1}} & & \\
0 & \longrightarrow & C_1^i & \xrightarrow{f^i} & C_2^i & \xrightarrow{g^i} & C_3^i & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d_1^i} & & \downarrow{\scriptstyle d_2^i} & & \downarrow{\scriptstyle d_3^i} & & \\
0 & \longrightarrow & C_1^{i+1} & \xrightarrow{f^{i+1}} & C_2^{i+1} & \xrightarrow{g^{i+1}} & C_3^{i+1} & \longrightarrow & 0
\end{array}$$

whose rows are admissible short exact sequences, and where d_2^i and δ_3^{i-1} are still open. Consider the diagram of topological A-modules

(8.6.8)
$$\begin{array}{ccc}
\mathrm{Ker}\,(d_3^i \circ g^i) & \xrightarrow{\overline{g}^i} & \mathrm{Ker}\, d_3^i \\
\downarrow{\scriptstyle \overline{d}_2^i} & & \downarrow \\
\mathrm{Ker}\, g^{i+1} & \longrightarrow & 0
\end{array}$$

deduced from the right bottom square of (8.6.7). In view of claim 8.2.4 we deduce that $\overline{g}^i$ and $\overline{d}_2^i$ are open maps. Moreover, f^i and f^{i+1} factor through admissible monomorphisms $C_1^i \to \mathrm{Ker}\,(d_3^i \circ g^i)$ and $C_1^{i+1} \to \mathrm{Ker}\, g^{i+1}$. Likewise, d_2^{i-1} and d_3^{i-1} factor through continuous maps $C_2^{i-1} \to \mathrm{Ker}\,(d_3^i \circ g^i)$ and $C_3^{i-1} \to \mathrm{Ker}\, d_3^i$. Thus, we may replace the bottom right square of (8.6.7) with (8.6.8), and assume from start that $C_3^{i+1} = 0$ as well. In this situation, g^{i-1} and f^{i+1}

are both isomorphisms of topological A-modules, and d_3^{i-1} is an open map, so we are reduced to showing :

Claim 8.6.9. Consider a diagram of topological A-modules

$$\begin{array}{ccccccccc} & & & & C_4 & & & & \\ & & & & \downarrow{\scriptstyle h} & & & & \\ 0 & \longrightarrow & C_1 & \xrightarrow{f} & C_2 & \xrightarrow{g} & C_3 & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle k} & & & & \\ & & & & C_5 & & & & \end{array}$$

such that :

(a) The horizontal row is an admissible short exact sequence.
(b) $g \circ h$ and k are open maps and $k \circ h = 0$.

Then $k \circ f$ is an open map.

Proof of the claim. After pulling back the short exact admissible sequence of (f, g) along the map $g \circ h$, we obtain the short exact sequence of topological modules

$$0 \to C_1 \xrightarrow{f'} C_2 \times_{C_3} C_4 \xrightarrow{g'} C_4 \to 0.$$

Notice that g' is still an admissible epimorphism, by claim 8.2.4. Likewise, f' is an admissible monomorphism, since its composition with the projection $\pi : C_2 \times_{C_3} C_4 \to C_2$ equals f, which is an admissible monomorphism by assumption. Moreover, since $g \circ h$ is open, the same holds for π, again by claim 8.2.4, and therefore also for $k' := k \circ \pi : C_2 \times_{C_3} C_4 \to C_5$. Furthermore, the pair $(h, \mathbf{1}_{C_4})$ defines a continuous section $s : C_4 \to C_2 \times_{C_3} C_4$ for g', whence a continuous isomorphism of A-modules

$$\omega : C_1 \times C_4 \xrightarrow{\sim} C_2 \times_{C_3} C_4 \qquad (x, y) \mapsto f'(x) + s(y) = (f(x) + h(y), y)$$

and we notice that ω is an isomorphism of topological A-modules : indeed, let us endow $f(C_1)$ with the topology induced by the inclusion into C_2, so that f factors through an isomorphism $u : C_1 \xrightarrow{\sim} f(C_1)$ of topological A-modules; then the inverse of ω is the continuous map

$$C_2 \times_{C_3} C_4 \to C_1 \times C_4 \qquad (a, b) \mapsto (u^{-1}(a - h(b)), b)$$

whence the assertion. Summing up, we deduce that the continuous map $\psi := k \circ \pi \circ \omega : C_1 \times C_4 \to C_5$ is open, and notice that $\psi(x, y) = k \circ f(x)$ for every $(x, y) \in C_1 \times C_4$, since $k \circ h = 0$. Lastly, let $U \subset C_1$ be any open subset; then $U \times C_4$ is open in $C_1 \times C_4$, and $\psi(U \times C_4) = k \circ f(U)$ is open in C_5, whence the claim. ◊

(ii): Define the A-module D and the A-linear map $h : \mathrm{Ker}\, d_2^{i+1} \to D$ as in (i), and endow D with the topology induced by the inclusion into C_3^{i+1}. Then h is continuous, and it is also an open map, since the same holds for $h \circ \delta_2^i = \delta_3^i \circ g^i : C_2^i \to D$ (lemma 8.2.5). We may then argue as in the proof of (i), to reduce to the case where $C_j^{i+2} = 0$ for $j = 1, 2, 3$, in which case d_1^i and d_3^i are both open maps, and we must show that the same holds for d_2^i.

Next, we pull back the admissible short exact sequence (f^i, g^i) (resp. (f^{i+1}, g^{i+1})) along the map g^i (resp. along the map $d_3^i \circ g^i$) to get the commutative diagram of topological A-modules:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_1^i & \xrightarrow{f'^i} & C_2^i \times_{C_3^i} C_2^i & \xrightarrow{g'^i} & C_2^i & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d_1^i} & & \downarrow{\scriptstyle d_2'^i} & & \| & & \\
0 & \longrightarrow & C_1^{i+1} & \xrightarrow{f'^{i+1}} & C_2^{i+1} \times_{C_3^{i+1}} C_2^i & \xrightarrow{g'^{i+1}} & C_2^i & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \pi^{i+1}} & & \downarrow{\scriptstyle d_3^i \circ g^i} & & \\
0 & \longrightarrow & C_1^{i+1} & \xrightarrow{f^{i+1}} & C_2^{i+1} & \xrightarrow{g^{i+1}} & C_3^{i+1} & \longrightarrow & 0.
\end{array} \tag{8.6.10}$$

Arguing as in the proof of claim 8.6.9 we easily see that the rows of (8.6.10) are still admissible short exact sequences. Moreover, the pair $(\mathbf{1}_{C_2^i}, \mathbf{1}_{C_2^i})$ (resp. $(d_2^i, \mathbf{1}_{C_2^i})$) gives a section s^i (resp. s^{i+1}) for g'^i (resp. for g'^{i+1}). Furthermore, π^{i+1} is an open map, by claim 8.2.4, and if $\pi^i : C_2^i \times_{C_3^{i+1}} C_2^i \to C_2^i$ denotes the projection, we have

$$d_2^i \circ \pi^i = \pi^{i+1} \circ d_2'^i.$$

Thus, in order to show that d_2^i is open, it suffices to check that the same holds for $d_2'^i$ (lemma 8.2.5). Summing up, we may replace the exact rows (f^i, g^i) and (f^{i+1}, g^{i+1}) by (f'^i, g'^i) and (f'^{i+1}, g'^{i+1}), and assume also from start that $C_3^i = C_3^{i+1}$ and $d_3^i = \mathbf{1}_{C_3^i}$, and moreover, that g^i and g^{i+1} admit continuous sections s^i and respectively s^{i+1}, such that $d_2^i \circ s^i = s^{i+1}$. Arguing as in the proof of claim 8.6.9 we deduce isomorphisms of topological A-modules

$$\omega^k : C_1^k \times C_3^i \xrightarrow{\sim} C_2^k \qquad (x, y) \mapsto f^k(x) + s^k(y) \qquad \text{for } k = i, i+1$$

fitting into a commutative diagram

$$\begin{array}{ccc}
C_1^i \times C_3^i & \xrightarrow{\omega^i} & C_2^i \\
\downarrow{\scriptstyle d_1^i \times \mathbf{1}_{C_3^i}} & & \downarrow{\scriptstyle d_2^i} \\
C_1^{i+1} \times C_3^i & \xrightarrow{\omega^{i+1}} & C_2^{i+1}
\end{array}$$

and since d_1^i is an open map, the same holds for $d_1^i \times \mathbf{1}_{C_3^i}$, whence the assertion.

(iii): Arguing as in the proof of (i), we reduce to the case where $C_j^{i+2} = 0$ for every $j = 1, 2, 3$, in which case d_1^i is an open map. Then, we define $\overline{C}_j^i$ for $j = 1, 2$ as in the proof of (i), so that d_j^i factors through a continuous map $\overline{d}_j^i : \overline{C}_j^i \to C_j^{i+1}$ for $j = 1, 2$. Notice that $\overline{d}_1^i$ is an open map, and δ_2^{i-1} induces an open map $\mathrm{Coker}\, f^{i-1} \to \mathrm{Ker}\, \overline{d}_2^i$. We may then replace C_j^i by $\overline{C}_j^i$ for $j = 1, 2$, and assume from start that $C_1^{i-1} = 0$, in which case we obtain a commutative diagram (8.6.7) whose rows are admissible short exact sequences, and where d_1^i and δ_2^{i-1} are open maps. Next, we may argue again as in the proof of (i), to reduce to the case where $C_3^{i+1} = 0$ as well, in which case we have to show that d_3^{i-1} is an open map. To this aim, we consider the cartesian diagram

$$\begin{array}{ccc}
C_1^i \times_{C_2^{i+1}} C_2^i & \xrightarrow{\pi_2^i} & C_2^i \\
\downarrow{\scriptstyle \pi_1^i} & & \downarrow{\scriptstyle d_2^i} \\
C_1^i & \xrightarrow{f^{i+1} \circ d_1^1} & C_2^{i+1}
\end{array}$$

and we notice that the pair $(\mathbf{1}_{C_1^i}, f^i)$ defines a continuous section $s : C_1^i \to C_1^i \times_{C_2^{i+1}} C_2^i$ for π_1^i. Likewise, the pair consisting of the zero map $\mathrm{Ker}\, d_2^i \to C_1^i$ and the inclusion map $\mathrm{Ker}\, d_2^i \to C_2^i$

defines a continuous map $t : \mathrm{Ker}\, d_2^i \to C_1^i \times_{C_2^{i+1}} C_2^i$, whence an isomorphism of topological A-modules

$$\omega : C_1^i \times \mathrm{Ker}\, d_2^i \xrightarrow{\sim} C_1^i \times_{C_2^{i+1}} C_2^i \qquad (x, y) \mapsto s(x) + t(y) = (x, f^i(x) + y).$$

Furthermore, under the current assumptions, f^{i+1} is an isomorphism of topological A-modules, so $f^{i+1} \circ d_1^i$ is an open map, and then the same holds for π_2^i, according to claim 8.2.4. Lastly, notice the commutative diagram

$$\begin{array}{ccccccc}
C_1^i \times C_2^{i-1} & \xrightarrow{\pi_2^{i-1}} & C_2^{i-1} & & \xrightarrow{\quad g^{i-1} \quad} & & C_3^{i-1} \\
\Big\downarrow {\scriptstyle \mathbf{1}_{C_1^i} \times \delta_2^{i-1}} & & & & & & \Big\downarrow {\scriptstyle d_3^{i-1}} \\
C_1^i \times \mathrm{Ker}\, d_2^i & \xrightarrow{\omega} & C_1^i \times_{C_3^{i+1}} C_2^i & \xrightarrow{\pi_2^i} & C_2^i & \xrightarrow{g^i} & C_3^i
\end{array}$$

where π_2^{i-1} is the projection. Since δ_2^{i-1} is an open map, the same holds for $\mathbf{1}_{C_1^i} \times \delta_2^{i-1}$, and then also for $d_3^{i-1} \circ g^{i-1} \circ \pi_2^{i-1}$. By lemma 8.2.5, we conclude that d_3^{i-1} is open as well, as required. $\square$

8.6.11. Consider an object $C^{\bullet\bullet}$ of $\mathsf{C}_2(A\text{-}\mathsf{TopMod})$, *i.e.* a double complex of topological A-modules whose horizontal and vertical differentials

$$d_h^{p,q} : C^{p,q} \to C^{p+1,q} \qquad d_v^{p,q} : C^{p,q} \to C^{p,q+1}$$

are continuous maps, and therefore induce continuous maps

$$\delta_h^{p,q} : C^{p,q} \to \mathrm{Ker}\, d_h^{p+1,q} \qquad \delta_v^{p,q} : C^{p,q} \to \mathrm{Ker}\, d_v^{p,q+1} \qquad \text{for every } p, q \in \mathbb{Z}.$$

With this notation, we have :

Corollary 8.6.12. *In the situation of* (8.6.11)*, suppose moreover that :*

(a) $C^{p,q} = 0$ *whenever* $p < 0$ *or* $q < 0$.
(b) *The maps* $\delta_v^{p,q}$ *are open for every* $p, q \in \mathbb{Z}$.
(c) *The maps* $\delta_h^{p,q}$ *are open for every* $p \in \mathbb{Z}$ *and every* $q > 0$.

Then the maps $\delta_h^{p,0}$ *are open for every* $p \in \mathbb{Z}$.

Proof. To ease notation, set

$$B_v^{pq} := \mathrm{Im}\, d_v^{p,q-1} \qquad Z_v^{pq} := \mathrm{Ker}\, d_v^{p,q} \qquad H_v^{p,q} := \mathrm{Coker}\, \delta_h^{p,q} \qquad \text{for every } p, q \in \mathbb{Z}$$

and denote by $d_B^{p,q} : B_v^{p,q} \to B^{p+1,q}$ and $d_Z^{p,q} : Z_v^{p,q} \to Z^{p+1,q}$ the restrictions of $d_h^{p,q}$, for every $p, q \in \mathbb{Z}$. Then $(B_v^{\bullet,q}, d_B^{\bullet,q})$ and $(Z_v^{\bullet,q}, d_Z^{\bullet,q})$ are complexes of topological A-modules, and we get admissible short exact sequences of complexes

$$0 \to (Z_v^{\bullet,q}, d_Z^{\bullet,q}) \to (C^{\bullet,q}, d_h^{\bullet,q}) \to (B_v^{\bullet,q+1}, d_B^{\bullet,q+1}) \to 0$$
$$0 \to (B_v^{\bullet,q}, d_B^{\bullet,q}) \to (Z_v^{\bullet,q}, d_Z^{\bullet,q}) \to H_v^{\bullet,q} \to 0.$$

Let also

$$\delta_B^{p,q} : B_v^{p,q} \to \mathrm{Ker} d_B^{p+1,q} \qquad \delta_Z^{p,q} : Z_v^{p,q} \to \mathrm{Ker} d_Z^{p+1,q}$$

be the continuous maps induced by $d_B^{p,q}$ and respectively $d_Z^{p,q}$, for every $p, q \in \mathbb{Z}$.

Claim 8.6.13. For every p, q the following holds :

(d) If $\delta_Z^{p,q}$ is an open map, the same holds for $\delta_B^{p,q}$.
(e) If $\delta_B^{p-1,q+1}$ is an open map, the same holds for $\delta_Z^{p,q}$, provided $q > 0$.

Proof of the claim. Indeed, $H_v^{\bullet,q}$ is complex of discrete topological A-modules, for every $q \in \mathbb{Z}$, due to assumption (b), so its differentials $d_H^{\bullet,q}$ trivially induce open maps $H_v^{p,q} \to \mathrm{Ker}\, d_H^{p+1,q}$ for every q; then (d) follows from proposition 8.6.6(i). Likewise, (e) follows from assumption (b) and proposition 8.6.6(i). ◊

We now show that, for every $n \in \mathbb{Z}$, the maps $\delta_B^{p,q}$ are open for every $p, q \in \mathbb{Z}$ such that $p + q = n$ and $q > 0$. We argue by induction on p. For $p < -1$, assumption (a) says that $B_v^{p,q} = B_v^{p+1,q} = 0$, so the assertion trivially holds. Suppose then that $p \geq -1$, and that we already know that $\delta_B^{p-1,q+1}$ is open; from claim 8.6.13 we deduce that $\delta_B^{p,q}$ is also open, as stated. Especially, we see that $\delta_B^{p,1}$ is an open map, for every $p \in \mathbb{Z}$. Lastly, notice that (a) and (b) also imply that the topology of $Z_v^{p,0}$ is discrete for every $p \in \mathbb{Z}$; then we apply proposition 8.6.6(ii) to the admissible short exact sequence $0 \to Z_v^{\bullet,0} \to C^{\bullet,0} \to B_v^{\bullet,1}$ to conclude. □

8.6.14. Let A be any topological ring, $(C^\bullet, d^\bullet)$ any complex of topological A-modules, $\varphi^\bullet : C^\bullet \to C^\bullet$ an endomorphism of $C^\bullet$ and $h^\bullet$ a chain homotopy from $\mathbf{1}_{C^\bullet}$ to $\varphi^\bullet$ in the category $\mathsf{C}(A\text{-}\mathsf{TopMod})$, so that $\varphi^i : C^i \to C^i$ and $h^i : C^i \to C^{i-1}$ are continuous A-linear maps, for every $i \in \mathbb{Z}$.

Lemma 8.6.15. *In the situation of* (8.6.14)*, fix $i \in \mathbb{Z}$ and suppose that $\varphi^i = 0$. Then we have :*

(i) d^{i-1} *induces an open and surjective map* $\delta^i : C^{i-1} \to \mathrm{Ker}\, d^i$.

(ii) $\mathrm{Ker}\, d^i$ *is a direct summand of the topological A-module C^i* (i.e. *the inclusion map* $\mathrm{Ker}\, d^i \to C^i$ *admits a continuous left inverse).*

Proof. The assumption means that

$$d^{i-1} \circ h^i + h^{i+1} \circ d^i = \mathbf{1}_{C^i}.$$

Now, set $e^i := d^{i-1} \circ h^i$; we compute

$$(\mathbf{1}_{C^i} - e^i) \circ e^i = h^{i+1} \circ d^i \circ d^{i-1} \circ h^i = 0$$

and therefore e^i is an idempotent continuous endomorphism of C^i. According to example 3.7.36, it follows that e^i and $\mathbf{1}_{C^i} - e^i$ induce a natural isomorphism of topological A-modules

$$\mathrm{Ker}\,(e^i) \oplus \mathrm{Ker}\,(\mathbf{1}_{C^i} - e^i) \xrightarrow{\sim} C^i. \qquad (8.6.16)$$

Notice as well that $d^i \circ e^i = 0$, whence $d^i = d^i \circ (\mathbf{1}_{C^i} - e^i)$, and we deduce easily that

$$\mathrm{Ker}\,(\mathbf{1}_{C^i} - e^i) = \mathrm{Ker}\, d^i$$

whence (ii). By the same token, we see that e^i restricts to the identity map on $\mathrm{Ker}\, d^i$, *i.e.* δ^i admits a continuous right inverse, given by the restriction $\mathrm{Ker}\, d^i \to C^{i-1}$ of h^i. Taking into account lemma 8.2.5, we then get also (i). □

8.6.17. *Extensions of topological rings.* Let A be any topological ring whose topology is linear; we shall consider for any A-algebra C, the category

$$\mathsf{Exal}_A(C)$$

whose objects are all short exact sequences of A-modules

$$\Sigma \quad : \quad 0 \to M \to E \xrightarrow{\psi} C \to 0 \qquad (8.6.18)$$

such that E is an A-algebra (with A-module structure given by the structure map $A \to E$), and ψ is a map of A-algebras. The morphisms in $\mathsf{Exal}_A(C)$ are the commutative ladders of A-modules whose central vertical arrow g is a map of A-algebras :

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \longrightarrow & E' & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle g} & & \| & & \\ 0 & \longrightarrow & M'' & \longrightarrow & E'' & \longrightarrow & C & \longrightarrow & 0. \end{array} \qquad (8.6.19)$$

For any topological A-algebra C whose topology is linear, we shall consider the category

$$\mathsf{Exaltop}_A(C)$$

whose objects are the short exact sequences of A-modules (8.6.18) such that E is a topological A-algebra whose topology is linear (and again, with A-module structure given by the structure map $A \to E$), and ψ is a continuous and open map of topological A-algebras, whose kernel M is a discrete topological space, for the topology induced from E. The morphisms in $\mathsf{Exaltop}_A(C)$ are the commutative ladders (8.6.19) such that g is a continuous map of topological A-algebras.

8.6.20. Now, let A be as in (8.6.17), C any topological A-algebra whose topology is linear, $\varphi : A \to C$ the structure morphism, and $(I_\lambda \mid \lambda \in \Lambda)$ (resp. $(J_{\lambda'} \mid \lambda' \in \Lambda')$) a collection of open ideals of A (resp. of C), which is a fundamental system of open neighborhoods of 0. Let $\Lambda'' \subset \Lambda \times \Lambda'$ be the subset of all (λ, λ') such that $\varphi(I_\lambda) \subset J_{\lambda'}$; the set Λ'' is partially ordered, by declaring that $(\lambda, \lambda') \leq (\mu, \mu')$ if and only if $I_\mu \subset I_\lambda$ and $J_{\mu'} \subset J_{\lambda'}$. Set $A_\lambda := A/I_\lambda$ for every $\lambda \in \Lambda$ (resp. $C_{\lambda'} := C/J_{\lambda'}$ for every $\lambda' \in \Lambda'$); for $(\lambda, \lambda'), (\mu, \mu') \in \Lambda''$ with $(\lambda, \lambda') \leq (\mu, \mu')$, the surjection $\pi_{\mu'\lambda'} : C_{\mu'} \to C_{\lambda'}$ induces a functor

$$\mathsf{Exal}_{A_\lambda}(C_{\lambda'}) \to \mathsf{Exal}_{A_\mu}(C_{\mu'}) \qquad \Sigma \mapsto \Sigma * \pi_{\mu'\lambda'}$$

(see [75, §2.5.5]) and clearly the rule $(\lambda, \lambda') \mapsto \mathsf{Exal}_{A_\lambda}(C_{\lambda'})$ yields a pseudo-functor

$$\mathsf{E} : (\Lambda'', \leq) \to \mathbf{Cat}.$$

Moreover, we have a pseudo-cocone :

$$\mathsf{E} \Rightarrow \mathsf{Exaltop}_A(C) \tag{8.6.21}$$

defined as follows. To any $(\lambda, \lambda') \in \Lambda''$ and any object $\Sigma_{\lambda,\lambda'} : 0 \to M \to E_{\lambda'} \to C_{\lambda'} \to 0$ of $\mathsf{Exal}_{A_\lambda}(C_{\lambda'})$, one assigns the extension (8.6.18) obtained by pulling back $\Sigma_{\lambda,\lambda'}$ along the projection $\pi_{\lambda'} : C \to C_{\lambda'}$. Let $\beta : E \to E_{\lambda'}$ be the induced projection; we endow E with the linear topology defined by the fundamental system of all open ideals of the form $\psi^{-1}J \cap \beta^{-1}J'$ where J (resp. J') ranges over the set of open ideals of C (resp. over the set of all ideals of $E_{\lambda'}$). With this topology, it is easily seen that both ψ and the induced structure map $A \to E$ are continuous ring homomorphisms. Moreover, if $J \subset J_{\lambda'}$, then $\psi^{-1}J \cap \beta^{-1}J' = J \times (J' \cap M)$, from which it follows that ψ is an open map. Furthermore, $M \cap \beta^{-1}0 = 0$, which shows that M is discrete, for the topology induced by the inclusion map $M \to E$. Summing up, we have attached to $\Sigma_{\lambda,\lambda'}$ a well defined object $\Sigma_{\lambda,\lambda'} * \pi_{\lambda'}$ of $\mathsf{Exaltop}_A(C)$, and it is easily seen that the rule $\Sigma_{\lambda,\lambda'} \mapsto \Sigma_{\lambda,\lambda'} * \pi_{\lambda'}$ is functorial in $\Sigma_{\lambda,\lambda'}$, and for $(\lambda, \lambda) \leq (\mu, \mu')$ in Λ'', there is a natural isomorphism in $\mathsf{Exaltop}_A(C)$:

$$\Sigma_{\lambda,\lambda'} * \pi_{\lambda'} \xrightarrow{\sim} (\Sigma_{\lambda,\lambda'} * \pi_{\mu'\lambda'}) * \pi_{\mu'}$$

(details left to the reader).

Lemma 8.6.22. *The pseudo-cocone* (8.6.21) *induces an equivalence of categories :*

$$\beta : 2\text{-}\operatorname*{colim}_{\Lambda''} \mathsf{E} \xrightarrow{\sim} \mathsf{Exaltop}_A(C).$$

Proof. Let Σ as in (8.6.18) be any object of $\mathsf{Exaltop}_A(C)$. By assumption, there exists an open ideal $J \subset E$ such that $M \cap J = 0$. Since ψ is an open map, there exists $\lambda' \in \Lambda'$ such that $J_{\lambda'} \subset \psi(J)$, and after replacing J by $J \cap \psi^{-1}J_{\lambda'}$, we may assume that $\psi(J) = J_{\lambda'}$. Likewise, if $\varphi_E : A \to E$ is the structure morphism, there exists $\lambda \in \Lambda$ such that $I_\lambda \subset \varphi_E^{-1}J$, and it follows that the induced extension

$$\Sigma_{\lambda,\lambda'} : 0 \to M \to E/J \to C_{\lambda'} \to 0$$

is an object of $\mathsf{Exal}_{A_\lambda}(C_{\lambda'})$. We notice :

Claim 8.6.23. There exists a natural isomorphism $\Sigma \xrightarrow{\sim} \Sigma_{\lambda,\lambda'} * \pi_{\lambda'}$ in $\mathsf{Exaltop}_A(C)$.

Proof of the claim. (i): By construction, ψ and the projection $\pi_J : E \to E/J$ define a unique morphism $\gamma : E \to (E/J) \times_{C_{\lambda'}} C$ of A-algebras, restricting to the identity map on M (which is an ideal in both of these A-algebras). It is clear that γ is an isomorphism, and therefore it yields a natural isomorphism $\Sigma \stackrel{\sim}{\to} \Sigma_{\lambda,\lambda'} * \pi_{\lambda'}$ in $\mathsf{Exal}_A(C)$. It remains to check that γ is continuous and open. For the continuity, it suffices to remark that, for every ideal $I \subset E/J$ and every $\mu' \in \Lambda'$, the ideals $\gamma^{-1}(I \times_{C_{\lambda'}} C) = \pi_J^{-1} I$ and $\gamma^{-1}(E/J \times_{C_{\lambda'}} J_{\mu'}) = \psi^{-1} J_{\mu'}$ are open in E, which is obvious, since J is an open ideal and ψ is continuous. Lastly, let $I \subset E$ be any open ideal such that $I \subset J \cap \psi^{-1} J_{\lambda'}$; since ψ is an open map, it is easily seen that $\gamma(I) = 0 \times \psi(I)$ is an open ideal of $(E/J) \times_{C_{\lambda'}} C$, so γ is open. $\Diamond$

From claim 8.6.23 we see already that β is essentially surjective. It also follows easily that β is full. Indeed, consider any morphism $s : \Sigma' \to \Sigma''$ of $\mathsf{Exaltop}_A(C)$ as in (8.6.19), and pick an open ideal $J'' \subset E''$ with $J \cap M'' = 0$; set $J' := g^{-1} J''$, and notice that $J' \cap M' = 0$. Moreover, if the image of J'' in C equals $J_{\lambda'}$ for some $\lambda' \in \Lambda'$, then clearly the same holds for the image of J' in C. Therefore, in this case the foregoing construction yields objects $\Sigma'_{\lambda,\lambda'}$ and $\Sigma''_{\lambda,\lambda'}$ of $\mathsf{Exal}_{A_\lambda}(C_{\lambda'})$ (for a suitable $\lambda \in \Lambda$), whose middle terms are respectively E'/J' and E''/J'', and s descends to a morphism $s_{\lambda'} : \Sigma'_{\lambda,\lambda'} \to \Sigma''_{\lambda,\lambda'}$, whose middle term is the map $g_{\lambda'} : E'/J' \to E''/J''$ induced by g. By inspecting the proof of claim 8.6.23, we deduce a commutative diagram

$$\begin{array}{ccc} E' & \xrightarrow{\sim} & (E'/J') \times_{C_{\lambda'}} C \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g_{\lambda'} \times_{C_{\lambda'}} C} \\ E'' & \xrightarrow{\sim} & (E''/J'') \times_{C_{\lambda'}} C \end{array}$$

whose horizontal arrows are the maps that define the isomorphisms $\Sigma' \stackrel{\sim}{\to} \Sigma'_{\lambda,\lambda'} * \pi_{\lambda'}$ and $\Sigma'' \stackrel{\sim}{\to} \Sigma''_{\lambda,\lambda'} * \pi_{\lambda'}$ in $\mathsf{Exaltop}_A(C)$. It follows easily that $s_\lambda * \pi_{\lambda'} = s$, whence the assertion. Lastly, the faithfulness of β is immediate, since the projections $\pi_{\lambda'}$ are surjective maps. $\Box$

8.6.24. Let A be as in 8.6.17, and C any A-algebra (resp. any topological A-algebra); we denote by

$$\mathsf{nilExal}_A(C) \qquad (\text{ resp. } \mathsf{nilExaltop}_A(C)\ )$$

the full subcategory of $\mathsf{Exal}_A(C)$ (resp. of $\mathsf{Exaltop}_A(C)$) whose objects are the *nilpotent extensions* of C, *i.e.* those extensions (8.6.18), where M is a nilpotent ideal of E. Moreover, in the situation of (8.6.20), clearly E restricts to a pseudo-functor

$$\mathsf{nilE} : (\Lambda'', \leq) \to \mathbf{Cat} \qquad (\lambda, \lambda') \mapsto \mathsf{nilExal}_{A_\lambda}(C_{\lambda'}).$$

Also, (8.6.21) restricts to a pseudo-cocone on nilE, and lemma 8.6.22 immediately implies an equivalence of categories

$$(8.6.25) \qquad 2\text{-}\operatorname*{colim}_{\Lambda''} \mathsf{nilE} \stackrel{\sim}{\to} \mathsf{nilExaltop}_A(C).$$

Proposition 8.6.26. *Let A and C be as in* (8.6.20). *The following holds :*

(i) *Suppose that $J^2_{\lambda'}$ is open in C, for every $\lambda' \in \Lambda'$. Then the forgetful functor*

$$(8.6.27) \qquad \mathsf{nilExaltop}_A(C) \to \mathsf{nilExal}_A(C)$$

is fully faithful.

(ii) *Suppose additionally, that :*
 (a) *C is noetherian, and $I \subset C$ is an ideal such that the topology of C is I-adic.*
 (b) *I^2_λ is open in A, for every $\lambda \in \Lambda$.*
 Then the essential image of (8.6.27) *is the (full) subcategory of all nilpotent extensions* (8.6.18) *such that the C-module M/M^2 is annihilated by a power of I.*

Proof. (i): The functor is obviously faithful, and in light of (8.6.25), we come down to the following situation. We have a commutative ladder of extensions of A-algebras

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & E & \xrightarrow{\psi} & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \pi_{\lambda'}} & & \\ 0 & \longrightarrow & N & \longrightarrow & E' & \longrightarrow & C_{\lambda'} & \longrightarrow & 0 \end{array}$$

for some $\lambda' \in \Lambda'$, whose top (resp. bottom) row is an object of $\mathsf{nilExaltop}_A(C)$ (resp. of $\mathsf{nilExal}_{A_\lambda}(C_{\lambda'})$, for some $\lambda \in \Lambda$), and we need to show that the kernel of g contains an open ideal. To this aim, we remark :

Claim 8.6.28. For any open ideal $J \subset E$, the ideal J^2 is open as well.

Proof of the claim. Recall that M is a discrete A-module for the topology induced by E; then, after replacing J by a smaller open ideal, we may assume that $J \cap M = 0$. We have $I := \psi(J^2) = \psi(J)^2$, so the assumption in (i) say that I is open in C, and therefore $\psi^{-1}I = J^2 \oplus M$ is open in E, so finally $J \cap \psi^{-1}I = J^2$ is open in E, as stated. ◊

Now, set $J := \psi^{-1}J_{\lambda'}$; then J is an open ideal of E, and clearly $g(I) \subset N$. Say that $N^k = 0$; then $g(I^k) = 0$, and I^k is an open ideal of E, by claim 8.6.28, whence the contention.

(ii): It is easily seen that, for every extension (8.6.18) in the essential image of (8.6.27), we must have $I^k(M/M^2) = 0$ for every sufficiently large $k \geq 0$ (details left to the reader). Conversely, consider a nilpotent extension Σ as in (8.6.18), with $I^k(M/M^2) = 0$ and $M^t = 0$ for some $k, t \in \mathbb{N}$. Pick a finite system $\mathbf{f} := (f_1, \dots, f_r)$ of elements of E whose images in B form a system of generators for I^k; notice that the ideal $(\mathbf{f}^n)B$ is open in B, and $(\mathbf{f}^n)M = 0$ for every $n \geq t$. There follows an inverse system of exact sequences (notation of remark 7.8.1(ii))

$$H_1(\mathbf{f}^n, B) \to M \xrightarrow{\beta_n} E/(\mathbf{f}^n)E \to B/(\mathbf{f}^n)B \to 0 \qquad \text{for every } n \geq t$$

with transition maps induced by the morphisms $\boldsymbol{\varphi}_\mathbf{f}$ of (7.8.20). As B is noetherian, lemma 7.8.44 and remark 7.8.45 imply that the system $(H_1(\mathbf{f}^n, B) \mid n \in \mathbb{N})$ is essentially zero. Therefore, the same holds for the inverse system $(\mathrm{Ker}\,\beta_n \mid n > 0)$. However, the transition map $\mathrm{Ker}\,\beta_{n+1} \to \mathrm{Ker}\,\beta_n$ is obviously injective for every $n \geq t$, so we conclude that β_n is injective for some sufficiently large integer n. For such n, we obtain a nilpotent extension

$$0 \to M \to E/(\mathbf{f}^n)E \to B/(\mathbf{f}^n)B \to 0. \tag{8.6.29}$$

Lastly, let $\varphi : A \to B$ be the structure map, and pick $\lambda \in \Lambda$ such that $I_\lambda \subset \varphi^{-1}I^k$; it is easily seen that $I^t_\lambda M = 0$ and $\varphi(I^s_\lambda) \subset (\mathbf{f}^n)B$ for $s \in \mathbb{N}$ large enough. Therefore, some sufficiently large power of I_λ annihilates $E/(\mathbf{f}^n)E$; under our assumption (b), such power of I_λ contains another open ideal I_μ, so (8.6.29) is an object of $\mathsf{nilExal}_{A_\mu}(B/(\mathbf{f}^n)B)$ whose image in $\mathsf{nilExaltop}_A(B)$ agrees with Σ. □

Proposition 8.6.30. *Let A be a topological ring (whose topology is linear), B a noetherian A-algebra, $I \subset B$ an ideal, and suppose that :*

(a) *The structure map $A \to B$ is continuous for the I-adic topology on B.*
(b) *For every open ideal $J \subset A$, the ideal J^2 is also open.*

Then the following conditions are equivalent :

(c) *B (with its I-adic topology) is a formally smooth A-algebra.*
(d) *$\Omega^1_{B/A} \otimes_B B/I$ is a projective B/I-module, and $H_1(\mathbb{L}_{B/A} \otimes_B B/I) = 0$.*

Proof. More generally, let A and C be as in (8.6.20), and M a *discrete* C-module, *i.e.* a C-module annihilated by an open ideal; we denote by $\mathrm{Exaltop}_A(C, M)$ the C-module of square zero topological A-algebra extensions of C by M. Likewise, for every $(\lambda, \lambda') \in \Lambda''$, and every

$C_{\lambda'}$-module M, let $\mathrm{Exal}_{A_\lambda}(C_{\lambda'}, M)$ be the $C_{\lambda'}$-module of isomorphism classes of square zero A_λ-algebra extensions of $C_{\lambda'}$ by M. For $(\lambda, \lambda') \leq (\mu, \mu')$, we get a natural map of $C_{\mu'}$-modules

(8.6.31) $$\mathrm{Exal}_{A_\lambda}(C_{\lambda'}, M) \to \mathrm{Exal}_{A_\mu}(C_{\mu'}, M) \qquad \text{for every } C_{\lambda'}\text{-module } M$$

and (8.6.25) implies a natural isomorphism :

$$\operatorname*{colim}_{(\lambda,\lambda')\in\Lambda''} \mathrm{Exal}_{A_\lambda}(C_{\lambda'}, M) \xrightarrow{\sim} \mathrm{Exaltop}_A(C, M)$$

which is well defined for every discrete C-module M. By inspecting the definitions, and taking into account [63, Ch.0, Prop.19.4.3], it is easily seen that C is a formally smooth A-algebra if and only if $\mathrm{Exaltop}_A(C, M) = 0$ for every such discrete C-module M. We also have a natural C-linear map :

(8.6.32) $$\mathrm{Exaltop}_A(C, M) \to \mathrm{Exal}_A(C, M) \qquad \text{for every discrete } C\text{-module } M$$

and proposition 8.6.26(i) shows that (8.6.32) is an injective map, provided $J^2_{\lambda'}$ is an open ideal of C, for every $\lambda' \in \Lambda'$. Moreover, for $C := B$ (with its I-adic topology), the map (8.6.32) is an isomorphism, by virtue of proposition 8.6.26(ii).

Now, recall the natural isomorphism of C_λ-modules ([103, Ch.III, Th.1.2.3])

$$\mathrm{Exal}_A(B, M) \xrightarrow{\sim} \mathrm{Ext}^1_B(\mathbb{L}_{B/A}, M) \qquad \text{for every } B\text{-module } M.$$

Combining with the foregoing, we deduce a natural B-linear isomorphism

$$\mathrm{Exaltop}_A(B, M) \xrightarrow{\sim} \mathrm{Ext}^1_B(\mathbb{L}_{B/A}, M)$$

for every B-module M annihilated by some ideal I^k. Summing up, B is a formally smooth A-algebra if and only if $\mathrm{Ext}^1_B(\mathbb{L}_{B/A}, M)$ vanishes for every discrete B-module M. Set $B_0 := B/I$; by considering the I-adic filtration on a given M, a standard argument shows that the latter condition holds if and only if it holds for every B_0-module M, and in turns, this is equivalent to the vanishing of $\mathrm{Ext}^1_{B_0}(\mathbb{L}_{B/A} \otimes_B B_0, M)$ for every B_0-module M. This last condition is equivalent to (d), as stated. □

We conclude this section with a list of topological corollaries of the conditions $(\mathrm{a})_{\mathbf{f}} - (\mathrm{f})_{\mathbf{f}}$ that were introduced in (7.8.21).

8.6.33. Let A be a ring, $\mathbf{f} := (f_1, \dots, f_r)$ a finite sequence of elements of A that generate an ideal $I \subset A$. Let also $(C_\bullet, d_\bullet)$ be any bounded above complex of flat A-modules such that, for every $j \in \mathbb{Z}$ the annihilator ideal of $H_j C_\bullet$ contains a power of I. Denote also by $A^\wedge$ the I-adic completion of A, and by $C^\wedge_\bullet$ the complex whose term (resp. differential) in degree j is the I-adic completion of C_j (resp. of d_j), for every $j \in \mathbb{Z}$. Lastly, set $Q_\bullet := \mathrm{Cone}(A[0] \to A^\wedge[0])$, the cone of the morphism given by the completion map, and let $\mathbf{f}^\wedge$ be the image in $A^\wedge$ of the sequence $\mathbf{f}$.

Corollary 8.6.34. *In the situation of* (8.6.33)*, the following holds :*

(i) *If A satisfies condition* $(\mathrm{a})_{\mathbf{f}}$ *of* (7.8.21)*, the natural maps*

$$C_\bullet \to A^\wedge \otimes_A C_\bullet \to C^\wedge_\bullet$$

are quasi-isomorphisms.

(ii) *The following conditions are equivalent :*
   (a) *A satisfies condition* $(\mathrm{a})_{\mathbf{f}}$.
   (b) *$A^\wedge$ satisfies condition* $(\mathrm{a})_{\mathbf{f}^\wedge}$ *and* $Q_\bullet \overset{\mathbf{L}}{\otimes}_A A/I[0]$ *is acyclic.*

Proof. (i): According to [163, Th.3.5.8], for every $i \in \mathbb{Z}$ we have a natural short exact sequence

$$0 \to L := \lim_{n\in\mathbb{N}}{}^1 H_{i+1}(C_\bullet \otimes_A A/I^n) \to H_i(C^\wedge) \xrightarrow{\omega} \lim_{n\in\mathbb{N}} H_i(C_\bullet \otimes_A A/I^n) \to 0.$$

On the other hand, the complex $C_\bullet$ fulfills condition $(\mathrm{f})_{\mathbf{f}}$ of (7.8.21), so the kernel and cokernel of the natural morphism of inverse systems $(H_i(C_\bullet) \mid n \in \mathbb{N}) \to (H_i(C_\bullet \otimes_A A/I^n) \mid n \in \mathbb{N})$ are both essentially zero, hence the induced map

$$H_i(C_\bullet) \to \lim_{n\in\mathbb{N}} H_i(C_\bullet \otimes_A A/I^n)$$

is an isomorphism for every $i \in \mathbb{Z}$, and $L = 0$ ([163, Prop.3.5.7]). Hence, ω is an isomorphism, and it is easily seen that the resulting isomorphism $H_i(C^\wedge) \stackrel{\sim}{\to} H_i(C)$ is the inverse of the map induced by the completion map $C_\bullet \to C_\bullet^\wedge$, hence the latter is a quasi-isomorphism. Next we remark :

Claim 8.6.35. Let B be any ring, $\mathbf{g}$ a finite sequence of elements of B that generates an ideal J, and $K_\bullet$ a bounded above complex of B-modules. The following conditions are equivalent :

(a) $K_\bullet \overset{\mathbf{L}}{\otimes}_B B/J[0]$ is acyclic.
(b) $\mathbf{K}_\bullet(\mathbf{g}, K_\bullet)$ is acyclic.

Proof of the claim. (a)⇒(b): For every $q \in \mathbb{N}$ we consider the spectral sequences

$$E^2_{pq} := H_p(K_\bullet \overset{\mathbf{L}}{\otimes}_B H_q\mathbf{K}_\bullet(\mathbf{g})[0]) \Rightarrow H_{p+q}(\mathbf{g}, K_\bullet)$$
$$F(q)^2_{ij} := \mathrm{Tor}_i^{B/J}(H_j(K_\bullet \overset{\mathbf{L}}{\otimes}_B B/J[0]), H_q\mathbf{K}_\bullet(\mathbf{g})) \Rightarrow E^2_{i+j,q} \quad \text{for every } q \in \mathbb{Z}.$$

Assumption (a) implies that $F(q)^2_{ij} = 0$ for every $i, j, q \in \mathbb{Z}$, so that $E^2_{pq} = 0$ for every $p \in \mathbb{Z}$, whence (b). Conversely, if (b) holds, we show – by induction on n – that $H_n(K_\bullet \overset{\mathbf{L}}{\otimes}_B B/J[0]) = 0$ for every $n \in \mathbb{Z}$. First, since $K_\bullet$ is bounded above, we may find $n_0 \in \mathbb{Z}$ such that the assertion holds for every $n \le n_0$. Next, let $k > n_0$, and suppose that the assertion is already known for every $n < k$. It follows that $F(q)^2_{ij} = 0$ whenever $j < k$, and clearly the same holds also whenever $i < 0$; thus $F(q)^2_{0k} = E^2_{kq}$ for every $q \in \mathbb{Z}$. By the same token, we deduce that $E^2_{pq} = 0$ for every $p < k$, and clearly the same holds whenever $q < 0$. Summing up, it follows that $E^2_{k0} = H_k(\mathbf{g}, K_\bullet) = 0$; however, $F(0)^2_{0k} = H_k(K_\bullet \overset{\mathbf{L}}{\otimes}_B B/J[0])$, whence the contention. ◊

Claim 8.6.36. Suppose that A satisfies condition $(\mathrm{a})_{\mathbf{f}}$ of (7.8.21), let $k \in \mathbb{N}$ be any integer, and N any A/I^k-module. The completion map $A \to A^\wedge$ induces an isomorphism

$$N \stackrel{\sim}{\to} A^\wedge \otimes_A N \qquad \text{and} \qquad \mathrm{Tor}_i^A(A^\wedge, N) = 0 \qquad \text{for every } i > 0.$$

Proof of the claim. Notice that, since $\mathbf{K}_j(\mathbf{f})$ is a free A-module of finite rank for every $j \in \mathbb{Z}$, the natural map $A^\wedge \otimes_A \mathbf{K}_\bullet(\mathbf{f}) \to \mathbf{K}_\bullet(\mathbf{f})^\wedge$ is an isomorphism. But we have already shown that the completion map $\mathbf{K}_\bullet(\mathbf{f}) \to \mathbf{K}_\bullet(\mathbf{f})^\wedge$ is a quasi-isomorphism, hence $\mathbf{K}_\bullet(\mathbf{f}, Q_\bullet)$ is acyclic, and in light of claim 8.6.35 we see that the same holds for $Q_\bullet \overset{\mathbf{L}}{\otimes}_A A/I[0]$. Let M be any A/I-module; from the change of rings spectral sequence

$$\mathrm{Tor}_i^{A/I}(H_j(Q_\bullet \otimes_A A/I[0]), M) \Rightarrow H_{i+j}(Q_\bullet \overset{\mathbf{L}}{\otimes}_A M[0])$$

we deduce that $Q_\bullet \overset{\mathbf{L}}{\otimes}_A M[0]$ is also acyclic. Lastly, by considering, for $t = 0, \dots, k-1$ the distinguished triangles

$$Q_\bullet \overset{\mathbf{L}}{\otimes}_A I^tN/I^{t+1}N[0] \to Q_\bullet \overset{\mathbf{L}}{\otimes}_A N/I^{t+1}N[0] \to Q_\bullet \overset{\mathbf{L}}{\otimes}_A N/I^tN[0] \to Q_\bullet \overset{\mathbf{L}}{\otimes}_A I^tN/I^{t+1}N[1]$$

a simple induction shows that $Q_\bullet \overset{\mathbf{L}}{\otimes}_A N[0]$ is acyclic as well; the latter assertion is equivalent to the claim. ◊

Now, let us consider the spectral sequence

$$E^2_{pq} := \mathrm{Tor}^A_p(A^\wedge, H_q C_\bullet) \Rightarrow H_{p+q}(A^\wedge[0] \overset{\mathbf{L}}{\otimes}_A C_\bullet)$$

and notice that $A^\wedge[0] \overset{\mathbf{L}}{\otimes}_A C_\bullet = A^\wedge \otimes_A C_\bullet$, since $C_\bullet$ is a complex of flat A-modules. By assumption, for every $q \in \mathbb{Z}$ there exists $k \in \mathbb{N}$ such that $I^k \cdot H_q C_\bullet = 0$; taking into account claim 8.6.36 we conclude that $E^2_{pq} = 0$ whenever $p > 0$, and $E^2_{0q} = H_q C_\bullet$ for every $q \in \mathbb{Z}$; it is then easily seen that the resulting isomorphism $H_q C_\bullet \xrightarrow{\sim} H_q(A^\wedge \otimes_A C_\bullet)$ is the map induced by the completion map $A \to A^\wedge$.

(ii.a)⇒(ii.b): We have already remarked in the proof of (i) that if A satisfies condition $(\mathrm{a})_{\mathbf{f}}$, then $Q_\bullet \overset{\mathbf{L}}{\otimes}_A A/I[0]$ is acyclic. Next, define A_r and $\beta_{\mathbf{f}} : A_r \to A$ as in (7.8.21), and consider the spectral sequence

$$E(n)^2_{ij} := H_i(\mathrm{Tor}^{A_r}_j(A_r/I^n_r, A) \overset{\mathbf{L}}{\otimes}_A Q_\bullet) \Rightarrow H_{i+j}(A_r/I^n_r \overset{\mathbf{L}}{\otimes}_{A_r} Q_\bullet) \qquad \text{for every } n \in \mathbb{N}.$$

In light of claim 8.6.36 we see that $E(n)^2_{ij} = 0$ for every $i \in \mathbb{Z}$ and $j, n \in \mathbb{N}$, so $A_r/I^n_r \overset{\mathbf{L}}{\otimes}_{A_r} Q_\bullet$ is acyclic, and therefore the completion map $A \to A^\wedge$ induces quasi-isomorphisms

$$A_r/I^n_r \otimes_{A_r} A \xrightarrow{\sim} A_r/I^n_r \otimes_{A_r} A^\wedge \qquad \text{for every } n \in \mathbb{N} \tag{8.6.37}$$

so $A^\wedge$ satisfies $(\mathrm{a})_{\mathbf{f}}$.

(ii.b)⇒(ii.a): Arguing as in the proof of claim 8.6.36 we see that $N \overset{\mathbf{L}}{\otimes}_A Q_\bullet$ is acyclic for every $k \in \mathbb{N}$ and every A/I^k-module N. Then we find again $E(n)^2_{ij} = 0$ for every $i \in \mathbb{Z}$ and every $j, n \in \mathbb{N}$, as well as the quasi-isomorphisms (8.6.37), and the assertion follows. □

Corollary 8.6.38. *In the situation of* (7.8.35)*, set* $Z := \mathrm{Spec}\, A/I$*, and suppose that* $U := \mathrm{Spec}\, A \setminus Z$ *is an affine scheme. Then* $\overline{A}$ *satisfies condition* $(\mathrm{c})^{\mathrm{un}}_{\overline{\mathbf{f}}}$ *of* (7.8.21).

Proof. Endow A with its I-adic topology, and $A_U := \mathscr{O}_U(U)$ with the f-adic topology $\mathscr{T}_U$ provided by proposition 8.3.30(i). Clearly, the restriction map $\rho_U : A \to A_U$ factors through $\overline{A}$, so the latter is an open subring of A_U. Now, by definition there is no prime ideal of U that contains all the elements $\overline{f}_1, \dots, \overline{f}_r$; since U is affine by assumption, it follows that there exist $g_1, \dots, g_r \in A_U$ such that $\sum_{i=1}^r g_i \cdot \overline{f}_i = 1$ in A_U. Next, since $\overline{A}$ is a ring of definition of A_U, and $\overline{I} := I/J$ an ideal of adic definition, it follows that there exists $k \in \mathbb{N}$ such that $g_i \cdot \overline{I}^k \subset \overline{A}$ for $i = 1, \dots, r$. It follows easily that the scalar multiplication by g_i induces an $\overline{A}$-linear map $\overline{I}^{n+k} \to \overline{I}^n$ for every $n \in \mathbb{N}$, and the latter in turn induces a map of complexes

$$\psi_{i,n} : \mathbf{K}_\bullet(\overline{\mathbf{f}}, \overline{I}^{n+k}) \to \mathbf{K}_\bullet(\overline{\mathbf{f}}, \overline{I}^n) \qquad \text{for every } n \in \mathbb{N} \text{ and } i = 1, \dots, r.$$

On the other hand, scalar multiplication by $\overline{f}_i$ induces a map of complexes

$$\varphi_{i,n} : \mathbf{K}_\bullet(\overline{\mathbf{f}}, \overline{I}^n) \to \mathbf{K}_\bullet(\overline{\mathbf{f}}, \overline{I}^n) \qquad \text{for every } n \in \mathbb{N} \text{ and } i = 1, \dots, r$$

that equals the zero morphism in the homotopy category (see lemma 7.8.2(i)). Summing up, we conclude that $\sum_{i=1}^r \varphi_{i,n} \circ \psi_{i,n}$ is the zero morphism $\mathbf{K}_\bullet(\overline{\mathbf{f}}, \overline{I}^{n+k}) \to \mathbf{K}_\bullet(\overline{\mathbf{f}}, \overline{I}^n)$ in the homotopy category, and on the other hand it coincides with the inclusion map. The assertion follows. □

Remark 8.6.39. (i) Let A be a ring, $J \subset A$ an ideal of finite type; endow A with its J-adic topology, denote by $A^\wedge$ the completion of A, and set $X_A := \mathrm{Spec}\, A$, $X_{A^\wedge} := \mathrm{Spec}\, A^\wedge$. Let also

$U \subset X_A$ be an affine open subset containing the analytic locus of X_A (see definition 8.3.28), and set $U^\wedge := U \times_{X_A} X_{A^\wedge}$. Moreover, let

$$A_U := \mathscr{O}_{X_A}(U) \quad A_{U^\wedge} := \mathscr{O}_{X_{A^\wedge}}(U^\wedge) \quad \tau_A := \mathrm{Ker}(A \to A_U) \quad \tau_{A^\wedge} := \mathrm{Ker}(A^\wedge \to A_{U^\wedge})$$

and denote by $(\tau_A A^\wedge)^c$ the topological closure of $\tau_A A^\wedge$ in $A^\wedge$. Then we have

$$\tau_{A^\wedge} \subset (\tau_A A^\wedge)^c.$$

Indeed, endow A_U with the f-adic topology characterized by proposition 8.3.30(i), and let $A_U^\wedge$ be the completion of A_U; then A/τ_A is a ring of definition of A_U, hence its completion $(A/\tau_A)^\wedge = A^\wedge/(\tau_A A^\wedge)^c$ is a ring of definition of $A_U^\wedge$, and the natural map $A^\wedge/(\tau_A A^\wedge)^c \otimes_A A_U \to A_U^\wedge$ is an isomorphism (propositions 8.2.13(iii) and 8.3.33(i,ii)). On the other hand, $A_{U^\wedge} = A^\wedge \otimes_A A_U$, so we get a commutative diagram of rings :

$$\begin{array}{ccc} A^\wedge & \longrightarrow & A^\wedge/(\tau_A A^\wedge)^c \\ \downarrow & & \downarrow \\ A_{U^\wedge} & \longrightarrow & A_U^\wedge \end{array}$$

whose right vertical arrow is injective, whence the assertion.

(ii) In the situation of (i), say furthermore that $X_A \setminus U = \mathrm{Spec}\, A/I$, for some ideal $I \subset A$ generated by a finite sequence $\mathbf{f} := (f_1, \dots, f_r)$ of elements of A, and suppose that $\tau_A \cdot I^k = 0$ for some $k \in \mathbb{N}$. Then $\tau_A \cap I^n = 0$ for some $n \in \mathbb{N}$. Indeed, by virtue of corollary 8.6.38 and proposition 7.8.36, the ring A satisfies condition $(\mathrm{c})_{\mathbf{f}}^{\mathrm{un}}$ of (7.8.21), and the proof of proposition 7.8.36 shows that the sought identity follows from condition $(\mathrm{c})_{\mathbf{f}}$. In particular, τ_A is a discrete subset of A, and the natural map $\tau_A \to \tau_A A^\wedge$ is an isomorphism; then, τ_A is (naturally identified with) a discrete subset of $A^\wedge$ as well, and (i) implies that the natural map $\tau_A \to \tau_{A^\wedge}$ is an isomorphism as well.

(iii) Consider now the special case where $J = bA$ for some $b \in A$. Then we claim more precisely that $\mathrm{Ann}_{A^\wedge}(b^n)$ is the topological closure of the image of $\mathrm{Ann}_A(b^n)$ in $A^\wedge$, for every $n \in \mathbb{N}$. Indeed, let $x \in \mathrm{Ann}_{A^\wedge}(b^n)$; for every $k \in \mathbb{N}$ we may find $x_k \in A$ and $y_k \in A^\wedge$ such that $x = x_k + b^k y_k$ in $A^\wedge$. Then $b^n x_k \in (b^{n+k} A^\wedge) \cap A = b^{n+k} A$; say that $b^n x_k = b^{n+k} z_k$ with $z_k \in A$. Hence the sequence $(x_k - b^k z_k \mid k \in \mathbb{N})$ lies in $\mathrm{Ann}_A(b^n)$ and converges b-adically to x, whence the contention.

Corollary 8.6.40. *In the situation of* (8.6.33)*, let B be a ring, $\varphi : B \to A$ a ring homomorphism, $K \subset B$ and $J \subset I$ two ideals. Set $M := K \otimes_B A$, and denote by*

$$\psi : M \to A$$

the A-linear map induced by φ. Suppose moreover that A satisfies condition $(\mathrm{a})_{\mathbf{f}}$ *of* (7.8.21)*, and the image of ψ is open in the I-adic topology of A. Then we have :*

(i) *There exists $n \in \mathbb{N}$ such that $I^n M \cap \mathrm{Ker}\, \psi = 0$.*

(ii) *A is J-adically complete and separated if and only if the same holds for M.*

Proof. (i): Notice first that $\mathrm{Ker}\, \psi = \mathrm{Tor}_1^B(B/K, A)$. Especially, $\mathrm{Ker}\, \psi$ is naturally a B/K-module. By assumption, there exists $k \in \mathbb{N}$ such that $I^k \subset \mathrm{Im}\, \psi = \varphi(K) \cdot A$; we deduce that $\mathrm{Ker}\, \psi$ is also an A/I^k-module. Next, we consider the short exact sequences

$$0 \to \mathrm{Ker}\, \psi \to M \to \mathrm{Im}\, \psi \to 0$$
$$0 \to \mathrm{Im}\, \psi \to A \to A/\mathrm{Im}\, \psi \to 0.$$

There follows exact sequences of inverse systems

$$T_\bullet := (\mathrm{Tor}_1^{A_r}(A_r/I_r^n, \mathrm{Im}\, \psi) \mid n \in \mathbb{N}) \to K_\bullet := (A/I^n \otimes_A \mathrm{Ker}\, \psi \mid n \in \mathbb{N}) \to (M/I^n M \mid n \in \mathbb{N})$$
$$T'_\bullet := (\mathrm{Tor}_2^{A_r}(A_r/I_r^n, A/\mathrm{Im}\, \psi) \mid n \in \mathbb{N}) \to T_\bullet \to T''_\bullet := (\mathrm{Tor}_1^{A_r}(A_r/I_r^n, A) \mid n \in \mathbb{N}).$$

Now, since A satisfies condition $(\mathrm{a})_{\mathbf{f}}$, the inverse system $T''_\bullet$ is essentially zero, and the same holds for $T'_\bullet$, by virtue of lemma 7.8.24; hence, $T_\bullet$ is essentially zero, by lemma 7.8.19. Moreover, $K_n = \mathrm{Ker}\,\psi$ for every $n \geq k$, and the transition maps $K_{n+1} \to K_n$ are the identities. Then the assertion follows by a simple diagram chase : we leave the details to the reader.

(ii): Since $\mathrm{Im}\,\psi$ is open in A for the I-adic topology, it is also open in A for the J-adic topology, and the latter agrees with the topology induced by the J-adic topology of A; therefore, A is J-adically complete and separated if and only if the same holds for $\mathrm{Im}\,\psi$. On the other hand, the J-adic topology on $\mathrm{Im}\,\psi$ also agrees with the quotient topology induced by the J-adic topology of M. Moreover, (ii) implies that the J-adic topology of M induces the discrete topology on $\mathrm{Ker}\,\psi$. Then, proposition 8.2.13(i,v) yields a short exact sequence of J-adic completions :

$$0 \to \mathrm{Ker}\,\psi \to M^\wedge \to (\mathrm{Im}\,\psi)^\wedge \to 0$$

which shows that $\mathrm{Im}\,\psi$ is J-adically complete and separated if and only if the same holds for M, whence (ii). □

9. Complements of commutative algebra

This chapter is a miscellanea of results of commutative algebra that shall be needed in the rest of the treatise.

9.1. **Valuation theory.**

This section is a selection of topics in valuation theory, and will be complemented by the section 9.2, devoted to Huber's theory of the valuation spectrum.

Definition 9.1.1. Recall that an *ordered abelian group* is a datum $(\Gamma, \cdot, 1, \leq)$ consisting of an abelian group $(\Gamma, \cdot, 1)$ and a total ordering $\leq$ on Γ (see example 1.1.6(iii)) such that

$$\gamma \leq \gamma' \quad \Rightarrow \quad \gamma \cdot \gamma'' \leq \gamma' \cdot \gamma'' \qquad \text{for every } \gamma, \gamma', \gamma'' \in \Gamma.$$

(i) We denote by $\Gamma^+ \subset \Gamma$ the submonoid of all $\gamma \in \Gamma$ such that $\gamma \leq 1$.

(ii) A subgroup Δ of the ordered abelian group Γ is said to be *convex* if it satisfies the following condition. If $\delta \in \Delta^+ := \Delta \cap \Gamma^+$, and $\gamma \in \Gamma$ is any element with $1 \geq \gamma \geq \delta$, then $\gamma \in \Delta$. The *spectrum* of Γ is the set of all convex subgroups of Γ, denoted

$$\mathrm{Spec}\,\Gamma.$$

(iii) The *convex rank* of Γ, denoted

$$c.\mathrm{rk}(\Gamma) \in \mathbb{N} \cup \{\infty\}$$

is the supremum over the lengths r of the chains $0 \subset \Delta_1 \subsetneq \cdots \subsetneq \Delta_r := \Gamma$ of convex subgroups of Γ. The *rational rank* of Γ is

$$\mathrm{rk}(\Gamma) := \dim_{\mathbb{Q}}(\Gamma \otimes_{\mathbb{Z}} \mathbb{Q}) \in \mathbb{N} \cup \{\infty\}.$$

(iv) A morphism of ordered abelian groups is a group homomorphism that is an order-preserving map for the underlying totally ordered sets.

Remark 9.1.2. Let $(\Gamma, \cdot, 1, \leq)$ be any ordered abelian group.

(i) It is easily seen that Γ is a torsion-free abelian group. Moreover, we have quite generally

$$c.\mathrm{rk}(\Gamma) \leq \mathrm{rk}(\Gamma)$$

(details left to the reader).

(ii) Notice as well that there exists a unique structure of ordered abelian group on $\Gamma \otimes_{\mathbb{Z}} \mathbb{Q}$ such that the natural map $\Gamma \to \Gamma \otimes_{\mathbb{Z}} \mathbb{Q}$ is a morphism of ordered groups. Namely, we have $\gamma \otimes q \leq 1$ in $\Gamma \otimes_{\mathbb{Z}} \mathbb{Q}$ whenever $\gamma \leq 1$ in Γ and $q \geq 0$. Moreover, for every $q \in \mathbb{Q}$ we have a well defined group endomorphism

$$\Gamma \otimes_{\mathbb{Z}} \mathbb{Q} \to \Gamma \otimes_{\mathbb{Z}} \mathbb{Q} \qquad (\gamma \otimes r) \mapsto (\gamma \otimes r)^q := \gamma \otimes qr.$$

(iii) A subgroup Δ of Γ is convex if and only if there exists an ordered abelian group structure on Γ/Δ such that the projection $\Gamma \to \Gamma/\Delta$ is a morphism of ordered abelian groups. Then the ordered abelian group structure with this property is unique. Moreover, notice that the intersection of an arbitrary family of convex subgroups of Γ is still convex. Especially, for every subgroup Δ of Γ, there exists a minimal convex subgroup of Γ containing Δ, called the *convex hull* of Δ; explicitly, it is the subgroup of all elements $\gamma \in \Gamma$ such that there exists $\delta \in \Delta^+$ with $\delta \leq \gamma \leq \delta^{-1}$.

(iv) There is a natural inclusion-reversing bijection

$$\mathrm{Spec}\,\Gamma \xrightarrow{\sim} \mathrm{Spec}\,\Gamma^+ \qquad \Delta \mapsto \Gamma^+ \setminus \Delta^+$$

(notation of definitions 6.1.10(i) and 9.1.1(ii)). Indeed, it is easily seen that $\Gamma^+ \setminus \Delta^+$ is a prime ideal of Γ^+, for every convex subgroup Δ of Γ. Conversely, for any prime ideal $\mathfrak{p} \subset \Gamma^+$, the subset $\Delta(\mathfrak{p})^+ := \Gamma^+ \setminus \mathfrak{p}$ is a submonoid, and we claim that the subgroup $\Delta(\mathfrak{p})$ generated by $\Delta(\mathfrak{p})^+$ is convex. Indeed, suppose that $\gamma \in \Delta(\mathfrak{p})^+$ and $\gamma' \in \Gamma^+$ is any other element such that $\gamma \leq \gamma'$; then $\beta := \gamma \cdot \gamma'^{-1} \in \Gamma^+$, therefore γ' cannot lie in $\mathfrak{p}$, for otherwise the same would hold for $\gamma = \beta \cdot \gamma'$. The contention follows.

(v) Let $f : \Gamma \to \Gamma'$ be a morphism of ordered abelian groups. Then f induces a mapping

$$\mathrm{Spec}\,\Gamma' \to \mathrm{Spec}\,\Gamma \qquad \Delta \mapsto f^{-1}\Delta.$$

For instance, if $\Delta \subset \Gamma$ is any convex subgroup, and $\pi : \Gamma \to \Gamma/\Delta$ the projection, then $\mathrm{Spec}\,\pi : \mathrm{Spec}\,\Gamma/\Delta \to \mathrm{Spec}\,\Gamma$ is an injective map, whose image is the subset of all convex subgroups of Γ containing Δ. In the same vein, notice that the inclusion map $i : \Gamma \to \Gamma \otimes_{\mathbb{Z}} \mathbb{Q}$ induces a bijection

$$\mathrm{Spec}\,i : \mathrm{Spec}\,\Gamma \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} \mathrm{Spec}\,\Gamma$$

(details left to the reader). Moreover, if f as above is injective, then $\mathrm{Spec}\,f$ is surjective : indeed, if $\Delta \subset \Gamma$ is any convex subgroup, let Δ' be the convex hull of $f(\Delta)$ in Γ'; then it is easily seen that $f^{-1}\Delta' = \Delta$.

(vi) If $c.\mathrm{rk}(\Gamma) = 1$, we may find an injective morphism of ordered abelian groups

$$\rho : (\Gamma, \cdot, 1, \leq) \to (\mathbb{R}, +, 0, \leq).$$

Indeed, pick an element $\gamma \in \Gamma$ with $\gamma > 1$. For every $\delta \in \Gamma$ and every positive integer n, there exists a largest integer $k(n)$ such that $\gamma^{k(n)} < \delta^n$. Then $(k(n)/n \mid n \in \mathbb{N})$ is a Cauchy sequence and we let $\rho(\gamma) := \lim_{n\to\infty} k(n)/n$. One verifies easily that ρ is an order-preserving group homomorphism, and since the convex rank of Γ equals one, it follows that ρ is injective.

(vii) To any $\gamma \in \Gamma^+$ we may attach two convex subgroups

$$o(\gamma) := \bigcap_{n\in\mathbb{N}} \{\delta \in \Gamma \mid \gamma \leq \delta^n \leq \gamma^{-1}\} \qquad \text{and} \qquad O(\gamma) := \bigcup_{n\in\mathbb{N}} \{\delta \in \Gamma \mid \gamma^n \leq \delta \leq \gamma^{-n}\}.$$

Clearly $o(\gamma) \subset O(\gamma)$, and if $\gamma \neq 1$, the quotient $Q(\gamma) := O(\gamma)/o(\gamma)$ has convex rank one (for its natural ordered group structure as in (iii)). Indeed, suppose that δ, μ are two elements of $O(\gamma)$ with classes $\overline{\delta}, \overline{\mu} \in Q(\gamma)_+$ and $\overline{\delta} \neq 1$; the assumptions mean that there exist $n, k \in \mathbb{N}$ such that $\delta^k < \gamma$ and $\gamma^n \leq \mu \leq \gamma^{-n}$. Then $\delta^{kn} \leq \mu$; thus, the only convex subgroups of $Q(\gamma)$ are $Q(\gamma)$ and $\{1\}$, as claimed.

(viii) Standard examples of ordered abelian groups are $(\mathbb{Q}, +, 0, \leq)$ and $(\mathbb{R}, +, 0, \leq)$; the latter is also isomorphic to the ordered group $(\mathbb{R}_{>0}, \cdot, 1, \leq)$ of strictly positive real numbers. We shall also use the notation : $\mathbb{Q}_{>0} := \mathbb{Q} \cap \mathbb{R}_{>0}$, $\mathbb{R}_+ := \mathbb{R}_{>0} \cup \{0\}$ and $\mathbb{Q}_+ := \mathbb{Q}_{>0} \cup \{0\}$.

Definition 9.1.3. Let A be any ring, $(\Gamma, \cdot, 1, \leq)$ any ordered abelian group; we extend the ordering and the composition law of Γ to $\Gamma_\circ := \Gamma \cup \{0\}$ (cp. remark 4.8.14(i)), by the rule

$$0 < \gamma \qquad \text{and} \qquad 0 \cdot \gamma = 0 = \gamma \cdot 0 \qquad \text{for every } \gamma \in \Gamma.$$

Moreover, for every $q \in \mathbb{Q}_+$ we extend to $(\Gamma \otimes_{\mathbb{Z}} \mathbb{Q})_\circ$ the q-th power operation of remark 9.1.2(ii), by declaring that $0^0 := 1$ and $0^q := 0$ for every strictly positive $q \in \mathbb{Q}$.

(i) A Γ-*valued semi-norm* on A is a mapping

$$v : A \to \Gamma_\circ$$

such that :

- $v(0) = 0$ and $v(1) \leq 1$.
- $v(a - b) \leq \max(v(a), v(b))$ for every $a, b \in A$.
- $v(ab) \leq v(a) \cdot v(b)$ for every $a, b \in A$.

The *value group* of v is the subgroup

$$\Gamma_v$$

of Γ generated by $\mathrm{Im}\,(v) \setminus \{0\}$. The *rank* of v is the convex rank of its value group.

(ii) We say that the semi-norm v is *power-multiplicative* if we have :

$$v(a^n) = v(a)^n \qquad \text{for every } a \in A \text{ and every integer } n > 0.$$

(iii) We say that the semi-norm v is a *valuation* if we have :

$$v(1) = 1 \qquad \text{and} \qquad v(ab) = v(a) \cdot v(b) \qquad \text{for every } a, b \in A.$$

(iv) We say that v is a *norm* if $v(a) \neq 0$ for every $a \neq 0$ in A.

(v) We say that the semi-norm v is *real-valued* if its value group is a subgroup of $\mathbb{R}_{>0}$.

(vi) Let Γ and Γ' be two ordered abelian groups, v (resp. v') a Γ-valued (resp. Γ'-valued) semi-norm on A. We say that v is *equivalent* to v' if there exist an ordered abelian group Γ'' and a semi-norm $v'' : A \to \Gamma''$, together with injective maps of ordered abelian groups $\varphi : \Gamma'' \to \Gamma$ and $\varphi' : \Gamma'' \to \Gamma'$ such that $\varphi_\circ \circ v'' = v$ and $\varphi'_\circ \circ v'' = v'$ (notation of remark 4.8.14(i)).

Remark 9.1.4. Let A and Γ be as in definition 9.1.3, and $|\cdot|_A$ any Γ-valued semi-norm on A.

(i) Notice that $|1|_A = |1 \cdot 1|_A \leq |1|_A^2 \leq |1|_A$, whence $|1|_A = |1|_A^2$, so that either $|1|_A = 1$ or $|1|_A = 0$. In case $|1|_A = 0$, then $|a|_A = |a \cdot 1|_A \leq |a|_A \cdot 0 = 0$, and in this case $|\cdot|_A$ is the trivial semi-norm with constant value 0.

(ii) We have $|-a|_A = |a|_A$ for every $a \in A$. Indeed, $|-a|_A = |0 - a|_A \leq \max(|0|_A, |a|_A) = |a|_A$; after replacing a by $-a$ we get also $|a|_A \leq |-a|_A$, whence the claim.

(iii) Let $a, b \in A$ be any two elements such that $|a|_A > |b|_A$. Then we have $|a + b|_A = |a|_A$. Indeed, suppose that $|a + b|_A < |a|_A$; taking into account (ii) we get

$$|a|_A \leq \max(|a + b|_A, |-b|_A) < |a|_A$$

which is absurd.

(iv) Let $S \subset A$ be any multiplicative subset such that $|s|_A \neq 0$ for every $s \in S$. If $|\cdot|_A$ is a valuation, there exists a unique valuation

$$|\cdot|_{S^{-1}A} : S^{-1}A \to \Gamma_\circ$$

whose composition with the localization map $A \to S^{-1}A$ agrees with $|\cdot|_A$.

(v) Notice that the subsets

$$\mathrm{Ker}\,(|\cdot|_A) := \{a \in A \mid |a|_A = 0\} \subset (A, |\cdot|_A)^+ := \{a \in A \mid |a|_A \leq 1\}$$

are respectively an ideal and a subring of A. Clearly $|\cdot|_A$ is the composition of the projection

$$A \to A/\mathrm{Ker}\,(|\cdot|_A)$$

and a unique Γ-valued norm on $A/\mathrm{Ker}\,(|\cdot|_A)$. Moreover, if $|\cdot|_A$ is a valuation, then $\mathrm{Ker}\,(|\cdot|_A)$ is a prime ideal of A called the *support* of $|\cdot|_A$, and we define the *residue field* of $|\cdot|_A$ as

$$\kappa(|\cdot|_A) := \mathrm{Frac}\, A/\mathrm{Ker}\,(|\cdot|_A).$$

In light of (iv), in this case $|\cdot|_A$ factors uniquely through the projection $A \to \kappa(|\cdot|_A)$ and a valuation $|\cdot|_\kappa$ on $\kappa(|\cdot|_A)$, called the *residual valuation of* $|\cdot|_A$. Conversely, if $\mathfrak{p}$ is any prime ideal of A, and $|\cdot|_{\kappa(\mathfrak{p})}$ is any valuation on the residue field $\kappa(\mathfrak{p})$ of $\mathfrak{p}$, then the composition of $|\cdot|_{\kappa(\mathfrak{p})}$ with the projection $A \to \kappa(\mathfrak{p})$ is a valuation on A.

(vi) With the notation of (v), we claim that if $|\cdot|_A$ is a power-multiplicative semi-norm, then the subring $(A, |\cdot|_A)^+$ is integrally closed in A. Indeed, let $a \in A$ be any element that satisfies an identity of the type

$$a^n + b_1 a^{n-1} + \cdots + b_n = 0 \qquad \text{with } b_1, \dots, b_n \in (A, |\cdot|_A)^+.$$

It follows that $|a|_A^n = |a^n|_A \leq \max(|b_i|_A \cdot |a|_A^{n-i} \mid i = 1, \dots, n)$. Hence, say that $|a|_A^n \leq |b_i| \cdot |a|_A^{n-i}$ for some $i \leq n$; if $|a| = 0$, obviously $a \in (A, |\cdot|_A)^+$. Otherwise, we deduce $|a|_A^i \leq |b_i|_A \leq 1$, whence $|a|_A \leq 1$, and the claim follows.

(vii) Let $\mathfrak{p} \subset A$ be any prime ideal. Then there exists a unique (up to equivalence) rank zero valuation on A with support $\mathfrak{p}$, defined by the rule :

$$|x| := \begin{cases} 0 & \text{if } x \in \mathfrak{p} \\ 1 & \text{otherwise.} \end{cases}$$

Any valuation of this type is called a *trivial valuation* on A.

Example 9.1.5. (i) Let A be any ring, P a monoid, $(\Gamma, \cdot, 1, \leq)$ an ordered abelian group, and $\varphi : P \to \Gamma_\circ$ a morphism of monoids. We deduce a mapping (notation of (4.8.50))

$$v_\varphi : A[P] \to \Gamma_\circ \qquad \sum_{x \in P} a_x \cdot x \mapsto \max(\varphi(x) \mid x \in P,\ a_x \neq 0)$$

(where the maximum of the empty set is taken to be $0 \in \Gamma_\circ$, so that $w_\varphi(0) = 0$). It is easily seen that v_φ is a Γ-valued seminorm on $A[P]$. Also, v_φ is a norm if and only if $\varphi(P) \subset \Gamma$.

(ii) Suppose moreover that A is a reduced ring, P is integral and P^{gp} is a torsion-free abelian group; then v_φ is valuation. Indeed, consider the first the case where A is a domain; then the same holds for $A[P]$, since the latter is a subring of $A[P^{\mathrm{gp}}]$, which in turn is the filtered union of its subrings $A[G]$, with G ranging over all free abelian subrings of finite rank contained in P^{gp}. Now, let

$$a := \sum_{x \in P} a_x \cdot x \qquad b := \sum_{x \in P} b_x \cdot x$$

be any two non-zero elements of $A[P]$, and set $\alpha := v_\varphi(a)$ and $\beta := v_\varphi$; let also $a' := \sum_{\varphi(x) = \alpha} a_x \cdot x$, and $b' := \sum_{\varphi(x) = \beta} b_x \cdot x$. It is easily seen that $v_\varphi(ab) = v_\varphi(a'b')$; on the other hand we have $a'b' \neq 0$ since $A[P]$ is a domain, so we must have $v_\varphi(a'b') = \alpha\beta$, whence the assertion. If A is any reduced ring, we have a natural inclusion $A \to \prod_{\mathfrak{p}} A_{\mathfrak{p}}$, where $\mathfrak{p}$ ranges over all minimal prime ideals of A, and each factor $A_{\mathfrak{p}}$ is a field. Then we get likewise an inclusion $A[P] \to \prod_{\mathfrak{p}} A_{\mathfrak{p}}[P]$, and the assertion is easily reduced to the foregoing case : details left to the reader.

9.1.6. Real-valued semi-norms, of course, occur frequently in algebraic and analytic questions. The datum $(A, |\cdot|_A)$ of a ring and a real valued semi-norm $|\cdot|_A$ is usually called a *(real valued) semi-normed ring*. If $|\cdot|_A$ is power-multiplicative, one also says that $(A, |\cdot|_A)$ is a *uniform semi-normed ring*. A *morphism of semi-normed rings* $f : (A, |\cdot|_A) \to (B, |\cdot|_B)$ is a ring homomorphism $f : A \to B$ for which there exists a real number $C \geq 0$ such that

(9.1.7) $$|f(a)|_B \leq C \cdot |a|_A \qquad \text{for every } a \in A.$$

Clearly a composition of morphisms of semi-normed ring is again a morphism of semi-normed rings, hence the semi-normed rings and their morphisms form a category $\mathscr{S}$, and we let $\mathsf{u}.\mathscr{S} \subset$

$\mathscr{S}$ be the full subcategory whose objects are the uniform semi-normed rings. Notice that if f verifies (9.1.7) and $(B, |\cdot|_B)$ is a uniform semi-normed ring, then we have

$$|f(a)|_B^n = |f(a^n)|_B \leq C \cdot |a^n| \leq C \cdot |a|^n \qquad \text{for every } n \in \mathbb{N} \text{ and } a \in A$$

whence $|f(a)|_B \leq C^{1/n} \cdot |a|_A$ for every such n and a; especially, (9.1.7) holds with $C \leq 1$.

Lemma 9.1.8. *Let $(A, |\cdot|)$ be any (real valued) semi-normed ring. Then the following holds :*

(i) *The sequence $(|a^n|^{1/n} \mid n \in \mathbb{N})$ converges in $\mathbb{R}$ for every $a \in A$. We set*

$$|a|^* := \lim_{n \to +\infty} |a^n|^{1/n}.$$

(ii) *The mapping $|\cdot|^* : A \to \mathbb{R}$ is a power-multiplicative semi-norm on A.*

(iii) *$|a|^* \leq |a|$ for every $a \in A$.*

(iv) *The rule : $(A, |\cdot|) \mapsto (A, |\cdot|^*)$ defines a left adjoint for the inclusion functor $\mathsf{u}.\mathscr{S} \to \mathscr{S}$.*

Proof. (i): Fix $a \in A$, and set $\nu_n := |a^n|^{1/n}$ for every integer $n > 0$. Suppose first that $\nu_k = 0$ for some $k \in \mathbb{N}$; it follows easily that $\nu_n = 0$ for every $n \geq k$, whence the assertion, in this case. Hence, we may assume that $\nu_n > 0$ for every $n \in \mathbb{Z}$; we notice that

$$\nu_{kn} \leq \nu_n \qquad \text{for every } k, n > 0$$

and therefore

$$\nu_{kn+r} \leq (|a^{kn}| \cdot |a^r|)^{1/(kn+r)} \leq \nu_n^{1-1/(k+1)} \cdot |a^r|^{1/(nk+r)}$$

for every $k, n, r \in \mathbb{N}$ with $0 \leq r < n$ and $k > 0$. Now, for a given integer $n > 0$ and a real number $\varepsilon > 0$, pick $k \in \mathbb{N}$ large enough, so that $|a^r|^{1/(nk)} > 1 - \varepsilon$ for every $r = 0, \dots, n-1$, and $1/(k+1) < \varepsilon$; it follows that

$$\nu_t \leq \nu_n^{1-\varepsilon} \cdot (1 - \varepsilon) \qquad \text{for every integer } t \geq kn.$$

Consequently

$$\limsup_{n \to +\infty} \nu_n = \liminf_{n \to +\infty} \nu_n$$

whence the assertion.

(ii): Fix $a, b \in A$ and a real number $\varepsilon > 0$; by (i), there exists $k_0 \in \mathbb{N}$ such that

$$|a^k|^{1/k} < |a|^* \cdot (1 + \varepsilon) \qquad \text{and} \qquad |b^k|^{1/k} < |b|^* \cdot (1 + \varepsilon) \qquad \text{for every } k > k_0.$$

Pick an integer $t \geq 2k_0$, and large enough, so that

$$|a^i|^{1/n}, |b^i|^{1/n} < 1 + \varepsilon \qquad \text{for every } n \geq t \text{ and every } i \leq k_0.$$

Now, we have :

$$|(a+b)^n|^{1/n} \leq \max\{|a^i|^{1/n} \cdot |b^{n-i}|^{1/n} \mid i = 0, \dots, n\}.$$

On the other hand, we may assume that $|a|^* \geq |b|^*$; set also

$$M := \max((|a|^* \cdot (1 + \varepsilon))^{1-\varepsilon}, |a|^* \cdot (1 + \varepsilon)).$$

Then, for every $n \in \mathbb{N}$ such that $n \geq t$ and $k_0/n \leq \varepsilon$, we get

$$|a^i|^{1/n} \cdot |b^{n-i}|^{1/n} \leq \begin{cases} |a|^* \cdot (1 + \varepsilon) & \text{whenever } i, n - i > k_0 \\ M \cdot (1 + \varepsilon) & \text{otherwise.} \end{cases}$$

We immediately deduce that $|a + b|^* \leq \max(|a|^*, |b|^*)$ for every $a, b \in A$.

The inequality $|ab|^* \leq |a|^* \cdot |b|^*$ for every $a, b \in A$ is an immediate consequence of the corresponding property for $|\cdot|$, and the power-multiplicative condition follows easily from the definition of $|\cdot|^*$.

(iii) follows easily by remarking that $|a^n| \leq |a|^n$ for every $a \in A$.

(iv): Let $(B, |\cdot|_B)$ be a uniform semi-normed ring, and $f : (A, |\cdot|) \to (B, |\cdot|_B)$ a morphism of semi-normed rings, so that (9.1.7) holds for some $C \geq 0$. We get : $|f(a)|_B = |f(a^n)|_B^{1/n} \leq$

$C^{1/n} \cdot |a^n|^{1/n}$ for every $a \in A$ and $n \in \mathbb{N}$. After taking the limit for $n \to +\infty$, it follows that $|f(a)|_B \leq |a|^*$ for every $a \in A$, so that $f : (A, |\cdot|^*) \to (B, |\cdot|_B)$ is a morphism in $\mathsf{u}.\mathscr{S}$. The assertion follows immediately. □

Example 9.1.9. (i) Let A be any ring, $A_0 \subset A$ a subring, $I \subset A_0$ an ideal. We define, for every $n \in \mathbb{N}$, the A_0-submodule of A

$$I^{-n} := \{a \in A \mid aI^n \subset A_0\}$$

and we consider the *order function associated with I*

$$\nu_I : A \to \mathbb{Z} \cup \{\pm\infty\} \qquad a \mapsto \sup\{n \in \mathbb{Z} \mid a \in I^n\}$$

(where the supremum of the empty set is taken to be $-\infty$). It is easily seen that

- $\nu_I(a + b) \geq \min(\nu_I(a), \nu_I(b))$ for every $a, b \in A$
- $\nu_I(ab) \geq \nu_I(a) + \nu_I(b)$ for every $a, b \in A$
- $\nu_I(a) \geq 0$ if and only if $a \in A_0$.

Let also

$$A(I) := \{a \in A \mid \nu_I(a) \neq -\infty\}.$$

It is easily seen that $A(I)$ is a subring of A. Moreover, fix any real number $\rho \in]0, 1[$, and set

$$|a|_I := \rho^{\nu_I(a)} \in \mathbb{R}_{\geq 0}. \qquad \text{for every } a \in A(I)$$

(with the convention that $\rho^{+\infty} := 0$). It is clear that $|\cdot|_I$ is a semi-norm on $A(I)$. The corresponding power-multiplicative semi-norm $|\cdot|_I^*$ provided by lemma 9.1.8 is sometimes called the *asymptotic Samuel function* of I.

(ii) In the situation of (i), let $J \subset A_0$ be another ideal, with $I \subset J$. Then it is easily seen that, for every $a \in A$, we have either $\nu_J(a) \geq \nu_I(a) \geq 0$, or else $0 > \nu_I(a) \geq \nu_J(a)$. Therefore

$$A(J) \subset A(I)$$

and the two foregoing conditions can be unified in the following assertion. For every $a \in A(J)$ we have either $\nu_J(a) = 0$, in which case $\nu_I(a) = 0$ as well, or else

$$0 \leq \nu_I(a)/\nu_J(a) \leq 1.$$

We deduce a corresponding inequality for asymptotic Samuel functions : namely, for every $a \in A(J)$ with $|a|_J^* \neq 0$ we have either $|a|_J^* = 1$, in which case $|a|_I^* = 1$ as well, or else

$$0 \leq \frac{\log |a|_I^*}{\log |a|_J^*} \leq 1.$$

(iii) For instance, for any integer $n > 0$ we have $A(I) = A(I^n)$ and

$$\nu_{I^n}(a^n) \geq \nu_I(a) \geq n \cdot \nu_{I^n}(a) \qquad \text{for every } a \in A$$

from which it follows easily that

$$|a|_{I^n}^* = (|a|_I^*)^{1/n} \qquad \text{for every } a \in A(I).$$

(iv) Let I and I' two ideals of A_0, and suppose that the I-adic topology on A_0 agrees with the I'-adic topology. Combining (ii) and (iii) we deduce that

$$A(I) = A(I')$$

and for every $a \in A(I)$ we have $|a|_I^* = 1$ (resp. $|a|_I^* = 0$) if and only if $|a|_{I'}^* = 1$ (resp. $|a|_{I'}^* = 0$); also, there exists a real number $C \geq 0$ (independent of a) such that

$$0 \leq \frac{\log |a|_I^*}{\log |a|_{I'}^*}, \frac{\log |a|_{I'}^*}{\log |a|_I^*} \leq C \qquad \text{whenever } |a|_I^* \neq 1, 0.$$

(v) We notice as well that the asymptotic Samuel function is independent of the subring A_0. Namely, in the situation of (i), suppose that A'_0 is another subring of A such that I is also an ideal of A'_0. Then we may set

$$I'^n := I^n \qquad \text{for every } n > 0\text{, and} \qquad I'^{-k} := \{a \in A \mid aI^k \subset A'_0\} \qquad \text{for every } k \geq 0$$

(where $I^0 := A_0$). Using the system of powers $(I^n \mid n \in \mathbb{Z})$ we define the function ν_I and the subring $A(I)$ as in (i), and with the system of powers $(I'^n \mid n \in \mathbb{Z})$ we may define likewise the function $\nu_{I'}$ and the subring $A(I')$. After fixing $\rho \in]0, 1[$, we then may define correspondingly the semi-norms $|\cdot|_I$ and $|\cdot|_{I'}$, as well as the associated power multiplicative semi-norms $|\cdot|^*_I$ and $|\cdot|^*_{I'}$. Obviously, for every $a \in A$ we have $\nu_I(a) > 0$ if and only $\nu_{I'}(a) > 0$, and $\nu_I(a) = \nu_{I'}(a)$ for every $a \in A$ such that $\nu_I(a) > 0$. Moreover, suppose that $\nu_I(a) = -n$ for some $n \in \mathbb{N}$; this means especially that $aI^n \subset A_0$, so that $aI^{n+1} \subset I \subset A'_0$, whence $\nu_I(a) \geq -n - 1$. It follows easily that

$$A(I) = A(I') \qquad \text{and} \qquad |a|^*_I = |a|^*_{I'} \qquad \text{for every } a \in A(I).$$

(vi) It follows especially from (v) that the subring $(A(I), |\cdot|^*_I)^+$ of $A(I)$ is independent of A_0 (notation of remark 9.1.4(v)), and moreover, if $B \subset A$ is any subring such that I is also an ideal of B, then $B \subset (A(I), |\cdot|^*_I)^+$. More generally, we have :

Proposition 9.1.10. *In the situation of example* 9.1.9(i)*, consider any subring B of A with*

$$I \subset B \subset (A(I), |\cdot|^*_I)^+.$$

Then the following holds :

(i) *$A(I) \subset A(IB)$ and $|a|^*_I = |a|^*_{IB}$ for every $a \in (A(I), |\cdot|^*_I)^+$.*
(ii) *Suppose moreover that I is a finitely generated ideal of A_0. Then $A(I) = A(IB)$ and $|a|^*_I = |a|^*_{IB}$ for every $a \in A(I)$.*

Proof. (i): Arguing as in example 9.1.9(v), we easily see that $\nu_{IB}(a) \geq \nu_I(a)$ for every $a \in A$ such that $\nu_I(a) > 0$, and $\nu_{IB}(a) \geq \nu_I(a) - 1$ if $\nu(a) \leq 0$. This already implies the stated inclusion of subrings, as well as the inequality

$$|a|^*_{IB} \leq |a|^*_I \qquad \text{for every } a \in A(I). \tag{9.1.11}$$

Now, suppose that $|a|^*_{IB} = \rho^c$ for some real number $c > 0$. Set $n(k) := \nu_{IB}(a^k)$ for every $k \in \mathbb{N}$. It follows that $\lim_{k\to+\infty} n(k)/k = c$, and especially $n(i) > 0$ for every sufficiently large $i \in \mathbb{N}$. It suffices to check that $|a^i|^*_I \leq \rho^{ic}$, so we may replace a by a^i for such sufficiently large i, and assume from start that $\nu_{IB}(a) \geq 0$. In this situation, for every $k \in \mathbb{N}$ we may then write $a^k = \sum_{j=1}^r a_i b_i$, for some $a_1, \dots, a_r \in I^{n(k)}$ and $b_1, \dots, b_r \in B$. Since by assumption $|b_i|^*_I \leq 1$ for every $i \leq r$, taking into account lemma 9.1.8(ii,iii) we deduce that

$$|a^k|^*_I \leq \sup(|x|^*_I \mid x \in I^{n(k)}) \leq \sup(|x|_I \mid x \in I^{n(k)}) \leq \rho^{n(k)}$$

whence $|a|^*_I \leq \rho^{n(k)/k}$ for every $k \in \mathbb{N}$, and therefore $|a|^*_I \leq |a|^*_{IB}$.

(ii): Indeed, we know already that $A(I) \subset A(IB)$, and to show the converse inclusion, pick any finite system of generators $(t_1, \dots, t_r)$ for I, and let $a \in A(IB)$ be any element. This means that $a \cdot (IB)^n \subset B \subset A(I)$ for some $n \in \mathbb{N}$. Hence we may find $m \in \mathbb{N}$ such that

$$a \cdot t_1^{k_1} \cdots t_r^{k_r} \cdot I^m \subset A_0 \qquad \text{for every } (k_1, \dots, k_r) \in \mathbb{N}^r \text{ with } k_1 + \cdots + k_r = n$$

whence $a \cdot I^{m+n} \subset A_0$, so $a \in A(I)$, as stated. Next, to conclude, in light of (9.1.11), it suffices to show that $|a|^*_I \leq |a|^*_{IB}$, for every $a \in A(I)$. However, in case $|a|^*_{IB} = \rho^c$ for some real number $c > 0$, we may argue as in the proof of (i) to deduce the sought inequality. We may therefore assume that $|a|^*_{IB} \leq \rho^{-c}$ for some real number $c > 0$, and we have to show that $|a|^*_I \leq \rho^{-c}$ as well. Define $n(k)$ as in the proof of (i), for every $k \in \mathbb{N}$. Notice that, if $n(k) > 0$

for some $k \in \mathbb{N}$, we have $|a^k|_{IB} \leq \rho$, whence $|a|^*_{IB} \leq \rho^{1/k}$, in which case the assertion is already known. Hence, we may further assume that $n(k) \leq 0$ for every $k \in \mathbb{N}$, in which case

$$\lim_{k\to+\infty} -n(k)/k \leq c \qquad \text{and} \qquad a^k \cdot (IB)^{-n(k)} \subset B \qquad \text{for every } k \in \mathbb{N}.$$

Pick any integers $p, q > 0$ such that $c < p/q$; it follows easily that

$$a^{nq} \cdot (IB)^{np-1} \subset B \qquad \text{for every sufficiently large } n \in \mathbb{N}.$$

Fix such a sufficiently large $n \in \mathbb{N}$; we deduce

$$x(k_\bullet) := a^{nq} \cdot t_1^{k_1} \cdots t_r^{k_r} \in IB \qquad \text{for every } k_\bullet := (k_1, \dots, k_r) \in \mathbb{N}^r \text{ with } k_1 + \cdots + k_r = np$$

in which case $|x(k_\bullet)|^*_{IB} < 1$, and therefore $|x(k_\bullet)|^*_I < 1$ as well, by the foregoing, for every such $k_\bullet$. Then we may find $m \in \mathbb{N}$ such that

$$a^{nmq} \cdot t_1^{mk_1} \cdots t_r^{mk_r} \in A_0 \qquad \text{for every } k_\bullet := (k_1, \dots, k_r) \in \mathbb{N}^r \text{ with } k_1 + \cdots + k_r = np.$$

However, let $J \subset A_0$ be the ideal generated by $t_1^m, \dots, t_r^m$, and notice that

$$I^{(m-1)(r+nps)+1} \subset J^{nps} \qquad \text{for every } s \in \mathbb{N}.$$

We conclude that

$$a^{nmqs} \cdot I^{(m-1)(r+nps)+1} \subset A_0 \qquad \text{for every } s \in \mathbb{N}$$

from which we see that

$$\lim_{s\to+\infty} -\nu_I(a^{nmqs})/(nmqs) \leq \lim_{s\to+\infty} ((m-1)(r+nps)+1)/(nmqs) < p/q$$

and since the rational number p/q can be taken arbitrary close to c, the sought inequality follows. □

Definition 9.1.12. Let F be a field.

(i) We say that a subring $V \subset F$ is a *valuation ring of* F if the following holds. For every $x \in F^\times$ we have either $x \in V$ or $x^{-1} \in V$.

(ii) Let $(A, \mathfrak{m}_A)$, $(B, \mathfrak{m}_B)$ be two local subrings of F. We say that B *dominates* A if $A \subset B$ and $A \cap \mathfrak{m}_B = \mathfrak{m}_A$.

Remark 9.1.13. (i) Let K be any field; notice that any semi-norm, and especially, any valuation on K is necessarily a norm.

(ii) Let $|\cdot|_K$ be any such valuation, and let $K^+ := (K, |\cdot|_K)^+$, as in remark 9.1.4(v). Then it is easily seen that K^+ is a valuation ring of K. Also, the invertible elements of of K^+ are precisely those $x \in K$ such that $|x|_K = 1$, and the set $\{x \in K \mid |x|_K < 1\}$ is the unique maximal ideal of K^+; especially, K^+ is a local domain.

(iii) Furthermore, K^+ is integrally closed. Indeed, let $x \in K$ be integral over K^+, so that $x^n + y_1x^{n-1} + \cdots + y_n = 0$ for some $y_1, \dots, y_n \in K^+$, and suppose, by way of contradiction, that $x \notin K^+$, *i.e.* that $|x|_K > 1$; it follows easily that $|x^n|_K > |y_ix^{n-i}|_K$ for every $i = 1, \dots, n$, so that remark 9.1.4(iii) yields $|x^n + y_1x^{n-1} + \cdots + y_n|_K = |x^n|_K \neq 0$, which is absurd.

(iv) Conversely, if V is any valuation ring of K, set $\Gamma := K^\times/V^\times$. Thus, Γ is an abelian group, and we define an ordering on Γ as follows. For given classes $\overline{x}, \overline{y} \in \Gamma$, we declare that $\overline{x} \leq \overline{y}$ if and only if $y^{-1}x \in V$. Then it is easily seen that $(\Gamma, \leq)$ is an ordered abelian group, and the projection $K^\times \to \Gamma$ is the restriction to $K^\times$ of a well defined valuation $|\cdot|_K$ of K, such that $V = (K, |\cdot|_K)^+$, so the maximal ideal $\mathfrak{m}_V$ of V and the subset $V^\times$ are also described as in (ii) (details left to the reader).

(v) If the valuation $|\cdot|_K$ is a surjective map, we say that $(K, |\cdot|_K)$ is a *valued field*. In view of (iv), we see that a valuation ring of K is the same as the datum of the equivalence class (in the sense of definition 9.1.3(vi)) of a valued field $(K, |\cdot|_K)$.

(vi) Let A be any ring, $|\cdot|$ and $|\cdot|'$ two valuations on A; in light of (v), we see that the following conditions are equivalent :

(a) the valuations $|\cdot|$ and $|\cdot|'$ are equivalent
(b) $|\cdot|$ and $|\cdot|'$ have the same support and $(\kappa(|\cdot|), |\cdot|_\kappa)^+ = (\kappa(|\cdot|'), |\cdot|'_\kappa)^+$
(c) for every $a, b \in A$ we have $|a| \leq |b|$ if and only if $|a|' \leq |b|'$.

(vii) Let $(K, |\cdot|)$ be any valued field, and Γ the value group of $|\cdot|$. There is an inclusion-reversing bijection

$$\mathrm{Spec}\,\Gamma \xrightarrow{\sim} \mathrm{Spec}\,K^+$$

that assigns to every convex subgroup $\Delta \subset \Gamma$ the prime ideal

$$\mathfrak{p}(\Delta) := \{x \in K^+ \mid \gamma > |x| \text{ for every } \gamma \in \Delta\}$$

and whose inverse assigns, to every prime ideal $\mathfrak{p} \subset K^+$, the convex subgroup

$$\Delta(\mathfrak{p}) := \{\gamma \in \Gamma \mid \gamma, \gamma^{-1} > |x| \text{ for every } x \in \mathfrak{p}\}.$$

Indeed, it is easily seen that $\mathfrak{p}(\Delta)$ is a prime ideal, and we check that $\Delta(\mathfrak{p})$ is a convex subgroup. First, say that $\gamma, \gamma' \in \Delta(\mathfrak{p})$; if either $\gamma > 1$ or $\gamma' > 1$, it is clear that $\gamma \cdot \gamma' > |x|$ for every $x \in \mathfrak{p}$. If $\gamma, \gamma' \leq 1$, we may find $y, y' \in K^+$ such that $\gamma = |y|$ and $\gamma' = |y'|$, and by assumption, neither y nor y' lie in $\mathfrak{p}$, so also $y \cdot y' \notin \mathfrak{p}$, and therefore $\gamma \cdot \gamma' > |x|_K$ for every $x \in \mathfrak{p}$. Hence, $\Delta(\mathfrak{p})$ is a subgroup of Γ, and the convexity of $\Delta(\mathfrak{p})$ is obvious. Next, set $\mathfrak{q} := \mathfrak{p}(\Delta)$; directly from the definitions we see that $\Delta \subset \Delta(\mathfrak{q})$. Conversely, let $\gamma \in \Gamma^+ \setminus \Delta$; then $\gamma = |y|$ for some $y \in K^+$, and since $\gamma < \delta$ for every $\delta \in \Delta$, we get $y \in \mathfrak{q}$, so that $\gamma \notin \Delta_\mathfrak{q}$. Hence, $\Delta = \Delta(\mathfrak{q})$, and especially, the rule $\mathfrak{p} \mapsto \Delta(\mathfrak{p})$ is surjective from $\mathrm{Spec}\,K^+$ to $\mathrm{Spec}\,\Gamma$; but it is easily seen that this rule is also injective, whence the contention.

Definition 9.1.14. Let $(K, |\cdot|_K)$ be any valued field, and Γ the value group of $|\cdot|_K$.

(i) For every $\gamma \in \Gamma^+$, set $U_\gamma := \{x \in K^+ \mid |x|_K < \gamma\}$. The *valuation topology* of K^+ is the unique linear topology $\mathscr{T}_{K^+}$ that admits the family $\{U_\gamma \mid \gamma \in \Gamma^+\}$ as a fundamental system of open neighborhoods of 0.
(ii) In view of remark 8.3.2(i) there exists a unique ring topology $\mathscr{T}_K$ on K such that :
 (a) K^+ is an open subring of K
 (b) $\mathscr{T}_K$ induces on K^+ the valuation topology $\mathscr{T}_{K^+}$.
 We call $\mathscr{T}_K$ the *valuation topology* of K.
(iii) We say that $(K, |\cdot|_K)$ is a *Tate valued field*, if the valuation topology of K is f-adic and not discrete.
(iv) Let A be any ring, and v any valuation on A. We say that v is a *Tate valuation* if the residual valued field $(\kappa(v), \overline{v})$ is a Tate valued field.

Remark 9.1.15. The class of Tate valuations was originally introduced by R.Huber in [98], where they are called *microbial valuations*.

Proposition 9.1.16. *With the notation of definition* 9.1.14*, let* $(K^{+\wedge}, \mathscr{T}^\wedge_{K^+})$ *(resp.* $(K^\wedge, \mathscr{T}^\wedge_K)$*) be the completion of the topological ring* $(K^+, \mathscr{T}_{K^+})$ *(resp.* $(K, \mathscr{T}_K)$*). We have :*

(i) *Let* $\mathscr{T}'$ *be any topology on* K^+ *that is linear and separated. Then either* $\mathscr{T}' = \mathscr{T}_{K^+}$*, or else* $\mathscr{T}'$ *is the discrete topology.*
(ii) $(K, |\cdot|_K)$ *is a Tate valued field if and only if* $(K, \mathscr{T}_K)$ *is a Tate topological ring.*
(iii) $K^{+\wedge}$ *is a valuation ring and* $\mathscr{T}^\wedge_{K^+}$ *is the valuation topology on* $K^{+\wedge}$*.*
(iv) $K^\wedge$ *is the field of fractions of* K^+*, and* $\mathscr{T}^\wedge_K$ *is the valuation topology on* $K^\wedge$*.*
(v) *The valuation* $|\cdot|_K : K \to \Gamma_\circ$ *extends uniquely to a valuation* $|\cdot|^\wedge_K : K^\wedge \to \Gamma_\circ$ *whose valuation ring is* $K^{+\wedge}$*.*

Proof. (i): Let $x \in K^+ \setminus \{0\}$ be any element; since $\mathscr{T}'$ is separated, there exists an ideal $I \subset K^+$ that is open for the topology $\mathscr{T}'$, and such that $x \notin I$. Then $I \subset U_\gamma$, with $\gamma := |x|_K$ (notation of definition 9.1.14(i)), so $\mathscr{T}'$ is finer than $\mathscr{T}_{K^+}$. If $\mathscr{T}'$ is not the discrete topology, there exists $y \in I \setminus \{0\}$, and therefore $U_\delta \subset I$, with $\delta := |y|_K$, so in this case $\mathscr{T}' = \mathscr{T}_{K^+}$.

(ii): Suppose that $\mathscr{T}_K$ is f-adic; then, since K^+ is a bounded subset of K, the valuation topology $\mathscr{T}_{K^+}$ must be adic with a finitely generated ideal of adic definition I (proposition 8.3.18(ii)). Such I is necessarily principal, and $I \neq 0$, if $\mathscr{T}_K$ is not discrete; in this case, any generator of I is a topologically nilpotent unit in K. Conversely, if $a \in K$ is a topologically nilpotent unit for the valuation topology of K, then $a \in K^+$, and it is easily seen that the valuation topology of K^+ agrees with the K^+a-adic topology, so $\mathscr{T}_{K^+}$ is adic and $\mathscr{T}_K$ is f-adic.

(iii): Let $a_\bullet := (a_i \mid i \in I)$ be any Cauchy net in K (indexed by a filtered ordered set I); for every $\gamma \in \Gamma$, pick $i(\gamma) \in I$ such that $|a_j - a_k|_K < \gamma$ for every $j, k \geq i(\gamma)$. Suppose first that $|a_{i(\gamma)}|_K \leq \gamma$ for every $\gamma \in \Gamma$; then $|a_j|_K \leq \gamma$ for every $j \geq i(\gamma)$ and every $\gamma \in \Gamma$, *i.e.* $a_\bullet$ converges to 0, and we set $|a_\bullet|_K^\wedge := 0$. Otherwise, there exists $\delta \in \Gamma$ such that $|a_{i(\delta)}|_K > \delta$ and then $|a_j|_K = \delta$ for every $j \geq i(\delta)$, in which case we set $|a_\bullet|_K^\wedge := |a_{i(\delta)}|_K$. It is easily seen that $|a_\bullet|_K^\wedge$ depends only on the class of $a_\bullet$ in $K^\wedge$, and if $b_\bullet$ is any other Cauchy sequence, then

$$|(a_i b_i \mid i \in I)|_K^\wedge = |a_\bullet|_K^\wedge \cdot |b_\bullet|_K^\wedge \qquad \text{and} \qquad |(a_i + b_i \mid i \in I)|_K^\wedge \leq \max(|a_\bullet|_K^\wedge, |b_\bullet|_K^\wedge)$$

(details left to the reader). It follows already that $K^\wedge$ is a domain. Moreover, we have $|a_\bullet|_K^\wedge \leq 1$ if and only if the class of $a_\bullet$ is equivalent to the class of a Cauchy net in K^+. Next, suppose that $1 \geq |a_\bullet|_K^\wedge \geq |b_\bullet|_K^\wedge > 0$, in which case there exists $i_0 \in I$ such that

$$|a_j|_K \geq |b_j|_K \qquad \text{and} \qquad |a_j|_K = \beta := |a_\bullet|_K^\wedge \qquad \text{for every } j \geq i_0$$

and set $v_j := 1$ for every $j \in I$ such that $j < i_0$ and $v_j := a_j^{-1} b_j$ for every $j \geq i_0$. We claim that $v_\bullet := (v_j \mid j \in I)$ is a Cauchy net in K^+ : indeed, for any $\gamma \in \Gamma$, pick $i(\gamma) \in I$ such that $|a_j - a_k|_K, |b_j - b_k|_R < \gamma$ for every $j, k \geq i(\gamma)$; a simple calculation shows that $|v_j - v_k| < \gamma \cdot \beta^2$ whenever $j, k \geq \max(i_0, i(\gamma))$, as required. Now, since $(v_j a_j \mid j \in I)$ and $b_\bullet$ represent the same element in $K^{+\wedge}$, it follows that the class of $a_\bullet$ divides the class of $b_\bullet$ in $K^{+\wedge}$ (the quotient is the class of $v_\bullet$). Lastly, (i) easily implies that $\mathscr{T}_{K^+}^\wedge$ is the valuation topology on $K^{+\wedge}$, whence the assertion.

(v): Furthermore, notice that if the class x of $a_\bullet$ in $K^\wedge$ is not 0, we have $x = a_{i(\delta)} \cdot y$, with y an invertible element of $K^{+\wedge}$, so we must have $|x|' = |a_{i(\delta)}|_K$ for every valuation $|\cdot|'$ on $K^\wedge$ that extends $|\cdot|_K$.

(iv): By the same token, we have $x^{-1} = a_{i(\delta)}^{-1} \cdot y^{-1}$, which shows that K is a field, and concludes the proof. □

Proposition 9.1.17. *Let $(K, |\cdot|_K)$ be a valued field such that the valuation ring K^+ is henselian, K^{s} a separable closure of K, and $(K^\wedge, |\cdot|_K^\wedge, \mathscr{T}_K^\wedge)$ the completion of $(K, |\cdot|_K, \mathscr{T}_K)$. Then :*

(i) *The valuation ring $K^{\wedge +}$ of $K^\wedge$ is henselian.*
(ii) *The ring $K^{\wedge\mathrm{s}} := K^\wedge \otimes_K K^{\mathrm{s}}$ is a field, and is a separable closure of $K^\wedge$.*
(iii) *Especially, we have a natural group isomorphism :*

$$\mathrm{Gal}(K^{\mathrm{s}}/K) \xrightarrow{\sim} \mathrm{Gal}(K^{\wedge\mathrm{s}}/K^\wedge) \qquad \sigma \mapsto K^\wedge \otimes_K \sigma.$$

Proof. (i): The assertion follows immediately from the more general :

Claim 9.1.18. Let $(A_\bullet, J_\bullet) := (A_\lambda, J_\lambda) \mid \lambda \in \mathrm{Ob}(\Lambda))$ be a system of henselian pairs indexed by any small category, with transition morphisms $f_\varphi : A_\lambda \to A_\mu$ such that $f_\varphi(J_\lambda) \subset J_\mu$ for every morphism $\varphi : \lambda \to \mu$ of Λ. Let A (resp. J) be the limit of the system $A_\bullet$ (resp. of the system $J_\bullet$); then the pair (A, J) is henselian.

Proof of the claim. Let $\mathscr{A}$ be the category whose objects are all the pairs (B, I), where B is a ring and $I \subset B$ an ideal, and whose morphisms $(B, I) \to (B', I')$ are all the ring homomorphisms $f : B \to B'$ with $f(I) \subset I'$. Denote by $\mathscr{A}^{\mathrm{h}}$ the full subcategory of $\mathscr{A}$ whose objects are the henselian pairs. Then the inclusion functor $\mathscr{A}^{\mathrm{h}} \to \mathscr{A}$ admits a left adjoint, which assigns to every pair (B, I) its henselization $(B^{\mathrm{h}}, I^{\mathrm{h}})$. It is easily seen that $\mathscr{A}$ is complete, so the claim follows directly from corollary 1.3.26. ◊

(ii): Let K^{a} be an algebraic closure of K containing K^{s} and contained in an algebraic closure $K^{\wedge\mathrm{a}}$ of $K^{\wedge}$, and notice that the valuation $|\cdot|_K$ extends uniquely to a valuation $|\cdot|_E$ on every subextension $E \subset K^{\mathrm{a}}$, since K^+ is henselian ([75, Rem.6.1.12(iv)]). Especially, for every such E and every K-algebra homomorphism $\sigma : E \to K^{\mathrm{a}}$ we must have

$$|a|_E = |\sigma(a)|_{K^{\mathrm{a}}} \qquad \text{for every } a \in E.$$

Claim 9.1.19. (Krasner's lemma) With the foregoing notation, let $a \in K^{\mathrm{s}}$, $b \in K^{\mathrm{a}}$, and suppose that for every homomorphism $\sigma : K[a] \to K^{\mathrm{s}}$ of K-algebras with $\sigma(a) \neq a$ we have

$$|b - a|_{K^{\mathrm{s}}} < |\sigma(a) - a|_{K^{\mathrm{s}}}.$$

Then $K[a] \subset K[b]$.

Proof of the claim. It suffices to show that for every $K[b]$-algebra homomorphism $\tau : K[a, b] \to K^{\mathrm{a}}$ we have $\tau(a) = a$. However, suppose that $\tau(a) \neq a$ for such a map τ; we deduce that

$$|b - a|_{K^{\mathrm{a}}} < |\tau(a) - a|_{K^{\mathrm{s}}} \qquad \text{and} \qquad |b - \tau(a)|_{K^{\mathrm{a}}} = |\tau(b - a)|_{K^{\mathrm{a}}} = |b - a|_{K^{\mathrm{a}}}$$

whence :

$$|\tau(a) - a|_{K^{\mathrm{s}}} = |\tau(a) - b + b - a|_{K^{\mathrm{s}}} \leq \max(|\tau(a) - b|_{K^{\mathrm{a}}}, |b - a|_{K^{\mathrm{a}}}) < |\tau(a) - a|_{K^{\mathrm{s}}}$$

a contradiction. ◊

Let now $K^{\wedge} \subset E^{\wedge}$ be a non-trivial finite Galois extension with $E^{\wedge} \subset K^{\wedge\mathrm{a}}$, and pick $a \in E^{\wedge}$ such that $E^{\wedge} = K^{\wedge}[a]$. Let $P(X) \in K^{\wedge}[X]$ be the minimal polynomial of a over $K^{\wedge}$; in light of (i) and [75, Rem.6.1.12(iv)], the valuation $|\cdot|^{\wedge}_K$ of $K^{\wedge}$ extends uniquely to a valuation of $|\cdot|^{\wedge}_E$ of $E^{\wedge}$; with this notation, we set

$$G := \mathrm{Gal}(E^{\wedge}/K^{\wedge}) \qquad \gamma := \min(|\sigma(a) - a|^{\wedge}_E \mid \sigma \in G \setminus \mathbf{1}_{E^{\wedge}}) \in |E^{\wedge}|^{\wedge}_E.$$

Say that $P = X^n + c_1 X^{n-1} + \cdots + c_n$ and pick a polynomial $Q(X) \in K[X]$ of degree n, say $Q = X^n + d_1 X^{n-1} + \cdots + d_n$ such that

$$|c_i - d_i|^{\wedge}_K < \gamma^n / |a|^{n-i}_E \qquad \text{for } i = 1, \dots, n.$$

It follows easily that $|Q(a)|^{\wedge}_E < \gamma^n$. Then, say that $Q(X) = \prod_{i=1}^n (X - b_i)$ for some $b_1, \dots, b_n \in K^{\mathrm{a}}$; we deduce that $|a - b| < \gamma$ for some $b \in \{b_1, \dots, b_n\}$, in which case claim 9.1.19 implies that $E^{\wedge} \subset K^{\wedge}[b]$, and since P and Q have the same degree, we see that Q is irreducible over $K^{\wedge}$ and $E^{\wedge} = K^{\wedge}[b]$. Especially, b is separable over $K^{\wedge}$, so the roots of Q are all distinct, and therefore b is also separable over K. Summing up, we have already shown that $K^{\wedge}K^{\mathrm{s}} \subset K^{\wedge\mathrm{a}}$ is a separable closure of $K^{\wedge}$. It remains to check that the natural map $K^{\wedge} \otimes_K K^{\mathrm{s}} \to K^{\wedge}K^{\mathrm{s}}$ is injective. To this aim, let $K \subset E$ be any finite Galois extension with $E \subset K^{\mathrm{s}}$, and denote by $\mathrm{tr}_{E/K} : E \to K$ the K-linear trace map, so that $\mathrm{tr}_{E/K}(a) = \sum_{\sigma \in \mathrm{Gal}(E/K)} \sigma(a)$. Clearly $\mathrm{tr}_{E/K}$ is continuous for the valuation topologies $\mathscr{T}_E$ and $\mathscr{T}_K$ of E and K; on the other hand, since E is separable over K, the trace map induces a perfect pairing $E \otimes_K E \to K : x \otimes y \mapsto \mathrm{tr}_{E/K}(xy)$ for every $x, y \in E$. Hence, let $x_1, \dots, x_n$ be a basis of the K-vector space E, and $y_1, \dots, y_n \in E$ the unique elements such that $\mathrm{tr}_{E/K}(x_i y_j) = \delta_{ij}$ for every $i, j = 1, \dots, n$. There follow K-linear isomorphisms

$$K^{\oplus n} \xrightarrow{\varphi} E \xrightarrow{\psi} K^{\oplus n} \qquad \text{such that} \qquad \psi \circ \varphi = \mathbf{1}_{K^{\oplus n}}$$

where $\varphi(a_1,\dots,a_n) := \sum_{i=1}^n a_i x_i$ and $\psi(b) := (\mathrm{tr}_{E/K}(by_1),\dots,\mathrm{tr}_{E/K}(by_n))$ for every $b \in E$ and every $(a_1,\dots,a_n) \in K^{\oplus n}$. Clearly both φ and ψ are continuous for the topology $\mathscr{T}_E$ and the product topology on $K^{\oplus n}$, hence they are both homeomorphisms. Let $(E^\wedge, \mathscr{T}_E^\wedge)$ be the completion of $(E, \mathscr{T}_E)$; we deduce a commutative diagram of $K^\wedge$-linear maps

$$\begin{array}{ccc} K^\wedge \otimes_K E & \xrightarrow{K^\wedge \otimes_K \psi} & K^\wedge \otimes_K K^{\oplus n} \\ {\scriptstyle\mu}\downarrow & & \downarrow \\ E^\wedge & \xrightarrow{\psi^\wedge} & (K^\wedge)^{\oplus n} \end{array}$$

where $\psi^\wedge$ is the completion of ψ and μ is the multiplication map; especially both horizontal arrows are bijections, and the same holds for the right vertical arrow. Then μ is bijective as well, whence the contention.

(iii) is an immediate consequence of (ii). □

Remark 9.1.20. The left adjoint property of the topological henselization $A \mapsto A^{\mathrm{h}}$ implies that for every f-adic ring A, the identity map $\mathbf{1}_{A^{\mathrm{h}}}$ is the unique continuous A-algebra endomorphism of A^{h}. However, A^{h} in general will admit also non-continuous A-algebra endomorphisms. For instance, let K be an algebraically closed field of characteristic zero, and set

$$F_0 := K[\mathbb{Q}_+] = K[T^{1/n} \mid n \in \mathbb{N} \setminus \{0\}] \qquad F := \mathrm{Frac}\, F_0.$$

Let also $v : F_0 \to \mathbb{Q}_\circ$ be the valuation of F_0 associated with the inclusion map $\mathbb{Q}_+ \to \mathbb{Q}$ as in example 9.1.5 (where $\mathbb{Q}$ is endowed with its standard ordering). Then $F^+ := \kappa(v)^+$ is a valuation ring of F, and (F, v) is a Tate valued field. Clearly the value group of v is $\mathbb{Q}$ and the residue field is K. Let $\mathfrak{m}_F \subset F^+$ be the maximal ideal, and $(F^{+\mathrm{h}}, \mathfrak{m}_F^{\mathrm{h}})$ the henselization of the pair $(F^+, \mathfrak{m})$; then $F^{+\mathrm{h}}$ is a valuation ring with the same value group $\mathbb{Q}$, and the same residue field K ([75, Lemma 6.2.5]). Moreover, the topological henselization F^{h} of the Tate valued field F is the field of fractions of $F^{+\mathrm{h}}$. It then follows easily from [75, Prop.6.2.12] that F^{h} is algebraically closed; on the other hand, it is easily seen that F is not algebraically closed, so there exists a non-trivial automorphism of F^{h} that fixes F.

Theorem 9.1.21. *Let K be a field, $A \subset K$ a subring, and $\mathfrak{p} \subset A$ a prime ideal of A. Then there exists a valuation ring $(V, \mathfrak{m}_V)$ of K such that*

$$A \subset V \qquad \textit{and} \qquad \mathfrak{p} = \mathfrak{m}_V \cap A.$$

Proof. We may replace A by $A_\mathfrak{p}$, and assume from start that A is a local ring. Denote by $\mathscr{F}$ the set of all subrings B of K containing A and such that $1 \notin \mathfrak{p}B$; we endow $\mathscr{F}$ with the partial ordering given by inclusion of subrings. Then $A \in \mathscr{F}$, and if $\mathscr{G}$ is a non-empty totally ordered subset of $\mathscr{F}$, then $\bigcup_{B \in \mathscr{G}} B$ still lies in $\mathscr{F}$. By Zorn's lemma, it follows that $\mathscr{F}$ admits a maximal element V. First, we claim that V is local. Indeed, by assumption $1 \notin \mathfrak{p}V$, so $\mathfrak{p}V$ is contained in a maximal ideal $\mathfrak{m}$ of V, and $1 \notin \mathfrak{p}V_\mathfrak{m}$, so $V_\mathfrak{m} \in \mathscr{F}$, and therefore $V = V_\mathfrak{m}$, since V is maximal. Let $\mathfrak{m}_V$ be the maximal ideal of V; by construction, $\mathfrak{p} \subset \mathfrak{m}_V \cap A$, and since $\mathfrak{p}$ is maximal in A, we get $\mathfrak{p} = \mathfrak{m}_V \cap A$. It remains only to check that V is a valuation ring of K. Thus, say that $x \in K$ and $x \notin V$; by maximality of V, we have $V[x] \notin \mathscr{F}$, so $1 \in \mathfrak{p}V[x]$, *i.e.* there exists an identity of the form $1 = a_0 + a_1 x + \cdots + a_n x^n$ for some $a_0, \dots, a_n \in \mathfrak{p}V \subset \mathfrak{m}_V$. Since $1 - a_0 \in V^\times$, we deduce a relation of the form

(9.1.22) $$1 = b_1 x + \cdots + b_n x^n \qquad \text{where } b_1, \dots, b_n \in \mathfrak{m}_V.$$

Likewise, if $x^{-1} \notin V$, we find an identity of the form

(9.1.23) $$1 = c_1 x^{-1} + \cdots + c_m x^{-m} \qquad \text{where } c_1, \dots, c_m \in \mathfrak{m}_V$$

and we may assume that $m, n \in \mathbb{N}$ are the minimal integers for which there exist identities (9.1.22) and (9.1.23). Up to swapping the roles of x and x^{-1}, we may also assume that $n \geq m$. In this case we deduce

$$1 = b_1 x + \cdots + b_{n-1}x^{n-1} + b_n \cdot (c_1 x^{n-1} + \cdots + c_m x^{n-m})$$

which is another identity of the type (9.1.22), but with a strictly smaller value of n, a contradiction. We conclude that either $x \in V$ or $x^{-1} \in V$, as required. □

9.1.24. Let K be any field, and denote by $\mathscr{L}_K$ the set of all local subrings of K. We endow $\mathscr{L}_K$ with a partial ordering, by declaring that, for every $A, B \in \mathscr{L}_K$ we have $A \leq B$ if and only if B dominates A (see definition 9.1.12(ii)).

Corollary 9.1.25. *With the notation of* (9.1.24)*, the following holds :*

(i) *For every $A \in \mathscr{L}_K$ there exists a maximal element of $\mathscr{L}_K$ that dominates A.*
(ii) *An element of $\mathscr{L}_K$ is maximal if and only if it is a valuation ring of K.*

Proof. (i): Let $(B_i \mid i \in I)$ be a non-empty totally ordered subset of $\mathscr{L}_K$ such that $A \leq B_i$ for every $i \in I$, and set $B := \bigcup_{i \in I} B_i$. It is easily seen that $B \in \mathscr{L}_K$, and $A \leq B$. Then the assertion follows from Zorn's lemma.

(ii): Let A be any maximal element of $\mathscr{L}_K$; by theorem 9.1.21 there exists a valuation ring V of K that dominates A, so $A = V$. Conversely, if $(V, \mathfrak{m}_V)$ is a valuation ring of K and $(A, \mathfrak{m}_A)$ any element of $\mathscr{L}_K$ that dominates V, then we must have $A = V$: indeed, if the latter fails, there exists $a \in A \setminus V$, so that $a^{-1} \in \mathfrak{m}_V \subset \mathfrak{m}_A$, which is absurd. □

9.1.26. Henceforth, and until the end of this section, we let $(K, |\cdot|)$ be a given valued field, whose valuation ring (resp. maximal ideal, resp. residue field, resp. value group) shall be denoted K^+ (resp. $\mathfrak{m}_K$, resp. κ, resp. Γ). The following result is due to Nagata ([134, Th.3]).

Proposition 9.1.27. *Let A be an essentially finitely presented K^+-algebra (definition* 7.3.52*), and M a finitely generated K^+-flat A-module. Then M is a finitely presented A-module.*

Proof. Let us write $A = S^{-1}B$ for some finitely presented K^+-algebra B, and some multiplicative subset $S \subset B$; then we may find a finitely generated B-module M_B with an isomorphism $\varphi : S^{-1}M_B \xrightarrow{\sim} M$ of A-modules. Let M'_B be the image of M_B in $M_B \otimes_{K^+} K$; since M is K^+-flat, φ induces an isomorphism $S^{-1}M'_B \to M$. It then suffices to show that M'_B is a finitely presented B-module; hence we may replace A by B and M by M'_B, and assume that A is a finitely presented K^+-algebra.

We are further easily reduced to the case where $A = K^+[T_1, \dots, T_n]$ is a free polynomial K^+-algebra. Pick a finite system of generators Σ for M; define increasing filtrations $\mathrm{Fil}_\bullet A$ and $\mathrm{Fil}_\bullet M$ on A and M, by letting $\mathrm{Fil}_k A$ be the K^+-submodule of all polynomials $P(T_1, \dots, T_n) \in A$ of total degree $\leq k$, and setting $\mathrm{Fil}_k M := \mathrm{Fil}_k A \cdot \Sigma \subset M$ for every $k \in \mathbb{N}$. We consider the Rees algebra $\mathsf{R}(\underline{A})_\bullet$ and the Rees module $\mathsf{R}(\underline{M})_\bullet$ associated with these filtrations as in definition 7.9.1(iii,iv). Say that Σ is a subset of cardinality N; we obtain an A-linear surjection $\varphi : A^{\oplus N} \to M$, and we set $M'_k := (\mathrm{Fil}_k A)^{\oplus N} \cap \mathrm{Ker}\,\varphi$ for every $k \in \mathbb{N}$. Notice that the resulting graded K^+-module $M'_\bullet := \bigoplus_{k \in \mathbb{N}} M'_k$ is actually a $\mathsf{R}(\underline{A})_\bullet$-module and we get a short exact sequence of graded $\mathsf{R}(\underline{A})_\bullet$-modules

$$C_\bullet \quad : \quad 0 \to M'_\bullet \to \mathsf{R}(\underline{A})_\bullet^{\oplus N} \to \mathsf{R}(\underline{M})_\bullet \to 0.$$

Notice also that the K^+-modules $\mathsf{R}(\underline{A})_k^{\oplus N}$ and $\mathsf{R}(\underline{M})_k$ are torsion-free and finitely generated, hence they are free of finite rank (see [75, Rem.6.1.12(ii)]); then the same holds for M'_k, for every $k \in \mathbb{N}$. It follows that the complex $C_\bullet \otimes_{K^+} \kappa$ is still short exact. On the other hand, $\mathsf{R}(\underline{A})_\bullet$ is a K^+-algebra of finite type (see example 7.9.5), hence $R := \mathsf{R}(\underline{A})_\bullet \otimes_{K^+} \kappa$ is a noetherian ring; therefore, $M'_\bullet \otimes_{K^+} \kappa$ is a graded R-module of finite type, say generated by the system of homogenenous elements $\{\overline{x}_1, \dots, \overline{x}_t\}$. Lift these elements to a system $\Sigma' := \{x_1, \dots, x_t\}$ of homogeneous elements of $M'_\bullet$, and denote by $M''_\bullet \subset M'_\bullet$ the $\mathsf{R}(\underline{A})_\bullet$-submodule generated by Σ'. By construction, $M''_k \otimes_{K^+} \kappa = M'_k \otimes_{K^+} \kappa$, hence $M''_k = M'_k$ for every $k \in \mathbb{N}$, by Nakayama's lemma. In other words, Σ' is a system of generators for $M'_\bullet$; it follows easily that the image of Σ' in $\mathrm{Ker}\,\varphi$ is also a system of generator for the latter A-module, and the proposition follows. □

Corollary 9.1.28. *Every essentially finitely presented K^+-algebra is a coherent ring.*

Proof. One reduces easily to the case of a free polynomial K^+-algebra $K^+[T_1, \dots, T_n]$, in which case the assertion follows immediately from proposition 9.1.27 : the details shall be left to the reader. □

Lemma 9.1.29. *If A if a finitely presented K^+-algebra, the subset $\mathrm{Min}\,A \subset \mathrm{Spec}\,A$ of minimal prime ideals is finite.*

Proof. We begin with the following :

Claim 9.1.30. The assertion holds when A is a flat K^+-algebra.

Proof of the claim. Indeed, in this case, by the going-down theorem ([126, Th.9.5]) the minimal primes lie in $\mathrm{Spec}\,A \otimes_{K^+} K$, which is a noetherian ring. ◊

In general, A is defined over a $\mathbb{Z}$-subalgebra $R \subset K^+$ of finite type. Let $L \subset K$ be the field of fractions of R, and set $L^+ := L \cap K^+$; then L^+ is a valuation ring of finite rank, and there exists a finitely presented L^+-algebra A' such that $A \simeq A' \otimes_{L^+} K^+$. For every prime ideal $\mathfrak{p} \subset L^+$, let $\kappa(\mathfrak{p})$ be the residue field of $L^+_\mathfrak{p}$, and set :

$$A'(\mathfrak{p}) := A' \otimes_{L^+} \kappa(\mathfrak{p}) \qquad K^+(\mathfrak{p}) := K^+ \otimes_{L^+} \kappa(\mathfrak{p}).$$

Since $\mathrm{Spec}\,L^+$ is a finite set, it suffices to show that the subset $\mathrm{Min}\,A \otimes_{L^+} \kappa(\mathfrak{p})$ is finite for every $\mathfrak{p} \in \mathrm{Spec}\,L^+$. However, $A \otimes_{L^+} \kappa(\mathfrak{p}) \simeq A'(\mathfrak{p}) \otimes_{\kappa(\mathfrak{p})} K^+(\mathfrak{p})$, so this ring is a flat $K^+(\mathfrak{p})$-algebra of finite presentation. Let I be the nilradical of $K^+(\mathfrak{p})$; since $K^+(\mathfrak{p})/I$ is a valuation ring, claim 9.1.30 applies with A replaced by $A \otimes_{K^+} K^+(\mathfrak{p})/I$, and concludes the proof. □

Lemma 9.1.31. *Let $f : A \to B$ be a ring homomorphism of essentially finite type, M a B-module, and C a B-algebra. The following holds :*

(i) *If M is finitely presented as an A-module, then it is finitely presented as a B-module.*
(ii) *If C is finitely presented as a B-module, then it is finitely presented as a B-algebra.*
(iii) *If C is finitely presented (resp. essentially finitely presented) as an A-algebra, then it is finitely presented (resp. essentially finitely presented) as a B-algebra.*

Proof. (i): For any ring homomorphism $g : R \to S$ and any S-module N, let us denote by $N_{[g]}$ the R-module obtained from N by restriction of scalars along g. Set $C := B \otimes_A B$, and let $\nabla : C \to B$ be the ring homomorphism such that $\nabla(b \otimes b') := bb'$ for every $b, b' \in B$; notice that if B is a localization of $A[b_1, \dots, b_k]$, then $I := \mathrm{Ker}(\nabla)$ is the ideal generated by $(b_i \otimes 1 - 1 \otimes b_i \mid i = 1, \dots, k)$. Let also $B \xrightarrow{j_1} C \xleftarrow{j_2} B$ be the ring homomorphisms such that $j_1(b) := b \otimes 1$ and $j_2(b) := 1 \otimes b$ for every $b \in B$. Since $M_{[f]}$ is finitely presented, the same holds for the B-module $B \otimes_A M_{[f]} = (C_{[j_2]} \otimes_B M)_{[j_1]}$. The short exact sequence of C-modules

$$\Sigma := (0 \to I \to C \xrightarrow{\nabla} B \to B_{[\nabla]} \to 0)$$

induces a right exact sequence of B-modules

$$(\Sigma_{[j_2]} \otimes_B M)_{[j_1]} := (I_{[j_2]} \otimes_B M \to B \otimes_A M_{[f]} \to (B_{[\nabla \circ j_2]} \otimes_B M)_{[j_1]} \to 0)$$

and clearly $\nabla \circ j_2 = \mathbf{1}_B$, so that $(B_{[\nabla \circ j_2]} \otimes_B M)_{[j_1]} = M$. Moreover, if $M_{[f]} = Ax_1 + \cdots + Ax_n$, then the image of $I_{[j_2]} \otimes_B M$ in $B \otimes_A M_{[f]}$ is the B-submodule generated by $(b_i \otimes x_j - 1 \otimes b_i x_j \mid i = 1, \dots, k; j = 1, \dots, n)$, whence the assertion.

(ii): Let $c_1, \dots, c_n$ be a finite system of generators of the B-module C, with $c_1 = 1$; we let

$$\varphi : B[X_1, \dots, X_n] \to C$$

be the unique surjective homomorphism of B-algebras such that $\varphi(X_i) = c_i$ for $i = 1, \dots, n$. For every $i, j = 1, \dots, n$ with $i \leq j$ there exist $b_{ij1}, \dots, b_{ijn} \in B$ such that $c_i c_j = \sum_{k=1}^n b_{ijk} c_k$, and we let $J \subset B[X_1, \dots, X_n]$ be the ideal generated by

$$\{X_i X_j - \textstyle\sum_{k=1}^n b_{ijk} X_k \mid 1 \leq i \leq j \leq n\} \cup \{1 - X_1\}.$$

Clearly φ factors through a surjective map $\overline{\varphi} : B[X_1, \dots, X_n]/J \to C$. Moreover, it is easily seen that $B[X_1, \dots, X_n] = J + N$, where $N \subset B[X_1, \dots, X_n]$ is the free B-submodule of rank n generated by $X_1, \dots, X_n$. The restriction $N \to C$ of $\overline{\varphi}$ is then a surjective B-linear map, whose kernel is therefore a B-module of finite type $N' \subset N$; it follows that $\mathrm{Ker}(\varphi) = J + N'$ is an ideal of finite type, whence the assertion.

(iii): Suppose first that C is essentially finitely presented as an A-algebra, *i.e.* for some $k \in \mathbb{N}$ there exists a surjective homomorphism of A-algebras $\varphi : A[X_1, \dots, X_k] \to C'$ such that $\mathrm{Ker}(\varphi)$ is a finitely generated ideal, and $C = S^{-1}C'$ for some subset $S \subset A[X_1, \dots, X_k]$. Let $A' := S^{-1}A[X_1, \dots, X_k]$, $B' := S^{-1}B[X_1, \dots, X_k]$, and denote by $\psi : B' \to C$ be the unique homomorphism of B-algebras such that $\psi(X_i) = \varphi(X_i)$ for $i = 1, \dots, k$, and $g : A' \to B'$ the unique homomorphism of A-algebras such that $g(X_i) = X_i$ for $i = 1, \dots, k$. Then g is essentially of finite presentation, and C is an A'-module of finite presentation, by restriction of scalars along $\psi \circ g$, so it is a B'-module of finite presentation via ψ, by (i); hence, C is a finitely presented B'-algebra, by (ii), whence the assertion, in this case. Lastly, if C is finitely presented as an A-algebra, we may assume take $S = \{1\}$ in the foregoing, so that $A' = A[X_1, \dots, X_k]$ and $B' = B[X_1, \dots, X_k]$, and the same argument then shows that C is a finitely presented B-algebra. □

Lemma 9.1.32. *Let $(A, \mathfrak{m}_A)$ be a local ring, $K^+ \to A$ a local and essentially finitely presented ring homomorphism. Then there exists a local morphism $\varphi : V \to A$ of essentially finitely presented K^+-algebras, such that the following holds :*

(i) *V is a valuation ring, its maximal ideal $\mathfrak{m}_V$ is generated by the image of $\mathfrak{m}_K$, and the map $K^+ \to V$ induces an isomorphism $\Gamma \xrightarrow{\sim} \Gamma_V$ of value groups.*
(ii) *φ induces a finite field extension $V/\mathfrak{m}_V \to A/\mathfrak{m}_A$.*

Proof. Set $X := \mathrm{Spec}\, A$, $S := \mathrm{Spec}\, K^+$ and let $x \in X$ be the closed point. Pick $a_1, \dots, a_d \in A$ whose classes in $\kappa(x)$ form a transcendence basis over $\kappa(s)$. The system $(a_i \mid i \leq d)$ defines a factorization of the morphism $\mathrm{Spec}\, \varphi : X \to S$ as a composition $X \xrightarrow{g} Y := \mathbb{A}^d_S \xrightarrow{h} S$, such that $\xi := g(x)$ is the generic point of $h^{-1}(s) \subset Y$. The morphism g is essentially finitely presented (lemma 9.1.31(iii)) and moreover :

Claim 9.1.33. The stalk $\mathscr{O}_{Y,\xi}$ is a valuation ring with value group Γ.

Proof of the claim. Indeed, set $B := K^+[T_1, \dots, T_d]$; one sees easily that $V := \mathscr{O}_{X,\xi}$ is the valuation ring of the Gauss valuation $|\cdot|_B : \mathrm{Frac}(B) \to \Gamma \cup \{0\}$ such that

$$\Big| \sum_{\alpha \in \mathbb{N}^d} a_\alpha T^\alpha \Big|_B = \max\{|a_\alpha| \mid \alpha \in \mathbb{N}^d\}$$

(where $a_\alpha \in K^+$ and $T^\alpha := T_1^{\alpha_1} \cdots T_r^{\alpha_r}$ for all $\alpha := (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$, and $a_\alpha = 0$ except for finitely many $\alpha \in \mathbb{N}^d$). ◊

In view of claim 9.1.33, to conclude the proof it suffices to notice that $\kappa(x)$ is a finite extension of $\kappa(\xi)$. □

Proposition 9.1.34. *Let $\varphi : K^+ \to A$ and $\psi : A \to B$ be two essentially finitely presented ring homomorphisms (definition* 7.3.52*). Then :*

(i) *If ψ is integral, B is a finitely presented A-module.*
(ii) *If A is local, φ is local and flat, and $A/\mathfrak{m}_K A$ is a field, then A is a valuation ring and φ induces an isomorphism $\Gamma \xrightarrow{\sim} \Gamma_A$ of value groups.*

Proof. (i): The assumption means that there exist a finitely presented A-algebra C and a multiplicative system $T \subset C$ such that $B = T^{-1}C$. Let $I := \bigcup_{t\in T} \mathrm{Ann}_C(t)$. Then I is the kernel of the localization map $C \to B$; the latter is integral by hypothesis, hence for every $t \in T$ there is a monic polynomial $P(X) \in C[X]$, say of degree n, such that $P(t^{-1}) = 0$ in B, hence $t't^n \cdot P(t^{-1}) = 0$ in C, for some $t' \in T$, *i.e.* $t'(1 - ct) = 0$ holds in C for some $c \in C$, in other words, the image of t is already a unit in C/I, so that $B = C/I$. Then lemma 8.1.64 says that $V(I) \subset \mathrm{Spec}\, C$ is closed under generizations, hence $V(I)$ is open, by corollary 8.1.52 and lemma 9.1.29, and finally I is finitely generated, by lemma 8.1.61(ii). So B is a finitely presented A-algebra. Then (i) follows from the well known :

Claim 9.1.35. Let R be any ring, S a finitely presented and integral R-algebra. Then S is a finitely presented R-module.

Proof of the claim. Let us pick a presentation : $S = R[x_1, \dots, x_n]/(f_1, \dots, f_m)$. By assumption, for every $i \leq n$ we can find a monic polynomial $P_i(T) \in R[T]$ such that $P_i(x_i) = 0$ in S. Let us set $S' := R[x_1, \dots, x_n]/(P_1(x_1), \dots, P_n(x_n))$; then S' is a free R-module of finite rank, and there is an obvious surjection $S' \to S$ of R-algebras, the kernel of which is generated by the images of the polynomials $f_1, \dots, f_m$. The claim follows. ◊

(ii): By lemma 9.1.32, we may assume from start that the residue field of A is a finite extension of κ. We can write $A = C_\mathfrak{p}$ for a K^+-algebra C of finite presentation, and a prime ideal $\mathfrak{p} \subset C$ containing $\mathfrak{m}_K$; under the stated assumptions, $A/\mathfrak{m}_K A$ is a finite field extension of κ, hence $\mathfrak{p}$ is a maximal ideal of C, and moreover $\mathfrak{p}$ is isolated in the fibre $g^{-1}(s)$ of the structure morphism $g : Z := \mathrm{Spec}\, C \to \mathrm{Spec}\, K^+$. It then follows from [65, Ch.IV, Cor.13.1.4] that, up to shrinking Z, we may assume that g is a quasi-finite morphism ([60, Ch.II, §6.2]). Let $K^{\mathrm{h}+}$ be the henselization of K^+, and set $C^{\mathrm{h}} := C \otimes_{K^+} K^{\mathrm{h}+}$. There exists precisely one prime ideal $\mathfrak{q} \subset C^{\mathrm{h}}$ lying over $\mathfrak{p}$, and $C^{\mathrm{h}}_\mathfrak{q}$ is a flat, finite and finitely presented $K^{\mathrm{h}+}$-algebra by [66, Ch.IV, Th.18.5.11], so $C^{\mathrm{h}}_\mathfrak{q}$ is henselian, by [66, Ch.IV, Prop.18.5.6(i)], and then $C^{\mathrm{h}}_\mathfrak{q}$ is the henselization of A, since the natural map $A \to C^{\mathrm{h}}_\mathfrak{q}$ is ind-étale and induces an isomorphism $C^{\mathrm{h}}_\mathfrak{q} \otimes_{K^+} \kappa \simeq \kappa(x)$. Finally, $C^{\mathrm{h}}_\mathfrak{q}$ is also a valuation ring with value group Γ, by virtue of [75, Lemma 6.1.13 and Rem.6.1.12(vi)]. To conclude the proof of (ii), it now suffices to remark :

Claim 9.1.36. Let V be a valuation ring, $\varphi : R \to V$ a faithfully flat morphism. Then R is a valuation ring and φ induces an injection $\varphi_\Gamma : \Gamma_R \to \Gamma_V$ of the respective value groups.

Proof of the claim. Clearly R is a domain, since it is a subring of V. To show that R is a valuation ring, it then suffices to prove that, for any two ideals $I, J \subset R$, we have either $I \subset J$ or $J \subset I$. However, since V is a valuation ring, we know already that either $I \cdot V \subset J \cdot V$ or $J \cdot V \subset I \cdot V$; since φ is faithfully flat, the assertion follows. By the same token, we deduce that an ideal $I \subset R$ is equal to R if and only if $I \cdot V = V$, which implies that φ_Γ is injective (see [75, §6.1.11]). □

9.2. **Huber's theory of the valuation spectrum.** In this section we present the first elements of Huber's theory of the valuation spectrum of an arbitrary ring, for which the original reference is his Habilitationschrift [98]. Let A be any ring. We denote by

$$\mathrm{Spv}\, A$$

the set of equivalence classes of valuations on A. For every $a, b \in A$ we set

$$R_A\Big(\frac{a}{b}\Big) := \{v \in \mathrm{Spv}\, A \mid v(a) \leq v(b) \neq 0\}$$

and more generally, with every $n \in \mathbb{N}$ and every $a_1, \dots, a_n, b_1, \dots, b_n \in A$ we associate the *rational subset*

$$R_A\Big(\frac{a_1}{b_1}, \dots, \frac{a_n}{b_n}\Big) := \bigcap_{i=1}^{n} R_A\Big(\frac{a_i}{b_i}\Big)$$

and we denote $\mathscr{B}_A$ the boolean algebra generated by the system of rational subsets of $\mathrm{Spv}\, A$. Explicitly, every element of $\mathscr{B}_A$ is a finite union of subsets of the form

$$\{v \in \mathrm{Spv}\, A \mid v(a_1) < v(b_1), \dots, v(a_n) < v(b_n),\ v(c_1) \leq v(d_1), \dots, v(c_m) \leq v(d_m)\}$$

where $n, m \in \mathbb{N}$ are arbitrary integers, and $a_1, b_1, \dots, a_n, b_n, c_1, d_1, \dots, c_m, d_m \in A$ are arbitrary elements.

Definition 9.2.1. For every ring A, we endow $\mathrm{Spv}\, A$ with the topology $\mathscr{T}_A$ with basis given by the rational subsets. The pair $(\mathrm{Spv}\, A, \mathscr{T}_A)$ will be called the *valuation spectrum* of A.

Theorem 9.2.2. *For every ring A, the topological space $(\mathrm{Spv}\, A, \mathscr{T}_A)$ is spectral, and $\mathscr{B}_A$ is the set of all constructible subsets of $\mathrm{Spv}\, A$.*

Proof. We endow $\mathrm{Spv}\, A$ with the topology $\mathscr{T}_{\mathrm{cons}}$ with basis $\mathscr{B}_A$. Endow as well the set $\{0, 1\}$ with its discrete topology, and the set $\mathscr{P} := \{0, 1\}^{A \times A}$ with the product topology. We attach to every valuation v of A a mapping $\varphi_v : A \times A \to \{0, 1\}$, by declaring that $\varphi_v(a, b) = 1$ if and only if $v(a) \geq v(b)$. If v, w are any two valuations on A, then it is clear that $\varphi_v = \varphi_w$ if and only if v is equivalent to w, so that the rule $v \mapsto \varphi_v$ yields a well defined injective map

$$\varphi : \mathrm{Spv}\, A \to \mathscr{P}$$

and it is easily seen that the topology on $\mathrm{Spv}\, A$ induced by $\mathscr{P}$ via φ agrees with $\mathscr{T}_{\mathrm{cons}}$.

Claim 9.2.3. The image of φ is the subset of $\mathscr{P}$ of all mappings $f : A \times A \to \{0, 1\}$ fulfilling the following conditions for every $a, b, c \in A$:

(a) we have either $f(a, b) = 1$ or $f(b, a) = 1$
(b) if $f(a, b) = f(b, c) = 1$, then $f(a, c) = 1$
(c) we have either $f(a, a + b) = 1$ or $f(b, a + b) = 1$
(d) if $f(a, b) = 1$, then $f(ac, bc) = 1$
(e) if $f(ac, bc) = 1$ and $f(0, c) = 0$, then $f(a, b) = 1$
(f) $f(0, 1) = 0$.

Proof of the claim. It is clear that φ_v fulfills these conditions for every $v \in \mathrm{Spv}\, A$. Conversely, let f be a mapping fulfilling conditions (a)–(f); from (f), (a) and (d) we deduce :

(g) $f(x, 0) = 1$ for every $x \in A$.

Set $\mathfrak{p} := \{a \in A \mid f(0, a) = 1\}$. We show that $\mathfrak{p}$ is an ideal of A. Indeed, from (a) it follows that $0 \in \mathfrak{p}$. Next, say that $a, b \in \mathfrak{p}$; from (c) we may assume that $f(a, a + b) = 1$, and then (b) implies that $a + b \in \mathfrak{p}$. Lastly, if $a \in \mathfrak{p}$ and $b \in A$, then (d) implies that $ab \in \mathfrak{p}$, as required. Moreover, (e) and (f) imply that $\mathfrak{p}$ is a prime ideal, and we set $\kappa := \mathrm{Frac}\, A/\mathfrak{p}$. We notice :

(h) $f(a + x, a) = f(a, a + x) = 1$ for every $a \in A$ and $x \in \mathfrak{p}$.

Indeed, suppose that $f(a + x, a) = 0$ for such a and x; then (c) implies that $f(-x, a) = 1$, and combining with (b) we get $f(0, a) = 1$. On the other hand, (g) says that $f(a + x, 0) = 1$, and combining again with (b) we derive $f(a + x, a) = 1$, a contradiction. To derive the second identity, set $b := a + x$, so that $f(b - x, b) = 1$, by the foregoing, as required.

We can now show that f factors through a mapping

$$\overline{f} : A/\mathfrak{p} \times A/\mathfrak{p} \to \{0, 1\}.$$

Indeed, let $a, b \in A$ and $c \in \mathfrak{p}$ be arbitrary elements, and assume first that $f(a, b) = 1$; we need to show that $f(a + c, b) = 1$, and this follows from (b) and (h). Similarly, we get $f(a, b + c) = 1$.

Lastly, if $f(a,b) = 0$ we must have $f(a+c,b) = 0$, for otherwise the foregoing would give $f(a,b) = f(a+c-c,b) = f(a+c,b) = 1$, a contradiction; similarly, we see that $f(a,b+c) = 0$ in this case. This shows that $\overline{f}$ is well defined, and by construction we have

(i) $\overline{f}(0,x) = 0$ for every $x \in A/\mathfrak{p} \setminus \{0\}$.

Now, for every $x \in \kappa$, write $x = a^{-1}b$ for some $a,b \in A/\mathfrak{p}$, with $a \neq 0$; we say that x is *f-positive* if $\overline{f}(a,b) = 1$. From (i) and (e) it follows easily that this condition does not depend on the choice of a and b, and we let $V \subset \kappa$ be the subset of all f-positive elements. Suppose that $x,y \in V$, and write $x = a^{-1}b$, $y = a^{-1}b'$ for some $a,b,b' \in A/\mathfrak{p}$ with $a \neq 0$; this means that $\overline{f}(a,b) = \overline{f}(a,b') = 1$, and by virtue of (c) we may assume that $\overline{f}(b,b+b') = 1$. Then (b) implies that $\overline{f}(a,b+b') = 1$, so $x+y \in V$. Likewise, (d) implies $\overline{f}(aa',ba') = 1$ and $\overline{f}(ba',bb') = 1$, and then (b) yields $f(aa',bb') = 1$, so $xy \in V$. Also, we have either $\overline{f}(-1,1) = 1$ or $\overline{f}(1,-1) = 1$ by (a), so $-1 \in V$; summing up, we conclude that V is a subring of κ, and (a) implies that V is a valuation ring of κ. Now, let w be the valuation of κ attached to V as in remark 9.1.13(iv), and set $v := w \circ \pi$, where $\pi : A \to \kappa$ is the natural map; it is easily seen that $\varphi_v = f$, whence the claim. ◊

Claim 9.2.3 implies that the image of φ is a closed subset of the quasi-compact and separated topological space $\mathscr{P}$, so $(\mathrm{Spv}\,A, \mathscr{T}_{\mathrm{cons}})$ is quasi-compact and separated as well. Now, it is clear that every rational subset of $\mathrm{Spv}\,A$ is open and closed in the topology $\mathscr{T}_{\mathrm{cons}}$. Moreover, it follows easily from remark 9.1.13(vi) that $(\mathrm{Spv}\,A, \mathscr{T}_A)$ is a T_0 topological space. Then the theorem follows from proposition 8.1.43. □

Remark 9.2.4. (i) Let A be any ring. Quite generally, we have

$$R_A\Big(\frac{f_1}{f_0},\dots,\frac{f_n}{f_0}\Big) \cap R_A\Big(\frac{g_1}{g_0},\dots,\frac{g_m}{g_0}\Big) = R_A\Big(\frac{f_i g_j}{f_0 g_0} \mid i=0,\dots,n,\ j=0,\dots,m\Big)$$

for every sequences $f_\bullet := (f_0,\dots,f_n)$ and $g_\bullet := (g_0,\dots,g_m)$ of elements of A.

(ii) Every ring homomorphism $f : A \to B$ induces a mapping

$$\mathrm{Spv}\,f : \mathrm{Spv}\,B \to \mathrm{Spv}\,A \qquad v \mapsto v \circ f$$

and obviously we have

$$(\mathrm{Spv}\,f)^{-1} R_A\Big(\frac{a}{b}\Big) = R_B\Big(\frac{f(a)}{f(b)}\Big) \qquad \text{for every } a,b \in A$$

so that $\mathrm{Spv}\,f$ is a continuous spectral map $(\mathrm{Spv}\,B, \mathscr{T}_B) \to (\mathrm{Spv}\,A, \mathscr{T}_A)$ (remark 8.1.20(iii)).

(iii) Moreover, we have a natural *support map*

$$\sigma_A : \mathrm{Spv}\,A \to \mathrm{Spec}\,A \qquad v \mapsto \mathrm{Ker}\,v$$

and notice that

$$\sigma_A^{-1}(\mathrm{Spec}\,A_s) = R_A\Big(\frac{s}{s}\Big) \qquad \text{for every } s \in A$$

so also σ_A is spectral. Furthermore, it is easily seen that σ_A restricts to a homeomorphism

$$(\mathrm{Spv}\,A)_0 \xrightarrow{\sim} \mathrm{Spec}\,A$$

where $(\mathrm{Spv}\,A)_0$ is the subset of all rank zero valuations on A, endowed with the topology induced by the inclusion map into $\mathrm{Spv}\,A$. Notice as well that, by remark 9.1.4(iv,v), we have a natural homeomorphism

$$\sigma_A^{-1}(\mathfrak{p}) \xrightarrow{\sim} \mathrm{Spv}\,\kappa(\mathfrak{p}) \qquad \text{for every } \mathfrak{p} \in \mathrm{Spec}\,A \tag{9.2.5}$$

where the fibre $\sigma_A^{-1}(\mathfrak{p})$ is endowed with the topology induced by $\mathrm{Spv}\,A$ via the inclusion map.

(iv) Let $S \subset A$ be any multiplicative system, and $i : A \to S^{-1}A$ the localization map; taking into account remark 9.1.4(iv), it is easily seen that $\mathrm{Spv}\, i$ is injective, and

$$\mathrm{Im}\,\mathrm{Spv}\, i = \bigcap_{s\in S} R_A\Big(\frac{s}{s}\Big).$$

Moreover, we have

$$R_{S^{-1}A}\Big(\frac{s^{-1}a}{t^{-1}b}\Big) = (\mathrm{Spv}\, i)^{-1} R_A\Big(\frac{at}{bs}\Big) \qquad \text{for every } a, b \in R \text{ and } s, t \in S$$

so the topology of $\mathrm{Spv}\, S^{-1}A$ is induced by the topology of $\mathrm{Spv}\, A$, via the mapping $\mathrm{Spv}\, i$, and therefore the latter identifies $\mathrm{Spv}\, S^{-1}A$ with a pro-constructible subset of $\mathrm{Spv}\, A$.

(v) For any ring A, we set

$$\mathrm{Spv}^+ A := \{v \in \mathrm{Spv}\, A \mid v(a) \leq 1 \text{ for every } a \in A\}.$$

Hence, $\mathrm{Spv}^+ A$ is a pro-constructible subset of $\mathrm{Spv}\, A$, and we endow it with the topology induced by the inclusion map into $\mathrm{Spv}\, A$, so that $\mathrm{Spv}^+ A$ is a spectral space (theorem 9.2.2 and corollary 8.1.42). For every $v \in \mathrm{Spv}^+ A$, let also

$$\sigma_A^+(v) := \{a \in A \mid v(a) < 1\}.$$

It is easily seen that $\sigma_A^+(v)$ is a prime ideal of A, so we get a well defined map

$$\sigma_A^+ : \mathrm{Spv}^+ A \to \mathrm{Spec}\, A.$$

Notice that σ_A^+ is surjective : indeed, for every prime ideal $\mathfrak{p}$, the trivial valuation $v_\mathfrak{p}$ with support equal to $\mathfrak{p}$ lies in $\mathrm{Spv}^+ A$, and $\sigma_A^+(v_\mathfrak{p}) = \mathfrak{p}$. Moreover, we have

$$(\sigma_A^+)^{-1}(\mathrm{Spec}\, A_s) = R_A\Big(\frac{1}{s}\Big) \cap \mathrm{Spv}^+ A \qquad \text{for every } s \in A$$

so σ_A^+ is continuous and spectral (corollary 8.1.42). In many references, the prime ideal $\sigma_A^+(v)$ is called the *center* of the valuation v. Notice also that, for every ring homomorphism $f : A \to B$, the map $\mathrm{Spv}\, f$ restricts to a continuous spectral map

$$\mathrm{Spv}^+ f : \mathrm{Spv}^+ B \to \mathrm{Spv}^+ A.$$

9.2.6. Let X be any scheme. A *valuation of* X is a pair (x, v) consisting of a point $x \in X$ and the equivalence class of a valuation v of the residue field $\kappa(x)$; then we say that x is the *support* of (x, v). We denote by

$$\mathrm{Spv}\, X$$

the set of valuations of X, and we endow it with the coarsest topology $\mathscr{T}_X$ containing the subsets

$$U\Big(\frac{a}{b}\Big) := \{(x, v) \in \mathrm{Spv}\, X \mid x \in U,\ v(a(x)) \leq v(b(x)) \neq 0\}$$

where U ranges over all the open subsets of X, and a, b are arbitrary elements of $\mathscr{O}_X(U)$ (and $a(x), b(x)$ are the images of a and b in $\kappa(x)$). The topological space $(\mathrm{Spv}\, X, \mathscr{T}_X)$ is called the *valuation spectrum of* X. Any morphism of schemes $f : Y \to X$ induces a mapping

$$\mathrm{Spv}\, f : \mathrm{Spv}\, Y \to \mathrm{Spv}\, X \qquad (y, w) \mapsto (f(y), w \circ f_y^\flat)$$

where $f_y^\flat : \kappa(f(y)) \to \kappa(y)$ is the ring homomorphism induced by the morphism of structure sheaves $f^\flat : \mathscr{O}_X \to f_*\mathscr{O}_Y$ associated with f. Notice that

$$(\mathrm{Spv} f)^{-1} U\Big(\frac{a}{b}\Big) = f^{-1} U\Big(\frac{f_U^\flat(a)}{f_U^\flat(b)}\Big)$$

for every U, a and b as in the foregoing, so $\mathrm{Spv}\, f$ is a continuous mapping. In the same vein, consider the *support map*

$$\sigma_X : \mathrm{Spv}\, X \to X \qquad (x, v) \to x.$$

Clearly $\sigma_X^{-1} U = U(\frac{1}{1})$ for every open subset $U \subset X$, so σ_X is continuous as well.

Remark 9.2.7. The construction of (9.2.6) generalizes that of (9.2) : indeed, let A be any ring; we have a natural mapping

$$\text{(9.2.8)} \qquad \mathrm{Spv}\, A \to \mathrm{Spv}(\mathrm{Spec}\, A) \qquad v \mapsto (\mathrm{Ker}\, v, \overline{v})$$

where, for any valuation v on A, we let $\overline{v}$ be the residual valuation of v (see remark 9.1.4(v)). In light of remarks 9.1.13(vi) and 9.2.4(iv) it is easily seen that (9.2.8) is a homeomorphism (details left to the reader). Taking into account theorem 9.2.2, we see that $\mathrm{Spv}\, X$ is locally spectral for every scheme X, and if X is quasi-compact and quasi-separated, then $\mathrm{Spv}\, X$ is spectral (lemma 8.1.15(iv)). Moreover, remark 9.2.4(ii) and proposition 8.1.16(iii) imply that $\mathrm{Spv}\, f$ is a spectral map, for every morphism of schemes $f : Y \to X$.

Proposition 9.2.9. *Let $f : A \to B$ and $g : A \to A'$ be two ring homomorphisms, and set $B' := A' \otimes_A B$. Then the induced continuous map of topological spaces*

$$\varphi : \mathrm{Spv}\, B' \to \mathrm{Spv}\, A' \times_{\mathrm{Spv} A} \mathrm{Spv}\, B$$

is surjective.

Proof. Let $v' \in \mathrm{Spv}\, A'$ and $w \in \mathrm{Spv}\, B$ be two elements such that $v := \mathrm{Spv}(g)(v') = \mathrm{Spv}(f)(w)$, and denote by

$$\overline{v} \in \mathrm{Spv}\, \kappa(v) \qquad \overline{v}' \in \mathrm{Spv}\, \kappa(v') \qquad \overline{w} \in \mathrm{Spv}\, \kappa(w)$$

the residual valuations of v, v', and w; the maps f and g induce field extensions $\kappa(v) \to \kappa(v')$ and $\kappa(v) \to \kappa(w)$, whence a ring homomorphism $h : B' \to C := \kappa(v') \otimes_{\kappa(v)} \kappa(w)$. Suppose that $u \in \mathrm{Spv}\, C$ is an element whose image in $\mathrm{Spv}\, \kappa(v') \times_{\mathrm{Spv}\, \kappa(v)} \mathrm{Spv}\, \kappa(w)$ equals the pair $(\overline{v}', \overline{w})$; then $\varphi \circ \mathrm{Spv}(h)(u) = (v', w)$. Thus, we are reduced to the case where A, A' and B are three fields. To ease notation, set $A^+ := \kappa(v)^+$, and define likewise A'^+ and B^+; let as well $B'^+ := A'^+ \otimes_{A^+} B^+$. Since the maps $A^+ \to A'^+$ and $A^+ \to B^+$ are flat, it is easily seen that the induced map $B'^+ \to B'$ is injective. Pick any prime ideal $\mathfrak{p} \in \mathrm{Spec}\, B'^+$ whose image in $\mathrm{Spec}\, A'^+$ (resp. in $\mathrm{Spec}\, B^+$) is the (unique) closed point; the induced map $B'^+_\mathfrak{p} \to B'_\mathfrak{p}$ is still injective, and especially, $B'_\mathfrak{p} \neq 0$, so we may find a maximal ideal $\mathfrak{m}$ of $B'_\mathfrak{p}$, and we denote by $\kappa(\mathfrak{m})$ the residue field of $\mathfrak{m}$. The image of $B'^+_\mathfrak{p}$ in $\kappa(\mathfrak{m})$ is a local ring D, and by corollary 9.1.25 we may find a valuation ring $(V, \mathfrak{m}_V)$ of $\kappa(\mathfrak{m})$ that dominates D. Lastly, V yields a well defined point $u \in \mathrm{Spv}\, B'$, and a simple inspection shows that $\varphi(u) = (v', w)$, as required. □

9.2.10. We wish next to investigate the specializations in the valuation spectrum. Thus, let A be any ring, $v : A \to \Gamma_\circ$ a valuation. The convex subgroup of Γ_v generated by $\mathrm{Im}\,(v) \setminus \Gamma_\circ^+$ is called the *characteristic subgroup* of v, and shall be denoted

$$c\Gamma_v.$$

Now, let $\Delta \subset \Gamma$ be any convex subgroup, and $\pi : \Gamma \to \Gamma/\Delta$ the projection; we associate with Δ two mappings

$$v_\Delta : A \to (\Gamma/\Delta)_\circ \qquad a \mapsto \pi_\circ \circ v(a)$$

$$v^\Delta : A \to \Delta_\circ \qquad a \mapsto \begin{cases} v(a) & \text{if } v(a) \in \Delta \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 9.2.11. *With the notation of* (9.2.10)*, the following holds :*

(i) *v_Δ is a valuation on A.*

(ii) v^Δ *is a valuation if and only if* $c\Gamma_v \subset \Delta$.

Proof. (i) is clear. For (ii), suppose first that $c\Gamma_v \subset \Delta$, pick any $a, b \in A$, and let us show that $v(a+b)^\Delta \leq \max(v(a)^\Delta, v(b)^\Delta)$. This is clear if both $v(a), v(b) \in \Delta_\circ$, so we may assume that $v(b) \notin \Delta_\circ$, in which case notice that $v(b) < 1$, since $c\Gamma_v \subset \Delta$. Now, if $v(a) \in \Delta$, then we must have $v(b) < v(a)$, as Δ is convex; therefore, $v(a+b) = v(a)$ (remark 9.1.4(iii)), and the contention follows easily in this case. Lastly, if neither of $v(a), v(b)$ lies in Δ, the foregoing yields $v(a), v(b) < 1$, hence $v(a+b) < 1$, and moreover we may assume that $v(a+b) \leq v(a)$; then $v(a+b)$ cannot lie in Δ, as the latter is convex, so $v(a+b)^\Delta = 0$, and the assertion follows also in this case. Next, let us check that $v(ab)^\Delta = v(a)^\Delta \cdot v(b)^\Delta$. Again, this is clear if both $v(a), v(b) \in \Delta_\circ$, hence suppose that $v(b) \notin \Delta_\circ$; if $v(a) \in \Delta_\circ$, then it follows easily that $v(ab) \notin \Delta$, in which case the assertion holds. If neither of $v(a), v(b)$ is in $\Delta_\circ$, we have already remarked that $v(a), v(b) < 1$, hence $v(ab) < v(a) < 1$, so $v(ab)$ cannot lie in Δ, as the latter is convex. Thus, the assertion holds also in this case.

Conversely, suppose that $c\Gamma_v \not\subset \Delta$; then there exists $a \in A$ such that $v(a) > 1$ but $v(a) \notin \Delta$. Set $b := 1 - a$; it follows that $v(b) = v(a)$, and $v(a+b) = 1 \in \Delta$, so $v(a+b)^\Delta = 1$ but $v(a)^\Delta = v(b)^\Delta = 0$, and therefore v^Δ is not a semi-norm. □

Example 9.2.12. In the situation of example 9.1.5, suppose that A is a domain and P is integral with P^{gp} torsion-free, so that v_φ is a valuation on $A[P]$, and let $\Delta \subset \Gamma$ be a convex subgroup.

(i) Lemma 9.2.11(ii) implies that v_φ^Δ is a valuation on $A[P]$ if and only if $\varphi(P) \subset \Gamma_\circ^+ \cup \Delta$. Moreover, if the latter condition holds, the mapping

$$\varphi^\Delta : P \to \Gamma_\circ \qquad x \mapsto \begin{cases} \varphi(x) & \text{if } \varphi(x) \in \Delta \\ 0 & \text{otherwise.} \end{cases}$$

is a morphism of monoids, and $v_\varphi^\Delta = v_{\varphi^\Delta}$.

(ii) Furthermore, $(v_\varphi)_\Delta = v_{\pi\circ\varphi}$, where $\pi : \Gamma \to \Gamma/\Delta$ is the projection.

Remark 9.2.13. With the notation of (9.2.10), we notice :

(i) Let $\overline{v} : \kappa(v) \to \Gamma_\circ$ be the residual valuation of v, and $\kappa(v)^+$ the valuation ring of $\overline{v}$; the correspondence of remark 9.1.13(vii) assigns to Δ a prime ideal $\mathfrak{p}(\Delta) \subset \kappa(v)^+$, and we have

$$\mathrm{Ker}\, v_\Delta = \mathrm{Ker}\, v \qquad \kappa(v_\Delta)^+ = \kappa(v)^+_{\mathfrak{p}(\Delta)} = \overline{v}^{-1}(\Gamma^+ \cdot \Delta_\circ).$$

(ii) Let $\pi : A \to \kappa(v)$ be the natural map; then $c\Gamma_v \subset \Delta$ if and only if $\pi(A) \subset \kappa(v)^+_{\mathfrak{p}(\Delta)}$.

(iii) Suppose now that $c\Gamma_v \subset \Delta$; then :

$$\kappa(v^\Delta) \subset \mathrm{Frac}\, \kappa(v)^+/\mathfrak{p}(\Delta) \qquad \text{and} \qquad \kappa(v^\Delta)^+ = \kappa(v^\Delta) \cap \kappa(v)^+/\mathfrak{p}(\Delta).$$

(iv) The valuation v_Δ is a generization of v in $\mathrm{Spv}\, A$. Indeed, if $a, b \in A$ are any two elements such that $v(a) \leq v(b)$, then clearly $v_\Delta(a) \leq v_\Delta(b)$, whence the claim.

(v) Also, if $c\Gamma_v \subset \Delta$, then v^Δ is a specialization of v in $\mathrm{Spv}\, A$: indeed, if $v^\Delta(a) \leq v^\Delta(b)$, it is easily seen that $v(a) \leq v(b)$.

Definition 9.2.14. Let A be a ring, and $v, w \in \mathrm{Spv}\, A$ any two elements.

(i) We say that w is a *primary specialization* of v in $\mathrm{Spv}\, A$, if there exists a convex subgroup $\Delta \subset \Gamma_v$ containing $c\Gamma_v$, and such that w is (equivalent to) v^Δ. In this case, we also say that v is a *primary generization* of w.

(ii) We say that w is a *secondary generization* of v if there exists a convex subgroup $\Delta \subset \Gamma_v$ such that w is (equivalent to) v_Δ. In this case, we also say that v is a *secondary specialization* of w.

(iii) We say that w is a *generalized primary specialization* of v, if w is either a primary specialization of v, or else $c\Gamma_v = \{1\}$ and w is a trivial (*i.e.* rank zero) valuation of A with $\mathrm{Ker}\, v^{\{1\}} \subset \mathrm{Ker}\, w$. In this case, w is a specialization of $v^{\{1\}}$: see remark 9.2.4(iii).

Recall that specializations of points define a partial ordering on $\mathrm{Spv}\, A$: see remark 8.1.45(ii). Let v be any semi-norm on A, and $I \subset A$ any ideal; for the following lemma, we shall say that I is *v-convex*, if the following holds. For every $a \in A$ such that there exists $b \in I$ with $v(a) \leq v(b)$, we have $a \in I$ as well.

Lemma 9.2.15. *Let A be any ring, $v, w, u \in \mathrm{Spv}\, A$ any three elements. We have :*

(i) *If w is a primary specialization of v and u is a primary specialization of w, then u is a primary specialization of v.*
(ii) *If w is a secondary generization of v and u is a secondary generization of w, then u is a secondary generization of v.*
(iii) *Suppose that w is a generization of v. Then w is a secondary generization of v if and only if $\mathrm{Ker}\, w = \mathrm{Ker}\, v$.*
(iv) *The primary specializations (resp. the secondary generizations) of v form a totally ordered subset of the partially ordered set $\mathrm{Spv}\, A$.*
(v) *The supports of the primary specializations of v are the v-convex prime ideals of A.*

Proof. (i) and (ii) are clear from the explicit description in (9.2.10).

(iii): Suppose that w is a generization of v whose support equals that of v, and let $\overline{v}$ and $\overline{w}$ be the respective residual valuations on $\kappa := \kappa(v) = \kappa(w)$; from (9.2.5) we deduce that $\overline{w}$ is a generization of $\overline{v}$ in $\mathrm{Spv}\, \kappa$, which means that $\kappa(\overline{v})^+ \subset \kappa(\overline{w})^+$. Then, let $\Delta \subset \Gamma_v$ be the image of $(\kappa(w)^+)^\times$; it follows easily that Δ is a convex subgroup, and taking into account remark 9.1.13(iv), we see that Γ_w is naturally identified with Γ_v/Δ, and $w = v_\Delta$.

(iv): Indeed, the explicit description of (9.2.10) implies more precisely that the partially ordered set of secondary generizations of v is naturally identified with $\mathrm{Spec}\, \Gamma_v$, where the latter is totally ordered by inclusion (see definition 9.1.1(ii)). Likewise, we have a surjective order-preserving map from $\mathrm{Spec}\, \Gamma_v/c\Gamma_v$ onto the set of primary specializations of v.

(v): Let $\Delta \subset \Gamma_v$ be a convex subgroup containing $c\Gamma_v$; let also $a \in A$ and $b \in \mathrm{Ker}(v^\Delta)$ with $v(a) \leq v(b)$. Then $v(b) \notin \Delta$, and in particular $v(b) < 1$, since $c\Gamma_v \subset \Delta$; since Δ is convex, it follows that $v(a) \notin \Delta$, hence $a \in \mathrm{Ker}(v^\Delta)$. This shows that $\mathrm{Ker}(v^\Delta)$ is v-convex. Conversely, suppose that the prime ideal $\mathfrak{p} \subset A$ is v-convex. Hence, $v(x) \neq 0$ for every $x \in A \setminus \mathfrak{p}$, and we let $\Delta \subset \Gamma_v$ be the subgroup generated by $v(A \setminus \mathfrak{p})$. We claim that $\gamma > v(x)$ for every $\gamma \in \Delta$ and every $x \in \mathfrak{p}$. Indeed, suppose that this condition fails for some $\gamma \in \Delta$ and $x \in \mathfrak{p}$; since $A \setminus \mathfrak{p}$ is a submonoid of $(A, \cdot)$, we may write $\gamma = v(a) \cdot v(b)^{-1}$ for some $a, b \in A \setminus \mathfrak{p}$, and we get $v(a) \leq v(bx)$, which is absurd, since $\mathfrak{p}$ is v-convex. Now, let $\Delta' \subset \Gamma_v$ be the smallest convex subgroup containing Δ; it is easily seen that we still have $\gamma > v(x)$ for every $\gamma \in \Delta'$ and every $x \in \mathfrak{p}$. On the other hand, by construction we have $v(A \setminus \mathfrak{p}) \subset \Delta'$, so $\mathfrak{p} = \{x \in A \mid \gamma > v(x) \text{ for every } \gamma \in \Delta'\}$. Moreover, say that $a \in A$ and $v(a) \geq 1$; then $a \in A \setminus \mathfrak{p}$, since $\mathfrak{p}$ is v-convex, and consequently $v(a) \in \Delta'$, *i.e.* the characteristic subgroup of v lies in Δ', so that $v^{\Delta'} \in \mathrm{Spv}\, A$, and to conclude, it suffices to remark that $\mathrm{Ker}\, v^{\Delta'} = \mathfrak{p}$. □

Proposition 9.2.16. *Let A be any ring, v a valuation of A, and $\mathfrak{p} \subset A$ a prime ideal. We have :*

(i) *If $\mathfrak{p} \subset \mathrm{Ker}\, v$, there exists a primary generization w of v with $\mathrm{Ker}\, w = \mathfrak{p}$. Especially, the map σ_A of remark 9.2.4(iii) is generizing.*
(ii) *If $v \in \mathrm{Spv}^+ A$, then every primary specialization and every primary generization of v in $\mathrm{Spv}\, A$ lies also in $\mathrm{Spv}^+ A$, and if w is any primary specialization (or generization) of v, then $\sigma_A^+(w) = \sigma_A^+(v)$ (notation of remark 9.2.4(v)).*
(iii) *If $v \in \mathrm{Spv}^+ A$ and $\mathfrak{p}$ contains the center of v, there exists a secondary specialization $w \in \mathrm{Spv}^+ A$ of v with center equal to $\mathfrak{p}$. Especially, the map σ_A^+ is specializing.*

Proof. (i): The valuation v factors through the projection $A \to A' := A/\mathfrak{p}$ and a valuation v' of A'; after replacing A and v by A' and v', we may then assume that A is a domain and $\mathfrak{p} = 0$. In this case, let $\mathfrak{q} := \mathrm{Ker}\, v \subset A$; according to theorem 9.1.21 there exists a valuation ring

$(V, \mathfrak{m}_V)$ of $K := \mathrm{Frac}\, A$ such that $A \subset V$ and $\mathfrak{q} = \mathfrak{m}_V \cap A$; by the same token, there exists a valuation ring $(\overline{W}, \mathfrak{m}_{\overline{W}})$ of $V/\mathfrak{m}_V$ such that $\overline{W} \cap \kappa(\mathfrak{q}) = \kappa(v)^+$. Let $\pi_V : V \to V/\mathfrak{m}_V$ be the projection; then $W := \pi_V^{-1}(\overline{W})$ is a valuation ring of K; let w_K be the unique valuation of K with $\kappa(w_K)^+ = W$, and w the restriction of w_K to the subring A. Notice that $\mathfrak{m}_V = \mathrm{Ker}(W \to \overline{W})$ is a prime ideal of W; then the bijection of remark 9.1.13(vii) assigns to $\mathfrak{m}_V$ a convex subgroup $\Delta(\mathfrak{m}_V)$ such that $\mathfrak{m}_V = \{x \in W \mid \gamma < w_K(x) \text{ for every } \gamma \in \Delta(\mathfrak{m}_V)\}$. Especially, we see that $\mathfrak{m}_W$ is a w_K-convex ideal, hence $\mathfrak{q}$ is w-convex. Let $\Delta \subset \Gamma_w$ be the convex subgroup generated by $w(A \setminus \mathfrak{q})$; the proof of lemma 9.2.15(v) shows that $c\Gamma_w \subset \Delta$ and $\mathrm{Ker}(w^\Delta) = \mathfrak{q}$. We need to check that w^Δ is equivalent to v. Notice first that for every $a, b \in A$ we have :

$$w(a) \leq w(b) \Leftrightarrow a \in bW \qquad \text{and} \qquad v(a) \leq v(b) \Leftrightarrow \pi_V(a) \in \pi_V(b)\overline{W}. \tag{9.2.17}$$

Now, suppose that $w^\Delta(a) \leq w^\Delta(b)$, and let us show that $v(a) \leq v(b)$. Indeed, if $w^\Delta(b) \neq 0$, we have $w^\Delta \in R_A\left(\frac{a}{b}\right)$, whence $w \in R_A\left(\frac{a}{b}\right)$, so $w(a) \leq w(b)$, and thus $v(a) \leq v(b)$, in light of (9.2.17). If $w^\Delta(b) = 0$, we have as well $w^\Delta(a) = 0$, and then $v(a) = v(b) = 0$, since $\mathrm{Ker}(v) = \mathrm{Ker}(w^\Delta)$. Conversely, suppose that $v(a) \leq v(b)$; if $v(b) = 0$, we see as in the foregoing that $w^\Delta(a) = w^\Delta(b) = 0$. Thus, let $v(b) \neq 0$, and suppose by contradiction that $w^\Delta(a) > w^\Delta(b)$; we deduce that $w^\Delta(a), w^\Delta(b) \neq 0$, hence $a, b \neq 0$ and $w(a) < w(b)$. Let $\mathfrak{m}_W = \pi_V^{-1}(\mathfrak{m}_{\overline{W}})$ be the maximal ideal of W; we get $b \in a \cdot \mathfrak{m}_W$, so that $\pi_V(b) \in \pi_V(a) \cdot \mathfrak{m}_{\overline{W}}$, *i.e.* $v(a) > v(b)$, against our assumption. We may now apply remark 9.1.13(vi) to conclude.

(ii) is clear from the definitions.

(iii): Let $V \subset \kappa(v)$ be the valuation ring of the residual valuation of v, and $\mathfrak{m}_V$ the maximal ideal of V; then v induces a ring homomorphism $\pi : A \to V$ such that $\mathfrak{q} := \pi^{-1}\mathfrak{m}_V$ is the center of v. We deduce an injective ring homomorphism $\overline{A} := A/\mathfrak{q} \to \kappa := V/\mathfrak{m}_V$, and we denote by $\overline{\mathfrak{m}} \subset \overline{A}$ the image of $\mathfrak{m}$. By corollary 9.1.25, there exists a valuation ring $\overline{W}$ of κ that dominates the image of $\overline{A}_{\overline{\mathfrak{m}}}$, and we denote by $W \subset V$ the preimage of $\overline{W}$. Then W is a valuation ring of $\kappa(v)$, and it corresponds to a secondary specialization of v in $\mathrm{Spv}^+ A$. □

Proposition 9.2.18. *Let A be any ring, and $v \in \mathrm{Spv}\, A$ any element. Then, every specialization of v is a secondary specialization of a generalized primary specialization of v.*

Proof. Let w be a specialization of v. If $c\Gamma_v = \{1\}$ and $v(a) \geq 1$ for every $a \in A \setminus \mathrm{Ker}\, w$, then $\mathrm{Ker}\, v^{\{1\}} \subset \mathrm{Ker}\, w$, so the trivial semi-norm u with support $\mathrm{Ker}\, w$ is a generalized primary specialization of v, and w is a secondary specialization of u. We may therefore assume that either $c\Gamma_v \neq \{1\}$, or $v(a) < 1$ for some $a \in A \setminus \mathrm{Ker}\, w$.

Claim 9.2.19. Let $a, b \in A$ be any two elements with $v(a) \leq v(b)$, $w(a) \neq 0$ and $w(b) = 0$. Then $v(a) = v(b) \neq 0$.

Proof of the claim. The assumptions imply that $w \in R_A(\frac{b}{a})$, and since v is a generization of w, it follows that $v \in R_A(\frac{b}{a})$ as well, whence the claim. ◊

Claim 9.2.20. $\mathrm{Ker}\, w$ is v-convex.

Proof of the claim. Indeed, let $x, y \in A$ be two elements with

$$v(x) \leq v(y) \qquad \text{and} \qquad w(y) = 0 \tag{9.2.21}$$

and suppose, by way of contradiction, that $w(x) \neq 0$; by claim 9.2.19 it follows that

$$v(x) = v(y) \neq 0. \tag{9.2.22}$$

Consider first the case where $c\Gamma_v \neq \{1\}$, and pick $a \in A$ with $v(a) > 1$; combining with (9.2.21) we get $v(x) \leq v(ay)$ and $w(ay) = 0$, so claim 9.2.19 yields $v(ay) = v(x)$, which contradicts (9.2.22). Next, if $v(a) < 1$ for some $a \in A$ such that $w(a) \neq 0$, we get $v(ax) \leq v(y)$

and $w(ax) \neq 0$, again by combining with (9.2.21), and claim 9.2.19 yields $v(ax) = v(y)$, again a contradiction. $\diamondsuit$

By claim 9.2.20 and lemma 9.2.15(iv,v), there exists a unique primary specialization u of v with $\operatorname{Ker} u = \operatorname{Ker} w$, and in view of lemma 9.2.15(iii), it suffices to show that w is a specialization of u. To this aim, let $a, b \in A$ be any two elements such that $w \in R_A(\frac{a}{b})$; since v is a generization of w, we have $v \in R_A(\frac{a}{b})$ as well, and since u is a primary specialization of v, we deduce $u(a) \leq u(b)$. Lastly, since $\operatorname{Ker} u = \operatorname{Ker} w$ and $w(b) \neq 0$, we get $u(b) \neq 0$, so $u \in R_A(\frac{a}{b})$, and the assertion follows. □

Lemma 9.2.23. *Let A be a ring, $v, w \in \operatorname{Spv} A$ be two elements, such that w is a primary specialization of v. We have :*

(i) *For every secondary specialization v' of v there exists a unique secondary specialization w' of w such that w' is a primary specialization of v'.*
(ii) *For every secondary specialization w' of w there exists a secondary specialization v' of v such that w' is a primary specialization of v'.*
(iii) *For every secondary generization v' of v there exists a unique secondary generization w' of w such that w' is a generalized primary specialization of v'.*
(iv) *For every secondary generization w' of w there exists a secondary generization v' of v such that w' is a primary specialization of v'.*

Proof. (i): The uniqueness of w' follows from lemma 9.2.15(iii,iv). For the existence, let Δ' be a convex subgroup of $\Gamma_{v'}$ such that $v = v'_{\Delta'}$, and Σ a convex subgroup of $\Gamma_v = \Gamma_{v'}/\Delta'$ containing $c\Gamma_v$ and such that $w = v^{\Sigma}$. Denote by Σ' the unique convex subgroup of $\Gamma_{v'}$ such that $\Delta' \subset \Sigma'$ and $\Sigma'/\Delta' = \Sigma$; then $c\Gamma_{v'} \subset \Sigma'$ and we may take $w' := (v')^{\Sigma'}$.

(ii): By remark 9.2.13(ii), there exists a prime ideal $\mathfrak{p} \subset \kappa(v)^+$ such that $\kappa(w) \subset K := \operatorname{Frac} \kappa(v)^+/\mathfrak{p}$ and $\kappa(w)^+ = \kappa(w) \cap K^+$, where $K^+ := \kappa(v)^+/\mathfrak{p}$. On the other hand, $\kappa(w') = \kappa(w)$, and $\kappa(w')^+ \subset \kappa(w)^+$. Let $\mathfrak{m}_K$, $\mathfrak{m}_w$ and $\mathfrak{m}_{w'}$ be the maximal ideals of respectively K^+, $\kappa(w)^+$ and $\kappa(w')^+$; it follows easily that $\mathfrak{m}_K \cap \kappa(w) = \mathfrak{m}_w \subset \mathfrak{m}_{w'}$. Then $\mathfrak{m}_w$ is an ideal of $\kappa(w')^+$, and we have injective ring homomorphisms

$$B := \kappa(w')^+/\mathfrak{m}_w \to \kappa(w)^+/\mathfrak{m}_w \to K^+/\mathfrak{m}_K.$$

According to corollary 9.1.25, we may then find a valuation ring $\overline{V}$ of $K^+/\mathfrak{m}_K$ that dominates B. Since B is a valuation ring, we easily deduce that $B = \overline{V} \cap \operatorname{Frac} B$. Let $V \subset K^+$ be the preimage of $\overline{V}$; then V is a valuation ring of K such that $\kappa(w')^+ = V \cap \kappa(w')$ (details left to the reader). Let $V' \subset \kappa(v)^+$ be the preimage of V; then V' is a valuation ring of $\kappa(v)$, and we let v' be the unique valuation of A with $\operatorname{Ker} v' = \operatorname{Ker} v$ and $\kappa(v')^+ = V'$; it is easily seen that v' is the sought secondary specialization of v.

(iii): The uniqueness of w' follows from lemma 9.2.15(iii,iv), and the existence follows from proposition 9.2.18.

(iv): Let Δ be a convex subgroup of Γ_v containing $c\Gamma_v$ and such that $w = v^{\Delta}$, and Σ a convex subgroup of Δ such that $w' = w_{\Sigma}$. Set $v' := v_{\Sigma}$; then $\Delta' := \Delta/\Sigma$ is a convex subgroup of $\Gamma_{v'} = \Gamma_v/\Sigma$ that contains $c\Gamma_{v'}$, and $w' = (v')^{\Delta'}$. □

Proposition 9.2.24. *Let A be a ring, and $v \in \operatorname{Spv} A$ any element. Then every specialization of v is a primary specialization of a secondary specialization of v.*

Proof. Let w be a specialization of v. If w is a secondary specialization of a primary specialization of v, the assertion follows from lemma 9.2.23(ii). In light of proposition 9.2.18, we may then assume that the characteristic subgroup of v is trivial and $\operatorname{Ker} v^{\{1\}} \subset \operatorname{Ker} w$. In this case, proposition 9.2.16(i) yields a primary generization w' of w with $\operatorname{Ker} w' = \operatorname{Ker} v^{\{1\}}$. Clearly

w' is a secondary specialization of $v^{\{1\}}$, so lemma 9.2.23(ii) ensures that we may find a secondary specialization v' of v which is also a primary generization of w', and then w is a primary specialization of v' (lemma 9.2.15(i)). □

Lemma 9.2.25. *Let $f : A \to B$ be any ring homomorphism, and $v, w \in \mathrm{Spv}\, B$ any two elements. We have :*

- (i) *If w is a primary (resp. secondary) specialization of v, then $\mathrm{Spv}(f)(w)$ is a primary (resp. secondary) specialization of $\mathrm{Spv}(f)(v)$.*
- (ii) *If $\mathrm{Spec}\, f$ is an open immersion, and $\mathrm{Spv}(f)(w)$ is a primary (resp. secondary) specialization of $\mathrm{Spv}(f)(v)$, then w is a primary (resp. secondary) specialization of v.*
- (iii) *$\mathrm{Spv}\, f$ restricts to a surjection from the set of secondary specializations (resp. generizations) of v onto the set of secondary specializations (resp. generizations) of $\mathrm{Spv}(f)(v)$.*
- (iv) *Suppose that w is a primary specialization of v, and denote by P (resp. Q) the set of primary specializations of v in $\mathrm{Spv}\, B$ (resp. of $\mathrm{Spv}(f)(v)$ in $\mathrm{Spv}\, A$) that are also primary generizations of w in $\mathrm{Spv}\, B$ (resp. of $\mathrm{Spv}(f)(w)$ in $\mathrm{Spv}\, A$). Then $\mathrm{Spv}\, f$ restricts to a surjection $P \to Q$.*

Proof. (i) is immediate from the definitions.

(ii): Set $v' := \mathrm{Spv}(f)(v)$; if $\mathrm{Spec}\, f$ is an open immersion, f induces an isomorphism $\kappa(v') \xrightarrow{\sim} \kappa(v)$, and similarly for w. The assertion is an immediate consequence.

(iii): Clearly, f induces an injective map $i : \Gamma_{v'} \to \Gamma_v$, and $\mathrm{Spec}\, i$ is surjective, by remark 9.1.2(v), so $\mathrm{Spv}\, f$ maps surjectively the secondary generizations of v onto the ones of v'. Next, let w' be a secondary specialization of v' in $\mathrm{Spv}\, A$; then $\kappa(w') = \kappa(v')$ and $\kappa(w')^+$ is a valuation ring of $\kappa(v')$ contained in $\kappa(v')^+$. Let $\mathfrak{m} \subset \kappa(v)^+$ be the maximal ideal, so that $\mathfrak{m}' := \mathfrak{m} \cap \kappa(v')$ is the maximal ideal of $\kappa(v')^+$ and a prime ideal of $\kappa(w')^+$; then $K^+ := \kappa(w')^+/\mathfrak{m}'$ is a valuation ring of $K := \kappa(v')^+/\mathfrak{m}'$, and by corollary 9.1.25 there exists a valuation ring V of $\kappa(v)^+/\mathfrak{m}$ that dominates K^+. The preimage of V in $\kappa(v)^+$ is a valuation ring of $\kappa(v)$ and the corresponding valuation w of B is a secondary specialization of v with $\mathrm{Spv}(f)(w) = w'$.

(iv): Again, since the induced morphism of ordered groups i is injective, the assertion follows easily from remark 9.1.2(v). □

Theorem 9.2.26. *Let $f : A \to B$ be a flat ring homomorphism, and $v \in \mathrm{Spv}\, B$ any element. Then the following holds :*

- (i) *$\mathrm{Spv}\, f$ restricts to a surjection from the set of primary generizations of v in $\mathrm{Spv}\, B$ to the set of primary generizations of $\mathrm{Spv}(f)(v)$ in $\mathrm{Spv}\, A$.*
- (ii) *$\mathrm{Spv}\, f$ is generizing.*

Proof. (i): Let t be any primary generization of $s := \mathrm{Spv}(f)(v)$. According to remark 9.2.13(i), there exists a prime ideal $\mathfrak{p} \subset \kappa(t)^+$ such that

- the image of the natural map $A \to \kappa(t)$ lies in $A' := \kappa(t)^+_{\mathfrak{p}}$
- $\kappa(s)$ is naturally identified with a subfield of the residue field $\overline{\kappa} := A'/\mathfrak{p}A'$ of A'
- under this identification, we have $\kappa(s)^+ = \kappa(s) \cap (\kappa(t)^+/\mathfrak{p})$.

With this notation, let $h : A \to A'$ the resulting map, and

$$\overline{t} : \kappa(t) = \mathrm{Frac}\, A' \to \Gamma_{t\circ} \qquad \text{and} \qquad \overline{s} : \overline{\kappa} \to \Gamma_{s\circ}$$

be respectively the residual valuation of t and the valuation of $\overline{\kappa}$ such that $\kappa(\overline{s})^+ = \kappa(t)^+/\mathfrak{p}$. Then, $\overline{t}$ and $\overline{s}$ yield elements t' and respectively s' of $\mathrm{Spv}\, A'$ such that

$$s = \mathrm{Spv}(h)(s') \qquad \text{and} \qquad t = \mathrm{Spv}(h)(t').$$

Let $f' : A' \to B' := A' \otimes_A B$ be the induced map; by proposition 9.2.9 we may find $v' \in \mathrm{Spv}\, B'$ whose image in $\mathrm{Spv}\, A'$ (resp. in $\mathrm{Spv}\, B$) equals s' (resp. v), and in light of lemma 9.2.25(i), it suffices to exhibit a primary generization w' of v' in $\mathrm{Spv}\, B'$ with $\mathrm{Spv}(f')(w') = t'$. However, since f' is flat, there exists a prime ideal $\mathfrak{r} \subset \mathfrak{q} := \mathrm{Ker}\, v'$ in B', such that $f'^{-1}\mathfrak{r} = 0$ ([126, Th.9.5]); we set $B'' := B'_\mathfrak{q}/\mathfrak{r}B'_\mathfrak{q}$ and we let $f'' : A' \to B''$ be the resulting map. There exists a unique valuation v'' of B'' whose image in $\mathrm{Spv}\, B'$ equals v', and it suffices to exhibit a primary generization w'' of v'' in $\mathrm{Spv}\, B''$ such that $\mathrm{Spv}(f'')(w'') = t'$. To this aim, set $K'' := \mathrm{Frac}\, B''$; by corollary 9.1.25 we may find a valuation ring V of K'' that dominates B''. Let κ'' be the residue field of V, and denote by R the image of $\kappa(v'')^+$ in κ''; we may find a valuation ring V' of κ'' that dominates R (corollary 9.1.25). Denote by $V'' \subset V$ the preimage of V'. Then V'' is a valuation ring of K'', and we may take for w'' the unique valuation of B'' with $\kappa(w'')^+ = V''$.

(ii) is an immediate consequence of (i), lemma 9.2.25(iii) and proposition 9.2.24. □

Theorem 9.2.27. *Let $f : A \to B$ be an integral ring homomorphism, $v \in \mathrm{Spv}\, B$ any element. The following holds :*

(i) *If $v' \in \mathrm{Spv}\, B$ is a specialization of v with $\mathrm{Spv}(f)(v') = \mathrm{Spv}(f)(v)$, then $v = v'$.*
(ii) *$\mathrm{Spv}\, f$ restricts to a bijection from the set of primary specializations of v in $\mathrm{Spv}\, B$ to the set of primary specializations of $\mathrm{Spv}(f)(v)$ in $\mathrm{Spv}\, A$.*
(iii) *$\mathrm{Spv}\, f$ is specializing.*
(iv) *If f is injective, $\mathrm{Spv}\, f$ is surjective.*
(v) *If A is integral and normal, B is integral and f is injective, then $\mathrm{Spv}\, f$ restricts to a surjection from the set of primary generizations of v in $\mathrm{Spv}\, B$ onto the set of primary generizations of $\mathrm{Spv}(f)(v)$ in $\mathrm{Spv}\, A$.*

Proof. Set $w := \mathrm{Spv}(f)(v)$, and let Γ_v and Γ_w be the value groups of v and w.

(i): Let $\mathfrak{p}$ and $\mathfrak{q}$ be the supports of v and respectively v'; under the current assumptions, we have $f^{-1}\mathfrak{p} = f^{-1}\mathfrak{q}$, and $\mathfrak{q}$ is a specialization of $\mathfrak{p}$ in $\mathrm{Spec}\, B$; then $\mathfrak{q} = \mathfrak{q}'$, by [126, Th.9.3(ii)]. Taking into account proposition 9.2.24, we conclude that v' is a secondary specialization of v. However, since f is integral, it induces an isomorphism $\Gamma_v \otimes_\mathbb{Z} \mathbb{Q} \xrightarrow{\sim} \Gamma_w \otimes_\mathbb{Z} \mathbb{Q}$. On the other hand, we have a natural bijection between $\mathrm{Spec}\, \Gamma_v$ and the set S_v of secondary specializations of v in $\mathrm{Spv}\, B$, and likewise for the set S_w of secondary specializations of w; taking into account remark 9.1.2(v), we deduce that f induces a bijection $S_v \xrightarrow{\sim} S_w$, whence the assertion.

(ii): We remark :

Claim 9.2.28. Under the assumptions of the theorem, let $b \in B$ be any element. Then there exists $a \in A$ such that $v(b) \leq w(a)$.

Proof of the claim. Since f is integral, we may find $a_1, \dots, a_n \in A$ such that $b^n + a_1 b^{n-1} + \cdots + a_n = 0$. It follows that $v(b)^n \leq v(b)^i \cdot w(a_{n-i})$ for some $i < n$. If $v(b) = 0$, there is nothing to prove; otherwise, we get $v(b)^{n-i} \leq w(a_{n-i})$. If $v(b) \leq 1$, there is nothing to prove; otherwise, we deduce that $v(b) \leq w(a_{n-i})$, as required. ◊

From claim 9.2.28 it follows easily that $c\Gamma_w = c\Gamma_v \cap \Gamma_w$, whence the assertion.

(iii) follows immediately from (ii), proposition 9.2.24 and lemma 9.2.25(iii).

(iv): Notice first that $\mathrm{Spec}\, f$ is surjective, under these assumptions. Indeed, let $\mathfrak{p} \subset A$ be any prime ideal, and pick a minimal prime ideal $\mathfrak{q} \subset A$ contained in $\mathfrak{p}$; the induced ring homomorphism $f_\mathfrak{q} : A_\mathfrak{q} \to B_\mathfrak{q}$ is still injective, so $B \neq 0$; now, if $\mathfrak{q}' \subset B_\mathfrak{q}$ is any prime ideal, then $f_\mathfrak{q}^{-1}(\mathfrak{q}') = \mathfrak{q}$, and by [126, Th.9.4(i)] there exists a specialization $\mathfrak{p}'$ of $\mathfrak{q}'$ in $\mathrm{Spec}\, B$ such that $\mathrm{Spec}\,(f)(\mathfrak{p}') = \mathfrak{p}$. Now, let $w \in \mathrm{Spv}\, A$ be any element, and $\mathfrak{p}' \in \mathrm{Spec}\, B$ any prime ideal such that $f^{-1}\mathfrak{p}' = \mathfrak{p} := \mathrm{Ker}\, w$; it follows easily from theorem 9.1.21 that there exists a valuation w' of the residue field $\kappa(\mathfrak{p}')$ whose restriction to $\kappa(\mathfrak{p})$ is equivalent to v.

(v): Let w' be a primary generization of w in $\mathrm{Spv}\, A$. We need to exhibit a primary generization v' of v in $\mathrm{Spv}\, B$ such that $\mathrm{Spv}(f)(v') = w'$. To this aim, pick a normal algebraic field extension E of $\mathrm{Frac}\, A$ containing $\mathrm{Frac}\, B$, and denote by C the integral closure of A in E. By (iv), we may find $w'', v'' \in \mathrm{Spv}\, C$ such that $w''_{|A} = w'$ and $v''_{|B} = v$. By (ii), we may find a primary specialization t of w'' in $\mathrm{Spv}\, C$ such that $t_{|A} = w$. By construction, $t_{|A} = v''_{|A}$, so there exists an automorphism h of the A-algebra C such that $\mathrm{Spv}(h)(t) = v''$ ([34, Ch.V, §2, no.3, Prop.6 and Ch.VI, §8, no.6, Cor.1]). It is easily seen that $v' := \mathrm{Spv}(h)(w'')_{|B}$ will do. □

Theorem 9.2.29. *Let $f : A \to B$ be a finitely presented ring homomorphism, and $T \subset \mathrm{Spv}\, B$ any constructible subset. Then $\mathrm{Spv}(f)(T)$ is a constructible subset of $\mathrm{Spv}\, A$.*

Proof. We may assume that there exist elements $a_1, b_1, \dots, a_n, b_n, c_1, d_1, \dots, c_m, d_m \in B$ such that T is the set of all $v \in \mathrm{Spv}\, B$ with $v(a_i) < v(b_i)$ and $v(c_j) \leq v(d_j)$ for every $i = 1, \dots, n$ and $j = 1, \dots, m$. We remark :

Claim 9.2.30. The theorem holds if $\mathrm{Spec}\, f$ is a closed immersion.

Proof of the claim. Indeed, in this case f is a surjective map, whose kernel is a finitely generated ideal $I \subset A$. Pick a finite system $x_1, \dots, x_r \in A$ of generators of I, and for every $i = 1, \dots, n$, $j = 1, \dots, m$ choose elements $a'_i, b'_i, c'_j, d'_j \in A$ whose images equal respectively a_i, b_i, c_j, d_j in B. Then, $\mathrm{Spv}(f)(T)$ is the constructible subset of $\mathrm{Spv}\, A$ of all valuations v of A such that $v(x_k) = 0$ for every $k = 1, \dots, r$, and $v(a'_i) < v(b'_i)$ and $v(c'_j) \leq v(d'_j)$ for every $i = 1, \dots, n$ and $j = 1, \dots, m$. ◊

Claim 9.2.31. (i) We may assume that A and B are $\mathbb{Z}$-algebras of finite type.

(ii) For every $\mathbb{Z}$-algebra R of finite type, $\mathrm{Spv}^+ R$ is a rational subset of $\mathrm{Spv}\, R$ (notation of remark 9.2.4(v)).

Proof of the claim. (i): Indeed, we may write A as the union of the filtered system $(A_\lambda \mid \lambda \in \Lambda)$ of its $\mathbb{Z}$-subalgebras of finite type, and then there exists $\lambda \in \Lambda$ such that $B = B_\lambda \otimes_{A_\lambda} A$ for a ring homomorphism $f_\lambda : A_\lambda \to B_\lambda$ of finite type. Moreover, after replacing λ by a larger index, we may assume that there exist elements $a'_1, b'_1, \dots, a'_n, b'_n, c'_1, d'_1, \dots, c'_m, d'_m \in B_\lambda$ such that $a'_i \otimes 1 = a_i$, $b_i = b'_i \otimes 1$, $c_j = c'_j \otimes 1$ and $d_j = d'_j \otimes 1$ in B, for every $i = 1, \dots, n$ and $j = 1, \dots, m$. We then let $T' \subset \mathrm{Spv}\, B_\lambda$ be the constructible subset of all $v \in \mathrm{Spv}\, B_\lambda$ such that $v(a'_i) < v(b'_i)$ and $v(c'_j) \leq v(d'_j)$ for every $i = 1, \dots, n$ and $j = 1, \dots, m$. There follows a commutative diagram of topological spaces

$$\begin{array}{ccc} \mathrm{Spv}\, B & \xrightarrow{\pi_B} & \mathrm{Spv}\, B_\lambda \\ {\scriptstyle \mathrm{Spv}\, f}\downarrow & & \downarrow{\scriptstyle \mathrm{Spv}\, f_\lambda} \\ \mathrm{Spv}\, A & \xrightarrow{\pi_A} & \mathrm{Spv}\, A_\lambda \end{array}$$

clearly $T = \pi_B^{-1} T'$, and it follows easily from proposition 9.2.9 that

$$\pi_A^{-1}(\mathrm{Spv}(f_\lambda)(T')) = \mathrm{Spv}(f)(T).$$

Since π_A is spectral (remark 9.2.4(ii)), we are then reduced to showing that the theorem holds for the map f_λ.

(ii): Pick a finite system $x_1, \dots, x_r$ of elements of R such that $R = \mathbb{Z}[x_1, \dots, x_r]$; then it is easily seen that $\mathrm{Spv}^+ R = R_A(\frac{x_1}{1}, \dots, \frac{x_r}{1})$. ◊

Henceforth, we assume that A and B are $\mathbb{Z}$-algebras of finite type, and we set

$$C := B[X_i, Y_j \mid i = 1, \dots, n,\ j = 1, \dots, m]/I$$

where I is the ideal generated by the system $(a_i - X_i b_i, c_j - Y_j d_j \mid i = 1, \dots, n,\ j = 1, \dots, m)$. We let $T' \subset \mathrm{Spv}\, C$ be the constructible subset of all $w \in \mathrm{Spv}\, C$ such that

$$0 < w(b_i) \qquad w(X_i) < 1 \qquad w(Y_j) \leq 1 \qquad \text{for } 1 \leq i \leq n \text{ and } 1 \leq j \leq m$$

and denote by $g : B \to C$ the natural ring homomorphism.

Claim 9.2.32. $\mathrm{Spv}(g)(T') = T$.

Proof of the claim. Let $w \in T'$ be any element, and set $v := \mathrm{Spv}(g)(w)$; we may regard w as a valuation of $B[X_1, \dots, X_n, Y_1, \dots, Y_m]$ such that $w(b_i) > 0$ and $w(a_i) = w(X_i) \cdot w(b_i)$ for every $i = 1, \dots, n$, and $w(c_j) = w(Y_j) \cdot w(d_j)$ for every $j = 1, \dots, m$. It follows easily that $v \in T$. Conversely, let $v \in T$ be any element, and set $D := C \otimes_B \kappa(v)$. Moreover, set $\Sigma := \{1 \leq j \leq m \mid v(d_j) = 0\}$. A simple inspection yields an isomorphism

$$h : D \xrightarrow{\sim} \kappa(v)[Y_j \mid j \in \Sigma]$$

of $\kappa(v)$-algebras, such that :

- $h(X_i) = h(a_i) \cdot h(b_i)^{-1}$ for $i = 1, \dots, n$
- $h(Y_j) = Y_j$ for $j \in \Sigma$
- $h(Y_j) = h(c_j) \cdot h(d_j)^{-1}$ for every $j \in \{1, \dots, m\} \setminus \Sigma$.

Let $(V, \mathfrak{m}_V)$ be any valuation ring of $\mathrm{Frac}\, D$ containing $\kappa(v)^+[Y_j \mid j \in \Sigma]$ and such that $\mathfrak{m}_V \cap \kappa(v)$ is the maximal ideal of $\kappa(v)^+$ (theorem 9.1.21); then V corresponds to a valuation $\overline{w}$ of D whose restriction to $\kappa(v)$ is equivalent to the residual valuation $\overline{v}$ of v. So, $\overline{w}$ induces a valuation w of C with $\mathrm{Spv}(g)(w) = v$, and by construction we have $w(X_i) = v(a_i) \cdot v(b_i)^{-1} < 1$ for every $i = 1, \dots, n$; likewise, we get $w(Y_j) = v(c_j) \cdot v(d_j)^{-1} \leq 1$ for $j \in \Sigma$, and $w(Y_j) \leq 1$ for $j \in \{1, \dots, m\} \setminus \Sigma$. The claim follows. ◊

In view of claim 9.2.32, we may replace B by C, and T by T', after which we may assume:

(C) There exist finite sequences $a_\bullet := (a_1, \dots, a_n)$, $b_\bullet := (b_0, b_1, \dots, b_m)$ of elements of B such that
$$T = R_B\Big(\frac{b_0}{b_0}, \frac{b_1}{1}, \dots, \frac{b_m}{1}\Big) \setminus \Big(R_B\Big(\frac{1}{a_1}\Big) \cup \dots \cup R_B\Big(\frac{1}{a_n}\Big)\Big).$$

We shall argue by induction on the dimension of B. Since B is noetherian, $\mathrm{Spec}\, B$ has finitely many irreducible components $Z_1, \dots, Z_k$, and

$$\mathrm{Spv}\, B = \bigcup_{i=1}^{k} \mathrm{Spv}\, Z_i$$

(notation of (9.2.6)); moreover, since the inclusion map $Z_i \to \mathrm{Spec}\, B$ is spectral, the subset $\mathrm{Spv}\, Z_i \cap T$ is constructible in $\mathrm{Spv}\, Z_i$, for every $i = 1, \dots, k$. Thus, it suffices to show that the restriction $\mathrm{Spv}\, Z_i \to \mathrm{Spv}\, A$ of $\mathrm{Spv}\, f$ maps constructible subsets to constructible subsets, for every $i = 1, \dots, k$. We may therefore assume from start that $\mathrm{Spec}\, B$ is also irreducible and reduced, *i.e.* that B is a domain. Then, the topological closure W of $\mathrm{Spec}(f)(\mathrm{Spec}\, B)$ in $\mathrm{Spec}\, A$ is irreducible (lemma 8.1.3(v)); by virtue of claim 9.2.30, it suffices to show that the induced map $\mathrm{Spv}\, B \to \mathrm{Spv}\, W$ sends constructible subsets to constructible subsets. We may then assume as well that A is a domain, and f is injective. In this situation, if $\dim\, B = 0$, we see that A and B are finite fields, and the assertion is obvious. Hence, we suppose henceforth that $d := \dim\, B > 0$, and that the theorem has already been shown for every $\mathbb{Z}$-algebra B of finite type of dimension $< d$.

Pick finite subsets $I \subset A$ and $J \subset B$ such that $A = \mathbb{Z}[I]$ and $B = A[J]$ and $I \cap J = \varnothing$. For every pair of subsets $I' \subset I$ and $J' \subset J$ consider the constructible open subsets

$$S_A(I') := R_A\Big(\frac{x}{1} \mid x \in I'\Big) \cap R_A\Big(\frac{1}{x} \mid x \in I \setminus I'\Big) \subset \mathrm{Spv}\, A$$
$$S_B(I', J') := R_B\Big(\frac{x}{1} \mid x \in I' \cup J'\Big) \cap R_B\Big(\frac{1}{x} \mid x \in (I \cup J) \setminus (I' \cup J')\Big) \subset \mathrm{Spv}\, B.$$

Clearly $\mathrm{Spv}\, A = \bigcup_{I' \subset I} S_A(I')$, and likewise for $\mathrm{Spv}\, B$, so it suffices to show that the subset $\mathrm{Spv}(f)(T \cap S_B(I', J'))$ is constructible in $S_A(I')$, for any such I' and J' (lemma 8.1.19(iii,x.a)). Now, for given I' and J', let :

- $A_1 := \mathbb{Z}[I' \cup \{x^{-1} \mid x \in I \setminus I'\}] \subset \mathrm{Frac}\, A$
- $B_1 := A_1[J' \cup \{x^{-1} \mid x \in J \setminus J'\} \cup a_\bullet \cup b_\bullet] \subset \mathrm{Frac}\, B$
- $A_2 := A \cdot A_1 \subset \mathrm{Frac}\, A$ and $B_2 := B \cdot B_1 \subset \mathrm{Frac}\, B$.

The natural inclusions maps $A \to A_2 \leftarrow A_1$ and $B \to B_2 \leftarrow B_1$ are localizations, so $\dim B_1 = \dim B_2 = d$ ([65, Ch.IV, Prop.10.6.1(ii)]), and f extends to an injective ring homomorphism $f_2 : A_2 \to B_2$, which in turn restricts to a map $f_1 : A_1 \to B_1$. So we get a commutative diagram of continuous maps

$$\begin{array}{ccccc} \mathrm{Spv}\, B_1 & \longleftarrow & \mathrm{Spv}\, B_2 & \longrightarrow & \mathrm{Spv} B \\ {\scriptstyle \mathrm{Spv}\, f_1}\downarrow & & {\scriptstyle \mathrm{Spv}\, f_2}\downarrow & & \downarrow{\scriptstyle \mathrm{Spv}\, f} \\ \mathrm{Spv}\, A_1 & \longleftarrow & \mathrm{Spv}\, A_2 & \longrightarrow & \mathrm{Spv} A \end{array}$$

whose horizontal arrows are quasi-compact open immersions. Moreover, $S_A(I')$ and $S_B(I', J')$ lie respectively in the images of $\mathrm{Spv}\, A_2$ and $\mathrm{Spv}\, B_2$, and their images in $\mathrm{Spv}\, A_1$ and $\mathrm{Spv}\, B_1$ are the subsets

$$Z_A(I') := \{v \in \mathrm{Spv}^+ A_1 \mid v(x^{-1}) \neq 0 \text{ for every } x \in I \setminus I'\}$$
$$Z_B(I', J') := \{v \in \mathrm{Spv}^+ B_1 \mid v(x^{-1}) \neq 0 \text{ for every } x \in (I \cup J) \setminus (I' \cup J')\}.$$

Hence, we may replace A by A_1, B by B_1, and suppose that

$$T = \Big(R_B\Big(\frac{b_0}{b_0}\Big) \setminus R_B\Big(\frac{1}{b_1}\Big) \cup \cdots \cup R_B\Big(\frac{1}{b_n}\Big)\Big) \cap \mathrm{Spv}^+ B \qquad \text{for certains } b_0, \dots, b_n \in B.$$

Next, by [86, Partie I, Th.5.2.2] we may find an integer $r > 0$ and elements $t_1, \dots, t_r \in A \setminus \{0\}$ such that, for every $i = 1, \dots, r$ the inclusion map of subrings of $B'_i := B[t_i^{-1}]$

$$A_i := A\Big[\frac{t_1}{t_i}, \dots, \frac{t_r}{t_i}\Big] \to B_i := B\Big[\frac{t_1}{t_i}, \dots, \frac{t_r}{t_i}\Big]$$

is flat. We consider the constructible subsets

$$S_0 := \{v \in \mathrm{Spv}\, B \mid v(t_i) = 0 \text{ for } i = 1, \dots, r\} \qquad S_i := R_B\Big(\frac{t_1}{t_i}, \dots, \frac{t_r}{t_i}\Big) \qquad (i = 1, \dots, r).$$

Since $\mathrm{Spv}\, B = \bigcup_{i=0}^r S_i$, it suffices to show that $\mathrm{Spv}(f)(T \cap S_i)$ is constructible in $\mathrm{Spv}\, A$ for every $i = 0, \dots, r$. However, $T \cap S_0$ is a constructible subset of $\mathrm{Spv}\, B/I$, where $I \subset B$ is the ideal generated by $t_1, \dots, t_r$; since $\dim B/I < \dim B$, the inductive assumption yields the assertion for $i = 0$. For $i = 1, \dots, r$, set $A'_i := A[t_i^{-1}]$; we get a commutative diagram of continuous maps

$$\begin{array}{ccccc} \mathrm{Spv}\, B'_i & \xrightarrow{\varphi'_i} & \mathrm{Spv}\, B_i & \xrightarrow{\varphi_i} & \mathrm{Spv}\, B \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spv}\, A'_i & \xrightarrow{\psi'_i} & \mathrm{Spv}\, A_i & \xrightarrow{\psi_i} & \mathrm{Spv}\, A \end{array}$$

and we notice that the maps φ'_i, ψ'_i, $\psi_i \circ \psi'_i$ and $\varphi_i \circ \varphi'_i$ are quasi-compact open immersions, and $T \cap S_i = \varphi_i(T_i)$, for the constructible subsets of $\mathrm{Spv}\, B'_i$

$$T_i := \Big(R_{B_i}\Big(\frac{t_i b_0}{t_i b_0}\Big) \setminus \Big(R_{B_i}\Big(\frac{1}{b_1}\Big) \cup \cdots \cup R_{B_i}\Big(\frac{1}{b_n}\Big)\Big)\Big) \cap \mathrm{Spv}^+ B_i \subset \varphi'_i(\mathrm{Spv}\, B'_i).$$

In light of lemma 8.1.19(iii,x.a), it then suffices to show that the image of T_i is constructible in $\mathrm{Spv}\, A_i$. Thus, we may replace A, B, b_0 and T respectively by A_i, B_i, $t_i b_0$ and T_i, and assume furthermore that f is a flat ring homomorphism.

Let $\varphi : \mathrm{Spec}\, B/b_0B \to \mathrm{Spec}\, A$ be the restriction of $\mathrm{Spec}\, f$; there exists a non-empty affine open subset $U \subset \mathrm{Spec}\, A$ such that the restriction $\varphi^{-1}U \to U$ of φ is a flat morphism of schemes ([64, Ch.IV, Th.6.9.1]). For every $\mathfrak{p} \in U$, we have a short exact sequence of A-modules

$$\mathscr{E} \quad : \quad 0 \to B_\mathfrak{p} \to B_\mathfrak{p} \to B_\mathfrak{p}/b_0B_\mathfrak{p} \to 0$$

and the induced sequence $\mathscr{E} \otimes_A \kappa(\mathfrak{p})$ is still exact, since $B_\mathfrak{p}/b_0B_\mathfrak{p}$ is a flat $A_\mathfrak{p}$-module (notation of definition 4.9.17(i)). In other words, we have :

(D) The image of b_0 in $B \otimes_A \kappa(\mathfrak{p})$ is a regular element, for every $\mathfrak{p} \in U$.

Set

$$Z := \mathrm{Spec}\, A \setminus U \qquad Z_B := \mathrm{Spec}(f)^{-1}Z \qquad U_B := \mathrm{Spec}(f)^{-1}U.$$

Then $\mathrm{Spv}\, Z$ is a constructible closed subset of $\mathrm{Spv}\, A$ and Z_B is an affine scheme of dimension $< d$; also, $T' := T \cap \mathrm{Spv}\, Z_B$ is a constructible subset of $\mathrm{Spv}\, Z_B$, so our inductive assumption says that $\mathrm{Spv}(f)(T')$ is constructible in $\mathrm{Spv}\, A$. It remains therefore only to show that the subset $\mathrm{Spv}(f)(T \cap \mathrm{Spv}\, U_B)$ is constructible in $\mathrm{Spv}\, A$.

Let $J \subset B$ be the ideal generated by $b_1, \dots, b_n$; by [63, Ch.IV, Th.1.8.4], the subset $W := \mathrm{Spec}(f)(\mathrm{Spec}\, B/J)$ is constructible in $\mathrm{Spec}\, A$, hence

$$\mathrm{Spv}^+ W := (\sigma_A^+)^{-1}(W)$$

is constructible in $\mathrm{Spv}^+ A$ (remark 9.2.4(v)). In light of claim 9.2.31(ii), to conclude the proof, it then suffices to show :

Claim 9.2.33. Under the current assumptions, we have

$$\mathrm{Spv}(f)(T \cap \mathrm{Spv}\, U_B) = \mathrm{Spv}^+ W \cap \mathrm{Spv}\, U.$$

Proof of the claim. A simple inspection shows that $\mathrm{Spv}(f)(T) \subset \mathrm{Spv}^+ W$, so it remains only to check that $\mathrm{Spv}^+ W \cap \mathrm{Spv}\, U \subset \mathrm{Spv}(f)(T \cap \mathrm{Spv}\, U_B)$. Now, let $v \in \mathrm{Spv}^+ W \cap \mathrm{Spv}\, U$ be any element, and let $\pi : A \to \kappa(v)$ be the induced map. Since $v \in \mathrm{Spv}^+ A$, we have $\mathrm{Im}\, \pi \subset V := \kappa(v)^+$, and we set $B' := B \otimes_A V$. Let also $\mathfrak{m}_V$ and η_V be respectively the closed point and the generic point of $\mathrm{Spec}\, V$; since $v \in \mathrm{Spv}\, U$, the image of η_V in $\mathrm{Spec}\, A$ lies in U. By construction, there exists $\mathfrak{p} \in W$ such that $\mathrm{Spec}\, f(\mathfrak{p})$ is the center of v, *i.e.* the image of $\mathfrak{m}_V$ in $\mathrm{Spec}\, A$ (see remark 9.2.4(v)). Then we may find $\mathfrak{p}' \in \mathrm{Spec}\, B'$ whose images in $\mathrm{Spec}\, B$ and $\mathrm{Spec}\, V$ equal respectively $\mathfrak{p}$ and $\mathfrak{m}_V$. We pick a minimal prime ideal $\mathfrak{q}'$ of B' contained in $\mathfrak{p}'$. Since f is flat, the same holds for the induced map $V \to B \otimes_A V$, and therefore the image of $\mathfrak{q}'$ in $\mathrm{Spec}\, V$ equals η_V ([126, Th.9.5]); consequently the image of $\mathfrak{q}'$ in $\mathrm{Spec}\, B$ lies in U_B. Let $\overline{b}_0 \in B'$ be the image of b_0; from condition (D) and [126, Th.6.1(ii), Th.6.5(iii)], we deduce that

$$\overline{b}_0 \notin \mathfrak{q}'. \tag{9.2.34}$$

Choose a valuation ring V' of the residue field $\kappa(\mathfrak{q}')$ that dominates the image of the local ring $B'_{\mathfrak{p}'}$ (corollary 9.1.25). Then, V' corresponds to a point $v' \in \mathrm{Spv}\, B$, and we notice that $\mathrm{Spv}\, f(v') = v$: indeed, the induced map $V \to \kappa(\mathfrak{q}')$ is injective, and V' dominates the image of this map, whence the assertion. To conclude the proof of the claim, it suffices to check that $v' \in T \cap \mathrm{Spv}\, U_B$. However, (9.2.34) already implies that $v'(b_0) \neq 0$. Moreover, since V' contains the image of B, we have $v(b_i) \leq 1$ for every $i = 1, \dots, m$, and since the image of $\mathfrak{p}'$ lies in the maximal ideal of V, we have $v(b_i) < 1$ for every $i = 1, \dots, n$. Thus, $v \in T$. Lastly, since the image of $\mathfrak{q}'$ in $\mathrm{Spec}\, B$ lies in U_B, we get $v \in \mathrm{Spv}\, U_B$ as well, as needed. □

Remark 9.2.35. Our proof of theorem 9.2.29 employs crucially Gruson and Raynaud's flattening technique ([86]). On the other hand, Huber in [98] recovers this theorem rather easily via "elimination of quantifiers" for the first order theory of algebraically closed valued fields with non-trivial valuation. However, the latter is of course a deep model theoretic result, so Huber's proof is hardly more elementary than ours.

We wish next to explain how the valuation spectrum can be applied to the study of integral closures of rings and ideals. We begin with the following :

Definition 9.2.36. Let $f : A \to B$ be any ring homomorphism, $I \subset A$ any ideal. The *integral closure of I in B* is the set denoted

$$\mathrm{i.c.}(I, B)$$

and consisting of all $x \in B$ that satisfy, for some $n \in \mathbb{N}$, an equation of the type :

$$x^n + f(a_1) \cdot x^{n-1} + \cdots + f(a_n) = 0 \qquad \text{where } a_j \in I^j \text{ for } j = 1, \dots, n. \tag{9.2.37}$$

Notice that for $I = A$, the set $\mathrm{i.c.}(A, B)$ is the usual integral closure of A in B. We have :

Lemma 9.2.38. *In the situation of definition* 9.2.36*, let $b \in B$ be any element.*

(i) *The following conditions are equivalent :*
  (a) $b \in \mathrm{i.c.}(I, B)$.
  (b) $bU^{-1} \in \mathrm{i.c.}(\mathsf{R}(A, I)_\bullet, \mathsf{R}(B, B)_\bullet)$ *(notation of example* 7.9.3*).*

(ii) *Especially,* $\mathrm{i.c.}(I, B)$ *is an ideal of* $\mathrm{i.c.}(A, B)$.

(iii) *For every prime ideal $\mathfrak{p} \in \mathrm{Spec}\, B$, let $\bar{b}_\mathfrak{p} \in \kappa(\mathfrak{p})$ denote the image of b (notation of definition* 4.9.17(i)*). The following conditions are equivalent :*
  (a) $b \in \mathrm{i.c.}(I, B)$.
  (b) $\bar{b}_\mathfrak{p} \in \mathrm{i.c.}(I, \kappa(\mathfrak{p}))$ *for every minimal prime ideal $\mathfrak{p}$ of B.*

Proof. Suppose that b satisfies an identity (9.2.37), and set $T := U^{-1}$ to ease notation; we get

$$(bT)^n + Tf(a_1) \cdot (bT)^{n-1} + \cdots + T^n f(a_n) = 0$$

where $T^j f(a_j) \in \mathsf{R}(A, I)_\bullet$ for $j = 1, \dots, n$. Conversely, suppose the bT is integral over $\mathsf{R}(A, I)_\bullet$, so that we have a monic polynomial $P(X) = X^n + u_1 X^{n-1} + \cdots + u_n$, with coefficients in $\mathsf{R}(A, I)_\bullet$, such that $P(bT) = 0$; for every $j = 1, \dots, n$, denote by v_j the component of u_j in degree n (for the grading of $\mathsf{R}(A, I)_\bullet$); since bT is homogeneous of degree 1, it follows that $(bT)^n + v_1 \cdot (bT)^{n-1} + \cdots + v_n = 0$, and letting $T = 1$, we deduce that b is integral over I. This proves (i), and assertion (ii) is an immediate consequence.

(iii): Clearly (iii.a)⇒(iii.b). Hence, suppose that (iii.b) holds, and for every minimal prime ideal $\mathfrak{p}$ of B, pick a monic polynomial $P_\mathfrak{p}(X) \in A[X]$, say of degree $n_\mathfrak{p}$, such that $P_\mathfrak{p}(\bar{b}_\mathfrak{p}) = 0$, and whose coefficient in degree $n_\mathfrak{p} - j$ lies in I^j, for $j = 1, \dots n_\mathfrak{p}$. For every such $\mathfrak{p}$, let $V_\mathfrak{p} \subset \mathrm{Spec}\, B$ denote the subset of all prime ideals containing $P_\mathfrak{p}(b)$. Hence, $Z := \bigcup_{\mathfrak{p} \in \mathrm{Min}(B)} V_\mathfrak{p}$ contains the set $\mathrm{Min}(B)$ of minimal prime ideals of B; but each $V_\mathfrak{p}$ is a constructible closed subset of $\mathrm{Spec}\, B$, hence $Z = \mathrm{Spec}\, B$, and there exists a finite subset $S \subset \mathrm{Min}(B)$ such that $\bigcup_{\mathfrak{p} \in S} V_\mathfrak{p} = \mathrm{Spec}\, B$ (theorem 8.1.34(i)). Set $P := \prod_{\mathfrak{p} \in S} P_\mathfrak{p}$; it follows that $P(b)$ lies in the nilpotent ideal of B, so $P^r(b) = 0$ for a sufficiently large $r \in \mathbb{N}$. However, let N be the degree of P^r; it is easily seen that the coefficient in degree $N - j$ of P^r lies in I^j, for every $j = 1, \dots, N$ (details left to the reader), whence (iii.a). □

Concerning integral closures of ideals, much more can be found in the treatise [102], a comprehensive reference for this subject. For our purposes, proposition 9.2.42 hereafter will suffice. For its proof, we shall employ the technical lemma 9.2.40, which shall be applied also later to the study of the adic spectrum of a topological ring, and for this reason, is stated here in wider generality than needed for our immediate purposes in this section.

9.2.39. Let B be any ring, $A \subset B$ a subring, I an ideal of A, and Σ, Σ' two sets; denote by $\mathscr{P}_I$ the subset of the free polynomial A-algebra

$$R[Z] \qquad \text{where} \qquad R := A[X_i, Y_j \mid i \in \Sigma,\ j \in \Sigma']$$

consisting of all polynomials P fulfilling the following conditions :

- We have $P = Z^n + \sum_{k=1}^n Z^{n-k} \cdot P_k(X_\bullet, Y_\bullet)$ for some $P_1, \dots, P_n \in R$ such that P_k is homogeneous of degree k for every $k = 1, \dots, n$ (for the standard $\mathbb{N}$-grading on R such that $X_i, Y_j \in \deg_1 R$ for every $i \in \Sigma$ and $j \in \Sigma'$).
- For every $k = 1, \dots, n$, denote by $P_k(0, Y_\bullet)$ the image of P_k by the evaluation map $R \to A[Y_j \mid j \in \Sigma']$ such that $X_i \mapsto 0$ for every $i \in \Sigma$; then $P_k(0, Y_\bullet) \in I[Y_i \mid j \in \Sigma']$.

Set as well

$$T := \{v \in \operatorname{Spv} B \mid v(a) \leq 1 \text{ and } v(b) < 1 \text{ for every } a \in A \text{ and } b \in I\}$$
$$T_0 := \{v \in T \mid \operatorname{Ker} v \text{ is a minimal prime ideal of } B\}.$$

Notice that T is a pro-constructible subset of $\operatorname{Spv} B$, and it contains all the primary generizations of its points.

Lemma 9.2.40. *In the situation of* (9.2.39)*, suppose moreover that I is contained in the Jacobson radical of A, and let also $(g_i \mid i \in \Sigma)$ and $(h_j \mid j \in \Sigma')$ be two systems of elements of B. Then for every $f \in B$ the following conditions are equivalent :*

(a) *For every $v \in T$ with $v(f) \neq 0$ there exists either $i \in \Sigma$ such that $v(f) \leq v(g_i)$ or else $j \in \Sigma'$ such that $v(f) < v(h_j)$.*
(b) *For every $v \in T_0$ with $v(f) \neq 0$ there exists either $i \in \Sigma$ such that $v(f) \leq v(g_i)$ or else $j \in \Sigma'$ such that $v(f) < v(h_j)$.*
(c) *There exists a polynomial $P \in \mathscr{P}_I$ such that $P(f, g_\bullet, h_\bullet) = 0$ in B.*

Proof. (c)⇒(a): Let $P \in \mathscr{P}_I$ such that (c) holds, and write $P = Z^n + \sum_{k=1}^n Z^{n-k} \cdot P_k(X_\bullet, Y_\bullet)$ for some $P_1, \dots, P_n \in R$. Fix $v \in T$; then there exists $k \leq n$ such that

$$v(f^n) \leq v(f^{n-k} \cdot P_k(g_\bullet, h_\bullet)).$$

If $v(f) = 0$, there is nothing to show; otherwise, we deduce that $v(f^k) \leq v(P_k(g_\bullet, h_\bullet))$. By assumption, P_k is homogeneous of degree k, so there exist two systems of non-negative integers $\nu_\bullet := (\nu_i \mid i \in \Sigma)$, $\mu_\bullet := (\mu_j \mid j \in \Sigma')$, and an element $a \in A$ such that the monomial $aX_\bullet^{\nu_\bullet}Y_\bullet^{\mu_\bullet}$ appears in P_k and

$$v(f^k) \leq v(a) \cdot v(g_\bullet^{\nu_\bullet}) \cdot v(h_\bullet^{\mu_\bullet}).$$

Especially, set $|\nu_\bullet| := \sum_{i\in\Sigma} \nu_i$, and define likewise $|\mu_\bullet|$; then $|\nu_\bullet| + |\mu_\bullet| = k$. Suppose first that $|\nu_\bullet| > 0$; if there exists $i \in I$ with $v(f) \leq v(g_i)$, we are done. Otherwise, we have $v(f^{|\nu_\bullet|}) > v(g_\bullet^{\nu_\bullet})$, and therefore $v(f^{|\mu_\bullet|}) < v(h_\bullet^{\mu_\bullet})$, since $v(a) \leq 1$. In this case, we easily get $v(f) < v(h_j)$ for some $j \in \Sigma'$. Lastly, if $|\nu_\bullet| = 0$, by assumption we must have $a \in I$, so that $v(a) < 1$, and therefore $v(f^k) < v(h_\bullet^{\mu_\bullet})$, so we conclude as in the previous case.

(a)⇒(b) is obvious.

(b)⇒(c): We suppose that (c) fails, and we exhibit $v \in T_0$ such that $v(f) \neq 0$ and $v(g_i) < v(f)$, $v(h_j) \leq v(f)$ for every $i \in \Sigma$ and $j \in \Sigma'$. To this aim, consider the subring

$$A' := A[g_i/f, h_j/f \mid i \in \Sigma,\ j \in \Sigma'] \subset B' := B[f^{-1}]$$

and the ideal J_0 of A' generated by the system $(g_i/f \mid i \in \Sigma)$; set $J := J_0 + IA'$.

Claim 9.2.41. If (c) fails, we have $J \neq A'$.

Proof of the claim. If $J = A'$, we may find an expression of the form

$$1 = \sum_{i\in\Sigma} \frac{g_i}{f} \cdot F_i\Big(\frac{g_\bullet}{f}, \frac{h_\bullet}{f}\Big) + \sum_{k=1}^{r} a_k \cdot G_k\Big(\frac{g_\bullet}{f}, \frac{h_\bullet}{f}\Big) \qquad \text{in } A'$$

for systems $(F_i(X_\bullet, Y_\bullet) \mid i \in \Sigma)$ and $(G_k(X_\bullet, Y_\bullet) \mid k = 1, \dots, r)$ of elements of R, with $F_i = 0$ except for finitely many values of $i \in \Sigma$, and a system $(a_1, \dots, a_r)$ of elements of I. We may then multiply both sides of this identity by f^{n+1} for a sufficiently large $n \in \mathbb{N}$, and obtain an identity in A of the form $P(f, g_\bullet, h_\bullet) = 0$, with

$$P(Z, X_\bullet, Y_\bullet) = Z^{n+1} - \sum_{i\in\Sigma} Z^n \cdot X_i \cdot F_i(X_\bullet/Z, Y_\bullet/Z) - \sum_{k=1}^{r} Z^{n+1} \cdot a_k \cdot G_k(X_\bullet/Z, Y_\bullet/Z).$$

A simple inspection shows that the coefficient of the monomial Z^{n+1} appearing in P is of the form $1 - a$ for some $a \in I$: namely, a is the sum of the terms a_k such that $G_k \in A$. However, $1 - a \in A^\times$, since I is contained in the Jacobson radical of A; to conclude the proof, it suffices to observe that $(1 - a)^{-1} \cdot P \in \mathscr{P}_I$: details left to the reader. ◊

By virtue of claim 9.2.41, we may find a maximal ideal $\mathfrak{m}$ of A' containing J, and we let $\mathfrak{p} \subset \mathfrak{m}$ be any minimal prime ideal of A'; then $A'_\mathfrak{p} \subset B'_\mathfrak{p}$ and we pick any minimal prime ideal $\mathfrak{q}$ of $B'_\mathfrak{p}$. Clearly $\mathfrak{q} \cap A'_\mathfrak{p} = \mathfrak{p}A'_\mathfrak{p}$, so the induced map $C := A'/\mathfrak{p} \to D := B'_\mathfrak{p}/\mathfrak{q}$ is still injective. Let $\overline{\mathfrak{m}} \subset C$ be the image of $\mathfrak{m}$; by corollary 9.1.25, we may find a valuation ring V of $\mathrm{Frac}\, D$ that dominates $C_{\overline{\mathfrak{m}}}$. The valuation ring V corresponds to an element $v \in T_0$ with the required properties. □

Proposition 9.2.42. *Let $i : A \to B$ be an injective ring homomorphism, J any ideal of A. Set*

$$\mathscr{V}_{B/A} := \mathrm{Spv}(i)^{-1}(\mathrm{Spv}^+ A) \qquad \mathscr{V}^0_{B/A} := \{v \in \mathscr{V}_{B/A} \mid \ker v \text{ is a minimal prime ideal of } B\}.$$

For every $v \in \mathscr{V}_{B/A}$, let also $\pi_v : B \to \kappa(v)$ be the natural map. Then we have :

$$\mathrm{i.c.}(J, B) = \bigcap_{v\in\mathscr{V}_{B/A}} \pi_v^{-1}(J \cdot \kappa(v)^+) = \bigcap_{v\in\mathscr{V}^0_{B/A}} \pi_v^{-1}(J \cdot \kappa(v)^+).$$

Proof. By applying lemma 9.2.40 with $I = 0$, $(g_i \mid i \in \Sigma)$ any (non-empty) set of generators for J, and $\Sigma' = \varnothing$, we see that $\mathrm{i.c.}(J, B)$ is the set of elements $f \in B$ such that the following holds. For every $v \in \mathscr{V}_{B/A}$ (respectively, for every $v \in \mathscr{V}^0_{B/A}$) there exists $i \in \Sigma$ with $v(f) \leq v(g_i)$. The proposition is an immediate consequence. □

Remark 9.2.43. For every ring homomorphism $f : A \to B$, every ideal $I \subset A$, and every $q \in \mathbb{Q}_+$, write $q = m/n$ for integers $m, n \in \mathbb{N}$, and let

$$\mathrm{i.c.}(I, B, q) := \{b \in B \mid b^n \in \mathrm{i.c.}(I^m, B)\}.$$

(i) Define $\mathscr{V}_{B/A} := \mathrm{Spv}(f)^{-1}(\mathrm{Spv}^+ A)$. We claim that $\mathrm{i.c.}(I, B, q)$ is the set of all $b \in B$ fulfilling the following condition. For every $v \in \mathscr{V}_{B/A}$ there exists $x \in I$ such that $v(b) \leq v(f(x))^q$ (see remark 9.1.2(ii)). Indeed, by proposition 9.2.42, we have :

$$\mathrm{i.c.}(I, B, q) = \{b \in B \mid \pi_v(b^n) \in I^m \cdot \kappa(v)^+ \text{ for every } v \in \mathscr{V}_{B/A}\}.$$

The latter condition can be restated as follows. For every $v \in \mathscr{V}_{B/A}$ there exists $x \in I^m \cdot \kappa(v)^+$ such that $v(b^n) \leq v(x)$. We may write such x as a sum $\sum_{i=1}^k f(x_i) \cdot y_i$ for certain $x_i \in I^m$, $y_i \in V$, and then $v(x) \leq v(f(x_i))$ for at least an index $i \leq r$. By the same token, we may assume that x_i is a product $a_1 a_2 \cdots a_m$ of elements of I, and then $v(x_i) \leq v(f(a_j)^m)$ for at least one index $j \leq m$.

(ii) Especially, (i) shows that $\mathrm{i.c.}(I, B, q)$ depends only on q (and not on the integers m, n). Notice as well that

(9.2.44) $$\mathrm{i.c.}(I^m, B) = \mathrm{i.c.}(I, B, m) \qquad \text{for every } m \in \mathbb{N}.$$

Moreover, a simple inspection of the definition reveals that $\mathrm{i.c.}(I, B, q) \subset \mathrm{i.c.}(A, B)$, and then the criterion of (i) also implies that $\mathrm{i.c.}(I, B, q)$ is an ideal of $\mathrm{i.c.}(A, B)$, for every $q \in \mathbb{Q}_+$. Furthermore, we have

(9.2.45) $$\mathrm{i.c.}(I, B, q) = \mathrm{i.c.}(\mathrm{i.c.}(I, A), B, q) \qquad \text{for every } q \in \mathbb{Q}_+.$$

Indeed, since $I \subset \mathrm{i.c.}(I,A)$, we have obviously $\mathrm{i.c.}(I,B,q) \subset \mathrm{i.c.}(\mathrm{i.c.}(I,A),B,q)$. Conversely, if $b \in \mathrm{i.c.}(\mathrm{i.c.}(I,A),B,q)$ and $v \in \mathscr{V}_{B/A}$ is any element, (i) says that there exists $x \in \mathrm{i.c.}(I,A)$ such that $v(b) \leq v(f(x))^q$; by the same token, there exists $y \in I$ such that $v(x) \leq v(y)$, so that $v(b) \leq v(f(y))^q$, which shows that $b \in \mathrm{i.c.}(I,B,q)$. Next, let us set

$$\overline{\mathrm{i.c.}}(I,B,r) := \bigcap_{\substack{q \in \mathbb{Q}_+ \\ q<r}} \mathrm{i.c.}(I,B,q) \qquad \text{for every } r \in \mathbb{R}_{>0}$$

and define as well $\overline{\mathrm{i.c.}}(I,B,0) := \mathrm{i.c.}(A,B)$. Denote also by $\mathscr{R}_{B/A}$ the set of all real valued valuations $|\cdot|$ on B that lie in $\mathscr{V}_{B/A}$. We have :

Lemma 9.2.46. *With the notation of remark* 9.2.43(ii)*, let $r \in \mathbb{R}_+$ be any real number, $b \in B$ any element, and denote by $A[b]$ the A-subalgebra of B generated by b. The following conditions are equivalent :*

(a) $b \in \overline{\mathrm{i.c.}}(I,B,r)$.

(b) $|b| \leq \sup_{x\in I} |f(x)|^r$ *for every element* $|\cdot|$ *of* $\mathscr{R}_{A[b]/A}$.

Proof. (a)⇒(b): The assumption means that $b \in \mathrm{i.c.}(I, A[b], q)$ for every non-negative rational number $q < r$. By remark 9.2.43(i), this implies that, for every $|\cdot|$ in $\mathscr{R}_{A[b]/A}$ and every such q, there exists $x \in I$ such that $|b| \leq |f(x)|^q$. Assertion (b) is an immediate consequence.

(b)⇒(a): Suppose first that $r = 0$. In this case, we have $|b| \leq 1$ for every $\mathscr{R}_{A[b]/A}$, and we need to show that $b \in \mathrm{i.c.}(A,B)$. Suppose, that the latter fails; then, by remark 9.2.43(i), there exists a valuation $|\cdot|$ in $\mathscr{V}_{A[b]/A}$ such that $|b| > 1$. It is easily seen that the characteristic subgroup of $|\cdot|$ equals $\Delta := O(|b|)$ (notation of (9.2.10) and remark 9.1.2(vii)), hence $|\cdot|^{\Delta}$ is still an element of $\mathscr{V}_{A[b]/A}$ and $|b|^{\Delta} > 1$ (lemma 9.2.11(ii)). We may therefore assume that the value group of $|\cdot|$ equals Δ, and $|\cdot|' := |\cdot|_{o(|b|)}$ is then a valuation of rank one (lemma 9.2.11(i) and remark 9.1.2(vii)). Lastly, $|\cdot|'$ is equivalent to an element of $\mathscr{R}_{A[b]/A}$, by remark 9.1.2(vi), and by construction we have $|b|' > 1$, a contradiction.

Next, suppose that $q > 0$, in which case – by remark 9.2.43(i) – it suffices to show that, for every $|\cdot|$ in $\mathscr{V}_{A[b]/A}$ and every non-negative rational number $q < r$, there exists $x \in I$ such that $|b| \leq |f(x)|^q$. However, by the foregoing case we know already that $|b| \leq 1$ for every such $|\cdot|$.

• If $|b| = 0$, there is nothing to prove, so suppose now that $0 < |b| < 1$, and set $\Delta := O(|b|)$, $\Delta' := o(|b|)$; notice also that the characteristic subgroup of $|\cdot|$ is $\{1\}$, hence $|\cdot|^{\Delta}$ lies still in $\mathscr{V}_{A[b]/A}$ (lemma 9.2.11(ii)) and $|b|^{\Delta} = |b|$. The same then holds for $|\cdot|' := |\cdot|^{\Delta}_{\Delta'}$ (lemma 9.2.11(i)), and as in the foregoing, we see that the latter valuation is equivalent to an element of $\mathscr{R}_{A[b]/A}$. Moreover, since $|b| \notin \Delta'$, we still have $|b|' < 1$. Assumption (b) then implies that $|b|' \leq \sup_{x\in I} |f(x)|'^r$. Explicitly, for every real number $\varepsilon > 0$ there exists $x_\varepsilon \in I$ such that $|f(x_\varepsilon)|'^r > |b|' - \varepsilon$. Then $|f(x_\varepsilon)|'^q > t_\varepsilon := (|b|' - \varepsilon)^{r/q}$, and since $q/r < 1$ and $|b|' < 1$, we may find ε small enough, so that $t_\varepsilon > |b|'$, and consequently $|f(x_\varepsilon)|'^q > |b|'$. It follows that $|f(x_\varepsilon)|^q > |b|$, as required.

• Lastly, suppose that $|b| = 1$. In this case, we must exhibit some $x \in I$ with $|f(x)| = 1$. Suppose that no such element exist, and let $\pi : A[b] \to \kappa := \kappa(|\cdot|)$ be the projection (notation of remark 9.1.4(v)); in the current situation, the image of π lies in the valuation ring κ^+ of the induced valuation $|\cdot|_\kappa$, and the assumption means that $\pi(I)$ lies in the maximal ideal $\mathfrak{m}$ of κ^+. Then, denote by $|\cdot|'_\kappa$ the trivial valuation on $\kappa^+/\mathfrak{m}$; the composition of $|\cdot|'_\kappa$ with the induced projection $A \to \kappa^+/\mathfrak{m}$ gives an element $|\cdot|'_A$ of $\mathscr{R}_{A[b]/A}$ such that $|b|'_A = 1$ and $|f(x)|'_A = 0$ for every $x \in I$, contradicting assumption (b). □

For future reference, we point out the following "approximation lemma" for integral closures:

Lemma 9.2.47. *Let A be any ring, $a_\bullet := (a_1, \dots, a_n)$ and $b_\bullet := (b_1, \dots, b_n)$ two sequences of elements of A, and $q > 1$ any rational number such that*

$$b_i - a_i \in \text{i.c.}(I, A, q) \qquad \textit{for every } i = 1, \dots, n.$$

Denote by $I \subset A$ (resp. $J \subset A$) the ideal generated by $a_\bullet$ (resp. by $b_\bullet$), and suppose that :

(a) *either, the radical of I equals the radical of J*
(b) *or else, I is contained in the Jacobson radical of A.*

Then $\text{i.c.}(I, A) = \text{i.c.}(J, A)$.

Proof. Since $\text{i.c.}(I, A, q) \subset \text{i.c.}(I, A)$, it is clear that $\text{i.c.}(J, A) \subset \text{i.c.}(I, A)$. To show the converse inclusion, set

$$r_v := \max(v(a_1), \dots, v(a_n)) \qquad s_v := \max(v(b_1), \dots, v(b_n)) \qquad \text{for every } v \in \operatorname{Spv} A^+.$$

By virtue of proposition 9.2.42, it suffices to check that $s_v \geq r_v$ for every such v. Now, suppose first that (a) holds, and say that $r_v = v(a_i)$ for some $i \leq n$; under our assumptions, remark 9.2.43(i) implies that $v(b_j - a_j) \leq r_v^q$ for every $j = 1, \dots, n$. If $r_v = 1$, the ideal I is not contained in the center of v (see remark 9.2.4(v)), and then the same holds for J, so that $s_v = 1$ as well. If $r_v = 0$, we get $v(b_j - a_j) = 0$ for every $j \leq n$, whence $s_v = 0$ as well. Lastly, if $0 < s_v < 1$, we get $s_v^q < s_v$, so that $v(a_i) = v(b_i)$, and thus $s_v \geq r_v$ again.

Next, suppose that (b) holds, and let $v \in \operatorname{Spv}^+ A$ be any valuation; again, we need to show that $s_v \geq r_v$, and by virtue of proposition 9.2.16(iii) we may assume that the center $\mathfrak{p}$ of v is a maximal ideal of A; especially $I \subset \mathfrak{p}$, in which case $r_v < 1$, and we argue as in the previous case, to conclude. □

We shall continue in section 9.3 the study of the class of ideals considered in remark 9.2.43. We shall be especially interested in the case where $A = B$ is a perfect topological $\mathbb{F}_p$-algebra: see lemma 9.3.74, which in turn shall be useful for our investigation of the finer topological properties of the ring $W(A)$ of Witt vectors over A.

9.3. **Witt vectors.** This reviews the ring of p-typical Witt vectors associated with an arbitrary ring; the basic reference is [34, Ch.IX, §1], though our treatment is self-contained and pays greater attention to the topological aspects of the theory.

Henceforth we fix a prime number p and we set $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$.

Definition 9.3.1. Let M be the free monoid with basis $\{X_i, Y_i \mid i \in \mathbb{N}\}$, and consider the morphism of monoids

$$\deg : M \to \mathbb{N}^{\oplus 2} \qquad \text{such that} \qquad \deg(X_i) := (p^i, 0) \quad \deg(Y_i) := (0, p^i) \qquad \text{for every } i \in \mathbb{N}.$$

Let $P \in \mathbb{Z}[M] = \mathbb{Z}[X_i, Y_i \mid i \in \mathbb{N}]$ be any polynomial; we can write uniquely $P = \sum_{i=1}^k n_i P_i$ for integers $n_1, \dots, n_k \in \mathbb{Z} \setminus \{0\}$ and distinct monomials $P_1, \dots, P_k \in M$, and we say that :

- P is *bihomogeneous of degree* $(a, b) \in \mathbb{N}^{\oplus 2}$, if $\deg(P_i) = (a, b)$ for every $i = 1, \dots, k$
- P is *homogeneous of total degree* $d \in \mathbb{N}$, if $\deg(P_i) = (a_i, b_i)$ with integers $a_i, b_i \in \mathbb{N}$ such that $a_i + b_i = d$, for every $i = 1, \dots, k$.

9.3.2. Following [34, Ch.IX, §1, n.1] one defines, for every $n \in \mathbb{N}$, the *n-th Witt polynomial*

$$\boldsymbol{\omega}_n(X_0, \dots, X_n) := \sum_{i=0}^{n} p^i X_i^{p^{n-i}} \in \mathbb{Z}[X_0, \dots, X_n].$$

Notice the inductive relations :

$$\begin{aligned} \boldsymbol{\omega}_{n+1}(X_0, \dots, X_{n+1}) &= \boldsymbol{\omega}_n(X_0^p, \dots, X_n^p) + p^{n+1} X_{n+1} \\ &= X_0^{p^{n+1}} + p \cdot \boldsymbol{\omega}_n(X_1, \dots, X_{n+1}). \end{aligned} \tag{9.3.3}$$

Let A be any ring; for every $n \in \mathbb{N}$, and every $\underline{a} := (a_n \mid n \in \mathbb{N}) \in A^{\mathbb{N}}$, the element

$$\boldsymbol{\omega}_n(\underline{a}) := \boldsymbol{\omega}_n(a_0, \dots, a_n) \in A$$

is called the *n-th ghost component* of $\underline{a}$, and we consider the *ghost map*

$$\boldsymbol{\omega}_A : A^{\mathbb{N}} \to A^{\mathbb{N}} \qquad \underline{a} \mapsto (\boldsymbol{\omega}_n(\underline{a}) \mid n \in \mathbb{N}).$$

Lemma 9.3.4. *Let A be a ring, $(J_n \mid n \in \mathbb{N})$ a sequence of ideals of A, such that*

$$pJ_n + J_n^p \subset J_{n+1} \subset J_n \qquad \textit{for every } n \in \mathbb{N}.$$

Let also $x, y \in A$ be any two elements, and $\underline{a}, \underline{b} \in A^{\mathbb{N}}$ any two sequences. The following holds :

(i) *If $x \equiv y \pmod{J_0}$, we have $x^{p^n} \equiv y^{p^n} \pmod{J_n}$ for every $n \in \mathbb{N}$.*

(ii) *If $a_i \equiv b_i \pmod{J_m}$ for every $i = 0, \dots, n$, then $\boldsymbol{\omega}_i(\underline{a}) \equiv \boldsymbol{\omega}_i(\underline{b}) \pmod{J_{m+i}}$ for every $i = 0, \dots, n$.*

Proof. (i) is left to the reader, and (ii) follows easily from (i). □

Proposition 9.3.5. *With the notation of* (9.3.2)*, we have :*

(i) *If p is a regular (resp. invertible) element in A, then $\boldsymbol{\omega}_A$ is injective (resp. bijective).*

(ii) *Suppose that A admits a ring endomorphism $\varphi : A \to A$ such that*

$$\varphi(a) \equiv a^p \pmod{pA} \qquad \textit{for every } a \in A.$$

Then $\operatorname{Im} \boldsymbol{\omega}_A = \{\underline{b} \in A^{\mathbb{N}} \mid \varphi(b_n) \equiv b_{n+1} \pmod{p^{n+1}A}$ for every $n \in \mathbb{N}\}$. Especially, $\operatorname{Im} \boldsymbol{\omega}_A$ is a subring of $A^{\mathbb{N}}$.

Proof. (i): Indeed, (9.3.3) shows that the condition $\boldsymbol{\omega}_n(\underline{a}) = \underline{b}$ is equivalent to the identities

$$b_0 = a_0 \qquad \text{and} \qquad b_n = \boldsymbol{\omega}_{n-1}(a_0^p, \dots, a_{n-1}^p) + p^n a_n \qquad \text{for every } n \geq 1$$

whence the assertion.

(ii): An easy induction argument reduces the assertion to the following :

Claim 9.3.6. Let $n > 0$ be any integer, and $a_0, \dots, a_{n-1}, b_n \in A$ any sequence of elements. Set $b_{n-1} := \boldsymbol{\omega}_{n-1}(a_0, \dots, a_{n-1})$. Then the following conditions are equivalent :

(a) There exists $a_n \in A$ such that $b_n = \boldsymbol{\omega}_n(a_0, \dots, a_n)$.

(b) $\varphi(b_{n-1}) \equiv b_n \pmod{p^n A}$.

Proof of the claim. We apply lemma 9.3.4(ii) to the filtration $(p^n A \mid n \in \mathbb{N})$, to deduce that

$$\begin{aligned}
\boldsymbol{\omega}_n(a_0, \dots, a_{n-1}, x) &\equiv \boldsymbol{\omega}_{n-1}(a_0^p, \dots, a_{n-1}^p) && \pmod{p^n A} \qquad \text{(by (9.3.3))} \\
&\equiv \boldsymbol{\omega}_{n-1}(\varphi(a_0), \dots, \varphi(a_{n-1})) && \pmod{p^n A} \\
&\equiv \varphi(b_{n-1}) && \pmod{p^n A}
\end{aligned}$$

for every $x \in A$. The claim is an immediate consequence. □

9.3.7. Take now $A := \mathbb{Z}[X_i, Y_i \mid i \in \mathbb{N}]$, and let $\varphi : A \to A$ be the endomorphism such that $\varphi(X_i) = X_i^p$ and $\varphi(Y_i) = Y_i^p$ for every $i \in \mathbb{N}$. Clearly φ fulfills the condition of proposition 9.3.5(ii); from proposition 9.3.5 and (9.3.3) we deduce that there exist polynomials

$$S_n, P_n \in \mathbb{Z}[X_0, \dots, X_n, Y_0, \dots, Y_n] \quad I_n \in \mathbb{Z}[X_0, \dots, X_n] \quad F_n \in \mathbb{Z}[X_0, \dots, X_{n+1}]$$

uniquely characterized, for every $n \in \mathbb{N}$, by the identities :

$$\begin{aligned}
\boldsymbol{\omega}_n(S_0, \dots, S_n) &= \boldsymbol{\omega}_n(X_0, \dots, X_n) + \boldsymbol{\omega}_n(Y_0, \dots, Y_n) \\
\boldsymbol{\omega}_n(P_0, \dots, P_n) &= \boldsymbol{\omega}_n(X_0, \dots, X_n) \cdot \boldsymbol{\omega}_n(Y_0, \dots, Y_n) \\
\boldsymbol{\omega}_n(I_0, \dots, I_n) &= -\boldsymbol{\omega}_n(X_0, \dots, X_n) \\
\boldsymbol{\omega}_n(F_0, \dots, F_n) &= \boldsymbol{\omega}_{n+1}(X_0, \dots, X_{n+1}).
\end{aligned}$$

Remark 9.3.8. (i) Notice that for every $n \in \mathbb{N}$, the polynomial ω_n is bihomogeneous of degree $(p^n, 0)$. By a simple induction argument, it follows easily that S_n is homogeneous of total degree p^n, and P_n (resp. I_n, resp. F_n) is bihomogeneous (resp. homogeneous) of degree (p^n, p^n) (resp. p^n, resp. p^{n+1}), for every $n \in \mathbb{N}$.

(ii) For example, a simple calculation yields the formulae :

$$\begin{aligned}
&S_0 = X_0 + Y_0 && S_1 = X_1 + Y_1 - \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} X_0^i Y_0^{p-i} \\
&P_0 = X_0 Y_0 && P_1 = X_0^p Y_1 + X_1 Y_0^p + pX_1 Y_1 \\
&I_0 = -X_0 && I_1 = -X_1 - \varepsilon \cdot X_0^p \\
&F_0 = X_0^p + pX_1 && F_1 = (1 - p^{p-1})X_1^p + pX_2 - \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} X_0^{pi}(pX_1)^{p-i}.
\end{aligned}$$

where $\varepsilon = 1$ if $p = 2$, and $\varepsilon = 0$ if $p > 2$. Also, since $\omega_n(0, \dots, 0, X_n) = p^n X$, we see that

$$(9.3.9)\qquad \begin{aligned} S_n(0,\dots,0,X_n,0,\dots,0,Y_n) &= X_n + Y_n \\ P_n(0,\dots,0,X_n,0,\dots,0,Y_n) &= p^n X_n Y_n \end{aligned} \qquad \text{for every } n \in \mathbb{N}.$$

(iii) A simple induction shows that :

$$(9.3.10)\qquad F_n \equiv X_n^p \pmod{p\mathbb{Z}[X_0,\dots,X_{n+1}]} \qquad \text{for every } n \in \mathbb{N}.$$

Indeed, for $n = 0$ the assertion is clear from (ii). Suppose that $n > 0$, and that the assertion is already known for every integer $< n$, and to ease notation, set $A := \mathbb{Z}[X_0, \dots, X_{n+1}]$ and $\boldsymbol{\omega}_n(F) := \boldsymbol{\omega}_n(F_0, \dots, F_n)$; from (9.3.3) we get

$$\boldsymbol{\omega}_n(F) \equiv \boldsymbol{\omega}_n(X_0^p, \dots, X_n^p) \equiv \boldsymbol{\omega}_{n-1}(X_0^{p^2}, \dots, X_{n-1}^{p^2}) + p^n X_n^p \pmod{p^{n+1}A}$$

On the other hand, the inductive assumption, together with lemma 9.3.4(i,ii) implies that

$$\boldsymbol{\omega}_n(F) \equiv \boldsymbol{\omega}_{n-1}(F_0^p, \dots, F_{n-1}^p) + p^n F_n \equiv \boldsymbol{\omega}_{n-1}(X_0^{p^2}, \dots, X_{n-1}^{p^2}) + p^n F_n \pmod{p^{n+1}A}$$

therefore $p^n F_n \equiv p^n X_n^p \pmod{p^{n+1}A}$, and the assertion follows.

(iv) Proposition 9.3.5(ii) seems to be due to B.Dwork and J.Dieudonné (see [121, §VII.4]). Together with (9.3.7), it yields a simplification of the original argument of E.Witt for the construction of the rings that bear his name, as we proceed now to explain.

9.3.11. Let now $(A, +, \cdot, \mathscr{T})$ be an arbitrary topological ring (see definition 8.3.1(i)). We endow $A^{\mathbb{N}}$ with the topology $\mathscr{T}_{W(A)}$ defined as the product of the topologies $\mathscr{T}$ on each copy of A, and for every $\underline{a} := (a_n \mid n \in \mathbb{N}), \underline{b} := (b_n \mid n \in \mathbb{N})$ in $A^{\mathbb{N}}$, we set :

$$\begin{aligned}
S_A(\underline{a}, \underline{b}) &:= (S_n(a_0, \dots, a_n, b_0, \dots, b_n) \mid n \in \mathbb{N}) \\
P_A(\underline{a}, \underline{b}) &:= (P_n(a_0, \dots, a_n, b_0, \dots, b_n) \mid n \in \mathbb{N}) \\
I_A(\underline{a}) &:= (I_n(a_0, \dots, a_n) \mid n \in \mathbb{N}).
\end{aligned}$$

There follow identities :

$$\begin{aligned}
\boldsymbol{\omega}_A(S_A(\underline{a}, \underline{b})) &= \boldsymbol{\omega}_A(\underline{a}) + \boldsymbol{\omega}_A(\underline{b}) \\
\boldsymbol{\omega}_A(P_A(\underline{a}, \underline{b})) &= \boldsymbol{\omega}_A(\underline{a}) \cdot \boldsymbol{\omega}_A(\underline{b}) \\
\boldsymbol{\omega}_A(I_A(\underline{a})) &= -\boldsymbol{\omega}_A(\underline{a})
\end{aligned}$$

for every $\underline{a}, \underline{b} \in A^{\mathbb{N}}$. Moreover, for any other topological ring B, and any continuous ring homomorphism $\psi : B \to A$, let $\psi^{\mathbb{N}} : B^{\mathbb{N}} \to A^{\mathbb{N}}$ be the induced continuous map; clearly we have the identities

$$(9.3.12)\qquad S_A \circ (\psi^{\mathbb{N}} \times \psi^{\mathbb{N}}) = \psi^{\mathbb{N}} \circ S_B \quad P_A \circ (\psi^{\mathbb{N}} \times \psi^{\mathbb{N}}) = \psi^{\mathbb{N}} \circ P_B \quad I_A \circ \psi^{\mathbb{N}} = \psi^{\mathbb{N}} \circ I_B.$$

Theorem 9.3.13. *With the notation of* (9.3.11)*, the set* $A^{\mathbb{N}}$*, endowed with the addition law* $S_A : A^{\mathbb{N}} \times A^{\mathbb{N}} \to A^{\mathbb{N}}$ *and product law* $P_A : A^{\mathbb{N}} \times A^{\mathbb{N}} \to A^{\mathbb{N}}$ *is a commutative topological ring, whose zero element and unit element are respectively the sequences*

$$\underline{0}_A := (0, 0, \dots) \qquad \textit{and} \qquad \underline{1}_A := (1, 0, 0, \dots).$$

Moreover, the opposite $-\underline{a}$ *(that is, with respect to the addition law* S_A*) of any* $\underline{a} \in A^{\mathbb{N}}$*, is the element* $I_A(\underline{a})$*.*

Proof. It is clear that S_A, P_A and I_A are continuous mapping for the topology $\mathscr{T}_{W(A)}$, hence it remains only to prove that $(A^{\mathbb{N}}, S_A, P_A)$ is a ring. Pick any surjective ring homomorphism $\psi : B \to A$, with $B = \mathbb{Z}[M]$ for some free monoid M, and endow B with the discrete topology; then $\psi^{\mathbb{N}}$ is also surjective, and in light of (9.3.12), it suffices to prove the theorem for B. Also, the p-Frobenius endomorphism of M induces an endomorphism φ of B that lifts the Frobenius endomorphism of B/pB. We may then replace A by B, and assume from start that p is a regular element of A, and $\varphi : A \to A$ is a given lift of the Frobenius endomorphism of A/pA. In this case, proposition 9.3.5(i,ii) implies that $\boldsymbol{\omega}_A$ maps bijectively $A^{\mathbb{N}}$ onto a subring of $A^{\mathbb{N}}$, and the identities of (9.3.7) tell us that the ring structure induced on $A^{\mathbb{N}}$ via this identification, is precisely the one given by the addition law S_A and multiplication law P_A, and also the zero element and the unit are the required ones, since $\boldsymbol{\omega}_A(\underline{0}_A) = 0 \in A^{\mathbb{N}}$ and $\boldsymbol{\omega}_A(\underline{1}_A) = 1 \in A^{\mathbb{N}}$. By the same token, it is clear that $-\underline{a}$ is computed by $I_A(\underline{a})$ in the resulting ring. □

Definition 9.3.14. Let $(A, \mathscr{T})$ be any topological ring. The topological ring

$$(A^{\mathbb{N}}, S_A, P_A, \underline{0}_A, \underline{1}_A, \mathscr{T}_{W(A)})$$

is called *the ring of Witt vectors associated with* A, and is denoted by $W(A, \mathscr{T})$. If no ambiguities are likely to arise, we often omit reference to $\mathscr{T}$, and write simply $W(A)$.

9.3.15. By construction, the ghost map is a continuous ring homomorphism

$$\boldsymbol{\omega}_A : W(A) \to A^{\mathbb{N}}$$

(where $A^{\mathbb{N}}$ is endowed with termwise addition and multiplication). Henceforth, the addition and multiplication of elements $\underline{a}, \underline{b} \in W(A)$ shall be denoted simply in the usual way : $\underline{a} + \underline{b}$ and $\underline{a} \cdot \underline{b}$, and the neutral element of addition and multiplication shall be denoted respectively 0 and 1. The rule $A \mapsto W(A)$ is a functor from the category of topological rings (and continuous ring homomorphisms) to itself; indeed, if $\psi : A \to B$ is any continuous ring homomorphism, the map $\psi^{\mathbb{N}}$ yields a continuous ring homomorphism

$$W(\psi) : W(A) \to W(B) \qquad \underline{a} \mapsto (\psi(a_n) \mid n \in \mathbb{N}).$$

and moreover one has the identity :

$$\psi^{\mathbb{N}} \circ \boldsymbol{\omega}_A = \boldsymbol{\omega}_B \circ W(\psi).$$

Remark 9.3.16. (i) More generally, if A is not necessarily unital ring, the proof of theorem 9.3.13 shows that the datum $(A^{\mathbb{N}}, S_A, P_A, \underline{0}_A)$ is a non-unital ring $W(A)$, that we shall also call the ring of Witt vectors associated with A. Then clearly the rule $A \mapsto W(A)$ extends to an endofunctor of the category of (associative, commutative) non-unital rings.

(ii) If A is a unital ring, and $f : A \to B, g : A \to C$ two unital A-algebras, the natural ring homomorphisms $B \to B \otimes_A C \leftarrow C$ induce a map of $W(A)$-algebras

$$WB \otimes_{WA} WC \to W(B \otimes_A C). \tag{9.3.17}$$

Namely, it is the unique $W(A)$-linear map such that $\underline{b} \otimes \underline{c} \mapsto (P_n(\underline{b} \otimes 1, 1 \otimes \underline{c}) \mid n \in \mathbb{N})$ for every $\underline{b} \in W(B)$ and $\underline{c} \in W(C)$, with $\underline{b} \otimes 1 := (b_n \otimes 1 \mid n \in \mathbb{N})$, and likewise for $1 \otimes \underline{c}$.

(iii) More generally, let A be a unital ring; a *non-unital* A*-algebra* is the datum of an A-module B and a (commutative, associative) non-unital ring structure on B such that $a \cdot (bb') =$

$(ab) \cdot b'$ for every $a \in A$ and every $b, b' \in B$. To every non-unital A-algebra B, we attach a unital A-algebra whose underlying A-module is $A \oplus B$, whose structure map $A \to A \oplus B$ is the natural inclusion, and with

$$(a, b) \cdot (a', b') := (aa', ab' + a'b + bb').$$

(iv) With this terminology, we may then extend (ii) to the case where B and C are non-unital A-algebras : in this case, the homomorphism (9.3.17) of non-unital rings is obtained as follows. Recall that the polynomial P_n is bihomogeneous of degree (p^n, p^n) for every $n \in \mathbb{N}$ (remark 9.3.8(i)); hence there exists $N \in \mathbb{N}$ such that $P_n = \sum_{i=1}^{N} r_i P_{n,i}$, with $r_i \in \mathbb{Z}$, and where each $P_{n,i}$ is a monomial of bidegree (p^n, p^n), *i.e.* there exists $M_i \in \mathbb{N}$ with $P_{n,i} = \prod_{j=0}^{M_i} X_j^{d_j} Y_j^{e_j}$ and $\sum_{j=0}^{M_i} d_j p^j = \sum_{j=0}^{M_i} e_j p^j = p^n$. Then we set $Q_n(\underline{b}, \underline{c}) := \sum_{i=1}^{N} r_i P_{n,i}(\underline{b}, \underline{c})$, with $P_{n,i}(\underline{b}, \underline{c}) := \prod_{j=0}^{M_i} b_j^{d_j} \otimes \prod_{j=0}^{M_i} c_j^{e_j}$ for every $i = 1, \dots, N$ and $\underline{b}, \underline{c}$ as in (ii); lastly, (9.3.17) is given by the rule : $\underline{b} \otimes \underline{c} \mapsto (Q_n(\underline{b}, \underline{c}) \mid n \in \mathbb{N})$. In order to check that (9.3.17) is a map of non-unital rings, endow $B' := A \oplus B$ and $C' := A \oplus C$ of the unital A-algebra structures described in (iii), and notice that (9.3.17) is the restriction of the map of unital $W(A)$-algebras $WB' \otimes_{WA} WC' \to W(B' \otimes_A C')$ of (ii).

9.3.18. Next, one defines two continuous mappings $W(A) \to W(A)$, by the rule :

$$F_A(\underline{a}) := (F_n(a_0, \dots, a_{n+1}) \mid n \in \mathbb{N})$$
$$V_A(\underline{a}) := (0, a_0, a_1, \dots).$$

F_A and V_A are often called, respectively, the *Frobenius* and *Verschiebung* maps ([34, Ch.IX, §1, n.5]). Consider also the mappings $f_A, v_A : A^{\mathbb{N}} \to A^{\mathbb{N}}$ such that

$$f_A(\underline{a}) := (a_1, a_2, \dots) \qquad v_A(\underline{a}) := (0, pa_0, pa_1, \dots) \qquad \text{for every } \underline{a} := (a_0, a_1, \dots) \in A^{\mathbb{N}}.$$

Then f_A is a continuous ring endomorphism of $A^{\mathbb{N}}$, and v_A is a continuous endomorphism of the additive group of $A^{\mathbb{N}}$, and the basic identity characterizing $(F_n \mid n \in \mathbb{N})$ can be written as :

$$\boldsymbol{\omega}_A \circ F_A = f_A \circ \boldsymbol{\omega}_A. \tag{9.3.19}$$

On the other hand, (9.3.3) yields the identity :

$$\boldsymbol{\omega}_A \circ V_A = v_A \circ \boldsymbol{\omega}_A. \tag{9.3.20}$$

Let $\psi : A \to B$ be a continuous ring homomorphism; directly from the definitions we obtain :

$$W(\psi) \circ F_A = F_B \circ W(\psi) \qquad W(\psi) \circ V_A = V_B \circ W(\psi). \tag{9.3.21}$$

Proposition 9.3.22. *With the notation of* (9.3.18)*, we have :*

(i) *The map F_A is a continuous endomorphism of the ring $W(A)$, and V_A is a continuous endomorphism of the additive group underlying $W(A)$.*

(ii) *Moreover one has the identities :*

$$\begin{aligned} F_A \circ V_A &= p \cdot \mathbf{1}_{W(A)} \\ V_A(\underline{a} \cdot F_A(\underline{b})) &= V_A(\underline{a}) \cdot \underline{b} \quad \textit{for every } \underline{a}, \underline{b} \in W(A). \end{aligned} \tag{9.3.23}$$

(iii) *F_A is a lifting of the Frobenius endomorphism of $W(A)/pW(A)$,* i.e. *:*

$$F_A(\underline{a}) \equiv \underline{a}^p \pmod{pW(A)} \qquad \textit{for every } \underline{a} \in W(A).$$

Proof. (i): Arguing as in the proof of theorem 9.3.13, we reduce to the case where p is a regular element in A and the Frobenius endomorphism of A/pA lifts to a ring endomorphism $\varphi : A \to A$, in which case $\boldsymbol{\omega}_A$ is an injective ring homomorphism (proposition 9.3.5(i)). Then the assertion follows immediately from (9.3.19) and (9.3.20). By the same token, we also get the identities of (ii). Likewise, (iii) is reduced to the identity

$$f_A(\underline{a}) - \underline{a}^p \in p \cdot \operatorname{Im} \boldsymbol{\omega}_A \qquad \text{for every } \underline{a} := (a_n \mid n \in \mathbb{N}) \in \operatorname{Im} \boldsymbol{\omega}_A$$

(here $\underline{a}^p = (a_n^p \mid n \in \mathbb{N})$ denotes the p-power map for the ring structure of $A^{\mathbb{N}}$). Now, we have

$$(9.3.24) \quad \varphi(a_n) \equiv a_n^p \pmod{pA} \qquad \varphi(a_n) \equiv a_{n+1} \pmod{p^{n+1}A} \qquad \text{for every } n \in \mathbb{N}$$

by proposition 9.3.5(ii), hence $a_n^p - a_{n+1} \in pA$ for every $n \in \mathbb{N}$. Set $b_n := p^{-1} \cdot (a_n^p - a_{n+1})$; again by proposition 9.3.5(ii), it suffices to check that $\varphi(b_n) \equiv b_{n+1} \pmod{p^{n+1}A}$, that is

$$\varphi(a_n^p - a_{n+1}) \equiv a_{n+1}^p - a_{n+2} \pmod{p^{n+2}A} \qquad \text{for every } n \in \mathbb{N}.$$

However, from (9.3.24) and lemma 9.3.4(i) we deduce that $\varphi(a_n)^p \equiv a_{n+1}^p \pmod{p^{n+2}A}$, whence the contention : details left to the reader. □

9.3.25. Let $n \in \mathbb{N}$ be any integer; it follows easily from (9.3.23) that

$$V_n(A) := \mathrm{Im}\, V_A^n$$

is an ideal of $W(A)$, and we define the ring of *n-truncated Witt vectors* of A as the quotient

$$W_n(A) := W(A)/V_n(A)$$

which we endow with the topology $\mathscr{T}_{W_n(A)}$ induced by the projection $\pi_n : W(A) \to W_n(A)$.

Remark 9.3.26. (i) More generally, if A is a not necessarily unital ring, then $V_n(A)$ is a well-defined ideal of the non-unital ring $W(A)$ (see remark 9.3.16(i)). Hence, the quotient $W_n(A)$ is a well-defined non-unital ring in this case, for every $n \in \mathbb{N}$.

(ii) Likewise, in the situation of remark 9.3.16(iii), we have a well-defined homomorphism of non-unital A-algebras $W_nB \otimes_{W_nA} WC \to W_n(B \otimes_A C)$, for every $n \in \mathbb{N}$.

Lemma 9.3.27. *With the notation of* (9.3.25)*, we have :*

(i) $\underline{a} = (a_0, \dots, a_{m-1}, 0, \dots) + V_A^m \circ f_A^m(\underline{a})$ *for every* $\underline{a} \in W(A)$ *(where the addition is taken in the additive group of* $W(A)$*).*

(ii) *The projection* $W(A) \to A^n \;:\; (a_i \mid i \in \mathbb{N}) \mapsto (a_0, \dots, a_{n-1})$ *factors through* π_n *and a bijection* $W_n(A) \stackrel{\sim}{\to} A^n$*, and* $\mathscr{T}_{W_n(A)}$ *corresponds to the product topology of* A^n*, under this identification.*

Proof. (i): Arguing as in the proof of theorem 9.3.13, we may reduce to the case where p is a regular element in A, in which case $\boldsymbol{\omega}_A$ is an injective ring homomorphism, and therefore it suffices to show that

$$\boldsymbol{\omega}_A(\underline{a}) = \boldsymbol{\omega}_A(a_0, \dots, a_{m-1}, 0, \dots) + \boldsymbol{\omega}_A \circ V_A^m \circ f_A^m(\underline{a}).$$

The latter follows by a simple inspection : details left to the reader. Assertion (ii) follows easily from (i). □

Remark 9.3.28. (i) In view of lemma 9.3.27(ii), we get therefore an inverse system of topological rings $(W_n(A) \mid n \in \mathbb{N})$, whose limit is $W(A)$. Clearly the ghost components descend to well-defined continuous ring homomorphisms :

$$\overline{\boldsymbol{\omega}}_m : W_n(A) \to A \qquad \text{for every } m < n.$$

Especially, the map $\overline{\boldsymbol{\omega}}_0 : W_1(A) \to A$ is an isomorphism of topological rings.

(ii) A direct inspection of the construction shows that the endofunctor W of the category of topological rings commutes with all limits.

(iii) In the same vein, if the topological ring A is the limit of a system $(A_i \mid i \in I)$ of topological rings, then F_A is the limit of the corresponding system of map $(F_{A_i} \mid i \in I)$, and likewise for V_A.

(iv) Let A be any topological ring, I any ideal of the monoid $(A, \cdot)$, and $r \in \mathbb{N}$ any integer. In light of remark 9.3.8(i), it is easily seen that the sets

$$W(I, r) := \{(a_n \mid n \in \mathbb{N}) \in W(A) \mid a_n \in I^{p^n}A \text{ for every } n \leq r\} \quad \text{and} \quad W(I) := \bigcap_{n \in \mathbb{N}} W(I, n)$$

are ideals of $W(A)$: details left to the reader. Moreover, we have the inclusions :

(9.3.29) $$W(I^c, r) \subset W(I, r)^c \quad \text{and} \quad W(I^c) \subset W(I)^c$$

(where, for any topological space X and every subset $S \subset X$ we denote by S^c the topological closure of S in X). Indeed, by definition we have $W(I, r) := \prod_{n\in\mathbb{N}} J_n$, with $J_n := I^{p^n}A$ for every $n \leq r$ and $J_n := A$ for $n > r$, so that $W(I, r)^c = \prod_{n\in\mathbb{N}} J_n^c$, and it suffices to recall that $(I^c)^{p^n} \subset (I^{p^n})^c$ for every $n \in \mathbb{N}$. The same argument applies to $W(I)$.

Especially, if $I^{p^n}A$ is a closed ideal in the topology of A for every $n \in \mathbb{N}$, then it follows that $W(I, r)$ is a closed ideal of $W(A)$, and the same holds also for $W(I)$.

Furthermore, if I, J are any two ideals of the monoid $(A, \cdot)$, and $r \in \mathbb{N}$ is any integer, it follows easily from remark 9.3.8(i) that

$$W(I, r) \cdot W(J, r) \subset W(IJ, r) \qquad \text{and} \qquad W(I) \cdot W(J) \subset W(IJ).$$

We warn the reader that the notation $W(I)$ might also refer to the non-unital Witt ring associated to the non-unital ring I, as defined in remark 9.3.26; these two meanings of $W(I)$ agree if and only if $I^2 = I$. However, in practice the ambiguity should be harmless, since non-unital Witt rings only intervene once in our text, in the statement and proof of proposition 14.7.18.

Proposition 9.3.30. *Let A be a topological ring, and $n \in \mathbb{N}$. We have :*

(i) *With the notation of remark* 9.3.28(i)*, the ring homomorphism*

$$\pi_n : W_{n+1}(A) \to A^{n+1} \qquad \underline{a} \mapsto (\overline{\boldsymbol{\omega}}_0(\underline{a}), \overline{\boldsymbol{\omega}}_1(\underline{a}), \dots, \overline{\boldsymbol{\omega}}_n(\underline{a}))$$

is integral with nilpotent kernel.

(ii) $p^n A^{n+1} \subset \operatorname{Im} \pi_n \subset \{(x_0, \dots, x_n) \in A^{n+1} \mid x_i \equiv x_0^{p^i} \pmod{pA}$ *for* $i = 1, \dots, n\}$.

Proof. (i): Denote by $B \subset A^{n+1}$ the integral closure of the image of π_n. Clearly B contains the $n+1$ canonical idempotents $e_0, \dots, e_n$ of A^{n+1}. Next, a simple inspection shows that

$$\pi_n(ae_0) = (a, a^p, \cdots, a^{p^n}) \qquad \text{for every } a \in A.$$

It follows that $a^{p^i}e_i = e_i \cdot \pi_n(ae_0) \in B$, and then also $ae_i \in B$, for every $a \in A$ and every $i \leq n$. This implies that $B = A^{n+1}$, *i.e.* π_n is integral, as asserted. Next, the sequence of ideals $(V_i(A) \mid i \in \mathbb{N})$ induces a descending filtration $(\overline{V}_i \mid i = 0, \dots, n)$ on $W_{n+1}(A)$, and we notice:

Claim 9.3.31. For every $i = 0, \dots, n-1$, we have an isomorphism of $W_{n+1}(A)$-modules

$$A \xrightarrow{\sim} \overline{V}_i/\overline{V}_{i+1} \qquad a \mapsto ae_i \pmod{\overline{V}_{i+1}}$$

for the $W_{n+1}(A)$-module structure on A induced by $\overline{\boldsymbol{\omega}}_i$.

Proof of the claim. It is clear that this map is bijective. To show that it is also $W_{n+1}(A)$-linear, notice that $P_j(X_0, \dots, X_j, 0, \dots, 0) = 0$ for every $j < i$ (remark 9.3.8(i)), so that

$$\boldsymbol{\omega}_i(0, \dots, 0, P_i(X_0, \dots, X_i, 0, \dots, 0, Y_i)) = \boldsymbol{\omega}_i(X_0, \dots, X_i) \cdot \boldsymbol{\omega}_i(0, \dots, 0, Y_i)$$

by (9.3.7). This translates as the identity

$$p^i \cdot P_i(X_0, \dots, X_i, 0, \dots, 0, Y_i) = \boldsymbol{\omega}_i(X_0, \dots, X_i) \cdot p^i Y_i$$

whence $P_i(X_0, \dots, X_i, 0, \dots, 0, Y_i) = \boldsymbol{\omega}_i(X_0, \dots, X_i) \cdot Y_i$, which implies the claim. ◊

Set $I := \operatorname{Ker} \pi_n$; from claim 9.3.31 we deduce that $I \cdot \overline{V}_i \subset \overline{V}_{i+1}$ for every $i = 0, \dots, n$, whence $I^{n+1} = 0$ in $W_{n+1}(A)$, which concludes the proof of (i).

(ii): The second inclusion follows by direct inspection of the Witt polynomials. To show the first inclusion, consider any sequence $(a_0, \dots, a_n) \in A^{n+1}$; we need to find a solution $(x_0, \dots, x_n) \in W_{n+1}$ for the system of polynomial equations :

$$x_0^{p^i} + px_1^{p^{i-1}} + \cdots + p^i x_i = p^n a_i \qquad \text{for } i = 0, \dots, n$$

and we claim that there exists a solution that fulfills as well the further condition : $x_i \in p^{n-i}A$ for $i = 0, \dots, n$. The proof proceeds by a simple induction on $i \leq n$, which we leave to the reader. □

Corollary 9.3.32. *Let A be any topological ring such that $p^k A = 0$ for some $k \in \mathbb{N}$. Then* $\mathrm{Ker}\,(\overline{\omega}_0 : W_{n+1}A \to A)$ *is nilpotent for every $n \in \mathbb{N}$.*

Proof. Let $\underline{a} \in W_{n+1}A$ be an element such that $\overline{\omega}_0(\underline{a}) = 0$, and define π_n as in proposition 9.3.30(i); in view of (9.3.3), we see that $\pi_n(\underline{a}) = (0, pb_1, \dots, pb_n)$ for some $b_1, \dots, b_n \in A$; hence $\underline{a}^k \in \mathrm{Ker}\,\pi_n$, and it suffices to invoke proposition 9.3.30(i) to conclude. □

Lemma 9.3.33. *Let $(A, \mathscr{T})$ be a topological ring, with separated completion $(A^\wedge, \mathscr{T}^\wedge)$.*

(i) *If the topology $\mathscr{T}$ is discrete, the topology $\mathscr{T}_{W(A)}$ agrees with the linear topology defined by the filtration $(V_m(A) \mid m \in \mathbb{N})$.*
(ii) *The topology $\mathscr{T}$ is separated (resp. complete) if and only if the same holds for $\mathscr{T}_{W(A)}$.*
(iii) *If $\mathscr{T}$ is a linear topology, then the same holds for $\mathscr{T}_{W(A)}$.*
(iv) *Denote also by $W(A, \mathscr{T})^\wedge$ the separated completion of $W(A, \mathscr{T})$. The natural map $W(A, \mathscr{T}) \to W(A^\wedge, \mathscr{T}^\wedge)$ factors uniquely via an isomorphism of topological rings*

$$W(A, \mathscr{T})^\wedge \xrightarrow{\sim} W(A^\wedge, \mathscr{T}^\wedge).$$

Proof. (i) follows easily from lemma 9.3.27(ii) : details left to the reader.

(iii): Let $(I_\lambda \mid \lambda \in \Lambda)$ be a fundamental system of open ideals of A, and for every $\lambda \in \Lambda$ and every $r \in \mathbb{N}$, denote by $\pi_{\lambda,r} : W(A) \to W_r(A/I_\lambda)$ the projection; then it is easily seen that the family of ideals $(\mathrm{Ker}\,\pi_{\lambda,r} \mid \lambda \in \Lambda,\ r \in \mathbb{N})$ is a fundamental system of open neighborhoods of $0 \in W(A)$. The assertion is an immediate consequence.

(iv) In light of corollary 8.2.16(ii), we are reduced to checking that $W_n(A^\wedge)$ is the separated completion of $W_n(A)$. The latter is established by induction on $n \in \mathbb{N}$, by applying proposition 8.2.13(i,v) to the exact sequences of topological groups :

$$0 \to V_n(A)/V_{n+1}(A) \to W_{n+1}(A) \to W_n(A) \to 0$$

where $V_n(A)/V_{n+1}(A)$ is endowed with the topology induced by the inclusion into $W_{n+1}(A)$, and is thus isomorphic to $(A, \mathscr{T})$, by virtue of claim 9.3.31.

(ii): $\mathscr{T}$ is separated (resp. complete) if and only if the completion map $A \to A^\wedge$ is injective (resp. surjective), if and only if the induced map $W(A, \mathscr{T}) \to W(A^\wedge, \mathscr{T}^\wedge)$ is injective (resp. surjective), and by virtue of (iv), the latter holds if and only if the completion map $W(A, \mathscr{T}) \to W(A, \mathscr{T})^\wedge$ is injective (resp. surjective), whence the assertion. □

9.3.34. *Teichmüller mapping.* The projection $\boldsymbol{\omega}_0$ admits a continuous set-theoretic section

$$\tau_A : A \to W(A) \qquad a \mapsto (a, 0, 0, \dots).$$

For every $a \in A$, we call $\tau_A(a)$ the *Teichmüller representative* of a. Clearly, for any continuous ring homomorphism $\psi : A \to B$ we have :

$$\tau_B \circ \psi = W(\psi) \circ \tau_A. \tag{9.3.35}$$

Proposition 9.3.36. *With the notation of* (9.3.34)*, we have :*

(i) $\tau_A(a) \cdot \underline{b} = (a^{p^n} b_n \mid n \in \mathbb{N})$ *for every $a \in A$ and $\underline{b} := (b_n \mid n \in \mathbb{N}) \in W(A)$.*
(ii) *Moreover, τ_A is a multiplicative map,* i.e. *the following identity holds :*

$$\tau_A(a) \cdot \tau_A(b) = \tau_A(a \cdot b) \qquad \text{for every } a, b \in A$$

(iii) $\underline{a} = \sum_{n=0}^{\infty} V_A^n(\tau_A(a_n))$ *for every $\underline{a} := (a_n \mid n \in \mathbb{N}) \in W(A)$, where the convergence of the series is relative to the topology $\mathscr{T}_{W(A)}$.*

Proof. (i): Since P_n is bihomogeneous of degree (p^n, p^n) (remark 9.3.8(i)), we have

$$P_n(X_0, 0, \dots, 0, Y_0, \dots, Y_n) = X_0^{p^n} \cdot P_n(1, 0, \dots, 0, Y_0, \dots, Y_n) = X_0^{p^n} \cdot Y_n$$

where the second identity follows after recalling that $\underline{1}_A := (1, 0, \dots)$ is the unit of $W(A)$. The assertion follows immediately. (ii) is a special case of (i), and (iii) follows easily from lemma 9.3.27(i). □

Example 9.3.37. (i) For any $n \in \mathbb{N}$, let $(a_0, \dots, a_n) \in W_{n+1}(\mathbb{Z})$ be the sequence that represents the element p^n. Since $\overline{\omega}_i$ is a ring homomorphism, we have

$$p^n = a_0^{p^i} + pa_1^{p^{i-1}} + \cdots + p^i a_i \qquad \text{for every } i = 0, \dots, n.$$

A simple induction then shows that

$$a_i \equiv p^{n-i} \pmod{p^{n-i+1}} \qquad \text{for every } i = 0, \dots, n.$$

(ii) Let A be any topological ring, $n, i \in \mathbb{N}$ any integers; set

$$k := \min(i, n) \qquad \text{and} \qquad \overline{V}_{n,j}(A) := V_j(A)/V_{n+1}(A) \qquad \text{for every } j \leq n.$$

Let also $\overline{V} : W_n(A) \to W_{n+1}(A)$ be the $\mathbb{Z}$-linear map induced by V. We claim that

$$\sum_{j=0}^{k} p^{i-j}\overline{V}_{n,j}(A) = J_{i,n} := \{(a_0, \dots, a_n) \in W_{n+1}(A) \mid a_j \in p^{i-j}A \text{ for every } j \leq k\}.$$

For the proof, we argue by induction on n and i, and notice first that $J_{i,n}$ is an ideal of $W_{n+1}(A)$ for every n and i, due to remark 9.3.8(i). If $n = 0$ or $i = 0$, there is nothing to prove. Suppose then that $n, i > 0$ and that the assertion is known for all strictly smaller values of n and i; from (i) we see that $p^i W_{n+1}(A) \subset J_{i,n}$, and the inductive assumption shows that

$$\sum_{j=1}^{k} p^{i-j}\overline{V}_j(A) = \overline{V}\Big(\sum_{j=0}^{k-1} p^{i-j-1}\overline{V}_{n-1,j}(A)\Big) = \overline{V}(J_{i-1,n-1}) \subset J_{i,n}$$

whence $\sum_{j=0}^{k} p^{i-j}\overline{V}_{n,j}(A) \subset J_{i,n}$. Conversely, say that $\underline{a} := (a_0, \dots, a_n) \in J_{i,n}$, so that $a_0 = p^i b_0$ for some $b_0 \in A$; due to lemma 9.3.27(i) and proposition 9.3.36(i) we may write

$$\underline{a} = \tau(a_0) + \overline{V}(a_1, \dots, a_n) = \tau(b_0) \cdot (p^i, 0, \dots, 0) + \overline{V}(a_1, \dots, a_n)$$

and notice that $\overline{V}(a_1, \dots, a_n) \in \overline{V}(J_{i-1,n-1}) \subset \sum_{j=1}^{k} p^{i-j}\overline{V}_{j,n}(A)$. Also, from (i) we see that $(p^i, 0, \dots, 0) = p^i + \overline{V}(b_1, \dots, b_n)$, where again $(b_1, \dots, b_n) \in J_{i-1,n-1}$. Summing up, we get $J_{i,n} \subset \sum_{j=0}^{k} p^{i-j}\overline{V}_{j,n}(A)$, as required.

(iii) Especially, (ii) implies that for every ring A we have

$$p^i W_{n+1}(A) \subset J_{i,n} \subset p^{i-n} W_{n+1}(A) \qquad \text{for every } i, n \in \mathbb{N} \text{ with } i \geq n.$$

If $\mathscr{T}_p$ denotes the p-adic topology on A, it follows that the topology of $W_{n+1}(A, \mathscr{T}_p)$ agrees with the p-adic topology of the underlying ring $W_{n+1}(A)$, for every $n \in \mathbb{N}$.

Example 9.3.38. (i) Let $S \subset \mathbb{Z}$ be any subset, and A any ring (which we may regard as a discrete topological ring). Then the localization map $j : A \to S^{-1}A$ induces a ring isomorphism

$$S^{-1}W_{n+1}(A) \xrightarrow{\sim} W_{n+1}(S^{-1}A) \qquad \text{for every } n \in \mathbb{N}.$$

Indeed, let us show first that multiplication by every $s \in S$ is a bijection on $W_{n+1}(S^{-1}A)$. To this aim, we consider the descending filtration $(\overline{V}_i A \mid i = 0, \dots, n)$ as in the proof of proposition 9.3.30(i); we are then easily reduced to checking that multiplication by s is bijective on $\overline{V}_i(A)/\overline{V}_{i+1}(A)$ for $i = 0, \dots, n-1$. But according to claim 9.3.31, the latter quotient is isomorphic to $S^{-1}A$, for the $W_{n+1}(S^{-1}A)$-module structure induced on $S^{-1}A$ by $\overline{\omega}_i$, whence the contention. Thus, $W_{n+1}(j)$ extends to a well defined ring homomorphism $\lambda : S^{-1}W_{n+1}A \to$

$W_{n+1}(S^{-1}A)$. To see that the latter is an isomorphism, we notice that λ is a homomorphism of filtered rings for the filtrations $S^{-1}\overline{V}_\bullet A$ on $S^{-1}W_{n+1}A$ and $\overline{V}_\bullet(S^{-1}A)$ on $W_{n+1}(S^{-1}A)$; then it suffices to check that the associated graded ring homomorphism $\mathrm{gr}_\bullet\lambda$ is bijective, and the latter assertion follows again from claim 9.3.31.

(ii) If $p \in A^\times$, a simple inspection shows that ghost map $\boldsymbol{\omega}_A : W(A) \to A^{\mathbb{N}}$ is a ring homomorphism, and likewise, $W_{n+1}A \xrightarrow{\sim} A^{n+1}$ in this case, for every $n \in \mathbb{N}$. Combining with (i), we deduce that for every ring A, the localization $A \to A[p^{-1}]$ induces a ring isomorphism

$$W_{n+1}(A)[p^{-1}] \xrightarrow{\sim} (A[p^{-1}])^{n+1} \qquad \text{for every } n \in \mathbb{N}.$$

(iii) In the same vein, let $S \subset A$ be any multiplicative subset, and denote by $\tau_A(S) \subset W_{n+1}A$ the image of the subset $\{\tau_A(s) \mid s \in S\} \subset W(A)$ (notation of (9.3.34)). Then the localization map $j : A \to S^{-1}A$ induces a ring isomorphism

$$\lambda : \tau_A(S)^{-1}W_{n+1}(A) \xrightarrow{\sim} W_{n+1}(S^{-1}A) \qquad \text{for every } n \in \mathbb{N}.$$

Indeed, from proposition 9.3.36(ii) we see that the image of $\tau_A(S)$ is invertible in $W_{n+1}(S^{-1}A)$, so λ is well defined. Next, proposition 9.3.36(i) easily implies that λ is surjective. Lastly, the kernel of λ is $\tau_A(S)^{-1}\mathrm{Ker}\, W_{n+1}(j)$; but clearly $\mathrm{Ker}\, W_{n+1}(j)$ is the ideal of all sequences $\underline{a} := (a_0, \dots, a_n)$ such that for every $i = 0, \dots, n$ there exists $s_i \in S$ with $s_i a_i = 0$. Then $\tau_A(s_0 \cdots s_n) \cdot \underline{a} = 0$ in $W_{n+1}A$; this shows that $\mathrm{Ker}\,\lambda = 0$, and concludes the proof.

9.3.39. *Let now A be a topological $\mathbb{F}_p$-algebra.* In this case (9.3.10) yields the identity :

$$F_A(\underline{a}) = (a_n^p \mid n \in \mathbb{N}) \qquad \text{for every } \underline{a} := (a_n \mid n \in \mathbb{N}) \in W(A). \tag{9.3.40}$$

As an immediate consequence we get :

$$p \cdot \underline{a} = V_A \circ F_A(\underline{a}) = F_A \circ V_A(\underline{a}) = (0, a_0^p, a_1^p, \dots) \qquad \text{for every } \underline{a} \in W(A). \tag{9.3.41}$$

Especially, we have

$$p = (0, 1, 0, \dots) \qquad \text{in } W(A). \tag{9.3.42}$$

Moreover, let $\Phi_A : A \to A$ be the Frobenius endomorphism; (9.3.40) also implies the identity :

$$F_A \circ \tau_A = \tau_A \circ \Phi_A. \tag{9.3.43}$$

Proposition 9.3.44. *Let A be any $\mathbb{F}_p$-algebra. We have :*

- (i) *The p-adic topology on $W(A)$ agrees with the $V_1(A)$-adic topology, and both are finer than the topology $\mathscr{T}'$ defined by the filtration $(V_n(A) \mid n \in \mathbb{N})$.*
- (ii) *Moreover, $W(A)$ is also complete and separated for the p-adic topology.*
- (iii) $W(A)^\times = \boldsymbol{\omega}_0^{-1}(A^\times)$.
- (iv) $V^i(\tau_A(b)) \cdot V^j(\tau_A(b)) = V^{i+j}(\tau_A(a^{p^j}b^{p^i}))$ *for every $a, b \in A$ and every $i, j \in \mathbb{N}$.*

Proof. (i): It is clear from (9.3.41) that the p-adic topology on $W(A)$ is finer than both $\mathscr{T}'$ and the V_1-adic topology. Conversely, for every $\underline{a}, \underline{b} \in W(A)$ we have

$$V_A(\underline{a}) \cdot V_A(\underline{b}) = V_A(\underline{a} \cdot F_A V_A(\underline{b})) = p \cdot V_A(\underline{a} \cdot \underline{b})$$

whence $V_1^2 \subset pA$, which says that the V_1-adic topology is finer than the p-adic topology.

(ii): Since $W(A)$ is complete and separated for the linear topology $\mathscr{T}'$, the assertion follows from lemma 8.3.12.

(iii): Clearly $W(A)^\times \subset \boldsymbol{\omega}_0^{-1}(A^\times)$. For the converse, notice that the ideal $V_1(A)$ is contained in the Jacobson radical ideal of $W(A)$, due to (i),(ii) and remark 8.3.10(v). Therefore, the induced map $\mathrm{Spec}\,\boldsymbol{\omega}_0 : \mathrm{Spec}\, A \to \mathrm{Spec}\, W(A)$ restricts to a bijection on the sets of maximal ideals of A and $W(A)$, and the assertion follows easily.

(iv): Without loss of generality, we may assume that $i \leq j$; then we compute

$$\begin{aligned} V^i(\tau_A(b)) \cdot V^j(\tau_A(b)) &= V^i(\tau_A(a) \cdot F^i V^j(\tau_A(b))) && \text{(by (9.3.23))} \\ &= V^i(\tau_A(a) \cdot p^i \cdot V^{j-i}(\tau_A(b))) && \text{(again by (9.3.23))} \\ &= p^i \cdot V^i(\tau_A(a) \cdot V^{j-i}(\tau_A(b))) && \text{(by proposition 9.3.22(i))} \\ &= p^i \cdot V^i(V^{j-i}(\tau_A(a^{p^{j-i}} b))) && \text{(by proposition 9.3.36(i))} \\ &= p^i \cdot V^j(\tau_A(a^{p^{j-i}} b)) && \text{(by (9.3.41))} \\ &= V^{i+j}(\tau_A(a^{p^j} b^{p^i})) \end{aligned}$$

as stated. □

9.3.45. Suppose additionally that A is *perfect*, *i.e.* that Φ_A is an automorphism of the topological ring A (especially, it is a homeomorphism). It follows from (9.3.40) that F_A is an automorphism of $W(A)$. Hence, in view of (9.3.23) and (9.3.41) :

$$p^n \cdot W(A) = V_n(A) \qquad \text{for every } n \in \mathbb{N}.$$

So, the p-adic filtration coincides with the filtration $(V_n(A) \mid n \in \mathbb{N})$ and with the $V_1(A)$-adic filtration. Especially, the 0-th ghost component descends to an isomorphism

$$\overline{\boldsymbol{\omega}}_0 : W(A)/pW(A) \xrightarrow{\sim} A.$$

Also, the identity of proposition 9.3.36(iii) can be written in the form :

$$\text{(9.3.46)} \qquad \underline{a} = \sum_{n=0}^{\infty} p^n \cdot \tau_A(a_n^{p^{-n}}) \qquad \text{for every } \underline{a} := (a_n \mid n \in \mathbb{N}) \in W(A).$$

Proposition 9.3.47. *Let A be a reduced $\mathbb{F}_p$-algebra, $\underline{a} := (a_n \mid n \in \mathbb{N}) \in W(A)$ any element.*

(i) *Consider the following conditions :*
 (a) *The ideal $J := \sum_{n \in \mathbb{N}} a_n A$ is not contained in any minimal prime ideal of A.*
 (b) *$\underline{a}$ is a regular element of $W(A)$.*
 Then (a)⇒(b)*, and if A has only finitely many minimal prime ideals,* (b)⇒(a).

(ii) *$W(A)$ is reduced, and if A is a domain, the same holds for $W(A)$.*

(iii) *If A is a perfect field, then $W(A)$ is a complete discrete valuation ring of mixed characteristic $(0, p)$, maximal ideal $pW(A)$, and residue field A.*

(iv) *If $A \subset B$ is an inclusion of perfect $\mathbb{F}_p$-algebras, then $W(A) \subset W(B)$, and*

$$p^n W(A) = W(A) \cap p^n W(B) \qquad \textit{for every } n \in \mathbb{N}.$$

(v) *If A is perfect, and $\sum_{i=0}^{k} a_i A = A$ for some $k \in \mathbb{N}$, we have*

$$\underline{a} W(A) \cap p^n W(A) = p^{n-k} \cdot (\underline{a} W(A) \cap p^k W(A)) \qquad \textit{for every } n \in \mathbb{N}$$

and moreover, the ideal $\underline{a} W(A)$ is a closed subset for the p-adic topology on $W(A)$.

Proof. Notice that the assertions are independent of the topology of A and B, hence we may assume that both these topologies are discrete.

(iii): We know already that $W(A)$ is p-adically complete, by proposition 9.3.44(ii). Also, it follows easily from (9.3.41) that p is regular in $W(A)$, and taking into account proposition 9.3.44(iii) we see that every element of $W(A)$ can be written uniquely in the form $p^n \cdot \underline{u}$, for some $n \in \mathbb{N}$ and an invertible element $\underline{u} \in W(A)$. The assertion follows immediately.

(ii): Suppose first that A is a domain, and we prove that the same holds for $W(A)$. To this aim, let B be any perfect field containing A; since $W(A)$ is a subring of $W(B)$, it suffices to check that $W(B)$ is a domain, which is clear from (iii).

Next, let $\mathrm{Min}\, A$ be the set of all minimal prime ideals of A; if A is reduced, the natural map $j : A \to B := \prod_{\mathfrak{p}\in\mathrm{Min}\, A} A_\mathfrak{p}$ is injective, and therefore $W(A)$ is a subring of $W(B) = \prod_{\mathfrak{p}\in\mathrm{Min}\, A} W(A_\mathfrak{p})$. To prove that $W(A)$ is reduced, it then suffices to show that the same holds for every factor $W(A_\mathfrak{p})$, but this is already known, since each such $A_\mathfrak{p}$ is a field.

(i): Suppose that (a) holds; arguing as in the proof of (ii), we reduce to checking that the image of $\underline{a}$ is regular in $W(A_\mathfrak{p})$, for every $\mathfrak{p} \in \mathrm{Min}\, A$. But under condition (a), it is clear that the image of $\underline{a}$ does not vanish in $W(A_\mathfrak{p})$ for any such $\mathfrak{p}$. Thus, in this case (ii) implies that (b) holds. Conversely, suppose that $\mathrm{Min}\, A = \{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$ is a finite set, and say that $J \subset \mathfrak{p}_1$. Pick any $f \in \bigcap_{i=2}^r \mathfrak{p}_i \setminus \mathfrak{p}_1$; since j is injective, it is easily seen that $\mathrm{Ann}_A(f) = \mathfrak{p}_1$ (details left to the reader). By proposition 9.3.36(i), we then see that $\underline{a} \cdot \tau_A(f) = 0$, which contradicts (b).

(iv): It is clear that $W(A) \subset W(B)$, and also that $V_n(A) = W(A) \cap V_n(B)$, so the assertion follows from (9.3.45).

(v): Say that $x\underline{a} = p^n y$ for some $x, y \in W(A)$; by (i), p is regular in $W(A)$, so for the proof of the stated equality it suffices to show that $x \in p^{n-k}W(A)$. However, we have already noticed that the map j is injective, and therefore $p^k W(A) = W(A) \cap p^k W(B)$, by (iv). Hence, it suffices to show that the image of x lies in $p^{n-k}W(A_\mathfrak{p})$ for every such $\mathfrak{p}$. Moreover, $A_\mathfrak{p}$ is a perfect field for every $\mathfrak{p} \in \mathrm{Min}\, A$, and $\sum_{i=0}^k a_i \cdot A_\mathfrak{p} = A_\mathfrak{p}$, so we may replace from start A by any $A_\mathfrak{p}$, in which case $W(A)$ is a discrete valuation ring with p as uniformizer, by (iii), and notice that by assumption the elements $a_0, \dots, a_k$ cannot all vanish, therefore

$$\underline{a} \notin p^{k+1}W(A) \qquad \text{and} \qquad x \cdot \underline{a} \in p^n W(A)$$

whence the assertion : details left to the reader. Lastly, the foregoing implies that the p-adic topology on $\underline{a}W(A)$ agrees with the topology induced from the p-adic topology on $W(A)$. On the other hand, (i) says that $\underline{a}$ is regular on $W(A)$, hence scalar multiplication by a induces a bijection $W(A) \xrightarrow{\sim} \underline{a}W(A)$ that identifies the p-adic topology of $W(A)$ with the p-adic topology of $\underline{a}W(A)$. Since $W(A)$ is p-adically complete and separated (proposition 9.3.44(ii)), it follows that the same holds for $\underline{a}W(A)$. Summing up, $\underline{a}W(A)$ is complete and separated for the topology induced from the p-adic topology of $W(A)$, and consequently it is a closed subset in $W(A)$, for this topology. □

Example 9.3.48. Let $(A, \mathscr{T})$ be a topological $\mathbb{F}_p$-algebra.

(i) If A is perfect and $\mathscr{T}$ is the discrete topology, then it follows from (9.3.45) and proposition 9.3.44(i) that $\mathscr{T}_{W(A)}$ agrees with the p-adic topology.

(ii) By inspecting the proof of theorem 8.2.8(i), it is easily seen that if $(A, \mathscr{T})$ is perfect, the same holds for its separated completion.

(iii) For instance, we have a natural isomorphism of topological rings :

$$\mathbb{Z}_p \xrightarrow{\sim} W(\mathbb{F}_p)$$

(where $\mathbb{F}_p$ is endowed with the discrete topology, and $\mathbb{Z}_p$ with the p-adic topology). Indeed, (i) says that the topology of $W(\mathbb{F}_p)$ agrees with the p-adic topology, hence the unique ring homomorphism $\mathbb{Z} \to W(\mathbb{F}_p)$ extends to a unique continuous map $\psi : \mathbb{Z}_p \to W(\mathbb{F}_p)$. Since p is a regular element of $W(\mathbb{F}_p)$ (proposition 9.3.47(i)), the map is injective. Moreover, if we endow both rings with their p-adic filtrations, the induced map $\mathrm{gr}_\bullet\psi : \mathrm{gr}_\bullet\mathbb{Z}_p \to \mathrm{gr}_\bullet W(\mathbb{F}_p)$ on associated graded rings is easily seen to be an isomorphism. Then the assertion follows from [34, Ch.III, §2, n.8, Cor.2].

(iv) It follows from (iii) that the structure map $\beta : \mathbb{F}_p \to A$ induces a natural structure $W(\beta)$ of topological $\mathbb{Z}_p$-algebra on $W(A)$.

(v) Let $A := \mathbb{F}_p[T^{1/p^\infty}]$, the universal perfect $\mathbb{F}_p$-algebra in one generator (notation of example 4.8.55(iii)), and endow A with its discrete topology. Then $W(A)$ is the p-adic completion

$$\mathbb{Z}_p\{T^{1/p^\infty}\}$$

of $\mathbb{Z}_p[T^{1/p^\infty}]$. The latter can be described as the ring of all power series

$$\sum_{n\in\mathbb{N}} a_n T^{\lambda_n} \tag{9.3.49}$$

where $(\lambda_n \mid n \in \mathbb{N})$ is any sequence of elements of $\mathbb{N}[1/p]$, and $(a_n \mid n \in \mathbb{N})$ is any sequence of elements of $\mathbb{Z}_p$ that converges p-adically to 0. For the proof, we argue as in (iii): we have a unique map of $\mathbb{Z}_p$-algebras $\psi : \mathbb{Z}_p[T^{1/p^\infty}] \to W(A)$ such that $\psi(T^\lambda) := \tau_A(T^\lambda)$ for every $\lambda \in \mathbb{N}[1/p]$ (for the $\mathbb{Z}_p$-algebra structure on $W(A)$ provided by (iv)). Then ψ extends by continuity to a unique map of topological $\mathbb{Z}_p$-algebras $\psi^\wedge : \mathbb{Z}_p\{T^{1/p^\infty}\} \to W(A)$, and if we endow both rings with their p-adic filtrations, it is easily seen that the resulting map of associated graded rings $\mathrm{gr}^\bullet\psi^\wedge$ is bijective, so the same holds for $\psi^\wedge$, again by [34, Ch.III, §2, n.8, Cor.3].

9.3.50. *The lifting property.* In the situation of lemma 9.3.4, endow A (resp. A/J_n, for every $n \in \mathbb{N}$) with the linear topology $\mathscr{T}$ defined by the system of ideals $(J_n \mid n \in \mathbb{N})$ (resp. with the discrete topology) and let $\pi_n : A \to A/J_n$ and $\pi'_n : A/J_n \to A/J_0$ be the natural projections, for every $n \in \mathbb{N}$. Let also R be a topological $\mathbb{F}_p$-algebra, and $\overline{\varphi} : R \to A/J_0$ a continuous ring homomorphism. We say that a continuous mapping $\varphi_n : R \to A/J_n$ is a *lifting of* $\overline{\varphi}$ if we have:

$$\varphi_n(x^p) = \varphi_n(x)^p \qquad \text{for every } x \in R\text{, and} \qquad \pi'_n \circ \varphi_n = \overline{\varphi}.$$

Lemma 9.3.51. *With the notation of* (9.3.50)*, the following holds :*

(i) *If φ_n and φ'_n are two liftings of $\overline{\varphi}$, then φ_n and φ'_n agree on R^{p^n}.*

(ii) *If R is a perfect topological $\mathbb{F}_p$-algebra, there exists a unique lifting $\varphi_n : R \to A/J_n$ of $\overline{\varphi}$. We have $\varphi_n(1) = 1$ and $\varphi_n(xy) = \varphi_n(x)\cdot\varphi_n(y)$ for every $x, y \in R$.*

(iii) *Let R be as in* (ii)*, and suppose that $(A, \mathscr{T})$ is complete and separated. Then :*

  (a) *There exists a unique continuous mapping $\varphi : R \to A$ such that*

$$\pi_0 \circ \varphi = \overline{\varphi} \qquad \textit{and} \qquad \varphi(x^p) = \varphi(x)^p \quad \textit{for every } x \in R.$$

  (b) *Moreover, $\varphi(1) = 1$ and $\varphi(xy) = \varphi(x)\cdot\varphi(y)$ for every $x, y \in R$.*

  (c) *If furthermore, A is an $\mathbb{F}_p$-algebra, then φ is a homomorphism of topological rings.*

Proof. (i) is an easy consequence of lemma 9.3.4(i).

(ii): The uniqueness follows from (i). For the existence, pick any mapping $\sigma : A/J_0 \to A/J_n$ such that $\pi'_n \circ \sigma = \mathbf{1}_{A/J_0}$, and notice that the mapping $\psi := \sigma \circ \overline{\varphi} : R \to A/J_n$ is continuous, and $\pi'_n \circ \psi = \overline{\varphi}$. We let :

$$\varphi_n(x) := \psi(x^{p^{-n}})^{p^n} \qquad \text{for every } x \in R.$$

Using lemma 9.3.4(i) one verifies easily that φ_n does not depend on the choice of ψ. Especially, define $\psi' : R \to A/J_n$ by the rule : $x \mapsto \psi(x^{p^{-1}})^p$ for every $x \in R$. Clearly $\pi'_n \circ \psi' = \overline{\varphi}$ as well, hence :

$$\varphi_n(x^p) = \psi'(x^{p^{1-n}})^{p^n} = \psi(x^{p^{-n}})^{p^{n+1}} = \varphi_n(x)^p \qquad \text{for every } x \in R$$

as claimed. If we choose σ so that $\sigma(1) = 1$, we obtain $\varphi_n(1) = 1$. Finally, since $\overline{\varphi}$ is a ring homomorphism we have $\psi(x)\cdot\psi(y) \equiv \psi(xy) \pmod{J_0}$ for every $x, y \in R$. Hence

$$\psi(x)^{p^n}\cdot\psi(y)^{p^n} \equiv \psi(xy)^{p^n} \pmod{J_n}$$

(again by lemma 9.3.4(i)) and finally $\varphi_n(x^{p^n})\cdot\varphi_n(y^{p^n}) = \varphi_n((xy)^{p^n})$, which implies the last stated identity, since R is perfect.

(iii): The existence and uniqueness of φ follow from (ii). It remains only to show that $\varphi(x) + \varphi(y) = \varphi(x+y)$ in case A is an $\mathbb{F}_p$-algebra. It suffices to check the latter identity on the projections onto A/J_n, for every $n \in \mathbb{N}$, in which case one argues analogously to the foregoing proof of the multiplicative property for φ_n : the details shall be left to the reader. □

Proposition 9.3.52. *In the situation of* (9.3.50)*, suppose that R is a perfect topological $\mathbb{F}_p$-algebra and $(A, \mathscr{T})$ is complete and separated. Then we have :*

(i) *For every $n \in \mathbb{N}$ there exists a unique ring homomorphism $v_n : W_{n+1}(A/J_0) \to A/J_n$ such that the following diagram commutes :*

$$\begin{array}{ccc} W_{n+1}(A) & \xrightarrow{\boldsymbol{\omega}_n} & A \\ {\scriptstyle W_{n+1}(\pi_0)}\downarrow & & \downarrow{\scriptstyle \pi_n} \\ W_{n+1}(A/J_0) & \xrightarrow{v_n} & A/J_n. \end{array}$$

(ii) *Let $u_n := v_n \circ W_{n+1}(\overline{\varphi}) \circ \overline{F}_R^{-n}$, where $\overline{F}_R : W_{n+1}(R) \to W_{n+1}(R)$ is induced by the Frobenius automorphism of $W(R)$. Then we have a commutative diagram :*

$$\begin{array}{ccc} W_{n+2}(R) & \xrightarrow{u_{n+1}} & A/J_{n+1} \\ \downarrow & & \downarrow{\scriptstyle \vartheta} \\ W_{n+1}(R) & \xrightarrow{u_n} & A/J_n \end{array} \qquad \textit{for every } n \in \mathbb{N}$$

where the vertical maps are the natural projections.

(iii) *There exists a unique ring homomorphism u such that the following diagram commutes:*

$$\begin{array}{ccc} W(R) & \xrightarrow{u} & A \\ {\scriptstyle \boldsymbol{\omega}_0}\downarrow & & \downarrow{\scriptstyle \pi_0} \\ R & \xrightarrow{\overline{\varphi}} & A/J_0. \end{array}$$

Furthermore, u is continuous for the topology $\mathscr{T}_{W(R)}$, and we have :

(9.3.53) $$u(\underline{a}) = \sum_{n\in\mathbb{N}} p^n \cdot \varphi(a_n^{p^{-n}}) \qquad \textit{for every } \underline{a} := (a_n \mid n \in \mathbb{N}) \in W(R)$$

with $\varphi : R \to A$ the unique continuous mapping characterized as in lemma 9.3.51(iii)*.*

Proof. (i): We have to check that $\boldsymbol{\omega}_n(a_0, \dots, a_n) \in J_n$ whenever $a_0, \dots, a_n \in J_0$, which is clear from lemma 9.3.4(ii).

(ii): Since $\overline{F}_R$ is an automorphism, it boils down to verifying :

Claim 9.3.54. $\vartheta \circ v_{n+1}(\overline{\varphi}(a_0), \dots, \overline{\varphi}(a_{n+1})) = v_n(\overline{\varphi}(F_0(a_0, a_1)), \dots, \overline{\varphi}(F_n(a_0, \dots, a_{n+1})))$ for every $(a_0, \dots, a_{n+1}) \in W_{n+1}(A)$.

Proof of the claim. Directly from (9.3.21) we derive :

$$(\overline{\varphi}(F_0(a_0, a_1)), \dots, \overline{\varphi}(F_n(a_0, \dots, a_{n+1}))) = (F_0(\overline{\varphi}(a_0), \overline{\varphi}(a_1)), \dots, F_n(\overline{\varphi}(a_0), \dots, \overline{\varphi}(a_n))).$$

Hence, it suffices to show that

$$\vartheta \circ v_{n+1}(b_0, \dots, b_{n+1}) = v_n(F_0(b_0, b_1), \dots, F_n(b_0, \dots, b_{n+1}))$$

for every $(b_0, \dots, b_{n+1}) \in W_{n+1}(A/J_0)$. The latter identity follows from (9.3.19). ◊

(iii): We take $u := \lim_{n\in\mathbb{N}} u_n$, where the maps u_n are as in (ii), for every $n \in \mathbb{N}$. With this definition, it is clear that u is continuous for the topology $\mathscr{T}_{W(R)}$. Next we remark that $\pi_0 \circ u \circ \tau_R = \overline{\varphi} \circ \boldsymbol{\omega}_0 \circ \tau_R = \overline{\varphi}$, hence :

(9.3.55) $$u \circ \tau_R = \varphi$$

by lemma 9.3.51(iii.a). Now (9.3.53) holds by (9.3.46), (9.3.55) and the continuity of u. □

Remark 9.3.56. (i) Let $\mathscr{N}_p$ be the full subcategory of $\mathbb{Z}$-TopAlg whose objects are the discrete topological rings in which p is nilpotent. For every perfect topological $\mathbb{F}_p$-algebra E and every topological ring W, we consider the functors

$$\overline{H}_E, H_W : \mathscr{N}_p \to \mathbf{Set}$$

such that

$$\overline{H}_E(A) := \mathrm{Hom}_{\mathbb{Z}\text{-}\mathsf{TopAlg}}(E, A/pA) \qquad H_W(A) := \mathrm{Hom}_{\mathbb{Z}\text{-}\mathsf{TopAlg}}(W, A)$$

for every $A \in \mathrm{Ob}(\mathscr{N}_p)$ (where A/pA is endowed with the discrete topology); to any continuous ring homomorphism $A \to A'$ of objects of $\mathscr{N}_p$, the functor $\overline{H}_E$ assigns the mapping

$$\overline{H}_E(\varphi) : H_E(A) \to H_E(A') \qquad (\overline{\psi} : E \to A/pA) \mapsto (\varphi \otimes_{\mathbb{Z}} \mathbb{F}_p) \circ \overline{\psi}$$

and likewise, $H_W(\varphi) : H_W(A) \to H_W(A')$ assigns to any $\psi : A \to A'$ the map $\varphi \circ \psi$. Then proposition 9.3.52(iii) says that we have a natural isomorphism of functors

$$H_{W(E)} \xrightarrow{\sim} \overline{H}_E \tag{9.3.57}$$

that assigns to every $A \in \mathrm{Ob}(\mathscr{N}_p)$ the bijection $H_{W(E)}(A) \xrightarrow{\sim} \overline{H}_E(A) : \varphi \mapsto \varphi \otimes_{\mathbb{Z}} \mathbb{F}_p$.

(ii) Suppose that E is a perfect topological $\mathbb{F}_p$-algebra whose topology is linear, complete and separated. Then the existence of an isomorphism of functors (9.3.57) characterizes $W(E)$ up to natural isomorphism, in the category of complete and separated topological rings whose topology is linear and coarser than the p-adic topology. Indeed, suppose that W' is any other such topological ring, endowed with an isomorphism of functors $H_{W'} \xrightarrow{\sim} \overline{H}_E$. There follows an isomorphism of functors $\omega : H_{W(E)} \xrightarrow{\sim} H_{W'}$. Now, let $I \subset W(E)$ be any open ideal; the quotient $W(E)/I$ is an object of $\mathscr{N}_p$, and the projection $\pi_I : W(E) \to W(E)/I$ is an element of $H_{W(E)}(W(E)/I)$, so we get a continuous ring homomorphism $\omega(\pi_I) : W' \to W(E)/I$. Moreover, if $J \subset I$ is another open ideal, the projection $\pi_{I,J} : W(E)/J \to W(E)/I$ is a morphism of $\mathscr{N}_p$, and we have

$$\pi_{I,J} \circ \omega(\pi_J) = H_{W'}(\pi_{I,J})(\omega(\pi_J)) = \omega(H_{W(E)}(\pi_{I,J})(\pi_J)) = \omega(\pi_I)$$

by the naturality of ω. Hence, we have a well defined inverse system $(\omega(\pi_I) \mid I \subset W(E))$ indexed by the cofiltered system of open ideals of $W(E)$; the limit of this system is a well defined continuous ring homomorphism $\alpha : W' \to W(E)$. Swapping the roles of W' and $W(E)$, the inverse natural transformation $\omega^{-1} : H_{W'} \xrightarrow{\sim} H_{W(E)}$ yields likewise a continuous ring homomorphism $\beta : W(E) \to W'$, as W' is complete and separated by assumption.

It remains to check that α and β are mutually inverse isomorphisms. To this aim, let $I' \subset W'$ be any open ideal, $\pi'_{I'} : W' \to W'/I'$ the projection, and set $I := \mathrm{Ker}\,(\omega^{-1}(\pi'_{I'}) : W(E) \to W'/I')$, so that $\omega^{-1}(\pi'_{I'})$ factors as the composition of π_I and a continuous ring homomorphism $\tau : W(E)/I \to W'/I'$. Notice that

$$\begin{aligned}\pi'_{I'} \circ \beta \circ \alpha &= \omega^{-1}(\pi'_{I'}) \circ \alpha = \tau \circ \pi_I \circ \alpha = \tau \circ \omega(\pi_I) = H_{W'}(\tau)(\omega(\pi_I)) \\ &= \omega(H_{W(E)}(\tau)(\pi_I)) = \omega(\tau \circ \pi_I) = \omega(\omega^{-1}(\pi'_{I'})) = \pi'_{I'}\end{aligned}$$

which implies that $\beta\circ\alpha = \mathbf{1}_{W'}$. Likewise one checks that $\beta\circ\alpha = \mathbf{1}_{W(E)}$, whence the contention.

(iii) Let E, E' and E'' be three perfect topological $\mathbb{F}_p$-algebras whose topologies are linear, complete and separated, and $E \to E'$, $E \to E''$ two continuous ring homomorphisms. It is clear that $(E' \otimes_E E'', \mathscr{T}^{\otimes}_{E',E''})$ is a perfect topological $\mathbb{F}_p$-algebra (notation of (8.3.7)), hence the same holds for $E' \widehat{\otimes}_E E''$ (example 9.3.48(ii)), and in light of (8.3.7) and (i), we get natural isomorphisms of functors

$$H_{W(E' \widehat{\otimes}_E E'')} \xrightarrow{\sim} H_{W(E')} \times_{H_{W(E)}} H_{W(E'')} \xleftarrow{\sim} H_{W(E') \widehat{\otimes}_{W(E)} W(E'')}$$

whence, by (ii), a natural isomorphism of topological rings

$$W(E' \widehat{\otimes}_E E'') \xrightarrow{\sim} W(E') \widehat{\otimes}_{W(E)} W(E''). \tag{9.3.58}$$

Proposition 9.3.59. *Let A be any discrete and reduced $\mathbb{F}_p$-algebra. We have :*

(i) *$W(A)$, endowed with its p-adic topology, is a flat and topologically free $\mathbb{Z}_p$-module.*
(ii) *If A is perfect, and $(g_\lambda \mid \lambda \in \Lambda)$ is any basis of the $\mathbb{F}_p$-vector space underlying A, then $(\tau_A(g_\lambda) \mid \lambda \in \Lambda)$ is a topological basis of the topological $\mathbb{Z}_p$-module underlying $W(A)$.*

Proof. The element p is regular in $W(A)$, by proposition 9.3.47(i), so $W(A)$ is torsion-free, hence flat, as a $\mathbb{Z}_p$-module, for its natural $\mathbb{Z}_p$-module structure as in example 9.3.48(iv). In order to check both (i) and (ii), it then suffices to show the following general

Claim 9.3.60. Let R be any artinian ring, κ (resp. $\mathfrak{m}$) the residue field (resp. the maximal ideal) of R, and M any flat R-module. Let also $(f_\lambda \mid \lambda \in \Lambda)$ be any system of elements of M, whose image in $M_0 := M \otimes_R \kappa$ is a basis for the latter κ-vector space. Then the induced R-linear map

$$\psi : R^{\oplus\Lambda} \to M \qquad e_\lambda \mapsto f_\lambda \qquad \text{for every } \lambda \in \Lambda$$

is an isomorphism (here $(e_\lambda \mid \lambda \in \Lambda)$ is the canonical basis of $R^{\oplus\Lambda}$).

Proof of the claim. Endow $R^{\oplus\Lambda}$ and M with the $\mathfrak{m}$-adic filtrations, and let $\mathrm{gr}_\bullet R^{\oplus\Lambda}$ and $\mathrm{gr}_\bullet M$ be the associated graded κ-vector spaces. By virtue of [126, Th.22.3], the natural κ-linear map

$$\mathfrak{m}^n/\mathfrak{m}^{n+1} \otimes_\kappa M_0 \to \mathrm{gr}_n M$$

is an isomorphism for every $n \in \mathbb{N}$. This easily implies that the κ-linear map $\mathrm{gr}_\bullet : \mathrm{gr}_\bullet R^{\oplus\Lambda} \to \mathrm{gr}_\bullet M$ induced by ψ, is an isomorphism for every $n \in \mathbb{N}$. Then the assertion follows from [34, Ch.III, §2, n.8, Cor.3]. □

9.3.61. *Teichmüller representative of a sum.* We shall need a combinatorial identity expressing the Teichmüller representative of a finite sum of elements of a perfect topological $\mathbb{F}_p$-algebra A, as a series of terms, each of which is itself a Teichmüller representative. To state the result, let

$$\Sigma_n^{(k)} := \{(\sigma_0, \dots, \sigma_k) \in p^{-n}\mathbb{N}^{k+1} \setminus p^{1-n}\mathbb{N}^{k+1} \mid \sigma_0 + \cdots + \sigma_k = 1\} \qquad \text{for every } k, n \in \mathbb{N}.$$

and set $\Sigma^{(k)} := \bigcup_{n\in\mathbb{N}} \Sigma_n^{(k)}$ for every $k \in \mathbb{N}$. Also for every $a := (a_0, \dots, a_k) \in A^{k+1}$ and every $\sigma := (\sigma_0, \dots, \sigma_k) \in \Sigma^{(k)}$ set

$$a^\sigma := a_0^{\sigma_0} \cdots a_k^{\sigma_k}$$

(notice that the fractional powers $a_i^{\sigma_i}$ are well defined, since A is perfect).

Proposition 9.3.62. *For every integer $k \in \mathbb{N}$ the following holds :*

(i) *There exists a mapping $\Sigma^{(k)} \to \mathbb{Z}_p \ : \ \sigma \mapsto c_\sigma$ such that*

$$\tau_A(a_0 + \cdots + a_k) = \sum_{n\in\mathbb{N}} p^n \cdot \sum_{\sigma \in \Sigma_n^{(k)}} c_\sigma \cdot \tau_A(a^\sigma)$$

for every perfect $\mathbb{F}_p$-algebra A and every $a := (a_0, \dots, a_k) \in A^{k+1}$.

(ii) $c_\sigma = 1$ *for every $\sigma \in \Sigma_0^{(k)}$.*

Proof. To begin with, let us show the following :

Claim 9.3.63. With the notation of the proposition, for every $n \in \mathbb{N}$ we have :

$$(\tau_A(a_0)^{1/p^n} + \cdots + \tau_A(a_k)^{1/p^n})^{p^n} \equiv \tau_A(a_0 + \cdots + a_k) \pmod{p^n W(A)}.$$

Proof of the claim. Since τ_A is a splitting of $\boldsymbol{\omega}_0$, for every $n \in \mathbb{N}$ we have

$$\tau_A(a_0)^{1/p^n} + \cdots + \tau_A(a_k)^{1/p^n} \equiv \tau_A(a_0 + \cdots + a_k)^{1/p^n} \pmod{pW(A)}$$

and then the claim follows immediately from lemma 9.3.4(i). ◊

Now, consider first the case where $A = \mathbb{F}_p[T_0^{1/p^\infty}, \dots, T_k^{1/p^\infty}]$ is the universal perfect $\mathbb{F}_p$-algebra in the variables $T_0, \dots, T_k$ (see example 4.8.55(iii)). We endow A with the discrete topology. Set $T_\bullet := (T_0, \dots, T_k)$, and notice that the system of monomials $(T_\bullet^\mu \mid \mu \in \mathbb{N}[1/p]^{k+1})$ is a basis for the $\mathbb{F}_p$-vector space underlying A, so the system $(\tau_A(T_\bullet)^\mu \mid \mu \in \mathbb{N}[1/p]^{k+1})$ is a topological basis of the topologically free $\mathbb{Z}_p$-module underlying $W(A)$ (proposition 9.3.59(ii)). Hence, the image of the same system in $W(A)/p^nW(A)$ is a basis for the latter free $\mathbb{Z}/p^n\mathbb{Z}$-module, for every $n \in \mathbb{N}$. Consider the image S_n of $S := \tau_A(T_0 + \cdots + T_k)$ in $W(A)/p^nW(A)$; from claim 9.3.63 we see that S_n is uniquely a sum of products $c_\mu^{(n)} T_\bullet^\mu$ with $c_k^{(n)} \in \mathbb{Z}/p^n\mathbb{Z}$ and $T_\bullet^\mu$ a monomial of the above type, with the further condition that

$$c_\mu^{(n)} = 0 \qquad \text{unless} \qquad \mu \in \bigcup_{i=0}^{n} \Sigma_i^{(k)}.$$

Moreover, the uniqueness of the resulting expression means that the image of $c_\mu^{(m)}$ in $\mathbb{Z}/p^n\mathbb{Z}$ agrees with $c_\mu^{(n)}$, for every such μ and every $m \geq n$. Summing up, we get the sought identity for S, and (ii) follows as well. Next, if A is any perfect $\mathbb{F}_p$-algebra, there exists a unique (continuous) map of $\mathbb{F}_p$-algebras

$$\psi : \mathbb{F}_p[T_0^{1/p^\infty}, \dots, T_k^{1/p^\infty}] \to A \qquad \text{such that } \psi(T_i) = a_i \text{ for } i = 0, \dots, k.$$

Since the sought identity is already known for $T_\bullet$, we deduce it for a, by virtue of (9.3.35). □

9.3.64. Let A be any perfect $\mathbb{F}_p$-algebra, $\underline{b} := (b - n \mid n \in \mathbb{N}) \in W(A)$ any element, and set

$$\mathrm{Ann}_A(\underline{b}) := \{a \in A \mid \tau_A(a) \cdot \underline{b} = 0\}.$$

Corollary 9.3.65. *With the notation of* (9.3.64)*, the subset* $\mathrm{Ann}_A(\underline{b})$ *is a radical ideal of* A.

Proof. Let $a \in \mathrm{Ann}_A(\underline{b})$ for some $n \in \mathbb{N}$. We show first that $a^\mu \in \mathrm{Ann}_A(\underline{b})$ for every $\mu \in \mathbb{N}[1/p]$. Indeed, in view of proposition 9.3.36(i), the assumption means that $a^{p^n} b_n = 0$ for every $n \in \mathbb{N}$, and therefore $a^{p^{n-k}} b_n^{p^{-k}} = 0$ for every $k, n \in \mathbb{N}$, since A is perfect. Hence $a^{p^{n-k}} b_n = 0$ as well, for every $n, k \in \mathbb{N}$, which means that $\tau_A(a^{p^{-k}}) \cdot \underline{b} = 0$ for every $k \in \mathbb{N}$ (again by proposition 9.3.36(i)); the assertion is a straightforward consequence.

It then remains only to check that $\mathrm{Ann}_A(\underline{b})$ is an ideal of A. However, if $a, a' \in \mathrm{Ann}_A(\underline{b})$ and $x \in A$ is any element, it is clear that $ax \in \mathrm{Ann}_A(\underline{b})$, and proposition 9.3.62 easily implies that $a + a' \in \mathrm{Ann}_A(\underline{b})$ as well, as required. □

In the same vein, though the Teichmüller mapping is not a ring homomorphism, one can attach to it a continuous map on Zariski spectra, as explained in the following proposition.

Proposition 9.3.66. *Let E be any perfect $\mathbb{F}_p$-algebra; the following holds :*

(i) *For every $\mathfrak{p} \in \mathrm{Spec}\, W(E)$, the subset $\tau_E^{-1}\mathfrak{p}$ is a prime ideal of E.*

(ii) *The resulting map*

$$\mathrm{Spec}\, \tau_E : \mathrm{Spec}\, W(E) \to \mathrm{Spec}\, E \qquad \mathfrak{p} \mapsto \tau_E^{-1}\mathfrak{p}$$

is continuous and spectral.

(iii) *Let $\mathfrak{p} \in \mathrm{Spec}\, W(E)$ be any prime ideal, set $\mathfrak{q} := \mathrm{Spec}\, \tau_E(\mathfrak{p}) \in \mathrm{Spec}\, E$, and let $\pi_\mathfrak{p} : W(E) \to \kappa(\mathfrak{p})$, $\pi_\mathfrak{q} : E \to \kappa(\mathfrak{q})$ be the projections. If $\mathfrak{p}$ is a closed subset in the p-adic topology of $W(E)$, then τ_E induces a morphism of multiplicative monoids*

$$\tau_\mathfrak{p} : \kappa(\mathfrak{q}) \to \kappa(\mathfrak{p}) \qquad \textit{such that} \qquad \tau_\mathfrak{p} \circ \pi_\mathfrak{q} = \pi_\mathfrak{p} \circ \tau_E.$$

(iv) *Let E' be any other perfect $\mathbb{F}_p$-algebra, and $\varphi : E \to E'$ any continuous ring homomorphism. Then the resulting diagram commutes :*

$$\begin{array}{ccc} \mathrm{Spec}\, W(E') & \xrightarrow{\mathrm{Spec}\,\tau_{E'}} & \mathrm{Spec}\, E' \\ {\scriptstyle \mathrm{Spec}\, W(\varphi)} \downarrow & & \downarrow {\scriptstyle \mathrm{Spec}\,\varphi} \\ \mathrm{Spec}\, W(E) & \xrightarrow{\mathrm{Spec}\,\tau_E} & \mathrm{Spec}\, E. \end{array}$$

Proof. (i): Say that $x, y \in E$ and $\tau_E(xy) \in \mathfrak{p}$; since τ_E is a multiplicative map, it follows easily that either x or y lies in $\tau_E^{-1}\mathfrak{p}$. By the same token, if $x \in \tau_E^{-1}\mathfrak{p}$ and $y \in E$, then $xy \in \tau_E^{-1}\mathfrak{p}$. Next, suppose that $x, y \in \tau_E^{-1}\mathfrak{p}$; by proposition 9.3.62, we may write $\tau_E(x+y)$ as a p-adically convergent series $\sum_{n\in\mathbb{N}} p^n \cdot z_n$, where each z_n is a finite $\mathbb{Z}_p$-linear combination of terms of the form $\tau_E(x^\lambda y^{1-\lambda})$, with $\lambda, 1-\lambda \in \mathbb{N}[1/p]$. It follows easily that $z_n = \tau_E(x^{1/p})\cdot z'_n + \tau_E(y^{1/p})\cdot z''_n$, for suitable $z'_n, z''_n \in W(E)$. Summing up, we get

$$\tau_E(x+y) = a \cdot \tau_E(x^{1/p}) + b \cdot \tau_E(y^{1/p}) \qquad \text{where} \qquad a := \sum_{n\in\mathbb{N}} c_n z'_n \qquad b := \sum_{n\in\mathbb{N}} c_n z''_n.$$

However, by the foregoing we already know that $x^{1/p}, y^{1/p} \in \tau_E^{-1}\mathfrak{p}$, so $\tau_E(x+y) \in \mathfrak{p}$, whence the contention.

(ii): Let $f \in E$ any element; directly from the definition we get

$$(\mathrm{Spec}\,\tau_E)^{-1}(\mathrm{Spec}\, E_f) = \mathrm{Spec}\, A_{\tau_E(f)} \tag{9.3.67}$$

which says that $\mathrm{Spec}\,\tau_E$ is continuous and spectral.

(iii): We check first that τ_E descends to a morphism $\overline{\tau} : E/\mathfrak{q} \to W(E)/\mathfrak{p}$ of multiplicative monoids. To this aim, let $x \in E$ and $y \in \mathfrak{q}$ be any two elements; arguing as in the proof of (i), we see that $d := \tau_E(x+y) - \tau_E(x)$ can be written as p-adically convergent series $\sum_{n\in\mathbb{N}} d_n$, where $d_n \in p^n\tau_E(y^{1/p^n})W(E)$ for every $n \in \mathbb{N}$. We have already remarked that $\tau_E(y^{1/p^n}) \in \mathfrak{p}$ for every $n \in \mathbb{N}$, and since $\mathfrak{p}$ is p-adically closed in $W(E)$, it follows that $d \in \mathfrak{p}$ as well, whence the claim. Now, by construction $\overline{\tau}^{-1}(0) = \{0\}$, so $\overline{\tau}$ extends uniquely to a morphism of monoids $\tau_\mathfrak{p}$ as sought.

(iv) follows immediately from (9.3.35). □

9.3.68. *Angular powers.* For the further investigation of the ring of Witt vectors $W(A)$ of a given perfect topological $\mathbb{F}_p$-algebra A, it is useful to consider special types of ideals of A and $W(A)$ defined by certain combinatorial conditions that we proceed to explain.

• First, consider any *p-perfect* monoid P, *i.e.* such that the p-Frobenius endomorphism Φ_P of P is bijective. A basic example of p-perfect monoid is $\mathbb{N}[1/p] := S_p^{-1}\mathbb{N}$, the localization of $\mathbb{N}$ at its multiplicative subset $S_p := \{p^k \mid k \in \mathbb{N}\}$. Notice that this condition on P means that for every $x \in P$ and every $\lambda \in \mathbb{N}[1/p]$, the fractional power x^λ is well defined : indeed, we may write $\lambda = p^{-n}a$ for some $a, n \in \mathbb{N}$ and set $x^\lambda := \Phi_P^{-n}(x^a)$. It is easily seen that this definition is independent of the choice of a and n.

• For any ideal $I \subset P$ and every $\lambda = p^{-n}a \in \mathbb{Z}[1/p]$ we define the *angular power* :

$$I^{\langle\lambda\rangle} := \bigcup_{r\in\mathbb{N}} \Phi_P^{-n-r}(I^{p^r a}) \qquad \text{if } \lambda > 0, \text{ and} \qquad I^{\langle\lambda\rangle} := P \qquad \text{if } \lambda \leq 0.$$

Notice that $I^{\langle\lambda\rangle}$ is an increasing union of ideals of P, and a simple inspection shows that the definition does not depend on the choice of a and n. Explicitly, if $\lambda \geq 0$, the ideal $I^{\langle\lambda\rangle}$ is the subset of all products of the form $a_1^{\mu_1}\cdots a_k^{\mu_k}$, where $k \in \mathbb{N}$ is any integer, $a_1, \dots, a_k$ are arbitrary elements of I, and $\mu_1, \dots, \mu_k$ are arbitrary rational numbers in $\mathbb{N}[1/p]$ such that $\sum_{i=1}^k \mu_i = \lambda$. Moreover, if $S \subset I$ is any system of generators for I, then $I^{\langle\lambda\rangle}$ is generated by the set of all products as above, where $a_1, \dots, a_k \in S$.

Lemma 9.3.69. *In the situation of* (9.3.68), *the following holds :*

(i) *If* $(I_j \mid j \in J)$ *is any filtered family of ideals of* P, *we have*

$$(\bigcup_{j\in J} I_j)^{\langle\lambda\rangle} = \bigcup_{j\in J} I_j^{\langle\lambda\rangle} \qquad \text{for every } \lambda \in \mathbb{N}[1/p].$$

(ii) *For every ideal* $I \subset P$, *every* $n \in \mathbb{N}$ *and every* $\lambda, \mu \in \mathbb{N}[1/p]$ *we have*

(a) $(I^n)^{\langle\lambda\rangle} = (I^{\langle\lambda\rangle})^n = I^{\langle n\lambda\rangle}$.

(b) $I^{\langle\lambda\rangle} I^{\langle\mu\rangle} = I^{\langle\lambda+\mu\rangle}$.

(c) $(I^{\langle\lambda\rangle})^{\langle\mu\rangle} = I^{\langle\lambda\mu\rangle}$.

(iii) $I^{\langle\lambda\rangle} J^{\langle\lambda\rangle} = (IJ)^{\langle\lambda\rangle}$ *for every ideals* $I, J \subset P$ *and every* $\lambda \in \mathbb{N}[1/p]$.

(iv) *Suppose that* $I \subset P$ *is a non-empty finitely generated ideal generated by* r *elements of* P. *Then* $I^{\langle\lambda\rangle} \subset I^n$ *for every* $n \in \mathbb{N}$ *such that* $\lambda \geq n + r - 1$.

Proof. (i): This is clear from the explicit description of the angular power furnished in (9.3.68).

(ii.a): Indeed, say that $\lambda = p^k a$ for some $a, n \in \mathbb{N}$; we have

$$(I^n)^{\langle\lambda\rangle} = \bigcup_{r\in\mathbb{N}} \Phi_P^{-r-k}(I^{nap^r}) = I^{\langle n\lambda\rangle} = \bigcup_{r\in\mathbb{N}} (\Phi_P^{-r-k}(I^{ap^r})^n = (I^{\langle\lambda\rangle})^n.$$

(ii.b): Clearly $I^{\langle\lambda\rangle} I^{\langle\mu\rangle} \subset I^{\langle\lambda+\mu\rangle}$. For the converse inclusion, say that $a_1^{t_1} \cdots a_k^{t_k} \in I^{\langle\lambda+\mu\rangle}$, for given $a_1, \dots, a_k \in I$ and rational numbers $t_1, \dots, t_k$ in $\mathbb{N}[1/p]$ such that $\sum_{i=1}^k t_i = \lambda + \mu$. If $\mu = 0$, there is nothing to prove; otherwise, let $j < k$ be the largest integer such that $s := \sum_{i=0}^j t_i \leq \lambda$. Then $t_{j+1} > t'_{j+1} := \lambda - s$, and if we set $t''_{j+1} := t_{j+1} - t'_{j+1}$ we get

$$a_1^{t_1} \cdots a_{j+1}^{t'_{j+1}} \in I^{\langle\lambda\rangle} \qquad a_{j+1}^{t''_{j+1}} a_{j+2}^{t_{j+2}} \cdots a_k^{t_k} \in I^{\langle\mu\rangle}$$

whence the contention.

(ii.c): Say that $\lambda = p^{-n}a$ and $\mu = p^{-m}b$; taking into account (ii.a) we may compute

$$(I^{\langle\lambda\rangle})^{\langle\mu\rangle} = \bigcup_{r\in\mathbb{N}} \Phi_P^{-m-r}((I^{\langle\lambda\rangle})^{bp^r}) = \bigcup_{r\in\mathbb{N}} \Phi_P^{-m-r}((I^{bp^r})^{\langle\lambda\rangle}) = \bigcup_{r\in\mathbb{N}} \Phi_P^{-m-r}(\bigcup_{s\in\mathbb{N}} \Phi_P^{-n-s}(I^{abp^{r+s}}))$$

whence the sought identity.

(iii): Indeed, for $\lambda = p^{-n}a$ we may compute

$$I^{\langle\lambda\rangle} J^{\langle\lambda\rangle} = \bigcup_{r\in\mathbb{N}} \Phi_P^{-n-r}(I^{ap^r}) \cdot \bigcup_{r\in\mathbb{N}} \Phi_P^{-n-r}(J^{ap^r}) = \bigcup_{r\in\mathbb{N}} \Phi_P^{-n-r}(I^{ap^r} J^{ap^r})$$

whence the contention.

(iv): Let $(x_1, \dots, x_r)$ be such a system of generators for I, so that $I^{\langle\lambda\rangle}$ is generated by all the products of the form $x := x_1^{t_1} \cdots x_r^{t_r}$, where $t_1, \dots, t_r \in \mathbb{N}[1/p]$ and $t_1 + \cdots + t_r = \lambda$. Then, for every such product and every $i = 1, \dots, r$, let $a_i \in \mathbb{N}$ be the unique integer such that $t_i \in [a_i, a_i + 1[$, and set $s := a_1 + \cdots + a_r$; clearly $x \in I^s$. Suppose now that $\lambda \geq n + r - 1$; if $t_i \in \mathbb{N}$ for every $i = 1, \dots, r$, it follows that $s = \lambda \geq n$, since $r > 0$. Lastly, if there exists $i \leq r$ such that $t_i \notin \mathbb{N}$, then $s + r > \lambda$, whence $s \geq n$ again, and the assertion follows. □

Remark 9.3.70. (i) Let A be a ring, P a p-perfect monoid, as defined in (9.3.68), and $u : P \to (A, \cdot)$ a morphism of monoids. Then, for every ideal $I \subset P$ and every $\lambda \in \mathbb{Z}[1/p]$ we may consider the ideal $I^{\langle\lambda\rangle}A$ of the ring A, generated by $u(I^{\langle\lambda\rangle})$. We also define :

$$I^{\lfloor r\rfloor}A := \bigcap_{\substack{\mu\in\mathbb{Z}[1/p]\\ \mu<r}} I^{\langle\mu\rangle}A \qquad I^{\lceil s\rceil}A := \bigcup_{\substack{\mu\in\mathbb{Z}[1/p]\\ \mu>s}} I^{\langle\mu\rangle}A \qquad \text{for every } r \in \mathbb{R}.$$

From lemma 9.3.69(ii.a,ii.c) we deduce immediately :

$$(I^{\langle\lambda\rangle}A)^n = (I^{\langle\lambda\rangle})^n A = I^{\langle n\lambda\rangle}A \qquad I^{\langle\lambda\rangle} I^{\langle\mu\rangle} A = I^{\langle\lambda+\mu\rangle}A \qquad I^{\langle\lambda\rangle} J^{\langle\lambda\rangle} A = (IJ)^{\langle\lambda\rangle}A$$

for every ideals $I, J \subset P$ and every $\lambda, \mu \in \mathbb{N}[1/p]$ and every $n \in \mathbb{N}$.

(ii) If A is a perfect $\mathbb{F}_p$-algebra, we may take for u the identity map of the multiplicative monoid $P := (A, \cdot)$. In this case, notice that every ideal J of the ring A is also an ideal of the underlying monoid P, so the angular powers of J are well defined ideals of P; moreover :

$$(IA)^{\langle\lambda\rangle}A = I^{\langle\lambda\rangle}A \qquad \text{for every } \lambda \in \mathbb{N}[1/p] \text{ and every ideal } I \subset P$$

(details left to the reader). Combining with lemma 9.3.69(ii.c) we get

$$(I^{\langle\lambda\rangle}A)^{\langle\mu\rangle}A = I^{\langle\lambda\mu\rangle}A \qquad \text{for every } \lambda \in \mathbb{N}[1/p] \text{ and every ideal } I \subset P.$$

Furthermore, suppose that I is an ideal of the ring A generated (in the usual ring-theoretic sense) by a subset $S \subset A$ consisting of r elements, and let J_S be the ideal of P generated by S; then $I^n = J_S^n A$ and $I^{\langle\lambda\rangle}A = J_S^{\langle\lambda\rangle}A$ for every $n \in \mathbb{N}$ and $\lambda \in \mathbb{N}[1/p]$, so lemma 9.3.69(iv) yields

$$I^{\langle\lambda\rangle}A \subset I^n \qquad \text{for every } n \in \mathbb{N} \text{ and every } \lambda \in \mathbb{N}[1/p] \text{ such that } \lambda \geq n + r - 1.$$

(iii) If A is as in (ii), we may also take

$$P := (A, \cdot) \qquad \text{and} \qquad u := \tau_A : P \to W(A).$$

If $a_\bullet := (a_j \mid j \in J)$ is any system of elements of A generating an ideal I of the monoid P, we may regard we shall also use the notation

$$[a_\bullet]^{\langle\lambda\rangle} := I^{\langle\lambda\rangle}W(A).$$

(iv) For instance, if A is as in (ii), from lemma 9.3.69(ii.a) and (9.3.46) we see that

$$W(I^{\langle\lambda\rangle}, r) = \left\{\sum_{n=0}^{\infty} p^n \cdot \tau_A(b_n) \mid b_n \in I^{\langle\lambda\rangle}A \text{ for every } n \leq r\right\} \tag{9.3.71}$$

for every ideal $I \subset A$, every $\lambda \in \mathbb{N}[1/p]$ and every $r \in \mathbb{N}$ (notation of remark 9.3.28(iv)), and likewise we can describe $W(I^{\langle\lambda\rangle})$. Moreover, for every such I, notice that

$$p^k W(A) \cap W(I^{\langle\lambda\rangle}) = p^k W(I^{\langle\lambda\rangle}) \qquad \text{for every } \lambda \in \mathbb{N}[1/p] \text{ and every } k \in \mathbb{N}. \tag{9.3.72}$$

Indeed, say that $\underline{b} := (b_n \mid n \in \mathbb{N}) \in p^k W(A) \cap W(I^{\langle\lambda\rangle})$; in view of lemma 9.3.69(ii.a), this means that $b_n = 0$ for $n = 0, \dots, k-1$, and $b_n \in I^{\langle p^n\lambda\rangle}$ for every integer $n \geq k$. Clearly $\underline{c} := (b_{n+k}^{1/p^k} \mid n \in \mathbb{N}) \in W(I^{\langle\lambda\rangle})$, and $p^k \cdot \underline{c} = \underline{b}$, whence the claim. As an immediate consequence, we deduce that the 0-th ghost map $\boldsymbol{\omega}_0$ induces an A-linear isomorphism

$$W(I^{\langle\lambda\rangle})/pW(I^{\langle\lambda\rangle}) \xrightarrow{\sim} I^{\langle\lambda\rangle}A \qquad \text{for every } \lambda \in \mathbb{N}[1/p]. \tag{9.3.73}$$

Lemma 9.3.74. *Let A be any perfect $\mathbb{F}_p$-algebra, and $I, J \subset A$ two ideals. We have :*

(i) $I^{\lfloor r\rfloor}A = \overline{\text{i.c.}}(I, A, r)$ *for every $r \in \mathbb{R}_+$ (notation of remark 9.2.43).*

(ii) *Suppose that I and J are finitely generated, and let $(a_1, \dots, a_n)$ (resp. $(b_1, \dots, b_n)$) be a finite system of generators for I (resp. J). Let also $q > 1$ be a rational number with*

$$b_i - a_i \in \text{i.c.}(I, A, q) \qquad \textit{for every } i = 1, \dots, n$$

and suppose furthermore that

(a) *either, the radical of I equals the radical of J*

(b) *or else, I is contained in the Jacobson radical of A.*

Then $I^{\langle 1\rangle}A = J^{\langle 1\rangle}A$.

Proof. (i): We are easily reduced to showing that

$$I^{\langle\lambda\rangle}A \subset \text{i.c.}(I, A, \lambda) \subset I^{\langle\lambda'\rangle}A \qquad \text{for every } \lambda' \in \mathbb{N}[1/p] \text{ such that } \lambda' < \lambda.$$

To this aim, a direct inspection of the definition shows that

$$\Phi_A(\text{i.c.}(I, A, \lambda)) = \text{i.c.}(I, A, p\lambda) \qquad \text{and} \qquad \Phi_A(I^{\langle\lambda\rangle}A) = I^{\langle p\lambda\rangle}A \qquad \text{for every } \lambda \in \mathbb{N}[1/p]$$

so that we may replace λ by $p^n\lambda$ for a suitable $n \in \mathbb{N}$, after which we may assume that $\lambda \in \mathbb{N}$. Moreover, since the subset $S := \{k\lambda/p^n \mid k, n \in \mathbb{N}\}$ is dense in $\mathbb{N}[1/p]$, we may assume that $\lambda' \in S$. In this case, taking into account (9.2.44) and lemma 9.3.69(ii.a), we may further replace I by $I^{\langle\lambda\rangle}A$, λ by 1, and λ' by λ'/λ, so we reduce to checking that

$$I^{\langle 1\rangle}A \subset \text{i.c.}(I, A) \subset I^{\langle\mu\rangle}A \qquad \text{for every } \mu \in \mathbb{N}[1/p] \text{ such that } \mu < 1.$$

To this aim, say that $x \in I^{\langle 1\rangle}$; by definition, this means that $x^{p^n} \in I^{p^n}$ for some $n \in \mathbb{N}$, whence $x \in \text{i.c.}(I, A)$. Next, let $x \in \text{i.c.}(I, A)$, so that $x^n \in \sum_{i=0}^{n-1} x^i I^{n-i}$ for some $n \in \mathbb{N}$.

Claim 9.3.75. $x^m \in \sum_{i=0}^{n-1} x^i I^{m-i}$ for every $m \geq n$.

Proof of the claim. We argue by induction on $m - n$. If $m = n$, there is nothing to prove, so assume that the claim is already known for some $m \geq n$. In this case, we get

$$x^{m+1} \in \sum_{i=1}^{n} x^i I^{m+1-i} = x^n I^{m+1-n} + \sum_{i=1}^{n-1} x^i I^{m+1-i}.$$

Moreover, $x^n I^{m+1-n} \in \sum_{i=0}^{n-1} x^i I^{m+1-i}$, so the claim holds for $m + 1$. ◊

It follows from claim 9.3.75 that $x^{p^k} \in I^{p^k+1-n}$, whence $x \in I^{\langle 1-(n-1)/p^k\rangle}$ for every $k \in \mathbb{N}$ such that $p^k \geq n$, whence the contention.

(ii): Under the stated conditions, lemma 9.2.47 says that $\text{i.c.}(I, A) = \text{i.c.}(J, A)$. Together with (9.2.45), we deduce that

$$\text{i.c.}(I, A, q) = \text{i.c.}(J, A, q).$$

Hence $J \subset I + \text{i.c.}(I, A, q) \subset I^{\langle 1\rangle}A$ and likewise $I \subset J + \text{i.c.}(J, A, q) \subset J^{\langle 1\rangle}A$, whence the assertion. □

9.3.76. Let A be a perfect $\mathbb{F}_p$-algebra, $a_\bullet := (a_1, \dots, a_k)$ a finite system of elements of A, and $I \subset A$ (resp. $\mathscr{J} \subset W(A)$) the ideal generated by $a_\bullet$ (resp. by $\tau_A(a_1), \dots, \tau_A(a_k)$). Set

$$\mathscr{I} := pW(A) + \mathscr{J} \qquad W(\lambda) := W(I^{\langle\lambda\rangle}) \qquad \text{for every } \lambda \in \mathbb{N}[1/p]$$

(notation of remark 9.3.28(iv)). Lastly, denote by $\mathscr{T}_A$ (resp. $\mathscr{T}_{W(A)}$) the topology of A (resp. of $W(A, \mathscr{T}_A)$). We complement the generalities of lemma 9.3.69 with the following:

Proposition 9.3.77. *In the situation of* (9.3.76)*, the following holds :*

(i) $[a_\bullet]^{\langle\lambda\rangle} \subset W(\lambda) \subset [a_\bullet]^{\langle\lambda'\rangle}$ *for every* $\lambda' < \lambda$ *in* $\mathbb{N}[1/p]$ *(notation of remark* 9.3.70(iii)*).*

(ii) *If* $\mathscr{T}_A$ *agrees with the* I*-adic topology,* $\mathscr{T}_{W(A)}$ *agrees with the* $\mathscr{I}$*-adic topology.*

Proof. (ii): Notice that $\tau_A(b^\mu) = \tau_A(b)^\mu$ for every $b \in A$ and every $\mu \in \mathbb{N}[1/p]$. Taking into account (9.3.71) and proposition 9.3.62, we deduce that

$$W(\lambda, r) := W(I^{\langle\lambda\rangle}, r) = p^{r+1}W(A) + [a_\bullet]^{\langle\lambda\rangle} \qquad \text{every } r \in \mathbb{N}$$

which, in light of lemma 9.3.69(iv), shows that the linear topology induced by the cofiltered system of ideals $(W(\lambda, r) \mid r \in \mathbb{N},\ \lambda \in \mathbb{N}[1/p])$ agrees with the $\mathscr{I}$-adic topology. But by the same token we see that the latter family is a fundamental system of open neighborhoods of zero in $W(A)$, whence the contention.

(i): Since $W(\lambda, r)$ is an open ideal in the p-adic topology of $W(A)$ for every $r \in \mathbb{N}$, we see that $W(\lambda)$ is a closed ideal in $W(A)$ for the p-adic topology of $W(A)$. On the other hand, from (9.3.72) it follows that the p-adic topology of $W(\lambda)$ agrees with the topology induced from the p-adic topology of $W(A)$. Taking into account proposition 9.3.44(ii), we deduce that $W(\lambda)$ is complete and separated for its p-adic topology, and then (9.3.73) and claim 9.3.60 imply that $W(\lambda)$ is a topologically free $\mathbb{Z}_p$-module for this topology, and any lifting of an $\mathbb{F}_p$-basis of $I^{\langle\lambda\rangle}A$ yields a topological basis. However, clearly we may find :

- a subset $S \subset \mathbb{N}[1/p]^{\oplus k}$ consisting of sequences $(\mu_1, \dots, \mu_k)$ such that $\mu_1 + \cdots + \mu_k = \lambda$
- a system of elements $(b_\mu \mid \mu \in S)$ of A such that the family $(b_\mu \cdot a^\mu \mid \mu \in S)$ is an $\mathbb{F}_p$-basis of $I^{\langle\lambda\rangle}A$, where $a^\mu := a_1^{\mu_1} \cdots a_k^{\mu_k}$ for every $\mu \in S$

so that the system $(\tau_A(b_\mu \cdot a^\mu) \mid \mu \in S)$ is a topological $\mathbb{Z}_p$-basis of $W(\lambda)$. Now, clearly $[a_\bullet]^{\langle\lambda\rangle} \subset W(\lambda)$; to check the second inclusion, let $\underline{w} \in W(\lambda)$ be any element. By construction, there exists a unique system $(d_\mu \mid \mu \in S)$ of elements of $\mathbb{Z}_p$ such that

$$\underline{w} = \sum_{\mu \in S} d_\mu \cdot \tau_A(b_\mu \cdot a^\mu)$$

where the series converges in the p-adic topology of $W(A)$, and the set $\{\mu \in S \mid d_\mu \notin p^n\mathbb{Z}_p\}$ is finite for every $n \in \mathbb{N}$. Pick any $N \in \mathbb{N}$ such that $\lambda - \lambda' \geq kp^{-N}$, and for every $r \in \mathbb{R}$ define

$$\overline{r} \in p^{-N}\mathbb{Z} \qquad r^* \in [0, p^{-N}[\qquad \text{fulfilling the condition :} \qquad r = \overline{r} + r^*.$$

Then, for every $\mu \in S$ let also $\overline{\mu} := (\overline{\mu}_1, \dots, \overline{\mu}_k)$, $\mu^* := (\mu_1^*, \dots, \mu_k^*)$, and set

$$S' := \{\sigma \in p^{-N}\mathbb{N}^{\oplus k} \mid \lambda \geq \sigma_1 + \cdots + \sigma_k > \lambda'\}.$$

With this notation, it is easily seen that $\overline{\mu} \in S'$ for every $\mu \in S$. Hence, let also

$$c_\sigma := \sum_{\substack{\mu \in S \\ \overline{\mu} = \sigma}} d_\mu \cdot \tau_A(b_\mu \cdot a^{\mu^*}) \qquad \text{for every } \sigma \in S'$$

(notice that this series converges p-adically in $W(A)$, so c_σ is well defined); we get

$$\underline{w} = \sum_{\sigma \in S'} c_\sigma \cdot \tau_A(a^\sigma).$$

To conclude, it suffices to remark that S' is a finite set, and $a^\sigma \in I^{\langle\lambda'\rangle}A$ for every $\sigma \in S'$. □

Example 9.3.78. Let $A := \mathbb{F}_p[T^{1/p^\infty}]$, endow A with its T-adic topology $\mathscr{T}_T$, and let $(A^\wedge, \mathscr{T}_T^\wedge)$ be the completion of $(A, \mathscr{T}_T)$. According to lemma 9.3.33(iv), the topological ring $W(A^\wedge, \mathscr{T}_T^\wedge)$ is the completion of $W(A, \mathscr{T}_T)$. In light of example 9.3.48(v) and proposition 9.3.77(ii), the latter is naturally isomorphic to the (p,T)-adic completion of $\mathbb{Z}_p\{T^{1/p^\infty}\}$, or equivalently, the (p,T)-adic completion of $\mathbb{Z}_p[T^{1/p^\infty}]$. Explicitly, this is the ring of all power series (9.3.49) where $(\lambda_n \mid n \in \mathbb{N})$ is any sequence of elements of $\mathbb{N}[1/p]$, and $(a_n \mid n \in \mathbb{N})$ is any sequence of elements of $\mathbb{Z}_p$, with the property that

$$\lim_{n\to\infty} v_p(a_n) + \lambda_n = \infty$$

(where $v_p : \mathbb{Z}_p \to \mathbb{N} \cup \{\infty\}$ is the p-adic valuation).

Corollary 9.3.79. *Let $(\Lambda, \leq)$ be a filtered set with an initial element λ_0, and $A_\bullet := (A_\lambda \mid \lambda \in \Lambda)$ a system of perfect $\mathbb{F}_p$-algebras; let $I_0 \subset A_{\lambda_0}$ be an ideal of finite type. Endow A_{λ_0} and the colimit A of $A_\bullet$ with their I_0-adic topologies $\mathscr{T}_0$, respectively $\mathscr{T}_A$; also, endow the colimit $\mathscr{W}$ of the induced system of rings $(W(A_\lambda) \mid \lambda \in \Lambda)$ with the unique $\mathscr{W}$-linear topology $\mathscr{T}_\mathscr{W}$ such that the natural map $W(A_{\lambda_0}, \mathscr{T}_0) \to (\mathscr{W}, \mathscr{T}_\mathscr{W})$ is adic. Then the induced map of separated completions :*

$$(\mathscr{W}, \mathscr{T}_\mathscr{W})^\wedge \to W((A, \mathscr{T}_A)^\wedge)$$

is an isomorphism of topological rings.

Proof. Let $a_1, \dots, a_k$ be a finite set of generators of I_0, and denote by $\mathscr{I}_0 \subset W(A_{\lambda_0})$ the ideal generated by $p, \tau_{A_{\lambda_0}}(a_1), \dots, \tau_{A_{\lambda_0}}(a_k)$. Then the topologies of $W(A_{\lambda_0}, \mathscr{T}_0)$ and of $W(A, \mathscr{T}_A)$ are $\mathscr{I}_0$-adic, by proposition 9.3.77(ii). Moreover, the natural map $W(A, \mathscr{T}_A)^\wedge \to W((A, \mathscr{T}_A)^\wedge)$ is an isomorphism of topological rings (lemma 9.3.33(iv)). Clearly the natural continuous map

map $\varphi : (\mathscr{W}, \mathscr{T}_{\mathscr{W}}) \to W(A, \mathscr{T}_A)$ has dense image; thus, we are reduced to checking that the $\mathscr{I}_0$-adic topology of $W(A)$ induces the $\mathscr{I}_0$-adic topology on $\mathscr{W}$ via φ (theorem 8.2.8(iii)).

However, the proof of proposition 9.3.77(ii) shows that the $\mathscr{I}_0$-adic topology on $W(A)$ admits the fundamental system of open ideals $(W(I_0^{\langle\mu\rangle}A, r) \mid \mu \in \mathbb{N}[1/p], r \in \mathbb{N})$, (see remark 9.3.70(iv)), and likewise for the $\mathscr{I}_0$-adic topology on $W(A_{\lambda_0})$. Now, $\mathscr{W}$ is the set of equivalence classes $[\lambda, \underline{w}]$ of pairs $(\lambda, \underline{w})$ with $\lambda \in \Lambda$ and $\underline{w} := (w_n \mid n \in \mathbb{N}) \in W(A_\lambda)$; we have $\varphi([\lambda, \underline{w}]) \in W(I_0^{\langle\mu\rangle}A, r)$ if and only if there exists $\lambda' \geq \lambda$ such that the image of w_i lies in $I_0^{\langle\mu\rangle}A_{\lambda'}$ for $i = 0, \dots, r$, and the latter holds if and only if $[\lambda, \mu] \in W(I_0^{\langle\mu\rangle}, r) \cdot \mathscr{W}$, whence the assertion. $\square$

9.3.80. *Semi-norms on the Witt ring.* Let A be any topological ring, $|\cdot|_A$ a real-valued semi-norm on A (see definition 9.1.3(v)). For any non-empty subset $S \subset A$ we let

$$|S|_A := \sup_{a \in S} |a|_A \in \mathbb{R}_+ \cup \{+\infty\}.$$

Also, for every real number $\rho \in [0, 1]$ we consider the mapping

$$|\cdot|_\rho : W(A) \to \mathbb{R}_+ \cup \{+\infty\} \qquad (a_n \mid n \in \mathbb{N}) \mapsto \sup_{n \in \mathbb{N}} |a_n|_A^{p^{-n}} \cdot \rho^n.$$

For instance, notice that $|0|_\rho = 0$ and $|1|_\rho = 1$; more generally, we have $|\tau_A(a)|_\rho = |a|_A$ for every $a \in A$. We let

$$W(A, \rho) := \{\underline{a} \in W(A) \mid |\underline{a}|_\rho \in \mathbb{R}_+\}.$$

Lemma 9.3.81. *With the notation of* (9.3.80)*, the following holds :*

(i) $W(A, \rho)$ *is an additive subgroup of* $W(A)$ *for every* $\rho \in [0, 1]$.
(ii) *Both* $W(A, 0)$ *and* $W(A, 1)$ *are subrings of* $W(A)$.
(iii) $|\underline{a} - \underline{b}|_\rho \leq \max(|\underline{a}|_\rho, |\underline{b}|_\rho)$ *for every* $\underline{a}, \underline{b} \in W(A)$.
(iv) $|\underline{a} \cdot \underline{b}|_{\rho_1\rho_2} \leq |\underline{a}|_{\rho_1} \cdot |\underline{b}|_{\rho_2}$ *for every* $\underline{a}, \underline{b} \in W(A)$ *and every* $\rho_1, \rho_2 \in [0, 1]$.
(v) $\rho^p \cdot |F_A(\underline{a})|_{\rho^p} \leq |\underline{a}|_\rho^p$ *for every* $\underline{a} \in W(A)$ *and every* $\rho \in [0, 1]$.
(vi) $|V_A(\underline{a})|_{\rho^{1/p}} = \rho^{1/p} \cdot |\underline{a}|_\rho^{1/p}$ *for every* $\underline{a} \in W(A)$ *and every* $\rho \in [0, 1]$.
(vii) $|\tau_A(a) \cdot \underline{b}|_\rho \leq |a|_A \cdot |\underline{b}|_\rho$ *for every* $a \in A$ *and every* $\underline{b} \in W(A)$*, and if* $|\cdot|_A$ *is a valuation, this inequality is actually an equality.*

Proof. Let $\underline{a} := (a_n \mid n \in \mathbb{N}), \underline{b} := (b_n \mid n \in \mathbb{N})$ be any two elements of $W(A)$, and suppose that $M := \max(|\underline{a}|_\rho, |\underline{b}|_\rho) \in \mathbb{R}_+$. This condition means that

$$|a_n|_A, |b_n|_A \leq (M/\rho^n)^{p^n} \qquad \text{for every } n \in \mathbb{N}.$$

Set $\underline{c} := \underline{a} + \underline{b}$, $\underline{d} := \underline{a} \cdot \underline{b}$ and recall that $\underline{c} = (c_n \mid n \in \mathbb{N})$ (resp. $\underline{d} = (d_n \mid n \in \mathbb{N})$), where $c_n = S_n(a_0, \dots, a_n, b_0, \dots, b_n)$ (resp. $d_n = P_n(a_0, \dots, a_n, b_0, \dots, b_n)$: notation of (9.3.7)) for every $n \in \mathbb{N}$. By construction, S_n is a $\mathbb{Z}$-linear combination of monomials $Q(X_0, \dots, X_n, Y_0, \dots, Y_n)$ of total degree p^n (remark 9.3.8(i)); we claim that

$$r := |Q(a_0, \dots, a_n, b_0, \dots, b_n)|_A \leq (M/\rho^n)^{p^n} \qquad \text{for every such } Q.$$

Indeed, say that $Q = \prod_{i=0}^n X_i^{n_i} Y_i^{m_i}$; then $\sum_{i=0}^n (n_i + m_i)p^i = p^n$, and therefore

$$r \leq \prod_{i=0}^n (M/\rho^i)^{(n_i+m_i)p^i} = M^{p^n} \cdot \rho^{-t} \qquad \text{where } t := \sum_{i=0}^n i \cdot (n_i + m_i)p^i \leq np^n.$$

Since $\rho \leq 1$, the claim follows. Lastly, arguing with the polynomials I_n of (9.3.7) that are bihomogeneous of degree $(p^n, 0)$, we get as well $|-\underline{a}|_\rho \leq |\underline{a}|_\rho$, and therefore $|-\underline{a}|_\rho = |\underline{a}|_\rho$ for every $\underline{a} \in W(A)$. Assertions (i) and (iii) are immediate consequences.

(iv): Likewise, P_n is a $\mathbb{Z}$-linear combination of monomials R of bidegree (p^n, p^n), and a similar calculation shows that

$$|R(a_0, \dots, a_n, b_0, \dots, b_n)|_A \leq |\underline{a}|_{\rho_1} \cdot |\underline{b}|_{\rho_2} \cdot (\rho_1 \rho_2)^{-np^n} \qquad \text{for every such } R$$

(details left to the reader), whence the assertion. (ii) now follows from (iii) and (iv).

(v) is proven in the same way : for every $n \in \mathbb{N}$ one estimates $|F_n(a_0, \dots, a_{n+1})|$ by writing F_n as a sum of bihomogeneous monomials of degree $(p^{n+1}, 0)$: the details are left to the reader.

(vi): Set $M := |\underline{a}|_\rho$, fix a real number $\varepsilon > 0$, and pick $n \in \mathbb{N}$ such that $|a_n|_A^{p^{-n}} \cdot \rho^n > M - \varepsilon$. Then we have

$$|a_{k-1}|_A \leq (M/\rho^{k-1})^{p^{k-1}} = (M^{1/p})^{p^k} \cdot \rho^{-(k-1)p^{k-1}} \qquad \text{for every } k > 0$$

which is equivalent to

$$|a_{k-1}|_A^{p^{-k}} \cdot \rho^{k/p} \leq (\rho M)^{1/p} \qquad \text{for every } k > 0.$$

On the other hand, we have $|a_n|_A^{p^{-n-1}} \cdot \rho^{(n+1)/p} > (\rho(M-\varepsilon))^{1/p}$ whence the contention.

(vii) follows directly from proposition 9.3.36(i). □

Proposition 9.3.82. *In the situation of* (9.3.80)*, suppose that A is an $\mathbb{F}_p$-algebra. We have:*

(i) $|F_A(\underline{a})|_{\rho^p} = |\underline{a}|_\rho^p$ *for every $\underline{a} \in W(A)$ and every $\rho \in [0,1]$.*

(ii) *$W(A,\rho)$ is a subring of $W(A)$, and $|\cdot|_\rho$ restricts to a semi-norm on $W(A,\rho)$.*

(iii) *If moreover $|\cdot|_A$ is a valuation, the same holds for the restriction of $|\cdot|_\rho$ to $W(A,\rho)$.*

(iv) *If A is perfect and $|\cdot|_A$ is a valuation, we have*

$$\Big|\sum_{n\in\mathbb{N}} p^n \cdot \tau_A(b_n)\Big|_\rho = \sup_{n\in\mathbb{N}} |b_n|_A \cdot \rho^n \qquad \textit{for every sequence } (b_n \mid n \in \mathbb{N}) \textit{ of elements of } A.$$

Proof. Assertion (i) (resp. (iv)) follows easily from (9.3.40) (resp. from (9.3.46)).

(iii): For $\rho = 0$, we have $|\underline{a}|_\rho = |a_0|_A$ for every $\underline{a} := (a_n \mid n \in \mathbb{N}) \in W(A)$, so the assertion is clear in this case. Hence, suppose that $\rho \in]0,1]$, and let $\underline{a}, \underline{a}' \in W(A,\rho)$ be any two elements. It is clear that $\underline{a} \in W(A, \rho - \varepsilon)$ for every $\varepsilon \in [0, \rho]$, and moreover

$$\lim_{\varepsilon\to 0} |\underline{a}|_{\rho-\varepsilon} = |\underline{a}|_\rho. \tag{9.3.83}$$

Furthermore, it is easily seen that, if $\varepsilon > 0$, there exist minimal $N, N' \in \mathbb{N}$ such that

$$|a_N|_A^{p^{-N}} \cdot (\rho - \varepsilon)^N = |\underline{a}|_{\rho-\varepsilon} \qquad \text{and} \qquad |a'_{N'}|_A^{p^{-N'}} \cdot (\rho-\varepsilon)^{N'} < |\underline{a}'|_{\rho-\varepsilon}.$$

We have to check that $|\underline{a} \cdot \underline{a}'|_\rho = |\underline{a}|_\rho \cdot |\underline{a}'|_\rho$, and in light of (9.3.83) it suffices to show that $|\underline{a} \cdot \underline{a}'|_{\rho-\varepsilon} = |\underline{a}|_{\rho-\varepsilon} \cdot |\underline{a}'|_{\rho-\varepsilon}$ for every $\varepsilon \in [0,\rho]$. We may then replace ρ by $\rho - \varepsilon$, and assume from start that N, N' are the minimal integers such that

$$|a_N|_A^{p^{-N}} \cdot \rho^N = |\underline{a}|_\rho \qquad \text{and} \qquad |a'_{N'}|_A^{p^{-N'}} \cdot \rho^{N'} < |\underline{a}'|_\rho.$$

With this notation, write $\underline{a} = \underline{b} + V_A^N(\underline{c})$, with $\underline{b} := (a_0, \dots, a_{N-1}, 0, \dots)$, $\underline{c} := (a_N, a_{N+1}, \dots)$. By remark 9.1.4(iii) we have

$$|\underline{b}|_\rho < |\underline{a}|_\rho \qquad \text{and therefore} \qquad |\underline{a}|_\rho = |V_A^N(\underline{c})|_\rho$$

from which we deduce that $|\underline{b} \cdot \underline{a}'|_\rho < |\underline{a}|_\rho \cdot |\underline{a}'|_\rho$, so – again by remark 9.1.4(iii) – we are reduced to checking that $|V_A^N(\underline{c})|_\rho \cdot |\underline{a}'|_\rho = |V_A^N(\underline{c}) \cdot \underline{a}'|_\rho$. But proposition 9.3.22(ii) and lemma 9.3.81(v) yield

$$|V_A^N(\underline{c}) \cdot \underline{a}'|_\rho = |V_A^N(\underline{c} \cdot F_A^N(\underline{a}'))|_\rho = |\underline{c} \cdot F_A^N(\underline{a}')|_{\rho^{p^N}}^{1/p^N} \cdot \rho^N$$

and on the other hand

$$|V_A^N(\underline{c})|_\rho = |\underline{c}|_{\rho^{p^N}}^{1/p^N} \cdot \rho^N \qquad \text{and} \qquad |F_A^N(\underline{a}')|_{\rho^{p^N}}^{1/p^N} = |\underline{a}'|_\rho$$

by lemma 9.3.81(v) and (i). Say that $\underline{a}'' := F_A^N(\underline{a}')$; then notice that – in view of (9.3.40) – the integer N' is still the smallest one such that

$$|a''_{N'}|_A^{p^{-N'}} \cdot \rho^{N'} = |\underline{a}''|_\rho.$$

Summing up, we may replace $\underline{a}$ and $\underline{a}'$ by $\underline{c}$ and respectively $\underline{a}''$, and ρ by ρ^N, and assume from start that $N = 0$, *i.e.* that $|a_0|_A = |\underline{a}|_\rho$. Likewise, by repeating the same considerations on $\underline{a}'$, we may assume that $|a'_0|_A = |\underline{a}'|_\rho$. Lastly, notice that

$$|\underline{a} \cdot \underline{a}'|_\rho \geq |a_0 a'_0|_A = |a_0|_A \cdot |a'_0|_A = |\underline{a}|_\rho \cdot |\underline{a}'|_\rho$$

and the converse inequality is already known, by (ii), whence the contention.

(ii): Let $\underline{a} := (a_n \mid n \in \mathbb{N})$ and $\underline{b} := (b_n \mid n \in \mathbb{N})$ be any two elements of $W(A, \rho)$; we have to show that $|\underline{a} \cdot \underline{b}|_\rho \leq |\underline{a}|_\rho \cdot |\underline{b}|_\rho$. However, from propositions 9.3.36(iii) and 9.3.44(iv) we get

$$\underline{a} \cdot \underline{b} \equiv \sum_{i+j=n} V_A^{i+j}(\tau_A(a^{p^j} b^{p^i})) \pmod{V_A^{n+1}} \qquad \text{for every } n \in \mathbb{N}$$

and in view of lemma 9.3.81(iii) we are then easily reduced to the case where $\underline{a} = V_A^i(\tau_A(x))$ and $\underline{b} = V_A^j(\tau_A(y))$ for some $i, j \in \mathbb{N}$ and $x, y \in A$. We compute :

$$\begin{aligned} |V_A^{i+j}(\tau_A(x^{p^j} y^{p^i}))|_\rho &= \rho^{i+j} \cdot |\tau_A(x^{p^j} y^{p^i})|_{\rho^{p^{i+j}}}^{1/p^{i+j}} && \text{(by lemma 9.3.81(vi))} \\ &= \rho^{i+j} \cdot |x^{p^j} y^{p^i}|_A^{1/p^{i+j}} \\ &\leq \rho^{i+j} \cdot |x|_A^{1/p^i} \cdot |y|_A^{1/p^j} \\ &= |V_A^i(\tau_A(x))|_\rho \cdot |V_A^j(\tau_A(y))|_\rho && \text{(again by lemma 9.3.81(vi))} \end{aligned}$$

as required. □

Remark 9.3.84. (i) In spite of the inequalities of lemma 9.3.81(vii) and proposition 9.3.82(iv), the map $\tau_A : A \to W(A, \rho)$ is not necessarily continuous for the topology on $W(A, \rho)$ induced by the semi-norm $|\cdot|_\rho$, unless $\rho = 0$.

(ii) Normed Witt vectors similar to those introduced in (9.3.80) have been previously considered in the literature : see *e.g.* [53], [113], [127].

Lemma 9.3.85. *Let E be a perfect topological $\mathbb{F}_p$-algebra, $I \subset E$ any ideal, and denote by $\mathscr{R}_E$ the set of all real-valued valuations on E with $|E| = 1$ (notation of* (9.3.80)*). For every $r, r' \in \mathbb{R}_+$ we have :*

(i) $|I^{\lfloor r \rfloor}|_E = |I|_E^r$ *for every* $|\cdot|_E$ *in* $\mathscr{R}_E$.
(ii) $(I^{\lfloor r \rfloor})^{\lfloor r' \rfloor} = I^{\lfloor rr' \rfloor}$.
(iii) $I^{\lfloor r \rfloor} = (I^{\lfloor r \rfloor})^{\langle 1 \rangle}$.
(iv) $W(I^{\lfloor r \rfloor} E) = \{\underline{a} \in W(E) \mid |\underline{a}|_1 \leq |I|^r$ *for every* $|\cdot|$ *in* $\mathscr{R}_E\}$.

Proof. (i): Clearly $|I^{\lfloor r \rfloor}|_E \leq |I|_E^r$. For the converse inequality, fix any real number $\varepsilon > 0$, and pick $y \in I$ such that $|y|_E^r > |I|_E^r - \varepsilon$. Since $\mathbb{N}[1/p]$ is dense in $\mathbb{R}_+$, we may find $\lambda \in \mathbb{N}[1/p]$ such that $\lambda \geq r$ and $|y|_E^\lambda > |I|_E^r - \varepsilon$. It follows that $|y^\lambda| = |y|^\lambda \leq |I|^r$ for every $|\cdot|$ in $\mathscr{R}_E$, and $|y^\lambda|_E > |I|_E^r - \varepsilon$, so that $|I^{\lfloor r \rfloor}| > |I|_E^r - \varepsilon$, as needed.

(ii): Indeed, by lemmata 9.2.46 and 9.3.74(i), the ideal $(I^{\lfloor r \rfloor})^{\lfloor r' \rfloor}$ (resp. $I^{\lfloor rr' \rfloor}$) is the set of all $x \in E$ such that $|x| \leq |I^{\lfloor r \rfloor}|^{r'}$ (resp. $|x| \leq |I|^{rr'}$) for every $|\cdot|$ in $\mathscr{R}_E$. But from (i) we get $|I^{\lfloor r \rfloor}|^{r'} = |I|^{rr'}$ for every such $|\cdot|$, whence the assertion.

(iii): This is obvious for $r = 0$, and for $r > 0$, we have $I^{\lfloor r \rfloor} \subset (I^{\lfloor r \rfloor})^{\langle 1 \rangle} \subset (I^{\lfloor r \rfloor})^{\lfloor 1 \rfloor}$ by lemma 9.3.74(i), and then the assertion follows from (ii).

(iv) follows by combining (iii), (9.3.71) and proposition 9.3.82(iv). □

9.4. **Fontaine rings.** In this section we revisit and expand Fontaine and Wintenberger's theory of the *fields of norms* of a local field of characteristic zero, which we frame as a special case of a simple and very general construction that endows every topological $\mathbb{F}_p$-algebra A with a universal perfect cover $\mathbf{E}(A) \to A$; indeed, $\mathbf{E}$ shall be a right adjoint to the inclusion of the subcategory of perfect topological $\mathbb{F}_p$-algebras into the category of all topological $\mathbb{F}_p$-algebras. In section 16.1, this functor will play an important role in the study of perfectoid rings.

Definition 9.4.1. (i) A *topological monoid* is the datum $(P, \mu, \mathscr{T})$ of a monoid (P, μ) and a topology $\mathscr{T}$ on the set P, such that the composition law $\mu : P \times P \to P$ of P is a continuous map (where $P \times P$ is endowed with the product topology). A morphism of topological monoids $(P, \mu, \mathscr{T}) \to (P', \mu', \mathscr{T}')$ is a morphism of monoids $P \to P'$ that is continuous for the topologies $\mathscr{T}$ and $\mathscr{T}'$. Clearly the topological monoids and their morphisms form a category

$$\mathsf{TopMon}.$$

(ii) Let p be any prime integer. We say that $(P, \mu, \mathscr{T})$ is a *p-perfect* topological monoid if the p-Frobenius endomorphism $\boldsymbol{p}_P : P \to P$ is an isomorphism of topological monoids (see definition 4.8.40(ii)). This means that P is uniquely p-divisible, and $\boldsymbol{p}_P$ is a homeomorphism for the topology $\mathscr{T}$. We denote by

$$p\text{-}\mathsf{TopMon}$$

the full subcategory of TopMon whose objects are the p-perfect topological monoids.

Remark 9.4.2. Using the criterion of proposition 1.2.22(i) is easily seen that TopMon is a complete category : indeed, any family $((P_i, \mathscr{T}_i) \mid i \in I)$ of topological monoids admits a product P, which is just the product P of the underlying topological spaces, endowed with the unique composition law such that all the projections $P \to P_i$ are morphisms of monoids. Also, if $f, g : (P, \mathscr{T}) \to (P', \mathscr{T}')$ is any pair of morphisms in TopMon, the equalizer of f and g is the equalizer of the underlying morphisms of monoids, endowed with the topology induced from $\mathscr{T}$: details left to the reader. Likewise, one checks that TopMon is also cocomplete.

Proposition 9.4.3. *The inclusion functor* p-TopMon $\to$ TopMon *admits both a right and a left adjoint, denoted respectively*

$$\mathbf{E} : \mathsf{TopMon} \to p\text{-}\mathsf{TopMon} \qquad \textit{and} \qquad \mathbf{E}^* : \mathsf{TopMon} \to p\text{-}\mathsf{TopMon}.$$

Proof. Let $(P, \mathscr{T})$ be any topological monoid; we let $((P_n, \mathscr{T}_n); \varphi_n : P_{n+1} \to P_n \mid n \in \mathbb{N})$ be the system of topological monoids such that $(P_n, \mathscr{T}_n) := (P, \mathscr{T})$ and φ_n is the p-Frobenius endomorphism of P for every $n \in \mathbb{N}$. In light of remark 9.4.2 we may set

$$(\mathbf{E}(P), \mathscr{T}_{\mathbf{E}(P)}) := \lim_{n \in \mathbb{N}} (P_n, \mathscr{T}_n) \qquad (\mathbf{E}^*(P), \mathscr{T}^*_{\mathbf{E}(P)}) := \operatorname*{colim}_{n \in \mathbb{N}} (P_n, \mathscr{T}_n).$$

So $\mathbf{E}(P)$ is the set of all sequences $(a_n \mid n \in \mathbb{N})$ of elements $a_n \in P$ such that $a_n = a_{n+1}^p$ for every $n \in \mathbb{N}$. Any morphism of topological monoids $f : P \to Q$ induces a morphism of topological monoids

$$\mathbf{E}(f) : \mathbf{E}(P) \to \mathbf{E}(Q) \qquad (a_n \mid n \in \mathbb{N}) \mapsto (f(a_n) \mid n \in \mathbb{N}).$$

Moreover, clearly the Frobenius endomorphism $\boldsymbol{p}_{\mathbf{E}(P)}$ is an automorphism of $\mathbf{E}(P)$ with inverse $\mathbf{f}$ given explicitly by the rule :

$$\mathbf{f}(a_n \mid n \in \mathbb{N}) = (a_{n+1} \mid n \in \mathbb{N}) \qquad \text{for every } (a_n \mid n \in \mathbb{N}) \in \mathbf{E}(P).$$

In other words, $\mathbf{f}$ is the limit of the system of morphisms $(\mathbf{1}_P : P_n \to P_{n+1} \mid n \in \mathbb{N})$, so it is a continuous map, and therefore $\boldsymbol{p}_{\mathbf{E}(P)}$ is a homeomorphism. Furthermore, the projection to P_0 defines a natural morphism of topological monoids

$$\overline{u}_P : \mathbf{E}(P) \to P \qquad (a_n \mid n \in \mathbb{N}) \mapsto a_0 \tag{9.4.4}$$

and if P is already p-perfect, then clearly $\overline{u}_P$ is an isomorphism of topological monoids. Lastly, let $f : Q \to P$ be any morphism of topological monoids, such that Q is p-perfect. There follows a well defined morphism $f^\dagger := \mathbf{E}(f) \circ \overline{u}_Q^{-1} : Q \to \mathbf{E}(P)$ of topological monoids. Conversely, to any $g : Q \to \mathbf{E}(P)$ we may attach the morphism $g^\dagger := \overline{u}_P \circ g : Q \to P$. It is easily seen that the rules $f \mapsto f^\dagger$ and $g \mapsto g^\dagger$ are mutually inverse bijections, which concludes the construction of the right adjoint.

Next, let us check that the inverse g of $\boldsymbol{p}_{\mathbf{E}^*(P)}$ is continuous. Indeed, any element $\alpha \in \mathbf{E}^*(P)$ is the class of a pair (a, n), for some $n \in \mathbb{N}$ and some $a \in P_n$, and $\mathrm{g}(\alpha)$ is then the class of $(a, n+1)$; in other words, g is the colimit of the system of morphisms $(\mathbf{1}_P : P_n \to P_{n+1} \mid n \in \mathbb{N})$, whence the contention. This shows that $\mathbf{E}^*(P)$ is an object of p-TopMon, and obviously the rule $P \mapsto \mathbf{E}^*(P)$ yields a well defined functor as sought. Furthermore, the map $j_0 : P \to \mathbf{E}^*(P)$ defines a natural transformation which is an isomorphism in case P is already p-perfect. Now, the definition of an adjunction between $\mathbf{E}^*$ and the inclusion functor is, *mutatis mutandis*, as in the foregoing case : the details shall be left to the reader. □

Remark 9.4.5. (i) By construction, the natural transformation $P \mapsto \overline{u}_P$ defined in (9.4.4) is the counit of the adjunction exhibited by proposition 9.4.3. Likewise, the unit of this adjunction is the transformation $Q \mapsto \overline{u}_Q^{-1}$ for every p-perfect topological monoid Q.

(ii) Taking into account the triangular identities of (1.1.13), we deduce from (i) the identity

$$\mathbf{E}(\overline{u}_P) = \overline{u}_{\mathbf{E}(P)} : \mathbf{E}(\mathbf{E}(P)) \to \mathbf{E}(P)$$

and this map is an isomorphism of p-perfect topological monoids.

(iii) For every topological monoid P, we have

$$\mathbf{E}(P)^\times = \overline{u}_P^{-1}(P^\times).$$

Indeed, the inclusion $\mathbf{E}(P)^\times \subset \overline{u}_P^{-1}(P^\times)$ is obvious. For the converse, suppose that $x := (x_n \mid n \in \mathbb{N}) \in \mathbf{E}(P)$ and $\overline{u}_P(x) \in P^\times$. This means that $x_0 \in P^\times$, and since $x_{n+1}^p = x_n$ for every $n \in \mathbb{N}$, it follows that $x_n \in P^\times$ for every $n \in \mathbb{N}$, so $x \in \mathbf{E}(P)^\times$.

(iv) Likewise, it is easily seen that an element $x \in \mathbf{E}(P)$ is regular if and only if the same holds for $\overline{u}_P(x)$ (see example 4.8.36(i)).

Example 9.4.6. Endow the abelian group $\mathbb{Q}_p/\mathbb{Z}_p$ (resp. $\mathbb{Q}_p$) with its discrete topology (resp. with its usual p-adic topology); then there exists a natural isomorphism $\mathbf{E}(\mathbb{Q}_p/\mathbb{Z}_p) \xrightarrow{\sim} \mathbb{Q}_p$, that identifies the map $\overline{u}_{\mathbb{Q}_p/\mathbb{Z}_p}$ with the projection $\mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p$ (verification left to the reader).

9.4.7. Consider now the category of *topological* $\mathbb{F}_p$*-algebras*

$$\mathbb{F}_p\text{-}\mathsf{TopAlg}$$

(where $\mathbb{F}_p$ is endowed with its discrete topology : see definition 8.3.1(iii)). If R is an object of $\mathbb{F}_p$-TopAlg, the p-Frobenius of the multiplicative monoid $(R, \cdot)$ is the Frobenius endomorphism Φ_R of the ring R, and a simple inspection of the proof of proposition 9.4.3 reveals that the composition law of $\mathbf{E}(R, \cdot, \mathscr{T})$ is the multiplication map for a natural structure of topological ring. Moreover, for any morphism of topological $\mathbb{F}_p$-algebras $f : R \to S$, the map $\mathbf{E}(f) : \mathbf{E}(R) \to \mathbf{E}(S)$ is a morphism of topological $\mathbb{F}_p$-algebras. In other words, $\mathbf{E}$ lifts to a functor

$$\mathbf{E} : \mathbb{F}_p\text{-}\mathsf{TopAlg} \to \mathbb{F}_p\text{-}\mathsf{TopAlg}_{\mathrm{perf}} \qquad (9.4.8)$$

to the full subcategory of $\mathbb{F}_p$-TopAlg whose objects are the *perfect* topological $\mathbb{F}_p$-algebras (see (9.3.45)), and (9.4.8) commutes with the forgetful functors $(R, +, \cdot, \mathscr{T}) \mapsto (R, \cdot, \mathscr{T})$ to topological monoids. Furthermore, the map $\overline{u}_R$ of (9.4.4) is a morphism of topological $\mathbb{F}_p$-algebras, so the adjunction exhibited in the proof of proposition 9.4.3 lifts to an adjunction between the functor $\mathbf{E}$ of (9.4.8) and the forgetful functor $\mathbb{F}_p$-TopAlg$_{\mathrm{perf}} \to \mathbb{F}_p$-TopAlg.

The same argument shows that $\mathbf{E}^*$ lifts to a left adjoint to the forgetful functor

$$\mathbf{E}^* : \mathbb{F}_p\text{-}\mathsf{TopAlg} \to \mathbb{F}_p\text{-}\mathsf{TopAlg}_{\mathrm{perf}}.$$

Remark 9.4.9. (i) Suppose that $(R, \mathscr{T})$ is a topological $\mathbb{F}_p$-algebra whose topology $\mathscr{T}$ is linear. Then it is easily seen that a fundamental system of open neighborhoods of $0 \in \mathbf{E}(R, \mathscr{T})$ is given by the intersections of finitely many elements of the family of ideals

$$\Phi^n_{\mathbf{E}(R)}(\overline{u}_R^{-1}(I)) \qquad \text{for every } n \in \mathbb{N} \text{ and every open ideal } I \subset R.$$

Especially, the topology $\mathscr{T}_\mathbf{E}$ of $\mathbf{E}(R)$ is also linear. Moreover, if $\mathscr{T}$ is separated (resp. and complete) then the same holds for the topology of $\mathbf{E}(R)$, since the latter is a closed subset of $R^\mathbb{N}$, for the product topology : details left to the reader.

(ii) In the situation of (i), suppose moreover that $I \subset R$ is an ideal such that $\mathscr{T}$ agrees with the I-adic topology on R, and set $\mathscr{I} := \overline{u}_R^{-1}(I)$. Since $\Phi^n_{\mathbf{E}(R)}(\mathscr{I}) \subset \mathscr{I}^{p^n} \subset \overline{u}_R^{-1}(I^{p^n})$, we see that $\mathscr{T}_\mathbf{E}$ is the linear topology defined by the system of ideals $(\Phi^n_{\mathbf{E}(R)}(\mathscr{I}) \mid n \in \mathbb{N})$. Especially, if $\mathscr{I}$ is finitely generated, $\mathscr{T}_\mathbf{E}$ agrees with the $\mathscr{I}$-adic topology.

(iii) Suppose that the Frobenius endomorphism Φ_R of the topological $\mathbb{F}_p$-algebra R is surjective. Then clearly the same holds for $\overline{u}_R$. Moreover, if Φ_R is open and surjective, the same holds for $\overline{u}_R$. Indeed, any open subset of $\mathbf{E}(R)$ is a union of subsets of the form $U := \mathbf{E}(R) \cap \prod_{n\in\mathbb{N}} U_n$, where each U_n is open in R and there exists $k \in \mathbb{N}$ such that $U_n = A$ for every $n > k$; with this notation, since Φ_R is surjective, we have

$$\overline{u}_R(U) = \Phi^k_R(U_k \cap \Phi_R^{-1} U_{k-1} \cap \cdots \cap \Phi_R^{-k} U_0)$$

and this set is open in R, when Φ_R is an open map.

(iv) By virtue of proposition 1.3.25(iii), both the functor $\mathbf{E}$ for topological monoid and the one for topological $\mathbb{F}_p$-algebras commute with all limits. By the same token, the existence of the left adjoint $\mathbf{E}^*$ implies that the forgetful functor p-$\mathsf{TopMon} \to \mathsf{TopMon}$ also commutes with limits, and likewise for perfect topological $\mathbb{F}_p$-algebras.

(v) Let A be any $\mathbb{F}_p$-algebra, $I \subset A$ a finitely generated ideal, and denote by $\mathscr{T}_I$ (resp. $\mathscr{T}_d$) the I-adic topology (resp. the discrete topology) on A. Then $(A, \mathscr{T}_I)$ is a perfect topological $\mathbb{F}_p$-algebra if and only if the same holds for $(A, \mathscr{T}_d)$. Indeed, suppose that the Frobenius endomorphism Φ_A is bijective on A; in order to show that Φ_A is an automorphism of $(A, \mathscr{T})$, it suffices to prove that for every $k \in \mathbb{N}$ there exists $n(k) \in \mathbb{N}$ such that $I^{n(k)} \subset \Phi(I^n)$; but the latter is clear, since I is finitely generated (details left to the reader).

Theorem 9.4.10. *Let $(A, \mathscr{T}_A)$ be any complete and separated topological ring whose topology $\mathscr{T}_A$ is linear and coarser than the p-adic topology. Let also $I \subset A$ be a given ideal that is topologically nilpotent for the topology $\mathscr{T}_A$, and $\pi : A \to A/I$ the projection. Endow A/I with the quotient topology $\mathscr{T}_{A/I}$ induced by $\mathscr{T}_A$ via π, and denote by $((A/I)^\wedge, \mathscr{T}^\wedge_{A/I})$ the completion of $(A/I, \mathscr{T}_{A/I})$, and by $j : A \to (A/I)^\wedge$ the completion map. Suppose moreover that :*

(a) *either, I is a closed subset for the topology $\mathscr{T}_A$*

(b) *or else, I is finitely generated, and a closed subset for the p-adic topology of A.*

Then the maps

$$\mathbf{E}(\pi) : \mathbf{E}(A, \mathscr{T}_A) \to \mathbf{E}(A/I, \mathscr{T}_{A/I}) \qquad \mathbf{E}(j) : \mathbf{E}(A/I, \mathscr{T}_{A/I}) \to \mathbf{E}((A/I)^\wedge, \mathscr{T}^\wedge_{A/I})$$

are isomorphisms of topological monoids.

Proof. Let $(J_\lambda \mid \lambda \in \Lambda)$ be a family of ideals of A which is a fundamental system of open neighborhoods of 0 for the topology $\mathscr{T}_A$. Suppose first that I is closed for the topology $\mathscr{T}_A$, and set $\pi^\wedge := j \circ \pi : A \to (A/I)^\wedge$; in this case, j is injective, so the same holds for $\mathbf{E}(j)$, and to prove the theorem it suffices to check that $\mathbf{E}(\pi^\wedge)$ is an isomorphism. However, we have natural isomorphisms

$$A \stackrel{\sim}{\to} \lim_{\lambda\in\Lambda} A/J_\lambda \qquad (A/I)^\wedge \stackrel{\sim}{\to} \lim_{\lambda\in\Lambda} A/(I+J_\lambda)$$

whence, by remark 9.4.9(iv), induced isomorphisms of topological monoids

$$\mathbf{E}(A) \stackrel{\sim}{\to} \lim_{\lambda\in\Lambda} \mathbf{E}(A/J_\lambda) \qquad \mathbf{E}((A/I)^\wedge) \stackrel{\sim}{\to} \lim_{\lambda\in\Lambda} \mathbf{E}(A/(I+J_\lambda)).$$

Thus, it suffices to show that the induced map $\mathbf{E}(A/J_\lambda) \to \mathbf{E}(A/(I+J_\lambda))$ is an isomorphism of topological monoids for every $\lambda \in \Lambda$. In this case, we may then replace A by A/J_λ, and assume from start that the topology $\mathscr{T}_A$ is discrete, hence I is a nilpotent ideal.

Claim 9.4.11. The theorem holds when the topology $\mathscr{T}_A$ is discrete and $I = pA$.

Proof of the claim. In this case, the p-adic filtration is finite, hence the homomorphism of topological $\mathbb{F}_p$-algebras $\overline{u}_{A/pA} : \mathbf{E}(A/pA) \to A/pA$ lifts to a morphism of monoids $v : \mathbf{E}(A/pA) \to A$ (lemma 9.3.51(iii)). Explicitly, say that $p^k A = 0$ for some $k \in \mathbb{N}$, let $\overline{a}_\bullet := (\overline{a}_n \mid n \in \mathbb{N}) \in \mathbf{E}(A/pA)$ be any element, and pick $a_k \in A$ whose image in A/pA equals $\overline{a}_k$; then lemma 9.3.4(i) shows that $v(\overline{a}_\bullet) = a_k^{p^k}$. This description easily implies that v is continuous (for the discrete topology on A). Therefore v corresponds, by adjunction, to a unique morphism $v^\dagger := \mathbf{E}(v) \circ \overline{u}^{-1}_{\mathbf{E}(A/pA)} : \mathbf{E}(A/pA) \to \mathbf{E}(A)$ of p-perfect topological monoids. We have

$$\mathbf{E}(\pi) \circ v^\dagger = \mathbf{E}(\pi \circ v) \circ \overline{u}^{-1}_{\mathbf{E}(A/pA)} = \mathbf{E}(\overline{u}_{A/pA}) \circ \overline{u}^{-1}_{\mathbf{E}(A/pA)} = \mathbf{1}_{\mathbf{E}(A/pA)}$$

by remark 9.4.5(ii). Lastly, the foregoing explicit description of v implies that $v^\dagger \circ \mathbf{E}(\pi) = \mathbf{1}_{\mathbf{E}(A)}$ (details left to the reader), and the claim follows. ◊

Now, we have a commutative diagram of topological monoids

$$\begin{array}{ccc} \mathbf{E}(A) & \xrightarrow{\mathbf{E}(\pi')} & \mathbf{E}(A/pA) \\ {\scriptstyle \mathbf{E}(\pi)}\downarrow & & \downarrow{\scriptstyle \mathbf{E}(\pi'')} \\ \mathbf{E}(A/I) & \xrightarrow{\mathbf{E}(\pi''')} & \mathbf{E}(A/(pA+I)) \end{array}$$

where π', π'', π''' are the projections. By claim 9.4.11 we know already that $\mathbf{E}(\pi')$ and $\mathbf{E}(\pi''')$ are isomorphisms, hence it suffices to show that the same holds for $\mathbf{E}(\pi'')$. Thus, we may further replace A by A/pA, and assume from start that A is also an $\mathbb{F}_p$-algebra. In this case, the I-adic filtration on A fulfills the conditions of lemma 9.3.4, hence lemma 9.3.51(iii) applies and says that the map $\overline{u}_{A/I} : \mathbf{E}(A/I) \to A/I$ lifts to a morphism of monoids $w : \mathbf{E}(A/I) \to A$, and arguing as in the proof of claim 9.4.11 we see that w is continuous, so it yields, again by adjunction, a morphism of topological monoids $w^\dagger : \mathbf{E}(A/I) \to \mathbf{E}(A)$, which is seen to be an inverse for $\mathbf{E}(\pi)$, by the same argument as in the proof of claim 9.4.11. This concludes the proof, in case I is closed for the topology $\mathscr{T}_A$.

For the general case, denote by $\overline{I} \subset A$ the topological closure of A (for the topology $\mathscr{T}_A$), endow $A/\overline{I}$ with the quotient topology $\mathscr{T}_{A/\overline{I}}$ induced by the projection $A \to A/\overline{I}$, and let $((A/\overline{I})^\wedge, \mathscr{T}^\wedge_{A/\overline{I}})$ be the completion of $(A/\overline{I}, \mathscr{T}_{A/\overline{I}})$; denote also by $\pi' : (A/I)^\wedge \to (A/\overline{I})^\wedge$ and $\overline{\pi} : A \to (A/\overline{I})^\wedge$ the induced maps; we deduce morphisms of topological monoids $\mathbf{E}(A) \to \mathbf{E}((A/I)^\wedge) \to \mathbf{E}((A/\overline{I})^\wedge)$ whose composition equals $\mathbf{E}(\overline{\pi})$. Notice that $\overline{I}$ is still topological nilpotent, hence $\mathbf{E}(\overline{\pi})$ is an isomorphism, by the foregoing case. Also π' is an isomorphism, so the same holds for $\mathbf{E}(\pi')$ as well. We set $v := \mathbf{E}(\overline{\pi})^{-1} \circ \mathbf{E}(\pi') : \mathbf{E}((A/I)^\wedge) \to \mathbf{E}(A)$. Clearly $v \circ \mathbf{E}(\pi) = \mathbf{1}_{\mathbf{E}(A)}$, so it remains only to check that v is bijective, or equivalently, that the same holds for $\mathbf{E}(\pi)$. To this aim, let $\mathscr{T}'_A$ be the $(I+pA)$-adic topology on A, and notice that $\mathscr{T}'_A$ is still complete and separated (lemma 8.3.12); moreover, clearly both I and pA are topologically nilpotent ideals in $(A, \mathscr{T}'_A)$. Furthermore, notice that

$$\bigcap_{n\in\mathbb{N}}(I + (I + pA)^n) = \bigcap_{n\in\mathbb{N}}(I + p^n A) = I$$

since I is closed in the p-adic topology of A; thus, I is closed in the topology $\mathscr{T}'_A$, so the foregoing case shows that the map $\mathbf{E}(\pi) : \mathbf{E}(A, \mathscr{T}'_A) \to \mathbf{E}(A/I)$ is indeed bijective. □

Remark 9.4.12. Keep the situation of theorem 9.4.10.

(i) For every $(\overline{a}_n \mid n \in \mathbb{N}) \in \mathbf{E}(A/I)$ and every $n \in \mathbb{N}$ choose a representative $a_n \in A$ for the class $\overline{a}_n$. By inspecting the proof, we see that the inverse of $\mathbf{E}(\pi)$ is given by the rule :

$$(\overline{a}_n \mid n \in \mathbb{N}) \mapsto (\lim_{k\to\infty} a_{n+k}^{p^k} \mid n \in \mathbb{N}) \tag{9.4.13}$$

where the convergence is relative to the topology $\mathscr{T}_A$.

(ii) Suppose moreover, that $p \in I$; then $\mathbf{E}(A/I)$ is a topological $\mathbb{F}_p$-algebra, and the isomorphism $\mathbf{E}(\pi)$ can be used to transfer to $\mathbf{E}(A)$ the ring structure of $\mathbf{E}(A/I)$. In light of (i), we get the following expressions for the addition and multiplication laws on $\mathbf{E}(A)$ such that $\mathbf{E}(\pi)$ becomes an isomomorphism of topological rings :

$$(a_n \mid n \in \mathbb{N}) + (b_n \mid n \in \mathbb{N}) := (\lim_{k\to+\infty} (a_{n+k} + b_{n+k})^{p^k} \mid n \in \mathbb{N})$$
$$(a_n \mid n \in \mathbb{N}) \cdot (b_n \mid n \in \mathbb{N}) := (a_n b_n \mid n \in \mathbb{N})$$

where the limit is computed in the topology $\mathscr{T}_A$.

(iii) In the situation of (ii), suppose furthermore that A is a domain (resp. a field). Then theorem 9.4.10 implies immediately that the same holds for $\mathbf{E}(A/I)$ (and of course also for $\mathbf{E}(A)$, for its ring structure as in (ii)).

Corollary 9.4.14. *Let $(A, \mathscr{T})$ be a perfect topological $\mathbb{F}_p$-algebra, $I \subset A$ an open and topologically nilpotent ideal, and denote by $(A^\wedge, \mathscr{T}^\wedge)$ the separated completion of $(A, \mathscr{T})$. Then there is a natural isomorphism of topological $\mathbb{F}_p$-algebras*

$$\omega : (A^\wedge, \mathscr{T}^\wedge) \xrightarrow{\sim} \mathbf{E}(A/I) \qquad \textit{such that} \qquad \pi = \omega \circ \overline{u}_{A/I}$$

where $\pi : A^\wedge \to A/I$ is the natural map.

Proof. Let $I^\wedge$ be the topological closure of the image of I in $A^\wedge$, and endow $B := A^\wedge/I^\wedge$ with the quotient topology induced by the projection $\pi : A^\wedge \to B$; notice that the natural map $A/I \to B$ induces an isomorphism $\varphi : A/I \xrightarrow{\sim} B^\wedge$ on the respective separated completions. Then example 9.3.48(iii) and theorem 9.4.10 imply that the composition

$$A^\wedge \xrightarrow{\overline{u}_{A^\wedge}^{-1}} \mathbf{E}(A^\wedge) \xrightarrow{\mathbf{E}(\pi)} \mathbf{E}(B) \xrightarrow{\mathbf{E}(\varphi)} \mathbf{E}(A/I)$$

is an isomorphism, whence the corollary. □

9.4.15. Let now $(R, |\cdot|_R)$ be any valuation ring with value group Γ_R and residue characteristic p. Notice that the p-Frobenius map of Γ_R is injective, so we may regard $\mathbf{E}(\Gamma_R)$ as a subgroup of Γ_R. To ease notation, set $\mathbf{E} := \mathbf{E}(R/pR)$; also, let $\Delta := \Gamma_R$ in case R is an $\mathbb{F}_p$-algebra, and otherwise let $\Delta := O(|p|_R)$ (notation of remark 9.1.2(vii)). We define a mapping

$$|\cdot|_\mathbf{E} : \mathbf{E} \to \Gamma_{\mathbf{E}^\circ} \qquad \text{where } \Gamma_\mathbf{E} := \mathbf{E}(\Gamma_R) \cap \Delta$$

as follows. Let $\underline{a} := (a_n \mid n \in \mathbb{N}) \in \mathbf{E}$ be any element, and for every $n \in \mathbb{N}$ choose a representative $\widetilde{a}_n \in R$ for the class $a_n \in R/pR$. If $|\widetilde{a}_n|_R \leq |p|_R$ for every $n \in \mathbb{N}$, then we set $|\underline{a}|_\mathbf{E} := 0$; if there exists $n \in \mathbb{N}$ such that $|\widetilde{a}_n| > |p|_R$, then we set $|\underline{a}|_\mathbf{E} := |\widetilde{a}_n|_R^{p^n}$. One verifies easily that the definition is independent of all the choices : the details shall be left to the reader.

Lemma 9.4.16. *In the situation of* (9.4.15), *the pair* $(\mathbf{E}, |\cdot|_{\mathbf{E}})$ *is a valuation ring.*

Proof. The bijection of remark 9.1.13(vii) attaches to Δ the prime ideal $\mathfrak{p}(\Delta) = \bigcap_{n\in\mathbb{N}} p^n R$, and we set $R' := R/\mathfrak{p}(\Delta)$. Since the set of ideals of R' is naturally identified with the set of those ideals of R that contain $\mathfrak{p}(\Delta)$, it follows easily that R' is a valuation ring, and then it is clear that its valuation is induced by the primary specialization $|\cdot|_R^\Delta$ of $|\cdot|_R$ (notation of (9.2.10)). Notice moreover that $R'/pR' = R/pR$, so that $\mathbf{E}(R'/pR') = \mathbf{E}$. Summing up, we may replace R by R' after which we may assume from start that $\Gamma_R = \Delta$, and that the p-adic topology on R is separated. Moreover, notice that in this case the p-adic topology on R agrees with either the valuation topology (see definition 9.1.14) or with the discrete topology; hence the p-adic completion $R^\wedge$ of R is still a valuation ring (proposition 9.1.16(iii)). Then, since $R^\wedge/pR^\wedge = R/pR$, we may replace R by $R^\wedge$, and assume from start that R is p-adically complete, in which case remark 9.4.12(iii) already shows that $\mathbf{E}$ is a domain. Moreover, in this situation the isomorphism $\mathbf{E}(R) \stackrel{\sim}{\to} \mathbf{E}$ of theorem 9.4.10 identifies the map $|\cdot|_{\mathbf{E}}$ with the mapping

$$|\cdot|_{\mathbf{E}(R)} : \mathbf{E}(R) \to \Gamma_{R^\circ} \qquad (a_n \mid n \in \mathbb{N}) \mapsto |a_0|_R.$$

With this notation, clearly $|x_\bullet \cdot y_\bullet|_{\mathbf{E}(R)} = |x_\bullet|_{\mathbf{E}(R)} \cdot |y_\bullet|_{\mathbf{E}(R)}$ for every $x_\bullet, y_\bullet \in \mathbf{E}(R)$. By the same token, it is clear that $x_\bullet$ divides $y_\bullet$ in $\mathbf{E}(R)$ if and only if $|x_\bullet|_{\mathbf{E}(R)} \geq |y_\bullet|_{\mathbf{E}(R)}$ so $(\mathbf{E}, |\cdot|_{\mathbf{E}})$ is a valuation ring. □

The last topics of this section are the graded versions of the functors W and $\mathbf{E}$.

9.4.17. Let $(\Gamma, +, 0)$ be a monoid, $p \in \mathbb{N}$ a prime integer, $(A, \underline{B})$ a topological ring with Γ-graded structure, and suppose that the p-Frobenius endomorphism of Γ is injective. The assumption implies that the natural map $\Gamma \to \mathbf{E}^*(\Gamma)$ is injective (notation of proposition 9.4.3), hence we may regard $\underline{B}$ naturally as an $\mathbf{E}^*(\Gamma)$-graded $\mathbb{Z}$-algebra. We define

$$W(\underline{B}) := \bigoplus_{\gamma\in\mathbf{E}^*(\Gamma)} \mathrm{gr}_\gamma W(\underline{B}) \qquad \text{where} \qquad \mathrm{gr}_\gamma W(\underline{B}) := \prod_{n\in\mathbb{N}} \mathrm{gr}_{p^n\gamma} B \qquad \text{for every } \gamma \in \mathbf{E}^*(\Gamma).$$

From remark 9.3.8(i) it follows easily that, for every $\gamma, \mu \in \Gamma$, the addition and multiplication laws of $W(B)$ restrict respectively to maps

$$\mathrm{gr}_\gamma W(\underline{B}) \times \mathrm{gr}_\gamma W(\underline{B}) \to \mathrm{gr}_\gamma W(\underline{B}) \qquad \mathrm{gr}_\gamma W(\underline{B}) \times \mathrm{gr}_\mu W(\underline{B}) \to \mathrm{gr}_{\gamma+\mu} W(\underline{B}).$$

We may therefore define an addition law on $W(\underline{B})$ by taking the direct sum of the addition laws on each Γ-graded summand. Likewise, the map $x \mapsto -x$ on $W(B)$ restricts to a map $\mathrm{gr}_\gamma W(\underline{B}) \to \mathrm{gr}_\gamma W(\underline{B})$ for every $\gamma \in \Gamma$, so that $W(\underline{B})$ is an abelian group with this addition law. Next, clearly $1 \in \mathrm{gr}_0 W(\underline{B})$, so we obtain a ring structure on $W(\underline{B})$, by extending linearly the multiplication map already defined on each graded term. Thus, the multiplication map of $W(\underline{B})$ agrees with that of $W(B)$ on each pair of homogeneous elements, but it will be usually different, on non-homogeneous elements.

Lemma 9.4.18. *With the notation of* (9.4.17)*, the induced map*

$$\varphi : W(\underline{B}) \to W(B) \qquad (b_\gamma \mid \gamma \in \mathbf{E}^*(\Gamma)) \mapsto \sum_{\gamma\in\mathbf{E}^*(\Gamma)} b_\gamma$$

is injective (again, notice that φ agrees with the inclusion map on homogeneous terms, but it usually differs from it on non-homogeneous elements).

Proof. For every $r \in \mathbb{N}$ and $\gamma \in \mathbf{E}^*(\Gamma)$ define $\mathrm{gr}_\gamma W_r(\underline{B})$ as the image of $\mathrm{gr}_\gamma W(\underline{B})$ into $W_r(B)$, and set

$$W_r(\underline{B}) := \bigoplus_{\gamma\in\mathbf{E}^*(\Gamma)} \mathrm{gr}_\gamma W_r(\underline{B}).$$

We shall show, more precisely, that the resulting map $\varphi_r : W_r(\underline{B}) \to W_r(B)$ is a bijection. We argue by induction on r. For $r = 1$, the assertion follows by a simple inspection. Next, suppose that $r \geq 0$ and that φ_r is bijective; we deduce a commutative ladder with exact rows :

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bigoplus_{\gamma\in\mathbf{E}^*(\Gamma)} \mathrm{gr}_{p^r\gamma}B & \longrightarrow & W_{r+1}(\underline{B}) & \longrightarrow & W_r(\underline{B}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \psi_r} & & \downarrow{\scriptstyle \varphi_{r+1}} & & \downarrow{\scriptstyle \varphi_r} & & \\ 0 & \longrightarrow & W_{r+1}(B) \cap \overline{V}_r & \longrightarrow & W_{r+1}(B) & \longrightarrow & W_r(B) & \longrightarrow & 0 \end{array}$$

where $\overline{V}_r$ denotes the image of $V_r(B)$ in $W_{r+1}(B)$. It then suffices to check that ψ_r is a bijection; but (9.3.9) implies that ψ_r is none else than the restriction of the additive map V^r, and since $\mathbf{E}^*(\Gamma)$ is p-perfect, the source of ψ_r is just B, whence the contention. □

Due to lemma 9.4.18, we may regard $W(\underline{B})$ as a subring of $W(B)$. We also see that $W_r(B)$ inherits from $\underline{B}$ a natural $\mathbf{E}^*(\Gamma)$-graded $\mathbb{Z}$-algebra structure, for every $r \in \mathbb{N}$. Moreover, $\mathrm{gr}_0 W(\underline{B}) = W(\mathrm{gr}_0 B)$, so $W(\underline{B})$ is an $\mathbf{E}^*(\Gamma)$-graded $W(\mathrm{gr}_0 B)$-algebra. Set

$$W(A, \underline{B}) := (W(A), W(\underline{B}))$$

and let also $\mathbf{E}^*(\Gamma)_{(\boldsymbol{p})}$ be the object of the category $\mathbf{E}^*(\Gamma)/\mathbf{Mnd}$ given by the p-Frobenius automorphism $\boldsymbol{p}_{\mathbf{E}^*(\Gamma)} : \mathbf{E}^*(\Gamma) \to \mathbf{E}^*(\Gamma)_{(\boldsymbol{p})}$.

Proposition 9.4.19. *With the notation of* (9.4.17)*, the following holds :*

(i) *The pair $W(A, \underline{B})$ is an $\mathbf{E}^*(\Gamma)$-graded structure on $W(A)$ (see definition* 8.5.1(i)*).*

(ii) *The map $\boldsymbol{\omega}_0$ is a morphism of topological rings with $\mathbf{E}^*(\Gamma)$-graded structures*

$$\boldsymbol{\omega}_0 : W(A, \underline{B}) \to (A, \underline{B}).$$

(iii) *The Frobenius and Verschiebung maps of $W(A)$ (see* (9.3.18)*) induce respectively a morphism of topological rings with $\mathbf{E}^*(\Gamma)$-graded structures*

$$F_A : W(A, \underline{B}) \to (W(A), \mathbf{E}^*(\Gamma) \times_{\mathbf{E}^*(\Gamma)_{(\boldsymbol{p})}} W(\underline{B}))$$

and a continuous map of $\mathbf{E}^*(\Gamma)$-graded topological abelian groups

$$V_A : \mathbf{E}^*(\Gamma) \times_{\mathbf{E}^*(\Gamma)_{(\boldsymbol{p})}} W(\underline{B}) \to W(\underline{B})$$

(notation of definition 7.6.1(iv)*).*

Proof. (i): Indeed, a direct inspection shows that $\mathrm{gr}_\gamma W(\underline{B})$ is a closed subset in $W(A, \mathscr{T}_A)$, for every $\gamma \in \mathbf{E}^*(\Gamma)$. Next, notice that the topology of $W(B)$ agrees with the topology induced by $W(A)$ via the inclusion map $W(B) \to W(A)$. Hence, in order to check that the topology induced by $W(A)$ on $W(\underline{B})$ is linear and defined by a system of open graded ideals, it suffices to show that the same holds for the topology induced by $W(B)$. However, let $(J_\lambda \mid \lambda \in \Lambda)$ be any fundamental system of open graded ideals of B, and set $\mathscr{J}_{\lambda,r} := \mathrm{Ker}\,(W(B) \to W_r(B/J_\lambda))$ for every $\lambda \in \Lambda$ and $r \in \mathbb{N}$; the proof of lemma 9.3.33(iii) shows that $(\mathscr{J}_{\lambda,r} \mid \lambda \in \Lambda;\ r \in \mathbb{N})$ is a fundamental system of open ideals of $W(B)$, and we are reduced to checking that $\mathscr{J}_{\lambda,r} \cap W(\underline{B})$ is a graded ideal of $W(\underline{B})$, for every $\lambda \in \Lambda$ and $r \in \mathbb{N}$. To this aim, we consider the commutative diagram

$$\begin{array}{ccc} W(\underline{B}) & \xrightarrow{\ \varphi\ } & W(B) \\ \downarrow & & \downarrow \\ W_r(\underline{B}/J_\lambda) & \xrightarrow{\ \varphi_r\ } & W_r(B/J_\lambda) \end{array}$$

where φ_r is the isomorphism defined as in the proof of lemma 9.4.18, and the vertical arrows are the natural projections. We deduce that $\mathscr{J}_{\lambda,r} \cap W(\underline{B})$ is the kernel of the projection $W(\underline{B}) \to$

$W_r(\underline{B}/J_\lambda)$, and the assertion follows easily. It remains to check that $W(\underline{B})$ is a dense subset of $W(A)$, but this is clear, since φ_r is bijective for every $r \in \mathbb{N}$.

(ii) follows by a direct inspection of the construction.

(iii) follows easily from remark 9.3.8(i). □

We have the following graded version of proposition 9.3.52 :

Proposition 9.4.20. *Let $p \in \mathbb{N}$ be a prime integer, Γ a monoid whose p-Frobenius endomorphism is injective, $(R, \underline{S})$ and $(A, \underline{B})$ two topological rings with Γ-structures, and suppose that:*

(a) *R is a perfect topological $\mathbb{F}_p$-algebra, and A is complete and separated.*
(b) *$\underline{B}$ admits a fundamental system $(J_n \mid n \in \mathbb{N})$ of graded open ideals such that $pJ_n + J_n^p \subset J_{n+1} \subset J_n$ for every $n \in \mathbb{N}$.*

Then, for every morphism $\overline{\varphi} : (R, \underline{S}) \to (A, \underline{B})$ of topological rings with Γ-graded structures there exists a unique morphism $u : W(R, \underline{S}) \to (A, \underline{B})$ of topological rings with Γ-graded structures that makes commute the diagram

$$\begin{array}{ccc} W(R, \underline{S}) & \xrightarrow{u} & (A, \underline{B}) \\ {\scriptstyle \boldsymbol{\omega}_0}\downarrow & & \downarrow{\scriptstyle \pi} \\ (R, \underline{S}) & \xrightarrow{\overline{\varphi}} & (A, \underline{B})/J_0 \end{array}$$

where π is the natural projection.

Proof. We have to show that the unique continuous ring homomorphism $u : W(R) \to A$ provided by proposition 9.3.52(iii) restricts to a map of Γ-graded rings $W(\underline{S}) \to \underline{B}$. However, recall that $\mathrm{gr}_\gamma B$ is a closed subset of A, so it is complete and separated for the topology induced by $\underline{B}$, for every $\gamma \in \Gamma$, and hence also for the p-adic topology (lemma 8.3.12). Then, let $\varphi : R \to A$ be the unique continuous mapping characterized as in lemma 9.3.51(iii); by inspecting (9.3.53), we are reduced to checking that $\varphi(\mathrm{gr}_\gamma S) \subset \mathrm{gr}_\gamma B$ for every $\gamma \in \Gamma$. To this aim, pick any map $\psi : R \to A$ such that :

(a) $\pi \circ \psi = \overline{\varphi}$
(b) $\psi(\mathrm{gr}_\gamma S) \subset \mathrm{gr}_\gamma B$ for every $\gamma \in \Gamma$

and recall that $\varphi(r) = \lim_{n\to+\infty} \psi(r^{p^{-n}})^{p^n}$ for every $r \in R$. The assertion follows easily. □

9.4.21. Let Γ be a monoid, P a Γ-graded monoid. Clearly $\mathbf{E}(P)$ is naturally an $\mathbf{E}(\Gamma)$-graded monoid. Moreover, let $\underline{B} := (B, \mathrm{gr}_\bullet B)$ be a Γ-graded topological $\mathbb{Z}$-algebra; we define the Γ-graded monoid

$$B^* := \coprod_{\gamma\in\Gamma} \mathrm{gr}_\gamma B$$

whose composition law is induced by the multiplication map of B. The natural morphism of Γ-graded monoids $B^* \to B$ (which is not injective, unless $\Gamma = \{0\}$) induces a morphism of $\mathbf{E}(\Gamma)$-graded monoids

$$\mathbf{E}(B^*) \to \mathbf{E}(B).$$

Furthermore, if B is an $\mathbb{F}_p$-algebra we may define the $\mathbf{E}(\Gamma)$-graded subring of $\mathbf{E}(B)$

$$\mathbf{E}(\underline{B}) := \bigoplus_{\gamma\in\mathbf{E}(\Gamma)} \mathrm{gr}_\gamma \mathbf{E}(B^*)$$

which we endow with the topology induced from $\mathbf{E}(B)$, via the inclusion map $\mathbf{E}(\underline{B}) \to \mathbf{E}(B)$, and the rule $\underline{B} \mapsto \mathbf{E}(\underline{B})$ defines a functor from the category of Γ-graded topological $\mathbb{F}_p$-algebras to the category of perfect $\mathbf{E}(\Gamma)$-graded topological $\mathbb{F}_p$-algebras. It is easily seen that this functor is right adjoint to the forgetful functor $\underline{C} \mapsto \underline{C}_{/\Gamma}$ induced by $\overline{u}_\Gamma : \mathbf{E}(\Gamma) \to \Gamma$.

Proposition 9.4.22. *Let $(\Gamma, +, 0)$ be a monoid, $(A, \underline{B})$ a topological $\mathbb{F}_p$-algebra with Γ-graded structure, such that A is complete and separated, and the p-Frobenius endomorphism $\boldsymbol{p}_\Gamma$ of Γ is injective. We have :*

(i) $\mathbf{E}(A, \underline{B}) := (\mathbf{E}(A), \mathbf{E}(\underline{B}))$ *is an* $\mathbf{E}(\Gamma)$*-graded structure on* $\mathbf{E}(A)$.

(ii) *The pair* $(\overline{u}_A, \overline{u}_\Gamma)$ *is a morphism of topological* $\mathbb{F}_p$*-algebras with graded structures :*

$$\overline{u}_A : \mathbf{E}(A, \underline{B}) \to (A, \underline{B}).$$

(iii) *Let* Γ' *be another monoid whose* p*-Frobenius endomorphism is injective, and* $\Gamma \to \Gamma'$ *any morphism of monoids. Then the identity map of* $\mathbf{E}(A)$ *induces an isomorphism of topological rings with* Γ'*-graded structures*

$$\mathbf{E}(A, \underline{B}, \Gamma)_{/\Gamma'} \xrightarrow{\sim} \mathbf{E}((A, \underline{B}, \Gamma)_{/\Gamma'}).$$

Proof. Indeed, the definition immediately shows that $\mathrm{gr}_\gamma \mathbf{E}(B)$ is closed in the topology of $\mathbf{E}(A)$ for every $\gamma \in \Gamma$. Next, notice that – under the current assumptions – we may regard $\mathbf{E}(\Gamma)$ as a submonoid of Γ, and for every graded ideal $J \subset B$ and every $r \in \mathbb{N}$, set

$$J^* := \coprod_{\gamma \in \Gamma} \mathrm{gr}_\gamma J \qquad \text{and} \qquad \mathbf{E}(\underline{J}) := \bigoplus_{\gamma \in \Gamma} \mathrm{gr}_\gamma \mathbf{E}(J^*).$$

So, $\mathbf{E}(\underline{J})$ is a graded ideal of $\mathbf{E}(\underline{B})$, and from remark 9.4.9(i) it follows that the topology of $\mathbf{E}(A)$ induces on $\mathbf{E}(\underline{B})$ the linear topology defined by the cofiltered system of graded ideals $(\Phi^r_{\mathbf{E}(\underline{B})} \mathbf{E}(J) \mid J \subset B;\ r \in \mathbb{N})$, where J ranges over all the open graded ideals of B. It remains to check that $\mathbf{E}(\underline{B})$ is dense in $\mathbf{E}(A)$. To this aim, let $\underline{a} := (a_n \mid n \in \mathbb{N}) \in \mathbf{E}(A)$ be any element, and for every finite subset $S \subset \mathbf{E}(\Gamma)$ and every $n \in \mathbb{N}$, set $S(n) := \boldsymbol{p}_\Gamma^{-n}(S)$; for every such S, we define the element

$$\underline{a}_S := (a_{n,S(n)} \mid n \in \mathbb{N}) \in \mathbf{E}(\underline{B})$$

where the component $a_{n,S(n)} \in \bigoplus_{\gamma \in S(n)} \mathrm{gr}_\gamma B$ is defined as in remark 8.5.2(iii).

Claim 9.4.23. (i) $a_{n,\gamma} = 0$ for every $\gamma \in \Gamma \setminus \mathbf{E}(\Gamma)$ and every $n \in \mathbb{N}$.

(ii) $(a_{n,S(n)} \mid S \subset \mathbf{E}(\Gamma))$ is a Cauchy net in B whose limit is a_n, for every $n \in \mathbb{N}$.

(iii) $a_{n,S(n)} = a^p_{n+1,S(n+1)}$, for every $n \in \mathbb{N}$ and every finite subset $S \subset \mathbf{E}(\Gamma)$.

Proof of the claim. For every $n \in \mathbb{N}$, let $(a_{n,\gamma} \mid \gamma \in \Gamma) \in \prod_{\gamma \in \Gamma} B_\gamma$ be the sequence of canonical projections of a_n (see remark 8.5.2(iii)); in view of lemma 8.5.6(i) we see that $a_{n,p\gamma} = a^p_{n+1,\gamma}$ for every $\gamma \in \Gamma$, and $a_{n,\gamma} = 0$ if $\gamma \in \Gamma \setminus \boldsymbol{p}_\Gamma(\Gamma)$. Assertions (i) and (iii) are immediate consequences.

(ii) follows immediately from (i) and proposition 8.5.3(ii). ◊

From claim 9.4.23(ii,iii) we conclude that the system $(\underline{a}_S \mid S \subset \mathbf{E}(\Gamma))$ is a Cauchy net in $\mathbf{E}(\underline{B})$ whose limit is $\underline{a}$, as required.

(ii) is clear by inspecting the constructions.

(iii): Indeed, by example 8.5.7(iii), any such morphism will be necessarily an isomorphism, and conversely, consider the natural morphism $(A, \underline{B}, \Gamma) \to (A, \underline{B}, \Gamma)_{/\Gamma'}$ which induces a morphism $\mathbf{E}(A, \underline{B}, \Gamma) \to \mathbf{E}((A, \underline{B}, \Gamma)_{/\Gamma'})$ which in turns factors uniquely through a morphism as sought, again by example 8.5.7(iii). □

9.4.24. Let $(A, \underline{B})$ be a complete and separated topological ring with Γ-graded structure, whose topology is linear and coarser than the p-adic topology. Endow A/pA with the quotient topology induced by the projection $\pi_A : A \to A/pA$, and let $(A/pA)^\wedge$ be the separated completion of A/pA; by theorem 9.4.10, the map π_A induces an isomorphism of topological monoids

$$\mathbf{E}(\pi_A) : \mathbf{E}(A) \xrightarrow{\sim} \mathbf{E}(A/pA)$$

and the completion map $j : A/pA \to (A/pA)^\wedge$ induces an isomorphism of topological rings

$$\mathbf{E}(j) : \mathbf{E}(A/pA) \to \mathbf{E}((A/pA)^\wedge).$$

Especially, $\mathbf{E}(A/pA)$ is a complete and separated topological ring whose topology is linear (remark 9.4.9(i)), and we may transfer the ring structure of $\mathbf{E}(A/pA)$ onto $\mathbf{E}(A)$ (see remark 9.4.12(ii)), after which the latter can also be regarded as a complete and separated topological ring with a linear topology. Furthermore, suppose that the p-Frobenius map of Γ is injective, so that the topological rings with $\mathbf{E}(\Gamma)$-structures

$$((A/pA)^\wedge, \underline{B}_0) := ((A, \underline{B})/pB)^\wedge \qquad \text{and} \qquad \mathbf{E}((A/pA)^\wedge, \underline{B}_0)$$

are well defined, by proposition 9.4.22(i) (and remark 8.5.2(iv)), and we may transfer as well this graded structure onto $\mathbf{E}(A)$ and $\mathbf{E}(A/pA)$. We denote by

$$\mathbf{E}(A, \underline{B}) \qquad \text{and} \qquad \mathbf{E}(A/pA, \underline{B}/p\underline{B})$$

the resulting topological rings with $\mathbf{E}(\Gamma)$-graded structures, whose underlying topological rings are respectively $\mathbf{E}(A)$ and $\mathbf{E}(A/pA)$. Define B' as in remark 8.5.2(ii), and notice that $pB = pB' \cap B$ and $pA \subset A \cap pB'$, so that

$$pB = pA \cap B \tag{9.4.25}$$

and therefore the natural map $\mathbf{E}(B/pB) \to \mathbf{E}(A/pA)$ is injective.

Proposition 9.4.26. *In the situation of* (9.4.24)*, let also $\underline{C}$ (resp. $\underline{C}_0$) be the Γ-graded dense subring of* $\mathbf{E}(A)$ *(resp. of* $\mathbf{E}(A/pA)$*). With the notation of* (9.4.21) *we have :*

$$\underline{C}_0 = \mathbf{E}(\underline{B}/p\underline{B}) \qquad \textit{and} \qquad \underline{C} = \mathbf{E}(\underline{B}).$$

Proof. Notice that – as explained in remark 8.5.4(ii) – since A is complete and separated, $\mathrm{gr}_\gamma B_0$ is the maximal separated quotient of $\mathrm{gr}_\gamma(B/pB)$, for every $\gamma \in \Gamma$; it is also a closed subset of the complete ring $(A/pA)^\wedge$, so it is the separated completion of $\mathrm{gr}_\gamma(B/pB)$.

Now, from (9.4.25) we deduce that the canonical γ-projection of $(A, \underline{B})$ induces a continuous B_0/pB_0-linear map $\pi_\gamma : A/pA \to \mathrm{gr}_\gamma(B/pB)$ for every $\gamma \in \Gamma$, and in view of the injectivity of $\boldsymbol{p}_\Gamma$, proposition 8.5.3(ii) and lemma 8.5.6(i) easily imply that $\pi_{p\gamma}(a^p) = \pi_\gamma(a)^p$ in the Γ-graded $\mathbb{Z}$-algebra B/pB. To ease notation, set $D_\gamma := \mathrm{gr}_\gamma \mathbf{E}(\underline{B})$ and $D_{0,\gamma} := \mathrm{gr}_\gamma \mathbf{E}(\underline{B}/p\underline{B})$ for every $\gamma \in \mathbf{E}(\Gamma)$; there follows, for every such γ, a continuous $\mathbf{E}(\mathrm{gr}_0 B)$-linear map

$$\pi_{\mathbf{E},\gamma} : \mathbf{E}(A/pA) \to D_{0,\gamma} \qquad (a_n \mid n \in \mathbb{N}) \mapsto (\pi_{\gamma/p^n}(a_n) \mid n \in \mathbb{N}).$$

Let also $\pi^\wedge_\gamma : (A/pA)^\wedge \to \mathrm{gr}_\gamma B_0$ be the completion of π_γ, for every $\gamma \in \Gamma$; we obtain likewise a continuous $\mathbf{E}(\mathrm{gr}_0 B_0)$-linear map for every $\gamma \in \mathbf{E}(\Gamma)$

$$\pi^\wedge_{\mathbf{E},\gamma} : \mathbf{E}((A/pA)^\wedge) \to \mathrm{gr}_\gamma \mathbf{E}(\underline{B}_0) \qquad (a_n \mid n \in \mathbb{N}) \mapsto (\pi^\wedge_{\gamma/p^n}(a_n) \mid n \in \mathbb{N}).$$

However, $\pi^\wedge_{\mathbf{E},\gamma}$ is none else than the canonical γ-projection of $\mathbf{E}((A/pA)^\wedge, \underline{B}_0)$; indeed, it is easily seen that it coincides with the latter map on the dense subring $\mathbf{E}(\underline{B}_0)$, and $\mathrm{gr}_\gamma \mathbf{E}(\underline{B}_0)$ is separated, whence the claim. Moreover, by construction we have a commutative diagram

$$\begin{array}{ccc} \mathbf{E}(A/pA) & \xrightarrow{\prod_{\gamma\in\mathbf{E}(\Gamma)} \pi_{\mathbf{E},\gamma}} & \prod_{\gamma\in\mathbf{E}(\Gamma)} D_{0,\gamma} \\ {\scriptstyle \mathbf{E}(j)}\downarrow & & \downarrow{\scriptstyle \prod_{\gamma\in\mathbf{E}(\Gamma)} i_\gamma} \\ \mathbf{E}((A/pA)^\wedge) & \xrightarrow{\prod_{\gamma\in\mathbf{E}(\Gamma)} \pi^\wedge_{\mathbf{E},\gamma}} & \prod_{\gamma\in\mathbf{E}(\Gamma)} \mathrm{gr}_\gamma \mathbf{E}(\underline{B}_0) \end{array}$$

where $i_\gamma : D_{0,\gamma} \to \mathrm{gr}_\gamma \mathbf{E}(\underline{B}_0)$ is the restriction of $\mathbf{E}(j)$, for every $\gamma \in \Gamma$. Then, a simple inspection of this diagram shows that $\mathrm{gr}_\gamma C_0 \subset D_{0,\gamma}$ for every $\gamma \in \Gamma$, and the converse inclusion is obvious. Lastly, it is clear that $\mathbf{E}(\pi_A)(D_\gamma) \subset D_{0,\gamma}$ for every $\gamma \in \mathbf{E}(\Gamma)$. To show the converse inclusion, let $\bar{b}_\bullet := (\bar{b}_n \mid n \in \mathbb{N})$ be any element of $D_{0,\gamma}$, and for every $n \in \mathbb{N}$ pick

$b_n \in \mathrm{gr}_{\gamma/p^n} B$ whose image in $\mathrm{gr}_{\gamma/p^n}(B/pB)$ agrees with $\bar{b}_n$; according to remark 9.4.12(i), the element $\mathbf{E}(\pi_A)^{-1}(\bar{b}_\bullet)$ is the sequence $(b'_n \mid n \in \mathbb{N})$ with $b'_n := \lim_{k\to\infty} b_{n+k}^{p^k}$ for every $n \in \mathbb{N}$, where the convergence is relative to the topology of A. But since $\mathrm{gr}_\gamma B$ is closed in A for every $\gamma \in \Gamma$, we see that $b'_n \in \mathrm{gr}_{\gamma/p^n} B$ for every $n \in \mathbb{N}$, whence the claim. □

9.5. **Divided power modules and algebras.** This section is a review of the theory of divided power modules and algebras.

Definition 9.5.1. Let A be a ring, $k \in \mathbb{N}$ an integer, and $\underline{M} := (M_1, \dots, M_k)$ an object of the abelian category $(A\text{-}\mathbf{Mod})^k$, *i.e.* any sequence of A-modules. Let also Q be another A-module, and $\underline{d} := (d_1, \dots, d_k) \in \mathbb{N}^{\oplus k}$ any sequence of k integers.

(i) We denote by $F_{\underline{M}}, G_Q : A\text{-}\mathsf{Alg} \to \mathbf{Set}$ the functors given respectively by the rules :

$$B \mapsto (M_1 \times \cdots \times M_k) \otimes_A B \quad \text{and} \quad B \mapsto Q \otimes_A B \qquad \text{for every } A\text{-algebra } B.$$

Also, let $M_i^\vee := \mathrm{Hom}_A(M_i, A)$ for every $i = 1, \dots, k$, and set :

$$\mathrm{Sym}_A^\bullet(\underline{M}^\vee) := \bigotimes_{i=1}^k \mathrm{Sym}_A^\bullet(M_i^\vee)$$

which we view as a $\mathbb{N}^{\oplus k}$-graded A-algebra, with the grading given by the rule :

$$\mathrm{Sym}_A^{\underline{n}}(\underline{M}^\vee) := \bigotimes_{i=1}^k \mathrm{Sym}_A^{n_i}(M_i^\vee) \qquad \text{for every } \underline{n} := (n_1, \dots, n_k) \in \mathbb{N}^{\oplus k}.$$

(ii) A *homogeneous multipolynomial law* of degree $\underline{d}$ on the sequence of A-modules $\underline{M}$, with values in Q, is a natural transformation $\lambda : F_{\underline{M}} \to G_Q$, which we write also

$$\lambda : \underline{M} \rightsquigarrow Q$$

such that for every A-algebra B we have :

$$\lambda_B(b_\bullet x_\bullet) = b_1^{d_1} \cdots b_k^{d_k} \cdot \lambda_B(x_\bullet) \qquad \text{for every } b_\bullet \in B^{\oplus k} \text{ and every } x_\bullet \in F_{\underline{M}}(B) \tag{9.5.2}$$

where $b_\bullet x_\bullet := (b_1 x_1, \dots, b_k x_k)$ (and where we let $b^0 := 1$ for every $b \in B$).

(iii) The set of all homogeneous multipolynomial laws $\underline{M} \rightsquigarrow Q$ of degree $\underline{d}$ is denoted

$$\mathrm{Pol}_A^{\underline{d}}(\underline{M}, Q).$$

Remark 9.5.3. (i) Let $\underline{f} := (f_i : M'_i \to M_i \mid i = 1, \dots, k)$ be a sequence of A-linear maps, and set $\underline{M}' := (M'_1, \dots, M'_k)$; let also $g : Q \to Q'$ be another A-linear map. Suppose that $\lambda : \underline{M} \rightsquigarrow Q$ is a homogeneous multipolynomial law of degree d; then we obtain a law

$$g \circ \lambda \circ \underline{f} : \underline{M}' \rightsquigarrow Q'$$

of the same type, by the rule :

$$B \mapsto (g \otimes_A B) \circ \lambda_B \circ ((f_1 \times \cdots \times f_k) \otimes_A B) : (M'_1 \times \cdots \times M'_k) \otimes_A B \to Q' \otimes_A B$$

for every A-algebra B. This shows that the rule $(\underline{M}, Q) \mapsto \mathrm{Pol}_A^{\underline{d}}(\underline{M}, Q)$ is functorial in both arguments. However, it is not *a priori* obvious that this functor takes values in (essentially) small sets. The latter assertion will nevertheless be proven shortly.

(ii) Suppose that $\underline{M}' := (M'_1, \dots, M'_{k'})$ is another sequence of A-modules, Q' another A-module, and $\underline{d}' := (d'_1, \dots, d'_{k'}) \in \mathbb{N}^{\oplus k'}$ another sequence of integers. If we have two homogeneous polynomial laws $\lambda : \underline{M} \rightsquigarrow Q$ and $\lambda' : \underline{M}' \rightsquigarrow Q'$ of degrees respectively $\underline{d}$ and $\underline{d}'$, we can form the tensor product

$$\lambda \otimes \lambda' : (\underline{M}, \underline{M}') \rightsquigarrow Q \otimes_A Q'$$

which is the homogeneous multipolynomial law of degree $(\underline{d}, \underline{d}')$ given by the rule :

$$(\lambda \otimes \lambda')_B(x_1, \dots, x_k, x'_1, \dots, x'_{k'}) := \lambda_B(x_1, \dots, x_k) \otimes \lambda'_B(x'_1, \dots, x'_{k'})$$

for every A-algebra B and every $(x_1, \dots, x'_{k'}) \in (M_1 \times \cdots \times M'_{k'}) \otimes_A B$.

(iii) Let B be any A-algebra, and Q a B-module, which we also regard as an A-module, by restriction of scalars. Set also $N := M_1 \times \cdots \times M_k$, to ease notation. We have a bijection :

$$\mathrm{Pol}^{\underline{d}}_B(B \otimes_A \underline{M}, Q) \stackrel{\sim}{\to} \mathrm{Pol}^{\underline{d}}_A(\underline{M}, Q) \qquad (9.5.4)$$

natural with respect to B-linear maps $Q \to Q'$ and morphisms $\underline{M}' \to \underline{M}$ in $(A\text{-}\mathbf{Mod})^k$. Namely, to every homogeneous polynomial law $\lambda : B \otimes_A \underline{M} \rightsquigarrow Q$ of degree $\underline{d}$, we assign the homogeneous polynomial law $\lambda^\dagger : \underline{M} \rightsquigarrow Q$ that makes commute the diagrams :

$$\begin{array}{ccccc}
C \otimes_A N & & \xrightarrow{\quad\lambda^\dagger_C\quad} & & C \otimes_A Q \\
\downarrow{\scriptstyle \varphi \otimes_A N} & & & & \downarrow \\
C \otimes_A B \otimes_A N & \xrightarrow{\sim} & (C \otimes_A B) \otimes_B (B \otimes_A N) & \xrightarrow{\lambda_{C \otimes_A B}} & (C \otimes_A B) \otimes_B Q
\end{array}$$

whose right vertical arrow is the standard isomorphism, and where $\varphi : C \to C \otimes_A B$ is given by the rule : $c \mapsto c \otimes 1$ for every $c \in C$, for all A-algebras C. Indeed, the inverse of (9.5.4) is the mapping that assigns to every homogeneous polynomial law $\mu : M \rightsquigarrow Q$ of degree $\underline{d}$, the homogeneous polynomial law $\mu^\dagger : B \otimes_A \underline{M} \rightsquigarrow Q$ that makes commute the diagrams :

$$\begin{array}{ccc}
C \otimes_A N & \xrightarrow{\mu_C} & C \otimes_A Q \\
\downarrow & & \downarrow \\
C \otimes_B (B \otimes_A N) & \xrightarrow{\mu^\dagger_C} & C \otimes_B Q
\end{array} \qquad \text{for every } B\text{-algebra } C$$

whose left vertical arrow is the standard isomorphism, and whose right vertical arrow is the B-linear map such that $c \otimes q \mapsto c \otimes q$, for every $c \in C$ and $q \in Q$. We leave to the reader the verification of the identities : $(\lambda^\dagger)^\dagger = \lambda$ and $(\mu^\dagger)^\dagger = \mu$ for every such λ and μ.

9.5.5. In the situation of definition 9.5.1, let also $\mathscr{B}$ be any full subcategory of A-Alg such that $\mathrm{Ob}(\mathscr{B})$ contains all the free polynomial A-algebras of finite type, and denote by $j_{\mathscr{B}} : \mathscr{B} \to A$-$\mathsf{Alg}$ the inclusion functor. We consider also the set :

$$\mathrm{Pol}^{\underline{d}}_A(\underline{M}, Q)_{\mathscr{B}}$$

of all natural transformations $\lambda : F_{\underline{M}} \circ j_{\mathscr{B}} \Rightarrow G_Q \circ j_{\mathscr{B}}$ verifying condition (9.5.2) for every $B \in \mathrm{Ob}(\mathscr{B})$.

Proposition 9.5.6. *With the notation of* (9.5.5)*, the restriction map :*

$$\mathrm{Pol}^{\underline{d}}_A(\underline{M}, Q) \to \mathrm{Pol}^{\underline{d}}_A(\underline{M}, Q)_{\mathscr{B}} \qquad \lambda \mapsto \lambda * j_{\mathscr{B}}$$

is bijective for every $k \in \mathbb{N}$, every $\underline{d} \in \mathbb{N}^{\oplus k}$, every ring A, and all A-modules $M_1, \dots, M_k, Q$.

Proof. Let $\mathscr{B}_0$ (resp. $\mathscr{B}_1$) be the full subcategory of A-Alg whose objects are the free polynomial A-algebras of finite type (resp. the free polynomial A-algebras on arbitrary sets of indeterminates). To ease notation, set $N := M_1 \times \cdots \times M_k$; we show first :

Claim 9.5.7. The proposition holds for $\mathscr{B} = \mathscr{B}_1$.

Proof of the claim. Let us consider, for every A-algebra B, the standard simplicial resolution $P_B[\bullet] \to B$ arising from the cotriple associated to the forgetful functor A-$\mathsf{Alg} \to \mathbf{Set}$ and its left adjoint (see example 7.10.25(i)); hence the augmentation $\varepsilon_B : P_B[0] \to B$ is a surjective map of A-algebras, and the set underlying B is the coequalizer of the maps of A-algebras

$\partial_0, \partial_1 : P_B[1] \to P_B[0]$. If $\lambda : \underline{M} \rightsquigarrow Q$ is any homogeneous polynomial law of degree $\underline{d}$, the commutativity of the diagram

$$\begin{array}{ccc} P_B[1] \otimes_A N & \xrightarrow{\lambda_{P_B[1]}} & P_B[1] \otimes_A Q \\ {\scriptstyle \partial_0 \otimes_A N} \downarrow\downarrow {\scriptstyle \partial_1 \otimes_A N} & & {\scriptstyle \partial_0 \otimes_A Q} \downarrow\downarrow {\scriptstyle \partial_1 \otimes_A Q} \\ P_B[0] \otimes_A N & \xrightarrow{\lambda_{P_B[0]}} & P_B[0] \otimes_A Q \\ {\scriptstyle \varepsilon_B \otimes_A N} \downarrow & & \downarrow {\scriptstyle \varepsilon_B \otimes_A Q} \\ B \otimes_A N & \xrightarrow{\lambda_B} & B \otimes_A Q \end{array} \qquad (9.5.8)$$

and the surjectivity of $\varepsilon_B \otimes_A N$, already show that $\lambda_{P_B[0]}$ determines λ_B, whence the injectivity of our restriction map (as $P_B[0]$ is a free polynomial A-algebra). Next, let $\lambda : F_{\underline{M}} \circ j_{\mathscr{B}_1} \Rightarrow G_Q \circ j_{\mathscr{B}_1}$ be an element of $\mathrm{Pol}^{\underline{d}}_A(\underline{M}, Q)_{\mathscr{B}_1}$; since the tensor product is right exact, it is easily seen that the map $\varepsilon_B \otimes_A N$ identifies the set $B \otimes_A N$ with the coequalizer (in the category of sets) of the pair of maps $\partial_0 \otimes_A N, \partial_1 \otimes_A N : P_B[1] \otimes_A N \to P_B[0] \otimes_A N$ (details left to the reader). Since $P_B[0]$ and $P_B[1]$ are free polynomial A-algebras, in this situation the top square subdiagram of (9.5.8) is well defined and commutes, so there exists exactly one map of sets λ_B that makes commute also the bottom square subdiagram. It is easily seen that such λ_B fulfills (9.5.2), since the same holds for $\lambda_{P_B[0]}$, and since both ε_B and $\varepsilon_B \otimes_A N$ are surjective. Hence, we obtain a well defined map λ_B as required, for every A-algebra B, and it remains to check that the rule $B \mapsto \lambda_B$ defines a natural transformation $F_{\underline{M}} \Rightarrow G_Q$, *i.e.* that for every map $f : B \to B'$ of A-algebra, we have $(f \otimes_A Q) \circ \lambda_B = \lambda_{B'} \circ (f \otimes_A N)$. To this aim, we are easily reduced to showing the same identity for the map $g := f \circ \varepsilon_B : P_B[0] \to B'$; but since $P_B[0]$ is a free polynomial A-algebra, we may find a map of A-algebras $h : P_B[0] \to P_{B'}[0]$ such that $g = \varepsilon_{B'} \circ h$. Then, by a simple diagram chase, we are further reduced to checking the sought identity for the map h; but the latter holds by assumption, since both $P_B[0]$ and $P_{B'}[0]$ are free polynomial A-algebras. ◊

Next, we show that the proposition holds for $\mathscr{B} = \mathscr{B}_0$. In view of claim 9.5.7, we are reduced to checking that the corresponding restriction map $\mathrm{Pol}^{\underline{d}}_A(\underline{M}, Q)_{\mathscr{B}_1} \to \mathrm{Pol}^{\underline{d}}_A(\underline{M}, Q)_{\mathscr{B}_0}$ is bijective. To this aim, let us fix, for every free polynomial A-algebra B, a fitered system $(B_i \mid i \in I_B)$ of free polynomial A-subalgebras of B whose colimit is B. Now, if λ is any element of $\mathrm{Pol}^{\underline{d}}_A(\underline{M}, Q)_{\mathscr{B}_1}$, it is easily seen that λ_B must be the colimit of the induced system of maps of sets $(\lambda_{B_i} \mid i \in I_B)$. This shows already that our restriction map is injective. Lastly, let λ be any element of $\mathrm{Pol}^{\underline{d}}_A(\underline{M}, Q)_{\mathscr{B}_0}$; then conversely we may define, for every free polynomial A-algebra B, a map $\lambda_B : F_{\underline{M}}B \to G_Q B$, by taking the colimit of the system of maps $(\lambda_{B_i} \mid i \in I_B)$. It is easily seen that this map λ_B fulfills condition (9.5.2), for every such B : the details shall be left to the reader. It remains to check that the rule $B \mapsto \lambda_B$ defines a natural transformation. Hence, let $f : B \to B'$ be a map of free polynomial A-algebras; we need to show that $(f \otimes_A Q) \circ \lambda_B = \lambda_{B'} \circ (f \otimes_A N)$. However, let $(g_i : B_i \to B \mid i \in I_B)$ and $(g'_j : B'_j \to B' \mid j \in I_{B'})$ be the universal cocones; we are easily reduced to checking that the same identity holds for the map $f \circ g_i : B_i \to B'$, for every $i \in I_B$. But for every such i we may find $j \in I_{B'}$ such that $f \circ g_i$ factors through g'_j and a map $h : B_i \to B'_j$ of A-algebras, and we are further reduced to checking the sought identity for the map h. The latter holds by assumption, since $B_i, B'_j \in \mathrm{Ob}(\mathscr{B}_0)$.

Now, let $\mathscr{B}$ be any full subcategory of A-Alg containing $\mathscr{B}_0$; by the foregoing, it is clear that the restriction map $\mathrm{Pol}^{\underline{d}}_A(\underline{M}, Q) \to \mathrm{Pol}^{\underline{d}}_A(\underline{M}, Q)_{\mathscr{B}}$ is injective. Lastly, let λ be any element of $\mathrm{Pol}^{\underline{d}}_A(\underline{M}, Q)_{\mathscr{B}}$, and denote by λ_0 its image in $\mathrm{Pol}^{\underline{d}}_A(\underline{M}, Q)_{\mathscr{B}_0}$; by the foregoing, λ_0 is the restriciton of some $\lambda' \in \mathrm{Pol}^{\underline{d}}_A(\underline{M}, Q)$, and it remains to check that $\lambda'_B = \lambda_B$ for every $B \in \mathrm{Ob}(\mathscr{B})$. However, for every such B, and every $x \in F_{\underline{M}}B$, we may find $B_0 \in \mathrm{Ob}(\mathscr{B}_0)$, a map $f : B_0 \to B$ of A-algebras, and $y \in F_{\underline{M}}B_0$ with $f \otimes_A M(y) = x$; by construction, $\lambda_B(x) = \lambda_{B_0}(y) = \lambda'_{B_0}(y) = \lambda'_B(x)$, as required. □

Lemma 9.5.9. *In the situation of definition* 9.5.1*, suppose that the A-module M_i is free of finite rank, for every $i = 1, \dots, k$. Then, there exists a natural bijection*

$$\mathrm{Pol}^{\underline{d}}_A(\underline{M}, Q) \xrightarrow{\sim} Q \otimes_A \mathrm{Sym}^{\underline{d}}_A(\underline{M}^\vee).$$

Proof. Indeed, fix bases $(e_{i1}, \dots, e_{ir_i})$ for each A-module M_i, and denote by $(e^*_{i1}, \dots, e^*_{ir_i})$ the dual basis of $M_i^\vee$. For every $i = 1, \dots, k$ there is a natural (injective) map of A-algebras

$$\mathrm{Sym}^\bullet_A(M_i^\vee) \to \mathrm{Sym}^\bullet_A(\underline{M}^\vee) \quad : \quad s \mapsto 1 \otimes \cdots \otimes s \otimes \cdots \otimes 1$$

allowing to view naturally the e^*_{ij} as elements of $\mathrm{Sym}^\bullet_A(\underline{M}^\vee)$. Recall also that $\mathrm{Sym}^\bullet_A(M_i^\vee)$ is the free A-algebra $A[e^*_{i1}, \dots, e^*_{ir_i}]$ on the set $e^*_{i\bullet}$, for every $i = 1, \dots, k$. Let now $\lambda : \underline{M} \rightsquigarrow Q$ be a given homogeneous multipolynomial law of degree $\underline{d}$, and set

$$P_\lambda(e^*_{\bullet\bullet}) := \lambda_{\mathrm{Sym}^\bullet_A(\underline{M}^\vee)}\Big(\sum_{j=1}^{r_1} e_{1j} \otimes e^*_{1j}, \dots, \sum_{j=1}^{r_k} e_{kj} \otimes e^*_{kj}\Big) \in Q \otimes_A \mathrm{Sym}^\bullet_A(\underline{M}^\vee).$$

Notice that the terms in parenthesis are elements of $M_i \otimes_A M_i^\vee \subset M_i \otimes_A \mathrm{Sym}^\bullet_A(M_i^\vee)$ that do not depend on the chosen bases, since they correspond to the identity automorphisms of M_i, under the natural identification $M_i \otimes_A M_i^\vee \xrightarrow{\sim} \mathrm{End}_A(M_i)$. Now, let B be any A-algebra, and $b_{\bullet\bullet} := (b_{ij} \mid i = 1, \dots, k;\ j = 1, \dots, r_i)$ any sequence of elements of B. By considering the unique map of A-algebras $\varphi_{b_{\bullet\bullet}} : \mathrm{Sym}^\bullet_A(\underline{M}^\vee) \to B$ given by the rule : $e^*_{ij} \mapsto b_{ij}$ for every $i = 1, \dots, k$ and every $j = 1, \dots, r_i$, it is easily seen that

$$\lambda_B\Big(\sum_{j=1}^{r_1} b_{1j}e_{1j}, \dots, \sum_{j=1}^{r_k} b_{kj}e_{kj}\Big) = P_\lambda(b_{\bullet\bullet}) := (Q \otimes_A \varphi_{b_{\bullet\bullet}})(P_\lambda(e_{\bullet\bullet})) \in Q \otimes_A B.$$

In other words, λ is completely determined by the polynomial $P_\lambda(e^*_{\bullet\bullet})$ with coefficients in Q.

Next, take $B := \mathrm{Sym}^\bullet_A(\underline{M}^\vee)[Y_1, \dots, Y_k]$; the homogeneity condition on λ implies that

$$P_\lambda(Y_i e^*_{i\bullet} \mid i = 1, \dots, k) = \lambda_B\Big(Y_1 \cdot \sum_{j=1}^{r_1} e_{1j} \otimes e^*_{1j}, \dots, Y_k \cdot \sum_{j=1}^{r_k} e_{kj} \otimes e^*_{kj}\Big) = Y_1^{d_1} \cdots Y_k^{d_k} \cdot P(e^*_{\bullet\bullet})$$

which means that $P_\lambda \in Q \otimes_A \mathrm{Sym}^{\underline{d}}_A(\underline{M}^\vee)$. Conversely, it is clear that any element of $Q \otimes_A \mathrm{Sym}^{\underline{d}}_A(\underline{M}^\vee)$ yields a homogeneous law of degree $\underline{d}$, whence the lemma. □

Lemma 9.5.10. *Let $f_\bullet, g_\bullet : \underline{M}'' \to \underline{M}'$ be two morphisms of $(A\text{-}\mathbf{Mod})^k$, and $p_\bullet : \underline{M}' \to \underline{M}$ the projection of $\underline{M}'$ onto $\underline{M} :=$ $\mathrm{Coequal}(f_\bullet, g_\bullet)$. Suppose that for $i = 1, \dots, k$, the map*

$$h_i : M''_i \to M'_i \times_{(p_i,p_i)} M'_i \qquad x \mapsto (f_i(x), g_i(x)) \qquad \textit{for every } x \in M''_i$$

is surjective. Then the induced map of sets

$$\mathrm{Pol}^{\underline{d}}_A(\underline{M}, Q) \to \ \mathrm{Equal}\,(\mathrm{Pol}^{\underline{d}}_A(\underline{M}', Q) \rightrightarrows \mathrm{Pol}^{\underline{d}}_A(\underline{M}'', Q))$$

is bijective for every A-module Q and every $\underline{d} \in \mathbb{N}^{\oplus k}$ (notation of example 1.2.16(ii)*).*

Proof. For $i = 1, \dots, k$, let $q_i : M'_i \oplus M'_i \to M_i$ be the map such that $(x, y) \mapsto p_i(x - y)$ for every $x, y \in M'_i$. Then q_i is a surjection with kernel $M'_i \times_{M_i} M'_i$. If B is any A-algebra, it follows that the image of $B \otimes_A (\underline{M}' \times_{\underline{M}} \underline{M}')$ in $B \otimes_A (\underline{M}' \oplus \underline{M}')$ is $(B \otimes_A \underline{M}') \times_{B \otimes_A \underline{M}} (B \otimes_A \underline{M}')$, and on the other hand $B \otimes_A h_i$ is still surjective for every $i = 1, \dots, k$, so the same holds for each of the induced maps $h_{B,i} : B \otimes_A M''_i \to (B \otimes_A M'_i) \times_{B \otimes_A M_i} (B \otimes_A M'_i)$, and clearly $B \otimes_A p_\bullet$ is the projection of $B \otimes_A \underline{M}'$ onto the coequalizer of $B \otimes_A f_\bullet$ and $B \otimes_A g_\bullet$.

Now, let $\lambda : \underline{M}' \to Q$ be any homogeneous polynomial law of degree $\underline{d}$, and B any A-algebra. Since each $h_{B,i}$ is surjective, it follows that λ_B factors through a unique map of sets $\mu_B : F_{\underline{M}}B \to B \otimes_A Q$, and an easy inspection shows that the resulting rule : $B \mapsto \mu_B$ is a homogeneous polynomial law of degree $\underline{d}$. □

Remark 9.5.11. (i) Lemma 9.5.9 shows especially that, if $M_1, \dots, M_k$ are free A-modules of finite rank, the rule $Q \mapsto \mathrm{Pol}^{\underline{d}}_A(\underline{M}, Q)$ yields a functor with values in essentially small sets, and it is clear that the sum of two homogeneous multipolynomial laws of degree $\underline{d}$ is still a law of the same type; likewise, if we multiply such a law by an element of A, we get a new law of the same type. Up to isomorphism, we may therefore assume that this is a functor A-$\mathbf{Mod} \to A$-$\mathbf{Mod}$.

(ii) Let M be a free A-module of finite rank r, and fix any basis $e_1, \dots, e_r$ of M. The symmetric group S_d acts on the set $T := \{1, \dots, r\}^d$ by permutations :

$$\sigma(i_1, \dots, i_d) := (i_{\sigma^{-1}(1)}, \dots, i_{\sigma^{-1}(d)}) \qquad \text{for every } \sigma \in S_d \text{ and every } i_\bullet \in T.$$

For every such $i_\bullet$, let $[i_\bullet] \in S_d\backslash T$ be the class of $i_\bullet$, and denote by $e_{[i_\bullet]} \in \mathrm{Sym}^d_A M$ be the class of $e_{i_\bullet} := e_{i_1} \otimes \cdots \otimes e_{i_d}$. Clearly, if $i_\bullet, j_\bullet \in T$ and $[i_\bullet] = [j_\bullet]$, then $e_{[i_\bullet]} = e_{[j_\bullet]}$, so this notation is unambiguous, and it is easily seen that the system $(e_{[i_\bullet]} \mid [i_\bullet] \in S_d\backslash T)$ is a basis of $\mathrm{Sym}^d_A M$ (details left to the reader). Moreover, let N be any other free A-module of finite rank s, and fix as well a basis $f_1, \dots f_s$ of N; recall that we have a natural A-linear isomorphism

$$\omega_{M,N} : M^\vee \otimes_A N^\vee \xrightarrow{\sim} (M \otimes_A N)^\vee. \tag{9.5.12}$$

Namely, denote by $e^*_1, \dots, e^*_r$ and $f^*_1, \dots, f^*_s$ the corresponding dual bases; then $(e_i \otimes f_j \mid i = 1, \dots, r; j = 1, \dots, s)$ is a basis of $M \otimes_A N$, and we let likewise $((e_i \otimes f_j)^* \mid i = 1, \dots, r; j = 1, \dots, s)$ be its dual basis. With this notation, we have

$$\omega_{M,N}(e^*_i \otimes f^*_j) = (e_i \otimes f_j)^* \qquad \text{for every } i = 1, \dots, r \text{ and } j = 1, \dots, s.$$

It is easily seen that this isomorphism is independent of the choice of bases $e_\bullet$ and $f_\bullet$.

(iii) The lemma can be rephrased by saying that, under the stated assumptions, the A-module

$$\Gamma^{\underline{d}}_A(\underline{M}) := (\mathrm{Sym}^{\underline{d}}_A \underline{M}^\vee)^\vee$$

represents the functor A-$\mathbf{Mod} \to A$-$\mathbf{Mod}$ such that $Q \mapsto \mathrm{Pol}^{\underline{d}}_A(\underline{M}, Q)$ for every A-module Q.

(iv) The rule $(A, M) \mapsto \Gamma^{\underline{d}}_A(\underline{M})$ is functorial for sequences $\underline{M}$ of free A-modules of finite rank. Indeed, let $\underline{\varphi} : \underline{M} \to \underline{N}$ be a morphism of such sequences. For every degree $\underline{d}$ of length k, the induced map $\mathrm{Pol}^{\underline{d}}_A(\underline{N}, Q) \to \mathrm{Pol}^{\underline{d}}_A(\underline{M}, Q)$ corresponds to a map

$$\mathrm{Sym}^{\underline{d}}_A(\underline{N}^\vee) \otimes_A Q \to \mathrm{Sym}^{\underline{d}}_A(\underline{M}^\vee) \otimes_A Q \tag{9.5.13}$$

that can be worked out as follows. Pick bases $(e_{ij} \mid j = 1, \dots, r_i)$ for N_i and $(f_{ij} \mid j = 1, \dots, s_i)$ for M_i, for every $i = 1, \dots, k$, and denote as usual by $(e^*_{ij} \mid j = 1, \dots, r_i)$, $(f^*_{ij} \mid j = 1, \dots, s_i)$ the respective dual bases. Say that $\lambda : N \rightsquigarrow Q$ is a given homogeneous law of degree $\underline{d}$, and let $P_\lambda(e^*_{\bullet\bullet}) \in \mathrm{Sym}^{\underline{d}}_A(\underline{N}^\vee) \otimes_A Q$ be the corresponding homogeneous polynomial. By construction, the polynomial $P_{\lambda\circ\underline{\varphi}} \in \mathrm{Sym}^{\underline{d}}_A(\underline{M}^\vee) \otimes_A Q$ corresponding to $\lambda \circ \varphi$ is calculated as

$$\begin{aligned} P_{\lambda\circ\underline{\varphi}}(f^*_{\bullet\bullet}) &:= (\lambda\circ\underline{\varphi})_{\mathrm{Sym}^\bullet_A(\underline{M}^\vee)}\Big(\sum_{j=1}^{s_i} f_{ij}\otimes f^*_{ij} \mid i=1,\dots,k\Big) \\ &= \lambda_{\mathrm{Sym}^\bullet_A(\underline{M}^\vee)}\Big(\sum_{j=1}^{s_i} \varphi_i(f_{ij})\otimes f^*_{ij} \mid i=1,\dots,k\Big) \\ &= \lambda_{\mathrm{Sym}^\bullet_A(\underline{M}^\vee)}\Big(\sum_{j=1}^{r_i} e_{ij}\otimes \varphi_i^\vee(e^*_{ij}) \mid i=1,\dots,k\Big) \\ &= P_\lambda(\varphi_i^\vee(e^*_{i\bullet} \mid i=1,\dots,k)). \end{aligned}$$

In other words, (9.5.13) equals $\mathrm{Sym}^{\underline{d}}_A(\underline{\varphi}^\vee)\otimes_A \mathbf{1}_Q$, so if we let

$$\Gamma^{\underline{d}}_A(\underline{\varphi}) := \mathrm{Sym}^{\underline{d}}_A(\underline{\varphi}^\vee)^\vee$$

we obtain :

- a functor $\Gamma^{\underline{d}}_A(-)$ from the full subcategory $(A\text{-}\mathbf{Mod}^o)^k_{\mathrm{ff}}$ of $(A\text{-}\mathbf{Mod}^o)^k$ consisting of all sequences of free A-modules of finite rank, into the category of A-modules
- for every A-module Q, a well defined isomorphism of functors on this subcategory

$$(Q,\underline{M}) \mapsto (\mathrm{Pol}^{\underline{d}}_A(\underline{M},Q) \stackrel{\sim}{\to} \mathrm{Hom}_A(\Gamma^{\underline{d}}_A\underline{M},Q) \quad : \quad A\text{-}\mathbf{Mod}\times(A\text{-}\mathbf{Mod}^o)^k_{\mathrm{ff}} \to A\text{-}\mathbf{Mod}$$

(v) Notice that for every sequence of A-modules $\underline{M} := (M_1,\dots,M_k)$, every A-multinear map $\varphi : M_1\times\cdots\times M_k \to Q$ induces, for every A-algebra B, a B-multilinear map $\varphi_B : (B\otimes_A M_1)\times\cdots\times(B\otimes_A M_k) \to B\otimes_A Q$, and the rule : $B\mapsto \varphi_B$ for every such B is obviously a homogeneous polynomial law of degree $1_k := (1,\dots,1)$. We obtain therefore a natural transformation of functors :

$$\mathrm{Hom}_A(M_1\otimes_A\cdots\otimes_A M_k, Q) \to \mathrm{Pol}^{1_k}_A(\underline{M},Q) \qquad \text{for every } A\text{-module } Q. \tag{9.5.14}$$

In case each M_i is free of finite rank, this transformation must come from an A-linear map

$$\Gamma^{1_k}_A(\underline{M}) \to M_1\otimes_A\cdots\otimes_A M_k. \tag{9.5.15}$$

On the other hand, by construction, and in view of (9.5.12), we have a natural identification

$$\Gamma^{1_k}_A(\underline{M}) = (M_1^\vee\otimes_A\cdots\otimes_A M_k^\vee)^\vee \stackrel{\sim}{\to} M_1\otimes_A\cdots\otimes_A M_k$$

and by inspecting the proof of lemma 9.5.9, it is easily seen that this latter map coincides with (9.5.15) : the detailed verification shall be left to the reader.

The following result extends the above observations to arbitrary sequences $\underline{M}$ of A-modules.

Theorem 9.5.16. *Let $\underline{M}$ be any sequence as in definition* 9.5.1(i). *We have :*

(i) $\mathrm{Pol}^{\underline{d}}_A(\underline{M},Q)$ *is an essentially small set, for every A-module Q.*

(ii) *There exists a functor*

$$\Gamma^{\underline{d}}_A : (A\text{-}\mathbf{Mod})^k \to A\text{-}\mathbf{Mod} \tag{9.5.17}$$

with a natural isomorphism of A-module valued functors

$$\mathrm{Pol}^{\underline{d}}_A(-,Q) \stackrel{\sim}{\to} \mathrm{Hom}_A(\Gamma^{\underline{d}}_A(-),Q) \qquad \textit{for every } A\textit{-module } Q.$$

Proof. Suppose first that the A-modules $M_1,\dots,M_k$ are finitely presented, and for $i=1,\dots,k$, pick an A-linear surjection $f_i : L_i \to M_i$, with L_i free of finite rank. Let $g_i : L_i\oplus L_i \to M_i$ be the map given by the rule : $(l,l')\mapsto f_i(l-l')$ for every $(l,l')\in L_i\oplus L_i$; since M_i is finitely presented, $\mathrm{Ker}\, g_i$ is finitely generated, for every $i=1,\dots,k$, so we may find a free A-module

L'_i of finite rank, and a surjective A-linear map $L'_i \to \operatorname{Ker} g_i$. By composing with the inclusion $\operatorname{Ker} g_i \to L_i \oplus L_i$ and the two projections $L_i \oplus L_i \to L_i$, we deduce a presentation of M_i :

$$\text{(9.5.18)} \qquad \operatorname{Coker}(h_i^1, h_i^2 : L'_i \rightrightarrows L_i) \xrightarrow{\sim} M_i$$

such that the induced map $L'_i \to L_i \times_{M_i} L_i$ is surjective. Set $\underline{L} := (L_1, \dots, L_k)$, $\underline{L}' := (L'_1, \dots, L'_k)$ and $\underline{h}^j := (h_1^j, \dots, h_k^j)$ for $j = 1, 2$. By lemma 9.5.10, the presentations (9.5.18) induce a presentation of sets :

$$\operatorname{Pol}_A^{\underline{d}}(\underline{M}, Q) \xrightarrow{\sim} \operatorname{Equal}(\operatorname{Pol}_A^{\underline{d}}(\underline{L}, Q) \rightrightarrows \operatorname{Pol}_A^{\underline{d}}(\underline{L}', Q)) \qquad \text{for every } A\text{-module } Q.$$

Combining with lemma 9.5.9, it follows easily that the theorem holds for $\underline{M}$, with

$$\Gamma_A^{\underline{d}}(\underline{M}) := \operatorname{Coequal}(\Gamma_A^{\underline{d}}(\underline{h}^1), \Gamma_A^{\underline{d}}(\underline{h}^2) : \Gamma_A^{\underline{d}}(\underline{L}') \rightrightarrows \Gamma_A^{\underline{d}}(\underline{L}))$$

(details left to the reader). Lastly, let $\underline{M}$ be an arbitrary sequence. We may then find a filtered system of sequences $(\underline{M}_\sigma \mid \sigma \in \Sigma)$ consisting of finitely presented A-modules and with A-linear transition maps, whose colimit is isomorphic to $\underline{M}$. Since such colimits are preserved by arbitrary base changes, and commute with the forgetful functor to sets, it follows easily that the system induces an isomorphism of sets :

$$\operatorname{Pol}_A^{\underline{d}}(\underline{M}, Q) \xrightarrow{\sim} \lim_{\sigma \in \Sigma} \operatorname{Pol}_A^{\underline{d}}(\underline{M}_\sigma, Q) \qquad \text{for every } A\text{-module } Q$$

(details left to the reader). In view of the foregoing, the theorem then holds for $\underline{M}$, with

$$\Gamma_A^{\underline{d}}(\underline{M}) := \operatorname*{colim}_{\sigma \in \Sigma} \Gamma_A^{\underline{d}}(\underline{M}_\sigma).$$

Again, we invite the reader to spell out the details. □

9.5.19. The identity map of $\Gamma_A^{\underline{d}} \underline{M}$ corresponds to a homogeneous multipolynomial law

$$\boldsymbol{\lambda}_{\underline{M}}^{\underline{d}} : \underline{M} \rightsquigarrow \Gamma_A^{\underline{d}} \underline{M}$$

such that every other law $\underline{M} \rightsquigarrow Q$ homogeneous of the same degree, factors uniquely through $\boldsymbol{\lambda}_{\underline{M}}^{\underline{d}}$ and an A-linear map $\Gamma_A^{\underline{d}} \underline{M} \to Q$ (in the sense explained in remark 9.5.3(i)). Especially, if $\underline{M}' \in \operatorname{Ob}((A\text{-}\mathbf{Mod})^{k'})$ is another sequence, and $\underline{d}' \in \mathbb{N}^{\oplus k'}$ any sequence of integers, the tensor product $\boldsymbol{\lambda}_{\underline{M}}^{\underline{d}} \otimes \boldsymbol{\lambda}_{\underline{M}'}^{\underline{d}'}$ (remark 9.5.3(ii)) factors uniquely through $\boldsymbol{\lambda}_{\underline{M}, \underline{M}'}^{\underline{d}, \underline{d}'}$ and an A-linear map

$$\text{(9.5.20)} \qquad \Gamma_A^{\underline{d}, \underline{d}'}(\underline{M}, \underline{M}') \to \Gamma_A^{\underline{d}}(\underline{M}) \otimes_A \Gamma_A^{\underline{d}'}(\underline{M}').$$

Furthermore, in the situation of remark 9.5.3(iii), the bijection (9.5.4) translates as a bijection :

$$\operatorname{Hom}_B(\Gamma_B^{\underline{d}}(B \otimes_A \underline{M}), Q) \xrightarrow{\sim} \operatorname{Hom}_A(\Gamma_A^{\underline{d}} \underline{M}, Q) \xrightarrow{\sim} \operatorname{Hom}_B(B \otimes_A \Gamma_A^{\underline{d}} \underline{M}, Q)$$

natural with respect to B-linear maps $Q \to Q'$ and morphisms $\underline{M}' \to \underline{M}$ in $(A\text{-}\mathbf{Mod})^k$. Such a natural bijection then must come from an isomorphism of B-modules

$$\text{(9.5.21)} \qquad B \otimes_A \Gamma_A^{\underline{d}}(\underline{M}) \xrightarrow{\sim} \Gamma_B^{\underline{d}}(B \otimes_A \underline{M}) \qquad 1 \otimes \boldsymbol{\lambda}_{\underline{M}}^{\underline{d}}(m) \mapsto \boldsymbol{\lambda}_{B \otimes_A \underline{M}}^{\underline{d}}(1 \otimes m)$$

which is natural with respect to morphisms in $(A\text{-}\mathbf{Mod})^k$ and maps $B \to B'$ of A-algebras.

Corollary 9.5.22. *With the notation of* (9.5.19)*, we have :*

(i) *The functor* (9.5.17) *commutes with all filtered colimits.*
(ii) *The map* (9.5.20) *is an isomorphism of A-modules, for every $\underline{M}$, $\underline{M}'$, $\underline{d}$ and $\underline{d}'$.*
(iii) *If the A-modules $M_1, \dots, M_k$ are all free (resp. finitely presented, resp. finitely generated, resp. flat), then the same holds for $\Gamma_A^{\underline{d}}(\underline{M})$.*

(iv) *In the situation of lemma* 9.5.10*, the induced map*

$$\mathrm{Coequal}\,(\Gamma^d_A(f_\bullet), \Gamma^d_A(g_\bullet) : \Gamma^d_A(\underline{M}'') \rightrightarrows \Gamma^d_A(\underline{M}')) \;\stackrel{\sim}{\to} \Gamma^d_A(\underline{M})$$

is an isomorphism of A-modules.

(v) *For every $\underline{M} := (M_1, \dots, M_k) \in \mathrm{Ob}((A\text{-}\mathbf{Mod})^k)$, the natural transformation* (9.5.14) *is an isomorphism of functors, and therefore it induces a natural A-linear isomorphism:*

$$\omega^{1_k}_{\underline{M}} : \Gamma^{1_k}_A(\underline{M}) \stackrel{\sim}{\to} M_1 \otimes_A \cdots \otimes_A M_k.$$

(vi) *Set $0_k := (0, \dots, 0) \in \mathbb{N}^{\oplus k}$; we have a natural isomorphism of functors :*

$$\Gamma^{0_k}_A(\underline{M}) \stackrel{\sim}{\to} A \qquad \textit{for every } \underline{M} \in \mathrm{Ob}((A\text{-}\mathbf{Mod})^k)$$

that identifies $\boldsymbol{\lambda}^{0_k}_{\underline{M}}$ with the polynomial law $\underline{M} \rightsquigarrow A : x_\bullet \mapsto 1$ for all $B \in \mathrm{Ob}(A\text{-}\mathsf{Alg})$ and all $x_\bullet \in F_{\underline{M}}(B)$, and identifies $\Gamma^{0_k}_A f$ with $\mathbf{1}_A$, for every morphism f of $(A\text{-}\mathbf{Mod})^k$.

Proof. (i): It suffices to check that the natural map

$$\mathrm{Pol}^d_A(\operatorname*{colim}_{\sigma\in\Sigma} \underline{M}_\sigma, Q) \to \lim_{\sigma\in\Sigma} \mathrm{Pol}^d_A(\underline{M}_\sigma, Q)$$

is an isomorphism, for every filtered system of A-modules $(\underline{M}_\sigma \mid \sigma \in \Sigma)$ and every A-module Q; the latter assertion is obvious (details left to the reader).

(iii): The assertion concerning free modules of finite rank, and finitely generated or finitely presented modules follows directly from the explicit construction in lemma 9.5.9 and theorem 9.5.16. The assertion for flat modules follows from the assertion for free modules of finite rank and from (i), by means of Lazard's theorem [120, Ch.I, Th.1.2]. Moreover, from remark 9.5.11(iv), we see that if $\varphi : \underline{M} \to \underline{N}$ is a morphism of sequences of free A-modules of finite rank, such that φ_i is a split injection for every $i = 1, \dots, n$, then $\Gamma^d_A(\varphi)$ is also a split injective map of free A-modules; from this, and from (i), it follows easily that Γ^d_A transforms sequences of free A-modules (of possibly infinite rank) into free A-modules : details left to the reader.

(iv) follows formally from lemma 9.5.10 : details left to the reader.

(ii): Let $t : \Gamma^{d,d'}_A(\underline{M}, \underline{M}') \to Q$ be an A-linear map. To t we attach an A-linear map

$$t^* : \Gamma^d_A(\underline{M}) \otimes_A \Gamma^{d'}_A(\underline{M}') \to Q$$

as follows. Set $\tau := t \circ \boldsymbol{\lambda}^{d,d'}_{\underline{M},\underline{M}'} : (\underline{M}, \underline{M}') \rightsquigarrow Q$ (notation of remark 9.5.3(i)). To every A-algebra B, the law τ attaches the mapping

$$B \otimes_A \underline{M} \to \mathrm{Pol}^{d'}_B(B \otimes_A \underline{M}', B \otimes_A Q) \qquad x \mapsto \tau_{B,x}$$

where $(\tau_{B,x})_C(y) := \tau_C(1 \otimes x, y)$ for every $x \in B \otimes_A \underline{M}$, every B-algebra C and every $y \in C \otimes_A \underline{M}'$. In light of (iii), the latter is the same as a mapping

$$B \otimes_A \underline{M} \to \mathrm{Hom}_B(B \otimes_A \Gamma^{d'}_A(\underline{M}'), B \otimes_A Q) \qquad x \mapsto t_{B,x}$$

and notice that, for every $x \in B \otimes_A \underline{M}$, the B-linear map $t_{B,x}$ is characterized by the identity

$$t_{B,x}(\boldsymbol{\lambda}^{d'}_{\underline{M}',B}(y)) = (\tau_{B,x})_B(y) \qquad \text{for every } y \in B \otimes_A \underline{M}'. \tag{9.5.23}$$

We deduce a compatible system of mappings

$$B \otimes_A \Gamma^{d'}_A(\underline{M}') \to \mathrm{Pol}^d_B(B \otimes_A \underline{M}, B \otimes_A Q) \quad : \quad y \mapsto \tau^*_{B,y} \qquad \text{for every } A\text{-algebra } B$$

where $(\tau^*_{B,y})_C(x) := t_{C,x}(1 \otimes y)$ for every B-algebra C, every $x \in C \otimes_A \underline{M}$, and every $y \in B \otimes_A \Gamma^{d'}_A(\underline{M}')$. The C-linearity of $t_{C,x}$ implies that these latter mappings are B-linear (for every A-algebra B), and then they are the same as a compatible system of B-linear maps

$$B \otimes_A \Gamma^{d'}_A(\underline{M}') \to \mathrm{Hom}_B(B \otimes_A \Gamma^d_A(\underline{M}), B \otimes_A Q) \quad : \quad y \mapsto t^*_{B,y} \tag{9.5.24}$$

for every A-algebra B. Notice that, for every $y \in B \otimes_A \Gamma_A^{d'}(\underline{M}')$, the B-linear map $t^*_{B,y}$ is characterized by the identity :

$$t^*_{B,y}(\boldsymbol{\lambda}^d_{\underline{M},B}(x)) = (\tau^*_{B,y})_B(x) \qquad \text{for every } x \in B \otimes_A \underline{M}. \tag{9.5.25}$$

Obviously, the datum of a compatible system (9.5.24) is the same as that of a map t^* as sought.

Claim 9.5.26. $t^* \circ (9.5.20) = t$.

Proof of the claim. It suffices to show that $t^* \circ (9.5.20) \circ \boldsymbol{\lambda}^{d,d'}_{\underline{M},\underline{M}'} = \tau$. However, recall that $(9.5.20) \circ \boldsymbol{\lambda}^{d,d'}_{\underline{M},\underline{M}'} = \boldsymbol{\lambda}^d_{\underline{M}} \otimes \boldsymbol{\lambda}^{d'}_{\underline{M}'}$, so we are reduced to checking that $t^* \circ (\boldsymbol{\lambda}^d_{\underline{M}} \otimes \boldsymbol{\lambda}^{d'}_{\underline{M}'}) = \tau$. Hence, let B be any A-algebra, and $m \in B \otimes_A \underline{M}$, $m' \in B \otimes_A \underline{M}'$ any two elements. To ease notation, set $x := \boldsymbol{\lambda}^d_{\underline{M}}(m)$ and $y := \boldsymbol{\lambda}^{d'}_{\underline{M}'}(m')$. We compute :

$$\begin{aligned}
(t^* \circ (\boldsymbol{\lambda}^d_{\underline{M}} \otimes \boldsymbol{\lambda}^{d'}_{\underline{M}'}))_B(m, m') &= (B \otimes_A t^*)(x \otimes y) \\
&= t^*_{B,y}(x) \\
&= (\tau^*_{B,y})_B(m) && \text{(by (9.5.25))} \\
&= t_{B,m}(y) \\
&= (\tau_{B,m})_B(m') && \text{(by (9.5.23))} \\
&= \tau_B(m, m')
\end{aligned}$$

as claimed. ◊

Especially, if we let t be the identity map of $\Gamma_A^{d,d'}(\underline{M}, \underline{M}')$, claim 9.5.26 shows that the resulting t^* is a left inverse for (9.5.20). Now, if all the modules $M_1, \dots, M'_{k'}$ are free of finite rank, then lemma 9.5.9 shows that both of the A-modules appearing in (9.5.20) are free and have the same rank, so the assertion follows, in this case.

Next, suppose that each A-module M_i and M'_i is finitely presented; pick for $i = 1, \dots, k$ a finite presentation for M_i as in (9.5.18), and let $\underline{L} := (L_1, \dots, L_k)$ and $\underline{L}' := (L'_1, \dots, L'_k)$. By (iv) and naturality of (9.5.20), we get an induced commutative diagram with exact rows :

$$\begin{array}{ccccccc}
\Gamma_A^{\underline{d}}(\underline{L}', \underline{M}') & \rightrightarrows & \Gamma_A^{\underline{d}}(\underline{L}, \underline{M}') & \longrightarrow & \Gamma_A^{\underline{d}}(\underline{M}, \underline{M}') & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\Gamma_A^{\underline{d}}(\underline{L}') \otimes_A \Gamma_A^{\underline{d}}(\underline{M}') & \rightrightarrows & \Gamma_A^{\underline{d}}(\underline{L}) \otimes_A \Gamma_A^{\underline{d}}(\underline{M}') & \longrightarrow & \Gamma_A^{\underline{d}}(\underline{M}) \otimes_A \Gamma_A^{\underline{d}}(\underline{M}') & \longrightarrow & 0
\end{array}$$

so it suffices to prove the assertion for $(\underline{L}, \underline{M}')$ and $(\underline{L}', \underline{M}')$. Arguing likewise with a presentation for $\underline{M}'$, we reduce to the case where $\underline{M}$ and $\underline{M}'$ consist of free A-modules of finite rank, which has already been treated. Lastly, if $\underline{M}$ and $\underline{N}$ are arbitrary sequences, we may present them as filtered colimits of sequences of finitely presented A-modules; since tensor products commute with such colimits, we are done, by the foregoing case (details left to the reader).

(v): First, by a direct inspection of the constructions we get a commutative diagram :

$$\begin{array}{ccc}
\Gamma_A^{1_k}(\underline{M}) & \xrightarrow{\qquad\qquad} & \Gamma_A^1 M_1 \otimes_A \cdots \otimes_A \Gamma_A^1 M_k \\
 & {\scriptstyle \omega^{1_k}_{\underline{M}}} \searrow \qquad \swarrow {\scriptstyle \omega^1_{M_1} \otimes_A \cdots \otimes_A \omega^1_{M_k}} & \\
 & M_1 \otimes_A \cdots \otimes_A M_k &
\end{array}$$

whose top horizontal arrow is obtained from the isomorphisms (9.5.20) provided by (ii). Hence, we are reduced to the case where $k = 1$, so $\underline{M}$ is a single A-module M. Next, by virtue of (i), we may assume that M is an A-module of finite presentation, in which case we may pick a presentation (9.5.18) of M by free A-modules of finite rank, and by virtue of (iv) we are then further reduced to the case where M is free of finite rank, in which case the assertion is already known by remark 9.5.11(v).

(vi): Let $\lambda : \underline{M} \rightsquigarrow Q$ be any homogeneous polynomial law of degree 0_k; then :

$$\lambda_B(x_\bullet) = 0^0 \cdots 0^0 \cdot \lambda_B(x_\bullet) = \lambda_B(0x_1, \dots, 0x_k) = \lambda_B(0, \dots, 0)$$

for every A-algebra B and every $x_\bullet \in F_{\underline{M}}(B)$. In other words, λ_B is the constant map with value $\lambda_A(0, \dots, 0)$ for every such B. We have then a natural isomorphism of functors :

$$\mathrm{Pol}_A^{0_k}(\underline{M}, Q) \xrightarrow{\sim} Q \xrightarrow{\sim} \mathrm{Hom}_A(A, Q) \qquad \text{for every } A\text{-module } Q$$

which must come from an isomorphism $\Gamma_A^{0_k}\underline{M} \xrightarrow{\sim} A$, and the stated identifications of $\boldsymbol{\lambda}_{\underline{M}}^{0_k}$ and $\Gamma_A^{0_k} f$ follow by direct inspection. □

Remark 9.5.27. (i) Let $\underline{M}$ and $\underline{d}$ be as in (9.5.19), suppose that all the A-modules M_i are free of finite rank, and set $S(\underline{M}) := \mathrm{Sym}_A^\bullet(\underline{M}^\vee)$. Let Q be another A-module, and $\lambda : \underline{M} \rightsquigarrow Q$ a homogeneous multipolynomial law of degree $\underline{d}$. From lemma 9.5.9 we may extract the following calculation for the A-linear map $\lambda^\dagger : \Gamma_A^{\underline{d}}(\underline{M}) \to Q$ corresponding to λ. Set

$$P_\lambda := \lambda_{S(\underline{M})}(\mathbf{1}_{M_1}, \dots, \mathbf{1}_{M_k}) \in \Gamma_A^{\underline{d}}(\underline{M})^\vee \otimes_A Q.$$

Then $\lambda^\dagger$ is the image of P_λ under the natural identification $\Gamma_A^{\underline{d}}(\underline{M})^\vee \otimes_A Q \xrightarrow{\sim} \mathrm{Hom}_A(\Gamma_A^{\underline{d}}(\underline{M}), Q)$.

(ii) Let $\underline{M}$, $\underline{M}'$, $\underline{d}$ and $\underline{d}'$ be as in (9.5.19), suppose that all the A-modules M_i and M'_i are free of finite rank, and to ease notation let

$$\boldsymbol{\lambda} := \boldsymbol{\lambda}_{\underline{M}}^{\underline{d}} \qquad \boldsymbol{\lambda}' := \boldsymbol{\lambda}_{\underline{M}'}^{\underline{d}'} \qquad \lambda := \boldsymbol{\lambda} \otimes \boldsymbol{\lambda}'.$$

By (i), we compute

$$P_\lambda = \boldsymbol{\lambda}_{S(\underline{M},\underline{M}')}(\mathbf{1}_{M_1}, \dots, \mathbf{1}_{M_k}) \otimes \boldsymbol{\lambda}'_{S(\underline{M},\underline{M}')}(\mathbf{1}_{M'_1}, \dots, \mathbf{1}_{M'_{k'}}).$$

However, $\boldsymbol{\lambda}_{S(\underline{M},\underline{M}')}(\mathbf{1}_{M_1}, \dots, \mathbf{1}_{M_k})$ is the image of $P_{\boldsymbol{\lambda}}$ under the A-linear map induced by the A-algebra homomorphism $S(\underline{M}) \to S(\underline{M}, \underline{M}') = S(\underline{M}) \otimes_A S(\underline{M}')$ given by the rule $s \mapsto s \otimes 1$, for every $s \in S(\underline{M})$. Likewise we determine $\boldsymbol{\lambda}'_{S(\underline{M},\underline{M}')}(\mathbf{1}_{M'_1}, \dots, \mathbf{1}_{M'_{k'}})$, and we conclude that $P_\lambda = P_{\boldsymbol{\lambda}} \otimes P_{\boldsymbol{\lambda}'}$, whence $\lambda^\dagger = \boldsymbol{\lambda}^\dagger \otimes \boldsymbol{\lambda}'^\dagger$, under the natural identification

$$\mathrm{End}_A(\Gamma_A^{\underline{d},\underline{d}'}(\underline{M}, \underline{M}')) \xrightarrow{\sim} \mathrm{End}_A(\Gamma_A^{\underline{d}}(\underline{M})) \otimes_A \mathrm{End}_A(\Gamma_A^{\underline{d}'}(\underline{M}')).$$

But by definition, $(\boldsymbol{\lambda}_{\underline{M}}^{\underline{d}})^\dagger$ is the identity automorphism of $\Gamma_A^{\underline{d}}(\underline{M})$, for every sequence $\underline{M}$, and every degree $\underline{d}$. We see therefore that :

$$(\boldsymbol{\lambda}_{\underline{M}}^{\underline{d}} \otimes \boldsymbol{\lambda}_{\underline{M}'}^{\underline{d}'})^\dagger = (\boldsymbol{\lambda}_{\underline{M},\underline{M}'}^{\underline{d},\underline{d}'})^\dagger$$

which translates as saying that the A-linear map (9.5.20) is none else than the transpose of the natural isomorphism

$$\mathrm{Sym}_A^{\underline{d}}(\underline{M}^\vee) \otimes_A \mathrm{Sym}_A^{\underline{d}'}(\underline{M}'^\vee) \xrightarrow{\sim} \mathrm{Sym}_A^{\underline{d},\underline{d}'}(\underline{M}^\vee, \underline{M}'^\vee).$$

This gives an alternative proof of corollary 9.5.22(ii), in the case of free A-modules.

9.5.28. Let A be a ring, and $d \in \mathbb{N}$. For every A-module M, the symmetric group S_d acts on $\otimes_A^d M$ by permuting the tensor factors :

$$\sigma(m_1 \otimes \cdots \otimes m_d) := m_{\sigma^{-1}(1)} \otimes \cdots \otimes m_{\sigma^{-1}(d)} \qquad \text{for every } \sigma \in S_d \text{ and every } m_1, \dots, m_d \in M$$

(here $\otimes_A^0 M = A$ for every such M, and S_0 is the trivial group with one element). We set :

$$\mathrm{TS}_A^d M := (\otimes_A^d M)^{S_d}.$$

Clearly every A-linear map $f : M \to M'$ induces an A-linear map $\otimes^d_A(f) : \otimes^d_A M \to \otimes^d_A M'$ such that $m_1 \otimes \cdots m_d \mapsto f(m_1) \otimes \cdots \otimes f(m_d)$ for every $m_1, \dots, m_d \in M$, and $\otimes^d_A(f)$ restricts to a map $\mathrm{TS}^d_A(f) : \mathrm{TS}^d_A M \to \mathrm{TS}^d_A M'$, so we get a well defined functor

$$\mathrm{TS}^d_A : A\text{-}\mathbf{Mod} \to A\text{-}\mathbf{Mod} \qquad \text{for every } d \in \mathbb{N} \text{ and every ring } A.$$

Moreover, notice that the A-linear map

$$\otimes^d_A M \to \mathrm{TS}^d_A M \qquad x \mapsto \sum_{\sigma \in S_d} \sigma(x)$$

factors through an A-linear map

$$\alpha^d_M : \mathrm{Sym}^d_A M \to \mathrm{TS}^d_A M.$$

Let furthermore $\beta^d_M : \mathrm{TS}^d_A M \to \mathrm{Sym}^d_A M$ be the restriction of the projection $\otimes^d_A M \to \mathrm{Sym}^d_A M$.

Lemma 9.5.29. (i) *For every flat A-algebra B, the natural map*

$$B \otimes_A \mathrm{TS}^d_A M \to \mathrm{TS}^d_B(B \otimes_A M)$$

is an isomorphism.

(ii) *If M is free of finite rank r, then $\mathrm{TS}^d_A M$ is a free direct factor of $\otimes^d_A M$ of rank $\binom{r+d-1}{d}$.*

(iii) $\alpha^d_M \circ \beta^d_M = d! \cdot \mathbf{1}_{\mathrm{TS}^d_A M}$ *and* $\beta^d_M \circ \alpha^d_M = d! \cdot \mathbf{1}_{\mathrm{Sym}^d_A M}$.

(iv) *Let $e \in \mathbb{N}$ be another integer; if M is a flat A-module, the same holds for $\mathrm{TS}^d_A M$, and there exists a subgroup $W \subset S_{de}$ such that the natural map*

$$\mathrm{TS}^e_A(\mathrm{TS}^d_A M) \to \otimes^e_A(\otimes^d_A M) \xrightarrow{\sim} \otimes^{de}_A M$$

identifies $\mathrm{TS}^e_A(\mathrm{TS}^d_A M)$ with the submodule $(\otimes^{de}_A M)^W$ of $\otimes^{de}_A M$.

Proof. (i) and (iii) shall be left to the reader.

(ii): Fix a basis $e_1, \dots, e_r$ of M; then $\otimes^d_A M$ is free with basis $(e_{i_1} \otimes \cdots \otimes e_{i_d} \mid i_1, \dots, i_d \in \{1, \dots, j\})$. For every $i_\bullet \in T := \{1, \dots, r\}^d$, let also $S_d(i_\bullet) \subset T$ be the orbit of $i_\bullet$ under the S_d-action defined as in remark 9.5.11(ii); moreover, set $e_{i_\bullet} := e_{i_1} \otimes \cdots \otimes e_{i_d}$ for every such $i_\bullet$. Choose a set of representatives Δ for the quotient $S_d \backslash T$; it is then easily seen that $\mathrm{TS}^d_A M$ is free with basis :

$$[e_{i_\bullet}] := \sum_{j_\bullet \in S_d(i_\bullet)} e_{j_\bullet} \qquad \text{for all } i_\bullet \in \Delta$$

and the assertion follows by a direct computation that we leave to the reader.

(iv): In order to check that $\mathrm{TS}^d_A M$ is a flat A-module, since the functor TS^d_A commutes with filtered colimits, by Lazard's theorem ([120, Ch.I, Th.1.2]) we are reduced to the case where M is a free A-module of finite rank, in which case the assertion follows from (ii). Let $S_{d,e}$ be the group of permutations of $\Sigma := \{1, \dots, d\} \times \{1, \dots, e\}$ (of course, this is isomorphic to S_{de}). We have injective group homomorphisms :

$$S^e_d \xrightarrow{f} S_{d,e} \xleftarrow{g} S_e$$

where f assigns to every $(\sigma_1, \dots, \sigma_e) \in S^e_d$ the permutation such that $(a, b) \mapsto (\sigma_b(a), b)$ for every $(a, b) \in \Sigma$, and g assigns to every $\psi \in S_e$ the permutation such that $(a, b) \mapsto (a, \psi(b))$ for every such (a, b). Clearly $\mathrm{Im}(f) \cap \mathrm{Im}(g) = \{\mathbf{1}_\Sigma\}$, and for every $\psi \in S_e$, the inner automorphism of $S_{d,e}$ given by conjugation by $g(\psi)$ leaves $\mathrm{Im}(f)$ invariant. Thus, we can combine f and g into an injective group homomorphism

$$W := S^e_d \rtimes S_e \to S_{d,e} \qquad (\sigma, \psi) \mapsto f(\sigma) \circ g(\psi).$$

Claim 9.5.30. For $i = 1, 2$, let M_i be an A-module with a finite group Γ_i of A-linear automorphisms. If M_i and $M_{3-i}^{\Gamma_{3-i}}$ are flat A-modules for either $i = 1$ or $i = 2$, the natural map

$$M_1^{\Gamma_1} \otimes_A M_2^{\Gamma_2} \to (M_1 \otimes_A M_2)^{\Gamma_1 \times \Gamma_2}$$

is an isomorphism.

Proof of the claim. Say that M_2 and $M_1^{\Gamma_1}$ are flat A-modules. We have :

$$(M_1 \otimes_A M_2)^{\Gamma_1 \times \Gamma_2} = ((M_1 \otimes_A M_2)^{\Gamma_1 \times \mathbf{1}_{M_2}})^{\mathbf{1}_{M_1} \times \Gamma_2}$$

and the natural map $(M_1 \otimes_A M_2)^{\Gamma_1 \times \mathbf{1}_{M_2}} \to M_1^{\Gamma_1} \otimes_A M_2$ (resp. $(M_1^{\Gamma_1} \otimes_A M_2)^{\mathbf{1}_{M_1} \times \Gamma_2} \to M_1^{\Gamma_1} \otimes_A M_2^{\Gamma_2}$) is an isomorphism, since M_2 (resp. $M_1^{\Gamma_1}$) is a flat A-module. The assertion follows in this case; one argues likewise in case M_1 and $M_2^{\Gamma_2}$ are flat A-modules. ◊

Since $\mathrm{TS}^d_A M$ is a flat A-module, by claim 9.5.30, and a simple induction on e, we compute :

$$\mathrm{TS}^e_A(\mathrm{TS}^d_A M) = (\otimes^e_A(\otimes^d_A M)^{S_d})^{S_e} = ((\otimes^e_A(\otimes^d_A M))^{S_d^e})^{S_e} = ((\otimes^e_A(\otimes^d_A M))^W$$

as stated. □

9.5.31. For every $d > 0$ and every A-module M set $M^{(d)} := (M, \dots, M) \in \mathrm{Ob}((A\text{-}\mathbf{Mod})^d)$, and for every $\underline{e} \in \mathbb{N}^{\oplus d}$ let $|\underline{e}| := e_1 + \cdots + e_k$; we have a natural transformation of functors :

$$\mathrm{Pol}^{\underline{e}}_A(M^{(d)}, Q) \to \mathrm{Pol}^{|\underline{e}|}_A(M, Q) \qquad \text{for every } A\text{-module } Q$$

that assigns to every homogeneous polynomial law $\lambda : M^{(d)} \rightsquigarrow Q$ of degree $\underline{e}$ the polynomial law $\lambda^\dagger : M \rightsquigarrow Q$ such that

$$\lambda^\dagger_B(x) = \lambda_B(x, \dots, x) \qquad \text{for every } A\text{-algebra } B \text{ and every } x \in B \otimes_A M.$$

In view of corollary 9.5.22(ii), this transformation then comes from an A-linear map

$$\Gamma^{|\underline{e}|}_A M \to \Gamma^{\underline{e}}_A M^{(d)} \xrightarrow{\sim} \Gamma^{e_1}_A M \otimes_A \cdots \otimes_A \Gamma^{e_d}_A M \tag{9.5.32}$$

which is characterized as the unique A-linear map such that

$$(\boldsymbol{\lambda}^{\underline{e}}_{M^{(d)}})_B(x) \mapsto (\boldsymbol{\lambda}^{e_1}_M)_B(x) \otimes \cdots \otimes (\boldsymbol{\lambda}^{e_d}_M)_B(x)$$

for every A-algebra B and every $x \in B \otimes_A M$.

9.5.33. Consider next the special case where $\underline{e} = (e, \dots, e)$ for some $e \in \mathbb{N}$; then, every $\sigma \in S_d$ determines a natural isomorphism of functors :

$$\mathrm{Pol}^{\underline{e}}_A(M^{(d)}, Q) \xrightarrow{\sim} \mathrm{Pol}^{\underline{e}}_A(M^{(d)}, Q)$$

that assigns to every λ as in the foregoing, the polynomial law $\sigma(\lambda) : M^{(d)} \rightsquigarrow Q$ such that

$$\sigma(\lambda)_B(x_\bullet) = \lambda_B(x_{\sigma(1)}, \dots, x_{\sigma(d)}) \quad \text{for every } A\text{-algebra } B \text{ and every } x_\bullet \in (B \otimes_A M)^{\oplus d}.$$

We obtain in this fashion a natural S_d-action on the functor $\mathrm{Pol}^{\underline{e}}_A(M^{(d)}, -)$, which clearly translates as the permutation action on $\Gamma^{\underline{e}}_A(M^{(d)}) \xrightarrow{\sim} \otimes^d_A(\Gamma^e_A M)$, as defined in (9.5.28). Now, obviously for every homogeneous polynomial law $\lambda : M^{(d)} \rightsquigarrow Q$ and every $\sigma \in S_d$ we have $\sigma(\lambda)^\dagger = \lambda^\dagger$. This implies that (9.5.32) factors through an A-linear map :

$$t^{d,e}_M : \Gamma^{de}_A M \to \mathrm{TS}^d_A(\Gamma^e_A M) \qquad \text{for every } e \in \mathbb{N} \text{ and every } d > 0.$$

By corollary 9.5.22(v), the map $t^{d,1}_M$ can be regarded as an A-linear map

$$t^d_M : \Gamma^d_A M \to \mathrm{TS}^d_A M$$

and in view of lemma 9.5.29(i) and proposition 9.5.6, the corresponding homogeneous polynomial law $\tau^d_M : M \rightsquigarrow \mathrm{TS}^d_A M$ of degree d is characterized by the identities :

$$(\tau^d_M)_B(x) = x \otimes \cdots \otimes x \in \mathrm{TS}^d_B(B \otimes_A M) \quad \text{for every flat } A\text{-algebra } B \text{ and every } x \in B \otimes_A M.$$

Proposition 9.5.34. *Let $e, d \in \mathbb{N}$ with $d > 0$, and M a flat A-module.*

(i) *The map t^d_M is an isomorphism.*

(ii) *We have a commutative diagram :*

$$\begin{array}{ccc} \Gamma^{de}_A M & \xrightarrow{t^{d,e}_M} & \mathrm{TS}^d_A(\Gamma^e_A M) \\ {\scriptstyle t^{de}_M}\downarrow & & \downarrow{\scriptstyle \mathrm{TS}^d_A(t^e_M)} \\ \mathrm{TS}^{de}_A M & \xrightarrow{j} & \mathrm{TS}^d_A(\mathrm{TS}^e_A M) \end{array}$$

where j is the inclusion map (we identify $\mathrm{TS}^d_A(\mathrm{TS}^e_A M)$ with $(\otimes^{de}_A M)^W$ as in lemma 9.5.29(iv)).

(iii) *There exists a natural A-linear map $\alpha^{d,e}_M : \mathrm{TS}^d_A(\Gamma^e_A M) \to \Gamma^{de}_A M$ such that :*

$$\alpha^{d,e}_M \circ t^{d,e}_M = \frac{(de)!}{(e!)^d \cdot d!} \cdot \mathbf{1}_{\Gamma^{de}_A M}.$$

Proof. (i): Since both Γ^d_A and TS^d_A commute with filtered colimits of A-modules, Lazard's theorem ([120, Ch.I, Th.1.2]) reduces to the case where M is free of finite rank r; fix a basis $e_1, \dots, e_r$ of M. By (9.5.12) and a simple induction, we obtain a natural A-linear isomorphism:

(9.5.35) $$\otimes^d_A(M^\vee) \stackrel{\sim}{\to} (\otimes^d_A M)^\vee \qquad e^*_{i_1} \otimes \cdots \otimes e^*_{i_d} \mapsto (e_{i_1} \otimes \cdots \otimes e_{i_d})^*.$$

For such M, the split inclusion map $\mathrm{TS}^d_A M \to \otimes^d_A M$ induces a surjection

(9.5.36) $$(\otimes^d_A M)^\vee \to (\mathrm{TS}^d_A M)^\vee$$

and taking into account remark 9.5.11(iii), it is easily seen that the composition of (9.5.35) and (9.5.36) factors through an A-linear surjection

(9.5.37) $$(\Gamma^d_A M)^\vee \stackrel{\sim}{\to} \mathrm{Sym}^d_A(M^\vee) \to (\mathrm{TS}^d_A M)^\vee.$$

By lemma 9.5.29(ii), the A-modules $\mathrm{Sym}^d_A(M^\vee)$ and $(\mathrm{TS}^d_A M)^\vee$ are free of the same rank, so (9.5.37) is an isomorphism, and it suffices to check that the transpose of t^d_M is an inverse of (9.5.37). However, set $B := \mathrm{Sym}^\bullet_A(M^\vee)$, and recall that t^d_M is the image of $P(e^*_\bullet) := (\tau^d_M)_B(\mathbf{1}_M) \in \mathrm{Sym}^d_A(M^\vee) \otimes_A \mathrm{TS}^d_A M$ under the natural identification :

$$\mathrm{Sym}^d_A(M^\vee) \otimes_A \mathrm{TS}^d_A M \stackrel{\sim}{\to} \mathrm{Hom}_A(\mathrm{Sym}^d_A(M^\vee)^\vee, \mathrm{TS}^d_A M).$$

Set $T := \{1, \dots, r\}^d$, and fix a set of representatives Δ for $S_d \backslash T$; we compute :

$$P(e^*_\bullet) = (\tau^d_M)_B\Big(\sum_{i=1}^r e^*_i \otimes e_i\Big) = \sum_{i_\bullet \in T} e^*_{i_\bullet} \otimes e_{i_\bullet} = \sum_{i_\bullet \in \Delta} e^*_{[i_\bullet]} \otimes [e_{i_\bullet}]$$

where $([e_{i_\bullet}] \mid i_\bullet \in \Delta)$ is the basis of $\mathrm{TS}^d_A M$ exhibited in the proof of lemma 9.5.29(ii), and $e^*_{[i_\bullet]} \in \mathrm{Sym}^d_A(M^\vee)$ is defined as in remark 9.5.11(ii), for every $i_\bullet \in \Delta$. Hence, $P(e^*_\bullet)$ corresponds to the A-linear map $t^d_M : \mathrm{Sym}^d_A(M^\vee)^\vee \to \mathrm{TS}^d_A M$ such that $t^d_M(l) = \sum_{i_\bullet \in \Delta} l(e^*_{[i_\bullet]}) \cdot [e_{i_\bullet}]$ for every A-linear form $l : \mathrm{Sym}^d_A(M^\vee) \to A$. Consequently, $(t^d_M)^\vee : (\mathrm{TS}^d_A M)^\vee \to \mathrm{Sym}^d_A(M^\vee)$ is the map such that $(t^d_M)^\vee(l') = \sum_{i_\bullet \in \Delta} l'([e_{i_\bullet}]) \cdot e^*_{[i_\bullet]}$ for every A-linear form $l' : \mathrm{TS}^d_A M \to A$. Lastly, let $([e_{i_\bullet}]^* \mid i_\bullet \Delta)$ be the dual of the basis $([e_{i_\bullet}] \mid i_\bullet \in \Delta)$; a simple inspection shows that (9.5.37) is given by the rule : $e^*_{[i_\bullet]} \mapsto [e_{i_\bullet}]^*$ for every $i_\bullet \in \Delta$, and on the other hand $(t^d_M)^\vee([e_{i_\bullet}]^*) = e^*_{[i_\bullet]}$ for every such $i_\bullet$, as required.

(ii): It suffices to compare the two corresponding homogeneous polynomial laws $M \rightsquigarrow \mathrm{TS}^d_A(\mathrm{TS}^e_A M)$; which amounts to showing that $j \circ \tau^{de}_M = \mathrm{TS}^e(t^e_M) \circ \tau^{d,e}_M$, where $\tau^{d,e}_M : M \rightsquigarrow \mathrm{TS}^d_A(\Gamma^e_A M)$ is the homogeneous polynomial law corresponding to $t^{d,e}_M$. Then the assertion follows by inspecting the characterizations of $t^{d,e}_M$, t^{de}_M and t^e_M given in (9.5.31) and (9.5.33).

(iii): Let $a^{d,e}_M : (\otimes^{de}_A M)^W \to \mathrm{TS}^{de}_A M$ be the A-linear map such that : $x \mapsto \sum_{\sigma\in W\backslash S_d} \sigma(x)$ for every $x \in (\otimes^{de}_A M)^W$. It is easily seen that $a^{d,e}_M \circ j = \frac{(de)!}{(e!)^d\cdot d!} \cdot \mathbf{1}_{\mathrm{TS}^{de}_A M}$, where $j : \mathrm{TS}^{de}_A M \to (\otimes^{de}_A M)^W$ is the inclusion map. By virtue of lemma 9.5.29(iv), the maps $\mathrm{TS}^d_A(t^e_M)$ and t^{de}_M identify $a^{d,e}_M$ with an A-linear map $\alpha^{d,e}_A : \mathrm{TS}^d_A(\Gamma^e_A M) \to \Gamma^{de}_A M$, and the sought identity follows directly from (ii). □

9.5.38. Let $\underline{M}$ and $\underline{M}'$ be two sequences of A-modules, of the same length k. We set

$$\underline{M} \otimes_A \underline{M}' := (M_1 \otimes_A M'_1, \dots, M_k \otimes_A M'_k)$$

For every $\underline{d} \in \mathbb{N}^{\oplus k}$ we have a natural transformation of functors

$$\mathrm{Pol}^{\underline{d}}_A(\underline{M} \otimes_A \underline{M}', Q) \to \mathrm{Pol}^{\underline{d},\underline{d}}_A((\underline{M}, \underline{M}'), Q) \qquad \text{for every } A\text{-module } Q$$

that assigns to every homogeneous polynomial law $\lambda : \underline{M} \otimes_A \underline{M}' \rightsquigarrow Q$ of degree $\underline{d}$ the law $\lambda^\dagger : (\underline{M}, \underline{M}') \rightsquigarrow Q$ given by the rule :

$$\lambda^\dagger_B(m_\bullet, m'_\bullet) := \lambda_B(m_1 \otimes m'_1, \dots, m_k \otimes m'_k)$$

for every A-algebra B and every $(m_\bullet, m'_\bullet) \in F_{\underline{M},\underline{M}'}(B)$. This natural transformation must come from a map of A-modules $\Gamma^{\underline{d},\underline{d}}_A(\underline{M}, \underline{M}') \to \Gamma^{\underline{d}}_A(\underline{M} \otimes_A \underline{M}')$ which, by virtue of corollary 9.5.22(ii), is the same as a map natural in both $\underline{M}$ and $\underline{M}'$:

$$\varphi^{\underline{d}}_{\underline{M},\underline{M}'} : \Gamma^{\underline{d}}_A(\underline{M}) \otimes_A \Gamma^{\underline{d}}_A(\underline{M}') \to \Gamma^{\underline{d}}_A(\underline{M} \otimes_A \underline{M}').$$

Remark 9.5.39. (i) By direct inspection, we see that the map $B \otimes_A \varphi^{\underline{d}}_{\underline{M},\underline{M}'}$ is identified with $\varphi^{\underline{d}}_{B\otimes_A\underline{M},B\otimes_A\underline{M}'}$ for every A-algebra B, under the natural isomorphisms (9.5.21).

(ii) Under the identifications of (i), the map $\varphi^{\underline{d}}_{\underline{M},\underline{M}'}$ is characterized as the unique A-linear map such that

$$(\boldsymbol{\lambda}^{\underline{d}}_{\underline{M}})_B(m) \otimes (\boldsymbol{\lambda}^{\underline{d}}_{\underline{M}'})_B(m') \mapsto (\boldsymbol{\lambda}^{\underline{d}}_{\underline{M}\otimes_A\underline{M}'})_B(m \otimes m')$$

for every A-algebras B, every $m \in B\otimes_A(M_1\times\cdots\times M_k)$ and every $m' \in B\otimes_A(M'_1\times\cdots\times M'_k)$.

9.5.40. Let $\underline{M}$ and $\underline{M}'$ be two sequences of A-modules, of the same length k, and $\underline{e}$, $\underline{e}'$ two degrees of length k. If $\lambda : (\underline{M}, \underline{M}') \rightsquigarrow Q$ is any homogeneous law of degree $(\underline{e}, \underline{e}')$, then we can also regard λ as a homogeneous multipolynomial law of degree $\underline{e} + \underline{e}'$ on the sequence $\underline{M} \oplus \underline{M}'$ of length k. In this way, we obtain a natural mapping

$$\mathrm{Pol}^{\underline{e},\underline{e}'}_A((\underline{M}, \underline{M}'), Q) \to \mathrm{Pol}^{\underline{e}+\underline{e}'}_A(\underline{M} \oplus \underline{M}', Q) \qquad \text{for every } A\text{-module } Q \tag{9.5.41}$$

which, by transposition, corresponds to a unique A-linear map

$$\Gamma^{\underline{e}+\underline{e}'}_A(\underline{M} \oplus \underline{M}') \to \Gamma^{\underline{e},\underline{e}'}_A(\underline{M}, \underline{M}'). \tag{9.5.42}$$

Fix $\underline{d} \in \mathbb{N}^{\oplus k}$; in view of corollary 9.5.22(ii), we see that the sum of the maps (9.5.42) for all pairs $(\underline{e}, \underline{e}')$ with $\underline{e} + \underline{e}' = \underline{d}$, is a well defined A-linear map

$$\Delta^{\underline{d}}_{\underline{M},\underline{M}'} := \bigoplus_{\underline{e}+\underline{e}'=\underline{d}} \Delta^{\underline{e},\underline{e}'}_{\underline{M},\underline{M}'} : \Gamma^{\underline{d}}_A(\underline{M} \oplus \underline{M}') \to \bigoplus_{\underline{e}+\underline{e}'=\underline{d}} \Gamma^{\underline{e}}_A(\underline{M}) \otimes_A \Gamma^{\underline{e}'}_A(\underline{M}').$$

Proposition 9.5.43. *The map $\Delta^{\underline{d}}_{\underline{M},\underline{M}'}$ is an isomorphism, for every $\underline{M}$, $\underline{M}'$, and every degree $\underline{d}$.*

Proof. Arguing as in the proof of corollary 9.5.22(ii), we are easily reduced to the case where $\underline{M}$ and $\underline{M}'$ are sequences of free A-modules of finite rank. In this case, (9.5.42) can be computed as in remark 9.5.27(i); namely, it corresponds to the element

$$P \in \mathrm{Sym}^{\underline{e}+\underline{e}'}_A((\underline{M} \oplus \underline{M}')^\vee) \otimes_A \mathrm{Sym}^{\underline{e},\underline{e}'}_A(\underline{M}^\vee, \underline{M}'^\vee)^\vee$$

determined as follows. Take $Q := \Gamma_A^{\underline{e},\underline{e}'}(\underline{M},\underline{M}')$, and let $\lambda : \underline{M} \oplus \underline{M}' \rightsquigarrow \Gamma_A^{\underline{e},\underline{e}'}(\underline{M},\underline{M}')$ be the image of $\boldsymbol{\lambda}^{\underline{e},\underline{e}'}_{\underline{M},\underline{M}'}$ under (9.5.41). To ease notation, for every sequence $\underline{N}$ of A-modules, let $S(\underline{N}) := \mathrm{Sym}^{\bullet}_A(\underline{N}^\vee)$; then

$$P = \lambda_{S(\underline{M}\oplus\underline{M}')}(\mathbf{1}_{M_1\oplus M'_1}, \dots, \mathbf{1}_{M_k\oplus M'_k}).$$

Thus, we come down to evaluating the image of $\mathbf{1}_{M_i\oplus M'_i}$ under the natural identification

$$(M_i \oplus M'_i) \otimes_A S(M_i \oplus M'_i) \xrightarrow{\sim} (M_i \otimes_A S(M_i \oplus M'_i)) \times (M'_i \otimes_A S(M_i \oplus M'_i)).$$

for every $i = 1, \dots, k$. We easily find that $\mathbf{1}_{M_i\oplus M'_i} \mapsto (p_i, p'_i)$ under this identification, where $p_i : M_i \oplus M'_i \to M_i$ and $p'_i : M_i \oplus M'_i \to M'_i$ are the natural projections; so finally

$$P = (\boldsymbol{\lambda}^{\underline{e},\underline{e}'}_{\underline{M},\underline{M}'})_{S(\underline{M}\oplus\underline{M}')}(p_1, \dots, p_k, p'_1, \dots, p'_k). \tag{9.5.44}$$

Pick a basis $(e_{i1}, \dots, e_{ir_i})$ (resp. $(e'_{i1}, \dots, e'_{ir'_i})$) for each M_i (resp. M'_i), let $(e^*_{i1}, \dots, e^*_{ir_i})$ (resp. $(e'^*_{i1}, \dots, e'^*_{ir'_i})$)) be the dual basis, and for every $= 1, \dots, k$ and every $j = 1, \dots, r_i$ (resp. $j = 1, \dots, r'_i$) denote by ε_{ij} (resp. ε'_{ij}) the element of $(M_i \oplus M'_i)^\vee$ that agrees with e^*_{ij} on M_i and that vanishes on M'_i (resp. that agrees with e'^*_{ij} on M'_i and vanishes on M_i). Notice that

$$p_i = \sum_{j=1}^{r_i} \varepsilon^*_{ij} \otimes e_{ij} \qquad p'_i = \sum_{j=1}^{r'_i} \varepsilon'^*_{ij} \otimes e'_{ij} \qquad \text{for every } i = 1, \dots, k.$$

With this notation, it follows that the right-hand side of (9.5.44) is the image of

$$(\boldsymbol{\lambda}^{\underline{e},\underline{e}'}_{\underline{M},\underline{M}'})_{S(\underline{M},\underline{M}')}(\mathbf{1}_{M_1}, \dots, \mathbf{1}_{M_k}, \mathbf{1}_{M'_1}, \dots, \mathbf{1}_{M'_k}) = \mathbf{1}_{\mathrm{Sym}^{\underline{e},\underline{e}'}_A(\underline{M}^\vee,\underline{M}'^\vee)}$$

under the natural map

$$S(\underline{M},\underline{M}') \to S(\underline{M}\oplus\underline{M}') \quad : \quad e^*_{ij} \mapsto \varepsilon^*_{ij} \quad e'^*_{ij} \mapsto \varepsilon'^*_{ij}. \tag{9.5.45}$$

In other words, the sought P is none else than the restriction of (9.5.45) to the direct factor $\mathrm{Sym}^{\underline{e},\underline{e}'}_A(\underline{M}^\vee, \underline{M}'^\vee)$, and this restriction is an isomorphism onto a direct factor of the A-module $\mathrm{Sym}^{\underline{e}+\underline{e}'}_A(\underline{M}\oplus\underline{M}')$. By transposition, we see that (9.5.42) is identified with the projection onto a natural direct factor of $\Gamma^{\underline{e}+\underline{e}'}_A(\underline{M}\oplus\underline{M}')$, and a simple inspection shows that the sum of all these projections is the identity map of the latter A-module, whence the proposition. ☐

Remark 9.5.46. (i) Let $\underline{M}, \underline{M}'$, Q and λ be as in (9.5.40). The image of λ under (9.5.41) is characterized as the unique law $\varphi : \underline{M} \oplus \underline{M}' \rightsquigarrow Q$ of degree $\underline{e} + \underline{e}'$ such that $\varphi(m, m') = \lambda_B(m, m')$ for every A-algebra B and every $(m, m') \in B \otimes_A (\underline{M}\oplus\underline{M}')$. In turns, φ corresponds to the unique A-linear map $f : \Gamma^{\underline{e}+\underline{e}'}_A(\underline{M}\oplus\underline{M}') \to Q$ such that $(B \otimes_A f)(\boldsymbol{\lambda}^{\underline{e}+\underline{e}'}_{\underline{M}\oplus\underline{M}'})_B(m, m') = \lambda_B(m, m')$ for every B and (m, m') as above. Especially, if we take $\lambda := \lambda^{\underline{e},\underline{e}'}_{\underline{M},\underline{M}'}$, we see that (9.5.42) is characterized as the unique A-linear map such that

$$(\boldsymbol{\lambda}^{\underline{e}+\underline{e}'}_{\underline{M}\oplus\underline{M}'})_B(m, m') \mapsto (\boldsymbol{\lambda}^{\underline{e},\underline{e}'}_{\underline{M},\underline{M}'})_B(m, m')$$

for every B and (m, m') as above. Lastly, it follows that $\Delta^{\underline{e},\underline{e}'}_{\underline{M},\underline{M}'}$ is characterized as the unique A-linear map such that

$$(\boldsymbol{\lambda}^{\underline{e}+\underline{e}'}_{\underline{M}\oplus\underline{M}'})_B(m, m') \mapsto (\boldsymbol{\lambda}^{\underline{e}}_{\underline{M}})_B(m) \otimes (\boldsymbol{\lambda}^{\underline{e}'}_{\underline{M}'})_B(m')$$

for every A-algebra B and every $(m, m') \in B \otimes_A (\underline{M}\oplus\underline{M}')$.

(ii) In the same vein, let $f : \underline{M} \to \underline{N}$ be any morphism in $(A\text{-}\mathbf{Mod})^k$, and $\underline{d} \in \mathbb{N}^{\oplus k}$ any degree. Then we see that the induced map

$$\Gamma^{\underline{d}}_A(f) : \Gamma^{\underline{d}}_A(\underline{M}) \to \Gamma^{\underline{d}}_A(\underline{N})$$

is characterized as the unique A-linear map such that

$$(\boldsymbol{\lambda}^{\underline{d}}_{\underline{M}})_B(m) \mapsto (\boldsymbol{\lambda}^{\underline{d}}_{\underline{N}})_B(f(m))$$

for every A-algebra B, and every $m \in B \otimes_A \underline{M}$ (details left to the reader).

9.5.47. Let $\underline{M}$, $\underline{M}'$, $\underline{M}''$ be three sequences of A-modules of length k, and $\underline{d}$, $\underline{d}'$, $\underline{d}''$ three degrees, also of lengths k. We obtain a diagram of A-linear maps :

$$\begin{array}{ccc}
\Gamma^{\underline{e}+\underline{e}'+\underline{e}''}_A(\underline{M} \oplus \underline{M}' \oplus \underline{M}'') & \xrightarrow{\Delta^{\underline{e},\underline{e}'+\underline{e}''}_{\underline{M},\underline{M}'\oplus\underline{M}''}} & \Gamma^{\underline{e}}_A(\underline{M}) \otimes_A \Gamma^{\underline{e}'+\underline{e}''}_A(\underline{M}' \oplus \underline{M}'') \\
{\scriptstyle \Delta^{\underline{e}+\underline{e}',\underline{e}''}_{\underline{M}\oplus\underline{M}',\underline{M}''}} \downarrow & & \downarrow {\scriptstyle \Gamma^{\underline{e}}_A(\underline{M})\otimes_A \Delta^{\underline{e}',\underline{e}''}_{\underline{M}',\underline{M}''}} \\
\Gamma^{\underline{e}+\underline{e}'}_A(\underline{M} \oplus \underline{M}') \otimes_A \Gamma^{\underline{e}''}_A(\underline{M}'') & \xrightarrow{\Delta^{\underline{e},\underline{e}'}_{\underline{M},\underline{M}'}\otimes_A \Gamma^{\underline{e}''}_A(\underline{M}'')} & \Gamma^{\underline{e}}_A(\underline{M}) \otimes_A \Gamma^{\underline{e}'}_A(\underline{M}') \otimes_A \Gamma^{\underline{e}''}_A(\underline{M}'').
\end{array}$$

Proposition 9.5.48. *The diagram of* (9.5.47) *commutes.*

Proof. By remark 9.5.46(i), the map $\Delta^{\underline{e},\underline{e}'+\underline{e}''}_{\underline{M},\underline{M}'\oplus\underline{M}''}$ is characterized as the unique one such that

$$(\boldsymbol{\lambda}^{\underline{e}+\underline{e}'+\underline{e}''}_{\underline{M}\oplus\underline{M}'})_B(m, m', m'') \mapsto (\boldsymbol{\lambda}^{\underline{e}}_{\underline{M}})_B(m) \otimes (\boldsymbol{\lambda}^{\underline{e}'+\underline{e}''}_{\underline{M}\oplus\underline{M}'})_B(m', m'')$$

for every A-algebra B and every $(m, m', m'') \in B \otimes_A (\underline{M} \oplus \underline{M}' \oplus \underline{M}'')$, and $\Delta^{\underline{e}',\underline{e}''}_{\underline{M}',\underline{M}''}$ is the unique A-linear map such that

$$(\boldsymbol{\lambda}^{\underline{e}'+\underline{e}''}_{\underline{M}'\oplus\underline{M}''})_B(m', m'') \mapsto (\boldsymbol{\lambda}^{\underline{e}'}_{\underline{M}'})_B(m') \otimes (\lambda^{\underline{e}''}_{\underline{M}''})_B(m'')$$

for every A-algebra B and every $(m', m'') \in B \otimes_A (\underline{M}' \oplus \underline{M}'')$. Therefore, the composition of the top horizontal and right vertical arrows in the diagram is the unique A-linear map such that

$$(\boldsymbol{\lambda}^{\underline{e}+\underline{e}'+\underline{e}''}_{\underline{M}\oplus\underline{M}'})_B(m, m', m'') \mapsto (\boldsymbol{\lambda}^{\underline{e}}_{\underline{M}})_B(m) \otimes (\boldsymbol{\lambda}^{\underline{e}'}_{\underline{M}'})_B(m') \otimes (\lambda^{\underline{e}''}_{\underline{M}''})_B(m'')$$

for every B and (m, m', m'') as above. The same calculation can be carried out for the composition of the other two arrows, and clearly one obtains the same result. □

9.5.49. Let $\underline{M}$ be any sequence of A-modules of length k, and denote by $s : \underline{M} \oplus \underline{M} \to \underline{M}$ the addition map : $(m, m') \mapsto m + m'$ for every $m, m' \in \underline{M}$. By proposition 9.5.43, we have a unique system of A-linear maps

$$\mu^{\underline{e},\underline{e}'}_{\underline{M}} : \Gamma^{\underline{e}}_A(\underline{M}) \otimes_A \Gamma^{\underline{e}'}_A(\underline{M}) \to \Gamma^{\underline{e}+\underline{e}'}_A(\underline{M}) \qquad \text{for every } \underline{e}, \underline{e}' \in \mathbb{N}^{\oplus k}$$

such that, for every $\underline{d} \in \mathbb{N}^{\oplus k}$, the resulting diagram :

(9.5.50)
$$\begin{array}{ccc}
\Gamma^{\underline{d}}_A(\underline{M}^{\oplus 2}) & \xrightarrow{\Delta^{\underline{d}}_{\underline{M},\underline{M}}} & \bigoplus_{\underline{e}+\underline{e}'=\underline{d}} \Gamma^{\underline{e}}_A(\underline{M}) \otimes_A \Gamma^{\underline{e}'}_A(\underline{M}) \\
{\scriptstyle \Gamma^{\underline{d}}_A(s)} \downarrow & & \downarrow {\scriptstyle \bigoplus_{\underline{e}+\underline{e}'=\underline{d}} \mu^{\underline{e},\underline{e}'}_{\underline{M}}} \\
\Gamma^{\underline{d}}_A(\underline{M}) & = & \Gamma^{\underline{d}}_A(\underline{M})
\end{array}$$

commutes. Consider now, for every $\underline{e}, \underline{e}', \underline{e}'' \in \mathbb{N}^{\oplus k}$ the diagram of A-linear maps :

$$\begin{array}{ccccc}
\Gamma^{\underline{d}}_A(\underline{M}^{\oplus 3}) & \xrightarrow{\Delta^{\underline{d}}_{\underline{M}^{\oplus 2},\underline{M}}} & \bigoplus_{\underline{d}'+\underline{e}''=\underline{d}} \Gamma^{\underline{d}'}_A(\underline{M}^{\oplus 2}) \otimes_A \Gamma^{\underline{e}''}_A \underline{M} & \xrightarrow{\bigoplus_{\underline{d}'+\underline{e}''=\underline{d}} \Delta^{\underline{d}'}_{\underline{M},\underline{M}} \otimes_A \Gamma^{\underline{e}''}_A \underline{M}} & \bigoplus_{\underline{e}+\underline{e}'+\underline{e}''=\underline{d}} \Gamma^{\underline{e}}_A \underline{M} \otimes_A \Gamma^{\underline{e}'}_A \underline{M} \otimes_A \Gamma^{\underline{e}''}_A \underline{M} \\
\downarrow {\scriptstyle \Gamma^{\underline{d}}_A(s \oplus \mathbf{1}_{\underline{M}})} & & \downarrow {\scriptstyle \bigoplus_{\underline{d}'+\underline{e}''=\underline{d}} \Gamma^{\underline{d}'}_A(s) \otimes_A \Gamma^{\underline{e}''}_A \underline{M}} & & \downarrow {\scriptstyle \bigoplus_{\underline{e}+\underline{e}'+\underline{e}''=\underline{d}} \mu^{\underline{e},\underline{e}'}_{\underline{M}} \otimes_A \Gamma^{\underline{e}''}_A \underline{M}} \\
\Gamma^{\underline{d}}_A(\underline{M}^{\oplus 2}) & \xrightarrow{\Delta^{\underline{d}}_{\underline{M},\underline{M}}} & \bigoplus_{\underline{d}'+\underline{e}''=\underline{d}} \Gamma^{\underline{d}'}_A \underline{M} \otimes_A \Gamma^{\underline{e}''}_A \underline{M} & = & \bigoplus_{\underline{d}'+\underline{e}''=\underline{d}} \Gamma^{\underline{d}'}_A \underline{M} \otimes_A \Gamma^{\underline{e}''}_A \underline{M} \\
\downarrow {\scriptstyle \Gamma^{\underline{d}}_A(s)} & & \downarrow {\scriptstyle \bigoplus_{\underline{d}'+\underline{e}''=\underline{d}} \mu^{\underline{d}',\underline{e}''}_{\underline{M}}} & & \downarrow {\scriptstyle \bigoplus_{\underline{d}'+\underline{e}''=\underline{d}} \mu^{\underline{d}',\underline{e}''}_{\underline{M}}} \\
\Gamma^{\underline{d}}_A \underline{M} & = & \Gamma^{\underline{d}}_A \underline{M} & = & \Gamma^{\underline{d}}_A \underline{M}
\end{array}$$

whose top left square subdiagram commutes by naturality of $\Delta^{\underline{d}}$, and whose other three square subdiagrams commute by definition. We also have a similar diagram, whose two top horizontal arrows are $\Delta^{\underline{d}}_{\underline{M},\underline{M}^{\oplus 2}}$ and $\bigoplus_{\underline{e}+\underline{d}'=\underline{d}} \Gamma^{\underline{e}}_A \underline{M} \otimes_A \Delta^{\underline{d}'}_{\underline{M},\underline{M}}$, and whose two left (resp. right) vertical arrows are $\Gamma^{\underline{d}}_A(\mathbf{1}_{\underline{M}} \oplus s)$ and $\Gamma^{\underline{d}}_A(s)$ (resp. $\bigoplus_{\underline{e}+\underline{e}'+\underline{e}''=\underline{d}} \Gamma^{\underline{e}}_A \underline{M} \otimes_A \mu^{\underline{e}',\underline{e}''}_{\underline{M}}$ and $\bigoplus_{\underline{e}+\underline{d}'=\underline{d}} \mu^{\underline{e},\underline{d}'}_{\underline{M}}$). On the other hand, clearly $s \circ (s \oplus \mathbf{1}_{\underline{M}}) = s \circ (\mathbf{1}_{\underline{M}} \oplus s)$; combining with proposition 9.5.48, we finally conclude that the diagram

$$\begin{array}{ccc}
\Gamma^{\underline{e}}_A(\underline{M}) \otimes_A \Gamma^{\underline{e}'}_A(\underline{M}) \otimes_A \Gamma^{\underline{e}''}_A(\underline{M}) & \xrightarrow{\mu^{\underline{e},\underline{e}'}_{\underline{M}} \otimes_A \Gamma^{\underline{e}''}_A(\underline{M})} & \Gamma^{\underline{e}+\underline{e}'}_A(\underline{M}) \otimes_A \Gamma^{\underline{e}''}_A(\underline{M}) \\
\downarrow {\scriptstyle \Gamma^{\underline{e}}_A(\underline{M}) \otimes_A \mu^{\underline{e}',\underline{e}''}_{\underline{M}}} & & \downarrow {\scriptstyle \mu^{\underline{e}+\underline{e}',\underline{e}''}_{\underline{M}}} \\
\Gamma^{\underline{e}}_A(\underline{M}) \otimes_A \Gamma^{\underline{e}'+\underline{e}''}_A(\underline{M}) & \xrightarrow{\mu^{\underline{e},\underline{e}'+\underline{e}''}_{\underline{M}}} & \Gamma^{\underline{e}+\underline{e}'+\underline{e}''}_A(\underline{M})
\end{array}$$

commutes, for every $\underline{e}, \underline{e}', \underline{e}'' \in \mathbb{N}^{\oplus k}$. The foregoing shows that the maps $\mu^{\bullet\bullet}_{\underline{M}}$ define on

$$\Gamma^{\bullet}_A(\underline{M}) := \bigoplus_{\underline{d} \in \mathbb{N}^{\oplus k}} \Gamma^{\underline{d}}_A(\underline{M}).$$

a natural structure of associative $\mathbb{N}^{\oplus k}$-graded A-algebra. We call $\Gamma^{\bullet}_A(\underline{M})$ the *divided power envelope* of $\underline{M}$. Moreover, let $\sigma : \underline{M} \oplus \underline{M} \to \underline{M} \oplus \underline{M}$ be the map such that $\sigma(m, m') := (m', m)$ for all $m, m' \in \underline{M}$; remark 9.5.46(i) yields a commutative diagram :

$$\begin{array}{ccc}
\Gamma^{\underline{e}+\underline{e}'}_A(\underline{M} \oplus \underline{M}) & \xrightarrow{\Delta^{\underline{e},\underline{e}'}_{\underline{M},\underline{M}}} & \Gamma^{\underline{e}}_A(\underline{M}) \otimes_A \Gamma^{\underline{e}'}_A(\underline{M}) \\
\downarrow {\scriptstyle \Gamma^{\underline{e}+\underline{e}'}_A(\sigma)} & & \downarrow {\scriptstyle \omega^{\underline{e},\underline{e}'}_{\underline{M}}} \\
\Gamma^{\underline{e}+\underline{e}'}_A(\underline{M} \oplus \underline{M}) & \xrightarrow{\Delta^{\underline{e}',\underline{e}}_{\underline{M},\underline{M}}} & \Gamma^{\underline{e}'}_A(\underline{M}) \otimes_A \Gamma^{\underline{e}}_A(\underline{M})
\end{array}$$

for every $\underline{e}, \underline{e}' \in \mathbb{N}^{\oplus k}$, where $\omega^{\underline{e},\underline{e}'}_{\underline{M}}$ is the isomorphism that swaps the two tensor factors. Since obviously $s \circ \sigma = s$, it follows easily that :

$$\mu^{\underline{e}',\underline{e}}_{\underline{M}} \circ \omega^{\underline{e},\underline{e}'}_{\underline{M}} = \mu^{\underline{e},\underline{e}'}_{\underline{M}} \qquad \text{for every } \underline{e}, \underline{e}' \in \mathbb{N}^{\oplus k}$$

which means that the algebra $\Gamma^{\bullet}_A(\underline{M})$ is commutative. It is customary to use the notation

$$m^{[\underline{d}]} := (\boldsymbol{\lambda}^{\underline{d}}_{\underline{M}})_B(m) \in B \otimes_A \underline{M}$$

for every A-algebra B, every $m \in B \otimes_A \underline{M}$, and every $\underline{d} \in \mathbb{N}^{\oplus k}$. With this notation, the commutativity of (9.5.50) amounts to the identity

$$(a+b)^{[\underline{d}]} = \sum_{\underline{e}+\underline{e}'=\underline{d}} a^{[\underline{e}]} \cdot b^{[\underline{e}']} \tag{9.5.51}$$

for every A-algebra B, every $a, b \in B \otimes_A \underline{M}$, and every $\underline{d} \in \mathbb{N}^{\oplus k}$. Clearly, the above construction yields a well defined functor

$$(A\text{-}\mathbf{Mod})^k \to A\text{-}\mathsf{Alg} \quad : \quad \underline{M} \mapsto \Gamma^\bullet_A(\underline{M}) \tag{9.5.52}$$

and notice that, for every morphism $f : \underline{M} \to \underline{N}$ of sequences, we have

$$(\Gamma^\bullet_A f)(a^{[\underline{d}]}) = (fa)^{[\underline{d}]} = (\Gamma^\bullet_A f(a))^{[\underline{d}]}$$

for every degree $\underline{d}$, every A-algebra B, and every $a \in B \otimes_A \underline{M}$.

Theorem 9.5.53. *With the notation of* (9.5.49)*, we have :*

(i) *For every A-algebra B, there exists a natural isomorphism of $\mathbb{N}^{\oplus k}$-graded algebras :*

$$B \otimes_A \Gamma^\bullet_A(\underline{M}) \xrightarrow{\sim} \Gamma^\bullet_B(B \otimes_A \underline{M}).$$

(ii) *The functor* (9.5.52) *commutes with filtered colimits.*

(iii) *For any two sequences of A-modules $\underline{M}$ and $\underline{M}'$ of the same length k, there exists a natural isomorphism of $\mathbb{N}^{\oplus k}$-graded A-algebras :*

$$\Gamma^\bullet_A(\underline{M} \oplus \underline{M}') \xrightarrow{\sim} \Gamma^\bullet_A(\underline{M}) \otimes_A \Gamma^\bullet_A(\underline{M}')$$

(with the $\mathbb{N}^{\oplus k}$-grading of the tensor product defined in the obvious way).

Proof. (i) and (ii) follow immediately from the natural isomorphisms (9.5.21) and corollary 9.5.22(i), and (iii) follows directly from proposition 9.5.43. □

9.5.54. Consider the special case of a sequence $\underline{M}$ consisting of a single A-module M. From (9.5.51), by evaluating the expression $(2m)^{[d]} = 2^d \cdot m^{[d]}$, a simple induction on d shows that

$$d! \cdot m^{[d]} = m^d \qquad \text{in } \Gamma^\bullet_A(M)\text{, for every } m \in M. \tag{9.5.55}$$

Corollary 9.5.56. *In the situation of* (9.5.54)*, we have in $\Gamma^\bullet_A(M)$ the identity :*

$$m^{[i]} \cdot m^{[j]} = \binom{i+j}{i} \cdot m^{[i+j]} \qquad \textit{for every } m \in M \textit{ and every } i, j \in \mathbb{N}.$$

Proof. By considering the A-linear map $f : A \to M$ such that $f(1) = m$, we reduce to the case where $M = A$ and $m = 1$. By considering the unique ring homomorphism $\mathbb{Z} \to A$ we reduce further – in light of theorem 9.5.53(i) – to the case where $A = M = \mathbb{Z}$. However, $\Gamma^\bullet_\mathbb{Z}(\mathbb{Z})$ is a torsion-free $\mathbb{Z}$-module, by lemma 9.5.9, so the identity follows easily from (9.5.55). □

Corollary 9.5.57. *Let M be an A-module, and $N \subset M$ any submodule. The induced map*

$$\Gamma^\bullet_A M \to \Gamma^\bullet_A(M/N)$$

is surjection of A-algebras, with kernel generated by the image of $\Gamma^+_A N := \bigoplus_{d>0} \Gamma^d_A N$.

Proof. From corollary 9.5.22(iv) we get an isomorphism of A-algebras

$$\mathrm{Coequal}\,(\Gamma^\bullet_A(j_1), \Gamma^\bullet_A(j_2) : \Gamma^\bullet_A(M \times_{M/N} M) \rightrightarrows \Gamma^\bullet_A M)) \xrightarrow{\sim} \Gamma^\bullet_A(M/N)$$

where $j_1, j_2 : M \times_{M/N} M \to M$ are the natural maps. However, the isomorphism of A-modules $M \oplus N \xrightarrow{\sim} M \times_{M/N} M : (x, y) \mapsto x + y$ identifies j_1 (resp. j_2) with the map $M \oplus N \to M$ such that $(x, y) \mapsto x + y$ (resp. $(x, y) \mapsto x$). Let also $i_1 : M \to M \oplus N$ and $i_2 : N \to M \oplus N$ be the inclusion maps to the first and respectively second direct summand, so that $j_2 \circ i_1 = j_1 \circ i_1 = \mathbf{1}_M$, $j_2 \circ i_2 = 0$, and $l := j_1 \circ i_2 : N \to M$ is the inclusion map. Now, by direct inspection we see that $\Gamma^d_A(i_1)$ maps $\Gamma^d_A M$ to the direct summand $\Gamma^d_A M \otimes_A \Gamma^0_A N \stackrel{\sim}{\to} \Gamma^{(d,0)}_A(M,N)$, for every $d \in \mathbb{N}$, and the resulting map $\Gamma^d_A M \to \Gamma^{(d,0)}_A(M,N)$ is characterized as the unique one such that $x^{[d]} \mapsto (x,0)^{[d,0]}$ for every A-algebra B and every $x \in B \otimes_A M$. On the other hand, by construction we have $(x,y)^{[d,0]} \mapsto x^{[d]} \otimes y^{[0]} = x^{[d]} \otimes 1$ for every A-algebra B and every $(x,y) \in B \otimes_A (M \oplus N)$, under the natural identification $B \otimes_A \Gamma^0_A N \stackrel{\sim}{\to} B$ (corollary 9.5.22(vi) and remark 9.5.46(i)). Likewise we may characterize $\Gamma^d_A(i_2)$ for every $d \in \mathbb{N}$; hence, $\Gamma^\bullet_A(i_1)$ and $\Gamma^\bullet_A(i_2)$ are the maps of A-algebras :

$$\Gamma^\bullet_A M \to \Gamma^\bullet_A M \otimes_A \Gamma^\bullet_A N \quad x \mapsto x \otimes 1. \qquad\qquad \Gamma^\bullet_A N \to \Gamma^\bullet_A M \otimes_A \Gamma^\bullet_A N \quad y \mapsto 1 \otimes y.$$

Thus, let $d \in \mathbb{N}$ and $z \in \Gamma^d_A(M \oplus N)$, and write $z = \sum_{i=0}^d x_i \otimes y_{d-i}$ with $x_i \in \Gamma^i_A M$ and $y_i \in \Gamma^i_A N$ for $i = 0, \dots, d$. It follows that

$$\Gamma^\bullet(j_1)(z) = \sum_{i=0}^d x_i \cdot \Gamma^{d-i}_A(l)(y_{d-i}) \qquad \text{and} \qquad \Gamma^\bullet(j_2)(z) = x_d \cdot \Gamma^0_A(l)(y_0)$$

so finally, the kernel of the projection onto $\Gamma^\bullet_A(M/N)$ is the A-module generated by all products $x \cdot \Gamma^i_A(l)(y)$, for every $i \in \mathbb{N}$, every $j \in \mathbb{N} \setminus \{0\}$, every $x \in \Gamma^i_A M$ and every $y \in \Gamma^j_A N$. ☐

9.5.58. For every $d, e \in \mathbb{N}$ with $d > 0$, we have a natural triangle of A-linear maps :

$$\begin{array}{ccc} \Gamma^{de}_A M & \xrightarrow{t^{d,e}_M} & \mathrm{TS}^d_A(\Gamma^e_A M) \\ & \nwarrow^{\gamma^{d,e}_M} \quad \swarrow_{\beta^{d,e}_M} & \\ & \mathrm{Sym}^d_A(\Gamma^e_A M) & \end{array} \qquad \text{for every } A\text{-module } M$$

where $\beta^{d,e}_M$ is the restriction of the projection $\otimes^d_A(\Gamma^e_A M) \to \mathrm{Sym}^d_A(\Gamma^e_A M)$, the map $\gamma^{d,e}_M$ is induced by the multiplication map of the A-algebra $\Gamma^\bullet_A M$, and $t^{d,e}_M$ is defined as in (9.5.33).

Remark 9.5.59. If $e = 1$, the diagram of (9.5.58) becomes the triangle of A-linear maps :

$$\begin{array}{ccc} \Gamma^{d}_A M & \xrightarrow{t^{d}_M} & \mathrm{TS}^d_A M \\ & \nwarrow^{\gamma^{d}_M} \quad \swarrow_{\beta^{d}_M} & \\ & \mathrm{Sym}^d_A M & \end{array} \qquad \text{for every } A\text{-module } M$$

where t^d_M and β^d_M are defined as in (9.5.33) and (9.5.28), and with $\gamma^d_M := \gamma^{d,1}_M$.

Proposition 9.5.60. (i) *With the notation of* (9.5.58)*, we have :*

$$\gamma^{d,e}_M \circ \beta^{d,e}_M \circ t^{d,e}_M = \frac{(de)!}{(e!)^d} \cdot \mathbf{1}_{\Gamma^{de}_A M}.$$

(ii) *If M is a flat A-module, let $\alpha^{d,e}_M$ be the map of proposition* 9.5.34(iii)*; we have :*

$$\gamma^{d,e}_M \circ \beta^{d,e}_M = d! \cdot \alpha^{d,e}_M.$$

(iii) *Define the map α^d_M as in* (9.5.28)*; with the notation of remark* 9.5.59*, we have :*

$$\gamma^d_M \circ \beta^d_M \circ t^d_M = d! \cdot \mathbf{1}_{\Gamma^d_A M} \qquad t^d_M \circ \gamma^d_M = \alpha^d_M \qquad \beta^d_M \circ t^d_M \circ \gamma^d_M = d! \cdot \mathbf{1}_{\mathrm{Sym}^d_A M}.$$

Proof. (i): It suffices to check that the homogeneous polynomial law $M \rightsquigarrow \Gamma^{de}_A M$ corresponding to $\gamma^{d,e}_M \circ \beta^{d,e}_M \circ t^{d,e}_M$ equals $\frac{(de)!}{(e!)^d} \cdot \boldsymbol{\lambda}^{de}_M$, *i.e.* that for every A-algebra B we have :

$$\gamma^{d,e}_M \circ \beta^{d,e}_M \circ t^{d,e}_M((\boldsymbol{\lambda}^{de}_M)_B(m)) = \frac{(de)!}{(e!)^d} \cdot (\boldsymbol{\lambda}^{de}_M)_B(m) \qquad \text{for every } m \in B \otimes_A M.$$

By proposition 9.5.6, it even suffices to check the above identity for all flat A-algebras B; then, by considering the A-linear map $A \to M$ such that $1 \mapsto m$, and by invoking the naturality of the maps of (9.5.58), and their compatibility with the flat base change $A \to B$ ((9.5.21) and lemma 9.5.29(i)) we are then reduced to the case where $M = A = B$ and $m = 1$. However, recall that if M is a flat A-module, the isomorphisms t^{de}_M and t^e_M identify $t^{d,e}_M$ with the inclusion map $\mathrm{TS}^{de}_A M \to (\otimes^{de}_A M)^W$ (where $W \subset S_d$ is the subgroup $(S_d)^e \ltimes S_e$), and $\beta^{d,e}_M$ with the restriction $\mathrm{TS}^d_A(\mathrm{TS}^e_A M) \to \mathrm{Sym}^d_A(\mathrm{TS}^e_A M)$ of the projection map $\otimes^d_A(\mathrm{TS}^e_A M) \to \mathrm{Sym}^d_A(\mathrm{TS}^e_A M)$ (proposition 9.5.34); hence, we are reduced to checking that :

$$(9.5.61) \qquad t^{de}_A \circ \gamma^{d,e}_A \circ \mathrm{Sym}^d_A(t^e_A)^{-1}(1 \otimes \cdots \otimes 1) = \frac{(de)!}{(e!)^d} \cdot (1 \otimes \cdots \otimes 1).$$

Now, let M be more generally a free A-module of rank r, fix a basis $e_1, \dots, e_r$ of M, and let $e^*_1, \dots, e^*_r$ be the dual basis. Let also $p_1, p_2 : M \oplus M \to M$ be the two projections, and set $\varepsilon^*_{j,i} := e^*_i \circ p_j : M \oplus M \to A$ for $j = 1, 2$ and $i = 1, \dots, r$. Moreover, set :

$$l(i_\bullet) := k \qquad \text{for every } k \in \mathbb{N} \text{ and every } i_\bullet \in T_k := \{1, \dots, r\}^k.$$

Let $e^*_{[i_\bullet]} \in \mathrm{Sym}^k_A(M^\vee)$ be the class of $e^*_{i_1} \otimes \cdots \otimes e^*_{i_k}$, for every $i_\bullet \in T_k$, and define likewise $\varepsilon^*_{[j,i_\bullet]} \in \mathrm{Sym}^k_A((M \oplus M)^\vee)$ for $j = 1, 2$. Notice that $\mathrm{Sym}^d_A((M \oplus M)^\vee)$ admits the basis :

$$(\varepsilon^*_{[1,i_\bullet]} \otimes \varepsilon^*_{[2,j_\bullet]} \mid l(i_\bullet) + l(j_\bullet) = d) \qquad \text{for every } d \in \mathbb{N}.$$

The proof of proposition 9.5.43 identifies $\Delta^{k,k'}_{M,M} : \Gamma^{k+k'}_A(M \oplus M) \to \Gamma^{k,k'}_A(M, M)$ with the transpose of the A-linear map

$$\mathrm{Sym}^{(k,k')}_A(M^\vee, M^\vee) \to \mathrm{Sym}^{k+k'}_A((M \oplus M)^\vee) \qquad e^*_{[i_\bullet]} \otimes e^*_{[j_\bullet]} \mapsto \varepsilon^*_{[1,i_\bullet]} \otimes \varepsilon^*_{[2,j_\bullet]}$$

for every $k, k' \in \mathbb{N}$. Furthermore, for every $d \in \mathbb{N}$ choose a set of representatives Δ_d for the quotient $S_d \backslash T_d$, define the basis $([e_{i_\bullet}] \mid i_\bullet \in \Delta_d)$ of $\mathrm{TS}^d_A M$ as in the proof of lemma 9.5.29(ii), and let $([e_{i_\bullet}]^* \mid i_\bullet \in \Delta_d)$ be its dual basis. The proof of proposition 9.5.34 shows that the transpose of t^d_M is the A-linear map

$$(\mathrm{TS}^d_A M)^\vee \xrightarrow{\sim} \mathrm{Sym}^d_A(M^\vee) \qquad [e_{i_\bullet}]^* \mapsto e^*_{[i_\bullet]} \qquad \text{for every } i_\bullet \in \Delta_d.$$

Notice as well that $(\varepsilon^*_{1,i}, \varepsilon^*_{2,i} \mid i = 1, \dots, r)$ is a basis of $(M \oplus M)^\vee$, and let $(\varepsilon_{1,i}, \varepsilon_{2,i} \mid i = 1, \dots, r)$ be the dual basis of $M \oplus M$; for every $d \in \mathbb{N}$ and every $a_\bullet \in T'_d := (\{1, 2\} \times \{1, \dots, r\})^d$ we set $\varepsilon_{a_\bullet} := \varepsilon_{a_1} \otimes \cdots \otimes \varepsilon_{a_d} \in \otimes^d_A(M \oplus M)$. Then, for every $k, k' \in \mathbb{N}$ with $d = k + k'$, and every pair of sequences $i_\bullet \in T_k$, $j_\bullet \in T_{k'}$, let $\langle i_\bullet, j_\bullet \rangle \in T'_d$ be the sequence $((1, i_1), \dots, (1, i_k), (2, j_1), \dots (2, j_{k'}))$; we may choose a set of representatives Δ'_d for the quotient $S_d \backslash T'_d$, consisting of sequences of this form, and we obtain as in the proof of lemma 9.5.29(ii), a basis $([\varepsilon_{\langle i_\bullet, j_\bullet \rangle}] \mid \langle i_\bullet, j_\bullet \rangle \in \Delta'_d)$ of $\mathrm{TS}^d_A(M \oplus M)$. Summing up, we conclude that the isomorphisms t^d_M, t^k_M and $t^{k'}_M$ identify $\Delta^{k,k'}_{M,M}$ with the A-linear map

$$\mathrm{TS}^d_A(M \oplus M) \to \mathrm{TS}^k_A(M) \otimes_A \mathrm{TS}^{k'}_A(M) \qquad [\varepsilon_{\langle i_\bullet, j_\bullet \rangle}] \mapsto \begin{cases} [e_{i_\bullet}] \otimes [e_{j_\bullet}] & \text{if } l(i_\bullet) = k \\ 0 & \text{otherwise.} \end{cases}$$

Next, let $s : M \oplus M \to M$ be the addition map; by naturality of the maps $t^d_\bullet$, we get a commutative diagram :

$$\begin{array}{ccccc} \Gamma^d_A(M \oplus M) & \xrightarrow{t^d_{M \oplus M}} & \mathrm{TS}^d_A(M \oplus M) & \longrightarrow & \otimes^d_A(M \oplus M) \\ \downarrow{\scriptstyle \Gamma^d_A(s)} & & \downarrow{\scriptstyle \mathrm{TS}^d_A(s)} & & \downarrow{\scriptstyle \otimes^d_A(s)} \\ \Gamma^d_A M & \xrightarrow{t^d_M} & \mathrm{TS}^d_A M & \longrightarrow & \otimes^d_A M \end{array}$$

whose unlabeled horizontal arrows are the inclusion maps. It is easily seen that $\otimes^d_A(s)$ is given by the rule : $\varepsilon_{\langle i_\bullet, j_\bullet\rangle} \mapsto e_{(i_\bullet, j_\bullet)}$ for every $\langle i_\bullet, j_\bullet\rangle \in T'_d$, with $(i_\bullet, j_\bullet) := (i_1, \dots, i_k, j_1, \dots, j_{k'})$ (where $k := l(i_\bullet)$ and $k' := l(j_\bullet)$). Then, for every $k \in \mathbb{N}$ and every sequence $i_\bullet \in T_k$, set

$$X(i_\bullet, u) := \{t \in \{1, \dots, k\} \mid i_t = u\} \qquad \text{for every } u = 1, \dots, r.$$

Let $N_u(i_\bullet)$ be the cardinality of $X(i_\bullet, u)$, and for every $i_\bullet \in T_k$ and $j_\bullet \in T_{k'}$ define

$$C(i_\bullet, j_\bullet) := \prod_{u=1}^{r} \binom{N_u(i_\bullet) + N_u(j_\bullet)}{N_u(i_\bullet)}.$$

A direct calculation that we leave to the reader then shows that $\mathrm{TS}^d_A(s)$ is given by the rule :

$$[\varepsilon_{\langle i_\bullet, j_\bullet\rangle}] \mapsto C(i_\bullet, j_\bullet) \cdot [e_{(i_\bullet, j_\bullet)}] \qquad \text{for every } \langle i_\bullet, j_\bullet\rangle \in \Delta'_d.$$

In view of the foregoing, we deduce that the isomorphisms t^d_M, t^k_M and $t^{k'}_M$ identify the map $\mu^{k,k'}_M$ of (9.5.49) with the map

$$\mathrm{TS}^k_A M \otimes_A \mathrm{TS}^{k'}_A M \to \mathrm{TS}^d_A M \qquad \text{such that} \qquad [e_{i_\bullet}] \otimes [e_{j_\bullet}] \mapsto C(i_\bullet, j_\bullet) \cdot [e_{(i_\bullet, j_\bullet)}]$$

for every $i_\bullet \in \Delta_k$ and $j_\bullet \in \Delta_{k'}$. Notice also that

$$N_u(i_\bullet) + N_u(j_\bullet) = N_u(i_\bullet, j_\bullet) \qquad \text{for every } u = 1, \dots, r.$$

Hence, for every $d, e \in \mathbb{N}$ with $d \geq 2$, let $i_{1,\bullet}, \dots, i_{d,\bullet} \in T_e$, and set

$$\binom{a}{b_1, \dots, b_d} := \frac{a!}{b_1! \cdots b_d!} \qquad \text{for every } a, b_1, \dots, b_d \in \mathbb{N}.$$

Define also inductively the sequence $(i_{1,\bullet}, \dots, i_{k,\bullet}) \in T_{ke}$ for every $k = 2, \dots, d$ as follows. For $k = 2$ it is already defined; for $k > 2$, we set $(i_{1,\bullet}, \dots, i_{k,\bullet}) := ((i_{1,\bullet}, \dots, i_{k-1,\bullet}), i_{k,\bullet})$. With this notation, we conclude that the map $t^{de}_M \circ \gamma^{d,e}_M \circ \mathrm{Sym}^d_A(t^e_A)^{-1}$ is given by the rule :

$$[e_{i_{1,\bullet}}] \otimes \cdots \otimes [e_{i_{d,\bullet}}] \mapsto \prod_{u=1}^{r} \binom{N_u(i_{1,\bullet}, \dots, i_{d,\bullet})}{N_u(i_1), \dots, N_u(i_d)} \cdot [e_{(i_{1,\bullet}, \dots, i_{d,\bullet})}]. \tag{9.5.62}$$

The sought identity (9.5.61) follows at once.

(ii): As usual, we are easily reduced to the case where M is free of some finite rank r, in which case the isomorphism $\mathrm{TS}^d_A(t^e_M)$ and lemma 9.5.29(iv) identify again $\beta^{d,e}_M$ with a map :

$$(\otimes^{de}_A M)^W \xrightarrow{\sim} \mathrm{TS}^d_A(\mathrm{TS}^e_A M) \to \mathrm{Sym}^d_A(\mathrm{TS}^e_A M). \tag{9.5.63}$$

Now, fix again a basis $e_1, \dots, e_r$ of M; for every $i_{\bullet\bullet} := (i_{1,\bullet}, \dots, i_{d,\bullet}) \in T^d_e = T_{de}$, let $W(i_{\bullet\bullet})$ be the orbit of $i_{\bullet\bullet}$ under the natural W-action on T_{de}, and set

$$[e_{i_{\bullet\bullet}}]_W := \sum_{j_{\bullet\bullet} \in W(i_{\bullet\bullet})} e_{j_{\bullet\bullet}}$$

(where, as usual, $e_{j_{\bullet\bullet}} := e_{j_{1,1}} \otimes \cdots \otimes e_{j_{d,e}} \in \otimes^{de}_A M$ for every $j_{\bullet\bullet} \in T^d_e$). Fix also a set $\Delta_{d,e}$ of representatives for the quotient $W\backslash T^d_e$; it is easily seen that $(\otimes^{de}_A M)^W$ admits the basis $([e_{i_{\bullet\bullet}}]_W \mid i_{\bullet\bullet} \in \Delta_{d,e})$. Moreover, let $[i_{1,\bullet}], \dots, [i_{d,\bullet}] \in S_e\backslash T_e$ be the classes of the sequences $i_{1,\bullet}, \dots, i_{d,\bullet}$, and for every set Σ, let $c(\Sigma)$ be the cardinality of Σ. With this notation, a simple inspection shows that (9.5.63) is given by the rule :

$$[e_{(i_{1,\bullet}, \dots, i_{d,\bullet})}]_W \mapsto c(S_d([i_{1,\bullet}], \dots, [i_{d,\bullet}])) \cdot e_{[i_{1,\bullet}]} \otimes \cdots \otimes e_{[i_{d,\bullet}]} \qquad \text{for every } i_{\bullet\bullet} \in \Delta_{d,e}$$

where $S_d([i_{1,\bullet}], \dots, [i_{d,\bullet}])$ denotes the orbit of $([i_{1,\bullet}], \dots, [i_{d,\bullet}]) \in (S_e\backslash T_e)^d$ under the natural S_d-action. Combining with (9.5.62), we see that $\gamma^{d,e}_M \circ \beta^{d,e}_M$ is naturally identified with the A-linear map

$$(\otimes^{de}_A M)^W \to \mathrm{TS}^{de}_A M \tag{9.5.64}$$

given by the rule :

$$[e_{i_{\bullet\bullet}}]_W \mapsto c(S_d([i_{1,\bullet}],\dots,[i_{d,\bullet}]))\cdot\prod_{u=1}^r \binom{N_u(i_{\bullet\bullet})}{N_u(i_1),\dots,N_u(i_d)}\cdot[e_{i_{\bullet\bullet}}] \qquad \text{for every } i_{\bullet\bullet}\in\Delta_{d,e}.$$

Let $S_{de}(i_{\bullet\bullet})$ be the orbit of $i_{\bullet\bullet}$ under the natural S_{de}-action on T_{de}, and notice that :

$$[e_{i_{\bullet\bullet}}] = \sum_{j_{\bullet\bullet}\in W\backslash S_{de}(i_{\bullet\bullet})} [e_{j_{\bullet\bullet}}]_W \qquad \text{for every } i_{\bullet\bullet}\in T_d^e$$

where $W\backslash S_{de}(i_{\bullet\bullet})$ is the quotient of $S_{de}(i_{\bullet\bullet})$ under the natural W-action. Notice as well that the cardinalities of the stabilizers of $i_{\bullet\bullet}$ under the S_{de}-action and the S_e^d-action are respectively:

$$\frac{(de)!}{c(S_{de}(i_{\bullet\bullet}))} = \prod_{u=1}^r N_u(i_{\bullet\bullet})! \qquad \text{and} \qquad \frac{(e!)^d}{c(S_e^d(i_{\bullet\bullet}))} = \prod_{u=1}^r\prod_{j=1}^d N_u(i_{j,\bullet})!$$

whence :

$$\frac{(de)!}{(e!)^d}\cdot\frac{c(S_e^d(i_{\bullet\bullet}))}{c(S_{de}(i_{\bullet\bullet}))} = \prod_{u=1}^r\binom{N_u(i_{\bullet\bullet})}{N_u(i_1),\dots,N_u(i_d)}.$$

On the other hand, we have : $S_d([i_{1,\bullet}],\dots,[i_{d,\bullet}]) = S_e^d\backslash W(i_{\bullet\bullet})$, whence :

$$c(S_d([i_{1,\bullet}],\dots,[i_{d,\bullet}])) = \frac{c(W(i_{\bullet\bullet}))}{c(S_e^d(i_{\bullet\bullet}))}.$$

Summing up, we see that (9.5.64) is given by the rule :

$$[e_{i_{\bullet\bullet}}]_W \mapsto \frac{(de)!}{(e!)^d}\cdot\frac{c(W(i_{\bullet\bullet}))}{c(S_{de}(i_{\bullet\bullet}))}\cdot\sum_{j_{\bullet\bullet}\in W\backslash S_{de}(i_{\bullet\bullet})}[e_{j_{\bullet\bullet}}]_W \qquad \text{for every } i_{\bullet\bullet}\in\Delta_{d,e}.$$

However, for every $i_{\bullet\bullet}\in T_{de}$ we may compute :

$$\begin{aligned}\frac{c(W(i_{\bullet\bullet}))}{c(S_{de}(i_{\bullet\bullet}))}\cdot\sum_{j_{\bullet\bullet}\in W\backslash S_{de}(i_{\bullet\bullet})}[e_{j_{\bullet\bullet}}]_W = \frac{1}{c(S_{de}(i_{\bullet\bullet}))}\cdot\sum_{j_{\bullet\bullet}\in S_{de}(i_{\bullet\bullet})}[e_{j_{\bullet\bullet}}]_W &= \frac{1}{c(S_{de})}\cdot\sum_{\sigma\in S_{de}}[e_{\sigma(i_{\bullet\bullet})}]_W \\ &= \frac{c(W)}{c(S_{de})}\cdot\sum_{\sigma\in W\backslash S_{de}}[e_{\sigma(i_{\bullet\bullet})}]_W\end{aligned}$$

and notice that $c(W)/c(S_{de}) = \frac{(e!)^d\cdot d!}{(de)!}$. The sought identity now follows by a direct inspection of the definition of $\alpha_M^{d,e}$.

(iii): The first stated identity is a special case of (i). Next, recall that β_M^d is the restriction of the projection $\pi_M^d : \otimes_A^d M \to \mathrm{Sym}_A^d M$; the second identity will follow immediately from :

Claim 9.5.65. $t_M^d\circ\gamma_M^d\circ\pi_M^d(x) = \sum_{\sigma\in S_d}\sigma(x)$ for every $x\in\otimes_A^d M$.

Proof of the claim. We are easily reduced to the case where M is free of some finite rank r, and as usual we fix a basis $e_1,\dots,e_r$. In this case, π_M^d is the A-linear map such that $e_{i_\bullet}\mapsto e_{[i_\bullet]}$ for every $i_\bullet\in T_d$. Moreover, by the foregoing, the map $t_M^d\circ\gamma_M^d$ is given by the rule :

$$e_{[i_\bullet]}\mapsto\prod_{u=1}^r N_u(i_\bullet)!\cdot[e_{i_\bullet}] = \sum_{\sigma\in S_d}e_{\sigma(i_\bullet)} \qquad \text{for every } i_\bullet\in T_d.$$

The claim follows at once. ◊

The third stated identity follows from the second one and lemma 9.5.29(iii). □

9.5.66. Let now A be a ring, $d \in \mathbb{N}$, and for every A-algebra B and every B-module M, let $\mu_{B,M} : B \otimes_A M \to B$ be the scalar multiplication map, *i.e.* the A-linear map such that $b \otimes m \mapsto bm$ for every $b \in B$ and every $m \in M$. We deduce an A-linear map

$$\mu^d_{B,M} : \Gamma^d_A(B) \otimes_A \Gamma^d_A(M) \xrightarrow{\varphi^d_{B,M}} \Gamma^d_A(B \otimes_A M) \xrightarrow{\Gamma^d_A(\mu_{B,M})} \Gamma^d_A(M)$$

for every such B and M, where $\varphi^d_{B,M}$ is defined as in (9.5.38). By virtue of remarks 9.5.39(ii) and 9.5.46(ii), the system of such maps is uniquely characterized by the identities :

$$\mu^d_{B,M}(b^{[d]} \otimes m^{[d]}) = (bm)^{[d]} \qquad (9.5.67)$$

for every A-algebra B, every B-module M, and every $b \in B$ and $m \in M$. Especially, taking $M := B$, it follows easily that $\mu^d_{B,B}$ furnishes the multiplication map for a well defined A-algebra structure on $\Gamma^d_A B$, for every such B and d. Moreover, $\mu^d_{B,M}$ is the scalar multiplication map for a well defined $\Gamma^d_A B$-module structure on $\Gamma^d_A M$, for every B-module M. Furthermore, if $f : B \to C$ is any map of A-algebras, then clearly the same holds for $\Gamma^d_A(f)$, and if $g : M \to N$ is any morphism of B-modules, then $\Gamma^d_A(g)$ is a $\Gamma^d_A B$-linear map. Summing up, we obtain for every $d \in \mathbb{N}$, every ring A, and every A-algebra B, natural functors :

$$\Gamma^d_A : A\text{-}\mathsf{Alg} \to A\text{-}\mathsf{Alg} \qquad \Gamma^d_A : B\text{-}\mathbf{Mod} \to \Gamma^d_A B\text{-}\mathbf{Mod}.$$

Notice also that $\mathrm{TS}^d_A B$ is an A-subalgebra of the A-algebra $\otimes^d_A B$.

Proposition 9.5.68. (i) *The natural bijection* (9.5.21) *is an isomorphism of C-algebras*

$$C \otimes_A \Gamma^d_A B \xrightarrow{\sim} \Gamma^d_C(C \otimes_A B) \qquad \textit{for every } A\textit{-algebra } C.$$

(ii) *The map $t^d_B : \Gamma^d_A B \to \mathrm{TS}^d_A B$ of* (9.5.33) *is an integral radicial morphism of A-algebras, and $\mathrm{Ker}(t^d_B)$ lies in the nilradical of $\Gamma^d_A B$, for every $d \in \mathbb{N}$ and every A-algebra B.*

(iii) *If B is an A-algebra of finite type, the same holds for $\Gamma^d_A B$.*

(iv) *Let $f : B_1 \to B_2$ be an integral map of A-algebras; we have :*

(a) *The maps $\mathrm{TS}^d_A f$ and $\Gamma^d_A f$ are integral.*

(b) *If moreover f is radicial, the same holds for $\Gamma^d_A f$.*

(c) *if $\mathrm{Ker}\, f$ lies in the nilradical of B_1, then $\mathrm{Ker}\,(\Gamma^d_A f)$ lies in the nilradical of $\Gamma^d_A B_1$.*

Proof. (i) follows easily from the characterization (9.5.67) of the map $\mu^d_{B,B}$.

(ii): Again, the characterization (9.5.67) of $\mu^d_{B,B}$ easily implies that t^d_B is a map of A-algebras. Next, let us check that for every algebraically closed field K and every ring homomorphism $A \to K$, the map t^d_B induces a bijection :

$$\mathrm{Hom}_{A\text{-}\mathsf{Alg}}(\mathrm{TS}^d_A B, K) \xrightarrow{\sim} \mathrm{Hom}_{A\text{-}\mathsf{Alg}}(\Gamma^d_A B, K). \qquad (9.5.69)$$

To this aim, set $B_K := K \otimes_A B$, and consider the commutative diagram

$$\begin{array}{ccc} K \otimes_A \Gamma^d_A B & \xrightarrow{K \otimes_A t^d_B} & K \otimes_A \mathrm{TS}^d_A B \\ \downarrow & & \downarrow \varphi \\ \Gamma^d_K(B_K) & \xrightarrow{t^d_{B_K}} & \mathrm{TS}^d_K B_K \end{array}$$

whose left vertical arrow is the isomorphism of (i), and where φ is the restriction of the natural isomorphism $K \otimes_A (\otimes^d_A B) \xrightarrow{\sim} \otimes^d_K B_K$. On the one hand, (9.5.69) is naturally identified with the analogous map $\mathrm{Hom}_{K\text{-}\mathsf{Alg}}(\mathrm{TS}^d_K B_K, K) \xrightarrow{\sim} \mathrm{Hom}_{K\text{-}\mathsf{Alg}}(\Gamma^d_K B_K, K)$ induced by $K \otimes_A t_B$, and on the other hand, $t^d_{B_K}$ is an isomorphism, by proposition 9.5.34(i); hence, we are reduced to checking that φ induces a bijection

$$\mathrm{Hom}_{K\text{-}\mathsf{Alg}}(\mathrm{TS}^d_K B_K, K) \xrightarrow{\sim} \mathrm{Hom}_{K\text{-}\mathsf{Alg}}(K \otimes_A \mathrm{TS}^d_A B, K) \xrightarrow{\sim} \mathrm{Hom}_{A\text{-}\mathsf{Alg}}(\mathrm{TS}^d_A B, K).$$

To this aim, we consider the commutative diagram :

$$\begin{array}{ccc} \mathrm{TS}^d_A B & \xrightarrow{j_B} & \otimes^d_A B \\ \downarrow & & \downarrow \\ \mathrm{TS}^d_K B_K & \xrightarrow{j_{B_K}} & \otimes^d_K B_K \end{array}$$

where j_B and j_{B_K} are the inclusion maps, which are integral, and according to [34, Ch.5, §2, n.2, Cor. of Th.2] induce bijections

$$\mathrm{Hom}_{A\text{-}\mathsf{Alg}}(\otimes^d_A B, K)/G \xrightarrow{\sim} \mathrm{Hom}_{A\text{-}\mathsf{Alg}}(\mathrm{TS}^d_A B, K)$$
$$\mathrm{Hom}_{K\text{-}\mathsf{Alg}}(\otimes^d_K B_K, K)/G \xrightarrow{\sim} \mathrm{Hom}_{K\text{-}\mathsf{Alg}}(\mathrm{TS}^d_K B_K, K)$$

for the S_d-actions on $\otimes^d_A B$ and on $\otimes^d_K B_K$ as in (9.5.28). It then suffices to remark that the natural map $\otimes^d_A B \to \otimes^d_K B_K$ induces a bijection $\mathrm{Hom}_{K\text{-}\mathsf{Alg}}(\otimes^d_K B_K, K) \xrightarrow{\sim} \mathrm{Hom}_{A\text{-}\mathsf{Alg}}(\otimes^d_A B, K)$. This shows already that t^d_B is radicial and that $\mathrm{Ker}\, t^d_B$ lies in the nilradical of $\Gamma^d_A B$. To conclude, it suffices to check that $j_B \circ t^d_B$ is integral. To this aim, choose a surjective map of A-algebras $P \to B$ from a free polynomial A-algebra; since the induced map $\otimes^d_A P \to \otimes^d_A B$ is surjective, it suffices to check the assertion for the composition of t^d_P and the inclusion map $\mathrm{TS}^d_A P \to \otimes^d_A P$. But the latter is integral, and t^d_P is an isomorphism, by proposition 9.5.34(i).

(iii): Let $f : A[T_\bullet] \to B$ be a surjection of A-algebras, from a free polynomial A-algebra $A[T_\bullet]$ of finite type. As $\Gamma^d_A f$ is surjective (corollary 9.5.22(iv)), it suffices to check the assertion for $\Gamma^d_A A[T_\bullet]$. Next, the isomorphism $A \otimes_{\mathbb{Z}} \Gamma^d_{\mathbb{Z}} \mathbb{Z}[T_\bullet] \xrightarrow{\sim} \Gamma^d_A A[T_\bullet]$ of lemma 9.5.29(i) reduces to the case where $A = \mathbb{Z}$, and especially, A is noetherian. Then, notice that $\otimes^d_{\mathbb{Z}} \mathbb{Z}[T_\bullet]$ is a $\mathbb{Z}$-algebra of finite type, and by (ii) and proposition 9.5.34(i), $\Gamma^d_{\mathbb{Z}} \mathbb{Z}[T_\bullet]$ is isomorphic to $\mathrm{TS}^d_A \mathbb{Z}[T_\bullet]$. Since the $\otimes^d_{\mathbb{Z}} \mathbb{Z}[T_\bullet]$ is integral over $\mathrm{TS}^d_A \mathbb{Z}[T_\bullet]$ is, the assertion follows from [12, Prop.7.8].

(iv): In light of (ii) we have a commutative diagram of A-algebras :

$$\begin{array}{ccccc} \Gamma^d_A B_1 & \xrightarrow{t^d_{B_1}} & \mathrm{TS}^d_A B_1 & \xrightarrow{j_1} & \otimes^d_A B_1 \\ {\scriptstyle \Gamma^d_A f}\downarrow & & \downarrow{\scriptstyle \mathrm{TS}^d_A f} & & \downarrow{\scriptstyle \otimes^d_A f} \\ \Gamma^d_A B_2 & \xrightarrow{t^d_{B_2}} & \mathrm{TS}^d_A B_2 & \xrightarrow{j_2} & \otimes^d_A B_2 \end{array}$$

where j_1 and j_2 are the inclusion maps. In order to check that $\mathrm{TS}^d_A f$ is integral, it suffices then to show that the same holds for $(\otimes^d_A f) \circ j_1$; however, j_1 is integral, and the same holds for $\otimes^d_A f$, since f is integral, whence the assertion. Combining with (ii), we deduce that $(\mathrm{TS}^d_A f) \circ t^d_{B_1}$ is integral; hence, let $x \in \Gamma^d_A B_2$ be any element, and pick a monic polynomial $P \in (\Gamma^d_A B_1)[T]$ such that $P(t_{B_2} x) = 0$. Set $Q := P - T$, and notice that $Q(x) \in \mathrm{Ker}(t^d_{B_2})$; by (ii), it follows that $Q(x)^n = 0$ for some $n \in \mathbb{N}$, so x is integral over $\Gamma^d_A B_1$. This completes the proof of (iv.a).

Next, recall that f is radicial (resp. that $\mathrm{Ker}\, f$ lies in the nilradical $\mathscr{N}$ of B_1) if and only if, for every algebraically closed field K, the induced map $\mathrm{Hom}_{\mathbb{Z}\text{-}\mathsf{Alg}}(B_2, K) \to \mathrm{Hom}_{\mathbb{Z}\text{-}\mathsf{Alg}}(B_1, K)$ is injective (resp. surjective). It follows easily that if f is radicial (resp. if $\mathrm{Ker}\, f \subset \mathscr{N}$), then the same holds for $\otimes^d_A f$, and arguing as in the proof of (ii), we deduce that the same holds also for $\mathrm{TS}^d_A f$. By the same token, (ii) implies that $t^d_{B_i}$ induces a bijection $\mathrm{Hom}_{\mathbb{Z}\text{-}\mathsf{Alg}}(\mathrm{TS}^d_A B_i, K) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}\text{-}\mathsf{Alg}}(\Gamma^d_A B_i, K)$ for $i = 1, 2$ and every algebraically closed field K. Assertions (iv.b) and (iv.c) follow immediately. □

Definition 9.5.70. With the notation of (9.5.66), we call a *norm map of degree* d for the A-algebra B, any map of A-algebras

$$\Gamma^d_A B \to A.$$

Remark 9.5.71. Let B be any A-algebra, $d \in \mathbb{N}$, and $\nu : \Gamma^d_A B \to A$ a norm map of degree d.

(i) In light of (9.5.67), we see that the datum of ν is equivalent to that of a system of morphisms of multiplicative monoids

$$N_C : C \otimes_A B \to C \qquad \text{for every } A\text{-algebra } C$$

natural with respect to maps of A-algebras $C \to C'$, and such that

$$N_C(c \otimes b) = c^d \cdot N_C(1 \otimes b) \qquad \text{for every } c \in C \text{ and } b \in B.$$

Indeed, from ν we obtain such a compatible system $N_\bullet$ by the rule :

$$N_C := (C \otimes_A \nu) \circ (\boldsymbol{\lambda}^d_B)_C \qquad \text{for every } A\text{-algebra } C$$

and conversely, every such compatible system $N_\bullet$ yields an A-linear map $\nu : \Gamma^d_A B \to A$, by the universal property of $\Gamma^d_A B$, and the condition that N_C is a map of multiplicative monoids implies easily that ν is a map of A-algebras (details left to the reader).

(ii) The isomorphism of proposition 9.5.68(i) identifies naturally $C \otimes_A \nu$ with a norm map $\nu_C : \Gamma^c_C(C \otimes_A B) \to C$ of degree d. Indeed, if $N_\bullet$ (resp. $N^C_\bullet$) is the compatible system of morphisms of multiplicative monoids associated with ν (resp. with ν_C), then $N^C_\bullet$ is the just the restriction of $N_\bullet$ to the category of C-algebras.

Example 9.5.72. (i) Let A be a ring, B an A-algebra whose underlying A-module is projective of constant finite rank d. Then, for every other A-algebra C, the C-module underlying the C-algebra $C \otimes_A B$ is projective of constant finite rank d. Every $x \in C \otimes_A B$ induces a C-linear endomorphism $\mu_x : C \otimes_A B \to C \otimes_A B$ such that $x' \mapsto xx'$ for every $x' \in C \otimes_A B$. The determinant $N_C(x) := \det(\mu_x)$ is an element of C, and the construction of $N_C : C \otimes_A B \to C$ is natural for maps $C \to C'$ of A-algebras, so we get a compatible system $N_\bullet$ of maps of sets for every A-algebra C. In view of remark 9.5.71(i), $N_\bullet$ corresponds to a unique norm map

$$\nu_{B/A} : \Gamma^d_A B \to A \qquad \text{such that} \qquad (C \otimes_A \nu_{B/A})(x^{[d]}) = \det(\mu_x)$$

for every A-algebra C and every $x \in C \otimes_A B$.

(ii) Let A be a normal domain, K its field of fractions, and B the integral closure of A in a finite field extension E of K, of degree d. Then, every free polynomial A-algebra $A[T_\bullet]$ is again a normal domain, and $B[T_\bullet] = B \otimes_A A[T_\bullet]$ is the integral closure of $A[T_\bullet]$ in the finite field extension $E(T_\bullet)$ of $K(T_\bullet)$. With this notation, the norm map $N_{K[T_\bullet]} : E[T_\bullet] = K[T_\bullet] \otimes_K E \to K[T_\bullet]$ is well defined as in (i), and we claim that $N_{K[T_\bullet]}(B[T]) \subset A[T]$. Indeed, recall that $\det(\mu_x) = (-1)^d \cdot \chi_x(0)$, where $\chi_x \in K(T_\bullet)[X]$ is the characteristic polynomial of the endomorphism μ_x of $E(T_\bullet)$, for every $x \in E(T_\bullet)$; let also $m_x \in K(T_\bullet)[X]$ be the minimal polynomial of x over $K(T_\bullet)$. Since χ_x and m_x have the same set of roots Σ, it suffices to check that every root of m_x is integral over $A[T_\bullet]$, when $x \in B[T_\bullet]$. However, let L be any normal extension of $K(T_\bullet)$ containing $E(T_\bullet)$; then Σ is the orbit of x under the action of the Galois group $\mathrm{Gal}(L/K(T_\bullet))$, whence the assertion. Thus, for every free polynomial A-algebra $A[T_\bullet]$ the restriction of $N_{K[T_\bullet]}$ is a well defined morphism of multiplicative monoids

$$N'_{A[T_\bullet]} : B[T_\bullet] \to A[T_\bullet]$$

and it is clear that the rule $A[T_\bullet] \mapsto N'_{A[T_\bullet]}$ is natural for maps of free polynomial A-algebras, since the same holds for the rule $K[T_\bullet] \mapsto N_{K[T_\bullet]}$. By virtue of proposition 9.5.6, it follows that $N'_\bullet$ extends uniquely to a well defined homogeneous polynomial law $N' : B \rightsquigarrow A$ of degree d, and it is easily seen that N'_C is still a morphism of multiplicative monoids, for every A-algebra C (details left to the reader); whence finally, a norm map of degree d for the A-algebra B :

$$\nu'_{B/A} : \Gamma^d_A B \to A.$$

(iii) Let A be a ring, G a finite group of automorphisms of A, and $H \subset G$ a subgroup with $[G : H] = d$. For every A^G-algebra B, we get an induced action of G on $B \otimes_{A^G} A$, and if B is a flat A^G-algebra, the natural maps :

$$B \to (B \otimes_{A^G} A)^G \qquad B \otimes_{A^G} A^H \to (B \otimes_{A^G} A)^H$$

are isomorphisms (details left to the reader). Fix a system of representatives $\Sigma \subset G$ for the classes of G/H; for every flat A^G-algebra B, we then get a map of multiplicative monoids :

$$N''_B : B \otimes_{A^G} A^H \to B \qquad x \mapsto \prod_{g \in \Sigma} g(x)$$

which is clearly natural for every map of flat A^G-algebras. By proposition 9.5.6, the rule $B \mapsto N''_B$ then extends uniquely to a well defined homogeneous polynomial law $N'' : A^H \rightsquigarrow A^G$ of degree d, whence again a norm map of degree d for the A^G-algebra A^H :

$$\nu''_{A^H/A^G} : \Gamma^d_{A^G} A^H \to A^G.$$

(iv) The following example generalizes all the previous ones. Consider a ring A, an injective ring homomorphism $A \to A'$, a projective A'-module P of constant rank d, and a map of associative A-algebras $\rho : B \to \mathrm{End}_{A'}(P')$. For every flat A-algebra C, let moreover

$$\rho_C := C \otimes_A \rho : C \otimes_A B \to C \otimes_A \mathrm{End}_{A'}(P') \xrightarrow{\sim} \mathrm{End}_{C \otimes_A A'}(C \otimes_A P')$$

and suppose that $N'''_C(b) := \det(\rho_C(b)) \in C$ for every such C and every $b \in C \otimes_A B$. By proposition 9.5.6, the rule : $C \mapsto N'''_C$ for every such C extends uniquely to a homogeneous polynomial law $N''' : B \rightsquigarrow A$ of degree d, and clearly N_C is a morphism of multiplicative monoids for every flat A-algebra C, hence also for any A-algebra C (details left to the reader); whence finally a norm map of degree d

$$\nu'''_{P,\rho} : \Gamma^d_A B \to A.$$

9.5.73. Let A be a ring, B an A-algebra, $d \in \mathbb{N}$, and $\nu : \Gamma^d_A B \to A$ a norm map of degree d. Let also $N : B \rightsquigarrow A$ be the homogeneous polynomial law of degree d associated with ν, as in remark 9.5.71(i). We set

$$\chi_b(T) := N_{C[T]}(T - b) \in C[T] \qquad \text{for every } A\text{-algebra } C \text{ and every } b \in C \otimes_A B.$$

Let moreover $\nu_{C[T]} := C[T] \otimes_A \Gamma^d_A B \to C[T]$ for every such C, and notice that

$$\chi_b(T) = \nu_{C[T]}((T - b)^{[d]})$$

and $(T - b)^{[d]} = \sum_{i=0}^d T^{[i]} \cdot (-b)^{[d-i]} = \sum_{i=0}^d T^i \cdot 1^{[i]} \cdot (-b)^{[d-i]}$ in $C[T] \otimes_A \Gamma^\bullet_A B$, by virtue of (9.5.51). Since $(-b)^{[0]} = 1$, and $\nu_{C[T]}(1^{[d]}) = N_{C[T]}(1) = 1$, we may compute :

$$\chi_b(T) = \sum_{i=0}^{d} T^i \cdot \nu_{C[T]}(1^{[i]} \cdot (-b)^{[d-i]}) = T^d + \sum_{i=0}^{d-1} T^i \cdot \nu_{C[T]}(1^{[i]} \cdot (-b)^{[d-i]}).$$

We get therefore a well defined homogeneous polynomial law of degree d :

$$B \rightsquigarrow B \qquad b \mapsto \chi_b(b) \in C \otimes_A B \qquad \text{for every } A\text{-algebra } C \text{ and every } b \in C \otimes_A B$$

and we define the *characteristic map* of ν as the associated A-linear map $\chi^\dagger : \Gamma^d_A B \to B$.

Definition 9.5.74. With the notation of (9.5.73), we say that ν is a *Cayley-Hamilton norm map* if $\chi^\dagger = 0$, *i.e.* if $\chi_b(b) = 0$ for every A-algebra C and every $b \in C \otimes_A B$.

Example 9.5.75. (i) In the situation of example 9.5.72(iv), notice that, by the theorem of Cayley-Hamilton, we have $\rho_C(\chi_b(b)) = 0$ for every flat A-algebra C and every $b \in C \otimes_A B$. Thus, if ρ_C is injective for every such C, the norm map $\nu'''_{P,\rho}$ is Cayley-Hamilton.

(ii) Especially, the norm maps $\nu_{B/A}, \nu'_{B/A}$ and ν''_{A^H/A^G} of example 9.5.72(i,ii,iii) are Cayley-Hamilton.

Remark 9.5.76. (i) Let $\varphi : B' \to B$ be a map of A-algebras, $\nu : \Gamma^d_A B \to A$ a norm map of degree d, and set $\nu' := \nu \circ \Gamma^d_A(\varphi)$. Let also $\chi^\dagger$ and $\chi'^\dagger$ be the characteristic maps of ν, respectively ν'. By a direct inspection of the constructions we get a commutative diagram :

$$\begin{array}{ccc} \Gamma^d_A B' & \xrightarrow{\chi'^\dagger} & B' \\ {\scriptstyle \Gamma^d_A\varphi}\downarrow & & \downarrow{\scriptstyle \varphi} \\ \Gamma^d_A B & \xrightarrow{\chi^\dagger} & B \end{array}$$

(details left to the reader). In particular, if φ is surjective and ν' is Cayley-Hamilton, the same holds for ν, since $\Gamma^d_A(\varphi)$ is surjective (corollary 9.5.22(iv)).

(ii) In particular, let $\nu : \Gamma^d_A B \to A$ be a norm map of degree d, and $I \subset B$ the ideal generated by the image of the characteristic map $\chi^\dagger$ of ν; let also $\pi : B \to B/I$ be the projection, and suppose that ν factors through $\Gamma^d_A(\pi)$ and a norm map $\overline{\nu} : \Gamma^d_A(B/I) \to A$. Then, if $\overline{\chi}^\dagger$ denotes the characteristic map of $\overline{\nu}$, we have $\overline{\chi}^\dagger \circ \Gamma^d_A(\pi) = \pi \circ \chi^\dagger = 0$, by (i), whence $\overline{\chi}^\dagger = 0$, since $\Gamma^d_A(\pi)$ is a surjection (corollary 9.5.22(iv)). In other words, in this situation, $\overline{\nu}$ is Cayley-Hamilton.

(iii) In the situation of (i), let also C be any A-algebra, and $\nu_C : \Gamma^d_C(C \otimes_A B) \to C$ the norm map induced by ν as in remark 9.5.71(ii). Then we get a commutative diagram of C-modules :

$$\begin{array}{ccc} C \otimes_A \Gamma^d_A B & \xrightarrow{\sim} & \Gamma^d_C(C \otimes_A B) \\ & {\scriptstyle C\otimes_A \chi^\dagger}\searrow \quad \swarrow{\scriptstyle \chi^{C\dagger}} & \\ & C \otimes_A B & \end{array}$$

where $\chi^{C\dagger}$ is the characteristic map of ν_C, and the top horizontal arrow is the isomorphism of proposition 9.5.68(i). Especially, if ν is a Cayley-Hamilton norm map, the same holds for ν_C.

Theorem 9.5.77. *Let $\nu : \Gamma^d_A B \to A$ be a norm map of degree d, and $I \subset B$ the ideal generated by the image of the characteristic map $\chi^\dagger$ of ν. Then ν is the composition of the projection $\Gamma^d_A B \to \Gamma^d_A(B/I)$ and a Cayley-Hamilton norm map $\overline{\nu} : \Gamma^d_A(B/I) \to A$.*

Proof. By remark 9.5.76(ii), it suffices to show that ν factors through the projection onto $\Gamma^d_A(B/I)$ and a norm map $\overline{\nu} : \Gamma^d_A(B/I) \to A$. We first deal with the following special case :

Claim 9.5.78. The theorem holds if A is an algebraically closed field.

Proof of the claim. In this case, B is trivially a flat A-algebra, so ν is naturally identified with a map of A-algebras $\mathrm{TS}^d_A B \to A$ (propositions 9.5.68(ii) and 9.5.34(i)). The latter extends to a map of A-algebras $\otimes^d_A B \to A$, which is the same as a system of d maps of A-algebras $\nu_1, \dots, \nu_d : B \to A$. Let $\rho : B \to \mathrm{End}_A(A^{\oplus d})$ be the morphism of associative A-algebras that maps each $b \in B$ to the diagonal $d \times d$ matrix $\rho(b)$ with $\rho(b)_{ii} = \nu'_i(b)$ for $i = 1, \dots, d$. Working out the definitions, we see that ν corresponds to the homogeneous polynomial law $B \rightsquigarrow A$ arising from ρ, as in example 9.5.72(iv). Let $J := \mathrm{Ker}\,\rho$, so that ρ factors through the projection $\pi : B \to B/J$ and a map of associative A-algebras $\rho' : B/J \to \mathrm{End}_A(A^{\oplus d})$; then ρ' induces, again by example 9.5.72(iv), a norm map $\nu' : \Gamma^d_A(B/J) \to A$ such that $\nu = \nu' \circ \Gamma^d_A(\pi)$, and ν' is Cayley-Hamilton, by virtue of example 9.5.75(i). By remark 9.5.76(i), it follows that $I \subset J$, whence the required factorization of ν. ◊

Now, choose a free polynomial A-algebra P, a surjective map $\varphi : P \to B$ of A-algebras, and set $\nu^P := \nu \circ \Gamma^d_A(\varphi) : \Gamma^d_A P \to A$; remark 9.5.76(i) yields the identity :

$$\varphi \circ \chi^{P\dagger} = \chi^\dagger \circ \Gamma^d_A(\varphi)$$

where $\chi^\dagger : \Gamma^d_A B \to B$ and $\chi^{P\dagger} : \Gamma^d_A P \to P$ are the characteristic maps of ν and ν^P. Let $J \subset P$ be the ideal generated by the image of $\chi^{P\dagger}$; since $\Gamma^d_A(\varphi)$ is surjective (corollary 9.5.22(iv)), it follows that $\varphi(J) = I$. Let $\overline{\varphi} : P/J \to B/I$ be the map induced by φ, and $\pi : P \to P/J$ the projection; it follows easily from corollary 9.5.57 that $\mathrm{Ker}\,\Gamma^d_A(\overline{\varphi})$ is the image of $\mathrm{Ker}\,\Gamma^d_A(\varphi)$, under the map $\Gamma^d_A(\pi) : \Gamma^d_A P \to \Gamma^d_A(P/J)$. Now, suppose that ν^P factors through $\Gamma^d_A(\pi)$ and a norm map $\overline{\nu}^P : \Gamma^d_A(P/J) \to A$; then we deduce that ν factors through the projection $\Gamma^d_A B \to \Gamma^d_A(B/I)$ and a norm map $\overline{\nu} : \Gamma^d_A(B/I) \to A$.

Thus, it suffices to show the theorem for P, and we may assume from start that $B = A[T_\bullet]$ is a free polynomial A-algebra. In this case, set $A_0 := \Gamma^d_{\mathbb{Z}}(\mathbb{Z}[T_\bullet])$; taking into account the isomorphism of proposition 9.5.68(i), we get a norm map of degree d :

$$\nu_0 : \Gamma^d_{A_0}(A_0[T_\bullet]) \xrightarrow{\sim} A_0 \otimes_{\mathbb{Z}} \Gamma^d_{\mathbb{Z}}(\mathbb{Z}[T_\bullet]) = A_0 \otimes_{\mathbb{Z}} A_0 \xrightarrow{\mu_{A_0}} A_0$$

where μ_{A_0} is the multiplication map of the ring A_0. By the same token, ν is naturally identified with a map of A-algebras $A \otimes_{\mathbb{Z}} A_0 \to A$ and we get therefore an induced A_0-algebra structure on A (indeed, the set of norm maps of degree d of the A-algebra $A[T_\bullet]$ is in natural bijection with the set of all such A_0-algebra structures). Then, a simple inspection yields a commutative diagram of A-algebras :

$$\begin{array}{ccc} A \otimes_{A_0} \Gamma^d_{A_0}(A_0[T_\bullet]) & \xrightarrow{\ \omega\ } & \Gamma^d_A(A[T_\bullet]) \\ & {\scriptstyle A\otimes_{A_0}\nu_0} \searrow \quad \swarrow {\scriptstyle \nu} & \\ & A & \end{array}$$

where ω is again the natural isomorphism of proposition 9.5.68(i). Let $\chi_0^\dagger : \Gamma^d_{A_0}(A_0[T_\bullet]) \to A_0$ be the characteristic map of ν_0, and $I_0 \subset A_0[T_\bullet]$ the ideal generated by the image of $\chi_0^\dagger$; from remark 9.5.76(iii) we see that the image of $A \otimes_{A_0} I_0$ in $A[T_\bullet]$ is the ideal I. Let $\pi_0 : A_0[T_\bullet] \to A_0[T_\bullet]/I_0$ be the projection; by corollary 9.5.57, the kernel of $\Gamma^d_{A_0}(\pi_0)$ is the image of $\Gamma^d_{A_0}(I_0)$, so the kernel of ν is the image of $A \otimes_{A_0} \Gamma^d_{A_0}(I_0) \simeq \Gamma^d_A(A \otimes_{A_0} I_0)$, and the latter is also the kernel of the projection $\Gamma^d_A(A[T_\bullet]) \to \Gamma^d_A(A[T_\bullet]/I)$, again by corollary 9.5.57. Summing up, we see that if ν_0 factors through $\Gamma^d_{A_0}(\pi_0)$ and a norm map $\overline{\nu}_0 : \Gamma^d_{A_0}(A_0[T_\bullet]/I_0) \to A_0$, then ν is the composition of the projection onto $\Gamma^d_A(A[T_\bullet]/I)$ and a norm map $\overline{\nu} : \Gamma^d_A(A[T_\bullet]/I) \to A$.

Thus, we are further reduced to the case where $A = A_0$, and in particular A is a domain, since the same clearly holds for $\mathrm{TS}^d_{A_0}(A_0[T_\bullet])$ (propositions 9.5.68(ii) and 9.5.34(i)). Next, let K be an algebraic closure of the field of fractions of A, and set $B_K := K \otimes_A B$ and $\nu_K := K \otimes_A \nu$. Since $\Gamma^d_A B$ is a flat A-module (corollary 9.5.22(iii)), the induced map $\Gamma^d_A B \to K \otimes_A \Gamma^d_A B \xrightarrow{\sim} \Gamma^d_K B_K$ is injective, and ν_K is naturally identified with a norm map $\Gamma^d_K B_K \to K$. Let $\chi^\dagger_K$ be the characteristic map of ν_K; arguing as in the foregoing, we see that the ideal of B_K generated by the image of $\chi^\dagger_K$ is $I_K := K \otimes_A I$. Hence, if ν_K is the composition of the projection $\Gamma^d_K B_K \to \Gamma^d_K(B_K/I_K)$ and a norm map $\overline{\nu}_K : \Gamma^d_K(B_K/I_K) \to K$, then ν factors through the projection onto $\Gamma^d_A(B/I)$ and a norm map $\Gamma^d_A(B/I) \to A$. Thus, it suffices to show the theorem for the norm map ν_K, and this is already known by claim 9.5.78. $\square$

Remark 9.5.79. Theorem 9.5.77 admits a non-commutative generalization in the theory of determinants of algebras : see [47, Lemma 1.21].

9.6. **Regular rings.** In a later section, we shall need a criterion for the regularity of a certain type of extensions of regular local rings. Such a criterion is stated incorrectly in [63, Ch.0, Th.22.5.4]; our first task is to supply a corrected version of *loc.cit.*

Proposition 9.6.1. *Let K be a field, and L a field extension of K. We have :*

(i) $H_i\mathbb{L}_{L/K} = 0$ *for every* $i > 1$.

(ii) *If L is a separable extension of K, then* $H_i\mathbb{L}_{L/K} = 0$ *for every* $i > 0$.

(iii) (Cartier's equality). *If L is an extension of finite type of K, then*

$$\dim_L H_1\mathbb{L}_{L/K} < +\infty \qquad \textit{and} \qquad \dim_L \Omega^1_{L/K} - \dim_L H_1\mathbb{L}_{L/K} = \mathrm{tr.deg}[L : K].$$

Proof. (ii): By [103, Ch.II, (1.2.3.4)], we may assume that L is an extension of K of finite type, in which case L is a separable algebraic extension of a subfield $L_0 \subset L$, which is a purely transcendantal extension of K of finite type. Since L is an étale L_0-algebra, and since L_0 is a localization of a smooth K-algebra, we then have

$$\text{(9.6.2)} \qquad \mathbb{L}_{L/L_0} \xrightarrow{\sim} 0 \qquad \text{and} \qquad \mathbb{L}_{L_0/K} \xrightarrow{\sim} \Omega^1_{L/K_0}[0] \qquad \text{in } \mathsf{D}(L\text{-}\mathbf{Mod})$$

([103, Ch.III, Prop.3.1.1 and Prop.3.1.2] and [103, Ch.II, Cor.2.3.1.1]). However, the tower of field extensions $K \subset L_0 \subset L$ yields a distinguished triangle :

$$\text{(9.6.3)} \qquad \mathbb{L}_{L/L_0}[1] \to \mathbb{L}_{L_0/K} \otimes_{L_0} L \to \mathbb{L}_{L/K} \to \mathbb{L}_{L/L_0} \qquad \text{in } \mathsf{D}(L\text{-}\mathbf{Mod})$$

([103, Ch.II, Prop.2.1.2]) whence the assertion.

(i): Let K_0 be the smallest subfield of K; so K_0 is either $\mathbb{Q}$ or $\mathbb{F}_p$, for some prime integer p, and in either case, it is a perfect field. By (i), we then have $H_i\mathbb{L}_{K/K_0} = H_i\mathbb{L}_{L/K_0} = 0$ for every $i > 0$. However, the tower of field extensions $K_0 \subset K \subset L$ yields a distinguished triangle :

$$\mathbb{L}_{L/K}[1] \to \mathbb{L}_{K/K_0} \otimes_K L \to \mathbb{L}_{L/K_0} \to \mathbb{L}_{L/K}$$

([103, Ch.II, Prop.2.1.2]) whence the assertion.

(iii): Suppose first that L is a separable extension of K, and pick $L_0 \subset L$ as in the proof of (ii), so $L_0 = \mathrm{Frac}\, K[T_1, \dots, T_r]$, with $r := \mathrm{tr.deg}[L : K]$. From (9.6.2) and (9.6.2), we get :

$$\dim_L \Omega^1_{L/K} = \dim_L H_0\mathbb{L}_{L/K} = \dim_{L_0} \Omega^1_{L_0/K} = \dim_{L_0} L_0 \otimes_{K[T_1,\dots,T_r]} \Omega^1_{K[T_1,\dots,T_r]/K} = r$$

which establishes the sought identity, as in this case we have $H_1\mathbb{L}_{L/K} = 0$, by (ii).

Next, suppose that K is of finite type over its smallest subfield K_0. We have $H_1\mathbb{L}_{L/K_0} = 0$, by (ii); then the tower of field extensions $K_0 \to K \to L$ yields an exact sequence

$$0 \to H_1\mathbb{L}_{L/K} \to \Omega^1_{K/K_0} \otimes_K L \to \Omega^1_{L/K_0} \to \Omega^1_{L/K} \to 0$$

([103, Ch.II, Prop.2.1.2]), and on the other hand :

$$\dim_K \Omega^1_{K/K_0} = \mathrm{tr.deg}[K : K_0] \qquad \text{and} \qquad \dim_L \Omega^1_{L/K_0} = \mathrm{tr.deg}[L : K_0]$$

by the foregoing case. The assertion follows straightforwardly, in this case.

Lastly, let K be an arbitrary field, and write K as the union of the filtered family $(K_\lambda \mid \lambda \in \Lambda)$ of its subfields that are finitely generated over K_0. Choose $x_1, \dots, x_t \in L$ such that $K(x_1, \dots, x_t) = L$. Let $\varphi : K[X_1, \dots, X_t] \to L$ be the map of K-algebras given by the rule : $X_i \mapsto x_i$ for $i = 1, \dots, t$. Then $I := \mathrm{Ker}\,\varphi$ is a finitely generated ideal, so we may find $\lambda \in \Lambda$ and an ideal $I_\lambda \subset K_\lambda[X_1, \dots, X_t]$ such that $I = I_\lambda \otimes_{K_\lambda} K$ (details left to the reader). Denote by L_λ the field of fractions of $K_\lambda[T_1, \dots, T_t]/I_\lambda$; it is easily seen that $L_\lambda \otimes_{K_\lambda} K$ is a domain, and its field of fractions is a K-algebra naturally isomorphic to L. Especially, we have $\mathrm{tr.deg}[L_\lambda : K_\lambda] = \mathrm{tr.deg}[L : K]$, and on the other hand, there is a natural isomorphism :

$$\mathbb{L}_{L_\lambda/K_\lambda} \otimes_{L_\lambda} L \xrightarrow{\sim} \mathbb{L}_{L/K} \qquad \text{in } \mathsf{D}^-(L\text{-}\mathbf{Mod})$$

([103, Ch.II, Prop.2.2.1, Cor.2.3.1.1]). Then the sought identity for the extension $K \subset L$ follows from the same identity for the extension $K_\lambda \subset L_\lambda$. The latter is already known, by the previous case. Likewise, the argument shows that $\dim_L H_1\mathbb{L}_{L/K} < +\infty$. □

9.6.4. Consider a local ring A, with maximal ideal $\mathfrak{m}_A$, and residue field $k_A := A/\mathfrak{m}_A$ of characteristic $p > 0$. From [63, Ch.0, Th.20.5.12(i)] we get a complex of k_A-vector spaces

$$(9.6.5) \qquad 0 \to \mathfrak{m}_A/(\mathfrak{m}_A^2 + pA) \xrightarrow{d_A} \Omega^1_{A/\mathbb{Z}} \otimes_A k_A \to \Omega^1_{k_A/\mathbb{Z}} \to 0.$$

Lemma 9.6.6. *The complex* (9.6.5) *is exact.*

Proof. Set $(A_0, \mathfrak{m}_{A_0}) := (A/pA, \mathfrak{m}_A/pA)$ and $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$; the induced sequence of ring homomorphisms $\mathbb{F}_p \to A_0 \to k_A$ yields a distinguished triangle ([103, Ch.II, §2.1.2.1]) :

$$\mathbb{L}_{A_0/\mathbb{F}_p} \overset{\mathbf{L}}{\otimes}_{A_0} k_A \to \mathbb{L}_{k_A/\mathbb{F}_p} \to \mathbb{L}_{k_A/A_0} \to \mathbb{L}_{A_0/\mathbb{F}_p} \overset{\mathbf{L}}{\otimes}_{A_0} k_A[1].$$

However, we have $H_i\mathbb{L}_{k_A/\mathbb{F}_p} = 0$ for every $i > 0$, by proposition 9.6.1(ii). We deduce a short exact sequence

$$0 \to H_1\mathbb{L}_{k_A/A_0} \to \Omega^1_{A_0/\mathbb{F}_p} \otimes_{A_0} k_A \to \Omega^1_{k_A/\mathbb{F}_p} \to 0$$

which, under the natural identifications

$$\Omega^1_{k_A/\mathbb{Z}} \xrightarrow{\sim} \Omega^1_{k_A/\mathbb{F}_p} \qquad \Omega^1_{A/\mathbb{Z}} \otimes_A k_A \xrightarrow{\sim} \Omega^1_{A_0/\mathbb{F}_p} \otimes_{A_0} k_A \qquad H_1\mathbb{L}_{k_A/A_0} \xrightarrow{\sim} \mathfrak{m}_{A_0}/\mathfrak{m}^2_{A_0}$$

([103, Ch.III, Cor.1.2.8.1]) becomes (9.6.5), up to replacing d_A by $-d_A$ ([103, Ch.III, Prop.1.2.9]; details left to the reader). ☐

9.6.7. Keep the situation of (9.6.4), and define

$$A_2 := A \times_{k_A} W_2(k_A)$$

where $W_2(k_A)$ is the ring of 2-truncated Witt vectors of k_A, as in (9.3.25), which is augmented over k_A, via the ghost component map $\overline{\omega}_0 : W_2(k_A) \to k_A$. Notice that

$$V_1(k_A) := \mathrm{Ker}\,\overline{\omega}_0$$

is an ideal of $W_2(k_A)$ such that $V_1(k_A)^2 = 0$; especially, it is naturally a k_A-vector space, with addition and scalar multiplication given respectively by the rules :

$$(0, a) + (0, b) := (0, a + b) \qquad x \cdot (0, a) := (x, 0) \cdot (0, a) = (0, x^p \cdot a)$$

for every $a, b, x \in k_A$. In other words, the map

$$(9.6.8) \qquad V_1(k_A) \to k_A^{1/p} \qquad (0, a) \mapsto a^{1/p}$$

is an isomorphism of k_A-vector spaces, and we also see that $V_1(k_A)$ is a one-dimensional $k_A^{1/p}$-vector space, with scalar multiplication given by the rule : $a\cdot(0, b) := (0, a^p b)$ for every $a \in k_A^{1/p}$ and $b \in k_A$. By construction, we have a natural exact sequence of A_2-modules :

$$(9.6.9) \qquad 0 \to V_1(k_A) \to A_2 \xrightarrow{\pi} A \to 0.$$

Especially, A_2 is a local ring, and π is a local ring homomorphism inducing an isomorphism on residue fields.

9.6.10. It is easily seen that the rule $(A, \mathfrak{m}_A) \mapsto A_2$ defines a functor on the category p-**Local** of local rings with residue field of characteristic $p > 0$, and local ring homomorphisms, to the category of (commutative, unital) rings. Hence, let us set

$$(A, \mathfrak{m}_A) \mapsto \overline{\Omega}_A := \Omega^1_{A_2/\mathbb{Z}} \otimes_{A_2} A.$$

It follows that the rule $(A, \mathfrak{m}_A) \mapsto (A, \overline{\Omega}_A)$ yields a functor

$$p\text{-}\mathbf{Local} \to \mathsf{Alg.Mod}$$

to the category whose objects are the pairs (B, M) where B is a ring, and M is a B-module (the morphisms $(B, M) \to (B', M')$ are the pairs (φ, ψ) where $\varphi : B \to B'$ is a ring homomorphism, and $\psi : B' \otimes_B M \to M$ is a B'-linear map : cp. [75, Def.2.5.22(ii)]).

In view of (9.6.9) and [63, Ch.0, Th.20.5.12(i)] we get an exact sequence of A-modules

$$V_1(k_A) \to \overline{\Omega}_A \to \Omega^1_{A/\mathbb{Z}} \to 0$$

whence, after tensoring with k_A, a sequence of k_A-vector spaces

$$0 \to V_1(k_A) \xrightarrow{j_A} \overline{\Omega}_A \otimes_A k_A \xrightarrow{\rho} \Omega^1_{A/\mathbb{Z}} \otimes_A k_A \to 0 \qquad (9.6.11)$$

which is right exact by construction.

Proposition 9.6.12. *With the notation of* (9.6.10)*, we have :*

(i) *If* $p \notin \mathfrak{m}_A^2$*, then the sequence* (9.6.11) *is exact.*

(ii) *If* $p \in \mathfrak{m}_A^2$*, then* $\mathrm{Ker}\, j_A = \{(0, a^p) \mid a \in k_A\}$*. Especially, the isomorphism* (9.6.8) *identifies* $\mathrm{Ker}\, j_A$ *with the subfield* k_A *of* $k_A^{1/p}$*.*

Proof. In view of lemma 9.6.6, the kernel of ρ is naturally identified with the kernel of the natural map

$$\mathfrak{m}_{A_2}/(\mathfrak{m}_{A_2}^2 + pA_2) \to \mathfrak{m}_A/(\mathfrak{m}_A^2 + pA).$$

On the other hand, it is easily seen that $\mathfrak{m}_{A_2} = \mathfrak{m}_A \oplus V_1(k_A)$, and under this identification, the multiplication law of A_2 restricts on $\mathfrak{m}_{A_2}$ to the mapping given by the rule : $(a, b) \cdot (a', b') := (aa', 0)$ for every $a, a' \in \mathfrak{m}_A$ and $b, b' \in V_1(k_A)$. There follows a natural isomorphism

$$\mathfrak{m}_{A_2}/\mathfrak{m}_{A_2}^2 \xrightarrow{\sim} (\mathfrak{m}_A/\mathfrak{m}_A^2) \oplus V_1(k_A) \qquad (9.6.13)$$

which identifies the map $\mathfrak{m}_{A_2}/\mathfrak{m}_{A_2}^2 \to \mathfrak{m}_A/\mathfrak{m}_A^2$ deduced from π (notation of (9.6.9)), with the projection on the first factor. Moreover, by inspecting the definitions, we easily get a commutative diagram

$$\begin{array}{ccc} V_1(k_A) & \xrightarrow{j_A} & \overline{\Omega}_A \otimes_A k_A \\ \downarrow & & \uparrow{\scriptstyle d_{A_2}} \\ \mathfrak{m}_{A_2}/\mathfrak{m}_{A_2}^2 & \longrightarrow & \mathfrak{m}_{A_2}/(\mathfrak{m}_{A_2}^2 + pA_2) \end{array}$$

whose bottom horizotal arrow is the quotient map, and whose left vertical arrow is the inclusion map of the direct summand $V_1(k_A)$ resulting from (9.6.13).

The map $A_2 \to \mathfrak{m}_{A_2}/\mathfrak{m}_{A_2}^2$ given by the rule : $a \mapsto pa \pmod{\mathfrak{m}_{A_2}^2}$ factors (uniquely) through a k_A-linear map

$$t_{A_2} : k_A \to \mathfrak{m}_{A_2}/\mathfrak{m}_{A_2}^2 \qquad a \pmod{\mathfrak{m}_A} \mapsto pa \pmod{\mathfrak{m}_{A_2}^2}$$

and likewise we may define a map $t_A : k_A \to \mathfrak{m}_A/\mathfrak{m}_A^2$. The snake lemma then gives an induced map $\partial : \mathrm{Ker}\, t_A \to V_1(k_A)$, and in view of the foregoing, the proposition follows from :

Claim 9.6.14. (i) If $p \notin \mathfrak{m}_A^2$, then t_A is injective.

(ii) If $p \in \mathfrak{m}_A^2$, then $\mathrm{Im}\, \partial = \{(0, a^p) \mid a \in k_A\}$.

Proof of the claim. (i) is obvious.

(ii): By virtue of (9.3.23), we have $p = (p, (0, 1))$ in A_2, and if $(a, y) \in A_2$ is any element, then $p \cdot (a, y) = (pa, (0, a^p))$, so in case $p \in \mathfrak{m}_A^2$, the map t_{A_2} is given by the rule :

$$a \mapsto (pa, (0, \overline{a}^p)) = (0, (0, \overline{a}^p)) \in (\mathfrak{m}_A/\mathfrak{m}_A^2) \oplus V_1(k_A) \qquad \text{for every } a \in A.$$

By the same token, in this case t_A is the zero map. The claim follows straightforwardly. □

9.6.15. Keep the notation of (9.6.10), and assume that $p \notin \mathfrak{m}_A^2$, so (9.6.11) is a k_A-extension of $\Omega^1_{A/\mathbb{Z}} \otimes_A k_A$ by $V_1(k_A)$, by virtue of proposition 9.6.12; hence, (9.6.11) $\otimes_{k_A} k_A^{1/p}$ is a $k_A^{1/p}$-extension of the corresponding $k_A^{1/p}$-vector spaces. Recall now that $V_1(k_A)$ is naturally a $k_A^{1/p}$-vector space of dimension one, and let $\psi_A : V_1(k_A) \otimes_{k_A} k_A^{1/p} \to V_1(k_A)$ be the scalar multiplication. By push out along ψ_A, we obtain therefore an extension $\psi_A *$ (9.6.11) $\otimes_{k_A} k_A^{1/p}$ fitting into a commutative ladder with exact rows :

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V_1(k_A) \otimes_{k_A} k_A^{1/p} & \longrightarrow & \overline{\Omega}_A \otimes_A k_A^{1/p} & \longrightarrow & \Omega^1_{A/\mathbb{Z}} \otimes_A k_A^{1/p} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \psi_A} & & \downarrow{\scriptstyle \boldsymbol{\psi}_A} & & \| & & \\ 0 & \longrightarrow & V_1(k_A) & \xrightarrow{\mathbf{j}_A} & \boldsymbol{\Omega}_A & \longrightarrow & \Omega^1_{A/\mathbb{Z}} \otimes_A k_A^{1/p} & \longrightarrow & 0 \end{array}$$

(see [75, §2.5.5]). We consider now the mapping :

$$\boldsymbol{d}_A : A \to \boldsymbol{\Omega}_A \qquad a \mapsto \boldsymbol{\psi}_A(d(a, \tau_{k_A}(\overline{a})) \otimes 1) \qquad \text{for every } a \in A$$

where :

- $\overline{a} \in k_A$ is the image of a in k_A
- τ_{k_A} is the Teichmüller mapping (see (9.3.34)), so $\tau_{k_A}(\overline{a}) = (\overline{a}, 0) \in W_2(k_A)$
- $d : A_2 \to \Omega_{A_2/\mathbb{Z}}$ is the universal derivation of A_2.

Since τ_{k_A} is multiplicative, the map $\boldsymbol{d}_A$ satisfies Leibniz's rule, *i.e.* we have

$$\boldsymbol{d}_A(ab) = \overline{a} \cdot \boldsymbol{d}_A(b) + \overline{b} \cdot \boldsymbol{d}_A(a) \qquad \text{for every } a, b \in A.$$

However, $\boldsymbol{d}_A$ is not quite a derivation, since additivity fails for τ_{k_A}, hence also for $\boldsymbol{d}_A$. Instead, from remark 9.3.8(ii), and recalling that $(p-1)! = -1$ in $\mathbb{F}_p$, we get the identity :

$$\begin{aligned} \tau_{k_A}(\overline{a} + \overline{b}) &= \tau_{k_A}(\overline{a}) + \tau_{k_A}(\overline{b}) - \sum_{i=1}^{p-1} \Big(0, \frac{\overline{a}^i}{i!} \cdot \frac{\overline{b}^{p-i}}{(p-i)!}\Big) \\ &= \tau_{k_A}(\overline{a}) + \tau_{k_A}(\overline{b}) - \sum_{i=1}^{p-1} \frac{\overline{a}^{i/p}}{i!} \cdot \frac{\overline{b}^{1-i/p}}{(p-i)!} \cdot p \end{aligned}$$

for every $\overline{a}, \overline{b} \in k_A$. On the other hand, notice that

$$\begin{aligned} \boldsymbol{\psi}_A(d(0, x \cdot p) \otimes 1) &= \mathbf{j}_A \circ \psi_A(x \cdot p \otimes 1) \\ &= x \cdot \mathbf{j}_A \circ \psi_A(p \otimes 1) \\ &= x \cdot \boldsymbol{\psi}_A(d(0,p) \otimes 1) \\ &= x \cdot \boldsymbol{\psi}_A(d(p - (p,0)) \otimes 1) \\ &= -x \cdot \boldsymbol{d}_A(p) \end{aligned}$$

for every $x \in k_A^{1/p}$, whence :

$$\boldsymbol{d}_A(a+b) = \boldsymbol{d}_A(a) + \boldsymbol{d}_A(b) + \sum_{i=1}^{p-1} \frac{\overline{a}^{i/p}}{i!} \cdot \frac{\overline{b}^{1-i/p}}{(p-i)!} \cdot \boldsymbol{d}_A(p) \qquad \text{for every } a, b \in A.$$

9.6.16. Especially, notice we do have $\boldsymbol{d}_A(a+b) = \boldsymbol{d}_A(a) + \boldsymbol{d}_A(b)$ in case either a or b lies in $\mathfrak{m}_A$. Hence, $\boldsymbol{d}_A$ restricts to an additive map $\mathfrak{m}_A \to \boldsymbol{\Omega}_A$, and Leibniz's rule implies that the latter descends to a k_A-linear map

$$\overline{\boldsymbol{d}}_A : \mathfrak{m}_A/\mathfrak{m}_A^2 \to \boldsymbol{\Omega}_A.$$

Proposition 9.6.17. *Let $(A, \mathfrak{m}_A)$ be a local ring with residue field $k_A := A/\mathfrak{m}_A$ of characteristic $p > 0$, and such that $p \notin \mathfrak{m}_A^2$. Then there exists a natural exact sequence of $k_A^{1/p}$-vector spaces :*

$$0 \to \mathfrak{m}_A/\mathfrak{m}_A^2 \otimes_{k_A} k_A^{1/p} \xrightarrow{\overline{\boldsymbol{d}}_A \otimes k_A^{1/p}} \boldsymbol{\Omega}_A \to \Omega^1_{k_A/\mathbb{Z}} \otimes_{k_A} k_A^{1/p} \to 0.$$

Proof. By inspecting the constructions in (9.6.15), we obtain a commutative ladder of k_A-vector spaces, with exact rows :

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \dfrac{pA}{\mathfrak{m}_A^2 \cap pA} \otimes_{k_A} k_A^{1/p} & \longrightarrow & \dfrac{\mathfrak{m}_A}{\mathfrak{m}_A^2} \otimes_{k_A} k_A^{1/p} & \longrightarrow & \dfrac{\mathfrak{m}_A}{\mathfrak{m}_A^2 + pA} \otimes_{k_A} k_A^{1/p} & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle i} & & \Big\downarrow{\scriptstyle \overline{\boldsymbol{d}}_A \otimes_{k_A} k_A^{1/p}} & & \Big\downarrow{\scriptstyle d_A \otimes_{k_A} k_A^{1/p}} & & \\
0 & \longrightarrow & V_1(k_A) & \xrightarrow{\ \mathbf{j}_A\ } & \boldsymbol{\Omega}_A & \longrightarrow & \Omega^1_{A/\mathbb{Z}} \otimes_A k_A^{1/p} & \longrightarrow & 0
\end{array}$$

such that $d_A \otimes_{k_A} k_A^{1/p}$ is injective with cokernel naturally isomorphic to $\Omega^1_{k_A/\mathbb{Z}} \otimes_{k_A} k_A^{1/p}$ (lemma 9.6.6). By the snake lemma, it then suffices to show that i is an isomorphism; but since both the source and target of the latter map are one-dimensional $k_A^{1/p}$-vector spaces, we come down to checking that i is not the zero map. But a simple inspection yields $i(p \otimes 1) = (0, 1)$. $\square$

9.6.18. We wish next to show how the rule $A \mapsto \boldsymbol{\Omega}_A$ extends to a functor

$$p\text{-}\mathbf{Local} \to \mathsf{Alg.Mod} \qquad (A, \mathfrak{m}_A) \mapsto (k_A^{1/p}, \boldsymbol{\Omega}_A).$$

Namely, if $p \notin \mathfrak{m}_A^2$, we let $\boldsymbol{\Omega}_A$ be the $k_A^{1/p}$-module introduced in (9.6.15), and if $p \in \mathfrak{m}_A^2$, we set

$$\boldsymbol{\Omega}_A := \Omega^1_{A/\mathbb{Z}} \otimes_A k_A^{1/p}.$$

To any local ring homomorphism $\varphi : A \to B$ we attach the $k_A^{1/p}$-linear map

$$\boldsymbol{\Omega}_\varphi : \boldsymbol{\Omega}_A \to \boldsymbol{\Omega}_B$$

defined as follows :

• In case $p \notin \mathfrak{m}_B^2$, then obviously $p \notin \mathfrak{m}_A^2$. Denote by $\overline{\varphi} : k_A^{1/p} \to k_B^{1/p}$ the map induced by φ. By (9.6.10), we have an induced A-linear map $\overline{\Omega}_\varphi \otimes_A \overline{\varphi} : \overline{\Omega}_A \otimes_A k_A^{1/p} \to \overline{\Omega}_B \otimes_B k_B^{1/p}$. Also, the naturality of the ghost component map $\overline{\omega}_0$ yields a k_A-linear map $V_1(\varphi) : V_1(k_A) \to V_1(k_B)$, and indeed, $V_1(\varphi)$ is even $k_A^{1/p}$-linear. Moreover, for every $a \in k_A$ and $b \in k_A^{1/p}$ we may compute

$$\begin{aligned}
\boldsymbol{\psi}_B(\overline{\Omega}_\varphi(j_A(0,a)) \otimes \overline{\varphi}(b)) &= \boldsymbol{\psi}_B(j_B(V_1(\varphi)(0,a)) \otimes \overline{\varphi}(b)) \\
&= \mathbf{j}_B \circ \psi_B(V_1(\varphi)(0,a) \otimes \overline{\varphi}(b)) \\
&= \mathbf{j}_B \circ V_1(\varphi) \circ \psi_A((0,a) \otimes b)
\end{aligned}$$

so the pair $(\boldsymbol{\psi}_B \circ (\overline{\Omega}_\varphi \otimes \overline{\varphi}), \mathbf{j}_B \circ V_1(\varphi))$ determines a unique $k_A^{1/p}$-linear map $\boldsymbol{\Omega}_\varphi : \boldsymbol{\Omega}_A \to \boldsymbol{\Omega}_B$ as required.

• In case $p \in \mathfrak{m}_A^2$, then $p \in \mathfrak{m}_B^2$, and we set $\boldsymbol{\Omega}_\varphi := \Omega^1_\varphi \otimes_A \overline{\varphi}$, where $\Omega^1_\varphi : \Omega^1_{A/\mathbb{Z}} \to \Omega^1_{B/\mathbb{Z}}$ is the natural A-linear map.

• Lastly, if $p \notin \mathfrak{m}_A^2$ but $p \in \mathfrak{m}_B^2$, we let $\boldsymbol{\Omega}_\varphi$ be the composition of the natural projection $\boldsymbol{\Omega}_A \to \Omega^1_A \otimes_A k_A^{1/p}$ and $\Omega^1_\varphi \otimes_A \overline{\varphi}$.

If $\varphi' : B \to C$ is another morphism in p-**Local**, it is easy to verify that $\boldsymbol{\Omega}_{\varphi' \circ \varphi} = \boldsymbol{\Omega}_{\varphi'} \circ \boldsymbol{\Omega}_\varphi$, so we do get a functor as sought (details left to the reader).

9.6.19. In the same vein, we may extend the map $\boldsymbol{d}_A$ to a natural transformation of functors. Namely, if $p \notin \mathfrak{m}_A^2$, we let $\boldsymbol{d}_A$ be the map introduced in (9.6.15), and if $p \in \mathfrak{m}_A^2$, we let $\boldsymbol{d}_A : A \to \boldsymbol{\Omega}_A$ be the map induced by the universal derivation $A \to \Omega^1_{A/\mathbb{Z}}$. Then it is easily seen that every local ring homomorphism $\varphi : A \to B$ induces a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\boldsymbol{d}_A} & \boldsymbol{\Omega}_A \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\boldsymbol{\Omega}_\varphi} \\ B & \xrightarrow{\boldsymbol{d}_B} & \boldsymbol{\Omega}_B. \end{array}$$

Proposition 9.6.20. *Let $(A, \mathfrak{m}_A)$, $(B, \mathfrak{m}_B)$ be two local rings with residue characteristic $p > 0$, and $\varphi : A \to B$ a local ring homomorphism. We have :*

(i) *Suppose that φ is formally smooth for the $\mathfrak{m}_A$-adic topology on A and the $\mathfrak{m}_B$-adic topology on B. Then φ and $\boldsymbol{\Omega}_\varphi$ induce injective maps*

$$(\mathfrak{m}_A/\mathfrak{m}_A^2) \otimes_{k_A} k_B \to \mathfrak{m}_B/\mathfrak{m}_B^2 \qquad \boldsymbol{\Omega}_A \otimes_{k_A^{1/p}} k_B^{1/p} \to \boldsymbol{\Omega}_B.$$

(ii) *Suppose that*
   (a) $\mathfrak{m}_A \cdot B = \mathfrak{m}_B$.
   (b) *The induced residue field extension $k_A \to k_B$ is algebraic and separable.*
   (c) *φ is flat.*

Then $\boldsymbol{\Omega}_\varphi$ induces an isomorphism of $k_B^{1/p}$-vector spaces :

$$\boldsymbol{\Omega}_A \otimes_A B \xrightarrow{\sim} \boldsymbol{\Omega}_B.$$

(iii) *If $B = A/\mathfrak{m}_A^2$ and φ is the natural projection, then $\boldsymbol{\Omega}_\varphi$ is an isomorphism.*
(iv) *The functor $\boldsymbol{\Omega}_\bullet$ and the natural transformation $\boldsymbol{d}_\bullet$ commute with all filtered colimits.*

Proof. (ii): Assumption (a) and the flatness of φ imply that the induced map

$$(\mathfrak{m}_A/\mathfrak{m}_A^2) \otimes_{k_A} k_B \to \mathfrak{m}_B/\mathfrak{m}_B^2$$

is an isomorphism ([126, Th.22.3]), so $p \in \mathfrak{m}_A^2$ if and only if $p \in \mathfrak{m}_B^2$. On the other hand, under assumption (b), we have

$$k_B^{1/p} = k_A^{1/p} \cdot k_B. \tag{9.6.21}$$

Now, consider first the case where $p \in \mathfrak{m}_A^2$; then $\boldsymbol{\Omega}_\varphi = \Omega^1_\varphi \otimes_A \overline{\varphi}$ (notation of (9.6.18)), so the assertion follows from (9.6.21) together with the following

Claim 9.6.22. If $p \in \mathfrak{m}_A^2$, then Ω^1_φ induces a k_B-linear isomorphism

$$\Omega^1_{A/\mathbb{Z}} \otimes_A k_B \to \Omega^1_{B/\mathbb{Z}} \otimes_B k_B.$$

Proof of the claim. Under the standing assumptions, (9.6.4) gives a commutative ladder with short exact rows

$$\begin{array}{ccccccccc} 0 & \to & (\mathfrak{m}_A/\mathfrak{m}_A^2) \otimes_{k_A} k_B & \to & \Omega^1_{A/\mathbb{Z}} \otimes_A k_B & \to & \Omega^1_{k_A/\mathbb{Z}} \otimes_{k_A} k_B & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathfrak{m}_B/\mathfrak{m}_B^2 & \to & \Omega^1_{B/\mathbb{Z}} \otimes_B k_B & \to & \Omega^1_{k_B/\mathbb{Z}} & \to & 0 \end{array} \tag{9.6.23}$$

and we have already remarked that the left vertical arrow is an isomorphism; the same holds for the right vertical arrow, due to assumption (b). The claim follows. ◊

The case where $p \notin \mathfrak{m}_B^2$ is treated likewise : one argue as in the proof of claim 9.6.22, except that, instead of appealing to (9.6.5), one invokes proposition 9.6.17 and the naturality of $\overline{\boldsymbol{d}}_\bullet$ established in (9.6.19) : the details shall be left to the reader.

(iii): For the case where $p \in \mathfrak{m}_A^2$, we consider again the induced ladder (9.6.23) : under the standing assumptions, clearly the first and third vertical arrows are isomorphisms, so the same holds for the middle one.

For the case where $p \notin \mathfrak{m}_A^2$, one argues again likewise : the ladder (9.6.23) is replaced by the corresponding ladder for $\boldsymbol{\Omega}_\bullet$, provided by proposition 9.6.17 and the naturality of $\overline{\boldsymbol{d}}_\bullet$: the details shall be left to the reader.

(iv): Let $((A_\lambda, \mathfrak{m}_\lambda) \mid \lambda \in \Lambda)$ be a filtered system of local rings and local ring homomorphisms. Notice that if $p \in \mathfrak{m}_\lambda^2$ for some $\lambda \in \Lambda$, then $p \in \mathfrak{m}_\mu^2$ for every $\mu \geq \lambda$ in Λ, and since Λ is filtered, we may then replace Λ with Λ/λ, and assume that $p \in \mathfrak{m}_\mu$ for every $\mu \in \Lambda$. In this case, the contention follows straightforwardly from the corresponding assertion for the functor $\Omega^1_\bullet$ and the universal derivation $d_\bullet$.

It remains to consider the case where $p \notin \mathfrak{m}_\lambda$ for every $\lambda \in \Lambda$. For this, one applies proposition 9.6.17, which reduces again the contention to the previous case (details left to the reader).

(i): We consider first the following special case :

Claim 9.6.24. Let $n \in \mathbb{N}$ be any integer, and $\mathfrak{q} \subset A[T_1, \dots, T_n]$ any prime ideal such that $\mathfrak{q} \cap A = \mathfrak{m}_A$. Set $R := A[T_1, \dots, T_n]_\mathfrak{q}$ and denote by $\mathfrak{m}_R$ and k_R respectively the maximal ideal and the residue field of R. Then :

- (i) $\dim_{k_R} \mathfrak{m}_R/\mathfrak{m}_R^2 = n + \dim_{k_A} \mathfrak{m}_A/\mathfrak{m}_A^2$.
- (ii) The natural map $\gamma : (\mathfrak{m}_A/\mathfrak{m}_A^2) \otimes_{k_A} k_R \to \mathfrak{m}_R/\mathfrak{m}_R^2$ is injective.
- (iii) The natural map $\Omega^1_{A/\mathbb{Z}} \otimes_A R \to \Omega^1_{R/\mathbb{Z}}$ is injective, and its image is a direct summand of $\Omega^1_{R/\mathbb{Z}}$.
- (iv) The natural map $\boldsymbol{\Omega}_A \otimes_{k_A^{1/p}} k_R^{1/p} \to \boldsymbol{\Omega}_R$ is injective.

Proof of the claim. (i): By proposition 9.6.1(i), we have $H_i\mathbb{L}_{k_R/k_A} = 0$ for every $i > 1$; then the transitivity triangle for the cotangent complex relative to the maps $A \to k_A \to k_R$ yields a short exact sequence of k_R-vector spaces :

$$0 \to H_1\mathbb{L}_{k_A/A} \otimes_{k_A} k_R \xrightarrow{\alpha} H_1\mathbb{L}_{k_R/A} \to H_1\mathbb{L}_{k_R/k_A} \to 0$$

([103, Ch.II, §2.1.2.1]). Likewise, since $H_i\mathbb{L}_{R/A} = 0$ for every $i > 0$ ([103, Ch.II, Cor.1.2.6.3]), the sequence of maps $A \to R \to k_R$ yields an exact sequence of k_R-vector spaces :

$$0 \to H_1\mathbb{L}_{k_R/A} \xrightarrow{\beta} H_1\mathbb{L}_{k_R/R} \to \Omega^1_{R/A} \otimes_R k_R \to \Omega_{k_R/A} \to 0.$$

However, we have natural isomorphisms

$$H_1\mathbb{L}_{k_A/A} \xrightarrow{\sim} \mathfrak{m}_A/\mathfrak{m}_A^2 \qquad H_1\mathbb{L}_{k_R/R} \xrightarrow{\sim} \mathfrak{m}_R/\mathfrak{m}_R^2$$

([103, Ch.III, Cor.1.2.8.1]), and clearly $\dim_{k_R} \Omega_{R/A} \otimes_R k_R = n$. Thus :

$$\dim_{k_A} \mathfrak{m}_A/\mathfrak{m}_A^2 = \dim_{k_R} H_1\mathbb{L}_{k_R/A} - \dim_{k_R} H_1\mathbb{L}_{k_R/k_A}$$
$$\dim_{k_R} \mathfrak{m}_R/\mathfrak{m}_R^2 = \dim_{k_R} H_1\mathbb{L}_{k_R/A} + n - \dim_{k_R} \Omega_{k_R/k_A}.$$

Taking into account the identity

$$\dim_{k_R} H_1\mathbb{L}_{k_R/k_A} = \dim_{k_R} \Omega_{k_R/k_A}$$

provided by proposition 9.6.1(iii), the assertion follows.

(ii): Notice that the composition of α and β yields an injective map $(\mathfrak{m}_A/\mathfrak{m}_A^2) \otimes_{k_A} k_R \to \mathfrak{m}_R/\mathfrak{m}_R^2$ so it suffices to check that this composition equals γ. However, let

$$\Sigma \quad : \quad H_1\mathbb{L}_{k_R/R} \xrightarrow{d} H_0 k_R \otimes_R \mathbb{L}_{R/A}$$

be the boundary map of the transitivity triangle

$$k_R \otimes_R \mathbb{L}_{R/A} \to \mathbb{L}_{k_R/A} \to \mathbb{L}_{k_R/R} \to k_R \otimes_R \sigma\mathbb{L}_{R/A} \tag{9.6.25}$$

arising from the sequence $A \to R \to k_R$; we regard Σ as a complex placed in degrees $[-1, 0]$. Then, clearly, the triangle $\tau^{\geq -1}$(9.6.25) is naturally isomorphic in $\mathsf{D}(k_R\text{-}\mathbf{Mod})$ to the triangle

$$k_R \otimes_R H_0\mathbb{L}_{R/A}[0] \to \Sigma \to H_1\mathbb{L}_{k_R/R}[1] \to k_R \otimes_R H_0\mathbb{L}_{R/A}[1]. \tag{9.6.26}$$

According to [103, III.1.2.9.1], the complex Σ is naturally isomorphic to the complex

$$\Theta \quad : \quad \mathfrak{m}_R/\mathfrak{m}_R^2 \xrightarrow{-d_{R/A}} k_R \otimes_R \Omega^1_{R/A}$$

(placed in degrees $[-1, 0]$) where $d_{R/A}$ is induced by the universal derivation $R \to \Omega^1_{R/A}$. Likewise, we have natural isomorphisms

$$\tau^{\geq -1}\mathbb{L}_{k_R/R} \xrightarrow{\sim} \mathfrak{m}_R/\mathfrak{m}_R^2[1] \qquad \tau^{\geq -1}\mathbb{L}_{R/A} \xrightarrow{\sim} \Omega^1_{R/A}[0] \tag{9.6.27}$$

and under these identifications, 9.6.26 is the obvious triangle deduced from Θ. Especially, the map β is naturally identified with the inclusion map $\mathrm{Ker}\, d_{R/A} \to \mathfrak{m}_R/\mathfrak{m}_R^2$.

Likewise, there exists a natural identification

$$\tau^{\geq -1}\mathbb{L}_{k_A/A} \xrightarrow{\sim} \mathfrak{m}_A/\mathfrak{m}_A^2[1] \qquad \text{in } \mathsf{D}(k_A\text{-}\mathbf{Mod}) \tag{9.6.28}$$

as well as a natural map of complexes

$$k_R \otimes_{k_A} (\mathfrak{m}_A/\mathfrak{m}_A^2)[1] \to \Theta \tag{9.6.29}$$

deduced from γ. To conclude, it suffices to check that the map

$$\tau^{\geq -1}k_R \otimes_A \mathbb{L}_{k_A/A} \to \tau^{\geq -1}\mathbb{L}_{k_R/A} \tag{9.6.30}$$

coming from the transitivity triangle for the sequence $A \to k_A \to k_R$, corresponds to the morphism (9.6.29), under the identification (9.6.28) and the previous identification of $\tau^{\geq -1}\mathbb{L}_{k_R/A}$ with Θ. To this aim, it suffices to compare the maps obtained by applying to these two morphisms the functor $\mathrm{Ext}^1_{k_R}(-, M[0])$, for arbitrary k_R-modules M. Now, recall that there exists a natural isomorphism

$$\mathrm{Ext}^1_{k_R}(\tau^{\geq -1}\mathbb{L}_{k_R/R}, M[0]) \xrightarrow{\sim} \mathrm{Exal}_R(k_R, M) \qquad \text{for every } k_R\text{-module } M.$$

As explained in [103, III.1.2.8], under the identification (9.6.27), this becomes the following k_R-linear isomorphism

$$\mathrm{Hom}_{k_R}(\mathfrak{m}_R/\mathfrak{m}_R^2, M) \xrightarrow{\sim} \mathrm{Exal}_R(k_R, M) \qquad \varphi \mapsto \varphi * U \tag{9.6.31}$$

(notation of [75, §2.5.5]), where

$$U \quad : \quad 0 \to \mathfrak{m}_R/\mathfrak{m}_R^2 \to R/\mathfrak{m}_R^2 \to k_R \to 0$$

is the natural extension. There is a natural surjection

$$\mathrm{Hom}_{k_R}(\mathfrak{m}_R/\mathfrak{m}_R^2, M) \to \mathrm{Ext}^1_{k_R}(\Theta, M[0])$$

and the foregoing implies that the natural isomorphism

$$\mathrm{Ext}^1_{k_R}(\tau^{\geq -1}\mathbb{L}_{k_R/A}, M[0]) \xrightarrow{\sim} \mathrm{Exal}_A(k_R, M)$$

is also realized as in (9.6.31) : given a class c in $\mathrm{Ext}^1_{k_R}(\Theta, M|0])$, take an arbitrary representative $\varphi : \mathfrak{m}_R/\mathfrak{m}_R^2 \to M$, and the correspondence associates with c the class of the push out $\varphi * U$.

By the same token, we have a natural isomorphism

$$\mathrm{Ext}^1_{k_R}(\tau^{\geq -1}k_R \otimes_{k_A} \mathbb{L}_{k_A/A}, M(0]) \xrightarrow{\sim} \mathrm{Exal}_A(k_A, M) \qquad \text{for every } k_R\text{-module } M$$

and on the one hand, the map $\mathrm{Ext}^1_{k_R}((9.6.30), M[0])$ is identified naturally with the map

$$\mathrm{Exal}_A(k_R, M) \to \mathrm{Exal}_A(k_A, M)$$

given by pull back along the inclusion map $\iota : k_A \to k_R$. On the other hand, by [103, III.1.2.8], the identification (9.6.27) induces the isomorphism

$$\mathrm{Hom}_{k_A}(\mathfrak{m}_A/\mathfrak{m}_A^2, M) \stackrel{\sim}{\to} \mathrm{Exal}_A(k_A, M) \qquad \varphi' \mapsto \varphi' * U'$$

where

$$U' \quad : \quad 0 \to \mathfrak{m}_A/\mathfrak{m}_A^2 \to A/\mathfrak{m}_A^2 \to k_A \to 0$$

is the natural extension. So finally, the assertion boils down to the identity

$$(\varphi \circ \gamma) * U' = \varphi * U * \iota \qquad \text{in } \mathrm{Exal}_A(k_A, M)$$

for every k_R-module M and every $\varphi : \mathfrak{m}_R/\mathfrak{m}_R^2 \to M$. We leave the verification as an exercise for the reader.

(iii) is a standard calculation (more precisely, the complement of $\Omega^1_{A/\mathbb{Z}} \otimes_A R$ in $\Omega^1_{R/\mathbb{Z}}$ is the free R-module generated by $dT_1, \dots, dT_n$).

(iv): If $p \in \mathfrak{m}_A^2$, the assertion follows from (iii). Suppose then that $p \notin \mathfrak{m}_A^2$. By inspecting the constructions, we get a natural commutative ladder of $k_R^{1/p}$-vector spaces

$$\begin{array}{ccccccccc}
0 & \longrightarrow & V_1(k_A) \otimes_{k_A^{1/p}} k_R^{1/p} & \longrightarrow & \boldsymbol{\Omega}_A \otimes_{k_A^{1/p}} k_R^{1/p} & \longrightarrow & \Omega^1_{A/\mathbb{Z}} \otimes_A k_R^{1/p} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V_1(k_R) & \longrightarrow & \boldsymbol{\Omega}_R & \longrightarrow & \Omega^1_{R/\mathbb{Z}} \otimes_R k_R^{1/p} & \longrightarrow & 0
\end{array}$$

whose central vertical arrow is the map of the claim. However, it is easily seen that the left vertical arrow is an isomorphism, so the assertion follows from (iii). ◊

Now, pick a free polynomial A-algebra C and a surjective map $C \to B$ of A-algebras; let $\mathfrak{q} \subset C$ be the preimage of $\mathfrak{m}_B$, set $R := C_\mathfrak{q}$, and let $\mathfrak{m}_R$ be the maximal ideal of R. Since B is formally smooth over A for the topologies of the maximal ideals, the induced surjection $R/\mathfrak{m}_R^2 \to B/\mathfrak{m}_B^2$ admits a section $B/\mathfrak{m}_B^2 \to R/\mathfrak{m}_R^2$ which is also a local map of A-algebras. In view of (iii), there follow commutative diagrams of k_B-linear and respectively $k_B^{1/p}$-linear maps

$$\begin{array}{ccc}
(\mathfrak{m}_A/\mathfrak{m}_A^2) \otimes_{k_A} k_B & \longrightarrow & \mathfrak{m}_B/\mathfrak{m}_B^2 \\
 & \searrow \quad \swarrow & \\
 & \mathfrak{m}_R/\mathfrak{m}_R^2 &
\end{array}
\qquad
\begin{array}{ccc}
\boldsymbol{\Omega}_A \otimes_{k_A} k_B^{1/p} & \longrightarrow & \boldsymbol{\Omega}_B \\
 & \searrow \quad \swarrow & \\
 & \boldsymbol{\Omega}_R &
\end{array}$$

which reduce the assertion to the case where $B = R$. In this case, write C as the filtered colimit of the system $(C_\lambda \mid \lambda \in \Lambda)$ of its free polynomial A-subalgebras of finite type; for each $\lambda \in \Lambda$, let $\mathfrak{q}_\lambda := \mathfrak{q} \cap C_\lambda$ and $R_\lambda := (C_\lambda)_{\mathfrak{q}_\lambda}$. Clearly R is the filtered colimit of the system $(R_\lambda \mid \lambda \in \Lambda)$, so the assertion follows from (iv) and claim 9.6.24(ii). □

9.6.32. Let $(A, \mathfrak{m}_A)$ be any object of p-**Local**, and $k_A := A/\mathfrak{m}_A$. Consider a finite sequence $f_1, \dots, f_n$ of elements of A, and $e_1, \dots, e_n \in \mathbb{N}$ with $e_i > 1$ for every $i = 1, \dots, n$. Set

$$C := A[T_1, \dots, T_n]/(T_1^{e_1} - f_1, \dots, T_n^{e_n} - f_n).$$

Fix a prime ideal $\mathfrak{n} \subset C$ such that $\mathfrak{n} \cap A = \mathfrak{m}_A$, and let $B := C_\mathfrak{n}$. So the induced map $A \to B$ is a local ring homomorphism; we denote by $\mathfrak{m}_B$ the maximal ideal of B, and set $k_B := B/\mathfrak{m}_B$. Also, let $\nu := \dim_{k_A} E$, where $E \subset \boldsymbol{\Omega}_A$ is the $k_A^{1/p}$-vector space spanned by $\boldsymbol{d}_A f_1, \dots, \boldsymbol{d}_A f_n$.

Theorem 9.6.33. *In the situation of* (9.6.32)*, suppose moreover that :*

(a) $f_i \in \mathfrak{m}_A$*, for every* $i \leq n$ *such that* p *does not divide* e_i*.*
(b) $\mathfrak{m}_A/\mathfrak{m}_A^2$ *is a finite dimensional* k_A*-vector space.*

Then $\mathfrak{m}_B/\mathfrak{m}_B^2$ *is a finite dimensional* k_B*-vector space, and we have :*

$$\dim_{k_B} \mathfrak{m}_B/\mathfrak{m}_B^2 = n + \dim_{k_A} \mathfrak{m}_A/\mathfrak{m}_A^2 - \nu.$$

Proof. Let $\mathfrak{q} \subset A[T_1, \dots, T_n]$ be the preimage of $\mathfrak{n}$, set $R := A[T_1, \dots, T_n]_\mathfrak{q}$, and denote by $\mathfrak{p}$ the maximal ideal of R, and by $F \subset \mathfrak{p}/\mathfrak{p}^2$ the k_B-vector space spanned by $T_1^{e_1} - f_1, \dots, T_n^{e_n} - f_n$. Clearly, we have a short exact sequence

$$0 \to F \to \mathfrak{p}/\mathfrak{p}^2 \to \mathfrak{m}_B/\mathfrak{m}_B^2 \to 0.$$

Suppose first that $p \notin \mathfrak{m}_A^2$, in which case $p \notin \mathfrak{p}^2$, by claim 9.6.24(ii), hence $\boldsymbol{\Omega}_R$ is defined by (9.6.15). On the other hand, notice that $\boldsymbol{d}(T_i^{e_i}) = \boldsymbol{d}(T_i^{e_i} - f_i) + \boldsymbol{d}(f_i)$ in $\boldsymbol{\Omega}_R$, since $T_i^{e_i} - f_i \in \mathfrak{p}$. Now, if e_i is a multiple of p, Leibniz's rule yields $\boldsymbol{d}(T_i^{e_i}) = e_i \cdot T_i^{e_i - 1}\boldsymbol{d}(T_i) = 0$. If e_i is not a multiple of p, we have $f_i \in \mathfrak{m}_A$ by assumption; hence $T_i \in \mathfrak{p}$ and therefore $\boldsymbol{d}(T_i^{e_i}) = 0$ again, since $e_i > 1$. In either case, we find

$$\boldsymbol{d}(f_i) = -\overline{\boldsymbol{d}}(T_i^{e_i} - f_i) \qquad \text{in } \boldsymbol{\Omega}_R\text{, for every } i = 1, \dots, n.$$

In view of proposition 9.6.17, it follows that

$$\dim_{k_B} \mathfrak{m}_B/\mathfrak{m}_B^2 = \dim_{k_B} \mathfrak{p}/\mathfrak{p}^2 - \dim_{k_B^{1/p}} E'$$

where $E' \subset \boldsymbol{\Omega}_R$ is the $k_B^{1/p}$-vector space spanned by $\boldsymbol{d}(f_1), \dots, \boldsymbol{d}(f_n)$. Then the assertion follows from claim 9.6.24(i,iv).

Lastly, suppose $p \in \mathfrak{m}_A^2$, so that $p \in \mathfrak{p}^2$ as well. Arguing as in the foregoing case, we see that $\dim_{k_B} F$ equals the dimension of the k_B-vector subspace of $\Omega^1_{R/\mathbb{Z}} \otimes_R k_B$ spanned by $df_1, \dots, df_n$, and in view of claim 9.6.24(iii), the latter equals ν, whence the contention. □

Corollary 9.6.34. *In the situation of theorem* 9.6.33*, the following conditions are equivalent :*

(a) A *is a regular local ring, and* $\nu = n$.

(b) B *is a regular local ring.*

Proof. Suppose first that (b) holds. Since the map $A \to B$ is faithfully flat, it is easily seen that A is noetherian, and then [63, Ch.0, Prop.17.3.3(i)] shows already that A is a regular local ring. Moreover, C is clearly a finite A-algebra, therefore $\dim_{k_B} \mathfrak{m}_B/\mathfrak{m}_B^2 = \dim B \leq \dim A = \dim_{k_A} \mathfrak{m}_A/\mathfrak{m}_A^2$. Theorem 9.6.33 then implies that $\nu = n$, so (a) holds.

Next, suppose that (a) holds. We apply theorem 9.6.33 as in the foregoing, to deduce that $\dim B = \dim A = \dim_{k_B} \mathfrak{m}_B/\mathfrak{m}_B^2$, whence (b) : details left to the reader. □

Remark 9.6.35. (i) Keep the notation of corollary 9.6.34. In [63, Ch.0, Th.22.5.4] it is asserted that condition (b) is equivalent to the following :

(a') A is regular and the space $E \subset \Omega^1_{A/\mathbb{Z}} \otimes_A k_A$ spanned by $df_1, \dots, df_n$ has dimension n.

When $p \in \mathfrak{m}_A^2$, this condition (a') agrees with our condition (a), so in this case of course we do have (a')⇔(b). However, the latter equivalence fails in general, in case $p \notin \mathfrak{m}_A^2$: the mistake is found in [63, Ch.0, Rem.22.4.8], which is false. The implication (a')⇒(b) does remain true in all cases : this is easily deduced from theorem 9.6.33, since the image of $\boldsymbol{d}_A(f)$ in $\Omega^1_{A/\mathbb{Z}} \otimes_{k_A} k_A^{1/p}$ agrees with $df \otimes 1$, for every $f \in A$. The proof of *loc.cit.* is correct for $p \in \mathfrak{m}_A^2$, which is the only case that is used in the proof of corollary 9.6.34.

(ii) Moreover, in the situation of corollary 9.6.34, suppose that B is a regular local ring. Then the sequence

$$(\boldsymbol{d}_B f_1^{1/e_1}, \dots, \boldsymbol{d}_B f_n^{1/e_n})$$

spans a $k_B^{1/p}$-vector subspace of $\boldsymbol{\Omega}_B$ of dimension n. Indeed, consider the ring

$$C := B[T_1, \dots, T_n]/(T_1^p - f_1^{1/e_1}, \dots, T_n^p - f_n^{1/e_n}).$$

Corollary 9.6.34 applies to the extension $A \subset C$, the sequence $(f_1, \dots, f_n)$, and the sequence of integers $(pe_1, \dots, pe_n)$, so C is a regular local ring. But the same corollary applies as well to the extension $B \subset C$, the sequence $(f_1^{1/e_1}, \dots, f_n^{1/e_n})$, and the sequence of integers $(p, \dots, p)$, and yields the assertion.

(iii) Let us say that the sequence $(f_1, \dots, f_n)$ is *maximal in* A if

$$(\boldsymbol{d}_A f_1, \dots, \boldsymbol{d}_A f_n)$$

is a basis of the $k_A^{1/p}$-vector space $\boldsymbol{\Omega}_A$. Then, in the situation of (ii), we claim that the sequence $(f_1, \dots, f_n)$ is maximal in A if and only if the sequence $(f_1^{1/e_1}, \dots, f_n^{1/e_n})$ is maximal in B. Indeed, under the current assumptions we have $\dim_{k_A} \Omega^1_{k_A/\mathbb{Z}} = \dim_{k_B} \Omega^1_{k_B/\mathbb{Z}}$, since these integers are equal to the transcendence degree of k_A (and k_B) over $\mathbb{F}_p$, and on the other hand $\dim_{k_A} \mathfrak{m}_A/\mathfrak{m}_A^2 = \dim_{k_B} \mathfrak{m}_B/\mathfrak{m}_B^2$, since A and B are regular local rings of the same dimension; then the assertion follows from (ii), lemma 9.6.6 and proposition 9.6.17 (details left to the reader).

(iv) Likewise, consider any morphism $\varphi : A \to B$ in **Local** fulfilling conditions (a),(b) and (c) of proposition 9.6.20(ii). Then clearly the sequence $(f_1, \dots, f_n)$ is maximal in A if and only if the sequence $(\varphi(f_1), \dots, \varphi(f_n))$ is maximal in B.

9.6.36. Let $p > 0$ be a prime integer, and A an $\mathbb{F}_p$-algebra. Denote by $\Phi_A : A \to A$ the *Frobenius endomorphism* of A, given by the rule : $a \mapsto a^p$ for every $a \in A$. For every A-module M, we let $M_{(\Phi)}$ be the A-module obtained from M via restriction of scalars along the map Φ_A (that is, $a \cdot m := a^p m$ for every $a \in A$ and $m \in M$). Notice that Φ_A is an A-linear map $A \to A_{(\Phi)}$. Theorem 9.6.37, and part (i) of the following theorem 9.7.26 are due to E.Kunz.

Theorem 9.6.37. *Let A be a noetherian local $\mathbb{F}_p$-algebra. Then the following conditions are equivalent :*

(i) *A is regular.*
(ii) *Φ_A is a flat ring homomorphism.*
(iii) *There exists $n > 0$ such that Φ_A^n is a flat ring homomorphism.*

Proof. (i)⇒(ii): Let $A^\wedge$ be the completion of A, and $f : A \to A^\wedge$ the natural map. Clearly

$$f \circ \Phi_A = \Phi_{A^\wedge} \circ f.$$

Since f is faithfully flat, it follows that Φ_A is flat if and only if the same holds for $\Phi_{A^\wedge}$, so we may replace A by $A^\wedge$, and assume from start that A is complete, hence $A = k[[T_1, \dots, T_d]]$, for a field k of characteristic p, and $d = \dim A$ ([63, Ch.0, Th.19.6.4]). Then, it is easily seen that

$$\Phi_A(A) = A^p = k^p[[T_1^p, \dots, T_d^p]].$$

Set $B := k[[T_1^p, \dots, T_d^p]]$; the ring A is a free B-module (of rank p^d), hence it suffices to check that the inclusion map $A^p \to B$ is flat. However, denote by $\mathfrak{m}$ the maximal ideal of A^p; clearly B is an $\mathfrak{m}$-adically ideal-separated A-module (see [126, p.174, Def.]), hence it suffices to check that $B/\mathfrak{m}^k B$ is a flat $A^p/\mathfrak{m}^k$-module for every $k > 0$ ([126, Th.22.3]). The latter is clear, since $k[T_1^p, \dots, T_d^p]$ is a flat $k^p[T_1^p, \dots, T_d^p]$-module.

(ii)⇒(iii) is obvious.

(iii)⇒(i): Notice first that $\operatorname{Spec} \Phi_A^n$ is the identity map on the topological space underlying $\operatorname{Spec} A$; especially, Φ_A^n is flat if and only if it is faithfully flat, and the latter condition implies that Φ_A^n is injective. We easily deduce that if (iii) holds, then A is reduced. Now, consider quite generally, any finite system $x_\bullet := (x_1, \dots, x_t)$ of elements of A, and let $I \subset A$ be the ideal generated by $x_\bullet$; we shall say that $x_\bullet$ is a system of *independent* elements, if I/I^2 is a free A/I-module of rank n. We show first the following :

Claim 9.6.38. Let $y, z, x_2, \dots, x_t$ be a family of elements of A, such that $x_\bullet := (yz, x_2, \dots, x_t)$ is a system of independent elements, and denote by $J \subset A$ the ideal generated by $x_\bullet$. We have :

(i) $(y, x_2, \dots, x_t)$ is a system of independent elements of A.
(ii) If $\mathrm{length}_A A/J$ is finite, then

$$\mathrm{length}_A A/J = \mathrm{length}_A A/(y, x_2, \dots, x_t) + \mathrm{length}_A A/(z, x_2, \dots, x_t).$$

Proof of the claim. (i): Suppose $a_1 y + a_2 x_2 + \dots + a_t x_t = 0$ is a linear relation with $a_1, \dots, a_t \in A$, and let $I \subset A$ be the ideal generated by $y, x_2, \dots, x_t$. We have to show that $a_1, \dots, a_t \in I$. However, as $a_1 yz + a_2 z x_2 + \dots + a_t z x_t = 0$, it follows by assumption, that a_1 lies in $J \subset I$. Write $a_1 = b_1 yz + b_2 x_2 + \dots + b_t x_t$; then

$$b_1 y^2 z + (a_2 + b_2 y) x_2 + \dots + (a_t + b_t y) x_t = 0.$$

Therefore $a_i + b_i y \in J$ for $i = 2, \dots, t$, and therefore $a_2, \dots, a_t \in I$, as required.

(ii): It suffices to show that the natural map of A-modules :

$$A/(z, x_2, \cdots, x_t) \to I/J \qquad a \mapsto ay + J$$

is an isomorphism. However, the surjectivity is immediate. To show the injectivity, suppose that $ay \in J$, *i.e.* $ay = b_1 yz + b_2 x_2 + \dots + b_t x_t$ for some $b_1, \dots, b_t \in A$; we deduce that $(b_1 z - a) yz + b_2 z x_2 + \dots + b_t z x_t = 0$, hence $a - b_1 z \in J$ by assumption, so a lies in the ideal generated by $z, x_2, \dots, x_t$, as required. ◊

Now, set $q := p^n$, and $A_\nu := A^{q^\nu} \subset A$ for every integer $\nu > 0$; pick a minimal system $x_\bullet := (x_1, \dots, x_t)$ of generators of the maximal ideal $\mathfrak{m}_A$ of A, and notice that, since A is reduced, $\Phi_A^{n\nu}$ induces an isomorphism $A \to A_\nu$, hence $x_\bullet^{(\nu)} := (x_1^{q^\nu}, \dots, x_t^{q^\nu})$ is a minimal system of generators for the maximal ideal $\mathfrak{m}_\nu$ of A_ν. Set as well $I_\nu := \mathfrak{m}_\nu A$; since the inclusion map $A_\nu \to A$ is flat by assumption for every $\nu > 0$, we have a natural isomorphism of A-modules

$$(\mathfrak{m}_\nu/\mathfrak{m}_\nu^2) \otimes_{A_\nu} A \xrightarrow{\sim} I_\nu/I_\nu^2.$$

On the other hand, set $k_\nu := A_\nu/\mathfrak{m}_\nu$; by Nakayama's lemma, $\dim_{k_\nu} \mathfrak{m}_\nu/\mathfrak{m}_\nu^2 = t$, so I_ν/I_ν^2 is a free A-module of rank t, *i.e.* $x_\bullet^{(\nu)}$ is an independent system of elements of A. From claim 9.6.38(ii) and a simple induction, we deduce that

$$\mathrm{length}_A A/I_\nu = \mathrm{length}_{A^\wedge} A^\wedge/I_\nu A^\wedge = q^{\nu t} \qquad \text{for every } \nu > 0 \tag{9.6.39}$$

(where the first equality holds, since I_ν is an open ideal in the $\mathfrak{m}_A$-adic topology of A). According to [63, Ch.0, Th.19.9.8] (and its proof), $A^\wedge$ contains a field isomorphic to $k_0 := A/\mathfrak{m}_A$, and the inclusion map $k_0 \to A^\wedge$ extends to a surjective ring homomorphism $k_0[[X_1, \dots, X_t]] \to A^\wedge$, such that $X_i \mapsto x_i$ for $i = 1, \dots, t$. Denote by J the kernel of this surjection; in view of (9.6.39), we have

$$\mathrm{length}_{A^\wedge} k_0[[X_1, \dots, X_t]]/(J, X_1^{q^\nu}, \dots, X_t^{q^\nu}) = q^{\nu t}$$

which means that $J \subset (X_1^{q^\nu}, \dots, X_t^{q^\nu})$ for every $\nu > 0$. We conclude that $J = 0$, and $A^\wedge = k_0[[X_1, \dots, X_t]]$ is regular, so the same holds for A. □

As an application, let us point out the following combinatorial version of Kunz's theorem :

Theorem 9.6.40. *Let P be a monoid and $k > 1$ an integer; suppose that $P^\sharp$ is fine and the Frobenius endomorphism $\boldsymbol{k}_P : P \to P$ is flat (definition* 4.8.40(ii)*). Then $P^\sharp$ is a free monoid.*

Proof. First, we remark that $\boldsymbol{k}_P^\sharp : P^\sharp \to P^\sharp$ is still flat (corollary 6.1.49(i)). Moreover, $\boldsymbol{k}_P^\sharp$ is injective. Indeed, suppose that $x^k = y^k \cdot u$ for some $x, y \in P$ and $u \in P^\times$; from theorem 6.1.42 we deduce that there exist $b_1, b_2, t \in P$ such that

$$b_1 x = b_2 y \qquad 1 = b_1^k t \qquad u = b_2^k t.$$

Especially, $b_1, b_2 \in P^\times$, so the images of x and y agree in $P^\sharp$. Hence, we may replace P by $P^\sharp$, and assume that $\boldsymbol{k}_P$ is flat and injective, in which case $\mathbb{Z}[\boldsymbol{k}_P] : \mathbb{Z}[P] \to \mathbb{Z}[P]$ is flat (theorem 6.2.3), integral and injective, hence it is faithfully flat. Now, let R be the colimit of the system of rings $(R_n \mid n \in \mathbb{N})$, where $R_n := \mathbb{Z}[P]$, and the transition map $R_n \to R_{n+1}$ is $\mathbb{Z}[\boldsymbol{k}_P]$ for every $n \in \mathbb{N}$. The induced map $j : R_0 \to R$ is still faithfully flat; moreover, let p be any prime divisor of k, and notice that $j \circ \boldsymbol{p}_P = j$ (where $\boldsymbol{p}_P$ is the p-Frobenius map). It follows that $\boldsymbol{p}_P$ is flat and injective as well, so $\mathbb{F}_p[\boldsymbol{p}_P] : \mathbb{F}_p[P] \to \mathbb{F}_p[P]$ is a flat ring homomorphism (again, by theorem 6.2.3), and then the same holds for the induced map $\mathbb{F}_p[[\boldsymbol{p}_P]] : \mathbb{F}_p[[P]] \to \mathbb{F}_p[[P]]$. By theorem 9.6.37, we deduce that $\mathbb{F}_p[[P]]$ is a regular local ring, with maximal ideal $\mathfrak{m} := \mathbb{F}_p[[\mathfrak{m}_P]]$; notice that the images of the elements of $\mathfrak{m}_P \backslash \mathfrak{m}_P^2$ yield a basis for the $\mathbb{F}_p$-vector space $\mathfrak{m}/\mathfrak{m}^2$. Say that $\mathfrak{m}_P \backslash \mathfrak{m}_P^2 = \{x_1, \dots, x_s\}$; it follows that the continuous ring homomorphism

$$\mathbb{F}_p[[T_1, \dots, T_s]] \to \mathbb{F}_p[[P]] \qquad T_i \mapsto x_i \qquad \text{for } i = 1, \dots, s$$

is an isomorphism. From the discussion of (6.4.25), we immediately deduce that $P \simeq \mathbb{N}^{\oplus s}$, as required. □

Next, we wish to present a characterization of regular local rings via the cotangent complex, borrowed from [3], which shall be used in the following section on excellent rings.

Lemma 9.6.41. *Let A be a ring, $\mathbf{f} := (f_1, \dots, f_n)$ a completely secant sequence of elements of A, that generates an ideal $I \subset A$, and set $A_0 := A/I$. Then there is a natural isomorphism*

$$\mathbb{L}_{A_0/A} \xrightarrow{\sim} I/I^2[1] \qquad \textit{in } \mathsf{D}(A_0\text{-}\mathbf{Mod})$$

and I/I^2 is a free A_0-module of rank n.

Proof. The sequence $\mathbf{f}$ is quasi-regular, by proposition 7.8.15, so the second assertion follows from remark 7.8.16(ii). Next, set $R := \mathbb{Z}[T_1, \dots, T_n]$, and let $u : R \to A$ be the unique ring homomorphism such that $u(T_i) = a_i$ for $i = 1, \dots, n$. We get a cocartesian diagram of rings :

$$\begin{array}{ccc} R & \xrightarrow{u} & A \\ {\scriptstyle p}\downarrow & & \downarrow \\ \mathbb{Z} & \longrightarrow & A_0 \end{array}$$

with p the standard augmentation such that $p(T_1) = \cdots = p(T_n) = 0$; also, $\mathrm{Tor}_i^R(\mathbb{Z}, A_0) = 0$ for every $i > 0$, by lemma 7.8.14(i). By [103, Ch.II, Prop.2.2.1] there follows a natural isomorphism:

$$A_0 \otimes_{\mathbb{Z}} \mathbb{L}_{\mathbb{Z}/R} \xrightarrow{\sim} \mathbb{L}_{A_0/A} \qquad \text{in } \mathsf{D}(A_0\text{-}\mathbf{Mod}).$$

Thus, we are reduced to prove the sought isomorphism for the ring R and the regular sequence $(T_1, \dots, T_n)$, in which case $I = \operatorname{Ker} p$, and $R/I = \mathbb{Z}$. However, by [103, Ch.II, Prop.2.1.2], the sequence of ring homomorphisms $\mathbb{Z} \to R \xrightarrow{p} \mathbb{Z}$ induces a distinguished triangle

$$\mathbb{L}_{\mathbb{Z}/R}[-1] \to \mathbb{L}_{R/\mathbb{Z}} \otimes_R \mathbb{Z} \to \mathbb{L}_{\mathbb{Z}/\mathbb{Z}} \to \mathbb{L}_{\mathbb{Z}/R} \qquad \text{in } \mathsf{D}(\mathbb{Z}\text{-}\mathbf{Mod})$$

and to conclude, it suffices to recall that $\mathbb{L}_{\mathbb{Z}/\mathbb{Z}} \xrightarrow{\sim} 0$, and $\mathbb{L}_{R/\mathbb{Z}} \xrightarrow{\sim} \Omega^1_{R/\mathbb{Z}}[0]$, by virtue of [103, Ch.II, Prop.1.2.4.4]. □

Proposition 9.6.42. *Let $(A, \mathfrak{m})$ be a local noetherian ring, $a \in \mathfrak{m}$, and $A_0 := A/aA$. The following conditions are equivalent :*

(a) *a is a regular element of A.*
(b) *$H_2\mathbb{L}_{A_0/A} = 0$, and aA/a^2A is a free A_0-module of rank one.*

Proof. For any ring R, any non-invertible element $x \in R$, and any $n \in \mathbb{N}$, set $R_n := R/x^{n+1}R$, and consider the R_0-linear map

$$\beta_{x,n} : R_0 \to x^n R/x^{n+1}R \qquad (a \bmod xR) \mapsto (x^n a \bmod x^{n+1}R).$$

Notice that the sequence (x) is R-quasi-regular if and only if $\beta_{x,n}$ is an isomorphism for every $n \in \mathbb{N}$ (see definition 7.8.13). We remark :

Claim 9.6.43. Suppose that R is a noetherian local ring, denote by κ_R the residue field of R, and let $n > 0$ be any given integer. The following conditions are equivalent :

(c) $\beta_{x,n}$ is an isomorphism.
(d) $x^n \neq 0$ and $\mathrm{Tor}_1^{R_0}(x^n R/x^{n+1}R, \kappa_R) = 0$.
(e) $x^n \neq 0$ and the surjection $R_0 \to \kappa_R$ induces a surjective map

$$H_2(\mathbb{L}_{R_{n-1}/R} \otimes_{R_{n-1}} R_0) \to H_2(\mathbb{L}_{R_{n-1}/R} \otimes_{R_{n-1}} \kappa_R).$$

Proof of the claim. It is easily seen that (c)⇒(d).

Conversely, if (d) holds, notice that $x^n R/x^{n+1}R \neq 0$, as $\bigcap_{n\in\mathbb{N}} x^n R = 0$ ([126, Th.8.10(i)]). Hence $\kappa_R \otimes_R \beta_{x,n}$ is an isomorphism of one-dimensional κ_R-vector spaces. On the other hand, under assumption (d), the natural map $\kappa_R \otimes_R \mathrm{Ker}\,\beta_{x,n} \to \mathrm{Ker}(\kappa_R \otimes_R \beta_{x,n})$ is an isomorphism. By Nakayama's lemma, we conclude that $\mathrm{Ker}\,\beta_{x,n} = 0$, *i.e.* (c) holds.

Next, recall that $H_0(\mathbb{L}_{R_{n-1}/R}) = 0$ for every $n \in \mathbb{N}$ ([103, Ch.II, Prop.1.2.4.2]); then the Künneth spectral sequence ([163, Th.5.6.4]) yields natural isomorphisms of R_0-modules

$$H_1(\mathbb{L}_{R_{n-1}/R} \otimes_{R_{n-1}} M) \xrightarrow{\sim} x^n R/x^{2n}R \otimes_{R_{n-1}} M \xrightarrow{\sim} x^n R/x^{n+1}R \otimes_{R_0} M$$

for every R_0-module M ([103, Ch.III, Cor.1.2.8.1]). Denote by $\mathfrak{m}_0 \subset R_0$ the maximal ideal; there follows a left exact sequence

$$0 \to \mathrm{Tor}_1^{R_0}(x^n R/x^{n+1}R, \kappa_R) \to H_1(\mathbb{L}_{R_{n-1}/R} \otimes_{R_{n-1}} \mathfrak{m}_0) \to H_1(\mathbb{L}_{R_{n-1}/R} \otimes_{R_{n-1}} R_0)$$

which shows that (d)⇔(e) (details left to the reader). ◊

Claim 9.6.44. Let $f : R \to R'$ be a ring homomorphism, and $x' := f(x)$. Suppose that $\beta_{x,n}$ and $\beta_{x',n}$ are bijections, for some $n \in \mathbb{N}$, and set $R'_n := R'/x'^{n+1}R'$. Then the induced morphism

$$\mathbb{L}_{R_0/R_n} \otimes_{R_0} R'_0 \to \mathbb{L}_{R'_0/R'_n}$$

is an isomorphism in $\mathsf{D}(R'_0\text{-}\mathbf{Mod})$.

Proof of the claim. If $n = 0$, there is nothing to prove, hence assume that $n > 0$. We remark that, for every $i = 0, \dots, n$, we have an exact complex

$$R_n \xrightarrow{x^{i+1}} R_n \xrightarrow{x^{n-i}} R_n.$$

Indeed, for $i = 0$, this results immediately from the assumption that $\mathrm{Ker}\,\beta_{n,x} = 0$. Suppose that $i > 0$, and that the assertion is known for $i - 1$; then, if $yx^{n-i} = x^{n+1}z$ for some $y, z \in R$, the inductive hypothesis implies that $y = x^i u$ for some $u \in R$. It follows that $x^n(u - zx) = 0$, and then $u - zx \in xR$, again since $\mathrm{Ker}\,\beta_{x,n} = 0$; thus, $u \in xR$, and $y \in x^{i+1}R$, as asserted.

We deduce that the R_n-module R_0 admits a free resolution

$$\Sigma \quad : \quad \cdots \to R_n \xrightarrow{x} R_n \xrightarrow{x^n} R_n \xrightarrow{x} R_n \to R_0.$$

The same argument applies to R' and its element x', and yields a corresponding free resolution Σ' of the R'_n-module R'_0. Clearly $\Sigma' \otimes_{R_n} R'_n = \Sigma$, *i.e.* the natural morphism $R_0 \overset{\mathbf{L}}{\otimes}_{R_n} R'_n \to R'_0$ is an isomorphism in $\mathsf{D}(R'_0\text{-}\mathbf{Mod})$. The claim then follows from [103, Ch.II, Prop.2.2.1]. ◊

With these preliminaries, we may now return to the situation of the proposition : first, lemma 9.6.41 says that (a)⇒(b). For the converse, in light of proposition 7.8.15, it suffices to show that $\beta_{a,n} : A_0 \to a^n A/a^{n+1}A$ is bijective for every $n \in \mathbb{N}$. We shall argue by induction on n.

For $n = 0$, there is nothing to prove. Assume that $n > 0$, and that the assertion is known for $n - 1$. Let $\mathfrak{n} \subset A[T]$ be the (unique) maximal ideal containing T, and set $B := A[T]_{\mathfrak{n}}$. We let $f : B \to A$ be the map of A-algebras given by the rule : $T \mapsto a$. Define as usual $B_n := B/T^{n+1}B$ and $A_n := A/a^{n+1}A$ for every $n \in \mathbb{N}$. Clearly $\beta_{T,n-1} : B_0 \to T^{n-1}B/T^nB$ is bijective, and the same holds for $\beta_{a,n-1}$, by inductive assumption. Then, claim 9.6.44 says that the induced morphism $\mathbb{L}_{B_0/B_{n-1}} \otimes_{B_0} A_0 \to \mathbb{L}_{A_0/A_{n-1}}$ is an isomorphism in $\mathsf{D}(A_0\text{-}\mathbf{Mod})$. Denote by κ the residue field of A and B; by considering the Künneth spectral sequence

$$E^2_{pq} := \mathrm{Tor}_p^{A_0}(H_q\mathbb{L}_{A_0/A}, \kappa) \Rightarrow H_{p+q}(\mathbb{L}_{A_0/A} \otimes_{A_0} \kappa)$$

([163, Th.5.6.4]) we get $H_2(\mathbb{L}_{A_0/A} \otimes_{A_0} \kappa) = 0$, by virtue of (b). Consequently, the commutative diagram of ring homomorphisms

$$\begin{array}{ccccc} B & \longrightarrow & B_{n-1} & \longrightarrow & B_0 \\ \downarrow & & \downarrow & & \downarrow \\ A & \longrightarrow & A_{n-1} & \longrightarrow & A_0 \end{array}$$

induces a commutative ladder with exact rows ([103, Ch.II, Prop.2.1.2]) :

$$\begin{array}{ccccc} H_3(\mathbb{L}_{B_0/B_{n-1}} \otimes_{B_0} \kappa) & \longrightarrow & H_2(\mathbb{L}_{B_{n-1}/B} \otimes_{B_{n-1}} \kappa) & \longrightarrow & H_2(\mathbb{L}_{B_0/B} \otimes_{B_0} \kappa) \\ \downarrow & & \downarrow & & \downarrow \\ H_3(\mathbb{L}_{A_0/A_{n-1}} \otimes_{A_0} \kappa) & \longrightarrow & H_2(\mathbb{L}_{A_{n-1}/A} \otimes_{A_{n-1}} \kappa) & \longrightarrow & 0 \end{array}$$

whose left vertical arrow is an isomorphism. It follows that the central vertical arrow is surjective. Consider now the commutative diagram induced by the maps $B_0 \to A_0 \to \kappa$:

$$\begin{array}{ccc} H_2(\mathbb{L}_{B_{n-1}/B} \otimes_{B_{n-1}} B_0) & \longrightarrow & H_2(\mathbb{L}_{B_{n-1}/B} \otimes_{B_{n-1}} \kappa) \\ \downarrow & & \downarrow \\ H_2(\mathbb{L}_{A_{n-1}/A} \otimes_{A_{n-1}} A_0) & \longrightarrow & H_2(\mathbb{L}_{A_{n-1}/A} \otimes_{A_{n-1}} \kappa). \end{array} \qquad (9.6.45)$$

We have just seen that the right vertical arrow of (9.6.45) is surjective, and the same holds for its top horizontal arrow, in light of claim 9.6.43. Thus, finally, the bottom horizontal arrow is surjective as well, so $\beta_{a,n}$ is an isomorphism (claim 9.6.43), and the proposition is proved. □

Theorem 9.6.46. *Let $(A, \mathfrak{m})$ be a local ring, and $I \subset \mathfrak{m}$ an ideal; denote by κ the residue field of A, and set $A_0 := A/I$. Consider the following conditions :*

(a) *Every minimal system of generators of I is a regular sequence of elements of A.*
(b) *I is generated by a regular sequence of A.*
(c) *The natural morphism $\mathbb{L}_{A_0/A} \to I/I^2[1]$ is an isomorphism in $\mathsf{D}(A_0\text{-}\mathbf{Mod})$, and I/I^2 is a flat A_0-module.*
(d) *$H_n(\mathbb{L}_{A_0/A} \otimes_{A_0} \kappa) = 0$ for every $n \geq 2$.*
(e) *$H_2(\mathbb{L}_{A_0/A} \otimes_{A_0} \kappa) = 0$.*

Then (a)⇒(b)⇒(c)⇒(d) ⇒(e)*, and if A is noetherian,* (e)⇒(a).

Proof. Clearly (a)⇒(b) and (d)⇒(e); moreover, (b)⇒(c), by lemma 9.6.41 and proposition 7.8.15. To see that (c)⇒(d), we invoke the Künneth spectral sequence ([163, Th.5.6.4]) :

$$E^2_{pq} := \mathrm{Tor}_p^{A_0}(H_q\mathbb{L}_{A_0/A}, \kappa) \Rightarrow H_{p+q}(\mathbb{L}_{A_0/A} \otimes_{A_0} \kappa). \qquad (9.6.47)$$

Lastly, let us check that (e)⇒(a) when A is noetherian. Thus, let $(a_1, \dots, a_n)$ be a minimal system of generators for I; recall that $n = \dim_\kappa I \otimes_A \kappa$. We argue by induction on n. If $n = 0$, there is nothing to show. If $n=1$, recall that $H_0\mathbb{L}_{A_0/A}=0$ and $H_1\mathbb{L}_{A_0/A}=I/I^2$ ([103, Ch.II, Prop.1.2.4.2 and Ch.III, Cor.1.2.8.1]); then from the spectral sequence (9.6.47) we deduce that

$$\mathrm{Tor}_1^{A_0}(I/I^2,\kappa)=E^2_{1,1}=E^\infty_{1,1}=0$$

so I/I^2 is a flat A_0-module, by the local flatness criterion ([126, Th.22.3]). Then, again from (9.6.47) it follows easily that

$$H_2(\mathbb{L}_{A_0/A})\otimes_{A_0}\kappa=E^2_{0,2}=E^\infty_{0,2}=0$$

and by [103, Ch.II, Cor.2.3.7], the A_0-module $H_2\mathbb{L}_{A_0/A}$ is of finite type, so that $H_2\mathbb{L}_{A_0/A}=0$, by Nakayama's lemma; we may then conclude with proposition 9.6.42.

Next, suppose that $n>1$, and that the assertion is already known when $\dim_\kappa I\otimes_A\kappa<n$. Set $B:=A/a_1A$ and $J:=IB$. The ring homomorphisms $A\to B\to A_0$ induce exact sequences

$$0\to H_2(\mathbb{L}_{A_0/B}\otimes_{A_0}\kappa)\to H_1(\mathbb{L}_{B/A}\otimes_B\kappa)\to I\otimes_A\kappa\to J\otimes_B\kappa\to 0$$
$$H_3(\mathbb{L}_{A_0/B}\otimes_{A_0}\kappa)\to H_2(\mathbb{L}_{B/A}\otimes_B\kappa)\to 0$$

([103, Ch.II, Prop.2.1.2 and Ch.III, Cor.1.2.8.1]). However, clearly J admits a generating system of length $n-1$, hence $n':=\dim_\kappa J\otimes_B\kappa<n$. But $\dim_\kappa H_1(\mathbb{L}_{B/A}\otimes_B\kappa)=1$, so we have necessarily $n'=n-1$ and $H_2(\mathbb{L}_{A_0/B}\otimes_{A_0}\kappa)=0$. By inductive assumption, the sequence $(\overline{a}_2,\dots,\overline{a}_n)$ of the images in B of $(a_2,\dots,a_n)$, is then regular. By virtue of lemma 9.6.41 and proposition 7.8.15, it follows that $H_3(\mathbb{L}_{A_0/B}\otimes_{A_0}\kappa)=0$, whence, $H_2(\mathbb{L}_{B/A}\otimes_B\kappa)=0$, so a_1 is a regular element in A, and finally, $(a_1,\dots,a_n)$ is a regular sequence, as stated. □

Corollary 9.6.48. *Let $(A,\mathfrak{m})$ be a local noetherian ring, with residue field κ. The following conditions are equivalent :*

(a) *A is regular.*
(b) *The natural morphism $\mathbb{L}_{\kappa/A}\to\mathfrak{m}/\mathfrak{m}^2[1]$ is an isomorphism in $\mathsf{D}(\kappa\text{-}\mathbf{Mod})$.*
(c) *$H_2\mathbb{L}_{\kappa/A}=0$.*

Proof. It follows immediately, by invoking theorem 9.6.46 with $I:=\mathfrak{m}$. □

We conclude this section with a desingularization result which was first proved in [19, Chap.II Prop.3.2] and can also be found in [118, Lemma 3.10]; it is moreover closely related to [123, Prop.1.2.8].

9.6.49. Let $(A,\mathfrak{m})$ be a local noetherian ring, $\mathfrak{p}\subset A$ a prime ideal with $\dim A/\mathfrak{p}=1$. We define inductively as follows, an inverse system of A-schemes :

$$\cdots\to X_2\xrightarrow{\pi_1}X_1\xrightarrow{\pi_0}X_0$$

and for every $n\in\mathbb{N}$ a reduced subscheme $C_n\subset X_n$, with $\pi_n(C_{n+1})=C_n$. Set $X_0:=\mathrm{Spec}\,A$ and $C_0:=\mathrm{Spec}\,A/\mathfrak{p}$. For every $n\in\mathbb{N}$, let $Z_n\subset C_n$ be the pre-image of $\mathfrak{m}$, endowed with its reduced subscheme structure, and $\mathscr{I}_n\subset\mathscr{O}_{X_n}$ the ideal defining Z_n; we let $\pi_n:X_{n+1}\to X_n$ be the blow-up of $\mathscr{I}_n$, and $C_{n+1}\subset X_{n+1}$ the strict transform of C_n. For every $n\in\mathbb{N}$, let also $i_n:C_n\to X_n$ be the closed immersion, and set $\mathscr{J}_n:=\mathrm{Ker}(\mathscr{O}_{X_n}\to i_{n*}\mathscr{O}_{C_n})$.

Proposition 9.6.50. *In the situation of* (9.6.49)*, suppose that the normalization $(A/\mathfrak{p})^\nu$ of $A/\mathfrak{p}$ is a finite $A/\mathfrak{p}$-algebra. Then there exists $n_0\in\mathbb{N}$ such that for every $n\geq n_0$ we have:*

(i) *C_n is a regular scheme.*
(ii) *$\mathscr{J}^m_{n,z}/\mathscr{J}^{m+1}_{n,z}$ is a torsion-free $\mathscr{O}_{C_n,z}$-module, for every $z\in Z_n$ and every $m\in\mathbb{N}$.*
(iii) *If moreover, $A_\mathfrak{p}$ is a regular ring, the same holds for $\mathscr{O}_{X_n,z}$, for every $z\in Z_n$.*

Proof. (i): Recall that the restriction $\overline{\pi}_n:C_{n+1}\to C_n$ of π_n is the blowup of the ideal $\overline{\mathscr{I}}_n\subset\mathscr{O}_{C_n}$ defining Z_n, for every $n\in\mathbb{N}$; we notice :

Claim 9.6.51. $\overline{\pi}_n$ is a finite morphism, for every $n\in\mathbb{N}$.

Proof of the claim. Clearly, C_{n+1} is an integral scheme, and $\overline{\pi}_n$ is projective and induces an isomorphism $\overline{\pi}_n^{-1}(C_n \setminus Z_n) \xrightarrow{\sim} C_n \setminus Z_n$. Let $y \in C_{n+1}$ be any point with $z := \overline{\pi}_n(y) \in Z_n$; the induced map $\mathscr{O}_{C_n,z} \to \mathscr{O}_{C_{n+1},y}$ is injective, and induces an isomorphism of the respective fields of fractions. Moreover, $\dim \mathscr{O}_{C_{n+1},y} \geq 1$, hence the transcendence degree of the residue field extension $\kappa(z) \to \kappa(y)$ is 0, by [126, Th.15.5]. By [126, Th.5.6], this implies that $\overline{\pi}_n^{-1}(z)$ is a $\kappa(z)$-scheme of dimension 0 and of finite type, for every $z \in Z_n$, so $\overline{\pi}_n^{-1}(x)$ is a finite set, for every $x \in C_n$. Then the claim follows from [65, Ch.IV, Th.8.11.1]. $\Diamond$

From claim 9.6.51, we have $C_n = \operatorname{Spec} R_n$, for some finite $A/\mathfrak{p}$-algebra R_n lying in the normalization $(A/\mathfrak{p})^\nu$ of $A/\mathfrak{p}$. Since $A/\mathfrak{p}$ is noetherian and $(A/\mathfrak{p})^\nu$ is a finite $A/\mathfrak{p}$-algebra, it follows that there exists $n_0 \in \mathbb{N}$ such that $\overline{\pi}_n$ is an isomorphism for every $n \geq n_0$. The latter means that $\overline{\mathscr{I}}_n$ is an invertible ideal for every such n, whence the assertion.

(ii,iii): By (i) we find $k \in \mathbb{N}$ such that $C_k = \operatorname{Spec}(A/\mathfrak{p})^\nu$. We then get an inverse system

$$\cdots \to X_k(z) \times_{X_k} X_{k+2} \to X_k(z) \times_{X_k} X_{k+1} \to X_k(z) \qquad \text{for every } z \in Z_k$$

(notation of definition 4.9.17(iii)) whose transition morphism $X_k(z) \times_{X_k} \pi_{k+n}$ is the blow-up of the restriction of $\mathscr{I}_{k+n}$, for every $n \in \mathbb{N}$. Since Z_k has finitely many points, it suffices therefore to prove the assertions for A replaced by $A_z := \mathscr{O}_{X_k,z}$ and $\mathfrak{p}$ replaced by the prime ideal $\mathfrak{p}_z \subset A_z$ such that $\operatorname{Spec} A_z/\mathfrak{p}_z = C_k(z)$, for every $z \in Z_k$. Thus, we may assume that $A/\mathfrak{p}$ is regular, and $\overline{\pi}_n$ is an isomorphism $C_{n+1} \xrightarrow{\sim} C_n$, so that Z_n has a unique point z_n, for every $n \in \mathbb{N}$. Let

$$A_n := \mathscr{O}_{X_n,z_n} \qquad \mathfrak{m}_n := \mathscr{I}_{n,z_n} \qquad \mathfrak{p}_n := \mathscr{J}_{n,z_n} \qquad \text{for every } n \in \mathbb{N}$$

so that $X_n(z_n) = \operatorname{Spec} A_n$. For given $n \in \mathbb{N}$, let $x_1, \dots, x_k$ be a finite system of generators of $\mathfrak{p}_n$, and $x_0 \in \mathfrak{m}_n$ any element whose image in $A_n/\mathfrak{p}_n$ generates the ideal $\mathfrak{m}_n/\mathfrak{p}_n$, so that the system $x_0, \dots, x_k$ generates the ideal $\mathfrak{m}_n$. We have already remarked that $X_n(z_n) \times_{X_n} \pi_n$ is the blow-up of the ideal $(\mathscr{I}_n)_{|X_n(z_n)}$; hence, $X_n(z_n) \times_{X_n} X_{n+1}$ admits the affine open covering $U_0 \cup \dots \cup U_k$, where $U_i := \operatorname{Spec} B_i$, with $B_i := A_n[\frac{x_0}{x_i}, \dots, \frac{x_k}{x_i}]$ for every $i = 0, \dots, k$.

Claim 9.6.52. (i) $C_{n+1} \subset U_0$.

(ii) $x_1/x_0, \dots, x_k/x_0$ generate a prime ideal $\mathfrak{q}_n \subset B_0$.

(iii) $\mathfrak{n}_n := B_0 x_0 + \mathfrak{q}_n$ is a maximal ideal of B_0.

(iv) The induced maps $A_n/\mathfrak{p}_n \to B_0/\mathfrak{q}_n \to A_{n+1}/\mathfrak{p}_{n+1}$ are isomorphisms.

Proof of the claim. (i): The universal property of the blow-up $X_n(z_n) \times_{X_n} \pi_n$ says that for $i = 0, \dots, k$, every morphism of schemes $f : T \to X_n(z_n)$ such that $f^{-1}(\mathscr{I}_n)_{|X_n(z_n)} \cdot \mathscr{O}_T$ is invertible and generated by the image of x_i, factors uniquely through U_i. Especially, the projection $X_n(z_n) \times_{X_n} C_{n+1} \to X_n(z_n)$ factors through U_0, whence (i).

(ii,iv): From (i) we get a commutative diagram of schemes

$$\begin{array}{ccc} X_n(z_n) \times_{X_n} C_{n+1} & \longrightarrow & U_0 \\ \downarrow & & \downarrow \\ X_n(z_n) \times_{X_n} C_n & \longrightarrow & X_n(z_n) \end{array}$$

whose horizontal arrows are closed immersions, and whose left vertical arrow is an isomorphism. Hence, the induced morphism

$$X_n(z_n) \times_{X_n} C_{n+1} \to U_0 \times_{X_n} C_n = \operatorname{Spec} B_0/\mathfrak{p}_n B_0 \tag{9.6.53}$$

is a closed immersion. However, $X_n(z_n) \times_{X_n} C_{n+1} = \operatorname{Spec} A_{n+1}/\mathfrak{p}_{n+1}$, and notice that x_0 annihilates the ideal $\mathfrak{q}_n/\mathfrak{p}_n B_0$ of $B_0/\mathfrak{p}_n B_0$; since x_0 is a regular element of $A_{n+1}/\mathfrak{p}_{n+1}$, it follows that the map $B_0/\mathfrak{p}_n B_0 \to A_{n+1}/\mathfrak{p}_{n+1}$ corresponding to (9.6.53) factors through a surjection $f : B_0/\mathfrak{q}_n \to A_{n+1}/\mathfrak{p}_{n+1}$. On the other hand, the unique map of A_n-algebras $A_n \to B_0/\mathfrak{q}_n$

factors through a surjection $g : A_n/\mathfrak{p}_n \to B_0/\mathfrak{q}_n$, and by assumption $f \circ g$ is an isomorphism, whence both (ii) and (iv).

(iii): From (iv) we get surjections $A_n/\mathfrak{m}_n \to B_0/\mathfrak{n}_{n+1} \to A_{n+1}/\mathfrak{m}_{n+1}$, whence the claim. $\diamondsuit$

In light of claim 9.6.52(ii,iv), and since A_{n+1} is a localization of B_0, we see that $\mathfrak{p}_{n+1}$ is generated by the images of $x_1/x_0, \dots, x_k/x_0$; whence, for every $m \in \mathbb{N}$, an $A_n/\mathfrak{p}_n$-linear map

$$\varphi_{n,m} : \mathfrak{p}_n^m/\mathfrak{p}_n^{m+1} \to \mathfrak{p}_{n+1}^m/\mathfrak{p}_{n+1}^{m+1} \qquad (y \mod \mathfrak{p}_n^{m+1}) \mapsto (x_0^{-m}y \mod \mathfrak{p}_{n+1}^{m+1})$$

whose image contains a set of generators for the $A_{n+1}/\mathfrak{p}_{n+1}$-module $\mathfrak{p}_{n+1}^m/\mathfrak{p}_{n+1}^{m+1}$. Since the natural map $A_n/\mathfrak{p} \to A_{n+1}/\mathfrak{p}_{n+1}$ is an isomorphism, it follows that $\varphi_{n,m}$ is surjective, for every $m \in \mathbb{N}$. Moreover, set $R_n := \bigoplus_{m\in\mathbb{N}} \mathfrak{p}_n^m/\mathfrak{p}_n^{m+1}$ for every $n \in \mathbb{N}$; it is easily seen that the direct sum of the maps $\varphi_{n,m}$ yields a surjective map of graded $A_n/\mathfrak{p}_n$-algebras :

$$\varphi_n : R_n \to R_{n+1}.$$

Claim 9.6.54. (i) $A_n[x_0^{-1}] \otimes_{A_n} \varphi_n$ is an isomorphism.
(ii) $\mathrm{Ann}_{R_n}(\mathfrak{m}_n) \subset \mathrm{Ker}\,\varphi_n$.

Proof of the claim. (i): We have already remarked that A_{n+1} is a localization of B_0, and $\mathfrak{p}_{n+1} = \mathfrak{q}_n A_{n+1}$; hence $\mathfrak{p}_{n+1}^m/\mathfrak{p}_{n+1}^{m+1} = A_{n+1} \otimes_{B_0} \mathfrak{q}_n^m/\mathfrak{q}_n^{m+1}$, so that $\varphi_{n,m}$ is a composition :

$$\mathfrak{p}_n^m/\mathfrak{p}_n^{m+1} \xrightarrow{\alpha_m} \mathfrak{q}_n^m/\mathfrak{q}_n^{m+1} \xrightarrow{\beta_m} \mathfrak{p}_{n+1}^m/\mathfrak{p}_{n+1}^{m+1} \qquad \text{for every } m \in \mathbb{N}$$

where α_m is given by the rule : $(y \mod \mathfrak{p}_n^{m+1}) \mapsto (x_0^{-m}y \mod \mathfrak{q}_n^{m+1})$, for every $y \in \mathfrak{p}_n^m$, and β_m is the localization. Since the localization $B_0 \to A_{n+1}$ induces an isomorphism $B_0/\mathfrak{q}_n \xrightarrow{\sim} A_{n+1}/\mathfrak{p}_{n+1}$ (claim 9.6.52(iii)), the map β_m is bijective for every $m \in \mathbb{N}$. Thus, we need to check that $A_n[x_0^{-1}] \otimes_{A_n} \alpha_m$ is bijective for every $m \in \mathbb{N}$. However, clearly $A_n[1/x_0] = B_0[1/x_0]$, and $\mathfrak{p}_n[1/x_0] = \mathfrak{q}_n[1/x_0]$ in $B_0[1/x_0]$, whence the assertion.

(ii): We come down to checking that the annihilator of x_0 in $\mathfrak{p}_m^n/\mathfrak{p}_m^{n+1}$ lies in $\mathrm{Ker}\,\varphi_{m,n}$ for every $m \in \mathbb{N}$. Hence, let $y \in \mathfrak{p}_m^n$ whose class $\overline{y} \in \mathfrak{p}_m^n/\mathfrak{p}_m^{n+1}$ is annihilated by x_0; the latter means that there exist polynomials $P, Q \in A_n[X_1, \dots, X_k]$ homogeneous of degree n and respectively $n+1$, such that $y = P(x_1, \dots, x_k)$, and $x_0 y = Q(x_1, \dots, x_k)$. It follows that

$$x_0^{m+1} P(x_1/x_0, \dots, x_k/x_0) = x_0^{m+1} Q(x_1/x_0, \dots, x_k/x_0) \qquad \text{in } B_0$$

whence $t := P(x_1/x_0, \dots, x_k/x_0) = Q(x_1/x_0, \dots, x_k/x_0)$ in B_0, since x_0 is regular in this ring. But $\alpha_m(\overline{y})$ is precisely the class of t in $\mathfrak{q}_m^n/\mathfrak{q}_m^{n+1}$, so $\alpha_m(\overline{y}) = 0$, whence the assertion. $\diamondsuit$

Now, set $J_r := \mathrm{Ann}_{R_0}(\mathfrak{m}_0^r)$ for every $r \in \mathbb{N}$; since R_0 is noetherian, there exists $r \in \mathbb{N}$ such that $J_r = J_{r+1}$. Also, clearly $\mathrm{Ann}_{R_n}(\mathfrak{m}_n^i) = \mathrm{Ann}_{R_n}(\mathfrak{m}_0^i)$ for every $i, n \in \mathbb{N}$. In light of claim 9.6.54, it then follows easily that $\mathrm{Ker}\,(\varphi_n \circ \cdots \circ \varphi_0) = J_r$ for every $n \geq r$, and $R_0/J_r \xrightarrow{\sim} R_n$ is then a torsion-free $A_0/\mathfrak{p}_0$-algebra for every $n \geq r$, whence (ii).

It follows in particular that $\mathfrak{p}_n^m/\mathfrak{p}_n^{m+1}$ is a free $A_n/\mathfrak{p}_n$-module of finite rank for every $n \geq r$ and every $m \in \mathbb{N}$, since it is torsion-free, and since $A_n/\mathfrak{p}_n$ is a discrete valuation ring. According to [66, Ch.IV, Cor.19.1.2], proposition 7.8.15 and remark 7.8.16(i,ii), in order to show (iii) it then suffices to check that the natural map of graded $A_n/\mathfrak{p}_n$-algebras

$$\psi : \mathrm{Sym}^\bullet_{A_n}(\mathfrak{p}_n/\mathfrak{p}_n^2) \to R_n$$

is an isomorphism for every $n \geq r$. Now, by the same token, $\mathrm{Sym}^m_{A_n}(\mathfrak{p}_n/\mathfrak{p}_n^2)$ is a free $A_n/\mathfrak{p}_n$-module of finite rank, for every $n \geq r$ and every $m \in \mathbb{N}$; Since ψ is surjective, it then suffices to check that these modules have the same rank, for every such n and m. However, notice that the structure map $A \to A_n$ induces an isomorphism $A_\mathfrak{p} \xrightarrow{\sim} (A_n)_{\mathfrak{p}_n}$, for every $n \in \mathbb{N}$, hence $(A_n)_{\mathfrak{p}_n}$ is a regular ring for every such n, under the assumptions of (iii). Invoking again [66, Ch.IV, Cor.19.1.2], proposition 7.8.15 and remark 7.8.16, we deduce that the localization $\psi_{\mathfrak{p}_n}$ of ψ is an isomorphism for every $n \geq r$, whence the contention. $\square$

Definition 9.6.55. Let A be a noetherian ring. We say that A is a *Nagata ring*, if for every $\mathfrak{p} \in \operatorname{Spec} A$ and every finite field extension $\kappa(\mathfrak{p}) \subset L$, the integral closure of $A/\mathfrak{p}$ in L is a finite A-module. (Such rings are called *universally japanese* in [63, Ch.0, Déf.23.1.1].)

Corollary 9.6.56. *Let* $(A, \mathfrak{m})$ *be a Nagata local domain such that the regular locus of* $\operatorname{Spec} A$ *is an open subset. Then there exist a regular local domain* $(A', \mathfrak{m}')$ *and an injective local ring homomorphism* $A \to A'$ *of essentially finite type, such that :*

(i) *The induced map* $\operatorname{Frac}(A) \xrightarrow{\sim} \operatorname{Frac}(A')$ *is an isomorphism.*
(ii) *The induced map* $A/\mathfrak{m} \to A'/\mathfrak{m}'$ *is a finite field extension.*

Proof. Recall that the regular locus U of $X := \operatorname{Spec} A$ is the set of all $\mathfrak{p} \in X$ such that $A_\mathfrak{p}$ is a regular ring. If $U = \operatorname{Spec} A$, we may take $A' = A$; hence, suppose that U is a proper subset of X. Since A is noetherian, we may find $\mathfrak{p} \in U$ such that $U \cap \operatorname{Spec} A/\mathfrak{p} = \{\mathfrak{p}\}$. Especially, $\{\mathfrak{p}\}$ is an open subset of $\operatorname{Spec} A/\mathfrak{p}$, hence there exists $f \in A \setminus \mathfrak{p}$ such that $\operatorname{Spec}(A/\mathfrak{p})_f = \{\mathfrak{p}\}$, so that $(A/\mathfrak{p})_f$ is a field. By [63, Ch.0, Cor.16.3.3], it follows that $\dim A/\mathfrak{p} = 1$. Then the assertion follows immediately from proposition 9.6.50(iii) and claim 9.6.51. □

9.7. **Regular morphisms and excellent rings.** Recall that a morphism of schemes $f : X \to Y$ is called *regular*, if it is flat, and for every $y \in Y$, the fibre $f^{-1}(y)$ is locally noetherian and geometrically regular ([64, Ch.IV, Déf.6.8.1]).

Lemma 9.7.1. *Let* $f : X \to Y$ *and* $g : Y \to Z$ *be two morphisms of locally noetherian schemes. We have :*

(i) *If* f *and* g *are regular, then the same holds for* $g \circ f$.
(ii) *If* $g \circ f$ *is regular, and* f *is faithfully flat, then* g *is regular.*

Proof. (i): Clearly $h := g \circ f$ is flat. Let $z \in Z$ be any point, and K any finite extension of $\kappa(z)$. Set

$$X' := h^{-1}(z) \times_{\kappa(z)} K \qquad \text{and} \qquad Y' := g^{-1}(z) \times_{\kappa(z)} K.$$

It is easily seen that the induced morphism $f' : X' \to Y'$ is regular. Moreover, for every $y \in Y'$, the local ring $\mathscr{O}_{Y',y'}$ is regular, since g is regular. Then the assertion follows from :

Claim 9.7.2. Let $A \to B$ be a flat and local ring homomorphism of local noetherian rings. Denote by $\mathfrak{m}_A \subset A$ the maximal ideal, and suppose that both A and $B_0 := B/\mathfrak{m}_A B$ are regular. Then B is regular.

Proof of the claim. On the one hand, $\dim B = \dim A + \dim B_0$ ([63, Ch.IV, Cor.6.1.2]). On the other hand, let $a_1, \dots, a_n$ be a minimal generating system for $\mathfrak{m}_A$, and $b_1, \dots, b_m$ a system of elements of the maximal ideal $\mathfrak{m}_B$ of B, whose images in B_0 is a minimal generating system for $\mathfrak{m}_B/\mathfrak{m}_A B$. By Nakayama's lemma, it is easily seen that the system $a_1, \dots, a_n, b_1, \dots, b_m$ generates the ideal $\mathfrak{m}_B$. Since A and B_0 are regular, $n = \dim A$ and $m = \dim B$, so $n + m = \dim B$, and the claim follows. ◊

(ii): Clearly g is flat. Then the assertion follows easily from [63, Ch.0, Prop.17.3.3(i)] : details left to the reader. □

Definition 9.7.3. Let A be a noetherian ring.

(i) We say that A is a *G-ring*, if the formal fibres of $\operatorname{Spec} A$ are geometrically regular, *i.e.* for every $\mathfrak{p} \in \operatorname{Spec} A$, the natural morphism $\operatorname{Spec} A_\mathfrak{p}^\wedge \to \operatorname{Spec} A_\mathfrak{p}$ from the spectrum of the $\mathfrak{p}$-adic completion of A, is regular : see [64, Ch.IV, §7.3.13].
(ii) We say that A is *quasi-excellent*, if A is a G-ring, and moreover the following holds. For every prime ideal $\mathfrak{p} \subset A$, and every finite radicial extension K' of the field of fractions K of $B := A/\mathfrak{p}$, there exists a finite B-subalgebra B' of K' such that the field of fractions of B' is K', and the *regular locus* of $\operatorname{Spec} B'$ is an open subset (the latter is the set of all prime ideals $\mathfrak{q} \subset B'$ such that $B'_\mathfrak{q}$ is a regular ring).

(iii) We say that A is *catenary* if any two saturated chains $(\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_n)$, $(\mathfrak{q}_0 \subset \cdots \subset \mathfrak{q}_m)$ of prime ideals of A, with $\mathfrak{p}_0 = \mathfrak{q}_0$ and $\mathfrak{p}_n = \mathfrak{q}_m$, have the same length (so $n = m$) ([63, Ch.0, (16.1.4)]). We say that A is *universally catenary* if every A-algebra of finite type is catenary.

(iv) We say that A is *excellent*, if it is quasi-excellent and universally catenary ([64, Ch.IV, Déf.7.8.2]).

Lemma 9.7.4. *Let A be a noetherian ring.*

(i) *If A is quasi-excellent, then A is a Nagata ring (see definition* 9.6.55*).*

(ii) *If A is a G-ring, then every A-algebra of essentially finite type is a G-ring.*

(iii) *Suppose that the natural morphism* $\mathrm{Spec}\, A^\wedge_\mathfrak{m} \to \mathrm{Spec}\, A_\mathfrak{m}$ *is regular for every maximal ideal* $\mathfrak{m} \subset A$*. Then A is a G-ring.*

(iv) *If A is a local G-ring, then A is quasi-excellent.*

(v) *If A is a complete local ring, then A is excellent.*

Proof. (i): This is [64, Ch.IV, Cor.7.7.3].

(ii): The assertion for localizations is obvious, and the assertion on A-algebras of finite type follows from [64, Ch.IV, Th.7.4.4(ii)].

(iv) follows from [64, Ch.IV, Th.6.12.7, Prop.7.3.18, Th.7.4.4(ii)].

(v): In light of (iv), it suffices to remark that every complete noetherian local ring is universally catenary ([64, Ch.IV, Prop.5.6.4] and [63, Ch.0, Th.19.8.8(i)]) and is a G-ring ([63, Ch.0, Th.22.3.3, Th.22.5.8, and Prop.19.3.5(iii)]).

(iii): Let us remark, more generally :

Claim 9.7.5. Let $\varphi : A \to B$ be a faithfully flat ring homomorphism of noetherian rings, such that $f := \mathrm{Spec}\,\varphi$ is regular. If B is a G-ring, the same holds for A.

Proof of the claim. In light of (ii), we easily reduce to the case where both A and B are local, φ is a local ring homomorphism, and it suffices to show that the natural morphism $\pi_A : \mathrm{Spec}\, A^\wedge \to \mathrm{Spec}\, A$ is regular (where $A^\wedge$ is the completion of A). Consider the commutative diagram :

$$(9.7.6) \qquad \begin{array}{ccc} \mathrm{Spec}\, B^\wedge & \xrightarrow{f^\wedge} & \mathrm{Spec}\, A^\wedge \\ {\scriptstyle \pi_B}\downarrow & & \downarrow{\scriptstyle \pi_A} \\ \mathrm{Spec}\, B & \xrightarrow{f} & \mathrm{Spec}\, A. \end{array}$$

By assumption, π_B is a regular morphism; Then the same holds for $f \circ \pi_B = \pi_A \circ f^\wedge$ (lemma 9.7.1(i)). However, it is easily seen that the induced map $\varphi^\wedge : A^\wedge \to B^\wedge$ is still a local ring homomorphism, hence $f^\wedge$ is faithfully flat, so the claim follows from lemma 9.7.1(ii). ◊

Now, in order to prove (iii), it suffices to check that $A_\mathfrak{m}$ is a G-ring for every maximal ideal $\mathfrak{m} \subset A$. In view of our assumption, the latter assertion follows from claim 9.7.5 and (v). □

Proposition 9.7.7. *Let A be a ring, B a noetherian A-algebra of finite (Krull) dimension, $n \in \mathbb{N}$ an integer, and suppose that $H_k(\mathbb{L}_{B/A} \otimes_B \kappa(\mathfrak{p})) = 0$ for every prime ideal $\mathfrak{p} \subset B$ and every $k = n, \dots, n + \dim B$. Then $H_n(\mathbb{L}_{B/A} \otimes_B M) = 0$ for every B-module M.*

Proof. Let us start out with the following more general :

Claim 9.7.8. Let A be a ring, B a noetherian A-algebra, $\mathfrak{p} \in \mathrm{Spec}\, B$ a prime ideal, $n \in \mathbb{N}$ an integer, and suppose that the following two conditions hold :

(a) $H_n(\mathbb{L}_{B/A} \otimes_B \kappa(\mathfrak{p})) = 0$.

(b) $H_{n+1}(\mathbb{L}_{B/A} \otimes_B B/\mathfrak{q}) = 0$ for every proper specialization $\mathfrak{q}$ of $\mathfrak{p}$ in $\mathrm{Spec}\, B$.

Then, $H_n(\mathbb{L}_{B/A} \otimes_B B/\mathfrak{p}) = 0$.

Proof of the claim. Let $b \in B \setminus \mathfrak{p}$ be any element, and set $M := B/(\mathfrak{p} + bB)$. Since B is noetherian, M admits a finite filtration $M_0 \subset M_1 \subset \cdots \subset M_n := M$ such that, for every $i = 0, \ldots, n-1$, the subquotient M_{i+1}/M_i is isomorphic to $B/\mathfrak{q}$, for some proper specialization $\mathfrak{q}$ of $\mathfrak{p}$ ([126, Th.6.4]). From (b), and a simple induction, we deduce that $H_{n+1}(\mathbb{L}_{B/A} \otimes_B M) = 0$. Whence, by considering the short exact sequence of B-modules

$$0 \to B/\mathfrak{p} \xrightarrow{b} B/\mathfrak{p} \to M \to 0$$

we see that scalar multiplication by b is an injective map on the B-module $H_n(\mathbb{L}_{B/A} \otimes_B B/\mathfrak{p})$. Since this holds for every $b \in B \setminus \mathfrak{p}$, we conclude that the natural map

$$H_n(\mathbb{L}_{B/A} \otimes_B B/\mathfrak{p}) \to H_n(\mathbb{L}_{B/A} \otimes_B B/\mathfrak{p}) \otimes_B B_{\mathfrak{p}} = H_n(\mathbb{L}_{B/A} \otimes_B \kappa(\mathfrak{p}))$$

is injective. Then the assertion follows from (a). ◊

Now, let $\mathfrak{p} \subset B$ be any prime ideal; the assumption, together with claim 9.7.8 and a simple induction on $d := \dim B/\mathfrak{p}$, shows that

$$H_k(\mathbb{L}_{B/A} \otimes_B B/\mathfrak{p}) = 0 \qquad \text{for every } k = n, \ldots, n + \dim B - d.$$

For any B-module M, set $H(M) := H_n(\mathbb{L}_{B/A} \otimes_B M)$. Especially, we get $H(B/\mathfrak{p}) = 0$, for any prime ideal $\mathfrak{p} \subset B$. Since B is noetherian, it follows easily that $H(M) = 0$ for any B-module M of finite type (details left to the reader). Next, if M is arbitrary, we may write it as the union of the filtered family $(M_i \mid i \in I)$ of its submodules of finite type; since $H(M)$ is the colimit of the induced system $(H(M_i) \mid i \in I)$, we see that $H(M) = 0$, as sought. □

Corollary 9.7.9. *Let $A \to B$ be a homomorphism of noetherian rings. Then the following conditions are equivalent :*

(a) *$\Omega^1_{B/A}$ is a flat B-module, and $H_i\mathbb{L}_{B/A} = 0$ for every $i > 0$.*
(b) *$\Omega^1_{B/A}$ is a flat B-module, and $H_1\mathbb{L}_{B/A} = 0$.*
(c) *The induced morphism of schemes $f : \mathrm{Spec}\, B \to \mathrm{Spec}\, A$ is regular.*
(d) *$H_1(\mathbb{L}_{B/A} \otimes_B \kappa(x)) = 0$ for every $x \in \mathrm{Spec}\, B$.*

Proof. Set $X := \mathrm{Spec}\, B$ and $Y := \mathrm{Spec}\, A$; let $x \in X$, and $y := f(x)$. We remark :

Claim 9.7.10. The following conditions are equivalent :

(e) the map on stalks $\mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ is formally smooth for the preadic topologies defined by the maximal ideals.
(f) f is flat at the point x, and the $\kappa(y)$-algebra $\mathscr{O}_{f^{-1}(y),x}$ is geometrically regular.
(g) $H_1(\mathbb{L}_{B/A} \otimes_B \kappa(x)) = 0$.
(h) $H_i(\mathbb{L}_{B/A} \otimes_B \kappa(x)) = 0$ for every $i > 0$.

Proof of the claim. The equivalence (e)⇔(f) follows from [63, Ch.0, Th.19.7.1]. Next, to ease notation, set $A' := \mathscr{O}_{X,x}$ and $B' := \mathscr{O}_{Y,y}$; we have a natural identification

$$\mathbb{L}_{B/A} \otimes_B \kappa(x) \xrightarrow{\sim} \mathbb{L}_{B'/A'} \otimes_{B'} \kappa(x)$$

([103, Ch.II, Cor.2.3.1.1]) from which we get (e)⇔(g), by virtue of proposition 8.6.30. Obviously, (h)⇒(g). Conversely, suppose that (g) holds, and set $A'' := \kappa(x)$ and $B'' := A'' \otimes_{A'} B'$; by the foregoing, we know already that B' is flat A'-algebra, so we get an isomorphism

$$\mathbb{L}_{B'/A'} \otimes_{B'} \kappa(x) \xrightarrow{\sim} \mathbb{L}_{B''/A''} \otimes_{B''} \kappa(x)$$

([103, Ch.II, Prop.2.2.1]). Moreover, B'' is a regular local ring; in view of corollary 9.6.48, the sequence of ring homomorphisms $A'' \to B'' \to \kappa(x)$ then yields an isomorphism

$$H_i(\mathbb{L}_{B''/A''} \otimes_{B''} \kappa(x)) \xrightarrow{\sim} H_i\mathbb{L}_{\kappa(x)/A''} \qquad \text{for every } i > 1$$

([103, Ch.II, Prop.2.1.2]). Combining with proposition 9.6.1(i), we get (h). ◊

From claim 9.7.10, we already see that (a)⇒(b)⇒(c)⇔(d). Lastly, suppose that (d) holds; in order to show that $\Omega^1_{B/A}$ is a flat B-module, it suffices to check that $\mathrm{Tor}^B_1(H_0\mathbb{L}_{B/A}, M) = 0$ for every B-module M. To this aim, recall the Künneth spectral sequence

$$E^2_{pq} := \mathrm{Tor}^B_p(H_q\mathbb{L}_{B/A}, M) \Rightarrow H_{p+q}(\mathbb{L}_{B/A} \otimes_B M).$$

It is easily seen that $E^2_{1,0} = E^\infty_{1,0}$, so we are reduced to proving that $H_1(\mathbb{L}_{B/A} \otimes_B M) = 0$ for every such M. Then, proposition 9.7.7 further reduces to checking that $H_i(\mathbb{L}_{B/A} \otimes_B \kappa(x)) = 0$ for every $i > 0$. The latter follows from (d) and claim 9.7.10. Likewise, taking $M := B$, and arguing with proposition 9.7.7 and claim 9.7.10, we also deduce that $H_i\mathbb{L}_{B/A} = 0$ for every $i > 0$, whence (a). □

Corollary 9.7.11. *Let A be a local noetherian ring, $A^\wedge$ the completion of A. Then the following conditions are equivalent :*

(a) *A is quasi-excellent.*
(b) *$\Omega^1_{A^\wedge/A}$ is a flat $A^\wedge$-module, and $H_1\mathbb{L}_{A^\wedge/A} = 0$.*
(c) *$\Omega^1_{A^\wedge/A}$ is a flat $A^\wedge$-module, and $H_i\mathbb{L}_{A^\wedge/A} = 0$ for every $i > 0$.*

Proof. Taking into account lemma 9.7.4(iii), this is a special case of corollary 9.7.9. □

Proposition 9.7.12. *Let $p > 0$ be a prime integer, A a regular local and excellent $\mathbb{F}_p$-algebra, B a local noetherian A-algebra, $\mathfrak{m}_B$ the maximal ideal of B, and M a B-module of finite type. Then the B-module $\Omega^1_{A/\mathbb{F}_p} \otimes_A M$ is separated for the $\mathfrak{m}_B$-preadic topology.*

Proof. Let $\varphi : A \to B$ be the structure morphism, and set $\mathfrak{p} := \varphi^{-1}\mathfrak{m}_B$; the localization $A_\mathfrak{p}$ is still excellent (lemma 9.7.4(ii,iv)) and regular, and clearly $\Omega^1_{A_\mathfrak{p}/\mathbb{F}_p} \otimes_{A_\mathfrak{p}} M = \Omega^1_{A/\mathbb{F}_p} \otimes_A M$, hence we may replace A by $A_\mathfrak{p}$, and assume that φ is local. Let $A^\wedge$ and $B^\wedge$ be the completions of A and B, set $M^\wedge := B^\wedge \otimes_B M$, and notice that both of the natural maps $\mathbb{F}_p \to A$ and $A \to A^\wedge$ are regular. In view of corollary 9.7.9, it follows that both of the natural B-linear maps

$$\Omega^1_{A/\mathbb{F}_p} \otimes_A M \to \Omega^1_{A/\mathbb{F}_p} \otimes_A M^\wedge \qquad \Omega^1_{A/\mathbb{F}_p} \otimes_A M \to \Omega^1_{A^\wedge/\mathbb{F}_p} \otimes_{A^\wedge} M^\wedge$$

are injective (to see the injectivity of the second map, one applies the transitivity triangle arising from the sequence of ring homomorphisms $\mathbb{F}_p \to A \to A^\wedge$: details left to the reader). Thus, we may assume that A is complete. Now, for every ring R, and every integer $m \in \mathbb{N}$, set

$$R_{(m)} := R[[T_1, \dots, T_m]] \qquad R_{\langle m\rangle} := R[[T_1^p, \dots, T_m^p]] \subset R_{(m)}. \tag{9.7.13}$$

With this notation, we have an isomorphism $A \xrightarrow{\sim} \kappa_{(d)}$ of $\mathbb{F}_p$-algebras, where κ is the residue field of A, and $d := \dim A$ ([63, Ch.0, Th.19.6.4]).

Claim 9.7.14. Let K be any field of characteristic p, and $m \in \mathbb{N}$ any integer. We have :

(i) There exists a cofiltered system $(K^\lambda \mid \lambda \in \Lambda)$ of subfields of K such that $[K : K^\lambda]$ is finite for every $\lambda \in \Lambda$, and $\bigcap_{\lambda\in\Lambda} K^\lambda = K^p$.
(ii) For every system $(K^\lambda \mid \lambda \in \Lambda)$ fulfilling the condition of (i), the following holds :
   (a) $\Omega^1_{K_{(m)}/K^\lambda_{\langle m\rangle}}$ is a free $K_{(m)}$-module of finite rank, for every $\lambda \in \Lambda$, and the rule
   $$M \mapsto \Omega_m(M) := \lim_{\lambda\in\Lambda} (\Omega^1_{K_{(m)}/K^\lambda_{\langle m\rangle}} \otimes_{K_{(m)}} M)$$
   defines an exact functor $K_{(m)}$-$\mathbf{Mod} \to K_{(m)}$-$\mathbf{Mod}$.
   (b) The natural map
   $$\eta_m(M) : \Omega^1_{K_{(m)}/\mathbb{F}_p} \otimes_{K_{(m)}} M \to \Omega_m(M)$$
   is injective for every $K_{(m)}$-module M.
   (c) Let F (resp. F^λ) denote the field of fractions of $K_{(m)}$ (resp. of $K^\lambda_{\langle m\rangle}$, for every $\lambda \in \Lambda$); then $\bigcap_{\lambda\in\Lambda} F^\lambda = F^p$.

Proof of the claim. (i) and (ii.c) follow from [63, Ch.0, Prop.21.8.8] (and its proof).

(ii.a): For given $\lambda \in \Lambda$, say that $x_1, \dots, x_r$ is a p-basis of K over K^λ; then it is easily seen that $x_1, \dots, x_r, T_1, \dots, T_m$ is a p-basis of $K_{(m)}$ over $K^\lambda_{\langle m\rangle}$ (see [63, Ch.0, Déf.21.1.9]). According to [63, Ch.0, Cor.21.2.5], it follows that $\Omega^1_{K_{(m)}/K^\lambda_{\langle m\rangle}}$ is the free $K_{(m)}$-module of finite type with basis $dx_1, \dots, dx_r, dT_1, \dots, dT_m$. Moreover, say that $K^\mu \subset K^\lambda$, and let $x_{r+1}, \dots, x_s$ be a p-basis of K^λ over K^μ; then $x_1, \dots, x_s$ is a p-basis of K over K^μ ([63, Ch.0, Lemme 21.1.10]), so the induced map

$$\Omega^1_{K_{(m)}/K^\mu_{\langle m\rangle}} \to \Omega^1_{K_{(m)}/K^\lambda_{\langle m\rangle}}$$

is a projection onto a direct factor, and the assertion follows easily.

(ii.b): We are easily reduced to the case where M is a $K_{(m)}$-module of finite type, and in light of (ii.a), we may further assume that M is a cyclic $K_{(m)}$-module. Next, we remark that, due to (ii.c), the natural map

$$\Omega^1_{F/\mathbb{F}_p} \to \lim_{\lambda\in\Lambda} \Omega^1_{F/F^\lambda}$$

is injective ([63, Ch.0, Th.21.8.3]); in other words, $\eta_m(F)$ is injective. In order to show the injectivity of $\eta_m(M)$, it then suffices to check that the functor $M \mapsto \Omega^1_{K_{(m)}/\mathbb{F}_p} \otimes_{K_{(m)}} M$ is exact. The latter holds by virtue of corollary 9.7.9, since the (unique) morphism of schemes $\mathrm{Spec}\, K_{(m)} \to \mathrm{Spec}\, \mathbb{F}_p$ is obviously regular. This completes the proof for $m = 0$. Suppose now that $m > 0$, and that the injectivity of $\eta_n(M)$ is already known for every $n < m$ and every $K_{(n)}$-module M. By the foregoing, it remains to check that $\eta_m(K_{(m)}/I)$ is injective, for every non-zero ideal $I \subset K_{(m)}$. Pick any non-zero $f \in I$, and set $R := K_{(m-1)}$; according to [34, Ch.VII, n.7, Lemme 3] and [34, Ch.VII, n.8, Prop.6], there exist an automorphism σ of the ring $K_{(m)}$, and elements $g \in R[T_m]$, $u \in K^\times_{(m)}$ such that $\sigma(f) = u\cdot g$, and $g = T^d_m + a_1 T^{d-1}_m + \dots + a_d$ for some $d \geq 0$ and certain elements $a_1, \dots, a_d$ of the maximal ideal of R. However, set $M' := K_{(m)}/\sigma(I)$; in view of the commutative diagram of $\mathbb{F}_p$-modules :

$$\begin{array}{ccc}
\Omega^1_{K_{(m)}/\mathbb{F}_p} \otimes_{K_{(m)}} M & \xrightarrow{\eta_m(M)} & \Omega_m(M) \\
\downarrow & & \downarrow \\
\Omega^1_{K_{(m)}/\mathbb{F}_p} \otimes_{K_{(m)}} M' & \xrightarrow{\eta_m(M')} & \Omega_m(M')
\end{array}$$

(whose vertical arrows are induced by σ) we see that $\eta_m(M)$ is injective, if and only if the same holds for $\eta_m(M')$. Hence, we may replace I by $\sigma(I)$, and assume that $g \in I$. In this case, set also $R^\lambda := K^\lambda_{\langle m-1\rangle}$ for every $\lambda \in \Lambda$, and notice that the natural maps

$$R[T_m]/g^p R[T_m] \to K_{(m)}/g^p K_{(m)} \qquad R^\lambda[T^p_m]/g^p R^\lambda[T^p_m] \to K^\lambda_{\langle m\rangle}/g^p K^\lambda_{\langle m\rangle}$$

are bijective; there follows a commutative diagram of $K_{(m)}$-modules :

$$\begin{array}{ccc}
\Omega^1_{R[T_m]/\mathbb{F}_p} \otimes_{R[T_m]} M & \longrightarrow & \Omega^1_{K_{(m)}/\mathbb{F}_p} \otimes_{K_{(m)}} M \\
{\scriptstyle \alpha^\lambda\otimes_{R[T_m]}M}\downarrow & & \downarrow{\scriptstyle \eta^\lambda_m\otimes_{K_{(m)}}M} \\
\Omega^1_{R[T_m]/R^\lambda} \otimes_{R[T_m]} M & \longrightarrow & \Omega^1_{K_{(m)}/K^\lambda_{\langle m\rangle}} \otimes_{K_{(m)}} M
\end{array}$$

whose horizontal arrows are isomorphisms, and $\eta_m(M) = \lim_{\lambda\in\Lambda} \eta^\lambda_m \otimes_{K_{(m)}} M$. On the other hand, for every $\lambda \in \Lambda$ we have a commutative ladder of $R[T_m]$-modules with exact rows :

$$\Sigma_\lambda \quad : \quad \begin{array}{ccccccccc} 0 & \to & \Omega^1_{R/\mathbb{F}_p} \otimes_R R[T_m] & \to & \Omega^1_{R[T_m]/\mathbb{F}_p} & \to & \Omega^1_{R[T_m]/R} & \to & 0 \\ & & \downarrow{\scriptstyle \eta^\lambda_{m-1}\otimes_R R[T_m]} & & \downarrow{\scriptstyle \alpha^\lambda} & & \| & & \\ 0 & \to & \Omega^1_{R/R^\lambda} \otimes_R R[T_m] & \to & \Omega^1_{R[T_m]/R^\lambda} & \to & \Omega^1_{R[T_m]/R} & \to & 0 \end{array}$$

and notice that the rows of $\Sigma_\lambda \otimes_{R[T_m]} M$ are still short exact, for every $\lambda \in \Lambda$. By inductive assumption, $\lim_{\lambda\in\Lambda} \eta^\lambda_{m-1} \otimes_R M$ is an injective map; we deduce that the same holds for $\lim_{\lambda\in\Lambda} \alpha^\lambda \otimes_{R[T]} M$, and the claim follows. ♢

Take $K := \kappa$, and pick any cofiltered system $(K^\lambda \mid \lambda \in \Lambda)$ as provided by claim 9.7.14(i); in light of claim 9.7.14(ii.b), it now suffices to show that the resulting $\Omega_d(M)$ is a separated B-module, for every noetherian $\kappa_{(d)}$-algebra B and every B-module M of finite type. However, $\Omega_d(M)$ is a submodule of $\prod_{\lambda\in\Lambda}(\Omega^1_{K_{(m)}/K^\lambda_{\langle m\rangle}} \otimes_{K_{(m)}} M)$, hence we are reduced to checking that each direct factor of the latter B-module is separated. But in view of claim 9.7.14(ii.a), we see that each such factor is a finite direct sum of copies of M, so finally we come down to the assertion that M is separated for the $\mathfrak{m}_B$-adic topology, which is well known. □

Theorem 9.7.15. *Let $\varphi : A \to B$ be a local ring homomorphism of local noetherian rings. Suppose that A is quasi-excellent, and φ is formally smooth for the preadic topologies defined by the maximal ideals. Then* $\mathrm{Spec}\,\varphi$ *is regular.*

Proof. This is the main result of [4]. We begin with the following general remark :

Claim 9.7.16. Let R be a local noetherian ring, $\mathfrak{m}_R \subset R$ the maximal ideal, $H : R\text{-}\mathbf{Mod} \to R\text{-}\mathbf{Mod}$ an additive functor, and suppose that

(a) H is R-linear, *i.e.* $H(t \cdot \mathbf{1}_M) = t \cdot H(\mathbf{1}_M)$ for every R-module M, and every $t \in R$.
(b) H is *semi-exact*, *i.e.* for any short exact sequence $0 \to M' \to M \to M'' \to 0$ of R-modules, the induced sequence $H(M') \to H(M) \to H(M'')$ is exact.
(c) H commutes with filtered colimits.
(d) $H(M)$ is separated for the $\mathfrak{m}_R$-adic topology, for every R-module M of finite type.
(e) $H(R/\mathfrak{m}_R) = 0$.

Then $H(M) = 0$ for every R-module M.

Proof of the claim. Since H commutes with filtered colimits, it suffices to show that $H(M) = 0$ for every finitely generated R-module M, and since H is semi-exact, a simple induction reduces further to the case where M is a cyclic R-module. Let now $\mathscr{F}$ be the family of all ideals I of R such that $H(R/I) \neq 0$, and suppose, by way of contradiction, that $\mathscr{F} \neq \varnothing$; pick a maximal element J of $\mathscr{F}$, and set $M := R/J$. By assumption, $J \neq \mathfrak{m}_R$; thus, let $t \in R$ be a non-invertible element with $t \notin J$; we get an exact sequence

$$H(M) \xrightarrow{t} H(M) \to H(B_{(m+n)}/(J + tB_{(m+n)}))$$

whose third term vanishes, by the maximality of J. On the other hand, $\bigcap_{n\in\mathbb{N}} t^n H(M) = 0$, since $H(M)$ is separated. Thus $H(M) = 0$, contradicting the choice of J, and the claim follows. ♢

Denote by κ the residue field of A, and for every $m, n \in \mathbb{N}$, let $\varphi_{m,n} : A_{(m)} \to B_{(m+n)}$ be the composition of $\varphi_{(m)} : A_{(m)} \to B_{(m)}$ (the T-adic completion of $\varphi \otimes_A A[T_1, \dots, T_m]$) with the natural inclusion map $B_{(m)} \to B_{(m+n)}$ (notation of (9.7.13)).

Claim 9.7.17. In the situation of the theorem, suppose furthermore that A is either a field or a complete discrete valuation ring of mixed characteristic. Then, the morphism $\mathrm{Spec}\,\varphi_{m,n}$ is regular for every $m, n \in \mathbb{N}$.

Proof of the claim. Set

$$H(M) := H_1(\mathbb{L}_{B_{(m+n)}/A_{(m)}} \otimes_{B_{(m+n)}} M).$$

We shall consider separately three different cases :

• Suppose first that A is a field of characteristic $p > 0$. According to corollary 9.7.9, it suffices to show that $H(M)$ vanishes for every $B_{(m+n)}$-module M. Notice that the natural map $\mathbb{F}_p \to B_{(m+n)}$ is regular; from the distinguished triangle ([103, Ch.II, Prop.2.1.2])

$$\mathbb{L}_{A_{(m)}/\mathbb{F}_p} \otimes_{A_{(m)}} B_{(m+n)} \to \mathbb{L}_{B_{(m+n)}/\mathbb{F}_p} \to \mathbb{L}_{B_{(m+n)}/A_{(m)}} \to \mathbb{L}_{A_{(m)}/\mathbb{F}_p} \otimes_{A_{(m)}} B_{(m+n)}[1]$$

and corollary 9.7.9, we deduce an injective $B_{(m+n)}$-linear map

$$H(M) \to \Omega^1_{A_{(m)}/\mathbb{F}_p} \otimes_{A_{(m)}} M$$

from which it follows that if M is a $B_{(m+n)}$-module of finite type, $H(M)$ is a separated $B_{(m+n)}$-module, for the preadic topology defined by the maximal ideal of $B_{(m+n)}$ (proposition 9.7.12). Since φ is formally smooth, $\varphi_{m,n}$ is also formally smooth for the preadic topologies defined by the maximal ideals of $A_{(m)}$ and $B_{(m+n)}$; from proposition 8.6.30, we see that $H(M) = 0$, if M is the residue field of $B_{(m+n)}$. Then the assertion follows from claim 9.7.16.

• Next, suppose that A is either a field of characteristic zero, or a complete discrete valuation ring of mixed characteristic (so either $\kappa = A$, or else κ is a field of positive characteristic). According to corollary 9.7.9, it suffices to show that $H(M)$ vanishes for $M = \kappa(\mathfrak{q})$, where $\mathfrak{q} \subset B_{(m+n)}$ is any prime ideal. Fix such $\mathfrak{q}$, and set $\mathfrak{p} := \mathfrak{q} \cap A_{(m)}$; if $\mathfrak{p} = 0$, then M is a K-algebra, where K is the field of fractions of $A_{(m)}$; now, K is a field of characteristic zero, and $B' := B_{(m+n)} \otimes_{A_{(m)}} K$ is a regular local K-algebra, so the induced morphism $\mathrm{Spec}\,B' \to \mathrm{Spec}\,K$ is regular; since $H(M) = H_1(\mathbb{L}_{B'/K} \otimes_{B'} M)$, the assertion follows from corollary 9.7.9. Notice that this argument applies especially to the case where $m = 0$; for the general case, we argue by induction on m. Hence, suppose that $n \in \mathbb{N}$, $m > 0$, and that the assertion is already known for $\varphi_{n,m-1}$.

Consider first the case where A is a discrete valuation ring, and $\mathfrak{p}$ contains the maximal ideal of A, and set $\overline{B} := B \otimes_A \kappa$. Since $\varphi_{m,n}$ is flat, and since M is a $\overline{B}_{(m+n)}$-module, we have a natural isomorphism

$$H(M) \xrightarrow{\sim} H_1(\mathbb{L}_{\overline{B}_{(m+n)}/\kappa_{(m)}} \otimes_{\overline{B}_{(m+n)}} M).$$

Then the sought vanishing follows from the foregoing, since $\overline{B}$ is a formally smooth κ-algebra (for the preadic topology of its maximal ideal).

Lastly, suppose that either A is a field, or $\mathfrak{p}$ does not contain the maximal ideal of A (and $\mathfrak{p} \neq 0$). In either of these two cases, we may find $f \in \mathfrak{p}$ whose image in $\kappa_{(m)}$ is not zero. According to [34, Ch.VII, n.7, Lemme 3] and [34, Ch.VII, n.8, Prop.6], we may find an automorphism σ of the A-algebra $A_{(m)}$ and elements $g \in A_{(m-1)}[T_m]$, $u \in A^\times_{(m)}$ such that $\sigma(f) = u \cdot g$, and $g = T^d_m + a_1 T^{d-1}_m + \cdots + a_d$ for some $d \geq 0$ and certain elements $a_1, \dots, a_d$ of the maximal ideal of $A_{(m-1)}$. Denote by $\sigma' : B_{(m)} \xrightarrow{\sim} B_{(m)}$ the T-adic completion of $\sigma \otimes_A B$, and let $\sigma_B : B_{(m+n)} \xrightarrow{\sim} B_{(m+n)}$ be the T-adically continuous automorphism that restricts to σ' on $B_{(m)}$, and such that $\sigma_B(T_i) = T_i$ for $i = m+1, \dots, m+n$. Set $M' := B_{(m+n)}/\sigma_B(\mathfrak{q})$; by construction, we have a commutative diagram of A-algebras

$$\begin{array}{ccc} A_{(m)} & \xrightarrow{\varphi_{m,n}} & B_{(m+n)} \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\sigma_B} \\ A_{(m)} & \xrightarrow{\varphi_{m,n}} & B_{(m+n)} \end{array}$$

inducing an isomorphism

$$\mathbb{L}_{B_{(m+n)}/A_{(n)}} \otimes_{B_{(m+n)}} M \stackrel{\sim}{\to} \mathbb{L}_{B_{(m+n)}/A_{(n)}} \otimes_{B_{(m+n)}} M' \qquad \text{in } \mathsf{D}(A\text{-}\mathbf{Mod}).$$

Thus, we may replace M by M', and assume from start that $g \in \mathfrak{p}$. In this case, set $B' := B[[T_{m+1}, \dots, T_{m+n}]]$, and notice that both of the natural maps

$$A_{(m-1)}[T_m]/gA_{(m-1)}[T_m] \to A_{(m)}/gA_{(m)} \qquad B'_{(m-1)}[T_m]/gB'_{(m-1)}[T_m] \to B_{(m+n)}/gB_{(m+n)}$$

are isomorphisms. Since both $\varphi_{m,n}$ and the map $A_{(m-1)}[T_m] \to B'_{(m-1)}[T_m]$ induced by φ are flat ring homomorphisms, there follows a natural isomorphism of $B_{(m+n)}$-modules :

$$\begin{aligned} \mathbb{L}_{B_{(m+n)}/A_{(n)}} \otimes_{B_{(m+n)}} M &\stackrel{\sim}{\to} \mathbb{L}_{B'_{(m-1)}[T_m]/A_{(m-1)}[T_m]} \otimes_{B'_{(m-1)}[T_m]} M \\ &\stackrel{\sim}{\to} \mathbb{L}_{B'_{(m-1)}/A_{(m-1)}} \otimes_{B'_{(m-1)}} M \end{aligned}$$

([103, Ch.II, Prop.2.2.1]). However, the resulting map $A_{(m-1)} \to B'_{(m-1)}$ is none else than $\varphi_{m-1,n}$, up to a relabeling of the variables; the vanishing of $H(M)$ then follows from the inductive assumption (and from corollary 9.7.9). $\diamondsuit$

Claim 9.7.18. In the situation of the theorem, suppose furthermore that A and B are complete, and let M be a B-module of finite type. Then $H_1(\mathbb{L}_{B/A} \otimes_B M)$ is a B-module of finite type.

Proof of the claim. Let $\overline{f} : A_0 \to \kappa$ be a surjective ring homomorphism, with A_0 a Cohen ring ([63, Ch.0, Th.19.8.6(ii)]), and set $\overline{B} := B \otimes_A \kappa$; in light of [63, Ch.0, Lemme 19.7.1.3], there exists a flat local, complete and noetherian A_0-algebra B_0 fitting into a cocartesian diagram :

$$\begin{array}{ccc} A_0 & \xrightarrow{\psi} & B_0 \\ {\scriptstyle\overline{f}}\downarrow & & \downarrow{\scriptstyle\overline{f}_B} \\ \kappa & \longrightarrow & \overline{B}. \end{array}$$

Denote by κ_B the residue field of B; there follow natural isomorphisms of B-modules :

$$H_1(\mathbb{L}_{B/A} \otimes_B \kappa_B) \stackrel{\sim}{\to} H_1(\mathbb{L}_{\overline{B}/\kappa} \otimes_{\overline{B}} \kappa_B) \stackrel{\sim}{\to} H_1(\mathbb{L}_{B_0/A_0} \otimes_{B_0} \kappa_B)$$

([103, Ch.II, Prop.2.2.1]) which, according to proposition 8.6.30, imply that ψ is formally smooth (for the preadic topologies defined by the maximal ideals). By [63, Ch.0, Th.19.8.6(i)], $\overline{f}$ lifts to a ring homomorphism $f : A_0 \to A$, whence a commutative diagram

$$\begin{array}{ccc} A_0 & \xrightarrow{\psi} & B_0 \\ {\scriptstyle\varphi\circ f}\downarrow & & \downarrow{\scriptstyle\overline{f}_B} \\ B & \longrightarrow & \overline{B}. \end{array}$$

Then, by [63, Ch.0, Cor.19.3.11], the map $\overline{f}_B$ lifts to a ring homomorphism $f_B : B_0 \to B$. Notice now that both f and f_B are local maps and induce isomorphisms on the residue fields; it follows easily that, for suitable $m, n \in \mathbb{N}$, they extend to surjective maps g and g_B fitting into a commutative diagram

$$\begin{array}{ccc} A_{0,(m)} & \xrightarrow{\psi_{(m,n)}} & B_{0,(m+n)} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g_B} \\ A & \xrightarrow{\varphi} & B \end{array}$$

whence exact sequences ([103, Ch.II, Prop.2.1.2])

$$H_1(\mathbb{L}_{B_{0,(m+n)}/A_{0,(m)}} \otimes_{B_{0,(m+n)}} M) \to H_1(\mathbb{L}_{B/A_{0,(m)}} \otimes_B M) \to H_1(\mathbb{L}_{B/B_{0,(m+n)}} \otimes_B M)$$
$$H_1(\mathbb{L}_{B/A_{0,(m)}} \otimes_B M) \to H_1(\mathbb{L}_{B/A} \otimes_B M) \to \Omega^1_{A/A_{0,(m)}} \otimes_A M = 0.$$

However, claim 9.7.17 applies to ψ, and together with corollary 9.7.9, it implies that the first module of the first of these sequences vanishes; on the other hand, it is easily seen that the third B-module of the same sequence is finitely generated, so the same holds for the middle term. By inspecting the second exact sequence, the claim follows. ◊

We may now conclude the proof of the theorem : let $A^\wedge$ and $B^\wedge$ be the completions of A and B; since φ is formally smooth, the same holds for its completion $\varphi^\wedge : A^\wedge \to B^\wedge$. First, we show that $\operatorname{Spec}\varphi^\wedge$ is regular; to this aim, it suffices to check that $H^\wedge(M) := H_1(\mathbb{L}_{B^\wedge/A^\wedge} \otimes_{B^\wedge} M)$ vanishes for every $B^\wedge$-module M (corollary 9.7.9). However, we know already that $H^\wedge(M)$ is a $B^\wedge$-module of finite type, if the same holds for M (claim 9.7.18); especially, for such M, $H^\wedge(M)$ is separated for the adic topology of $B^\wedge$ defined by the maximal ideal. Moreover, $H^\wedge(M) = 0$, if M is the residue field of $B^\wedge$ (proposition 8.6.30). Then the assertion follows from claim 9.7.16. Lastly, the natural morphism $\operatorname{Spec} A^\wedge \to \operatorname{Spec} A$ is regular by assumption, hence the same holds for the induced morphism $\operatorname{Spec} B^\wedge \to \operatorname{Spec} A$ (lemma 9.7.1(i)). Finally, since $B^\wedge$ is a faithfully flat B-algebra, we conclude that $\operatorname{Spec}\varphi$ is regular, by virtue of lemma 9.7.1(ii). □

Proposition 9.7.19. *Let A be a noetherian local ring, and $\varphi : A \to B$ an ind-étale local ring homomorphism. Then :*

(i) *A is quasi-excellent if and only if the same holds for B.*
(ii) *If A is excellent, the same holds for B.*

Proof. Set $f := \operatorname{Spec}\varphi$, and $f^\wedge := \operatorname{Spec}\varphi^\wedge$, where $\varphi^\wedge : A^\wedge \to B^\wedge$ is the map of complete local rings obtained from φ. Notice first that, for every $y \in \operatorname{Spec} A$, and every $x \in f^{-1}(y)$, the stalk $\mathscr{O}_{f^{-1}y,x}$ is an ind-étale $\kappa(y)$-algebra, *i.e.* is a separable algebraic extension of $\kappa(y)$, hence $f^{-1}(y)$ is geometrically regular, and therefore f is regular and faithfully flat. Moreover, f is the limit of a cofiltered system of formally étale morphisms, hence it is formally étale; *a fortiori*, B is also formally smooth over A for the preadic topologies, so $B^\wedge$ is formally smooth over $A^\wedge$ for the preadic topologies, and consequently $f^\wedge$ is a regular morphism (theorem 9.7.15).

(i): If B is quasi-excellent, claim 9.7.5 and lemma 9.7.4(iv) imply that A is quasi-excellent. Next, assume that A is quasi-excellent; we have a commutative diagram (9.7.6), in which π_A is a regular morphism, and then the same holds for $\pi_A \circ f^\wedge = f \circ \pi_B$ (lemma 9.7.1(i)). Now, let $y \in \operatorname{Spec} B$ be any point, and set $z := f(y)$; we know that $(f \circ \pi_B)^{-1}(z)$ is a geometrically regular affine scheme, so write it as the spectrum of a noetherian ring C. The stalk $E := \mathscr{O}_{f^{-1}(z),y}$ is a separable algebraic field extension of $\kappa(z)$, hence $\pi_B^{-1}(y) \simeq \operatorname{Spec} C \otimes_B E$, the spectrum of a localization of C, so it is again geometrically regular, *i.e.* π_B is a regular morphism, and we conclude by lemma 9.7.4(iii,iv).

(ii): By (i), it remains only to show that B is universally catenary, provided the same holds for A. To this aim, it suffices to apply [66, Ch.IV, Lemma 18.7.5.1]. □

Proposition 9.7.20. *Let $p > 0$ be a prime integer, and A a noetherian $\mathbb{F}_p$-algebra whose Frobenius endomorphism Φ_A is a finite map (see* (9.6.36)*), and for every prime ideal $\mathfrak{p} \subset A$, set $\kappa(\mathfrak{p}) := A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$ (the residue field of the point $\mathfrak{p} \in \mathrm{Spec}\, A$). Then*

$$\dim A_\mathfrak{p}/\mathfrak{q}A_\mathfrak{p} = \dim_{\kappa(\mathfrak{p})} \Omega^1_{\kappa(\mathfrak{p})/\mathbb{F}_p} - \dim_{\kappa(\mathfrak{q})} \Omega^1_{\kappa(\mathfrak{q})/\mathbb{F}_p}$$

for every pair of prime ideals $\mathfrak{q} \subset \mathfrak{p} \subset A$.

Proof. For every $\mathbb{F}_p$-algebra R, let $\Phi_R : R \to R$ be the Frobenius endomorphism. We notice :

Claim 9.7.21. Let A be an $\mathbb{F}_p$-algebra such that Φ_A is a finite map. We have :

(i) For every A-algebra B of essentially finite type, Φ_B is finite as well.

(ii) Suppose moreover that A is noetherian, $I \subset A$ is any ideal, and let $A^\wedge_I$ be the I-adic completion of A. Then :

  (a) $\Phi_{A^\wedge_I} = \mathbf{1}_{A^\wedge_I} \otimes_A \Phi_A$ is finite, and $\Omega^1_{A^\wedge_I/\mathbb{F}_p} = A^\wedge_I \otimes_A \Omega^1_{A/\mathbb{F}_p}$.

  (b) If A is reduced, the same holds for $A^\wedge_I$.

(iii) Suppose that $(A, \mathfrak{m}_A)$ is a local noetherian domain, and denote by K (resp. κ) the field of fractions (resp. the residue field) of A. Then

$$\dim_K \Omega^1_{K/\mathbb{F}_p} = \dim_\kappa \Omega^1_{\kappa/\mathbb{F}_p} + \dim A. \tag{9.7.22}$$

Proof of the claim. (i): Suppose first that $B = A[X]$; then it is easily seen that $\Phi_B = \Phi_A \otimes_{\mathbb{F}_p} \Phi_{\mathbb{F}_p[X]}$, whence the contention, in this case. By an easy induction, we deduce that the claim holds as well for any free polynomial A-algebra of finite type. Next, let $I \subset A$ be any ideal; since $\Phi_A(I) \subset I$, it is easily seen that $\Phi_{A/I} = \Phi_A \otimes_A A/I$, so $\Phi_{A/I}$ is finite, and therefore the assertion holds for any A-algebra of finite type. Lastly, let $S \subset A$ be any multiplicative subset; since $\Phi_A(S) \subset S$, it is easily seen that

$$\Phi_{S^{-1}A} = S^{-1}\Phi_A \tag{9.7.23}$$

and the assertion follows.

(ii.a): Set $B := A_{(\Phi)}$; by assumption, $\Phi_A : A \to B$ is a finite A-linear map, hence its I-adic completion $(\Phi_A)^\wedge_I : A^\wedge_I \to B^\wedge_I$ equals $\mathbf{1}_{A^\wedge_I} \otimes_A \Phi_A$ ([126, Th.8.7]). Say that I is generated by r elements of A; then it is easily seen that $I^{(p-1)r+1} \subset \Phi_A(I) \subset I$, so we have a natural $A^\wedge_I$-linear identification on I-adic completions :

$$B^\wedge_I \xrightarrow{\sim} (A^\wedge_I)_{(\Phi)} \tag{9.7.24}$$

and under this identification, $(\Phi_A)^\wedge_I = \Phi_{A^\wedge_I}$, whence the first assertion of (ii.a). Next, we notice that (9.7.24) yields a natural identification $\Omega^1_{A^\wedge_I/\mathbb{F}_p} = \Omega^1_{B^\wedge_I/\mathbb{F}_p}$, and on the other hand, the sequence of ring homomorphisms

$$\mathbb{F}_p \to A^\wedge_I \xrightarrow{\Phi_{A^\wedge_I}} B^\wedge_I$$

yields natural identifications :

$$\Omega^1_{B^\wedge_I/\mathbb{F}_p} = \Omega^1_{B^\wedge_I/A^\wedge_I} = A^\wedge_I \otimes_A \Omega^1_{B/A} = A^\wedge_I \otimes_A \Omega^1_{B/\mathbb{F}_p} = B^\wedge_I \otimes_B \Omega^1_{B/\mathbb{F}_p} = A^\wedge_I \otimes_A \Omega^1_{A/\mathbb{F}_p}$$

where the last equality is induced by (9.7.24) and the identity map $A \xrightarrow{\sim} B$. This completes the proof of the second assertion.

(ii.b): Since A is reduced, Φ_A is injective, and then the same holds for its completion $(\Phi_A)^\wedge_I$. But we have seen that the latter is naturally identified with $\Phi_{A^\wedge_I}$, whence the claim.

(iii): Notice that $\Omega^1_{K/\mathbb{F}_p}$ is a K-vector space of finite dimension, since its dimension equals $[K : K^p]$ ([63, Ch.0, Th.21.4.5]), and the latter is finite, by (i); the same argument applies to the κ-vector space $\Omega^1_{\kappa/\mathbb{F}_p}$. Denote by $\delta(A)$ the difference between the left and right hand-side of (9.7.22); we have to show that $\delta(A) = 0$. Let $A^\wedge$ be the $\mathfrak{m}_A$-adic completion of A, and $\mathfrak{p} \subset A^\wedge$

any minimal prime ideal such that $\dim A^\wedge/\mathfrak{p} = d := \dim A$. In view of (ii.b), $L := (A^\wedge)_\mathfrak{p}$ is a field; taking into account (ii.a), we get

$$\Omega^1_{L/\mathbb{F}_p} = L \otimes_{A^\wedge} \Omega^1_{A^\wedge/\mathbb{F}_p} = L \otimes_A \Omega^1_{A/\mathbb{F}_p} = L \otimes_K \Omega^1_{K/\mathbb{F}_p}.$$

Hence $\dim_L \Omega^1_{L/\mathbb{F}_p} = \dim_K \Omega^1_{K/\mathbb{F}_p}$, so we see that

(9.7.25) $\qquad \delta(A) = \delta(A^\wedge/\mathfrak{p}) \qquad$ for any prime ideal $\mathfrak{p} \subset A^\wedge$ such that $\dim A^\wedge/\mathfrak{p} = d$.

We may then replace A by $A^\wedge/\mathfrak{p}$, after which we may assume that A is a complete local domain. In this case, A contains a field mapping isomorphically onto κ, and there is a finite map of κ-algebras $A' := \kappa[[T_1, \dots, T_d]] \to A$ ([126, Th.29.4(iii)]). Denote by K' the field of fractions of A', and notice that $[K : K'] = [K : K^p] \cdot [K^p : K'] = [K : K^p] \cdot [K^p : K'^p]$; on the other hand, Φ_K induces an isomorphism $K \xrightarrow{\sim} K^p$, hence have as well $[K : K'] = [K^p : K'^p]$, so $[K : K^p] = [K' : K'^p]$ and finally :

$$\dim_{K'} \Omega^1_{K'/\mathbb{F}_p} = \dim_K \Omega^1_{K/\mathbb{F}_p}$$

([63, Ch.0, Cor.21.2.5]). Since $\dim A' = \dim A$, we conclude that $\delta(A) = \delta(A')$. Hence, it suffices to check that $\delta(A') = 0$. Furthermore, set $B := \kappa[T_1, \dots, T_d]$, and let $\mathfrak{m} \subset B$ be the maximal ideal generated by $T_1, \dots, T_d$; we have $A' = B^\wedge_\mathfrak{m}$, so (9.7.25) further reduces to showing that $\delta(B_\mathfrak{m}) = 0$, which shall be left as an exercise for the reader. ◊

Now, in view of claim 9.7.21(i), we may replace A by $A_\mathfrak{p}/\mathfrak{q}A_\mathfrak{p}$, and assume from start that A is a local domain, that $\mathfrak{q} = 0$ and that $\mathfrak{p}$ is the maximal ideal; in this case, the sought identity is given by claim 9.7.21(iii). □

Theorem 9.7.26. *Let $p > 0$ be a prime integer, and A a noetherian $\mathbb{F}_p$-algebra. We have :*

(i) *If the Frobenius endomorphism Φ_A of A is a finite map, then A is excellent.*
(ii) *Conversely, suppose that A is a local Nagata ring, with residue field k, and that $[k : k^p]$ is finite. Then Φ_A is a finite ring homomorphism.*

Proof. (i): We reproduce the proof from [125, Appendix, Th.108].

Claim 9.7.27. If Φ_A is finite, then A is quasi-excellent.

Proof of the claim. We check first the openness condition for the regular loci. To this aim, in light of claim 9.7.21(i), we may assume that A is an integral domain, and it suffices to prove that the regular locus of $\operatorname{Spec} A$ is an open subset. Set $B := A_{(\Phi)}$ (notation of (9.6.36)), and let $\mathfrak{p} \subset A$ be any prime ideal; by theorem 9.6.37 and (9.7.23), the ring $A_\mathfrak{p}$ is regular if and only if $(\Phi_A)_\mathfrak{p}$ is a flat ring homomorphism, if and only if $B_\mathfrak{p}$ is a flat $A_\mathfrak{p}$-module. Since, by assumption, B is a finite A-module, the latter holds if and only if $B_\mathfrak{p}$ is a free $A_\mathfrak{p}$-module. Then, the assertion follows from [126, Th.4.10].

Next, we show that A is a G-ring. By claim 9.7.21(i), we may assume that A is local, and we let $A^\wedge$ be the completion of A. From claim 9.7.21(ii.a) we see that the natural map

$$A_{(\Phi)} \overset{\mathbf{L}}{\otimes}_A A^\wedge \to (A^\wedge)_{(\Phi)}$$

is an isomorphism in $\mathsf{D}(A\text{-}\mathbf{Mod})$. Then the assertion follows from [75, Lemma 6.5.13(i)] and corollary 9.7.11. ◊

Now, it follows easily from proposition 9.7.20 and claim 9.7.21(i), that A is universally catenary; combining with claim 9.7.27, we obtain (i).

(ii): The assertion to prove is that $A_{(\Phi)}$ is a finite A-module. To this aim, denote by $I \subset A$ the nilradical ideal; we remark :

Claim 9.7.28. It suffices to show that $(A/I)_{(\Phi)}$ is a finite A-module.

Proof of the claim. Indeed, suppose that $(A/I)_{(\Phi)}$ is a finite A-module; we shall deduce, by induction on s, that $(A/I^s)_{(\Phi)}$ is a finite A-module, for every $s > 0$. Since A is noetherian, we have $I^t = 0$ for $t > 0$ large enough, so the claim will follow. The assertion for $s = 1$ is our assumption. Suppose therefore that $s > 1$, and that we already know the assertion for $s - 1$. Notice that I^s/I^{s-1} is a finite A/I-module, hence a finite $(A/I)_{(\Phi)}$-module, by our assumption. Since $(A/I^{s-1})_{(\Phi)}$ is a finite A-module, we deduce easily that the assertion holds for s, as required. $\diamondsuit$

In view of claim 9.7.28, we may replace A by A/I (which is obviously still a Nagata ring), and assume from start that A is reduced. In this case, let $\{\mathfrak{p}_1, \dots, \mathfrak{p}_t\}$ be the set of minimal prime ideals of A; we have a commutative diagram of ring homomorphisms :

$$\begin{array}{ccc} A & \xrightarrow{\Phi_A} & A \\ \downarrow & & \downarrow \\ \prod_{i=1}^t A/\mathfrak{p}_i & \xrightarrow{\prod_{i=1}^t \Phi_{A/\mathfrak{p}_i}} & \prod_{i=1}^t A/\mathfrak{p}_i \end{array}$$

whose vertical arrows are injective and finite maps. Suppose now that $(A/\mathfrak{p}_i)_{(\Phi)}$ is a finite $A/\mathfrak{p}_i$-module for every $i = 1, \dots, t$. Then $\prod_{i=1}^t (A/\mathfrak{p}_i)_{(\Phi)}$ is a finite A-module, and therefore the same holds for its A-submodule $A_{(\Phi)}$. Thus, we are reduced to checking the sought assertion for each of the quotients $A/\mathfrak{p}_i$ (which are still Nagata rings), and hence we may assume from start that A is a domain. In this case, denote by K the field of fractions of A; by the definition of Nagata ring, we are further reduced to checking that $\Phi_K : K \to K$ is a finite field extension, *i.e.* that $[K : K^p]$ is finite. Denote $A^\wedge$ the completion of A; we remark :

Claim 9.7.29. (i) The endomorphism $\Phi_{A^\wedge}$ is a finite map.

(ii) $\mathrm{Spec}\, K \otimes_A A^\wedge$ is a geometrically reduced K-scheme.

Proof of the claim. (i): Indeed, $A^\wedge$ is a quotient of a power series ring $B := k[[T_1, \dots, T_r]]$ ([63, Ch.0, Th.19.8.8(i)]), hence it suffices to show that Φ_B is finite. However, $B^p = k^p[[T_1^p, \dots, T_r^p]]$, and $[k : k^p]$ is finite by assumption, whence the claim.

(ii): This is [64, Ch.IV, Th.7.6.4]. $\diamondsuit$

Let $\mathfrak{p} \subset A^\wedge$ be any minimal prime ideal; it follows from claim 9.7.29(ii) that $L := (A^\wedge)_\mathfrak{p}$ is a separable field extension of K ([64, Ch.IV, Prop.4.6.1]), so the natural map

$$\Omega^1_{K/\mathbb{F}_p} \otimes_K L \to \Omega^1_{L/\mathbb{Z}}$$

is injective ([63, Ch.0, Cor.20.6.19(i)]); on the other hand, claim 9.7.29(i) implies that $[L : L^p]$ is finite, *i.e.* $\Omega^1_{L/\mathbb{F}_p}$ is a finite-dimensional L-vector space ([63, Ch.0, Th.21.4.5]). We conclude that $\Omega^1_{K/\mathbb{F}_p}$ is a finite-dimensional K-vector space, whence the contention, again by *loc.cit.* $\square$

Part (i) of proposition 9.7.31 was first observed in [14, Th.1.1]. Also, the class of ring homomorphisms introduced by definition 9.7.30(ii) has been extensively studied in [13], in which one can find most of the results that follow, and much more.

Definition 9.7.30. Let $f : (A, \mathfrak{m}_A) \to (B, \mathfrak{m}_B)$ be a local ring homomorphism of local noetherian rings, and denote by $B^\wedge$ the $\mathfrak{m}_B$-adic completion of B.

(i) A *Cohen factorization* of f is the datum of a complete local noetherian ring $(C, \mathfrak{m}_C)$ and ring homomorphisms $g : A \to C$, $h : C \to B^\wedge$ such that $C/\mathfrak{m}_A C$ is a regular ring, g is faithfully flat, h is surjective, and $f = h \circ g$.

(ii) We say that f is a *homomorphism of complete intersection* if it admits a Cohen factorization as in (i), such that $\mathrm{Ker}\, h$ is generated by a regular sequence of elements of $\mathfrak{m}_C$.

Proposition 9.7.31. *Let $f : (A, \mathfrak{m}_A) \to (B, \mathfrak{m}_B)$ be a local homomorphism of complete local noetherian rings, and denote by κ_A and κ_B the residue fields of A and B. The following holds :*

(i) *f admits a Cohen factorization.*

(ii) *If $H_1\mathbb{L}_{\kappa_B/\kappa_A}$ is a finite dimensional κ_B-vector space, we may find a Cohen factorization $A \xrightarrow{g} (C, \mathfrak{m}_C) \xrightarrow{h} B$ of f, where g is formally smooth for the $\mathfrak{m}_A$-adic and $\mathfrak{m}_C$-adic topologies.*

Proof. (i): Let p be the characteristic of κ_A and κ_B. If $p = 0$, we have fields $k_A \subset A, k_B \subset B$ such that the induced maps $k_A \to \kappa_A$ and $k_B \to \kappa_B$ are isomorphisms ([126, Th.28.3]); if $p > 0$, we have complete discrete valuation rings V_A, V_B and local ring homomorphisms $V_A \to A$, $V_B \to B$ inducing isomorphisms $V_A/pV_A \xrightarrow{\sim} \kappa_A$, $V_B/pV_B \xrightarrow{\sim} \kappa_B$ ([126, Th.29.2]). In the first case, set $R_0 := \mathbb{Q}$, $R_A := k_A$ and $R_B := k_B$; in the second case, set $R_0 := \mathbb{Z}_{p\mathbb{Z}}$, $R_A := V_A$, and $R_B := V_B$. In either case, let k_A and k_B be the residue fields of R_A and R_B, and notice that R_A and R_B are formally smooth R_0-algebras for the p-adic topologies ([126, Th.26.9 and Th.28.10]). Fix finite systems of generators $x_1, \dots, x_n$ for $\mathfrak{m}_A$, and $y_1, \dots, y_m$ for $\mathfrak{m}_B$; we may assume that $m \geq n$, and $f(x_i) = y_i$ for $i = 1, \dots, n$. Set

$$A_0 := R_A[[X_1, \dots, X_n]] \qquad B_0 := R_B[[Y_1, \dots, Y_m]]$$

and let $\mathfrak{m}_{A_0} \subset A_0$, $\mathfrak{m}_{B_0} \subset B_0$ be the maximal ideals; we deduce surjective maps

$$\varphi_A : A_0 \to A \qquad \varphi_B : B_0 \to B$$

such that $\varphi_A(X_i) = x_i$ and $\varphi_B(Y_j) = y_j$ for every $i = 1, \dots, n$ and $j = 1, \dots, m$. Obviously $J := \mathrm{Ker}\,\varphi_B \subset \mathfrak{m}_{B_0}$, and B_0 is a complete local ring, so that B_0 is J-adically complete and separated (lemma 8.3.12). We then get a commutative diagram of rings :

$$\begin{array}{ccccc} R_0 & \xrightarrow{i_B} & R_B & \xrightarrow{j_B} & B_0 \\ {\scriptstyle i_A}\downarrow & & & & \downarrow{\scriptstyle \varphi_B} \\ R_A & \xrightarrow{j_A} & A & \xrightarrow{f} & B \end{array}$$

and by [63, Ch.0, Cor.19.3.11] there exists a local ring homomorphism

$$\psi : R_A \to B_0 \qquad \text{such that} \qquad \psi \circ i_A = j_B \circ i_B \qquad \text{and} \qquad \varphi_B \circ \psi = f \circ j_A.$$

Then ψ extends uniquely to a local ring homomorphism

$$g_0 : A_0 \to B_0 \qquad \text{such that} \qquad g_0(X_1) = Y_1, \dots, g_0(X_n) = Y_n.$$

As f is continuous for the $\mathfrak{m}_A$-adic and $\mathfrak{m}_B$-adic topologies, the resulting diagram commutes :

$$\begin{array}{ccc} A_0 & \xrightarrow{\varphi_A} & A \\ {\scriptstyle g_0}\downarrow & & \downarrow{\scriptstyle f} \\ B_0 & \xrightarrow{\varphi_B} & B. \end{array}$$

Set $C := B_0 \otimes_{A_0} A$; then f factors through $g := g_0 \otimes_{A_0} A : A \to C$ and a map $h : C \to B$.

Claim 9.7.32. g_0 is a flat ring homomorphism.

Proof of the claim. Set $X_\bullet := (X_1, \dots, X_n)$ and $Y_\bullet := (Y_1, \dots, Y_n)$; as $X_\bullet$ is a regular sequence in A_0, the Koszul complex $\mathbf{K}(X_\bullet, A_0)$ is a resolution of the quotient R_A of A_0 (proposition 7.8.7); by the same token, $B_0 \otimes_{A_0} \mathbf{K}_\bullet(X_\bullet, A_0) = \mathbf{K}_\bullet(Y_\bullet, B_0)$ is a resolution of $B_0 \otimes_{A_0} R_A$, since $Y_\bullet$ is a regular sequence in B_0. This shows that $\mathrm{Tor}_1^{A_0}(B_0, R_A) = 0$, and clearly $B_0 \otimes_{A_0} R_A$ is a flat R_A-algebra; then the assertion follows from the local flatness criterion ([126, Th.22.3]). $\Diamond$

Clearly g_0 is a local map, hence it is faithfully flat, by claim 9.7.32, so the same holds for g. Moreover, $C/\mathfrak{m}_A C = B_0 \otimes_{A_0} k_A = k_B[[Y_{n+1}, \dots, Y_m]]$ is a regular ring, and clearly h is surjective, as the same holds for φ_B. Then $A \xrightarrow{g} C \xrightarrow{h} B$ is the sought Cohen factorization of f.

(ii): According to [63, Ch.0, Th.19.7.1, Th.19.7.2, Th.22.2.6] there exists a complete local ring $(C_0, \mathfrak{m}_{C_0})$, and a local ring homomorphism $g_0 : A \to C_0$ formally smooth for the $\mathfrak{m}_A$-adic and $\mathfrak{m}_{C_0}$-adic topologies, with an isomorphism of κ_A-algebras $C_0/\mathfrak{m}_{C_0} \xrightarrow{\sim} \kappa_B$. Let $\pi_B : B \to \kappa_B$ and $\pi_{C_0} : C \to \kappa_B$ be the natural projections; we obtain a commutative diagram :

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g_0}\downarrow & & \downarrow{\scriptstyle \pi_B} \\ C_0 & \xrightarrow{\pi_{C_0}} & \kappa_B \end{array}$$

and by [63, Ch.0, Cor.19.3.11] there exists a ring homomorphism $h_0 : C_0 \to B$ such that $\pi_B \circ h_0 = \pi_{C_0}$ and $h_0 \circ g_0 = f$. Pick $y_1, \dots, y_m$ as in the proof of (i), set $C := C_0[[Y_1, \dots, Y_m]]$, let $\mathfrak{m}_C \subset C$ be the maximal ideal, and $h : C \to B$ the unique continuous ring homomorphism such that $h_{|C_0} = h_0$ and $h(Y_i) = y_i$ for $i = 1, \dots, m$. Then h is surjective, and C is a formally smooth A-algebra ([63, Prop.19.3.5(i)]), for the $\mathfrak{m}_A$-adic and $\mathfrak{m}_C$-adic topologies. □

Proposition 9.7.33. *Let $f : (A, \mathfrak{m}_A) \to (B, \mathfrak{m}_B)$ be a local ring homomorphism of local noetherian rings, and k_B the residue field of B. The following conditions are equivalent :*

(a) *For every Cohen factorization $A \xrightarrow{g} C \xrightarrow{h} B^\wedge$ of f, the kernel of h is generated by a regular sequence.*
(b) *f is a homomorphism of complete intersection.*
(c) $H_i(\mathbb{L}_{B/A} \otimes_B k_B) = 0$ *for every $i > 1$.*
(d) $H_2(\mathbb{L}_{B/A} \otimes_B k_B) = 0$.

Proof. Obviously (a)⇒(b) and (c)⇒(d).

(b)⇒(c): First, since $B^\wedge$ is a flat B-algebra, by [103, Ch.II, Prop.2.2.1] we have :

$$\mathbb{L}_{B^\wedge/B} \otimes_{B^\wedge} k_B \xrightarrow{\sim} \mathbb{L}_{k_B/k_B} \xrightarrow{\sim} 0 \qquad \text{in } \mathsf{D}(k_B\text{-}\mathbf{Mod}).$$

Next, from the sequence of maps $A \xrightarrow{f} B \to B^\wedge$ we get a distinguished triangle :

$$\mathbb{L}_{B/A} \otimes_B k_B \xrightarrow{\beta} \mathbb{L}_{B^\wedge/A} \otimes_{B^\wedge} k_B \to \mathbb{L}_{B^\wedge/B} \otimes_{B^\wedge} k_B = 0$$

([103, Ch.II, Prop.2.1.2]), so β is an isomorphism in $\mathsf{D}(k_B\text{-}\mathbf{Mod})$. We may therefore replace B by $B^\wedge$, and assume from start that B is complete and separated. In this case, from a given Cohen factorization $A \xrightarrow{g} C \xrightarrow{h} B$, we get a distinguished triangle

$$\mathbb{L}_{C/A} \otimes_C k_B \to \mathbb{L}_{B/A} \otimes_B k_B \to \mathbb{L}_{B/C} \otimes_B k_B$$

and if $\mathrm{Ker}\, h$ is generated by a regular sequence, we have as well $H_i(\mathbb{L}_{B/C} \otimes_B k_B) = 0$ for every $i > 1$, by theorem 9.6.46. We are thus reduced to checking that $H_i(\mathbb{L}_{C/A} \otimes_C k_B) = 0$ for every $i > 1$. Let also k_A be the residue field of A, and set $C' := k_A \otimes_A C$; since C is a flat A-algebra, invoking again [103, Ch.II, Prop.2.2.1] we get a natural identification :

$$\text{(9.7.34)} \qquad \mathbb{L}_{C/A} \otimes_C k_B \xrightarrow{\sim} \mathbb{L}_{C'/k_A} \otimes_{C'} k_B.$$

Hence it suffices to show that $H_i(\mathbb{L}_{C'/k_A} \otimes_{C'} k_B) = 0$ for every $i > 1$. Recall that the smallest subfield k_0 of k_A is perfect, so C' is a geometrically regular k_0-algebra, whence :

$$\text{(9.7.35)} \qquad H_i(\mathbb{L}_{C'/k_0} \otimes_{C'} k_B) = 0 \qquad \text{for every } i > 0$$

(claim 9.7.10). By the same token, we have $H_i(\mathbb{L}_{k_A/k_0}) = 0$ for every $i > 0$. Lastly, the sequence of ring homomorphisms $k_0 \to k_A \to C'$ yields a distinguished triangle :

$$\mathbb{L}_{k_A/k_0} \otimes_{k_A} k_B \to \mathbb{L}_{C'/k_0} \otimes_{C'} k_B \to \mathbb{L}_{C'/k_A} \otimes_{C'} k_B$$

from which, combining with the foregoing, we get the sought vanishing.

(d)⇒(a): Arguing as in the foregoing, we may assume that $B = B^\wedge$; then, a given Cohen factorization $A \to C \to B$ yields an exact sequence

$$H_2(\mathbb{L}_{B/A} \otimes_B k_B) \to H_2(\mathbb{L}_{B/C} \otimes_B k_B) \xrightarrow{\partial} H_1(\mathbb{L}_{C/A} \otimes_C k_B)$$

([103, Ch.II, Prop.2.1.2]) and in view of theorem 9.6.46, we are reduced to checking that the boundary map ∂ is the zero map. To this aim, we consider the commutative diagram of rings :

$$\begin{array}{ccccc} \mathbb{Z} & \longrightarrow & C & \longrightarrow & B \\ \downarrow & & \| & & \| \\ A & \longrightarrow & C & \longrightarrow & B \end{array}$$

from which, by the functoriality property of the transitivity triangle ([103, Ch.II, (2.1.1)]), we deduce a commutative diagram of k_B-modules :

$$\begin{array}{ccc} H_2(\mathbb{L}_{B/C} \otimes_B k_B) & \xrightarrow{\partial} & H_1(\mathbb{L}_{C/\mathbb{Z}} \otimes_C k_B) \\ \| & & \downarrow{\scriptstyle \varphi} \\ H_2(\mathbb{L}_{B/C} \otimes_B k_B) & \xrightarrow{\partial} & H_1(\mathbb{L}_{C/A} \otimes_C k_B) \end{array}$$

which further reduces to checking that φ is the zero map. Now, define k_A, k_0 and C' as in the foregoing; we get a commutative diagram of k_B-modules :

$$\begin{array}{ccc} H_1(\mathbb{L}_{C/\mathbb{Z}} \otimes_C k_B) & \longrightarrow & H_1(\mathbb{L}_{C'/k_0} \otimes_{C'} k_B) \\ {\scriptstyle \varphi}\downarrow & & \downarrow \\ H_1(\mathbb{L}_{C/A} \otimes_C k_B) & \longrightarrow & H_1(\mathbb{L}_{C'/k_A} \otimes_{C'} k_B) \end{array}$$

whose bottom horizontal arrow is bijective, by virtue of the natural identification (9.7.34), and whose top horizontal arrow is the zero map, by (9.7.35), whence the assertion. □

Corollary 9.7.36. *Let $(A, \mathfrak{m}_A) \xrightarrow{f} (B, \mathfrak{m}_B) \xrightarrow{g} (C, \mathfrak{m}_C)$ be two local homomorphisms of noetherian local rings. The following holds :*

(i) *If f and g are of complete intersection, the same holds for $g \circ f$.*
(ii) *If g and $g \circ f$ are of complete intersection, the same holds for f.*
(iii) *If f is formally smooth for the $\mathfrak{m}_A$-adic and $\mathfrak{m}_B$-adic topologies, and $g \circ f$ is of complete intersection, then g is of complete intersection.*
(iv) *If f is formally smooth for the $\mathfrak{m}_A$-adic and $\mathfrak{m}_B$-adic topologies, or else surjective with kernel generated by a regular sequence of $\mathfrak{m}_A$, then f is of complete intersection.*

Proof. (i): Let k_C be the residue field of C; we consider the induced distinguished triangle :

(9.7.37) $$\mathbb{L}_{B/A} \otimes_B k_C \to \mathbb{L}_{C/A} \otimes_C k_C \to \mathbb{L}_{C/B} \otimes_C k_C$$

([103, Ch.II, Prop.2.1.2]). By proposition 9.7.33, the assumptions of (i) amount to the vanishing of $H_2(\mathbb{L}_{B/A} \otimes_B k_C)$ and $H_2(\mathbb{L}_{C/B} \otimes_C k_C)$, whence the vanishing of $H_2(\mathbb{L}_{C/A} \otimes_C k_C)$, which yields the assertion, again by proposition 9.7.33.

(ii): We argue likewise with (9.7.37) : by proposition 9.7.33, the assumptions imply that $H_2(\mathbb{L}_{C/A} \otimes_C k_C) = H_3(\mathbb{L}_{C/B} \otimes_C k_C) = 0$, whence $H_2(\mathbb{L}_{B/A} \otimes_B k_C) = 0$, which yields the assertion, again by proposition 9.7.33.

(iii): By proposition 9.7.33, the assumption on $g \circ f$ yields $H_2(\mathbb{L}_{C/A} \otimes_C k_C) = 0$, and by claim 9.7.10, the assumption on f yields $H_1(\mathbb{L}_{B/A} \otimes_B k_C) = 0$. Then, from (9.7.37) we get $H_2(\mathbb{L}_{C/B} \otimes_C k_C) = 0$, which yields the assertion, as usual by proposition 9.7.33.

(iv): It follows immediately from theorem 9.6.46, claim 9.7.10, and proposition 9.7.33. □

Lemma 9.7.38. *Let $f : (A, \mathfrak{m}_A) \to (B, \mathfrak{m}_B)$ be a local ring homomorphism of complete intersection of local noetherian rings, M an A-module of finite type, and $L_\bullet$ a complex of free A-modules of finite rank. The following holds :*

(i) *If $H_n(B \otimes_A L_\bullet) = 0$ for some $n \in \mathbb{Z}$, then $H_n(L_\bullet) = 0$.*

(ii) *If $\mathrm{Tor}_1^A(M, B) = 0$, then $\mathrm{Tor}_i^A(M, B) = 0$ for every $i > 0$.*

Proof. (i): Pick a Cohen factorization $A \xrightarrow{g} C \xrightarrow{h} B^\wedge$ of f, so that $\mathrm{Ker}\, h$ is generated by a regular sequence of C; since g is faithfully flat, it suffices to check that $H_n(C \otimes_A L_\bullet) = 0$, and since $B^\wedge$ is a flat B-algebra, we have $H_n(B^\wedge \otimes_A L_\bullet) = 0$. Thus, we may replace A by C, B by $B^\wedge$, and $L_\bullet$ by $C \otimes_A L_\bullet$, and assume from start that f is surjective, and its kernel is generated by a regular sequence $x_\bullet := (x_1, \dots, x_k)$ of $\mathfrak{m}_A$. Next, arguing by induction on the length k of $x_\bullet$, we are easily reduced to showing the following. If $x \in \mathfrak{m}_A$ is a regular element, and $H_n(L_\bullet/xL_\bullet) = 0$, then $H_n(L_\bullet) = 0$. However, since L_i is a free A-module for every $i \in \mathbb{Z}$, we have a short exact sequence of complexes :

$$0 \to L_\bullet \xrightarrow{x \cdot \mathbf{1}_{L_\bullet}} L_\bullet \to L_\bullet/xL_\bullet \to 0$$

whence a short exact sequence of A-modules :

$$H_n(L_\bullet) \xrightarrow{x \cdot \mathbf{1}_{H_n(L_\bullet)}} H_n(L_\bullet) \to H_n(L_\bullet/xL_\bullet) = 0.$$

But $H_n(L_\bullet)$ is an A-module of finite type, so $H_n(L_\bullet) = 0$, by Nakayama's lemma.

(ii): Since $B^\wedge$ is a faithfully flat B-algebra, we have a natural identification

$$B^\wedge \otimes_B \mathrm{Tor}_i^A(M, B) \xrightarrow{\sim} \mathrm{Tor}_i^A(M, B^\wedge) \qquad \text{for every } i \in \mathbb{N}$$

and it suffices to check that $B^\wedge \otimes_B \mathrm{Tor}_i^A(M, B) = 0$ for every $i > 0$. Moreover, the induced map $A \to B^\wedge$ is still of complete intersection, by corollary 9.7.36(i,iv). Thus, we may replace B by $B^\wedge$, and assume that B is $\mathfrak{m}_B$-adically complete, in which case f admits a Cohen factorization $(A, \mathfrak{m}_A) \xrightarrow{g} (C, \mathfrak{m}_C) \xrightarrow{h} (B, \mathfrak{m}_B)$ such that $\mathrm{Ker}\, h$ is generated by a regular sequence of $\mathfrak{m}_C$, and since g is faithfully flat, we have a natural identification

$$C \otimes_A \mathrm{Tor}_i^A(M, B) \xrightarrow{\sim} \mathrm{Tor}_i^C(C \otimes_A M, B) \qquad \text{for every } i \in \mathbb{N}.$$

Again, it suffices to check that $C \otimes_A \mathrm{Tor}_i^A(M, B) = 0$ for every $i > 0$, and h is of complete intersection, by corollary 9.7.36(iv). Thus, we may replace f by h and M by $A^\wedge \otimes_A M$, and assume that f is surjective, with kernel generated by a regular sequence $x_\bullet := (x_1, \dots, x_k)$ of $\mathfrak{m}_A$. Now, in view of lemma 7.8.14(ii), proposition 7.8.15 and remark 7.8.16(i), the assumption of (ii) is equivalent to the vanishing of $H_1(x_\bullet, M)$. By the same token, the latter implies that $x_\bullet$ is completely secant on M. Then the assertion follows, by invoking again lemma 7.8.14(ii). □

9.7.39. Let $\mathscr{L}$ be the category whose objects are the local noetherian rings, and whose morphisms are the local ring homomorphisms. For every $(B, \mathfrak{m}_B) \in \mathrm{Ob}(\mathscr{L})$, denote by

$$\mathrm{Hom}_{\mathscr{L}}(-, B) : \mathscr{L} \to \mathbf{Set}$$

the presheaf on $\mathscr{L}$ represented by $(B, \mathfrak{m}_B)$, and for every $k \in \mathbb{N}$, set $B_k := B/\mathfrak{m}_B^{k+1}$; the projection $\pi_k : B \to B_k$ induces a natural transformation

$$\pi_{k*} : \mathrm{Hom}_{\mathscr{L}}(-, B) \to \mathrm{Hom}_{\mathscr{L}}(-, B_k) \qquad (f : A \to B) \mapsto (\pi_k \circ f : A \to B_k).$$

Proposition 9.7.40. *With the notation of* (9.7.39)*, for every $(B, \mathfrak{m}_B) \in \mathrm{Ob}(\mathscr{L})$ there exists $n \in \mathbb{N}$, and a family of subsets*

$$(\Sigma_{B,A} \subset \mathrm{Hom}_{\mathscr{L}}(A, B_n) \mid (A, \mathfrak{m}_A) \in \mathrm{Ob}(\mathscr{L}))$$

such that for every $(A, \mathfrak{m}_A) \in \mathrm{Ob}(\mathscr{L})$ we have :

$$\pi_{n*,A}^{-1}(\Sigma_{B,A}) = \{f \in \mathrm{Hom}_{\mathscr{L}}(A, B) \mid f \text{ is a morphism of complete intersection}\}.$$

Proof. By claim 7.9.18, for every $i > 0$ we have $c_i \in \mathbb{N}$ such that the projection $B_{n+c_i} \to B_n$ induces the zero map

$$\mathrm{Tor}_i^B(B_{n+c_i}, B_1) \to \mathrm{Tor}_i^B(B_n, B_1) \qquad \text{for every } n \in \mathbb{N}.$$

Notice that $B_0 = k_B$, the residue field of B. Then, by [3, Ch.X, Cor.13], for every $i > 0$ there exists $d_i \in \mathbb{N}$ such that the projection $B_{n+d_i} \to B_n$ induces the zero map

$$H_i(\mathbb{L}_{B_{n+d_i}/B} \otimes_B k_B) \to H_i(\mathbb{L}_{B_n/B} \otimes_B k_B) \qquad \text{for every } n \in \mathbb{N}.$$

Now, for every $n \in \mathbb{N}$ and every morphism $f : (A, \mathfrak{m}_A) \to (B, \mathfrak{m}_B)$ of $\mathscr{L}$, the sequence of ring homomorphisms $A \xrightarrow{f} B \xrightarrow{\pi_n} B_n$ induces a distinguished triangle ([103, Ch.II, Prop.2.1.2])

$$\mathbb{L}_{B/A} \otimes_B k_B \to \mathbb{L}_{B_n/A} \otimes_B k_B \to \mathbb{L}_{B_n/B} \otimes_B k_B$$

whence, for every $i \in \mathbb{N}$, an exact sequence of k_B-modules

$$H_{i+1}(\mathbb{L}_{B_n/B} \otimes_B k_B) \to H_i(\mathbb{L}_{B/A} \otimes_B k_B) \xrightarrow{\varphi_{i,n}} H_i(\mathbb{L}_{B_n/A} \otimes_B k_B) \to H_i(\mathbb{L}_{B_n/B} \otimes_B k_B).$$

Taking into account lemma 7.8.19, it follows easily that the the induced inverse systems

$$\mathrm{Ker}\,\varphi_{i,\bullet} := (\mathrm{Ker}\,\varphi_{i,n} \mid n \in \mathbb{N}) \qquad \text{and} \qquad \mathrm{Coker}\,\varphi_{i,\bullet} := (\mathrm{Coker}\,\varphi_{i,n} \mid n \in \mathbb{N})$$

are uniformly essentially zero, for every $i > 0$. Say that e and e' are the steps of $\mathrm{Ker}\,\varphi_{2,\bullet}$ and respectively $\mathrm{Coker}\,\varphi_{2,\bullet}$ (see definition 7.8.18(ii)), and notice that these steps depend only on B (and not on A). Then, a simple diagram chase shows that $\varphi_{2,n}$ is injective for every $n \geq e$, and :

$$\mathrm{Im}(\varphi_{2,e}) = \mathrm{Im}(\psi_{e,e'} : H_2(\mathbb{L}_{B_{e+e'}/A} \otimes_B k_B) \to H_2(\mathbb{L}_{B_e/A} \otimes_B k_B))$$

so that $\varphi_{2,e}$ induces an isomorphism $H_2(\mathbb{L}_{B/A} \otimes_B k_B) \xrightarrow{\sim} \mathrm{Im}(\psi_{e,e'})$. Now, if $g : A \to B$ is another local map, denote by B' the A-algebra whose underlying ring is B, and whose structure map is g; as in the foregoing, we set $B'_n := B'/\mathfrak{m}_B^{n+1}$ for every $n \in \mathbb{N}$. Then, from the sequence of maps $A \xrightarrow{\pi_{e+e'} \circ g} B'_{e+e'} \to B'_e$ we get a corresponding map

$$\psi'_{e,e'} : H_2(\mathbb{L}_{B'_{e+e'}/A} \otimes_{B'} k_B) \to H_2(\mathbb{L}_{B'_e/A} \otimes_{B'} k_B)$$

and since the steps e and e' depend only on the underlying ring $B = B'$, we deduce as well an isomorphism

$$H_2(\mathbb{L}_{B'/A} \otimes_{B'} k_B) \xrightarrow{\sim} \mathrm{Im}(\psi'_{e,e'}).$$

Lastly, if $\pi_{e+e'} \circ f = \pi_{e+e'} \circ g$, then $\psi_{e,e'} = \psi'_{e,e'}$, whence an isomorphism

$$H_2(\mathbb{L}_{B'/A} \otimes_{B'} k_B) \xrightarrow{\sim} H_2(\mathbb{L}_{B'/A} \otimes_{B'} k_B)$$

and especially, g is of complete intersection, if and only if the same holds for f, again by proposition 9.7.33. Hence, the proposition holds with $n := e + e'$. □

Theorem 9.7.41. *Let $(A, \mathfrak{m}_A)$ be a quasi-excellent local ring, $(B, \mathfrak{m}_B)$ an excellent henselian local ring, and $f : A \to B$ a local ring homomorphism. Let k_A and k_B be the residue fields of A and B, and suppose that $H_1\mathbb{L}_{k_B/k_A}$ is a finite dimensional k_B-vector space. Then B is the colimit of a filtered system $(B_\lambda \mid \lambda \in \Lambda)$ of local A-algebras of essentially finite type, such that:*

(i) *the transition map $B_\lambda \to B_\mu$ is local of complete intersection, for every $\mu \geq \lambda$ in Λ*

(ii) *the universal cocone $(B_\lambda \to B \mid \lambda \in \Lambda)$ consists of local maps of complete intersection.*

Moreover, if the $\mathfrak{m}_B$-adic completion $B^\wedge$ of B is a domain, then B_λ is a domain for every $\lambda \in \Lambda$.

Proof. Assertion (i) follows from (ii) and corollary 9.7.36(ii). Hence, let us show (ii).

• Suppose first that B is $\mathfrak{m}_B$-adically complete and separated. Let $A^\wedge$ be the $\mathfrak{m}_A$-adic completion of A, and $j_A : A \to A^\wedge$ the completion map; by proposition 9.7.31(ii), the completion $f^\wedge : A^\wedge \to B$ of f admits a Cohen factorization $A^\wedge \xrightarrow{g^\wedge} (C, \mathfrak{m}_C) \xrightarrow{h} B$, where $g^\wedge$ is formally smooth for the $\mathfrak{m}_A$-adic and $\mathfrak{m}_C$-adic topologies. Then the same holds for $g := g^\wedge \circ j_A : A \to C$, and since A is quasi-excellent, the induced morphism $\mathrm{Spec}\, C \to \mathrm{Spec}\, A$ is regular (theorem 9.7.15). Then, by [138, Th.1.8], C is the colimit of a filtered system $(C_\lambda \mid \lambda \in \Lambda)$ of smooth A-algebras. For every $\lambda \in \Lambda$, let $\mathfrak{p}_\lambda \subset C_\lambda$ be the preimage of $\mathfrak{m}_C$; clearly C is also the filtered colimit of the induced system of local A-algebras of essentially finite type $(D_\lambda := C_{\lambda,\mathfrak{p}_\lambda} \mid \lambda \in \Lambda)$, with local transition maps. By corollary 9.7.36(iv), the map g is of complete intersection, and by corollary 9.7.36(iii), the same then holds for each map $D_\lambda \to C$ of the universal cocone. Now, pick an exact complex of C-modules of finite type

$$\Sigma_\bullet \qquad : \qquad C^{\oplus m} \xrightarrow{u} C^{\oplus n} \xrightarrow{v} C \xrightarrow{h} B.$$

We may then find $\lambda \in \Lambda$, and D_λ-linear maps

$$D_\lambda^{\oplus m} \xrightarrow{u_\lambda} D_\lambda^{\oplus n} \xrightarrow{v_\lambda} D_\lambda \qquad \text{such that} \qquad C \otimes_{D_\lambda} u_\lambda = u \quad \text{and} \quad C \otimes_{D_\lambda} v_\lambda = v$$

and we set $u_\mu := D_\mu \otimes_{D_\lambda} u_\lambda$, $v_\mu := D_\mu \otimes_{D_\lambda} v_\lambda$, and $B_\lambda := \mathrm{Coker}\, v_\lambda$ for every $\mu \in \Lambda$ with $\mu \geq \lambda$. Since $v \circ u = 0$, we may find then also find $\mu \geq \lambda$ such that $v_\mu \circ u_\mu = 0$. After replacing Λ by a cofinal subset, we thus obtain a compatible system of complexes :

$$\Sigma_{\lambda,\bullet} \qquad : \qquad D_\lambda^{\oplus m} \xrightarrow{u_\lambda} D_\lambda^{\oplus n} \xrightarrow{v_\lambda} D_\lambda \xrightarrow{h_\lambda} B_\lambda \qquad \text{for every } \lambda \in \Lambda$$

whose colimit is $\Sigma_\bullet$, and clearly the induced filtered system $(B_\lambda \mid \lambda \in \Lambda)$, whose colimit is B, consists of local A-algebras, and its transition maps are local homomorphisms of A-algebras. By construction, $\Sigma_\bullet = C \otimes_{D_\lambda} \Sigma_{\lambda,\bullet}$, so $\Sigma_{\lambda,\bullet}$ is exact for every $\lambda \in \Lambda$, by lemma 9.7.38(i). It follows easily that :

$$\mathrm{Tor}_1^{D_\lambda}(B_\lambda, C) = 0 \qquad \text{for every } \lambda \in \Lambda$$

and then we have $\mathrm{Tor}_i^{D_\lambda}(B_\lambda, C) = 0$ for every $\lambda \in \Lambda$ and every $i > 0$ (lemma 9.7.38(ii)). By [103, Ch.II, Prop.2.2.1] we deduce, for every $\lambda \in \Lambda$, a natural identification

$$\mathbb{L}_{B/B_\lambda} \otimes_B k_B \xrightarrow{\sim} \mathbb{L}_{C/D_\lambda} \otimes_C k_B \qquad \text{in } \mathsf{D}(k_B\text{-}\mathbf{Mod})$$

and taking into account proposition 9.7.33, we finally obtain (ii).

• Suppose next that $(B, \mathfrak{m}_B)$ is henselian, and let $j_B : B \to B^\wedge$ be the completion map. By the foregoing, $B^\wedge$ is the colimit of a filtered system $(\mathsf{B}_\lambda \mid \lambda \in \Lambda)$ of local A-algebras of essentially finite type, and such that the universal cocone

$$(g_\lambda : \mathsf{B}_\lambda \to B^\wedge \mid \lambda \in \Lambda)$$

consists of maps of complete intersection. For every $k \in \mathbb{N}$, let $\pi_k : B^\wedge \to B^\wedge/\mathfrak{m}_B^{k+1}B^\wedge$ be the projection, and for every $\lambda \in \Lambda$, pick $n_\lambda \in \mathbb{N}$ such that every local ring homomorphism $h : \mathsf{B}_\lambda \to B^\wedge$ with $\pi_{n_\lambda} \circ h = \pi_{n_\lambda} \circ g_\lambda$ is of complete intersection (proposition 9.7.40).

Let then A-$\mathsf{Alg}_{\mathrm{eft}}$ be the full subcategory of A-Alg whose objects are the A-algebras of essentially finite type, and $\varphi : A\text{-}\mathsf{Alg}_{\mathrm{eft}} \to A\text{-}\mathsf{Alg}$ the inclusion functor; moreover, let $\mathscr{C}$ be the full subcategory of $\varphi A\text{-}\mathsf{Alg}_{\mathrm{eft}}/B$ whose set of objects consists of all the maps of A-algebras $h : \mathsf{B}_\lambda \to B$, where $\lambda \in \Lambda$ is any element, and $\pi_{n_\lambda} \circ j_B \circ h = \pi_{n_\lambda} \circ g_\lambda$. For any such h, the map $j_B \circ h$ is of complete intersection, due to the choice of n_λ, and the same holds for j_B (corollary 9.7.36(iv)), and then also for h, by corollary 9.7.36(ii). It is also easily seen that the category $\varphi A\text{-}\mathsf{Alg}_{\mathrm{eft}}/B$ is filtered, and that B represents the colimit of the source functor

$$\varphi A\text{-}\mathsf{Alg}_{\mathrm{eft}}/B \to A\text{-}\mathsf{Alg} \qquad (g : C \to B) \mapsto C$$

By virtue of lemma 1.5.7(iii.b) and proposition 1.5.21(ii), we are thus reduced to checking :

Claim 9.7.42. For every A-algebra of essentially finite type C, and every morphism $g : C \to B$ of A-algebras, there exist a morphism $h : \mathsf{B}_\lambda \to B$ of $\mathscr{C}$, and a morphism of A-algebras $l : C \to \mathsf{B}_\lambda$ such that $h \circ l = g$.

Proof of the claim. Since A is noetherian, C is an A-algebra of essentially of finite presentation; hence, we may find $\lambda \in \Lambda$ and a morphism $l : C \to \mathsf{B}_\lambda$ of A-algebras such that $g_\lambda \circ l = j_B \circ g$. We regard B and B_λ as C-algebras, via the maps g and respectively l, and $B^\wedge$ as a B-algebra, via j_B; the maps j_B and g_λ then define a map of B-algebras $t : B \otimes_C \mathsf{B}_\lambda \to B^\wedge$, and notice that $B \otimes_C \mathsf{B}_\lambda$ is a B-algebra of essentially finite type. According to [138, Th.1.3], we may then find a map of B-algebras $t' : B \otimes_C \mathsf{B}_\lambda \to B$ such that $\pi_{n_\lambda} \circ j_B \circ t' = \pi_{n_\lambda} \circ t$. It is easily seen that the resulting map $h : \mathsf{B}_\lambda \to B$ with $h(b) := t'(1 \otimes b)$ for every $b \in \mathsf{B}_\lambda$, will do. ◊

- Lastly, suppose that $B^\wedge$ is a domain, and for any given $\lambda \in \Lambda$, choose a Cohen factorization $B_\lambda \xrightarrow{g_\lambda} B'_\lambda \xrightarrow{h_\lambda} B^\wedge$, with $I := \operatorname{Ker} h_\lambda$ generated by a regular sequence; thus, the graded ring $\mathrm{gr}_\bullet B'_\lambda$ associated with the I-adic filtration on B'_λ is isomorphic to a polynomial algebra $B^\wedge[T_1, \dots, T_n]$ (proposition 7.8.15). Hence $\mathrm{gr}_\bullet B'_\lambda$ is a domain, and since the I-adic filtration is separated on B'_λ ([126, Th.8.10(i)]), B'_λ is a domain; on the other hand, g_λ is faithfully flat, hence injective, so B_λ is a domain. □

9.8. **Normalization, weak normalization and p-root closure.** Recall that for an inclusion of rings $A \subset B$, we denote by $\mathrm{i.c.}(A, B)$ the integral closure of A into B (see definition 7.6.3(i)); recall moreover that

(9.8.1) $$\mathrm{i.c.}(S^{-1}A, S^{-1}B) = S^{-1}\mathrm{i.c.}(A, B) \qquad \text{for every subset } S \subset A.$$

More generally, let X be a scheme, $\mathscr{A} \subset \mathscr{B}$ an inclusion of quasi-coherent $\mathscr{O}_X$-algebras; then the rule : $U \mapsto \mathscr{C}(U) := \mathrm{i.c.}(\mathscr{A}(U), \mathscr{B}(U))$ for every affine open subset $U \subset X$ defines a subpresheaf $\mathscr{C} \subset \mathscr{B}$ on the site of affine open subsets of X, and (9.8.1) easily implies that $\mathscr{C}$ is a sheaf on this site, so it extend uniquely on the Zariski site of X to a quasi-coherent $\mathscr{O}_X$-subalgebra of $\mathscr{B}$, which we call the *integral closure of $\mathscr{A}$ in $\mathscr{B}$*, and denote

$$\mathrm{i.c.}(\mathscr{A}, \mathscr{B}).$$

Also, for every ring A, we let $\mathscr{N}_A$ be the nilradical of A, and set $A_{\mathrm{red}} := A/\mathscr{N}_A$, the maximal reduced quotient of A. Every ring homomorphism $f : A \to A'$ induces a ring homomorphism $f_{\mathrm{red}} : A_{\mathrm{red}} \to A'_{\mathrm{red}}$. Recall that

$$\mathscr{N}_{S^{-1}A} = S^{-1}A \qquad \text{and} \qquad S^{-1}(A_{\mathrm{red}}) = (S^{-1}A)_{\mathrm{red}} \qquad \text{For every subset } S \subset A.$$

Arguing as in the foregoing, we deduce that for every quasi-coherent $\mathscr{O}_X$-algebra $\mathscr{A}$, the rule : $U \mapsto \mathscr{N}_{\mathscr{A}(U)}$ for every affine open subset $U \subset X$ extends uniquely to a quasi-coherent ideal

$$\mathscr{N}_{\mathscr{A}} \subset \mathscr{A}$$

called the *nilradical of $\mathscr{A}$*. The $\mathscr{O}_X$-algebra $\mathscr{A}_{\mathrm{red}} := \mathscr{A}/\mathscr{N}_{\mathscr{A}}$ is quasi-coherent, and every morphism $\varphi : \mathscr{A} \to \mathscr{A}'$ of quasi-coherent $\mathscr{O}_X$-algebras induces a morphism $\varphi_{\mathrm{red}} : \mathscr{A}_{\mathrm{red}} \to \mathscr{A}'_{\mathrm{red}}$.

Lemma 9.8.2. *Let A be a Krull domain, F the field of fractions of A, B a flat A-algebra, and suppose that $B \otimes_A \kappa(\mathfrak{p})$ is reduced, for every prime ideal $\mathfrak{p} \in \operatorname{Spec} A$ of height one. Then B is integrally closed in $B \otimes_A F$.*

Proof. See [126, §12] for the basic generalities on Krull domains; especially, [126, Th.12.6] asserts that if A is a Krull domain, the natural sequence of A-modules :

$$\mathscr{E} \quad : \quad 0 \to A \to F \to \bigoplus_{\mathrm{ht}\,\mathfrak{p}=1} F/A_{\mathfrak{p}} \to 0$$

is short exact, where $\mathfrak{p} \in \operatorname{Spec} A$ ranges over the prime ideals of height one. By flatness, $\mathscr{E} \otimes_A B$ is still exact, hence $B = \bigcap_{\mathrm{ht}\,\mathfrak{p}=1} B \otimes_A A_{\mathfrak{p}}$ (where the intersection takes place in $B \otimes_A F$). We are therefore reduced to the case where A is a discrete valuation ring. Let t denote a chosen generator of the maximal ideal of A, and suppose that $x \in B \otimes_A F$ is integral over B, so that $x^n + b_1 x^{n-1} + \cdots + b_n = 0$ in $B \otimes_A F$, for some $b_1, \dots, b_n \in B$; let also $r \in \mathbb{N}$ be the minimal integer such that we have $x = t^{-r}b$ for some $b \in B$. We have to show that $r = 0$; to this aim, notice that $b^n + t^r b_1 b^{n-1} + \cdots + t^{rn} b_n = 0$ in B; if $r > 0$, it follows that the image $\bar{b}$ of b in B/tB satisfies the identity : $\bar{b}^n = 0$, therefore $\bar{b} = 0$, since by assumption, B/tB is reduced. Thus $b = tb'$ for some $b' \in B$, and $x = t^{1-r}b'$, contradicting the minimality of r. □

The following result generalizes [64, Ch.IV, Prop.6.14.4] and [66, ChIV, Prop.18.12.15].

Proposition 9.8.3. *Let $f : X \to S$ be a flat morphism of schemes with geometrically reduced fibres, $\mathscr{B}_1 \subset \mathscr{B}_2$ two quasi-coherent $\mathscr{O}_S$-algebras, and $\mathscr{B}_3 := \mathrm{i.c.}(\mathscr{B}_1, \mathscr{B}_2)$. Suppose that :*

(a) *either f is locally finitely presented,*
(b) *or else, S is locally noetherian.*

Then $f^*\mathscr{B}_3 = \mathrm{i.c.}(f^*\mathscr{B}_1, f^*\mathscr{B}_2)$ and $f^*(\mathscr{B}_1)_{\mathrm{red}} = f^*(\mathscr{B}_{1,\mathrm{red}})$.

Proof. The assertion is local on both S and X, hence we may assume that $S = \mathrm{Spec}\, A$ and $X = \mathrm{Spec}\, A'$, for a ring A and an A-algebra A'. Then also $\mathscr{B}_1$ and $\mathscr{B}_2$ are the quasi-coherent algebras associated with A-algebras $B_1 \subset B_2$, and $\mathscr{B}_3$ is the quasi-coherent algebra associated with $B_3 := \mathrm{i.c.}(B_1, B_2)$. Suppose first that condition (a) holds, and write A as the colimit of a filtered system $(A_\lambda \mid \lambda \in \Lambda)$ of noetherian subrings; we remark :

Claim 9.8.4. Let $A_\bullet := (A_\lambda \mid \lambda \in \Lambda)$ be a filtered system of rings, A the colimit of $A_\bullet$, and $X \to S := \mathrm{Spec}\, A$ a flat morphism of schemes of finite presentation, with geometrically reduced fibres. Then for some $\lambda \in \Lambda$ there exists a flat morphism of schemes with geometrically reduced fibres $X_\lambda \to S_\lambda$, and an isomorphism of S-schemes $S \times_{S_\lambda} X_\lambda \xrightarrow{\sim} X$.

Proof of the claim. By [65, Ch.IV, Th.8.8.2(ii)], we may find $\lambda \in \Lambda$, a finitely presented morphism $f_\lambda : X_\lambda \to S_\lambda$, and an isomorphism of S-schemes : $X \xrightarrow{\sim} X_\lambda \times_{S_\lambda} S$. We set $S_\mu := \mathrm{Spec}\, A_\mu, X_\mu := X_\lambda \times_{S_\lambda} S_\mu$, and $f_\mu := S_\mu \times_{S_\lambda} f_\lambda : X_\mu \to S_\mu$ for every $\mu \in \Lambda$ with $\mu \geq \lambda$; after replacing Λ by a cofinal subset, we may then assume that f_μ is defined for every $\mu \in \Lambda$, and moreover that each such f_μ is flat ([65, Ch.IV, Cor.11.2.6.1]). For every $\mu \in \Lambda$, let $Z_\mu \subset S_\mu$ be the subset of all $s \in S_\mu$ such that the fibre $f_\mu^{-1}(s)$ is not geometrically reduced; by [65, Ch.IV, Th.9.7.7], Z_μ is a constructible subset of S_μ. By assumption, we have

$$\bigcap_{\mu \in \Lambda} p_\mu^{-1} Z_\mu = \varnothing$$

and it is clear that $p_\mu^{-1} Z_\mu \subset p_\lambda^{-1} Z_\lambda$ whenever $\mu \geq \lambda$. Then, $Z_\mu = \varnothing$ for some $\mu \in \Lambda$ ([63, Ch.IV, Prop.1.8.2, Cor.1.9.8]), hence we may replace Λ by a still smaller cofinal subset, and achieve that all the f_λ have geometrically reduced fibres. ◊

By claim 9.8.4, we find $\lambda \in \Lambda$ and a flat ring homomorphism $\varphi_0 : A_\lambda \to A'_0$ such that :

- all the fibres of $\mathrm{Spec}(\varphi_0)$ are geometrically reduced
- there exists an isomorphism of A-algebras $A' \xrightarrow{\sim} A \otimes_{A_\lambda} A'_0$.

We need to show that $A' \otimes_A B_3 = \mathrm{i.c.}(A' \otimes_A B_1, A' \otimes_A B_2)$ and $A' \otimes_A B_{1,\mathrm{red}} = (A' \otimes_A B_1)_{\mathrm{red}}$. However, in view of the natural identifications : $A' \otimes_A B_i \xrightarrow{\sim} A'_0 \otimes_{A_\lambda} B_i$ for $i = 1, 2, 3$, we may replace A and A' by A_λ and A'_0, and assume that A is noetherian, *i.e.* that condition (b) holds. For the first identity, arguing as in the proof of [64, Ch.IV, Prop.6.14.4 and Cor.6.14.5], we reduce to the following two separate cases :

(a') B_1 is a domain of finite type over A and B_2 is a finite extension of $\mathrm{Frac}(B_1)$
(b') B_1 and B_2 are fields, and B_1 is algebraically closed in B_2 (so $B_3 = B_1$ in this case).

In case (a') holds, B_3 is a Krull domain ([133, Th.33.10]), and the assertion follows from lemma 9.8.2. In case (b'), we may replace A by B_1 and A' by $A' \otimes_A B_1$, and assume that A is a field, A' is a geometrically reduced A-algebra, $B := B_2$ is a field extension of A such that A is algebraically closed in B, and we need to check that A' is integrally closed in $A' \otimes_A B$. Now, pick a basis $(b_i \mid i \in I)$ for the A-vector space B, with $b_{i_0} = 1$ for some $i_0 \in I$, and say that $\sum_{i\in J} a_i \otimes b_i \in A' \otimes_A B$ is integral over A' (for some finite subset $J \subset I$); we need to check that $a_i = 0$ for every $i \in J \setminus \{i_0\}$. However, since A' is reduced, it suffices to check that the image of a_i vanishes in $\kappa(\mathfrak{p})$, for every minimal prime ideal $\mathfrak{p}$ of A'. Thus, we may further assume that A' is a field, in which case it is a separable (not necessarily algebraic) extension of A, since it is a geometrically reduced A-algebra. Then, $A' \otimes_A B$ is a domain, according to [39, Ch.V, §17, n.2, Cor.], and A' is algebraically closed in $\mathrm{Frac}(A' \otimes_A B)$, by [39, Ch.V, §17, Exerc.2(b)].

For the second identity, notice that $A' \otimes_A \mathscr{N}_{B_1} \subset \mathscr{N}_{A'\otimes_A B_1}$; hence it suffices to check that if B_1 is reduced, the same holds for $A' \otimes_A B_1$. Moreover, we may assume that B_1 is an A-algebra of finite type, in which case B_1 is noetherian, and it has finitely many minimal prime ideals $\mathfrak{q}_1, \dots, \mathfrak{q}_n$, and since B_1 is reduced, the natural map $B_1 \to \prod_{i=1}^n B_{\mathfrak{q}_i}$ is injective. Then, the same holds for the induced map $A' \otimes_A B_1 \to \prod_{i=1}^n A' \otimes_A B_{\mathfrak{q}_i}$. However, $B_{\mathfrak{q}_i}$ is a field, and $A' \otimes_A B_{\mathfrak{q}_i}$ is reduced for every $i = 1, \dots, n$, since A' has geometrically reduced fibres over A; the assertion follows. □

Definition 9.8.5. Let $f : X' \to X$ be a morphism of schemes, ξ' a geometric point of X' localized at $x' \in X'$, and set $\xi := f(\xi')$. We shall say that f is *pro-smooth at the point* x' if the induced map of strictly henselian local rings

$$\mathscr{O}_{X(\xi),\xi} \to \mathscr{O}_{X'(\xi'),\xi'}$$

is ind-smooth (*i.e.* a filtered colimit of smooth ring homomorphisms).

Corollary 9.8.6. *Let $f : X' \to X$ be a morphism of schemes, ξ' a geometric point of X', $\mathscr{R} \subset \mathscr{S}$ a monomorphism of quasi-coherent $\mathscr{O}_X$-algebras, and*

$$\varphi : f^*\mathrm{i.c.}(\mathscr{R}, \mathscr{S}) \to \mathrm{i.c.}(f^*\mathscr{R}, f^*\mathscr{S})$$

the induced morphism. Suppose that f is pro-smooth at the support x' of ξ'; then we have :

(i) *f is flat at the point x'.*
(ii) *$\varphi_{\xi'}$ is an isomorphism.*

Proof. (i): Set $x := f(x'), \xi := f(\xi')$. We have a commutative diagram of local S-schemes

$$\begin{array}{ccc} X'(\xi') & \xrightarrow{f_{(\xi')}} & X(\xi) \\ {\scriptstyle\psi'}\downarrow & & \downarrow{\scriptstyle\psi} \\ X'(x') & \xrightarrow{f_{(x')}} & X(x) \end{array}$$

whose vertical arrows are pro-étale morphisms, and whose top horizontal arrow is pro-smooth by assumption. Especially, since ψ' is faithfully flat, the same holds for $f_{(x')}$, *i.e.* f is flat at x'.

(ii): We may assume that $X = \mathrm{Spec}\, A$ for some ring A. Let $g : X'(\xi') \to X'$ be the natural morphism. It suffices to show that $g^*\varphi$ is an isomorphism. However, $g^*\mathrm{i.c.}(f^*\mathscr{R}, f^*\mathscr{S}) = \mathrm{i.c.}(g^*f^*\mathscr{R}, g^*f^*\mathscr{S})$ by proposition 9.8.3, and $g^*\varphi$ is the induced morphism

$$(f \circ g)^*\mathrm{i.c.}(\mathscr{R}, \mathscr{S}) \to \mathrm{i.c.}((f \circ g)^*\mathscr{R}, (f \circ g)^*\mathscr{S}).$$

Notice that $f \circ g$ is induced by an ind-smooth ring homomorphism $A \to \mathscr{O}_{X',\xi'}$, so we may invoke again proposition 9.8.3 to conclude. □

9.8.7. Recall that a morphism of schemes $\varphi : X \to Y$ is called *universally injective* or *radicial* if for every field K, the induced map on K-valued points $X(K) \to Y(K)$ is injective ([59, Ch.I, Def.3.5.4]). It is easily seen that the latter holds if and only if φ is injective and the residue field extension $\kappa(x) \to \kappa(\varphi(x))$ is purely inseparable for every $x \in X$. In the rest of this section we study certain classes of ring homomorphisms $f : A \to B$ such that $\mathrm{Spec}\, f$ is radicial, in which case we just say that f is *radicial*. We shall see that f is radicial if and only if the kernel I_f of the multiplication law $B \otimes_A B \to B$ lies in the nilradical $\mathscr{N}_{B\otimes_A B}$ (corollary 9.8.13(i)).

Lemma 9.8.8. (i) *Let S be a scheme, and $\varphi : Y \to X$ and $\varphi' : Y' \to X'$ two surjective morphisms of S-schemes. Then $\varphi \times_S \varphi' : Y \times_S Y' \to X \times_S X'$ is surjective.*

(ii) *Let $f : A \to B$ be a ring homomorphism such that* $\mathrm{Spec}\, f$ *is surjective; then* $\mathrm{Ker}\, f \subset \mathscr{N}_A$.

(iii) *Let A be a ring, $f : B \to C$ and $f' : B' \to C'$ two integral homomorphisms of A-algebras with* $\mathrm{Ker}\, f \subset \mathscr{N}_B$ *and* $\mathrm{Ker}\, f' \subset \mathscr{N}_{B'}$. *Then $f \otimes_A f' : B \otimes_A B' \to C \otimes_A C'$ is integral and its kernel lies in $\mathscr{N}_{B\otimes_A B'}$.*

Proof. (i): (See [59, Ch.I, Prop.3.5.2(i)]). We have $\varphi \times_S \varphi' = (\varphi \times_S X') \circ (Y \times_S \varphi')$, so it suffices to show that both $\varphi \times_S X'$ and $Y \times_S \varphi'$ are surjective. But notice that the natural isomorphism $Y \times_S X' \xrightarrow{\sim} Y \times_X (X \times_S X')$ identifies $\varphi \times_S X'$ with $\varphi \times_X (X \times_S X')$, and similarly for $Y \times_S \varphi'$; hence, we are reduced to checking if $\varphi : Y \to X$ is a surjective morphism of schemes, and $\psi : Z \to X$ is any X-scheme, then the base change $\varphi \times_X Z : Y \times_X Z \to Z$ is surjective. Thus, let $y \in Y$ and $z \in Z$ with $x := \varphi(y) = \psi(z)$, and let $\kappa(z) \leftarrow \kappa(x) \to \kappa(y)$ be the induced residue field extensions; it suffices to check that $\mathrm{Spec}\,\kappa(y) \times_{\mathrm{Spec}\,\kappa(x)} \mathrm{Spec}\,\kappa(z) \neq \varnothing$, which is clear, since $\kappa(y) \otimes_{\kappa(x)} \kappa(z) \neq 0$.

(ii): Let $a \in \mathrm{Ker}\, f$, and $\mathfrak{p} \in \mathrm{Spec}\, A$; by assumption there exists $\mathfrak{q} \in \mathrm{Spec}\, B$ with $\mathfrak{p} = f^{-1}\mathfrak{q}$, so that $a \in \mathfrak{p}$, whence the assertion.

(iii): Obviously $f \otimes_A f'$ is integral. Since $(B \otimes_A B')_{\mathrm{red}} = (B_{\mathrm{red}} \otimes_{A_{\mathrm{red}}} B'_{\mathrm{red}})_{\mathrm{red}}$, and similarly for $(C \otimes_A C')_{\mathrm{red}}$, we may replace f and f' by f_{red} and f'_{red}, and assume from start that f and f' are injective. Then, by the going up theorem, both $\mathrm{Spec}\, f$ and $\mathrm{Spec}\, f'$ are surjective, hence the same holds for $\mathrm{Spec}\, f \otimes_A f'$, by (i), and then the assertion follows from (ii). □

The following proposition 9.8.9 is borrowed from [20, Exp.XII, Lemma 2.6], and its corollary 9.8.13(ii) can also be easily deduced from [161, Th.1].

Proposition 9.8.9. *Let $f : A \to B$ be an injective and finite ring homomorphism of noetherian rings. Then f is the composition of a finite sequence of injective ring homomorphisms*

$$B_n := A \xrightarrow{f_n} B_{n-1} \xrightarrow{f_{n-1}} B_{n-2} \to \cdots \to B_0 := B$$

such that the induced morphism of schemes $\mathrm{Spec}\, f_i : \mathrm{Spec}\, B_i \to B_{i+1}$ *is an effective epimorphism in the category of schemes, for every $i = 1, \dots, n$.*

Proof. The assertion means that f_i identifies B_{i+1} with the equalizer of the two natural ring homomorphisms $(g_{i,k} : B_i \to B_i \otimes_{B_{i+1}} B_i \mid k = 1, 2)$, with $g_{i,1}(b) := b \otimes 1$ and $g_{i,2}(b) := 1 \otimes b$ for $i = 1, \dots, n$ and every $b \in B_{i+1}$. Then let us define inductively $B_0 := B$, and $B_{i+1} := \mathrm{Ker}(h_{i,1} - h_{i,2})$ for every $i \in \mathbb{N}$, where $h_{i,1}, h_{i,2} : B_i \to B_i \otimes_A B_i$ are the analogous maps.

Claim 9.8.10. For every $i \in \mathbb{N}$, the induced map $B_i \otimes_A B_i \to B_i \otimes_{B_{i+1}} B_i$ is an isomorphism.

Proof of the claim. The map in question is clearly surjective. We compute in $B_i \otimes_A B_i$:

$$cb \otimes b' = (c \otimes 1) \cdot (b \otimes b') = (1 \otimes c) \cdot (b \otimes b') = b \otimes cb' \qquad \text{for every } b, b' \in B_i \text{ and } c \in B_{i+1}$$

which easily implies that the map in question is also injective. ◊

Due to claim 9.8.10, it remains only to check that $B_n = A$ for sufficiently large $n \in \mathbb{N}$. To this aim, set $M_i := B_i/A$, and let $S_i \subset \mathrm{Spec}\, A$ be the support of M_i, for every $i \in \mathbb{N}$. Then $S_{i+1} \subset S_i$ for every $i \in \mathbb{N}$, and since A is noetherian, there exists $n \in \mathbb{N}$ such that $S := S_n = S_i$ for every $i \geq n$. Thus, it suffices to check that $S = \varnothing$. If the latter fails, let $\mathfrak{p}$ be a maximal point of the closed subset S; a simple induction on $i \in \mathbb{N}$ shows that $(B_{i+1})_\mathfrak{p}$ is the equalizer of the localizations $(h_{i,1})_\mathfrak{p}, (h_{i,2})_\mathfrak{p} : B_{i,\mathfrak{p}} \otimes_{A_\mathfrak{p}} B_{i,\mathfrak{p}}$, for every $i \in \mathbb{N}$. Hence, we may replace f by its localization $f_\mathfrak{p} : A_\mathfrak{p} \to B_\mathfrak{p}$, and assume from start that $(A, \mathfrak{p})$ is local, and the support of M_n is the maximal ideal of A, hence the A-module M_n has finite length, so there exists $i \geq n$

such that $M_i = M_{i+1}$, *i.e.* $B_i = B_{i+1}$. Set $\kappa(\mathfrak{p}) := A/\mathfrak{p}$ and $B_i(\mathfrak{p}) := B_i/\mathfrak{p}B_i$; it follows that each of the maps $B_i \to B_i \otimes_A B_i$ is an isomorphism, hence the same holds for the induced maps $B_i(\mathfrak{p}) \to B_i(\mathfrak{p}) \otimes_{\kappa(\mathfrak{p})} B_i(\mathfrak{p})$, so $\dim_{\kappa(\mathfrak{p})} B_i(\mathfrak{p}) = 1$, and therefore $f : A \to B_i$ induces an isomorphism $\kappa(\mathfrak{p}) \xrightarrow{\sim} B_i(\mathfrak{p})$. By Nakayama's lemma, we conclude that $A = B_i$, whence $M_i = 0$ and $S_i = \varnothing$, a contradiction. □

9.8.11. Let $p \in \mathbb{Z}$ be a prime integer. For every $\mathbb{F}_p$-algebra R, we let R^{perf} be the inductive limit of the system of $\mathbb{F}_p$-algebras $(R_n \mid n \in \mathbb{N})$ with $R_n := R$ for every $n \in \mathbb{N}$, and with transition map given by the Frobenius endomorphism $\Phi_R : R_n \to R_{n+1}$ of R, for every such n. The rule $R \mapsto R^{\mathrm{perf}}$ yields a left adjoint for the forgetful functor from the category of perfect $\mathbb{F}_p$-algebras to the category of all $\mathbb{F}_p$-algebras. Moreover, by virtue of [75, Th.3.5.13(ii)], any étale homomorphism $R \to S$ of $\mathbb{F}_p$-algebras induces an isomorphism

$$R^{\mathrm{perf}} \otimes_R S \xrightarrow{\sim} S^{\mathrm{perf}}. \qquad (9.8.12)$$

Especially, for every subset $\Sigma \subset R$, we have

$$(\Sigma^{-1}R)^{\mathrm{perf}} = \Sigma^{-1}R^{\mathrm{perf}}.$$

Corollary 9.8.13. *Let $f : A \to B$ be a ring homomorphism.*

(i) *The following conditions are equivalent :*

(a) *f is a radicial homomorphism.*

(b) *$I_f \subset \mathscr{N}_{B\otimes_A B}$ (notation of* (9.8.7)*).*

(ii) *Let moreover $p \in \mathbb{N}$ be a prime integer, and suppose that A and B are $\mathbb{F}_p$-algebras, and f is integral and injective. Then* (a) *and* (b) *are also equivalent to :*

(c) *f induces an isomorphism $f^{\mathrm{perf}} : A^{\mathrm{perf}} \xrightarrow{\sim} B^{\mathrm{perf}}$.*

Proof. (a)⇔(b): Set $X := \mathrm{Spec}\,A$, $Y := \mathrm{Spec}\,B$, and let $\pi : Y \times_X Y \to X$ be the projection. With this notation, $\mathrm{Spec}\,B \otimes_A B/I_f$ is the diagonal closed subset $\Delta_{Y/X} \subset Y \times_X Y$. Suppose then that (b) holds; it follows already that $\Delta_{Y/X} = Y \times_X Y$, so $\varphi := \mathrm{Spec}\,f : Y \to X$ is injective. Moreover, let $y \in Y$ be any point, set $x := \varphi(y)$, and let $\kappa(x)$ and $\kappa(y)$ be the residue fields of x and respectively y; then $Z(y) := \mathrm{Spec}\,\kappa(y) \otimes_{\kappa(x)} \kappa(y) = \pi^{-1}(y)$ consists of a unique point of $Y \times_X Y$, which means that the field extension $\kappa(x) \subset \kappa(y)$ is algebraic and purely inseparable, therefore (a) holds. Conversely, if (a) holds, then φ is injective, and $Z(y)$ consists of a unique point, so that $\Delta_{Y/X} = Y \times_X Y$, whence (b).

(b)⇔(c): Since the natural map $R^{\mathrm{perf}} \to (R_{\mathrm{red}})^{\mathrm{perf}}$ is an isomorphism for every $\mathbb{F}_p$-algebra R, and since f is radicial if and only if the same holds for the induced map $f_{\mathrm{red}} : A_{\mathrm{red}} \to B_{\mathrm{red}}$, *we may assume that A and B are reduced*, in which case (c) means that for every $b \in B$ there exists $n \in \mathbb{N}$ with $b^{p^n} \in A$. Now, let us write B as the filtered colimit of the system $(f_\lambda : A \to B_\lambda \mid \lambda \in \Lambda)$ of its finite A-subalgebras. It follows easily that (c) holds for f if and only if it holds for every f_λ. On the other hand, $\mathrm{Spec}\,f$ is the limit of the system of maps $(\mathrm{Spec}\,f_\lambda \mid \lambda \in \Lambda)$, and each $\mathrm{Spec}\,f_\lambda$ is surjective; it follows easily that $\mathrm{Spec}\,f$ is injective if and only if the same holds for each $\mathrm{Spec}\,f_\lambda$. Moreover, it is clear that for any field K, and any filtered system $E_\bullet := (E_\lambda \mid \lambda \in \Lambda)$ of field extension of K, the colimit of $E_\bullet$ is algebraic and purely inseparable over K if and only if the same holds for every E_λ; summing up, we see that f is radicial if and only if the same holds for every f_λ, hence, in view of (i), it suffices to show the equivalence (b)⇔(c) for each f_λ, and *we may assume that f is a finite ring homomorphism.*

Next, let $(A_\lambda \mid \lambda \in \Lambda)$ be the filtered system of all $\mathbb{Z}$-subalgebras of A of finite type; also, choose any finite subset $\Sigma \subset B$ such that $B = A[\Sigma]$. Notice that (c) holds if and only if for every $b \in \Sigma$ there exists $n \in \mathbb{N}$ such that $b^{p^n} \in A$; then the latter condition holds if and only if there exists $\lambda \in \Lambda$ and $n \in \mathbb{N}$ such that $b^{p^n} \in A_\lambda$ for every $b \in \Sigma$. Moreover, there exists a cofinal subset $\Lambda' \subset \Lambda$ such that the restriction $f_\lambda : A_\lambda \to B_\lambda := A_\lambda[\Sigma]$ is a finite ring homomorphism for every $\lambda \in \Lambda'$. Furthermore, recall that I_f is the ideal generated by $(b \otimes 1 - 1 \otimes b \mid b \in \Sigma)$, and likewise for the ideal I_{f_λ}, for every $\lambda \in \Lambda$. It follows that $I_f \subset \mathscr{N}_{B \otimes_A B}$ if and only if $I_{f_\lambda} \subset \mathscr{N}_{B_\lambda \otimes_{A_\lambda} B_\lambda}$ for some $\lambda \in \Lambda'$. Summing up, it then suffices to check the equivalence (b)$\Leftrightarrow$(c) for each map f_λ, and therefore *we may assume that* A *and* B *are noetherian*.

Next, pick a decomposition $f = f_1 \circ \cdots \circ f_n$ as in proposition 9.8.9; since $\mathrm{Spec}\, f_i$ is a surjection for every $i = 1, \dots, n$, clearly f is radicial if and only if the same holds for each f_i. Similarly, $f^{\mathrm{perf}} = f_1^{\mathrm{perf}} \circ \cdots \circ f_n^{\mathrm{perf}}$ is an isomorphism if and only if f_i^{perf} is an isomorphism for every $i = 1, \dots, n$. Thus, it suffices to show the equivalence (b)$\Leftrightarrow$(c) for each f_i, and therefore *we may assume that* $\mathrm{Spec}\, f$ *is an effective epimorphism of schemes*. Now, (b) says that for every $b \in B$ there exists $n \in \mathbb{N}$ with

$$(b \otimes 1 - 1 \otimes b)^{p^n} = b^{p^n} \otimes 1 - 1 \otimes b^{p^n} = 0.$$

By assumption, the latter holds if and only if $b^{p^n} \in A$, whence the contention. □

Remark 9.8.14. (i) Let $f : A \to B$ an injective ring homomorphism, and denote by $\mathscr{E}$ the set of all A-subalgebras $B' \subset B$ such that the induced map $f' : A \to B'$ is integral and radicial; recall that if $B' = A[\Sigma]$ for a subset $\Sigma \subset B'$, then $I_{f'}$ is generated by the system $(b \otimes 1 - 1 \otimes b \mid b \in \Sigma)$.

(ii) In view of corollary 9.8.13(i), it follows easily that if $B', B'' \in \mathscr{E}$, then also $B' \cdot B'' \in \mathscr{E}$.

(iii) Moreover, for every system $(B_\lambda \mid \lambda \in \Lambda)$ of elements of $\mathscr{E}$ filtered by inclusion, we have $\bigcup_{\lambda \in \Lambda} B_\lambda \in \mathscr{E}$.

(iv) Furthermore, if $B' \in \mathscr{E}$, and $B'' \subset B'$ is any A-subalgebra, then $B'' \in \mathscr{E}$ as well.

Definition 9.8.15. From remark (9.8.14)(ii,iii), it follows that $\mathscr{E}$ is a filtered set, and the subring $C := \bigcup_{B' \in \mathscr{E}} B'$ is the largest element of $\mathscr{E}$; we call C the *weak normalization* of A in B.

Remark 9.8.16. (i) Consider a commutative diagram of rings :

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ {\scriptstyle g_A}\downarrow & & \downarrow{\scriptstyle g_B} \\ A' & \xrightarrow{i'} & B' \end{array}$$

where i and i' are injective ring homomorphisms. Denote by C (resp. C') the weak normalization of A in B (resp. of A' in B'). Then $g_B(C) \subset C'$. Indeed, let $x \in C$; by remark 9.8.14(iv), the induced map $A \to A[x]$ is radicial, hence $x \otimes 1 - 1 \otimes x$ is nilpotent in $A[x] \otimes_A A[x]$, so that its image $g_B(x) \otimes 1 - 1 \otimes g_B(x)$ is nilpotent in $A'[x] \otimes_{A'} A'[x]$, and therefore the induced map $A' \to A'[x]$ is radicial, whence the claim.

(ii) Let $f : A \to B$ be an injective and integral ring homomorphism; let j_1 (resp. j_2) be the ring homomorphism $B \to B \otimes_A B$ given by the rule : $b \mapsto b \otimes 1$ (resp. $b \mapsto 1 \otimes b$) for every $b \in B$. For $i = 1, 2$ let also $j'_i : B \to (B \otimes_A B)_{\mathrm{red}}$ be the composition of j_i with the projection $B \otimes_A B \to (B \otimes_A B)_{\mathrm{red}}$. Then the weak normalization of A in B is the equalizer of j'_1 and j'_2. Indeed, if $b \in B$ and $A[b]$ is a radicial A-algebra, then $b \otimes 1 - 1 \otimes b$ is nilpotent in $A[b] \otimes_A A[b]$, hence also in $B \otimes_A B$, so $j'_1(b) = j'_2(b)$. Conversely, if the latter identity holds, $b \otimes 1 - 1 \otimes b$ is nilpotent in $B \otimes_A B$; but notice that the map $A[b] \to B$ is still injective and integral, so the kernel of the induced map $A[b] \otimes_A A[b] \to B \otimes_A B$ lies in the nilradical (lemma 9.8.8(iii)). It follows that $b \otimes 1 - 1 \otimes b$ is nilpotent in $A[b] \otimes_A A[b]$, so $A[b]$ is a radicial A-algebra, as required.

Lemma 9.8.17. *Let* $f_\bullet := (f_\lambda : A_\lambda \to B_\lambda \mid \lambda \in \Lambda)$ *be a filtered system of injective ring homomorphisms; denote by* $f : A \to B$ *the colimit of* $f_\bullet$*, and for every* $\lambda \in \Lambda$ *let* C_λ *be the weak normalization of* A_λ *in* B_λ*. Then, the system of rings* $(B_\lambda \mid \lambda \in \Lambda)$ *restricts to a filtered system of rings* $C_\bullet := (C_\lambda \mid \lambda \in \Lambda)$*, whose colimit* C *is the weak normalization of* A *in* B*.*

Proof. Let $f' : A \to C$ be the map induced by f, and for every $\lambda \in \Lambda$, let $f'_\lambda : A_\lambda \to C_\lambda$ be the map induced by f_λ. By corollary 9.8.13(i) we know that $I_{f'_\lambda} \subset \mathscr{N}_{C_\lambda \otimes_{A_\lambda} C_\lambda}$; on the other hand, it is easily seen that $I_{f'}$ is the colimit of the system of ideals $(I_{f'_\lambda} \mid \lambda \in \Lambda)$, hence $I_{f'} \subset \mathscr{N}_{C\otimes_A C}$. Moreover, since C_λ is an integral A_λ-algebra for every $\lambda \in \Lambda$, it is easily seen that C is an integral A-algebra. Invoking again corollary 9.8.13(i), we deduce that C lies in the weak normalization C' of A in B. Conversely, if $x \in C'$, then in particular x lies in the integral closure C'' of A in B, hence we find $\lambda \in \Lambda$ and $x_\lambda \in B_\lambda$ whose class in B agrees with x, and such that x_λ is integral over A_λ; by corollary 9.8.13(i), we have $x \otimes 1 - 1 \otimes x \in \mathscr{N}_{C'\otimes_A C'}$, and therefore $x \otimes 1 - 1 \otimes x \in \mathscr{N}_{C''\otimes_A C''}$ as well. Hence, after replacing λ by some $\lambda' \geq \lambda$, we may assume that $x_\lambda \otimes 1 - 1 \otimes x_\lambda \in \mathscr{N}_{C''_\lambda \otimes_{A_\lambda} C''_\lambda}$, where C''_λ denotes the integral closure of A_λ in B_λ. Now, the inclusion $A_\lambda[x] \to C''_\lambda$ is integral and injective, hence the kernel of the induced ring homomorphism $A_\lambda[x] \otimes_{A_\lambda} A_\lambda[x] \to C''_\lambda \otimes_{A_\lambda} C''_\lambda$ lies in the nilradical (lemma 9.8.8(iii)). Summing up, we conclude that $x_\lambda \otimes 1 - 1 \otimes x_\lambda$ is already nilpotent in $A_\lambda[x] \otimes_{A_\lambda} A_\lambda[x]$, so $x_\lambda \in C_\lambda$, and finally $x \in C$. This shows that $C' = C$, as required. □

Proposition 9.8.18. *Let $f : A \to B$ be an injective ring homomorphism, C the weak normalization of A in B, and A' a flat A-algebra such that the induced morphism* $\mathrm{Spec}\, A' \to \mathrm{Spec}\, A$ *has geometrically reduced fibres. Suppose that :*

- *either A' is a finitely presented A-algebra*
- *or else, A is noetherian.*

Then $A' \otimes_A f : A' \to B' := A' \otimes_A B$ is an injective ring homomorphism, and the weak normalization of A' in B' is $A' \otimes_A C$.

Proof. Since A' is a flat A-algebra, $A' \otimes_A f$ is still injective, as stated. Now, let A^ν (resp. A'^ν) be the integral closure of A in B (resp. of A' in B'); then $A'^\nu = A' \otimes_A A^\nu$ (proposition 9.8.3). We may then replace B by A^ν, and assume that B is an integral A-algebra. In this case, notice that for every A-algebra R we have $(A' \otimes_A R)_{\mathrm{red}} = A' \otimes_A R_{\mathrm{red}}$ (proposition 9.8.3); then the assertion follows easily from remark 9.8.16(ii). □

9.8.19. Let $f : A \to B$ be an integral and injective ring homomorphism, $I \subset A$ an ideal; set $X := \mathrm{Spec}\, A$, $Y := \mathrm{Spec}\, B$, $Z := \mathrm{Spec}\, A/I$, $U := X \setminus Z$, and notice that $\varphi := \mathrm{Spec}\, f : Y \to X$ is surjective by the going up theorem, hence the same holds for its restriction $\varphi^{-1}Z \to Z$, and therefore the kernel of $A/I \otimes_A f : A/I \to B/IB$ lies in the nilradical (lemma 9.8.8(ii)), *i.e.* the induced map $(A/I)_{\mathrm{red}} \to (B/IB)_{\mathrm{red}}$ is injective and integral as well.

Lemma 9.8.20. *In the situation of* (9.8.19)*, suppose that φ restricts to a radicial morphism of schemes $\varphi_{|U} : \varphi^{-1}U \to U$, and let C (resp. $\overline{C}$) be the weak normalization of A in B (resp. of $(A/I)_{\mathrm{red}}$ in $(B/IB)_{\mathrm{red}}$). Then we have :*

$$C = B \times_{(B/IB)_{\mathrm{red}}} \overline{C}.$$

Proof. By remark 9.8.16(i), the projection $B \to (B/IB)_{\mathrm{red}}$ maps C into $\overline{C}$, whence the inclusion $C \subset C' := B \times_{(B/IB)_{\mathrm{red}}} \overline{C}$. For the converse inclusion, we need to show that the induced map $g : A \to C'$ is radicial. To this aim, it suffices to check that the same holds for $A/I \otimes_A g : A/I \to C'/IC$ and for $A[a^{-1}] \otimes_A g : A[a^{-1}] \to C'[a^{-1}]$, for every $a \in I$. However, for any such a, the localization $C'[a^{-1}]$ is the fibre product $B[a^{-1}] \times_{(B/IB)_{\mathrm{red}}[a^{-1}]} \overline{C}[a^{-1}]$, and since $(B/IB)_{\mathrm{red}}[a^{-1}] = \overline{C}[a^{-1}] = 0$, we see that $C'[a^{-1}] = B[a^{-1}]$; but f induces a radicial map $A[a^{-1}] \to B[a^{-1}]$, since $\varphi_{|U}$ is radicial. It remains thus only to check that C/IC is a radicial A/I-algebra; to this aim, notice that the kernel of the surjection $B \to (B/IB)_{\mathrm{red}}$ is the radical J of the ideal $IB \subset B$. Then J is also the kernel of the surjection $C' \to \overline{C}$; on the other hand, since $IB \subset J \subset C$, we have $I^2B \subset IC \subset IB \subset J$, and clearly J is also the radical of the ideal I^2B of C. Hence, C/IC is a radicial A/I-algebra if and only if the same holds for the A/I-algebra C/J; but $C/J = \overline{C}$, whence the contention. □

Proposition 9.8.21. *Let B be a ring, $A \subset B$ a subring, and G a finite group of automorphisms of B such that $g(A) = A$ for every $g \in G$. Denote by C (resp. D) the integral closure (resp. the weak normalization) of A in B. Then we have :*

(i) *C^G is the integral closure of A^G in B^G.*
(ii) *D^G is the weak normalization of A^G in B^G.*

Proof. (i): Clearly $\mathrm{i.c.}(A^G, B^G) \subset C^G$. To show the reverse inclusion, let $x \in C^G$, so that $x \in B^G$, and x is integral over A; but since G is a finite group, A is integral over A^G (details left to the reader), hence x is integral over A^G, as required.

(ii): Let D' be the weak normalization of A^G in B^G; we have $D' \subset D$ by remark 9.8.16(i), and then obviously $D' \subset D^G$. For the reverse inclusion, we come down to checking that the inclusion map $A^G \to D^G$ is radicial. To this aim, notice first :

Claim 9.8.22. (i) $g(D) = D$ for every $g \in G$.

(ii) Let K be any algebraically closed field, and $E \subset B$ any subring such that $g(E) = E$ for every $g \in G$. Then the set $\mathrm{Hom}_{\mathbb{Z}\text{-}\mathsf{Alg}}(E, K)$ inherits from E a natural G-action, and the map $\mathrm{Hom}_{\mathbb{Z}\text{-}\mathsf{Alg}}(E, K) \to \mathrm{Hom}_{\mathbb{Z}\text{-}\mathsf{Alg}}(E^G, K) : \varphi \mapsto \varphi_{|A^G}$ factors through a bijection

$$\mathrm{Hom}_{\mathbb{Z}\text{-}\mathsf{Alg}}(E, K)/G \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}\text{-}\mathsf{Alg}}(E^G, K).$$

Proof of the claim. (i): We apply remark 9.8.16(i) with $A' := A$, $B' := B$, $g_B := g$, $g_A := g_{|A}$, and with i and i' the inclusion maps; then we get $g(D) \subset D$, and similarly $g^{-1}(D) \subset D$, whence the claim.

(ii): The injectivity follows easily from [34, Ch.5, §2, n.2, Cor. of Th.2], and the surjectivity is clear, since the inclusion map $E^G \to E$ is integral. ◊

Now, the radicial map $A \to D$ induces a bijection $\mathrm{Hom}_{\mathbb{Z}\text{-}\mathsf{Alg}}(D, K) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}\text{-}\mathsf{Alg}}(A, D)$ for every algebraically closed field K; in view of claim 9.8.22(i), this map is obviously G-equivariant, hence it induces a bijection on the quotients by the respective G-actions. Then, claim 9.8.22(ii) implies that the map $A^G \to D^G$ also induces a bijection $\mathrm{Hom}_{\mathbb{Z}\text{-}\mathsf{Alg}}(D^G, K) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}\text{-}\mathsf{Alg}}(A^G, D)$ for every such K, whence the contention. □

Proposition 9.8.23. *In the situation of remark* 9.8.16(i)*, let D (resp. D') be the integral closure of A in B (resp. of A' in B'), and suppose that we have norm maps $\nu_A : \Gamma^d_A A' \to A$ and $\nu_B : \Gamma^d_B B' \to B$ of some degree $d \in \mathbb{N}$, that make commute the induced diagram :*

$$\begin{array}{ccc} \Gamma^d_A A' & \longrightarrow & \Gamma^d_B B' \\ {\scriptstyle \nu_A}\downarrow & & \downarrow{\scriptstyle \nu_B} \\ A & \xrightarrow{\ i\ } & B \end{array}$$

(see definition 9.5.70*). Then there exist unique norm maps $\nu_C : \Gamma^d_C C' \to C$ and $\nu_D : \Gamma^d_D D' \to D$ of degree d, that make commute the resulting diagram :*

$$\begin{array}{ccccccc} \Gamma^d_A A' & \longrightarrow & \Gamma^d_C C' & \longrightarrow & \Gamma^d_D D' & \longrightarrow & \Gamma^d_B B' \\ {\scriptstyle \nu_A}\downarrow & & {\scriptstyle \nu_C}\downarrow & & \downarrow{\scriptstyle \nu_D} & & \downarrow{\scriptstyle \nu_B} \\ A & \longrightarrow & C & \longrightarrow & D & \longrightarrow & B. \end{array}$$

Moreover, if ν_B is Cayley-Hamilton, the same holds for ν_C and ν_D (see definition 9.5.74*).*

Proof. We need to show that that the image of the composition of ν_B with the natural map $\Gamma^d_D D' \to \Gamma^d_B B'$ lies in D; to this aim, it suffices to check that the natural map $\Gamma^d_A A' \to \Gamma^d_D D'$ is integral. However, the latter is the composition of the integral map $\Gamma^d_A A' \to D \otimes_A \Gamma^d_A D'$ (proposition 9.5.68(iv.a)), the isomorphism $D \otimes_A \Gamma^d_A D' \xrightarrow{\sim} \Gamma^d_D(D \otimes_A D')$ (proposition 9.5.68(i)) and the surjection $\Gamma^d_D(D \otimes_A D') \to \Gamma^d_D D'$ (corollary 9.5.22(iv)), whence the contention.

Likewise, we need to check that the image of the composition of ν_B with the natural map $\Gamma^d_C C' \to \Gamma^d_B B'$ lies in C. To this aim, notice that the natural map $\Gamma^d_A A' \to \Gamma^d_A C'$ is integral, radicial, and its kernel lies in the nilradical of $\Gamma^d_A A'$ (proposition 9.5.68(iv.b,iv.c)); the same holds for the natural map $\Gamma^d_A C' \to C \otimes_A \Gamma^d_A C' \xrightarrow{\sim} \Gamma^d_C(C \otimes_A C')$. By the same token, since the natural map $C \otimes_A C' \to C'$ is surjective, and its kernel lies in the nilradical of $C \otimes_A C'$, then the same holds for the induced map $\Gamma^d_C(C \otimes_A C') \to \Gamma^d_C C'$. Summing up, we conclude that the natural map $\Gamma^d_A A' \to \Gamma^d_C C'$ is integral, radicial, and its kernel J lies in the nilradical of $\Gamma^d_A A'$; we have therefore a commutative diagram of rings :

$$\begin{array}{ccc} \Gamma^d_A(A')/J & \longrightarrow & \Gamma^d_C C' \\ \downarrow & & \downarrow \\ A & \xrightarrow{\ i\ } & B \end{array}$$

whose top horizontal arrow is integral, radicial and injective. Then the contention follows from remark 9.8.16(i). Lastly, let $\chi^\dagger_B, \chi^\dagger_C, \chi^\dagger_D$ be the characteristic maps of ν_B, ν_C and ν_D (see (9.5.73)); by inspecting the definitions, we get a commutative diagram :

$$\begin{array}{ccccc} \Gamma^d_C C' & \longrightarrow & \Gamma^d_D D' & \longrightarrow & \Gamma^d_B B' \\ {\scriptstyle \chi^\dagger_C}\downarrow & & {\scriptstyle \chi^\dagger_D}\downarrow & & \downarrow{\scriptstyle \chi^\dagger_B} \\ C' & \longrightarrow & D' & \longrightarrow & B' \end{array}$$

whence the last assertion of the proposition. □

Definition 9.8.24. (i) Let R be a ring, $S \subset R$ a subring, and $p > 0$ a prime integer. We say that *S is p-root closed in R*, if for every $x \in R$ such that $x^p \in S$, we have $x \in S$.

(ii) Let R be a ring and $p > 0$ a prime integer; if $(S_\lambda \mid \lambda \in \Lambda)$ is any family of p-root closed subrings of R, then clearly $\bigcap_{\lambda \in \Lambda} S_\lambda$ is also p-root closed in R. Hence, for every subring $S \subset R$ there exists a smallest p-root closed subring S' of R containing S. We say that S' is the *p-root closure of S in R*.

Remark 9.8.25. (i) Let R be a ring, $S \subset R$ a subring; consider the increasing countable system $(S_n \mid n \in \mathbb{N})$ of subrings of R defined inductively as follows : $S_0 := S$, and $S_{n+1} := S_n[\Sigma_n]$ for every $n \in \mathbb{N}$, where $\Sigma_n := \{x \in R \mid x^p \in S_n\}$. Then it is easily seen that $\bigcup_{n \in \mathbb{N}} S_n$ is the p-root closure of S in R.

(ii) In the situation of remark 9.8.16(i), let D (resp. D') be the p-root closure of A in B (resp. of A' in B'). Define the system of subrings $(A_n \mid n \in \mathbb{N})$ of B (resp. $(A'_n \mid n \in \mathbb{N})$ of B') as in (i); a simple induction shows that $g_B(A_n) \subset A'_n$ for every $n \in \mathbb{N}$, so g_B restricts to a ring homomorphism $D \to D'$.

Lemma 9.8.26. (i) *Let R be a ring, $a \in R$ a regular element such that $pR \subset a^p R$. Then R is p-root closed in $R[a^{-1}]$ if and only if the Frobenius endomorphism of $R/a^p R$ induces an injective ring homomorphism $\overline{\Phi} : R/aR \to R/a^p R$.*

(ii) *Let R be a topological ring, $S \subset R$ an open subring, T the p-root closure of S in R; endow S and T with the topologies induced by R, and denote by $R^\wedge$, $S^\wedge$ and $T^\wedge$ the respective completions. Then $T^\wedge$ is the p-root closure in $R^\wedge$ of $S^\wedge$.*

(iii) *Let $R_\bullet := (R_\lambda \mid \lambda \in \Lambda)$ be a system of rings, with transition morphisms $f_{\lambda\mu} : R_\lambda \to R_\mu$ for every $\lambda, \mu \in \Lambda$ with $\mu \geq \lambda$. For every $\lambda \in \Lambda$, let also $S_\lambda \subset R_\lambda$ be a subring, such that $f_{\lambda\mu}(S_\lambda) \subset S_\mu$ for every such λ, μ. Denote by $D_\lambda \subset R_\lambda$ the p-root closure of S_λ in R_λ, for every $\lambda \in \Lambda$. The following holds :*

(a) *Let R and S be the limits of $R_\bullet$ and respectively of $S_\bullet := (S_\lambda \mid \lambda \in \Lambda)$. Then the limit of the induced system $D_\bullet := (D_\lambda \mid \lambda \in \Lambda)$ is the p-root closure of S in R.*

(b) *Suppose moreover that* Λ *is filtered, and let* R' *and* S' *be the colimits of* $R_\bullet$ *and respectively of* $S_\bullet$*. Then the colimit of* $D_\bullet$ *is the* p*-root closure of* S' *in* R'*.*

(iv) *Let* B *be a ring,* $A \subset B$ *a subring,* C *the* p*-root closure of* A *in* B*, and* $S \subset A$ *a multiplicative subset. Then* $S^{-1}C$ *is the* p*-root closure of* $S^{-1}A$ *in* $S^{-1}B$*.*

Proof. (i): Suppose first that $\overline{\Phi}$ is injective; let $x \in R[a^{-1}]$ be any element such that $x^p \in R$, and suppose, by way of contradiction, that $x \notin R$. There exists a smallest $m \in \mathbb{N}$ such that $y := a^m x \notin R$ and $a^{m+1}x \in R$. Hence, $y^p = a^{pm}x^p \in R$, and therefore $(a^{m+1}x)^p = a^p y^p \in a^p R$. Thus, the class of $a^{m+1}x$ lies in $\mathrm{Ker}\,\overline{\Phi}$, *i.e.* $a^{m+1}x = az$ for some $z \in R$; since a is regular, it follows that $z = a^m x \in R$, contradicting the choice of m.

Conversely, suppose that R is integrally closed in $R[a^{-1}]$, and let $x \in R$ be an element whose class $\overline{x} \in R/aR$ lies in $\mathrm{Ker}\,\overline{\Phi}$; this means that there exists $y \in R$ such that $x^p = a^p y$. Hence, $(x/a)^p \in R$, so that $x/a \in R$, and therefore $\overline{x} = 0$.

(ii): Notice that $S^\wedge \cdot T = T^\wedge$, hence $T^\wedge$ lies in the p-root closure of $S^\wedge$ in $R^\wedge$. Hence, we are reduced to checking that S is p-root closed in R if and only if $S^\wedge$ is p-root closed in $R^\wedge$. Suppose first that S is p-root closed in R, and let $x \in R^\wedge$ be any element such that $x^p \in S^\wedge$. Denote by $j : R \to R^\wedge$ the completion map. Pick a Cauchy net $(x_\lambda \mid \lambda \in \Lambda)$ in R whose limit is x; then $(x_\lambda^p \mid \lambda \in \Lambda)$ is a Cauchy net converging to x^p, and since $S^\wedge$ is open in $R^\wedge$ (corollary 8.2.17(i)), we may replace Λ by a cofinal subset, and assume that $j(x_\lambda)^p \in S^\wedge$ for every $\lambda \in \Lambda$. Then $x_\lambda^p \in S$ (corollary 8.2.17(ii)), whence $x_\lambda \in S$ for every $\lambda \in \Lambda$, and finally $x \in S^\wedge$, as required. Conversely, if $S^\wedge$ is p-root closed in $R^\wedge$, and $x \in R$ is any element with $x^p \in S$, we have $j(x)^p \in S^\wedge$, so $j(x) \in S$, whence $x \in S$, again by corollary 8.2.17(ii).

The proof of (iii) shall be left to the reader.

(iv): Let $b \in B, s \in S$ such that $(b/s)^p \in S^{-1}C$; it follows easily that $t^p b^p \in C$ for some $t \in S$, hence $tb \in C$, and consequently $b/s \in S^{-1}C$. This shows that $S^{-1}C$ is p-root closed in $S^{-1}B$, hence it contains the p-root closure C' of $S^{-1}A$ in $S^{-1}B$. On the other hand, by remark 9.8.25, the localization $B \to S^{-1}B$ restricts to a ring homomorphism $C \to C'$; since C' is an $S^{-1}A$-algebra, we must then have $S^{-1}C \subset C'$. □

9.8.27. Let X be a scheme, $\mathscr{A} \to \mathscr{B}$ a monomorphism of quasi-coherent $\mathscr{O}_X$-algebras; for every affine open subset $U \subset X$, let $\mathscr{C}(U)$ (resp. $\mathscr{D}(U)$) be the weak normalization (resp. the p-root closure) of $\mathscr{A}(U)$ in $\mathscr{B}(U)$; by remark 9.8.16(i), the rule : $U \mapsto \mathscr{C}(U)$ (resp. $U \mapsto \mathscr{D}(U)$) defines a subpresheaf of $\mathscr{O}_X$-algebras $\mathscr{C} \subset \mathscr{B}$ (resp. $\mathscr{D} \subset \mathscr{B}$) on the site of affine open subsets of X, and it follows easily from proposition 9.8.18 (resp. from lemma 9.8.26(iv)) that $\mathscr{C}$ (resp. $\mathscr{D}$) is a sheaf on this site, so it extends uniquely to a subsheaf of $\mathscr{B}$ on the Zariski site of X; moreover, $\mathscr{C}$ and $\mathscr{D}$ are quasi-coherent $\mathscr{O}_X$-algebras. We call $\mathscr{C}$ and $\mathscr{D}$ respectively the *weak normalization* and the *p-root closure* of $\mathscr{A}$ in $\mathscr{B}$.

Proposition 9.8.28. *Let* $f : A \to B$ *be an injective ring homomorphism,* $p \in \mathbb{N}$ *a prime integer, and suppose that* f *induces an isomorphism* $A[1/p] \xrightarrow{\sim} B[1/p]$*. Let* A^ν *(resp.* C_0*, resp.* C_1*) be the integral closure (resp. the weak normalization, resp. the* p*-root closure) of* A *in* B*; denote by* $(A/pA)_{\mathrm{red}} \xrightarrow{j} (A^\nu/pA^\nu)_{\mathrm{red}} \xleftarrow{\pi} A^\nu$ *the natural maps, and set*

$$C_2 := \{b \in B \mid \textit{there exists } n \in \mathbb{N} \textit{ such that } b^{p^n} \in A\}$$
$$D := \{x \in (A^\nu/pA^\nu)_{\mathrm{red}} \mid \textit{there exists } n \in \mathbb{N} \textit{ such that } x^{p^n} \in \mathrm{Im}(j)\}.$$

Then j *is injective, and* $C_0 = C_1 = C_2 = \pi^{-1}D$*.*

Proof. One argues as in (9.8.19) to see that j is injective.

Now, the inclusion $C_2 \subset C_1$ is obvious, and the identity $C_0 = \pi^{-1}D$ follows easily from lemma 9.8.20. By remark 9.8.25, we may write C_1 as the increasing union of the family $A_0 \subset A_1 \subset A_2 \subset \cdots$ of subrings of B, defined inductively by $A_0 := A$, and $A_{i+1} := A_i[\Sigma_i]$ for every

$i \in \mathbb{N}$, where $\Sigma_i := \{b \in B \mid b^p \in A_i\}$. By remark 9.8.14(iii), in order to check that $C_1 \subset C_0$, it will therefore suffice to show that the inclusion map $A_i \to A_{i+1}$ is radicial for every $i \in \mathbb{N}$. However, notice that $A_i[1/p] = A_{i+1}[1/p]$ for every such i, hence we are reduced to checking that the induced map $A_i/pA_i \to A_{i+1}/pA_{i+1}$ is radicial, which is clear by construction.

Lastly, let us show that $C_0 \subset C_2$. To this aim, let us write f as the colimit of a filtered system of injective ring homomorphisms $(f_\lambda : A_\lambda \to B_\lambda \mid \lambda \in \Lambda)$, where A_λ and B_λ are $\mathbb{Z}$-subalgebras of finite type of A and respectively B, for every $\lambda \in \Lambda$; since f induces an isomorphism $A[1/p] \xrightarrow{\sim} B[1/p]$, we may also assume that f_λ induces an isomorphism $A_\lambda[1/p] \xrightarrow{\sim} B_\lambda[1/p]$ for every $\lambda \in \Lambda$ (details left to the reader). For every such λ, let $C_{2,\lambda}$ be the subset of all $b \in B_\lambda$ such that $b^{p^n} \in A_\lambda$ for some $n \in \mathbb{N}$, and let $C_{0,\lambda}$ be the weak normalization of A_λ in B_λ. Clearly C_2 is the filtered union of the resulting system $(C_{2,\lambda} \mid \lambda \in \Lambda)$; on the other hand, C_0 is the colimit of the system of subrings $(C_{0,\lambda} \mid \lambda \in \Lambda)$, by lemma 9.8.17. Hence, we are reduced to checking that $C_{0,\lambda} \subset C_{2,\lambda}$ for every $\lambda \in \Lambda$, and *we may assume that A and B are noetherian*.

Let now $b \in C_0$, and set $B' := A[b]$; we have to check that $b^{p^n} \in A$ for some $n \in \mathbb{N}$, and obviously $A[1/p] = B'[1/p]$, so we may replace B by B' and *suppose that f is finite and radicial*. In this case, pick a decomposition $f = f_1 \circ \cdots \circ f_n$ as in proposition 9.8.9; again, it suffices to check the assertion for each f_i, hence *we may assume that* $\mathrm{Spec}\, f$ *is an effective epimorphism of schemes* (and still radicial). Thus, for every $b \in B$ the element $c := b \otimes 1 - 1 \otimes b$ is nilpotent in $B \otimes_A B$ (corollary 9.8.13(i)), and moreover notice that the two natural maps $B[1/p] \to B \otimes_A B[1/p]$ are isomorphisms, since $A[1/p] = B[1/p]$, hence the same holds for the multiplication law $B \otimes_A B[1/p] \to B[1/p]$, and the therefore image of c vanishes in $B \otimes_A B[1/p]$, *i.e.* $p^k c = 0$ in $B \otimes_A B$ for some $k \in \mathbb{N}$, and after replacing k by a larger integer, we may assume that $c^k = 0$ as well. We write

$$b^{p^n} \otimes 1 = 1 \otimes b^{p^n} + \sum_{j=1}^{p^n} \binom{p^n}{j} (1 \otimes b^{p^n - j}) \cdot c^j = 1 \otimes b^{p^n} + \sum_{j=1}^{k-1} \binom{p^n}{j} (1 \otimes b^{p^n - j}) \cdot c^j$$

and recall that the p-adic valuation of $\binom{p^n}{j}$ equals $n - v_p(j)$ for $j = 1, \dots, p^n$, where $v_p(j)$ denotes the p-adic valuation of j. In particular, with $n := 2k$, we see that $\binom{p^n}{j}$ is divisible by p^k for every $j = 1 \dots, k-1$, so finally $b^{p^n} \otimes 1 = 1 \otimes b^{p^n}$ in $B \otimes_A B$, and consequently $b^{p^n} \in A$, since $\mathrm{Spec}\, f$ is an effective epimorphism. □

Proposition 9.8.29. *Let us resume the situation of remark* 9.8.16(i)*, and let $a \in A$ be any element and $n \in \mathbb{N}$ any integer; also, denote by A^ν (resp. A'^ν) the integral closure of A in B (resp. of A' in B'). The following holds :*

(i) *If g_A and g_B are injective, $aB' \subset \mathrm{Im}\, g_B$ and $a^n A' \subset \mathrm{Im}\, g_A$, then :*

$$aA'^\nu \subset g_B(A^\nu) \qquad \textit{and} \qquad aC' \subset g_B(C).$$

(ii) *If g_A and g_B are surjective, and* $a\mathrm{Ker}\, g_B = 0$*, then :*

$$a \cdot g_B^{-1} A'^\nu \subset A^\nu \qquad \textit{and} \qquad a \cdot g_B^{-1} C' \subset C.$$

(iii) *If g_A and g_B are surjective, and* $\mathrm{Ker}\, g_B \subset \mathscr{N}_B$ *(notation of* (9.8.7)*), then :*

$$g_B^{-1} A'^\nu = A^\nu \qquad \textit{and} \qquad g_B^{-1} C' = C.$$

Proof. (i): Let $b \in A'^\nu$, and pick any monic polynomial $P := X^m + a'_1 X^{m-1} + \cdots + a'_m \in A'[X]$ such that $P(b^n) = 0$. Hence $a^{mn} \cdot P(b^n) = 0 = (ab)^{mn} + a^n a'_1 (ab)^{n(m-1)} + \cdots + a^{mn} a'_m = 0$, and by assumption $a^n a'_1, a^{2n} a'_2, \dots, a^{mn} a'_m \in g_A(A)$ and $ab \in g_B(B)$, so $ab \in g_B(A^\nu)$, whence the first stated inclusion. Next, set $C_0 := g_A(A) + aC'$; clearly C'_0 is an A-subalgebra of $g_B(B)$, and we need to check that the induced map $A \to C_0$ is radicial.

Claim 9.8.30. The kernel and cokernel of the induced map $C_0 \otimes_A C_0 \to C' \otimes_{A'} C'$ are annihilated by a^{n+2}.

Proof of the claim. By consruction, the cokernel of the inclusion $j : C_0 \to C'$ is annihilated by a; since the kernel of $C' \otimes_A j$ is a quotient of $\mathrm{Tor}_1^A(C', C'/C_0)$, it follows that both the kernel and cokernel of $C' \otimes_A j$ are annihilated by a. By the same token, the kernel and cokernel of $(C' \otimes_A C_0) \otimes_{C_0} j : C' \otimes_A C_0 \to C' \otimes_A C'$ are annihilated by a as well. Lastly, the natural map $\pi : C' \otimes_A C' \to C'' \otimes_{A'} C'$ is clearly surjective, and its kernel is generated by the elements of the form $a'x \otimes y - x \otimes a'y$, for all $a' \in A'$ and all $x, y \in C'$. By assumption, $a^n(a'x \otimes y - x \otimes a'y) = 0$ in $C' \otimes_A C'$, so $a^n \mathrm{Ker}\, \pi = 0$. The claim follows easily. ◊

Let $c \in aC'$ be any element; then $c \otimes 1 - 1 \otimes c$ is nilpotent in $C' \otimes_{A'} C'$ (corollary 9.8.13(i)); by claim 9.8.30, it follows that there exists $N \in \mathbb{N}$ with $a^{n+2}(c \otimes 1 - 1 \otimes c)^N = 0$ in $C_0 \otimes_A C_0$. Then, say that $c = ad$ for some $d \in C'$; it follows that $(c \otimes 1 - 1 \otimes c)^3 = a^3 d^3 \otimes 1 - 3a^2 d^2 \otimes ad + 3ad \otimes a^2 d^2 + 1 \otimes a^3 d^3$ is divisible by a in $C_0 \otimes_A C_0$, and finally $(c \otimes 1 - 1 \otimes c)^{N+3n+6} = 0$ in $C_0 \otimes_A C_0$, whence the contention.

(ii): Let b and P be as in the foregoing; suppose that $b = g_B(c)$ for some $c \in B$, and pick $a_1, \dots, a_m \in A$ with $g_A(a_i) = a'_i$ for $i = 1, \dots, m$. Set $Q := X^m + a_1 X^{m-1} + \cdots + a_m \in A[X]$; then $aQ(c) = 0$, and it follows easily that $ac \in A^\nu$, whence the first inclusion of (ii). Next, set $C_0 := A + a \cdot g_B^{-1} C'$; we need to show that the resulting ring homomorphism $A \to C_0$ is radicial. However, notice that the restriction $g_C : C_0 \to C$ of g_B has both kernel and cokernel annihilated by a; then by lemma 14.1.55 we find an A-linear map $h : C \to C_0$ with $h \circ g_C = a^2 \mathbf{1}_{C_0}$ and $g_C \circ h = a^2 \cdot \mathbf{1}_C$. Hence $(h \otimes_A h) \circ (g_C \otimes_A g_C) = a^4 \mathbf{1}_{C_0}$ and $(g_C \otimes_A g_C) \circ (h \otimes_A h) = a^4 \mathbf{1}_C$, so both the kernel and cokernel of $g_C \otimes_A g_C : C_0 \otimes_A C_0 \to C \otimes_A C = C \otimes_{A'} C$ are annihilated by a^4. We may now argue as in the proof of (i), to conclude that $c \otimes 1 - 1 \otimes c$ is nilpotent in $C_0 \otimes_A C_0$, for every $c \in a \cdot g_B^{-1} C'$, whence the contention.

(iii): Let b, c, P and Q be as in the proof of (ii); since $\mathrm{Ker}\, g_B \subset \mathscr{N}_B$, we have $Q(c)^n = 0$ for some $n \in \mathbb{N}$, so $c \in A^\nu$, whence the first identity of (iii). Next, set $C_0 := g_B^{-1} C'$; we need to check that the map $A \to C_0$ is radicial, or equivalently, that the same holds the induced map on reduced quotients $A_{\mathrm{red}} \to (C_0)_{\mathrm{red}}$. However, g_B induces an isomorphism $(C_0)_{\mathrm{red}} \xrightarrow{\sim} (C')_{\mathrm{red}}$, and by definition the map $A'_{\mathrm{red}} \to (C')_{\mathrm{red}}$ is radicial; since g_A induces a surjection $A_{\mathrm{red}} \to A'_{\mathrm{red}}$, the assertion follows easily. □

10. Cohomology and local cohomology of sheaves

10.1. Cohomology of topoi and topological spaces.

Definition 10.1.1. Let X be a topological space, $\mathscr{F}$ a sheaf on X.

(i) We say that $\mathscr{F}$ is *flabby* if the restriction map $\mathscr{F}(X) \to \mathscr{F}(U)$ is surjective for every open subset $U \subset X$.

(ii) We say that $\mathscr{F}$ is *qc-flabby* if the restriction map $\mathscr{F}(V) \to \mathscr{F}(U)$ is a surjection whenever $U \subset V$ are quasi-compact open subsets of X.

(iii) The *support* of $\mathscr{F}$ is the subset :

$$\mathrm{Supp}\, \mathscr{F} := \{x \in X \mid \mathscr{F}_x \neq \varnothing\} \subset X.$$

(iv) On the other hand, if $\mathscr{F}$ is an abelian sheaf, we define the *support* of $\mathscr{F}$ as the subset :

$$\mathrm{Supp}\, \mathscr{F} := \{x \in X \mid \mathscr{F}_x \neq 0\} \subset X.$$

Lemma 10.1.2. *Let X be a topological space, $(X_\lambda \mid \lambda \in \Lambda)$ a family of open subsets of X with $X = \bigcup_{\lambda \in \Lambda} X_\lambda$, and $\mathscr{F}$ a sheaf on X. For every $\lambda \in \Lambda$, let $\mathscr{F}_\lambda$ be the restriction of $\mathscr{F}$ to X_λ. Then $\mathscr{F}$ is flabby if and only if $\mathscr{F}_\lambda$ is flabby for every $\lambda \in \Lambda$.*

Proof. It is easily seen that if $\mathscr{F}$ is flabby, the same holds for every $\mathscr{F}_\lambda$. Conversely, suppose that $\mathscr{F}_\lambda$ is flabby for every $\lambda \in \Lambda$; let $U \subset X$ be any open subset, and $\sigma \in \mathscr{F}(U)$. For every subset $\Lambda' \subset \Lambda$ set $X_{\Lambda'} := \bigcup_{\lambda \in \Lambda'} X_\lambda$, and denote by $\mathscr{P}$ the set of all pairs (Λ', σ') where Λ' is a subset of Λ, and $\sigma' \in \mathscr{F}(U \cup X_{\Lambda'})$ is any section whose restriction to U agrees with

σ. We endow $\mathscr{P}$ with the partial order such that $(\Lambda', \sigma') \leq (\Lambda'', \sigma'')$ if and only if $\Lambda' \subset \Lambda''$ and the restriction of σ'' to $U \cup X_{\Lambda'}$ agrees with σ'. It is clear that if $((\Lambda_i, \sigma_i) \mid i \in I)$ is any totally ordered subset in $\mathscr{P}$, then there exists (Λ', σ') in $\mathscr{P}$ with $\Lambda' = \bigcup_{i \in I} \Lambda_i$ and such that $(\Lambda_i, \sigma_i) \leq (\Lambda', \sigma')$ for every $i \in I$. Also, $(\varnothing, \sigma) \in \mathscr{P}$. By Zorn's lemma, $\mathscr{P}$ admits then a maximal element (Λ', σ'), and it suffices to check that $\Lambda' = \Lambda$. Thus, suppose that $\lambda \in \Lambda \setminus \Lambda'$; set $\Lambda'' := \Lambda' \cup \{\lambda\}$, and let σ'' be the restriction of σ' to $U_\lambda := X_\lambda \cap (U \cup X_{\Lambda'})$. Since $\mathscr{F}_\lambda$ is flabby, there exists $\sigma_\lambda \in X_\lambda$ whose restriction to U_λ agrees with σ''. Then there exists $\tau \in \mathscr{F}(U \cup X_{\Lambda''})$ whose restriction to $U \cup X_{\Lambda'}$ agrees with σ' and whose restriction to X_λ agrees with σ_λ. By construction, we have $(\Lambda'', \tau) > (\Lambda', \sigma')$, a contradiction. □

10.1.3. Let X be a topological space, and consider a short exact sequence

$$\Sigma \qquad : \qquad 0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$$

of abelian sheaves on X. For every open subset $U \subset X$, the sequence Σ induces a complex :

$$\Sigma(U) \qquad : \qquad 0 \to \mathscr{F}'(U) \to \mathscr{F}(U) \to \mathscr{F}''(U) \to 0.$$

Lemma 10.1.4. *In the situation of* (10.1.3)*, the following holds :*

(i) *If $\mathscr{F}'$ is flabby, the sequence $\Sigma(U)$ is short exact.*
(ii) *If both $\mathscr{F}$ and $\mathscr{F}'$ are flabby, the same holds for $\mathscr{F}''$.*
(iii) *Suppose that X is locally coherent and quasi-separated. Then :*
   (a) *If $\mathscr{F}'$ is qc-flabby and U is quasi-compact, the sequence $\Sigma(U)$ is short exact.*
   (b) *If both $\mathscr{F}$ and $\mathscr{F}'$ are qc-flabby, the same holds for $\mathscr{F}''$.*

Proof. (i): Let $s'' \in \mathscr{F}''(U)$ be any section; we need to show that s'' is the image of an element of $\mathscr{F}(U)$. To this aim, let $\mathscr{P}$ be the set of all pairs (V, t), where $V \subset U$ is an open subset, and $t \in \mathscr{F}(V)$ is a section whose image in $\mathscr{F}''(V)$ agrees with the restriction of s''. Then $\mathscr{P}$ is partially ordered by declaring that $(V, t) \leq (V', t')$ if and only if $V \subset V'$ and t is the restriction of t' to V. Clearly, if $((V_\lambda, t_\lambda) \mid \lambda \in \Lambda)$ is a totally ordered subset of $\mathscr{P}$, then there exists a unique section t on $V := \bigcup_{\lambda \in \Lambda} V_\lambda$ such that $(V, t) \in \mathscr{P}$ and $(V_\lambda, t_\lambda) \leq (V, t)$ for every $\lambda \in \Lambda$. Moreover, $\mathscr{P} \neq \varnothing$, since $(\varnothing, 0) \in \mathscr{P}$. By Zorn's lemma, $\mathscr{P}$ admits therefore a maximal element (V, s), and it suffices to check that $V = U$. Thus, suppose that $x \in U \setminus V$; by assumption, we may find an open neighborhood V' of x in U, and $t \in \mathscr{F}(V')$ with $(V', t) \in \mathscr{P}$. Set $W := V \cap V'$, and let t_W (resp. s_W) be the restriction of t (resp. of s) to W; then $s_W - t_W \in \mathscr{F}'(W)$. Since $\mathscr{F}'$ is flabby, the same holds for its restriction $\mathscr{F}'_{|V}$ (lemma 10.1.2), hence we find $r \in \mathscr{F}'(V')$ whose restriction to W agrees with $s_W - t_W$; set $t' := r + t \in \mathscr{F}(V')$. Set $V'' := V \cup V'$; clearly the restriction of t' to W agrees with s_W, hence there exists $u \in \mathscr{F}(V'')$ whose restriction to V and V' agrees respectively with s and t'. By construction, we have $(V'', t') \in \mathscr{P}$ and $(V'', t') > (V, s)$, a contradiction.

(ii): In view of (i), we get a commutative ladder with short exact rows :

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{F}'(X) & \longrightarrow & \mathscr{F}(X) & \longrightarrow & \mathscr{F}''(X) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \rho'} & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho''} & & \\ 0 & \longrightarrow & \mathscr{F}'(U) & \longrightarrow & \mathscr{F}(U) & \longrightarrow & \mathscr{F}''(U) & \longrightarrow & 0 \end{array}$$

whence, by the snake lemma, a surjection : $\operatorname{Coker} \rho \to \operatorname{Coker} \rho''$. But $\operatorname{Coker} \rho = 0$, since $\mathscr{F}$ is flabby; so $\operatorname{Coker} \rho'' = 0$, whence the assertion.

The proof of (iii.a) is a variant of that of (i). Indeed, we may replace X by U, and thereby assume that X is quasi-compact. Then we have to check that every section $s'' \in \mathscr{F}''(X)$ is the image of an element of $\mathscr{F}(X)$. However, we can find a finite covering $(U_i \mid i = 1, \dots, n)$ of X, consisting of quasi-compact open subsets, such that $s''_{|U_i}$ is the image of a section $s_i \in \mathscr{F}(U_i)$. For every $k \leq n$, let $V_k := U_1 \cup \cdots \cup U_k$; we show by induction on k that $s''_{|V_k}$ is in the image of $\mathscr{F}(V_k)$; the lemma will follow for $k = n$. For $k = 1$ there is nothing to prove. Suppose that $k > 1$ and that the assertion is known for all $j < k$; hence we can find $t \in \mathscr{F}(V_{k-1})$ whose image is $s''_{|V_{k-1}}$. The difference $u := (t - s_k)_{|U_k \cap V_{k-1}}$ lies in the image of $\mathscr{F}'(U_k \cap V_{k-1})$; since $U_k \cap V_{k-1}$ is quasi-compact and $\mathscr{F}'$ is qc-flabby, u extends to a section of $\mathscr{F}'(U_k)$. We can then replace s_k by $s_k + u$, and assume that s_k and t agree on $U_k \cap V_{k-1}$, whence a section on $V_k = U_k \cup V_{k-1}$ with the sought property. Assertion (iii.b) is proved as (ii). □

Lemma 10.1.5. *Let $f : Y \to X$ be a continuous map of topological spaces, $\mathscr{F}$ a sheaf on X, and $\mathscr{G}$ a sheaf on Y. We have :*

- (i) *If f is an open immersion and $\mathscr{F}$ is qc-flabby on X, then $f^*\mathscr{F}$ is qc-flabby on Y.*
- (ii) *If $\mathscr{G}$ is flabby on Y, then $f_*\mathscr{G}$ is flabby on X, and $R^p f_*\mathscr{G} = 0$ for every $p > 0$.*
- (iii) *If f is quasi-compact and $\mathscr{G}$ is qc-flabby on Y, then $f_*\mathscr{G}$ is qc-flabby on X.*
- (iv) *If X and Y are locally coherent, f is quasi-compact and quasi-separated, and $\mathscr{G}$ is abelian and qc-flabby, then $R^p f_*\mathscr{G} = 0$ for every $p > 0$.*
- (v) *Let $\mathscr{A}$ be any sheaf of rings on X. Every injective $\mathscr{A}$-module is flabby.*
- (vi) *If X is locally noetherian, every qc-flabby sheaf on X is flabby.*

Proof. (i) and (iii) are trivial.

(ii): Clearly $f_*\mathscr{G}$ is flabby, and it remains to check that $R^p f_*\mathscr{G} = 0$ for every $p > 0$. To this aim, in view of remark 7.3.31(vi) and lemma 10.1.4(ii), it suffices to show that for every short exact sequence $0 \to \mathscr{G}' \to \mathscr{G} \to \mathscr{G}'' \to 0$ of flabby abelian sheaves on Y, the resulting sequence $0 \to f_*\mathscr{G}' \to f_*\mathscr{G} \to f_*\mathscr{G}'' \to 0$ is still short exact; the latter follows from 10.1.4(i).

(iv): The assertion is local on X, so we may assume that X is quasi-compact and quasi-separated, and then Y is also quasi-compact. Remark 7.3.31(vi) and lemma 10.1.4(iii.b) reduce to showing that, for every short exact sequence $0 \to \mathscr{G}' \to \mathscr{G} \to \mathscr{G}'' \to 0$ of qc-flabby abelian sheaves on Y, the resulting sequence $0 \to (f_*\mathscr{G}')_x \to (f_*\mathscr{G})_x \to (f_*\mathscr{G}'')_x \to 0$ is short exact, for every $x \in X$. However, every such x has a fundamental system of open neighborhoods consisting of coherent open subsets of X, so we come down to checking that the induced sequence $0 \to \mathscr{G}'(f^{-1}U) \to \mathscr{G}(f^{-1}U) \to \mathscr{G}''(f^{-1}U) \to 0$ is short exact, for every coherent open neighborhood U of x. But under the current assumptions, $f^{-1}U$ is quasi-compact and quasi-separated, so we conclude by lemma 10.1.4(iii.a).

(v): Let $j : U \to X$ be an open immersion, and $\mathscr{I}$ any injective $\mathscr{A}$-module; the counit of adjunction $j_!j^*\mathscr{A} \to \mathscr{A}$ is a monomorphism of $\mathscr{A}$-modules, so it induces a surjection

$$\varphi : \mathrm{Hom}_{\mathscr{A}}(\mathscr{A}, \mathscr{I}) \to \mathrm{Hom}_{\mathscr{A}}(j_!j^*\mathscr{A}, \mathscr{I}).$$

We have a natural identification : $\mathrm{Hom}_{\mathscr{A}}(\mathscr{A}, \mathscr{I}) \xrightarrow{\sim} \mathscr{I}(X)$; by adjunction, we have also a natural identification : $\mathrm{Hom}_{\mathscr{A}}(j_!j^*\mathscr{A}, \mathscr{I}) \xrightarrow{\sim} \mathrm{Hom}_{j^*\mathscr{A}}(j^*\mathscr{A}, j^*\mathscr{I}) \xrightarrow{\sim} \mathscr{I}(U)$. Under these identifications, φ corresponds to the restriction map $\mathscr{I}(X) \to \mathscr{I}(U)$, whence the contention.

(vi): Let $U \subset X$ be any open subset; we have to check that the restriction map $\mathscr{F}(X) \to \mathscr{F}(U)$ is surjective. To this aim, fix $s \in \mathscr{F}(U)$, and let S be the set of all pairs (V, s_V) where $V \subset X$ is an open subset containing U, and $s_V \in \mathscr{F}(V)$ is a section whose restriction to U agrees with s. We endow S with a partial ordering, by declaring that $(V, s_V) \leq (V', s_{V'})$ if $V \subset V'$ and the restriction of $s_{V'}$ to V agrees with s_V. A standard application of Zorn's lemma shows that S admits a maximal element (V, s_V), and it suffices to show that $V = X$. However, if the latter fails, there exists a noetherian open subset $U' \subset X$ which does not lie in V, and $U' \cap V$ is quasi-compact (remark 8.1.6(iv)); since $\mathscr{F}$ is qc-flabby, we may then find $s_{U'} \in \mathscr{F}(U')$ whose restriction to $U' \cap V$ agrees with the restriction of s_V. Then there exists a unique $s_{U' \cup V} \in \mathscr{F}(U' \cap V)$ whose restriction to U' and V agrees with $s_{U'}$ and respectively s_V; the pair $(U' \cup V, s_{U' \cup V})$ lies in S, contradicting the maximality of (V, s_V). □

10.1.6. Let Λ be a small cofiltered category. We call a *system of topological spaces indexed by* Λ, any functor (notation of (8.1.7))

$$X_\bullet : \Lambda \to \mathbf{Top} \qquad \lambda \mapsto X_u \qquad (\lambda \xrightarrow{u} \mu) \mapsto (\varphi_u : X_\lambda \to X_\mu)$$

We also call a *compatible system of sheaves on* $X_\bullet$ the datum of a family $(\mathscr{F}_\lambda \mid \lambda \in \mathrm{Ob}(\Lambda))$, where $\mathscr{F}_\lambda$ is a sheaf on X_λ for every $\lambda \in \mathrm{Ob}(\Lambda)$, together with *transition maps* :

$$\beta_u : \varphi_u^{-1}\mathscr{F}_\lambda \to \mathscr{F}_\mu \qquad \text{for every morphism } u : \mu \to \lambda \text{ in } \Lambda$$

such that $\beta_{\mathbf{1}_\lambda} = \mathbf{1}_{\mathscr{F}_\lambda}$ for every $\lambda \in \mathrm{Ob}(\Lambda)$, and for any two morphisms $\nu \xrightarrow{v} \mu \xrightarrow{u} \lambda$ of Λ, the following diagram commutes :

$$\begin{array}{ccc} \varphi_\nu^{-1} \circ \varphi_u^{-1}\mathscr{F}_\lambda & \xrightarrow{\sim} & \varphi_{u\circ v}^{-1}\mathscr{F}_\lambda \\ {\scriptstyle \varphi_\nu^{-1}\beta_u}\downarrow & & \downarrow{\scriptstyle \beta_{u\circ v}} \\ \varphi_\nu^{-1}\mathscr{F}_\mu & \xrightarrow{\beta_\nu} & \mathscr{F}_\nu. \end{array}$$

Lemma 10.1.7. *With the notation of* (10.1.6)*, let* X *be the limit of* $X_\bullet$*, and* $(\varphi_\lambda : X \to X_\lambda \mid \lambda \in \Lambda)$ *a universal cone. Suppose that either one of the following conditions holds :*

(a) $X_\bullet$ *is a constant functor with value* X *(so* $X_\lambda = X$ *for every* $\lambda \in \mathrm{Ob}(\Lambda)$*, and* $\varphi_u = \mathbf{1}_X$ *for every morphism* u *of* Λ*), and* X *is quasi-compact.*

(b) X_λ *is a sober and quasi-compact topological space, for every* $\lambda \in \Lambda$*.*

Then the natural map

$$\operatorname*{colim}_{\lambda\in\mathrm{Ob}(\Lambda^o)} \Gamma(X_\lambda, \mathscr{F}_\lambda) \to \Gamma(X, \mathscr{F}) \qquad \textit{with} \qquad \mathscr{F} := \operatorname*{colim}_{\lambda\in\mathrm{Ob}(\Lambda^o)} \varphi_\lambda^*\mathscr{F}_\lambda$$

is injective.

Proof. By virtue of proposition 1.5.21(ii), we may assume that Λ is a cofiltered partially ordered set, and we let $(\Lambda^o, \leq)$ be the opposite ordering on Λ (see example 1.1.6(iii)). We then come down to proving the following assertion. Let $\lambda \in \Lambda^o$ be any index, $s, s' \in \Gamma(X_\lambda, \mathscr{F}_\lambda)$ two sections whose images agree in $\Gamma(X, \mathscr{F})$, and for every $\mu \in \Lambda^o$ such that $\mu \geq \lambda$, denote by s_μ and s'_μ the images of s and s' in $\Gamma(X_\mu, \mathscr{F}_\mu)$; then there exists an index $\mu \geq \lambda$, such that $s_\mu = s'_\mu$. However, set

$$Z_\mu := \{x \in X_\mu \mid s_{\mu,x} \neq s'_{\mu,x}\} \qquad \text{for every } \mu \in \Lambda^o \text{ such that } \mu \geq \lambda$$

and endow Z_μ with the topology induced from X_μ. Now Z_μ is a closed subset of X_μ for every $\mu \geq \lambda$, and it is easily seen that $\psi_u(Z_\mu) \subset Z_\nu$ for every morphism $u : \mu \geq \nu$. Thus, $(Z_\mu \mid \mu \geq \lambda)$ is a cofiltered system of topological spaces, and the assumption on s and s' implies that the limit of this system is empty. If condition (a) holds, each Z_μ is a closed subset of the quasi-compact space X, so we deduce that $Z_\mu = \varnothing$ for some $\mu \geq \lambda$, whence $s_\mu = s'_\mu$ for such index μ. If (b) holds, each Z_μ is a sober space (lemma 8.1.3(iii)), and we conclude again that $Z_\mu = \varnothing$ for some $\mu \geq \lambda$, due to proposition 8.1.23(ii.a). □

10.1.8. Let Λ be a small cofiltered category; we consider the datum consisting of:

• Two systems $X_\bullet$ and $Y_\bullet$ of topological spaces indexed by Λ, such that X_λ and Y_λ are *locally spectral* for every $\lambda \in \Lambda$, with *quasi-compact and quasi-separated* transition maps :

$$(10.1.9) \qquad \varphi_u : X_\mu \to X_\lambda \qquad \psi_u : Y_\mu \to Y_\lambda \qquad \text{for every morphism } u : \mu \to \lambda \text{ in } \Lambda$$

• A compatible system of sheaves $\mathscr{F}_\bullet$ on $Y_\bullet$, as defined in (10.1.6).

• A natural transformation $g_\bullet : X_\bullet \Rightarrow Y_\bullet$, *i.e.* a system $g_\bullet := (g_\lambda : Y_\lambda \to X_\lambda \mid \lambda \in \mathrm{Ob}(\Lambda))$ of *quasi-compact and quasi-separated* continuous maps, such that

$$g_\lambda \circ \psi_u = \varphi_u \circ g_\mu \qquad \text{for every morphism } u : \mu \to \lambda.$$

The limits of $X_\bullet$, $Y_\bullet$ and $g_\bullet$ are representable by topological spaces and continuous maps

$$X := \lim_{\Lambda} X_\bullet \qquad Y := \lim_{\Lambda} Y_\bullet \qquad g := \lim_{\Lambda} g_\bullet : Y \to X$$

and the universal cones consist of *continuous spectral maps*

$$\varphi_\lambda : X \to X_\lambda \qquad \psi_\lambda : Y \to Y_\lambda \qquad \text{for every } \lambda \in \Lambda.$$

Proposition 10.1.10. *In the situation of* (10.1.8)*, the following holds :*

(i) *If Y_λ is spectral for every $\lambda \in \mathrm{Ob}(\Lambda)$, the natural map*

$$\operatorname*{colim}_{\lambda \in \mathrm{Ob}(\Lambda^o)} \Gamma(Y_\lambda, \mathscr{F}_\lambda) \to \Gamma(Y, \mathscr{F}) \qquad \textit{with} \qquad \mathscr{F} := \operatorname*{colim}_{\lambda \in \mathrm{Ob}(\Lambda^o)} \psi^*_\lambda \mathscr{F}_\lambda$$

is a bijection.

(ii) *If $\mathscr{F}_\bullet$ is a system of abelian sheaves, then the natural morphisms :*

$$\operatorname*{colim}_{\lambda \in \mathrm{Ob}(\Lambda^o)} \varphi^*_\lambda R^i g_{\lambda *} \mathscr{F}_\lambda \to R^i g_* \mathscr{F}$$

are isomorphisms for every $i \in \mathbb{N}$.

Proof. By virtue of proposition 1.5.21(ii), we may assume that Λ is a cofiltered partially ordered set, and we let $(\Lambda^o, \leq)$ be the opposite ordering on Λ (see example 1.1.6(iii)). The injectivity of the map in (i) is already known, by lemma 10.1.7. Let us then verify the surjectivity. In case $Y = \varnothing$, we know that $Y_\lambda = \varnothing$ for some $\lambda \in \Lambda^o$, so the assertion holds. Hence, we may assume that Y is not empty. Now, let $s \in \Gamma(Y, \mathscr{F})$ be any section; since

$$\mathscr{F}_y = \operatorname*{colim}_{\lambda \in \Lambda^o} \mathscr{F}_{\lambda, \psi_\lambda(y)} \qquad \text{for every } y \in Y$$

we may find, for every $y \in Y$, an index $\lambda(y) \in \Lambda^o$, a quasi-compact open neighborhood U_y of $\psi_{\lambda(y)}(y)$ in $Y_{\lambda,y}$ and a section $s(y) \in \Gamma(U_y, \mathscr{F}_\lambda)$ such that the image of $s(y)$ in $\mathscr{F}_y$ equals s_y. Then, for each $y \in Y$, we may find a quasi-compact open neighborhood U'_y of y in $\psi^{-1}_{\lambda(y)} U_y$ such that the image of $s(y)$ in U'_y agrees with the restriction of s. The family $(U'_y \mid y \in Y)$ is an open covering of Y, and since Y is quasi-compact (theorem 8.1.34(i)), we may find a finite (non-empty) subset $S \subset Y$ such that $(U'_y \mid y \in S)$ already covers Y. Then, for each $y \in S$ we may find an index $\mu(y) \geq \lambda(y)$ and a quasi-compact open subset U''_y of $\psi^{-1}_{\mu(y)\lambda(y)} U_{\lambda(y)}$ such that $\psi^{-1}_{\mu(y)} U''_y = U'_y$ (corollary 8.1.40(ii.a)). Next, since Λ is cofiltered, we may find $\mu \in \Lambda^o$ such that $\mu \geq \mu(y)$ for every $y \in S$, and after replacing U''_y by $\psi^{-1}_{\mu,\mu(y)} U''_y$, and $s(y)$ by its image in $\Gamma(\psi^{-1}_{\mu,\mu(y)} U''_y, \mathscr{F}_\mu)$ for every $y \in S$, we may assume that $\mu(y)$ is a single index μ, independent of $y \in S$. Now, since $\bigcup_{y \in S} \psi^{-1}_\mu U''_y = Y$, there exists $\nu \geq \mu$ such that $\bigcup_{y \in S} \psi^{-1}_{\nu\mu} U''_y = Y_\mu$ (corollary 8.1.40(ii.b)), so we may replace μ by ν, each U''_y by its preimage in Y_ν, and $s(y)$ by its image in $\Gamma(\psi^{-1}_{\nu\mu} U''_y, \mathscr{F}_\nu)$, and further assume that $(U''_y \mid y \in S)$ is a covering of Y_μ. Lastly, notice that – by construction – for every $y, y' \in S$, the images of $s(y)$ and $s(y')$ agree in $\Gamma(\psi^{-1}_\mu(U_y \cap U_{y'}), \mathscr{F})$. Since the injectivity of the map in (i) has already been established, we deduce that, for every $y, y' \in S$ there exists $\lambda(y, y') \geq \mu$ such that the images of $s(y)$ and $s(y')$ agree in $\Gamma(\psi^{-1}_{\lambda(y,y'),\mu}(U''_y \cap U''_{y'}), \mathscr{F}_{\lambda(y,y')})$. Since Λ is cofiltered, we may then find $\lambda \in \Lambda^o$ such that $\lambda \geq \lambda(y, y')$ for every $y, y' \in S$. Now, let $V_y := \psi^{-1}_{\lambda\mu} U''_y$ and denote by $s'(y)$ the image of $s(y)$ in $\Gamma(V_y, \mathscr{F}_\lambda)$, for every $y \in S$; by construction we have $s'(y)_{|V_y \cap V_{y'}} = s'(y)_{|V_y \cap V_{y'}}$ for every $y, y' \in S$, so the system $(s'(y) \mid y \in S)$ corresponds to a well defined section $s' \in \Gamma(Y_\lambda, \mathscr{F}_\lambda)$, whose image in $\Gamma(Y, \mathscr{F})$ is finally the original section s.

(ii): We notice :

Claim 10.1.11. Suppose that $\mathscr{F}_\lambda$ is qc-flabby for every $\lambda \in \Lambda^o$. Then the same holds for $\mathscr{F}$.

Proof of the claim. Indeed, let $U \subset V$ be any two quasi-compact open subsets of Y; we need to show that the restriction map

$$\Gamma(V, \mathscr{F}) \to \Gamma(U, \mathscr{F})$$

is onto. By corollary 8.1.40(ii) we can find $\lambda \in \Lambda^o$ and quasi-compact open subsets $U_\lambda \subset V_\lambda \subset Y_\lambda$ such that $U = \psi_\lambda^{-1} U_\lambda$, and likewise for V. Let us set $U_\mu := \psi_{\mu\lambda}^{-1} U_\lambda$ for every $\mu \in \Lambda^o$ such that $\mu \geq \lambda$, and likewise define V_μ. Then, up to replacing Λ^o by the cofinal subset $\{\mu \in \Lambda^o \mid \mu \geq \lambda\}$, we may assume that $U_\mu \subset V_\mu$ are defined for every $\mu \in \Lambda^o$. By (i) the natural map

$$\operatorname*{colim}_{\lambda \in \Lambda^o} \Gamma(U_\lambda, \mathscr{F}_\lambda) \to \Gamma(U, \mathscr{F})$$

is bijective for every $n \in \mathbb{N}$, and likewise for V, whence the claim. ◊

Now, we pick for each $\lambda \in \Lambda^o$ an injective resolution $\mathscr{F}_\lambda \to \mathscr{I}_\lambda^\bullet$, having care to construct $\mathscr{I}_\lambda^\bullet$ functorially, so that $\mathscr{F}_\bullet$ extends to a compatible system of complexes $(\mathscr{F}_\lambda \to \mathscr{I}_\lambda^\bullet \mid \lambda \in \Lambda^o)$. From claim 10.1.11, we know that $\mathscr{I}^\bullet := \operatorname{colim}_{\lambda \in \Lambda^o} \psi_\lambda^* \mathscr{I}_\lambda^\bullet$ is a complex of qc-flabby sheaves; then lemma 10.1.5(iv) and remark 7.3.31(vi) yield a natural isomorphism :

$$g_* \mathscr{I}^\bullet \xrightarrow{\sim} Rg_* \mathscr{F} \qquad \text{in } \mathsf{D}(\mathbb{Z}_X\text{-}\mathbf{Mod}).$$

To conclude, let $V \to X$ be any quasi-compact open immersion, which as usual we see as the limit of a system $(V_\lambda \to X_\lambda \mid \lambda \in \Lambda)$ of quasi-compact open immersions; we come down to checking that the natural map

$$\Gamma(V, \operatorname*{colim}_{\lambda \in \Lambda^o} \varphi_\lambda^* g_{\lambda *} \mathscr{I}_\lambda^\bullet) \to \Gamma(V, g_{\infty *} \mathscr{I}^\bullet) = \Gamma(g^{-1}V, \mathscr{I}^\bullet)$$

is an isomorphism of complexes. However, $g^{-1}V$ is the limit of the system $(g_\lambda^{-1} V_\lambda \mid \lambda \in \Lambda)$ of spectral topological spaces and quasi-compact quasi-separated maps, so (i) naturally identifies both complexes with

$$\operatorname*{colim}_{\lambda \in \Lambda^o} \Gamma(g_\lambda^{-1} V_\lambda, \mathscr{I}_\lambda^\bullet)$$

whence the contention. □

Remark 10.1.12. In the situation of (10.1.8), suppose that $X_\bullet$ and $Y_\bullet$ are two systems of *schemes*, that the systems $\varphi_\bullet$ and $\psi_\bullet$ of (10.1.9) consist of *affine* morphisms of schemes, and $g_\bullet$ is a system of *quasi-compact and quasi-separated* morphisms of schemes. By [65, Ch.IV, Prop.8.2.3] the limits (in the category of schemes) of the systems $X_\bullet$, $Y_\bullet$ (resp. $g_\bullet$) are representable by schemes (resp. by a morphism of schemes) whose underlying topological spaces (resp. continuous map) agrees with X and Y (resp. g), and the resulting φ_λ, ψ_λ are affine morphisms of schemes, for every $\lambda \in \mathrm{Ob}(\Lambda)$. Hence, proposition 10.1.10 applies especially to such inverse systems.

10.1.13. Let $(X, \mathscr{A})$ be a ringed space, and denote by $\mathbb{Z}_X$ the sheaf of rings associated with the constant presheaf with values $\mathbb{Z}$; then $\mathbb{Z}_X\text{-}\mathbf{Mod}$ is the category of abelian sheaves on X, and the natural morphism $\mathbb{Z}_X \to \mathscr{A}$ induces a forgetful functor

$$\varphi_{\mathscr{A}} : \mathsf{D}(\mathscr{A}\text{-}\mathbf{Mod}) \to \mathsf{D}(\mathbb{Z}_X\text{-}\mathbf{Mod}).$$

Furthermore, let $f : (Y, \mathscr{B}) \to (X, \mathscr{A})$ be a morphism of ringed spaces. For every $\mathscr{B}$-module $\mathscr{F}$, the direct image $f_* \mathscr{F}$ is naturally an $\mathscr{A}$-module, and we notice that the derived functor $Rf_* : \mathsf{D}^+(\mathscr{B}\text{-}\mathbf{Mod}) \to \mathsf{D}^+(\mathscr{A}\text{-}\mathbf{Mod})$ commutes with the forgetful functors $\varphi_{\mathscr{A}}, \varphi_{\mathscr{B}}$; indeed, every injective $\mathscr{B}$-module is a flabby $\mathbb{Z}_Y$-module (lemma 10.1.5(v)), and the latter are acyclic for the functor $f_* : \mathbb{Z}_Y\text{-}\mathbf{Mod} \to \mathbb{Z}_X\text{-}\mathbf{Mod}$ (lemma 10.1.5(ii)), so the assertion follows easily.

One defines as in example 7.1.16(v) a *total* Hom *cochain complex*, which is a functor :

$$\mathscr{H}om^\bullet_{\mathscr{A}} : \mathsf{C}(\mathscr{A}\text{-}\mathbf{Mod}) \times \mathsf{C}(\mathscr{A}\text{-}\mathbf{Mod})^o \to \mathsf{C}(\mathscr{A}\text{-}\mathbf{Mod})$$

on the category of complexes of $\mathscr{A}$-modules. Recall that, for any two complexes $M_\bullet$, $N^\bullet$, and any object U of X, the group of n-cocycles in $\mathscr{H}om^\bullet_{\mathscr{A}}(M_\bullet, N^\bullet)(U)$ is naturally isomorphic to $\mathrm{Hom}_{\mathsf{C}(\mathscr{A}_{|U}\text{-}\mathbf{Mod})}(M_{\bullet|U}, N^\bullet_{|U}[n])$, and $H^n\mathscr{H}om^\bullet_{\mathscr{A}}(M_\bullet, N^\bullet)(U)$ is naturally isomorphic to the group of homotopy classes of maps $M_{\bullet|U} \to N^\bullet_{|U}[n]$ (see example 7.1.15(i)). We also set :

$$(10.1.14)\qquad \mathrm{Hom}^\bullet_{\mathscr{A}} := \Gamma \circ \mathscr{H}om^\bullet_{\mathscr{A}} : \mathsf{C}(\mathscr{A}\text{-}\mathbf{Mod}) \times \mathsf{C}(\mathscr{A}\text{-}\mathbf{Mod})^o \to \mathsf{C}(\Gamma(\mathscr{A})\text{-}\mathbf{Mod}).$$

The bifunctor $\mathscr{H}om^\bullet_{\mathscr{A}}$ admits a right derived functor :

$$R\mathscr{H}om^\bullet_{\mathscr{A}} : \mathsf{D}^+(\mathscr{A}\text{-}\mathbf{Mod}) \times \mathsf{D}(\mathscr{A}\text{-}\mathbf{Mod})^o \to \mathsf{D}(\mathscr{A}\text{-}\mathbf{Mod})$$

for whose construction we refer to [163, §10.7]. Likewise, one has a derived functor $R\mathrm{Hom}^\bullet_{\mathscr{A}}$ for (10.1.14), and there are natural isomorphisms of $\mathscr{A}$-modules :

$$(10.1.15)\qquad H^iR\mathrm{Hom}^\bullet_{\mathscr{A}}(M^\bullet, N^\bullet) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{D}^+(\mathscr{A}\text{-}\mathbf{Mod})}(M^\bullet, N^\bullet[i])$$

for every $i \in \mathbb{Z}$, and every bounded below complexes $M^\bullet$ and $N^\bullet$ of $\mathscr{A}$-modules.

Lemma 10.1.16. *Let $(X, \mathscr{A})$ be a ringed space, $\mathscr{B}$ an $\mathscr{A}$-algebra. Then :*

(i) *If $\mathscr{I}$ is an injective $\mathscr{A}$-module, $\mathscr{H}om_{\mathscr{A}}(\mathscr{F}, \mathscr{I})$ is flabby for every $\mathscr{A}$-module $\mathscr{F}$, and $\mathscr{H}om_{\mathscr{A}}(\mathscr{B}, \mathscr{I})$ is an injective $\mathscr{B}$-module.*

(ii) *There is a natural isomorphism of bifunctors :*

$$R\Gamma \circ R\mathscr{H}om^\bullet_{\mathscr{A}} \xrightarrow{\sim} R\mathrm{Hom}^\bullet_{\mathscr{A}}.$$

(iii) *The forgetful functor $\mathsf{D}^+(\mathscr{B}\text{-}\mathbf{Mod}) \to \mathsf{D}^+(\mathscr{A}\text{-}\mathbf{Mod})$ admits the right adjoint :*

$$\mathsf{D}^+(\mathscr{A}\text{-}\mathbf{Mod}) \to \mathsf{D}^+(\mathscr{B}\text{-}\mathbf{Mod}) \quad : \quad K^\bullet \mapsto R\mathscr{H}om^\bullet_{\mathscr{A}}(\mathscr{B}, K^\bullet).$$

Proof. (i) : Let $j : U \subset X$ be any open immersion; a section $s \in \mathscr{H}om_{\mathscr{A}}(\mathscr{F}, \mathscr{I})(U)$ is a map of $\mathscr{A}_{|U}$-modules $s : \mathscr{F}_{|U} \to \mathscr{I}_{|U}$. We deduce a map of $\mathscr{A}$-modules $j_!s : j_!\mathscr{F}_{|U} \to j_!\mathscr{I}_{|U} \to \mathscr{I}$; since $\mathscr{I}$ is injective, $j_!s$ extends to a map $\mathscr{F} \to \mathscr{I}$, as required. Next, let $\mathscr{B}$ be an $\mathscr{A}$-algebra; we recall the following :

Claim 10.1.17. The functor

$$\mathscr{A}\text{-}\mathbf{Mod} \to \mathscr{B}\text{-}\mathbf{Mod} \quad : \quad \mathscr{F} \mapsto \mathscr{H}om_{\mathscr{A}}(\mathscr{B}, \mathscr{F})$$

is right adjoint to the forgetful functor.

Proof of the claim. We have to exhibit a natural bijection

$$\mathrm{Hom}_{\mathscr{A}}(N, M) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{B}}(N, \mathrm{Hom}_{\mathscr{A}}(\mathscr{B}, M))$$

for every $\mathscr{A}$-module M and $\mathscr{B}$-module N. This is given by the following rule. To an $\mathscr{A}$-linear map $t : N \to M$ one assigns the $\mathscr{B}$-linear map $t' : N \to \mathrm{Hom}_{\mathscr{A}}(\mathscr{B}, M)$ such that $t'(x)(b) = t(x \cdot b)$ for every $x \in N(U)$ and $b \in \mathscr{B}(V)$, where $V \subset U$ are any two open subsets of X. To describe the inverse of this transformation, it suffices to remark that $t(x) = t'(x)(1)$ for every local section x of N. ◊

Since the forgetful functor is exact, the second assertion of (i) follows immediately from claim 10.1.17. Assertion (ii) follows from (i) and [163, Th.10.8.2].

(iii): Let $K^\bullet$ (resp. $L^\bullet$) be a bounded below complex of $\mathscr{B}$-modules (resp. of injective $\mathscr{A}$-modules); then :

$$\begin{aligned}\mathrm{Hom}_{\mathsf{D}^+(\mathscr{A}\text{-}\mathbf{Mod})}(K^\bullet, L^\bullet) &= H^0\mathrm{Hom}^\bullet_{\mathsf{C}(\mathscr{A}\text{-}\mathbf{Mod})}(K^\bullet, L^\bullet)\\ &= H^0\mathrm{Hom}^\bullet_{\mathsf{C}(\mathscr{B}\text{-}\mathbf{Mod})}(K^\bullet, \mathscr{H}om_{\mathscr{A}}(\mathscr{B}, L^\bullet))\\ &= \mathrm{Hom}_{\mathsf{D}^+(\mathscr{B}\text{-}\mathbf{Mod})}(K^\bullet, \mathscr{H}om_{\mathscr{A}}(\mathscr{B}, L^\bullet))\end{aligned}$$

where the last identity follows from (i) and (10.1.15). □

Theorem 10.1.18 (Trivial duality)**.** *Let $f : (Y, \mathscr{B}) \to (X, \mathscr{A})$ be a morphism of ringed spaces. For every $M_\bullet \in \mathrm{Ob}(\mathsf{D}^-(\mathscr{A}\text{-}\mathbf{Mod}))$ and $N^\bullet \in \mathrm{Ob}(\mathsf{D}^+(\mathscr{B}\text{-}\mathbf{Mod}))$, there are natural isomorphisms :*

(i) $R\mathrm{Hom}^\bullet_{\mathscr{A}}(M_\bullet, Rf_*N^\bullet) \xrightarrow{\sim} R\mathrm{Hom}^\bullet_{\mathscr{B}}(Lf^*M_\bullet, N^\bullet)$ *in* $\mathsf{D}^+(\mathscr{A}(X)\text{-}\mathbf{Mod})$.

(ii) $R\mathscr{H}om^\bullet_{\mathscr{A}}(M_\bullet, Rf_*N^\bullet) \xrightarrow{\sim} Rf_*R\mathscr{H}om^\bullet_{\mathscr{B}}(Lf^*M_\bullet, N^\bullet)$ *in* $\mathsf{D}^+(\mathscr{A}\text{-}\mathbf{Mod})$.

Proof. One applies lemma 10.1.16(ii) to deduce (i) from (ii).

(ii) : By standard adjunctions, for any two complexes $M_\bullet$ and $N^\bullet$ as in the proposition, we have a natural isomorphism of total Hom cochain complexes :

$$\mathscr{H}om^\bullet_{\mathscr{A}}(M_\bullet, f_*N^\bullet) \xrightarrow{\sim} f_*\mathscr{H}om^\bullet_{\mathscr{B}}(f^*M_\bullet, N^\bullet). \tag{10.1.19}$$

Now, let us choose a flat resolution $P_\bullet \xrightarrow{\sim} M_\bullet$ of $\mathscr{A}$-modules, and an injective resolution $N^\bullet \xrightarrow{\sim} I^\bullet$ of $\mathscr{B}$-modules. In view of (10.1.19) and lemma 10.1.16(i) we have natural isomorphisms :

$$Rf_*R\mathscr{H}om^\bullet_{\mathscr{B}}(Lf^*M_\bullet, N^\bullet) \xrightarrow{\sim} f_*\mathscr{H}om^\bullet_{\mathscr{B}}(f^*P_\bullet, I^\bullet) \xleftarrow{\sim} \mathscr{H}om^\bullet_{\mathscr{A}}(P_\bullet, f_*I^\bullet)$$

in $\mathsf{D}^+(\mathscr{A}\text{-}\mathbf{Mod})$. It remains to show that the natural map

$$\mathscr{H}om^\bullet_{\mathscr{A}}(P_\bullet, f_*I^\bullet) \to R\mathscr{H}om^\bullet_{\mathscr{A}}(P_\bullet, f_*I^\bullet) \tag{10.1.20}$$

is an isomorphism. However, we have two spectral sequences :

$$E^1_{pq} := H_p\mathscr{H}om^\bullet_{\mathscr{A}}(P_q, f_*I^\bullet) \Rightarrow H_{p+q}\mathscr{H}om^\bullet_{\mathscr{A}}(P_\bullet, f_*I^\bullet)$$
$$F^1_{pq} := H_pR\mathscr{H}om^\bullet_{\mathscr{A}}(P_q, f_*I^\bullet) \Rightarrow H_{p+q}R\mathscr{H}om^\bullet_{\mathscr{A}}(P_\bullet, f_*I^\bullet)$$

and (10.1.20) induces a natural map of spectral sequences :

$$E^1_{pq} \to F^1_{pq} \tag{10.1.21}$$

Consequently, it suffices to show that (10.1.21) is an isomorphism for every $p, q \in \mathbb{N}$, and therefore we may assume that $P_\bullet$ consists of a single flat $\mathscr{A}$-module placed in degree zero. A similar argument reduces to the case where $I_\bullet$ consists of a single injective $\mathscr{B}$-module sitting in degree zero. To conclude, it suffices to show :

Claim 10.1.22. Let P be a flat $\mathscr{A}$-module, I an injective $\mathscr{B}$-module. Then the natural map :

$$\mathscr{H}om_{\mathscr{A}}(P, f_*I)[0] \to R\mathscr{H}om^\bullet_{\mathscr{A}}(P, f_*I)$$

is an isomorphism.

Proof of the claim. $R^i\mathscr{H}om^\bullet_{\mathscr{A}}(P, f_*I)$ is the sheaf associated with the presheaf on X :

$$U \mapsto R^i\mathrm{Hom}^\bullet_{\mathscr{A}_{|U}}(P_{|U}, f_*I_{|U}).$$

Since $I_{|U}$ is an injective $\mathscr{B}_{|U}$-module, it suffices therefore to show that $R^i\mathrm{Hom}^\bullet_{\mathscr{A}}(P, f_*I) = 0$ for $i > 0$. However, recall that there is a natural isomorphism :

$$R^i\mathrm{Hom}^\bullet_{\mathscr{A}}(P, f_*I) \simeq \mathrm{Hom}_{\mathsf{D}(\mathscr{A}\text{-}\mathbf{Mod})}(P, f_*I[-i]).$$

Since the homotopy category $\mathsf{Hot}(\mathscr{A}\text{-}\mathbf{Mod})$ admits a left calculus of fractions, we deduce a natural isomorphism :

$$R^i\mathrm{Hom}^\bullet_{\mathscr{A}}(P, f_*I) \simeq \operatorname*{colim}_{E_\bullet \to P} \mathrm{Hot}_{\mathscr{A}\text{-}\mathbf{Mod}}(E_\bullet, f_*I[-i]) \tag{10.1.23}$$

where the colimit ranges over the family of quasi-isomorphisms $E_\bullet \to P$. We may furthermore restrict the colimit in (10.1.23) to the subfamily of all such $E_\bullet \to P$ where $E_\bullet$ is a bounded above complex of flat $\mathscr{A}$-modules, since this subfamily is cofinal. Every such map $E_\bullet \to P$ induces a commutative diagram :

$$\begin{array}{ccc} \mathrm{Hom}_{\mathscr{A}}(P, f_*I[-i]) & \longrightarrow & \mathrm{Hot}_{\mathscr{A}\text{-}\mathbf{Mod}}(E_\bullet, f_*I[-i]) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathscr{B}}(f^*P, I[-i]) & \longrightarrow & \mathrm{Hot}_{\mathscr{B}\text{-}\mathbf{Mod}}(f^*E_\bullet, I[-i]) \end{array}$$

whose vertical arrows are isomorphisms. Since $E_\bullet$ and P are complexes of flat $\mathscr{A}$-modules, the induced map $f^*E_\bullet \to f^*P$ is again a quasi-isomorphism; therefore, since I is injective, the bottom horizontal arrow is an isomorphism, hence the same holds for the top horizontal one, and the claim follows easily. □

10.1.24. Suppose that $\mathscr{F}$ is an abelian group object in a given topos X, and let $(\mathrm{Fil}^n\mathscr{F} \mid n \in \mathbb{N})$ be a descending filtration by abelian subobjects of $\mathscr{F}$, such that $\mathrm{Fil}^0\mathscr{F} = \mathscr{F}$. Set $\mathscr{F}^n := \mathscr{F}/\mathrm{Fil}^{n+1}\mathscr{F}$ for every $n \in \mathbb{N}$, and denote by $\mathrm{gr}^\bullet\mathscr{F}$ the graded abelian object associated with the filtered object $\mathrm{Fil}^\bullet\mathscr{F}$; then we have :

Lemma 10.1.25. *In the situation of* (10.1.24)*, fix $q \in \mathbb{N}$ and suppose that the natural map :*

$$\mathscr{F} \to R\lim_{k\in\mathbb{N}} \mathscr{F}^k$$

is an isomorphism in $\mathsf{D}(\mathbb{Z}_X\text{-}\mathbf{Mod})$. Then, the following conditions are equivalent :

(a) $H^q(X, \mathrm{Fil}^n\mathscr{F}) = 0$ *for every* $n \in \mathbb{N}$.

(b) *The natural map* $H^q(X, \mathrm{Fil}^n\mathscr{F}) \to H^q(X, \mathrm{gr}^n\mathscr{F})$ *vanishes for every* $n \in \mathbb{N}$.

Proof. Obviously we have only to show that (b) implies (a). Moreover, let us set :

$$(\mathrm{Fil}^n\mathscr{F})^k := \mathrm{Fil}^n\mathscr{F}/\mathrm{Fil}^{k+1}\mathscr{F} \qquad \text{for every } n, k \in \mathbb{N} \text{ with } k \geq n.$$

There follows, for every $n \in \mathbb{N}$, a short exact sequence of inverse systems of sheaves :

$$0 \to ((\mathrm{Fil}^n\mathscr{F})^k \mid k \geq n) \to (\mathscr{F}^k \mid k \geq n) \to \mathscr{F}^{n-1} \to 0$$

(where the right-most term is the constant inverse system which equals $\mathscr{F}^{n-1}$ in all degrees, with transition maps given by the identity morphisms); whence a distinguished triangle :

$$R\lim_{k\in\mathbb{N}} (\mathrm{Fil}^n\mathscr{F})^k \to R\lim_{k\in\mathbb{N}} \mathscr{F}^k \to \mathscr{F}^{n-1}[0] \to R\lim_{k\in\mathbb{N}} (\mathrm{Fil}^n\mathscr{F})^k[1]$$

which, together with our assumption on $\mathscr{F}$, easily implies that the natural map $\mathrm{Fil}^n\mathscr{F} \to R\lim_{k\in\mathbb{N}}(\mathrm{Fil}^n\mathscr{F})^k$ is an isomorphism in $\mathsf{D}(\mathbb{Z}_X\text{-}\mathbf{Mod})$. Summing up, we may replace the datum $(\mathscr{F}, \mathrm{Fil}^\bullet\mathscr{F})$ by $(\mathrm{Fil}^n\mathscr{F}, \mathrm{Fil}^{\bullet+n}\mathscr{F})$, and reduce to the case where $n = 0$.

The inverse system $(\mathscr{F}^n \mid n \in \mathbb{N})$ defines an abelian group object of the topos $X^{\mathbb{N}}$ (see [75, §7.3.4]); whence a spectral sequence:

$$E_2^{pq} := \lim_{n\in\mathbb{N}}{}^p\, H^q(X, \mathscr{F}^n) \Rightarrow H^{p+q}(X^{\mathbb{N}}, \mathscr{F}^\bullet) \simeq H^{p+q}(X, R\lim_{n\in\mathbb{N}} \mathscr{F}^n). \tag{10.1.26}$$

By [163, Cor.3.5.4] we have $E_2^{pq} = 0$ whenever $p > 1$, and, in view of our assumptions, (10.1.26) decomposes into short exact sequences :

$$0 \to \lim_{n\in\mathbb{N}}{}^1\, H^{q-1}(X, \mathscr{F}^n) \xrightarrow{\alpha_q} H^q(X, \mathscr{F}) \xrightarrow{\beta_q} \lim_{n\in\mathbb{N}} H^q(X, \mathscr{F}^n) \to 0 \qquad \text{for every } q \in \mathbb{N}$$

where β_q is induced by the natural system of maps $(\beta_{q,n} : H^q(X, \mathscr{F}) \to H^q(X, \mathscr{F}^n) \mid n \in \mathbb{N})$.

Claim 10.1.27. If (b) holds, then $\beta_q = 0$.

Proof of the claim. Suppose that (b) holds; we consider the long exact cohomology sequence associated with the short exact sequence :

$$0 \to \mathrm{Fil}^{n+1}\mathscr{F} \to \mathrm{Fil}^n\mathscr{F} \to \mathrm{gr}^n\mathscr{F} \to 0$$

to deduce that the natural map $H^q(X, \mathrm{Fil}^{n+1}\mathscr{F}) \to H^q(X, \mathrm{Fil}^n\mathscr{F})$ is onto for every $n \in \mathbb{N}$; hence the same holds for the natural map $H^q(X, \mathrm{Fil}^n\mathscr{F}) \to H^q(X, \mathscr{F})$. By considering the long exact cohomology sequence attached to the short exact sequence :

$$0 \to \mathrm{Fil}^n\mathscr{F} \to \mathscr{F} \to \mathscr{F}^n \to 0 \tag{10.1.28}$$

we then deduce that $\beta_{q,n}$ vanishes for every $n \in \mathbb{N}$, which implies the claim. ◊

Claim 10.1.29. If (b) holds, then the inverse system $(H^{q-1}(X, \mathscr{F}^n) \mid n \in \mathbb{N})$ has surjective transition maps.

Proof of the claim. By considering the long exact cohomology sequence associated with the short exact sequence : $0 \to \mathrm{gr}^n \mathscr{F} \to \mathscr{F}^{n+1} \to \mathscr{F}^n \to 0$, we are reduced to showing that the boundary map $\partial_{q-1} : H^{q-1}(X, \mathscr{F}^n) \to H^q(X, \mathrm{gr}^n \mathscr{F})$ vanishes for every $n \in \mathbb{N}$. However, comparing with (10.1.28), we find that ∂_{q-1} factors through the natural map $H^q(X, \mathrm{Fil}^n \mathscr{F}) \to H^q(X, \mathrm{gr}^n \mathscr{F})$, which vanishes if (b) holds. ◊

The lemma follows from claims 10.1.27 and 10.1.29, and [163, Lemma 3.5.3]. □

Our next result generalizes to arbitrary coherent spaces a classical theorem from [84], where it was stated only for noetherian spaces. We need first to introduce a general truncation construction for abelian sheaves, that shall be useful also in section 11.2.

10.1.30. Let $(X, \mathscr{A})$ be a ringed topological space, $\varphi : V \to X$ a locally closed immersion, denote by $\overline{V}$ the topological closure of V in X, and set $\partial V := \overline{V} \setminus V$, which is also a closed subset of X. Let also $i : \overline{V} \to X$ and $j : V \to \overline{V}$ be the induced closed and respectively open immersions. We set

$$\varphi_! := i_* \circ j_! : \varphi^* \mathscr{A}\text{-}\mathbf{Mod} \to \mathscr{A}\text{-}\mathbf{Mod} \qquad t_V := \varphi_! \circ \varphi^* : \mathscr{A}\text{-}\mathbf{Mod} \to \mathscr{A}\text{-}\mathbf{Mod}$$

and we call $\varphi_!$ (resp. t_V) the functor of *extension by zero from V to X* (resp. of *truncation outside V*). Notice that we have natural identifications

$$t_V \mathscr{F}_x \xrightarrow{\sim} \begin{cases} \mathscr{F}_x & \text{if } x \in V \\ 0 & \text{otherwise.} \end{cases} \tag{10.1.31}$$

Lemma 10.1.32. *In the situation of* (10.1.30)*, set $W := X \setminus V$, denote by $\psi : W \to X$ the resulting locally closed immersion, and let $\mathscr{F}$ be an $\mathscr{A}$-module such that $\psi^* \mathscr{F} = 0$. Then there exists a unique natural isomorphism*

$$f_{\mathscr{F}} : t_V \mathscr{F} \xrightarrow{\sim} \mathscr{F}$$

such that $f_{\mathscr{F},x}$ is the natural identification (10.1.31)*, for every $x \in V$.*

Proof. The uniqueness of $f_{\mathscr{F}}$ is clear. For the existence, set $\mathscr{G} := i^* \mathscr{F}$; it suffices to exhibit natural isomorphisms

$$\mathscr{F} \xrightarrow{\sim} i_* \mathscr{G} \qquad \text{and} \qquad j_! \varphi^* \mathscr{F} \xrightarrow{\sim} \mathscr{G}$$

which induce the corresponding natural identifications on the stalks. So we may assume that φ is either an open or closed immersion. In case φ is open, we obtain a natural map $f : \varphi_! \mathscr{F} \to \mathscr{F}$ as follows. Recall that $\varphi_! \mathscr{F}$ is the sheaf associated with the presheaf $\mathscr{H}$ such that $\mathscr{H}(U) = \mathscr{F}(U)$ for $U \subset V$, and $\mathscr{H}(U) = 0$ otherwise; then clearly there is a natural monomorphism $\mathscr{H} \to \mathscr{F}$ in the category of presheaves on X, and the latter factors uniquely through a map as sought. By construction, f_x is the identity endomorphism of $\mathscr{F}_x$, for every $x \in V$, and f_x is also trivially an isomorphism for $x \notin V$, since in this case the stalks of both source and target of f_x vanish. So f is indeed an isomorphism with the required properties.

Lastly, suppose φ is a closed immersion; for any open subset T of V, let C_T be the cofiltered set of all open subsets U of X such that $U \cap V = T$, and set

$$\varphi^{-1} \mathscr{H}(T) := \operatorname*{colim}_{U \in C_T^o} \mathscr{H}(U) \qquad \text{for every presheaf of abelian groups } \mathscr{H} \text{ on } X.$$

The rule $T \mapsto \varphi^{-1} \mathscr{H}(T)$ for every open subset T of V yields a well defined abelian presheaf on V, functorial in $\mathscr{H}$ (the resulting functor φ^{-1} from abelian presheaves on X to abelian presheaves on V is left adjoint to the direct image functor φ_* : cp. remark 1.3.6(i)). Now, suppose that $\mathscr{H}$ is a $\mathscr{A}$-module such that $\mathscr{H}_{|X \setminus V} = 0$; in this case, it is easily seen that $\varphi^{-1} \mathscr{H}$ is a $\varphi^* \mathscr{A}$-module, and moreover the transition map $\mathscr{H}(U) \to \mathscr{H}(U')$ is an isomorphism, for every $U, U' \in C_T$ such that $U' \subset U$. Especially, this applies with $\mathscr{H} := \mathscr{F}$, so we get a natural isomorphism

$$f_U : \mathscr{F}(U) \xrightarrow{\sim} i^{-1}\mathscr{F}(U \cap V) = i_* i^{-1}\mathscr{F}(U) \qquad \text{for every open subset } U \subset X$$

and the system of such maps f_U is the sought isomorphism $\mathscr{F} \xrightarrow{\sim} i_* i^* \mathscr{F}$. □

Proposition 10.1.33. *Let V and V' be two locally closed subsets of X. The following holds :*

(i) *We have a unique isomorphism of functors :*

$$\tau^{V,V'} : t_V \circ t_{V'} \xrightarrow{\sim} t_{V \cap V'}$$

which induces, for every $\mathscr{A}$-module $\mathscr{F}$ and every $x \in V \cap V'$ a commutative diagram

$$\begin{array}{ccc} t_V \circ t_{V'}\mathscr{F}_x & \xrightarrow{\tau_x^{V,V'}} & t_{V\cap V''}\mathscr{F}_x \\ \alpha\downarrow & & \downarrow\beta \\ t_{V'}\mathscr{F}_x & \xrightarrow{\gamma} & \mathscr{F}_x \end{array} \tag{10.1.34}$$

where α, β and γ are the natural identifications of (10.1.31).

(ii) *Let $V'' \subset X$ be a third locally closed subset. Then the diagram*

$$\begin{array}{ccc} t_V \circ t_{V'} \circ t_{V''} & \xrightarrow{t_V * \tau^{V',V''}} & t_V \circ t_{V'\cap V''} \\ \tau^{V,V'} * t_{V''}\downarrow & & \downarrow\tau^{V,V'\cap V''} \\ t_{V\cap V'} \circ t_{V''} & \xrightarrow{\tau^{V\cap V',V''}} & t_{V\cap V'\cap V''} \end{array}$$

commutes.

(iii) *Suppose that :*
  (a) *either $V \subset V'$ and the resulting immersion $V \to V'$ is closed*
  (b) *or else $V' \subset V$ and the resulting immersion $V' \to V$ is open.*

Then there exists a unique morphism of functors

$$c^{V,V'} : t_{V'} \to t_V$$

which induces, for every $\mathscr{A}$-module $\mathscr{F}$ and every $x \in V \cap V'$, a commutative diagram

$$\begin{array}{ccc} t_{V'}\mathscr{F}_x & \xrightarrow{c_x^{V,V'}} & t_V\mathscr{F}_x \\ & \searrow \quad \swarrow & \\ & \mathscr{F}_x & \end{array}$$

whose downward arrows are the natural identifications (10.1.31).

(iv) *In the situation of* (iii), *let V'' be a third locally closed subset of X. Then the resulting diagram*

$$\begin{array}{ccc} t_{V''} \circ t_{V'} & \xrightarrow{t_{V''} * c^{V,V'}} & t_{V''} \circ t_V \\ \tau^{V'',V'}\downarrow & & \downarrow\tau^{V'',V} \\ t_{V''\cap V'} & \xrightarrow{c^{V''\cap V,V''\cap V'}} & t_{V''\cap V} \end{array}$$

commutes.

(v) *Let $V \subset V' \subset V''$ (resp. $V'' \subset V' \subset V$) be a chain of closed (resp. open) immersions of locally closed subsets of X. Then*

$$c^{V,V'} \circ c^{V',V''} = c^{V,V''}.$$

Proof. (i): The uniqueness is clear. For the existence, set $W := V \cap V''$, and let $\varphi : W \to X$ and $\psi : X \setminus W \to X$ be the locally closed immersions; by inspecting the definition, we find natural isomorphisms

$$\varphi^* \circ t_V \circ t_{V'}\mathscr{F} \xrightarrow{\sim} \varphi^*\mathscr{F} \xleftarrow{\sim} \varphi^* t_W \mathscr{F} \qquad \text{for every } \mathscr{A}\text{-module } \mathscr{F}$$

which induce on stalks the isomorphisms $\varphi^*(\gamma \circ \alpha)$ and respectively $\varphi^*\beta$. There follows an isomorphism $\varphi_! f : t_W \circ t_V \circ t_{V'}\mathscr{F} \xrightarrow{\sim} t_{V''} \circ t_{V''}\mathscr{F}$. On the other hand, it is easily seen that $\psi^* \circ t_V \circ t_{V'}\mathscr{F}$ and $\psi^* \circ t_W \mathscr{F}$ both vanish, so lemma 10.1.32 naturally identifies $\varphi_! f$ with an isomorphism $t_V \circ t_{V'}\mathscr{F} \xrightarrow{\sim} t_W\mathscr{F}$ with the sought property.

(ii): The commutativity of the diagram can be checked on stalks, and then it follows easily from the commutativity of (10.1.34) : details left to the reader.

(iii): Again, the uniqueness of $c^{V,V'}$ is clear. For the existence, consider first case (a), and write $V = V' \cap Z$ for some closed immersion $\varphi : Z \to X$. The unit of the adjunction (φ^*, φ_*) yields a natural morphism $\eta : \mathscr{F} \to t_Z\mathscr{F}$, whence a morphism $t_{V'}\eta : t_{V'}\mathscr{F} \to t_{V'} \circ t_Z\mathscr{F}$, which, by (i) is naturally identified with a morphism $t_{V'}\mathscr{F} \to t_V\mathscr{F}$ fulfilling the stated condition. A similar argument works in case (b) : we write $V' = V \cap U$ for some open immersion $\psi : U \subset X$ and we consider the counit $\varepsilon : t_U\mathscr{F} \to \mathscr{F}$ of the adjunction $(\psi_!, \psi^*)$; then (i) naturally identifies $t_V\varepsilon$ with a morphism with the sought properties.

(iv) and (v) can be checked on the stalks : the details shall be left to the reader. □

Theorem 10.1.35. *Let X be any coherent space, $\mathscr{F}$ any abelian sheaf on X. Then :*

(i) $H^i(X, \mathscr{F}) = 0$ *for every* $i > \dim X$.

(ii) *If X is spectral, we have more precisely*

$$H^i(X, \mathscr{F}) = 0 \qquad \text{for every } i > d_{\mathscr{F}} := \sup(\dim \overline{\{x\}} \mid x \in \operatorname{Supp} \mathscr{F}).$$

Proof. (Here, for any subset $T \subset X$, we denote by $\overline{T}$ the topological closure of T in X).

Claim 10.1.36. Let $d \in \mathbb{N}$ be any integer, and suppose that (i) holds for every spectral space X of dimension $\leq d$. Then (ii) holds for every spectral space X and every $\mathbb{Z}_X$-module $\mathscr{F}$ such that $d_{\mathscr{F}} \leq d$.

Proof of the claim. Indeed, let $\mathscr{S}$ be the set consisting of all pairs (U, s), where $U \subset X$ is a quasi-compact open subset, and $s \in \mathscr{F}(U)$ is any element. For any $(U, s) \in \mathscr{S}$, let $\operatorname{Supp}(s) := \{x \in U \mid s_x \neq 0\}$, notice that $\operatorname{Supp}(s)$ is a closed subset of U, and denote by $i_s : \operatorname{Supp}(s) \to U$ and $j_U : U \to X$ the resulting closed and respectively open immersions; the section s is the same as a morphism of $\mathbb{Z}_U$-modules $\mathbb{Z}_U \to j_U^*\mathscr{F}$ which factors through the epimorphism $\mathbb{Z}_U \to i_{s*}\mathbb{Z}_{\operatorname{Supp}(s)}$. By adjunction, there follows a unique morphism of $\mathbb{Z}_X$-modules $\varphi_{U,s} : j_{U!}i_{s*}\mathbb{Z}_{\operatorname{Supp}(s)} \to \mathscr{F}$. For any finite subset $S \subset \mathscr{S}$, we let

$$\varphi_S : \bigoplus_{(U,s) \in S} j_{U!}i_{s*}\mathbb{Z}_{\operatorname{Supp}(s)} \to \mathscr{F}$$

be the unique map of $\mathbb{Z}_X$-modules whose restriction to $j_{U!}i_{s*}\mathbb{Z}_{\operatorname{Supp}(s)}$ agrees with $\varphi_{U,s}$, for every $(U, s) \in S$. It is easily seen that $\mathscr{F}$ is the colimit of the filtered system of its subsheaves $\mathscr{F}_S := \operatorname{Im} \varphi_S$, for S ranging over all finite subsets of $\mathscr{S}$. By proposition 10.1.10(i) we deduce that $H^i(X, \mathscr{F})$ is the filtered colimit of the system of abelian groups $(H^i(X, \mathscr{F}_S) \mid S \subset \mathscr{S})$; it then suffices to show the assertion for each subsheaf $\mathscr{F}_S$. Thus, we may assume from start that there exists a finite subset $S \subset \mathscr{S}$ with $\mathscr{F} = \mathscr{F}_S$. In this case, we set

$$Z := \overline{\operatorname{Supp} \mathscr{F}} = \bigcup_{(U,s) \in S} \overline{\operatorname{Supp}(s)}$$

and we let $i_Z : Z \to X$ be the closed immersion. Clearly we have a natural isomorphism

$$H^i(X, \mathscr{F}) \xrightarrow{\sim} H^i(Z, i_Z^*\mathscr{F}) \qquad \text{for every } i \in \mathbb{N}$$

and Z is still a spectral space, by corollary 8.1.42. Hence, we may replace X by Z and $\mathscr{F}$ by $i_Z^*\mathscr{F}$, and assume from start that $Z = X$ (and clearly, we still have $\mathscr{F} = \mathscr{F}_S$). In view of our assumption, it then suffices to show that

$$\dim Z = \sup\Big(\dim \overline{\{x\}} \mid x \in \bigcup_{(U,s)\in S} \mathrm{Supp}(s)\Big).$$

To this aim, in light of remark 8.1.2(v), we are reduced to checking that $\dim \overline{\mathrm{Supp}(s)} = \sup(\dim \overline{\{x\}} \mid x \in \mathrm{Supp}(s))$ for every $(U,s) \in S$. However, notice that $\mathrm{Supp}(s)$ is pro-constructible in X, hence $\overline{\mathrm{Supp}(s)}$ is the set of all specializations in X of the points of $\mathrm{Supp}(s)$ (proposition 8.1.47(i)). Now the contention follows from remark 8.1.45(iii). $\Diamond$

(i): Let S_X be the sober space associated with X, and $f_X : X \to S_X$ the unit of adjunction (proposition 8.1.8); by remark 8.1.11, the space S_X is spectral; moreover the functor

$$f_X^* : \mathbb{Z}_{S_X}\text{-}\mathbf{Mod} \to \mathbb{Z}_X\text{-}\mathbf{Mod}$$

is an equivalence of categories, and the induced map $\Gamma(S_X, \mathscr{F}) \to \Gamma(X, f_X^*\mathscr{F})$ is a bijection for every $\mathbb{Z}_{S_X}$-module $\mathscr{F}$. It follows easily that the induced map

$$H^i(S_X, \mathscr{F}) \to H^i(X, f_X^*\mathscr{F})$$

is an isomorphism as well; thus, we may replace X by S_X, and assume from start that X is spectral.

We argue by induction on $d := \dim X$. If $d = 0$, X is a boolean space (see example 8.1.48), so every sheaf on X is qc-flabby, and the assertion follows from lemma 10.1.5(iv). We may then assume that $d > 0$ and that (i) is already known for every spectral space of dimension $< d$. Suppose, by way of contradiction, that $H^{d+1}(X, \mathscr{F}) \neq 0$ for some $\mathbb{Z}_X$-module $\mathscr{F}$, and pick any element $c \neq 0$ in this cohomology group. For any closed subset $Z \subset X$, let $i_Z : Z \to X$ be the corresponding closed immersion, denote by $\mathscr{Z}$ the system of all closed subsets $Z \subset X$ such that the image of c does not vanish in $H^{d+1}(Z, i_Z^*\mathscr{F})$, and endow $\mathscr{Z}$ with the partial ordering given by inclusion. Notice that each $Z \in \mathscr{Z}$ is also a spectral space (corollary 8.1.42), so we must have $\dim Z = d$, by inductive assumption.

Claim 10.1.37. (i) The partially ordered set $\mathscr{Z}$ admits minimal elements.

(ii) Every minimal element of $\mathscr{Z}$ is an irreducible closed subset of X.

Proof of the claim. (i): Let $(Z_i \mid i \in I)$ be any totally ordered non-empty subset of $\mathscr{Z}$; by Zorn's lemma, it suffices to check that $\bigcap_{i\in I} Z_i \in \mathscr{Z}$. However, notice that the inclusion maps $Z_i \to Z_j$ are spectral for every $i, j \in I$ with $Z_i \subset Z_j$ (lemma 8.1.19(x.e) and remark 8.1.20(iv)), so the latter assertion follows directly from proposition 10.1.10(i).

(ii): Let Z be any minimal element of $\mathscr{Z}$, and suppose, by way of contradiction, that $Z = Z_1 \cup Z_2$ for two non-empty closed subsets Z_1, Z_2 strictly contained in Z; notice that Z_1 and Z_2 are also spectral spaces (corollary 8.1.42). Pick any non-empty quasi-compact open subset $U \subset Z \setminus Z_1$, and set $Z'_1 := Z \setminus U$, $Z'_2 := \overline{U}$. Since $U \subset Z_2$, we have $Z'_2 \subset Z_2$ and $Z_1 \subset Z'_1$. Moreover, both Z'_1 (resp. Z'_2) is the set of all specializations of the points of $Z \setminus U$ (resp. of U); it follows that $Z'_3 := Z'_1 \cap Z'_2$ has dimension $< d$: indeed, any totally ordered sequence $x_1 < \cdots < x_k$ of elements of $(Z'_3, \leq)$ (with ordering given by specialization) can be extended to a totally ordered sequence of $(Z'_2, \leq)$, by adding a generization x_{k+1} of x_k with $x_{k+1} \in U$.

On the other hand, consider the sequence

$$(10.1.38) \qquad 0 \to t_{Z'_3}\mathscr{F} \to t_{Z'_1}\mathscr{F} \oplus t_{Z'_2}\mathscr{F} \to t_Z\mathscr{F} \to 0.$$

whose second and third arrows are given by the natural transformations provided by proposition 10.1.33(iii); by considering the stalks over the points of Z, it is easily seen that (10.1.38) is short exact, whence an exact sequence of abelian groups

$$H^d(Z'_3, i^*_{Z'_3}\mathscr{F}) \to H^{d+1}(Z, i^*_Z\mathscr{F}) \xrightarrow{\beta} H^{d+1}(Z'_1, i^*_{Z'_1}\mathscr{F}) \oplus H^{d+1}(Z'_2, i^*_{Z'_2}\mathscr{F}). \tag{10.1.39}$$

However, the first term of (10.1.39) vanishes by inductive assumption, hence β is injective; especially the image of c cannot vanish in both $H^{d+1}(Z'_1, i^*_{Z'_1}\mathscr{F})$ and $H^{d+1}(Z'_2, i^*_{Z'_2}\mathscr{F})$, contradicting the minimality of Z. ♢

By claim 10.1.37, we may replace X by a minimal element of $\mathscr{X}$, and assume from start that X is irreducible. In this case, denote by $\eta \in X$ the maximal point, and let $j_\eta : \{\eta\} \to X$ be the inclusion map; there follows a natural map of $\mathbb{Z}_X$-modules $f : \mathscr{F} \to j_{\eta*}j^*_\eta\mathscr{F}$ such that f_η is an isomorphism. Since $j^*_\eta\mathscr{F}$ is obviously qc-flabby on $\{\eta\}$, lemma 10.1.5(iii,iv) implies that $H^i(X, j_{\eta*}j^*_\eta\mathscr{F}) = 0$ for every $i > 0$. We deduce an exact sequence of abelian groups

$$H^{d+1}(X, \mathrm{Ker}\, f) \to H^{d+1}(X, \mathscr{F}) \to H^{d+1}(X, \mathrm{Im}\, f) \tag{10.1.40}$$

as well as an isomorphism of abelian groups

$$H^d(X, \mathrm{Coker}\, f) \xrightarrow{\sim} H^{d+1}(X, \mathrm{Im}\, f)$$

and by construction we have $d_{\mathrm{Ker}, f}, d_{\mathrm{Coker}\, f} < d$. From the inductive assumption and claim 10.1.36, it follows that the first and third terms of (10.1.40) vanish, so the same holds for the middle one, and the proof is concluded. □

Remark 10.1.41. A different proof of (a slight improvement of) theorem 10.1.35 is found in [147]. See also proposition 10.4.7(ii). Alternative proofs for both theorem 10.1.35 and proposition 10.4.7(ii.b,ii.c) are also found in [107, Tag 0A3G].

10.2. **Čech cohomology.** This section is a review of the standard constructions of resolutions via Čech complexes, for general topoi. We also include two classical applications to the cohomology of quasi-coherent modules on a scheme.

10.2.1. Let T be a topos, and $\mathfrak{U} := (U_i \mid i \in I)$ a family of objects of T; pick any final object 1_T of T, and set as well (notation of example 4.7.8(iii))

$$Y := \coprod_{i\in I} U_i \qquad U := \bigcup_{i\in I} \mathrm{Im}(U_i \to 1_T) \qquad Z := \complement U.$$

According to example 4.7.8(i,iii) there exist natural morphisms of topoi

$$j_U : T/U \to T \qquad \pi : T/Y \to T \qquad i : Z \to T.$$

Moreover, the functor $\pi^* : T \to T/Y$ admits a left adjoint $\pi_! : T/Y \to T$. Furthermore, let $\mathbb{Z}_T$ (resp. $\mathbb{Z}_X$, for every object X of T) be the constant ring object of T (resp. of T/X) associated with the ring $\mathbb{Z}$. The functor $\pi^* : \mathbb{Z}_T\text{-}\mathbf{Mod} \to \mathbb{Z}_Y\text{-}\mathbf{Mod}$ admits a left adjoint

$$\pi_! : \mathbb{Z}_Y\text{-}\mathbf{Mod} \to \mathbb{Z}_T\text{-}\mathbf{Mod}.$$

Then, according to (7.10.3), the adjoint pair $(\pi_!, \pi^*)$ determines a cotriple $(\perp^{\mathfrak{U}}, \eta, \mu)$, which in turns yields a functor

$$\mathbb{Z}_T\text{-}\mathbf{Mod} \to \widehat{s}.\mathbb{Z}_T\text{-}\mathbf{Mod} \qquad \mathscr{F} \mapsto \perp^{\mathfrak{U}}_\bullet \mathscr{F}.$$

(notation of definition 7.4.1(iv)). This construction can be described explicitly as follows. We attach to I the simplicial set $I^\bullet$ which, in every degree $n \in \mathbb{N}$ consists of the set of all mappings $[n] \to I$, where $[n] := \{0, \dots, n\}$; *i.e.*, this is the cartesian power I^{n+1} of I, whose elements are all the ordered sequences $\underline{t} := (t_0, \dots, t_n)$ of elements of I. Every morphism $\varphi : [n] \to [m]$ in the simplicial category Δ induces a map $\varphi^* : I^{m+1} \to I^{n+1}$ by the rule :

$$\underline{t} \mapsto \underline{t} \circ \varphi := (t_{\varphi 0}, \dots, t_{\varphi n}) \qquad \text{for every } \underline{t} \in I^{m+1}$$

and it is clear that the rules : $[n] \mapsto I^{n+1}$ and $\varphi \mapsto \varphi^*$ yield a well defined functor $\Delta^o \to \mathbf{Set}$. For every $\underline{t} \in I^{n+1}$, we fix as well an object $U_{\underline{t}}$ of T that represents the product $U_{t_0} \times \cdots \times U_{t_n}$, and denote by $j_{\underline{t}} : T/U_{\underline{t}} \to T$ the induced morphism. Notice that every $\varphi : [n] \to [m]$ in Δ and every $\underline{t} \in I^{m+1}$ induces a well defined morphism in T :

$$\nu_\varphi^{(\underline{t})} : U_{\underline{t}} \to U_{\varphi^*(\underline{t})}$$

namely, the unique morphism whose composition with the projection on the i-th factor of $U_{\varphi^*(\underline{t})}$ agrees with the projection onto the $\varphi(i)$-th factor of $U_{\underline{t}}$, for every $i = 0, \dots, n$. Then $\nu_\varphi^{(\underline{t})}$ in turns induces a morphism

$$u_\varphi^{(\underline{t})} : j_{\underline{t}!} j_{\underline{t}}^* \mathscr{F} \to j_{\varphi^*(\underline{t})!} j_{\varphi^*(\underline{t})}^* \mathscr{F}.$$

With this notation, we have

$$\perp_n^{\mathfrak{U}} \mathscr{F} := \bigoplus_{\underline{t} \in I^{n+1}} j_{\underline{t}!} j_{\underline{t}}^* \mathscr{F} \qquad \text{for every } n \in \mathbb{N}$$

and every morphism $\varphi : [n] \to [m]$ in Δ corresponds to the unique morphism of $\mathbb{Z}_T$-modules $u_\varphi : \perp_m^{\mathfrak{U}} \mathscr{F} \to \perp_n^{\mathfrak{U}} \mathscr{F}$ that makes commute the diagram

$$\begin{array}{ccc} j_{\underline{t}!} j_{\underline{t}}^* \mathscr{F} & \xrightarrow{u_\varphi^{(\underline{t})}} & j_{\varphi^*(\underline{t})!} j_{\varphi^*(\underline{t})}^* \mathscr{F} \\ \downarrow & & \downarrow \\ \perp_m^{\mathfrak{U}} \mathscr{F} & \xrightarrow{u_\varphi} & \perp_n^{\mathfrak{U}} \mathscr{F} \end{array} \qquad \text{for every } \underline{t} \in I^{m+1}.$$

where the vertical arrows are the inclusion maps. Moreover, $\perp_{-1}^{\mathfrak{U}} \mathscr{F} = \mathscr{F}$, and the augmentation $\varepsilon : \perp_0^{\mathfrak{U}} \mathscr{F} \to \perp_{-1}^{\mathfrak{U}} (\mathfrak{U})$ is the sum of the natural morphisms $j_{i!} j_i^* \mathscr{F} \to \mathscr{F}$ given by the counit for the adjunction $(j_{i!}, j_i^*)$, for every $i \in I$.

Definition 10.2.2. (i) With the notation of (10.2.1), the *augmented Čech resolution* associated with $\mathfrak{U}$ and the $\mathbb{Z}_T$-module $\mathscr{F}$ is the unnormalized chain complex of $\mathbb{Z}_T$-modules

$$(R_\bullet(\mathfrak{U}, \mathscr{F}), d_\bullet)$$

obtained from the augmented simplicial complex $\perp_\bullet^{\mathfrak{U}} (\mathfrak{U}, \mathbb{Z}_T)$ (see definition 7.4.24(i)).

(ii) To ease notation, we shall set also

$$R_\bullet(\mathfrak{U}) := R_\bullet(\mathfrak{U}, \mathbb{Z}_T) \qquad \text{and} \qquad \overline{R}_\bullet(\mathfrak{U}) := \tau_{\leq 0} R_\bullet(\mathfrak{U}).$$

Then the *augmented Čech complex of $\mathscr{F}$ relative to the covering* $\mathfrak{U}$ of U is the cochain complex

$$C^\bullet(\mathfrak{U}_\bullet, \mathscr{F}) := \mathrm{Hom}^\bullet_{\mathbb{Z}_T}(R_\bullet(\mathfrak{U}), \mathscr{F}[0]).$$

Hence, $R_\bullet(\mathfrak{U})$ is an object of $\mathsf{C}^{\leq 1}(\mathbb{Z}_T\text{-}\mathbf{Mod})$, and $C^\bullet(\mathfrak{U}_\bullet, \mathscr{F})$ is an object of $\mathsf{C}^{\geq -1}(\mathbb{Z}\text{-}\mathbf{Mod})$.

Lemma 10.2.3. *With the notation of definition* 10.2.2*, we have :*

(i) *For every $\mathbb{Z}_T$-module $\mathscr{F}$, the complex of $\mathbb{Z}_U$-modules $j_U^* R_\bullet(\mathfrak{U}, \mathscr{F})$ is acyclic.*

(ii) *The natural projection $\mathbb{Z}_T \to i_* \mathbb{Z}_Z$ induces a quasi-isomorphism :*

$$R_\bullet(\mathfrak{U}) \xrightarrow{\sim} i_* \mathbb{Z}_Z[-1] \qquad \textit{in } \mathsf{C}^-(\mathbb{Z}_T\text{-}\mathbf{Mod}).$$

(iii) *Suppose that $\mathfrak{U}$ is a* covering *of T (*i.e. *that $U = 1_T$ and $Z = \varnothing_T$). Then the differential d_0 of $R_\bullet(\mathfrak{U})$ induces a quasi-isomorphism*

$$\overline{R}_\bullet(\mathfrak{U}) \xrightarrow{\sim} \mathbb{Z}_T[0] \qquad \textit{in } \mathsf{C}^-(\mathbb{Z}_T\text{-}\mathbf{Mod}).$$

Proof. (i): The natural morphism $Y \to U$ is an epimorphism, hence it is a covering in the canonical topology of T, and therefore it suffices to check that $\pi^* R_\bullet(\mathfrak{U}, \mathscr{F})$ is acyclic. But proposition 7.10.5(i) says that $\pi^*(\perp^{\mathfrak{U}}_\bullet \mathscr{F})$ is homotopically trivial. Then the assertion follows from corollary 7.4.67(ii).

Assertions (ii) and (iii) are immediate consequences of (i). □

10.2.4. Keep the notation of (10.2.1), and choose any total ordering on I. We let $I^{n+1}_{\mathrm{alt}} \subset I^{n+1}$ be the subset of all injective order-preserving mappings $[n] \to I$, for every $n \in \mathbb{N}$, and we set

$$R^{\mathrm{alt}}_{-1}(\mathfrak{U}, \mathscr{F}) := \mathscr{F} \qquad R^{\mathrm{alt}}_n(\mathfrak{U}, \mathscr{F}) := \bigoplus_{\underline{t} \in I^{n+1}_{\mathrm{alt}}} j_{\underline{t}!} j^*_{\underline{t}} \mathscr{F} \qquad \text{for every } n \in \mathbb{N}.$$

Obviously $\varphi^*(I^{m+1}_{\mathrm{alt}}) \subset I^{n+1}_{\mathrm{alt}}$ for every *injective* morphism $\varphi : [n] \to [m]$ of Δ, so the differential d_n of $R_\bullet(\mathfrak{U}, \mathscr{F})$ restricts to a well defined morphism of $\mathbb{Z}_T$-modules $R^{\mathrm{alt}}_n(\mathfrak{U}, \mathscr{F}) \to R^{\mathrm{alt}}_{n-1}(\mathfrak{U}, \mathscr{F})$ for every $n \in \mathbb{N}$, and we obtain therefore a subcomplex

$$(R^{\mathrm{alt}}_\bullet(\mathfrak{U}, \mathscr{F}), d_\bullet).$$

We set as well $R^{\mathrm{alt}}_\bullet(\mathfrak{U}) := R^{\mathrm{alt}}_\bullet(\mathfrak{U}, \mathbb{Z}_T)$, and to every $\mathbb{Z}_T$-module $\mathscr{F}$ we assign its *alternating augmented Čech complex relative to the covering* $\mathfrak{U}$ of U, which is the cochain complex

$$C^\bullet_{\mathrm{alt}}(\mathfrak{U}_\bullet, \mathscr{F}) := \mathrm{Hom}^\bullet_{\mathbb{Z}}(R^{\mathrm{alt}}_\bullet(\mathfrak{U}), \mathscr{F}[0]).$$

We shall also consider the alternating variant of the *Čech resolution*

$$\overline{R}^{\mathrm{alt}}_\bullet(\mathfrak{U}) := \tau_{\leq 0} R^{\mathrm{alt}}_\bullet(\mathfrak{U}).$$

10.2.5. For every $n \in \mathbb{N}$, denote by S_{n+1} the permutation group of $[n]$; then, for any $\underline{t} \in I^{n+1}_{\mathrm{alt}}$, and any $\sigma \in S_{n+1}$, the sequence $\underline{t} \circ \sigma$ is an element of I^{n+1} such that $U_{\underline{t}\circ\sigma} = U_{\underline{t}}$, and there exists a unique isomorphism $\mu_{\underline{t}} : U_{\underline{t}\circ\sigma} \xrightarrow{\sim} U_{\underline{t}}$ whose composition with the projection on the i-th factor of $U_{\underline{t}}$ agrees with the projection on the $\sigma(i)$-th factor of $U_{\underline{t}\circ\sigma}$, for every $i \in [n]$. We set

$$\mathrm{Alt}^{(\underline{t}\circ\sigma)} := \mathrm{sign}(\sigma) \cdot v_{\underline{t}} : j_{(\underline{t}\circ\sigma)!} \mathbb{Z}_{U_{\underline{t}\circ\sigma}} \to j_{\underline{t}!} \mathbb{Z}_{U_{\underline{t}}}$$

where $v_{\underline{t}}$ is the isomorphism induced by $\mu_{\underline{t}}$, and where $\mathrm{sign}(\sigma)$ denotes the signature of σ. Then, for every $n \in \mathbb{N}$ there exists a unique morphism of $\mathbb{Z}_T$-modules

$$\mathrm{Alt}^{\mathfrak{U}}_n : R_n(\mathfrak{U}) \to R^{\mathrm{alt}}_n(\mathfrak{U})$$

whose kernel contains the direct summands $j_{\underline{t}!} \mathbb{Z}_{U_{\underline{t}}}$ such that the mapping $\underline{t} : [n] \to I$ is not injective, and that makes commute the diagram

$$\begin{array}{ccc} j_{(\underline{t}\circ\sigma)!} \mathbb{Z}_{U_{\underline{t}\circ\sigma}} & \xrightarrow{\mathrm{Alt}^{(\underline{t}\circ\sigma)}} & j_{\underline{t}!} \mathbb{Z}_{U_{\underline{t}}} \\ \downarrow & & \downarrow \\ R_n(\mathfrak{U}) & \xrightarrow{\mathrm{Alt}^{\mathfrak{U}}_n} & R^{\mathrm{alt}}_n(\mathfrak{U}) \end{array} \qquad \text{for every } \underline{t} \in I^{\mathrm{alt}}_n$$

(where the vertical arrows are the inclusion maps).

Proposition 10.2.6. *In the situation of* (10.2.5)*, denote by* $\iota^{\mathfrak{U}}_\bullet : R^{\mathrm{alt}}_\bullet(\mathfrak{U}) \to R_\bullet(\mathfrak{U})$ *the inclusion map, and suppose that* U_i *is a subobject of* 1_T*, for every* $i \in I$*. Then the following holds :*

(i) *The system* $(\mathrm{Alt}^{\mathfrak{U}}_{n-1} \mid n \in \mathbb{N})$ *defines an epimorphism of chain complexes*

$$\mathrm{Alt}^{\mathfrak{U}}_\bullet : R_\bullet(\mathfrak{U}) \to R^{\mathrm{alt}}_\bullet(\mathfrak{U})$$

whose kernel is independent of the choice of ordering on I.

(ii) $\mathrm{Alt}^{\mathfrak{U}}_\bullet \circ \iota^{\mathfrak{U}}_\bullet = \mathbf{1}_{R^{\mathrm{alt}}_\bullet(\mathfrak{U})}$. *and* $\iota^{\mathfrak{U}}_\bullet \circ \mathrm{Alt}^{\mathfrak{U}}_\bullet$ *is homotopically equivalent to* $\mathbf{1}_{R_\bullet(\mathfrak{U})}$.

(iii) *Especially,* $\iota^{\mathfrak{U}}_\bullet$ *and* $\mathrm{Alt}^{\mathfrak{U}}_\bullet$ *are mutually inverse isomorphisms in* $\mathsf{D}(\mathbb{Z}_X\text{-}\mathbf{Mod})$.

(iv) *The natural projection* $\mathbb{Z}_T \to i_*\mathbb{Z}_Z$ *induces a quasi-isomorphism :*

$$R^{\mathrm{alt}}_\bullet(\mathfrak{U}) \stackrel{\sim}{\to} i_*\mathbb{Z}_Z[-1] \qquad \textit{in } \mathsf{C}^-(\mathbb{Z}_T\text{-}\mathbf{Mod}).$$

(v) *Suppose that* $\mathfrak{U}$ *is a* covering *of* T *(*i.e. *that* $U = 1_T$ *and* $Z = \varnothing_T$*). Then the differential* d_0 *of* $R^{\mathrm{alt}}_\bullet(\mathfrak{U})$ *induces a quasi-isomorphism*

$$\overline{R}^{\mathrm{alt}}_\bullet(\mathfrak{U}) \stackrel{\sim}{\to} \mathbb{Z}_T[0] \qquad \textit{in } \mathsf{C}^-(\mathbb{Z}_T\text{-}\mathbf{Mod}).$$

Proof. (i): Fix an integer $n > 0$, a sequence $\underline{s} \in I^{n+1}$, and let $c \leq n+1$ be the cardinality of $\{s_0, \dots, s_n\}$. It suffices to check that the compositions $d_n \circ \mathrm{Alt}^{\mathfrak{U}}_n$ and $\mathrm{Alt}^{\mathfrak{U}}_{n-1} \circ d_n$ agree on $j_{\underline{s}!}\mathbb{Z}_{U_{\underline{s}}}$. In case $c < n$, it is easily seen that both these compositions vanish on such direct summand. In case $c = 1$ and $n = 0$, the assertion follows by a direct inspection. In case $c = n+1 > 1$, we may find a unique permutation $\sigma \in S_{n+1}$ and a unique $\underline{t} \in I^{n+1}_{\mathrm{alt}}$ such that $\underline{s} = \underline{t} \circ \sigma$, and we have to show that we have a commutative diagram

$$\begin{array}{ccc} j_{(\underline{t}\circ\sigma)!}\mathbb{Z}_{U_{\underline{t}\circ\sigma}} & \xrightarrow{\mathrm{Alt}^{(\underline{t}\circ\sigma)}} & j_{\underline{t}!}\mathbb{Z}_{U_{\underline{t}}} \\ {\scriptstyle d_{\underline{t}\circ\sigma}}\downarrow & & \downarrow{\scriptstyle d_{\underline{t}}} \\ R_{n-1}(\mathfrak{U}) & \xrightarrow{\mathrm{Alt}^{\mathfrak{U}}_{n-1}} & R_{n-1}(\mathfrak{U}') \end{array}$$

where $d_{\underline{t}}$ and $d_{\underline{t}\circ\sigma}$ denote the restrictions of the differentials of $R^{\mathrm{alt}}_\bullet(\mathfrak{U})$ and respectively $R_\bullet(\mathfrak{U})$. To this aim, for every $i = 0, \dots, n$ denote by $\beta^{(i)}_1$ the unique permutation of $[n]$ such that

$$\beta^{(i)}_1(\sigma(i)) = \sigma(i) \qquad \text{and} \qquad \beta^{(i)}_1(\sigma(j)) < \beta^{(i)}_1(\sigma(k)) \quad \text{for every } j,k \in [n] \setminus \{i\} \text{ with } j < k.$$

Also, we define a third permutation as follows :

- If $\sigma(i) < i$, let $\beta^{(i)}_2$ be the cycle $(i, i-1, \dots, \sigma(i))$ of length $i - \beta(i) + 1$
- If $\sigma(i) > i$, let $\beta^{(i)}_2$ be the cycle $(i, i+1, \dots, \sigma(i))$ of length $\sigma(i) - i + 1$
- If $\sigma(i) = i$, let $\beta^{(i)}_2$ be the identity map of $[n]$.

It is easily seen that in either cases

$$\sigma^{-1} = \beta^{(i)}_2 \circ \beta^{(i)}_1 \qquad \text{and} \qquad \mathrm{sign}(\beta^{(i)}_2) = (-1)^{i-\sigma(i)} \qquad \text{for every } i = 0, \dots, n.$$

Now, for every $i = 0, \dots, n$ let $\varepsilon_i : [n-1] \to [n]$ be the i-th face map (see (7.4.6)); we compute:

$$\begin{aligned} \mathrm{Alt}^{\mathfrak{U}}_{n-1} \circ d_{\underline{t}\circ\sigma} &= \mathrm{Alt}^{\mathfrak{U}}_{n-1} \circ \Big(\sum_{i=0}^n (-1)^i \cdot u^{(\underline{t}\circ\sigma)}_{\varepsilon_i}\Big) \\ &= \sum_{i=0}^n (-1)^i \cdot \mathrm{sign}(\beta^{(i)}_1) \cdot u^{(\underline{t}\circ\sigma)}_{\varepsilon_i} \\ &= \sum_{i=0}^n (-1)^{\sigma(i)} \cdot \mathrm{sign}(\sigma) \cdot u^{(\underline{t}\circ\sigma)}_{\varepsilon_i} \\ &= \sum_{i=0}^n (-1)^i \cdot \mathrm{sign}(\sigma) \cdot u^{(\underline{t})}_{\varepsilon_i} \\ &= \mathrm{sign}(\sigma) \cdot d_{\underline{t}} \end{aligned}$$

as required. Lastly, if $c = n$, the morphism $d_n \circ \mathrm{Alt}^{(\underline{t}\circ\sigma)}$ vanishes, and it remains to check that $\mathrm{Alt}^{\mathfrak{U}}_{n-1} \circ d_{\underline{s}} = 0$ as well. However, in this situation there exist exactly two distinct indices $i, j \in [n]$ such that $s_i = s_j$, and it follows already that $\mathrm{Alt}^{\mathfrak{U}}_{n-1} \circ u^{(\underline{s})}_{\varepsilon_k} = 0$ for every $k \neq i, j$.

Moreover, clearly there exists a unique $\underline{t} \in I^n_{\mathrm{alt}}$ such that $\{t_0, \dots, t_{n-1}\} = \{s_0, \dots, s_n\}$, and notice that the diagram :

$$\begin{array}{ccc} U_{\underline{s}} & \xrightarrow{\nu^{(\underline{s})}_{\varepsilon_i}} & U_{\varepsilon^*_i(\underline{s})} \\ {\scriptstyle \nu^{(\underline{s})}_{\varepsilon_j}} \downarrow & & \downarrow {\scriptstyle \mu_{\varepsilon^*_i(\underline{s})}} \\ U_{\varepsilon^*_j(\underline{s})} & \xrightarrow{\mu_{\varepsilon^*_j(\underline{s})}} & U_{\underline{t}} \end{array}$$

commutes (this is the only place where we use our assumption on the objects U_i). There follows a commutative diagram

$$\begin{array}{ccc} j_{\underline{s}!}\mathbb{Z}_{U_{\underline{s}}} & \xrightarrow{u^{(\underline{s})}_{\varepsilon_i}} & j_{\varepsilon^*_i(\underline{s})!}\mathbb{Z}_{U_{\varepsilon^*_i(\underline{s})}} \\ {\scriptstyle u^{(\underline{s})}_{\varepsilon_j}} \downarrow & & \downarrow {\scriptstyle v_{\varepsilon^*_i(\underline{s})}} \\ j_{\varepsilon^*_j(\underline{s})!}\mathbb{Z}_{U_{\varepsilon^*_j(\underline{s})}} & \xrightarrow{v_{\varepsilon^*_j(\underline{s})}} & U_{\underline{t}} \end{array}$$

Now, let β_i (resp. β_j) be the unique permutation of $[n]$ that fixes i (resp. j) and such that $\underline{s} \circ \beta_i$ (resp. $\underline{s} \circ \beta_j$) restricts to an order-preserving map on $[n] \setminus \{i\}$ (resp. on $[n] \setminus \{j\}$). Arguing as in the foregoing, we see that $\beta_i = \gamma \circ \beta_j$ for a cyclic permutation γ of length $|j - i|$, so that

$$\rho := (-1)^i \cdot \mathrm{sign}(\beta_i) = (-1)^{j+1} \cdot \mathrm{sign}(\beta_j).$$

Denote by $\mathbf{1}_{\underline{t}}$ the identity automorphism of $j_{\underline{t}!}\mathbb{Z}_{U_{\underline{t}}}$. Summing up, we get

$$\begin{aligned} \mathrm{Alt}^{\mathfrak{U}}_{n-1} \circ d_{\underline{s}} &= (-1)^i \cdot \mathrm{Alt}^{\mathfrak{U}}_{n-1} \circ u^{(\underline{s})}_{\varepsilon_i} + (-1)^j \cdot \mathrm{Alt}^{\mathfrak{U}}_{n-1} \circ u^{(\underline{s})}_{\varepsilon_j} \\ &= (-1)^i \cdot \mathrm{sign}(\beta_i) \cdot v_{\varepsilon^*_i(\underline{s})} \circ u^{(\underline{s})}_{\varepsilon_i} + (-1)^j \cdot \mathrm{sign}(\beta_j) \cdot v_{\varepsilon^*_j(\underline{s})} \circ u^{(\underline{s})}_{\varepsilon_j} \\ &= \rho \cdot (v_{\varepsilon^*_i(\underline{s})} \circ u^{(\underline{s})}_{\varepsilon_i} - v_{\varepsilon^*_j(\underline{s})} \circ u^{(\underline{s})}_{\varepsilon_j}) \\ &= 0 \end{aligned}$$

and the proof of (i) is concluded.

(ii): Let V be a universe such that T is V-small, and set $T' := T^\wedge_{\mathsf{V}}$; also, let J be the canonical topology on T. Then T' is a V-topos, and the Yoneda imbedding $h : T \to T'$ factors through an equivalence $T \stackrel{\sim}{\to} (T, J)^\sim_{\mathsf{U}}$ and the inclusion functor $(T, J)^\sim_{\mathsf{U}} \to T'$. Set $U'_i := h_{U_i} \in \mathrm{Ob}(T')$ for every $i \in I$, $U' := \bigcup_{i \in I} \mathrm{Im}(U'_i \to 1_{T'})$, and $\mathfrak{U}' := (U'_i \mid i \in I)$. Hence U' is a presheaf on T whose associated sheaf is isomorphic to h_U. Let $j_{U'} : T'/U' \to T'$ be the induced morphism of topoi; according to lemma 10.2.3, the chain complex $j^*_{U'}R_\bullet(\mathfrak{U}')$ is acyclic, hence the same holds for $j_{U'!}j^*_{U'}R_\bullet(\mathfrak{U}')$, and there follows a commutative diagram of complexes of $\mathbb{Z}_{T'}$-modules :

$$\begin{array}{ccc} \overline{R}_\bullet(\mathfrak{U}') & \longrightarrow & j_{U'!}\mathbb{Z}_{U'}[0] \\ {\scriptstyle \tau_{\leq 0}(\iota^{\mathfrak{U}'}_\bullet \circ \mathrm{Alt}^{\mathfrak{U}'}_\bullet)} \downarrow & & \| \\ \overline{R}_\bullet(\mathfrak{U}') & \longrightarrow & j_{U'!}\mathbb{Z}_{U'}[0] \end{array}$$

whose horizontal arrows are induced by the differential d_0 of $R_\bullet(\mathfrak{U}')$, and therefore are isomorphisms in the category $\mathsf{D}(\mathbb{Z}_{T'}\text{-}\mathbf{Mod})$, so that

$$\tau_{\leq 0}(\iota^{\mathfrak{U}'}_\bullet \circ \mathrm{Alt}^{\mathfrak{U}'}_\bullet) = \mathbf{1}_{\overline{R}_\bullet(\mathfrak{U}')} \qquad \text{in } \mathsf{D}(\mathbb{Z}_{T'}\text{-}\mathbf{Mod}).$$

However, from the explicit description of (10.2.1) it is easily seen that $\overline{R}_\bullet(\mathfrak{U}')$ is a complex of projective $\mathbb{Z}_{T'}$-modules, hence we have as well $\tau_{\leq 0}(\iota^{\mathfrak{U}'}_\bullet \circ \mathrm{Alt}^{\mathfrak{U}'}_\bullet) = \mathbf{1}_{\overline{R}_\bullet(\mathfrak{U}')}$ in $\mathsf{Hot}(\mathbb{Z}_{T'}\text{-}\mathbf{Mod})$, by theorem 7.3.25(iii), and then it follows as well that $\iota^{\mathfrak{U}'}_\bullet \circ \mathrm{Alt}^{\mathfrak{U}'}_\bullet = \mathbf{1}_{R_\bullet(\mathfrak{U}')}$ in $\mathsf{Hot}(\mathbb{Z}_{T'}\text{-}\mathbf{Mod})$. Next, notice that $R_\bullet(\mathfrak{U}')$ is a complex of abelian presheaves on T, whose associated complex of sheaves agrees with $R_\bullet(\mathfrak{U})$, and likewise the morphism of abelian sheaves associated with $\iota_\bullet^{\mathfrak{U}'} \circ \mathrm{Alt}_\bullet^{\mathfrak{U}'}$ agrees with $\iota_\bullet^{\mathfrak{U}} \circ \mathrm{Alt}_\bullet^{\mathfrak{U}}$. The assertion follows.

(iii) follows directly from (ii). Likewise, (iv) and (v) are immediate from (iii) and lemma 10.2.3(ii,iii). □

Remark 10.2.7. (i) In light of proposition 10.2.6(iv) we shall call $R_\bullet^{\mathrm{alt}}(\mathfrak{U})$ the *alternating augmented Čech resolution* associated with $\mathfrak{U}$. This name can be justified as follows. Notice that for every $\mathbb{Z}_T$-module $\mathscr{F}$ we have natural identifications

$$C^n(\mathfrak{U},\mathscr{F}) \stackrel{\sim}{\to} \prod_{\underline{t}\in I^{n+1}} \mathscr{F}(U_{\underline{t}}) \qquad C^n_{\mathrm{alt}}(\mathfrak{U},\mathscr{F}) \stackrel{\sim}{\to} \prod_{\underline{t}\in I^{n+1}_{\mathrm{alt}}} \mathscr{F}(U_{\underline{t}}) \tag{10.2.8}$$

for every $n \in \mathbb{N}$ (notation of definition 10.2.2(ii)). Under these identifications, $C^\bullet(\mathfrak{U},\mathscr{F})$ becomes the cochain complex

$$0 \to \Gamma(\mathscr{F}) \to \prod_{t\in I} \mathscr{F}(U_t) \to \cdots \to \prod_{\underline{t}\in I^{n+1}} \mathscr{F}(U_{\underline{t}}) \to \cdots \tag{10.2.9}$$

whose differential d^n is given by the following rule. First, d^{-1} assigns to each global section $s \in \Gamma(\mathscr{F})$ the system of its restrictions $(s_{|U_t} \mid t \in I)$. In case $n \geq 0$, we have

$$d^n(s_\bullet)_{\underline{t}} := \sum_{k=0}^{n+1} (-1)^{k+n+1} \cdot (s_{\underline{t}\circ\varepsilon_k})_{|U_{\underline{t}}} \qquad \text{for every } \underline{t} \in I^{n+2} \text{ and every } s_\bullet \in C^n(\mathfrak{U},\mathscr{F}).$$

(ii) Likewise, under the natural identifications of (i), the alternating augmented Čech complex $C^\bullet_{\mathrm{alt}}(\mathfrak{U},\mathscr{F})$ becomes the cochain complex

$$0 \to \Gamma(\mathscr{F}) \to \prod_{t\in I} \mathscr{F}(U_t) \to \cdots \to \prod_{\underline{t}\in I^{n+1}_{\mathrm{alt}}} \mathscr{F}(U_{\underline{t}}) \to \cdots$$

whose differentials are given by the same expressions as in (i).

(iii) Under the identification of (i), the morphisms $R_n(\mathfrak{U}) \to \mathscr{F}$ that factor through $\mathrm{Alt}_n^{\mathfrak{U}}$ correspond to the system of sections $f_\bullet := (f_{\underline{t}} \mid \underline{t} \in I^{n+1})$ with the following properties :

- $f_{\underline{t}} = 0$ whenever the map $\underline{t} : [n] \to I$ is not injective.
- $f_{\underline{t}\circ\sigma} = \mathrm{sign}(\sigma) \cdot f_{\underline{t}}$ for every injective map $\underline{t} : [n] \to I$ and every $\sigma \in S_{n+1}$.

Thus, under the identification (10.2.8), the submodule $\mathrm{Hom}_{\mathbb{Z}_X}(R_n^{\mathrm{alt}}(\mathfrak{U}),\mathscr{F})$ corresponds to the subgroup of all *alternating systems* $f_\bullet$ *of sections of* $\mathscr{F}$.

10.2.10. In the situation of (10.2.1), consider another family $\mathfrak{U}' := (U'_{i'} \mid i' \in I')$ of objects of T, indexed by a totally ordered set I', and set $U' := \bigcup_{i'\in I'} \mathrm{Im}(U'_{i'} \to 1_T)$ and $Z' := \complement U'$. Also, for every $n \in \mathbb{N}$, and every $\underline{s} \in I'^{n+1}$ set as usual $U'_{\underline{s}} := U'_{s_0} \times \cdots \times U'_{s_n}$, and denote by $j'_{\underline{s}} : T/U'_{\underline{s}} \to T$ the induced functor. Suppose that $\mathfrak{U}$ is a *refinement* of $\mathfrak{U}'$, *i.e.* there exists a mapping (that does not necessarily respect the orderings)

$$\tau : I \to I' \qquad \text{such that} \qquad \mathrm{Hom}_T(U_i, U'_{\tau(i)}) \neq \varnothing \qquad \text{for every } i \in I$$

and pick a morphism $\vartheta_i : U_i \to U'_{\tau(i)}$ for every $i \in I$. Then, for every $n \in \mathbb{N}$ and every $\underline{t} := (t_0,\dots,t_n) \in I^{n+1}$, we set $\tau(\underline{t}) := (\tau(t_0),\dots,\tau(t_n)) \in I'^{n+1}$, and notice that the morphism $\vartheta_{\underline{t}} := \vartheta_{t_0} \times \cdots \times \vartheta_{t_n} : U_{\underline{t}} \to U'_{\tau(\underline{t})}$ induces a morphism of $\mathbb{Z}_T$-modules

$$\varphi_{\underline{t}} : j_{\underline{t}!}\mathbb{Z}_{U_{\underline{t}}} \to j'_{\tau(\underline{t})!}\mathbb{Z}_{U'_{\tau(\underline{t})}}.$$

For every $n \in \mathbb{N}$, denote by $\varphi_n : R_n(\mathfrak{U}) \to R_n(\mathfrak{U}')$ the unique morphism of $\mathbb{Z}_T$-modules that makes commute the diagram

$$\begin{array}{ccc} j_{\underline{t}!}\mathbb{Z}_{U_{\underline{t}}} & \xrightarrow{\varphi(\underline{t})} & j'_{\tau(\underline{t})!}\mathbb{Z}_{U'_{\tau(\underline{t})}} \\ \downarrow & & \downarrow \\ R_n(\mathfrak{U}) & \xrightarrow{\varphi_n} & R_n(\mathfrak{U}') \end{array} \qquad \text{for every } \underline{t} \in I^{n+1}.$$

whose vertical arrows are the inclusion maps. Let as well $\varphi_{-1} := \mathbf{1}_{\mathbb{Z}_T}$. With this notation, it is clear that the system $(\varphi_{n-1} \mid n \in \mathbb{N})$ is a morphism of augmented simplicial complexes $R_\bullet(\mathfrak{U}) \to R_\bullet(\mathfrak{U}')$, and we denote

$$\varphi_\bullet : R_\bullet(\mathfrak{U}) \to R_\bullet(\mathfrak{U}')$$

the associated morphism of chain complexes.

10.2.11. Suppose next that $U'_{i'}$ is a subobject of 1_T, for every $i' \in I'$; then, on account of proposition 10.2.6(i) we may define

$$\varphi^{\mathrm{alt}}_\bullet := \mathrm{Alt}^{\mathfrak{U}'}_\bullet \circ \varphi_\bullet \circ \iota^{\mathfrak{U}}_\bullet : R^{\mathrm{alt}}_\bullet(\mathfrak{U}) \to R^{\mathrm{alt}}_\bullet(\mathfrak{U}').$$

The terms φ^{alt}_n can be described explicitly : first, it is clear that $\varphi^{\mathrm{alt}}_{-1} := \mathbf{1}_{\mathbb{Z}_T}$. Next, if $n \in \mathbb{N}$ and $\underline{t} \in I^{n+1}_{\mathrm{alt}}$ is any sequence, let

$$\varphi^{\mathrm{alt}}_{\underline{t}} : j_{\underline{t}!}\mathbb{Z}_{U_{\underline{t}}} \to R^{\mathrm{alt}}_n(\mathfrak{U}')$$

be the map given by the following rule. If $\tau(\underline{t})$ is also an injective sequence $[n] \to I'$, let $\sigma \in S_{n+1}$ be the unique permutation such that $\tau(\underline{t}) \circ \sigma \in I'^{n+1}_{\mathrm{alt}}$, and set $\varphi^{\mathrm{alt}}_{\underline{t}} := \mathrm{sign}(\sigma) \cdot \varphi_{(\underline{t})} : j_{\underline{t}!}\mathbb{Z}_{U_{\underline{t}}} \to \mathbb{Z}_{U'_{\tau(\underline{t})\circ\sigma}} \subset R^{\mathrm{alt}}_n(\mathfrak{U}')$. In case $\tau(\underline{t})$ is not injective, we let $\varphi^{\mathrm{alt}}_{\underline{t}}$ be the zero morphism. Then φ^{alt}_n is the sum of the $\varphi^{\mathrm{alt}}_{\underline{t}}$, for $\underline{t}$ ranging over all elements of I^{n+1}_{alt}. In case also U_i is a subobject of 1_T for every $i \in I$, it is easily seen that we get a commutative diagram

$$\begin{array}{ccc} R_\bullet(\mathfrak{U}) & \xrightarrow{\mathrm{Alt}^{\mathfrak{U}}_\bullet} & R^{\mathrm{alt}}_\bullet(\mathfrak{U}) \\ {\scriptstyle\varphi_\bullet}\downarrow & & \downarrow{\scriptstyle\varphi^{\mathrm{alt}}_\bullet} \\ R_\bullet(\mathfrak{U}') & \xrightarrow{\mathrm{Alt}^{\mathfrak{U}'}_\bullet} & R^{\mathrm{alt}}_\bullet(\mathfrak{U}'). \end{array}$$

Lemma 10.2.12. *In the situation of* (10.2.10)*, suppose that $\tau' : I \to I'$ is another mapping with* $\mathrm{Hom}_T(U_i, U'_{\tau'(i)}) \neq \varnothing$ *for every $i \in I$. Pick a system of morphisms $(\vartheta'_i : U_i \to U'_{\tau'(i)} \mid i \in I)$, and denote by $\varphi'_\bullet : R_\bullet(\mathfrak{U}) \to R_\bullet(\mathfrak{U}')$ the associated morphisms of chain complexes. We have :*

(i) *There exists a homotopy from $\varphi_\bullet$ to $\varphi'_\bullet$.*
(ii) *If $U'_{i'}$ is a subobject of 1_T for every $i' \in I'$, there is also a homotopy from $\varphi^{\mathrm{alt}}_\bullet$ to $\varphi'^{\mathrm{alt}}_\bullet$.*

Proof. Clearly it suffices to show (i). To this aim, for every $i \in I$ let $(\vartheta_i, \vartheta'_i) : U_i \to U'_{\tau(i)} \times U'_{\tau'(i)}$ be the unique morphism whose composition with the projection on the first (resp. second) factor equals ϑ_i (resp. ϑ'_i), and for every $n \in \mathbb{N}$, every $k = 0, \dots, n$, and every $\underline{t} \in I^{n+1}$, set

$$\tau(k, \underline{t}) := (\tau(t_0), \dots, \tau(t_k), \tau'(t_k), \dots, \tau'(t_n)) \in I^{n+2}$$
$$\vartheta^{(k)}_{\underline{t}} := \vartheta_{t_0} \times \cdots \times \vartheta_{t_{k-1}} \times (\vartheta_{t_k}, \vartheta'_{t_k}) \times \vartheta'_{t_{k+1}} \times \cdots \times \vartheta'_{t_n} : U_{\underline{t}} \to U'_{\tau(k,\underline{t})}.$$

Then $\vartheta^{(k)}_{\underline{t}}$ induces a natural morphism $h^{(k)}_{\underline{t}} : j_{\underline{t}!}\mathbb{Z}_{U_{\underline{t}}} \to j_{\tau(k,\underline{t})!}\mathbb{Z}_{U'_{\tau(k,\underline{t})}}$ and we set

$$h_{\underline{t}} := \sum_{k=0}^{n} (-1)^k \cdot h^{(k)}_{\underline{t}} : j_{\underline{t}!}\mathbb{Z}_{U_{\underline{t}}} \to R_{n+1}(\mathfrak{U}').$$

Clearly there exists a unique morphism of $\mathbb{Z}_T$-modules $h_n : R_n(\mathfrak{U}) \to R_{n+1}(\mathfrak{U}')$ whose restriction to $j_{\underline{t}!}\mathbb{Z}_{U_{\underline{t}}}$ agrees with $h_{\underline{t}}$ for every $\underline{t} \in I^{n+1}$. Let also $h_{-1} : R_{-1}(\mathfrak{U}) \to R_0(\mathfrak{U}')$ be the zero morphism; a direct calculation shows that the system $(h_{n-1} \mid n \in \mathbb{N})$ yields the sought homotopy : the details shall be left to the reader. □

Remark 10.2.13. (i) In the situation of (10.2.11), consider the case where $U_{i_0} = 1_T$ for some $i_0 \in I$. Then $\mathfrak{U}$ can be refined by itself in two different ways : we may take for $\tau : I \to I$ the identity map (with $\vartheta_i := \mathbf{1}_{U_i}$ for every $i \in I$), and we have also the map $\tau' : I \to I$ such that $\tau'(i) = i_0$ for every $i \in I$ (and then there is a unique choice of ϑ_i). Obviously, the morphism $\varphi_\bullet : R^{\mathrm{alt}}_\bullet(\mathfrak{U}) \to R^{\mathrm{alt}}_\bullet(\mathfrak{U})$ associated with τ is the identity map. For the morphism $\varphi'_\bullet : R^{\mathrm{alt}}_\bullet(\mathfrak{U}) \to R^{\mathrm{alt}}_\bullet(\mathfrak{U})$ associated with τ' it is easily seen that

$$\varphi'_{-1} = \mathbf{1}_{\mathbb{Z}_T} \qquad \text{and} \qquad \varphi'_n = 0 \qquad \text{for every } n > 0$$

and φ'_0 is the sum of the natural morphisms $j_{t!}\mathbb{Z}_{U_t} \to j_{i_0!}\mathbb{Z}_{U_{i_0}} = \mathbb{Z}_T$. The homotopy $h_\bullet$ from $\varphi_\bullet$ to $\varphi'_\bullet$ furnished by lemma 10.2.12 can be described as follows. First, we may suppose that i_0 is the maximal element of I. A simple inspection shows that $h_{-1} : R^{\mathrm{alt}}_{-1}(\mathfrak{U}) \to R^{\mathrm{alt}}_0(\mathfrak{U})$ is the zero map. Next, for every $n \in \mathbb{N}$ and every $\underline{t} \in I^{n+1}_{\mathrm{alt}}$, notice that $U_{\underline{t}} = U_{(\underline{t},i_0)}$, where $(\underline{t}, i_0)$ denotes the sequence $(t_0, \dots, t_n, i_0)$; it follows that we may write

$$h_n = \sum_{\underline{t} \in I^{n+1}_{\mathrm{alt}}} \rho(\underline{t}) \cdot h_{\underline{t}}$$

where $h_{\underline{t}} : j_{\underline{t}}\mathbb{Z}_{U_{\underline{t}}} \to j_{(\underline{t},i_0)!}\mathbb{Z}_{U_{(\underline{t},i_0)}}$ is the identity map, and $\rho(\underline{t})$ equals 0 if $t_n = i_0$, and equals 1 otherwise.

(ii) In the situation of (i), we deduce a homotopy from $\mathbf{1}_{R^{\mathrm{alt}}_\bullet(\mathfrak{U})}$ to the zero map. Indeed, let $h'_{-1} : \mathbb{Z}_T \to R_0(\mathfrak{U})$ be the composition of the identity map $\mathbb{Z}_T \to j_{i_0!}\mathbb{Z}_{U_{i_0}}$ and the inclusion map $j_{i_0!}\mathbb{Z}_{U_{i_0}} \to R_0(\mathfrak{U})$. Let also $h'_n := h_n$ for every $n \in \mathbb{N}$. A simple inspection shows that

$$\varphi'_0 = h'_{-1} \circ d_0$$

(where $d_0 : R_0(\mathfrak{U}) \to \mathbb{Z}_T$ is the differential of $R_\bullet(\mathfrak{U})$). It follows easily that $(h'_{n-1} \mid n \in \mathbb{N})$ is the sought homotopy.

(iii) Let $\mathscr{F}$ be any $\mathbb{Z}_T$-module. The homotopy $h'_\bullet$ of (i) induces a homotopy from the identity automorphism of the augmented Čech complex $C^\bullet_{\mathrm{alt}}(\mathfrak{U}, \mathscr{F})$ to its zero endomorphism. Under the natural identifications of remark 10.2.7(ii), this homotopy $h^\bullet$ can be described as follows.

- $h^0 : C^0(\mathfrak{U}, \mathscr{F}) \to \Gamma(\mathscr{F})$ is the projection onto the factor $\mathscr{F}(U_{i_0}) = \Gamma(\mathscr{F})$.
- For every $n \geq 0$, the map $h^{n+1} : C^{n+1}(\mathfrak{U}, \mathscr{F}) \to C^n(\mathfrak{U}, \mathscr{F})$ is the projection that sends to zero every factor $\mathscr{F}(U_{t_0,\dots,t_n})$ with $t_n < i_0$, and whose restriction to every factor $\mathscr{F}(U_{t_0,\dots,t_{n-1},i_0})$ is the identity map $\mathscr{F}(U_{t_0,\dots,t_{n-1},i_0}) \xrightarrow{\sim} \mathscr{F}(U_{t_0,\dots,t_{n-1}})$.

Remark 10.2.14. (i) In the situation of (10.2.4), it is clear that the rule $\mathscr{F} \mapsto R^{\mathrm{alt}}_\bullet(\mathfrak{U}, \mathscr{F})$ defines a functor from $\mathbb{Z}_T$-modules to chain complexes of $\mathbb{Z}_T$-modules. The latter then extends naturally to a functor from the category of chain complexes of $\mathbb{Z}_T$-modules to the category of double complexes of $\mathbb{Z}_T$-modules :

$$\mathsf{C}(\mathbb{Z}_T\text{-}\mathbf{Mod}) \to \mathsf{C}_2(\mathbb{Z}_T\text{-}\mathbf{Mod}) \qquad \mathscr{F}_\bullet \mapsto R^{\mathrm{alt}}_\bullet(\mathfrak{U}, \mathscr{F}_\bullet).$$

Likewise, $\perp^{\mathfrak{U}}_\bullet$ extends to a functor from augmented simplicial $\mathbb{Z}_T$-modules to augmented bisimplicial $\mathbb{Z}_T$-modules :

$$\widehat{s}.\mathbb{Z}_T\text{-}\mathbf{Mod} \to \widehat{s}.(\widehat{s}.\mathbb{Z}_T\text{-}\mathbf{Mod}) \qquad \mathscr{F}_\bullet \mapsto \perp^{\mathfrak{U}}_\bullet \mathscr{F}_\bullet.$$

(ii) Let now $\mathfrak{U}' := (U'_{i'} \mid i' \in I')$ be another family of objects of T. Then we may consider the augmented bisimplicial $\mathbb{Z}_T$-module

$$\perp^{\mathfrak{U},\mathfrak{U}'}_{\bullet\bullet} := \perp^{\mathfrak{U}'}_\bullet (\perp^{\mathfrak{U}}_\bullet \mathbb{Z}_T)$$

and we notice that

$$(\perp_{\bullet\bullet}^{\mathfrak{U},\mathfrak{U}'})^{\Delta} = \perp_{\bullet}^{\mathfrak{U}\times\mathfrak{U}'} \qquad \text{where } \mathfrak{U}\times\mathfrak{U}' := (U_i \times U'_{i'} \mid (i,i') \in I \times I').$$

In case both U_i and $U'_{i'}$ are subobjects of 1_T for every $i \in I$ and $i' \in I'$, we may also define the chain double complex

$$R_{\bullet\bullet}^{\mathrm{alt}}(\mathfrak{U},\mathfrak{U}') := R_{\bullet}^{\mathrm{alt}}(\mathfrak{U}',(R_{\bullet}^{\mathrm{alt}}(\mathfrak{U}))$$

and we notice that (notation of example 7.1.16(i))

$$R_{\bullet\bullet}^{\mathrm{alt}}(\mathfrak{U},\mathfrak{U}') = R_{\bullet}^{\mathrm{alt}}(\mathfrak{U}) \boxtimes_A R_{\bullet}^{\mathrm{alt}}(\mathfrak{U}').$$

10.2.15. In the situation of (10.2.1), let $\mathscr{A}$ be any T-ring; we consider the functor

$$\mathscr{A}\text{-}\mathbf{Mod} \to \mathsf{C}(\Gamma(\mathscr{A})\text{-}\mathbf{Mod}) \qquad \mathscr{F} \mapsto \overline{C}^{\bullet}(\mathfrak{U},\mathscr{F}) := \mathrm{Hom}_{\mathbb{Z}}(\overline{R}_{\bullet}(\mathfrak{U}),\mathscr{F}[0])$$

that assigns to every $\mathscr{A}$-module $\mathscr{F}$ its *Čech complex*, and we define as well the *Čech cohomology functor of $\mathscr{F}$* relative to the covering $\mathfrak{U}$, by setting :

$$H^i(\mathfrak{U},\mathscr{F}) := H^i\overline{C}^{\bullet}(\mathfrak{U},\mathscr{F}) \qquad \text{for every } i \in \mathbb{N}.$$

Suppose now that $\mathfrak{U}$ is a covering of T; since $\mathscr{F}$ represents a sheaf for the canonical topology on T, we easily see that the system of restriction maps $(\Gamma(\mathscr{F}) \to \mathscr{F}(U_i) \mid i \in I)$ induces a natural isomorphism

$$\Gamma(\mathscr{F}) \xrightarrow{\sim} H^0(\mathfrak{U},\mathscr{F}). \tag{10.2.16}$$

On the other hand, let $K^\bullet$ be any bounded below complex of $\mathscr{A}$-modules; lemma 10.2.3(iii) yields natural isomorphisms in $\mathsf{D}^+(\mathscr{A}\text{-}\mathbf{Mod})$ and in $\mathsf{D}^+(\Gamma(\mathscr{A})\text{-}\mathbf{Mod})$

$$R\mathscr{H}\!om^{\bullet}_{\mathbb{Z}}(\overline{R}_{\bullet}(\mathfrak{U}),K^\bullet) \xrightarrow{\sim} R\underline{\Gamma}K^\bullet \qquad R\mathrm{Hom}^{\bullet}_{\mathbb{Z}}(\overline{R}_{\bullet}(\mathfrak{U}),K^\bullet) \xrightarrow{\sim} R\Gamma K^\bullet.$$

Hence, after fixing an injective resolution $K^\bullet \xrightarrow{\sim} \mathscr{I}^\bullet$ we get a natural isomorphism

$$R\Gamma K^\bullet \xrightarrow{\sim} \mathrm{Tot}\,\overline{C}^{\bullet}(\mathfrak{U},\mathscr{I}^\bullet) \qquad \text{in } \mathsf{D}^+(\Gamma(\mathscr{A})\text{-}\mathbf{Mod}).$$

Thus, for every $p \in \mathbb{Z}$ let also define a presheaf $\mathscr{H}^p(K^\bullet)$ on T by the rule

$$\mathscr{H}^p(K^\bullet)(U) := R^p\Gamma(U,K^\bullet) \qquad \text{for every object } U \text{ of } T.$$

We may regard $\mathscr{H}^p(K^\bullet)$ as an object of the V-topos $T^{\wedge}_{\mathsf{V}}$, for some universe V such that T is V-small, and the family $\mathfrak{U}$ can be regarded as a system of objects of $T^{\wedge}_{\mathsf{V}}$, via the Yoneda imbedding $T \to T^{\wedge}_{\mathsf{V}}$. Then, the Čech cohomology functors of $\mathscr{H}^p(K^\bullet)$ relative to the covering $\mathfrak{U}$ are also well defined, and we deduce a 2-spectral sequence

$$E(\mathfrak{U})_2^{p,q} := H^p(\mathfrak{U},\mathscr{H}^q(K^\bullet)) \Rightarrow R^{p+q}\Gamma K^\bullet. \tag{10.2.17}$$

Notice also that

$$\mathscr{H}^0(\mathscr{F}[0]) = \mathscr{F} \qquad \text{for every } \mathbb{Z}_T\text{-module } \mathscr{F}$$

whence, natural maps

$$\Psi^p(\mathfrak{U},\mathscr{F}) : H^p(\mathfrak{U},\mathscr{F}) \to R^p\Gamma\mathscr{F} \qquad \text{for every } p \in \mathbb{N}$$

such that $\Psi^0(\mathfrak{U},\mathscr{F})$ is always an isomorphism (in fact, it is the inverse of the isomorphism (10.2.16)), and $\Psi^1(\mathfrak{U},\mathscr{F})$ is always an injective map.

10.2.18. In case U_i is a subobject of 1_T for every $i \in I$, we may repeat the considerations of (10.2.15) with the *alternating Čech complex*

$$\overline{C}^\bullet_{\mathrm{alt}}(\mathfrak{U}, \mathscr{F}) := \mathrm{Hom}_{\mathbb{Z}}(\overline{R}^{\mathrm{alt}}_\bullet(\mathfrak{U}), \mathscr{F}[0]).$$

Namely, let us set

$$H^i_{\mathrm{alt}}(\mathfrak{U}, \mathscr{F}) := H^i\overline{C}^\bullet_{\mathrm{alt}}(\mathfrak{U}, \mathscr{F}) \qquad \text{for every } i \in \mathbb{N}.$$

Then, proposition 10.2.6(ii) (together with example 7.1.25(ii)) yields a natural isomorphism

$$H^i_{\mathrm{alt}}(\mathfrak{U}, \mathscr{F}) \xrightarrow{\sim} H^i(\mathfrak{U}, \mathscr{F}) \qquad \text{for every } i \in \mathbb{N}$$

and if $\mathfrak{U}$ is a covering of T, obviously we get therefore a natural isomorphism

$$(10.2.19) \qquad R\Gamma K^\bullet \xrightarrow{\sim} \mathrm{Tot}\, \overline{C}^\bullet_{\mathrm{alt}}(\mathfrak{U}, \mathscr{I}^\bullet) \qquad \text{in } \mathsf{D}^+(\Gamma(\mathscr{A})\text{-}\mathbf{Mod}).$$

for any injective resolution $K^\bullet \xrightarrow{\sim} \mathscr{I}^\bullet$, and a 2-spectral sequence

$$(10.2.20) \qquad F(\mathfrak{U})^{p,q}_2 := H^p_{\mathrm{alt}}(\mathfrak{U}, \mathscr{H}^q(K^\bullet)) \Rightarrow R^{p+q}\Gamma K^\bullet.$$

Taking $K^\bullet := \mathscr{F}[0]$, there follows a natural map

$$\Psi^p_{\mathrm{alt}}(\mathfrak{U}, \mathscr{F}) : H^p_{\mathrm{alt}}(\mathfrak{U}, \mathscr{F}) \to R^p\Gamma\mathscr{F} \qquad \text{for every } p \in \mathbb{N}$$

and again, $\Psi^0_{\mathrm{alt}}(\mathfrak{U}, \mathscr{F})$ is an isomorphism, whereas $\Psi^1_{\mathrm{alt}}(\mathfrak{U}, \mathscr{F})$ is always an injective map. The following immediate corollary is often useful :

Corollary 10.2.21. *In the situation of* (10.2.15)*, suppose furthermore that*

$$\mathscr{H}^q(\mathscr{F}[0])(U_{\underline{t}}) = 0 \qquad \textit{for every integer } q > 0 \textit{ and every } \underline{t} \in I^q.$$

Then we have :

(i) *The map $\Psi^p(\mathfrak{U}, \mathscr{F})$ is an isomorphism for every $p \in \mathbb{N}$.*
(ii) *If U_i is a subobject of 1_T for every $i \in \mathbb{N}$, then the map $\Psi^p_{\mathrm{alt}}(\mathfrak{U}, \mathscr{F})$ is an isomorphism for every $p \in \mathbb{N}$.*

Proof. The assumption of (i) (resp. (ii)) implies that $E(\mathfrak{U})^{p,q}_2 = 0$ (resp. $F(\mathfrak{U})^{p,q}_2 = 0$) for every $q > 0$, whence both contentions. ☐

10.2.22. Let now $C := (\mathscr{C}, J)$ be a small site whose finite non-empty products and fibre products are representable. Then, the finite products of $\mathscr{C}/X$ are representable, for every $X \in \mathrm{Ob}(\mathscr{C})$; for every such X, we denote by J_X the topology on $\mathscr{C}/X$ induced by J (see (4.7.2)).

Now, let S be either C or $(\mathscr{C}/X, J_X)$ for some $X \in \mathrm{Ob}(\mathscr{C})$, and denote by $\mathscr{J}(S)$ the set of all families of objects of S that generate a sieve covering the final object of $T := S^\sim$, for the canonical topology on T. Then $\mathscr{J}(S)$ is endowed with a partial ordering, by declaring that $\mathfrak{U} \leq \mathfrak{U}'$ if and only if the sieve generated by $\mathfrak{U}$ contains the sieve generated by $\mathfrak{U}'$, for every $\mathfrak{U}, \mathfrak{U}' \in \mathscr{J}(S)$. Notice that this is equivalent to saying that the family $\mathfrak{U}'$ is a refinement of $\mathfrak{U}$, when we regard $\mathfrak{U}$ and $\mathfrak{U}'$ as families of objects of the topos T. Especially, when $\mathfrak{U} \leq \mathfrak{U}'$, the discussion of (10.2.10) yields a map of complexes $\varphi_{\mathfrak{U},\mathfrak{U}'} : R_\bullet(\mathfrak{U}') \to R_\bullet(\mathfrak{U})$, and lemma 10.2.12 says that the homotopy class of $\varphi_{\mathfrak{U},\mathfrak{U}'}$ depends only on $\mathfrak{U}$ and $\mathfrak{U}'$, so we get a well defined functor

$$R_\bullet : \mathscr{J}(S)^o \to \mathsf{Hot}(\mathbb{Z}_T\text{-}\mathbf{Mod}) \qquad \mathfrak{U} \mapsto R_\bullet(\mathfrak{U}).$$

Now, let $\mathscr{F}$ be any abelian sheaf on S. We then get, for every $i \in \mathbb{N}$, an induced functor

$$H^i(-, \mathscr{F}) : \mathscr{J}(S) \to \mathbb{Z}\text{-}\mathbf{Mod} \qquad \mathfrak{U} \mapsto H^i(\mathfrak{U}, \mathscr{F}).$$

Notice now that $\mathscr{J}(S)$ is a small and filtered category; we may then define

$$\check{H}^i(S, \mathscr{F}) := \operatorname*{colim}_{\mathfrak{U} \in \mathscr{J}(S)} H^i(\mathfrak{U}, \mathscr{F})$$

and clearly the rule $\mathscr{F} \mapsto \check{H}^i(S, \mathscr{F})$ yields a well defined functor

$$\check{H}^i(S, -) : \mathbb{Z}_T\text{-}\mathbf{Mod} \to \mathbb{Z}\text{-}\mathbf{Mod} \qquad \text{for every } i \in \mathbb{N}.$$

In case $S = (\mathscr{C}/X, J_X)$, we shall denote this functor by $\check{H}(X, -)$. By the same token, it is also clear that we get as well a natural morphism of 2-spectral sequences

$$E(\mathfrak{U})_2^{\bullet\bullet} \to E(\mathfrak{U}')_2^{\bullet\bullet} \qquad \text{whenever } \mathfrak{U} \leq \mathfrak{U}'$$

for every bounded below complex $K^\bullet$ of $\mathbb{Z}_T$-modules whence, after taking colimits, a 2-spectral sequence

$$E_2^{p,q} := \check{H}^p(S, \mathscr{H}^q(K^\bullet)) \Rightarrow R^{p+q}\Gamma K^\bullet$$

that, in turn, gives us again natural maps, for every $\mathbb{Z}_T$-module $\mathscr{F}$

$$\check{\Psi}^p(S, \mathscr{F}) : \check{H}^p(S, \mathscr{F}) \to R^p\Gamma\mathscr{F} \qquad \text{for every } p \in \mathbb{N}$$

which, in case $S = (\mathscr{C}/X, J_X)$, shall be denoted simply $\check{\Psi}^p(X, \mathscr{F})$.

Lemma 10.2.23. *In the situation of* (10.2.22)*, let $\mathscr{F}$ be any $\mathbb{Z}_T$-module, and take $K^\bullet := \mathscr{F}[0]$. Then we have :*

(i) $E_2^{0,q} = 0$ *for every $q > 0$.*
(ii) *Both $\check{\Psi}^0(S, \mathscr{F})$ and $\check{\Psi}^1(S, \mathscr{F})$ are isomorphisms, and $\check{\Psi}^2(S, \mathscr{F})$ is injective.*

Proof. (i): Fix any injective resolution $\mathscr{F} \to \mathscr{I}^\bullet$, and let $\overline{s} \in E_2^{0,q}$ be any element; by definition, there exists a family $(U_i \mid i \in I)$ of objects of S that generates a sieve covering the final object of $S^\sim$, such that $\overline{s}$ is represented by an element

$$\overline{s}_\bullet \in \mathrm{Ker}\Big(\prod_{i\in I} H^q(\mathscr{I}^\bullet(U_i)) \xrightarrow{d_v^{0,q}} \prod_{(i,j)\in I^2} H^q(\mathscr{I}^\bullet(U_{(i,j)}))\Big)$$

where $d_v^{0,q}$ is the differential of the Čech complex in degree 0, as in remark 10.2.7(i). In turn, the class $\overline{s}_i \in H^q(\mathscr{I}^\bullet(U_i))$ is represented by a section $s_i \in \mathscr{I}^q(U_i)$ for every $i \in I$, such that $d_h^{0,q}(s_i) = 0$, where $d_h^{0,q} : \mathscr{I}^q(U_i) \to \mathscr{I}^{q+1}(U_i)$ denotes the differential in degree q of the complex $\mathscr{I}^\bullet$. Since the latter is exact in every degree $q > 0$, it follows that for every $i \in I$ there exist a family $(U_{i,\lambda} \to U_i \mid \lambda \in \Lambda_i)$ of objects of $\mathscr{C}/U_i$ that generates a sieve covering the final object of $(\mathscr{C}/U_i, J_{U_i})$, and for every $\lambda \in \Lambda_i$ a section $s_{i,\lambda} \in \mathscr{I}^{q-1}(U_{i,\lambda})$ such that $d_h^{0,q-1}(s_{i,\lambda}) = s_{i|U_{i,\lambda}}$. Set $\mathfrak{U} := \bigcup_{i\in I}\{U_{i,\lambda} \mid \lambda \in \Lambda_i\}$; then $\mathfrak{U}$ lies in $\mathscr{J}(S)$, and it is easily seen that the image of $\overline{s}$ vanishes in $H^0(\mathfrak{U}, \mathscr{H}^q(\mathscr{F}))$, whence the contention.

(ii) is an immediate consequence of (i). □

Theorem 10.2.24 (Cartan)**.** *In the situation of* (10.2.22)*, let $\mathscr{F}$ be any $\mathbb{Z}_T$-module such that*

$$\check{H}^i(X, \mathscr{F}) = 0 \qquad \textit{for every } X \in \mathrm{Ob}(\mathscr{C}) \textit{ and every } i > 0.$$

Then we have :

(i) *The map $\check{\Psi}^q(C, \mathscr{F})$ is an isomorphism for every $q \in \mathbb{N}$.*
(ii) *More precisely, the map $\Psi^q(\mathfrak{U}, \mathscr{F})$ is an isomorphism for every $q \in \mathbb{N}$ and every family $\mathfrak{U}$ of objects of $\mathscr{C}$ that covers the final object of T.*

Proof. (i): We argue by induction on $q \in \mathbb{N}$, and notice that the assertion for $q \leq 1$ is already known without any assumption on $\mathscr{F}$, by lemma 10.2.23(ii). Thus, let $i \geq 2$ and suppose that the assertion of the theorem is already known for every $q < i$, every site C and every abelian sheaf $\mathscr{F}$ on C. Notice that if $X \in \mathrm{Ob}(\mathscr{C})$ and $\varphi : Y \to X$ is any object of $\mathscr{C}/X$, we have a natural isomorphism of categories

$$(\mathscr{C}/X)/\varphi \xrightarrow{\sim} \mathscr{C}/Y$$

and the topology J_φ induced on $(\mathscr{C}/X)/\varphi$ by J_X agrees, under this identification, with the topology J_Y. Therefore, the site $\mathscr{C}/X$ fulfills the assumptions of the theorem, and by inductive assumption we deduce that $\check{\Psi}^q(X,\mathscr{F})$ is an isomorphism for every $q < i$, and therefore $R^q\Gamma(X,\mathscr{F}) = 0$ for every $X \in \mathrm{Ob}(\mathscr{C})$ and every $q = 1,\dots,i-1$. Summing up, we get $\mathscr{H}^q(\mathscr{F}) = 0$ whenever $1 \leq q < i$, whence $E_2^{pq} = 0$ for every $q = 1,\dots,i-1$ (notation of (10.2.22)). We also know that $E_2^{0q} = 0$ for every $q > 0$, by lemma 10.2.23(i); in this situation, it is easily seen that $E_2^{i,0} = E_\infty^{i0}$ and $E_\infty^{i-p,p} = 0$ for every $p > 0$, and then $\check{\Psi}^i(X,\mathscr{F})$ is indeed an isomorphism.

(ii) is an immediate consequence of (i) and corollary 10.2.21(i). □

10.2.25. We wish now to apply the foregoing results to the cohomology of quasi-coherent modules on a scheme. To begin with, let A be a ring, M an A-module; set $X := \mathrm{Spec}\, A$ and denote by $\mathscr{M}$ the quasi-coherent $\mathscr{O}_X$-module arising from M. Let also $\mathbf{f} := (f_1,\dots,f_r)$ be a sequence of elements of A, and for every integer $n \geq -1$ and every injective order-preserving map

$$\underline{t} : [n] \to \Sigma := \{1,\dots,r\} \qquad k \mapsto t_k$$

let $A_{\underline{t}} := A[f_{t_0}^{-1}\cdots f_{t_n}^{-1}]$ and $U_{\underline{t}} := \mathrm{Spec}\, A_{\underline{t}}$; we can describe as follows the $A_{\underline{t}}$-module $\mathscr{M}(U_{\underline{t}})$. For every such $\underline{t}$, consider the system of A-modules $((M_{\underline{t}}^{(k)},\varphi_k) \mid k \in \mathbb{N})$ such that $M_{\underline{t}}^{(k)} := M$ for every $k \in \mathbb{N}$, and the map $\varphi_k : M_{\underline{t}}^{(k)} \to M_{\underline{t}}^{(k+1)}$ is the scalar multiplication by $f_{t_0}\cdots f_{t_n}$ if $n \geq 0$, and it is $\mathbf{1}_M$ when $n = -1$. Then we have a natural identification

$$\mathscr{M}(U_{\underline{t}}) \xrightarrow{\sim} \operatorname*{colim}_{k\in\mathbb{N}} M_{\underline{t}}^{(k)} \qquad \text{for every } n \geq -1 \text{ and every } \underline{t} : [n] \to \Sigma. \tag{10.2.26}$$

Moreover, for every face map $\varepsilon_i : [n-1] \to [n]$ in Δ, we can describe as follows the restriction map $\rho_{i,\underline{t}} : \mathscr{M}(U_{\underline{t}\circ\varepsilon_i}) \to \mathscr{M}(U_{\underline{t}})$. We consider the morphism of directed systems

$$\rho_i^{(\bullet)} : M_{\varepsilon_i^*(\underline{t})}^{(\bullet)} \to M_{\underline{t}}^{(\bullet)} \qquad \text{such that} \qquad \rho_i^{(k)} := f_{t_i}^k \cdot \mathbf{1}_M \quad \text{for every } k \in \mathbb{N}.$$

Then, under the identifications (10.2.26), the map $\rho_{i,\underline{t}}$ corresponds to the map

$$\operatorname*{colim}_{k\in\mathbb{N}} \rho_i^{(k)} : \operatorname*{colim}_{k\in\mathbb{N}} M_{\underline{t}\circ\varepsilon_i)}^{(k)} \to \operatorname*{colim}_{k\in\mathbb{N}} M_{\underline{t}}^{(k)}.$$

For every $n \geq -1$ let $\Sigma_{\mathrm{alt}}^{n+1}$ be the set of all injective order-preserving maps $[n] \to \Sigma$, and set

$$D_{(k)}^n := \mathrm{Hom}_{\mathbf{Set}}(\Sigma_{\mathrm{alt}}^{n+1}, M) \qquad \text{for every } k \in \mathbb{N}$$

which we endow with the A-module structure inherited from M; consider also the map

$$d_{(k)}^n : D_{(k)}^n \to D_{(k)}^{n+1} \qquad (\mu : \Sigma_{\mathrm{alt}}^{n+1} \to M) \mapsto \Big(\underline{t} \mapsto \sum_{i=0}^{n+1} (-1)^i \cdot f_{t_i}^k \cdot \mu(\underline{t}\circ\varepsilon_i)\Big)$$

for every $k \in \mathbb{N}$. Then the system $(D_{(k)}^\bullet, d_{(k)}^\bullet)$ is a well defined complex of A-modules, for every $k \in \mathbb{N}$. Lastly, for every $k \in \mathbb{N}$ we consider the morphism of complexes

$$D_{(k)}^\bullet \to D_{(k+1)}^\bullet \qquad (\mu : \Sigma_{\mathrm{alt}}^{n+1} \to M) \mapsto (\underline{t} \mapsto f_{t_0}\cdots f_{t_n} \cdot \mu(\underline{t})).$$

Summing up, and comparing with remark 10.2.7(i) we obtain a natural identification

$$C_{\mathrm{alt}}^\bullet(\mathfrak{U},\mathscr{M}) \xrightarrow{\sim} \operatorname*{colim}_{k\in\mathbb{N}} D_{(k)}^\bullet \qquad \text{where } \mathfrak{U} := (U_1,\dots,U_r).$$

On the other hand, notice that, for every $n \geq -1$, the free A-module $\Lambda_A^{n+1} A^{\oplus r}$ admits the standard basis $(e_{t_0} \wedge \cdots \wedge e_{t_n} \mid \underline{t} \in \Sigma_{\mathrm{alt}}^{n+1})$, so we have as well a natural identification

$$\mathrm{Hom}_A(\Lambda_A^{n+1} A^{\oplus r}, M) \xrightarrow{\sim} D_{(k)}^n \qquad \text{for every } n \geq -1$$

and a direct inspection reveals that, under this latter identification, the differential $d_{(k)}^n$ of $D_{(k)}^\bullet$ corresponds to the differential $\mathrm{Hom}_A(d_{\mathbf{f}^k,n}, M)$, where $\mathbf{f}^k := (f_1^k,\dots,f_r^k)$, and $d_{\mathbf{f}^k,n}$ is the differential in degree n of the Koszul complex of the sequence $\mathbf{f}^k$ (see remark 7.8.1(ii)). Thus, we get finally a natural isomorphism of complexes of A-modules

$$C^\bullet_{\mathrm{alt}}(\mathfrak{U},\mathscr{M}) \xrightarrow{\sim} \operatorname*{colim}_{k\in\mathbb{N}} \mathbf{K}^\bullet(\mathbf{f}^k, M)[1] \tag{10.2.27}$$

where the transition maps $\mathbf{K}^\bullet(\mathbf{f}^k, M) \to \mathbf{K}^\bullet(\mathbf{f}^{k+1}, M)$ are the morphisms $\varphi_{\mathbf{f}}$ of (7.8.20). As a corollary, we get the following classical result of Grothendieck :

Theorem 10.2.28. *In the situation of* (10.2.25)*, the following holds :*

(i) *The natural map $M[0] \to R\Gamma\,\mathscr{M}$ is an isomorphism in $\mathsf{D}(\mathscr{O}_X\text{-}\mathbf{Mod})$.*

(ii) *Let Y be any separated scheme, $\mathscr{F}^\bullet$ a bounded below complex of quasi-coherent $\mathscr{O}_Y$-modules, and $\mathfrak{U}$ any affine open covering of Y. Then the natural map*

$$\operatorname{Tot}\overline{C}^\bullet_{\mathrm{alt}}(\mathfrak{U},\mathscr{F}^\bullet) \to R\Gamma\mathscr{F}^\bullet$$

is an isomorphism in $\mathsf{D}^+(\mathscr{O}_Y(Y)\text{-}\mathbf{Mod})$.

(iii) *Especially, if Y is a separated scheme that can be covered by r affine open subsets, then $H^i(Y,\mathscr{F}) = 0$ for every $i \geq r$, and every quasi-coherent $\mathscr{O}_Y$-module $\mathscr{F}$.*

Proof. (i): Since the affine open subsets of X of the form $\operatorname{Spec} A[f^{-1}]$ for $f \in A$ are a basis of the topology of X that is closed under finite intersections, theorem 10.2.24 reduces to showing that

$$H^0(\mathfrak{U},\mathscr{M}) = M \qquad \text{and} \qquad H^i(\mathfrak{U},\mathscr{M}) = 0 \qquad \text{for every } i > 0$$

where $\mathfrak{U} := (U_i,\dots,U_r)$ is the finite affine open covering of X associated with a sequence $\mathbf{f} := (f_1,\dots,f_r)$ as in (10.2.25). However, the condition $X = \bigcup_{i=1}^r U_i$ is equivalent to saying that the ideal generated by the system $\mathbf{f}$ is A, in which case the same holds for the ideal generated by $\mathbf{f}^k$, for every $k \in \mathbb{N}$, and then lemma 7.8.2(iii) says that the complex $H^\bullet(\mathbf{f}^k, M)$ is homotopically trivial for every $k \in \mathbb{N}$. Taking into account (10.2.27), the contention follows.

(ii): Pick any resolution $\mathscr{F}^\bullet \xrightarrow{\sim} \mathscr{I}^\bullet$ by a bounded below complex of injective $\mathscr{O}_Y$-modules; in view of (10.2.19), it suffices to show that the natural map of double complexes

$$\overline{C}^\bullet_{\mathrm{alt}}(\mathfrak{U},\mathscr{F}^\bullet) \to \overline{C}^\bullet_{\mathrm{alt}}(\mathfrak{U},\mathscr{I}^\bullet)$$

induces a quasi-isomorphism on total complexes. However, notice that the open subset $U_{\underline{t}} \subset Y$ is affine, for every $n \in \mathbb{N}$ and every $\underline{t} \in I^{n+1}_{\mathrm{alt}}$ (notation of (10.2.4)); hence, it suffices to check that the induced map $\mathscr{F}^\bullet(V) \to \mathscr{I}^\bullet(V)$ is a quasi-isomorphism, for every affine open subset $V \subset Y$. To this aim, consider the spectral sequence

$$E_1^{p,q} := H^p(V,\mathscr{F}^q) \Rightarrow H^{p+q}\mathscr{I}^\bullet(V).$$

Since V is affine, we have $E_1^{p,q} = 0$ for every $p > 0$, due to (i); on the other hand, the differential $d_1^{0,q} : E_1^{0,q} \to E_1^{0,q+1}$ is nothing else than the differential $d^q(V) : \mathscr{F}^q(V) \to \mathscr{F}^{q+1}(V)$, for every $q \in \mathbb{Z}$, whence the claim.

(iii): Indeed, if $\mathfrak{U}$ is such a covering, then the complex $\overline{C}^\bullet_{\mathrm{alt}}(\mathfrak{U},\mathscr{F})$ is an object of the category $\mathsf{C}^{[0,r-1]}(\mathscr{O}_Y(Y)\text{-}\mathbf{Mod})$, so the assertion follows from (ii). ☐

We conclude this section with an application of Čech cohomology to the computation of sheaf cohomology after change of base scheme; much more can be found in [62, Ch.III, §6].

10.2.29. Namely, consider a diagram of schemes

$$\begin{array}{ccccc} X_2 & \longrightarrow & X_1 & \longrightarrow & X_0 \\ {\scriptstyle\pi_2}\downarrow & & {\scriptstyle\pi_1}\downarrow & & \downarrow{\scriptstyle\pi_0} \\ S_2 & \longrightarrow & S_1 & \xrightarrow{\varphi} & S_0 \end{array}$$

whose two square subdiagram are cartesian, and bounded above complexes of quasi-coherent $\mathscr{O}_{X_0}$-modules $\mathscr{F}_0^\bullet$ and quasi-coherent $\mathscr{O}_{S_i}$-modules $\mathscr{G}_i^\bullet$ for $i = 1, 2$. Then, for every $q \in \mathbb{N}$ there exists a quasi-coherent $\mathscr{O}_{X_1}$-module

$$\mathscr{T}or_q^{S_0}(\mathscr{G}_1^\bullet, \mathscr{F}_0^\bullet)$$

characterized, up to unique isomorphism, by the following properties ([62, Ch.III, §6.5]) :

(a) For every affine open subsets $V_0 \subset S_0$, $V \subset S_1 \times_{S_0} V_0$ and $U \subset X_0 \times_{S_0} V_0$, there exists an isomorphism of $\mathscr{O}_{X_1}(V \times_{S_0} U)$-modules

$$\omega_{V,U} : \Gamma(V \times_{S_0} U, \mathscr{T}or_q^{S_0}(\mathscr{G}_1^\bullet, \mathscr{F}_0^\bullet)) \xrightarrow{\sim} H_q(\mathscr{G}_1^\bullet(V) \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{S_0}(V_0)} \mathscr{F}_0^\bullet(U)).$$

(b) For every V_0, V, U as in (a) and every inclusion of affine open subsets $V'_0 \subset V_0$, $V' \subset V \cap (S_1 \times_{S_0} V'_0)$ and $U' \subset U \cap (X_0 \times_{S_0} V'_0)$, the resulting diagram commutes :

$$\begin{array}{ccc}
\Gamma(V \times_{S_0} U, \mathscr{T}or_q^{S_0}(\mathscr{G}_1^\bullet, \mathscr{F}_0^\bullet)) & \xrightarrow{\omega_{V,U}} & H_q(\mathscr{G}_1^\bullet(V) \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{S_0}(V_0)} \mathscr{F}_0^\bullet(U)) \\
\downarrow & & \downarrow \\
\Gamma(V' \times_{S_0} U', \mathscr{T}or_q^{S_0}(\mathscr{G}_1^\bullet, \mathscr{F}_0^\bullet)) & \xrightarrow{\omega_{V',U'}} & H_q(\mathscr{G}_1^\bullet(V') \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{S_0}(V'_0)} \mathscr{F}_0^\bullet(U'))
\end{array}$$

whose left vertical arrow is the restriction map of the sheaf $\mathscr{T}or_q^{S_0}(\mathscr{G}_1^\bullet, \mathscr{F}_0^\bullet)$, and whose right vertical arrow is the map $H_q(\rho' \overset{\mathbf{L}}{\otimes}_\rho \rho'')$ associated with the restriction maps $\rho : \mathscr{O}_{S_0}(V_0) \to \mathscr{O}_{S_0}(V'_0)$, $\rho' : \mathscr{G}^\bullet(V) \to \mathscr{G}^\bullet(V')$ and $\rho'' : \mathscr{F}_0^\bullet(U) \to \mathscr{F}_0^\bullet(U')$.

Proposition 10.2.30. *In the situation of* (10.2.29)*, the following holds :*

(i) *There exists a natural (homological)* 2*-spectral sequence*

$$\mathscr{T}or_p^{S_1}(\mathscr{G}_2^\bullet, \mathscr{T}or_q^{S_0}(\mathscr{O}_{S_1}[0], \mathscr{F}_0^\bullet)) \Rightarrow \mathscr{T}or_{p+q}^{S_0}(\mathscr{G}_2^\bullet, \mathscr{F}_0^\bullet).$$

(ii) *Suppose that π_0 is quasi-compact and separated, $\mathscr{F}_0$ is a bounded complex, and both S_1 and S_0 are affine. Set $A_i := \mathscr{O}_{S_i}(S_i)$ for $i = 0, 1$. Then there exist two natural spectral sequences :*

$$E_2^{pq} := H^{-p}(X_1, \mathscr{T}or_q^{S_0}(\mathscr{G}^\bullet, \mathscr{F}_0^\bullet)) \Rightarrow H_{p+q}(R\Gamma\mathscr{G}^\bullet \overset{\mathbf{L}}{\otimes}_{A_0} R\Gamma\mathscr{F}_0^\bullet)$$
$$F_2^{pq} := H_p(R\Gamma\mathscr{G}^\bullet \overset{\mathbf{L}}{\otimes}_{A_0} H^{-q}(X_0, \mathscr{F}_0^\bullet)[0]) \Rightarrow H_{p+q}(R\Gamma\mathscr{G}^\bullet \overset{\mathbf{L}}{\otimes}_{A_0} R\Gamma\mathscr{F}_0^\bullet).$$

Proof. (i): This is obtained from the standard change of base ring spectral sequence, which is natural in all arguments, and therefore globalizes immediately to the sheaf-theoretic situation considered here : the details shall be left to the reader.

(ii): Under our assumptions, X_0 is quasi-compact and separated, hence we may find a finite affine covering $\mathfrak{U}$ of X_0, and by theorem 10.2.28(ii), the alternating Čech complex for $\mathfrak{U}$ computes the cohomology of $\mathscr{F}_\bullet^\bullet$; especially the complex $R\Gamma\mathscr{F}_0^\bullet$ is bounded. Moreover, we may find a bounded above Cartan-Eilenberg resolution $\mathscr{L}^\bullet \xrightarrow{\sim} \varphi_*\mathscr{G}^\bullet$ consisting of quasi-coherent flat $\mathscr{O}_{S_0}$-modules, and $H_i(R\Gamma\mathscr{G}^\bullet \overset{\mathbf{L}}{\otimes}_{A_0} R\Gamma\mathscr{F}_0^\bullet)$ is naturally isomorphic to the quasi-coherent $\mathscr{O}_{S_1}$-module associated with the A_1-module

$$H_i((\mathrm{Tot}\,\overline{C}^\bullet_{\mathrm{alt}}(\mathfrak{U}, \mathscr{F}_0^\bullet)) \otimes_{A_0} \mathscr{L}^\bullet(S_0))$$

for every $i \in \mathbb{Z}$. Then the first spectral sequence is the standard spectral sequence attached to the double complex $(\mathrm{Tot}\,\overline{C}^\bullet_{\mathrm{alt}}(\mathfrak{U}, \mathscr{F}_0^\bullet)) \boxtimes_{A_0} \mathscr{L}^\bullet(S_0)$; similarly one obtains the second spectral sequence : details left to the reader. □

10.3. **Quasi-coherent modules.** For any scheme X, we denote by

$$\mathscr{O}_X\text{-}\mathbf{Mod} \qquad \mathscr{O}_X\text{-}\mathbf{Mod}_{\mathrm{qcoh}} \qquad \mathscr{O}_X\text{-}\mathbf{Mod}_{\mathrm{coh}} \qquad \mathscr{O}_X\text{-}\mathbf{Mod}_{\mathrm{lfft}}$$

the category of all (resp. of quasi-coherent, resp. of coherent, resp. of locally free of finite type) $\mathscr{O}_X$-modules. Recall that there is a natural functor

$$\mathscr{O}_X(X)\text{-}\mathbf{Mod} \to \mathscr{O}_X\text{-}\mathbf{Mod}_{\mathrm{qcoh}} \qquad M \mapsto M^\sim \tag{10.3.1}$$

that assigns to every $\mathscr{O}_X(X)$-module the quasi-coherent module $M^\sim$ such that

$$M^\sim(U) := M \otimes_{\mathscr{O}_X(X)} \mathscr{O}_X(U) \qquad \text{for every affine open subset } U \subset X.$$

This functor is an equivalence, if X is affine. We notice :

Corollary 10.3.2. *Let A be a ring, $M^\bullet$ (resp. $N_\bullet$) a bounded below (resp. above) complex of A-modules. Set $X := \operatorname{Spec} A$ and denote by $M^{\bullet\sim}$ (resp. $N_\bullet^\sim$) the associated complex of quasi-coherent $\mathscr{O}_X$-modules. We have :*

(i) *The natural map*

$$R\mathrm{Hom}^\bullet_A(N_\bullet, M^\bullet) \to R\mathrm{Hom}^\bullet_{\mathscr{O}_X}(N_\bullet^\sim, M^{\bullet\sim})$$

is an isomorphism in $\mathsf{D}^+(A\text{-}\mathbf{Mod})$.

(ii) *If A is coherent and $N_\bullet$ is a complex of finitely presented A-modules, then the natural morphism*

$$R\mathrm{Hom}^\bullet_A(N_\bullet, M^\bullet)^\sim \to R\mathscr{H}om^\bullet_{\mathscr{O}_X}(N_\bullet^\sim, M^{\bullet\sim})$$

is an isomorphism in $\mathsf{D}(\mathscr{O}_X\text{-}\mathbf{Mod})$.

Proof. (i): We apply the trivial duality theorem 10.1.18 to the unique morphism

$$f : (X, \mathscr{O}_X) \to (\{\mathrm{pt}\}, A)$$

of ringed spaces, where $\{\mathrm{pt}\}$ denotes the one-point space. Since f is flat and all quasi-coherent $\mathscr{O}_X$-modules are f_*-acyclic, the assertion follows easily.

(ii): Due to (i), it suffices to show the following assertion. For every $f \in A$, the natural map

$$R\mathrm{Hom}^\bullet_A(N_\bullet, M^\bullet) \otimes_A A_f \to R\mathrm{Hom}^\bullet_{A_f}(N_{\bullet,f}, M^\bullet_f)$$

is an isomorphism in $\mathsf{D}(A_f\text{-}\mathbf{Mod})$. To this aim, notice that, since A is coherent, we may find a resolution $P_\bullet$ of $N_\bullet$ consisting of free A-modules of finite type (details left to the reader); then we come down to checking that the natural map $\mathrm{Hom}_A(P_i, M^j) \otimes_A A_f \to \mathrm{Hom}_{A_f}(P_{i,f}, M^j_f)$ is an isomorphism, which is obvious. □

We denote by

$$\mathsf{D}(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{qcoh}} \qquad \mathsf{D}(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}}$$

the full triangulated subcategory of $\mathsf{D}(\mathscr{O}_X\text{-}\mathbf{Mod})$ consisting of the complexes $K^\bullet$ such that $H^i K^\bullet$ is a quasi-coherent (resp. coherent) $\mathscr{O}_X$-module for every $i \in \mathbb{Z}$. As usual, we shall use also the variants $\mathsf{D}^+(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$ (resp. $\mathsf{D}^-(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$, resp. $\mathsf{D}^b(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$, resp. $\mathsf{D}^{[a,b]}(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$) consisting of all objects of $\mathsf{D}(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$ whose cohomology vanishes in sufficiently large negative degree (resp. sufficiently large positive degree, resp. outside a bounded interval, resp. outside the interval $[a, b]$), and likewise for the corresponding subcategories of $\mathsf{D}(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}}$. Obviously, (10.3.1) induces a natural functor

$$\mathsf{D}(\mathscr{O}_X(X)\text{-}\mathbf{Mod}) \to \mathsf{D}(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{qcoh}} \qquad M^\bullet \mapsto M^{\bullet\sim}.$$

Proposition 10.3.3. *Let $f : Y \to X$ be a flat morphism of schemes, with X locally coherent (see definition* 8.1.58(i)*), and $K_\bullet \in \mathrm{Ob}(\mathsf{D}^-(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}})$ any complex. We have :*

(i) $R\mathscr{H}om^\bullet_{\mathscr{O}_X}(K_\bullet, L^\bullet) \in \mathrm{Ob}(\mathsf{D}^+(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}})$ *for every* $L^\bullet \in \mathrm{Ob}(\mathsf{D}^+(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}})$.

(ii) *For every* $L^\bullet \in \mathrm{Ob}(\mathsf{D}^+(\mathscr{O}_X\text{-}\mathbf{Mod}))$ *the natural morphism*

$$f^*R\mathscr{H}om^\bullet_{\mathscr{O}_X}(K_\bullet, L^\bullet) \to R\mathscr{H}om^\bullet_{\mathscr{O}_Y}(f^*K_\bullet, f^*L^\bullet) \qquad (10.3.4)$$

is an isomorphism in $\mathsf{D}^+(\mathscr{O}_X\text{-}\mathbf{Mod})$).

Proof. (i): The assertion is local on X, so we may assume that $X = \mathrm{Spec}\, A$, for some coherent ring A. In this case, we may find a bounded below (resp. above) complex $M^\bullet$ (resp. $N_\bullet$) of finitely presented A-modules, with isomorphisms $K^\bullet \xrightarrow{\sim} M^{\bullet\sim}$, $L_\bullet \xrightarrow{\sim} N^\sim_\bullet$ in $\mathsf{D}(\mathscr{O}_X\text{-}\mathbf{Mod})$. By virtue of corollary 10.3.2(ii), we are then reduced to checking that $R\mathrm{Hom}^\bullet_A(N_\bullet, M^\bullet)$ lies in $\mathsf{D}(A\text{-}\mathbf{Mod}_{\mathrm{coh}})$. But this is easily seen, since we may find a resolution of $N_\bullet$ by a bounded above complex of free A-modules of finite type (details left to the reader).

(ii): Again, the question is local on X, so we may assume that $K_\bullet$ admits a resolution by a bounded above complex $P_\bullet$ of free $\mathscr{O}_X$-modules of finite type. Since f is flat, we have natural convergent spectral sequences

$$E_1^{pq} := f^*R^p\mathscr{H}om^\bullet_{\mathscr{O}_X}(P_q, L^\bullet) \Rightarrow f^*R^{p+q}\mathscr{H}om^\bullet_{\mathscr{O}_X}(K_\bullet, L^\bullet)$$
$$F_1^{pq} := R^p\mathscr{H}om^\bullet_{\mathscr{O}_Y}(f^*P_q, f^*L^\bullet) \Rightarrow R^{p+q}\mathscr{H}om^\bullet_{\mathscr{O}_Y}(f^*K_\bullet, f^*L^\bullet)$$

as well as a morphism of spectral sequences $E_\bullet^{\bullet\bullet} \to F_\bullet^{\bullet\bullet}$, such that $H^\bullet$(10.3.4) is a morphism of filtered $\mathscr{O}_Y$-modules, for the two finite filtrations induced by these spectral sequences on their abutments. It then suffices to check that the induced morphism $E_\infty^{pq} \to F_\infty^{pq}$ is an isomorphism for every $p, q \in \mathbb{Z}$, and this in turn will follow, if we show that the morphism $E_1^{pq} \to F_1^{pq}$ is an isomorphism; but the latter assertion is obvious. ☐

Corollary 10.3.5. *Let* $f : Y \to X$ *be a quasi-compact and quasi-separated morphism of schemes,* $\mathscr{F}$ *a flat quasi-coherent* $\mathscr{O}_X$*-module, and* $\mathscr{G}$ *any* $\mathscr{O}_Y$*-module. Then the natural map*

$$\mathscr{F} \otimes_{\mathscr{O}_X} Rf_*\mathscr{G} \to Rf_*(f^*\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{G}) \qquad (10.3.6)$$

is an isomorphism in $\mathsf{D}(\mathscr{O}_X\text{-}\mathbf{Mod})$.

Proof. The assertion is local on X, hence we may assume that $X = \mathrm{Spec}\, A$ for some ring A, and $\mathscr{F} = M^\sim$ for some flat A-module M. By [120, Ch.I, Th.1.2], M is the colimit of a filtered family of free A-modules of finite rank; in view of proposition 10.1.10(ii), we may then assume that $M = A^{\oplus n}$ for some $n \geq 0$, in which case the assertion is obvious. ☐

Remark 10.3.7. Notice that, for an affine morphism $f : Y \to X$, the map (10.3.6) is an isomorphism for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ and any quasi-coherent $\mathscr{O}_Y$-module $\mathscr{G}$. The details shall be left to the reader.

Corollary 10.3.8. *Consider a cartesian diagram of schemes*

$$\begin{array}{ccc} Y' & \xrightarrow{g'} & Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{g} & X \end{array}$$

such that f *is quasi-compact and quasi-separated, and* g *is flat. Then the natural map*

$$g^*Rf_*\mathscr{G} \to Rf'_*g'^*\mathscr{G} \qquad (10.3.9)$$

is an isomorphism in $\mathsf{D}(\mathscr{O}_{X'}\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$*, for every quasi-coherent* $\mathscr{O}_Y$*-module* $\mathscr{G}$.

Proof. We easily reduce to the case where both X and X' are affine, hence g is an affine morphism. In this case, it suffices to show that g_*(10.3.9) is an isomorphism, since $Rf_*\mathscr{G}$ lies in

$\mathsf{D}(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$ and $Rf'_*g'^*\mathscr{G}$ lies in $\mathsf{D}(\mathscr{O}_{X'}\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$ ([2, Th.5.6]). However, we have a commutative diagram

$$\begin{array}{ccccc} g_*(g^*Rf_*\mathscr{G}) & & \xrightarrow{\quad\alpha\quad} & & Rf_*\mathscr{G}\otimes_{\mathscr{O}_X} g_*\mathscr{O}_{X'} \\ \downarrow & & & & \downarrow \\ g_*(Rf'_*g'^*\mathscr{G}) & \xrightarrow{\sim} & Rf_*g'_*(g'^*\mathscr{G}) & \xrightarrow{\beta} & Rf_*(\mathscr{G}\otimes_{\mathscr{O}_Y} g'_*\mathscr{O}_{Y'}) \end{array}$$

whose left vertical arrow is g_*(10.3.9) and whose right vertical arrow is the natural isomorphism provided by corollary 10.3.5 (applied to the flat $\mathscr{O}_Y$-module $\mathscr{F} := g'_*\mathscr{O}_{Y'}$). Also, α and β are the natural maps obtained as in [59, Ch.0, (5.4.10)]; it is easily seen that these are isomorphisms, for all affine morphisms g and g'. The assertion follows. □

Remark 10.3.10. Corollary 10.3.8 is also proved in [2, Th.6.7]. Notice that the corollary fails if $\mathscr{G}$ is not quasi-coherent. For a counterexample, let $R := k[x,y]_\mathfrak{m}$, where k is a field and $\mathfrak{m} \subset k[x,y]$ is the maximal ideal generated by x and y; *i.e.* R is the local ring at the origin of the affine plane over k. Let $R^\wedge$ be the completion of R, $X := \mathrm{Spec}\, R$, $X' := \mathrm{Spec}\, R^\wedge$, and set $Y := X\setminus\{\mathfrak{m}R\}$ and $Y' := Y\times_X X'$. Let moreover $\mathfrak{p}\in Y$ be the prime ideal generated by x, and $\mathfrak{p}'$ its inverse image in Y'; take for $\mathscr{G}$ the $\mathscr{O}_Y$-module obtained by extension by zero of the restriction to $\{\mathfrak{p}\}$ of the structure sheaf of $\mathscr{O}_Y$. With this notation, suppose that the map (10.3.9) induces an isomorphism on the stalks at the closed points; in cohomological degree zero, this comes down to saying that the natural map $f : (R^\wedge)_\mathfrak{p} \to (R^\wedge)_{\mathfrak{p}'}$ is an isomorphism. However, notice that the restriction $R_\mathfrak{p} \to (R^\wedge)_{\mathfrak{p}'}$ of f induces a homeomorphism $\mathrm{Spec}\,(R^\wedge)_{\mathfrak{p}'} \xrightarrow{\sim} \mathrm{Spec}\, R_\mathfrak{p}$. On the other hand, we can exhibit a height one prime ideal $\mathfrak{q}\in\mathrm{Spec}\,(R^\wedge)_\mathfrak{p}$ whose image in $\mathrm{Spec}\, R_\mathfrak{p}$ is the generic point. Indeed, let $a(X)\in x\cdot k[[x]]$ be a power series that is transcendental over $k(x)$; the kernel of the surjective continuous map of $k[[x]]$-algebras $g : R^\wedge = k[[x,y]] \to k[[x]]$ such that $g(y) := a(x)$ is the ideal I generated by $y - a(x)$, hence the latter is a prime ideal of $R^\wedge$, and we take $\mathfrak{q} := I_\mathfrak{p}$. To see that the preimage of $\mathfrak{q}$ in $\mathrm{Spec}\, R_\mathfrak{p}$ is the generic point, it suffices to notice that the restriction $R_\mathfrak{p} \to k[[x]]$ of g is injective, as $a(x)$ is transcendental over $k(x)$.

10.3.11. There are obvious forgetful functors:

$$\iota_X : \mathscr{O}_X\text{-}\mathbf{Mod}_{\mathrm{qcoh}} \to \mathscr{O}_X\text{-}\mathbf{Mod} \qquad R\iota_X : \mathsf{D}(\mathscr{O}_X\text{-}\mathbf{Mod}_{\mathrm{qcoh}}) \to \mathsf{D}(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$$

and we wish to exhibit right adjoints to these functors. To this aim, suppose first that X is affine; then we may consider the functor:

$$\mathsf{qcoh}_X : \mathscr{O}_X\text{-}\mathbf{Mod} \to \mathscr{O}_X\text{-}\mathbf{Mod}_{\mathrm{qcoh}} \qquad \mathscr{F} \mapsto \mathscr{F}(X)^\sim$$

If $\mathscr{G}$ is a quasi-coherent $\mathscr{O}_X$-module, then clearly $\mathsf{qcoh}_X\mathscr{G} \simeq \mathscr{G}$; moreover, for any other $\mathscr{O}_X$-module $\mathscr{F}$, there is a natural bijection:

$$\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{G},\mathscr{F}) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{O}_X(X)}(\mathscr{G}(X),\mathscr{F}(X)).$$

It follows easily that qcoh_X is the sought right adjoint.

10.3.12. Slightly more generally, let U be *quasi-affine*, *i.e.* a quasi-compact open subset of an affine scheme, and choose a quasi-compact open immersion $j : U \to X$ into an affine scheme X. In this case, we may define

$$\mathsf{qcoh}_U : \mathscr{O}_U\text{-}\mathbf{Mod} \to \mathscr{O}_X\text{-}\mathbf{Mod} \qquad \mathscr{F} \mapsto (\mathsf{qcoh}_X j_*\mathscr{F})_{|U}.$$

Since $j_* : \mathscr{O}_U\text{-}\mathbf{Mod} \to \mathscr{O}_X\text{-}\mathbf{Mod}$ is right adjoint to $j^* : \mathscr{O}_X\text{-}\mathbf{Mod} \to \mathscr{O}_U\text{-}\mathbf{Mod}$, we have a natural isomorphism: $\mathrm{Hom}_{\mathscr{O}_U}(\mathscr{G},\mathscr{F}) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{O}_X}(j_*\mathscr{G}, j_*\mathscr{F})$ for every $\mathscr{O}_U$-modules $\mathscr{G}$ and $\mathscr{F}$. Moreover, if $\mathscr{G}$ is quasi-coherent, the same holds for $j_*\mathscr{G}$ ([59, Ch.I, Cor.9.2.2]), whence a natural isomorphism $\mathscr{G} \simeq \mathsf{qcoh}_U\mathscr{G}$, by the foregoing discussion for the affine case. Summing up, this shows that qcoh_U is a right adjoint to ι_U, and especially it is independent, up to unique isomorphism, of the choice of j.

10.3.13. Next, suppose that X is quasi-compact and quasi-separated. We choose a finite covering $\mathfrak{U} := (U_i \mid i \in I)$ of X, consisting of affine open subsets, and set $U := \coprod_{i\in I} U_i$, the X-scheme which is the disjoint union (*i.e.* categorical coproduct) of the schemes U_i. We denote by $\mathfrak{U}_\bullet$ the simplicial covering such that $\mathfrak{U}_n := U \times_X \cdots \times_X U$, the $(n+1)$-th power of U, with the face maps given by the natural projections, and degeneracies induced by the diagonal map $U \to U \times_X U$; let also $\pi_n : \mathfrak{U}_n \to X$ be the natural morphism, for every $n \in \mathbb{N}$. Clearly we have $\pi_{n-1} \circ \partial_i = \pi_n$ for every face morphism $\partial_i : \mathfrak{U}_n \to \mathfrak{U}_{n-1}$. The simplicial scheme $\mathfrak{U}_\bullet$ (with the Zariski topology on each scheme $\mathfrak{U}_n$) can also be regarded as a fibred topos over the category Δ^o (notation of [75, §2.2]); then the datum $\mathscr{O}_{\mathfrak{U}_\bullet} := (\mathscr{O}_{\mathfrak{U}_n} \mid n \in \mathbb{N})$ consisting of the structure sheaves on each $\mathfrak{U}_n$ and the natural maps $\partial_i^* \mathscr{O}_{\mathfrak{U}_{n-1}} \to \mathscr{O}_{\mathfrak{U}_n}$ for every $n > 0$ and every $i = 0, \dots, n$ (and similarly for the degeneracy maps), defines a ring in the associated topos $\mathrm{Top}(\mathfrak{U}_\bullet)$ (see [75, §3.3.15]). We denote by $\mathscr{O}_{\mathfrak{U}_\bullet}\text{-}\mathbf{Mod}$ the category of $\mathscr{O}_{\mathfrak{U}_\bullet}$-modules in the topos $\mathrm{Top}(\mathfrak{U}_\bullet)$. The family $(\pi_n \mid n \in \mathbb{N})$ induces a morphism of topoi

$$\pi_\bullet : \mathrm{Top}(\mathfrak{U}_\bullet) \to s.X$$

where $s.X$ is the topos $\mathrm{Top}(X_\bullet)$ associated with the constant simplicial scheme $X_\bullet$ (with its Zariski topology) such that $X_n := X$ for every $n \in \mathbb{N}$ and such that all the face and degeneracy maps are $\mathbf{1}_X$. Clearly the objects of $s.X$ are nothing else than the cosimplicial Zariski sheaves on X. Especially, if we view a $\mathscr{O}_X$-module $\mathscr{F}$ as a constant cosimplicial $\mathscr{O}_X$-module, we may define the *augmented cosimplicial Čech $\mathscr{O}_X$-module*

$$\mathscr{F} \to \mathscr{C}^\bullet(\mathfrak{U}, \mathscr{F}) := \pi_{\bullet *} \circ \pi_\bullet^* \mathscr{F} \qquad (10.3.14)$$

associated with $\mathscr{F}$ and the covering $\mathfrak{U}$; in every degree $n \in \mathbb{N}$ this is defined by the rule :

$$\mathscr{C}^n(\mathfrak{U}, \mathscr{F}) := \pi_{n*}\pi_n^* \mathscr{F}$$

and the coface operators ∂^i are induced by the face morphisms ∂_i in the obvious way.

Lemma 10.3.15. (i) *The augmented complex* (10.3.14) *is aspherical for every $\mathscr{O}_X$-module $\mathscr{F}$.*

(ii) *If $\mathscr{F}$ is an injective $\mathscr{O}_X$-module, then* (10.3.14) *is homotopically trivial.*

Proof. (See also *e.g.* [79, Th.5.2.1].) The proof relies on the following alternative description of the cosimplicial Čech $\mathscr{O}_X$-module. Consider the adjoint pair :

$$(\pi^*, \pi_*) : \mathscr{O}_{\mathfrak{U}}\text{-}\mathbf{Mod} \rightleftarrows \mathscr{O}_X\text{-}\mathbf{Mod}$$

arising from our covering $\pi : \mathfrak{U} \to X$. Let $(\top := \pi_* \circ \pi^*, \eta, \mu)$ be the associated triple (see (7.10.3)), $\mathscr{F}$ any $\mathscr{O}_X$-module; we leave to the reader the verification that the resulting augmented cosimplicial complex $\mathscr{F} \to \top^\bullet\mathscr{F}$ – as defined in (7.10.2) – is none else than the augmented Čech complex (10.3.14). Thus, for every $\mathscr{O}_X$-module $\mathscr{F}$, the augmented complex $\pi^*\mathscr{F} \to \pi^*\mathscr{C}^\bullet(\mathfrak{U}, \mathscr{F})$ is homotopically trivial (proposition 7.10.5); since π is a covering morphism, (i) follows. Furthermore, by the same token, the augmented complex $\pi_*\mathscr{G} \to \mathscr{C}^\bullet(\mathfrak{U}, \pi_*\mathscr{G})$ is homotopically trivial for every $\mathscr{O}_{\mathfrak{U}}$-module $\mathscr{G}$; especially we may take $\mathscr{G} := \pi^*\mathscr{I}$, where $\mathscr{I}$ is an injective $\mathscr{O}_X$-module. On the other hand, when $\mathscr{I}$ is injective, the unit of adjunction $\mathscr{I} \to \top\mathscr{I}$ is split injective; hence the augmented complex $\mathscr{I} \to \mathscr{C}^\bullet(\mathfrak{U}, \mathscr{I})$ is a direct summand of the homotopically trivial complex $\top\mathscr{I} \to \mathscr{C}^\bullet(\mathfrak{U}, \top\mathscr{I})$, and (ii) follows. □

Notice now that the schemes $\mathfrak{U}_n$ are quasi-affine for every $n \in \mathbb{N}$, hence the functors $\mathsf{qcoh}_{\mathfrak{U}_n}$ are well defined as in (10.3.12), and indeed, the rule : $(\mathscr{F}_n \mid n \in \mathbb{N}) \mapsto (\mathsf{qcoh}_{\mathfrak{U}_n}\mathscr{F}_n \mid n \in \mathbb{N})$ yields a functor :

$$\mathsf{qcoh}_{\mathfrak{U}_\bullet} : \mathscr{O}_{\mathfrak{U}_\bullet}\text{-}\mathbf{Mod} \to \mathscr{O}_{\mathfrak{U}_\bullet}\text{-}\mathbf{Mod}.$$

This suggests to introduce a *quasi-coherent cosimplicial Čech $\mathscr{O}_X$-module* :

$$\mathsf{q}\mathscr{C}^\bullet(\mathfrak{U},\mathscr{F}) := \pi_{\bullet *}\circ \mathsf{qcoh}_{\mathfrak{U}_\bullet}\circ \pi_\bullet^*\mathscr{F}$$

for every $\mathscr{O}_X$-module $\mathscr{F}$ (regarded as a constant cosimplicial module in the usual way). More plainly, this is the cosimplicial $\mathscr{O}_X$-module such that :

$$\mathsf{q}\mathscr{C}^n(\mathfrak{U},\mathscr{F}) := \pi_{n*}\circ \mathsf{qcoh}_{\mathfrak{U}_n}\circ \pi_n^*\mathscr{F} \qquad \text{for every } n\in\mathbb{N}.$$

According to [59, Ch.I, Cor.9.2.2], the $\mathscr{O}_X$-modules $\mathsf{q}\mathscr{C}^n(\mathfrak{U},\mathscr{F})$ are quasi-coherent for all $n\in\mathbb{N}$. Finally, we define the functor :

$$\mathsf{qcoh}_X : \mathscr{O}_X\text{-}\mathbf{Mod}\to\mathscr{O}_X\text{-}\mathbf{Mod}_{\mathrm{qcoh}} \qquad \mathscr{F}\mapsto \mathrm{Equal}(\mathsf{q}\mathscr{C}^0(\mathfrak{U},\mathscr{F}) \overset{\partial^0}{\underset{\partial^1}{\rightrightarrows}} \mathsf{q}\mathscr{C}^1(\mathfrak{U},\mathscr{F}))$$

Proposition 10.3.16. (i) *In the situation of* (10.3.13)*, the functor* qcoh_X *is right adjoint to* ι_X.

(ii) *Let Y be any other quasi-compact and quasi-separated scheme, $f : X\to Y$ any morphism. Then the induced diagram of functors:*

$$\begin{array}{ccc} \mathscr{O}_X\text{-}\mathbf{Mod} & \xrightarrow{f_*} & \mathscr{O}_Y\text{-}\mathbf{Mod} \\ {\scriptstyle \mathsf{qcoh}_X}\downarrow & & \downarrow{\scriptstyle \mathsf{qcoh}_Y} \\ \mathscr{O}_X\text{-}\mathbf{Mod}_{\mathrm{qcoh}} & \xrightarrow{f_*} & \mathscr{O}_Y\text{-}\mathbf{Mod}_{\mathrm{qcoh}} \end{array}$$

commutes up to a natural isomorphism of functors.

Proof. For every $n\in\mathbb{N}$ and every $\mathscr{O}_{\mathfrak{U}_n}$-module $\mathscr{H}$, the counit of the adjunction yields a natural map of $\mathscr{O}_{\mathfrak{U}_n}$-modules: $\mathsf{qcoh}_{\mathfrak{U}_n}\mathscr{H}\to\mathscr{H}$. Taking $\mathscr{H}$ to be $\pi_n^*\mathscr{F}$ on $\mathfrak{U}_n$ (for a given $\mathscr{O}_X$-module $\mathscr{F}$), these maps assemble to a morphism of cosimplicial $\mathscr{O}_X$-modules :

$$\mathsf{q}\mathscr{C}^\bullet(\mathfrak{U},\mathscr{F})\to\mathscr{C}^\bullet(\mathfrak{U},\mathscr{F}) \tag{10.3.17}$$

and it is clear that (10.3.17) is an isomorphism whenever $\mathscr{F}$ is quasi-coherent. Let now $\varphi : \mathscr{G}\to\mathscr{F}$ be a map of $\mathscr{O}_X$-modules, with $\mathscr{G}$ quasi-coherent; after applying the natural transformation (10.3.17) and forming equalizers, we obtain a commutative diagram :

$$\begin{array}{ccc} \mathsf{qcoh}_X\mathscr{G} & \xrightarrow{\sim} & \mathscr{G} \\ {\scriptstyle \mathsf{qcoh}_X\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ \mathsf{qcoh}_X\mathscr{F} & \longrightarrow & \mathscr{F} \end{array}$$

from which we see that the rule: $\varphi\mapsto\mathsf{qcoh}_X\varphi$ establishes a natural injection:

$$\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{G},\mathscr{F})\to\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{G},\mathsf{qcoh}_X\mathscr{F}) \tag{10.3.18}$$

and since $\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{G},(10.3.17))$ is an isomorphism of cosimplicial $\mathscr{O}_X$-modules, (10.3.18) is actually a bijection, whence (i).

(ii) is obvious, since both $\mathsf{qcoh}_Y\circ f_*$ and $f_*\circ\mathsf{qcoh}_X$ are right adjoint to $f^*\circ\iota_Y=\iota_X\circ f^*$. □

Remark 10.3.19. The right adjoint functor qcoh_X was first constructed in [20, Exp.II, Lemma 3.2], for X quasi-compact and quasi-separated; a construction valid for all schemes X is found in [107, Tag 0781].

10.3.20. In the situation of (10.3.13), the functor qcoh_X is left exact, since it is a right adjoint, hence it gives rise to a left derived functor :

$$R\mathsf{qcoh}_X : \mathsf{D}^+(\mathscr{O}_X\text{-}\mathbf{Mod}) \to \mathsf{D}^+(\mathscr{O}_X\text{-}\mathbf{Mod}_{\mathrm{qcoh}}).$$

Proposition 10.3.21. *Let X be a quasi-compact and quasi-separated scheme. Then :*

(i) *$R\mathsf{qcoh}_X$ is right adjoint to $R\iota_X$.*
(ii) *Suppose moreover, that X is* semi-separated, i.e. *such that the diagonal morphism $X \to X \times_{\mathbb{Z}} X$ is affine. Then the unit of the adjunction $(R\iota_X, R\mathsf{qcoh}_X)$ is an isomorphism of functors.*

Proof. (i): The exactness of the functor ι_X implies that qcoh_X preserves injectives; the assertion is a formal consequence : the details shall be left to the reader.

(ii): Let $\mathscr{F}^\bullet$ be any complex of quasi-coherent $\mathscr{O}_X$-modules; we have to show that the natural map $\mathscr{F}^\bullet \to R\mathsf{qcoh}_X\mathscr{F}^\bullet$ is an isomorphism. Using a Cartan-Eilenberg resolution $\mathscr{F}^\bullet \xrightarrow{\sim} \mathscr{I}^{\bullet\bullet}$ we deduce a spectral sequence ([163, §5.7])

$$E_1^{pq} := R^p\mathsf{qcoh}_X H^q\mathscr{F}^\bullet \Rightarrow R^{p+q}\mathsf{qcoh}_X\mathscr{F}^\bullet$$

which easily reduces to checking the assertion for the cohomology of $\mathscr{F}^\bullet$, so we may assume from start that $\mathscr{F}^\bullet$ is a single $\mathscr{O}_X$-module placed in degree zero. Let us then choose an injective resolution $\mathscr{F} \xrightarrow{\sim} \mathscr{I}^\bullet$ (that is, in the category of all $\mathscr{O}_X$-modules); we have to show that $H^p\mathsf{qcoh}_X\mathscr{I}^\bullet = 0$ for $p > 0$. We deal first with the following special case :

Claim 10.3.22. Assertion (i) holds if X is affine.

Proof of the claim. Indeed, in this case, the chosen injective resolution of $\mathscr{F}$ yields a long exact sequence $0 \to \mathscr{F}(X) \to \mathscr{I}^\bullet(X)$, and therefore a resolution $\mathsf{qcoh}_X\mathscr{F} := \mathscr{F}(X)^\sim \to \mathsf{qcoh}_X\mathscr{I}^\bullet := \mathscr{I}^\bullet(X)^\sim$. ◊

For the general case, we choose any affine covering $\mathfrak{U}$ of X and we consider the cochain complex of cosimplicial complexes $\mathsf{q}\mathscr{C}^\bullet(\mathfrak{U}, \mathscr{I}^\bullet)$.

Claim 10.3.23. For any injective $\mathscr{O}_X$-module $\mathscr{I}$, the augmented complex:

$$\mathsf{qcoh}_X\mathscr{I} \to \mathsf{q}\mathscr{C}^\bullet(\mathfrak{U}, \mathscr{I})$$

is homotopically trivial.

Proof of the claim. It follows easily from proposition 10.3.16(ii) that

$$\mathsf{q}\mathscr{C}^\bullet(\mathfrak{U}, \mathscr{F}) \simeq \mathsf{qcoh}_X(\mathscr{C}^\bullet(\mathfrak{U}, \mathscr{F}))$$

for any $\mathscr{O}_X$-module $\mathscr{F}$. Then the claim follows from lemma 10.3.15(ii). ◊

We have a spectral sequence :

$$E_1^{pq} := H^p\mathsf{q}\mathscr{C}^\bullet(\mathfrak{U}, \mathscr{I}^q) \Rightarrow \mathrm{Tot}^{p+q}(\mathsf{q}\mathscr{C}^\bullet(\mathfrak{U}, \mathscr{I}^\bullet))$$

and it follows from claim 10.3.23 that $E_1^{pq} = 0$ whenever $p > 0$, and $E^{0q} = \mathsf{qcoh}_X\mathscr{I}^q$ for every $q \in \mathbb{N}$, so the spectral sequence $E^{\bullet\bullet}$ degenerates, and we deduce a quasi-isomorphism

$$\mathsf{qcoh}_X\mathscr{I}^\bullet \xrightarrow{\sim} \mathrm{Tot}^\bullet(\mathsf{q}\mathscr{C}^\bullet(\mathfrak{U}, \mathscr{I}^\bullet)). \tag{10.3.24}$$

On the other hand, for fixed $q \in \mathbb{N}$, we have $\mathsf{q}\mathscr{C}^q(\mathfrak{U}, \mathscr{I}^\bullet) = \pi_{n*}\mathsf{qcoh}_{\mathfrak{U}_n}\pi_n^*\mathscr{I}^\bullet$; since X is semi-separated, $\mathfrak{U}_n$ is affine, so the complex $\mathsf{qcoh}_{\mathfrak{U}_n}\pi_n^*\mathscr{I}^\bullet$ is a resolution of $\mathsf{qcoh}_{\mathfrak{U}_n}\pi_n^*\mathscr{F} = \pi_n^*\mathscr{F}$, by claim 10.3.22. Furthermore, $\pi_n : \mathfrak{U}_n \to X$ is an affine morphism, so $\mathsf{q}\mathscr{C}^q(\mathfrak{U}, \mathscr{I}^\bullet)$ is a resolution of $\pi_{n*}\pi_n^*\mathscr{F}$. Summing up, we see that $\mathrm{Tot}^\bullet(\mathsf{q}\mathscr{C}^\bullet(\mathfrak{U}, \mathscr{I}^\bullet))$ is quasi-isomorphic to $\mathscr{C}^\bullet(\mathfrak{U}, \mathscr{F})$, which is a resolution of $\mathscr{F}$, by lemma 10.3.15(i). Combining with (10.3.24), we deduce (ii). □

Theorem 10.3.25. *Let X be a quasi-compact semi-separated scheme. The forgetful functor*

$$R\iota_X : \mathsf{D}^+(\mathscr{O}_X\text{-}\mathbf{Mod}_{\mathrm{qcoh}}) \to \mathsf{D}^+(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$$

is an equivalence of categories, whose quasi-inverse is the restriction of $R\mathsf{qcoh}_X$.

Proof. By proposition 10.3.21(ii) we know already that the composition $R\mathsf{qcoh}_X \circ R\iota_X$ is a self-equivalence of $\mathsf{D}^+(\mathscr{O}_X\text{-}\mathbf{Mod}_{\mathrm{qcoh}})$. For every complex $\mathscr{F}^\bullet$ in $\mathsf{D}^+(\mathscr{O}_X\text{-}\mathbf{Mod})$, the counit of adjunction $\varepsilon_{\mathscr{F}^\bullet} : R\iota_X \circ R\mathsf{qcoh}_X\mathscr{F}^\bullet \to \mathscr{F}^\bullet$ can be described as follows. Pick an injective resolution $\alpha : \mathscr{F}^\bullet \to \mathscr{I}^\bullet$; then $\varepsilon_{\mathscr{F}^\bullet}$ is defined by the diagram :

$$\mathsf{qcoh}_X\mathscr{I}^\bullet \xrightarrow{\varepsilon^\bullet} \mathscr{I}^\bullet \xleftarrow{\alpha} \mathscr{F}^\bullet$$

where, for each $n \in \mathbb{N}$, the map $\varepsilon^n : \mathsf{qcoh}_X\mathscr{I}^n \to \mathscr{I}^n$ is the counit of the adjoint pair $(\iota_X, \mathsf{qcoh}_X)$. It suffices then to show that $\varepsilon^\bullet$ is a quasi-isomorphism, when $\mathscr{I}^\bullet$ is an object of $\mathsf{D}^+(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$. To this aim, we may further choose $\mathscr{I}^\bullet$ of the form $\mathrm{Tot}^\bullet(\mathscr{I}^{\bullet\bullet})$, where $\mathscr{I}^{\bullet\bullet}$ is a Cartan-Eilenberg resolution of $\mathscr{F}^\bullet$ (see [163, §5.7]). We then deduce a spectral sequence :

$$E_2^{pq} := R^p\mathsf{qcoh}_X H^q\mathscr{F}^\bullet \Rightarrow R^{p+q}\mathsf{qcoh}_X\mathscr{F}^\bullet.$$

The double complex $\mathscr{I}^{\bullet\bullet}$ also gives rise to a similar spectral sequence F_2^{pq}, and clearly $F_2^{pq} = 0$ whenever $p > 0$, and $F^{0q} = H^q\mathscr{F}^\bullet$. Furthermore, the counit of adjunction $\varepsilon^{\bullet\bullet} : \mathsf{qcoh}_X\mathscr{I}^{\bullet\bullet} \to \mathscr{I}^{\bullet\bullet}$ induces a morphism of spectral sequences $\omega^{pq} : E_2^{pq} \to F_2^{pq}$. Consequently, in order to prove that the $\varepsilon_{\mathscr{F}^\bullet}$ is a quasi-isomorphism, it suffices to show that ω^{pq} is an isomorphism for every $p, q \in \mathbb{N}$. This comes down to the assertion that $R^p\mathsf{qcoh}_X\mathscr{G} = 0$ for every quasi-coherent $\mathscr{O}_X$-module $\mathscr{G}$, and every $p > 0$. However, we have $\mathscr{G} = R\iota_X\mathscr{G}$, so that $R\mathsf{qcoh}_X\mathscr{G} = R\mathsf{qcoh}_X \circ R\iota_X\mathscr{G}$, and then the contention follows from proposition 10.3.21(ii). ☐

Remark 10.3.26. Theorem 10.3.25 appeared in [158, Prop.B.16] and [20, Exp.II, Prop.3.5] (the latter is stated for separated schemes, but the proof works in the semi-separated case as well). The theorem holds also for any locally noetherian scheme X, by virtue of [87, Ch.II, Cor.7.19] (whose proof uses the structure of injective $\mathscr{O}_X$-modules, rather than the functor qcoh_X).

Lemma 10.3.27. *Let X be a quasi-compact and quasi-separated scheme, U a quasi-compact open subset of X, $\mathscr{H}$ a quasi-coherent $\mathscr{O}_X$-module, $\mathscr{G}$ a finitely presented quasi-coherent $\mathscr{O}_U$-module, and $\varphi : \mathscr{G} \to \mathscr{H}_{|U}$ an $\mathscr{O}_U$-linear map. Then:*

- (i) *There exist a finitely presented quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ on X, and a $\mathscr{O}_X$-linear map $\psi : \mathscr{F} \to \mathscr{H}$, such that $\mathscr{F}_{|U} = \mathscr{G}$ and $\psi_{|U} = \varphi$.*
- (ii) *Especially, every finitely presented quasi-coherent $\mathscr{O}_U$-module extends to a finitely presented quasi-coherent $\mathscr{O}_X$-module.*

Proof. (i): Let $(V_i \mid i = 1, \dots, n)$ be a finite affine open covering of X. For every $i = 0, \dots, n$, let us set $U_i := U \cup V_1 \cup \cdots \cup V_i$; we construct, by induction on i, a family of finitely presented quasi-coherent $\mathscr{O}_{U_i}$-modules $\mathscr{F}_i$, and morphisms $\psi_i : \mathscr{F}_i \to \mathscr{H}_{|U_i}$ such that $\mathscr{F}_{i+1|U_i} = \mathscr{F}_i$ and $\psi_{i+1|U_{i+1}} = \psi_i$ for every $i < n$. For $i = 0$ we have $U_0 = U$, and we set $\mathscr{F}_0 := \mathscr{G}$, $\psi_0 := \varphi$. Suppose that $i < n$ and that $\mathscr{F}_i$ and ψ_i have already been given. Since X is quasi-separated, the same holds for U_{i+1}, and the immersion $j : U_i \to U_{i+1}$ is quasi-compact; it follows that $j_*\mathscr{F}_i$ and $j_*\mathscr{H}_{|U_i}$ are quasi-coherent $\mathscr{O}_{U_i}$-modules ([59, Ch.I, Prop.9.4.2(i)]). We let

$$\mathscr{M} := j_*\mathscr{F}_i \times_{j_*\mathscr{H}_{|U_i}} \mathscr{H}_{|U_{i+1}}.$$

Then $\mathscr{M}$ is a quasi-coherent $\mathscr{O}_{U_{i+1}}$-module admitting a map $\mathscr{M} \to \mathscr{H}_{|U_{i+1}}$, and such that $\mathscr{M}_{|U_i} = \mathscr{F}_i$. We can then find a filtered family of quasi-coherent $\mathscr{O}_{V_{i+1}}$-modules of finite presentation $(\mathscr{M}_\lambda \mid \lambda \in \Lambda)$, whose colimit is $\mathscr{M}_{|V_{i+1}}$. Since $\mathscr{F}_i$ is finitely presented and $U_i \cap V_{i+1}$ is quasi-compact, there exists $\lambda \in \Lambda$ such that the induced morphism $\beta : \mathscr{M}_{\lambda|U_i \cap V_{i+1}} \to \mathscr{F}_{i|U_i \cap V_{i+1}}$ is an epimorphism. Notice moreover that the restriction of $\mathscr{K} := \mathrm{Ker}(\mathscr{M}_\lambda \to \mathscr{M}_{|V_{i+1}})$ to $U_i \cap V_{i+1}$

coincides with $\mathrm{Ker}(\beta)$; since $\mathrm{Ker}(\beta)$ is of finite type, and since $U_i \cap V_{i+1}$ is quasi-compact, there exists an $\mathscr{O}_{V_{i+1}}$-submodule of finite type $\mathscr{N}$ of $\mathscr{K}$ such that $\mathscr{N}_{|U_i\cap V_{i+1}} = \mathrm{Ker}(\beta)$. Set $\overline{\mathscr{M}}_\lambda := \mathscr{M}_\lambda/\mathscr{N}$; then β descends to an isomorphism $\overline{\beta} : \overline{\mathscr{M}}_\lambda \xrightarrow{\sim} \mathscr{F}_{i|U_i\cap V_{i+1}}$, and the map $\mathscr{M}_\lambda \to \mathscr{M}_{|V_{i+1}}$ factors through $\overline{\mathscr{M}}_\lambda$. We can thus define $\mathscr{F}_{i+1}$ by gluing $\mathscr{F}_i$ and $\overline{\mathscr{M}}_\lambda$ along $\overline{\beta}$; likewise, ψ_i and the induced map $\overline{\mathscr{M}}_\lambda \to \mathscr{M}_{|V_{i+1}} \to \mathscr{H}_{|V_{i+1}}$ glue to a map ψ_{i+1} as required. Clearly the pair $(\mathscr{F} := \mathscr{F}_n, \psi := \psi_n)$ is the sought extension of $(\mathscr{G}, \varphi)$.

(ii): Let 0_X be the final object in the category of $\mathscr{O}_X$-modules; to extend a finitely presented quasi-coherent $\mathscr{O}_U$-module $\mathscr{G}$, it suffices to apply (i) to the unique map $\mathscr{G} \to 0_{X|U}$. □

For ease of reference, we point out the following simple consequence of lemma 10.3.27.

Corollary 10.3.28. *Let U be a quasi-affine scheme, $\mathscr{E}$ a locally free $\mathscr{O}_U$-module of finite type. Then we may find integers $n, m \in \mathbb{N}$ and a left exact sequence of $\mathscr{O}_U$-modules :*

$$0 \to \mathscr{E} \to \mathscr{O}_U^{\oplus m} \to \mathscr{O}_U^{\oplus n}.$$

Proof. Set $\mathscr{E}^\vee := \mathscr{H}om_{\mathscr{O}_U}(\mathscr{E}, \mathscr{O}_U)$, and notice that $\mathscr{E}^\vee$ is a locally free $\mathscr{O}_U$-module of finite type, especially it is quasi-coherent and of finite presentation. By assumption, U is a quasi-compact open subset of an affine scheme X, hence $\mathscr{E}^\vee$ extends to a quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ of finite presentation (lemma 10.3.27(ii)); we have $\mathscr{F} = F^\sim$, for some finitely presented A-module F; we choose a presentation of F as the cokernel of an A-linear map $A^{\oplus n} \to A^{\oplus m}$, for some $m, n \in \mathbb{N}$, whence a presentation of $\mathscr{E}^\vee$ as the cokernel of a morphism $\mathscr{O}_U^{\oplus n} \to \mathscr{O}_U^{\oplus m}$, and after dualizing again, we get the sought left exact sequence. □

10.3.29. Let X be a scheme; for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, we consider the full subcategory $C/\mathscr{F}$ of the category $\mathscr{O}_X\text{-}\mathbf{Mod}_{\mathrm{qcoh}}/\mathscr{F}$ whose objects are all the maps $\varphi : \mathscr{G} \to \mathscr{F}$ such that $\mathscr{G}$ is a finitely presented $\mathscr{O}_X$-module (notation of (1.1.24)). We denote by

$$\iota_{\mathscr{F}} : C/\mathscr{F} \to \mathscr{O}_X\text{-}\mathbf{Mod}_{\mathrm{qcoh}} \qquad (\mathscr{G} \to \mathscr{F}) \mapsto \mathscr{G}$$

the restriction of the source functor. The first observation is:

Lemma 10.3.30. *$C/\mathscr{F}$ is a filtered category.*

Proof. Since $C/\mathscr{F}$ always admits the initial object $0 \to \mathscr{F}$, its set of objects is non-empty. Thus, according to remark 1.2.21(i), it suffices to check that $C/\mathscr{F}$ is directed and satisfies the coequalizing condition. However, say that $\varphi_i : \mathscr{G}_i \to \mathscr{F}$ $(i = 1, 2)$ are any two objects of $C/\mathscr{F}$; then obviously there is a unique morphism $\mathscr{G}_1 \oplus \mathscr{G}_2 \to \mathscr{F}$ whose restriction to $\mathscr{G}_i$ agrees with φ_i for $i = 1, 2$.

Next, suppose that $\beta_1, \beta_2 : \mathscr{G} \to \mathscr{G}'$ are two morphisms of finitely presented $\mathscr{O}_X$-modules and $\psi : \mathscr{G}' \to \mathscr{F}$ an object of $C/\mathscr{F}$ such that $\psi \circ \beta_1 = \psi \circ \beta_2$. Set $\mathscr{G}'' := \mathrm{Coker}\,(\beta_1 - \beta_2)$; it is easily seen that $\mathscr{G}''$ is a finitely presented $\mathscr{O}_X$-module and ψ factors through a unique morphism $\mathscr{G}'' \to \mathscr{F}$, whence the lemma. □

Proposition 10.3.31. *With the notation of* (10.3.29)*, suppose that X is quasi-compact and quasi-separated. Then, for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, the induced map*

$$\operatorname*{colim}_{C/\mathscr{F}} \iota_{\mathscr{F}} \to \mathscr{F}$$

is an isomorphism.

Proof. Let $\mathscr{F}'$ be such colimit; clearly there is a natural map $\mathscr{F}' \to \mathscr{F}$, and we have to show that it is an isomorphism. To this aim, we can check on the stalk over the points $x \in X$, hence we come down to showing :

Claim 10.3.32. Let $\mathscr{F}$ be any quasi-coherent $\mathscr{O}_X$-module. Then :

(i) For every $s \in \mathscr{F}_x$ there exist a finitely presented quasi-coherent $\mathscr{O}_X$-module $\mathscr{G}$, a morphism $\varphi : \mathscr{G} \to \mathscr{F}$ and $t \in \mathscr{G}_x$ such that $\varphi_x(t) = s$.

(ii) For every map $\varphi : \mathscr{G} \to \mathscr{F}$ with $\mathscr{G}$ finitely presented and quasi-coherent, and every $s \in \operatorname{Ker} \varphi_x$, there exists a commutative diagram of quasi-coherent $\mathscr{O}_X$-modules :

$$\begin{array}{ccc} \mathscr{G} & & \\ {\scriptstyle\psi}\downarrow & \searrow{\scriptstyle\varphi} & \\ \mathscr{H} & \longrightarrow & \mathscr{F} \end{array}$$

with $\mathscr{H}$ finitely presented and $\psi_x(s) = 0$.

Proof of the claim. (i): Let $U \subset X$ be an open subset such that s extends to a section $s_U \in \mathscr{F}(U)$; we deduce a map $\varphi_U : \mathscr{O}_U \to \mathscr{F}_{|U}$ by the rule $a \mapsto a \cdot s_U$ for every $a \in \mathscr{O}_U(U)$. In view of lemma 10.3.27(i), the pair $(\varphi_U, \mathscr{O}_U)$ extends to a pair $(\varphi, \mathscr{G})$ with the sought properties.

(ii): We apply (i) to the quasi-coherent $\mathscr{O}_X$-module $\operatorname{Ker} \varphi$, to find a finitely presented quasi-coherent $\mathscr{O}_X$-module $\mathscr{G}'$, a morphism $\beta : \mathscr{G}' \to \operatorname{Ker} \varphi$ and $t \in \mathscr{H}_x$ such that $\beta_x(t) = s$. Then we let $\psi : \mathscr{G} \to \mathscr{H} := \operatorname{Coker}(\mathscr{G}' \xrightarrow{\beta} \operatorname{Ker} \varphi \to \mathscr{G})$ be the natural map. By construction, $\mathscr{H}$ is finitely presented, $\psi_x(s) = 0$ and clearly φ factors through ψ. □

10.3.33. Let X be a coherent scheme, $U \subset X$ a quasi-compact open subset, $a, b \in \mathbb{Z}$ any two integers with $a \leq b$, and $\mathscr{H}^\bullet$ any object of $\mathsf{D}^{[a,b]}(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$ such that $\mathscr{H}^\bullet_{|U}$ lies in $\mathsf{D}^{[a,b]}(\mathscr{O}_U\text{-}\mathbf{Mod})_{\mathrm{coh}}$. Let also $\mathscr{F}_i \subset H^i\mathscr{H}^\bullet$ be a quasi-coherent $\mathscr{O}_X$-submodule of finite type, for every $i = a, \dots, b$.

Proposition 10.3.34. *In the situation of* (10.3.33)*, the following holds :*

(i) *There exists an object $\mathscr{G}^\bullet$ of $\mathsf{D}^{[a,b]}(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}}$, with a morphism $\varphi^\bullet : \mathscr{G}^\bullet \to \mathscr{H}^\bullet$ in $\mathsf{D}(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$ such that :*
  (a) *$\varphi^\bullet_{|U}$ is an isomorphism.*
  (b) *$\mathscr{F}_i \subset \operatorname{Im} H^i\varphi^\bullet$ for every $i = a, \dots, b$.*

(ii) *If X is noetherian, we can find $\varphi^\bullet$ as in* (i) *such that $H^i\varphi^\bullet$ is a monomorphism, for every $i = a, \dots, b$.*

Proof. To start out, for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ such that $\mathscr{F}_{|U}$ is a coherent $\mathscr{O}_U$-module, define the category $C/\mathscr{F}$ as in (10.3.29), and consider the full subcategory $C'/\mathscr{F}$ (resp. $C''/\mathscr{F}$) of $C/\mathscr{F}$ whose objects are all the maps $\varphi : \mathscr{G} \to \mathscr{F}$ such that $\varphi_{|U}$ is an isomorphism (resp. an epimorphism).

Claim 10.3.35. $C'/\mathscr{F}$ and $C''/\mathscr{F}$ are filtered categories.

Proof of the claim. The proof of lemma 10.3.30 applies *verbatim* to show that both $C'/\mathscr{F}$ and $C''/\mathscr{F}$ satisfy the coequalizing condition of definition 1.2.19(v), and it also proves that $C''/\mathscr{F}$ is directed. Moreover, lemma 10.3.27(i) says that the sets of objects of these two categories are non-empty, which already implies that $C''/\mathscr{F}$ is filtered, by remark 1.2.21(i). By the same token, it remains only to check that $C'/\mathscr{F}$ satisfies the coequalizing condition. Hence, let $\varphi_i : \mathscr{G}_i \to \mathscr{F}$ $(i = 1, 2)$ be any two objects of $C'/\mathscr{F}$, and set $\mathscr{G}' := \mathscr{G}_1 \times_{\mathscr{F}} \mathscr{G}_2$; then $\mathscr{G}'$ is a quasi-coherent $\mathscr{O}_X$-module, and $\mathscr{G}'_{|U}$ is isomorphic to $\mathscr{F}_{|U}$, especially it is coherent. Then pick any object $\psi : \mathscr{G}'' \to \mathscr{G}'$ of the category $C''/\mathscr{G}'$, and denote by $\psi_i : \mathscr{G}'' \to \mathscr{G}_i$ $(i = 1, 2)$ the two morphisms induced by ψ; by construction we have $\varphi_1 \circ \psi_1 = \varphi_2 \circ \psi_2$. Hence, set $\mathscr{G} := \mathscr{G}_1 \amalg_{\mathscr{G}''} \mathscr{G}_2$; we get a unique morphism $\varphi : \mathscr{G} \to \mathscr{F}$ such that both φ_1 and φ_2 factor through φ and $\varphi_{|U}$ is an isomorphism. Since $\mathscr{G}$ is a finitely presented $\mathscr{O}_X$-module, we are done. ◇

Claim 10.3.36. With the foregoing notation, the following holds :

(i) $C'/\mathscr{F}$ is a cofinal subcategory of $C/\mathscr{F}$.
(ii) For every object $K^\bullet$ of $\mathsf{D}^b(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}}$ and every $c \in \mathbb{Z}$, the natural map

$$\operatorname*{colim}_{C/\mathscr{F}} \mathrm{Hom}_{\mathsf{D}(\mathscr{O}_X\text{-}\mathbf{Mod})}(K^\bullet, \iota_{\mathscr{F}}[c]) \to \mathrm{Hom}_{\mathsf{D}(\mathscr{O}_X\text{-}\mathbf{Mod})}(K^\bullet, \mathscr{F}[c]) \tag{10.3.37}$$

is an isomorphism (notation of (10.3.29)).

Proof of the claim. (i): Since U is quasi-compact, it is easily seen that $C''/\mathscr{F}$ is cofinal in $C/\mathscr{F}$, so we are reduced to checking that $C'/\mathscr{F}$ is cofinal in $C''/\mathscr{F}$. Hence, let $\psi : \mathscr{G}' \to \mathscr{F}$ be any object of $C''/\mathscr{F}$; since $\mathscr{O}_X$ is coherent, $\mathscr{K} := \mathrm{Ker}\,\psi_{|U}$ is a coherent $\mathscr{O}_U$-module, therefore we may extend $\mathscr{K}$ to a coherent $\mathscr{O}_X$-module $\mathscr{K}'$ and the identity map of $\mathscr{K}$ to a morphism $\psi' : \mathscr{K}' \to \mathrm{Ker}\,\psi$ of $\mathscr{O}_X$-modules (lemma 10.3.27(i)). Set $\mathscr{G} := \mathscr{G}'/\psi'(\mathscr{K}')$; the induced map $\mathscr{G} \to \mathscr{F}$ is an object of $C'/\mathscr{F}$, whence the contention.

(ii): We have two spectral sequences

$$E_2^{pq} := \operatorname*{colim}_{C/\mathscr{F}} R^p\Gamma \circ R^q\mathscr{H}om^\bullet_{\mathscr{O}_X}(K^\bullet, \iota_{\mathscr{F}}[c]) \Rightarrow \operatorname*{colim}_{C/\mathscr{F}} \mathrm{Hom}_{\mathsf{D}(\mathscr{O}_X\text{-}\mathbf{Mod})}(K^\bullet, \iota_{\mathscr{F}}[c+p+q])$$
$$F_2^{pq} := R^p\Gamma \circ R^q\mathscr{H}om^\bullet_{\mathscr{O}_X}(K^\bullet, \mathscr{F}[c]) \Rightarrow \mathrm{Hom}_{\mathsf{D}(\mathscr{O}_X\text{-}\mathbf{Mod})}(K^\bullet, \mathscr{F}[c+p+q])$$

and a morphism of spectral sequences $E_\bullet^{\bullet\bullet} \to F_\bullet^{\bullet\bullet}$, such that (10.3.37) is a morphism of filtered abelian groups, for the two finite filtrations induced by these spectral sequences on their abutments. Since the functors $R^p\Gamma$ commute with filtered colimits, we deduce that it suffices to show that the natural morphism

$$\operatorname*{colim}_{C/\mathscr{F}} R^q\mathscr{H}om^\bullet_{\mathscr{O}_X}(K^\bullet, \iota_{\mathscr{F}}[c])) \to R^q\mathscr{H}om^\bullet_{\mathscr{O}_X}(K^\bullet, \mathscr{F}[c])$$

is an isomorphism for every $q \in \mathbb{Z}$. Then, a standard *dévissage* argument further reduces to the case where $K^\bullet$ is concentrated in a single degree, so we come down to checking that the functor $\mathscr{F} \mapsto R^q\mathscr{H}om^\bullet_{\mathscr{O}_X}(\mathscr{G}, \mathscr{F})$ commutes with filtered colimits of quasi-coherent $\mathscr{O}_X$-modules, for every coherent $\mathscr{O}_X$-module $\mathscr{G}$ and every $q \in \mathbb{Z}$. To this aim we may assume that X is affine, in which case – since $\mathscr{O}_X$ is coherent – $\mathscr{G}$ admits a resolution $\mathscr{L}^\bullet$ consisting of free $\mathscr{O}_X$-modules of finite rank; the latter are acyclic for the functor $\mathscr{H}om_{\mathscr{O}_X}$, so we are left with the assertion that the functor $\mathscr{F} \mapsto H^q\mathscr{H}om_{\mathscr{O}_X}(\mathscr{L}^\bullet, \mathscr{F})$ commutes with filtered colimits, which is clear. ◊

Claim 10.3.38. The proposition holds if $a = b$.

Proof of the claim. From proposition 10.3.31 and claim 10.3.36(i) we deduce that, for every $x \in X$ we may find an object $\varphi : \mathscr{G} \to H^a\mathscr{H}$ such that $\mathscr{F}_{a,x} \subset \mathrm{Im}\,\varphi_x$, and therefore $\mathscr{F}_{a|U_x} \subset \mathrm{Im}\,\varphi_{|U_x}$ for some open neighborhood U_x of x in X. Since X is quasi-compact and $C/\mathscr{F}$ is filtered, assertion (i) follows easily. Next, suppose that a morphism $\varphi : \mathscr{G} \to H^a\mathscr{H}$ has already been exhibited that fulfills conditions (i.a) and (i.b), and set $\mathscr{G}' := \mathrm{Im}\,\varphi$; to show assertion (ii), it suffices to remark that if X is noetherian, $\mathscr{G}'$ is still a coherent $\mathscr{O}_X$-module. ◊

We proceed now by induction on $b - a$, the case where $a = b$ having been covered by claim 10.3.38. Thus, suppose that $a < b$, and that the proposition is already known for every object of $\mathsf{D}^{[a+1,b]}(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$ fulfilling the stated condition. We have a distinguished triangle

$$\mathscr{F}[-a] \to \mathscr{H}^\bullet \to \tau^{\geq a+1}\mathscr{H}^\bullet \xrightarrow{\alpha^\bullet} \mathscr{F}[1-a] \qquad \text{with } \mathscr{F} := H^a\mathscr{H}^\bullet$$

in $\mathsf{D}(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$ (where $\tau^{\geq a+1}$ denotes the truncation functor : see remark 7.3.14(iv)). By inductive assumption, we may find an object $\mathscr{G}'^\bullet$ of $\mathsf{D}^{[a+1,b]}(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}}$ and a morphism $\varphi'^\bullet : \mathscr{G}'^\bullet \to \tau^{\geq a+1}\mathscr{H}^\bullet$ in $\mathsf{D}(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$ which restricts to an isomorphism on U and such that $\mathscr{F}_i \subset \mathrm{Im}\,H^i\varphi'^\bullet$ for every $i = a+1, \dots, b$. There follows a morphism of distinguished triangles :

$$\begin{array}{ccccccc} \mathscr{F}[-a] & \longrightarrow & \mathscr{H}'^\bullet & \longrightarrow & \mathscr{G}'^\bullet & \longrightarrow & \mathscr{F}[1-a] \\ \| & & \downarrow{\scriptstyle\beta^\bullet} & & \downarrow{\scriptstyle\varphi'^\bullet} & & \| \\ \mathscr{F}[-a] & \longrightarrow & \mathscr{H}^\bullet & \longrightarrow & \tau^{\geq a+1}\mathscr{H}^\bullet & \xrightarrow{\alpha^\bullet} & \mathscr{F}[1-a] \end{array}$$

for some object $\mathscr{H}'^\bullet$ of $\mathsf{D}^{[a,b]}(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$, and clearly $\beta^\bullet$ restricts to an isomorphism on U and $H^a\beta^\bullet$ is an isomorphism. We may then replace $\mathscr{H}^\bullet$ by $\mathscr{H}'^\bullet$, and assume that $\tau^{\geq a+1}\mathscr{H}^\bullet$ lies in $\mathsf{D}^{[a+1,b]}(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}}$.

By claim 10.3.36, we may find an object $\psi : \mathscr{G} \to \mathscr{F}$ of $C'/\mathscr{F}$ such that $\alpha^\bullet$ factors through $\psi[1-a]$ and a morphism $\omega^\bullet : \tau^{\geq a+1}\mathscr{H}^\bullet \to \mathscr{G}[1-a]$ in $\mathsf{D}(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}}$. Moreover, by claim 10.3.38 we may assume that $\mathscr{F}_a \subset \mathrm{Im}\,\psi$, and also that ψ is a monomorphism, if X is noetherian. Then $\mathrm{Cone}\,\omega^\bullet[-1]$ lies in $\mathsf{D}^{[a,b]}(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}}$, and the induced morphism $\mathrm{Cone}\,\omega^\bullet[-1] \to \mathscr{H}^\bullet$ will do. □

Corollary 10.3.39. *Let X be a coherent scheme, $U \subset X$ a quasi-compact open subset. Then the induced functor*

$$\mathsf{D}^{[a,b]}(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}} \to \mathsf{D}^{[a,b]}(\mathscr{O}_U\text{-}\mathbf{Mod})_{\mathrm{coh}}$$

is essentially surjective, for every $a, b \in \mathbb{Z}$ with $a \leq b$.

Proof. Let $j : U \to X$ be the open immersion, $\mathscr{F}^\bullet$ an object of $\mathsf{D}^{[a,b]}(\mathscr{O}_U\text{-}\mathbf{Mod})_{\mathrm{coh}}$. By [61, Ch.III, Prop.1.4.10, Cor.1.4.12] and a standard spectral sequence argument, it is easily seen that

$$\mathscr{H}^\bullet := \tau^{\geq a} \circ \tau^{\leq b} Rj_*\mathscr{F}^\bullet$$

lies in $\mathsf{D}^{[a,b]}(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$ and $\mathscr{H}^\bullet_{|U} = \mathscr{F}^\bullet$. Then the assertion follows immediately from proposition 10.3.34(i). □

10.4. **Depth and cohomology with supports.** This section introduces and studies local cohomology and the closely related notion of depth, in the context of arbitrary schemes (whereas the usual references restrict to the case of locally noetherian schemes).

10.4.1. To begin with, let $(X, \mathscr{A})$ be any ringed topological space, $i : Z \to X$ a closed immersion, set $U := X \setminus Z$ and denote by $j : U \to X$ the resulting open immersion. One defines the functor

$$\underline{\Gamma}_Z : \mathscr{A}\text{-}\mathbf{Mod} \to \mathscr{A}\text{-}\mathbf{Mod} \qquad \mathscr{F} \mapsto \mathrm{Ker}\,(\mathscr{F} \to j_*j^*\mathscr{F})$$

as well as its composition with the global section functor

$$\Gamma_Z : \mathscr{A}\text{-}\mathbf{Mod} \to \mathscr{A}(X)\text{-}\mathbf{Mod} \qquad \mathscr{F} \mapsto \Gamma(X, \underline{\Gamma}_Z\mathscr{F}).$$

It is clear that $\underline{\Gamma}_Z$ and Γ_Z are left exact functors, hence they give rise to right derived functors

$$R\underline{\Gamma}_Z : \mathsf{D}^+(\mathscr{A}\text{-}\mathbf{Mod}) \to \mathsf{D}^+(\mathscr{A}\text{-}\mathbf{Mod}) \qquad R\Gamma_Z : \mathsf{D}^+(\mathscr{A}\text{-}\mathbf{Mod}) \to \mathsf{D}^+(\mathscr{A}(X)\text{-}\mathbf{Mod})$$

such that the natural morphism $\underline{\Gamma}_Z\mathscr{F} \to R^0\underline{\Gamma}_Z\mathscr{F}$ is an isomorphism, for every $\mathscr{A}$-module $\mathscr{F}$ (and likewise for $R^0\Gamma_Z$). Moreover, suppose that $\mathscr{F} \to \mathscr{I}^\bullet$ is a resolution of $\mathscr{F}$ by a complex of injective $\mathscr{A}$-modules; since injective sheaves are flabby (lemma 10.1.5(v)), we obtain a short exact sequence of complexes

$$0 \to \underline{\Gamma}_Z\mathscr{I}^\bullet \to \mathscr{I}^\bullet \to j_*j^*\mathscr{I}^\bullet \to 0$$

whence a natural exact sequence of $\mathscr{A}$-modules :

(10.4.2) $$0 \to \underline{\Gamma}_Z\mathscr{F} \to \mathscr{F} \to j_*j^*\mathscr{F} \to R^1\underline{\Gamma}_Z\mathscr{F} \to 0 \qquad \text{for every } \mathscr{A}\text{-module } \mathscr{F}$$

and natural isomorphisms :

(10.4.3) $$R^{q-1}j_*j^*\mathscr{F} \xrightarrow{\sim} R^q\underline{\Gamma}_Z\mathscr{F} \qquad \text{for all } q > 1.$$

Lemma 10.4.4. *In the situation of* (10.4.1)*, let* $\mathscr{B} \to \mathscr{A}$ *be any morphism of sheaves of rings on* X*. The following holds :*

(i) *Every flabby* $\mathscr{A}$*-module is* $\underline{\Gamma}_Z$*-acyclic.*

(ii) *The functor* $R\underline{\Gamma}_Z$ *commutes with the forgetful functor* $\mathsf{D}^+(\mathscr{A}\text{-}\mathbf{Mod}) \to \mathsf{D}^+(\mathscr{B}\text{-}\mathbf{Mod})$.

(iii) *Suppose that* X *is locally coherent and quasi-separated, and that* Z *is a constructible closed subset of* X*. Then :*

(a) *Every qc-flabby* $\mathscr{A}$*-module is* $\underline{\Gamma}_Z$*-acyclic.*

(b) *For every* $i \in \mathbb{N}$ *the functor* $R^i\underline{\Gamma}_Z$ *commutes with filtered colimits of* $\mathscr{A}$*-modules.*

Proof. (i): First of all, if $\mathscr{F}$ is flabby on X, the natural morphism $\mathscr{F} \to j_*j^*\mathscr{F}$ is obviously an epimorphism; together with (10.4.2), this implies that $R^1\underline{\Gamma}_Z\mathscr{F} = 0$ when $\mathscr{F}$ is flabby. Next, (10.4.3), together with the fact that flabby $\mathscr{A}$-modules are acyclic for direct image functors such as j_* (lemma 10.1.5(ii)), implies that $R^i\underline{\Gamma}_Z\mathscr{F} = 0$ also for $i > 1$, whence the contention.

Assertion (iii.a) is checked in the same way : first one notices that, under the stated assumptions, the morphism $\mathscr{F} \to j_*j^*\mathscr{F}$ is an epimorphism also for all qc-flabby sheaves on X, which again gives the vanishing of $R^1\underline{\Gamma}_Z\mathscr{F}$; for the higher derived functors one then appeals to (10.4.3) and lemma 10.1.5(iv).

(iii.b): Suppose first that $i > 1$; in this case, the assertion follows from (10.4.3) and the following more general :

Claim 10.4.5. In the situation of (iii), the functor R^ij_* commutes with filtered colimits, for every $i \in \mathbb{N}$.

Proof of the claim. The claim can be checked at the stalks at the points $x \in X$, hence we may assume that X is quasi-compact, in which case the same holds for U. Now, on the one hand, $R^qj_*j^*\mathscr{F}$ is the sheaf associated with the presheaf given by the rule : $U' \mapsto H^q(U \cap U', \mathscr{F})$ for every open subset $U' \subset X$, so $(R^qj_*j^*\mathscr{F})_x$ is also the stalk at x of this presheaf. On the other hand, x admits a fundamental system of open neighborhoods consisting of quasi-compact open subsets, and if U' is such a neighborhood of x, the subset $U' \cap U$ is also quasi-compact, and the functor $\mathscr{F} \mapsto H^q(U \cap U', \mathscr{F})$ commutes with filtered colimits, by proposition 10.1.10(ii). The claim follows by combining these two observations. ◊

In case $i \leq 1$, we argue similarly with (10.4.2), which reduces to checking that both functors $\mathscr{F} \mapsto \mathscr{F}_x$ and $\mathscr{F} \mapsto (j_*j^*\mathscr{F})_x$ commute with filtered colimits; this is trivial for the former functor, and for the latter it is a special case of claim 10.4.5.

Lastly, (ii) is an immediate consequence of (i) and remark 7.3.31(vi), since any injective $\mathscr{A}$-module is a flabby $\mathscr{B}$-module (lemma 10.1.5(v)). □

10.4.6. In the situation of (10.4.1), let X' be the amalgamated sum in the following cocartesian diagram of topological spaces :

$$\begin{array}{ccc} U & \xrightarrow{\ j\ } & X \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j'_1} \\ X & \xrightarrow{\ j'_2\ } & X' \end{array}$$

and let $\pi : X' \to X$ be the unique continuous map with $\pi \circ j'_i = \mathbf{1}_X$ for $i = 1, 2$. We notice :

Proposition 10.4.7. *In the situation of* (10.4.6)*, let also* $\mathscr{F}$ *be any sheaf on* X*. Then we have :*

(i) *If* $\mathscr{F}$ *is flabby, the same holds for* $\pi^*\mathscr{F}$*.*

(ii) *If moreover* $\mathscr{F}$ *is an* $\mathscr{A}$*-module, the following holds :*

(a) *We have a natural isomorphism in* $\mathsf{D}^+(\mathscr{A}(X)\text{-}\mathbf{Mod})$ *:*

$$R\Gamma\mathscr{F} \oplus R\Gamma_Z\mathscr{F} \xrightarrow{\sim} R\Gamma(\pi^*\mathscr{F}).$$

(b) *If* X *is coherent and* Z *is constructible in* X*, then* $R^i\Gamma_Z\mathscr{F} = 0$ *for every* $i > \dim X$*.*

(c) *If X is spectral and Z is constructible in X, define $d_{\mathscr{F}}$ as in theorem* 10.1.35(ii). *Then*

$$R^i\Gamma_Z\mathscr{F} = 0 \qquad \textit{for every } i > d_{\mathscr{F}}.$$

Proof. (i): Set $V_i := j'_i(X)$ for $i = 1, 2$; then X' admits the open covering $X' = V_1 \cup V_2$, and the restriction of $\pi^*\mathscr{F}$ is flabby on V_1 and V_2. Then the assertion follows from lemma 10.1.2.

(ii.a): For $i = 1, 2$ and every $\mathscr{A}$-module $\mathscr{G}$, we have a natural isomorphism of $\mathscr{A}(X)$-modules

$$\psi_{\mathscr{G},i} : \Gamma(X, \mathscr{F}) \xrightarrow{\sim} \Gamma(V_i, \mathscr{G}).$$

Moreover, for every $(\sigma, \tau) \in \Gamma\mathscr{G} \oplus \Gamma_Z\mathscr{G}$ there exists a unique section $\varphi_{\mathscr{G}}(\sigma, \tau) \in \Gamma(\pi^*\mathscr{G})$ whose restriction to V_1 agrees with $\psi_{\mathscr{G},1}(\sigma)$ and whose restriction to V_2 agrees with $\psi_{\mathscr{G}_2}(\sigma + \tau)$ (details left to the reader). The rule $(\sigma, \tau) \mapsto \varphi_{\mathscr{G}}(\sigma, \tau)$ yields a natural isomorphism

$$\varphi_{\mathscr{G}} : \Gamma\mathscr{G} \oplus \Gamma_Z\mathscr{G} \xrightarrow{\sim} \Gamma(\pi^*\mathscr{G}).$$

Now, let $\mathscr{F} \to \mathscr{G}^\bullet$ be a resolution of $\mathscr{F}$ by a complex of injective $\mathscr{A}$-modules; by lemma 10.1.5(v), the complex $\pi^*\mathscr{G}^\bullet$ is a resolution of $\pi^*\mathscr{F}$ by flabby $\pi^*\mathscr{A}$-modules. Combining with lemma 10.4.4(i) and remark 7.3.31(vi), we deduce that $\varphi_{\mathscr{G}^\bullet} : \Gamma\mathscr{G}^\bullet \oplus \Gamma_Z\mathscr{G}^\bullet \xrightarrow{\sim} \Gamma(\pi^*\mathscr{G}^\bullet)$ is the sought natural isomorphism.

Lastly, if X is coherent (resp. spectral) and Z is constructible, the open subsets V_1 and V_2 are coherent (resp. spectral) and retro-compact in X'; by lemma 8.1.15(iv), in this case X' is coherent (resp. X' is spectral, and it is easily seen that $d_{\pi^*\mathscr{F}} = d_{\mathscr{F}}$). Then (ii.b) and (ii.c) follow directly from (ii.a) and theorem 10.1.35. □

10.4.8. In the situation of (10.1.8), suppose that the maps g and g_λ, for every $\lambda \in \mathrm{Ob}(\Lambda)$ are open immersions, and set

$$Z := X \backslash Y \qquad Z_\lambda := X_\lambda \backslash Y_\lambda \qquad \text{for every } \lambda \in \mathrm{Ob}(\Lambda).$$

Consider also a compatible system $(\mathscr{G}_\lambda \mid \lambda \in \mathrm{Ob}(\Lambda))$ of sheaves on $X_\bullet$, where $\mathscr{G}_\lambda$ is a $\mathbb{Z}_{X_\lambda}$-module for every $\lambda \in \Lambda$, with transition maps

$$\varphi_u^{-1}\mathscr{G}_\lambda \to \mathscr{G}_\mu \qquad \text{for every morphism } u : \mu \to \lambda \text{ in } \Lambda$$

and set

$$\mathscr{G} := \operatorname*{colim}_{\lambda \in \mathrm{Ob}(\Lambda)} \varphi_\lambda^*\mathscr{G}_\lambda.$$

Notice that for every $\lambda \in \mathrm{Ob}(\Lambda)$ and every morphism $u : \mu \to \lambda$ in Λ we have

$$\varphi_u^{-1}Z_\lambda \subset Z_\mu \qquad \text{and} \qquad \varphi_\lambda^{-1}Z_\lambda \subset Z.$$

There follows a system of natural morphisms

$$\tau_u : \varphi_u^*\underline{\Gamma}_{Z_\lambda}\mathscr{G}_\lambda \to \underline{\Gamma}_{Z_\mu}\mathscr{G}_\mu \qquad \text{and} \qquad \tau_\lambda : \varphi_\lambda^*\underline{\Gamma}_{Z_\lambda}\mathscr{G}_\lambda \to \underline{\Gamma}_Z\mathscr{G}$$

amounting to a co-cone $\tau_\bullet$ with vertex $\underline{\Gamma}_Z\mathscr{G}$.

Proposition 10.4.9. *In the situation of* (10.4.8)*, the co-cone $\tau_\bullet$ induces a natural isomorphism*

$$\operatorname*{colim}_{\lambda \in \mathrm{Ob}(\Lambda)} \varphi_\lambda^* \, R^i\underline{\Gamma}_{Z_\lambda}\mathscr{G}_\lambda \xrightarrow{\sim} R^i\underline{\Gamma}_Z\mathscr{G} \qquad \textit{for every } i \in \mathbb{N}.$$

Proof. From proposition 10.1.10(ii) we deduce a natural isomorphism

$$\begin{aligned}\operatorname*{colim}_{\lambda \in \mathrm{Ob}(\Lambda)} \varphi_\lambda^* \, \underline{\Gamma}_{Z_\lambda}\mathscr{G}_\lambda &= \operatorname*{colim}_{\lambda \in \mathrm{Ob}(\Lambda)} \varphi_\lambda^* \, \mathrm{Ker}\,(\mathscr{G}_\lambda \to g_{\lambda*}g_\lambda^*\mathscr{G}_\lambda) \\ &\xrightarrow{\sim} \mathrm{Ker}\,(\mathscr{G} \to g_*g^*\mathscr{G}) = \underline{\Gamma}_Z\mathscr{G}.\end{aligned}$$

Now, let us choose a compatible system of injective resolutions $\mathscr{G}_\lambda \to \mathscr{I}_\lambda^\bullet$, so that the transition maps for the system $(\mathscr{G}_\lambda \mid \lambda \in \mathrm{Ob}(\Lambda))$ extend to a system of transition morphisms of complexes $(\mathscr{I}_\lambda^\bullet \mid \lambda \in \mathrm{Ob}(\Lambda))$. On the one hand, claim 10.1.11 says that

$$\mathscr{I}^\bullet := \operatorname*{colim}_{\lambda\in\mathrm{Ob}(\Lambda)} \varphi_\lambda^* \mathscr{I}_\lambda^\bullet$$

is a qc-flabby resolution of $\mathscr{G}$. On the other hand, the foregoing yields a natural isomorphism

$$\operatorname*{colim}_{\lambda\in\mathrm{Ob}(\Lambda)} \varphi_\lambda^* R^i\underline{\Gamma}_{Z_\lambda}\mathscr{G}_\lambda \xrightarrow{\sim} H^i\underline{\Gamma}_Z\mathscr{I}^\bullet.$$

Then the assertion follows from lemma 10.4.4(iii.a). □

Corollary 10.4.10. *Let $(X,\mathscr{A})$ be a ringed topological space with X locally spectral, $Z \subset X$ a constructible closed subset, $x \in Z$ any point, and denote by $j_x : X(x) \to X$ the inclusion map (notation of definition* 8.1.44(iii)*). We have a natural isomorphism of functors*

$$j_x^* R\underline{\Gamma}_Z \xrightarrow{\sim} R\underline{\Gamma}_{Z(x)} \qquad \textit{in } \mathsf{D}^+(j_x^*\mathscr{A}\text{-}\mathbf{Mod}).$$

Proof. We may assume that X is spectral, in which case $X(x)$ is the limit of the cofiltered system of all quasi-compact open neighborhoods of x in X, and if U is any such neighborhood, the inclusion map $U \to X$ is quasi-compact and quasi-separated. The sought morphism $\varphi^\bullet$ is then deduced from a co-cone constructed as in (10.4.8), and it suffices to check that $H^i\varphi^\bullet$ is an isomorphism on the underlying $\mathbb{Z}_{X(x)}$-modules for every $i \in \mathbb{Z}$, so the assertion is reduced to proposition 10.4.9. □

10.4.11. In the situation of (10.4.1), notice the natural isomorphism of $\mathscr{A}$-modules :

$$\mathscr{H}om_{\mathbb{Z}}(i_*\mathbb{Z}_Z, \mathscr{F}) \xrightarrow{\sim} \underline{\Gamma}_Z\mathscr{F} \qquad \text{for every } \mathscr{A}\text{-module } \mathscr{F}$$

whence a natural isomorphism of $\mathscr{A}(X)$-modules :

$$\mathrm{Hom}_{\mathbb{Z}}(i_*\mathbb{Z}_Z, \mathscr{F}) \xrightarrow{\sim} \Gamma_Z\mathscr{F} \qquad \text{for every } \mathscr{A}\text{-module } \mathscr{F}.$$

Now, let $\mathfrak{U} := (U_i \mid i \in I)$ be any open covering of U, indexed by a set I, and fix any total ordering on I. By proposition 10.2.6(iv), we get natural isomorphisms in $\mathsf{D}^+(\mathscr{A}\text{-}\mathbf{Mod})$ and $\mathsf{D}^+(\mathscr{A}(X)\text{-}\mathbf{Mod})$

$$R\mathscr{H}om^\bullet_{\mathbb{Z}}(R^{\mathrm{alt}}_\bullet(\mathfrak{U})[1], K^\bullet) \xrightarrow{\sim} R\underline{\Gamma}_Z K^\bullet \qquad R\mathrm{Hom}^\bullet_{\mathbb{Z}}(R^{\mathrm{alt}}_\bullet(\mathfrak{U})[1], K^\bullet) \xrightarrow{\sim} R\Gamma_Z K^\bullet$$

for every bounded below complex $K^\bullet$ of $\mathscr{A}$-modules. These isomorphisms allow us to compute $R\Gamma_Z K^\bullet$ as a *Čech cohomology functor*, as follows. Clearly, the rule : $\mathscr{F} \mapsto C^\bullet_{\mathrm{alt}}(\mathfrak{U}, \mathscr{F})$ defines a functor $\mathscr{A}\text{-}\mathbf{Mod} \to \mathsf{C}(\mathscr{A}(X)\text{-}\mathbf{Mod})$ (notation of definition 10.2.2(ii)). Now, pick any resolution $K^\bullet \xrightarrow{\sim} \mathscr{I}^\bullet$ by a bounded below complex of injective $\mathscr{A}$-modules; the foregoing yields a natural isomorphism :

$$R\Gamma_Z K^\bullet \xrightarrow{\sim} \mathrm{Tot}\, C^\bullet_{\mathrm{alt}}(\mathfrak{U}, \mathscr{I}^\bullet)[-1] \qquad \text{in } \mathsf{D}^+(\mathscr{A}(X)\text{-}\mathbf{Mod}).$$

10.4.12. We define a bifunctor

$$\mathscr{H}om^\bullet_Z := \underline{\Gamma}_Z \circ \mathscr{H}om^\bullet_{\mathscr{A}} : \mathsf{C}(\mathscr{A}\text{-}\mathbf{Mod}) \times \mathsf{C}(\mathscr{A}\text{-}\mathbf{Mod})^o \to \mathsf{C}(\mathscr{A}\text{-}\mathbf{Mod}).$$

The bifunctor $\mathscr{H}om^\bullet_Z$ admits a right derived functor :

$$R\mathscr{H}om^\bullet_Z : \mathsf{D}^+(\mathscr{A}\text{-}\mathbf{Mod}) \times \mathsf{D}(\mathscr{A}\text{-}\mathbf{Mod})^o \to \mathsf{D}(\mathscr{A}\text{-}\mathbf{Mod}).$$

The construction can be outlined as follows. First, for a fixed complex $K_\bullet$ of $\mathscr{A}$-modules, one can consider the right derived functor of the functor $\mathscr{G} \mapsto \mathscr{H}om^\bullet_Z(K_\bullet, \mathscr{G})$, which is denoted $R\mathscr{H}om^\bullet_Z(K_\bullet, -) : \mathsf{D}^+(\mathscr{A}\text{-}\mathbf{Mod}) \to \mathsf{D}(\mathscr{A}\text{-}\mathbf{Mod})$. Next, one verifies that every quasi-isomorphism $K_\bullet \to K'_\bullet$ induces an isomorphism of functors : $R\mathscr{H}om^\bullet_Z(K'_\bullet, -) \xrightarrow{\sim} R\mathscr{H}om^\bullet_Z(K_\bullet, -)$, hence the natural transformation $K_\bullet \mapsto R\mathscr{H}om^\bullet_Z(K_\bullet, -)$ factors through

$\mathsf{D}(\mathscr{A}\text{-}\mathbf{Mod})$, and this is the sought bifunctor $R\mathscr{H}om^\bullet_Z$. In case $Z = X$, one recovers the functor $R\mathscr{H}om^\bullet_\mathscr{A}$ of (10.1.13).

Lemma 10.4.13. *In the situation of* (10.4.1)*, the following holds :*

(i) *The functor $\underline{\Gamma}_Z$ is right adjoint to i_*i^*. More precisely, for any two $\mathscr{A}$-modules $\mathscr{F}$ and $\mathscr{G}$ we have natural isomorphisms :*
 (a) $\mathscr{H}om_\mathscr{A}(\mathscr{F}, \underline{\Gamma}_Z\mathscr{G}) \xrightarrow{\sim} \mathscr{H}om_\mathscr{A}(i_*i^*\mathscr{F}, \mathscr{G})$.
 (b) $R\mathscr{H}om^\bullet_\mathscr{A}(\mathscr{F}, R\underline{\Gamma}_Z\mathscr{G}) \xrightarrow{\sim} R\mathscr{H}om^\bullet_\mathscr{A}(i_*i^*\mathscr{F}, \mathscr{G})$.

(ii) *If $\mathscr{F}$ is an injective (resp. flabby) $\mathscr{A}$-module on X, then the same holds for $\underline{\Gamma}_Z\mathscr{F}$.*

(iii) *There are natural isomorphisms of bifunctors :*

$$R\mathscr{H}om^\bullet_Z \xrightarrow{\sim} R\underline{\Gamma}_Z \circ R\mathscr{H}om^\bullet_\mathscr{A} \xrightarrow{\sim} R\mathscr{H}om^\bullet_\mathscr{A}(-, R\underline{\Gamma}_Z-)$$

(iv) *If $W \subset X$ is any other closed subset, there is a natural isomorphism of functors :*

$$R\underline{\Gamma}_{W\cap Z} \xrightarrow{\sim} R\underline{\Gamma}_W \circ R\underline{\Gamma}_Z.$$

(v) *If $f : (Y, \mathscr{B}) \to (X, \mathscr{A})$ is any morphism of ringed spaces, there is a natural isomorphism of functors :*

$$Rf_* \circ R\underline{\Gamma}_{f^{-1}Z} \xrightarrow{\sim} R\underline{\Gamma}_Z \circ Rf_* : \mathsf{D}^+(\mathscr{B}\text{-}\mathbf{Mod}) \to \mathsf{D}^+(\mathscr{A}\text{-}\mathbf{Mod}).$$

(vi) *There is a natural isomorphism of bifunctors on $\mathsf{D}^+(\mathscr{A}\text{-}\mathbf{Mod})^o \times \mathsf{D}^+(\mathscr{A}\text{-}\mathbf{Mod})$:*

$$R\mathscr{H}om^\bullet_Z(R\underline{\Gamma}_Z-, -) \xrightarrow{\sim} R\mathscr{H}om^\bullet_\mathscr{A}(R\underline{\Gamma}_Z-, -).$$

Proof. (i): To establish the isomorphism (a), one uses the short exact sequence

$$0 \to j_!j^*\mathscr{F} \to \mathscr{F} \to i_*i^*\mathscr{F} \to 0$$

to show that any map $\mathscr{F} \to \underline{\Gamma}_Z\mathscr{G}$ factors through $\mathscr{H} := i_*i^*\mathscr{F}$. Conversely, since $j_*j^*\mathscr{H} = 0$, it is clear that every map $\mathscr{H} \to \mathscr{G}$ must factor through $\underline{\Gamma}_Z\mathscr{G}$. The isomorphism (b) is derived easily from (a) and (ii).

(iii): According to [163, Th.10.8.2], the first isomorphism in (iii) is deduced from lemmata 10.4.4(i) and 10.1.16(ii). The second isomorphism in (iii) follows from [163, Th.10.8.2], and assertion (ii) concerning injective sheaves.

(ii): Suppose first that $\mathscr{F}$ is injective; let $\mathscr{G}_1 \to \mathscr{G}_2$ a monomorphism of $\mathscr{A}$-modules and $\varphi : \mathscr{G}_1 \to \underline{\Gamma}_Z\mathscr{F}$ any $\mathscr{A}$-linear map. By (i) we deduce a map $i_*i^*\mathscr{G}_1 \to \mathscr{F}$, which extends to a map $i_*i^*\mathscr{G}_2 \to \mathscr{F}$, by injectivity of $\mathscr{F}$. Finally, (i) again yields a map $\mathscr{G}_2 \to \underline{\Gamma}_Z\mathscr{F}$ extending φ.

Next, if $\mathscr{F}$ is flabby, let $V \subset X$ be any open subset, and $s \in \underline{\Gamma}_Z\mathscr{F}(V)$; since $s_{|V\setminus Z} = 0$, we can extend s to a section $s' \in \underline{\Gamma}_Z\mathscr{F}(V \cup (X \setminus Z))$. Since $\mathscr{F}$ is flabby, s' extends to a section $s'' \in \mathscr{F}(X)$; however, by construction s'' vanishes on the complement of Z.

(iv): Clearly $\underline{\Gamma}_{W\cap Z} = \underline{\Gamma}_W \circ \underline{\Gamma}_Z$, so the claim follows easily from (ii) and [163, Th.10.8.2].

(v): Since flabby $\mathscr{B}$-modules are acyclic for f_* and $\underline{\Gamma}_{f^{-1}Z}$ (lemmata 10.1.5(ii) and 10.4.4(i)), one can apply (ii) and [163, Th.10.8.2] to identify both functors with $R(\underline{\Gamma}_Z \circ f_*)$.

(vi): Let $I^\bullet$ and $J^\bullet$ be two bounded below complexes of injective $\mathscr{A}$-modules. Clearly any morphism $\underline{\Gamma}_Z I^\bullet \to J^\bullet$ vanishes on U, whence the assertion. □

Definition 10.4.14. Let X be a topological space, and $K^\bullet \in \mathsf{D}^+(\mathbb{Z}_X\text{-}\mathbf{Mod})$.

(i) The *depth of $K^\bullet$ along Z* is

$$\mathrm{depth}_Z K^\bullet := \sup\{n \in \mathbb{Z} \mid R^i\underline{\Gamma}_Z K^\bullet = 0 \text{ for all } i < n\} \in \mathbb{Z} \cup \{+\infty\}.$$

From lemma 10.4.13(ii,iv) and the Grothendieck spectral sequence ([126, Th.5.8.3]), we see that

$$\mathrm{depth}_W K^\bullet \geq \mathrm{depth}_Z K^\bullet \qquad \text{whenever } W \subset Z. \tag{10.4.15}$$

(ii) Let $\Phi := (Z_\lambda \mid \lambda \in \Lambda)$ be a family of closed subsets of X, and suppose that Φ is cofiltered by inclusion. Due to (10.4.15) it is reasonable to define the *depth of $K^\bullet$ along* Φ as

$$\mathrm{depth}_\Phi K^\bullet := \sup\{\mathrm{depth}_{Z_\lambda} K^\bullet \mid \lambda \in \Lambda\} \in \mathbb{Z} \cup \{+\infty\}.$$

10.4.16. Let now X be a scheme and $i : Z \to X$ a closed immersion of schemes, such that Z is constructible in X. Then the open immersion $j : X\backslash Z \to X$ is quasi-compact and separated, hence the functors $R^q j_*$ preserve the subcategories of quasi-coherent modules, for every $q \in \mathbb{N}$ ([61, Ch.III, Prop.1.4.10]). In light of (10.4.3) we deduce that $R^q\underline{\Gamma}_Z$ restricts to a functor :

$$R^q\underline{\Gamma}_Z : \mathscr{O}_X\text{-}\mathbf{Mod}_{\mathrm{qcoh}} \to \mathscr{O}_X\text{-}\mathbf{Mod}_{\mathrm{qcoh}} \qquad \text{for every } q \in \mathbb{N}.$$

Lemma 10.4.17. *In the situation of* (10.4.16)*, the following holds :*

(i) *The functor $R\underline{\Gamma}_Z$ restricts to a triangulated functor :*

$$R\underline{\Gamma}_Z : \mathsf{D}^+(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{qcoh}} \to \mathsf{D}^+(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$$

(notation of (10.3.11)*).*

(ii) *Let $f : Y \to X$ be an affine morphism of schemes, $K^\bullet$ any object of $\mathsf{D}^+(\mathscr{O}_Y\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$. Then we have the identity :*

$$\mathrm{depth}_{f^{-1}Z} K^\bullet = \mathrm{depth}_Z Rf_* K^\bullet.$$

(iii) *Let $f : Y \to X$ be a flat morphism of schemes, $K^\bullet$ any object of $\mathsf{D}^+(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$. Then the natural map*

$$f^* R\underline{\Gamma}_Z K^\bullet \to R\underline{\Gamma}_{f^{-1}Z} f^* K^\bullet$$

is an isomorphism in $\mathsf{D}^+(\mathscr{O}_Y\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$. Especially, we have

$$\mathrm{depth}_{f^{-1}Z} f^* K^\bullet \geq \mathrm{depth}_Z K^\bullet$$

and if f is faithfully flat, the inequality is actually an equality.

(iv) *Let $K^\bullet$ be any object of $\mathsf{D}^+(\mathscr{O}_X\text{-}\mathbf{Mod})$, and $\mathscr{F}$ a flat quasi-coherent $\mathscr{O}_X$-module. Then the natural map*

$$\mathscr{F} \otimes_{\mathscr{O}_X} R\underline{\Gamma}_Z K^\bullet \to R\underline{\Gamma}_Z(\mathscr{F} \otimes_{\mathscr{O}_X} K^\bullet)$$

is an isomorphism in $\mathsf{D}(\mathscr{O}_X\text{-}\mathbf{Mod})$.

Proof. (i): Indeed, if $K^\bullet$ is an object of $\mathsf{D}^+(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$, we have a spectral sequence :

$$E_2^{pq} := R^p\underline{\Gamma}_Z H^q K^\bullet \Rightarrow R^{p+q}\underline{\Gamma}_Z K^\bullet.$$

(This spectral sequence is obtained from a Cartan-Eilenberg resolution of $K^\bullet$: see *e.g.* [163, §5.7].) Hence $R^\bullet\underline{\Gamma}_Z K^\bullet$ admits a finite filtration whose subquotients are quasi-coherent; then the lemma follows from [61, Ch.III, Prop.1.4.17].

(ii): To start with, a spectral sequence argument as in the foregoing shows that Rf_* restricts to a functor $\mathsf{D}^+(\mathscr{O}_Y\text{-}\mathbf{Mod})_{\mathrm{qcoh}} \to \mathsf{D}^+(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$, and moreover we have natural identifications : $R^i f_* K^\bullet \xrightarrow{\sim} f_* H^i K^\bullet$ for every $i \in \mathbb{Z}$ and every object $K^\bullet$ of $\mathsf{D}^+(\mathscr{O}_Y\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$. In view of lemma 10.4.13(v), we derive natural isomorphisms : $R^i\underline{\Gamma}_Z Rf_* K^\bullet \xrightarrow{\sim} Rf_* R^i\underline{\Gamma}_{f^{-1}Z} K^\bullet$ for every object $K^\bullet$ of $\mathsf{D}^+(\mathscr{O}_Y\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$. The assertion follows easily.

(iii): A spectral sequence argument as in the proof of (i) allows us to assume that $K^\bullet = \mathscr{F}[0]$ for some quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$. In this case, let $j_X : X\backslash Z \to X$ and $j_Y : Y\backslash f^{-1}Z \to Y$ be the open immersions; considering the exact sequences f^*(10.4.2) and f^*(10.4.3), we reduce to showing that the natural map $f^* Rj_{X*} j_X^* \mathscr{F} \to Rj_{Y*} j_Y^* f^* \mathscr{F}$ is an isomorphism in $\mathsf{D}^+(\mathscr{O}_Y\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$. The latter assertion follows from corollary 10.3.8.

(iv): The map in question is obtained as follows. We may assume that $K^\bullet$ is a complex of injective $\mathscr{O}_X$-modules, and let $i : \underline{\Gamma}_Z K^\bullet \to K^\bullet$ be the inclusion map; clearly $\mathscr{F} \otimes_{\mathscr{O}_X} i$ factors through the inclusion map $\underline{\Gamma}_Z(\mathscr{F} \otimes_{\mathscr{O}_X} K^\bullet) \to \mathscr{F} \otimes_{\mathscr{O}_X} K^\bullet$, whence the sought map

$\omega_{\mathscr{F}} : \mathscr{F} \otimes_{\mathscr{O}_X} \underline{\Gamma}_Z K^\bullet \to \underline{\Gamma}_Z(\mathscr{F} \otimes_{\mathscr{O}_X} K^\bullet)$. Now, the assertion is local on X, hence we may assume that $X = \mathrm{Spec}\, A$ for some ring A, and $\mathscr{F} = F^\sim$ for some flat A-module F. Then F is the colimit of a filtered system $(L_\lambda \mid \lambda \in \Lambda)$ of free A-modules of finite rank ([120, Ch.I, Th.1.2]), and by virtue of lemma 10.4.4(iii.b), the map $\omega_{\mathscr{F}}$ is the colimit of the system of induced maps $(\omega_{L_\lambda^\sim} \mid \lambda \in \Lambda)$. Thus, we are reduced to the case where F is a free A-module of finite rank, in which case the assertion is clear. □

Proposition 10.4.18. *Let X be an affine scheme, $Z \subset X$ any closed subset, $\mathscr{F}^\bullet$ a bounded below complex of quasi-coherent $\mathscr{O}_X$-modules, and $\mathfrak{U} := (U_t \mid t \in I)$ a family of affine open subsets of X such that $\bigcup_{t\in I} U_t = X \setminus Z$. There exists a natural isomorphism*

$$R\Gamma_Z \mathscr{F}^\bullet \xrightarrow{\sim} \mathrm{Tot}\, C^\bullet_{\mathrm{alt}}(\mathfrak{U}, \mathscr{F}^\bullet)[-1] \qquad \textit{in } \mathsf{D}^+(\mathscr{O}_X(X)\text{-}\mathbf{Mod})$$

(where $C^\bullet_{\mathrm{alt}}$ denotes the augmented alternating Čech complex).

Proof. Pick any resolution $\mathscr{F}^\bullet \xrightarrow{\sim} \mathscr{I}^\bullet$ by a bounded below complex of injective $\mathscr{O}_X$-modules; by virtue of (10.4.11) it suffices to show that the natural map of double complexes

$$C^\bullet_{\mathrm{alt}}(\mathfrak{U}, \mathscr{F}^\bullet) \to C^\bullet_{\mathrm{alt}}(\mathfrak{U}, \mathscr{I}^\bullet)$$

induces a quasi-isomorphism on total complexes. To this aim, we can argue as in the proof of theorem 10.2.28(ii) : the details shall be left to the reader (the condition that X is affine, is needed here, since we are dealing with the augmented Čech complex, whereas in the proof of theorem 10.2.28(ii) only the ordinary version intervenes, so for the proof of the latter it is only required that the intersection of any finite system of members of $\mathfrak{U}$ is affine). □

10.4.19. Let X be a scheme; for every $x \in X$, we consider the cofiltered family $\Phi(x)$ of all non-empty constructible closed subsets of $X(x) := \mathrm{Spec}\, \mathscr{O}_{X,x}$; clearly $\bigcap_{Z\in\Phi(x)} Z = \{x\}$. Let $K^\bullet$ be any object of $\mathsf{D}^+(\mathbb{Z}_X\text{-}\mathbf{Mod})$; we are interested in the quantities :

$$\delta(x, K^\bullet) := \mathrm{depth}_{\{x\}} K^\bullet(x) \qquad \text{and} \qquad \delta'(x, K^\bullet) := \mathrm{depth}_{\Phi(x)} K^\bullet(x)$$

(notation of definition 4.9.17(iii)). Especially, we wish to compute the depth of a complex $K^\bullet$ along a closed subset $Z \subset X$, in terms of the local invariants $\delta(x, K^\bullet)$ or $\delta'(x, K^\bullet)$, evaluated at the points $x \in Z$. This shall be achieved by theorem 10.4.21. To begin with, we remark :

Lemma 10.4.20. *With the notation of* (10.4.19)*, we have :*

(i) $\delta(x, K^\bullet) \geq \delta'(x, K^\bullet)$ *for every $x \in X$ and every $K^\bullet \in \mathrm{Ob}(\mathsf{D}^+(\mathbb{Z}_X\text{-}\mathbf{Mod}))$.*
(ii) *If the topological space $|X|$ underlying X is locally noetherian, the inequality of* (i) *is actually an equality.*
(iii) *If $\mathscr{F}$ is a flat quasi-coherent $\mathscr{O}_X$-module, then $\delta'(x, \mathscr{F}) \geq \delta'(x, \mathscr{O}_X)$.*

Proof. (i) follows easily from (10.4.15). Likewise, (ii) follows likewise from (10.4.15) and the fact that if $|X|$ is locally noetherian, $\{x\}$ is the smallest element of the family $\Phi(x)$.

(iii): According to [120, Ch.I, Th.1.2], the $\mathscr{O}_{X(x)}$-module $\mathscr{F}(x)$ is the colimit of a filtered system $(\mathscr{L}_\lambda \mid \lambda \in \Lambda)$ of free $\mathscr{O}_{X(x)}$-modules of finite rank. In view of lemma 10.4.4(iii.b), it is easily seen that

$$\mathrm{depth}_Z \mathscr{F} \geq \inf_{\lambda\in\Lambda} \mathrm{depth}_Z \mathscr{L}_\lambda = \mathrm{depth}_Z \mathscr{O}_X$$

for every closed constructible subset $Z \subset X$. The assertion follows. □

Theorem 10.4.21. *With the notation of* (10.4.19)*, let $Z \subset X$ be any closed constructible subset, $K^\bullet$ any object of* $\mathsf{D}^+(\mathbb{Z}_X\text{-}\mathbf{Mod})$*. Then*

$$\mathrm{depth}_Z K^\bullet = \inf \{\delta(x, K^\bullet) \mid x \in Z\} = \inf \{\delta'(x, K^\bullet) \mid x \in Z\}.$$

Proof. Let $d := \mathrm{depth}_Z K^\bullet$; in light of corollary 10.4.10, it is clear that $\delta(x, K^\bullet) \geq d$ for all $x \in Z$, hence, in order to prove the first identity it suffices to show :

Claim 10.4.22. Suppose $d < +\infty$. Then there exists $x \in Z$ such that $R^d\Gamma_{\{x\}}K^\bullet(x) \neq 0$.

Proof of the claim. By definition of d, we can find an open subset $V \subset X$ and a non-zero section $s \in \Gamma(V, R^d\underline{\Gamma}_Z K^\bullet)$. The support of s is a closed subset $S \subset Z$. Let x be a maximal point of S. From lemma 10.4.13(iv) we deduce a spectral sequence

$$E_2^{pq} := R^p\underline{\Gamma}_{\{x\}}R^q\underline{\Gamma}_{Z\cap X(x)}K^\bullet(x) \Rightarrow R^{p+q}\underline{\Gamma}_{\{x\}}K^\bullet(x)$$

and corollary 10.4.10 implies that $E_2^{pq} = 0$ whenever $q < d$, therefore :

$$R^d\underline{\Gamma}_{\{x\}}K^\bullet(x) \simeq R^0\underline{\Gamma}_{\{x\}}R^d\underline{\Gamma}_{Z\cap X(x)}K^\bullet(x).$$

However, the image of s in $R^d\underline{\Gamma}_{Z\cap X(x)}K^\bullet(x)$ is supported precisely at x, as required. ◇

Finally, since Z is constructible, $Z \cap X(x) \in \Phi(x)$ for every $x \in X$, hence $\delta'(x, K^\bullet) \geq d$. Then the second identity follows from the first and lemma 10.4.20(i). □

Corollary 10.4.23. *Let $Z \subset X$ be a closed and constructible subset, $\mathscr{F}$ a $\mathbb{Z}_X$-module, and $j : X\setminus Z \to X$ the induced open immersion. Then the natural map $\mathscr{F} \to j_*j^*\mathscr{F}$ is an isomorphism if and only if $\delta(x, \mathscr{F}) \geq 2$, if and only if $\delta'(x, \mathscr{F}) \geq 2$ for every $x \in Z$.* □

10.4.24. Let now X be an affine scheme, say $X := \mathrm{Spec}\, A$ for some ring A, and $Z \subset X$ a constructible closed subset. In this case, we wish to show that the depth of a complex of quasi-coherent $\mathscr{O}_X$-modules along Z can be computed in terms of Ext functors on the category of A-modules. This is the content of the following :

Proposition 10.4.25. *In the situation of* (10.4.24)*, let N be any finitely presented A-module such that* $\mathrm{Supp}\, N = Z$*, and $M^\bullet$ any bounded below complex of A-modules. Then :*

$$\mathrm{depth}_Z M^{\bullet\sim} = \sup\{n \in \mathbb{Z} \mid \mathrm{Ext}^j_A(N, M^\bullet) = 0 \textit{ for all } j < n\}.$$

Proof. ($M^{\bullet\sim}$ is the complex of quasi-coherent $\mathscr{O}_X$-modules determined by $M^\bullet$, as in (10.3)).

Claim 10.4.26. There is a natural isomorphism in $\mathsf{D}^+(A\text{-}\mathbf{Mod})$:

$$R\mathrm{Hom}^\bullet_A(N, M^\bullet) \xrightarrow{\sim} R\mathrm{Hom}^\bullet_{\mathscr{O}_X}(N^\sim, R\underline{\Gamma}_Z M^{\bullet\sim}).$$

Proof of the claim. Let $i : Z \to X$ be the closed immersion. The assumption on N implies that $N^\sim = i_*i^*N^\sim$; hence lemma 10.4.13(i.b) yields a natural isomorphism

$$(10.4.27) \qquad R\mathrm{Hom}^\bullet_{\mathscr{O}_X}(N^\sim, M^{\bullet\sim}) \xrightarrow{\sim} R\mathrm{Hom}^\bullet_{\mathscr{O}_X}(N^\sim, R\underline{\Gamma}_Z M^{\bullet\sim}).$$

The claim follows by combining corollary 10.3.2(i) and (10.4.27). ◇

From claim 10.4.26 we see already that $\mathrm{Ext}^j_A(N, M^\bullet) = 0$ whenever $j < \mathrm{depth}_Z M^{\bullet\sim}$. Suppose that $\mathrm{depth}_Z M^{\bullet\sim} = p < +\infty$; in this case, claim 10.4.26 gives an isomorphism :

$$\mathrm{Ext}^p_A(N, M^\bullet) \simeq \mathrm{Hom}_{\mathscr{O}_X}(N^\sim, R^p\underline{\Gamma}_Z M^{\bullet\sim}) \simeq \mathrm{Hom}_A(N, R^p\Gamma_Z M^{\bullet\sim})$$

where the last isomorphism holds by lemma 10.4.17. To conclude the proof, we have to exhibit a non-zero map from N to $Q := R^p\Gamma_Z M^{\bullet\sim}$. Let $F_0(N) \subset A$ denote the Fitting ideal of N; this is a finitely generated ideal, whose zero locus coincides with the support of N. More precisely, $\mathrm{Ann}_A(N)^r \subset F_0(N) \subset \mathrm{Ann}_A(N)$ for all sufficiently large integers $r > 0$. Let now $x \in Q$ be any non-zero element, and $f_1, \dots, f_k$ a finite system of generators for $F_0(N)$; since x vanishes on $X\setminus Z$, the image of x in Q_{f_i} is zero for every $i \leq k$, *i.e.* there exists $n_i \geq 0$ such that $f_i^{n_i}x = 0$ in Q. It follows easily that $F_0(N)^n \subset \mathrm{Ann}_A(x)$ for a sufficiently large integer n, and therefore x defines a map $\varphi : A/F_0(N)^n \to Q$ by the rule : $a \mapsto ax$. According to [75, Lemma 3.2.21], we can find a finite filtration $0 = J_m \subset \cdots \subset J_1 \subset J_0 := A/F_0(N)^n$ such that each J_i/J_{i+1} is quotient of a direct sum of copies of N. Let $s \leq m$ be the smallest integer such that $J_s \subset \mathrm{Ker}\,\varphi$. By restriction, φ induces a non-zero map $J_{s-1}/J_s \to Q$, whence a non-zero map $N^{(S)} \to Q$, for some set S, and finally a non-zero map $N \to Q$, as required. □

Remark 10.4.28. Notice that the existence of a finitely presented A-module N with $\mathrm{Supp}\, N = Z$ is equivalent to the constructibility of Z.

10.4.29. In the situation of (10.4.24), let $I \subset A$ be a finitely generated ideal such that $V(I) = Z$, and $M^\bullet$ a bounded below complex of A-modules. Then it is customary to set :

$$\mathrm{depth}_I M^\bullet := \mathrm{depth}_Z M^{\bullet\sim}$$

and the depth along Z of $M^{\bullet\sim}$ is also called the *I-depth* of $M^\bullet$. Moreover, if A is a local ring with maximal ideal $\mathfrak{m}_A$, we shall often use the standard notation :

$$\mathrm{depth}_A M^\bullet := \mathrm{depth}_{\{\mathfrak{m}_A\}} M^{\bullet\sim}$$

and this invariant is often called briefly the *depth* of $M^\bullet$. With this notation, theorem 10.4.21 can be rephrased as the identity :

$$\mathrm{depth}_I M^\bullet = \inf_{\mathfrak{p}\in V(I)} \mathrm{depth}_{A_\mathfrak{p}} M^\bullet_\mathfrak{p}. \qquad (10.4.30)$$

10.4.31. *Depth sensitivity of the Koszul complex.* Let now $\mathbf{f} := (f_1, \dots, f_r)$ be a finite system of generators of the ideal $I \subset A$, and $Z := \mathrm{Spec}\, A/I$. In this generality, the Koszul complex $\mathbf{K}^\bullet(\mathbf{f})$ of remark 7.8.1(ii) is not a resolution of the A-module A/I, and thus $H^\bullet(\mathbf{f}, M)$ (for an A-module M) does not necessarily agree with $\mathrm{Ext}^\bullet_A(A/I, M)$. Nevertheless, we have :

Proposition 10.4.32. (i) *In the situation of* (10.4.31), *let* $M^\bullet \in \mathrm{Ob}(\mathsf{D}^+(A\text{-}\mathbf{Mod}))$*; then :*

$$\mathrm{depth}_I M^\bullet = d := \sup\{i \in \mathbb{Z} \mid H^j(\mathbf{f}, M^\bullet) = 0 \textit{ for all } j < i\}.$$

(ii) *If* $d < +\infty$*, there are natural A-linear isomorphisms :*

$$H^d(\mathbf{f}, M^\bullet) \xrightarrow{\sim} \mathrm{Hom}_A(A/I, R^d\Gamma_Z M^{\bullet\sim}) \xrightarrow{\sim} \mathrm{Ext}^d_A(A/I, M^\bullet).$$

(iii) *Moreover we have a natural isomorphism in* $\mathsf{D}(A\text{-}\mathbf{Mod})$ *:*

$$\operatorname*{colim}_{m\in\mathbb{N}} \mathbf{K}^\bullet(\mathbf{f}^m, M^\bullet) \xrightarrow{\sim} R\Gamma_Z M^{\bullet\sim}$$

where the transition maps in the colimit are the maps $\boldsymbol{\varphi}^\bullet_{\mathbf{f}^n}$ *of* (7.8.20).

Proof. (i): Let $B := \mathbb{Z}[t_1, \dots, t_r] \to A$ be the ring homomorphism defined by the rule : $t_i \mapsto f_i$ for $i = 1, \dots, r$; we denote $\psi : X \to Y := \mathrm{Spec}\, B$ the induced morphism, $J \subset B$ the ideal generated by the system $\mathbf{t} := (t_i \mid i = 1, \dots, r)$, and set $W := V(J) \subset Y$. From lemma 10.4.13(v) we deduce a natural isomorphism in $\mathsf{D}(\mathscr{O}_Y\text{-}\mathbf{Mod})$:

$$\psi_* R\underline{\Gamma}_Z M^{\bullet\sim} \xrightarrow{\sim} R\underline{\Gamma}_W \psi_* M^{\bullet\sim}. \qquad (10.4.33)$$

Hence we are reduced to showing :

Claim 10.4.34. $d = \mathrm{depth}_J \psi_* M^\bullet$.

Proof of the claim. Notice that $\mathbf{t}$ is a regular sequence in B, hence $\mathrm{Ext}^\bullet_B(B/J, \psi_* M^\bullet) \simeq H^\bullet(\mathbf{t}, \psi_* M^\bullet)$; by proposition 10.4.25, there follows the identity :

$$\mathrm{depth}_J \psi_* M^\bullet = \sup\{n \in \mathbb{N} \mid H^i(\mathbf{t}, \psi_* M^\bullet) = 0 \text{ for all } i < n\}.$$

Then the assertion follows from lemma 7.8.2(iv). ◊

(ii): From lemma 10.4.13(i.b) and corollary 10.3.2(i) we derive a natural isomorphism :

$$R\mathrm{Hom}^\bullet_B(B/J, \psi_* M^\bullet) \xrightarrow{\sim} R\mathrm{Hom}^\bullet_{\mathscr{O}_Y}((B/J)^\sim, R\underline{\Gamma}_W \psi_* M^{\bullet\sim}).$$

However, due to (10.4.33) and claim 10.4.34 we may compute :

$$R^d\mathrm{Hom}^\bullet_{\mathscr{O}_Y}((B/J)^\sim, R\underline{\Gamma}_W \psi_* M^{\bullet\sim}) \xrightarrow{\sim} \mathrm{Hom}_B(B/J, R^d\Gamma_W \psi_* M^{\bullet\sim}) \xrightarrow{\sim} \mathrm{Hom}_A(A/I, R^d\Gamma_Z M^{\bullet\sim}).$$

The first claimed isomorphism easily follows. Similarly, one applies lemma 10.4.13(i.b) and corollary 10.3.2(i) to compute $\mathrm{Ext}^\bullet_A(A/I, M^\bullet)$, and deduces the second isomorphism of (ii) using (i). Assertion (iii) follows immediately from (10.2.27) and proposition 10.4.18. □

Corollary 10.4.35. *Let A be a noetherian ring, $I \subset A$ an ideal, $M^\bullet$ an object of $\mathsf{D}^+(A\text{-}\mathbf{Mod})$, and set $Z := \mathrm{Spec}\, A/I$. We have a natural isomorphism*

$$\operatorname*{colim}_{n\in\mathbb{N}} R^i\mathrm{Hom}^\bullet_A(A/I^n, M^\bullet) \xrightarrow{\sim} R^i\Gamma_Z M^{\bullet\sim} \qquad \textit{for every } i \in \mathbb{Z}.$$

Proof. This is obtained by combining proposition 10.4.32(iii) and remark 7.8.45. □

Corollary 10.4.36. *Let A be a noetherian ring, M an A-module, and $\mathbf{f} := (f_1, \dots, f_r)$ a finite sequence of elements of A that generates an ideal $I \subset A$. Suppose that :*

(a) $\mathrm{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) \geq r$ *for every* $\mathfrak{p} \in Z := \mathrm{Spec}\, A/I$.
(b) *Z has pure codimension r in* $\mathrm{Spec}\, A$.

Then $H^i(\mathbf{f}, M) = 0$ for every $i < r$, and for every $\mathfrak{p} \in Z$ we have :

$$\mathrm{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}/IM_\mathfrak{p}) = \mathrm{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) - r \qquad \textit{and} \qquad \dim A_\mathfrak{p}/I_\mathfrak{p} = \dim(A_\mathfrak{p}) - r.$$

Proof. The stated vanishing for $H^i(\mathbf{f}, M)$ follows immediately from (10.4.30) and proposition 10.4.32(i). Next, we have $\dim A_\mathfrak{p}/I_\mathfrak{p} \geq \dim(A_\mathfrak{p}) - r$ for every $\mathfrak{p} \in Z$, by [126, Th.13.6(ii)], and the opposite inequality follows easily from (b).

For a given $\mathfrak{p} \in Z$, set $\delta_\mathfrak{p} := \mathrm{depth}_{A_\mathfrak{p}} M_\mathfrak{p}$; pick a sequence $\mathbf{f}' := (f_{r+1}, \dots, f_{r+n})$ in A whose image in $A_\mathfrak{p}$ generates $\mathfrak{p}A_\mathfrak{p}$, and set $\mathbf{f}'' := (f_1, \dots, f_{r+n})$ and $\overline{M} := M/IM$. By proposition 10.4.32(i) and lemma 7.8.2(vi), if $\delta_\mathfrak{p} = +\infty$, we have $H_i(\mathbf{f}'', M_\mathfrak{p}) = 0$ for every $i \in \mathbb{N}$, and otherwise $r \leq \delta_\mathfrak{p} \leq n$, and we have

$$H_{n+r-\delta_\mathfrak{p}}(\mathbf{f}'', M_\mathfrak{p}) \neq 0 \qquad \text{and} \qquad H_i(\mathbf{f}'', M_\mathfrak{p}) = 0 \qquad \text{for every } i > n + r - \delta_\mathfrak{p}.$$

By the same token, $H_i(\mathbf{f}, M_\mathfrak{p}) = 0$ for every $i > 0$, *i.e.* the natural map $\mathbf{K}_\bullet(\mathbf{f}, M_\mathfrak{p}) \to \overline{M}_\mathfrak{p}[0]$ is an isomorphism in $\mathsf{D}(A_\mathfrak{p}\text{-}\mathbf{Mod})$, so we get isomorphisms :

$$\mathbf{K}_\bullet(\mathbf{f}'', M_\mathfrak{p}) \xrightarrow{\sim} \mathbf{K}_\bullet(\mathbf{f}')\otimes_{A_\mathfrak{p}}\mathbf{K}_\bullet(\mathbf{f}, M_\mathfrak{p}) \xrightarrow{\sim} \mathbf{K}_\bullet(\mathbf{f}')\otimes_{A_\mathfrak{p}}\overline{M}_\mathfrak{p}[0] = \mathbf{K}_\bullet(\mathbf{f}', \overline{M}_\mathfrak{p}) \qquad \text{in } \mathsf{D}(A_\mathfrak{p}\text{-}\mathbf{Mod}).$$

Summing up, we see that, if $\delta_\mathfrak{p} = +\infty$, then $H_i(\mathbf{f}', \overline{M}_\mathfrak{p}) = 0$ for every $i \in \mathbb{N}$, and otherwise :

$$H^j(\mathbf{f}', \overline{M}_\mathfrak{p}) = H_{n-j}(\mathbf{f}', \overline{M}_\mathfrak{p}) = H_{n-j}(\mathbf{f}'', M_\mathfrak{p}) = 0 \qquad \text{whenever } n - j > n + r - \delta_\mathfrak{p}$$

and $H^{\delta_\mathfrak{p}-r}(\mathbf{f}', \overline{M}_\mathfrak{p}) = H_{n+r-\delta_\mathfrak{p}}(\mathbf{f}', \overline{M}_\mathfrak{p}) \neq 0$, which implies that $\mathrm{depth}_{A_\mathfrak{p}}(\overline{M}_\mathfrak{p}) = \delta_\mathfrak{p} - r$, again by proposition 10.4.32(i). □

Corollary 10.4.37. *In the situation of* (10.4.29)*, let B be a faithfully flat A-algebra. Then :*

$$\mathrm{depth}_{IB} B \otimes_A M^\bullet = \mathrm{depth}_I M^\bullet.$$

Proof. It is a special case of lemma 10.4.17(iii). Alternatively, one remarks that

$$\mathbf{K}^\bullet(\mathbf{g}, M^\bullet \otimes_A B) \simeq \mathbf{K}^\bullet(\mathbf{g}, M^\bullet) \otimes_A B \qquad \text{for every finite sequence of elements } \mathbf{g} \text{ in } A$$

which allows us to apply proposition 10.4.32(iii). □

Theorem 10.4.38. *Let $A \to B$ be a local homomorphism of local rings, and suppose that the maximal ideals $\mathfrak{m}_A$ and $\mathfrak{m}_B$ of A and B are finitely generated. Let also M be an A-module and N a B-module which is flat over A. Then we have :*

$$\mathrm{depth}_B(M \otimes_A N) = \mathrm{depth}_A M + \mathrm{depth}_{B/\mathfrak{m}_A B}(N/\mathfrak{m}_A N).$$

Proof. Let $\mathbf{f} := (f_1, \dots, f_r)$ and $\overline{\mathbf{g}} := (\overline{g}_1, \dots, \overline{g}_s)$ be finite systems of generators for $\mathfrak{m}_A$ and respectively $\overline{\mathfrak{m}}_B$, the maximal ideal of $B/\mathfrak{m}_A B$. We choose an arbitrary lifting of $\overline{\mathbf{g}}$ to a finite system $\mathbf{g}$ of elements of $\mathfrak{m}_B$; then $(\mathbf{f}, \mathbf{g})$ is a system of generators for $\mathfrak{m}_B$. We have a natural identification of complexes of B-modules :

$$\mathbf{K}_\bullet(\mathbf{f}, \mathbf{g}) \xrightarrow{\sim} \mathrm{Tot}_\bullet(\mathbf{K}_\bullet(\mathbf{f}) \otimes_A \mathbf{K}_\bullet(\mathbf{g}))$$

whence natural isomorphisms :

$$\mathbf{K}^\bullet((\mathbf{f}, \mathbf{g}), M \otimes_A N) \xrightarrow{\sim} \mathrm{Tot}^\bullet(\mathbf{K}^\bullet(\mathbf{f}, M) \otimes_A \mathbf{K}^\bullet(\mathbf{g}, N)).$$

A standard spectral sequence, associated with the filtration by rows, converges to the cohomology of this total complex, and since N is a flat A-module, $\mathbf{K}^\bullet(\mathbf{g}, N)$ is a complex of flat A-modules, so that the E_1 term of this spectral sequence is found to be :

$$E_1^{pq} \simeq H^q(\mathbf{f}, M) \otimes_A \mathbf{K}^p(\mathbf{g}, N)$$

and the differentials $d_1^{pq} : E_1^{pq} \to E_1^{p+1,q}$ are induced by the differentials of the complex $\mathbf{K}^\bullet(\mathbf{g}, N)$. Set $\kappa_A := A/\mathfrak{m}_A$; notice that $H^q(\mathbf{f}, M)$ is a κ_A-vector space, hence

$$H^q(\mathbf{f}, M) \otimes_A H^\bullet(\mathbf{g}, N) \simeq H^\bullet(H^q(\mathbf{f}, M) \otimes_{\kappa_A} \mathbf{K}^\bullet(\overline{\mathbf{g}}, N/\mathfrak{m}_A N))$$

and consequently :

$$E_2^{pq} \simeq H^q(\mathbf{f}, M) \otimes_{\kappa_A} H^p(\overline{\mathbf{g}}, N/\mathfrak{m}_A N).$$

Especially :

(10.4.39) $\qquad E_2^{pq} = 0$ if either $p < d := \mathrm{depth}_A M$ or $q < d' := \mathrm{depth}_{B/\mathfrak{m}_A B}(N/\mathfrak{m}_A N)$.

Hence $H^i((\mathbf{f}, \mathbf{g}), M \otimes_A N) = 0$ for all $i < d + d'$ and moreover, if d and d' are finite, $H^{d+d'}((\mathbf{f}, \mathbf{g}), M \otimes_A N) \simeq E_2^{dd'} \neq 0$. Now the sought identity follows readily from proposition 10.4.32(i). $\square$

Remark 10.4.40. Let d and d' be as in (10.4.39), and suppose that d and d' are finite. By inspection of the proof, we see that – under the assumptions of theorem 10.4.38 – a natural isomorphism has been exhibited :

$$\mathrm{Ext}_A^{d+d'}(B/\mathfrak{m}_B, M \otimes_A N) \simeq \mathrm{Ext}_A^d(A/\mathfrak{m}_A, M) \otimes_A \mathrm{Ext}_{B/\mathfrak{m}_A B}^{d'}(B/\mathfrak{m}_B, N/\mathfrak{m}_A N).$$

Definition 10.4.41. Let $(A, \mathfrak{m})$ be a local ring, and $M_\bullet \in \mathrm{Ob}(\mathsf{D}^b(A\text{-}\mathbf{Mod}))$. The *residual Tor dimension* of $M_\bullet$ is $\mathrm{rTor.dim}_A(M_\bullet) := \sup\{i \in \mathbb{Z} \mid H_i(M_\bullet \overset{\mathbf{L}}{\otimes}_A A/\mathfrak{m}) \neq 0\} \in \mathbb{Z} \cup \{\pm\infty\}$.

Remark 10.4.42. (i) If A is a noetherian local ring, and M is any A-module of finite type, then $\mathrm{rTor.dim}_A(M)$ equals the projective dimension of M : see [126, §19, Lemma 1].

(ii) Our next result generalizes [15, Th.4.1].

Theorem 10.4.43. *Let A be a local noetherian ring, and $M_\bullet \in \mathrm{Ob}(\mathsf{D}^b(A\text{-}\mathbf{Mod}))$. Then :*

(i) $\mathrm{depth}_A(M_\bullet) = +\infty$ *if and only if* $\mathrm{rTor.dim}_A(M_\bullet) = -\infty$.

(ii) *If* $\mathrm{rTor.dim}_A(M_\bullet) \in \mathbb{Z}$, *we have the* Auslander-Buchsbaum identity *:*

$$\mathrm{depth}_A(M_\bullet) + \mathrm{rTor.dim}_A(M_\bullet) = \mathrm{depth}_A A.$$

Proof. (ii): Let $d := \dim A$, and pick a *system of parameters* $\mathbf{f}$ of A, *i.e.* a sequence $(f_1, \dots, f_d)$ of d elements of A that generates a primary ideal. Using a Cartan-Eilenberg resolution of the Koszul complex $\mathbf{K}_\bullet(\mathbf{f})$ (see remark 7.8.1(ii)), we get a spectral sequence (see [163, §5.7])

$$E^2_{pq} := H_p(M_\bullet \overset{\mathbf{L}}{\otimes}_A H_q(\mathbf{f}, A)) \Rightarrow H_{p+q}(\mathbf{f}, M_\bullet).$$

Now, by proposition 10.4.32(i) and lemma 7.8.2(v), we have :

$$\mathrm{depth}_A A = d - e_A \quad \text{where } e_A := \inf\{i \in \mathbb{Z} \mid H_j(\mathbf{f}, A) = 0 \text{ for all } i > j\}$$

$$\mathrm{depth}_A M_\bullet = d - e_{M_\bullet} \quad \text{where } e_{M_\bullet} := \inf\{i \in \mathbb{Z} \mid H_j(\mathbf{f}, M_\bullet) = 0 \text{ for all } i > j\}.$$

Thus, we have to show that if $t_{M_\bullet} := \mathrm{rTor.dim}_A(M_\bullet) \in \mathbb{Z}$, then

$$e_{M_\bullet} = e_A + t_{M_\bullet}. \tag{10.4.44}$$

Claim 10.4.45. For every A-module $N \neq 0$ of finite length, we have :

$$H_{t_{M_\bullet}}(M_\bullet \overset{\mathbf{L}}{\otimes} N) \neq 0 \qquad \text{and} \qquad H_i(M_\bullet \overset{\mathbf{L}}{\otimes} N) = 0 \qquad \text{for every } i > t_{M_\bullet}.$$

Proof of the claim. There exists a finite filtration $0 := N_0 \subset N_1 \subset \cdots \subset N_{n-1} \subset N_n := N$ such that N_{i+1}/N_i is isomorphic to $A/\mathfrak{m}$ for $i = 0, \dots, n-1$. Hence, by a simple induction on the length n of the filtration, we are reduced to checking the following assertion : let $\Sigma_\bullet := (0 \to N' \to N \to A/\mathfrak{m} \to 0)$ be a short exact sequence of A-modules of finite length. The induced map $H_i(M_\bullet \overset{\mathbf{L}}{\otimes}_A N') \to H_i(M_\bullet \overset{\mathbf{L}}{\otimes}_A N)$ is an isomorphism for every $i \geq t_{M_\bullet}$. The latter follows easily, y considering the long exact homology sequence induced by the short exact sequence of complexes $M_\bullet \overset{\mathbf{L}}{\otimes}_A \Sigma_\bullet$. ◊

Now, since A is noetherian and $\mathbf{f}$ generates a primary ideal, $H_q(\mathbf{f}, A)$ is a module of finite length for every $q \in \mathbb{N}$ (lemma 7.8.2(i)). Claim 10.4.45 implies that $0 \neq E^2_{t_{M_\bullet}, e_A}$, and $E^2_{pq} = 0$ whenever either $p > t_{M_\bullet}$ or $q > e_A$, so that $E^2_{t_{M_\bullet}, e_A} = E^\infty_{t_{M_\bullet}, e_A}$, whence (10.4.44).

(i): If $t_{M_\bullet} \in \mathbb{Z}$, then (10.4.44) implies that $e_{M_\bullet} \in \mathbb{Z}$, since $e_A \in \mathbb{N}$; then in this case $\mathrm{depth}_A M_\bullet \in \mathbb{Z}$. If $t_{M_\bullet} = -\infty$, claim 10.4.45 says that $E^2_{pq} = 0$ for every $p, q \in \mathbb{Z}$, whence $H_i(\mathbf{f}, M) = 0$ for every $i \in \mathbb{Z}$, *i.e.* $\mathrm{depth}_A(M_\bullet) = +\infty$. Lastly, by proposition 10.4.32(iii), this condition holds if and only if $\mathrm{colim}_{m\in\mathbb{N}}\, \mathbf{K}^\bullet(\mathbf{f}^m, M^\bullet) = 0$ in $\mathsf{D}^b(A\text{-}\mathbf{Mod})$. Set

$$K_\bullet := \operatorname*{colim}_{m\in\mathbb{N}} \mathbf{K}_\bullet(\mathbf{f}^m) \qquad T_{m,\bullet} := t^{\leq -1}\mathbf{K}_\bullet(\mathbf{f}^m) \quad \text{for every } m \in \mathbb{N} \qquad T_\bullet := \operatorname*{colim}_{m\in\mathbb{N}} T_{m,\bullet}.$$

Then, in turn the latter means that $M^\bullet \otimes_A K_\bullet = 0$ in $\mathsf{D}^b(A\text{-}\mathbf{Mod})$ (lemma 7.8.2(v)). Set $k := A/\mathfrak{m}$; the short exact of complexes $0 \to A[0] \to K_\bullet \to T_\bullet \to 0$ induces a distinguished triangle in $\mathsf{D}^-(A\text{-}\mathbf{Mod})$:

$$M_\bullet \overset{\mathbf{L}}{\otimes}_A k[0] \to M_\bullet \overset{\mathbf{L}}{\otimes}_A K_\bullet \overset{\mathbf{L}}{\otimes}_A k[0] \to M_\bullet \overset{\mathbf{L}}{\otimes}_A T_\bullet \overset{\mathbf{L}}{\otimes}_A k[0] \to M_\bullet \overset{\mathbf{L}}{\otimes}_A k[1].$$

Notice that the transition maps $T_{m,\bullet} \to T_{m+1,\bullet}$ induce the zero maps $T_{m,\bullet} \otimes_A k \to T_{m+1,\bullet} \otimes_A k$ for every $m \in \mathbb{N}$; hence $T_\bullet \overset{\mathbf{L}}{\otimes}_A k[0] = 0$ in $\mathsf{D}(A\text{-}\mathbf{Mod})$, and therefore $M_\bullet \overset{\mathbf{L}}{\otimes}_A T_\bullet \overset{\mathbf{L}}{\otimes}_A k[0] = 0$ as well. Summing up, we conclude that if $\mathrm{depth}_A M_\bullet = +\infty$, then $M_\bullet \overset{\mathbf{L}}{\otimes}_A k[0] = 0$ in $\mathsf{D}(A\text{-}\mathbf{Mod})$, *i.e.* $t_{M^\bullet} = -\infty$. □

10.4.46. Let $f : X \to S$ be any morphism of schemes. Recall that an $\mathscr{O}_X$-module $\mathscr{F}$ is said to be *f-flat at a point* $x \in X$ if $\mathscr{F}_x$ is a flat $\mathscr{O}_{S,f(x)}$-module. We say that $\mathscr{F}$ is *f-flat over a point* $s \in S$ if $\mathscr{F}$ is f-flat at all points of $f^{-1}(s)$. Finally, we say that $\mathscr{F}$ is *f-flat* if $\mathscr{F}$ is f-flat at all the points of X ([59, Ch.0, §6.7.1]).

Corollary 10.4.47. *Let $f : X \to S$ be a finitely presented morphism of schemes, $\mathscr{F}$ a f-flat quasi-coherent $\mathscr{O}_X$-module of finite presentation, $\mathscr{G}$ any quasi-coherent $\mathscr{O}_S$-module, $x \in X$ any point, $i : f^{-1}f(x) \to X$ the natural morphism. Then :*

$$\delta'(x, \mathscr{F} \otimes_{\mathscr{O}_X} f^*\mathscr{G}) = \delta'(x, i^*\mathscr{F}) + \delta'(f(x), \mathscr{G}).$$

Proof. Set $s := f(x)$, and denote by

$$j_s : S(s) := \mathrm{Spec}\, \mathscr{O}_{S,s} \to S \qquad \text{and} \qquad f_s : X(s) := X \times_S S(s) \to S(s)$$

the induced morphisms. By inspecting the definitions, one checks easily that

$$\delta'(x, \mathscr{F} \otimes_{\mathscr{O}_X} f^*\mathscr{G}) = \delta'(x, \mathscr{F}_{|X(s)} \otimes_{\mathscr{O}_{X(s)}} f_s^*\mathscr{G}(s)) \qquad \text{and} \qquad \delta'(s, \mathscr{G}) = \delta'(s, \mathscr{G}(s))$$

(notation of definition 4.9.17(iii)). Thus, we may replace S by $S(s)$ and X by $X(s)$, and assume from start that S is a local scheme and s its closed point. Clearly we may also assume that X is affine and finitely presented over S. Then we can write S as the limit of a cofiltered family $(S_\lambda \mid \lambda \in \Lambda)$ of local noetherian schemes, with local transition maps, and we may assume that $f : X \to S$ is a limit of a cofiltered family $(f_\lambda : X_\lambda \to S_\lambda \mid \lambda \in \Lambda)$ of morphisms of finite type, such that $X_\mu = X_\lambda \times_{S_\lambda} S_\mu$ whenever $\mu \geq \lambda$. Likewise, we may descend $\mathscr{F}$ to a family $(\mathscr{F}_\lambda \mid \lambda \in \Lambda)$ of finitely presented quasi-coherent $\mathscr{O}_{X_\lambda}$-modules, such that $\mathscr{F}_\mu = \varphi^*_{\mu\lambda}\mathscr{F}_\lambda$ for every $\mu \geq \lambda$ (where $\varphi_{\mu\lambda} : X_\mu \to X_\lambda$ is the natural morphism). Furthermore, up to replacing Λ by some coinitial subset, we may assume that $\mathscr{F}_\lambda$ is f_λ-flat for every $\lambda \in \Lambda$ ([65, Ch.IV, Cor.11.2.6.1(ii)]). For every $\lambda \in \Lambda$, denote by $\psi_\lambda : S \to S_\lambda$ the natural morphism, so that $s_\lambda := \psi_\lambda(s)$ is the closed point of S_λ; for every constructible closed subset $Z_\lambda \subset S_\lambda$, the set $Z := \psi_\lambda^{-1} Z_\lambda$ is constructible and closed in S, and according to lemma 10.4.17(ii) we have $\mathrm{depth}_Z \mathscr{G} = \mathrm{depth}_{Z_\lambda} \psi_{\lambda *}\mathscr{G}$. On the other hand, every closed constructible subset $Z \subset S$ is of the form $\psi_\lambda^{-1} Z_\lambda$ for some $\lambda \in \Lambda$ and some Z_λ closed constructible in S_λ ([65, Ch.IV, Cor.8.3.11]), so that :

$$\delta'(s, \mathscr{G}) = \sup\{\delta'(s_\lambda, \psi_{\lambda *}\mathscr{G}) \mid \lambda \in \Lambda\}. \tag{10.4.48}$$

Similarly, for every $\lambda \in \Lambda$ let x_λ be the image of x in X_λ and denote by $i_\lambda : f_\lambda^{-1}(s_\lambda) \to X_\lambda$ the natural morphism. Notice that $f_\lambda^{-1}(s_\lambda)$ is a scheme of finite type over $\mathrm{Spec}\,\kappa(s_\lambda)$, especially it is noetherian, hence $\delta'(x_\lambda, i_\lambda^* \mathscr{F}_\lambda) = \delta(x_\lambda, i_\lambda^* \mathscr{F}_\lambda)$, by lemma 10.4.20(ii); by the same token we have as well the identity : $\delta'(x, i^*\mathscr{F}) = \delta(x, i^*\mathscr{F})$. Denote by $g_\lambda : f^{-1}(s) \to f_\lambda^{-1}(s_\lambda)$ and $g_{\lambda,x} : \mathrm{Spec}\,\mathscr{O}_{f^{-1}(s),x} \to \mathrm{Spec}\,\mathscr{O}_{f_\lambda^{-1}(s_\lambda),x_\lambda}$ the induced morphisms.

Claim 10.4.49. There exists $\lambda \in \Lambda$ such that $g_{\mu,x}^{-1}(x_\mu) = \{x\}$ for every $\mu \geq \lambda$.

Proof of the claim. Let Z be the topological closure of $\{x\}$ in $f^{-1}(s)$; we may find $\lambda \in \Lambda$ and a closed subset $Z_\lambda \subset f_\lambda^{-1}(s_\lambda)$ such that $Z = g_\lambda^{-1} Z_\lambda$ ([65, Ch.IV, Cor.8.3.11]). Since Z is irreducible and g_λ is surjective, it is easily seen that Z_λ is irreducible. Moreover, the topological closure Z'_λ of $\{x_\lambda\}$ in $f_\lambda^{-1}(s_\lambda)$ lies in Z_λ, and $g_\lambda^{-1} Z'_\lambda$ contains x; thus, $Z_\lambda = Z'_\lambda$. It follows that $g_{\lambda,x}^{-1}(x_\lambda) = \{x\}$, whence the assertion. ◊

Since $g_{\lambda,x}$ is faithfully flat, from claim 10.4.49 and corollary 10.4.37 we see that, after replacing Λ by a coinitial subset, we may assume :

$$\delta'(x_\lambda, i_\lambda^* \mathscr{F}_\lambda) = \delta'(x, i^*\mathscr{F}) \qquad \text{for every } \lambda \in \Lambda. \tag{10.4.50}$$

Claim 10.4.51. Let K be a field, X and Y two K-schemes, and set $Z := X \times_{\mathrm{Spec}\,K} Y$. Let also $p : Z \to X$ and $q : Z \to Y$ be the projections, $\mathscr{M}$ any $\mathscr{O}_X$-module, and $\mathscr{N}$ any $\mathscr{O}_Y$-module. Then we have :

$$\mathrm{Supp}\,(p^*\mathscr{M} \otimes_{\mathscr{O}_Z} q^*\mathscr{N}) = p^{-1}(\mathrm{Supp}\,\mathscr{M}) \cap q^{-1}(\mathrm{Supp}\,\mathscr{N}).$$

Proof of the claim. To ease notation, set $\mathscr{P} := p^*\mathscr{M} \otimes_{\mathscr{O}_Z} q^*\mathscr{N}$. Let $z \in Z$ be any point with $x := p(z) \in \mathrm{Supp}\,\mathscr{M}$ and $y := q(z) \in \mathrm{Supp}\,\mathscr{N}$; it suffices to show that $\mathscr{P}_z \neq 0$. Now, $\mathscr{O}_{Z,z}$ is a localization of $\mathscr{O}_{X,x} \otimes_K \mathscr{O}_{Y,y}$, at a prime ideal $\mathfrak{p}$, and $\mathscr{P}_z = (\mathscr{M}_x \otimes_K \mathscr{N}_y)_\mathfrak{p}$. We come down therefore to the following situation. Let A and B be two local K-algebras, $\mathfrak{m}_A \subset A$, $\mathfrak{m}_B \subset B$ the respective maximal ideals, $M \neq 0$ an A-module, $N \neq 0$ a B-modules, and $\mathfrak{p} \subset A \otimes_K B$ a prime ideal such that

$$\mathfrak{m}_A \otimes_K B + A \otimes_K \mathfrak{m}_B \subset \mathfrak{p}. \tag{10.4.52}$$

We have to check that $(M \otimes_K N)_\mathfrak{p} \neq 0$. To this aim, we may reduce easily to the case where M and N are cyclic modules, in which case there exist a surjective A-linear map $M \to \kappa(A) := A/\mathfrak{m}_A$ and a surjective B-linear map $N \to \kappa(B) := B/\mathfrak{m}_B$; we are then further reduced to showing that $(\kappa(A) \otimes_K \kappa(B))_\mathfrak{p} \neq 0$, which is ensured by (10.4.52). ◊

Next, notice that the local scheme $X(x)$ is the limit of the cofiltered system of local schemes $(X_\lambda(x_\lambda) \mid \lambda \in \Lambda)$. Let $Z_\lambda \subset X_\lambda(x_\lambda)$ be a closed constructible subset and set $Z := \varphi_\lambda^{-1} Z_\lambda$, where $\varphi_\lambda : X(x) \to X_\lambda(x_\lambda)$ is the natural morphism. Let Y_λ be the fibre product in the cartesian diagram of schemes :

$$\begin{array}{ccc} Y_\lambda & \xrightarrow{q_\lambda} & X_\lambda(x_\lambda) \\ {\scriptstyle p_\lambda}\downarrow & & \downarrow \\ S & \xrightarrow{\psi_\lambda} & S_\lambda. \end{array}$$

The morphism φ_λ factors through a unique morphism of S-schemes $\overline{\varphi}_\lambda : X(x) \to Y_\lambda$, and if $y_\lambda \in Y_\lambda$ is the image of the closed point x of $X(x)$, then $\overline{\varphi}_\lambda$ induces a natural identification $X(x) \stackrel{\sim}{\to} Y_\lambda(y_\lambda)$. To ease notation, let us set :

$$\mathscr{H} := \mathscr{F} \otimes_{\mathscr{O}_X} f^*\mathscr{G} \qquad \text{and} \qquad \mathscr{H}_\lambda := \mathscr{F}_\lambda \otimes_{\mathscr{O}_{X_\lambda}} f_\lambda^* \psi_{\lambda *}\mathscr{G}.$$

In view of corollary 10.4.10, there follows a natural isomorphism :

$$R\underline{\Gamma}_Z \mathscr{H}(x) \stackrel{\sim}{\to} \overline{\varphi}{}_\lambda^* R\underline{\Gamma}_{q_\lambda^{-1}(Z_\lambda)}(q_\lambda^* \mathscr{F}_\lambda(x_\lambda) \otimes_{\mathscr{O}_{Y_\lambda}} p_\lambda^* \mathscr{G}). \tag{10.4.53}$$

On the other hand, applying the projection formula (see remark 10.3.7) we get :

$$q_{\lambda *}(q_\lambda^* \mathscr{F}_\lambda(x_\lambda) \otimes_{\mathscr{O}_{Y_\lambda}} p_\lambda^* \mathscr{G}) \simeq \mathscr{H}_\lambda(x_\lambda). \tag{10.4.54}$$

Combining (10.4.53), (10.4.54) and lemma 10.4.13(v) we deduce :

$$R\underline{\Gamma}_Z \mathscr{H}(x) \simeq \varphi_\lambda^* R\underline{\Gamma}_{Z_\lambda} \mathscr{H}_\lambda(x_\lambda).$$

Notice that the foregoing argument applies also in case S is replaced by some S_μ for some $\mu \geq \lambda$ (and consequently X is replaced by X_μ); we arrive therefore at the inequality :

$$\delta'(x_\lambda, \mathscr{H}_\lambda) \leq \delta'(x_\mu, \mathscr{H}_\mu) \leq \delta'(x, \mathscr{H}) \qquad \text{whenever } \mu \geq \lambda. \tag{10.4.55}$$

Claim 10.4.56. $\delta'(x, \mathscr{H}) = \sup \{\delta'(x_\lambda, \mathscr{H}_\lambda) \mid \lambda \in \Lambda\}$.

Proof of the claim. Due to (10.4.55) we may assume – again after replacing Λ by a cofinal subset – that $\delta'(x_\lambda, \mathscr{H}_\lambda)$ is a constant $d \in \mathbb{N}$ independent of λ. Especially, $R^d\underline{\Gamma}_{Z_\lambda} \mathscr{H}_\lambda(x_\lambda) \neq 0$ for every $\lambda \in \Lambda$ and every closed constructible subset $Z_\lambda \subset X_\lambda(x_\lambda)$. We have to show that $R^d\underline{\Gamma}_Z \mathscr{H}(x) \neq 0$ for arbitrarily small constructible closed subsets $Z \subset X(x)$. However, given such Z, there exist $\lambda \in \Lambda$ and Z_λ closed constructible in $X_\lambda(x_\lambda)$ such that $Z = \varphi^{-1}(Z_\lambda)$ ([65, Ch.IV, Cor.8.3.11]). Say that:

$$S = \operatorname{Spec} A \qquad S_\lambda = \operatorname{Spec} A_\lambda \qquad X_\lambda(x_\lambda) = \operatorname{Spec} B_\lambda.$$

Hence $Y_\lambda \simeq \operatorname{Spec} A \otimes_{A_\lambda} B_\lambda$. Let also F_λ (resp. G) be a B_λ-module (resp. A-module) such that $F_\lambda^\sim \simeq \mathscr{F}_\lambda$ (resp. $G^\sim \simeq \mathscr{G}$). Then $\mathscr{H}_{|Y_\lambda} \simeq (F_\lambda \otimes_{A_\lambda} G)^\sim$. Let $\mathfrak{m}_\lambda \subset B_\lambda$ and $\mathfrak{n}_\lambda \subset A_\lambda$ be the maximal ideals. Up to replacing Z by a smaller subset, we may assume that $Z_\lambda = V(\mathfrak{m}_\lambda)$. To ease notation, set :

$$E_1 := \operatorname{Ext}^a_{A_\lambda}(\kappa(s_\lambda), G) \qquad E_2 := \operatorname{Ext}^b_{B_\lambda/\mathfrak{n}_\lambda B_\lambda}(\kappa(x_\lambda), F_\lambda/\mathfrak{n}_\lambda F_\lambda)$$

with $a = \delta'(s_\lambda, \psi_{\lambda *}\mathscr{G})$ and $b = d - a$. Proposition 10.4.32(i),(ii) and remark 10.4.40 say that

$$0 \neq E_3 := \operatorname{Ext}^d_{B_\lambda}(\kappa(x_\lambda), F_\lambda \otimes_{A_\lambda} G) \simeq E_1 \otimes_{\kappa(s_\lambda)} E_2.$$

To conclude, it suffices to show that $y_\lambda \in \operatorname{Supp} E_3$. However, claim 10.4.51 says that

$$\operatorname{Supp} E_3 = p_\lambda^{-1}(\operatorname{Supp} E_1) \cap q_\lambda^{-1}(\operatorname{Supp} E_2).$$

Now, $p_\lambda(y_\lambda) = s$, and clearly $s \in \operatorname{Supp} E_1$, since G is an A-module and s is the closed point of S; likewise, $q_\lambda(y_\lambda) = x_\lambda$ is the closed point of $X_\lambda(x_\lambda)$, therefore it is in the support of E_2. $\diamondsuit$

We can now conclude the proof : indeed, from theorem 10.4.38 we derive :

$$\delta'(x_\lambda, \mathscr{H}_\lambda) = \delta'(x, i_\lambda^* \mathscr{F}_\lambda) + \delta'(f(x), \psi_{\lambda *}\mathscr{G}) \qquad \text{for every } \lambda \in \Lambda$$

and then the corollary follows from (10.4.48), (10.4.50) and claim 10.4.56. □

10.4.57. Consider now a finitely presented morphism of schemes $f : X \to Y$, and a sheaf of ideals $\mathscr{I} \subset \mathscr{O}_X$ of finite type. Let $x \in X$ be any point, and set $y := f(x)$; notice that $f^{-1}(y)$ is an algebraic $\kappa(y)$-scheme, so that the local ring $\mathscr{O}_{f^{-1}(y),x}$ is noetherian. We let $\mathscr{I}(y) := i_y^{-1}\mathscr{I} \cdot \mathscr{O}_{f^{-1}(y)}$, which is a coherent sheaf of ideals of the fibre $f^{-1}(y)$. The *fibrewise $\mathscr{I}$-depth of X over Y at the point x* is the integer:

$$\mathrm{depth}_{\mathscr{I},f}(x) := \mathrm{depth}_{\mathscr{I}(y)_x}\mathscr{O}_{f^{-1}(y),x}.$$

In case $\mathscr{I} = \mathscr{O}_X$, the fibrewise $\mathscr{I}$-depth shall be called the *fibrewise depth* at the point x, and shall be denoted by $\mathrm{depth}_f(x)$. In view of (10.4.30) we have the identity:

$$\mathrm{depth}_{\mathscr{I},f}(x) = \inf\{\mathrm{depth}_f(z) \mid z \in V(\mathscr{I}(y)) \text{ and } x \in \overline{\{z\}}\}. \tag{10.4.58}$$

10.4.59. The fibrewise $\mathscr{I}$-depth can be computed locally as follows. For a given $x \in X$, set $y := f(x)$ and let $U \subset X$ be any affine open neighborhood of x such that $\mathscr{I}_{|U}$ is generated by a finite family $\mathbf{f} := (f_i)_{1\leq i\leq r}$, where $f_i \in \mathscr{I}(U)$ for every $i \leq r$. Then proposition 10.4.32(i) implies that :

$$\mathrm{depth}_{\mathscr{I},f}(x) = \sup\{n \in \mathbb{N} \mid H^i(\mathbf{K}^\bullet(\mathbf{f}, \mathscr{O}_X(U))_x \otimes_{\mathscr{O}_{Y,y}} \kappa(y)) = 0 \text{ for all } i < n\}. \tag{10.4.60}$$

Proposition 10.4.61. *In the situation of* (10.4.57)*, for every integer $d \in \mathbb{N}$ we define the subset:*

$$L_{\mathscr{I}}(d) := \{x \in X \mid \mathrm{depth}_{\mathscr{I},f}(x) \geq d\}.$$

Then the following holds:

(i) *$L_{\mathscr{I}}(d)$ is a constructible subset of X (see definition* 8.1.13(v)*).*

(ii) *Let $Y' \to Y$ be any morphism of schemes, set $X' := X \times_Y Y'$, and let $g : X' \to X$, $f' : X' \to Y'$ be the induced morphisms. Then:*

$$L_{g^*\mathscr{I}}(d) = g^{-1}L_{\mathscr{I}}(d) \qquad \textit{for every } d \in \mathbb{N}.$$

Proof. (ii): The identity can be checked on the fibres, hence we may assume that $Y = \operatorname{Spec} k$ and $Y' = \operatorname{Spec} k'$ for some fields $k \subset k'$. Let $x' \in X'$ be any point and $x \in X$ its image; since the map $\mathscr{O}_{X,x} \to \mathscr{O}_{X',x'}$ is faithfully flat, corollary 10.4.37 implies:

$$\mathrm{depth}_{g^*\mathscr{I},f'}(x') = \mathrm{depth}_{\mathscr{I},f}(x)$$

whence (ii).

(i): We may assume that $Y = \operatorname{Spec} A$, $X = \operatorname{Spec} B$, $\mathscr{I} = I^\sim$, where A is a ring, B is a finitely presented A-algebra, and $I \subset B$ is a finitely generated ideal, for which we choose a finite system of generators $\mathbf{f} := (f_i)_{1\leq i\leq r}$. We may also assume that A is reduced. Suppose first that A is noetherian; then, for every $i \in \mathbb{N}$ and $y \in Y$ let us set:

$$N_{\mathscr{I}}(i) = \bigcup_{y\in Y} \operatorname{Supp} H^i(\mathbf{K}^\bullet(\mathbf{f}, B) \otimes_A \kappa(y)).$$

Taking into account (10.4.30) and (10.4.60), we deduce:

$$L_{\mathscr{I}}(d) = \bigcap_{i=0}^{d-1}(X \setminus N_{\mathscr{I}}(i)).$$

It suffices therefore to show that each subset $N_{\mathscr{I}}(i)$ is constructible. For every $j \in \mathbb{N}$ we set:

$$\mathbf{Z}^j := \mathrm{Ker}(d^j : \mathbf{K}^j(\mathbf{f}, B) \to \mathbf{K}^{j+1}(\mathbf{f}, B)) \qquad \mathbf{B}^j := \mathrm{Im}(d^{j-1} : \mathbf{K}^{j-1}(\mathbf{f}, B) \to \mathbf{K}^j(\mathbf{f}, B)).$$

Using [65, Ch.IV, Cor. 8.9.5], one deduces easily that there exists an affine open subscheme $U \subset Y$, say $U = \operatorname{Spec} A'$ for some flat A-algebra A', such that $A' \otimes_A \mathbf{Z}^\bullet$, $A' \otimes_A \mathbf{B}^\bullet$ and $A' \otimes_A H^\bullet(\mathbf{K}^\bullet(\mathbf{f}, B))$ are flat A'-modules. By noetherian induction, we can then replace Y by U and X by $X \times_Y U$, and assume from start that $\mathbf{Z}^\bullet$, $\mathbf{B}^\bullet$ and $H^\bullet(\mathbf{K}^\bullet(\mathbf{f}, B))$ are flat A-modules. In such case, taking homology of the complex $\mathbf{K}^\bullet(\mathbf{f}, B)$ commutes with any base change; therefore:

$$N_{\mathscr{I}}(i) = \bigcup_{y \in Y} \operatorname{Supp} H^i(\mathbf{f}, B) \otimes_A \kappa(y) = \operatorname{Supp} H^i(\mathbf{f}, B)$$

whence the claim, since the support of a B-module of finite type is closed in X.

Finally, for a general ring A, we can find a noetherian subalgebra $A' \subset A$, an A'-algebra B' of finite type and a finitely generated ideal $I' \subset B'$ such that $B = A \otimes_{A'} B$ and $I = I'B$. Let $\mathscr{I}'$ be the sheaf of ideals on $X' := \operatorname{Spec} B'$ determined by I'; by the foregoing, $L_{\mathscr{I}'}(d)$ is a constructible subset of X'. Thus, the assertion follows from (ii), and the fact that a morphism of schemes is continuous for the constructible topology ([63, Ch.IV, Prop.1.8.2]). □

10.5. **Depth and associated primes.**

Definition 10.5.1. Let X be a scheme, $\mathscr{F}$ a quasi-coherent $\mathscr{O}_X$-module.

(i) Let $x \in X$ be any point; we say that x is *associated with* $\mathscr{F}$ if there exists $f \in \mathscr{F}_x$ such that the radical of the annihilator of f in $\mathscr{O}_{X,x}$ is the maximal ideal of $\mathscr{O}_{X,x}$. If x is a point associated with $\mathscr{F}$ and x is not a maximal point of $\operatorname{Supp} \mathscr{F}$, we say that x is an *imbedded point* for $\mathscr{F}$. We shall denote :

$$\operatorname{Ass} \mathscr{F} := \{x \in X \mid x \text{ is associated with } \mathscr{F}\}.$$

(ii) We say that the $\mathscr{O}_X$-module $\mathscr{F}$ *satisfies condition* S_1 if every associated point of $\mathscr{F}$ is a maximal point of X.

(iii) Likewise, if X is affine, say $X = \operatorname{Spec} A$, and M is any A-module, we denote by $\operatorname{Ass}_A M \subset X$ (or just $\operatorname{Ass} M$, if the notation is not ambiguous) the set of prime ideals associated with the $\mathscr{O}_X$-module $M^\sim$ arising from M. An associated (resp. imbedded) point of $M^\sim$ is also called an *associated* (resp. *imbedded*) *prime ideal* of M.
We say that M *satisfies condition* S_1 if the same holds for the $\mathscr{O}_X$-module $M^\sim$.

(iv) Let $x \in X$ be a point, $\mathscr{G}$ a quasi-coherent $\mathscr{O}_X$-submodule of $\mathscr{F}$. We say that $\mathscr{G}$ is a *x-primary submodule* of $\mathscr{F}$ if $\operatorname{Ass} \mathscr{F}/\mathscr{G} = \{x\}$. We say that $\mathscr{G}$ is a *primary submodule* of $\mathscr{F}$ if there exists a point $x \in X$ such that $\mathscr{G}$ is x-primary.

(v) We say that a submodule $\mathscr{G}$ of the $\mathscr{O}_X$-module $\mathscr{F}$ admits a *primary decomposition* if there exist primary submodules $\mathscr{G}_1, \dots, \mathscr{G}_n \subset \mathscr{F}$ such that $\mathscr{G} = \mathscr{G}_1 \cap \cdots \cap \mathscr{G}_n$.

Remark 10.5.2. (i) Our definition of associated point is borrowed from [86, Partie I, Déf.3.2.1], and it corresponds to Bourbaki's notion of *weakly associated prime* ([34, Ch.IV, §1, Exerc.17]), also called *weak Bourbaki prime* in other works. Bourbaki's definition of associated prime, in [34, Ch.IV, §1, no.1] (which is the same as in [85, Exp.III, Déf.1.1]), agrees with ours for modules over noetherian rings (see lemma 10.5.3(ii)), but in general the two notions diverge.

(ii) For a noetherian ring A and a finitely generated A-module M, our condition S_1 is the same as in [89]. It also agrees with that of [64, Ch.IV, Déf.5.7.2], in case $\operatorname{Supp} M = \operatorname{Spec} A$.

Lemma 10.5.3. *Let A be a ring, and M any A-module. The following holds :*

(i) $\operatorname{Ass} M$ *is the set of all* $\mathfrak{p} \in \operatorname{Spec} A$ *with the following property. There exists* $m \in M$*, such that* $\mathfrak{p}$ *is a maximal point of the closed subset* $\operatorname{Supp}(m)$ *(*i.e. $\mathfrak{p}$ *is the preimage of a minimal prime ideal of the ring* $A/\operatorname{Ann}_A(m)$*).*

(ii) *If* $\mathfrak{p} \in \operatorname{Ass} M$*, and* $\mathfrak{p}$ *is finitely generated, there exists* $m \in M$ *with* $\mathfrak{p} = \operatorname{Ann}_A(m)$.

(iii) *The natural map* $M \to \prod_{\mathfrak{p} \in \operatorname{Ass} M} M_{\mathfrak{p}}$ *is injective.*

Proof. (i): Indeed, suppose that $m \in M\setminus\{0\}$ is any element, and $\mathfrak{p}$ is a maximal point of $V(\mathrm{Ann}_A(m))$; denote by $m_\mathfrak{p} \in M_\mathfrak{p} := M \otimes_A A_\mathfrak{p}$ the image of m. Then :

$$\mathrm{Ann}_{A_\mathfrak{p}}(m_\mathfrak{p}) = \mathrm{Ann}_A(m) \otimes_A A_\mathfrak{p}. \tag{10.5.4}$$

Hence $A_\mathfrak{p}/\mathrm{Ann}_{A_\mathfrak{p}}(m_\mathfrak{p}) = (A/\mathrm{Ann}_A(m)) \otimes_A A_\mathfrak{p}$, and the latter is by assumption a local ring of Krull dimension zero; it follows easily that the radical of $\mathrm{Ann}_{A_\mathfrak{p}}(m_\mathfrak{p})$ is $\mathfrak{p}A_\mathfrak{p}$, *i.e.* $\mathfrak{p} \in \mathrm{Ass}_A M$.

Conversely, suppose that $\mathfrak{p} \in \mathrm{Ass}_A M$; then there exists $m_\mathfrak{p} \in M_\mathfrak{p}$ such that the radical of $\mathrm{Ann}_{A_\mathfrak{p}}(m_\mathfrak{p})$ equals $\mathfrak{p}A_\mathfrak{p}$. We may assume that $m_\mathfrak{p}$ is the image of some $m \in M$; then (10.5.4) implies that $V(\mathrm{Ann}_A(m)) \cap \mathrm{Spec}\, A_\mathfrak{p} = V(\mathrm{Ann}_{A_\mathfrak{p}}(m_\mathfrak{p})) = \{\mathfrak{p}\}$, so $\mathfrak{p}$ is a maximal point of $V(\mathrm{Ann}_A(m))$.

(ii): Suppose $\mathfrak{p} \in \mathrm{Ass}\, M$ is finitely generated; by definition, there exists $y \in M_\mathfrak{p}$ such that the radical of $I := \mathrm{Ann}_{A_\mathfrak{p}}(y)$ equals $\mathfrak{p}A_\mathfrak{p}$. Since $\mathfrak{p}$ is finitely generated, some power of $\mathfrak{p}A_\mathfrak{p}$ is contained in I. Let $t \in \mathbb{N}$ be the smallest integer such that $\mathfrak{p}^t A_\mathfrak{p} \subset I$, and pick any non-zero element x in $\mathfrak{p}^{t-1}A_\mathfrak{p}y$; then $\mathrm{Ann}_{A_\mathfrak{p}}(x) = \mathfrak{p}A_\mathfrak{p}$. After clearing some denominators, we may assume that $x \in M$. Let $a_1, \dots, a_r$ be a finite set of generators for $\mathfrak{p}$; we may then find $t_1, \dots, t_r \in A\setminus\mathfrak{p}$ such that $t_i a_i x = 0$ in M, for every $i \leq r$. Set $x' := t_1 \cdot \ldots \cdot t_r x$; then $\mathfrak{p} \subset \mathrm{Ann}_A(x')$; however $\mathrm{Ann}_{A_\mathfrak{p}}(x') = \mathfrak{p}A_\mathfrak{p}$, therefore $\mathfrak{p} = \mathrm{Ann}_A(x')$, whence the contention.

(iii): The assertion means that for every $m \in M \setminus \{0\}$ there exists $\mathfrak{p} \in \mathrm{Ass}\, M$ such that the image of m does not vanish in $M_\mathfrak{p}$. But according to (i), $\mathrm{Ass}\, M$ contains every maximal point $\mathfrak{p}$ of $\mathrm{Supp}(m)$. ☐

Lemma 10.5.5. *Let X be a scheme, $\mathscr{F}$ a quasi-coherent $\mathscr{O}_X$-module. Then :*

(i) $\mathrm{Ass}\,\mathscr{F} \subset \mathrm{Supp}\,\mathscr{F}$.
(ii) $\mathrm{Ass}\,\mathscr{F}_{|U} = U \cap \mathrm{Ass}\,\mathscr{F}$ *for every open subset* $U \subset X$.
(iii) $\mathscr{F} = 0$ *if and only if* $\mathrm{Ass}\,\mathscr{F} = \varnothing$.

Proof. (i) and (ii) are obvious. To show (iii), we may assume that X is affine, according to (ii). Then the assertion follows immediately from lemma 10.5.5(iii). ☐

Proposition 10.5.6. *Let X be a scheme, $\mathscr{F}$ a quasi-coherent $\mathscr{O}_X$-module. We have :*

(i) $\mathrm{Ass}\,\mathscr{F} = \{x \in X \mid \delta(x, \mathscr{F}) = 0\}$.
(ii) *If* $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}''$ *is an exact sequence of quasi-coherent $\mathscr{O}_X$-modules, then :*

$$\mathrm{Ass}\,\mathscr{F}' \subset \mathrm{Ass}\,\mathscr{F} \subset \mathrm{Ass}\,\mathscr{F}' \cup \mathrm{Ass}\,\mathscr{F}''.$$

(iii) *If $x \in X$ is any point, and $\mathscr{G}_1, \dots, \mathscr{G}_n$ (for some $n > 0$) are x-primary submodules of $\mathscr{F}$, then $\mathscr{G}_1 \cap \dots \cap \mathscr{G}_n$ is also x-primary.*
(iv) *Let $f : X \to Y$ be a finite morphism of schemes. Then*
  (a) *The natural map :*

$$\bigoplus_{x \in f^{-1}(y)} \Gamma_{\{x\}}\mathscr{F}_{|X(x)} \to \Gamma_{\{y\}} f_*\mathscr{F}_{|Y(y)}$$

  is a bijection for all $y \in Y$.
  (b) $\mathrm{Ass}\, f_*\mathscr{F} = f(\mathrm{Ass}\,\mathscr{F})$.
(v) *Suppose that $\mathscr{F}$ is the union of a filtered family $(\mathscr{F}_\lambda \mid \lambda \in \Lambda)$ of quasi-coherent $\mathscr{O}_X$-submodules. Then :*

$$\bigcup_{\lambda \in \Lambda} \mathrm{Ass}\,\mathscr{F}_\lambda = \mathrm{Ass}\,\mathscr{F}.$$

(vi) *Let $f : Y \to X$ be a flat morphism of schemes, and suppose that the topological space $|X|$ underlying X is locally noetherian. Then* $\mathrm{Ass}\, f^*\mathscr{F} \subset f^{-1}\mathrm{Ass}\,\mathscr{F}$.

Proof. (i) and (v) follow directly from the definitions.

(ii): Consider, for every point $x \in X$ the induced exact sequence of $\mathscr{O}_{X(x)}$-modules :

$$0 \to \underline{\Gamma}_{\{x\}}\mathscr{F}'_x \to \underline{\Gamma}_{\{x\}}\mathscr{F}_x \to \underline{\Gamma}_{\{x\}}\mathscr{F}''_x.$$

Then :

$$\mathrm{Supp}\,\underline{\Gamma}_{\{x\}}\mathscr{F}'_x \subset \mathrm{Supp}\,\underline{\Gamma}_{\{x\}}\mathscr{F}_x \subset \mathrm{Supp}\,\underline{\Gamma}_{\{x\}}\mathscr{F}'_x \cup \mathrm{Supp}\,\underline{\Gamma}_{\{x\}}\mathscr{F}''_x$$

which, in light of (i), is equivalent to the contention.

(iii): One applies (ii) to the natural injection : $\mathscr{F}/(\mathscr{G}_1 \cap \cdots \cap \mathscr{G}_n) \to \oplus_{i=1}^n \mathscr{F}/\mathscr{G}_i$.

(iv): We may assume that Y is a local scheme with closed point $y \in Y$. Let $x_1, \dots, x_n \in X$ be the finitely many points lying over y; for every $i, j \leq n$, we let

$$\pi_j : X(x_j) \to Y \qquad \text{and} \qquad \pi_{ij} : X(x_i) \times_X X(x_j) \to Y$$

be the natural morphisms. To ease notation, denote also by $\mathscr{F}_i$ (resp. $\mathscr{F}_{ij}$) the pull back of of $\mathscr{F}$ to $X(x_j)$ (resp. to $X(x_i) \times_X X(x_j)$). The induced morphism $U := X(x_1) \amalg \cdots \amalg X(x_n) \to X$ is faithfully flat, so descent theory yields an exact sequence of $\mathscr{O}_Y$-modules :

$$(10.5.7) \qquad 0 \longrightarrow f_*\mathscr{F} \longrightarrow \textstyle\prod_{j=1}^n \pi_{j*}\mathscr{F}_j \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} \prod_{i,j=1}^n \pi_{ij*}\mathscr{F}_{ij}.$$

where $p_1, p_2 : U \times_X U \to U$ are the natural morphisms.

Claim 10.5.8. The induced maps :

$$\Gamma_{\{y\}}p_1^*, \Gamma_{\{y\}}p_2^* : \prod_{j=1}^n \Gamma_{\{y\}}\pi_{j*}\mathscr{F}_j \to \prod_{i,j=1}^n \Gamma_{\{y\}}\pi_{ij*}\mathscr{F}_{ij}$$

coincide.

Proof of the claim. Indeed, it suffices to verify that they coincide after projecting onto each factor $\mathscr{G}_{ij} := \Gamma_{\{y\}}\pi_{ij*}\mathscr{F}_{ij}$. But this is clear from definitions if $i = j$. On the other hand, if $i \neq j$, the image of π_{ij} in Y does not contain y, so the corresponding factor $\mathscr{G}_{ij}$ vanishes. ◊

From (10.5.7) and claim 10.5.8 we deduce that the natural map

$$\Gamma_{\{y\}}f_*\mathscr{F} \to \prod_{j=1}^n \Gamma_{\{y\}}\pi_{j*}\mathscr{F}_j$$

is a bijection. Clearly $\Gamma_{\{y\}}\pi_{j*}\mathscr{F}_j = \Gamma_{\{x\}}\mathscr{F}_j$, so both assertions follow easily.

(vi): Let $x \in X$ be any point; since $|X|$ is locally noetherian, the subset $\{x\}$ is closed and constructible in $X(x)$, and then $f^{-1}(x)$ is closed and constructible in $Y \times_X X(x)$. Hence $\delta(y, f^*\mathscr{F}) \geq \delta(x, \mathscr{F})$ for every $y \in f^{-1}(x)$ (lemma 10.4.17(iii) and theorem 10.4.21). Then the assertion follows from (i). □

Corollary 10.5.9. *In the situation of corollary* 10.4.47*, suppose furthermore that $|S|$ is a locally noetherian topological space. Then we have :*

$$\mathrm{Ass}\,\mathscr{F} \otimes_{\mathscr{O}_X} f^*\mathscr{G} = \bigcup_{s \in \mathrm{Ass}\,\mathscr{G}} \mathrm{Ass}\,\mathscr{F} \otimes_{\mathscr{O}_{S,s}} \kappa(s).$$

Proof. Under the stated assumptions, it is easily seen that, for every $x \in X$ (resp. $s \in S$), the subset $\{x\}$ (resp. $\{s\}$) is constructible in $X(x)$ (resp. in $S(s)$). Then the assertion follows immediately from proposition 10.5.6(i) and theorem 10.4.21. □

Remark 10.5.10. (i) Actually, it can be shown that the extra assumption on $|S|$ in corollary 10.5.9 is superflous : see [86, Part I, Prop.3.4.3].

(ii) In the situation of proposition 10.5.6, one could also consider the set of points $x \in X$ such that $\delta'(x, \mathscr{F}) = 0$. This set contains $\mathrm{Ass}\,\mathscr{F}$, by proposition 10.5.6(i) and lemma 10.4.20(i).

Such points are called *attached primes* or *strong Krull primes* of $\mathscr{F}$ in some literature : see *e.g.* [68], [67] (though the terminology "attached prime" is used also for a different, unrelated notion: see the footnote on page 404 of [67]).

Corollary 10.5.11. *Let X be a quasi-compact and quasi-separated scheme, $\mathscr{F}$ a quasi-coherent $\mathscr{O}_X$-module and $\mathscr{G} \subset \mathscr{F}$ a quasi-coherent submodule. Then :*

(i) *For every quasi-compact open subset $U \subset X$ and every quasi-coherent primary $\mathscr{O}_U$-submodule $\mathscr{N} \subset \mathscr{F}_{|U}$, with $\mathscr{G}_{|U} \subset \mathscr{N}$, there exists a quasi-coherent primary $\mathscr{O}_X$-submodule $\mathscr{M} \subset \mathscr{F}$ such that $\mathscr{M}_{|U} = \mathscr{N}$ and $\mathscr{G} \subset \mathscr{N}$.*

(ii) *$\mathscr{G}$ admits a primary decomposition if and only if there exists a finite open covering $X = U_1 \cup \cdots \cup U_n$ consisting of quasi-compact open subsets, such that the submodules $\mathscr{G}_{|U_i} \subset \mathscr{F}_{|U_i}$ admit primary decompositions for every $i = 1, \dots, n$.*

Proof. Say that $\mathscr{N}$ is x-primary for some point $x \in U$, and set $Z := X \setminus U$. According to [59, Ch.I, Prop.9.4.2], we can extend $\mathscr{N}$ to a quasi-coherent $\mathscr{O}_X$-submodule $\mathscr{M}_1 \subset \mathscr{F}$; up to replacing $\mathscr{M}_1$ by $\mathscr{M}_1 + \mathscr{G}$, we may assume that $\mathscr{G} \subset \mathscr{M}_1$. Since $(\mathscr{F}/\mathscr{M}_1)_{|U} = \mathscr{F}_{|U}/\mathscr{N}$, it follows from lemma 10.5.5(ii) that $\mathrm{Ass}\,\mathscr{F}/\mathscr{M}_1 \subset \{x\} \cup Z$. Let $\overline{\mathscr{M}} := \underline{\Gamma}_Z(\mathscr{F}/\mathscr{M}_1)$, and denote by $\mathscr{M}$ the preimage of $\overline{\mathscr{M}}$ in $\mathscr{F}$. There follows a short exact sequence :

$$0 \to \overline{\mathscr{M}} \to \mathscr{F}/\mathscr{M}_1 \to \mathscr{F}/\mathscr{M} \to 0.$$

Clearly $R^1\underline{\Gamma}_Z\overline{\mathscr{M}} = 0$, whence a short exact sequence

$$0 \to \underline{\Gamma}_Z\overline{\mathscr{M}} \to \underline{\Gamma}_Z(\mathscr{F}/\mathscr{M}_1) \to \underline{\Gamma}_Z(\mathscr{F}/\mathscr{M}) \to 0.$$

We deduce that $\mathrm{depth}_Z(\mathscr{F}/\mathscr{M}) > 0$; then theorem 10.4.21 and proposition 10.5.6(i) show that $Z \cap \mathrm{Ass}\,(\mathscr{F}/\mathscr{M}) = \varnothing$, so that $\mathscr{M}$ is x-primary, as required.

(ii): We may assume that a covering $X = U_1 \cup \cdots \cup U_n$ is given with the stated property. For every $i \leq n$, let $\mathscr{G}_{|U_i} = \mathscr{N}_{i1} \cap \cdots \cap \mathscr{N}_{ik_i}$ be a primary decomposition; by (i) we may extend every $\mathscr{N}_{ij}$ to a primary submodule $\mathscr{M}_{ij} \subset \mathscr{F}$ containing $\mathscr{G}$. Then $\mathscr{G} = \bigcap_{i=1}^n \bigcap_{j=1}^{k_i} \mathscr{M}_{ij}$ is a primary decomposition of $\mathscr{G}$. □

10.5.12. Let A, B be two local rings, with A a domain, $f : A \to B$ a local ring homomorphism, M an f-flat B-module of finite presentation. Denote by K (resp. κ) the field of fractions (resp. the residue field) of A, and set

$$B_0 := B \otimes_A \kappa \qquad B_K := B \otimes_A K \qquad M_0 := M \otimes_A \kappa \qquad M_K := M \otimes_A K.$$

Proposition 10.5.13. *In the situation of* (10.5.12)*, suppose that :*

(a) *either, A and B are both noetherian rings*

(b) *or else, f is essentially of finite presentation.*

Let $\mathfrak{p} \in \mathrm{Ass}_{B_K} M_K$ be any point, and denote by $V(\mathfrak{p})$ the topological closure of $\{\mathfrak{p}\}$ in $\mathrm{Spec}\,B$*. Then all the maximal points of $V(\mathfrak{p}) \cap \mathrm{Spec}\,B_0$ lie in $\mathrm{Ass}_{B_0} M_0$.*

Proof. We consider first case (a). Via the inclusion map $\mathrm{Spec}\,B_K \to \mathrm{Spec}\,B$ we may regard $\mathfrak{p}$ as a prime ideal of B, and $\mathrm{Ass}_{B_K} M_K$ as a subset of $\mathrm{Ass}_B M$; by lemma 10.5.3(ii), there exists a cyclic B-submodule N of M such that $\mathrm{Ann}_B N = \mathfrak{p}$. Denote by $\mathfrak{m}_A$ the maximal ideal of A, endow M with its $\mathfrak{m}_A$-adic filtration $\mathrm{Fil}^\bullet M$, and N with the filtration $\mathrm{Fil}^\bullet N$ induced by $\mathrm{Fil}^\bullet M$. Let also $\mathrm{gr}^\bullet M$ and $\mathrm{gr}^\bullet N$ be the graded B_0-modules associated with these filtrations on M and N. Since M is f-flat, we have a B_0-linear isomorphism

$$\mathrm{gr}^i M \xrightarrow{\sim} (\mathfrak{m}_A^i/\mathfrak{m}_A^{i+1}) \otimes_\kappa M_0 \qquad \text{for every } i \in \mathbb{N}$$

which implies that $\mathrm{Ass}_B(\mathrm{gr}^i M) = \mathrm{Ass}_B M_0$ for every $i \in \mathbb{N}$. From this, proposition 10.5.6(ii) and an easy induction on $i \in \mathbb{N}$, we deduce that

$$\mathrm{Ass}_B(N/\mathrm{Fil}^i N) \subset \mathrm{Ass}_B(M/\mathrm{Fil}^i M) \subset \mathrm{Ass}_B M_0 \qquad \text{for every } i \in \mathbb{N}.$$

On the other hand, by the Artin-Rees lemma ([126, Th.8.5]), there exist $i, j \in \mathbb{N}$ such that

$$\mathfrak{m}_A^j N \subset \mathrm{Fil}^i N \subset \mathfrak{m}_A N$$

so that $\mathrm{Supp}\, N/\mathfrak{m}_A N = \mathrm{Supp}\, N/\mathrm{Fil}^i N$. However, lemma 10.5.3(i) implies that all the maximal points of $\mathrm{Supp}\, N/\mathrm{Fil}^i N$ lie in $\mathrm{Ass}_B(N/\mathrm{Fil}^i N)$. To conclude, it suffices to remark that the support of $N/\mathfrak{m}_A N$ equals $V(\mathfrak{p}) \cap \mathrm{Spec}\, B_0$.

Next, suppose that (b) holds, and say that $B = C_\mathfrak{q}$ for some finitely presented A-algebra C and some prime ideal $\mathfrak{q} \subset C$ whose preimage in A is the maximal ideal. We write A as the colimit of a filtered system $(A_\lambda \mid \lambda \in \Lambda)$ of noetherian local subrings, such that the transition maps $A_\mu \to A_\lambda$ are local ring homomorphisms for every $\lambda, \mu \in \Lambda$ with $\lambda \geq \mu$; then there exist $\mu \in \Lambda$ and an A_μ-algebra C_μ of finite type with an isomorphism of $C_\mu \otimes_{A_\mu} A \to C$ of A-algebras. For every $\lambda \geq \mu$, let $C_\lambda := C_\mu \otimes_{A_\mu} A_\lambda$, denote by $\mathfrak{q}_\lambda \subset C_\lambda$ the preimage of $\mathfrak{q}$ under the induced map $C_\lambda \to C$, and set $B_\lambda := (C_\lambda)_{\mathfrak{q}_\lambda}$. Clearly B is the colimit of the resulting filtered system of rings $(B_\lambda \mid \lambda \in \Lambda,\, \lambda \geq \mu)$, and the induced maps $A_\lambda \to B_\lambda$ are local ring homomorphisms, for every $\lambda \geq \mu$; therefore we may assume that there exists a finitely generated B_μ-module M_μ with a B-linear isomorphism $M_\mu \otimes_{B_\mu} B \xrightarrow{\sim} M$, and we may likewise define $M_\lambda := M_\mu \otimes_{B_\mu} B_\lambda$ for every $\lambda \geq \mu$. In view of [65, Ch.IV, Cor.11.2.6.1(i)], we may further assume that M_λ is a flat A_λ-module, for every $\lambda \geq \mu$, and after replacing Λ by a cofinal subset, we may assume that μ is the initial element of the partially ordered set Λ. Then, to ease notation, denote by K_λ (resp. κ_λ) the field of fractions (resp. the residue field) of A_λ, and set

$$\begin{array}{lll} B_{K,\lambda} := B_\lambda \otimes_{A_\lambda} K_\lambda & M_{K,\lambda} := M_\lambda \otimes_{A_\lambda} K_\lambda & C_{K,\lambda} := C_\lambda \otimes_{A_\lambda} K_\lambda \\ B_{0,\lambda} := B_\lambda \otimes_{A_\lambda} \kappa_\lambda & M_{0,\lambda} := M_\lambda \otimes_{A_\lambda} \kappa_\lambda & \end{array} \qquad \text{for every } \lambda \in \Lambda.$$

Notice that the resulting map $f_\lambda : C_{K,\lambda} \to C_K := C \otimes_A K$ induces an isomorphism

$$C_{K,\lambda} \otimes_{K_\lambda} K \xrightarrow{\sim} C_K.$$

So, f_λ is a flat ring homomorphism, hence the natural morphism $\varphi_{K,\lambda} : \mathrm{Spec}\, B_K \to \mathrm{Spec}\, B_{K,\lambda}$ is also flat, and proposition 10.5.6(vi) yields

$$\mathrm{Ass}_{B_K} M_K \subset \varphi_{K,\lambda}^{-1} \mathrm{Ass}_{B_{K,\lambda}}(M_{K,\lambda}) \qquad \text{for every } \lambda \in \Lambda. \tag{10.5.14}$$

Now, say that $\mathfrak{p} \in \mathrm{Ass}_{B_K} M_K$, and let $\mathfrak{p}_\lambda \subset B_{K,\lambda}$ be the preimage of $\mathfrak{p}$, for every $\lambda \in \Lambda$; we have $\mathfrak{p}_\lambda \in \mathrm{Ass}_{B_{K,\lambda}}(M_{K,\lambda})$ by virtue of (10.5.14), so in this situation the case (a) of the proposition applies and shows that

$$Z_\lambda := \mathrm{Max}(\mathrm{Spec}\, B_{0,\lambda}/\mathfrak{p}_\lambda B_{0,\lambda}) \subset Z'_\lambda := \mathrm{Ass}_{B_\lambda}(M_{0,\lambda}) \qquad \text{for every } \lambda \in \Lambda$$

(where, for a scheme Z, we have denoted $\mathrm{Max}(Z)$ the set of maximal points of Z). Notice moreover, that the natural morphism $\varphi_{0,\lambda} : \mathrm{Spec}\, B_0 \to \mathrm{Spec}\, B_{0,\lambda}$ is also flat, so we have

$$\mathrm{Max}(\mathrm{Spec}\, B_0/\mathfrak{p}_\lambda B_0) = \bigcup_{z \in Z_\lambda} \mathrm{Max}(\varphi_{0,\lambda}^{-1}(z)) \qquad \text{for every } \lambda \in \Lambda$$

by the going down theorem ([126, Th.9.5]). On the other hand, from [64, Ch.IV, Prop.4.2.7] we have

$$\bigcup_{z \in Z'_\lambda} \mathrm{Max}(\varphi_{0,\lambda}^{-1}(z)) = (\mathrm{Spec}\, B_0) \cap \mathrm{Ass}_{B_{0,\lambda} \otimes_{k_\lambda} \kappa}(M_{0,\lambda} \otimes_{\kappa_\lambda} \kappa) = \mathrm{Ass}_{B_0}(M_0)$$

for every $\lambda \in \Lambda$. Summing up, we are reduced to checking that there exists $\lambda \in \Lambda$ such that the surjection $B_0/\mathfrak{p}_\lambda B_0 \to B_0/\mathfrak{p} B_0$ is an isomorphism, *i.e.* such that $\mathfrak{p}_\lambda B_0 = \mathfrak{p} B_0$. However, clearly $\mathfrak{p} B_0$ is the filtered union of the system of subideals $(\mathfrak{p}_\lambda B_0 \mid \lambda \in \Lambda)$; but B_0 is a noetherian ring, so the claim is obvious. □

Lemma 10.5.15. *Let X be a quasi-separated and quasi-compact scheme, $\mathscr{F}$ a quasi-coherent $\mathscr{O}_X$-module of finite type. Then the submodule $0 \subset \mathscr{F}$ admits a primary decomposition if and only if the following conditions hold:*

(a) $\mathrm{Ass}\,\mathscr{F}$ *is a finite set.*

(b) *For every $x \in \mathrm{Ass}\,\mathscr{F}$ there is a x-primary ideal $I \subset \mathscr{O}_{X,x}$ such that the natural map*

$$\Gamma_{\{x\}}\mathscr{F}_x \to \mathscr{F}_x/I\mathscr{F}_x$$

is injective.

Proof. In view of corollary 10.5.11(i) we are reduced to the case where X is an affine scheme, say $X = \mathrm{Spec}\,A$, and $\mathscr{F} = M^\sim$ for an A-module M of finite type. Suppose first that 0 admits a primary decomposition :

$$0 = \bigcap_{i=1}^{k} N_i. \tag{10.5.16}$$

Then the natural map $M \to \oplus_{i=1}^k M/N_i$ is injective, hence

$$\mathrm{Ass}\,M \subset \mathrm{Ass}\bigoplus_{i=1}^{k} M/N_i \subset \bigcup_{i=1}^{k} \mathrm{Ass}\,M/N_i \tag{10.5.17}$$

by proposition 10.5.6(ii), and this shows that (a) holds. Next, if N_i and N_j are $\mathfrak{p}$-primary for the same prime ideal $\mathfrak{p} \subset A$, we may replace both of them by their intersection (proposition 10.5.6(iii)). Proceeding in this way, we achieve that the N_i appearing in (10.5.16) are primary submodules for pairwise distinct prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_k \subset A$. By (10.5.17) we have $\mathrm{Ass}\,M \subset \{\mathfrak{p}_1, \dots, \mathfrak{p}_k\}$. Suppose that $\mathfrak{p}_1 \notin \mathrm{Ass}\,M$, and set $Q := \mathrm{Ker}\,(M \to \oplus_{i=2}^k M/N_i)$. For every $j > 1$ we have $\Gamma_{\{\mathfrak{p}_j\}}(M/N_1)^\sim = 0$, therefore :

$$\Gamma_{\{\mathfrak{p}_j\}}Q^\sim_{\mathfrak{p}_j} = \mathrm{Ker}\,(\Gamma_{\{\mathfrak{p}_j\}}M^\sim_{\mathfrak{p}_j} \to \oplus_{i=1}^k (M/N_i)^\sim_{\mathfrak{p}_j}) = 0.$$

Hence $\mathrm{Ass}\,Q = \varnothing$, by proposition 10.5.6(i), and then $Q = 0$ by lemma 10.5.5(iii). In other words, we can omit N_1 from (10.5.16), and still obtain a primary decomposition of 0; iterating this argument, we may achieve that $\mathrm{Ass}\,M = \{\mathfrak{p}_1, \dots, \mathfrak{p}_k\}$. After these reductions, we see that

$$\Gamma_{\{\mathfrak{p}_j\}}(M/N_i)^\sim_{\mathfrak{p}_j} = 0 \qquad \text{whenever } i \neq j$$

and consequently :

$$\mathrm{Ker}\,(\varphi_j : \Gamma_{\{\mathfrak{p}_j\}}M^\sim_{\mathfrak{p}_j} \to (M/N_j)_{\mathfrak{p}_j}) = 0 \qquad \text{for every } j = 1, \dots, k.$$

Now, for given $j \leq k$, let $\overline{f}_1, \dots, \overline{f}_n$ be a system of non-zero generators for M/N_j; by assumption $I_i := \mathrm{Ann}_A(\overline{f}_i)$ is a $\mathfrak{p}_j$-primary ideal for every $i \leq n$. Hence $\mathfrak{q}_j := \mathrm{Ann}_A(M/N_j) = \bigcap_{i=1}^n I_i$ is $\mathfrak{p}_j$-primary as well; since φ_j factors through $(M/\mathfrak{q}_jM)_{\mathfrak{p}_j}$, we see that (b) holds.

Conversely, suppose that (a) and (b) hold. Say that $\mathrm{Ass}\,M = \{\mathfrak{p}_1, \dots, \mathfrak{p}_k\}$; for every $j \leq k$ we choose a $\mathfrak{p}_j$-primary ideal $\mathfrak{q}_j$ such that the map $\Gamma_{\{\mathfrak{p}_j\}}M^\sim_{\mathfrak{p}_j} \to M_{\mathfrak{p}_j}/\mathfrak{q}_jM_{\mathfrak{p}_j}$ is injective. Clearly $N_j := \mathrm{Ker}\,(M \to M_{\mathfrak{p}_j}/\mathfrak{q}_jM_{\mathfrak{p}_j})$ is a $\mathfrak{p}_j$-primary submodule of M; moreover, the induced map $\varphi : M \to \oplus_{j=1}^k M/N_j$ is injective, since $\mathrm{Ass}\,\mathrm{Ker}\,\varphi = \varnothing$ (proposition 10.5.6(ii) and lemma 10.5.5(iii)). In other words, $0 = \bigcap_{j=1}^k N_j$ is a primary decomposition of 0. □

Proposition 10.5.18. *Let X be a quasi-compact and quasi-separated scheme, and $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ a short exact sequence of quasi-coherent $\mathscr{O}_X$-modules of finite type such that*

(a) $\mathrm{Ass}\,\mathscr{F}' \cap \mathrm{Supp}\,\mathscr{F}'' = \varnothing$.

(b) *The submodules $0 \subset \mathscr{F}'$ and $0 \subset \mathscr{F}''$ admit primary decompositions.*

Then the submodule $0 \subset \mathscr{F}$ admits a primary decomposition.

Proof. The assumptions imply that $\mathrm{Ass}\,\mathscr{F}'$ and $\mathrm{Ass}\,\mathscr{F}''$ are finite sets, (lemma 10.5.15), hence the same holds for $\mathrm{Ass}\,\mathscr{F}$ (proposition 10.5.6(ii)). Given any point $x \in X$ and any x-primary ideal $I \subset \mathscr{O}_{X,x}$, we may consider the commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Gamma_{\{x\}}\mathscr{F}'_x & \longrightarrow & \Gamma_{\{x\}}\mathscr{F}_x & \longrightarrow & \Gamma_{\{x\}}\mathscr{F}''_x \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\ \mathrm{Tor}_1^{\mathscr{O}_{X,x}}(I,\mathscr{F}''_x) & \longrightarrow & \mathscr{F}'_x/I\mathscr{F}'_x & \longrightarrow & \mathscr{F}_x/I\mathscr{F}_x & \longrightarrow & \mathscr{F}''_x/I\mathscr{F}''_x \end{array}$$

Now, if $x \in \mathrm{Ass}\,\mathscr{F}'$, assumption (a) implies that $\Gamma_{\{x\}}\mathscr{F}''_x = 0 = \mathrm{Tor}_1^{\mathscr{O}_{X,x}}(I_x, \mathscr{F}''_x)$, and by (b) and lemma 10.5.15 we can choose I such that α is injective; a little diagram chase then shows that β is injective as well. Similarly, if $x \in \mathrm{Ass}\,\mathscr{F}''$, we have $\Gamma_{\{x\}}\mathscr{F}'_x = 0$ and we may choose I such that γ is injective, which implies again that β is injective. To conclude the proof it suffices to apply again lemma 10.5.15. □

Proposition 10.5.19. *Let Y be a quasi-compact and quasi-separated scheme, $f : X \to Y$ a finite morphism and $\mathscr{F}$ a quasi-coherent $\mathscr{O}_X$-module of finite type. Then the $\mathscr{O}_X$-submodule $0 \subset \mathscr{F}$ admits a primary decomposition if and only if the same holds for the $\mathscr{O}_Y$-submodule $0 \subset f_*\mathscr{F}$.*

Proof. Under the stated assumptions, we can apply the criterion of lemma 10.5.15. To start out, it is clear from proposition 10.5.6(iv.b) that $\mathrm{Ass}\,\mathscr{F}$ is finite if and only if $\mathrm{Ass}\,f_*\mathscr{F}$ is finite. Next, suppose that $0 \subset \mathscr{F}$ admits a primary decomposition, let $y \in \mathrm{Ass}\,f_*\mathscr{F}$ be any point and set $f^{-1}(y) := \{x_1, \dots, x_n\}$; for every $j \leq n$ we can find an x_j-primary ideal $I_j \subset \mathscr{O}_X$ such that the map $\Gamma_{\{x_j\}}\mathscr{F}_{x_j} \to \mathscr{F}_{x_j}/I_j\mathscr{F}_{x_j}$ is injective. Let I be the kernel of the natural map $\mathscr{O}_Y \to f_*(\mathscr{O}_X/(I_1 \cap \cdots \cap I_n))$. Then I is a quasi-coherent y-primary ideal of $\mathscr{O}_Y$ and we deduce a commutative diagram :

$$\begin{array}{ccc} \bigoplus_{j=1}^n \Gamma_{\{x_j\}}(\mathscr{F}_{x_j}) & \longrightarrow & \Gamma_{\{y\}}(f_*\mathscr{F}_y) \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} \\ \bigoplus_{j=1}^n \mathscr{F}_{x_j}/I\mathscr{F}_{x_j} & \longrightarrow & f_*\mathscr{F}_y/If_*\mathscr{F}_y \end{array} \tag{10.5.20}$$

whose horizontal arrows are isomorphisms, in view of proposition 10.5.6(iv.a), and where α is injective by construction. It follows that β is injective, so condition (b) of lemma 10.5.15 holds for the stalk $f_*\mathscr{F}_y$, and since $y \in \mathrm{Ass}\,f_*\mathscr{F}$ is arbitrary, we see that $0 \subset f_*\mathscr{F}$ admits a primary decomposition. Conversely, suppose that $0 \subset f_*\mathscr{F}$ admits a primary decomposition; then for every $y \in Y$ we can find a quasi-coherent y-primary ideal $I \subset \mathscr{O}_Y$ such that the map β of (10.5.20) is injective; hence α is injective as well, and again we deduce easily that $0 \subset \mathscr{F}$ admits a primary decomposition. □

Proposition 10.5.21. *Let X and $\mathscr{F}$ be as in lemma* 10.5.15, *$i : Z \to X$ a closed constructible immersion, and $U := X \setminus Z$. Suppose that :*

(a) *The $\mathscr{O}_U$-submodule $0 \subset \mathscr{F}_{|U}$ and the $\mathscr{O}_Z$-submodule $0 \subset i^*\mathscr{F}$ admit primary decompositions.*
(b) *The natural map $\underline{\Gamma}_Z\mathscr{F} \to i_*i^*\mathscr{F}$ is injective.*

Then the $\mathscr{O}_X$-submodule $0 \subset \mathscr{F}$ admits a primary decomposition.

Proof. We shall verify that conditions (a) and (b) of lemma 10.5.15 hold for $\mathscr{F}$. To check condition (a), it suffices to remark that $U \cap \mathrm{Ass}\,\mathscr{F} = \mathrm{Ass}\,\mathscr{F}_{|U}$ (which is obvious) and that $Z \cap \mathrm{Ass}\,\mathscr{F} \subset \mathrm{Ass}\,i^*\mathscr{F}$, which follows easily from our current assumption (b).

Next we check that condition (b) of *loc.cit.* holds. This is no problem for the points $x \in U \cap \mathrm{Ass}\,\mathscr{F}$, so suppose that $x \in Z \cap \mathrm{Ass}\,\mathscr{F}$. Moreover, we may also assume that X is affine, say $X = \operatorname{Spec} A$, so that $Z = V(J)$ for some ideal $J \subset A$. Due to proposition 10.5.19 we know that $0 \subset \mathscr{G} := i_*i^*\mathscr{F}$ admits a primary decomposition, hence we can find an x-primary ideal $I \subset A$ such that the natural map $\Gamma_{\{x\}}\mathscr{G} \to \mathscr{G}_x/I\mathscr{G}_x \simeq \mathscr{F}_x/(I+J)\mathscr{F}_x$ is injective. Clearly $I + J$ is again a x-primary ideal; since Z is closed and constructible, corollary 10.4.10 and our assumption (b) imply that the natural map $\Gamma_{\{x\}}\mathscr{F} \to \Gamma_{\{x\}}\mathscr{G}$ is injective, hence the same holds for the map $\Gamma_{\{x\}}\mathscr{F} \to \mathscr{F}_x/(I+J)\mathscr{F}_x$, as required. □

10.5.22. Let now A be a ring, $\mathfrak{p} \subset A$ any prime ideal, and $n \geq 1$ any integer; if $A/\mathfrak{p}^n$ does not admit imbedded primes, then the ideal $\mathfrak{p}^n$ is $\mathfrak{p}$-primary. In the presence of imbedded primes, this fails. For instance, we have the following :

Example 10.5.23. Let k be a field, $k[x, y]$ the free polynomial algebra in indeterminates x and y; consider the ideal $I := (xy, y^2)$, and set $A := k[x, y]/I$. Let $\overline{x}$ and $\overline{y}$ be the images of x and y in A; then $\mathfrak{p} := (\overline{y}) \subset A$ is a prime ideal, and $\mathfrak{p}^2 = 0$. However, $\mathfrak{m} := \operatorname{Ann}_A(\overline{y})$ is the maximal ideal generated by $\overline{x}$ and $\overline{y}$, so the ideal $0 \subset A$ is not $\mathfrak{p}$-primary.

There is however a natural sequence of $\mathfrak{p}$-primary ideals naturally attached to $\mathfrak{p}$. To explain this, let us remark, more generally, the following :

Lemma 10.5.24. *Let A be a ring, $\mathfrak{p} \subset A$ a prime ideal. Denote by $\varphi_\mathfrak{p} : A \to A_\mathfrak{p}$ the localization map. The rule :*

$$I \mapsto \varphi_\mathfrak{p}^{-1} I$$

induces a bijection from the set of $\mathfrak{p}A_\mathfrak{p}$-primary ideals of $A_\mathfrak{p}$, to the set of $\mathfrak{p}$-primary ideals of A.

Proof. Suppose that $I \subset A_\mathfrak{p}$ is $\mathfrak{p}A_\mathfrak{p}$-primary; since the natural map $A/\varphi_\mathfrak{p}^{-1}I \to A_\mathfrak{p}/I$ is injective, it is clear that $\varphi_\mathfrak{p}^{-1}I$ is $\mathfrak{p}$-primary. Conversely, suppose that $J \subset A$ is $\mathfrak{p}$-primary; we claim that $J = \varphi_\mathfrak{p}^{-1}(J_\mathfrak{p})$. Indeed, by assumption (and by lemma 10.5.5(iii)) we have $(A/J)_\mathfrak{q} = 0$ whenever $\mathfrak{q} \neq \mathfrak{p}$; it follows easily that the localization map $A/J \to A_\mathfrak{p}/J_\mathfrak{p}$ is an isomorphism, whence the contention. □

Definition 10.5.25. Keep the notation of lemma 10.5.24; for every $n \geq 0$ one defines the *n-th symbolic power* of $\mathfrak{p}$, as the ideal :

$$\mathfrak{p}^{(n)} := \varphi_\mathfrak{p}^{-1}(\mathfrak{p}^n A_\mathfrak{p}).$$

By lemma 10.5.24, the ideal $\mathfrak{p}^{(n)}$ is $\mathfrak{p}$-primary for every $n \geq 1$. More generally, for every A-module M, let $\varphi_{M,\mathfrak{p}} : M \to M_\mathfrak{p}$ be the localization map; then one defines the *$\mathfrak{p}$-symbolic filtration* on M, by the rule :

$$\operatorname{Fil}^{(n)}_\mathfrak{p} M := \varphi^{-1}_{M,\mathfrak{p}}(\mathfrak{p}^n M_\mathfrak{p}) \qquad \text{for every } n \geq 0.$$

The filtration $\operatorname{Fil}^{(\bullet)}_\mathfrak{p} M$ induces a (linear) topology on M, called the *$\mathfrak{p}$-symbolic topology*.

More generally, if $\Sigma \subset \operatorname{Spec} A$ is any subset, we define the *Σ-symbolic topology* on M, as the coarsest linear topology $\mathscr{T}_\Sigma$ on M such that $\operatorname{Fil}^{(n)}_\mathfrak{p} M$ is an open subset of $\mathscr{T}_\Sigma$, for every $\mathfrak{p} \in \Sigma$ and every $n \geq 0$. If Σ is finite, it is induced by the *Σ-symbolic filtration*, defined by the submodules :

$$\operatorname{Fil}^{(n)}_\Sigma M := \bigcap_{\mathfrak{p} \in \Sigma} \operatorname{Fil}^{(n)}_\mathfrak{p} M \qquad \text{for every } n \geq 0.$$

10.5.26. Let A be a ring, M an A-module, $I \subset A$ an ideal. We shall show how to characterize the finite subsets $\Sigma \subset \operatorname{Spec} A$ such that the Σ-symbolic topology on M agrees with the I-preadic topology (see theorem 10.5.35). Hereafter, we begin with a few preliminary observations. First, suppose that A is noetherian; then it is easily seen that, for every prime ideal $\mathfrak{p} \subset A$, and every $\mathfrak{p}A_\mathfrak{p}$-primary ideal $I \subset A_\mathfrak{p}$, there exists $n \in \mathbb{N}$ such that $\mathfrak{p}^n A_\mathfrak{p} \subset I$. From lemma 10.5.24 we deduce that every $\mathfrak{p}$-primary ideal of A contains a symbolic power of $\mathfrak{p}$; *i.e.*, every

$\mathfrak{p}$-primary ideal is open in the $\mathfrak{p}$-symbolic topology of A. More generally, let $\Sigma \subset \mathrm{Spec}\, A$ be any subset, M a finitely generated A-module, and $N \subset M$ a submodule; from the existence of a primary decomposition for N ([126, Th.6.8]), we see that N is open in the Σ-symbolic topology of M, whenever $\mathrm{Ass}\, M/N \subset \Sigma$. Especially, if $\mathrm{Ass}\, M/I^nM \subset \Sigma$ for every $n \in \mathbb{N}$, then the Σ-symbolic topology is finer than the I-preadic topology on M. On the other hand, clearly $I^nM \subset \mathrm{Fil}^{(n)}_{\mathfrak{p}} M$ for every $\mathfrak{p} \in \mathrm{Supp}\, M/IM$ and every $n \in \mathbb{N}$, so the I-preadic topology on M is finer than the $\mathrm{Supp}\, M/IM$-symbolic topology. Summing up, if we have :

$$\Sigma_0(M) := \bigcup_{n\in\mathbb{N}} \mathrm{Ass}\, M/I^nM \subset \Sigma \subset \mathrm{Supp}\, M/IM$$

then the Σ-symbolic topology on M agrees with the I-preadic topology. Notice that the above expression for $\Sigma_0(M)$ is a union of finite subsets ([126, Th.6.5(i)]), hence $\Sigma_0(M)$ is – *a priori* – at most countable; in fact, we shall show that $\Sigma_0(M)$ is finite. Indeed, for every $n \in \mathbb{N}$, set :

$$\mathrm{gr}^n_I A := I^n/I^{n+1} \qquad \mathrm{gr}^n_I M := I^nM/I^{n+1}M.$$

Then $\mathrm{gr}^\bullet_I A := \oplus_{n\in\mathbb{N}} \mathrm{gr}^n_I A$ is naturally a graded A/I-algebra, and $\mathrm{gr}^\bullet_I M := \oplus_{n\in\mathbb{N}} \mathrm{gr}^n_I M$ is a graded $\mathrm{gr}^\bullet_I A$-module. Let $\psi : \mathrm{Spec}\, \mathrm{gr}^\bullet_I A \to \mathrm{Spec}\, A/I$ be the natural morphism, and set :

$$\Sigma(M) := \psi(\mathrm{Ass}_{\mathrm{gr}^\bullet_I A}(\mathrm{gr}^\bullet_I M)).$$

Lemma 10.5.27. *With the notation of* (10.5.26)*, we have :*

$$\mathrm{Ass}_{A/I}(\mathrm{gr}^n_I M) \subset \Sigma(M) \qquad \textit{for every } n \in \mathbb{N}.$$

Proof. To ease notation, set $A_0 := A/I$ and $B := \mathrm{gr}^\bullet_I A$. Suppose that $\mathfrak{p} \in \mathrm{Ass}_{A_0}(\mathrm{gr}^n_I M)$; by lemma 10.5.3(i), there exists $m \in \mathrm{gr}^n_I M$ such that $\mathfrak{p}$ is the preimage of a minimal prime ideal of $A_0/\mathrm{Ann}_{A_0}(m)$. However, if we regard m as a homogeneous element of the B-module $\mathrm{gr}^\bullet_I M$, we have the obvious identity :

$$\mathrm{Ann}_{A_0}(m) = A_0 \cap \mathrm{Ann}_B(m) \subset B$$

from which we see that the induced map

$$A_0/\mathrm{Ann}_{A_0}(m) \to B/\mathrm{Ann}_B(m)$$

is injective, hence ψ restricts to a dominant morphism $V(\mathrm{Ann}_B(m)) \to V(\mathrm{Ann}_{A_0}(m))$. Especially, there exists $\mathfrak{q} \in V(\mathrm{Ann}_B(m))$ with $\psi(\mathfrak{q}) = \mathfrak{p}$; up to replacing $\mathfrak{q}$ by a generization, we may assume that $\mathfrak{q}$ is a maximal point of $V(\mathrm{Ann}_B(m))$, hence $\mathfrak{q}$ is an associated prime for $\mathrm{gr}^\bullet_I M$, again by lemma 10.5.3(i). □

10.5.28. An easy induction, starting from lemma 10.5.27, shows that $\mathrm{Ass}_A M/I^nM \subset \Sigma(M)$, for every $n \in \mathbb{N}$, therefore $\Sigma_0(M) \subset \Sigma(M)$. However, if A is noetherian, the same holds for $\mathrm{gr}^\bullet_I A$ (since the latter is a quotient of an A/I-algebra of finite type), hence $\Sigma(M)$ is finite, provided M is finitely generated ([126, Th.6.5(i)]), and *a fortiori*, the same holds for $\Sigma_0(M)$.

Remark 10.5.29. Another proof of the finiteness of $\Sigma_0(M)$ can be found in [42].

10.5.30. Next, we wish to show that actually there exists a *smallest* subset $\Sigma_{\min}(M) \subset \mathrm{Spec}\, A$ such that the $\Sigma_{\min}(M)$-symbolic topology on M agrees with I-preadic topology; after some simple reductions, this boils down to the following assertion. Let $\Sigma \subset \mathrm{Spec}\, A$ be a finite subset, and $\mathfrak{p}, \mathfrak{p}' \in \Sigma$ two elements, such that the Σ-symbolic topology on M agrees with both the $\Sigma \setminus \{\mathfrak{p}\}$-symbolic topology and the $\Sigma\setminus\{\mathfrak{p}'\}$-symbolic topology; then these topologies agree as well with the $\Sigma \setminus \{\mathfrak{p}, \mathfrak{p}'\}$-symbolic topology. Indeed, given any subset $\Sigma' \subset \mathrm{Spec}\, A$ with $\mathfrak{p} \in \Sigma'$, for the Σ'-symbolic and the $\Sigma' \setminus \{\mathfrak{p}\}$-symbolic topologies to agree, it is necessary and sufficient that, for every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that :

$$\mathrm{Fil}^{(m)}_{\Sigma'\setminus\{\mathfrak{p}\}} M \subset \mathrm{Fil}^{(n)}_{\mathfrak{p}} M \qquad \text{or, what is the same :} \qquad \mathrm{Fil}^{(m)}_{\Sigma'\setminus\{\mathfrak{p}\}} M_{\mathfrak{p}} \subset \mathfrak{p}^n M_{\mathfrak{p}}.$$

In the latter inclusion, we may then replace $\Sigma' \setminus \{\mathfrak{p}\}$ by the smaller subset $\Sigma' \cap \mathrm{Spec}\, A_{\mathfrak{p}} \setminus \{\mathfrak{p}\}$, without changing the two terms. Suppose now that $\mathfrak{p}' \notin \mathrm{Spec}\, A_{\mathfrak{p}}$; then if we apply the above, first with $\Sigma' := \Sigma$, and then with $\Sigma' := \Sigma \setminus \{\mathfrak{p}'\}$, we see that the Σ-symbolic topology agrees with the $\Sigma\setminus\{\mathfrak{p}\}$-symbolic topology, if and only if the $\Sigma\setminus\{\mathfrak{p}'\}$-symbolic topology agrees with the $\Sigma\setminus\{\mathfrak{p}, \mathfrak{p}'\}$-symbolic topology, whence the contention. In case $\mathfrak{p}' \in \mathrm{Spec}\, A_{\mathfrak{p}}$, we may assume that $\mathfrak{p} \neq \mathfrak{p}'$, otherwise there is nothing to prove; then we shall have $\mathfrak{p} \notin \mathrm{Spec}\, A_{\mathfrak{p}'}$, so the foregoing argument still goes through, after reversing the roles of $\mathfrak{p}$ and $\mathfrak{p}'$.

10.5.31. Finally, theorem 10.5.35 will characterize the subset $\Sigma_{\min}(M)$ as in (10.5.28). To this aim, for every prime ideal $\mathfrak{p} \subset A$, let $A^\wedge_{\mathfrak{p}}$ denote the $\mathfrak{p}$-adic completion of A; we set :

$$\mathrm{Ass}_A(I, M) := \{\mathfrak{p} \in \mathrm{Spec}\, A \mid \text{there exists } \mathfrak{q} \in \mathrm{Ass}_{A^\wedge_{\mathfrak{p}}} M \otimes_A A^\wedge_{\mathfrak{p}} \text{ such that } \sqrt{\mathfrak{q} + IA^\wedge_{\mathfrak{p}}} = \mathfrak{p}A^\wedge_{\mathfrak{p}}\}$$

(where, for any ideal $J \subset A^\wedge_{\mathfrak{p}}$ we denote by $\sqrt{J} \subset A^\wedge_{\mathfrak{p}}$ the radical of J, so the above condition selects the points $\mathfrak{q} \in \mathrm{Spec}\, A^\wedge_{\mathfrak{p}}$, such that $\overline{\{\mathfrak{q}\}} \cap V(IA^\wedge_{\mathfrak{p}}) = \{\mathfrak{p}A^\wedge_{\mathfrak{p}}\}$).

Lemma 10.5.32. *Let A be a noetherian ring, M an A-module. Then we have :*

(i) $\mathrm{depth}_{A_{\mathfrak{p}}} M \otimes_A A_{\mathfrak{p}} = \mathrm{depth}_{A^\wedge_{\mathfrak{p}}} M \otimes_A A^\wedge_{\mathfrak{p}}$ *for every* $\mathfrak{p} \in \mathrm{Spec}\, A$.
(ii) $\mathrm{Ass}_A(0, M) = \mathrm{Ass}_A M$.
(iii) $\mathrm{Ass}_A(I, M) \subset V(I) \cap \mathrm{Supp}\, M$.
(iv) *Suppose that M is the union of a filtered family $(M_\lambda \mid \lambda \in \Lambda)$ of A-submodules. Then :*

$$\mathrm{Ass}_A(I, M) = \bigcup_{\lambda \in \Lambda} \mathrm{Ass}_A(I, M_\lambda).$$

(v) $\mathrm{Ass}_A(I, M)$ *contains the maximal points of* $V(I) \cap \mathrm{Supp}\, M$.

Proof. (iii) is immediate, and in view of [126, Th.8.8], (i) is a special case of corollary 10.4.37.

(ii): By definition, $\mathrm{Ass}_A(0, M)$ consists of all the prime ideals $\mathfrak{p} \subset A$ such that $\mathfrak{p}A^\wedge_{\mathfrak{p}}$ is an associated prime ideal of $M \otimes_A A^\wedge_{\mathfrak{p}}$; in light of proposition 10.5.6(i), the latter condition holds if and only if $\mathrm{depth}_{A^\wedge_{\mathfrak{p}}} M \otimes_A A^\wedge_{\mathfrak{p}} = 0$, hence if and only if $\mathrm{depth}_{A_{\mathfrak{p}}} M \otimes_A A_{\mathfrak{p}} = 0$, by (i); to conclude, one appeals again to proposition 10.5.6(i).

(iv): In view of proposition 10.5.6(v), we have $\mathrm{Ass}_{A^\wedge_{\mathfrak{p}}} M \otimes_A A^\wedge_{\mathfrak{p}} = \bigcup_{\lambda \in \Lambda} \mathrm{Ass}_{A^\wedge_{\mathfrak{p}}} M_\lambda \otimes_A A^\wedge_{\mathfrak{p}}$; the contention is an immediate consequence.

(v): In view of (iv), we are easily reduced to the case where M is a finitely generated A-module, in which case $\mathrm{Supp}\, M = V(J)$ for some ideal $J \subset A$. Hence, suppose that $\mathfrak{p} \subset A$ is a maximal point of $\mathrm{Spec}\, A/(I + J)$, in other words, the preimage of a minimal prime ideal of $A/(I + J)$; notice that $\mathrm{Supp}\, M \otimes_A A^\wedge_{\mathfrak{p}} = V(JA^\wedge_{\mathfrak{p}})$. Hence we have $(J + I)A^\wedge_{\mathfrak{p}} \subset \mathfrak{q} + IA^\wedge_{\mathfrak{p}}$ for every $\mathfrak{q} \in \mathrm{Supp}\, M \otimes_A A^\wedge_{\mathfrak{p}}$, so the radical of $\mathfrak{q} + IA^\wedge_{\mathfrak{p}}$ equals $\mathfrak{p}A^\wedge_{\mathfrak{p}}$, as required. ☐

Lemma 10.5.33 (Chevalley's theorem)**.** *Let $(A, \mathfrak{m})$ be an $\mathfrak{m}$-adically complete local noetherian ring, M a finitely generated A-module, and $(\mathrm{Fil}^n M \mid n \in \mathbb{N})$ a descending filtration consisting of A-submodules of M. Then the following conditions are equivalent :*

(a) $\bigcap_{n \in \mathbb{N}} \mathrm{Fil}^n M = 0$.
(b) *For every $i \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that* $\mathrm{Fil}^n M \subset \mathfrak{m}^i M$.

Proof. Clearly (b)⇒(a), hence suppose that (a) holds. For every $i, n \in \mathbb{N}$, set

$$J_{i,n} := \mathrm{Im}(\mathrm{Fil}^n M \to M/\mathfrak{m}^i M).$$

For given $i \in \mathbb{N}$, the A-module $M/\mathfrak{m}^i M$ is artinian, hence there exists $n \in \mathbb{N}$ such that $J_i := J_{i,n} = J_{i,n'}$ for every $n' \geq n$. Assertion (b) then follows from the following :

Claim 10.5.34. If (a) holds, $J_i = 0$ for every $i \in \mathbb{N}$.

Proof of the claim. By inspecting the definition, it is easily seen that the natural $A/\mathfrak{m}^{i+1}$-linear map $J_{i+1} \to J_i$ is surjective for every $i \in \mathbb{N}$, hence we are reduced to showing that $J := \lim_{i\in\mathbb{N}} J_i$ vanishes. However, J is naturally a submodule of $\lim_{i\in\mathbb{N}} M/\mathfrak{m}^i M \simeq M$, and if $x \in M$ lies in J, then we have $x \in \mathrm{Fil}^n M + \mathfrak{m}^i M$ for every $i, n \in \mathbb{N}$. Since $\mathrm{Fil}^n M$ is a closed subset for the $\mathfrak{m}$-adic topology of M ([126, Th.8.10(i)]), we have $\bigcap_{i\in\mathbb{N}}(\mathrm{Fil}^n M + \mathfrak{m}^i M) = \mathrm{Fil}^n M$, for every $n \in \mathbb{N}$, hence $x \in \bigcap_{n\in\mathbb{N}} \mathrm{Fil}^n M$, which vanishes, if (a) holds. □

The following theorem generalizes [145, Ch.VIII, §5, Cor.5] and [88, Prop.7.1], and is closely related to [148, Th.3.2]; see also [51, Th.3.6].

Theorem 10.5.35. *Let A be a noetherian ring, $I \subset A$ an ideal, M a finitely generated A-module, and $\Sigma \subset \mathrm{Spec}\, A/I$ a finite subset. Then the Σ-symbolic topology on M agrees with the I-preadic topology if and only if* $\mathrm{Ass}_A(I, M) \subset \Sigma$.

Proof. Let $\mathfrak{p} \in \mathrm{Ass}_A(I, M)\setminus\Sigma$, and suppose, by way of contradiction, that the Σ-symbolic topology agrees with the I-adic topology, *i.e.* for every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that :

$$\mathrm{Fil}^{(m)}_\Sigma M \subset I^n M.$$

Let $X := \mathrm{Spec}\, A_\mathfrak{p}$ and $U := X\setminus\{\mathfrak{p}\}$; after localizing at the prime $\mathfrak{p}$, we deduce the inclusion :

$$\mathrm{Fil}^{(m)}_{\Sigma\cap U} M_\mathfrak{p} \subset I^n M_\mathfrak{p} \tag{10.5.36}$$

(cp. the discussion in (10.5.28)). Let also $\mathscr{M} := M^\sim$, the quasi-coherent $\mathscr{O}_X$-module associated to M; clearly we have $I^m M \subset \mathrm{Fil}^{(m)}_\mathfrak{q} M$ for every $\mathfrak{q} \in \mathrm{Spec}\, A/I$, hence (10.5.36) implies the inclusion :

$$\{x \in M_\mathfrak{p} \mid x_{|U} \in I^m\mathscr{M}(U)\} \subset I^n M_\mathfrak{p}.$$

Let $A^\wedge_\mathfrak{p}$ (resp. $M^\wedge_\mathfrak{p}$) be the $\mathfrak{p}$-adic completion of A (resp. of M), $f : X^\wedge := \mathrm{Spec}\, A^\wedge_\mathfrak{p} \to X$ the natural morphism, and set $U^\wedge := f^{-1}U = X^\wedge \setminus \{\mathfrak{p}A^\wedge_\mathfrak{p}\}$. Since f is faithfully flat, and U is quasi-compact, we deduce that :

$$\{x \in M^\wedge_\mathfrak{p} \mid x_{|U^\wedge} \in I^m f^*\mathscr{M}(U^\wedge)\} \subset I^n M^\wedge_\mathfrak{p}. \tag{10.5.37}$$

On the other hand, by assumption there exists $\mathfrak{q} \in \mathrm{Ass}_{A^\wedge_\mathfrak{p}} M^\wedge_\mathfrak{p}$ such that $\overline{\{\mathfrak{q}\}}\cap V(IA^\wedge_\mathfrak{p}) = \{\mathfrak{p}A^\wedge_\mathfrak{p}\}$, hence we may find $x \in M^\wedge_\mathfrak{p}$ whose support is $\overline{\{\mathfrak{q}\}}$ (lemma 10.5.3(ii)). It follows easily that the image of x vanishes in $f^*\mathscr{M}/I^m f^*\mathscr{M}(U)$ for all $m \in \mathbb{N}$, *i.e.* $x_{|U} \in I^m f^*\mathscr{M}(U)$ for every $m \in \mathbb{N}$, hence $x \in I^n M^\wedge_\mathfrak{p}$ for every $n \in \mathbb{N}$, in view of (10.5.37). However, $M^\wedge_\mathfrak{p}$ is separated for the $\mathfrak{p}$-adic topology, *a fortiori* also for the I-preadic topology, so $x = 0$, a contradiction.

Next, let $\mathfrak{p} \in \Sigma\setminus\mathrm{Ass}_A(I, M)$, set $\Sigma' := \Sigma\setminus\{\mathfrak{p}\}$, and suppose that the Σ-symbolic topology agrees with the I-adic topology; we have to prove that the latter agrees as well with the Σ'-symbolic topology. This amounts to showing that, for every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that :

$$\mathrm{Fil}^{(m)}_{\Sigma'} M \subset \mathrm{Fil}^{(n)}_\mathfrak{p} M \qquad \text{or, what is the same :} \qquad \mathrm{Fil}^{(m)}_{\Sigma'} M_\mathfrak{p} \subset \mathfrak{p}^n M_\mathfrak{p}.$$

We may write :

$$\mathrm{Fil}^{(m)}_{\Sigma'} M_\mathfrak{p} = \{x \in M_\mathfrak{p} \mid x_{|U} \in (\mathrm{Fil}^{(m)}_{\Sigma'} M_\mathfrak{p})^\sim(U)\} = \{x \in M_\mathfrak{p} \mid x_{|U} \in (\mathrm{Fil}^{(m)}_{\Sigma} M_\mathfrak{p})^\sim(U)\}$$

from which it is clear that the contention holds if and only if, for every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that :

$$\{x \in M_\mathfrak{p} \mid x_{|U} \in I^m\mathscr{M}(U)\} \subset \mathfrak{p}^n M_\mathfrak{p}.$$

Arguing as in the foregoing, we see that the latter condition holds if and only if :

$$\mathrm{Fil}^m M^\wedge_\mathfrak{p} := \{x \in M^\wedge_\mathfrak{p} \mid x_{|U^\wedge} \in I^m f^*\mathscr{M}(U^\wedge)\} \subset \mathfrak{p}^n M^\wedge_\mathfrak{p}.$$

In view of lemma 10.5.33, we are then reduced to showing the following :

Claim 10.5.38. $\bigcap_{m\in\mathbb{N}} \mathrm{Fil}^m M^\wedge_\mathfrak{p} = 0.$

Proof of the claim. Let x be an element in this intersection; for any $\mathfrak{q} \in U^\wedge \cap V(IA_\mathfrak{p}^\wedge)$, we have $x_\mathfrak{q} \in \bigcap_{m\in\mathbb{N}} \mathfrak{q}^m(M_\mathfrak{p}^\wedge)_\mathfrak{q}$, hence $x_\mathfrak{q} = 0$, by [126, Th.8.10(i)]. In other words, $\mathrm{Supp}(x) \cap U \cap V(I) = \varnothing$. Suppose that $x \neq 0$, and let $\mathfrak{q}$ be any maximal point of $\mathrm{Supp}(x)$; by lemma 10.5.3(i), $\mathfrak{q}$ is an associated prime, and the foregoing implies that $\{\mathfrak{q}\} \cap V(IA_\mathfrak{p}^\wedge) = \{\mathfrak{p}A_\mathfrak{p}^\wedge\}$, which contradicts the assumption that $\mathfrak{p} \notin \mathrm{Ass}_A(I, M)$. □

Corollary 10.5.39. *In the situation of theorem* 10.5.35*,* $\mathrm{Ass}_A(I, M)$ *is a finite set.*

Proof. We have already found a finite subset $\Sigma \subset \mathrm{Spec}\, A/I$ such that the Σ-symbolic topology on M agrees with the I-preadic topology (see (10.5.28)). The contention then follows straightforwardly from theorem 10.5.35. □

Example 10.5.40. Let k be a field with $\mathrm{char}\, k \neq 2$, and let $C \subset \mathbb{A}_k^2 := \mathrm{Spec}\, k[X, Y]$ be the nodal curve cut by the equation $Y^2 = X^2 + X^3$, so that the only singularity of C is the node at the origin $p := (0, 0) \in C$. Let $R := k[X, Y, Z]$, $A := R/(Y^2 - X^2 - X^3)$; denote by $\pi : \mathbb{A}_k^3 := \mathrm{Spec}\, R \to \mathbb{A}_k^2$ the linear projection which is dual to the inclusion $k[X, Y] \to R$, so that $D := \pi^{-1}C = \mathrm{Spec}\, A$. We define a morphism $\varphi : \mathbb{A}_k^1 := \mathrm{Spec}\, k[T] \to D$ by the rule : $T \mapsto (T^2 - 1, T(T^2 - 1), T)$ (*i.e.* φ is dual to the homomorphism of k-algebras such that $X \mapsto T^2 - 1$, $Y \mapsto T(T^2 - 1)$ and $Z \mapsto T$). Let $C' \subset D$ be the image of φ, with its reduced subscheme structure. It is easy to check that the restriction of π maps C' birationally onto C, so there are precisely two points $p'_0, p'_1 \in C'$ lying over p. Let $\mathfrak{n} := I(C') \subset A$, the prime ideal which is the generic point of the (irreducible) curve C'. We claim that the $\mathfrak{n}$-preadic topology on A does not agree with the $\mathfrak{n}$-symbolic topology. To this aim – in view of theorem 10.5.35 – it suffices to show that $\{p'_0, p'_1\} \subset \mathrm{Ass}_A(\mathfrak{n}, A)$. However, for any closed point $\mathfrak{p} \in \pi^{-1}(p)$, the $\mathfrak{p}$-adic completion $A_\mathfrak{p}^\wedge$ admits two distinct minimal primes, corresponding to the two branches of the nodal conic C at the node p, and the corresponding irreducible components of $B := \mathrm{Spec}\, A_\mathfrak{p}^\wedge$ meet along the affine line $V(Z)$. To see this, we may suppose that $\mathfrak{p} = (X, Y, Z)$, hence $A_\mathfrak{p}^\wedge \simeq k[[X, Y, Z]]/(Y^2 - X^2(1 + X))$, and notice that the latter is isomorphic to $k[[S, Y, Z]]/(Y^2 - S^2)$, via the isomorphism that sends $Y \mapsto Y$, $Z \mapsto Z$ and $S \mapsto X(1 + X)^{1/2}$ (the assumption on the characteristic of k ensures that $1 + X$ admits a square root in $k[[X]]$). Now, say that $\mathfrak{p} = p'_0$; then $C'_\mathfrak{p} := C' \cap B$ is contained in only one of the two irreducible components of B. Let $\mathfrak{q} \in B$ be the minimal prime ideal whose closure does not contain $C'_\mathfrak{p}$; then $\mathfrak{q} \in \mathrm{Ass} A_\mathfrak{p}^\wedge$ and $\overline{\{\mathfrak{q}\}} \cap C'_\mathfrak{p} = \{\mathfrak{p}A_\mathfrak{p}^\wedge\}$, therefore $p'_0 \in \mathrm{Ass}_A(\mathfrak{n}, A)$, as stated.

Example 10.5.41. Let A be an excellent normal ring, $I \subset A$ any ideal, and set $Z := V(I)$. Then $\mathrm{Ass}_A(I, A)$ is the set $\mathrm{Max}(Z)$ of all maximal points of Z. Indeed, $\mathrm{Max}(Z) \subset \mathrm{Ass}_A(I, A)$ by lemma 10.5.32(v). Conversely, suppose $\mathfrak{p} \in \mathrm{Ass}_A(I, A)$; the completion $A_\mathfrak{p}^\wedge$ is still normal ([126, Th.32.2(i)]), and therefore its only associated prime is 0, so the assumption means that the radical of $IA_\mathfrak{p}^\wedge$ is $\mathfrak{p}A_\mathfrak{p}^\wedge$. Equivalently, $\dim A_\mathfrak{p}^\wedge/IA_\mathfrak{p}^\wedge = 0$, so $\dim A_\mathfrak{p}/IA_\mathfrak{p} = 0$, which is the contention.

Definition 10.5.42. Let X be a noetherian scheme, $Y \subset X$ a closed subset, $\mathfrak{X}$ the formal completion of X along Y ([59, Ch.I, §10.8]), and $f : \mathfrak{X} \to X$ the natural morphism of locally ringed spaces. We say that the pair (X, Y) *satisfies the Lefschetz condition*, if for every open subset $U \subset X$ with $Y \subset U$, and every locally free $\mathscr{O}_U$-module $\mathscr{E}$ of finite type, the natural map

$$\Gamma(U, \mathscr{E}) \to \Gamma(\mathfrak{X}, f^*\mathscr{E})$$

is an isomorphism. In this case, we also say that $\mathrm{Lef}(X, Y)$ holds. (Cp. [85, Exp.X, §2].)

Lemma 10.5.43. *In the situation of definition* 10.5.42*, suppose that* $\mathrm{Lef}(X, Y)$ *holds, and let* $U \subset X$ *be any open subset such that* $Y \subset U$*. Then :*

(i) *The functor :*

$$\mathscr{O}_U\text{-}\mathbf{Mod}_{\mathrm{lfft}} \to \mathscr{O}_\mathfrak{X}\text{-}\mathbf{Mod}_{\mathrm{lfft}} \quad : \quad \mathscr{E} \mapsto f^*\mathscr{E}$$

is fully faithful (notation of (10.3)*).*

(ii) *Denote by* $\mathscr{O}_U\text{-}\mathsf{Alg}_{\mathrm{lfft}}$ *the category of* $\mathscr{O}_U$*-algebras, whose underlying* $\mathscr{O}_U$*-module is locally free of finite type, and define likewise* $\mathscr{O}_{\mathfrak{X}}\text{-}\mathsf{Alg}_{\mathrm{lfft}}$*. Then the functor :*

$$\mathscr{O}_U\text{-}\mathsf{Alg}_{\mathrm{lfft}} \to \mathscr{O}_{\mathfrak{X}}\text{-}\mathsf{Alg}_{\mathrm{lfft}} \quad : \quad \mathscr{A} \mapsto f^*\mathscr{A}$$

is fully faithful.

Proof. (i): Let $\mathscr{E}$ and $\mathscr{F}$ be any two locally free $\mathscr{O}_U$-modules of finite type. We have :

$$\mathrm{Hom}_{\mathscr{O}_U}(\mathscr{E},\mathscr{F}) = \Gamma(U, \mathscr{H}om_{\mathscr{O}_U}(\mathscr{E},\mathscr{F}))$$

and likewise we may compute $\mathrm{Hom}_{\mathscr{O}_{\mathfrak{X}}}(f^*\mathscr{E}, f^*\mathscr{F})$. However, the natural map :

$$f^*\mathscr{H}om_{\mathscr{O}_U}(\mathscr{E},\mathscr{F}) \to \mathscr{H}om_{\mathscr{O}_{\mathfrak{X}}}(f^*\mathscr{E}, f^*\mathscr{F})$$

is an isomorphism of $\mathscr{O}_{\mathfrak{X}}$-modules. The assertion follows.

(ii): An object of $\mathscr{O}_U\text{-}\mathsf{Alg}_{\mathrm{lfft}}$ is a locally free $\mathscr{O}_U$-module $\mathscr{A}$ of finite type, together with morphisms $\mathscr{A} \otimes_{\mathscr{O}_U} \mathscr{A} \to \mathscr{A}$ and $1_{\mathscr{A}} : \mathscr{O}_U \to \mathscr{A}$ of $\mathscr{O}_U$-modules, fulfilling the usual unitarity, commutativity and associativity conditions. An analogous description holds for the objects of $\mathscr{O}_{\mathfrak{X}}\text{-}\mathsf{Alg}_{\mathrm{lfft}}$, and for the morphisms of either category. Since $\mathscr{A} \otimes_{\mathscr{O}_U} \mathscr{A}$ is again locally free of finite type, the assertion follows easily from (i) : the details are left to the reader. □

Lemma 10.5.44. *Let A be a noetherian ring, $I \subset A$ an ideal, $U \subset \mathrm{Spec}\, A$ an open subset, $\mathfrak{U}$ the formal completion of U along $U \cap V(I)$. Consider the following conditions :*

(a) $\mathrm{Lef}(U, U \cap V(I))$ *holds.*

(b) *The natural map* $\rho_U : \Gamma(U, \mathscr{O}_U) \to \Gamma(\mathfrak{U}, \mathscr{O}_{\mathfrak{U}})$ *is an isomorphism.*

(c) *The natural map* $\rho : A \to \Gamma(\mathfrak{U}, \mathscr{O}_{\mathfrak{U}})$ *is an isomorphism.*

Then (c)⇒(b)⇔(a)*, and* (c) *implies that A is I-adically complete.*

Proof. Clearly (a)⇒(b), hence we assume that (b) holds, and we show (a). Let $V \subset U$ be an open subset with $U \cap V(I) \subset V$ and $\mathscr{E}$ a coherent locally free $\mathscr{O}_V$-module. As A is noetherian, V is quasi-compact, so we may find a left exact sequence $P_\bullet := (0 \to \mathscr{E} \to \mathscr{O}_V^{\oplus m} \to \mathscr{O}_V^{\oplus n})$ of $\mathscr{O}_V$-modules (corollary 10.3.28). Since the natural map of locally ringed spaces $f : \mathfrak{U} \to U$ is flat, the sequence f^*P is still left exact. Since the global section functors are left exact, there follows a ladder of left exact sequences :

$$\Gamma(V, P_\bullet) \to \Gamma(\mathfrak{U}, f^*P_\bullet)$$

which reduces the assertion to the case where $\mathscr{E} = \mathscr{O}_V$; the latter is covered by the following :

Claim 10.5.45. The natural map $\rho_V : \Gamma(V, \mathscr{O}_V) \to \Gamma(\mathfrak{U}, \mathscr{O}_{\mathfrak{U}})$ is an isomorphism.

Proof of the claim. The isomorphism ρ_U factors through ρ_V, hence the latter is a surjection. Suppose that $s \in \mathrm{Ker}\, \rho_V$, and $s \neq 0$; then we may find $x \in V$ such that the image s_x of s in $\mathscr{O}_{V,x}$ does not vanish. Moreover, we may find $a \in A$ whose image $a(x)$ in $\kappa(x)$ does not vanish, and such that as is the restriction of an element of A; especially, $as \in \Gamma(U, \mathscr{O}_U)$, and clearly the image of as in $\Gamma(V, \mathscr{O}_V)$ lies in $\mathrm{Ker}\, \rho_V$. Therefore, $as = 0$ in $\Gamma(U, \mathscr{O}_U)$; however the image as_x of as in $\mathscr{O}_{U,x}$ is non-zero by construction, a contradiction. This shows that ρ_V is injective, whence the claim. ◊

Finally, suppose that (c) holds; arguing as in the proof of claim 10.5.45, one sees that ρ_U is an isomorphism. Moreover, since $\Gamma(\mathfrak{U}, \mathscr{O}_{\mathfrak{U}}) \simeq \lim_{n \in \mathbb{N}} \Gamma(\mathfrak{U}, \mathscr{O}_{\mathfrak{U}}/I^n\mathscr{O}_{\mathfrak{U}})$, the morphism ρ factors through the natural A-linear map $i : A \to A^\wedge$ to the I-adic completion of A. The composition with ρ^{-1} yields an A-linear left inverse $s : A^\wedge \to A$ to i. Set $N := \mathrm{Ker}\, s$; clearly s is surjective, hence $A^\wedge \simeq A \oplus N$. It follows easily that $N/I^nN = 0$ for every $n \in \mathbb{N}$, especially $N \subset \bigcap_{n \in \mathbb{N}} I^n A^\wedge$. Therefore $N = 0$, since $A^\wedge$ is separated for the I-adic topology. □

Proposition 10.5.46. *Let $\varphi : A \to B$ be a flat homomorphism of noetherian rings, $I \subset A$ an ideal, $U \subset \mathrm{Spec}\, B$ an open subset. Set $f := \mathrm{Spec}\, \varphi : \mathrm{Spec}\, B \to \mathrm{Spec}\, A$, and assume that :*

(a) *B is complete for the IB-adic topology.*
(b) *For every $x \in V(I)$, we have : $\{y \in f^{-1}(x) \mid \delta(y, \mathscr{O}_{f^{-1}(x)}) = 0\} \subset U$.*
(c) *For every $x \in \mathrm{Ass}_A(I, A)$, we have : $\{y \in f^{-1}(x) \mid \delta(y, \mathscr{O}_{f^{-1}(x)}) \leq 1\} \subset U$.*

Then $\mathrm{Lef}(U, U \cap V(IB))$ *holds.*

Proof. Set $\Sigma := \mathrm{Ass}_A(I, A)$, and let $\mathfrak{U}$ be the formal completion of U along $V(IB)$; by theorem 10.5.35, the I-preadic topology on A agrees with the Σ-symbolic topology. Let also $\mathscr{J}$ be the family consisting of all ideals $J \subset A$ such that $\mathrm{Ass}\, A/J \subset \Sigma$; it follows that the natural maps :

$$B \to \lim_{J \in \mathscr{J}} B/JB \qquad \Gamma(\mathfrak{U}, \mathscr{O}_{\mathfrak{U}}) \to \lim_{J \in \mathscr{J}} \Gamma(U, \mathscr{O}_U/J\mathscr{O}_U)$$

are isomorphisms (see the discussion in (10.5.26)). In view of lemma 10.5.44, we are then reduced to showing :

Claim 10.5.47. The natural map $B/JB \to \Gamma(U, \mathscr{O}_U/J\mathscr{O}_U)$ is an isomorphism for every $J \in \mathscr{J}$.

Proof of the claim. Let $f : Y := \mathrm{Spec}\, B/JB \to X := \mathrm{Spec}\, A/J$ be the induced morphism; in view of corollary 10.4.23, it suffices to prove that $\delta(y, \mathscr{O}_Y) \geq 2$ whenever $y \in Y \setminus U$. Thus, set $x := f(y)$; by corollary 10.4.47 we have :

$$\delta(y, \mathscr{O}_Y) = \delta(y, \mathscr{O}_{f^{-1}(x)}) + \delta(x, \mathscr{O}_X).$$

Now, if $\delta(x, \mathscr{O}_X) = 1$, notice that $f(Y) \subset V(J) \subset V(I)$, by lemmata 10.5.3(i) and 10.5.32(iii); hence (b) implies the contention in this case. Lastly, if $\delta(x, \mathscr{O}_X) = 0$, then $x \in \mathrm{Ass}\, A/J$ by proposition 10.5.6, hence we use assumption (c) to conclude. □

10.6. **Cohomology of projective schemes.** We begin by recalling a few generalities on graded algebras and their homogeneous prime spectra; next we define the blow up of a scheme along a quasi-coherent sheaf of ideals, and we prove some basic results on the higher direct images of quasi-coherent modules under blow up morphisms.

10.6.1. Let $A := \oplus_{n \in \mathbb{N}} A_n$ be a $\mathbb{N}$-graded ring, and set $A_+ := \oplus_{n > 0} A_n$, which is an ideal of A. Following [60, Ch.II, (2.3.1)], one denotes by $\mathrm{Proj}\, A$ the set consisting of all *graded prime ideals* of A that do not contain A_+, and one endows $\mathrm{Proj}\, A$ with the topology induced from the Zariski topology of $\mathrm{Spec}\, A$. For every homogeneous element $f \in A_+$, set :

$$D_+(f) := D(f) \cap \mathrm{Proj}\, A$$

where as usual, $D(f) := \mathrm{Spec}\, A_f \subset \mathrm{Spec}\, A$. Clearly :

(10.6.2) $$D_+(fg) = D_+(f) \cap D_+(g) \qquad \text{for every homogeneous } f, g \in A_+.$$

The system of open subsets $D_+(f)$, for f ranging over the homogeneous elements of A_+, is a basis of the topology of $\mathrm{Proj}\, A$ ([60, Ch.II, Prop.2.3.4]), and obviously any system of homogeneous generators $(f_\lambda \mid \lambda \in \Lambda)$ for the ideal A_+ yields an open covering

$$\mathrm{Proj}\, A = \bigcup_{\lambda \in \Lambda} D_+(f_\lambda).$$

For any homogeneous element $f \in A_+$, let also $A_{(f)} \subset A_f$ be the subring consisting of all elements of degree zero (for the natural $\mathbb{Z}$-grading of A_f); in other words :

$$A_{(f)} := \sum_{k \in \mathbb{N}} A_k \cdot f^{-k} \subset A_f.$$

The topological space $\mathrm{Proj}\, A$ carries a sheaf of rings $\mathscr{O}$, with isomorphisms of ringed spaces :

$$\omega_f : (D_+(f), \mathscr{O}_{|D_+(f)}) \xrightarrow{\sim} \mathrm{Spec}\, A_{(f)} \qquad \text{for every homogeneous } f \in A_+.$$

and the system of isomorphisms ω_f is compatible, in an obvious way, with the inclusions :

$$j_{f,g} : D_+(fg) \subset D_+(f)$$

as in (10.6.2), and with the natural homomorphisms $A_{(f)} \to A_{(fg)}$. Especially, the locally ringed space $(\mathrm{Proj}\, A, \mathscr{O})$ is a separated scheme.

Example 10.6.3. For any $d \in \mathbb{N}$, and any ring R, take $A := R[T_0, \dots, T_d]$, endowed with its standard $\mathbb{N}$-grading such that $\mathrm{gr}_n A$ is the R-module generated by the monomials of total degree n, for every $n \in \mathbb{N}$. The scheme $\mathrm{Proj}\, A$ is the *projective d-dimensional space* over $\mathrm{Spec}\, R$, denoted

$$\mathbb{P}^d_R.$$

According to (10.6.1), it admits the standard affine covering $\mathbb{P}^d_R = \bigcup_{i=0}^d D_+(T_i)$. Set as well

$$\tau_{ij} := T_j/T_i \qquad \text{for every } i, j = 0, \dots, n.$$

Then clearly $A_i := A_{(T_i)} = R[\tau_{ij} \mid j = 0, \dots, d]$ is a free polynomial R-algebra in d variables, so that $U_i := D_+(T_i)$ is isomorphic to the d-dimensional affine space $\mathbb{A}^d_R$ over $\mathrm{Spec}\, R$, for every $i = 0, \dots, d$. For $i \neq j$, the intersection $U_{ij} := D_+(T_i T_j) = U_i \cap U_j$ corresponds, under these identifications, to the open subsets

$$\mathrm{Spec}\, A_i[\tau_{ij}^{-1}] \subset U_i \qquad \text{and} \qquad \mathrm{Spec}\, A_j[\tau_{ji}^{-1}] \subset U_j$$

so we get a commutative diagram of isomorphisms of R-algebras

$$\begin{array}{ccc} & \mathscr{O}_{\mathbb{P}^d_R}(U_{ij}) & \\ \swarrow & & \searrow \\ A_i[\tau_{ij}^{-1}] & \longrightarrow & A_j[\tau_{ji}^{-1}] \end{array}$$

whose downward arrows are induced by the maps ω_{T_i} and ω_{T_j} of (10.6.1), and where the horizontal arrow is given by the rule :

$$\tau_{ik} \mapsto \tau_{jk} \cdot \tau_{ji}^{-1} \qquad \text{for every } k = 0, \dots, d. \tag{10.6.4}$$

10.6.5. Let $A' := \oplus_{n \in \mathbb{N}} A'_n$ be another $\mathbb{N}$-graded ring, and $\varphi : A \to A'$ a homomorphism of graded rings (*i.e.* $\varphi(A_n) \subset A'_n$ for every $n \in \mathbb{N}$). Following [60, Ch.II, (2.8.1)], we let :

$$G(\varphi) := \mathrm{Proj}\, A' \setminus V(\varphi(A_+)).$$

This open subset of $\mathrm{Proj}\, A'$ is also the same as the union of all the open subsets of the form $D_+(\varphi(f))$, where f ranges over the homogeneous elements of A_+. The restriction to $G(\varphi)$ of $\mathrm{Spec}\, \varphi : \mathrm{Spec}\, A' \to \mathrm{Spec}\, A$, is a continuous map ${}^a\varphi : G(\varphi) \to \mathrm{Proj}\, A$. Moreover, we have the identity :

$${}^a\varphi^{-1}(D_+(f)) = D_+(\varphi(f)) \qquad \text{for every homogeneous } f \in A_+.$$

Furthermore, the homomorphism φ induces a homomorphism $\varphi_{(f)} : A_{(f)} \to A'_{(\varphi(f))}$, whence a morphism of schemes :

$$\Phi_f : D_+(\varphi(f)) \to D_+(f).$$

Let $g \in A_+$ be another homogeneous element; it is easily seen that :

$$j_{f,g} \circ \Phi_{fg} = (\Phi_f)_{|D_+(\varphi(fg))}.$$

It follows that the locally defined morphisms Φ_f glue to a unique morphism of schemes :

$$\mathrm{Proj}\, \varphi : G(\varphi) \to \mathrm{Proj}\, A$$

such that the diagram of schemes :

(10.6.6)
$$\begin{array}{ccc} \mathrm{Spec}\, A'_{(\varphi(f))} & \xrightarrow{\mathrm{Spec}\,\varphi_{(f)}} & \mathrm{Spec}\, A_{(f)} \\ {\scriptstyle \omega_{\varphi(f)}}\downarrow & & \downarrow{\scriptstyle \omega_f} \\ D_+(\varphi(f)) & \xrightarrow{(\mathrm{Proj}\,\varphi)_{|D_+(\varphi(f))}} & D_+(f) \end{array}$$

commutes for every homogeneous $f \in A_+$ ([60, Ch.II, Prop.2.8.2]). Lastly, we remark that, for every homogeneous element $f' \in A'_+$, the open subset $D_+(f')$ lies in $G(\varphi)$ if and only if f' lies in the radical of the ideal of A' generated by $\varphi(A_+)$ ([60, Ch.II, Cor.2.3.15]). Especially, $G(\varphi) = \mathrm{Proj}\, A'$ whenever $\varphi(A_+)$ generates the ideal A'_+.

10.6.7. To ease notation, set $Y := \mathrm{Proj}\, A$. Let $M := \oplus_{n\in\mathbb{Z}} M_n$ be a $\mathbb{Z}$-graded A-module (*i.e.* $A_k \cdot M_n \subset M_{k+n}$ for every $k \in \mathbb{N}$ and $n \in \mathbb{Z}$); for every homogeneous $f \in A_+$, denote by $M_{(f)} \subset M_f$ the submodule consisting of all elements of degree zero (for the natural $\mathbb{Z}$-grading of M_f). Clearly $M_{(f)}$ is an $A_{(f)}$-module in a natural way, whence a quasi-coherent $\mathscr{O}_{D_+(f)}$-module $M^\sim_{(f)}$; these modules glue to a unique quasi-coherent $\mathscr{O}_Y$-module $M^\sim$ ([60, Ch.II, Prop.2.5.2]). Especially, for every $n \in \mathbb{Z}$, let $A(n)$ be the $\mathbb{Z}$-graded A-module such that $A(n)_k := A_{n+k}$ for every $k \in \mathbb{Z}$ (with the convention that $A_k = 0$ if $k < 0$). We set :

$$\mathscr{O}_Y(n) := A(n)^\sim.$$

Any element $f \in A_n$ induces a natural isomorphism of $D_+(f)$-modules :

$$\mathscr{O}_Y(n)_{|D_+(f)} \xrightarrow{\sim} \mathscr{O}_{D_+(f)}$$

([60, Ch.II, Prop.2.5.7]). Hence, on the open subset

$$U_n(A) := \bigcup_{f\in A_n} D_+(f)$$

the sheaf $\mathscr{O}_Y(n)$ restricts to an invertible $\mathscr{O}_{U_n(A)}$-module. Especially, if A_1 generates the ideal A_+, the $\mathscr{O}_Y$-modules $\mathscr{O}_Y(n)$ are invertible, for every $n \in \mathbb{Z}$.

Example 10.6.8. Resume the notation of example 10.6.3. A direct inspection of the definitions shows that $A(n)_{(T_i)}$ is the A_i-module generated by all fractions of the form $T_0^{r_0}T_1^{r_1}\cdots T_d^{r_d}T_i^{-k}$, for every $k \in \mathbb{N}$ and every $(r_0, \dots, r_d) \in \mathbb{N}^{\oplus d+1}$ such that $r_0 + r_1 + \cdots + r_d = n + k$. In other words, $A(n)_{(T_i)}$ is the A_i-submodule of $A(n)_{T_i}$ generated by T_i^n. We see then directly that this is a free A_i-module of rank one. Moreover, we get a commutative diagram of isomorphisms

$$\begin{array}{ccc} & \mathscr{O}_{\mathbb{P}^d_R}(n)(U_{ij}) & \\ \swarrow & & \searrow \\ T_i^n A_i[\tau_{ij}^{-1}] & \longrightarrow & T_j^n A_j[\tau_{ji}^{-1}] \end{array}$$

whose downward arrows are induced by the restriction maps of the sheaf $\mathscr{O}_{\mathbb{P}^d_R}(n)$, and where the horizontal map is the $A_i[\tau_{ij}^{-1}]$-linear map given by the rule

$$T_i^n \mapsto T_j^n \cdot \tau_{ji}^n$$

and where the $A_i[\tau_{ij}^{-1}]$-module structure of $T_j^n A_j[\tau_{ji}^{-1}]$ is defined by restriction of scalars along the isomorphism $A_i[\tau_{ij}^{-1}] \xrightarrow{\sim} A_j[\tau_{ji}^{-1}]$ as in (10.6.4).

10.6.9. In the situation of (10.6.5), let $M := \oplus_{n\in\mathbb{Z}} M_n$ be a $\mathbb{Z}$-graded A-module. Then $M' := M \otimes_A A'$ is a $\mathbb{Z}$-graded A'-module, with the grading defined by the rule :

$$M'_n := \sum_{j+k=n} \mathrm{Im}(M_j \otimes_{\mathbb{Z}} A'_k \to M') \qquad \text{for every } n \in \mathbb{Z}$$

([60, Ch.II, (2.1.2)]). Then [60, Ch.II, Prop.2.8.8] yields a natural morphism of $\mathscr{O}_{G(\varphi)}$-modules:

$$\nu_M : (\mathrm{Proj}\,\varphi)^* M^{\sim} \to (M')^{\sim}_{|G(\varphi)}.$$

Moreover, set :

$$G_1(\varphi) := \bigcup_{f\in A_1} D_+(\varphi(f))$$

and notice that $G_1(\varphi) \subset U_1(A') \cap G(\varphi)$; by inspecting the proof of *loc.cit.* we see that the restriction $\nu_{M|G_1(\varphi)}$ is an isomorphism. Especially, ν_M is an isomorphism whenever A_1 generates the ideal A_+. It is also easily seen that $G_1(\varphi) = U_1(A')$ if $\varphi(A_+)$ generates A'_+.

For any $f \in A_1$, the restriction $(\nu_M)_{|D_+(\varphi(f))}$ can be described explicitly : namely, we have natural identifications

$$\omega^*_f(M^{\sim}_{|D_+(f)}) \xrightarrow{\sim} M^{\sim}_{(f)} \qquad \omega^*_{\varphi(f)}(M')^{\sim}_{|D_+(\varphi(f))} \xrightarrow{\sim} (M')^{\sim}_{(\varphi(f))}$$

and in view of (10.6.6), the morphism $(\nu_M)_{|D_+(\varphi(f))}$ is induced by the $A'_{(\varphi(f))}$-linear map :

$$M_{(f)} \otimes_{A_{(f)}} A'_{(\varphi(f))} \to M'_{(\varphi(f))}$$

given by the rule :

$$(m_k \cdot f^{-k}) \otimes (a'_j \cdot \varphi(f)^{-j}) \mapsto (m_k \otimes a'_j) \cdot \varphi(f)^{-j-k} \qquad \text{for all } k, j \in \mathbb{Z},\, m_k \in M_k,\, a'_j \in A'_j.$$

10.6.10. The foregoing results can be improved somewhat, in the following special situation. Let $R \to R'$ be a ring homomorphism, A a $\mathbb{N}$-graded R-algebra (hence the structure morphism $R \to A$ is a ring homomorphism $R \to A_0$); the ring $A' := R' \otimes_R A$ is naturally a $\mathbb{N}$-graded R'-algebra, and the induced map $\varphi : A \to A'$ is a homomorphism of graded rings. In this case, obviously $\varphi(A_+)$ generates the ideal A'_+, hence $G(\varphi) = \mathrm{Proj}\,A'$, and indeed, $\mathrm{Proj}\,\varphi$ induces an isomorphism of $\mathrm{Spec}\,R'$-schemes :

$$Y' \xrightarrow{\sim} \mathrm{Spec}\,R' \times_{\mathrm{Spec}\,R} Y$$

where again $Y := \mathrm{Proj}\,A$ and $Y' := \mathrm{Proj}\,A'$. Moreover, for every $\mathbb{Z}$-graded A-module M, the corresponding morphism ν_M is an isomorphism, regardless of whether or not A_1 generates A_+ ([60, Ch.II, Prop.2.8.10]). Especially, $\nu_{A(n)}$ is a natural identification ([60, Ch.II, Cor.2.8.11]) :

$$(\mathrm{Proj}\,\varphi)^* \mathscr{O}_Y(n) \xrightarrow{\sim} \mathscr{O}_{Y'}(n) \qquad \text{for every } n \in \mathbb{Z}.$$

10.6.11. Keep the notation of (10.6.1), and for every integer $d > 0$, set

$$A^{(d)}_n := A_{nd} \qquad \text{for every } n \in \mathbb{N}, \text{ and} \qquad A^{(d)} := \bigoplus_{n\in\mathbb{N}} A^{(d)}_n.$$

Clearly $A^{(d)}$ is an $\mathbb{N}$-graded ring, and a subring of A, but the inclusion map $j : A^{(n)} \to A$ is not a homomorphism of $\mathbb{N}$-graded rings. However, for every homogeneous element $f \in A_+$, the map j induces a natural identification of A-algebras :

$$A^{(d)}_{(f^d)} \xrightarrow{\sim} A_{(f)} \qquad (10.6.12)$$

which in turn yields a natural isomorphism of A_0-schemes ([60, Ch.II, Prop.2.4.7(i)])

$$\Omega^{(d)} : \mathrm{Proj}\,A \xrightarrow{\sim} \mathrm{Proj}\,A^{(d)} \qquad \text{for every } d > 0.$$

To ease notation, we shall let $Y := \mathrm{Proj}\,A$ and $Y^{(d)} := \mathrm{Proj}\,A^{(d)}$ for every integer $d > 0$. Likewise, if M is any $\mathbb{Z}$-graded A-module, we may consider that $\mathbb{Z}$-graded $A^{(d)}$-module $M^{(d)}$

whose homogeneous direct summand of degree n equals M_{nd}, for every $n \in \mathbb{Z}$. Then the inclusion map $M^{(d)} \to M$ induces natural identifications

(10.6.13) $$M^{(d)}_{(f^d)} \to M_{(f)} \qquad \text{for every homogeneous element } f \text{ of } A.$$

Indeed, it is easily seen that this map is surjective, and it is injective, since it is the restriction of the injective map $M^{(d)}_{f^d} \to M_{f^d}$. Moreover, if we endow $M_{(f)}$ with the $A^{(d)}_{(f^d)}$-module structure induced by (10.6.12), it is easily seen that (10.6.13) is an isomorphism of $A^{(d)}_{(f^d)}$-modules. Thus, we obtain a natural isomorphism of quasi-coherent $\mathscr{O}_{Y^{(d)}}$-modules

$$M^{(d)\sim} \xrightarrow{\sim} \Omega^{(d)}_* M^\sim.$$

Next, for every $n \in \mathbb{Z}$, let $M(n)$ be the $\mathbb{Z}$-graded A-module such that $M(n)_k := M_{n+k}$ for every $k \in \mathbb{Z}$ (with A-module structure deduced from that of M, in the obvious way). Clearly

$$M(nd)^{(d)} = M^{(d)}(n) \qquad \text{for every } n \in \mathbb{Z} \text{ and every } d > 0$$

whence an induced isomorphism of quasi-coherent $\mathscr{O}_{Y^{(d)}}$-modules

$$M^{(d)}(n)^\sim \xrightarrow{\sim} \Omega^{(d)}_* M(nd)^\sim \qquad \text{for every } n \in \mathbb{Z} \text{ and every } d > 0.$$

Especially, we have a natural identification

(10.6.14) $$\mathscr{O}_{Y^{(d)}}(n) \xrightarrow{\sim} \Omega^{(d)}_* \mathscr{O}_Y(nd) \qquad \text{for every } n \in \mathbb{Z} \text{ and every } d > 0.$$

10.6.15. Let X be a scheme, $\mathscr{A} := \oplus_{n\in\mathbb{N}}\mathscr{A}_n$ a $\mathbb{N}$-graded quasi-coherent $\mathscr{O}_X$-algebra on the Zariski site of X; we let $\mathscr{A}_+ := \oplus_{n>0}\mathscr{A}_n$. According to [60, Ch.II, Prop.3.1.2], there exists an X-scheme $\pi : \mathrm{Proj}\,\mathscr{A} \to X$, with natural isomorphisms of U-schemes :

$$\psi_U : U \times_X \mathrm{Proj}\,\mathscr{A} \xrightarrow{\sim} \mathrm{Proj}\,\mathscr{A}(U)$$

for every affine open subset $U \subset X$, and the system of isomorphisms ψ_U is compatible, in an obvious way, with inclusions $U' \subset U$ of affine open subsets. Especially, π is a separated morphism. For any integer $d > 0$, every $f \in \Gamma(X, \mathscr{A}_d)$ defines an open subset $D_+(f) \subset \mathrm{Proj}\,\mathscr{A}$, such that :

$$D_+(f) \cap \pi^{-1}U = D_+(f_{|U}) \subset \mathrm{Proj}\,\mathscr{A}(U) \qquad \text{for every affine open subset } U \subset X.$$

10.6.16. To ease notation, set $Y := \mathrm{Proj}\,\mathscr{A}$, and let again $\pi : Y \to X$ be the natural morphism. Let $\mathscr{M} := \oplus_{n\in\mathbb{Z}}\mathscr{M}_n$ be a $\mathbb{Z}$-graded $\mathscr{A}$-module, quasi-coherent as a $\mathscr{O}_X$-module; for every affine open subset $U \subset X$, the graded $\mathscr{A}(U)$-module $\mathscr{M}(U)$ yields a quasi-coherent $\mathscr{O}_{\pi^{-1}U}$-module $\mathscr{M}_U^\sim$, and every inclusion of affine open subsets $U' \subset U$ induces a natural isomorphism of $\mathscr{O}_{\pi^{-1}U'}$-modules : $\mathscr{M}^\sim_{U|U'} \xrightarrow{\sim} \mathscr{M}^\sim_{U'}$. Therefore the locally defined modules $\mathscr{M}^\sim_U$ glue to a well defined quasi-coherent $\mathscr{O}_Y$-module $\mathscr{M}^\sim$.

Especially, for every $n \in \mathbb{Z}$, denote by $\mathscr{A}(n)$ the $\mathbb{Z}$-graded $\mathscr{A}$-module such that $\mathscr{A}(n)_k := \mathscr{A}_{n+k}$ for every $k \in \mathbb{Z}$, with the convention that $\mathscr{A}_n = 0$ whenever $n < 0$. We set:

$$\mathscr{O}_Y(n) := \mathscr{A}(n)^\sim \qquad \text{and} \qquad \mathscr{M}^\sim(n) := \mathscr{M}^\sim \otimes_{\mathscr{O}_Y} \mathscr{O}_Y(n).$$

Denote by $U_n(\mathscr{A}) \subset Y$ the union of the open subsets $U_n(\mathscr{A}(U))$, for U ranging over the affine open subsets of X; from the discussion in (10.6.7), it clear that the restriction $\mathscr{O}_Y(n)_{|U_n(\mathscr{A})}$ is an invertible $\mathscr{O}_{U_n(\mathscr{A})}$-module. This open subset can be described as follows. For every $x \in X$, let :

(10.6.17) $$\mathscr{A}_n(x) := \mathscr{A}_{n,x} \otimes_{\mathscr{O}_{X,x}} \kappa(x) \qquad \text{and set} \qquad \mathscr{A}(x) := \oplus_{n\in\mathbb{N}}\mathscr{A}_n(x)$$

which is a $\mathbb{N}$-graded $\kappa(x)$-algebra; then :

(10.6.18) $$U_n(\mathscr{A}) = \{y \in Y \mid \mathscr{A}_n(\pi(y)) \not\subset \mathfrak{p}(y)\}$$

where $\mathfrak{p}(y) \subset \mathscr{A}(\pi(y))$ denotes the prime ideal corresponding to the point y.

10.6.19. Moreover, for every $\mathbb{Z}$-graded $\mathscr{A}$-module $\mathscr{M}$, and every $n \in \mathbb{Z}$, there exists a natural morphism of $\mathscr{O}_Y$-modules :

$$\mathscr{M}^\sim(n) \to \mathscr{M}(n)^\sim \tag{10.6.20}$$

where $\mathscr{M}(n)$ is the $\mathbb{Z}$-graded $\mathscr{A}$-module given by the rule : $\mathscr{M}(n)_k := \mathscr{M}_{n+k}$ for every $k \in \mathbb{N}$ ([60, Prop.3.2.16]). The restriction of (10.6.20) to the open subset $U_1(\mathscr{A})$ is an isomorphism ([60, Ch.II, Cor.3.2.8]). Especially, we have natural morphisms of $\mathscr{O}_Y$-modules :

$$\mathscr{O}_Y(n) \otimes_{\mathscr{O}_Y} \mathscr{O}_Y(m) \to \mathscr{O}_Y(n+m) \qquad \text{for every } n, m \in \mathbb{Z} \tag{10.6.21}$$

([60, Ch.II, Prop.3.2.6]) whose restrictions to $U_1(\mathscr{A})$ are isomorphisms. Furthermore, we have a natural morphism $\mathscr{M}_0 \to \pi_*\mathscr{M}^\sim$ of $\mathscr{O}_X$-modules ([60, Ch.II, (3.3.2.1)]), whence, by adjunction, a morphism of $\mathscr{O}_Y$-modules :

$$\pi^*\mathscr{M}_0 \to \mathscr{M}^\sim. \tag{10.6.22}$$

Applying (10.6.22) to the modules $\mathscr{M}_n = \mathscr{M}(n)_0$, and taking into account the isomorphism (10.6.20), we deduce a natural morphism of $\mathscr{O}_{U_1(\mathscr{A})}$-modules :

$$(\pi^*\mathscr{M}_n)_{|U_1(\mathscr{A})} \to \mathscr{M}^\sim(n)_{|U_1(\mathscr{A})} \tag{10.6.23}$$

which can be described as follows. Let $U \subset X$ be any affine open subset; for every $f \in \mathscr{A}_1(U)$, the restriction of (10.6.23) to $D_+(f) \subset \pi^{-1}U$ is given by the morphisms

$$\mathscr{M}_n(U) \otimes_{\mathscr{O}_X(U)} \mathscr{A}(U)_{(f)} \to \mathscr{M}(n)(U)_{(f)} := \sum_{k \in \mathbb{Z}} \mathscr{M}_{k+n}(U) \cdot f^{-k} \subset \mathscr{M}(U)_f.$$

induced by the scalar multiplication $\mathscr{M}_n \otimes_{\mathscr{O}_X} \mathscr{A}_k \to \mathscr{M}_{n+k}$. Especially, we have natural morphisms of $\mathscr{O}_Y$-modules :

$$\pi^*\mathscr{A}_n \to \mathscr{O}_Y(n) \qquad \text{for every } n \in \mathbb{N} \tag{10.6.24}$$

whose restrictions to $U_1(\mathscr{A})$ are epimorphisms. An inspection of the definition also shows that the diagram of $\mathscr{O}_Y$-modules :

$$\begin{array}{ccc} \pi^*\mathscr{A}_n \otimes_{\mathscr{O}_Y} \pi^*\mathscr{A}_m & \longrightarrow & \pi^*\mathscr{A}_{n+m} \\ \downarrow & & \downarrow \\ \mathscr{O}_Y(n) \otimes_{\mathscr{O}_Y} \mathscr{O}_Y(m) & \longrightarrow & \mathscr{O}_Y(n+m) \end{array} \tag{10.6.25}$$

commutes for every $n, m \in \mathbb{N}$, where the top horizontal arrow is induced by the graded multiplication $\mathscr{A}_n \otimes_{\mathscr{O}_X} \mathscr{A}_m \to \mathscr{A}_{n+m}$, the vertical arrows are the maps (10.6.24), and the bottom horizontal arrow is the map (10.6.21).

10.6.26. Next, let $\mathscr{A}' := \oplus_{n \in \mathbb{N}} \mathscr{A}'_n$ be another $\mathbb{N}$-graded quasi-coherent $\mathscr{O}_X$-algebra on the Zariski site of X, and $\varphi : \mathscr{A} \to \mathscr{A}'$ a morphism of graded $\mathscr{O}_X$-algebras; for every affine open subset $U \subset X$, we deduce a morphism $\varphi_U : \mathscr{A}(U) \to \mathscr{A}'(U)$ of graded $\mathscr{O}_X(U)$-algebras, whence an open subset $G(\varphi_U) \subset \operatorname{Proj} \mathscr{A}'(U)$ as in (10.6.5). If $V \subset U$ is a smaller affine open subset, the natural isomorphism

$$V \times_U \operatorname{Proj} \mathscr{A}'(U) \xrightarrow{\sim} \operatorname{Proj} \mathscr{A}'(V)$$

induces an identification $V \times_U G(\varphi_U) \xrightarrow{\sim} G(\varphi_V)$, hence there exists a well defined open subset $G(\varphi) \subset \operatorname{Proj} \mathscr{A}'$ such that the morphisms $\operatorname{Proj} \varphi_U$ glue to a unique morphism of X-schemes :

$$\operatorname{Proj} \varphi : G(\varphi) \to \operatorname{Proj} \mathscr{A}.$$

If $\mathscr{A}'_+$ is generated – locally on X – by $\varphi(\mathscr{A}_+)$, we have $G(\varphi) = \operatorname{Proj} \mathscr{A}'$.

Moreover, if $\mathscr{M}$ is a $\mathbb{Z}$-graded quasi-coherent $\mathscr{A}$-module, the morphisms $\nu_{\mathscr{M}(U)}$ assemble into a well defined morphism of $\mathscr{O}_{G(\varphi)}$-modules :

$$\nu_{\mathscr{M}} : (\mathrm{Proj}\,\varphi)^*\mathscr{M}^\sim \to (\mathscr{M}')^\sim_{|G(\varphi)}$$

where the grading of $\mathscr{M}' := \mathscr{M} \otimes_{\mathscr{A}} \mathscr{A}'$ is defined as in (10.6.9). Likewise, the union of the subsets $G_1(\varphi_U)$, for U ranging over the affine open subsets of X, is an open subset :

(10.6.27) $$G_1(\varphi) \subset U_1(\mathscr{A}') \cap G(\varphi)$$

such that the restriction $\nu_{\mathscr{M}|G_1(\varphi)}$ is an isomorphism. Especially, set $Y' := \mathrm{Proj}\,\mathscr{A}'$; we have a natural morphism :

(10.6.28) $$\nu_{\mathscr{A}(n)} : (\mathrm{Proj}\,\varphi)^*\mathscr{O}_Y(n) \to \mathscr{O}_{Y'}(n)_{|G(\varphi)}$$

which is an isomorphism, if $\mathscr{A}_1$ generates $\mathscr{A}_+$ locally on X. Again, we have $G_1(\varphi) = U_1(\mathscr{A}')$ whenever $\varphi(\mathscr{A}_+)$ generates $\mathscr{A}'_+$, locally on X.

10.6.29. The discussion in (10.6.10) implies that any morphism of schemes $f : X' \to X$ induces a natural isomorphism of X'-schemes ([60, Ch.II, Prop.3.5.3]) :

(10.6.30) $$\mathrm{Proj}\, f^*\mathscr{A} \xrightarrow{\sim} X' \times_X \mathrm{Proj}\,\mathscr{A}$$

and the description (10.6.18) implies that (10.6.30) restricts to an isomorphism :

$$U_n(f^*\mathscr{A}) \xrightarrow{\sim} X' \times_X U_n(\mathscr{A}) \qquad \text{for every } n \in \mathbb{N}.$$

Furthermore, set $Y' := \mathrm{Proj}\, f^*\mathscr{A}$, and let $\pi_Y : Y' \to Y$ be the morphism deduced from (10.6.30); the discussion in (10.6.10) implies as well that, for any $\mathbb{Z}$-graded quasi-coherent $\mathscr{A}$-module $\mathscr{M}$, there is a natural isomorphism :

$$(f^*\mathscr{M})^\sim \xrightarrow{\sim} \pi_Y^*\mathscr{M}^\sim$$

([60, Ch.II, Prop.3.5.3]). Especially, f induces a natural identification ([60, Ch.II, Cor.3.5.4]) :

(10.6.31) $$\mathscr{O}_{Y'}(n) \xrightarrow{\sim} \pi_Y^*\mathscr{O}_Y(n) \qquad \text{for every } n \in \mathbb{Z}.$$

10.6.32. Keep the notation of (10.6.16), and let $\mathscr{C}_X$ be the category whose objects are all the pairs $(\psi : Z \to X, \mathscr{L})$, where ψ is a morphism of schemes and $\mathscr{L}$ is an invertible $\mathscr{O}_Z$-module on the Zariski site of Z; the morphisms $(\psi : Z \to X, \mathscr{L}) \to (\psi' : Z' \to X, \mathscr{L}')$ are the pairs (β, h), where $\beta : Z \to Z'$ is a morphism of X-schemes, and $h : \beta^*\mathscr{L}' \xrightarrow{\sim} \mathscr{L}$ is an isomorphism of $\mathscr{O}_Z$-modules (with composition of morphisms defined in the obvious way). Consider the functor:

$$F_{\mathscr{A}} : \mathscr{C}_X^o \to \mathbf{Set}$$

which assigns to any object $(\psi, \mathscr{L})$ of $\mathscr{C}_X$, the set consisting of all homomorphisms of graded $\mathscr{O}_Z$-algebras :

$$g : \psi^*\mathscr{A} \to \mathrm{Sym}^\bullet_{\mathscr{O}_Z}\mathscr{L}$$

which are epimorphisms on the underlying $\mathscr{O}_Z$-modules (here $\mathrm{Sym}^\bullet_{\mathscr{O}_Z}\mathscr{L}$ denotes the symmetric $\mathscr{O}_Z$-algebra on the $\mathscr{O}_Z$-module $\mathscr{L}$); on a morphism (β, h) as in the foregoing, and an element $g' \in F_{\mathscr{A}}(\psi', \mathscr{L}')$, the functor acts by the rule :

$$F_{\mathscr{A}}(\beta, h)(g') := (\mathrm{Sym}^\bullet_{\mathscr{O}_Z} h) \circ \beta^* g'.$$

Lemma 10.6.33. *The object* $(\pi : U_1(\mathscr{A}) \to X, \mathscr{O}_Y(1)_{|U_1(\mathscr{A})})$ *of* $\mathscr{C}_X$ *represents the functor* $F_{\mathscr{A}}$.

Proof. Given an object $(\psi : Z \to X, \mathscr{L})$ of $\mathscr{C}_X$, and $g \in F_{\mathscr{A}}(\psi, \mathscr{L})$, set :

$$\mathbb{P}(\mathscr{L}) := \mathrm{Proj}\,(\mathrm{Sym}^\bullet_{\mathscr{O}_Z}\mathscr{L}).$$

According to [60, Ch.II, Cor.3.1.7, Prop.3.1.8(iii)], the natural morphism $\pi_Z : \mathbb{P}(\mathscr{L}) \to Z$ is an isomorphism, and clearly in this situation the natural maps (10.6.24) are isomorphisms :

$$(10.6.34) \qquad \pi_Z^*\mathscr{L}^{\otimes n} \xrightarrow{\sim} \mathscr{O}_{\mathbb{P}(\mathscr{L})}(n) \qquad \text{for every } n \in \mathbb{N}.$$

On the other hand, since g is an epimorphism, we have $G(g) = \mathbb{P}(\mathscr{L})$; taking (10.6.30) into account, we deduce a morphism of Z-schemes :

$$\mathrm{Proj}\,g : \mathbb{P}(\mathscr{L}) \to Y' := Z \times_X \mathrm{Proj}\,\mathscr{A}$$

which is the same as a morphism of X-schemes :

$$\mathbb{P}(g) : Z \to \mathrm{Proj}\,\mathscr{A}.$$

We need to show that the image of $\mathbb{P}(g)$ lies in the open subset $U_1(\mathscr{A})$; to this aim, we may assume that both X and Z are affine, say $X = \mathrm{Spec}\,R$, $Z = \mathrm{Spec}\,S$, in which case $\mathscr{A}$ is the quasi-coherent algebra associated to a $\mathbb{N}$-graded R-algebra A, $\mathscr{L}$ is the invertible module associated to a projective rank one S-module L, and $g : S \otimes_R A \to \mathrm{Sym}^\bullet_S L$ is a surjective homomorphism of R-algebras. Then locally on Z, $\mathscr{L}$ is generated by elements of the form $g(1 \otimes t)$, for some local sections t of $\mathscr{A}_1$, and up to replacing Z by an affine open subset, we may assume that $t \in A_1$ is an element such that $t' := g(1 \otimes t)$ generates the free S-module L. In this situation, we have $\mathbb{P}(\mathscr{L}) = D_+(t')$, and $\mathbb{P}(g)$ is the same as the morphism $\Phi_t : D_+(t') \to D_+(t)$ (notation of (10.6.5)); especially the image of $\mathbb{P}(g)$ lies in $U_1(\mathscr{A})$, as required.

Moreover, the isomorphism $\pi_Z : \mathbb{P}(\mathscr{L}) \xrightarrow{\sim} Z$ is induced by the natural identification:

$$(10.6.35) \qquad S = S[t']_{(t')}.$$

From this description, we also can extract an explicit expression for Φ_t; namely, it is induced by the map of R-algebras :

$$A_{(t)} \to S \qquad \text{such that} \qquad a_k \cdot t^{-k} \mapsto g(1 \otimes a_k) \cdot t'^{-k}$$

for every $k \in \mathbb{N}$, and every $a_k \in A_k$. Next, letting $n := 1$ in (10.6.28) and (10.6.31), we obtain a natural isomorphism of $\mathscr{O}_{\mathbb{P}(\mathscr{L})}$-modules :

$$\mathscr{O}_{\mathbb{P}(\mathscr{L})}(1) \xrightarrow{\sim} (\mathrm{Proj}\,g)^*\mathscr{O}_{Y'}(1) \xrightarrow{\sim} (\mathrm{Proj}\,g)^* \circ \pi_Y^*\mathscr{O}_Y(1) \xrightarrow{\sim} \pi_Z^* \circ \mathbb{P}(g)^*\mathscr{O}_Y(1)$$

(notice that, since by assumption g is an epimorphism, we have $G_1(g) = U_1(\mathrm{Sym}^\bullet_{\mathscr{O}_Z}\mathscr{L}) = \mathbb{P}(\mathscr{L})$, hence $\nu_{\psi^*\mathscr{A}(1)}$ is an isomorphism). Composition with (10.6.34) yields an isomorphism :

$$h(g) : \mathbb{P}(g)^*\mathscr{O}_Y(1) \xrightarrow{\sim} \mathscr{L}$$

of $\mathscr{O}_Z$-modules, whence a morphism in $\mathscr{C}_X$

$$(\mathbb{P}(g), h(g)) : (\psi, \mathscr{L}) \to (\pi_{|U_1(\mathscr{A})}, \mathscr{O}_Y(1)_{|U_1(\mathscr{A})}).$$

In case X and Z are affine, and $\mathscr{L}$ is associated to a free module L, generated by an element of the form $t' := g(1 \otimes t)$ as in the foregoing, we can describe explicitly $h(g)$; namely, a direct inspection of the construction shows that in this case $h(g)$ is induced by the map of S-modules

$$S \otimes_{A_{(t)}} A(1)_{(t)} \to L \quad : \quad s \otimes a_k \cdot t^{1-k} \mapsto s \cdot g(1 \otimes a_k) \cdot (t')^{1-k} \qquad \text{for every } s \in S, a_k \in A_k.$$

Conversely, let $\beta : Z \to U_1(\mathscr{A})$ be a morphism of X-schemes, and $h : \beta^*\mathscr{O}_Y(1)_{|U_1(\mathscr{A})} \xrightarrow{\sim} \mathscr{L}$ an isomorphism of $\mathscr{O}_Z$-modules. In view of the natural isomorphisms (10.6.21), we deduce, for every $n \in \mathbb{N}$, an isomorphism :

$$h^{\otimes n} : \beta^*\mathscr{O}_Y(n)_{|U_1(\mathscr{A})} \xrightarrow{\sim} \mathscr{L}^{\otimes n}.$$

Combining with the epimorphisms (10.6.24) :

$$\omega_n : (\pi^*\mathscr{A}_n)_{|U_1(\mathscr{A})} \to \mathscr{O}_Y(n)_{|U_1(\mathscr{A})}$$

we may define the epimorphism of $\mathscr{O}_Z$-modules :

$$g(\beta, h) := \bigoplus_{n\in\mathbb{N}} h^{\otimes n} \circ \beta^*(\omega_n) : \psi^*\mathscr{A} \to \mathrm{Sym}^\bullet_{\mathscr{O}_Z}\mathscr{L} \qquad (10.6.36)$$

which, in view of (10.6.25), is a homomorphism of graded $\mathscr{O}_Z$-algebras, *i.e.* $g(\beta, h) \in F(\psi, \mathscr{L})$. This homomorphism can be described explicitly, locally on Z : namely, say again that $X = \mathrm{Spec}\, R$, $Z = \mathrm{Spec}\, S$, $\mathscr{L} = L^\sim$ for a free S-module of rank one, and $\mathscr{A} = A^\sim$ for some $\mathbb{N}$-graded R-algebra A; suppose moreover that the image of β lies in an open subset $D_+(t) \subset U_1(\mathscr{A})$, for some $t \in A_1$. Then β comes from a ring homomorphism $\beta^\natural : A_{(t)} \to S$, h is an S-linear isomorphism $S \otimes_{A_{(t)}} A(1)_{(t)} \xrightarrow{\sim} L$, and $t' := h(1 \otimes t)$ is a generator of L; moreover, ω_n is the epimorphism deduced from the map :

$$A_n \otimes_R A_{(t)} \to A(n)_{(t)} \quad : \quad a_n \otimes b_k \cdot t^{-k} \mapsto a_n b_k \cdot t^{-k} \qquad \text{for every } a_n \in A_n,\ b_k \in A_k$$

By inspecting the construction, we see therefore that g is the direct sum of the morphisms :

$$g_n : S \otimes_R A_n \to L^{\otimes n} \quad : \quad s \otimes a_n \mapsto s \cdot \beta^\natural(a_n \cdot t^{-n}) \cdot t'^{\otimes n} \qquad \text{for every } s \in S,\ a_n \in A_n.$$

Finally, it is easily seen that the natural transformations :

$$g \mapsto (\mathbb{P}(g), h(g)) \qquad \text{and} \qquad (\beta, h) \mapsto g(\beta, h) \qquad (10.6.37)$$

are inverse to each other : indeed, the verification can be made locally on Z, hence we may assume that X and Z are affine, and $\mathscr{L}$ is free, in which case one may use the explicit formulae provided above. □

Definition 10.6.38. Let X be a scheme, $\mathscr{I}$ a quasi-coherent sheaf of ideals of $\mathscr{O}_X$.

(i) The *positive Rees algebra* of $\mathscr{I}$ is the quasi-coherent $\mathbb{N}$-graded $\mathscr{O}_X$-algebra

$$\mathsf{R}^+(\mathscr{I})_\bullet := \bigoplus_{n\in\mathbb{N}} \mathscr{I}^n$$

with multiplication law deduced in the obvious way from that of $\mathscr{O}_X$ (cp. definition 7.9.1(iii)).

(ii) The *blowing up* of $\mathscr{I}$ is the X-scheme

$$\mathrm{Proj}\, \mathsf{R}^+(\mathscr{I})_\bullet \to X.$$

Remark 10.6.39. (i) With the notation of definition 10.6.38, let $\pi : Y \to X$ be the blowing up of $\mathscr{I}$. Notice that $\mathscr{I}$ is an invertible $\mathscr{O}_X$-module if and only if it is generated, locally on X, by a regular section of $\mathscr{O}_X$, and then the natural map

$$\mathrm{Sym}^n_{\mathscr{O}_X}\mathscr{I} \to \mathscr{I}^n$$

is an isomorphism, for every $n \in \mathbb{N}$. Taking into account [60, Ch.II, Cor.3.1.7, Prop.3.1.8(iii)], it follows that if $\mathscr{I}$ is an invertible ideal, then the blowing up of $\mathscr{I}$ is an isomorphism.

(ii) Let $X' \to X$ be any flat morphism of schemes; in light of (10.6.29), we get a natural isomorphism from $X' \times_X Y \to X'$ to the blowing up of the quasi-coherent ideal $\mathscr{I}\mathscr{O}_{X'}$ of $\mathscr{O}_{X'}$.

(iii) Especially, let $U \subset X$ be the complement in X of the closed subscheme $\mathrm{Spec}\, \mathscr{O}_X/\mathscr{I}$. From (i) and (ii) we deduce that the restriction $\pi^{-1}U \to U$ of π is an isomorphism.

(iv) Notice that $\mathsf{R}^+(\mathscr{I})_\bullet$ is generated, locally on X, by its degree one direct summand $\mathscr{I}$, hence $\mathscr{O}_Y(1)$ is an invertible $\mathscr{O}_Y$-module. On the other hand, a simple inspection shows that

$$\mathsf{R}^+(\mathscr{I})_\bullet(n) = \mathscr{I}^n \cdot \mathsf{R}^+(\mathscr{I})_\bullet \qquad \text{for every } n \in \mathbb{N}$$

whence a natural identification

$$\mathscr{O}_Y(n) \xrightarrow{\sim} \mathscr{I}^n \cdot \mathscr{O}_Y \qquad \text{for every } n \in \mathbb{N}.$$

Proposition 10.6.40. *With the notation of definition* 10.6.38*, the blowing up* $Y \to X$ *of* $\mathscr{I}$ *is characterized, up to unique isomorphism of* X*-schemes, by the following two conditions :*

(i) *The sheaf of ideals* $\mathscr{I} \cdot \mathscr{O}_Y$ *is an invertible* $\mathscr{O}_Y$*-module.*
(ii) *For every* X*-scheme* Z *such that* $\mathscr{I} \cdot \mathscr{O}_Z$ *is an invertible* $\mathscr{O}_Z$*-module, there exists a unique morphism* $Z \to Y$ *of* X*-schemes.*

Proof. The uniqueness up to unique isomorphism of an X-scheme $\pi : Y \to X$ fulfilling conditions (i) and (ii) is clear. It remains to check that these conditions hold for $Y := \mathrm{Proj}\,\mathsf{R}^+(\mathscr{I})_\bullet$.

However, condition (i) follows immediately from remark 10.6.39(iv). Next, let $f : Z \to X$ be as in (ii), and set $\mathscr{L} := \mathscr{I} \cdot \mathscr{O}_Z$; the induced map $f^*\mathscr{O}_X \to \mathscr{O}_Z$ induces an epimorphism of $\mathscr{O}_Z$-modules

$$\varphi_n : f^*\mathscr{I}^n \to \mathscr{I}^n\mathscr{O}_Z \xrightarrow{\sim} \mathrm{Sym}^n_{\mathscr{O}_Z}\mathscr{L} \qquad \text{for every } n \in \mathbb{N}$$

and it is clear that the system $(\varphi_n \mid n \in \mathbb{N})$ amounts to a morphism of $\mathbb{N}$-graded $\mathscr{O}_Z$-algebras

$$\varphi_\bullet : f^*\mathsf{R}^+(\mathscr{I})_\bullet \to \mathrm{Sym}^\bullet_{\mathscr{O}_Z}\mathscr{L}$$

which in turn corresponds to a morphism $g : Z \to Y$ of X-schemes, by virtue of lemma 10.6.33. Lastly, let $g' : Z \to Y$ be any other morphism of X-schemes, and denote by $U \subset X$ (resp. $U' \subset Z$) the complement in X (resp. in Z) of the closed subset $\mathrm{Spec}\,\mathscr{O}_X/\mathscr{I}$ (resp. $\mathrm{Spec}\,\mathscr{O}_Z/\mathscr{I}\mathscr{O}_Z$); clearly $f(U') \subset U$, hence g and g' both restrict to morphisms $U' \to \pi^{-1}U$ of X-schemes. Then remark 10.6.39(iii) implies that

$$g_{|U'} = g'_{|U'}.$$

Next, since $\mathscr{I}\mathscr{O}_Z$ is generated, locally on Z, by a regular section of $\mathscr{O}_Z$, the open subset U' is *schematically dense* in Z ([65, Ch.IV, Déf.11.10.2]), and on the other hand, we know that π is a separated morphism (see (10.6.15)); in view of [65, Ch.IV, Prop.11.10.1] we deduce that $g = g'$, which concludes the verification of (ii). $\square$

10.6.41. We may generalize as follows remark 10.6.39(ii). Let $X \to X_0$ be a morphism of schemes, $\mathscr{I}_0 \subset \mathscr{O}_{X_0}$ a quasi-coherent ideal, and set $\mathscr{I} := \mathscr{I}_0\mathscr{O}_X$; let also $\pi_0 : Y_0 \to X_0$ and $\pi : Y \to X$ be the blowing up morphisms of $\mathscr{I}_0$ and resepctively $\mathscr{I}$. By proposition 10.6.40 there exists a unique morphism of schemes $\psi : Y \to Y_0$ that makes commute the diagram

(10.6.42)
$$\begin{array}{ccc} Y & \xrightarrow{\psi} & Y_0 \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi_0} \\ X & \longrightarrow & X_0. \end{array}$$

Then we have :

Lemma 10.6.43. *The morphism of* X*-schemes* $\overline{\psi} : Y \to Y' := Y_0 \times_{X_0} X$ *induced by* ψ *is a closed immersion, and identifies* Y *with* $\mathrm{Spec}\,\mathscr{O}_{Y'}/\mathscr{J}$*, where*

$$\mathscr{J} := \bigcup_{n\in\mathbb{N}} \mathrm{Ann}_{\mathscr{O}_{Y'}}(\mathscr{I}^n).$$

Proof. Let $Z \to X$ be any morphism of schemes such that $\mathscr{I}\mathscr{O}_Z$ is an invertible $\mathscr{O}_Z$-module; since $\mathscr{I} = \mathscr{I}_0\mathscr{O}_X$, proposition 10.6.40 yields a unique morphism of X_0-schemes $Z \to Y_0$, so there exists a unique morphism of X-schemes $g : Z \to Y'$ as well. Next, since $\mathscr{I}\mathscr{O}_Z$ is invertible, it is easily seen that the induced morphism $\mathscr{O}_{Y'} \to g_*\mathscr{O}_Z$ factors through $\mathscr{O}_{Y'}/\mathscr{J}$, whence a unique morphism of X-schemes $Z \to Y'' := \mathrm{Spec}\,\mathscr{O}_{Y'}/\mathscr{J}$. Lastly, by construction

$\mathscr{I}\mathscr{O}_{Y'}$ is a locally principal ideal of $\mathscr{O}_{Y'}$, and then it is clear that $\mathscr{I}\mathscr{O}_{Y''}$ is an invertible $\mathscr{O}_{Y''}$-module. Summing up, we see that the morphism $Y'' \to X$ enjoys the universal property that characterizes the blowing up of $\mathscr{I}$, whence the lemma. □

10.6.44. Let A be a ring, $r \geq 1$ an integer, $\mathbf{f} := (f_1, \dots, f_r)$ a sequence of elements of A, and $I \subset A$ the ideal generated by $\mathbf{f}$. We view A as an algebra over the ring $A_0 := \mathbb{Z}[T_1, \dots, T_r]$, via the ring homomorphism

$$g : A_0 \to A \qquad T_i \mapsto f_i \qquad i = 1, \dots, r$$

and denote by $I_0 \subset A_0$ the ideal generated by $(T_1, \dots, T_r)$. Set also $X := \operatorname{Spec} A$, $X_0 := \operatorname{Spec} A_0$, let $\mathscr{I} \subset \mathscr{O}_X$ (resp. $\mathscr{I}_0 \subset \mathscr{O}_{X_0}$) be the quasi-coherent sheaf of ideals arising from I (resp. from I_0), and $\pi : Y \to X$ (resp. $\pi_0 : Y_0 \to X_0$) the blowing up of $\mathscr{I}$ (resp. of $\mathscr{I}_0$). Clearly $\mathscr{I}_0\mathscr{O}_Y = \mathscr{I}$, whence a commutative diagram (10.6.42), with bottom horizontal arrow given by $\operatorname{Spec} g : X \to X_0$. Furthermore, notice that we have systems of monomorphisms of invertible $\mathscr{O}_Y$-modules :

$$\text{(10.6.45)} \qquad \mathscr{O}_{Y_0}(n+1) \to \mathscr{O}_{Y_0}(n) \qquad \mathscr{O}_Y(n+1) \to \mathscr{O}_Y(n) \qquad \text{for every } n \in \mathbb{N}$$

corresponding to the inclusion maps $\mathscr{I}^{n+1}\mathscr{O}_{Y_0} \to \mathscr{I}^n\mathscr{O}_{Y_0}$ and $\mathscr{I}^{n+1}\mathscr{O}_Y \to \mathscr{I}^n\mathscr{O}_Y$ under the natural identifications of remark 10.6.39(iv).

Proposition 10.6.46. *With the notation of* (10.6.44)*, suppose moreover that the sequence* $\mathbf{f}$ *is completely secant. Then, for every* $n \in \mathbb{N}$ *we have :*

(i) *The natural map* $I^n \to H^0(Y, \mathscr{O}_Y(n))$ *is an isomorphism.*
(ii) $H^p(Y, \mathscr{O}_Y(n)) = 0$ *for every* $p > 0$.

Proof. We consider first the ring A_0 and the sequence $\mathbf{f}_0 := (T_1, \dots, T_r)$; we endow A_0 with its standard $\mathbb{N}$-grading, such that $\operatorname{gr}_n A_0$ is the $\mathbb{Z}$-module generated by the monomials of total degree n, for every $n \in \mathbb{N}$. Let also $\mathsf{R}_0^+ \subset A_0[V]$ be the Rees algebra associated with the I_0-adic filtration of A_0, so that $Y_0 = \operatorname{Proj} \mathsf{R}_0^+$ (see definition 7.9.1(iii); but since here we have a descending filtration on A_0, we let $V := U^{-1}$, so $\operatorname{gr}_n \mathsf{R}_0^+ := V^n I_0^n$ for every $n \in \mathbb{N}$). We consider the morphism of $\mathbb{N}$-graded $\mathbb{Z}$-algebras

$$h : A_0 \to \mathsf{R}_0^+ \qquad T_i \mapsto VT_i \qquad i = 1, \dots, r.$$

Notice that $\varphi(A_{0+})$ generates the ideal R_{0+}^+ of R_0^+, hence $\varphi := \operatorname{Proj} h$ is well defined on the whole of Y_0 (see (10.6.5)), and we get a commutative diagram of morphisms of schemes

$$\begin{array}{ccc} Y_0 & \xrightarrow{\varphi} & \mathbb{P}^{r-1}_{\mathbb{Z}} \\ {\scriptstyle \pi_0}\downarrow & & \downarrow \\ X_0 & \longrightarrow & \operatorname{Spec} \mathbb{Z}. \end{array}$$

For every $i = 1, \dots, r$, set $U_i := D_+(T_i) \subset \mathbb{P}^{r-1}_{\mathbb{Z}}$, and $U'_i := \varphi^{-1}U_i$. Recall that there are natural isomorphisms $\omega_i : \mathscr{O}_{\mathbb{P}^{r-1}_{\mathbb{Z}}}(U_i) \xrightarrow{\sim} A_i := \mathbb{Z}[\tau_{ij} \mid j = 1, \dots, r]$ (notation of example 10.6.3); we wish to give a corresponding description of the A_i-algebra $B_i := \mathscr{O}_{Y_0}(U'_i)$. To this aim, notice that, by definition, every element of $B_i \subset \mathbb{Z}[T_1, \dots, T_r, V, (VT_i)^{-1}]$ is a finite sum of terms of the form

$$\text{(10.6.47)} \quad (VT_1)^{\alpha_1} \cdots (VT_r)^{\alpha_r} \cdot (VT_i)^{-k} \cdot P(T_1, \dots, T_r) = \tau_{i1}^{\alpha_1} \cdots \tau_{ir}^{\alpha_r} \cdot P(T_i\tau_{i1}, \dots, T_i\tau_{ir})$$

where $k \in \mathbb{N}$ is any integer, $P \in A$ any polynomial, and $\alpha_1 + \cdots + \alpha_r = k$; *i.e.*

$$B_i = A_i[T_i] \qquad \text{for every } i = 1, \dots, r.$$

Likewise, for every $n \in \mathbb{N}$, the elements of $\mathsf{R}_0^+(n)_{(T_i)}$ are the finite sums of terms (10.6.47) where $k \in \mathbb{N}$ is any integer, $P \in A$ any polynomial, and $\alpha_1 + \cdots + \alpha_r = n + k$; *i.e.*

$$\mathsf{R}_0^+(n)_{(T_i)} = (VT_i)^n B_i \qquad \text{for every } i = 1, \dots, r.$$

For every $i \neq j$, set also $U_{ij} := U_i \cap U_j$ and $U'_{ij} := \varphi^{-1} U_{ij}$, and recall that the isomorphisms ω_i induce isomorphisms $\omega_{ij} : \mathscr{O}_{\mathbb{P}^{r-1}_{\mathbb{Z}}}(U_{ij}) \xrightarrow{\sim} A_i[\tau_{ij}^{-1}]$, such that the composition

$$\psi_{ij} := \omega_{ji} \circ \omega_{ij}^{-1} : A_i[\tau_{ij}^{-1}] \xrightarrow{\sim} A_j[\tau_{ji}^{-1}]$$

is given by the rule (10.6.4). There follow natural identifications $\mathscr{O}_{Y_0}(U'_{ij}) \xrightarrow{\sim} B_i[\tau_{ij}^{-1}]$, whence a commutative diagram

$$\begin{array}{ccc} A_i[\tau_{ij}^{-1}] & \xrightarrow{\psi_{ij}} & A_j[\tau_{ji}^{-1}] \\ \downarrow & & \downarrow \\ B_i[\tau_{ij}^{-1}] & \xrightarrow{\psi'_{ij}} & B_j[\tau_{ji}^{-1}] \end{array}$$

whose vertical arrows are induced by the map $\mathscr{O}_{\mathbb{P}^{r-1}_{\mathbb{Z}}} \to \varphi_* \mathscr{O}_{Y_0}$ associated with φ, and where ψ'_{ij} is the isomorphism given by the rule :

$$T_i \mapsto T_j \cdot \tau_{ji}.$$

Likewise, we get a commutative diagram

$$\begin{array}{ccc} & \mathscr{O}_{Y_0}(n)(U'_{ij}) & \\ \swarrow & & \searrow \\ (VT_i)^n B_i[\tau_{ij}^{-1}] & \longrightarrow & (VT_j)^n B_j[\tau_{ji}^{-1}] \end{array}$$

whose downward arrows are induced by the restriction maps of the sheaf $\mathscr{O}_{Y_0}(n)$, and where the horizontal map is the $B_i[\tau_{ij}^{-1}]$-linear map given by the rule

$$(VT_i)^n \mapsto (VT_j)^n \cdot \tau_{ji}^n$$

and where the $B_i[\tau_{ij}^{-1}]$-module structure of $(VT_j)^n B_j[\tau_{ji}^{-1}]$ is defined by restriction of scalars along the isomorphism ψ'_{ij}. Comparing with example 10.6.8 we deduce a natural isomorphism of $\mathscr{O}_{\mathbb{P}^{r-1}_{\mathbb{Z}}}$-modules :

$$\varphi_* \mathscr{O}_{Y_0}(n) \xrightarrow{\sim} \bigoplus_{k \in \mathbb{N}} \mathscr{O}_{\mathbb{P}^{r-1}_{\mathbb{Z}}}(n+k) \qquad \text{for every } n \in \mathbb{N}.$$

On the other hand, from [61, Ch.III, Prop.2.1.12] we get isomorphisms of $\mathbb{N}$-graded $\mathbb{Z}$-modules:

$$H^p\Big(\mathbb{P}^{r-1}_{\mathbb{Z}}, \bigoplus_{k \in \mathbb{N}} \mathscr{O}_{\mathbb{P}^{r-1}_{\mathbb{Z}}}(k)\Big) = \begin{cases} 0 & \text{for every } p > 0 \\ A & \text{for } p = 0 \end{cases}$$

Now, since φ is affine, we have

$$H^p(Y_0, \mathscr{O}_{Y_0}(n)) = H^p(\mathbb{P}^{r-1}_{\mathbb{Z}}, \varphi_* \mathscr{O}_{Y_0}(n)) \qquad \text{for every } n \in \mathbb{N}$$

and assertion (ii) already follows. We also deduce an isomorphism of $\mathbb{N}$-graded $\mathbb{Z}$-modules

$$H^0(Y_0, \mathscr{O}_{Y_0}(n)) \xrightarrow{\sim} H^0\Big(\mathbb{P}^{r-1}_{\mathbb{Z}}, \bigoplus_{k \in \mathbb{N}} \mathscr{O}_{\mathbb{P}^{r-1}_{\mathbb{Z}}}(n+k)\Big) = I_0^n$$

where $\mathrm{gr}_k I_0^n := \mathrm{gr}_{n+k} A$, for every $k \in \mathbb{N}$. To conclude the proof in this case, it remains only to check that this identification is the inverse of the natural map of (i). To this aim, we consider the induced commutative diagram

$$\begin{array}{ccc} H^0(Y_0, \mathscr{O}_{Y_0}(n)) & \longrightarrow & H^0(\mathbb{P}^{r-1}_{\mathbb{Z}}, \varphi_* \mathscr{O}_{Y_0}(n)) \\ \downarrow & & \downarrow \\ H^0(U'_i, \mathscr{O}_{Y_0}(n)) & \longrightarrow & H^0(U_i, \varphi_* \mathscr{O}_{Y_0}(n)). \end{array}$$

Now, let $T^\alpha := T_1^{\alpha_1} \cdots T_r^{\alpha_r} \in \mathrm{gr}_k I^n$ be any monomial (for any $k \in \mathbb{N}$); by inspecting the foregoing notation, we see that the image of T^α in $H^0(U'_i, \mathscr{O}_{Y_0}(n))$ equals $(VT_i)^n T_i^k \tau_{i,1}^{\alpha_1} \cdots \tau_{i,r}^{\alpha_r}$, which maps to the section $T_i^{n+k} \tau_{i,1}^{\alpha_1} \cdots \tau_{i,r}^{\alpha_r}$ of $H^0(U_i, \varphi_* \mathscr{O}_{Y_0}(n))$. But the latter is also the image of T^α under the restriction map $H^0(\mathbb{P}^{r-1}_{\mathbb{Z}}, \mathscr{O}_{\mathbb{P}^{r-1}}(n+k)) \to H^0(U_i, \mathscr{O}_{\mathbb{P}^{r-1}}(n+k))$, whence the contention.

Next, let A, $\mathbf{f}$ and I as in (10.6.44), and we define likewise $\mathsf{R}^+ \subset A[V]$ as the positive Rees algebra associated with the I-adic filtration of A; we consider the homomorphism of $\mathbb{N}$-graded $\mathbb{Z}$-algebras

$$G : \mathsf{R}_0^+ \to \mathsf{R}^+ \qquad V^n x \mapsto V^n g(x) \qquad \text{for every } n \in \mathbb{N} \text{ and every } x \in I_0^n.$$

Notice that G is even a morphism of $\mathbb{N}$-graded A_0-algebras, for the A_0-algebra structure on A and R^+ induced by g. There follows a morphism of $\mathbb{N}$-graded A-algebras

$$h' : A \otimes_{A_0} \mathsf{R}_0^+ \to \mathsf{R}^+$$

whose restriction $\mathrm{gr}_n h'$ is induced by the natural map $A \otimes_{A_0} I_0^n \to I^n$, for every $n \in \mathbb{N}$. The latter is an isomorphism for every $n \in \mathbb{N}$, by virtue of lemma 7.8.23, and it is clear that the morphism ψ of (10.6.44) equals $\mathrm{Proj}\, h'$; in view of (10.6.10), we deduce that the diagram (10.6.42) is cartesian under the current assumptions, and moreover h' induces natural isomorphisms of $\mathscr{O}_Y$-modules

$$\psi^* \mathscr{O}_{Y_0}(n) \xrightarrow{\sim} \mathscr{O}_Y(n) \qquad \text{for every } n \in \mathbb{N}.$$

Denote by $\mathfrak{U}'$ the affine open covering $(U'_i \mid i = 1, \dots, r)$ of Y_0, and set $\psi^{-1}\mathfrak{U}' := (\psi^{-1} U'_i \mid i = 1, \dots, r)$. there follows a natural isomorphism of alternating Čech complexes

$$(10.6.48) \qquad \overline{C}{}^\bullet_{\mathrm{alt}}(\mathfrak{U}', \mathscr{O}_{Y_0}(n)) \otimes_{A_0} A \xrightarrow{\sim} \overline{C}{}^\bullet_{\mathrm{alt}}(\psi^{-1}\mathfrak{U}', \mathscr{O}_Y(n)) \qquad \text{for every } n \in \mathbb{N}.$$

Claim 10.6.49. The natural map $\overline{C}{}^\bullet_{\mathrm{alt}}(\mathfrak{U}', \mathscr{O}_{Y_0}(n)) \overset{\mathbf{L}}{\otimes}_{A_0} A \to \overline{C}{}^\bullet_{\mathrm{alt}}(\mathfrak{U}', \mathscr{O}_{Y_0}(n)) \otimes_{A_0} A$ is an isomorphism in $\mathsf{D}(A\text{-}\mathbf{Mod})$, for every $n \in \mathbb{N}$.

Proof of the claim. We consider the standard spectral sequence

$$E_1^{pq} := \mathrm{Tor}_{-q}^{A_0}(\overline{C}{}^p_{\mathrm{alt}}(\mathfrak{U}', \mathscr{O}_{Y_0}(n)), A) \Rightarrow H^{p+q}(\overline{C}{}^\bullet_{\mathrm{alt}}(\mathfrak{U}', \mathscr{O}_{Y_0}(n)) \overset{\mathbf{L}}{\otimes}_{A_0} A)$$

and notice that it suffices to check that $E_1^{pq} = 0$ for every $p \neq 0$, since in that case the abutment is naturally isomorphic to the cohomology of the complex $(E_1^{0,\bullet}, d_1^{0,\bullet})$, which is the same as $\overline{C}{}^\bullet_{\mathrm{alt}}(\mathfrak{U}', \mathscr{O}_{Y_0}(n)) \otimes_{A_0} A$. However, with the foregoing notation, we see that $\overline{C}{}^q_{\mathrm{alt}}(\mathfrak{U}', \mathscr{O}_{Y_0}(n))$ is a direct sum of finitely many A_0-modules, each of which is a localization of B_i, for some $i \leq r$, so we are reduced to checking that $\mathrm{Tor}_p^{A_0}(B_i, A) = 0$ for $i = 1, \dots, r$ and every $p > 0$. But in turn, B_i is the degree 0 summand of a $\mathbb{Z}$-graded A_0-module, which is a localization of R_0^+, so we are further reduced to checking that $\mathrm{Tor}_p^{A_0}(\mathsf{R}_0^+, A) = 0$ for every $p > 0$, or equivalently, that $\mathrm{Tor}_p^{A_0}(I_0^n, A) = 0$ for every $p > 0$ and every $n \in \mathbb{N}$. The latter follows from lemma 7.8.23. ◇

Combining (10.6.48), claim 10.6.49 and theorem 10.2.28(ii), we get a natural isomorphism :

$$\overline{C}{}^\bullet_{\mathrm{alt}}(\mathfrak{U}', \mathscr{O}_{Y_0}(n)) \overset{\mathbf{L}}{\otimes}_{A_0} A \xrightarrow{\sim} R\Gamma(Y, \mathscr{O}_Y(n)) \qquad \text{in } \mathsf{D}(A\text{-}\mathbf{Mod}), \text{ for every } n \in \mathbb{N}.$$

By the same token, we have a natural isomorphism $\overline{C}^\bullet_{\mathrm{alt}}(\mathfrak{U}', \mathscr{O}_{Y_0}(n)) \xrightarrow{\sim} R\Gamma(Y_0, \mathscr{O}_{Y_0}(n))$ in $\mathsf{D}(A_0\text{-}\mathbf{Mod})$, whence a standard spectral sequence for every $n \in \mathbb{N}$

$$E(n)_2^{pq} := \mathrm{Tor}^{A_0}_{-p}(H^q(Y_0, \mathscr{O}_{Y_0}(n)), A) \Rightarrow H^{p+q}(\overline{C}^\bullet_{\mathrm{alt}}(\mathfrak{U}', \mathscr{O}_{Y_0}(n)) \overset{\mathbf{L}}{\otimes}_{A_0} A)$$

and the previous case tells us that $E(n)_2^{pq} = 0$ for every $q > 0$ and every $n \in \mathbb{N}$, and moreover $E(n)_2^{p0} = \mathrm{Tor}^{A_0}_{-p}(I_0^n, A)$ for every $p \in \mathbb{Z}$ and every $n \in \mathbb{N}$. Lastly, by invoking again lemma 7.8.23 we see that $E(n)_2^{p0} = 0$ whenever $p \neq 0$, and $E(n)_2^{00} = I_0^n A = I^n$ for every $n \in \mathbb{N}$, which concludes the proof of the proposition. □

Theorem 10.6.50. *With the notation of* (10.6.44)*, the following conditions are equivalent :*

(a) *The ring A satisfies condition* $(\mathrm{a})^{\mathrm{un}}_{\mathbf{f}}$.
(b) *The inverse system* $(H^p(Y, \mathscr{O}_Y(n)) \mid n \in \mathbb{N})$ *deduced from the system of maps* (10.6.45) *is uniformly essentially zero for every $p > 0$, and the same holds for the kernel and cokernel of the natural morphism of inverse systems* $(I^n \to H^0(Y, \mathscr{O}_Y(n)) \mid n \in \mathbb{N})$.
(c) *There exists $n \in \mathbb{N}$ such that $I^n \cdot H^p(Y, \mathscr{O}_Y) = 0$ for every $p > 0$, and I^n annihilates the kernel and cokernel of the natural map $A \to H^0(Y, \mathscr{O}_Y)$.*

Proof. Define $I_0 \subset A$, the positive Rees algebras R_0^+, R^+, and the morphism $\pi_0 : Y_0 \to X_0 := \mathrm{Spec}\, A_0$ as in the proof of proposition 10.6.46. We set $Y' := Y_0 \times_{X_0} X$ and define $\mathscr{J} \subset \mathscr{O}_{Y'}$, $\psi : Y \to Y$ and $\overline{\psi} : Y \to Y'$ as in lemma 10.6.43, so that $\mathscr{J}$ is the kernel of the induced map $\overline{\psi}^\sharp : \mathscr{O}_{Y'} \to \overline{\psi}_* \mathscr{O}_Y$. We shall consider as well the condition :

(d) There exists $k \in \mathbb{N}$ such that $I^k \mathscr{J} = I^k \cdot \mathscr{T}or_i^{X_0}(\mathscr{O}_{Y_0}, \mathscr{O}_X) = 0$ for every $i > 0$.

Claim 10.6.51. If condition (a) holds, there exists $k \in \mathbb{N}$ such that

$$I^k \cdot \mathrm{Tor}_i^{A_0}(I_0^n, A) = I^k \cdot \mathrm{Ker}(I_0^n \otimes_{A_0} A \to I^n) = 0 \qquad \text{for every } n \in \mathbb{N} \text{ and every } i > 0.$$

Proof of the claim. For every integer $p > 0$, let $k(p)$ be the step of the uniformly essentially zero system $(\mathrm{Tor}_p^{A_0}(A_0/I_0^n, A) \mid n \in \mathbb{N})$. For $p > 1$ (resp. for $p = 1$), the integer $k(p)$ is also the step of the uniformly essentially zero system $(T^n_{p-1} := \mathrm{Tor}^{A_0}_{p-1}(I_0^n, A) \mid n \in \mathbb{N})$ (resp. the uniformly essentially zero system $(K^n := \mathrm{Ker}(I_0^n \otimes_{A_0} A \to I^n) \mid n \in \mathbb{N})$). Now, for every $a \in I_0^{k(p)}$, scalar multiplication by a on I_0^n factors through the inclusion map $I_0^{n+k(p)} \to I_0^n$, so it induces the zero map on T^n_{p-1} (resp. on K^n) for every $n \in \mathbb{N}$. Lastly, recall that the homological dimension of A_0 equals $r + 1$, so $T^n_p = 0$ for every $p > r + 1$. We may therefore choose $k := \max(k(1), \dots, k(r + 1))$. ◊

(a)⇒(d): Indeed, recall that we have a natural $\mathscr{O}_{Y'}(U \times_{X_0} X)$-linear isomorphism

$$\Gamma(U \times_{X_0} X, \mathscr{T}or_i^{X_0}(\mathscr{O}_{Y_0}, \mathscr{O}_X)) \xrightarrow{\sim} T_i(U) := \mathrm{Tor}_i^{A_0}(H^0(\mathscr{O}_{Y_0}(U), A)$$

for every affine open subsets $U \subset Y_0$. Since Y' is quasi-compact, and taking into account lemma 10.6.43, it will then suffice to find $k \in \mathbb{N}$ and a finite open covering $U_\bullet := (U_\lambda \mid \lambda \in \Lambda)$ of Y_0 such that

- $I^k \cdot T_i(U_\lambda) = 0$ for every $i > 0$ and every $\lambda \in \Lambda$
- $I^k \cdot \mathrm{Ker}(\overline{\psi}^\sharp_{U_\lambda} : \mathscr{O}_{Y_0}(U_\lambda) \otimes_{A_0} A \to \mathscr{O}_Y(\overline{\psi}^{-1} U_\lambda)) = 0$ for every $\lambda \in \Lambda$.

However, we may choose $U_\bullet$ such $\mathscr{O}_{Y_0}(U_\lambda)$ is a direct summand of a graded localization of R_0^+, for every $\lambda \in \Lambda$; then $T_i(U_\lambda)$ is a direct summand of a localization of $\mathrm{Tor}_i^{A_0}(\mathsf{R}_0^+, A)$, and $\overline{\psi}^\sharp_{U_\lambda}$ is a direct summand of a localization of the natural map $\mathsf{R}_0^+ \otimes_{A_0} A \to \mathsf{R}^+$. Thus, the assertion follows from claim 10.6.51.

(b)⇒(c): The natural identification $\mathscr{I}^n \mathscr{O}_Y \xrightarrow{\sim} \mathscr{O}_Y(n)$ of remark 10.6.39(iv) yields exact cohomology sequences

$$H^p(Y, \mathscr{O}_Y(n)) \xrightarrow{\alpha_n^p} H^p(Y, \mathscr{O}_Y) \to H^p(Y, \mathscr{O}_Y/\mathscr{I}^n \mathscr{O}_Y) \qquad \text{for every } p, n \in \mathbb{N}.$$

However, assumption (b) implies that for every $p > 0$ there exists $k \in \mathbb{N}$ such that α_n^p is the zero map, for every $n \geq k$; hence, for such k the term $H^p(Y, \mathscr{O}_Y)$ is a submodule of $H^p(Y, \mathscr{O}_Y/\mathscr{I}^k\mathscr{O}_Y)$, which is obviously annihilated by I^k. By the same token we get a commutative ladder with exact rows for every $n \in \mathbb{N}$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I^n & \longrightarrow & A & \longrightarrow & A/I^n & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \beta_n} & & \downarrow{\scriptstyle \gamma_n} & & \downarrow{\scriptstyle \delta_n} & & \\ 0 & \longrightarrow & H^0(Y, \mathscr{O}_Y(n)) & \longrightarrow & H^0(Y, \mathscr{O}_Y) & \longrightarrow & H^0(Y, \mathscr{O}_Y/\mathscr{I}^n\mathscr{O}_Y) & & \end{array}$$

whence an exact sequence

$$0 \to \mathrm{Ker}\,\beta_n \xrightarrow{\tau_n} \mathrm{Ker}\,\gamma_n \to \mathrm{Ker}\,\delta_n \to \mathrm{Coker}\,\beta_n \xrightarrow{\sigma_n} \mathrm{Coker}\,\gamma_n \to \mathrm{Coker}\,\delta_n.$$

However, assumption (b) implies that there exists $n \in \mathbb{N}$ such that τ_n and σ_n both vanish; since $\mathrm{Ker}\,\delta_n$ and $\mathrm{Coker}\,\delta_n$ are both annihilated by I^n, we conclude that (c) holds.

Next, we show that (d) implies both (a) and (b). To this aim, we remark :

Claim 10.6.52. Suppose that condition (d) holds; then we have :

(i) The inverse system $(\mathscr{T}_n^i := \mathscr{T}or_i^{X_0}(\mathscr{O}_{Y_0}(n), \mathscr{O}_X) \mid n \in \mathbb{N})$ induced by the system of morphisms (10.6.45) is uniformly essentially zero, for every $i > 0$.

(ii) Likewise, the inverse system $(\mathscr{J}(n) := \mathrm{Ker}(\mathscr{O}_{Y'}(n) \to \overline{\psi}_*\mathscr{O}_Y(n)) \mid n \in \mathbb{N})$ is uniformly essentially zero.

Proof of the claim. (i): There exists an affine open covering $(U_1, \dots, U_r)$ of Y_0 such that

$$(\mathscr{I}\mathscr{O}_{Y_0})_{|U_i} = f_i\mathscr{O}_{Y_0} \qquad \text{for every } i = 1, \dots, r$$

and f_i is a regular element of $\mathscr{O}_{Y_0}(U_i)$, whence an identification

$$(\mathscr{I}^n\mathscr{O}_{Y_0})_{|U_i} = f_i^n\mathscr{O}_{U_i} \xrightarrow{\sim} \mathscr{O}_{U_i} \qquad \text{for every } i = 1, \dots, r.$$

Under this identification, the restriction $\mathscr{O}_{Y_0}(n+1)_{|U_i} \to \mathscr{O}_{Y_0}(n)_{|U_i}$ of the morphism (10.6.45) corresponds to the scalar multiplication by f_i. Hence, the inverse system $((\mathscr{T}_n^i)_{|U_i} \mid n \in \mathbb{N})$ is naturally identified with the inverse system $(\mathscr{T}_n'^i \mid n \in \mathbb{N})$ with $\mathscr{T}_n'^i := \mathscr{T}or_i^{X_0}(\mathscr{O}_{U_i}, \mathscr{O}_X)$ for every $i, n \in \mathbb{N}$, with transition maps given by scalar multiplication by f_i, whence (i).

For the proof of (ii) one argues in the same way, using the fact that $I^k\mathscr{J} = 0$ for some $k \in \mathbb{N}$ whose existence is ensured by condition (d) : the details shall be left to the reader. ◊

We consider now the inverse systems of spectral sequences provided by proposition 10.2.30

$$E(n)_2^{pq} := H^p(Y_0 \times_{X_0} X, \mathscr{T}or_{-q}^{X_0}(\mathscr{O}_{Y_0}(n), \mathscr{O}_X)) \Rightarrow H^{p+q}(A \overset{\mathbf{L}}{\otimes}_{A_0} R\Gamma\mathscr{O}_{Y_0}(n)) \tag{10.6.53}$$

where, as usual, the transition maps $E(n+1)_2^{pq} \to E(n)_2^{pq}$ are induced by the morphisms (10.6.45). Clearly $E_2^{pq} = 0$ if either $p < 0$ or $q > 0$, and claim 10.6.52(i) implies that the inverse system $(E(n)_2^{pq} \mid n \in \mathbb{N})$ is uniformly essentially zero whenever $q < 0$. It follows already that the inverse system $(E(n)_\infty^{pq} \mid n \in \mathbb{N})$ is uniformly essentially zero whenever $p + q < 0$. On the other hand, proposition 10.6.46 shows that

$$H^{-i}(A \overset{\mathbf{L}}{\otimes}_{A_0} R\Gamma\mathscr{O}_{Y_0}(n)) = \mathrm{Tor}_i^{A_0}(A, I_0^n) \qquad \text{for every } i \in \mathbb{Z}.$$

Summing up, and taking into account lemma 7.8.19, a simple induction shows that the system $(\mathrm{Tor}_i^{A_0}(A, I_0^n) \mid n \in \mathbb{N})$ is uniformly essentially zero for every $i > 0$. Notice also that $E(n)_2^{p0} = H^p(Y', \mathscr{O}_{Y'}(n)$ for every $n, p \in \mathbb{N}$.

Claim 10.6.54. (i) The inverse system $(E(n)^{p0}_s) \mid n \in \mathbb{N})$ is uniformly essentially zero, for every $p > 0$ and every $s \geq 2$.

(ii) The kernel and cokernel of the natural morphism of inverse systems

$$h_\bullet : (A \otimes_{A_0} I_0^n \mid n \in \mathbb{N}) \to (E(n)^{00}_2 \mid n \in \mathbb{N})$$

are both uniformly essentially zero.

Proof of the claim. (i): We argue by descending induction on p : if $p \geq r$, we have $E(n)^{p0}_2 = 0$ for every $n \in \mathbb{N}$, by theorem 10.2.28(iii), hence the claim trivially holds in this case. Thus, suppose that $q < r$, and that the claim is already known for every $p > q$ and every $s \geq 2$. We show, by descending induction on $s \geq 2$, that $(E(n)^{q0}_s \mid n \in \mathbb{N})$ is uniformly essentially zero. Since $E(n)^{q-s,s-1}_2 = 0$ for every $s \geq 2$, we have $E(n)^{q-s,s-1}_s = E(n)^{q+s,1-s}_s = 0$ as well, whenever $q + s \geq r$, and therefore $E(n)^{q0}_s = E(n)^{q0}_\infty = 0$ for every $p > 0$ and every $s \geq r$. Thus, suppose that $2 \leq t < r$, and that the assertion is known for every $s > t$; we consider the exact sequence of inverse systems

$$0 \to (E(n)^{q0}_{t+1} \mid n \in \mathbb{N}) \to (E(n)^{q0}_t \mid n \in \mathbb{N}) \to (E(n)^{q+t,1-t}_t \mid n \in \mathbb{N}).$$

Since $1 - t < 0$, the third term is uniformly essentially zero, and the same holds for the first term, by inductive assumption; by lemma 7.8.19, we deduce that the same holds for the middle term, as required.

(ii): The map h_n is the composition of the projection $h'_n : A \otimes_{A_0} I_0^n \to E(n)^{00}_\infty$ with the injective map $h''_n : E(n)^{00}_\infty \to E(n)^{00}_2$ deduced from the spectral sequences (10.6.53). But since the inverse system $(E(n)^{p,-p}_2 \mid n \in \mathbb{N})$ is uniformly essentially zero for every $p > 0$, and vanishes for $p \geq r$, the same holds for the inverse systems $(E(n)^{p,-p}_\infty \mid n \in \mathbb{N})$, and consequently the kernel of $h'_\bullet$ is uniformly essentially zero. Likewise, it is easily seen that the cokernel of $h''_\bullet$ is uniformly essentially zero. ◊

Define $\mathscr{J}(n)$ as in claim 10.6.52(ii), and set $J(n)^p := H^p(Y', \mathscr{J}(n)$ for every $n, p \in \mathbb{N}$; we get an exact sequence of inverse systems

$$(J(n)^p \mid n \in \mathbb{N}) \to (E(n)^{p0}_2 \mid n \in \mathbb{N}) \xrightarrow{k^p_\bullet} (H^p(Y, \mathscr{O}_Y(n)) \mid n \in \mathbb{N}) \to (J(n)^{p+1} \mid n \in \mathbb{N})$$

for every $p \in \mathbb{N}$, and claims 10.6.52(ii) and 10.6.54(i) imply that the first and third terms are uniformly essentially zero, whenever $p > 0$. Therefore, the same holds for the middle term. Lastly, we consider the commutative diagram

$$\begin{array}{ccc} (A \otimes_{A_0} I_0^n \mid n \in \mathbb{N}) & \xrightarrow{m_\bullet} & (I^n \mid n \in \mathbb{N}) \\ {\scriptstyle h_\bullet}\downarrow & & \downarrow{\scriptstyle l_\bullet} \\ (E(n)^{00}_2 \mid n \in \mathbb{N}) & \xrightarrow{k^0_\bullet} & (H^0(Y, \mathscr{O}_Y(n)) \mid n \in \mathbb{N}) \end{array}$$

where $m_\bullet$ is given by the natural surjections, and $h_\bullet$ is defined as in claim 10.6.54. Since the inverse system $(J(n)^p \mid n \in \mathbb{N})$ is uniformly essentially zero for $p = 0, 1$, we see that the kernel and cokernel of $k^0_\bullet$ are both uniformly essentially zero. In view of claim 10.6.54(ii), it follows immediately that the kernel of $m_\bullet$ is uniformly essentially zero, and the same holds for the kernel and cokernel of $l_\bullet$, which concludes the verification of both (a) and (b).

Lastly, we check that (c) implies both (a) and (b). To this aim we consider, for every $n \in \mathbb{N}$, the spectral sequences provided by proposition 10.2.30(ii)

$$E(n)^{pq}_2 := \mathrm{Tor}^{A_0}_{-p}(I_0^n, H^q(Y, \mathscr{O}_Y)) \Rightarrow H^{p+q}(I_0^n \overset{\mathbf{L}}{\otimes}_{A_0} R\Gamma \mathscr{O}_Y)$$

$$F(n)^{pq}_2 := H^p(Y, \mathscr{T}\!or^{X_0}_{-q}(\mathscr{I}_0^n, \mathscr{O}_Y)) \Rightarrow H^{p+q}(I_0^n \overset{\mathbf{L}}{\otimes}_{A_0} R\Gamma \mathscr{O}_Y).$$

Moreover, for every $\in \mathbb{N}$ denote by

$$\varphi_n : \mathscr{T}\!or_0^{X_0}(\mathscr{O}_{Y_0}, \mathscr{I}_0^n) = \pi_0^* \mathscr{I}_0^n \to \mathscr{I}_0^n \mathscr{O}_{Y_0} = \mathscr{O}_{Y_0}(n)$$

the natural morphism, and notice that $\psi^* \pi_0^* \mathscr{I}_0^n = \mathscr{T}\!or_0^{X_0}(\mathscr{I}_0^n, \mathscr{O}_Y)$.

Claim 10.6.55. (i) The inverse system $(\operatorname{Ker}\varphi_n \mid n \in \mathbb{N})$ is uniformly essentially zero, and the same holds for the inverse system $(\mathscr{T}\!or_q^{X_0}(\mathscr{I}_0^n, \mathscr{O}_Y) \mid n \in \mathbb{N})$, for every $q > 0$.

(ii) The inverse system $(H^i(I_0^n \overset{\mathbf{L}}{\otimes}_{A_0} R\Gamma\mathscr{O}_Y) \mid n \in \mathbb{N})$ is uniformly essentially zero for $i \neq 0$.

(iii) The spectral sequences $E(n)_2^{\bullet\bullet}$ induce, for every $p \in \mathbb{Z}$, an epimorphism of inverse systems $(E(n)_2^{p0} \mid n \in \mathbb{N}) \to (E(n)_\infty^{p0} \mid n \in \mathbb{N})$ whose kernel is uniformly essentially zero.

(iv) The spectral sequences $F(n)_2^{\bullet\bullet}$ induce, for every $p \in \mathbb{Z}$, a monomorphism of inverse systems $(F(n)_\infty^{p0} \mid n \in \mathbb{N}) \to (F(n)_2^{p0} \mid n \in \mathbb{N})$ whose cokernel is uniformly essentially zero.

(v) The inverse systems $(E(n)_2^{p,-p} \mid n \in \mathbb{N})$ and $(F(n)_2^{-p,p} \mid n \in \mathbb{N})$ are uniformly essentially zero for $p < 0$, and vanish identically for $p > 0$.

Proof of the claim. (i): We look at the spectral sequence

$$G(n)_{pq}^2 := \mathscr{T}\!or_p^{Y_0}(\mathscr{O}_Y, \mathscr{T}\!or_q^{X_0}(\mathscr{O}_{Y_0}, \mathscr{I}_0^n)) \Rightarrow \mathscr{T}\!or_{p+q}^{X_0}(\mathscr{I}_0^n, \mathscr{O}_Y)$$

of proposition 10.2.30(i). Since Y_0 is a noetherian scheme, corollary 7.9.22(iii) implies that the inverse system $(\mathscr{T}\!or_q^{X_0}(\mathscr{O}_{Y_0}, \mathscr{I}_0^n) \mid n \in \mathbb{N})$ is uniformly essentially zero for every $q > 0$, so the same holds for the inverse system $(G(n)_2^{pq} \mid n \in \mathbb{N})$, for every $p \in \mathbb{N}$ and every $q > 0$. By the same token, the inverse system $(\operatorname{Ker}\varphi_n \mid n \in \mathbb{N})$ is uniformly essentially zero; since the functor

$$\mathscr{T}\!or_p^{Y_0}(\mathscr{O}_Y, -) : \mathsf{D}^-(\mathscr{O}_{Y_0}\text{-}\mathbf{Mod}_{\mathrm{qcoh}}) \to \mathsf{D}^-(\mathscr{O}_Y\text{-}\mathbf{Mod}_{\mathrm{qcoh}})$$

is triangulated for every $p \in \mathbb{Z}$, we get a short exact sequence

$$0 \to \mathscr{T}\!or_p^{Y_0}(\mathscr{O}_Y, \operatorname{Ker}\varphi_n) \to G(n)_{p0}^2 \to \mathscr{T}\!or_p^{Y_0}(\mathscr{O}_Y, \mathscr{O}_{Y_0}(n)) \to 0 \qquad \text{for every } p, n \in \mathbb{N}$$

which shows that the inverse system $(G(n)_{p0}^2 \mid n \in \mathbb{N})$ is uniformly essentially zero for every $p > 0$. The assertion now follows by a simple induction, using lemma 7.8.19.

(ii): Condition (c) implies that the inverse system $(E(n)_2^{pq} \mid n \in \mathbb{N})$ is uniformly essentially zero for $q > 0$, and clearly it vanishes identically when either $q < 0$ or $p > 0$. It also vanishes identically for $q \geq r$, due to theorem 10.2.28(iii). Summing up, we see that the filtration induced by the spectral sequence $E(n)_2^{\bullet\bullet}$ on its abutment $H^i := H^i(I_0^n \overset{\mathbf{L}}{\otimes}_{A_0} R\Gamma\mathscr{O}_Y)$ is finite for every $i \in \mathbb{Z}$, and the inverse systems of its graded subquotients $(E(n)_\infty^{pq} \mid n \in \mathbb{N})$ are uniformly essentially zero whenever $p+q > 0$, whence the assertion in case $i > 0$, after invoking as usual lemma 7.8.19. For $i < 0$, we argue likewise, using the spectral sequences $F(n)_2^{\bullet\bullet}$: indeed, we see again easily that the filtration induced on the abutment H^i is finite for every $i \in \mathbb{Z}$; moreover, if $p < 0$ the inverse system $(F(n)_2^{pq} \mid n \in \mathbb{N})$ is identically zero, and it is uniformly essentially zero if $q < 0$, by virtue of (i), whence the assertion.

(iii): For $p > 0$ we have $E(n)_2^{p0} = 0$ for every $n \in \mathbb{N}$, so the assertion is trivial in this case. For $p \leq 0$, notice that $E(n)_s^{p+s,1-s} = 0$ for every $s \geq 2$, hence $E(n)_s^{p0}$ is a quotient of $E(n)_2^{p0}$ for every $s \geq 2$, whence a surjection $E(n)_2^{p0} \to E(n)_\infty^{p0}$ as sought. Moreover, notice that the inverse system $(E(n)_2^{p-s,s-1} \mid n \in \mathbb{N})$ is uniformly essentially zero for every $s \geq 2$, due to condition (c) and lemma 7.8.24, and it vanishes identically for $s \geq r$. Then the same holds for the inverse system $(E(n)_s^{p-s,s-1} \mid n \in \mathbb{N})$, for every $s \geq 2$, and the assertion follows easily.

(iv): For $p < 0$ we have $F(n)_2^{p0} = 0$ for every $n \in \mathbb{N}$, so the assertion is trivial in this case. For $p \geq 0$, notice that $F(n)_s^{p-s,s-1} = 0$ for every $s \geq 2$, hence $F(n)_s^{p0}$ is a submodule of $F(n)_2^{p0}$ for every $s \geq 2$, whence a injection $F(n)_\infty^{p0} \to F(n)_2^{p0}$ as sought. Moreover, notice that the inverse system $(F(n)_2^{p+s,1-s} \mid n \in \mathbb{N})$ is uniformly essentially zero for every $s \geq 2$, due to (i), and it vanishes identically for $s \geq r$. Then the same holds for the inverse system $(F(n)_s^{p+s,1-s} \mid n \in \mathbb{N})$, for every $s \geq 2$, and the assertion follows easily.

(v): The assertion for $p > 0$ is obvious, and the case $p < 0$ for $(E(n)_2^{p,-p} \mid n \in \mathbb{N})$ follows from condition (c) and lemma 7.8.24, and for $(F(n)_2^{-p,p} \mid n \in \mathbb{N})$ follows from (i). ◊

Assumption (c) implies that the natural map $A \to H^0(Y, \mathscr{O}_Y)$ induces a morphism of inverse systems

$$(\mathrm{Tor}_{-p}^{A_0}(I_0^n, A) \mid n \in \mathbb{N}) \to (E(n)_2^{p0} \mid n \in \mathbb{N}) \qquad \text{for every } p \in \mathbb{Z}$$

whose kernel and cokernel are uniformly essentially zero. Combining with claim 10.6.55(ii,iii), it follows already that the inverse system $(\mathrm{Tor}_p^{A_0}(A/I_0^n, A) \mid n \in \mathbb{N})$ is uniformly essentially zero for every $p > 1$. Next, since $\mathscr{O}_{Y_0}(n)$ is a flat $\mathscr{O}_{Y_0}$-module for every $n \in \mathbb{N}$, we have an exact sequence of $\mathscr{O}_Y$-modules

$$0 \to \psi^* \mathrm{Ker}\, \varphi_n \to \mathscr{T}\!or_0^{X_0}(\mathscr{I}_0^n, \mathscr{O}_Y) \to \psi^* \mathscr{O}_{Y_0}(n) = \mathscr{O}_Y(n) \to 0$$

whence exact sequences

$$H^p(Y, \psi^* \mathrm{Ker}\, \varphi_n) \to F(n)_2^{p0} \to H^p(Y, \mathscr{O}_Y(n)) \to H^{p+1}(Y, \psi^* \mathrm{Ker}\, \varphi_n)$$

for every $p \in \mathbb{Z}$ and every $n \in \mathbb{N}$. Combining with claim 10.6.55(ii,iv), we deduce that the inverse system $(H^p(Y, \mathscr{O}_Y(n)) \mid n \in \mathbb{N})$ is uniformly essentially zero for every $p > 0$.

Lastly, claim 10.6.55(v) implies that $(E(n)_\infty^{p,-p} \mid n \in \mathbb{N})$ and $(F(n)_\infty^{-p,p} \mid n \in \mathbb{N})$ vanish identically for every $p > 0$, and are uniformly essentially zero inverse systems for $p < 0$. It follows that the spectral sequences $E(n)_2^{\bullet\bullet}$ and $F(n)_2^{\bullet\bullet}$ induce respectively a monomorphism and an epimorphism

$$(E(n)_\infty^{00} \mid n \in \mathbb{N}) \to (H^0(I_0^n \overset{\mathbf{L}}{\otimes}_{A_0} R\Gamma \mathscr{O}_Y) \mid n \in \mathbb{N}) \to (F(n)_\infty^{00} \mid n \in \mathbb{N})$$

and the kernel and cokernel of the composition of these morphisms are uniformly essentially zero inverse systems. Taking into account claim 10.6.55(iv) we get therefore a commutative diagram of inverse systems

$$\begin{array}{ccccc}
(E(n)_2^{00} \mid n \in \mathbb{N}) & \longrightarrow & (E(n)_\infty^{00} \mid n \in \mathbb{N}) & \longrightarrow & (F(n)_\infty^{00} \mid n \in \mathbb{N}) \\
\| & & & & \downarrow \\
(I_0^n \otimes_{A_0} H^0(Y, \mathscr{O}_Y) \mid n \in \mathbb{N}) & \xrightarrow{\beta_\bullet} & (H^0(Y, \mathscr{O}_Y(n) \mid n \in \mathbb{N}) & \longleftarrow & (F(n)_2^{00} \mid n \in \mathbb{N})
\end{array}$$

all whose arrows, except possibly $\beta_\bullet$, have uniformly essentially zero kernels and cokernels, but then also $\beta_\bullet$ does.

Claim 10.6.56. $\beta_\bullet$ is the map induced by the identification $\omega_n : \mathscr{I}^n \mathscr{O}_Y \xrightarrow{\sim} \mathscr{O}_Y(n)$ of remark 10.6.39(iv) and the natural map $\psi^* \pi_0^* \mathscr{I}_0^n \to \mathscr{I}^n \mathscr{O}_Y$.

Proof of the claim. Indeed, by construction, the map $F(n)_2^{00} \to H^0(Y, \mathscr{O}_Y(n))$ is obtained from the identification ω_n and the natural isomorphism $\mathscr{T}\!or_0^{X_0}(\mathscr{O}_{Y_0}, \mathscr{I}_0^n) \xrightarrow{\sim} \psi^* \pi_0^* \mathscr{I}_0^n$, so we are reduced to checking that the map

$$E(n)_2^{00} = I_0^n \otimes_{A_0} H^0(Y, \mathscr{O}_Y) \to F(n)_2^{00} = H^0(Y, \psi^* \pi_0^* \mathscr{I}_0^n)$$

deduced from the foregoing commutative diagram, agrees with the standard one that assigns to any element $x \otimes s$ its class in $H^0(Y, \psi^* \pi_0^* \mathscr{I}_0^n)$. To this aim, set $U_i := D_+(f_i) \subset Y$ for $i = 1, \dots, r$, and let $Y'' := \amalg_{i=1}^r U_i$, the affine X-scheme given by the disjoint union of the affine open subsets $U_1, \dots, U_r$. We have an obvious morphism $\nu : Y'' \to Y$ of schemes, that restricts to the inclusion map on each open subset U_i of Y'', and let also $\psi'' := \psi \circ \nu : Y'' \to Y_0$. We have as well two systems of spectral sequences $E''(n)_2^{\bullet\bullet}$ and $F''(n)_2^{\bullet\bullet}$, obtained as in the foregoing, after replacing Y by Y'' and $\mathscr{O}_Y$ by $\mathscr{O}_{Y''}$. By functoriality of the spectral sequences, there follows a commutative diagram

$$\begin{array}{ccc} E(n)_2^{00} = I_0^n \otimes_{A_0} H^0(Y, \mathscr{O}_Y) & \longrightarrow & F(n)_2^{00} = H^0(Y, \psi^* \pi_0^* \mathscr{I}_0^n) \\ \downarrow & & \downarrow \\ E''(n)_2^{00} = I_0^n \otimes_{A_0} H^0(Y'', \mathscr{O}_{Y''}) & \longrightarrow & F''(n)_2^{00} = H^0(Y'', \psi''^* \pi_0^* \mathscr{I}_0^n) \end{array}$$

and notice that the right vertical arrow is injective. Thus, it suffices to check that the bottom horizontal arrow is the expected map. However, we have $E''(n)_2^{00} = E''(n)_\infty^{00} = E''(n)_\infty^0 = F''(n)_\infty^0 = F''(n)_\infty^{00} = F''(n)_2^{00}$ for every $n \in \mathbb{N}$, and a direct inspection shows that the bottom horizontal map is the identity map, whence the claim. ◊

From claim 10.6.56 and condition (c), it follows already that the natural morphism of inverse systems $(I_0^n \otimes_{A_0} A \mid n \in \mathbb{N}) \to (H^0(Y, \mathscr{O}_Y(n)) \mid n \in \mathbb{N})$ has uniformly essentially zero kernel and cokernel. But the latter factors as the composition of the epimorphism of inverse systems $(I_0^n \otimes_{A_0} A \mid n \in \mathbb{N}) \to (I^n \mid n \in \mathbb{N})$ and the natural morphism $(I^n \mid n \in \mathbb{N}) \to (H^0(Y, \mathscr{O}_Y(n)) \mid n \in \mathbb{N})$. It follows that the kernels and cokernels of both of the latter morphisms are uniformly essentially zero, and the proof of the theorem is concluded. □

Remark 10.6.57. By direct inspection of the proof, we obtain the following refinement of theorem 10.6.50. For every $r \in \mathbb{N}$ there exists an integer $\nu(r)$ such that, if the ring A satisfies condition $(\mathrm{a})_{\mathbf{f}}^{\mathrm{un}}$ with step $\leq r$, then the inverse system $(H^p(Y, \mathscr{O}_Y(n)) \mid n \in \mathbb{N})$ is uniformly essentially zero with step $\leq \nu(r)$, and the same holds for the kernel and cokernel of the inverse system $(I^n \to H^0(Y, \mathscr{O}_Y(n)) \mid n \in \mathbb{N})$. The detailed verification shall be left to the reader.

11. Duality theory

11.1. **Duality for quasi-coherent modules.** Let $f : X \to Y$ be a morphism of schemes; theorem 10.1.18 falls short of proving that Lf^* is a left adjoint to Rf_*, since the former functor is defined on bounded above complexes, while the latter is defined on complexes that are bounded below. This deficency has been overcome by N.Spaltenstein's paper [154], where he shows how to extend the usual constructions of derived functors to unbounded complexes. On the other hand, in many cases one can also construct a *right adjoint* to Rf_*. This is the subject of Grothendieck's duality theory. We shall collect the statements that we need from that theory, and refer to the original source [87] for the rather elaborate proofs.

11.1.1. For any morphism of schemes $f : X \to Y$, let $\overline{f} : (X, \mathscr{O}_X) \to (Y, f_*\mathscr{O}_X)$ be the corresponding morphism of ringed spaces. The functor

$$\overline{f}_* : \mathscr{O}_X\text{-}\mathbf{Mod} \to f_*\mathscr{O}_X\text{-}\mathbf{Mod}$$

admits a left adjoint $\overline{f}^*$ defined as usual ([59, Ch.0, §4.3.1]). In case f is affine, $\overline{f}$ is flat and the unit and counit of the adjunction restrict to isomorphisms of functors :

$$\mathbf{1} \to \overline{f}_* \circ \overline{f}^* : f_*\mathscr{O}_X\text{-}\mathbf{Mod}_{\mathrm{qcoh}} \to f_*\mathscr{O}_X\text{-}\mathbf{Mod}_{\mathrm{qcoh}}$$
$$\overline{f}^* \circ \overline{f}_* \to \mathbf{1} : \mathscr{O}_X\text{-}\mathbf{Mod}_{\mathrm{qcoh}} \to \mathscr{O}_X\text{-}\mathbf{Mod}_{\mathrm{qcoh}}.$$

on the corresponding subcategories of quasi-coherent modules ([60, Ch.II, Prop.1.4.3]). It follows easily that, on these subcategories, $\overline{f}^*$ is also a right adjoint to $\overline{f}_*$.

Lemma 11.1.2. *Keep the notation of* (11.1.1)*, and suppose that f is affine. Then the functor :*

$$\overline{f}^* : \mathsf{D}^+(f_*\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{qcoh}} \to \mathsf{D}^+(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$$

is right adjoint to the functor $R\overline{f}_* : \mathsf{D}^+(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{qcoh}} \to \mathsf{D}^+(f_*\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$. Moreover, the unit and counit of the resulting adjunction are isomorphisms of functors.

Proof. By trivial duality (theorem 10.1.18), $\overline{f}^*$ is left adjoint to $R\overline{f}_*$ on $\mathsf{D}^+(\mathscr{O}_X\text{-}\mathbf{Mod})$; it suffices to show that the unit (resp. and counit) of this latter adjunction are isomorphisms for every object $K^\bullet$ of $\mathsf{D}^+(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$ (resp. $L^\bullet$ of $\mathsf{D}^+(f_*\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$). Concerning the unit, we can use a Cartan-Eilenberg injective resolution of $\overline{f}^*K^\bullet$, and a standard spectral sequence, to reduce to the case where $K^\bullet = \mathscr{F}[0]$ for a quasi-coherent $f_*\mathscr{O}_X$-module $\mathscr{F}$; however, the natural map $\overline{f}_* \circ \overline{f}^*\mathscr{F} \to R\overline{f}_* \circ \overline{f}^*\mathscr{F}$ is an isomorphism, so the assertion follows from (11.1.1). $\square$

Remark 11.1.3. (i) Even when both X and Y are affine, neither the unit nor the counit of adjunction in (11.1.1) is an isomorphism on the categories of all modules. For a counterexample concerning the unit, consider the case of a finite injection of domains $A \to B$, where A is local and B is semi-local (but not local); let $f : X := \operatorname{Spec} B \to \operatorname{Spec} A$ be the corresponding morphism, and denote by $\mathscr{F}$ the $f_*\mathscr{O}_X$-module supported on the closed point, with stalk equal to B. Then one verifies easily that $\overline{f}_* \circ \overline{f}^*\mathscr{F}$ is a direct product of finitely many copies of $\mathscr{F}$, indexed by the closed points of X. Concerning the counit, keep the same morphism f, let $U := X\setminus\{x\}$, where x is a closed point and set $\mathscr{F} := j_!\mathscr{O}_U$; then it is clear that $\overline{f}^* \circ \overline{f}_*\mathscr{F}$ is supported on the complement of the closed fibre of f.

(ii) Similarly, one sees easily that $\overline{f}^*$ is not a right adjoint to $\overline{f}_*$ on the category of all $f_*\mathscr{O}_X$-modules.

(iii) Incidentally, the analogous functor $\overline{f}^*_{\text{ét}}$ defined on the étale (rather than Zariski) site is a right adjoint to $\overline{f}_{\text{ét}*} : \mathscr{O}_{X,\text{ét}}\text{-}\mathbf{Mod} \to f_*\mathscr{O}_{X,\text{ét}}\text{-}\mathbf{Mod}$, and on these sites the unit and counit of the adjunctions are isomorphisms for all modules.

11.1.4. For the construction of the right adjoint $f^!$ to Rf_*, one considers first the case where f factors as a composition of a finite morphism followed by a smooth one, in which case $f^!$ admits a corresponding decomposition. Namely :

- In case f is smooth, we shall consider the functor :
$$f^\sharp : \mathsf{D}(\mathscr{O}_Y\text{-}\mathbf{Mod}) \to \mathsf{D}(\mathscr{O}_X\text{-}\mathbf{Mod}) \qquad K^\bullet \mapsto f^*K^\bullet \otimes_{\mathscr{O}_X} \Lambda^n_{\mathscr{O}_X}\Omega^1_{X/Y}[n]$$
where n is the locally constant relative dimension function of f.
- In case f is finite, we shall consider the functor :
$$f^\flat : \mathsf{D}^+(\mathscr{O}_Y\text{-}\mathbf{Mod}) \to \mathsf{D}^+(\mathscr{O}_X\text{-}\mathbf{Mod}) \qquad K^\bullet \mapsto \overline{f}^*R\mathscr{H}om^\bullet_{\mathscr{O}_Y}(f_*\mathscr{O}_X, K^\bullet).$$

If f is any quasi-projective morphism, such factorization can always be found locally on X, and one is left with the problem of patching a family of locally defined functors $((f_{|U_i})^! \mid i \in I)$, corresponding to an open covering $X = \bigcup_{i\in I} U_i$. Since such patching must be carried out in the derived category, one has to take care of many cumbersome complications.

11.1.5. Recall that a finite morphism $f : X \to Y$ is said to be *pseudo-coherent* if $f_*\mathscr{O}_X$ is a pseudo-coherent $\mathscr{O}_Y$-module. This condition is equivalent to the pseudo-coherence of f in the sense of [20, Exp.III, Déf.1.2].

Lemma 11.1.6. *Let $f : X \to Y$ be a finite morphism of schemes. Then :*

(i) *If f is finitely presented, the functor*
$$\mathscr{F} \mapsto \overline{f}^*\mathscr{H}om_{\mathscr{O}_Y}(f_*\mathscr{O}_X, \mathscr{F})$$
is right adjoint to $f_* : \mathscr{O}_X\text{-}\mathbf{Mod}_{\mathrm{qcoh}} \to \mathscr{O}_Y\text{-}\mathbf{Mod}_{\mathrm{qcoh}}$.

(ii) *If f is pseudo-coherent, the functor*
$$K^\bullet \mapsto \overline{f}^*R\mathscr{H}om^\bullet_{\mathscr{O}_Y}(f_*\mathscr{O}_X, K^\bullet) \quad : \quad \mathsf{D}^+(\mathscr{O}_Y\text{-}\mathbf{Mod})_{\mathrm{qcoh}} \to \mathsf{D}^+(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$$
is right adjoint to
$$Rf_* : \mathsf{D}^+(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{qcoh}} \to \mathsf{D}^+(\mathscr{O}_Y\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$$
(notation of (10.3)*).*

Proof. (i): Under the stated assumptions, $f_*\mathscr{O}_X$ is a finitely presented $\mathscr{O}_Y$-module (by claim 9.1.35), hence the functor

$$\mathscr{F} \mapsto \mathscr{H}om_{\mathscr{O}_Y}(f_*\mathscr{O}_X, \mathscr{F}) \quad : \quad \mathscr{O}_Y\text{-}\mathbf{Mod} \to f_*\mathscr{O}_X\text{-}\mathbf{Mod}$$

preserves the subcategories of quasi-coherent modules, hence it restricts to a right adjoint for the forgetful functor $f_*\mathscr{O}_X\text{-}\mathbf{Mod}_{\mathrm{qcoh}} \to \mathscr{O}_Y\text{-}\mathbf{Mod}_{\mathrm{qcoh}}$ (claim 10.1.17). Then the assertion follows from (11.1.1).

(ii) is analogous : since $f_*\mathscr{O}_X$ is pseudo-coherent, the functor

$$K^\bullet \mapsto R\mathscr{H}om^\bullet_{\mathscr{O}_Y}(f_*\mathscr{O}_X, K^\bullet) \quad : \quad \mathsf{D}^+(\mathscr{O}_Y\text{-}\mathbf{Mod}) \to \mathsf{D}^+(f_*\mathscr{O}_X\text{-}\mathbf{Mod})$$

preserves the subcategories of complexes with quasi-coherent homology, hence it restricts to a right adjoint for the forgetful functor $\mathsf{D}^+(f_*\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{qcoh}} \to \mathsf{D}^+(\mathscr{O}_Y\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$, by lemma 10.1.16(iii). To conclude, it then suffices to apply lemma 11.1.2. ☐

Proposition 11.1.7. *Suppose that* $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ *are morphisms of schemes. Then :*

(i) *If f and g are finite and f is pseudo-coherent, there is a natural isomorphism of functors on* $\mathsf{D}^+(\mathscr{O}_Z\text{-}\mathbf{Mod})$ *:*

$$\xi_{f,g} : (g \circ f)^\flat \xrightarrow{\sim} f^\flat \circ g^\flat.$$

(ii) *If f and $g \circ f$ are finite and pseudo-coherent, and g is smooth of bounded fibre dimension, there is a natural isomorphism of functors on* $\mathsf{D}^+(\mathscr{O}_Z\text{-}\mathbf{Mod})$ *:*

$$\zeta_{f,g} : (g \circ f)^\flat \xrightarrow{\sim} f^\flat \circ g^\sharp.$$

(iii) *If f, g and $h \circ g$ are finite and pseudo-coherent, and h is smooth of bounded fibre dimension, then the diagram of functors on* $\mathsf{D}^+(\mathscr{O}_W\text{-}\mathbf{Mod})$ *:*

$$\begin{array}{ccc} (h \circ g \circ f)^\flat & \xrightarrow{\zeta_{g\circ f,h}} & (g \circ f)^\flat \circ h^\sharp \\ {\scriptstyle \xi_{f,h\circ g}}\downarrow & & \downarrow{\scriptstyle \xi_{f,g}\circ h^\sharp} \\ f^\flat \circ (h \circ g)^\flat & \xrightarrow{f^\flat(\zeta_{g,h})} & f^\flat \circ g^\flat \circ h^\sharp \end{array}$$

commutes.

(iv) *If, moreover*

$$\begin{array}{ccc} Y & \xrightarrow{g} & Z \\ {\scriptstyle h'}\downarrow & & \downarrow{\scriptstyle h} \\ W' & \xrightarrow{j} & W \end{array}$$

is a cartesian diagram of schemes, such that h is smooth of bounded fibre dimension and j is finite and pseudo-coherent, then there is a natural isomorphism of functors :

$$\vartheta_{j,h} : g^\flat \circ h^\sharp \xrightarrow{\sim} h'^\sharp \circ j^\flat.$$

Proof. This is [87, Ch.III, Prop.6.2, 8.2, 8.6]. We check only (i). There is a natural commutative diagram of ringed spaces :

$$\begin{array}{ccccc} (X, \mathscr{O}_X) & \xrightarrow{f} & (Y, \mathscr{O}_Y) & \xrightarrow{g} & (Z, \mathscr{O}_Z) \\ & {\scriptstyle \overline{f}}\searrow & \uparrow{\scriptstyle \alpha} & {\scriptstyle \overline{g}}\searrow & \uparrow{\scriptstyle \gamma} \\ & & (Y, f_*\mathscr{O}_X) & & (Z, g_*\mathscr{O}_Y) \\ & & & {\scriptstyle \varphi}\searrow & \uparrow{\scriptstyle \beta} \\ & & & & (Z, (g \circ f)_*\mathscr{O}_X) \end{array}$$

where :

$$\varphi \circ \overline{f} = \overline{g \circ f}. \tag{11.1.8}$$

By claim 10.1.17, the (forgetful) functors α_*, β_* and γ_* admit right adjoints :

$$\begin{aligned}
\alpha^\flat &: \mathscr{O}_Y\text{-}\mathbf{Mod} \to f_*\mathscr{O}_X\text{-}\mathbf{Mod} && : \mathscr{F} \mapsto \mathscr{H}om_{\mathscr{O}_Y}(f_*\mathscr{O}_X, \mathscr{F}) \\
\beta^\flat &: g_*\mathscr{O}_Y\text{-}\mathbf{Mod} \to (g\circ f)_*\mathscr{O}_X\text{-}\mathbf{Mod} && : \mathscr{F} \mapsto \mathscr{H}om_{g_*\mathscr{O}_Y}((g\circ f)_*\mathscr{O}_X, \mathscr{F}) \\
\gamma^\flat &: \mathscr{O}_Z\text{-}\mathbf{Mod} \to g_*\mathscr{O}_Y\text{-}\mathbf{Mod} && : \mathscr{F} \mapsto \mathscr{H}om_{\mathscr{O}_Z}(g_*\mathscr{O}_Y, \mathscr{F}).
\end{aligned}$$

Likewise, $(\gamma\circ\beta)_*$ admits a right adjoint $(\gamma\circ\beta)^\flat$ and the natural identification

$$\gamma_* \circ \beta_* \xrightarrow{\sim} (\gamma\circ\beta)_*$$

induces a natural isomorphisms of functors :

$$(\gamma\circ\beta)^\flat := \mathscr{H}om_{\mathscr{O}_Z}((g\circ f)_*\mathscr{O}_X, -) \xrightarrow{\sim} \beta^\flat \circ \gamma^\flat. \tag{11.1.9}$$

(Notice that, in general, there might be several choices of such natural transformations (11.1.9), but for the proof of (iii) – as well as for the construction of $f^!$ for more general morphisms f – it is essential to make an explicit and canonical choice.)

Claim 11.1.10. (i) $\alpha_* \circ \varphi^* = \overline{g}^* \circ \beta_*$.

(ii) The natural commutative diagram of sheaves :

$$\begin{array}{ccc}
\overline{g}^{-1} g_*\mathscr{O}_Y & \longrightarrow & \mathscr{O}_Y \\
\downarrow & & \downarrow \\
\overline{g}^{-1}(g\circ f)_*\mathscr{O}_X & \longrightarrow & f_*\mathscr{O}_X
\end{array}$$

is cocartesian.

Proof of the claim. (i) is an easy consequence of (ii). Assertion (ii) can be checked on the stalks, hence we may assume that $Z = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$ and $X = \operatorname{Spec} C$ are affine schemes. Let $\mathfrak{p} \in Y$ be any prime ideal, and set $\mathfrak{q} := g(\mathfrak{p}) \in Z$; then $(\overline{g}^{-1}g_*\mathscr{O}_Y)_\mathfrak{p} = B_\mathfrak{q}$ (the $A_\mathfrak{q}$-module obtained by localizing the A-module B at the prime $\mathfrak{q}$). Likewise, $(\overline{g}^{-1}(g\circ f)_*\mathscr{O}_X)_\mathfrak{p} = C_\mathfrak{q}$ and $(f_*\mathscr{O}_X)_\mathfrak{p} = C_\mathfrak{p}$. Hence the claim boils down to the standard isomorphism : $C_\mathfrak{q} \otimes_{B_\mathfrak{q}} B_\mathfrak{p} \simeq C_\mathfrak{p}$. ◊

Let $I^\bullet$ be a bounded below complex of injective $\mathscr{O}_Z$-modules; by lemma 10.1.16(i), $\gamma^\flat I^\bullet$ is a complex of injective $g_*\mathscr{O}_Y$-modules; taking into account (11.1.9) we deduce a natural isomorphism of functors on $\mathsf{D}^+(\mathscr{O}_Z\text{-}\mathbf{Mod})$:

$$R(\gamma\circ\beta)^\flat \xrightarrow{\sim} R(\beta^\flat \circ \gamma^\flat) = R\beta^\flat \circ R\gamma^\flat. \tag{11.1.11}$$

Furthermore, (11.1.8) implies that $\overline{g \circ f}^* = \overline{f}^* \circ \varphi^*$. Combining with (11.1.11), we see that the sought $\xi_{f,g}$ is a natural transformation :

$$\overline{f}^* \circ \varphi^* \circ R\beta^\flat \circ R\gamma^\flat \to \overline{f}^* \circ R\alpha^\flat \circ \overline{g}^* \circ R\gamma^\flat$$

of functors on $\mathsf{D}^+(\mathscr{O}_Z\text{-}\mathbf{Mod})$. Hence (i) will follow from :

Claim 11.1.12. (i) There exists a natural isomorphism of functors :

$$g_*\mathscr{O}_Y\text{-}\mathbf{Mod} \to f_*\mathscr{O}_X\text{-}\mathbf{Mod} \quad : \quad \varphi^* \circ \beta^\flat \xrightarrow{\sim} \alpha^\flat \circ \overline{g}^*.$$

(ii) The natural transformation :

$$R(\alpha^\flat \circ \overline{g}^*) \to R\alpha^\flat \circ \overline{g}^*$$

is an isomorphism of functors : $\mathsf{D}^+(g_*\mathscr{O}_Y\text{-}\mathbf{Mod}) \to \mathsf{D}^+(f_*\mathscr{O}_X\text{-}\mathbf{Mod})$.

Proof of the claim. First we show how to construct a natural transformation as in (i); this is the same as exhibiting a map of functors : $\alpha_* \circ \varphi^* \circ \beta^\flat \to \overline{g}^*$. In view of claim 11.1.10(i), the latter can be defined as the composition of $\overline{g}^*$ and the counit of adjunction $\beta_* \circ \beta^\flat \to \mathbf{1}_{g_*\mathscr{O}_Y\text{-}\mathbf{Mod}}$.

Next, recall the natural transformation :

$$(11.1.13)\quad \overline{g}^*\mathscr{H}om_{g_*\mathscr{O}_Y}(\mathscr{F},\mathscr{G}) \to \mathscr{H}om_{\mathscr{O}_Y}(\overline{g}^*\mathscr{F},\overline{g}^*\mathscr{G}) \quad \text{for every } g_*\mathscr{O}_Y\text{-modules } \mathscr{F} \text{ and } \mathscr{G}$$

(defined so as to induce the pull-back map on global Hom functors). Notice that the natural map $\overline{g}^*\overline{g}_*\mathscr{F} \to \mathscr{F}$ (counit of adjunction) is an isomorphism for every quasi-coherent $\mathscr{O}_Y$-module $\mathscr{F}$. Especially, if $\mathscr{F} = g_*\mathscr{A}$ for a quasi-coherent $\mathscr{O}_Y$-module $\mathscr{A}$, then (11.1.13) takes the form :

$$(11.1.14)\qquad \overline{g}^*\mathscr{H}om_{g_*\mathscr{O}_Y}(\overline{g}_*\mathscr{A},\mathscr{G}) \to \mathscr{H}om_{\mathscr{O}_Y}(\mathscr{A},\overline{g}^*\mathscr{G}).$$

Furthermore, by inspecting the definitions (and the proof of claim 10.1.17), one verifies easily that the map $\varphi^*\circ\beta^\flat(\mathscr{G}) \to \alpha^\flat\circ\overline{g}^*(\mathscr{G})$ constructed above is the same as the map (11.1.14), taken with $\mathscr{A} = f_*\mathscr{O}_X$. Notice that, since f is pseudo-coherent, $f_*\mathscr{O}_X$ is even a finitely presented $\mathscr{O}_Y$-module; thus, in order to conclude the proof of (i), it suffices to show that (11.1.14) is an isomorphism for every finitely presented $\mathscr{O}_Y$-module $\mathscr{A}$ and every $g_*\mathscr{O}_Y$-module $\mathscr{G}$. To this aim, we may assume that Z and Y are affine; then we may find a presentation $\underline{\mathscr{E}} := (\mathscr{O}_Y^{\oplus p} \to \mathscr{O}_Y^{\oplus q} \to \mathscr{A} \to 0)$. We apply the natural transformation (11.1.14) to $\underline{\mathscr{E}}$, thereby obtaining a commutative ladder with left exact rows; then the five-lemma reduces the assertion to the case where $\mathscr{A} = \mathscr{O}_Y$, which is obvious. To prove assertion (ii) we may again assume that Y and Z are affine, in which case we can find a resolution $L_\bullet \to f_*\mathscr{O}_X$ consisting of free $\mathscr{O}_Y$-modules of finite rank. For a given bounded below complex $I^\bullet$ of injective $g_*\mathscr{O}_Y$-modules, choose a resolution $\overline{g}^*I^\bullet \to J^\bullet$ consisting of injective $\mathscr{O}_Y$-modules. We deduce a commutative ladder :

$$\begin{array}{ccccc}
\overline{g}^*\mathscr{H}om^\bullet_{g_*\mathscr{O}_Y}(\overline{g}_*\circ f_*\mathscr{O}_X, I^\bullet) & \xrightarrow{\mu_1} & \mathscr{H}om^\bullet_{\mathscr{O}_Y}(f_*\mathscr{O}_X,\overline{g}^*I^\bullet) & \xrightarrow{\mu_3} & \mathscr{H}om^\bullet_{\mathscr{O}_Y}(f_*\mathscr{O}_X, J^\bullet) \\
{\scriptstyle\lambda_1}\downarrow & & \downarrow & & \downarrow{\scriptstyle\lambda_2} \\
\overline{g}^*\mathscr{H}om^\bullet_{g_*\mathscr{O}_Y}(\overline{g}_*L_\bullet, I^\bullet) & \xrightarrow{\mu_2} & \mathscr{H}om^\bullet_{\mathscr{O}_Y}(L_\bullet,\overline{g}^*I^\bullet) & \xrightarrow{\mu_4} & \mathscr{H}om^\bullet_{\mathscr{O}_Y}(L_\bullet, J^\bullet).
\end{array}$$

Since $\overline{g}$ is flat, λ_1 is a quasi-isomorphism, and the same holds for λ_2. The maps μ_1 and μ_2 are of the form (11.1.14), hence they are quasi-isomorphisms, by the foregoing proof of (i). Finally, it is clear that μ_4 is a quasi-isomorphism as well, hence the same holds for $\mu_3 \circ \mu_1$, and (ii) follows. □

11.1.15. Let now $f : X \to Y$ be an *embeddable* morphism, *i.e.* such that it can be factored as a composition $f = g \circ h$ where $h : X \to Z$ is a finite pseudo-coherent morphism, $g : Z \to Y$ is smooth and separated, and the fibres of g have bounded dimension. One defines :

$$f^! := h^\flat \circ g^\sharp.$$

Lemma 11.1.16. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ be three embeddable morphisms of schemes, such that the compositions $g \circ f$, $h \circ g$ and $h \circ g \circ f$ are also embeddable. Then we have :*

(i) *There exists a natural isomorphism of functors*

$$\psi_{g,f} : (g \circ f)^! \xrightarrow{\sim} f^! \circ g^!.$$

Especially, $f^!$ is independent – up to a natural isomorphism of functors – of the choice of factorization as a finite morphism followed by a smooth morphism.

(ii) *The diagram of functors :*

$$\begin{array}{ccc} (h\circ g\circ f)^! & \xrightarrow{\psi_{h,g\circ f}} & (g\circ f)^!\circ h^! \\ \downarrow{\scriptstyle \psi_{h\circ g,f}} & & \downarrow{\scriptstyle \psi_{g,f}\circ h^!} \\ f^!\circ(h\circ g)^! & \xrightarrow{f^!(\psi_{h,g})} & f^!\circ g^!\circ h^! \end{array}$$

commutes.

Proof. The proof is a complicated verification, starting from proposition 11.1.7. See [87, Ch.III, Th.8.7] for details. ☐

11.1.17. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be two finite and pseudo-coherent morphisms of schemes. The map $f^\natural : \mathscr{O}_Y \to f_*\mathscr{O}_X$ induces a natural transformation :

$$\mathscr{H}om^\bullet_{\mathscr{O}_Z}(g_*(f^\natural), -) : \beta_* \circ (\gamma\circ\beta)^\flat \to \gamma^\flat \tag{11.1.18}$$

(notation of the proof of proposition 11.1.7(i)) and we wish to conclude this section by exhibiting another compatibility involving (11.1.18) and the isomorphism $\xi_{f,g}$. To this aim, we consider the composition of isomorphism of functors on $\mathsf{D}^+(\mathscr{O}_Z\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$:

$$Rf_* \circ \overline{g\circ f}^* \xrightarrow{①} \alpha_* \circ R\overline{f}_* \circ \overline{f}^* \circ \varphi^* \xrightarrow{②} \alpha_* \circ \varphi^* \xrightarrow{③} \overline{g}^* \circ \beta_* \tag{11.1.19}$$

where :

① is deduced from (11.1.8) and the decomposition $f = \alpha \circ \overline{f}$.
② is induced from the counit of adjunction $\overline{\varepsilon} : R\overline{f}_* \circ \overline{f}^* \to \mathbf{1}$ provided by lemma 11.1.2.
③ is the identification of claim 11.1.10.

We define a morphism of functors as a composition :

$$Rf_* \circ (g\circ f)^\flat = Rf_* \circ \overline{g\circ f}^* \circ R(\gamma\circ\beta)^\flat \xrightarrow{④} \overline{g}^* \circ \beta_* \circ R(\gamma\circ\beta)^\flat \xrightarrow{⑤} g^\flat \tag{11.1.20}$$

where :

④ is the isomorphism (11.1.19) $\circ\, R(\gamma\circ\beta)^\flat$.
⑤ is the composition of $\overline{g}^*$ and the morphism $\beta_* \circ R(\gamma\circ\beta)^\flat \to R\gamma^\flat$ deduced from (11.1.18).

We have a natural diagram of functors on $\mathsf{D}^+(\mathscr{O}_Z\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$:

$$\begin{array}{ccccc} R(g\circ f)_*\circ(g\circ f)^\flat & \xrightarrow{⑥} & Rg_*\circ Rf_*\circ(g\circ f)^\flat & \xrightarrow{⑨} & Rg_*\circ g^\flat \\ \downarrow{\scriptstyle R(g\circ f)_*(\xi_{f,g})} & & \downarrow{\scriptstyle Rg_*\circ Rf_*(\xi_{f,g})} & & \uparrow{\scriptstyle ⑧} \\ R(g\circ f)_*\circ(f^\flat\circ g^\flat) & \xrightarrow{⑦} & Rg_*\circ Rf_*\circ(f^\flat\circ g^\flat) & = & Rg_*\circ(Rf_*\circ f^\flat)\circ g^\flat \end{array} \tag{11.1.21}$$

where :

⑥ and ⑦ are induced by the natural isomorphism $R(g\circ f)_* \xrightarrow{\sim} Rg_* \circ Rf_*$.
⑧ is induced by the counit of the adjunction $\varepsilon : Rf_* \circ f^\flat \to \mathbf{1}$.
⑨ is $Rg_* \circ$ (11.1.20).

Lemma 11.1.22. *In the situation of* (11.1.17), *diagram* (11.1.21) *commutes.*

Proof. The diagram splits into left and right subdiagrams, and clearly the left subdiagram commutes, hence it remains to show that the right one does too; to this aim, it suffices to consider the simpler diagram :

$$\begin{array}{ccc} Rf_* \circ (g\circ f)^\flat & \longrightarrow & g^\flat \\ {\scriptstyle Rf_*(\xi_{f,g})}\downarrow & \nearrow & \\ Rf_* \circ f^\flat \circ g^\flat & & \end{array}$$

whose horizontal arrow is (11.1.20), and whose upward arrow is deduced from the counit ε. However, the counit ε, used in ⑧, can be expressed in terms of the counit $\overline{\varepsilon} : R\overline{f}_* \circ \overline{f}^* \to \mathbf{1}$, used in ②, therefore we are reduced to considering the diagram of functors on $\mathsf{D}^+(\mathscr{O}_Z\text{-}\mathbf{Mod})_{\mathrm{qcoh}}$:

$$\begin{array}{ccccc} \alpha_* \circ R\overline{f}_* \circ \overline{f}^* \circ \varphi^* \circ R(\gamma\circ\beta)^\flat & \xrightarrow{\ (a)\ } & \alpha_* \circ \varphi^* \circ R(\gamma\circ\beta)^\flat & \xrightarrow{\ (e)\ } & \overline{g}^* \circ \beta_* \circ R(\gamma\circ\beta)^\flat \\ & & \downarrow{\scriptstyle (c)} & & \\ \downarrow{\scriptstyle Rf_*(\xi_{f,g})} & & \alpha_* \circ \varphi^* \circ R\beta^\flat \circ R\gamma^\flat & & \downarrow{\scriptstyle (5)} \\ & & \downarrow{\scriptstyle (d)} & & \\ \alpha_* \circ R\overline{f}_* \circ \overline{f}^* \circ R\alpha^\flat \circ g^\flat & \xrightarrow{\ (b)\ } & \alpha_* \circ R\alpha^\flat \circ \overline{g}^* \circ R\gamma^\flat & \xrightarrow{\ (f)\ } & g^\flat = \overline{g}^* \circ R\gamma^\flat \end{array}$$

where :

- ⓐ and ⓑ are induced from $\overline{\varepsilon}$.
- ⓒ is induced by the isomorphism (11.1.11).
- ⓓ is induced by the isomorphism of claim 11.1.12(i).
- ⓔ is induced by the identity of claim 11.1.10.
- ⓕ is induced by the counit $\alpha_* \circ R\alpha^\flat \to \mathbf{1}$.

Now, by inspecting the construction of $\xi_{f,g}$ one checks that the left subdiagram of the latter diagram commutes; moreover, from claim 11.1.12(ii) (and from lemma 10.1.16(i)), one sees that the right subdiagram is obtained by evaluating on complexes of injective modules the analogous diagram of functors on $\mathscr{O}_Z$-$\mathbf{Mod}$:

$$\begin{array}{ccc} \alpha_* \circ \varphi^* \circ (\gamma\circ\beta)^\flat & \xrightarrow{\ \boxed{e}\ } & \overline{g}^* \circ \beta_* \circ (\gamma\circ\beta)^\flat \\ \downarrow{\scriptstyle \boxed{c}} & & \\ \alpha_* \circ \varphi^* \circ \beta^\flat \circ \gamma^\flat & & \downarrow{\scriptstyle \boxed{5}} \\ \downarrow{\scriptstyle \boxed{d}} & & \\ \alpha_* \circ \alpha^\flat \circ \overline{g}^* \circ \gamma^\flat & \xrightarrow{\ \boxed{f}\ } & \overline{g}^* \circ \gamma^\flat . \end{array}$$

Furthermore, by inspecting the proof of claim 11.1.12(i), one verifies that $\boxed{f} \circ \boxed{d}$ is the same as the composition

$$\alpha_* \circ \varphi^* \circ \beta^\flat \circ \gamma^\flat \xrightarrow{(i)} \overline{g}^* \circ \beta_* \circ \beta^\flat \circ \gamma^\flat \xrightarrow{(j)} \overline{g}^* \circ \gamma^\flat$$

where (j) is deduced from the counit $\beta_* \circ \beta^\flat \to \mathbf{1}$, and (i) is deduced from the identity of claim 11.1.10. Hence we come down to showing that the diagram :

$$\begin{array}{ccc} \beta_* \circ (\gamma \circ \beta)^\flat & \xrightarrow{(11.1.18)} & \gamma^\flat \\ {\scriptstyle \beta_*(11.1.9)} \downarrow & \nearrow {\scriptstyle (k)} & \\ \beta_* \circ \beta^\flat \circ \gamma^\flat & & \end{array}$$

commutes in $g_*\mathscr{O}_Y$-$\mathbf{Mod}$, where (k) is deduced from the counit $\beta_* \circ \beta^\flat \to \mathbf{1}$. This is an easy verification, that shall be left to the reader. $\square$

11.2. **Cousin complexes.** In this section we study a special type of complexes of abelian sheaves, called *Cousin complexes*, defined on a large class of topological spaces. These complexes play an important role in the further development of duality theory for locally coherent schemes : in many cases the techniques explained in this section will allow to exhibit canonical representatives for the dualizing complexes that shall be investigated at length in section 11.3.

11.2.1. We resume the notation of (10.1.30), so $(X, \mathscr{A})$ is a ringed topological space, $\varphi : V \to X$ a locally closed immersion, we denote by $\overline{V}$ the topological closure of V in X, and we set $\partial V := \overline{V} \setminus V$. In (10.4.1) we have introduced the functor $\Gamma_{\overline{V}}$ of sections with support in $\overline{V}$, and its sheaf-valued variant $\underline{\Gamma}_{\overline{V}}$. We now extend these definitions to the case of a general locally closed immersion, by setting

$$\underline{\Gamma}_V := \varphi_* \circ \varphi^* \circ \underline{\Gamma}_{\overline{V}} : \mathscr{A}\text{-}\mathbf{Mod} \to \mathscr{A}\text{-}\mathbf{Mod} \qquad \Gamma_V := \Gamma \circ \underline{\Gamma}_V : \mathscr{A}\text{-}\mathbf{Mod} \to \mathscr{A}(X)\text{-}\mathbf{Mod}.$$

The functor $\underline{\Gamma}_V$ admits a derived functor

$$R\underline{\Gamma}_V : \mathsf{D}^+(\mathscr{A}\text{-}\mathbf{Mod}) \to \mathsf{D}^+(\mathscr{A}\text{-}\mathbf{Mod})$$

obtained as usual by evaluating on injective resolutions of given bounded below complexes.

Remark 11.2.2. (i) We have the following explicit description of $\underline{\Gamma}_V$. Let $k : V \to X \setminus \partial V$ and $l : X \setminus \partial V \to X$ be respectively the closed and open immersions. Since $\varphi = l \circ k$, there exist natural isomorphisms

$$\underline{\Gamma}_V \xrightarrow{\sim} \varphi_* \circ k^* \circ l^* \circ \underline{\Gamma}_{\overline{V}} \xrightarrow{\sim} l_* \circ k_* \circ k^* \circ \underline{\Gamma}_V \circ l^* \xrightarrow{\sim} l_* \circ \underline{\Gamma}_V \circ l^*$$

where the last map is induced by the natural identification $k_* \circ k^* \circ \underline{\Gamma}_V \xrightarrow{\sim} \underline{\Gamma}_V$ provided by lemma 10.1.32. There follows a natural isomorphism

$$\underline{\Gamma}_V \mathscr{F}(S) \xrightarrow{\sim} \underline{\Gamma}_{\overline{V}} \mathscr{F}(S \setminus \partial V) \qquad \text{for every open subset } S \text{ of } X.$$

(ii) Notice that $\partial V = \varnothing$ if φ is a closed immersion; in this case, (i) says that the functor defined in (11.2.1) is naturally isomorphic to its namesake introduced in (10.4.1), so the current notation is consistent with that of section 10.4.

(iii) For a general locally closed subset V and any $\mathscr{A}$-module $\mathscr{F}$, the natural monomorphism $\iota_{\mathscr{F}} : \underline{\Gamma}_{\overline{V}} \mathscr{F} \to \mathscr{F}$ induces a natural transformation

$$\underline{\Gamma}_V \mathscr{F} \to \varphi_* \varphi^* \mathscr{F} \qquad (11.2.3)$$

which is a monomorphism for every such $\mathscr{F}$, since the functor $\varphi_* \varphi^*$ is left exact. Notice that if V is open in X, the map $\varphi^* \iota_{\mathscr{F}}$ is an isomorphism, so the same holds for (11.2.3).

Proposition 11.2.4. *In the situation of* (11.2.1)*, the functor* $\underline{\Gamma}_V$ *is right adjoint to* t_V.

Proof. For φ a closed immersion, remark 11.2.2(ii) reduces the assertion to lemma 10.4.13(i), since in this case $\varphi_! = \varphi_*$. The assertion is also clear in case φ is an open immersion, since in this case $\underline{\Gamma}_V$ is naturally isomorphic to the functor $\varphi_* \circ \varphi^*$, by remark 11.2.2(iii). Notice that $V = \overline{V} \cap (X \setminus \partial V)$. By proposition 10.1.33(i) (and by remark 1.1.17(i)) we are reduced to exhibiting a natural isomorphism of functors

$$\underline{\Gamma}_V \xrightarrow{\sim} \underline{\Gamma}_{X\setminus\partial V} \circ \underline{\Gamma}_{\overline{V}}.$$

But the existence of such an isomorphism is asserted already in remark 11.2.2(i). □

11.2.5. For our discussion, it will be important to know that in fact there exists a canonical adjunction for the pair $(t_V, \underline{\Gamma}_V)$. In view of the proof of proposition 11.2.4 (and of remark 1.1.17(i)), it suffices to exhibit a canonical pair (η^V, ε^V) of units and counits for this adjunction, in case φ is either an open or a closed immersion.

- Suppose first that φ is a closed immersion. Then we want natural transformations

$$\eta^V_{\mathscr{F}} : \mathscr{F} \to \underline{\Gamma}_V \circ \varphi_* \circ \varphi^* \mathscr{F} \qquad \varepsilon^V_{\mathscr{F}} : \varphi_* \circ \varphi^* \circ \underline{\Gamma}_V \mathscr{F} \to \mathscr{F}$$

related by the triangular identities of (1.1.13). However, remark 11.2.2(ii) yields a natural isomorphism $\eta_1 : \underline{\Gamma}_V \varphi_* \varphi^* \mathscr{F} \xrightarrow{\sim} \varphi_* \varphi^* \mathscr{F}$, and on the other hand, we have a canonical unit $\eta_2 : \mathscr{F} \to \varphi_* \varphi^* \mathscr{F}$ for the adjunction (φ^*, φ_*), so we take $\eta^V_{\mathscr{F}} := \eta_1^{-1} \circ \eta_2$. Likewise, let $\varepsilon_1 : \underline{\Gamma}_V \mathscr{F} \to \mathscr{F}$ be the natural monomorphism; we take $\varepsilon^V_{\mathscr{F}} := \varepsilon_1 \circ f_{\mathscr{F}}$, where $f_{\mathscr{F}} : t_V \circ \underline{\Gamma}_V \mathscr{F} \xrightarrow{\sim} \underline{\Gamma}_V \mathscr{F}$ is the isomorphism provided by lemma 10.1.32. The triangular identities may then be checked on the stalks : we leave the details to the reader.

- Next, if φ is an open immersion, we need to find natural transformations

$$\eta^V_{\mathscr{F}} : \mathscr{F} \to \varphi_* \circ \varphi^* \circ \varphi_! \circ \varphi^* \mathscr{F} \qquad \varepsilon^V_{\mathscr{F}} : \varphi_! \circ \varphi^* \circ \varphi_* \circ \varphi^* \mathscr{F} \to \mathscr{F}$$

again related by the same triangular identities. We take for $\eta^V_{\mathscr{F}}$ (resp. $\varepsilon^V_{\mathscr{F}}$) the composition of the natural unit $\mathscr{F} \to \varphi_* \varphi^* \mathscr{F}$ of the adjunction (φ^*, φ^*) (resp. the counit $\varphi_! \circ \varphi^* \mathscr{F} \to \mathscr{F}$ of the adjunction $(\varphi_!, \varphi^*)$), with the isomorphism

$$\varphi_* \varphi^* \mathscr{F} \xrightarrow{\sim} \varphi_* \varphi^* \varphi_! \varphi^* \mathscr{F} \qquad (\text{ resp. } \varphi_! \varphi^* \varphi_* \varphi^* \mathscr{F} \xrightarrow{\sim} \varphi_! \varphi^* \mathscr{F} \text{ })$$

deduced from the natural identification of $\varphi^* \circ \varphi_!$ (resp. of $\varphi^* \circ \varphi_*$) with the identity functor of $\varphi^* \mathscr{A}$-$\mathbf{Mod}$. The triangular identities are as usual checked on the stalks.

11.2.6. Due to proposition 11.2.4, and the subsequent observations in (11.2.5), all the constructions of proposition 10.1.33 have natural counterparts for the functors $\underline{\Gamma}_V$. Namely :

- If V and V' are any two locally closed subsets of X, there is a natural isomorphism

$$\beta_{V,V'} : \underline{\Gamma}_V \circ \underline{\Gamma}_{V'} \xrightarrow{\sim} \underline{\Gamma}_{V \cap V'}$$

and if $V'' \subset X$ is any other locally closed subset, the diagram of functors

$$\begin{array}{ccc} \underline{\Gamma}_V \circ \underline{\Gamma}_{V'} \circ \underline{\Gamma}_{V''} & \xrightarrow{\underline{\Gamma}_V * \beta_{V',V''}} & \underline{\Gamma}_V \circ \underline{\Gamma}_{V' \cap V''} \\ {\scriptstyle \beta_{V,V'} * \underline{\Gamma}_{V''}} \downarrow & & \downarrow {\scriptstyle \beta_{V,V' \cap V''}} \\ \underline{\Gamma}_{V \cap V'} \circ \underline{\Gamma}_{V''} & \xrightarrow{\beta_{V \cap V',V''}} & \underline{\Gamma}_{V \cap V' \cap V''} \end{array} \tag{11.2.7}$$

commutes.

- Moreover, suppose that the pair (V, V') satisfies either of the conditions (a), (b) of proposition 10.1.33(iii); then there exists a natural morphism

$$\gamma_{V,V'} : \underline{\Gamma}_V \to \underline{\Gamma}_{V'}.$$

which corresponds to $c^{V,V'}$ under the natural adjunction described in (11.2.5), and if V'' is any third locally closed subset of X, we get a commutative diagram

$$\begin{array}{ccc} \underline{\Gamma}_{V''} \circ \underline{\Gamma}_V & \xrightarrow{\underline{\Gamma}_{V''} * \gamma_{V,V'}} & \underline{\Gamma}_{V''} \circ \underline{\Gamma}_{V'} \\ {\scriptstyle \beta_{V'',V}} \downarrow & & \downarrow {\scriptstyle \beta_{V'',V'}} \\ \underline{\Gamma}_{V''\cap V} & \xrightarrow{\gamma_{V''\cap V, V''\cap V'}} & \underline{\Gamma}_{V''\cap V'}. \end{array}$$

- Lastly, in the situation of proposition 10.1.33(v) we have the identity :

$$\gamma_{V',V''} \circ \gamma_{V,V'} = \gamma_{V,V''} : \underline{\Gamma}_V \to \underline{\Gamma}_{V''}.$$

Remark 11.2.8. Let X be a topological space, $j : U \to X$ an open immersion, $\varphi : V \to X$ and $\varphi' : V' \to X$ two locally closed immersions fulfilling either of conditions (a),(b) of proposition 10.1.33(iii). The transformations $\gamma_{V,V'}$ can also be described explicitly, thanks to the following considerations.

(i) First, notice that the functors $t_\bullet$ on $\mathscr{A}$-modules, the unit $\eta^\bullet$ and the counit $\varepsilon^\bullet$ are all compatible with restriction to U, *i.e.* we have identities of functors on $j^*\mathscr{A}$-modules and of morphisms between such functors :

$$j^* * t_V = t_{U\cap V} \quad j^* * \eta^V = \varepsilon^{V\cap U} \quad j^* * \varepsilon^V = \eta^{V\cap U} \qquad \text{for every locally closed subset } V \subset X.$$

This is obvious for the truncation functors, and for the units and counits it follows by a simple inspection of the definition. Then, the uniqueness properties of the transformations $\tau^{\bullet\bullet}$ and $c^{\bullet\bullet}$ imply that the same compatibility with restriction to U holds for the latter.

(ii) It follows formally from (i) that the isomorphisms $\beta_{\bullet\bullet}$ and the transformations $\gamma_{\bullet\bullet}$ are likewise compatible with restriction to U. On the other hand, notice that the restriction map

$$\mathrm{Hom}_{\mathscr{A}\text{-}\mathbf{Mod}}(t_{V'}\mathscr{F}, \mathscr{G}) \to \mathrm{Hom}_{j^*\mathscr{A}\text{-}\mathbf{Mod}}((t_{V'}\mathscr{F})_{|U}, \mathscr{G}_{|U})$$

is a bijection for every $\mathscr{F}, \mathscr{G} \in \mathrm{Ob}(\mathscr{A}\text{-}\mathbf{Mod})$, whenever $V' \subset U$. We conclude that the restriction map

$$\mathrm{Hom}_{\mathscr{A}\text{-}\mathbf{Mod}}(\mathscr{F}, \underline{\Gamma}_{V'}\mathscr{G}) \to \mathrm{Hom}_{j^*\mathscr{A}\text{-}\mathbf{Mod}}(\mathscr{F}_{|U}, \underline{\Gamma}_{V'\cap U}\mathscr{G}_{|U})$$

is also bijective when $V' \subset U$. Letting $\mathscr{F} := \underline{\Gamma}_V\mathscr{G}$, we deduce that $\gamma_{V,V'}$ is determined by its restriction to U, which is just $\gamma_{V\cap U, V'\cap U}$.

(iii) Now, suppose first that $\psi : V \to V'$ is a closed immersion, and denote by $\overline{V}$ and $\overline{V}'$ the topological closures in X of V and respectively V'; notice that the unit of the adjunction (ψ^*, ψ_*) induces an isomorphism

$$\varphi'^* \circ \Gamma_{\overline{V}} \xrightarrow{\sim} \psi_* \circ \psi^* \circ \varphi'^* \circ \Gamma_{\overline{V}} \qquad (11.2.9)$$

since $(\varphi'^* \circ \Gamma_{\overline{V}}\mathscr{F})_{|V'\setminus V} = 0$ for every $\mathscr{A}$-module $\mathscr{F}$ (lemma 10.1.32). We claim that $\gamma_{V,V'}$ is the composition

$$\varphi_* \circ \varphi^* \circ \underline{\Gamma}_{\overline{V}} \xrightarrow{\sim} \varphi'_* \circ \psi_* \circ \psi^* \circ \varphi'^* \circ \underline{\Gamma}_{\overline{V}} \xrightarrow{\sim} \varphi'_* \circ \varphi'^* \circ \underline{\Gamma}_{\overline{V}} \to \varphi'_* \circ \varphi'^* \circ \underline{\Gamma}_{\overline{V}'}.$$

where the second map is the inverse of (11.2.9) and the third is deduced from $\gamma_{\overline{V},\overline{V}'} : \underline{\Gamma}_{\overline{V}} \to \underline{\Gamma}_{\overline{V}'}$. Indeed, notice that the foregoing construction is compatible with restriction to U; taking into account (iii), we may then replace X by its open subset $X \setminus \partial V'$, and assume that both φ and φ' are closed immersions. In this case, the assertion follows by direct inspection of the definition of the unit and counit of the adjunction $(t_\bullet, \underline{\Gamma}_\bullet)$, and of the definition of $t_\bullet$, in the case of closed immersions.

(iv) Likewise, if V' is an open subset of V, pick a factorization of $\varphi : V \to X$ as the composition of a closed immersion $i : V \to U$ and an open immersion $j : U \to X$. There is a unique open subset $U' \subset U$ such that $V' = U' \cap V$; denote by $j' : U' \to X$ the open immersion, and for any given $\mathscr{A}$-module $\mathscr{F}$, let $\eta_{\mathscr{F}} : \mathscr{F} \to j'_* j'^* \mathscr{F}$ be the unit of adjunction; according to remark 11.2.2, we may regard $\eta_{\mathscr{F}}$ as a natural transformation

$$\eta_{\mathscr{F}} : \mathscr{F} \to \underline{\Gamma}_{U'}\mathscr{F}$$

and we claim that $\gamma_{V,V'}$ is the composition

$$\underline{\Gamma}_V \xrightarrow{\underline{\Gamma}_V * \eta} \underline{\Gamma}_V \circ \underline{\Gamma}_{U'} \xrightarrow{\beta_{V,U'}} \underline{\Gamma}_{V'}.$$

For the proof, we may argue as in (iii) to see that $\gamma_{V,V'}$ is determined by its restriction to U', and since the proposed construction is compatible with restriction to U', we are then reduced to the case where $U' = U = X$ and therefore $V = V'$. However, clearly $c^{V,V}$ is the identity endomorphism of t_V, so it remains only to prove that the proposed map is the identity of $\underline{\Gamma}_V$, if $V = V'$. The latter can be checked easily, whence the claim. We could have used the above construction as a definition for $\gamma_{V,V'}$ in the case of open immersions, but the method we followed has the advantage of dispensing us from verifying that the resulting transformation is independent of the chosen factorization of φ.

11.2.10. Let $V \subset V'$ be a closed immersion of locally closed subsets of a topological space X; from remark 11.2.8(iii), it is clear that $\gamma_{V,V'}$ is a monomorphism. Moreover, we have :

Lemma 11.2.11. *In the situation of* (11.2.10)*, the sequence*

$$\Sigma_{V,V'} \quad : \quad 0 \to \underline{\Gamma}_V\mathscr{F} \xrightarrow{\gamma_{V,V'}} \underline{\Gamma}_{V'}\mathscr{F} \xrightarrow{\gamma_{V',V'\setminus V}} \underline{\Gamma}_{V'\setminus V}\mathscr{F}$$

is left exact for every $\mathscr{A}$-module $\mathscr{F}$.

Proof. Indeed, say that $V' = U \cap Z$ for some closed subset Z and open subset U; then $V = \overline{V} \cap U$, with $\overline{V}$ the topological closure of V in X, and in light of (11.2.6) we see that there is a natural isomorphism $\Sigma_{V,V'} \xrightarrow{\sim} \underline{\Gamma}_U(\Sigma_{\overline{V},Z})$. However, $\underline{\Gamma}_U$ is a left exact functor (since it admits a left adjoint), so it suffices to check that $\Sigma_{\overline{V},Z}$ is left exact. Thus, we may assume from start that V and V' are closed in X. By the same token, we see that $\Sigma_{V,V'} \simeq \underline{\Gamma}_{V'}(\Sigma_{V,X})$, so we may further assume that $V' = X$, in which case the sequence reduces to the initial segment of (10.4.2). □

Proposition 11.2.12. *With the notation of* (11.2.1)*, the following holds :*

(i) *If $\mathscr{F}$ is a flabby $\mathscr{A}$-module, the same holds for $\underline{\Gamma}_V\mathscr{F}$.*
(ii) *Every flabby $\mathscr{A}$-module is $\underline{\Gamma}_V$-acyclic.*
(iii) *If $\mathscr{B} \to \mathscr{A}$ is any morphism of sheaves of rings on X, the functor $R\underline{\Gamma}_V$ commutes with the forgetful functor* $\mathsf{D}^+(\mathscr{A}\text{-}\mathbf{Mod}) \to \mathsf{D}^+(\mathscr{B}\text{-}\mathbf{Mod})$.
(iv) *If $\varphi' : V' \to X$ is any other locally closed immersion, there exists a natural isomorphism of functors :*

$$R\underline{\Gamma}_{V\cap V'} \xrightarrow{\sim} R\underline{\Gamma}_V \circ R\underline{\Gamma}_{V'} \qquad \mathsf{D}^+(\mathscr{A}\text{-}\mathbf{Mod}) \to \mathsf{D}^+(\mathscr{A}\text{-}\mathbf{Mod}).$$

(v) *Suppose that X is locally spectral and quasi-separated, and V is a constructible locally closed subset of X, then we have :*
   (a) *Every qc-flabby $\mathscr{A}$-module is $\underline{\Gamma}_V$-acyclic.*
   (b) *For every $i \in \mathbb{N}$, the functor $R^i\underline{\Gamma}_V$ commutes with filtered colimits of $\mathscr{A}$-modules.*
   (c) *For every $x \in V$, denote by $j_x : X(x) \to X$ the inclusion map (notation of definition* 8.1.44(iii)*). Then there exists a natural isomorphism*

$$j_x^* R\underline{\Gamma}_V \xrightarrow{\sim} R\underline{\Gamma}_{V(x)} \qquad \textit{in } \mathsf{D}^+(j_x^*\mathscr{A}\text{-}\mathbf{Mod}).$$

Proof. (i): By lemmata 10.4.13(ii) and 10.4.4(i), we know already that both assertions (i) and (ii) hold in case φ is a closed immersion; for the general case, we may then assume that φ is an open immersion. However, in this case both φ_* and φ^* take flabby sheaves to flabby sheaves (lemmata 10.1.2 and 10.1.5(ii)), and flabby sheaves are acyclic for both of these functors (lemma 10.1.5(ii)), so the assertion follows from remark 11.2.2(iii).

(iii) is clear from (ii), as every injective $\mathscr{A}$-module is a flabby $\mathscr{B}$-module (lemma 10.1.5(v)).

(iv) follows formally from (i), (ii) and (11.2.6) : details left to the reader.

(v.a): Since the assertion is local on X, we may assume that X is spectral, in which case V is the intersection of a closed constructible subset and an open constructible subset. By (iv) and lemma 10.1.5(iii), it then suffices to consider separately the cases where V is either open or closed in X. By lemma 10.4.4(iii.b) we know already the assertions in case V is a closed constructible subset of X, and the case where V is open follows easily from lemma 10.1.5(i,iv).

(v.b): We reduce again to the case where V is the intersection of a closed constructible subset Z and an open constructible subset U. Since – by (ii) – the functor $\underline{\Gamma}_Z$ transforms injective $\mathscr{A}$-modules into $\underline{\Gamma}_U$-acyclic $\mathscr{A}$-modules, we have a convergent spectral sequence

$$R^i\underline{\Gamma}_U \circ R^j\underline{\Gamma}_Z\mathscr{F} \Rightarrow R^{i+j}\underline{\Gamma}_V\mathscr{F}.$$

It then suffices to consider separately the case where V is either open or closed. These cases follow respectively from lemma 10.4.4(iii.b) and claim 10.4.5.

(v.c): Arguing as in the proof of (v.b), we reduce to consider separately the cases where V is either open or closed. For V closed, the assertion is corollary 10.4.10. For V open, the assertion follows from proposition 10.1.10(ii) and remark 11.2.2(iii), by arguing as in the proof of corollary 10.4.10 : details left to the reader. ☐

Remark 11.2.13. (i) Using proposition 11.2.12, the various compatibilities that have been found so far beween the functors $\underline{\Gamma}_V$ extend straightforwardly to their derived extensions. Especially, diagram (11.2.7) still commutes, after replacing the functors $\underline{\Gamma}_\bullet$ by their derived extensions everywhere, and the natural transformations $\beta_{\bullet\bullet}$ by their derived versions, provided by proposition 11.2.12(iii).

(ii) Likewise, if $V \subset V'$ (resp. $V' \subset V$) is any closed (resp. open) immersion of locally closed subsets of X, by deriving the morphisms $\gamma_{V,V'}$ of (11.2.6) we get a natural transformation

$$R\underline{\Gamma}_V \to R\underline{\Gamma}_{V'} \qquad (\text{ resp. } R\underline{\Gamma}_{V'} \to R\underline{\Gamma}_V\ ) \qquad \text{in } \mathsf{D}^+(\mathscr{A}\text{-}\mathbf{Mod})$$

and the compatibilities between the functors $\gamma_{\bullet\bullet}$ and $\beta_{\bullet\bullet}$ explicited in (11.2.6) hold, *mutatis mutandis*, also for the functors $R\underline{\Gamma}_\bullet$.

Lemma 11.2.14. *In the situation of* (11.2.10)*, let $\mathscr{F}$ be any flabby $\mathscr{A}$-module. Then, for every open subset $U \subset X$, the sequence*

$$0 \to \underline{\Gamma}_V\mathscr{F}(U) \to \underline{\Gamma}_{V'}\mathscr{F}(U) \to \underline{\Gamma}_{V'\setminus V}\mathscr{F}(U) \to 0$$

induced by the sequence $\Sigma_{V,V'}$ of lemma 11.2.11*, is short exact.*

Proof. We may assume that $U = X$, and it suffices to show that the map $g : \Gamma_{V'}\mathscr{F} \to \Gamma_{V'\setminus V}\mathscr{F}$ deduced from $\gamma_{V',V'\setminus V}$ is surjective, whenever $\mathscr{F}$ is flabby. To this aim, write $V' \setminus V = U' \cap V'$ for some open subset $U' \subset X$; then g is naturally identified with the map

$$\Gamma_X \circ \underline{\Gamma}_{V'}\mathscr{F} \to \Gamma_{U'} \circ \underline{\Gamma}_{V'}\mathscr{F}$$

deduced from $\gamma_{X,U'}$. But $\underline{\Gamma}_{V'}\mathscr{F}$ is still flabby (proposition 11.2.12(i)) so we are reduced to the case where $V' = X$, in which case g is naturally identified with the restriction map $\mathscr{F}(X) \to \mathscr{F}(X\setminus V)$, whence the contention. ☐

11.2.15. Let $(X, \mathscr{A})$ be a ringed topological space. A *family of supports* on X is a set Φ consisting of closed subsets of X, such that :

- $\varnothing \in \Phi$.
- If $Z \in \Phi$ and $Z' \subset Z$ is any closed subset, then $Z' \in \Phi$ as well.
- If $Z_1, Z_2 \in \Phi$, then $Z_1 \cup Z_2 \in \Phi$ as well.

Now, let Φ and Φ' be any two families of supports on X; we let $\Phi|\Phi'$ be the set whose elements are all the pairs $\underline{Z} := (Z, Z')$, where Z and Z' are arbitrary elements of Φ and respectively Φ', such that $Z' \subset Z$. We endow $\Phi|\Phi'$ with a partial ordering, by declaring that, for given pairs $\underline{Z}_1 := (Z_1, Z'_1), \underline{Z}_2 := (Z_2, Z'_2)$ of $\Phi|\Phi'$, we have $\underline{Z}_1 \leq \underline{Z}_2$ if $Z_1 \subset Z_2$ and $Z'_1 \subset Z'_2$. It is easily seen that $\Phi|\Phi'$ is filtered. To every object $\underline{Z} \in \Phi|\Phi'$ we assign the functor $\underline{\Gamma}_{Z\setminus Z'}$, and to any pair $(\underline{Z}_1, \underline{Z}_2)$ of elements of $\Phi|\Phi'$ with $\underline{Z}_1 \leq \underline{Z}_2$, we attach the composition

$$\underline{\Gamma}_{Z_1\setminus Z'_1} \xrightarrow{\gamma_{Z_1\setminus Z'_1, Z_2\setminus Z'_1}} \underline{\Gamma}_{Z_2\setminus Z'_1} \xrightarrow{\gamma_{Z_2\setminus Z'_2, Z_2\setminus Z'_1}} \underline{\Gamma}_{Z_2\setminus Z'_2} \tag{11.2.16}$$

(notation of (11.2.6)). We then define the functor

$$\underline{\Gamma}_{\Phi|\Phi'} := \operatorname*{colim}_{(Z,Z')\in\Phi|\Phi'} \underline{\Gamma}_{Z\setminus Z'} : \mathscr{A}\text{-}\mathbf{Mod} \to \mathscr{A}\text{-}\mathbf{Mod}$$

where the transition maps in the colimit are the maps (11.2.16), whence a derived functor

$$R\underline{\Gamma}_{\Phi|\Phi'} : \mathsf{D}^+(\mathscr{A}\text{-}\mathbf{Mod}) \to \mathsf{D}^+(\mathscr{A}\text{-}\mathbf{Mod})$$

and we have a natural isomorphism of functors on $\mathscr{A}$-$\mathbf{Mod}$:

$$R^i\underline{\Gamma}_{\Phi|\Phi'} \xrightarrow{\sim} \operatorname*{colim}_{(Z,Z')\in\Phi|\Phi'} R^i\underline{\Gamma}_{Z\setminus Z'} \qquad \text{for every } i \in \mathbb{N}.$$

11.2.17. Keep the notation of (11.2.15). As a special case, we may take $\Phi' := \{\varnothing\}$, and we let

$$\underline{\Gamma}_\Phi := \underline{\Gamma}_{\Phi|\{\varnothing\}} : \mathscr{A}\text{-}\mathbf{Mod} \to \mathscr{A}\text{-}\mathbf{Mod}.$$

If Φ and Φ' are arbitrary family of supports, we have maps of partially ordered sets

$$\Phi \leftarrow \Phi|\Phi' \to \Phi' \quad : \quad Z \leftarrow\!\shortmid (Z, Z') \mapsto Z'.$$

Suppose moreover that $\Phi' \subset \Phi$; then both of these maps are cofinal, by lemma 1.5.7(i), and with lemma 11.2.10 and proposition 11.2.12(ii), we deduce a short exact sequence of complexes

$$0 \to \underline{\Gamma}_{\Phi'}\mathscr{I} \to \underline{\Gamma}_\Phi\mathscr{I} \to \underline{\Gamma}_{\Phi|\Phi'}\mathscr{I} \to 0 \tag{11.2.18}$$

for every flabby $\mathscr{A}$-module $\mathscr{I}$, whence a distinguished triangle of functors on $\mathsf{D}^+(\mathscr{A}\text{-}\mathbf{Mod})$:

$$R\underline{\Gamma}_{\Phi'} \to R\underline{\Gamma}_\Phi \to R\underline{\Gamma}_{\Phi|\Phi'} \to R\underline{\Gamma}_{\Phi'}[1].$$

11.2.19. Keep the notation of (11.2.15). Suppose now that X is sober and locally noetherian, and moreover that we are given a *weak codimension function* on X, *i.e.* a map

$$c : X \to \mathbb{Z}$$

such that $c(x) > c(y)$ whenever $x \neq y$ and x is a specialization of y in X. To the weak codimension function c we attach a system $(\Phi^p \mid p \in \mathbb{N})$ of families of supports, by the rule :

$$\Phi^p := \{Z \subset X \mid Z \text{ is closed in } X, \text{and } c(z) \geq p \text{ for every } z \in Z\} \qquad \text{for every } p \in \mathbb{N}.$$

For every $x \in X$, let also $j_x : X(x) \to X$ and $i_x : \{x\} \to X$ be the inclusion maps (notation of definition 8.1.44(iii)). Moreover, let us set $\Phi^p(x) := \{Z \cap X(x) \mid Z \in \Phi^p\}$, for every $p \in \mathbb{Z}$.

Lemma 11.2.20. *In the situation of* (11.2.19)*, the following holds for every $x \in X$:*

(i) *The topological space $X(x)$ is spectral, noetherian and finite dimensional, and the function c is bounded on $X(x)$.*
(ii) *There exists a natural isomorphism of functors on $\mathsf{D}^+(j_x^*\mathscr{A}\text{-}\mathbf{Mod})$:*

$$j_x^* R\underline{\Gamma}_{\Phi^p} \xrightarrow{\sim} R\underline{\Gamma}_{\Phi^p(x)} \qquad \textit{for every } p \in \mathbb{Z}.$$

(iii) *$(\Phi^p(x) \mid p \in \mathbb{Z})$ is the system of families of supports on $X(x)$ attached to the weak codimension function $c_{|X(x)}$.*

Proof. (i): Since the topology of $X(x)$ is induced from that of X, it is clear that $X(x)$ is noetherian, and we have already seen that it is spectral (remark 8.1.45(i)). Especially, $X(x)$ admits finitely many maximal points (remark 8.1.45(iii)); if $\eta_1, \dots, \eta_k$ are these maximal points, set $C := \max(c(\eta_1), \dots, c(\eta_k))$. Then it is clear that the restriction to $X(x)$ of the function c takes its values in the range $C, C+1, \dots, c(x)$, and it follows easily that the length of any descending sequence of points in $(X(x), \leq)$ is bounded by $c(x) - C$.

(ii) follows easily from proposition 11.2.12(v.c), since every closed subset of a locally noetherian topological space is constructible.

(iii): If Z is any closed subset of $X(x)$, denote by $\overline{Z}$ the topological closure of Z in X; since $X(x)$ is a pro-constructible subset of X, the same holds for $Z = X(x) \cap \overline{Z}$. Then $\overline{Z}$ is the set of specializations in X of the points of Z (proposition 8.1.47(i)). Especially, if $c(z) \geq p$ for every $z \in Z$, then $c(z) \geq p$ for every $z \in \overline{Z}$, and then clearly $Z \in \Phi^p(x)$. The assertion follows immediately. □

Proposition 11.2.21. *In the situation of* (11.2.19)*, we have a natural isomorphism of functors*

$$R^i\underline{\Gamma}_{\Phi^p|\Phi^{p+1}} \stackrel{\sim}{\to} \bigoplus_{c(x)=p} j_{x*} \circ R^i\underline{\Gamma}_{\{x\}} \circ j_x^* \qquad \text{for every } p \in \mathbb{Z} \text{ and } i \in \mathbb{N}.$$

Proof. Since the assertion is local on X, we may assume that X is noetherian. We consider first the case where $i = 0$, and we fix $p \in \mathbb{Z}$. Let $Z \in \Phi^p$ be any element; recall that Z has a finite number of irreducible components $Z_1, \dots, Z_k$ (proposition 8.1.4). Let η_i be the generic point of Z_i, so that $c(\eta_i) \geq p$ for every $i = 1, \dots, k$. For every $i, j = 1, \dots, k$ with $i \neq j$, set $Z_{ij} := Z_i \cap Z_j$; then, for every such i, j, every irreducible component of T of Z_{ij} and every $t \in T$, we must have $c(t) \geq c(\eta_T) > c(\eta_i)$, if η_T denotes the generic point of T. We conclude that $Z_{ij} \in \Phi^{p+1}$ for every $i, j = 1, \dots k$ with $i \neq j$. Set also

$$Z^{>p} := \bigcup_{c(\eta_i)>p} Z_i \cup \bigcup_{i\neq j} Z_{ij} \qquad \text{and} \qquad \Psi^{p+1} := \{(Z, Z') \in \Phi^p|\Phi^{p+1} \mid Z^{>p} \subset Z' \subset Z\}.$$

Since the subset Ψ^p is clearly cofinal in $\Phi^p|\Phi^{p+1}$, we deduce a natural isomorphism

$$\underline{\Gamma}_{\Phi^p|\Phi^{p+1}} \stackrel{\sim}{\to} \operatorname*{colim}_{(Z,Z')\in\Psi^p} \underline{\Gamma}_{Z\setminus Z'}.$$

However, say that $(Z, Z') \in \Psi^p$; we notice that the closed subset $V := Z\setminus Z'$ of $X\setminus Z'$ is the disjoint union of its irreducible components $V_1, \dots, V_t$, hence the topology induced by X on V is the coproduct (*i.e.* disjoint union) of the topologies induced by X on $V_1, \dots, V_t$, and there follows a natural isomorphism

$$\underline{\Gamma}_V \stackrel{\sim}{\to} \bigoplus_{i=1}^t \underline{\Gamma}_{V_i}. \tag{11.2.22}$$

Moreover, say that $\underline{Z}_1, \underline{Z}_2 \in \Psi^p$ and $\underline{Z}_1 \leq \underline{Z}_2$, and denote by $V_1, \dots, V_s$ (resp. $W_1, \dots, W_t$) the irreducible components of $V := Z_1\setminus Z_1'$ (resp. of $W := Z_2\setminus Z_2'$). A simple inspection of the definitions reveals that $s \leq t$ and – after reordering of the W_i – the subset W_i is open in V_i for every $i = 1, \dots, s$. Moreover, under the isomorphism (11.2.22), the transition map $\underline{\Gamma}_V \to \underline{\Gamma}_W$ is identified with the composition

$$\bigoplus_{i=1}^s \underline{\Gamma}_{V_i} \xrightarrow{\bigoplus_{i=1}^s \gamma_{V_i,W_i}} \bigoplus_{i=1}^s \underline{\Gamma}_{W_i} \to \bigoplus_{i=1}^t \underline{\Gamma}_{W_i}$$

(where the second map is the natural monomorphism). Hence, let Δ^p be the set of all locally closed subsets V of X fulfilling the following condition. There exists $x \in X$ with $c(x) = p$, and V is a non-empty open subset of the topological closure $\overline{\{x\}}$ of $\{x\}$ in X. We endow Δ^p with a partial ordering, by declaring that, for any $V, V' \in \Delta^p$ we have $V \leq V'$ if $V' \subset V$ (notice that Δ^p is usually not a filtered partially ordered set); from the foregoing, we get a natural isomorphism

$$\underline{\Gamma}_{\Phi|\Phi} \xrightarrow{\sim} \operatorname*{colim}_{V \in \Delta^p} \underline{\Gamma}_V$$

where, for every $V, V' \in \Delta^p$ such that $V \leq V'$, the transition map $\underline{\Gamma}_V \to \underline{\Gamma}_{V'}$ is again given by the transformation $\gamma_{V,V'}$ corresponding to the open immersion $V' \to V$.

Next, for every $x \in X$ such that $c(x) = p$, set $\Delta_x := \{V \in \Delta^p \mid x \in V\}$, and notice that the partially ordered set Δ^p is the coproduct (*i.e.* the disjoint union) of its partially ordered subsets Δ_x. We deduce a natural isomorphism

$$\underline{\Gamma}_{\Phi|\Phi} \xrightarrow{\sim} \bigoplus_{c(x)=p} \operatorname*{colim}_{V \in \Delta_x} \underline{\Gamma}_V.$$

Thus, we come down to checking the following :

Claim 11.2.23. For every $p \in \mathbb{Z}$ and every $x \in X$ such that $c(x) = p$, there exists a natural isomorphism of functors

$$\operatorname*{colim}_{V \in \Delta^p_x} \underline{\Gamma}_V \xrightarrow{\sim} j_{x*} \circ \underline{\Gamma}_{\{x\}} \circ j^*_x.$$

Proof of the claim. Let Σ_x be the set of all quasi-compact open neighborhoods of x in X, and endow Σ_x with the partial ordering such that, for every $U, U' \in \Sigma_x$ we have $U \leq U'$ if $U' \subset U$; then Σ_x is filtered, and there follows a morphism of partially ordered sets

$$\Sigma_x \to \Delta_x \qquad U \mapsto U \cap \overline{\{x\}}$$

and using lemma 1.5.7(i), it is easily seen that this map is cofinal. Now, for every $U \in \Sigma_x$, let $j_U : U \to X$ be the open immersion; we have a natural identification

$$\underline{\Gamma}_{U \cap \overline{\{x\}}} \xrightarrow{\sim} j_{U*} \circ j^*_U \circ \underline{\Gamma}_{\overline{\{x\}}}$$

whence a natural isomorphism

$$\operatorname*{colim}_{V \in \Delta_x} \underline{\Gamma}_V = \operatorname*{colim}_{U \in \Sigma_x} j_{U*} \circ j^*_U \circ \underline{\Gamma}_{\overline{\{x\}}}.$$

However, $X(x)$ is the limit of the filtered system of spectral spaces Σ_x, so the claim follows easily from proposition 10.1.10(ii) (details left to the reader). ◊

Next, let $i \in \mathbb{N}$ be arbitrary, let $\mathscr{F}$ be any $\mathscr{A}$-module, and pick any resolution $\mathscr{F} \to \mathscr{I}^\bullet$ consisting of injective $\mathscr{A}$-modules; by the foregoing case, we get a natural isomorphism of $\mathscr{A}$-modules

$$R^i\underline{\Gamma}_{\Phi^p|\Phi^{p+1}}\mathscr{F} \xrightarrow{\sim} \bigoplus_{c(x)=p} H^i(j_{x*} \circ \underline{\Gamma}_{\{x\}} \circ j^*_x \mathscr{I}^\bullet).$$

However, on the one hand, $j^*\mathscr{I}^\bullet$ is a complex of qc-flabby $j^*\mathscr{A}$-modules (claim 10.1.11); on the other hand, qc-flabby abelian sheaves are acyclic for the functor $\underline{\Gamma}_{\{x\}}$ (lemmata 10.4.4(iii.b) and 8.1.15(vi)), so we get a natural isomorphism

$$H^i(j_{x*} \circ \underline{\Gamma}_{\{x\}} \circ j^*_x \mathscr{I}^\bullet) \xrightarrow{\sim} H^i(j_{x*} \circ R\underline{\Gamma}_{\{x\}} \circ j^*_x \mathscr{F}) \qquad \text{for every } i \in \mathbb{N} \text{ and every } x \in X.$$

Lastly, notice that $R\underline{\Gamma}_{\{x\}} \circ j^*_x \mathscr{F}$ is a complex of $j^*\mathscr{A}$-modules supported on $\{x\}$, and the functor j_{x*} is exact on the full subcategory of $j^*\mathscr{A}$-**Mod** consisting of such sheaves; thus, we have a natural isomorphism

$$H^i(j_{x*} \circ R\underline{\Gamma}_{\{x\}} \circ j^*_x \mathscr{F}) \xrightarrow{\sim} j_{x*} \circ R^i\underline{\Gamma}_{\{x\}} \circ j^*_x \mathscr{F} \qquad \text{for every } i \in \mathbb{N} \text{ and every } x \in X.$$

Summing up, we obtain the proposition. □

Definition 11.2.24. Let $(X, \mathscr{A})$ be a ringed topological space, with X sober and locally noetherian, and c a weak codimension function on X.

(i) We say that an object $K^\bullet$ of $\mathsf{D}^+(\mathscr{A}\text{-}\mathbf{Mod})$ is a *c-Cohen-Macaulay complex*, if we have

$$R^i\underline{\Gamma}_{\{x\}}K^\bullet_{|X(x)} = 0 \qquad \text{for every } x \in X \text{ and every integer } i \neq c(x).$$

We denote by

$$\mathscr{A}\text{-}\mathsf{CM}^c$$

the full subcategory of $\mathsf{D}^+(\mathscr{A}\text{-}\mathbf{Mod})$ whose objects are the c-Cohen-Macaulay complexes.

(ii) We say that an object $C^\bullet$ of $\mathsf{C}^+(\mathscr{A}\text{-}\mathbf{Mod})$ is a *Cousin complex*, if there exists a family $(M_x \mid x \in X)$ where M_x is an $i_x^*\mathscr{A}$-module for every $x \in X$ (notation of (11.2.19)), such that

$$C^p = \bigoplus_{c(x)=p} i_{x*}M_x \qquad \text{for every } p \in \mathbb{Z}.$$

The full subcategory of $\mathsf{C}^+(\mathscr{A}\text{-}\mathbf{Mod})$ whose objects are the Cousin complexes is denoted

$$\mathscr{A}\text{-}\mathbf{Cousin}.$$

Lemma 11.2.25. *In the situation of definition* 11.2.24*, we have :*

(i) $R\underline{\Gamma}_{\{x\}} \circ j_x^* \circ i_{y*} = 0$ *for every* $x, y \in X$ *such that* $x \neq y$.

(ii) *If* $C^\bullet$ *is a Cousin complex of* $\mathscr{A}$*-modules, then* C^p *is qc-flabby, for every* $p \in \mathbb{Z}$.

Proof. (i): Notice that any $i_y^*\mathscr{A}$-module M is trivially qc-flabby, so the same holds for $i_{y*}M$ (lemma 10.1.5(iv)), and therefore also for $j_x^*j_{y*}M$ (claim 10.1.11). Since qc-flabby $\mathscr{A}$-modules are acyclic for $\underline{\Gamma}_{\{x\}}$ (lemma 10.4.4(iii.a)), we conclude that $R\underline{\Gamma}_{\{x\}}j_x^*i_{y*}$ is the derived functor of $\underline{\Gamma}_{\{x\}}j_x^*i_{y*}$; but it is easily seen that the latter vanishes, whence the assertion.

(ii): We have just seen that i_*M is qc-flabby, for every $i_x^*\mathscr{A}$-module M; on the other hand, all direct sums of qc-flabby $\mathscr{A}$-modules are qc-flabby (claim 10.1.11), whence the contention. $\square$

11.2.26. As j_x^* and $R^i\underline{\Gamma}_{\{x\}}$ commute with all direct sums (proposition 11.2.12(v.b)), lemma 11.2.25(i) implies that the localization $\mathsf{C}^+(\mathscr{A}\text{-}\mathbf{Mod}) \to \mathsf{D}^+(\mathscr{A}\text{-}\mathbf{Mod})$ restricts to a functor

$$F : \mathscr{A}\text{-}\mathbf{Cousin} \to \mathscr{A}\text{-}\mathsf{CM}^c.$$

Theorem 11.2.27. *The functor F of* (11.2.26) *is an equivalence of categories.*

Proof. We construct a quasi-inverse to F, as follows. First, we endow every object $C^\bullet$ of $\mathsf{C}(\mathscr{A}\text{-}\mathbf{Mod})$, with a descending filtration $\mathrm{Fil}^\bullet_\Phi C^\bullet$, by the rule

$$\mathrm{Fil}^p_\Phi C^\bullet := \underline{\Gamma}_{\Phi^p} C^\bullet$$

where $(\Phi^p \mid p \in \mathbb{Z})$ is the system of families of supports associated with the weak codimension function c, as in (11.2.19). We also denote by $\mathrm{gr}^\bullet_\Phi C^\bullet$ the resulting associated graded object of $\mathsf{C}(\mathscr{A}\text{-}\mathbf{Mod})$. Next, for every object $K^\bullet$ of $\mathsf{D}^+(\mathscr{A}\text{-}\mathbf{Mod})$ we choose a quasi-isomorphism $\beta^\bullet_K : K^\bullet \xrightarrow{\sim} \mathscr{I}^\bullet_K$, with $\mathscr{I}^\bullet_K$ a bounded below complex of injective $\mathscr{A}$-modules. Following (7.2.14), we denote $E(\mathscr{I}_K)^{\bullet\bullet}_\bullet$ the spectral sequence arising from the filtered complex $(\mathscr{I}^\bullet_K, \mathrm{Fil}^\bullet_\Phi \mathscr{I}^\bullet_K)$. In light of (11.2.18) we have a natural identification

$$\mathrm{gr}^p_\Phi \mathscr{I}^\bullet_K \xrightarrow{\sim} \underline{\Gamma}_{\Phi^p|\Phi^{p+1}} \mathscr{I}^\bullet_K$$

and combining with remark 7.2.16(ii), we get a natural isomorphism

$$E(\mathscr{I}_K)^{pq}_1 \xrightarrow{\sim} R^{p+q}\underline{\Gamma}_{\Phi^p|\Phi^{p+1}} K^\bullet.$$

Taking into account proposition 11.2.21 we deduce that

$$G(K)^\bullet := (E(\mathscr{I}_K)^{\bullet 0}_1, d(\mathscr{I}_K)^{\bullet 0}_1)$$

is a Cousin complex, for every $q \in \mathbb{Z}$. Moreover, every morphism $\varphi^\bullet : K^\bullet \to L^\bullet$ in the category $\mathsf{D}^+(\mathscr{A}\text{-}\mathbf{Mod})$ can be represented by a unique morphism $\mathscr{I}^\bullet_\varphi : \mathscr{I}^\bullet_K \to \mathscr{I}^\bullet_L$ in $\mathsf{Hot}(\mathscr{A}\text{-}\mathbf{Mod})$

(notation of definition 7.1.4(ii)), and notice that $\mathscr{I}^\bullet_\varphi$ is a morphism of filtered complexes; in light of (7.2.23) we deduce a well defined morphism $G(\varphi)^\bullet : G(K)^\bullet \to G(L)^\bullet$. In this way, we obtain a functor

$$G : \mathsf{D}^+(\mathscr{A}\text{-}\mathbf{Mod}) \to \mathscr{A}\text{-}\mathbf{Cousin}$$

and we claim that the restriction of G to the subcategory $\mathscr{A}\text{-}\mathsf{CM}^c$ is the sought quasi-inverse.

We check first that $G \circ F$ is naturally isomorphic to the identity endofunctor of $\mathscr{A}\text{-}\mathbf{Cousin}$. To this aim, let us remark :

Claim 11.2.28. Let $K^\bullet$ be any c-Cohen-Macaulay complex. We have :

(i) $E(\mathscr{I}_K)^{pq}_1 = 0$ for every $p, q \in \mathbb{Z}$ with $q \neq 0$.

(ii) If $K^\bullet = FC^\bullet$ for a Cousin complex $C^\bullet$, the map $\beta^\bullet_K$ induces a natural isomorphism

$$\mathrm{gr}^p_\Phi C^\bullet \xrightarrow{\sim} E(\mathscr{I}_K)^{p0}_1[-p] \xrightarrow{\sim} C^p[-p] \qquad \text{for every } p \in \mathbb{Z}.$$

Proof of the claim. (i): Indeed, from proposition 11.2.21 we get a natural isomorphism

$$H^{p+q}(\mathrm{gr}^p_\Phi \mathscr{I}^\bullet_C) \xrightarrow{\sim} \bigoplus_{c(x)=p} j_{x*} \circ R^{p+q}\underline{\Gamma}_{\{x\}} K^\bullet_{|X(x)} \tag{11.2.29}$$

which vanishes for $q \neq 0$, since $K^\bullet$ is c-Cohen-Macaulay.

(ii): Let $C^\bullet$ and $(M_x \mid x \in X)$ be as in definition 11.2.24(ii); from (11.2.18) and lemma 11.2.25(ii), it follows that the natural map

$$\mathrm{gr}^\bullet_\Phi C^\bullet \to \underline{\Gamma}_{\Phi^p|\Phi^{p+1}} C^\bullet$$

is an isomorphism in $\mathsf{C}^+(\mathscr{A}\text{-}\mathbf{Mod})$. Also, combining with claim 10.1.11, we see that $C^\bullet_{|X(x)}$ is a complex of qc-flabby $j^*_x\mathscr{A}$-modules. By lemma 10.4.4(iii.a) we deduce that the natural map $\underline{\Gamma}_{\{x\}} C^\bullet_{|X(x)} \to R\underline{\Gamma}_{\{x\}} C^\bullet_{|X(x)}$ is an isomorphism in $\mathsf{D}^+(\mathscr{A}\text{-}\mathbf{Mod})$, for every $x \in X$. Moreover, since the functor $\underline{\Gamma}_{\{x\}}$ commutes with arbitrary direct sums (lemma 10.4.4(iii.b)), lemma 11.2.25(i) says that

$$\underline{\Gamma}_{\{x\}} C^\bullet_{|X(x)} = j^*_x i_{x*} M_x[-c(x)] \qquad \text{for every } x \in X.$$

Taking into account (11.2.29) (and again, proposition 11.2.21), the assertion follows. ◊

Now, let $C^\bullet$ be a cousin complex, and set $D^\bullet := GF(C)^\bullet$. From claim 11.2.28 we obtain a natural isomorphism

$$\mathrm{gr}^p_\Phi \mathscr{I}^\bullet_C \xrightarrow{\sim} C^p[-p] \qquad \text{for every } p \in \mathbb{Z}.$$

We need to show that, under this identification, the differential d^p_D of $D^p := H^p(\mathrm{gr}^p \mathscr{I}^\bullet_C)$ corresponds to the differential d^p_C of $C^\bullet$. However, on the one hand we have a commutative ladder

$$\begin{array}{ccccccccc}
0 & \to & \mathrm{gr}^{p+1}_\Phi C^\bullet & \to & \mathrm{Fil}^p_\Phi C^\bullet/\mathrm{Fil}^{p+2}_\Phi C^\bullet & \to & \mathrm{gr}^p_\Phi C^\bullet & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathrm{gr}^{p+1}_\Phi \mathscr{I}^\bullet_C & \to & \mathrm{Fil}^p_\Phi \mathscr{I}^\bullet_C/\mathrm{Fil}^{p+2}_\Phi \mathscr{I}^\bullet_C & \to & \mathrm{gr}^p_\Phi \mathscr{I}^\bullet_C & \to & 0
\end{array}$$

and claim 11.2.28(ii) says that both the first and third vertical arrows are isomorphisms, so the top row of is isomorphic to the bottom row. Moreover, we see that the top row is the natural short exact sequence

$$0 \to C^{p+1}[-p-1] \to t^{\leq p+1} t^{\geq p} C^\bullet \to C^p[-p] \to 0$$

where $t^{\leq p+1}$ and $t^{\geq p}$ are the brutal truncation functors of (7.1.1). Then it is clear that the boundary map $H^p(C^p[-p]) \to H^{p+1}(C^{p+1}[-p-1])$ in degree p associated with this short exact sequence, is none else than the differential d^p_C. Now the contention follows from the naturality of the boundary map, together with remark 7.2.16(ii).

Next, let $K^\bullet$ be any c-Cohen-Macaulay complex, and set

$$(\mathscr{J}_K, \mathrm{Fil}^\bullet \mathscr{J}_K) := \mathrm{Déc}\,(\mathscr{I}_K, \mathrm{Fil}^\bullet_\Phi \mathscr{I}^\bullet_K)$$

(notation of (7.2.24)). By proposition 7.2.26(i,ii) and claim 11.2.28(i), we have a natural epimorphism

$$u^\bullet : E(\mathscr{J}_K)^\bullet_0 \to E(\mathscr{I}_K)^\bullet_1 \qquad \text{in } \mathsf{C}^+(\mathscr{A}\text{-}\mathbf{Mod})$$

which induces an isomorphism in $\mathsf{D}^+(\mathscr{A}\text{-}\mathbf{Mod})$. Moreover, from claim 11.2.28(i) we see that the subcomplex $(E(\mathscr{J}_K)^{p,n-p}_0 \mid n \in \mathbb{Z})$ lies in $\mathrm{Ker}\, u^\bullet$, for every $p \neq 0$, so $u^\bullet$ restricts to an isomorphism

$$E(\mathscr{J}_K)^{0,\bullet}_0 = \mathrm{gr}^0 \mathscr{J}^\bullet_K \xrightarrow{\sim} G(K)^\bullet \qquad \text{in } \mathsf{D}^+(\mathscr{A}\text{-}\mathbf{Mod}) \tag{11.2.30}$$

(see remark 7.2.16(i)). We point out :

Claim 11.2.31. $\mathrm{Fil}^1 \mathscr{J}^\bullet_K$ and $\mathscr{J}^\bullet_K/\mathrm{Fil}^0 \mathscr{J}^\bullet_K$ are acyclic.

Proof of the claim. The assertion can be checked on the stalks over the points of X; taking into account lemma 11.2.20(ii,iii), we may then replace X by $X(x)$, for any point $x \in X$, and assume from start that c is bounded on X (lemma 11.2.20(i)). In this case, it is clear that the filtration $\mathrm{Fil}^\bullet_\Phi \mathscr{I}^\bullet_K$ is finite, for every complex $K^\bullet$ of $\mathscr{A}$-modules, and then the same holds for the filtration $\mathrm{Fil}^\bullet \mathscr{J}^\bullet_K$ (see (7.2.24)). In this situation, a simple induction argument reduces to checking that the associated subquotients $\mathrm{gr}^p \mathscr{J}^\bullet_K$ are acyclic, for every $p \neq 0$. However, combining remarks 7.2.9(i) and 7.2.16(ii), proposition 7.2.26(ii) and claim 11.2.28(i) we may compute

$$H^q \mathrm{gr}^p \mathscr{J}^\bullet_K \xrightarrow{\sim} E(\mathscr{J}_K)^{p,q-p}_1 \xrightarrow{\sim} E(\mathscr{I}_K)^{p+q,-p}_2 = 0 \qquad \text{for every } p, q \in \mathbb{Z} \text{ with } p \neq 0$$

whence the claim. ◊

Claim 11.2.31 yields a natural isomorphism

$$\mathrm{gr}^0 \mathscr{J}^\bullet_K \xrightarrow{\sim} \mathscr{J}^\bullet_K = \mathscr{I}^\bullet_K \qquad \text{in } \mathsf{D}^+(\mathscr{A}\text{-}\mathbf{Mod})$$

and combining with (11.2.30), we finally get an isomorphism

$$K^\bullet \xrightarrow{\sim} FG(K)^\bullet \qquad \text{in } \mathsf{D}^+(\mathscr{A}\text{-}\mathbf{Mod}).$$

Now the theorem follows from proposition 1.1.20(i). □

11.3. **Duality over coherent schemes.** If X is a locally coherent scheme (definition 8.1.58(i)) and $\mathscr{F}$ is an $\mathscr{O}_X$-module, we define the *dual $\mathscr{O}_X$-module*

$$\mathscr{F}^\vee := \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{O}_X).$$

Moreover, for every morphism of $\mathscr{O}_X$-modules $\varphi : \mathscr{F} \to \mathscr{G}$, we denote by $\varphi^\vee : \mathscr{G}^\vee \to \mathscr{F}^\vee$ the induced (transpose) morphism. As usual, there is a natural morphism of $\mathscr{O}_X$-modules:

$$\beta_\mathscr{F} : \mathscr{F} \to \mathscr{F}^{\vee\vee}.$$

Definition 11.3.1. Let X be a locally coherent scheme, and $\mathscr{F}$ an $\mathscr{O}_X$-module.

(i) We say that $\mathscr{F}$ is *reflexive at a point* $x \in X$ if there exists an open neighborhood $U \subset X$ of x such that $\mathscr{F}_{|U}$ is a coherent $\mathscr{O}_U$-module, and $\beta_{\mathscr{F}|U}$ is an isomorphism.
(ii) We say that $\mathscr{F}$ is *reflexive* if it is reflexive at all points of X.
(iii) We denote by $\mathscr{O}_X$-$\mathbf{Rflx}$ the full subcategory of the category $\mathscr{O}_X$-$\mathbf{Mod}$, consisting of all the reflexive $\mathscr{O}_X$-modules. It contains $\mathscr{O}_X$-$\mathbf{Mod}_{\mathrm{lfft}}$ as a full subcategory (see (10.3)).
(iv) Suppose that $X = \mathrm{Spec}\, R$ for a coherent ring R, and $\mathscr{F} = M^\sim$ for an R-module M. Then we say that M is a *reflexive* R-module if $\mathscr{F}$ is a reflexive $\mathscr{O}_X$-module.

Lemma 11.3.2. *Let X be a locally coherent scheme, $x \in X$ any point, and $\mathscr{F}$ a coherent $\mathscr{O}_X$-module. We have :*

(i) $\mathscr{F}^\vee$ *is a coherent* $\mathscr{O}_X$*-module.*
(ii) *The following conditions are equivalent:*
  (a) $\mathscr{F}$ *is reflexive at the point* x.
  (b) $\mathscr{F}_x$ *is a reflexive* $\mathscr{O}_{X,x}$*-module.*
  (c) *The map* $\beta_{\mathscr{F},x} : \mathscr{F}_x \to (\mathscr{F}^{\vee\vee})_x$ *is an isomorphism.*

Proof. (i): The assertion is local on X, hence we may assume that there exists a presentation $\mathscr{O}_X^{\oplus n} \to \mathscr{O}_X^{\oplus m} \to \mathscr{F} \to 0$, whence – after taking duals – a left exact sequence $0 \to \mathscr{F}^\vee \to \mathscr{O}_X^{\oplus m} \to \mathscr{O}_X^{\oplus n}$. Since $\mathscr{O}_X$ is coherent, the assertion follows.
(ii) shall be left to the reader. □

Lemma 11.3.3. *Let* X *be a reduced locally coherent scheme, and* $\mathscr{F}$ *a quasi-coherent* $\mathscr{O}_X$*-module of finite type. Endow the set* $\operatorname{Max} X$ *of maximal points of* X *with the topology induced by* X*, and consider the following conditions :*
(a) *There exists a quasi-compact open immersion* $j : U \to X$ *with dense image, such that* $j^*\mathscr{F}$ *is a locally free* $\mathscr{O}_U$*-module.*
(b) *There exists a quasi-compact open immersion* $j : U \to X$ *with dense image, such that* $j^*\mathscr{F}$ *is a coherent* $\mathscr{O}_U$*-module.*
(c) *The rank function of* $\mathscr{F}$ *:*
$$X \to \mathbb{N} \qquad x \mapsto \dim_{\kappa(x)} \kappa(x) \otimes_{\mathscr{O}_{X,x}} \mathscr{F}_x$$
restricts to a locally constant function on $\operatorname{Max} X$.
(d) $\mathscr{F}^\vee$ *is a reflexive* $\mathscr{O}_X$*-module and* $\beta^\vee_{\mathscr{F}} \circ \beta_{\mathscr{F}^\vee} = \mathbf{1}_{\mathscr{F}^\vee}$.

Then (a)⇒(b)⇒(c)⇒(d) *and if* X *is coherent we have* (c)⇒(a) *as well.*

Proof. Let us check first that (c)⇒(a) in case X is coherent. Indeed, notice that j is quasi-compact if and only if U is retro-compact in X, and the latter holds if and only if U is open and constructible in X (lemma 8.1.19(iv.b,v.c)). It follows that the existence of j with the sought properties can be checked locally on X, hence we may assume that X is affine. By proposition 10.3.34(i) we may find a coherent $\mathscr{O}_X$-module $\mathscr{G}$ and an $\mathscr{O}_X$-linear epimorphism $\varphi : \mathscr{G} \to \mathscr{F}$. Now, fix any $\eta \in \operatorname{Max} X$ and write $\mathscr{K} := \operatorname{Ker} \varphi$ as the filtered union of the system $(\mathscr{K}_i \mid i \in I)$ of its quasi-coherent $\mathscr{O}_X$-submodules of finite type; since $\mathscr{O}_{X,\eta}$ is a field, there exists $i \in I$ such that $\mathscr{K}_{i,\eta} = \mathscr{K}_\eta$. Thus, after replacing $\mathscr{G}$ by $\mathscr{G}/\mathscr{K}_i$, we may assume that φ_η is an isomorphism. Moreover, by [126, Th.4.10] we may find an affine open neighborhood U_η of η in X, an integer $r \in \mathbb{N}$ and an isomorphism $\mathscr{G}_{|U_\eta} \stackrel{\sim}{\to} \mathscr{O}_{U_\eta}^{\oplus r}$, and by assumption (c) we may also assume that the rank function of $\mathscr{F}$ is constant on $\operatorname{Max} U_\eta$. In this case, φ_τ is an isomorphism for every $\tau \in \operatorname{Max} U_\eta$, so $\mathscr{K}_\tau = 0$ for every such point τ. Since X is reduced and $\mathscr{G}_{|U_\eta}$ is a locally free $\mathscr{O}_{U_\eta}$-module, it follows that $\mathscr{K}_{|U_\eta} = 0$. By proposition 8.1.59, we may then find a finite subset $\Sigma \subset \operatorname{Max} X$ such that $\bigcup_{\eta\in\Sigma} \operatorname{Max} U_\eta = \operatorname{Max} X$, and it is easily seen that the open subset $U := \bigcup_{\eta\in\Sigma} U_\eta$ fulfills condition (a).

(a)⇒(b) is obvious.

(b)⇒(c): Notice first that any U as in (b) contains $\operatorname{Max} X$, by proposition 8.1.47(iii). Then, since X is reduced, [126, Th.4.10] implies that, for every $\eta \in \operatorname{Max} X$, we may find an affine open neighborhood U_η of η in X such that $\mathscr{F}_{|U_\eta}$ is a locally free $\mathscr{O}_{U_\eta}$-module, whence (c).

(c)⇒(d): Let us show first that $\mathscr{F}^\vee$ is coherent, under condition (c). The assertion is local on X, hence we may assume that X is affine, in which case, as we have already seen, (c) implies that there exists a quasi-compact open immersion $j : U \to X$ fulfilling condition (a), and we also know that any such U contains $\operatorname{Max} X$. By proposition 10.3.34(i), we may then find a coherent $\mathscr{O}_X$-module $\mathscr{G}$ and an $\mathscr{O}_X$-linear epimorphism $\varphi : \mathscr{G} \to \mathscr{F}$ such that $j^*\varphi$ is an isomorphism. There follows a left exact sequence $0 \to \mathscr{F}^\vee \to \mathscr{G}^\vee \to (\operatorname{Ker} \varphi)^\vee$, and $\mathscr{G}^\vee$ is coherent, by lemma 11.3.2(i). Since $\operatorname{Max} X \subset U$, we have $(\operatorname{Ker} \varphi)_\eta = 0$ for every $\eta \in \operatorname{Max} X$, and since $\mathscr{O}_X$ is reduced, it follows easily that $(\mathrm{Ker}\,\varphi)^\vee = 0$, and the claim follows. Now, since X is reduced, the only associated points of $\mathscr{O}_X$ are the maximal points of X (*i.e.* $\mathscr{O}_X$ has no imbedded points). It follows easily that, for every quasi-coherent $\mathscr{O}_X$-module $\mathscr{G}$, the dual $\mathscr{G}^\vee$ satisfies condition S_1 (see definition 10.5.1(ii)). Next, rather generally, let $\mathscr{M}$ be any $\mathscr{O}_X$-module; directly from the definitions one derives the identity:

$$\beta^\vee_{\mathscr{M}} \circ \beta_{\mathscr{M}^\vee} = \mathbf{1}_{\mathscr{M}^\vee}. \tag{11.3.4}$$

It remains therefore only to show that $\beta^\vee_{\mathscr{F}}$ is a right inverse for $\beta_{\mathscr{F}^\vee}$ when $\mathscr{F}^\vee$ is coherent. Since $\mathscr{F}^{\vee\vee\vee}$ satisfies condition S_1, it suffices to check that, for every maximal point ξ, the induced map on stalks

$$\beta^\vee_{\mathscr{F},\xi} : \mathscr{F}^{\vee\vee\vee}_\xi \to \mathscr{F}^\vee_\xi$$

is a right inverse for $\beta_{\mathscr{F}^\vee,\xi}$. However, since $\mathscr{O}_{X,\xi}$ is a field, $\beta^\vee_{\mathscr{F},\xi}$ is a linear map of $\mathscr{O}_{X,\xi}$-vector spaces of the same finite dimension, hence it is an isomorphism, in view of (11.3.4). □

Remark 11.3.5. The following observation is often useful. Suppose that X is a locally coherent scheme, $\mathscr{F}$ an $\mathscr{O}_X$-module, reflexive at a given point $x \in X$. We can then choose a presentation $\mathscr{O}^{\oplus n}_{X,x} \to \mathscr{O}^{\oplus m}_{X,x} \to \mathscr{F}^\vee_x \to 0$, and after dualizing we deduce a left exact sequence

$$0 \to \mathscr{F}_x \to \mathscr{O}^{\oplus m}_{X,x} \xrightarrow{u} \mathscr{O}^{\oplus n}_{X,x}. \tag{11.3.6}$$

Especially, if $\mathscr{O}_{X,x}$ satisfies condition S_1 (in the sense of definition 10.5.1(iii)), then the same holds for $\mathscr{F}_x$. For the converse, suppose additionally that X is reduced, and let $x \in X$ be a point for which there exists a left exact sequence such as (11.3.6); then lemma 11.3.3 says that $\mathscr{F}$ is reflexive at x : indeed, $\mathscr{F}_x \simeq (\mathrm{Coker}\,u^\vee)^\vee$.

Lemma 11.3.7. (i) *Let $f : X \to Y$ be a flat morphism of locally coherent schemes. The induced functor $\mathscr{O}_Y$-$\mathbf{Mod} \to \mathscr{O}_X$-$\mathbf{Mod}$ restricts to a functor*

$$f^* : \mathscr{O}_Y\text{-}\mathbf{Rflx} \to \mathscr{O}_X\text{-}\mathbf{Rflx}.$$

- (ii) *Let X_0 be a quasi-compact and quasi-separated scheme, $(X_\lambda \mid \lambda \in \Lambda)$ a cofiltered family of coherent X_0-schemes with flat transition morphisms $\psi_{\lambda\mu} : X_\lambda \to X_\mu$ such that the structure morphisms $X_\lambda \to X_0$ are affine, and set $X := \lim_{\lambda\in\Lambda} X_\lambda$. Then:*
  - (a) *X is coherent.*
  - (b) *the natural functor: $2\text{-}\mathop{\mathrm{colim}}_{\lambda\in\Lambda^o} \mathscr{O}_{X_\lambda}$-$\mathbf{Rflx} \to \mathscr{O}_X$-$\mathbf{Rflx}$ is an equivalence.*
- (iii) *Suppose that f is surjective, and let $\mathscr{F}$ be any coherent $\mathscr{O}_Y$-module. Then $\mathscr{F}$ is reflexive if and only if $f^*\mathscr{F}$ is a reflexive $\mathscr{O}_X$-module.*

Proof. (i) follows easily from [75, Lemma 2.4.29(i.a)].

(ii): For every $\lambda \in \Lambda$, denote by $\psi_\lambda : X \to X_\lambda$ the natural morphism. Let $U \subset X$ be a quasi-compact open subset, $u : \mathscr{O}^{\oplus n}_U \to \mathscr{O}_U$ any morphism of $\mathscr{O}_U$-modules; by corollary 8.1.40(ii.a) there exist $\lambda \in \Lambda$ and a quasi-compact open subset $U_\lambda \subset X_\lambda$ such that $U = \psi^{-1}(U_\lambda)$. By [65, Ch.IV, Th.8.5.2(i)] we may then suppose that u descends to a homomorphism $u_\lambda : \mathscr{O}^{\oplus n}_{U_\lambda} \to \mathscr{O}_{U_\lambda}$, whose kernel is of finite type, since X_λ is coherent. Since the transition morphisms are flat, we have $\mathrm{Ker}\,u = \psi^*_\lambda(\mathrm{Ker}\,u_\lambda)$, whence (ii.a). Next, using [65, Ch.IV, Th.8.5.2] one sees easily that the functor of (ii.b) is fully faithful and moreover, every reflexive $\mathscr{O}_X$-module $\mathscr{F}$ descends to a coherent $\mathscr{O}_{X_\lambda}$-module $\mathscr{F}_\lambda$ for some $\lambda \in \Lambda$. For every $\mu \geq \lambda$ let $\mathscr{F}_\mu := \psi^*_{\mu\lambda}\mathscr{F}_\lambda$; since $\beta_{\mathscr{F}}$ is an isomorphism, *loc.cit.* shows that $\beta_{\mathscr{F}_\mu}$ is already an isomorphism for some $\mu \geq \lambda$, whence (ii.b).

(iii): By virtue of (i), we may assume that $f^*\mathscr{F}$ is reflexive, and we need to show that the same holds for $\mathscr{F}$. However, the natural map $f^*(\mathscr{F}^{\vee\vee}) \to (f^*\mathscr{F})^{\vee\vee}$ is an isomorphism (proposition 10.3.3(ii)), hence $f^*\beta_{\mathscr{F}} = \beta_{f^*\mathscr{F}}$ is an isomorphism. Since f is faithfully flat, we deduce that $\beta_{\mathscr{F}}$ is an isomorphism, as stated. □

Proposition 11.3.8. *Let X be a coherent scheme, $U \subset X$ a quasi-compact open subset. Then:*

(i) *If X is reduced, the restriction functor*

$$\mathscr{O}_X\text{-}\mathbf{Rflx} \to \mathscr{O}_U\text{-}\mathbf{Rflx}$$

is essentially surjective.

(ii) *Let $\mathscr{F}$ be a reflexive $\mathscr{O}_X$-module. If $\delta'(x, \mathscr{O}_X) \geq 2$ for all $x \in X \setminus U$, the natural map*

$$\mathscr{F} \to j_* j^* \mathscr{F}$$

is an isomorphism. Especially, the restriction functor of (i) *is an equivalence.*

Proof. (i): Given a reflexive $\mathscr{O}_U$-module $\mathscr{F}$, lemma 10.3.27(ii) says that we can find a finitely presented quasi-coherent $\mathscr{O}_X$-module $\mathscr{G}$ extending $\mathscr{F}^\vee$; since X is coherent, $\mathscr{G}$ is a coherent $\mathscr{O}_X$-module, hence the same holds for $\mathscr{G}^\vee$, which extends $\mathscr{F}$ and is reflexive in light of lemma 11.3.3.

(ii): Notice that the first assertion easily implies that the restriction functor of (i) is fully faithful, so the second assertion follows from the first together with (i).

Next, since the first assertion is local on X, we can suppose that there exists a left exact sequence $0 \to \mathscr{F} \to \mathscr{O}_X^{\oplus m} \to \mathscr{O}_X^{\oplus n}$ (see remark 11.3.5). Since the functor j_* is left exact, it then suffices to prove the contention for the sheaves $\mathscr{O}_X^{\oplus m}$ and $\mathscr{O}_X^{\oplus n}$, and thus we may assume from start that $\mathscr{F} = \mathscr{O}_X$. Then, since $X \setminus U$ is constructible, corollary 10.4.23 applies and yields the assertion. □

Corollary 11.3.9. *Let X be a locally coherent scheme, $f : X \to S$ a flat, locally finitely presented morphism, $j : U \to X$ a quasi-compact open immersion, $\mathscr{F}$ a reflexive $\mathscr{O}_X$-module. Suppose that*

(a) $\mathrm{depth}_f(x) \geq 1$ *for every point $x \in X \setminus U$, and*
(b) $\mathrm{depth}_f(x) \geq 2$ *for every maximal point η of S and every $x \in (X \setminus U) \cap f^{-1}(\eta)$.*
(c) $\delta'(s, \mathscr{O}_S) > 0$ *for every non-maximal point $s \in S$.*

Then the natural morphism $\mathscr{F} \to j_* j^* \mathscr{F}$ is an isomorphism.

Proof. Since f is flat, corollary 10.4.47 and our assumptions imply that $\delta'(x, \mathscr{O}_X) \geq 2$ for every $x \in X \setminus U$, so the assertion follows from proposition 11.3.8(ii). □

11.3.10. Let X be any scheme. Recall that the *rank* of an $\mathscr{O}_X$-module $\mathscr{F}$ of finite type, is the upper semicontinuous function:

$$\mathrm{rk}\,\mathscr{F} : X \to \mathbb{N} \qquad x \mapsto \dim_{\kappa(x)} \mathscr{F}_x \otimes_{\mathscr{O}_{X,x}} \kappa(x).$$

Clearly, if $\mathscr{F}$ is a is locally free $\mathscr{O}_X$-module of finite type, $\mathrm{rk}\,\mathscr{F}$ is a continuous function on X. The converse holds, provided X is a reduced scheme. Moreover, if $\mathscr{F}$ is of finite presentation, $\mathrm{rk}\,\mathscr{F}$ is a constructible function and there exists a dense open subset $U \subset X$ such that $\mathrm{rk}\,\mathscr{F}$ restricts to a continuous function on U. We denote by $\mathbf{Pic}\,X$ the full subcategory of $\mathscr{O}_X\text{-}\mathbf{Mod}_{\mathrm{lfft}}$ consisting of all the objects whose rank is constant equal to one (*i.e.* the *invertible* $\mathscr{O}_X$-modules). In case X is locally coherent, we shall also consider the category $\mathbf{Div}\,X$ of *generically invertible* $\mathscr{O}_X$-modules, defined as the full subcategory of $\mathscr{O}_X\text{-}\mathbf{Rflx}$ consisting of all objects which are locally free of rank one on a dense open subset of X. If X is locally coherent, $\mathbf{Pic}\,X$ is a full subcategory of $\mathbf{Div}X$.

Remark 11.3.11. (i) Let A be any integral domain, and set $X = \mathrm{Spec}\,A$. Classically, one has a notion of reflexive fractional ideal of A (see [126, p.80] or example 6.4.38). Suppose now that A is also coherent, in which case we have the notion of reflexive A-module of definition 11.3.1(iv). We claim that these two notions overlap on the subclass of reflexive fractional ideals of finite type: more precisely, let $\mathbf{Div}(A)$ be the full subcategory of $A\text{-}\mathbf{Mod}$ whose objects are the finitely generated reflexive fractional ideals of A. Then the essential image of the functor

(11.3.12) $$\mathbf{Div}(A) \to \mathscr{O}_X\text{-}\mathbf{Mod} \qquad M \mapsto M^\sim$$

is the category $\mathbf{Div}\, X$, and (11.3.12) yields an equivalence of $\mathbf{Div}(A)$ with the latter category. Indeed, let $\mathscr{F}$ be any generically invertible $\mathscr{O}_X$-module, and set $I := \mathscr{F}(X)$; if K denotes the field of fractions of A, then $\dim_K I \otimes_A K = 1$. Let us then fix a K-linear isomorphism $I \otimes_A K \xrightarrow{\sim} K$, and notice that the induced A-linear map $I \to K$ is injective, since $\mathscr{F}$ is S_1 (remark 11.3.5). We may then view I as a finitely generated A-submodule of K, and then it is clear that I is a fractional ideal of A. Moreover, on the one hand $\mathscr{F}^\vee$ is the coherent $\mathscr{O}_X$-module $\mathrm{Hom}_A(I, A)^\sim$; on the other hand, the natural map $\mathrm{Hom}_A(I, A) \to \mathrm{Hom}_K(I \otimes_A K, K) = K$ is injective, and its image is the fractional ideal I^{-1} (see (6.4.35)). We easily deduce that I is a reflexive fractional ideal, and conversely it is easily seen that every $\mathscr{O}_X$-module in the essential image of (11.3.12) is generically invertible; since this functor is also obviously fully faithful, the assertion follows.

(ii) Let A be a coherent integral domain, and denote by $\mathrm{coh.Div}(A)$ the set of all coherent reflexive fractional ideals of A. It is easily seen that $\mathrm{coh.Div}(A)$ is a submonoid of $\mathrm{Div}(A)$, for the natural monoid structure introduced in example 6.4.38. More generally, if X is a coherent integral scheme, we may define a sheaf of monoids $\mathscr{D}iv_X$ on X, as follows. First, to any affine open subset $U \subset X$, we assign the monoid $\mathscr{D}iv_X(U) := \mathrm{coh.Div}(\mathscr{O}_X(U))$. For each inclusion $j : U' \subset U$ of affine open subset, notice that the restriction map $\mathscr{O}_X(U) \to \mathscr{O}_X(U')$ is a flat ring homomorphism, hence it gives a flat morphism of monoids $\mathscr{O}_X(U) \setminus \{0\} \to \mathscr{O}_X(U') \setminus \{0\}$. We then have an induced map of monoids $\mathscr{D}iv_X(j) : \mathscr{D}iv_X(U) \to \mathscr{D}iv_X(U')$, by virtue of lemma 6.4.45(iv). It is easily seen that the resulting presheaf $\mathscr{D}iv_X$ is a sheaf on the site of all affine open subsets of X. By [59, Ch.0, §3.2.2], the latter extends uniquely to a sheaf of monoids on X, which we denote again $\mathscr{D}iv_X$. We set

$$\mathrm{Div}(X) := \mathscr{D}iv_X(X).$$

If X is normal and locally noetherian (or more generally, if X is a *Krull scheme*, *i.e.* $\mathscr{O}_X(U)$ is a Krull ring, for every affine open subset $U \subset X$) then $\mathscr{D}iv_X$ is an abelian sheaf, and $\mathrm{Div}(X)$ is an abelian group (proposition 6.4.42(i,iii)).

11.3.13. For future reference, it is useful to recall some preliminaries concerning the determinant functors defined in [119]. Let X be a scheme. We denote by $\mathbf{gr.Pic}\, X$ the category of graded invertible $\mathscr{O}_X$-modules. An object of $\mathbf{gr.Pic}\, X$ is a pair (L, α), where L is an invertible $\mathscr{O}_X$-module and $\alpha : X \to \mathbb{Z}$ is a continuous function. A homomorphism $h : (L, \alpha) \to (M, \beta)$ is a homomorphism of $\mathscr{O}_X$-modules $h : L \to M$ such that $h_x = 0$ for every $x \in X$ with $\alpha(x) \neq \beta(x)$. We denote by $\mathbf{gr.Pic}^* X$ the subcategory of $\mathbf{gr.Pic}\, X$ with the same objects, and whose morphisms are the isomorphisms in $\mathbf{gr.Pic}\, X$. Notice that $\mathbf{gr.Pic}\, X$ is a tensor category : the tensor product of two objects (L, α) and (M, β) is the pair $(L, \alpha) \otimes (M, \beta) := (L \otimes_{\mathscr{O}_X} M, \alpha + \beta)$. We denote by $\mathscr{O}_X\text{-}\mathbf{Mod}^*_{\mathrm{lfft}}$ the category whose objects are the locally free $\mathscr{O}_X$-modules of finite type, and whose morphisms are the $\mathscr{O}_X$-linear isomorphisms. The *determinant* is the functor:

$$\det : \mathscr{O}_X\text{-}\mathbf{Mod}^*_{\mathrm{lfft}} \to \mathbf{gr.Pic}^* X \qquad F \mapsto (\Lambda^{\mathrm{rk}\, F}_{\mathscr{O}_X} F, \mathrm{rk}\, F).$$

Let $\mathsf{D}(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{perf}}$ be the category of perfect complexes of $\mathscr{O}_X$-modules; recall that, by definition, every perfect complex is locally isomorphic to a bounded complex of locally free $\mathscr{O}_X$-modules of finite type. The category $\mathsf{D}(\mathscr{O}_X\text{-}\mathbf{Mod})^*_{\mathrm{perf}}$ is the subcategory of $\mathsf{D}(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{perf}}$ with the same objects, and whose morphisms are the isomorphisms (*i.e.* the quasi-isomorphisms of complexes). The main theorem of chapter 1 of [119] can be stated as follows.

Lemma 11.3.14. *(*[119, Th.1]*) With the notation of* (11.3.13) *there exists, for every scheme X, an extension of the determinant functor to a functor:*

$$\mathsf{det} : \mathsf{D}(\mathscr{O}_X\text{-}\mathbf{Mod})^*_{\mathrm{perf}} \to \mathbf{gr.Pic}\, X.$$

These determinant functors commute with every base change. □

Proposition 11.3.15. *Let X be a regular scheme. Then every reflexive generically invertible $\mathscr{O}_X$-module is invertible.*

Proof. The question is local on X, hence we may assume that X is affine. Let $\mathscr{F}$ be a generically invertible $\mathscr{O}_X$-module, and $U \subset X$ a dense open subset such that $\mathscr{F}_{|U}$ is invertible. Denote by $Z_1, \dots, Z_t$ the irreducible components of $Z := X \setminus U$ whose codimension in X equals one, and for every $i = 1, \dots, t$, let η_i be the maximal point of Z_i, and set $A_i := \mathscr{O}_{X,\eta_i}$. Since $\mathscr{F}$ is S_1 (remark 11.3.5), the stalk $\mathscr{F}_{\eta_i}$ is a torsion-free A_i-module of finite type for every $i \leq t$; however, A_i is a discrete valuation ring, hence $\mathscr{F}_{\eta_i}$ is a free A_i-module, necessarily of rank one, for $i = 1, \dots, t$. Since $\mathscr{F}$ is coherent, it follows that there exists an open neighborhood U_i of η_i in X, such that $\mathscr{F}_{|U_i}$ is a free $\mathscr{O}_{|U_i}$-module. Hence, we may replace U by $U \cup U_1 \cup \cdots \cup U_t$ and assume that every irreducible component of Z has codimension > 1, therefore $\delta'(x, \mathscr{O}_X) \geq 2$ for every $x \in Z$. Now, $\mathscr{F}[0]$ is a perfect complex by Serre's theorem ([163, Th.4.4.16]), so the invertible $\mathscr{O}_X$-module $\det \mathscr{F}$ is well defined (lemma 11.3.14). Let $j : U \to X$ be the open immersion; in view of proposition 11.3.8, we deduce natural isomorphisms

$$\mathscr{F} \xrightarrow{\sim} j_*j^*\mathscr{F} \xrightarrow{\sim} j_*\mathsf{det}(j^*\mathscr{F}[0]) \xrightarrow{\sim} j_*j^*\mathsf{det}\,\mathscr{F}[0] \xleftarrow{\sim} \mathsf{det}\,\mathscr{F}[0]$$

and the assertion follows. □

We wish now to introduce a notion of duality better suited to derived categories of $\mathscr{O}_X$-modules (over a scheme X). Hereafter we only carry out a preliminary investigation of such derived duality – the full development of which, will be the task of section 11.5.

Definition 11.3.16. Let X be a locally coherent scheme. A complex $\omega^\bullet$ in $\mathsf{D}^b(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}}$ (notation of (10.3)) is called *dualizing* if it fulfills the following two conditions :

(a) The functor :

$$\mathscr{D} : \mathsf{D}(\mathscr{O}_X\text{-}\mathbf{Mod})^o \to \mathsf{D}(\mathscr{O}_X\text{-}\mathbf{Mod}) \qquad C^\bullet \mapsto R\mathscr{H}om^\bullet_{\mathscr{O}_X}(C^\bullet, \omega^\bullet)$$

restricts to a *duality functor* : $\mathscr{D} : \mathsf{D}^b(\mathscr{O}_X\text{-}\mathbf{Mod})^o_{\mathrm{coh}} \to \mathsf{D}^b(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}}$.

(b) The natural transformation : $\eta_{C^\bullet} : C^\bullet \to \mathscr{D} \circ \mathscr{D}(C^\bullet)$ restricts to a *biduality isomorphism* of functors on the category $\mathsf{D}^b(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}}$.

Remark 11.3.17. Let X be a locally coherent scheme, $\omega^\bullet$ an object of $\mathsf{D}^b(\mathscr{O}\text{-}\mathbf{Mod})_{\mathrm{coh}}$, and define the functor $\mathscr{D}$ as in definition 11.3.16.

(i) A standard *dévissage* argument shows that $\omega^\bullet$ is dualizing on X if and only if $\mathscr{D}(\mathscr{F}[0])$ lies in $\mathsf{D}^b(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}}$, and the biduality map $\mathscr{F}[0] \to \mathscr{D} \circ \mathscr{D}(\mathscr{F}[0])$ is an isomorphism for every coherent $\mathscr{O}_X$-module $\mathscr{F}$.

(ii) Suppose that the complex $\omega^\bullet$ is dualizing on X. From the natural identification $\omega^\bullet \xrightarrow{\sim} R\mathscr{H}om^\bullet_{\mathscr{O}_X}(\mathscr{O}_X[0], \omega^\bullet)$, we deduce a natural isomorphism

$$R\mathscr{H}om^\bullet_{\mathscr{O}_X}(\omega^\bullet, \omega^\bullet) \xrightarrow{\sim} \mathscr{O}_X[0] \qquad \text{in } \mathsf{D}^b(\mathscr{O}_X\text{-}\mathbf{Mod}).$$

Example 11.3.18. Suppose that X is a noetherian regular scheme (*i.e.* all the stalks $\mathscr{O}_{X,x}$ are regular rings). In light of Serre's theorem [163, Th.4.4.16], every object of $\mathsf{D}^b(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}}$ is a perfect complex. It follows easily that the complex $\mathscr{O}_X[0]$ is dualizing. For more general schemes, the structure sheaf does not necessarily work, and the existence of a dualizing complex is a delicate issue. On the other hand, one may ask to what extent a complex is determined by the properties (a) and (b) of definition 11.3.16. Clearly, if $\omega^\bullet$ is dualizing on X, then so is any other complex of the form $\omega^\bullet \otimes_{\mathscr{O}_X} \mathscr{L}$, where $\mathscr{L}$ is an invertible $\mathscr{O}_X$-module. Also, any shift of $\omega^\bullet$ is again dualizing. Conversely, the following proposition 11.3.25 says that any two dualizing complexes are related in such manner, up to quasi-isomorphism.

Lemma 11.3.19. *Let $(X, \mathscr{O}_X)$ be any locally ringed space, and $P^\bullet$ and $Q^\bullet$ two objects of* $\mathsf{D}^-(\mathscr{O}_X\text{-}\mathbf{Mod})$ *with a quasi-isomorphism :*

$$P^\bullet \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_X} Q^\bullet \overset{\sim}{\to} \mathscr{O}_X[0].$$

Then there exist an invertible $\mathscr{O}_X$-module $\mathscr{L}$, a continuous function $\sigma : |X| \to \mathbb{Z}$ and quasi-isomorphisms:

$$P^\bullet \overset{\sim}{\to} \mathscr{L}[\sigma] \qquad \textit{and} \qquad Q^\bullet \overset{\sim}{\to} \mathscr{L}^{-1}[-\sigma].$$

Proof. It suffices to verify that, locally on X, the complexes $P^\bullet$ and $Q^\bullet$ are of the required form; indeed in this case X will be a disjoint union of open sets U_n on which $H^\bullet P^\bullet$ is concentrated in degree n and $H^\bullet Q^\bullet$ is concentrated in degree $-n$. Note that $\mathscr{O}_X$-$\mathbf{Mod}$ is equivalent to the product of the categories $\mathscr{O}_{U_n}$-$\mathbf{Mod}$ and that the derived category of a product of abelian categories is the product of the derived categories of the factors.

Claim 11.3.20. Let A be a (commutative) local ring, $K^\bullet$ and $L^\bullet$ two objects of $\mathsf{D}^-(A\text{-}\mathbf{Mod})$ with a quasi-isomorphism

$$K^\bullet \overset{\mathbf{L}}{\otimes}_A L^\bullet \overset{\sim}{\to} A[0]. \tag{11.3.21}$$

Then there exist $s \in \mathbb{Z}$ and quasi-isomorphisms :

$$K^\bullet \overset{\sim}{\to} A[s] \qquad L^\bullet \overset{\sim}{\to} A[-s].$$

Proof of the claim. Set :

$$i_0 := \max(i \in \mathbb{Z} \mid (H^i K^\bullet) \neq 0) \quad \text{and} \quad j_0 := \max(i \in \mathbb{Z} \mid (H^i L^\bullet) \neq 0).$$

We may assume that $K^i = 0$ for every $i > i_0$, and that $L^\bullet$ is a bounded above complex of free A-modules. Then we may find a filtered system $(K^\bullet_\lambda \mid \lambda \in \Lambda)$ of complexes of A-modules bounded from above, such that

- $H^{i_0}(K^\bullet_\lambda)$ is a finitely generated A-module, for every $\lambda \in \Lambda$;
- the colimit of the system $(K^\bullet_\lambda \mid \lambda \in \Lambda)$ (in the category of complexes of A-modules) is isomorphic to $K^\bullet$.

From (11.3.21) we get an isomorphism $H^0(K^\bullet \otimes_A L^\bullet) \overset{\sim}{\to} A$, and it follows easily that the natural map $H^0(K^\bullet_\lambda \otimes_A L^\bullet) \to H^0(K^\bullet \otimes_A L^\bullet)$ is surjective for some $\lambda \in \Lambda$. For such λ, we may then find a morphism in $\mathsf{D}^-(A\text{-}\mathbf{Mod})$:

$$A[0] \to K^\bullet_\lambda \overset{\mathbf{L}}{\otimes}_A L^\bullet \tag{11.3.22}$$

whose composition with the natural map $K^\bullet_\lambda \overset{\mathbf{L}}{\otimes}_A L^\bullet \to K^\bullet \overset{\mathbf{L}}{\otimes}_A L^\bullet$ is the inverse of (11.3.21). Hence

$$(11.3.22) \overset{\mathbf{L}}{\otimes}_A K^\bullet : K^\bullet \to (K^\bullet_\lambda \overset{\mathbf{L}}{\otimes}_A L^\bullet) \overset{\mathbf{L}}{\otimes}_A K^\bullet \overset{\sim}{\to} K^\bullet_\lambda \overset{\mathbf{L}}{\otimes}_A (L^\bullet \overset{\mathbf{L}}{\otimes}_A K^\bullet) \overset{\sim}{\to} K^\bullet_\lambda$$

is a right inverse of the natural morphism $K^\bullet_\lambda \to K^\bullet$ in $\mathsf{D}^-(A\text{-}\mathbf{Mod})$. Especially, the induced map $H^{i_0}K^\bullet_\lambda \to H^{i_0}K^\bullet$ is surjective, *i.e.* $H^{i_0}K^\bullet$ is a finitely generated A-module. Likewise, we see that $H^{j_0}L^\bullet$ is a finitely generated A-module. Now, notice that

$$H^k(K^\bullet \overset{\mathbf{L}}{\otimes}_A L^\bullet) \simeq \begin{cases} H^{i_0}K^\bullet \otimes_A H^{j_0}L^\bullet & \text{for } k = i_0 + j_0 \\ 0 & \text{for } k > i_0 + j_0. \end{cases}$$

From Nakayama's lemma it follows easily that $H^{i_0+j_0}(K^\bullet \overset{\mathbf{L}}{\otimes}_A L^\bullet) \neq 0$, and then our assumptions imply that $i_0 + j_0 = 0$ and $H^{i_0}K^\bullet \otimes_A H^{j_0}L^\bullet \simeq A$. One deduces easily that $H^{i_0}K^\bullet \simeq A \simeq H^{j_0}L^\bullet$ (see *e.g.* [75, Lemma 4.1.5]). Furthermore, we can find a complex $K^\bullet_1$ in $\mathsf{D}^{<i_0}(A\text{-}\mathbf{Mod})$ (resp. $L^\bullet_1$ in $\mathsf{D}^{<j_0}(A\text{-}\mathbf{Mod})$) such that :

$$K^\bullet \simeq A[-i_0] \oplus K^\bullet_1 \qquad (\text{resp. } L^\bullet \simeq A[-j_0] \oplus L^\bullet_1)$$

whence a quasi-isomorphism :

$$\varphi : A[0] \xrightarrow{\sim} K^\bullet \overset{\mathbf{L}}{\otimes}_A L^\bullet \xrightarrow{\sim} A[0] \oplus K_1^\bullet[-j_0] \oplus L_1^\bullet[-i_0] \oplus (K_1^\bullet \overset{\mathbf{L}}{\otimes}_A L_1^\bullet).$$

However, by construction φ^{-1} restricts to an isomorphism on the direct summand $A[0]$, therefore $K_1^\bullet \simeq 0 \simeq L_1^\bullet$ in $\mathsf{D}(A\text{-}\mathbf{Mod})$, and the claim follows. ◊

Now, for any point $x \in X$, let $i_x : \{x\} \to X$ be the inclusion map, and set $K_x^\bullet := i_x^* K^\bullet$ for every complex $K^\bullet$ of $\mathscr{O}_X$-modules. Notice that if $K^\bullet$ is a complex of flat $\mathscr{O}_X$-modules, then $K_x^\bullet$ is a complex of flat $\mathscr{O}_{X,x}$-modules ([9, Exp.V, Prop.1.6(1)]), therefore the rule $K^\bullet \mapsto K_x^\bullet$ yields a well-defined functor $\mathsf{D}^-(\mathscr{O}_X\text{-}\mathbf{Mod}) \to \mathsf{D}^-(\mathscr{O}_{X,x}\text{-}\mathbf{Mod})$, and moreover we have a natural isomorphism

$$K_x^\bullet \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{X,x}} L_x^\bullet \xrightarrow{\sim} (K^\bullet \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_X} L^\bullet)_x \qquad (11.3.23)$$

for every objects $K^\bullet, L^\bullet$ of $\mathsf{D}^-(\mathscr{O}_X\text{-}\mathbf{Mod})$. Especially, under the current assumptions, and in view of claim 11.3.20, we may find $s \in \mathbb{Z}$, and an isomorphism of $\mathscr{O}_{X,x}$-modules

$$H^s P_x^\bullet \otimes_{\mathscr{O}_{X,x}} H^{-s} Q_x^\bullet \xrightarrow{\sim} H^0(P_x^\bullet \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{X,x}} Q_x^\bullet) \xrightarrow{\sim} \mathscr{O}_{X,x}. \qquad (11.3.24)$$

Since $\mathscr{O}_{X,x}$ is local, we may thus find $a_x \in H^s P_x^\bullet$ and $b_x \in H^{-s} Q_x^\bullet$ such that (11.3.24) maps $a_x \otimes b_x$ to 1. Then a_x and b_x extend to local sections

$$a \in \Gamma(U, \mathrm{Ker}(d : P^s \to P^{s-1})) \qquad b \in \Gamma(U, \mathrm{Ker}(d : Q^{-s} \to P^{-s-1}))$$

on some neighborhood $U \subset X$ of x, and after shrinking U, we may assume that $a \otimes b$ gets mapped to 1, under the induced morphism of $\mathscr{O}_U$-modules

$$H^s P_{|U}^\bullet \otimes_{\mathscr{O}_U} H^{-s} Q_{|U}^\bullet \to H^0(P^\bullet \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_X} Q^\bullet)_{|U} \xrightarrow{\sim} \mathscr{O}_U.$$

Then we obtain a well defined morphism in $\mathsf{D}^-(\mathscr{O}_U\text{-}\mathbf{Mod})$

$$\varphi : \mathscr{O}_U[s] \to P_{|U}^\bullet \qquad (\text{resp.} \quad \psi : \mathscr{O}_U[-s] \to Q_{|U}^\bullet)$$

by the rule : $t \mapsto t \cdot a$ (resp. $t \mapsto t \cdot b$) for every local section t of $\mathscr{O}_U$. Again by claim 11.3.20 and (11.3.23) we deduce that $\varphi_y : \mathscr{O}_{U,y}[s] \to P_y^\bullet$ is a quasi-isomorphism for every $y \in U$ (and likewise for ψ_y); . *i.e.* φ and ψ are the sought isomorphisms in $\mathsf{D}^-(\mathscr{O}_U\text{-}\mathbf{Mod})$. □

Proposition 11.3.25. *Suppose that $\omega_1^\bullet$ and $\omega_2^\bullet$ are two dualizing complexes for the locally coherent scheme X. Then there exist an invertible $\mathscr{O}_X$-module $\mathscr{L}$ and a continuous function $\sigma : |X| \to \mathbb{Z}$ such that*

$$\omega_2^\bullet \simeq \omega_1^\bullet \otimes_{\mathscr{O}_X} \mathscr{L}[\sigma] \qquad \textit{in } \mathsf{D}^b(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}}.$$

Proof. Denote by $\mathscr{D}_1$ and $\mathscr{D}_2$ the duality functors associated with ω_1 and respectively ω_2. By assumption, we can find complexes $P^\bullet, Q^\bullet$ in $\mathsf{D}^b(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}}$ such that $\omega_2 \simeq \mathscr{D}_1(P^\bullet)$ and $\omega_1 \simeq \mathscr{D}_2(Q^\bullet)$, and therefore

$$\mathscr{D}_2(\mathscr{F}^\bullet) \simeq R\mathscr{H}om^\bullet_{\mathscr{O}_X}(\mathscr{F}^\bullet, \mathscr{D}_1(P^\bullet)) \simeq \mathscr{D}_1(\mathscr{F}^\bullet \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_X} P^\bullet)$$

for every object $\mathscr{F}^\bullet$ of $\mathsf{D}^b(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}}$ ([163, Th.10.8.7]).

Claim 11.3.26. Let $C^\bullet$ be an object of $\mathsf{D}^-(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}}$, such that $\mathscr{D}_1(C^\bullet)$ is in $\mathsf{D}^b(\mathscr{O}_X\text{-}\mathbf{Mod})$. Then $C^\bullet$ is in $\mathsf{D}^b(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}}$.

Proof of the claim. For given $m, n \in \mathbb{N}$, the natural maps: $\tau^{\leq -n} C^\bullet \xrightarrow{\alpha} C^\bullet \xrightarrow{\beta} \tau^{\geq -m} C^\bullet$ induce morphisms

$$\mathscr{D}_1(\tau^{\geq -m} C^\bullet) \xrightarrow{\mathscr{D}_1(\beta)} \mathscr{D}_1(C^\bullet) \xrightarrow{\mathscr{D}_1(\alpha)} \mathscr{D}_1(\tau^{\leq -n} C^\bullet)$$

Say that $\omega^\bullet \simeq \tau_{\geq a}\omega^\bullet$ for some integer $a \in \mathbb{N}$. Then $\mathscr{D}_1(\tau^{\leq -n}C^\bullet)$ lies in $\mathsf{D}^{\geq n+a}(\mathscr{O}_X\text{-}\mathbf{Mod})$. Since $\mathscr{D}_1(C^\bullet)$ is bounded, it follows that $\mathscr{D}_1(\beta) = 0$ for n large enough. Consider now the commutative diagram :

$$\begin{array}{ccc} \tau^{\geq -n}C^\bullet & \xrightarrow{\beta\circ\alpha} & \tau^{\geq -m}C^\bullet \\ \downarrow & & \downarrow \eta \\ \mathscr{D}_1 \circ \mathscr{D}_1(\tau^{\geq -n}C^\bullet) & \xrightarrow{\mathscr{D}_1\circ\mathscr{D}_1(\beta\circ\alpha)} & \mathscr{D}_1 \circ \mathscr{D}_1(\tau^{\geq -m}C^\bullet) \end{array}$$

Since $\tau^{\geq -m}C^\bullet$ is a bounded complex, η is an isomorphism in $\mathsf{D}(\mathscr{O}_X\text{-}\mathbf{Mod})$, so $\beta \circ \alpha = 0$ whenever n is large enough. Clearly this means that $C^\bullet$ is bounded, as claimed. $\diamondsuit$

Applying claim 11.3.26 to $C^\bullet := \mathscr{F}^\bullet \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_X} P^\bullet$ (which is in $\mathsf{D}^-(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}}$, since X is coherent) we see that the latter is a bounded complex, and by reversing the roles of ω_1 and ω_2 it follows that the same holds for $\mathscr{F}^\bullet \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_X} Q^\bullet$. We then deduce isomorphisms in $\mathsf{D}^b(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}}$

$$\mathscr{D}_1 \circ \mathscr{D}_2(\mathscr{F}^\bullet) \simeq \mathscr{F}^\bullet \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_X} P^\bullet \qquad \text{and} \qquad \mathscr{D}_2 \circ \mathscr{D}_1(\mathscr{F}^\bullet) \simeq \mathscr{F}^\bullet \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_X} Q^\bullet.$$

Letting $\mathscr{F}^\bullet := \mathscr{O}_X[0]$ we derive :

$$\mathscr{O}_X[0] \simeq \mathscr{D}_2 \circ \mathscr{D}_1 \circ \mathscr{D}_1 \circ \mathscr{D}_2(\mathscr{O}_X[0]) \simeq \mathscr{D}_2 \circ \mathscr{D}_1(P^\bullet) \simeq P^\bullet \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_X} Q^\bullet.$$

Then lemma 11.3.19 says that $P^\bullet \simeq \mathscr{E}[\tau]$ for an invertible $\mathscr{O}_X$-module $\mathscr{E}$ and a continuous function $\tau : |X| \to \mathbb{Z}$. Consequently : $\omega_2 \simeq \mathscr{E}^\vee[-\tau] \otimes_{\mathscr{O}_X} \omega_1$, so the proposition holds with $\mathscr{L} := \mathscr{E}^\vee$ and $\sigma := -\tau$. $\square$

Lemma 11.3.27. *Let $f : X \to Y$ be a morphism of locally coherent schemes, and $\omega^\bullet_Y$ a dualizing complex on Y. We have :*

(i) *If f is finite and finitely presented, then $f^!\omega^\bullet_Y$ is dualizing on X.*
(ii) *If Y is coherent and f is a quasi-compact open immersion, $f^*\omega^\bullet_Y$ is dualizing on X.*

Proof. (i): Denote by $\overline{f} : (X, \mathscr{O}_X) \to (Y, f_*\mathscr{O}_X)$ the morphism of ringed spaces deduced from f. For any object $C^\bullet$ of $\mathsf{D}^-(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}}$ we have natural isomorphisms :

$$\begin{aligned} \mathscr{D}(C^\bullet) := R\mathscr{H}om_{\mathscr{O}_X}(C^\bullet, f^!\omega^\bullet_Y) &\xrightarrow{\sim} R\mathscr{H}om_{\mathscr{O}_X}(C^\bullet, \overline{f}^* R\mathscr{H}om_{\mathscr{O}_Y}(f_*\mathscr{O}_X, \omega^\bullet_Y)) \\ &\xrightarrow{\sim} \overline{f}^* R\mathscr{H}om_{f_*\mathscr{O}_X}(f_*C^\bullet, R\mathscr{H}om_{\mathscr{O}_Y}(f_*\mathscr{O}_X, \omega^\bullet_Y)) \\ &\xrightarrow{\sim} \overline{f}^* R\mathscr{H}om_{\mathscr{O}_Y}(f_*C^\bullet, \omega^\bullet_Y). \end{aligned}$$

Hence, if $C^\bullet$ is in $\mathsf{D}^b(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}}$, the same holds for $\mathscr{D}(C^\bullet)$; especially, this applies to $f^!\omega_Y \simeq \mathscr{D}(\mathscr{O}_X[0])$, and we can compute :

$$\begin{aligned} \mathscr{D} \circ \mathscr{D}(C^\bullet) &\xrightarrow{\sim} \overline{f}^* R\mathscr{H}om_{\mathscr{O}_Y}(R\mathscr{H}om_{\mathscr{O}_Y}(f_*C^\bullet, \omega^\bullet_Y), \omega^\bullet_Y) \\ &\xrightarrow{\sim} \overline{f}^* f_*C^\bullet \xrightarrow{\sim} C^\bullet \end{aligned}$$

and by inspecting the definitions, one verifies that the resulting natural transformation $C^\bullet \to \mathscr{D} \circ \mathscr{D}(C^\bullet)$ is the biduality isomorphism. The claim follows.

(ii): Since $\mathscr{O}_Y$ is coherent, the natural map

$$f^* R\mathscr{H}om_{\mathscr{O}_Y}(\mathscr{G}[0], \omega^\bullet_Y) \to R\mathscr{H}om_{\mathscr{O}_X}(f^*\mathscr{G}[0], f^*\omega^\bullet_Y)$$

is an isomorphism in $\mathsf{D}^b(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}}$, for every $\mathscr{O}_Y$-module $\mathscr{G}$. Then the assertion follows easily from lemma 10.3.27(ii) and remark 11.3.17(i). $\square$

Proposition 11.3.28. *Let A be a noetherian local ring, κ its residue field, $M^\bullet$ a bounded below complex of A-modules of finite type, and set $X := \mathrm{Spec}\, A$. We have :*

(i) *The following conditions are equivalent*
  (a) *There exists* $c \in \mathbb{Z}$ *such that* $R\mathrm{Hom}^\bullet_A(\kappa[0], M^\bullet) \simeq \kappa[c]$.
  (b) *The complex of* $\mathscr{O}_X$*-modules* $M^{\bullet\sim}$ *arising from* $M^\bullet$ *is dualizing on* X.
(ii) *If the equivalent conditions of* (i) *hold, then* $M^\bullet$ *has finite injective dimension.*

Proof. (i): We show first that (a)⇒(b). To this aim, in view of remark 11.3.17(i) and corollary 10.3.2(ii), it suffices to check :

Claim 11.3.29. Assume (a). Then, for every finitely generated A-module N, we have :
(i) $D^\bullet(N) := R\mathrm{Hom}^\bullet_A(N[0], M^\bullet) \in \mathsf{D}^{\leq -c}(A\text{-}\mathbf{Mod})$ (notation of (7.1)).
(ii) The natural map

$$N[0] \to DD^\bullet(N) := R\mathrm{Hom}^\bullet_A(D^\bullet(N), M^\bullet)$$

is an isomorphism.

Proof of the claim. We first show the claim for $N = \kappa$, in which case (i) holds by assumption. To check (ii), let $M^\bullet \stackrel{\sim}{\to} I^\bullet$ be a resolution consisting of a bounded below complex of injective A-modules, so that

$$D^\bullet(\kappa[0], M^\bullet) \stackrel{\sim}{\to} \mathrm{Hom}^\bullet_A(\kappa[0], I^\bullet) \stackrel{\sim}{\to} I^\bullet[\mathfrak{m}]$$

where $\mathfrak{m} \subset A$ is the maximal ideal, and $I^k[\mathfrak{m}]$ denotes the submodule of $\mathfrak{m}$-torsion elements in I^k, for every $k \in \mathbb{Z}$. Under these isomorphisms, it is easily seen that the biduality map of (ii) is identified with the unique one

$$\kappa \to H := \mathrm{Hot}_{A\text{-}\mathbf{Mod}}(I^\bullet[\mathfrak{m}], I^\bullet)$$

that sends $1 \in \kappa$ to the inclusion map $j^\bullet : I^\bullet[\mathfrak{m}] \to I^\bullet$ (details left to the reader). Now, it is clear that any morphism $I^\bullet[\mathfrak{m}] \to I^\bullet$ in $\mathsf{Hot}(A\text{-}\mathbf{Mod})$ factors through $j^\bullet$, and on the other hand, our assumption implies that $H \simeq \kappa$. Hence, pick a morphism $f^\bullet : I^\bullet[\mathfrak{m}] \to I^\bullet$ representing a generator for the A-module H, and write $f^\bullet = j^\bullet \circ g^\bullet$ for some endomorphism $g^\bullet$ of $I^\bullet[\mathfrak{m}]$; in other words, $f^\bullet$ is the image of $j^\bullet$ under the A-linear map

$$\mathrm{Hot}_{A\text{-}\mathbf{Mod}}(g^\bullet, I^\bullet) : H \to H$$

so the class of $j^\bullet$ cannot vanish in H, and (ii) follows in this case.

Next, we shall argue by induction on $d := \dim \mathrm{Supp}\, N$. If $d = 0$, then N is an A-module of finite length, in which case we argue by induction on the length l of N. If $l = 1$, we have $N \simeq \kappa$, so the assertions are already known. Suppose $l > 1$, and that both (i) and (ii) are already known for all A-modules of length $< d$; we may find an A-submodule $N' \subset N$ such that both N' and $N'' := N/N'$ have length $< d$. From the inductive assumption for N' and N'', and the induced distiguished triangle

$$D^\bullet(N'') \to D^\bullet(N) \to D^\bullet(N') \to D^\bullet(N'')[1]$$

we deduce that (i) holds for N. Likewise, since (ii) is known for both N' and N'', using the 5-lemma we deduce easily that the same holds also for N.

Lastly, suppose that $d > 0$, and both (i) and (ii) are already known for all A-modules of finite type whose support has dimension $< d$. Let $N' := \Gamma_{\{\mathfrak{m}\}} N$; both (i) and (ii) are already known for N', so the same *dévissage* argument as in the foregoing reduces to showing the claim for N/N', *i.e.* we may assume that $\mathfrak{m} \notin \mathrm{Ass}\, N$. Thus, let $t \in \mathfrak{m}$ be any element such that the scalar multiplication map $t \cdot \mathbf{1}_N$ is injective, so we have a short exact sequence

$$0 \to N \xrightarrow{t^k} N \to N_k := N/t^k N \to 0 \qquad \text{for every } k > 0.$$

Notice that $\dim \mathrm{Supp}\, N_k < d$: indeed, if $\mathfrak{p}$ is a minimal element of $\mathrm{Supp}\, N$, then $\mathfrak{p} \in \mathrm{Ass}\, N$ ([126, Th.6.5(iii)]), hence $t \notin \mathfrak{p}$, and therefore $\mathfrak{p} \notin \mathrm{Supp}\, N_k$. By inductive assumption, both (i) and (ii) hold for N_k (for every $k > 0$), and by considering the induced distinguished triangle

$$D^\bullet(N_k) \to D^\bullet(N) \xrightarrow{t^k} D^\bullet(N) \to D^\bullet(N_k)[1]$$

we deduce that scalar multiplication by t is an epimorphism on $H^j D^\bullet(N)$ whenever $j \geq -c$; but the latter is an A-module of finite type, so it must vanish, by Nakayama' lemma, and we conclude that (i) holds for N. Furthermore, by the same token we get an exact sequence

$$H^i DD^\bullet(N) \xrightarrow{t^k} H^i DD^\bullet(N) \to 0 \qquad \text{for every } i \neq 0$$

whence $H^i DD(N) = 0$ for $i \neq 0$, again by Nakayama's lemma. Lastly, for every $k > 0$ consider the ladder with exact rows :

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N & \xrightarrow{t^k} & N & \longrightarrow & N_k & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha} & & \downarrow & & \\
0 & \longrightarrow & H^0 DD^\bullet(N) & \xrightarrow{t^k} & H^0 DD^\bullet(N) & \longrightarrow & H^0 DD^\bullet(N_k) & \longrightarrow & 0
\end{array}$$

whose right-most vertical arrow is an isomorphism, by inductive assumption. A simple diagram chase then yields

$$H^0 DD^\bullet(N) = t^k \cdot H^0 DD^\bullet(N) + \alpha(N)$$

so α is surjective, by Nakayama's lemma. To show the injectivity of α, let $x \in N$ be any non-zero element, and choose $k > 0$ such that $x \notin t^k N$, so the image of x does not vanish in N_k, whence necessarily $\alpha(x) \neq 0$, and the claim follows. ◊

(b)⇒(a): Let $i : \mathrm{Spec}\, \kappa \to X$ be the closed immersion; from corollary 10.3.2(ii) we know that $i^! M^{\bullet\sim}$ is the complex arising from the complex of κ-modules $R\mathrm{Hom}_A(\kappa[0], M^\bullet)$. On the other hand, $i^! M^{\bullet\sim}$ is dualizing on $\mathrm{Spec}\, \kappa$, by virtue of lemma 11.3.27(i); obviously, $\kappa[0]^\sim$ is also dualizing on the same scheme, so the assertion follows from proposition 11.3.25.

(ii): We notice more precisely :

Claim 11.3.30. Suppose that the conditions of (i) hold for $M^\bullet$, and moreover that $H^0 M^\bullet \neq 0$ and $H^i M^\bullet = 0$ for every $i < 0$. Then $c \leq 0$, and the injective dimension of $M^\bullet$ equals $-c$.

Proof of the claim. Under the assumptions of the claim, it is clear that $c \leq 0$. Pick a quasi-isomorphism $M^\bullet \xrightarrow{\sim} I^\bullet$, where $I^\bullet$ is an object of $\mathsf{C}^{[0,-c]}(A\text{-}\mathbf{Mod})$ such that I^j is an injective A-module for every $j < -c$. Notice that $I^0 \neq 0$, since $H^0 M^\bullet$ does not vanish. Let N be any A-module; a standard argument shows that

$$\mathrm{Ext}^1_A(N, I^c) \simeq R^{1-c}\mathrm{Hom}^\bullet_A(N[0], M^\bullet)$$

and the latter vanishes if N is finitely generated, by virtue of claim 11.3.29(i). It follows that I^c is injective as well (see *e.g.* [126, Th.B3]), so the injective dimension of $M^\bullet$ is $\leq -c$. But it is also $\geq -c$, due to condition (i.a). ◊

After replacing $M^\bullet$ by $M^\bullet[a]$ for a suitable $a \in \mathbb{Z}$, the assumptions of claim 11.3.30 can obviously be fulfilled, whence (ii). □

Let us recall the following :

Definition 11.3.31. Let X be a noetherian scheme.

(i) We say that X is *Gorenstein*, if $\mathscr{O}_X[0]$ is dualizing on X.

(ii) If $X = \mathrm{Spec}\, A$ is affine and Gorenstein, we also say that A is a *Gorenstein ring*.

Remark 11.3.32. (i) In view of example 11.3.18, every regular noetherian ring is Gorenstein.

(ii) Let A be any local noetherian ring, and κ the residue field of A. In light of proposition 11.3.28(i) (and of corollary 10.3.2(ii)), we see that A is Gorenstein if and only if there exists $c \in \mathbb{N}$ such that $R\mathrm{Hom}^\bullet_A(\kappa[0], A[0]) \simeq \kappa[c]$.

11.3.33. Let $A \to B$ be a flat homomorphism of local noetherian rings, κ_A (resp. κ_B) the residue field of A (resp. of B), and $K^\bullet$ a bounded complex of A-modules of finite type. Set $X := \mathrm{Spec}\, A$, $Y := \mathrm{Spec}\, B$, denote by $f : Y \to X$ the resulting morphism of local schemes, and let $K^{\bullet\sim}$ be the complex of $\mathscr{O}_X$-modules arising from $K^\bullet$. We have :

Proposition 11.3.34. *In the situation of* (11.3.33)*, the following conditions are equivalent :*

(a) *$f^*K^{\bullet\sim}$ is dualizing on Y.*
(b) *$K^{\bullet\sim}$ is dualizing on X and $B \otimes_A \kappa_A$ is a Gorenstein ring.*

Proof. Set $C := B \otimes_A \kappa_A$, $F := \mathrm{Spec}\, C$ and $Z := \mathrm{Spec}\, \kappa_B$, denote $i_1 : Z \to F$ and $i_2 : F \to Y$ the closed immersions, and let $i := i_2 \circ i_1$. In light of proposition 11.3.28(i) (and of corollary 10.3.2(ii)) we see that (a) holds if and only if there exists $c \in \mathbb{Z}$ such that $i^!(f^*K^{\bullet\sim}) \simeq \mathscr{O}_Z[c]$. However, recall that $i^! \simeq i_1^! \circ i_2^!$ (proposition 11.1.7(i)); invoking again proposition 11.3.28(i) we deduce that (a) holds if and only if $i_2^!(f^*K^{\bullet\sim})$ is dualizing on F. The latter is the complex of $\mathscr{O}_F$-modules arising from

$$(11.3.35) \qquad R\mathrm{Hom}^\bullet_B(C[0], B \otimes_A K^\bullet) \simeq B \otimes_A L^\bullet \quad \text{where} \quad L^\bullet := R\mathrm{Hom}^\bullet_A(\kappa_A[0], K^\bullet)$$

(proposition 10.3.3(ii) and corollary 10.3.2(ii)). Hence, suppose that (b) holds; from proposition 11.3.28(i) we conclude that $i_2^!(f^*K^{\bullet\sim}) \simeq \mathscr{O}_F[c]$ for some $c \in \mathbb{Z}$; the latter is dualizing on Z by assumption, whence (a). Conversely, suppose that (a) holds; from (11.3.35) we see that $B \otimes_A L^\bullet$ is a bounded complex of C-modules whose cohomology is a free C-module in every degree. We recall the following :

Claim 11.3.36. Let R be any ring, $M^\bullet$ a bounded complex of R-modules, such that $H^iM^\bullet$ is a projective R-module, for every $i \in \mathbb{Z}$. Then we have an isomorphism

$$M^\bullet \xrightarrow{\sim} \bigoplus_{i\in\mathbb{Z}} (H^iM^\bullet)[-i] \qquad \text{in } \mathsf{D}(R\text{-}\mathbf{Mod}).$$

Proof of the claim. We argue by induction on the cardinality c_M of $S_M := \{i \in \mathbb{Z} \mid H^iM^\bullet \neq 0\}$. If $c_M = 0$, the assertion is obvious. Suppose that $c_M > 0$, and that the assertion is known for every complex $P^\bullet$ as in the claim, with $c_P < c_M$. Let $a \in \mathbb{Z}$ be the largest element of S_M; after replacing $M^\bullet$ by its truncation $\tau^{\leq a}M^\bullet$, we may assume that $M^i = 0$ for every $i > a$. Then $H^aM^\bullet \simeq \mathrm{Coker}\,(d^{a-1} : M^{a-1} \to M^a)$, and since $H^aM^\bullet$ is a projective R-module, we may find an R-linear isomorphism $M \xrightarrow{\sim} (H^aM^\bullet) \oplus \mathrm{Im}\, d^{a-1}$. Denote by $P^\bullet$ the complex such that $P^i := M^i$ for every integer $i < a$, and such that $P^a := \mathrm{Im}\, d^{a-1}$, with differentials induced by those of $M^\bullet$; then $M^\bullet \simeq (H^aM^\bullet)[-a] \oplus P^\bullet$ in $\mathsf{D}(R\text{-}\mathbf{Mod})$, and $c_P < c_M$, so the assertion is known for $P^\bullet$, and the claim follows. ◊

In light of claim 11.3.36 and remark 11.3.17(ii), we easily see that $B \otimes_A L^\bullet \simeq C[a]$ in $\mathsf{D}^b(C\text{-}\mathbf{Mod})$ for some $a \in \mathbb{Z}$ (details left to the reader), and then clearly $L^\bullet \simeq k_A[c]$ in $\mathsf{D}^b(A\text{-}\mathbf{Mod})$. Taking into account proposition 11.3.28(i), assertion (b) follows. □

Proposition 11.3.37. *Let X be a noetherian scheme, $\mathscr{K}^\bullet$ an object of $\mathsf{D}^b(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}}$. The following conditions are equivalent :*

(a) *$\mathscr{K}^\bullet$ is dualizing on X.*
(b) *$\mathscr{K}^\bullet(x)$ is dualizing on $X(x)$, for every $x \in X$ (notation of definition* 4.9.17(iii)*).*

Proof. (b)⇒(a): We notice the following :

Claim 11.3.38. Let X be a noetherian scheme, $\mathscr{K}^\bullet$ an object of $\mathsf{D}^b(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}}$, and $x \in X$ any point. Suppose that the complex of $\mathscr{O}_{X(x)}$-modules $\mathscr{K}^\bullet(x)$ has finite injective dimension. Then, for every $\mathscr{F}^\bullet \in \mathsf{D}^b(\mathscr{O}_{X,x}\text{-}\mathbf{Mod})_{\mathrm{coh}}$ there exists an open neighborhood U of x in X such that $R\mathscr{H}om^\bullet_{\mathscr{O}_X}(\mathscr{F}^\bullet, \mathscr{K}^\bullet)_{|U}$ lies in $\mathsf{D}^b(\mathscr{O}_U\text{-}\mathbf{Mod})_{\mathrm{coh}}$.

Proof of the claim. The question is local on X, so we may assume that X is affine, say $X = \mathrm{Spec}\, A$ for a noetherian ring A; also, by theorem 10.3.25 we may assume that $\mathscr{K}^\bullet$ arises from a bounded complex $K^\bullet$ of A-modules of finite type (details left to the reader). Then, remark 11.3.17(i) and corollary 10.3.2(ii) reduce to checking that for every A-module M there exists an open neighborhood U of x in X, such that the A-modules $T^i(M) := H^i R\mathrm{Hom}^\bullet_A(M[0], K^\bullet)$ have support in $X \setminus U$, for every sufficiently large $i \in \mathbb{Z}$.

We shall argue by induction on the dimension d of the support of M; if $d = 0$, there is nothing to show, hence suppose that $d > 0$, and that the assertion is already known for every A-module whose support has dimension strictly less than d. By [126, 6.4] we may further reduce to the case where $M = A/\mathfrak{p}$, where $\mathfrak{p} \subset A$ is a prime ideal. Then, pick a finite system of generators $t_1, \dots, t_n \in A$ for $\mathfrak{p}$, and set $L_\bullet := \mathbf{K}_\bullet(t_1, \dots, t_n)$ (notation of remark 7.8.1(ii)). Now, $H_0 L_\bullet \simeq A/\mathfrak{p}$, and $H_i L_\bullet = 0$ for every $i > n$; moreover, lemma 7.8.2(ii) says that $H_i L_\bullet$ is an $A/\mathfrak{p}$-module of finite type, for every $i = 0, \dots, n$, and vanishes for $i \in \mathbb{Z}$ outside this range; therefore, we may find $f \in A \setminus \mathfrak{p}$ such that $(H_i L_\bullet)_f$ is a free $(A/\mathfrak{p})_f$-module of some finite rank $b(i) \in \mathbb{N}$, for every $i \in \mathbb{Z}$ ([126, Th.4.10(ii)]), and there exists an integer $m \leq n$ such that $b(m) \neq 0$ and $b(i) = 0$ for every $i > m$. By choosing an injective resolution of $K^\bullet$, we get a spectral sequence

$$E_2^{pq} := R^p\mathrm{Hom}^\bullet_A((H_q L_\bullet)[0], K^\bullet) \Rightarrow R^{p+q}\mathrm{Hom}^\bullet_A(L_\bullet, K^\bullet)$$

with differentials $d_r^{pq} : E_r^{pq} \to E_r^{p-r+1,q+r}$ for every $p, q \in \mathbb{Z}$ and every $r \geq 2$. Taking into account corollary 10.3.2(ii), we see that

$$(E_2^{pq})_f \simeq (E_2^{p,0})_f^{\oplus b(q)} \qquad \text{for every } p \in \mathbb{Z}. \tag{11.3.39}$$

By the same token, we also get a natural isomorphism

$$E_2^{p0}(x) := E_2^{p0} \otimes_A \mathscr{O}_{X,x} \simeq R^p\mathrm{Hom}^\bullet_{\mathscr{O}_{X(x)}}((A/\mathfrak{p})^\sim[0](x), \mathscr{K}^\bullet(x)) \qquad \text{for every } p \in \mathbb{Z}.$$

Since $\mathscr{K}^\bullet(x)$ has finite injective dimension, it follows that there exists an integer r such that $E_2^{p,0}(x) = 0$ whenever $p > r$. Since E_2^{pq} is an A-module of finite type for every $p, q \in \mathbb{Z}$, we deduce that there exists an open neighborhood U' of x in X, such that, for every $p = r+1, \dots, r+n$, the support of $E_2^{p,0}$ is contained in $X \setminus U'$ ([126, Th.4.10(i)]). In this case, the support of $(E_2^{pq})_f$ is also contained in $X \setminus U'$, for every $(p, q) \in \mathbb{Z}^{\oplus 2}$ with $p = r+1, \dots, r+n$.

Now, let $\mathfrak{q} \in U' \cap \mathrm{Spec}\, A_f$ be any prime ideal, and suppose that $(E_2^{p,0})_\mathfrak{q} \neq 0$ for some integer $p > r + n$. From (11.3.39), we deduce that $(E_2^{p,m})_\mathfrak{q} \neq 0$ as well, and it is easily seen that $(E_\infty^{p,m})_\mathfrak{q} = (E_2^{p,m})_\mathfrak{q}$. Especially

$$R^{p+m}\mathrm{Hom}^\bullet_A(L_\bullet, K^\bullet) \neq 0. \tag{11.3.40}$$

However, $L_\bullet$ is also a complex of free A-modules, therefore

$$R\mathrm{Hom}^\bullet_A(L_\bullet, K^\bullet) \simeq \mathrm{Hom}^\bullet_A(L_\bullet, K^\bullet) \qquad \text{in } \mathsf{D}(A\text{-}\mathbf{Mod}).$$

Thus, if we represent $K^\bullet$ via an object of $\mathsf{C}^{[a,b]}(A\text{-}\mathbf{Mod})$ (for some $a, b \in \mathbb{Z}$ with $a \leq b$: notation of (7.1)), then $R\mathrm{Hom}^\bullet_A(L_\bullet, K^\bullet)$ is represented by an object of $\mathsf{C}^{[a,b+n]}(A\text{-}\mathbf{Mod})$, and consequently, $R^i\mathrm{Hom}^\bullet_A(L_\bullet, K^\bullet) = 0$ for $i > b + n$. Taking into account (11.3.40), we conclude that $p \leq p + m \leq b + n$, so that

$$T^i(M)_\mathfrak{q} = 0 \qquad \text{whenever } i > n + \max(b, r) \text{ and } \mathfrak{q} \in U' \cap \mathrm{Spec}\, A_f. \tag{11.3.41}$$

On the other hand, from the short exact of A-modules

$$0 \to M \xrightarrow{f} M \to M/fM \to 0$$

we see that $T^i(M)/fT^i(M) \subset T^{i+1}(M/fM)$ for every $i \in \mathbb{Z}$; since the support of M/fM has dimension $< d$, the inductive assumption tells us that there exist $N \in \mathbb{N}$ and an open neighborhood U'' of x in X such that the support of $T^{i+1}(M/fM)$ lies in $X \setminus U''$, for every $i > N$. From this, Nakayama's lemma implies that

$$T^i(M)_{\mathfrak{q}} = 0 \qquad \text{for every } \mathfrak{q} \in \operatorname{Spec} A/fA \text{ and every } i > N.$$

Combining with (11.3.41), we conclude that $U := U' \cap U''$ will do. ◊

Since X is quasi-compact, assumption (b), claim 11.3.38 and proposition 11.3.28(ii) imply that the rule : $\mathscr{F}^\bullet \mapsto \mathscr{D}(\mathscr{F}^\bullet) := R\mathscr{H}om^\bullet_{\mathscr{O}_X}(\mathscr{F}^\bullet, \mathscr{K}^\bullet)$ takes objects of $\mathsf{D}^b(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}}$ to objects of the same category. Then, corollary 10.3.2(ii) and assumption (b) imply that the natural map $\mathscr{F}[0](x) \to \mathscr{D} \circ \mathscr{D}(\mathscr{F}[0])(x)$ is an isomorphism in $\mathsf{D}^b(\mathscr{O}_{X(x)}\text{-}\mathbf{Mod})$ for every $x \in X$ and every coherent $\mathscr{O}_X$-module $\mathscr{F}$, whence (a).

(a)⇒(b): Let $x \in X$ be any point, and $\mathscr{M}$ any coherent $\mathscr{O}_{X(x)}$-module; we may extend $\mathscr{M}$ to a coherent $\mathscr{O}_U$-module on some affine open neighborhood U of x in X, and then the latter can be further extended to a coherent $\mathscr{O}_X$-module $\mathscr{F}$ (lemma 10.3.27(ii)). According to corollary 10.3.2(ii), the natural map $\mathscr{D}(\mathscr{F}[0])(x) \to R\mathscr{H}om^\bullet_{\mathscr{O}_{X(x)}}(\mathscr{M}[0], \mathscr{K}^\bullet(x))$ is an isomorphism in $\mathsf{D}^b(\mathscr{O}_{X(x)}\text{-}\mathbf{Mod})$, whence (b). □

Corollary 11.3.42. *Let $f : Y \to X$ be a morphism of finite type between affine noetherian schemes, and $\omega^\bullet$ a dualizing complex on X. Then $f^!\omega^\bullet$ is a dualizing complex on Y.*

Proof. We may factor f as the composition of a closed immersion $g : Y \to Z$ followed by a smooth and affine morphism $h : Z \to X$. In light of lemmata 11.3.27(i) and 11.1.16(i), it then suffices to check that $h^!\omega^\bullet$ is dualizing on Z, so we may assume from start that f is smooth, in which case we are reduced to showing that $f^*\omega^\bullet$ is dualizing on Y. Let $y \in Y$ be any point, set $x := f(y)$, and denote by $f_y : Y(y) \to X(x)$ the morphism induced by f; taking into account proposition 11.3.37, we are further reduced to checking that $f_y^*(\omega^\bullet_{|X(x)})$ is dualizing on $Y(y)$. However, the latter assertion follows easily from proposition 11.3.34, remark 11.3.32(i) and [126, Th.28.7]. □

Corollary 11.3.43. *Let A be a noetherian ring, $x \in X := \operatorname{Spec} A$ any point, and $K^\bullet$ any object of $\mathsf{D}^b(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}}$. Suppose that the following holds :*

(a) *Every irreducible reduced closed subscheme $Z \subset X$ containing x admits a dense open subset which is Gorenstein (see definition* 11.3.31(i)*).*
(b) *$K^\bullet(x)$ is dualizing on $X(x)$.*

Then there exists an open neighborhood U of x in X, such that $K^\bullet_{|U}$ is dualizing on U.

Proof. We begin with the following :

Claim 11.3.44. For every reduced and irreducible closed subscheme Z of X containing x, there exists a dense open subset U_Z of Z such that $K^\bullet(z)$ is dualizing on $X(z)$, for every $z \in U_Z$.

Proof of the claim. Let $i_Z : Z \to X$ be the closed immersion, and set $i_{Z(x)} := i_Z \times_X X(x) : Z(x) \to X(x)$; in view of proposition 10.3.3(ii), we easily see that there is a natural isomorphism

$$i^!_{Z(x)} K^\bullet(x) \xrightarrow{\sim} (i^!_Z K^\bullet)(x) \qquad \text{in } \mathsf{D}(\mathscr{O}_{Z(x)}\text{-}\mathbf{Mod}).$$

Hence, due to (b) and lemma 11.3.27, $(i^!_Z K^\bullet)(x)$ is dualizing on $Z(x)$. Denote by η_Z the generic point of Z; it follows that $(i^!_Z K^\bullet)(\eta_Z)$ is dualizing on $Z(\eta_Z)$ (proposition 11.3.37), and therefore it is isomorphic to $\mathscr{O}_{Z(\eta_Z)}[c]$ in $\mathsf{D}(\mathscr{O}_{Z(\eta_Z)}\text{-}\mathbf{Mod})$, due to (a). On the other hand, claim

11.3.38 implies that there exists an open neighborhood U'_Z of x in Z, such that $(i^!_Z K^\bullet)_{|U'_Z}$ lies in $\mathsf{D}^b(\mathscr{O}_{U'_Z}\text{-}\mathbf{Mod})_{\mathrm{coh}}$. It then follows easily that there exists a smaller open subset $U_Z \subset U'_Z$ of Z, such that $(i^!_Z K^\bullet)_{|U_Z} \simeq \mathscr{O}_{U_Z}[c]$ in $\mathsf{D}(\mathscr{O}_{U'_E}\text{-}\mathbf{Mod})$ (details left to the reader). In view of (a), we may then further shrink U_Z, and assume that $(i^!_Z K^\bullet)_{|U_Z}$ is dualizing on U_Z. Now, let $y \in U_Z$ be any point, denote by $i_1 : \operatorname{Spec} \kappa(y) \to Z(y)$ and $i_2 : Z(y) \to X(y)$ the closed immersions, and set $i_y := i_2 \circ i_1$. By proposition 11.3.37, the complex $i^!_2 K^\bullet(y) \simeq (i^!_Z K^\bullet)(y)$ is dualizing on $Z(y)$, so $i^!_y K^\bullet(y) \simeq i^!_1(i^!_2 K^\bullet(y))$ is dualizing on $\operatorname{Spec} \kappa(y)$ (lemma 11.3.27(i)), and consequently $K^\bullet(y)$ is dualizing on $X(y)$ (proposition 11.3.28(i)). The claim follows. $\Diamond$

Denote by $\mathscr{Z}$ the set of all reduced and irreducible closed subschemes of X containing x, and for every $Z \in \mathscr{Z}$, let U_Z be the largest open subset of Z such that $K^\bullet(y)$ is dualizing on $X(y)$ for every $y \in U_Z$. The subset $U := \bigcup_{Z \in \mathscr{Z}} U_Z$ is ind-constructible in X, and it is clearly generizing (proposition 11.3.37). Thus, U is open in X ([63, Ch.IV, Th.1.10.1]), and to conclude the proof, it suffices – again, by virtue of proposition 11.3.37 – to check that $x \in U$. Suppose that the latter fails, and let $Z \subset X$ be the closure of $\{x\}$ in X, endowed with its reduced subscheme structure; by assumption, $Z \cap U = \varnothing$; on the other hand, $Z \in \mathscr{Z}$, so that $U_Z \neq \varnothing$, by claim 11.3.44. The contradiction proves the assertion. $\square$

Remark 11.3.45. Let A be a noetherian ring, and set $X := \operatorname{Spec} A$.

(i) If X admits a dualizing complex, then every irreducible reduced closed subscheme $Z \subset X$ admits a dense open subset U which is Gorenstein. Indeed, let η_Z be the generic point of Z; we know that Z admits a dualizing complex $\omega^\bullet_Z$ (lemma 11.3.27(i)), hence $\omega^\bullet_Z(\eta_Z)$ is dualizing on $Z(\eta_Z)$ (proposition 11.3.37). But then there exists an isomorphism $\mathscr{O}_{Z(\eta_Z)}[c] \xrightarrow{\sim} \omega^\bullet(\eta_Z)$ in $\mathsf{D}^b(\mathscr{O}_{Z(\eta_Z)}\text{-}\mathbf{Mod})$ for some $c \in \mathbb{Z}$ (proposition 11.3.25), and any such isomorphism extends to an isomorphism $\mathscr{O}_U[c] \xrightarrow{\sim} (\omega^\bullet_Z)_{|U}$ on some open subset $U \subset Z$; since $(\omega^\bullet_Z)_{|U}$ is dualizing on U (lemma 11.3.27(ii)), the assertion follows.

(ii) The observation in (i) shows that the converse of corollary 11.3.43 holds as well : if $x \in X$ is any point, and U any open neighborhood of x in X such that U admits a dualizing complex, then conditions (a) and (b) of corollary 11.3.43 holds for x.

(iii) If A is local and X admits a dualizing complex, then the *formal fibres* of A are Gorenstein, *i.e.*, if we denote $A^\wedge$ the completion of A and set $X^\wedge := \operatorname{Spec} A^\wedge$, then $X^\wedge \times_X \operatorname{Spec} \kappa(x)$ is a Gorenstein scheme, for every $x \in X$. Indeed, notice that $A^\wedge \otimes_A A/\mathfrak{p}$ is the completion of $A/\mathfrak{p}$, for every $\mathfrak{p} \in \operatorname{Spec} A$, and $\operatorname{Spec} A/\mathfrak{p}$ admits a dualizing complex, if X does (lemma 11.3.27(i)); hence we may replace A by its image in $\kappa(x)$ and assume from start that A is a local noetherian domain, and x the generic point of X. Let $\omega^\bullet$ be a dualizing complex on X, and $f : X^\wedge \to X$ the natural morphism; by proposition 11.3.34, the complex $f^*\omega^\bullet$ is dualizing on $X^\wedge$. On the other hand, by (i) there exists a non-empty open subset $U \subset X$ which is Gorenstein, and therefore $\omega^\bullet_{|U} \simeq \mathscr{O}_U[c]$ for some $c \in \mathbb{Z}$. Due to lemma 11.3.27(ii), it follows that $f^{-1}U$ is Gorenstein as well, and then the same holds for $f^{-1}(x)$, by proposition 11.3.37.

(iv) If A is local and complete, then X admits a dualizing complex. Indeed, in this case A is a quotient of a regular local ring ([126, Th.29.4(ii)]), so the assertion follows from lemma 11.3.27(i) and example 11.3.18.

Corollary 11.3.46. *Let X be any noetherian scheme, $\omega^\bullet$ a dualizing complex on X. We have :*

(i) *For every $x \in X$ there exists a unique $c \in \mathbb{Z}$ such that*

$$J(x) := R^c\Gamma_{\{x\}}\omega^\bullet_{|X(x)} \neq 0.$$

(ii) *For every $x \in X$, the A-module $J(x)$ is the injective hull of the residue field $\kappa(x)$.*

Proof. In view of proposition 11.3.37, we may assume that X is local and x is its closed point; say that $X = \operatorname{Spec} A$ for some local noetherian ring A, in which case $\omega^\bullet$ is the $\mathscr{O}_X$-module arising from a bounded complex $M^\bullet$ of A-modules of finite type. We notice :

Claim 11.3.47. Let $\mathfrak{m} \subset A$ be the maximal ideal. There exists a unique $c \in \mathbb{Z}$ such that

(i) $T^i(N) := R^i\mathrm{Hom}^\bullet_A(N, M^\bullet) = 0$ for every A-module N of finite type supported at $\{x\}$ and every $i \neq c$.

(ii) The map $T^c(A/\mathfrak{m}^n) \to T^c(A/\mathfrak{m}^{n+1})$ induced by the projection $A/\mathfrak{m}^{n+1} \to A/\mathfrak{m}^n$ is injective, for every $n \in \mathbb{N}$.

Proof of the claim. (i): By proposition 11.3.28(i), we know already that there exists $c \in \mathbb{N}$ such that $T^i(A/\mathfrak{m}) = 0$ if and only if $i \neq c$. We must therefore check that the claim holds for this value of c. Now, let N be any A-module of finite type supported at $\{x\}$; we may find a finite filtration $N_0 := 0 \subset N_1 \subset \cdots \subset N_k$ consisting of A-submodules, such that $N_{j+1}/N_j \simeq A/\mathfrak{m}$ for every $i = 0, \dots, k-1$. We deduce short exact sequences

$$T^i(A/\mathfrak{m}) \to T^i(N_{j+1}) \to T^i(N_j) \qquad \text{for every } i \in \mathbb{N} \text{ and every } j = 0, \dots, k-1$$

whence the contention, by a simple induction on j.

(ii) is similar : arguing as in the foregoing, we get an exact sequence

$$T^{c-1}(\mathfrak{m}^{n+1}/\mathfrak{m}^n) \to T^c(A/\mathfrak{m}^n) \to T^c(A/\mathfrak{m}^{n+1})$$

whence the assertion. ◊

On the other hand, corollary 10.4.35 yields a natural identification :

$$J(x) \xrightarrow{\sim} \operatorname*{colim}_{n \in \mathbb{N}} T^c(A/\mathfrak{m}^n) \qquad (11.3.48)$$

and then assertion (i) follows proposition 11.3.28(i) and claim 11.3.47.

(ii): Claim 11.3.47 also implies that T^c is an exact functor on the full subcategory of A-**Mod** whose objects are the A-modules supported at $\{x\}$. By proposition 7.11.31, remark 7.11.32(i,ii) and (11.3.48) we deduce already that $J(x)$ is an injective A-module. Next, let $\kappa(x)^\sim$ be the quasi-coherent $\mathscr{O}_X$-module arising from $\kappa(x)$; we have natural identifications

$$\begin{aligned}
\mathrm{Hom}_A(\kappa(x), J(x))[c] &\xrightarrow{\sim} R\mathrm{Hom}^\bullet_{\mathscr{O}_X}(\kappa(x)^\sim[0], R\underline{\Gamma}_{\{x\}}\omega^\bullet) && \text{(by corollary 10.3.2(i))}\\
&\xrightarrow{\sim} R\Gamma_{\{x\}} \circ R\mathscr{H}om^\bullet_{\mathscr{O}_X}(\kappa(x)^\sim[0], \omega^\bullet) && \text{(by lemma 10.4.13(iii))}\\
&\xrightarrow{\sim} R\mathrm{Hom}^\bullet_{\mathscr{O}_X}(\kappa(x)^\sim[0], \omega^\bullet) && (\text{as } \mathrm{Supp}\,\kappa(x) = \{x\})\\
&\xrightarrow{\sim} R\mathrm{Hom}^\bullet_A(\kappa(x)[0], M^\bullet) && \text{(by corollary 10.3.2(i))}\\
&\xrightarrow{\sim} \kappa(x)[c] && \text{(by proposition 11.3.28(i)).}
\end{aligned}$$

Then the assertion follows immediately from theorem 7.11.22. □

11.3.49. Let X be a noetherian scheme that admits a dualizing complex $\omega^\bullet$. According to corollary 11.3.46, we get a well defined function

$$c_X : |X| \to \mathbb{Z}$$

by assigning to each $x \in X$ the unique $c_X(x) \in \mathbb{Z}$ such that $R^{c_X(x)}\Gamma_{\{x\}}\omega^\bullet_{|X(x)}$ does not vanish. With this notation, we may state :

Lemma 11.3.50. *In the situation of* (11.3.49)*, let $x, y \in X$ be any two points, such that x is an immediate specialization of y. Then*

$$c_X(x) = c_X(y) + 1.$$

Proof. By proposition 11.3.37 we may replace X by $X(x)$, and ω by $\omega_{|X(x)}$, and assume from start that X is local, and x is its closed point. Let $Z \subset X$ be the topological closure of $\{y\}$, endowed with its reduced closed subscheme structure. If we denote $i : Z \to X$ the corresponding closed immersion, then $i^!\omega$ is dualizing on Z (lemma 11.3.27(i)). Fix $z \in Z$, let

$i_z : Z(z) \to X(z)$ be the induced closed immersion, and set $c := c_X(z)$; we get natural isomorphisms :

$$
\begin{aligned}
R\Gamma_{\{z\}}(i^!\omega^\bullet)_{|Z(z)} &\xrightarrow{\sim} R\Gamma_{\{z\}} \circ R\mathscr{H}om^\bullet_{\mathscr{O}_{X(z)}}(i_{z,*}\mathscr{O}_{Z(y)}, \omega^\bullet_{|X(z)}) && \text{(by proposition 10.3.3(ii))} \\
&\xrightarrow{\sim} R\mathrm{Hom}^\bullet_{\mathscr{O}_{X(z)}}(i_{z,*}\mathscr{O}_{Z(y)}, R\underline{\Gamma}_{\{z\}}\omega^\bullet_{|X(z)}) && \text{(by lemma 10.4.13(iii))} \\
&\xrightarrow{\sim} \mathrm{Hom}^\bullet_{\mathscr{O}_{X(z)}}(i_{z,*}\mathscr{O}_{Z(y)}, R^c\underline{\Gamma}_{\{z\}}\omega^\bullet_{|X(z)})[-c] && \text{(by corollary 11.3.46)}
\end{aligned}
$$

which shows that

$$c_Z(z) = c_X(z)$$

provided c_Z is defined via $i^!\omega^\bullet$. Hence, we may replace X by Z, and assume additionally that $X = \mathrm{Spec}\, A$, where A is a one-dimensional local noetherian domain, and y is the generic point of X. In this case, $\omega^\bullet$ is the complex of $\mathscr{O}_X$-module arising from a bounded complex $M^\bullet$ of A-modules of finite type, and clearly

$$R^{c_X(y)}\Gamma_{\{y\}}\omega^\bullet_{|X(y)} \xrightarrow{\sim} H^{c_X(y)}(\omega^\bullet_y) \simeq H^{c_X(y)}(M^\bullet) \otimes_A K$$

where K is the field of fractions of A. To compute $c_X(x)$, we apply proposition 10.4.32(iii); to this aim, let f be any non-zero element of the maximal ideal of A; we get

$$R^i\Gamma_{\{x\}}\omega^\bullet \xrightarrow{\sim} \operatorname*{colim}_{n\in\mathbb{N}} H^i(f^n, M^\bullet) \qquad \text{for every } i \in \mathbb{Z}$$

(notation of remark 7.8.1(iii)). By inspecting the definitions, we find a ladder with exact rows

$$
\begin{array}{ccccc}
H^iM^\bullet & \xrightarrow{f^n} & H^iM^\bullet & \longrightarrow & H^{i+1}(f^n, M^\bullet) \\
\| & & \downarrow{\scriptstyle f} & & \downarrow \\
H^iM^\bullet & \xrightarrow{f^{n+1}} & H^iM^\bullet & \longrightarrow & H^{i+1}(f^{n+1}, M^\bullet)
\end{array}
\qquad \text{for every } i \in \mathbb{Z} \text{ and } n \in \mathbb{N}
$$

whence – after taking colimits – an exact sequence

$$H^iM^\bullet \xrightarrow{\alpha_i} (H^iM^\bullet) \otimes_A K \xrightarrow{\beta_i} R^{i+1}\Gamma_{\{x\}}M^\bullet \qquad \text{for every } i \in \mathbb{Z}$$

where α_i is induced by the inclusion map $A \to K$. However, if $(H^iM^\bullet) \otimes_A K \neq 0$, clearly α_i cannot be surjective, so $\mathrm{Im}\,\beta_i \neq 0$, and the lemma follows. □

Example 11.3.51. Let K be any field, and A a K-algebra of finite type. Set $S := \mathrm{Spec}\, K$, $X := \mathrm{Spec}\, A$, and denote by $f : X \to S$ the structure morphism. Since $\mathscr{O}_S[0]$ is dualizing on S, the complex $\omega^\bullet := f^!\mathscr{O}_S[0]$ is dualizing on X (corollary 11.3.42). In this case, the function c_X associated with ω_X as in (11.3.49) can be determined explicitly as follows. First, we may find a closed immersion $i : X \to Y := \mathbb{A}^n_S$ for some n, and if $p : Y \to S$ is the structure morphism, the proof of lemma 11.3.50 shows that $c_X = c_Y \circ i$, where c_Y is the function of the same type associated with $g^!\mathscr{O}_S[0]$. Hence, we may assume that $X = \mathbb{A}^n_S$, in which case $\omega^\bullet \simeq \mathscr{O}_X[n]$. Then X is regular ([126, Th.28.7]), so for every $x \in X$ we may compute :

$$c_X(x) = \mathrm{depth}_{\{x\}}\mathscr{O}_{X(x)} - n = n - \mathrm{tr.\,deg}(\kappa(x)/K) - n = -\mathrm{tr.\,deg}(\kappa(x)/K).$$

Theorem 11.3.52. *Let A be a noetherian ring, and suppose that $X := \mathrm{Spec}\, A$ admits a dualizing complex. Then A is universally catenary.*

Proof. In light of corollary 11.3.42, it suffices to show that A is catenary. But this follows easily from lemma 11.3.50 (details left to the reader). □

11.3.53. In the situation of (11.3.49), lemma 11.3.50 shows that c_X is a weak codimension function, with the terminology of (11.2.19). Then corollary 11.3.46(i) means precisely that $\omega^\bullet$ is a c_X-Cohen-Macaulay complex of $\mathscr{O}_X$-modules on X (see definition 11.2.24(i)). By theorem 11.2.27, we may then find a Cousin complex of $\mathscr{O}_X$-modules $\mathscr{R}^\bullet_\omega$, unique up to isomorphism of complexes, such that $\omega^\bullet$ is isomorphic to $\mathscr{R}^\bullet_\omega$ in $\mathsf{D}^+(\mathscr{O}_X\text{-}\mathbf{Mod})$. We call $\mathscr{R}^\bullet_\omega$ the *residual complex* arising from $\omega^\bullet$. By corollary 11.3.46(ii) we see that

$$\mathscr{R}^p_\omega = \bigoplus_{c_X(x)=p} E(x)^\sim \qquad \text{for every } p \in \mathbb{Z}$$

where $E(x)$ is an injective hull of the A-module $\kappa(x)$, for every $x \in X$ (notation of (10.3)).

11.3.54. For every coherent scheme X, consider the category

$$\mathsf{Dual}_X$$

whose objects are the pairs $(U, \omega^\bullet_U)$, where $U \subset X$ is an open subset, and $\omega^\bullet_U$ is a dualizing complex on X. The morphisms $(U, \omega^\bullet_U) \to (U', \omega^\bullet_{U'})$ are the pairs (j, β), where $j : U \to U'$ is an inclusion map of open subsets of X, and $\beta : j^*\omega^\bullet_{U'} \xrightarrow{\sim} \omega^\bullet_U$ is an isomorphism in $\mathsf{D}^b(\mathscr{O}_U\text{-}\mathbf{Mod})$. Let X_{Zar} denote the full subcategory of Sch/X whose objects are the (Zariski) open subsets of X; it follows easily from lemma 11.3.27(ii), that the forgetful functor

$$(11.3.55) \qquad \mathsf{Dual}_X \to X_{\mathrm{Zar}} \qquad (U, \omega^\bullet_U) \mapsto U$$

is a fibration (see definition 3.1.2(ii)) With this notation, we have :

Proposition 11.3.56. *Every descent datum for the fibration* (11.3.55) *is effective.*

Proof. Let $((U_i, \omega^\bullet_i); \beta_{ij} \mid i, j = 1, \dots, n)$ be a descent datum for the fibration (11.3.55); this means that each $(U_i, \omega^\bullet_i)$ is an object of Dual_X, and if let $U_{ij} := U_i \cap U_j$ for every $i, j \leq n$, then

$$\beta_{ij} : \omega^\bullet_{i|U_{ij}} \xrightarrow{\sim} \omega^\bullet_{j|U_{ij}}$$

are isomorphisms in $\mathsf{D}^b(\mathscr{O}_{U_{ij}}\text{-}\mathbf{Mod})$, fulfilling a suitable cocycle condition (see (3.5.22)).

We shall show, by induction on $k = 1, \dots, n$, that there exists a dualizing complex $\omega^\bullet_{X_k}$ on $X_k := U_1 \cup \cdots \cup U_k$, such that the descent datum $((U_i, \omega^\bullet_i); \beta_{ij} \mid i, j = 1, \dots, k)$ is isomorphic to the descent datum relative to the family $(U_i \mid i = 1, \dots, k)$ determined by $(X_k, \omega^\bullet_{X_k})$.

For $k = 1$, there is nothing to prove. Next, suppose that $k > 1$, and $\omega^\bullet_{X_{k-1}}$ with the sought properties has already been constructed. If $k - 1 = n$, we are done; otherwise, let $V_k := X_k \cap U_{k+1}$, set $\mathfrak{U}_k := (U_{i,k+1} \mid i = 1, \dots, k)$, and $C^\bullet := R\mathscr{H}om^\bullet_{\mathscr{O}_{V_k}}(\omega^\bullet_{X_k|V_k}, \omega^\bullet_{k+1|V_k})$. The system of morphisms $(\beta_{i,k+1} \mid i = 1, \dots, k)$ determines a class c_k in the Čech cohomology group $H^0(\mathfrak{U}_k, C^\bullet)$. However, we have a spectral sequence (see lemma 10.3.15) :

$$E^{pq}_2 := H^p(\mathfrak{U}_k, H^q C^\bullet) \Rightarrow H^{p+q} R\mathrm{Hom}^\bullet_{\mathscr{O}_{V_k}}(\omega^\bullet_{X_k|V_k}, \omega^\bullet_{k+1|V_k}).$$

Claim 11.3.57. $E^{pq}_2 = 0$ for every $q < 0$.

Proof of the claim. More precisely, we check that $H^q C^\bullet = 0$ for every $q < 0$. Indeed, by lemma 11.3.27(ii), that both $\omega^\bullet_{X_k|V_k}$ and $\omega^\bullet_{k+1|V_k}$ are dualizing on V_k. Moreover, on $U_{i,k+1}$ they restrict to isomorphic objects of $\mathsf{D}^b(\mathscr{O}_{U_{i,k+1}}\text{-}\mathbf{Mod})$, for every $i = 1, \dots, k$. It follows easily from proposition 11.3.25 that there exists an invertible $\mathscr{O}_{V_k}$-module $\mathscr{L}$ and an isomorphism

$$\omega^\bullet_{X_k|V_k} \xrightarrow{\sim} \omega^\bullet_{k+1|V_k} \otimes_{\mathscr{O}_{V_k}} \mathscr{L} \qquad \text{in } \mathsf{D}^b(\mathscr{O}_{V_k}\text{-}\mathbf{Mod}).$$

Therefore, the biduality map induces an isomorphism

$$R\mathscr{H}om^\bullet_{\mathscr{O}_{V_k}}(\omega^\bullet_{X_k|V_k}, \omega^\bullet_{k+1|V_k}) \xrightarrow{\sim} \mathscr{L}[0]$$

whence the claim. ◊

Claim 11.3.57 implies that c_k corresponds to an element of $H^0R\mathrm{Hom}^\bullet_{\mathscr{O}_{V_k}}(\omega^\bullet_{X_k|V_k}, \omega^\bullet_{k+1|V_k})$, and by construction, it is clear that this global section is an isomorphism $c_k : \omega^\bullet_{X_k|V_k} \xrightarrow{\sim} \omega^\bullet_{k+1|V_k}$ in $\mathsf{D}^b(\mathscr{O}_{V_k}\text{-}\mathbf{Mod})$. By definition, c_k is represented by a diagram of quasi-isomorphisms :

$$\omega^\bullet_{X_k|V_k} \leftarrow T^\bullet \rightarrow \omega^\bullet_{k+1|V_k} \qquad \text{in } \mathsf{C}(\mathscr{O}_{V_k}\text{-}\mathbf{Mod}) \tag{11.3.58}$$

for some complex $T^\bullet$. Let $T_!$ be the $\mathscr{O}_{X_{k+1}}$-module obtained as extension by zero of T, and define likewise $(\omega^\bullet_{X_k})_!$ and $(\omega^\bullet_{k+1})_!$. We let $\omega^\bullet_{X_{k+1}}$ be the cone of the map of complexes

$$T_! \rightarrow (\omega^\bullet_{X_k})_! \oplus (\omega^\bullet_{k+1})_!$$

deduced from (11.3.58). An easy inspection shows that this complex is dualizing on X_{k+1}, and it fulfills the stated condition. □

Remark 11.3.59. The proof of proposition 11.3.56 is a special case of a general technique for "glueing perverse sheaves" developed in [18].

In the study of duality for derived categories of modules, the role of reflexive modules is taken by the more general class of Cohen-Macaulay modules. There is a version of this theory for noetherian regular schemes, and a relative variant, for smooth morphisms. Let us begin by recalling the following :

Definition 11.3.60. Let $(A, \mathfrak{m}_A)$ be a local ring, M an A-module.

(i) Let $\mathscr{M}$ be the set of all finitely generated A-submodules M. The *dimension* of M is

$$\dim_A M := \sup(\dim \operatorname{Supp} M' \mid M' \in \mathscr{M}).$$

(ii) Suppose that the open subset $\operatorname{Spec} A \setminus \{\mathfrak{m}_A\}$ is quasi-compact, and $\operatorname{depth}_A M < +\infty$. If $\dim_A M = \operatorname{depth}_A M$, we say that M is a *Cohen-Macaulay* A-module. The category

$$A\text{-}\mathsf{CM}$$

is the full subcategory of A-$\mathbf{Mod}$ consisting of all finitely presented Cohen-Macaulay A-modules. For every $n \in \mathbb{N}$, we let A-CM_n be the full subcategory of A-CM whose objects are the Cohen-Macaulay A-modules of dimension n.

(iii) If A is a Cohen-Macaulay A-module, we say that A is a *Cohen-Macaulay* local ring.

(iv) Let X be a scheme, $\mathscr{F}$ a quasi-coherent $\mathscr{O}_X$-module. We say that $\mathscr{F}$ is a *Cohen-Macaulay* $\mathscr{O}_X$-module, if $\mathscr{F}_x$ is a Cohen-Macaulay $\mathscr{O}_{X,x}$-module, for every $x \in X$. We say that X is a *Cohen-Macaulay* scheme, if $\mathscr{O}_X$ is a Cohen-Macaulay $\mathscr{O}_X$-module.

(v) Let B be another local ring, $\varphi : A \rightarrow B$ a local ring homomorphism. The category

$$\varphi\text{-}\mathsf{CM}$$

of *φ-Cohen-Macaulay modules* is the full subcategory of B-$\mathbf{Mod}$ whose objects are all the finitely presented B-modules N that are φ-flat, and such that $N/\mathfrak{m}_A N$ is a Cohen-Macaulay B-module. For every $n \in \mathbb{N}$, we let φ-CM_n be the full subcategory of φ-CM whose objects are the φ-Cohen-Macaulay modules N with $\dim_B N/\mathfrak{m}_A N = n$.

(vi) Let $f : X \rightarrow Y$ be a locally finitely presented morphism of schemes, $\mathscr{F}$ a quasi-coherent $\mathscr{O}_X$-module, $x \in X$ any point, and set $y := f(x)$. We say that $\mathscr{F}$ is *f-Cohen-Macaulay* at the point x, if $\mathscr{F}_x$ is a $f^\natural_x$-Cohen-Macaulay module. We say that f is *Cohen-Macaulay* at the point x, if $\mathscr{O}_X$ is f-Cohen-Macaulay at the point x (cp. [64, Ch.IV, Déf.6.8.1]).

(vii) Let f and $\mathscr{F}$ be as in (vi). We say that $\mathscr{F}$ is f-Cohen-Macaulay (resp. that f is Cohen-Macaulay) if $\mathscr{F}$ (resp. $\mathscr{O}_X$) is f-Cohen-Macaulay at every point $x \in X$.

(viii) Let f and $\mathscr{F}$ be as in (vi). The *f-Cohen-Macaulay locus* of $\mathscr{F}$ is the subset

$$CM(f,\mathscr{F}) \subset X$$

consisting of all $x \in X$ such that f is Cohen-Macaulay at x. The subset $CM(f,\mathscr{O}_X)$ is also called the *Cohen-Macaulay locus of f* and is denoted briefly $CM(f)$.

Remark 11.3.61. Let $(A,\mathfrak{m}_A)$ be any local ring; set $X := \mathrm{Spec}\, A$, and for every A-module M, let $M^\sim$ be the quasi-coherent $\mathscr{O}_X$-module arising from M. Then, with the notation of theorem 10.1.35(i), we have

$$\dim_A M = d_{M^\sim} \qquad \text{for every } A\text{-module } M.$$

For the proof, let $\mathscr{M}$ be the set of all finitely generated A-submodules of M. Clearly $d_{M^\sim} = \sup(d_{M'^\sim} \mid M' \in \mathscr{M})$, so we are reduced to the case where M is finitely generated. Then $\mathrm{Supp}\, M$ is a closed subset of X, and $\dim_A M = \dim(\mathrm{Supp}\, M)$; moreover, clearly $\mathrm{Supp}\, M = \mathrm{Supp}\, M^\sim$, and the assertion folows easily.

Lemma 11.3.62. *Let $f : X \to Y$ be a locally finitely presented morphism of schemes, and $\mathscr{F}$ a finitely presented $\mathscr{O}_X$-module such that* $\mathrm{Supp}\,\mathscr{F} = X$. *We have :*

(i) *$CM(f,\mathscr{F})$ is an open subset of X,*
(ii) *The restriction $CM(f) \to Y$ of f is locally equidimensional.*
(iii) *Let $g : X' \to X$ be another morphism of schemes, $x' \in X'$ a point such that g is étale at x'. Then $g(x') \in CM(f,\mathscr{F})$ if and only if $x' \in CM(f \circ g, g^*\mathscr{F})$.*
(iv) *If $x \in CM(f)$, then $\delta'(x,\mathscr{O}_X) = \delta'(f(x),\mathscr{O}_Y) + \dim \mathscr{O}_{f^{-1}(fx),x}$ (notation of* (10.4.19)*).*

Proof. (i) and (ii) follow easily from [65, Ch.IV, Prop.15.4.3] and [64, Ch.IV, Prop.2.3.4].

(iii) follows from corollary 10.4.37. Lastly, (iv) is an immediate consequence of corollary 10.4.47. □

Remark 11.3.63. (i) The terminology "locally equidimensional" follows [66, $\mathrm{Err}_{\mathrm{IV}}35$], which modifies the terminology "equidimensional" of [65, Ch.IV, Déf.13.3.2]. Hence, a morphism of schemes $f : X \to Y$ locally of finite type is locally equidimensional, if it verifies the equivalent conditions of [65, Ch.IV, Prop.13.3.1] at every point $x \in X$. However, the *caveat* here is that the said conditions are *not* quite equivalent as stated : they become equivalent if one omits the words "contenant y" in line 2 of *loc.cit.* (with unchanged proof). So, the correct definition of "locally equidimensional" is in terms of the cited proposition thus amended.

(ii) A counterexample to [65, Ch.IV, Prop.13.3.1] is given by the identity morphism $X \to X$ of a scheme X with a point x such that the union of the irreducible components of X passing through x does not contain an open neighborhood of x in X (*e.g.* take X whose underlying topological space is zero-dimensional and not discrete); then $\mathbf{1}_X$ satisfies condition b) of [65, Ch.IV, Prop.13.3.1], but does not satisfy conditions a),a′) and a″).

Proposition 11.3.64. *Let A be a regular local ring of dimension d, and M a finitely generated Cohen-Macaulay A-module. Then :*

(i) $\mathrm{Ext}^i_A(M,A) = 0$ *for every $i \neq c := d - \dim M$.*
(ii) *The A-module $\mathscr{D}(M) := \mathrm{Ext}^c_A(M,A)$ is Cohen-Macaulay.*
(iii) *The natural map $M \to \mathrm{Ext}^c(\mathscr{D}(M),A)$ is an isomorphism, and* $\mathrm{Supp}\,\mathscr{D}(M) = \mathrm{Supp}\, M$.
(iv) *For every $n \in \mathbb{N}$, we have an equivalence of categories :*

$$\mathscr{D} : A\text{-}\mathsf{CM}_n \to A\text{-}\mathsf{CM}_n^o \qquad N \mapsto \mathrm{Ext}^{d-n}_A(N,A).$$

Proof. (i): According to [63, Ch.0, Prop.17.3.4], the projective dimension of M equals c, so we may find a minimal free resolution $0 \to L_c \to \cdots \to L_1 \to L_0 \to M$ for M of length c. Hence, we need only prove the sought vanishing for every $i < c$. Set $X := \mathrm{Spec}\, A$,

$Z := \mathrm{Supp}\, M \subset X$. By virtue of proposition 10.4.25, it suffices to show that $\mathrm{depth}_Z \mathscr{O}_X \geq c$. In light of (10.4.30), this comes down to showing that $\mathscr{O}_{X,z}$ is a local ring of depth $\geq c$, for every $z \in Z$. The latter holds, since $\mathscr{O}_{X,z}$ is a regular local ring of dimension $\geq c$ ([63, Ch.0, Cor.16.5.12]).

(ii): From (i) we deduce that $L^\vee_\bullet := (L^\vee_0 \to \cdots \to L^\vee_n)$, together with its natural augmentation $L^\vee_n \to \mathscr{D}(M)$, is a free resolution of $\mathscr{D}(M)$; especially, the projective dimension of $\mathscr{D}(M)$ is $\leq c$, and therefore

$$\mathrm{depth}_A \mathscr{D}(M) \geq \dim M$$

again by [63, Ch.0, Prop.17.3.4]. On the other hand, it follows easily from [163, Prop.3.3.10] that $\mathrm{Supp}\, \mathscr{D}(M) \subset \mathrm{Supp}\, M$, so $\mathscr{D}(M)$ is Cohen-Macaulay.

(iii): From the proof of (ii) we see that $R\mathrm{Hom}_A(\mathscr{D}(M), A)$ is computed by the complex $L^{\vee\vee}_\bullet = L_\bullet$, whence the first assertion. Invoking again [163, Prop.3.3.10], we deduce that $\mathrm{Supp}\, M \subset \mathrm{Supp}\, \mathscr{D}(M)$; since the converse inclusion is already known, these two supports coincide.

(iv) follows straightforwardly from (ii) and (iii). □

Corollary 11.3.65. *Let X be a regular noetherian scheme, and $j : Y \to X$ a closed immersion. We have :*

(i) *Y admits a dualizing complex $\omega^\bullet_Y$.*

(ii) *If Y is a Cohen-Macaulay scheme, then we may find a dualizing complex $\omega^\bullet_Y$ that is concentrated in degree 0. Moreover, $H^0(\omega^\bullet_Y)$ is a Cohen-Macaulay $\mathscr{O}_Y$-module.*

Proof. Indeed, lemma 11.3.27(i) shows that the complex $\omega^\bullet_Y := j^* R\mathscr{H}om_{\mathscr{O}_X}(j_*\mathscr{O}_Y, \mathscr{O}_X)$ will do. According to proposition 11.3.64(i,ii), this complex fulfills the condition of (ii), up to a suitable shift. □

11.3.66. Let $f : X \to S$ be a smooth quasi-compact morphism of schemes, $\mathscr{F}$ a finitely presented quasi-coherent $\mathscr{O}_X$-module, $x \in X$ any point, and $s := f(x)$. Set

$$A := \mathscr{O}_{S,s} \qquad B := \mathscr{O}_{X,x} \qquad F := \mathscr{F}_x \qquad B_0 := B \otimes_A \kappa(s) \qquad F_0 := F \otimes_A \kappa(s).$$

Theorem 11.3.67. *In the situation of* (11.3.66)*, suppose that F is a $f^\natural_x$-Cohen-Macaulay module. Then we may find an open neighborhood $U \subset X$ of x in X such that the following holds:*

(i) *There exists a finite resolution of the $\mathscr{O}_U$-module $\mathscr{F}_{|U}$, of length $n := \dim B_0 - \dim_B F_0$*

$$\Sigma_\bullet \quad : \quad 0 \to \mathscr{L}_n \to \mathscr{L}_{n-1} \to \cdots \to \mathscr{L}_0 \to \mathscr{F}_{|U} \to 0$$

consisting of free $\mathscr{O}_U$-modules of finite rank. Moreover, $\Sigma_\bullet$ is universally $\mathscr{O}_S$-exact, i.e. *for every coherent $\mathscr{O}_S$-module $\mathscr{G}$, the complex $\Sigma_\bullet \otimes_{\mathscr{O}_X} f^*\mathscr{G}$ is still exact.*

(ii) *The complex $K^\bullet := R\mathscr{H}om^\bullet_{\mathscr{O}_U}(\mathscr{F}_{|U}, \mathscr{O}_U)$ is concentrated in degree n.*

(iii) *The natural map $\mathscr{F}_{|U} \to R\mathscr{H}om_{\mathscr{O}_U}(K^\bullet, \mathscr{O}_U)$ is an isomorphism in* $\mathsf{D}(\mathscr{O}_U\text{-}\mathbf{Mod})$.

(iv) *The B-module $G := H^n K^\bullet_x$ is $f^\natural_x$-Cohen-Macaulay, and* $\mathrm{Supp}\, G = \mathrm{Supp}\, F$.

Proof. (i): According to proposition 7.3.53, the B-module F admits a minimal free resolution

$$\Sigma_{x,\bullet} \quad : \quad \cdots \to L_2 \xrightarrow{d_2} L_1 \xrightarrow{d_1} L_0 \xrightarrow{\varepsilon} F \to 0$$

that is universally A-exact; especially, $d_i(L_i)$ is a flat A-module, for every $i \in \mathbb{N}$. It also follows that $\Sigma_{x,\bullet} \otimes_A \kappa(s)$ is a minimal free resolution of the B_0-module F_0. Since the latter is Cohen-Macaulay, and B_0 is a regular local ring ([66, Ch.IV, Th.17.5.1]), the projective dimension of the B_0-module F_0 equals n ([63, Ch.0, Prop.17.3.4]), therefore $d_n(L_n) \otimes_A \kappa(s)$ is a free B_0-module, so $d_n(L_n)$ is a flat B-module (lemma 7.11.35); then we deduce that it is actually a free B-module, as it is finitely presented. Since $\Sigma_{x,\bullet}$ is minimal, we conclude that $L_i = 0$ for every $i > n$. We may now extend $\Sigma_{x,\bullet}$ to a finite resolution $\Sigma_\bullet$ of $\mathscr{F}_{|U}$ by free $\mathscr{O}_U$-modules on some open neighborhood U of x, and after replacing U by a smaller neighborhood of U, we may assume that $\mathscr{F}_{|U}$ is $f_{|U}$-flat ([65, Ch.IV, Th.11.3.1]), hence $\Sigma_\bullet$ is universally $\mathscr{O}_S$-exact, as stated.

(ii): Let $L_\bullet := (L_n \to \cdots \to L_0)$ be the complex obtained after omitting F from the resolution $\Sigma_{x,\bullet}$ (where L_0 is placed in degree 0). Then $K^\bullet$ is isomorphic to $L_\bullet^\vee := \mathrm{Hom}_B(L_\bullet, B)$. On the other hand, the universal exactness property of $\Sigma_{x,\bullet}$ implies that

$$\mathrm{Hom}_{B_0}(L_\bullet \otimes_A \kappa(s), B_0) = L_\bullet^\vee \otimes_A \kappa(s)$$

computes $R\mathrm{Hom}^\bullet_{B_0}(F_0, B_0)$. In view of proposition 11.3.64(i), we have $\mathrm{Ext}^i_{B_0}(F_0, B_0) = 0$ for every $i \neq n$. From this, a repeated application of [65, Ch.IV, Prop.11.3.7] shows that $L_\bullet^\vee$ is concentrated in degree n as well, and after shrinking U, we may assume that (ii) holds (details left to the reader).

(iii): Let $\mathscr{L}_\bullet := (\mathscr{L}_n \to \cdots \to \mathscr{L}_0)$ be the complex obtained by omitting $\mathscr{F}_{|U}$ form the resolution $\Sigma_\bullet$; the proof of (iii) shows that $R\mathscr{H}om_{\mathscr{O}_U}(K^\bullet, \mathscr{O}_U)$ is computed by $\mathscr{L}_\bullet^{\vee\vee} = \mathscr{L}_\bullet$, whence the contention.

(iv): By the same token, [65, Ch.IV, Prop.11.3.7] implies that $\mathrm{Coker}\, d_i^\vee$ is a flat A-module, for every $i = 1, \dots, n$. Especially, G is a flat A-module, and the complex $L_\bullet^\vee[-n]$, with its natural augmentation $L_n^\vee \to G$, is a universally A-exact and free resolution of the B-module G. Especially, the projective dimension of G is $\leq n$, therefore

$$\mathrm{depth}_B G \geq \dim_B F_0 \qquad (11.3.68)$$

by [63, Ch.0, Prop.17.3.4]. From (iii) we also see that the induced map

$$F_\mathfrak{p} \to \mathrm{Ext}^n_{B_\mathfrak{p}}(G_\mathfrak{p}, B_\mathfrak{p})$$

is an isomorphism, for every prime ideal $\mathfrak{p} \subset B$; therefore $\mathrm{Supp}\, F \subset \mathrm{Supp}\, G$. Symmetrically, the same argument yields $\mathrm{Supp}\, G \subset \mathrm{Supp}\, F$. Hence the supports of F and G agree. Lastly, combining with (11.3.68) we see that G is Cohen-Macaulay. □

11.3.69. Keep the notation of (11.3.66), and let $\varphi := f_x^\natural : A \to B$. Theorem 11.3.67(iii,iv) implies that, for every $n \in \mathbb{N}$, the functor

$$\mathscr{D}_\varphi : \varphi\text{-}\mathsf{CM}_n \to \varphi\text{-}\mathsf{CM}_n^o \qquad M \mapsto \mathrm{Ext}_B^{\dim B_0 - n}(M, B)$$

is an equivalence, and the natural map $M \to \mathscr{D}_\varphi \circ \mathscr{D}_\varphi(M)$ is an isomorphism, for every φ-Cohen-Macaulay module M. We shall see later also a relative variant of corollary 11.3.65, in a more special situation (see proposition 11.5.5).

11.4. **Schemes over a valuation ring.** Throughout this section, $(K, |\cdot|)$ is a valued field, whose valuation ring (resp. maximal ideal, resp. residue field, resp. value group) shall be denoted K^+ (resp. $\mathfrak{m}_K$, resp. κ, resp. Γ). Also, we let

$$S := \mathrm{Spec}\, K^+ \qquad \text{and} \qquad S_{/b} := \mathrm{Spec}\, K^+/bK^+ \qquad \text{for every } b \in \mathfrak{m}_K$$

(so $S_{/0} = S$) and we denote by $s := \mathrm{Spec}\, \kappa$ (resp. by $\eta := \mathrm{Spec}\, K$) the closed (resp. generic) point of S. More generally, for every S-scheme X we let as well

$$X_{/b} := X \times_S S_{/b} \qquad \text{for every } b \in \mathfrak{m}_K.$$

A basic fact, that follows immediately from corollary 9.1.28, is that every finitely presented S-scheme is coherent (this can also be deduced easily from [86, Part I, Th.3.4.6]).

Lemma 11.4.1. *Let X be an irreducible scheme, and $f : X \to S$ a flat morphism of schemes of finite type, $x \in X$, and $y := f(x)$. The following holds :*

(i) *The scheme $f^{-1}(y)$ is equidimensional of dimension* $\dim f^{-1}(\eta)$.
(ii) *The local ring $\mathscr{O}_{f^{-1}(y),x}$ is equidimensional.*

Proof. (i): After replacing K by the residue field $\kappa(y)$, and K^+ by its image in $\kappa(y)$ (which is a valuation ring of $\kappa(y)$: see [126, Th.10.1(iii)]), the assertion follows from [65, Ch.IV, Cor.13.1.6 and Lemme 14.3.10].

(ii) follows easily from (i) and [64, Ch.IV, Prop.5.2.1]. □

Proposition 11.4.2. *Let $f : X \to S$ be a flat and finitely presented morphism, $\mathscr{F}$ a coherent $\mathscr{O}_X$-module, $x \in X$ any point, $y := f(x)$, and suppose that $f^{-1}(y)$ is a regular scheme (this holds* e.g. *if f is a smooth morphism). Let also $n := \dim \mathscr{O}_{f^{-1}(y),x}$. Then:*

(i) $\mathrm{proj.dim}_{\mathscr{O}_{X,x}} \mathscr{F}_x \leq n+1$ *(here we set* $\mathrm{proj.dim}_{\mathscr{O}_{X,x}} 0 := 0$*).*

(ii) *If $\mathscr{F}$ is f-flat at the point x, then* $\mathrm{proj.dim}_{\mathscr{O}_{X,x}} \mathscr{F}_x \leq n$.

(iii) *If $\mathscr{F}$ is reflexive at the point x, then* $\mathrm{proj.dim}_{\mathscr{O}_{X,x}} \mathscr{F}_x \leq \max(0, n-1)$.

(iv) *If f has regular fibres, and $\mathscr{F}$ is reflexive and generically invertible (see* (11.3.10)*), then $\mathscr{F}$ is invertible.*

(v) *All the fibres of the induced morphism $f_x : X(x) \to S(y)$ are regular and irreducible.*

Proof. To start out, we may assume that X is affine, and then it follows easily from lemma 11.3.7(ii.a) that $\mathscr{O}_X$ is coherent.

(ii): Since $\mathscr{O}_X$ is coherent, we can find a possibly infinite resolution

$$\cdots \to E_2 \to E_1 \to E_0 \to \mathscr{F}_x \to 0$$

by free $\mathscr{O}_{X,x}$-modules E_i ($i \in \mathbb{N}$) of finite rank; set $E_{-1} := \mathscr{F}_x$ and $L := \mathrm{Im}(E_n \to E_{n-1})$. It suffices to show that the $\mathscr{O}_{X,x}$-module L is free. For every $\mathscr{O}_{X,x}$-module M we shall let $M(y) := M \otimes_{\mathscr{O}_{X,x}} \mathscr{O}_{f^{-1}(y),x}$. Since L and $\mathscr{F}_x$ are torsion-free, hence flat K^+-modules, the induced sequence of $\mathscr{O}_{f^{-1}(y),x}$-modules:

$$0 \to L(y) \to E_{n-1}(y) \to \cdots \to E_1(y) \to E_0(y) \to \mathscr{F}_x(y) \to 0$$

is exact; since $f^{-1}(y)$ is a regular scheme, $L(y)$ is a free $\mathscr{O}_{f^{-1}(y),x}$-module of finite rank. Since L is also flat as a K^+-module, [65, Ch.IV, Prop.11.3.7] and Nakayama's lemma show that any set of elements of L lifting a basis of $L(y)$, is a free basis of L.

(i): Locally on X we can find an epimorphism $\mathscr{O}_X^{\oplus n} \to \mathscr{F}$, whose kernel $\mathscr{G}$ is again a coherent $\mathscr{O}_X$-module, since $\mathscr{O}_X$ is coherent. Clearly it suffices to show that $\mathscr{G}$ admits, locally on X, a finite free resolution of length $\leq n$, which holds by (ii), since $\mathscr{G}$ is f-flat.

(iii): Suppose $\mathscr{F}$ is reflexive at x; by remark 11.3.5 we can find a left exact sequence

$$0 \to \mathscr{F}_x \to \mathscr{O}_{X,x}^{\oplus m} \xrightarrow{\alpha} \mathscr{O}_{X,x}^{\oplus n}.$$

It follows that $\mathrm{proj.dim}_{\mathscr{O}_{X,x}} \mathscr{F}_x = \max(0, \mathrm{proj.dim}_{\mathscr{O}_{X,x}}(\mathrm{Coker}\,\alpha) - 2) \leq \max(0, n-1)$, by (i).

(iv): Suppose that $\mathscr{F}$ is generically invertible, let $j : U \to X$ be the maximal open immersion such that $j^*\mathscr{F}$ is an invertible $\mathscr{O}_U$-module, and set $Z := X \setminus U$. Under the current assumptions, X is reduced; it follows easily that U is the set where $\mathrm{rk}\,\mathscr{F} = 1$, and that j is quasi-compact (since the rank function is constructible : see (11.3.10)). It follows from (iii) that $Z \cap f^{-1}(y)$ has codimension ≥ 2 in $f^{-1}(y)$, for every $y \in S$. Hence the conditions of corollary 11.3.9 are fulfilled, and furthermore $\mathscr{F}[0]$ is a perfect complex by (ii), so the invertible $\mathscr{O}_X$-module $\det \mathscr{F}$ is well defined (lemma 11.3.14). Then (iv) follows from the natural isomorphisms :

$$\mathscr{F} \xrightarrow{\sim} j_*j^*\mathscr{F} \xrightarrow{\sim} j_*\mathsf{det}(j^*\mathscr{F}[0]) \xrightarrow{\sim} j_*j^*\mathsf{det}\,\mathscr{F}[0] \xleftarrow{\sim} \mathsf{det}\,\mathscr{F}[0].$$

(v): Let $y' \in S(y)$ be any point; we need to check that $f_x^{-1}(y')$ is an irreducible regular scheme, and after replacing K^+ by a quotient, we may assume that y' is the generic point of S; after further replacing K^+ by a localization, we may also assume that y is the closed point of S. Set $A := \mathscr{O}_{X,x}$; by assumption, $f_x^{-1}(y)$ is a regular scheme, hence $\mathfrak{p} := \mathfrak{m}_K A$ is a prime ideal of A, so that $A_\mathfrak{p}$ is a valuation ring (proposition 9.1.34(ii)), and especially, it is a domain. Moreover, [65, Ch.IV, Prop.11.3.7] implies that for every $h \in A \setminus \mathfrak{p}$, scalar multiplication by h

is injective on A; thus, the localization map $A \to A_\mathfrak{p}$ is injective as well, so A is a domain. This already proves that $f_x^{-1}(y') = \operatorname{Spec} K \otimes_{K^+} A$ is irreducible. Next, let $\mathfrak{q} \subset A$ be any prime ideal with $\mathfrak{q} \cap K^+ = \{0\}$, and M any $A_\mathfrak{q}$-module of finite type; pick a finitely presented A-module N with an isomorphism $N_\mathfrak{q} \xrightarrow{\sim} M$. By (i), there exists a finite resolution $L_\bullet$ of N, consisting of free A-modules of finite rank; then $A_\mathfrak{q} \otimes_A L_\bullet$ is a finite resolution of M by free $A_\mathfrak{q}$-modules of finite rank. By Serre's theorem [126, Th.19.2], this shows that $A_\mathfrak{q}$ is regular, as required. □

Proposition 11.4.3. *Let X be a scheme, $\mathscr{E}$ a locally free $\mathscr{O}_X$-module of finite rank > 1 at every point of X, and denote by $\pi : \mathbb{P}(\mathscr{E}) \to X$ the structure morphism of the projective X-scheme associated with $\mathscr{E}$. Then the map*

$$H^0(X, \mathbb{Z}) \oplus \operatorname{Pic} X \to \operatorname{Pic} \mathbb{P}(\mathscr{E}) \qquad (r, \mathscr{L}) \mapsto \mathscr{O}_{\mathbb{P}(\mathscr{E})}(r) \otimes \pi^*\mathscr{L}$$

is an isomorphism of abelian groups.

Proof. To begin with, let us recall the following well known :

Claim 11.4.4. Let $f : Y \to T$ be a proper morphism of noetherian schemes, $\mathscr{F}$ a coherent f-flat $\mathscr{O}_Y$-module, $t \in T$ a point, $p \in \mathbb{N}$ an integer, and denote $i_t : f^{-1}(t) \to Y$ the natural immersion. Suppose that $H^{p+1}(f^{-1}(t), i_t^*\mathscr{F}) = 0$. Then the natural map

$$(R^p f_* \mathscr{F})_t \to H^p(f^{-1}(t), i_t^* \mathscr{F})$$

is surjective.

Proof of the claim. In light of [61, Ch.III, Prop.1.4.15], we easily reduce to the case where T is a local scheme, say $T = \operatorname{Spec} A$ for some noetherian local ring A, and t is the closed point of T. Set $k := \kappa(t)$, and for every $q \in \mathbb{N}$, consider the functor

$$F_{-q} : A\text{-}\mathbf{Mod} \to A\text{-}\mathbf{Mod} \qquad M \mapsto H^0(T, R^p f_*(f^* M^\sim \otimes_{\mathscr{O}_Y} \mathscr{F}))$$

(notation of (10.3)). Since $\mathscr{F}$ is f-flat, the system $(F_{-q} \mid q \in \mathbb{N})$ defines a homological functor. The assertion is that $F_{-p-1}(k) = 0$, in which case [62, Ch.III, Cor.7.5.3] (together with [61, Ch.III, Th.3.2.1 and Th.4.1.5]) says that $F_{-p-1}(M) = 0$ for every A-module M. Hence, F_{-p-1} is trivially an exact functor, and it follows that the natural map $F_{-p}(A) \to F_{-p}(k)$ is surjective ([62, Ch.III, Prop.7.5.4]), which is the claim. ◊

Now, the injectivity of the stated map is clear (details left to the reader). For the surjectivity, let $\mathscr{G}$ be an invertible $\mathscr{O}_{\mathbb{P}(\mathscr{E})}$-module, $x \in X$ any point, and $U \subset X$ an affine open neighborhood of x in X such that $\mathscr{E}_{|U}$ is a free $\mathscr{O}_X$-module of constant rank $n+1$ (for some $n \in \mathbb{N} \setminus \{0\}$), so that we have an isomorphism of U-schemes $\mathbb{P}^n_U \xrightarrow{\sim} \mathbb{P}(\mathscr{E}_{|U})$, where $\mathbb{P}^n_U$ denotes the projective scheme of relative dimension n over U. Write U as the limit of a filtered system $(U_\lambda \mid \lambda \in \Lambda)$ of affine noetherian schemes, and for every $\lambda \in \Lambda$, let $p_\lambda : \mathbb{P}^n_U \to \mathbb{P}^n_{U_\lambda}$ be the induced morphism, and $\pi_\lambda : \mathbb{P}^n_{U_\lambda} \to U_\lambda$ the projection; for some $\mu \in \Lambda$, we may find an invertible $\mathscr{O}_{\mathbb{P}^n_{U_\mu}}$-module $\mathscr{G}_\mu$ with an isomorphism $\mathscr{G}_{|\mathbb{P}^n_U} \xrightarrow{\sim} p_\mu^* \mathscr{G}_\mu$. Let $y \in U_\mu$ be the image of x, and $i_y : \mathbb{P}^n_{\kappa(y)} \to \mathbb{P}^n_{U_\mu}$ the natural immersion; we have $i_y^* \mathscr{G}_\mu \simeq \mathscr{O}_{\mathbb{P}^n_{\kappa(y)}}(r)$ for some $r \in \mathbb{Z}$. Set $\mathscr{F} := \mathscr{G}_\mu(-r)$, and notice that $H^1(\mathbb{P}^n_{\kappa(y)}, i_y^* \mathscr{F}) = 0$; moreover, $H := H^0(\mathbb{P}^n_{\kappa(y)}, i_y^* \mathscr{F})$ is a one-dimensional $\kappa(y)$-vector space. From claim 11.4.4, it follows that there exist an open neighborhood V_μ of y in U_μ, and a section $s \in H^0(\pi_\mu^{-1} V_\mu, \mathscr{F})$ mapping to a generator of H. The section s defines a map of $\mathscr{O}_{\mathbb{P}^n_{V_\mu}}$-modules $\varphi : \mathscr{O}_{\mathbb{P}^n_{V_\mu}}(r) \to \mathscr{G}_{\mu|\pi_\mu^{-1} V_\mu}$. Furthermore, since $i_y^* \mathscr{F} \simeq \mathscr{O}_{\mathbb{P}^n_{\kappa(y)}}$, it is easily seen that φ restricts to an isomorphism on some open subset of the form $\pi_\mu^{-1} W_\mu$, where $W_\mu \subset V_\mu$ is an affine open neighborhood of y in U_μ. Then $W := W_\mu \times_{U_\mu} U$ is an affine open neighborhood of x in X, and φ induces an isomorphism $\mathscr{O}_{\mathbb{P}^n_W}(r) \xrightarrow{\sim} \mathscr{G}_{|\pi^{-1} W}$ of $\mathscr{O}_{\mathbb{P}^n_W}$-modules.

Summing up, since $x \in X$ is arbitrary, we may find an affine open covering $X = \bigcup_{i \in I} U_i$, and for every $i \in I$ an integer r_i and an isomorphism $\varphi_i : \mathscr{O}_{\mathbb{P}(\mathscr{E}_i)}(r_i) \xrightarrow{\sim} \mathscr{G}_{|\mathbb{P}(\mathscr{E}_i)}$ of $\mathscr{O}_{\mathbb{P}(\mathscr{E}_i)}$-modules, with $\mathscr{E}_i := \mathscr{E}_{|U_i}$. Then, for every $i, j \in I$ such that $U_{ij} := U_i \cap U_j \neq \varnothing$, the composition of the restriction of φ_i followed by the restriction of φ_j^{-1}, is an isomorphism of $\mathscr{O}_{\mathbb{P}(\mathscr{E}_{ij})}$-modules

$$\varphi_{ij} : \mathscr{O}_{\mathbb{P}(\mathscr{E}_{ij})}(r_i) \xrightarrow{\sim} \mathscr{O}_{\mathbb{P}(\mathscr{E}_{ij})}(r_j) \qquad \text{where } \mathscr{E}_{ij} := \mathscr{E}_{|U_{ij}}.$$

This already implies that $r_i = r_j$ whenever $U_{ij} \neq \varnothing$, hence the rule : $x \mapsto r_i$ if $x \in U_i$ yields a well defined continuous map $r : X \to \mathbb{Z}$. We are therefore easily reduced to the case where r is constant on X, and after tensoring by $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(-r)$, we may even assume that $r = 0$.

In this case, each φ_i is an isomorphism $\varphi_i : \pi_i^*\mathscr{L}_i \xrightarrow{\sim} \mathscr{G}_{|\mathbb{P}(\mathscr{E}_i)}$, where $\pi_i : \mathbb{P}(\mathscr{E}_i) \to U_i$ denotes the restriction of π, and $\mathscr{L}_i$ is an invertible $\mathscr{O}_{U_i}$-module. Then, for each $i, j \in I$ let also $\pi_{ij} : \mathbb{P}(\mathscr{E}_{ij}) \to U_{ij}$ be the restriction of π; we get unique isomorphisms of $\mathscr{O}_{\mathbb{P}(\mathscr{E}_{ij})}$-modules

$$\varphi_{ij} : \pi_{ij}^*\mathscr{L}_{i|U_{ij}} \xrightarrow{\sim} \pi_{ij}^*\mathscr{L}_{j|U_{ij}} \qquad \text{such that} \qquad \varphi_{j|U_{ij}} \circ \varphi_{ij} = \varphi_{i|U_{ij}} \qquad \text{for all } i, j \in I$$

whence a unique system of isomorphisms of $\mathscr{O}_{U_{ij}}$-modules :

$$\psi_{ij} : \mathscr{L}_{i|U_{ij}} \xrightarrow{\sim} \mathscr{L}_{j|U_{ij}} \qquad \text{such that} \qquad \varphi_{ij} = \pi_{ij}^*\psi_{ij} \qquad \text{for every } i, j \in I.$$

Now, for every $i, j, k \in I$ let $U_{ijk} := U_{ij} \cap U_k$ and $\mathscr{E}_{ijk} := \mathscr{E}_{|U_{ijk}}$; it is easily seen that

$$\varphi_{jk|\mathbb{P}(\mathscr{E}_{ijk})} \circ \varphi_{ij|\mathbb{P}(\mathscr{E}_{ijk})} = \varphi_{ik|\mathbb{P}(\mathscr{E}_{ijk})} \quad \text{whence} \quad \psi_{jk|U_{ijk}} \circ \psi_{ij|U_{ijk}} = \psi_{ik|U_{ijk}}.$$

Hence, there exists an invertible $\mathscr{O}_X$-module $\mathscr{L}$ with a system of isomorphisms :

$$(\lambda_i : \mathscr{L}_{|U_i} \xrightarrow{\sim} \mathscr{L}_i \mid i \in I) \qquad \text{such that} \qquad \psi_{ij} \circ \lambda_{i|U_{ij}} = \lambda_{j|U_{ij}} \qquad \text{for every } i, j \in I.$$

It follows that the system of induced isomorphisms :

$$\varphi_i \circ \pi_i^*(\lambda_i) : (\pi^*\mathscr{L})_{|\mathbb{P}(\mathscr{E}_i)} \xrightarrow{\sim} \mathscr{G}_{|\mathbb{P}(\mathscr{E}_i)}$$

glues to yield an isomorphism of $\mathscr{O}_{\mathbb{P}(\mathscr{E})}$-modules $\pi^*\mathscr{L} \xrightarrow{\sim} \mathscr{G}$, and that concludes the proof. □

Corollary 11.4.5. *Let $f : X \to S$ be a smooth morphism of finite presentation, set $\mathbb{G}_{m,X} := X \times_S \operatorname{Spec} K^+[T, T^{-1}]$, and denote by $\pi : \mathbb{G}_{m,X} \to X$ the natural morphism. Then the map*

$$\operatorname{Pic} X \to \operatorname{Pic} \mathbb{G}_{m,X} \qquad \mathscr{L} \mapsto \pi^*\mathscr{L}$$

is an isomorphism.

Proof. To start out, the map is injective, since π admits a section. Next, let $j : \mathbb{G}_{m,X} \to \mathbb{P}^1_X$ be the natural open immersion, and denote again by $\pi : \mathbb{P}^1_X \to X$ the natural projection. According to theorem 10.3.25 and proposition 11.4.2(i), every invertible $\mathscr{O}_{\mathbb{G}_{m,X}}$-module extends to an invertible $\mathscr{O}_{\mathbb{P}^1_X}$-module (namely the determinant of a coherent extension). Since $j^*\mathscr{O}_{\mathbb{P}^1_X}(n) \simeq \mathscr{O}_{\mathbb{G}_{m,X}}$, the claim follows from proposition 11.4.3. □

Remark 11.4.6. The proof of proposition 11.4.3 follows [131, p.20] and [132, Lecture 13, Prop.3]. Of course, statements related to corollary 11.4.5 abound in the literature, and more general results are available: see *e.g.* [16] and [156].

Proposition 11.4.7. *Let $f : X \to S$ be a smooth morphism of finite type, $j : U \to X$ an open immersion, and $Z := X \setminus U$; suppose that X and U are quasi-compact and quasi-separated. Then:*

- (i) *Every invertible $\mathscr{O}_U$-module extends to an invertible $\mathscr{O}_X$-module.*
- (ii) *Suppose furthermore, that:*
  - (a) *For every point $y \in S$, the codimension of $Z \cap f^{-1}(y)$ in $f^{-1}(y)$ is ≥ 1, and*
  - (b) *The codimension of $Z \cap f^{-1}(\eta)$ in $f^{-1}(\eta)$ is ≥ 2.*

  Then the restriction functors:

(11.4.8) $$\mathscr{O}_X\text{-}\mathbf{Rflx} \to \mathscr{O}_U\text{-}\mathbf{Rflx} \qquad \mathbf{Pic}\, X \to \mathbf{Pic}\, U$$

are equivalences.

Proof. (i): First let us notice :

Claim 11.4.9. X is a finite disjoint union of normal integral schemes.

Proof of the claim. Since X is quasi-compact, the assertion is easily reduced to the case where $X = \mathrm{Spec}\, A$, for a smooth K^+-algebra A. Then A is integrally closed in $A_K := K \otimes_{K^+} A$, by proposition 9.8.3, and A_K is smooth over K, so it is a product of finitely many normal domains. The idempotents corresponding to this decomposition are integral over A, hence they lie in A. Thus, A is itself a product of finitely many normal domains, which is the claim. ◊

In view of claim 11.4.9, we may assume from start that X is integral, and then U is either empty (in which case there is nothing to prove), or dense in X, in which case the assertion follows from propositions 11.3.8(i) and 11.4.2(iv).

(ii): We begin with the following:

Claim 11.4.10. Under the assumptions of (ii), the functors (11.4.8) are faithful.

Proof of the claim. Since f is smooth, all the stalks of $\mathscr{O}_X$ are reduced ([66, Ch.IV, Prop.17.5.7]). Since every point of Z is specialization of a point of U, the claim follows easily from remark 11.3.5. ◊

Under assumptions (a) and (b), the conditions of corollary 11.3.9 are fulfilled, and one deduces that (11.4.8) are full functors, hence fully faithful, in view of claim 11.4.10. The essential surjectivity of the restriction functor for reflexive $\mathscr{O}_X$-modules is already known, by proposition 11.3.8(i). □

Lemma 11.4.11. *Let $f : X \to S$ be a flat, finitely presented morphism, and denote by $U \subset X$ the maximal open subset such that the restriction $f_{|U} : U \to S$ is smooth. Suppose that the following two conditions hold:*

(a) *For every point $y \in S$, the fibre $f^{-1}(y)$ is geometrically reduced.*

(b) *The generic fibre $f^{-1}(\eta)$ is geometrically normal.*

Then the restriction functor induces an equivalence of categories (notation of (11.3.10)*) :*

$$\mathbf{Div}\, X \xrightarrow{\sim} \mathbf{Pic}\, U. \tag{11.4.12}$$

Proof. Let $\mathscr{F}$ be any generically invertible reflexive $\mathscr{O}_X$-module; it follows from proposition 11.4.2(iv) that $\mathscr{F}_{|U}$ is an invertible $\mathscr{O}_U$-module, so the functor (11.4.12) is well defined. Let $y \in S$ be any point; condition (a) says in particular that the fibre $f^{-1}(y)$ is generically smooth over $\kappa(y)$ ([126, Th.28.7]). Then it follows from [66, Ch.IV, Th.17.5.1] that $U \cap f^{-1}(y)$ is a dense open subset of $f^{-1}(y)$, and if $x \in f^{-1}(y)\backslash U$, then the depth of $\mathscr{O}_{f^{-1}(y),x}$ is ≥ 1, since the latter is a reduced local ring of dimension ≥ 1. Similarly, condition (b) and Serre's criterion [126, Ch.8, Th.23.8] say that for every $x \in f^{-1}(\eta)\backslash U$, the local ring $\mathscr{O}_{f^{-1}(\eta),x}$ has depth ≥ 2. Furthermore, [65, Ch.IV, Prop.9.9.4] implies that the open immersion $j : U \to X$ is quasi-compact; summing up, we see that all the conditions of corollary 11.3.9 are fulfilled, so that $\mathscr{F} = j_*j^*\mathscr{F}$ for every reflexive $\mathscr{O}_X$-module $\mathscr{F}$. This already means that (11.4.12) is fully faithful. To conclude, it suffices to invoke proposition 11.3.8(i). □

11.4.13. Let $K(\mathsf{T})$ be the fraction field of the free polynomial K-algebra $K[\mathsf{T}]$. For every $\gamma \in \Gamma$ one can define an extension of $|\cdot|$ to a *Gauss valuation* $|\cdot|_{0,\gamma} : K(\mathsf{T}) \to \Gamma$ ([75, Ex.6.1.4(iii)]). If $f(\mathsf{T}) := \sum_{i=0}^{d} a_i \mathsf{T}^i$ is any polynomial, then $|f(\mathsf{T})|_{0,\gamma} = \max\{|a_i| \cdot \gamma^i \mid i = 0, \dots, d\}$. We let $V(\gamma)$ be the valuation ring of $|\cdot|_{0,\gamma}$, and set

$$R(\gamma) := K[\mathsf{T}, \mathsf{T}^{-1}] \cap V(1) \cap V(\gamma).$$

Since $K[\mathsf{T}, \mathsf{T}^{-1}]$ is a Dedekind domain, $R(\gamma)$ is an intersection of valuation rings of the field $K(\mathsf{T})$, hence it is a normal domain. By inspecting the definition we see that $R(\gamma)$ consists of all the elements of the form $f(\mathsf{T}) := \sum_{i=-n}^{n} a_i \mathsf{T}^i$, such that $|a_i| \leq 1$ and $|a_i| \cdot \gamma^i \leq 1$ for every $i = -n, \dots, n$. Suppose now that $\gamma \leq 1$, and choose $c \in K^+$ with $|c| = \gamma$; then every such $f(\mathsf{T})$ can be written uniquely in the form $\sum_{i=0}^{n} a_i \mathsf{T}^i + \sum_{j=1}^{n} b_j (c\mathsf{T}^{-1})^j$, where $a_i, b_j \in K^+$ for every $i, j \leq n$. Conversely, every such expression yields an element of $R(\gamma)$. In other words, we obtain a surjection of K^+-algebras $K^+[\mathsf{X}, \mathsf{Y}] \to R(\gamma)$ by the rule: $\mathsf{X} \mapsto \mathsf{T}$, $\mathsf{Y} \mapsto c\mathsf{T}^{-1}$. Obviously the kernel of this map contains the ideal $(\mathsf{XY} - c)$, and we leave to the reader the verification that the induced map

$$K^+[\mathsf{X}, \mathsf{Y}]/(\mathsf{XY} - c) \to R(\gamma) \qquad (11.4.14)$$

is indeed an isomorphism.

11.4.15. Throughout the following discussion, we shall assume that $\gamma \leq 1$. Every $\delta \in \Gamma$ with $\gamma \leq \delta \leq 1$, determines a prime ideal $\mathfrak{p}(\delta) := \{f \in R(\gamma) \mid |f|_{0,\delta} < 1\} \subset R(\gamma)$, such that $\mathfrak{m}_K R(\gamma) \subset \mathfrak{p}(\delta)$. Then it is easy to see that $R(\gamma)_{\mathfrak{p}(\delta)} \subset V(\delta)$, and moreover :

$$R(\gamma)_{\mathfrak{p}(\gamma)} = V(\gamma) \qquad R(\gamma)_{\mathfrak{p}(1)} = V(1)$$

since $V(1)$ (resp. $V(\gamma)$) is already a localization of $K^+[\mathsf{T}]$ (resp. of $K^+[c\mathsf{T}^{-1}]$). In case $\gamma < 1$, (11.4.14) implies that $R(\gamma) \otimes_{K^+} \kappa \simeq \kappa[\mathsf{X}, \mathsf{Y}]/(\mathsf{XY})$, and it follows easily that $\mathfrak{p}(1)$ and $\mathfrak{p}(\gamma)$ correspond to the two minimal prime ideals of $\kappa[\mathsf{X}, \mathsf{Y}]/(\mathsf{XY})$. In case $\gamma = 1$, we have $R(1) \otimes_{K^+} \kappa \simeq \kappa[\mathsf{X}, \mathsf{X}^{-1}]$, and again $\mathfrak{p}(1)$ corresponds to the generic point of $\mathrm{Spec}\, \kappa[\mathsf{X}, \mathsf{X}^{-1}]$. Notice that the natural morphism

$$f_\gamma : \mathbb{T}_K(\gamma) := \mathrm{Spec}\, R(\gamma) \to S$$

restricts to a smooth morphism $f_\gamma^{-1}(\eta) \to \mathrm{Spec}\, K$; moreover the closed fibre $f_\gamma^{-1}(s)$ is geometrically reduced. Notice also that $\mathbb{T}_K(\gamma) \times_S \mathrm{Spec}\, E^+ \simeq \mathbb{T}_E(\gamma)$ for every extension of valued fields $K \subset E$. In the following, we will write just $\mathbb{T}(\gamma)$ in place of $\mathbb{T}_K(\gamma)$, unless we have to deal with more than one base ring.

Proposition 11.4.16. *Keep the notation of* (11.4.15)*, and let $g : X \to \mathbb{T}(\gamma)$ be an étale morphism, $\mathscr{F}$ a coherent $\mathscr{O}_X$-module. Set $h := f_\gamma \circ g : X \to S$ and denote by $i_s : h^{-1}(s) \to X$ the natural morphism. Then $\mathscr{F}$ is reflexive at the point $x \in h^{-1}(s)$ if and only if the following three conditions hold:*

(a) *$\mathscr{F}$ is h-flat at the point x.*
(b) *$\mathscr{F}_x \otimes_{K^+} K$ is a reflexive $\mathscr{O}_{X,x} \otimes_{K^+} K$-module.*
(c) *The $\mathscr{O}_{h^{-1}(s),x}$-module $i_s^* \mathscr{F}_x$ satisfies condition S_1 (see definition* 10.5.1(iii)*).*

Proof. Suppose that $\mathscr{F}$ is reflexive at the point x; then it is easy to check that (a) and (b) hold. We prove (c): by remark 11.3.5 we can find a left exact sequence

$$0 \to \mathscr{F}_x \xrightarrow{\alpha} \mathscr{O}_{X,x}^{\oplus m} \xrightarrow{\beta} \mathscr{O}_{X,x}^{\oplus n}.$$

Then $\mathrm{Im}\, \beta$ is a flat K^+-module, since it is a submodule of the flat K^+-module $\mathscr{O}_{X,x}^{\oplus n}$; hence $\alpha \otimes_{K^+} \mathbf{1}_\kappa : i_s^* \mathscr{F}_x \to \mathscr{O}_{h^{-1}(s),x}^{\oplus m}$ is still injective, so we are reduced to showing that $h^{-1}(s)$ is a reduced scheme, which follows from [66, Ch.IV, Prop.17.5.7] and the fact that $f_\gamma^{-1}(s)$ is reduced.

Conversely, suppose that conditions (a)–(c) hold.

Claim 11.4.17. Let ξ be the generic point of an irreducible component of $h^{-1}(s)$. Then:

(i) $\mathscr{F}_x$ is a torsion-free $\mathscr{O}_{X,x}$-module.
(ii) $\mathscr{O}_{X,\xi}$ is a valuation ring.
(iii) Suppose that the closure of ξ contains x. Then $\mathscr{F}_\xi$ is a free $\mathscr{O}_{X,\xi}$-module of finite rank.

Proof of the claim. (i): By (a), the natural map $\mathscr{F}_x \to \mathscr{F}_x \otimes_{K^+} K$ is injective; since (b) implies that $\mathscr{F}_x \otimes_{K^+} K$ is a torsion-free $\mathscr{O}_{X,x}$-module, the same must then hold for $\mathscr{F}_x$.

(ii) follows from proposition 9.1.34(ii).

(iii): Suppose that $x \in \overline{\{\xi\}}$. We derive easily from (i) that $\mathscr{F}_\xi$ is a torsion-free $\mathscr{O}_{X,\xi}$-module, so the assertion follows from (ii) and [34, Ch.VI, §3, n.6, Lemma 1]. ♢

By (b), the morphism $\beta_{\mathscr{F},x} \otimes_{K^+} \mathbf{1}_K : \mathscr{F}_x \otimes_{K^+} K \to \mathscr{F}_x^{\vee\vee} \otimes_{K^+} K$ is an isomorphism (notation of (11.3)); since $\mathscr{F}$ is h-flat at x, we deduce easily that $\beta_{\mathscr{F},x}$ is injective and $C := \operatorname{Coker} \beta_{\mathscr{F},x}$ is a torsion K^+-module. To conclude, it remains only to show:

Claim 11.4.18. C is a flat K^+-module.

Proof of the claim. In view of lemma 7.11.35, it suffices to show that $\operatorname{Tor}_1^{K^+}(C, \kappa(s)) = 0$. However, from the foregoing we derive a left exact sequence

$$0 \longrightarrow \operatorname{Tor}_1^{K^+}(C, \kappa(s)) \longrightarrow \mathscr{F}_x \otimes_{K^+} \kappa(s) \xrightarrow{\beta_{\mathscr{F},x} \otimes_{K^+} \kappa(s)} \mathscr{F}_x^{\vee\vee} \otimes_{K^+} \kappa(s).$$

We are thus reduced to showing that $\beta_{\mathscr{F},x} \otimes_{K^+} \kappa(s)$ is an injective map. In view of condition (c), it then suffices to prove that $\beta_{\mathscr{F},\xi}$ is an isomorphism, whenever ξ is the generic point of an irreducible component of $h^{-1}(s)$ containing x. The latter assertion holds by virtue of claim 11.4.17(iii). □

11.4.19. For a given $\rho \in \Gamma$, let us pick $a \in K \setminus \{0\}$ such that $|a| = \rho$; we define the fractional ideal $I(\rho) \subset K[\mathsf{T}, \mathsf{T}^{-1}]$ as the $R(\gamma)$-submodule generated by T and a. The module $I(\rho)$ determines a quasi-coherent $\mathscr{O}_{\mathbb{T}(\gamma)}$-module $\mathscr{I}(\rho)$.

Lemma 11.4.20. *With the notation of* (11.4.19)*:*

(i) *$\mathscr{I}(\rho)$ is a reflexive $\mathscr{O}_{\mathbb{T}(\gamma)}$-module for every $\rho \in \Gamma$.*

(ii) *There exists a short exact sequence of $R(\gamma)$-modules:*

$$0 \to I(\rho^{-1}\gamma) \to R(\gamma)^{\oplus 2} \to I(\rho) \to 0.$$

Proof. To start with, let $a \in K \setminus \{0\}$ with $|a| = \rho$.

Claim 11.4.21. If either $\rho \geq 1$ or $\rho \leq \gamma$, then $I(\rho)$ and $I(\rho^{-1}\gamma)$ are rank one, free $R(\gamma)$-modules.

Proof of the claim. If $\rho \geq 1$ (resp. $\rho \leq \gamma$) then $\rho^{-1}\gamma \leq \gamma$ (resp. $\rho^{-1}\gamma \geq 1$), hence it suffices to show that $I(\rho)$ is free of rank one, in both cases. Suppose first that $\rho \geq 1$; in this case, multiplication by a^{-1} yields an isomorphism of $R(\gamma)$-modules $I(\rho) \xrightarrow{\sim} R(\gamma)$. Next, suppose that $\rho \leq \gamma$. Then $I(\rho)$ is the ideal (T, a), where $a = c \cdot b$ for some $b \in K^+$ and $|c| = \gamma$. Therefore $I(\rho) = (\mathsf{T}, \mathsf{T} \cdot (c\mathsf{T}^{-1}) \cdot b) = \mathsf{T} \cdot (1, c\mathsf{T}^{-1}b) = \mathsf{T}R(\gamma)$, and again $I(\rho)$ is a free $R(\gamma)$-module of rank one. ♢

In view of claim 11.4.21, we may assume that $\gamma < \rho < 1$. Let (e_1, e_2) be the canonical basis of the free $R(\gamma)$-module $R(\gamma)^{\oplus 2}$; we consider the $R(\gamma)$-linear surjection $\pi : R(\gamma)^{\oplus 2} \to I(\rho)$ determined by the rule: $e_1 \mapsto \mathsf{T}$, $e_2 \mapsto a$. Clearly $\operatorname{Ker} \pi$ contains the submodule $M(\rho) \subset R(\gamma)^{\oplus 2}$ generated by:

$$f_1 := ae_1 - \mathsf{T}e_2 \quad \text{and} \quad f_2 := c\mathsf{T}^{-1}e_1 - ca^{-1}e_2.$$

Let $\mathscr{N}$ be the quasi-coherent $\mathscr{O}_{\mathbb{T}(\gamma)}$-module associated with $N := R(\gamma)^{\oplus 2}/M(\rho)$.

Claim 11.4.22. With the foregoing notation:

(i) N is a flat K^+-module.

(ii) $K^+[c^{-1}] \otimes_{K^+} N$ is a free $K^+[c^{-1}] \otimes_{K^+} R(\gamma)$-module of rank one.

(iii) $K \otimes_{K^+} M(\rho) = K \otimes_{K^+} \operatorname{Ker} \pi$.

Proof of the claim. (i): We let $\mathrm{gr}_\bullet R(\gamma)$ be the T-adic grading on $R(\gamma)$ (*i.e.* $\mathrm{gr}_i R(\gamma) = \mathsf{T}^i \cdot K \cap R(\gamma)$ for every $i \in \mathbb{Z}$), and we define a compatible grading on $R(\gamma)^{\oplus 2}$ by setting: $\mathrm{gr}_i R(\gamma)^{\oplus 2} := (\mathrm{gr}_{i-1} R(\gamma) \cdot e_1) \oplus (\mathrm{gr}_i R(\gamma) \cdot e_2)$ for every $i \in \mathbb{Z}$. Since f_1 and f_2 are homogeneous elements, we deduce by restriction a grading $\mathrm{gr}_\bullet M(\rho)$ on $M(\rho)$, and a quotient grading $\mathrm{gr}_\bullet N$ on N, whence a short exact sequence of graded K^+-modules:

$$0 \to \mathrm{gr}_\bullet M(\rho) \xrightarrow{\mathrm{gr}_\bullet j} \mathrm{gr}_\bullet R(\gamma)^{\oplus 2} \to \mathrm{gr}_\bullet N \to 0.$$

However, by inspecting the definitions, it is easy to see that $\mathrm{gr}_\bullet j$ is a split injective map of free K^+-modules, hence $\mathrm{gr}_\bullet N$ is a free K^+-module, and then the same holds for N.

(ii) is easy and shall be left to the reader.

(iii): Similarly, one checks easily that $K \otimes_{K^+} I(\rho)$ is a free $K \otimes_{K^+} R(\gamma)$-module of rank one; then, by (ii) the quotient map $K \otimes_{K^+} N \to K \otimes_{K^+} I(\rho)$ is necessarily an isomorphism, whence the assertion. ◊

Claim 11.4.23. $\mathscr{N}$ is a reflexive $\mathscr{O}_{\mathbb{T}(\gamma)}$-module.

Proof of the claim. Since $\mathscr{N}$ is coherent, it suffices to show that $\mathscr{N}_x$ is a reflexive $\mathscr{O}_{\mathbb{T}(\gamma),x}$-module, for every $x \in \mathbb{T}(\gamma)$ (lemma 11.3.2(ii)). Let $y := f_\gamma(x)$; we may then replace $\mathscr{N}$ by its restriction to $\mathbb{T}(\gamma) \times_S S(y)$, which allows to assume that $y = s$ is the closed point of S. In this case, we can apply the criterion of proposition 11.4.16 to the morphism $f_\gamma : \mathbb{T}(\gamma) \to S$. We already know from claim 11.4.22(i) that $\mathscr{N}$ is f_γ-flat. Moreover, by claim 11.4.22(ii) we see that the restriction of $\mathscr{N}$ to $f_\gamma^{-1}(\eta)$ is reflexive. As $\gamma < \rho < 1$, by inspecting the definition and using the presentation (11.4.14), we deduce an isomorphism :

$$\kappa \otimes_{K^+} N \simeq \overline{R}^{\oplus 2}/(\mathsf{X}e_2, \mathsf{Y}e_1) \simeq (\overline{R}/\mathsf{X}\overline{R}) \oplus (\overline{R}/\mathsf{Y}\overline{R})$$

where $\overline{R} := \kappa[\mathsf{X}, \mathsf{Y}]/(\mathsf{XY}) \simeq \kappa \otimes_{K^+} R(\gamma)$. Thus, the $\overline{R}$-module $\kappa \otimes_{K^+} N$ satisfies condition S_1, whence the claim. ◊

It follows from claim 11.4.22(i,iii) that the quotient map $N \to I(\rho)$ is an isomorphism, so $\mathscr{I}(\rho)$ is reflexive, by claim 11.4.23. Next, let us define an $R(\gamma)$-linear surjection $\pi' : R(\gamma)^{\oplus 2} \to M(\rho)$ by the rule: $e_i \mapsto f_i$ for $i = 1, 2$. One checks easily that $\mathrm{Ker}\, \pi'$ contains the submodule generated by the elements $ca^{-1}e_1 - Te_2$ and $ce_1 - aTe_2$, and the latter is none else than the module $M(\rho^{-1}\gamma)$, according to our notation (notice that $\gamma < \rho^{-1}\gamma < 1$). We deduce a surjection of torsion-free $R(\gamma)$-modules $I(\rho^{-1}\gamma) \xrightarrow{\sim} R(\gamma)^{\oplus 2}/M(\rho^{-1}\gamma) \to M(\rho)$, which induces an isomorphism after tensoring by K, therefore $I(\rho^{-1}\gamma) \xrightarrow{\sim} M(\rho)$, which establishes (ii). □

11.4.24. Let $\mathbb{T}(\gamma)_{\mathrm{sm}} \subset \mathbb{T}(\gamma)$ be the largest open subset which is smooth over S. Set $S_\gamma := \mathrm{Spec}\, K^+/cK^+$; it is easy to see that $f_\gamma^{-1}(S \setminus S_\gamma) \subset \mathbb{T}(\gamma)_{\mathrm{sm}}$, and for every $y \in S_\gamma$, the difference $f_\gamma^{-1}(y) \setminus \mathbb{T}(\gamma)_{\mathrm{sm}}$ consists of a single point.

Proposition 11.4.25. *Let $\Delta(\gamma) \subset \Gamma$ be the smallest convex subgroup containing γ. Then there is a natural isomorphism of groups:*

$$\mathrm{Pic}\, \mathbb{T}(\gamma)_{\mathrm{sm}} \xrightarrow{\sim} \Delta(\gamma)/\gamma^{\mathbb{Z}}.$$

Proof. We consider the affine covering of $\mathbb{T}(\gamma)_{\mathrm{sm}}$ consisting of the two open subsets

$$U := \mathrm{Spec}\, K^+[\mathsf{T}, \mathsf{T}^{-1}] \qquad \text{and} \qquad V := \mathrm{Spec}\, K^+[c\mathsf{T}^{-1}, c^{-1}\mathsf{T}]$$

with intersection $U \cap V = \mathrm{Spec}\, K^+[c^{-1}, \mathsf{T}, \mathsf{T}^{-1}]$. We notice that both U and V are S-isomorphic to $\mathbb{G}_{m,S}$, and therefore

$$\mathrm{Pic}\, U = \mathrm{Pic}\, V = 0 \tag{11.4.26}$$

by corollary 11.4.5. From (11.4.26), a standard computation yields a natural isomorphism:

$$\mathrm{Pic}\, \mathbb{T}(\gamma)_{\mathrm{sm}} \xrightarrow{\sim} \mathscr{O}_{\mathbb{T}(\gamma)}(U)^\times \backslash \mathscr{O}_{\mathbb{T}(\gamma)}(U \cap V)^\times / \mathscr{O}_{\mathbb{T}(\gamma)}(V)^\times.$$

(Here, for a ring A, the notation $A^\times$ means the invertible elements of A.) However, $\mathscr{O}_{\mathbb{T}(\gamma)}(U)^\times = (K^+)^\times \cdot (c\mathsf{T}^{-1})^{\mathbb{Z}}$, $\mathscr{O}_{\mathbb{T}(\gamma)}(U \cap V)^\times = (K^+[c^{-1}])^\times \cdot \mathsf{T}^{\mathbb{Z}}$ and $\mathscr{O}_{\mathbb{T}(\gamma)}(V) = (K^+)^\times \cdot \mathsf{T}^{\mathbb{Z}}$, whence the contention. □

11.4.27. Proposition 11.4.25 establishes a natural bijection between the set of isomorphism classes of invertible $\mathscr{O}_{\mathbb{T}(\gamma)_{\mathrm{sm}}}$-modules and the set :

$$]\gamma, 1] := \{\rho \in \Gamma \mid \gamma < \rho < 1\} \cup \{1\}.$$

On the other hand, lemma 11.4.11 yields a natural bijection between $\mathrm{Pic}\, \mathbb{T}(\gamma)_{\mathrm{sm}}$ and the set of isomorphism classes of generically invertible reflexive $\mathscr{O}_{\mathbb{T}(\gamma)}$-modules. Furthermore, lemma 11.4.20 provides already a collection of such reflexive modules, and by inspection of the proof, we see that the family of sheaves $\mathscr{I}(\rho)$ is really parametrized by the subset $]\gamma, 1]$ (since the other values of ρ correspond to free $\mathscr{O}_{\mathbb{T}(\gamma)}$-modules of rank one). The two parametrizations are essentially equivalent. Indeed, let $a \in K \setminus \{0\}$ be any element such that $\rho := |a| \in]\gamma, 1]$. With the notation of the proof of proposition 11.4.25, we can define isomorphisms

$$\varphi : \mathscr{O}_U \xrightarrow{\sim} \mathscr{I}(\rho)_{|U} \qquad \psi : \mathscr{O}_V \xrightarrow{\sim} \mathscr{I}(\rho)_{|V}$$

by letting: $\varphi(1) := \mathsf{T}$ and $\psi(1) := a$. To verify that φ is an isomorphism, it suffices to remark that T is a unit on U, so $\mathscr{I}(\rho)_{|U} = \mathsf{T}\mathscr{O}_U = \mathscr{O}_U$. Likewise, on V we can write $\mathsf{T} = a \cdot (a^{-1}c) \cdot (c^{-1}\mathsf{T})$, so $\mathscr{I}(\rho)_{|V} = a\mathscr{O}_V$, and ψ is an isomorphism. Hence $\mathscr{I}(\rho)$ is isomorphic to the (unique) $\mathscr{O}_{\mathbb{T}(\gamma)}$-module whose global sections consist of all the pairs $(s_U, s_V) \in \mathscr{O}_U(U) \times \mathscr{O}_V(V)$, such that $\mathsf{T}^{-1}s_{U|U\cap V} = a^{-1}s_{V|U\cap V}$. Clearly, under the bijection of proposition 11.4.25, the invertible sheaf $\mathscr{I}(\rho)_{|\mathbb{T}(\gamma)_{\mathrm{sm}}}$ corresponds to the class of ρ in $\Delta(\gamma)/\gamma^{\mathbb{Z}}$. In particular, this shows that the reflexive $\mathscr{O}_{\mathbb{T}(\gamma)}$-modules $\mathscr{I}(\rho)$ are pairwise non-isomorphic for $\rho \in]\gamma, 1]$, and that every reflexive generically invertible $\mathscr{O}_{\mathbb{T}(\gamma)}$-module is isomorphic to one such $\mathscr{I}(\rho)$.

11.4.28. When $\gamma < 1$ and $t \in \mathbb{T}(\gamma)$ is the singular point of the closed fibre, the discussion of (11.4.27) also applies to describe the set $\overline{\mathrm{coh.Div}}(\mathscr{O}_{\mathbb{T}(\gamma),t})$ of isomorphism classes of coherent reflexive fractional ideals of $\mathscr{O}_{\mathbb{T}(\gamma),t}$ (remark 11.3.11(ii)). Indeed, any such module M extends to a reflexive $\mathscr{O}_{\mathbb{T}(\gamma)}$-module (first one uses lemma 11.3.7(ii.b) to extend M to some quasi-compact open subset $U \subset \mathbb{T}(\gamma)$, and then one may extend to the whole of $\mathbb{T}(\gamma)$, via proposition 11.3.8)(i). Hence $M \simeq \mathscr{I}(\rho)_t$ for some $\rho \in]\gamma, 1]$. It follows already that $\mathrm{coh.Div}(\mathscr{O}_{\mathbb{T}(\gamma),t})$ is naturally an abelian group, with multiplication law given by the rule :

$$(M, N) \mapsto M \odot N := j_*j^*(M \otimes_{\mathscr{O}_{\mathbb{T}(\gamma),t}} N) \qquad \text{for any two classes } M, N \in \mathrm{Div}(\mathscr{O}_{\mathbb{T}(\gamma),t})$$

where $j : \mathbb{T}(\gamma)_{\mathrm{sm}} \cap \mathrm{Spec}\, \mathscr{O}_{\mathbb{T}(\gamma),t} \to \mathrm{Spec}\, \mathscr{O}_{\mathbb{T}(\gamma),t}$ is the natural open immersion. Indeed, $M \odot N$ is reflexive (by proposition 11.3.8(i) and corollary 11.3.9), and the composition law $\odot$ is clearly associative and commutative, with $\mathscr{O}_{\mathbb{T}(\gamma),t}$ as neutral element; moreover, for any $\rho \in \Delta(\gamma)$, the class of $\mathscr{I}(\rho)_t$ admits the inverse $j_*((j^*\mathscr{I}(\rho)_t)^\vee)$. Furthermore, the modules $\mathscr{I}(\rho)_t$ are pairwise non-isomorphic for $\rho \in]\gamma, 1]$. Indeed, using the group law $\odot$, the assertion follows once we know that $\mathscr{I}(\rho)_t$ is not trivial, whenever $\rho \in]\gamma, 1[$. However, from the presentation of lemma 11.4.20(ii) one sees that $\mathscr{I}(\rho)_t \otimes_{\mathscr{O}_{\mathbb{T}(\gamma),t}} \kappa(t)$ is a two-dimensional $\kappa(t)$-vector space, so everything is clear. Moreover, a simple inspection shows that the composition law $\odot$ thus defined, agrees with the composition law of the monoid $\mathrm{coh.Div}(\mathscr{O}_{T(\gamma),t})$ given in remark 11.3.11(ii). Summing, we get a natural group isomorphism :

$$\overline{\mathrm{coh.Div}}(\mathscr{O}_{\mathbb{T}(\gamma),t}) \xrightarrow{\sim} \Delta(\gamma)/\gamma^{\mathbb{Z}}.$$

The following theorem generalizes this classification to reflexive modules of arbitrary generic rank.

Theorem 11.4.29. *Let $g : X \to \mathbb{T}(\gamma)$ be a pro-étale morphism, $x \in X$ any point, M a reflexive $\mathscr{O}_{X,x}$-module. Then there exist $\rho_1, \dots, \rho_n \in]\gamma, 1]$ and an isomorphism of $\mathscr{O}_{X,x}$-modules:*

$$M \xrightarrow{\sim} \bigoplus_{i=1}^{n} g^*\mathscr{I}(\rho_i)_x.$$

Proof. Using lemma 11.3.7(ii.b), we are easily reduced to the case where g is étale. Set $t := g(x)$; first of all, if $t \in \mathbb{T}(\gamma)_{\mathrm{sm}}$, then X is smooth over S at the point t, and consequently M (resp. $\mathscr{I}(\rho)_t$) is a free $\mathscr{O}_{X,x}$-module (resp. $\mathscr{O}_{\mathbb{T}(\gamma),t}$-module) of finite rank (by proposition 11.4.2(iii)), so the assertion is obvious in this case. Hence we may assume that $\gamma < 1$ and t is the unique point in the closed fibre of $\mathbb{T}(\gamma) \setminus \mathbb{T}(\gamma)_{\mathrm{sm}}$. Next, let $K^{\mathrm{sh}+}$ be the strict henselization of K^+; denote by $h : \mathbb{T}_{K^{\mathrm{sh}}}(\gamma) \to \mathbb{T}_K(\gamma)$ the natural map, and set $X' := X \times_{\mathbb{T}_K(\gamma)} \mathbb{T}_{K^{\mathrm{sh}}}(\gamma)$. Choose also a point $x' \in X'$ lying over x, and let $t' \in \mathbb{T}_{K^{\mathrm{sh}}}(\gamma)$ be the image of x'; then t' is the unique point of $\mathbb{T}_{K^{\mathrm{sh}}}(\gamma)$ with $h(t') = t$. We have a commutative diagram of ring homomorphisms:

$$\begin{array}{ccc} \mathscr{O}_{\mathbb{T}(\gamma),t} & \xrightarrow{g_x^\natural} & \mathscr{O}_{X,x} \\ {\scriptstyle h_t^\natural}\downarrow & & \downarrow \\ \mathscr{O}_{\mathbb{T}_{K^{\mathrm{sh}}}(\gamma),t'} & \longrightarrow & \mathscr{O}_{X',x'} \end{array}$$

whence an essentially commutative diagram of functors:

$$\begin{array}{ccc} \mathscr{O}_{\mathbb{T}(\gamma),t}\text{-}\mathbf{Rflx} & \xrightarrow{g_x^*} & \mathscr{O}_{X,x}\text{-}\mathbf{Rflx} \\ {\scriptstyle h_t^*}\downarrow & & \downarrow{\scriptstyle \beta} \\ \mathscr{O}_{\mathbb{T}_{K^{\mathrm{sh}}}(\gamma),t'}\text{-}\mathbf{Rflx} & \xrightarrow{\alpha} & \mathscr{O}_{X',x'}\text{-}\mathbf{Rflx}. \end{array}$$

Since K and K^{sh} have the same value group, the discussion in (11.4.28) shows that h_t^* induces bijections on the isomorphism classes of generically invertible modules. On the other hand, proposition 7.11.12 implies that the functor β induces injections on isomorphism classes. Consequently, in order to prove the theorem, we may replace the pair (X, x) by (X', x'), and assume from start that $K = K^{\mathrm{sh}}$.

In terms of the presentation 11.4.14 we can write $\mathbb{T}(\gamma) \times_S \operatorname{Spec} \kappa(s) = Z_1 \cup Z_2$, where Z_1 (resp. Z_2) is the reduced irreducible component on which X (resp. Y) vanishes. By inspecting the definitions, it is easy to check that $Z_1 \simeq \mathbb{A}^1_\kappa \simeq Z_2$ as κ-schemes. Let ξ_i be the generic point of Z_i, for $i = 1, 2$; clearly $\{t\} = Z_1 \cap Z_2$, hence $W_i := X \times_{\mathbb{T}(\gamma)} Z_i$ is non-empty and étale over Z_i, so $\mathscr{O}_{W_i,x}$ is an integral domain, hence $g^{-1}(\xi_i)$ contains exactly one point ζ_i that specializes to x, for both $i = 1, 2$. To ease notation, let us set $A := \mathscr{O}_{X,x}$. Then $\operatorname{Spec} A \otimes_{K^+} \kappa$ consists of exactly three points, namely x, ζ_1 and ζ_2. Set $M(\zeta_i) := M \otimes_A \kappa(\zeta_i)$, and notice that $n := \dim_{\kappa(\zeta_1)} M(\zeta_1) = \dim_{\kappa(\zeta_2)} M(\zeta_2)$, since M restricts to a locally free module over $(\operatorname{Spec} A)_{\mathrm{sm}}$, the largest essentially smooth open S-subscheme of $\operatorname{Spec} A$, which is connected. We choose a basis $\overline{e}_1, \dots, \overline{e}_n$ (resp $\overline{e}'_1, \dots, \overline{e}'_n$) for $M(\zeta_1)^\vee$ (resp. $M(\zeta_2)^\vee$), which we can then lift to a system of sections $e_1, \dots, e_n \in M^\vee_{\zeta_1} := M^\vee \otimes_A \mathscr{O}_{X,\zeta_1}$ (and likewise we construct a system $e'_1, \dots, e'_n \in M^\vee_{\zeta_2}$). Let $\mathfrak{p}_i \subset A$ be the prime ideal corresponding to ζ_i ($i = 1, 2$); after multiplication by an element of $A \setminus \mathfrak{p}_i$, we may assume that $e_1, \dots, e_n \in M^\vee$ (and likewise for $e'_1, \dots, e'_n$). Finally, we set $e''_i := \mathsf{Y}e_i + \mathsf{X}e'_i$ for every $i = 1, \dots, n$; it is clear that the system $(e''_1, \dots, e''_n)$ induces bases of $M(\zeta_i)^\vee$ for both $i = 1, 2$. We wish to consider the map:

$$j : M \to A^{\oplus n} \qquad m \mapsto (e''_1(m), \dots, e''_n(m)).$$

Set $C := \operatorname{Coker} j$, $I := \operatorname{Ann}_A C$, and $B := A/I$.

Claim 11.4.30. (i) The maps $M_{\zeta_i} \to \mathscr{O}^{\oplus n}_{X,\zeta_i}$ induced by j are isomorphisms.
(ii) $\mathrm{Ker}\, j = \mathrm{Ker}\, j \otimes_A \mathbf{1}_\kappa = 0$.
(iii) B is a finitely presented K^+-module, and C is a free K^+-module of finite rank.

Proof of the claim. (i): Using Nakayama's lemma, one deduces easily that these maps are surjective; since M_{ζ_i} is a free $\mathscr{O}_{X,\zeta_i}$-module of rank n, they are also necessarily injective.

(ii): Since $\mathscr{O}_{X,x}$ is normal, $\{0\}$ is its only associated prime; then the injectivity of j (resp. of $j \otimes_A \mathbf{1}_\kappa$) follows from (i), and the fact that M (resp. $M \otimes_A \kappa$) satisfies condition S_1, by remark 11.3.5 (resp. by proposition 11.4.16).

(iii): First of all, since A is coherent, I is a finitely generated ideal of A. Let $f : \mathrm{Spec}\, B \to \mathrm{Spec}\, K^+$ be the natural morphism. Since C is a finitely presented A-module, its support Z is a closed subset of $\mathrm{Spec}\, A$. From (i) we see that $Z \cap \mathrm{Spec}\, A \otimes_{K^+} \kappa \subset \{x\}$; on the other hand, Z is also the support of the closed subscheme $\mathrm{Spec}\, B$ of $\mathrm{Spec}\, A$. Therefore

$$f^{-1}(s) \cap \mathrm{Spec}\, B \subset \{x\}. \tag{11.4.31}$$

Since K^+ is henselian, it follows easily from (11.4.31) and [66, Ch.IV, Th.18.5.11(c″)] that B is a finite K^+-algebra of finite presentation, hence also a finitely presented K^+-module (claim 9.1.35). Since C is a finitely presented B-module, we conclude that C is finitely presented as K^+-module, as well. From (ii) we deduce a short exact sequence: $0 \to M \to A^{\oplus n} \to C \to 0$, and then the long exact Tor sequence yields: $\mathrm{Tor}^A_1(C, \kappa) = 0$ (cp. the proof of claim 11.4.18). We conclude by [34, Ch.II, §3, n.2, Cor.2 of Prop.5]. ◊

Claim 11.4.32. The composed morphism: $\mathrm{Spec}\, B \to \mathrm{Spec}\, A \to \mathbb{T}(\gamma)$ is a closed immersion.

Proof of the claim. Let $\mathfrak{p} \subset R(\gamma)$ be the maximal ideal corresponding to t, so that $\mathscr{O}_{\mathbb{T}(\gamma),t} = R(\gamma)_\mathfrak{p}$. From claim 11.4.30(iii) we see that the natural morphism $\psi : R(\gamma)_\mathfrak{p} \to B$ is finite. Moreover, $A/\mathfrak{p}A \simeq R(\gamma)/\mathfrak{p}$, hence $\psi \otimes_{R(\gamma)} \mathbf{1}_{R(\gamma)/\mathfrak{p}}$ is a surjection. By Nakayama's lemma we deduce that ψ is already a surjection, *i.e.* the induced morphism $\mathrm{Spec}\, B \to \mathrm{Spec}\, R(\gamma)_\mathfrak{p}$ is a closed immersion. Let $J_\mathfrak{p} := \mathrm{Ker}\, \psi$, $J := R(\gamma) \cap J_\mathfrak{p}$ and $D := R(\gamma)/J$. We are reduced to showing that the induced map $D \to D_\mathfrak{p}$ is an isomorphism. However, let $e \in R(\gamma) \setminus \mathfrak{p}$; since $D_\mathfrak{p} \simeq B$ is finite over K^+, we can find a monic polynomial $P[T] \in K^+[T]$ such that $P(e^{-1}) = 0$, therefore an identity of the form $1 = e \cdot Q(e)$ holds in $D_\mathfrak{p}$ for some polynomial $Q(T) \in K^+[T]$. But then the same identity holds already in the subring D, *i.e.* the element e is invertible in D, and the claim follows. ◊

Now, C is a finitely generated B-module, hence also a finitely generated $R(\gamma)$-module, due to claim 11.4.32. We construct a presentation of C in the following way. First of all, we have a short exact sequence of $R(\gamma) \otimes_{K^+} R(\gamma)$-modules:

$$\underline{E} \;:\; 0 \to \Delta \to R(\gamma) \otimes_{K^+} R(\gamma) \xrightarrow{\mu} R(\gamma) \to 0$$

where μ is the multiplication map. The homomorphism $R(\gamma) \to R(\gamma) \otimes_{K^+} R(\gamma) : a \mapsto 1 \otimes a$ fixes an $R(\gamma)$-module structure on every $R(\gamma) \otimes_{K^+} R(\gamma)$-module (the *right* $R(\gamma)$-module structure), and clearly $\underline{E}$ is split exact, when regarded as a sequence of $R(\gamma)$-modules via this homomorphism. Moreover, in terms of the presentation (11.4.14), the $R(\gamma) \otimes_{K^+} R(\gamma)$-module Δ is generated by the elements $\mathsf{X} \otimes 1 - 1 \otimes \mathsf{X}$ and $\mathsf{Y} \otimes 1 - 1 \otimes \mathsf{Y}$. Let n be the rank of the free K^+-module C (claim 11.4.30(iii)); there follows an exact sequence

$$\underline{E} \otimes_{R(\gamma)} C \;:\; 0 \to \Delta \otimes_{R(\gamma)} C \to R(\gamma)^{\oplus n} \to C \to 0$$

which we may and do view as a short exact sequence of $R(\gamma)$-modules, via the *left* $R(\gamma)$-module structure induced by the restriction of scalars $R(\gamma) \to R(\gamma) \otimes_{K^+} R(\gamma) : a \mapsto a \otimes 1$. The elements $\mathsf{X}, \mathsf{Y} \in R(\gamma)$ act as K^+-linear endomorphisms on C; one can then find bases $(b_i \mid i = 1, \dots, n)$ and $(b'_i \mid i = 1, \dots, n)$ of C, and elements $a_1, \dots, a_n \in K^+ \setminus \{0\}$ such that

$\mathsf{X}b_i = a_i b'_i$ for every $i \leq n$. Since $\mathsf{XY} = c$ in $R(\gamma)$, it follows that $\mathsf{Y}b'_i = ca_i^{-1}b_i$ for every $i \leq n$. With this notation, it is clear that $\Delta \otimes_{R(\gamma)} C$, with its left $R(\gamma)$-module structure, is generated by the elements:

$$\mathsf{X} \otimes b_i - 1 \otimes a_i b'_i \quad \text{and} \quad \mathsf{Y} \otimes b'_i - 1 \otimes ca_i^{-1}b_i \quad (i = 1, \dots, n).$$

For every $i \leq n$, let F_i be the $R(\gamma)$-module generated freely by elements $(\varepsilon_i, \varepsilon'_i)$, and $\Delta_i \subset F_i$ the submodule generated by $\mathsf{X}\varepsilon_i - a_i\varepsilon'_i$ and $\mathsf{Y}\varepsilon'_i - ca_i^{-1}\varepsilon_i$. Moreover, let us write $b'_i = \sum_{j=1}^n u_{ij}b_j$ with unique $u_{ij} \in K^+$, let F be the free $R(\gamma)$-module with basis $(e_i \mid i = 1, \dots, n)$, and define $\varphi : F \to \oplus_{i=1}^n F_i$ by the rule: $e_i \mapsto \varepsilon'_i - \sum_{j=1}^n u_{ij}\varepsilon_j$ for every $i \leq n$. We deduce a right exact sequence of $R(\gamma)$-modules:

$$F \oplus \bigoplus_{i=1}^n \Delta_i \xrightarrow{\psi_1} \bigoplus_{i=1}^n F_i \xrightarrow{\psi_2} C \to 0$$

where:

$$\begin{aligned} &\psi_1(f, d_1, \dots, d_n) = \varphi(f) + (d_1, \dots, d_n) && \text{for every } f \in F \text{ and } d_i \in \Delta_i \\ &\psi_2(\varepsilon_i) = b_i \quad \text{and} \quad \psi_2(\varepsilon'_i) = b'_i && \text{for every } i = 1, \dots, n. \end{aligned}$$

Claim 11.4.33. ψ_1 is injective.

Proof of the claim. Let L be the field of fractions of $R(\gamma)$; since the domain of ψ_1 is a torsion-free K^+-module, it suffices to verify that $\psi \otimes_{R(\gamma)} \mathbf{1}_L$ is injective. However, on the one hand $C \otimes_{R(\gamma)} L = 0$, and on the other hand, each $\Delta_i \otimes_{R(\gamma)} L$ is an L-vector space of dimension one, so the claim follows by comparing dimensions. ◊

By inspecting the definitions and the proof of lemma 11.4.20, one sees easily that

$$\Delta_i \simeq I(|a_i^{-1}c|) \quad \text{for every } i \leq n. \tag{11.4.34}$$

Moreover, by remark (7.3.50)(iii), there exist $p \in \mathbb{N}$ and an $R(\gamma)$-linear isomorphism:

$$\operatorname{Ker} \psi_2 \xrightarrow{\sim} R(\gamma)^{\oplus p} \oplus \operatorname{Syz}^1_{R(\gamma)} C. \tag{11.4.35}$$

On the other hand, remark (7.3.50)(iii) and claim 11.4.30(ii) also shows that there exist $q \in \mathbb{N}$ and an A-linear isomorphism:

$$M \xrightarrow{\sim} A^{\oplus q} \oplus \operatorname{Syz}^1_A C. \tag{11.4.36}$$

Combining (11.4.35) and (11.4.36) and using lemma 7.3.51, we deduce an $R(\gamma)^{\mathrm{h}}$-linear isomorphism:

$$\begin{aligned} (R(\gamma)^{\mathrm{h}})^{\oplus q} \oplus (R(\gamma)^{\mathrm{h}} \otimes_{R(\gamma)} \operatorname{Ker} \psi_2) &\xrightarrow{\sim} (R(\gamma)^{\mathrm{h}})^{\oplus p+q} \oplus (R(\gamma)^{\mathrm{h}} \otimes_{R(\gamma)} \operatorname{Syz}^1_{R(\gamma)} C) \\ &\xrightarrow{\sim} (R(\gamma)^{\mathrm{h}})^{\oplus p+q} \oplus \operatorname{Syz}^1_{R(\gamma)^{\mathrm{h}}}(R(\gamma)^{\mathrm{h}} \otimes_{R(\gamma)} C) \\ &\xrightarrow{\sim} (R(\gamma)^{\mathrm{h}})^{\oplus p+q} \oplus (R(\gamma)^{\mathrm{h}} \otimes_A \operatorname{Syz}^1_A C) \\ &\xrightarrow{\sim} (R(\gamma)^{\mathrm{h}})^{\oplus p} \oplus (R(\gamma)^{\mathrm{h}} \otimes_A M). \end{aligned}$$

By claim 11.4.33 and (11.4.34) it follows that $(R(\gamma)^{\mathrm{h}})^{\oplus p} \oplus (R(\gamma)^{\mathrm{h}} \otimes_A M)$ is a direct sum of modules of the form $R(\gamma)^{\mathrm{h}} \otimes_{R(\gamma)} I(\rho_i)$, for various $\rho_i \in \Gamma$ (recall that $I(1) = R(\gamma)$). Notice that every $I(\rho_i)$ is generically of rank one, hence indecomposable. Then it follows from corollary 7.11.11 that $R(\gamma)^{\mathrm{h}} \otimes_A M$ is a direct sum of various indecomposable $R(\gamma)^{\mathrm{h}}$-modules of the form $R(\gamma)^{\mathrm{h}} \otimes_A g^*\mathscr{I}(\rho_i)_x$. Finally, we apply proposition 7.11.12 to conclude the proof of the theorem. □

The rest of this section shall be concerned with some results that hold in the special case where the valuation of K has rank one.

Theorem 11.4.37. *Suppose that K is a valued field of rank one. Let $f : X \to S$ be a finitely presented morphism, $\mathscr{F}$ a coherent $\mathscr{O}_X$-module. Then :*

(i) *Every coherent proper submodule $\mathscr{G} \subset \mathscr{F}$ admits a primary decomposition.*

(ii) $\mathrm{Ass}\,\mathscr{F}$ *is a finite set.*

Proof. By lemma 10.5.15, assertion (ii) follows from (i). Using corollary 10.5.11(ii) we reduce easily to the case where X is affine, say $X = \mathrm{Spec}\,A$ for a finitely presented K^+-algebra A, and $\mathscr{F} = M^\sim$, $\mathscr{G} = N^\sim$ for some finitely presented A-modules $N \subset M \neq 0$. By considering the quotient M/N, we further reduce the proof to the case where $N = 0$. Let us choose a closed imbedding $i : X \to \mathrm{Spec}\,B$, where $B := K^+[T_1, \dots, T_r]$ is a free polynomial K^+-algebra; by proposition 10.5.19, the submodule $0 \subset \mathscr{F}$ admits a primary decomposition if and only if the submodule $0 \subset i_*\mathscr{F}$ does. Thus, we may replace A by B, and assume that $A = K^+[T_1, \dots, T_r]$, in which case we shall argue by induction on r. Let ξ denote the maximal point of $f^{-1}(s)$; by claim 9.1.33, $V := \mathscr{O}_{X,\xi}$ is a valuation ring with value group Γ. Then [75, Lemma 6.1.14] says that

$$M_\xi \simeq V^{\oplus m} \oplus (V/b_1V) \oplus \cdots \oplus (V/b_kV)$$

for some $m \in \mathbb{N}$ and elements $b_1, \dots, b_k \in \mathfrak{m}_K \setminus \{0\}$. After clearing some denominators, we may then find a map $\varphi : M' := A^{\oplus m} \oplus (A/b_1A) \oplus \cdots \oplus (A/b_kA) \to M$ whose localization φ_ξ is an isomorphism.

Claim 11.4.38. $\mathrm{Ass}\,A/bA = \{\xi\}$ whenever $b \in \mathfrak{m}_K \setminus \{0\}$.

Proof of the claim. Clearly $\mathrm{Ass}\,A/bA \subset f^{-1}(s)$. Let $x \in f^{-1}(s)$ be a non-maximal point. Thus, the prime ideal $\mathfrak{p} \subset A$ corresponding to x is not contained in $\mathfrak{m}_K A$, *i.e.* there exists $a \in \mathfrak{p}$ such that $|a|_A = 1$. Suppose by way of contradiction, that $x \in \mathrm{Ass}\,A/bA$; then we may find $c \in A$ such that $c \notin bA$ but $a^n c \in bA$ for some $n \in \mathbb{N}$. The conditions translate respectively as the inequalities :

$$|c|_A > |b|_A \qquad \text{and} \qquad |a^n|_A \cdot |c|_A = |a^n c|_A \leq |b|_A$$

which are incompatible, since $|a^n|_A = 1$. ◊

Since $(\mathrm{Ker}\,\varphi)_\xi = 0$, claim 11.4.38 and proposition 10.5.6(ii) imply that $\mathrm{Ass}\,\mathrm{Ker}\,\varphi = \varnothing$; therefore $\mathrm{Ker}\,\varphi = 0$, by lemma 10.5.5(iii). Moreover, the submodule $N_1 := A^{\oplus m}$ (resp. $N_2 := (A/b_1A) \oplus \cdots \oplus (A/b_kA)$) of M' is either ξ-primary (resp. $\{0\}$-primary), or else equal to M'; in the first case, $0 \subset M'$ admits the primary decomposition $0 = N_1 \cap N_2$, and in the second case, either 0 is primary, or else $M' = 0$ has the empty primary decomposition. Now, if $r = 0$ then $M = M'$ and we are done. Suppose therefore that $r > 0$ and that the theorem is known for all integers $< r$. Set $M'' := \mathrm{Coker}\,\varphi$; clearly $\xi \notin \mathrm{Supp}\,M''$, so by proposition 10.5.18 we are reduced to showing that the submodule $0 \subset M''$ admits a primary decomposition. We may then replace M by M'' and assume from start that $\xi \notin \mathrm{Supp}\,\mathscr{F}$. Let $\mathscr{F}^t \subset \mathscr{F}$ be the K^+-torsion submodule; clearly $\mathscr{F}/\mathscr{F}^t$ is a flat K^+-module, hence it is a finitely presented $\mathscr{O}_X$-module (proposition 9.1.27), so $\mathscr{F}^t$ is a finitely generated $\mathscr{O}_X$-module, by [37, §1, n.4, Prop.6]. Hence we may find a non-zero $a \in \mathfrak{m}_K$ such that $a\mathscr{F}^t = 0$, *i.e.* the natural map $\mathscr{F}^t \to \mathscr{F}/a\mathscr{F}$ is injective. Denote by $i : Z := V(\mathrm{Ann}\,\mathscr{F}/a\mathscr{F}) \to X$ the closed immersion.

Claim 11.4.39. There exists a finite morphism $g : Z \to \mathbb{A}^{r-1}_S$.

Proof of the claim. Since $\xi \notin \mathrm{Supp}\,\mathscr{F}$, we have $I \not\subset \mathfrak{m}_K A$, *i.e.* we can find $b \in I$ such that $|b|_A = 1$. Set $W := V(a, b) \subset X$; it suffices to exhibit a finite morphism $W \to \mathbb{A}^{r-1}_S$. By [33, §5.2.4, Prop.2], we can find an automorphism $\sigma : A/aA \to A/aA$ such that $\sigma(b) = u \cdot b'$, where u is a unit, and $b' \in (K^+/aK^+)[T_1, \dots, T_r]$ is of the form $b' = T_r^k + \sum_{j=0}^{k-1} a_j T_r^j$, for some $a_0, \dots, a_{k-1} \in (K^+/aK^+)[T_1, \dots, T_{r-1}]$. Hence the ring $A/(aA + bA) = A/(aA + b'A)$ is finite over $K^+[T_1, \dots, T_{r-1}]$, and the claim follows. ◊

Claim 11.4.40. The $\mathscr{O}_X$-submodule $0 \subset \mathscr{F}/a\mathscr{F}$ admits a primary decomposition.

Proof of the claim. Let $\mathscr{G} := (\mathscr{F}/a\mathscr{F})_{|Z}$. By our inductive assumption the submodule $0 \subset g_*\mathscr{G}$ on $\mathbb{A}^{r-1}_S$ admits a primary decomposition; a first application of proposition 10.5.19 then shows that the submodule $0 \subset \mathscr{G}$ on Z admits a primary decomposition, and a second application proves the same for the submodule $0 \subset i_*\mathscr{G} = \mathscr{F}/a\mathscr{F}$. ◊

Finally, let $j : Y := V(a) \subset X$ be the closed immersion, and set $U := X \setminus Y$. By construction, the natural map $\underline{\Gamma}_Y\mathscr{F} \to j_*j^*\mathscr{F}$ is injective. Furthermore, U is an affine noetherian scheme, so [126, Th.6.8] ensures that the $\mathscr{O}_U$-submodule $0 \subset \mathscr{F}_{|U}$ admits a primary decomposition. The same holds for $j^*\mathscr{F}$, in view of claim 11.4.40. To conclude the proof, it remains only to invoke proposition 10.5.21. □

Corollary 11.4.41. *Let K be a valued field of rank one, $f : X \to S$ a finitely presented morphism, $\mathscr{F}$ a coherent $\mathscr{O}_X$-module, and $\underline{\mathscr{I}} := (\mathscr{I}_\lambda \mid \lambda \in \Lambda)$ a cofiltered family such that :*

(a) *$\mathscr{I}_\lambda \subset \mathscr{O}_X$ is a coherent ideal for every $\lambda \in \Lambda$.*
(b) *$\mathscr{I}_\lambda \cdot \mathscr{I}_\mu \in \underline{\mathscr{I}}$ whenever $\mathscr{I}_\lambda, \mathscr{I}_\mu \in \underline{\mathscr{I}}$.*

Then the following holds :

(i) *$\mathscr{F}_\lambda := \mathrm{Ann}_{\mathscr{F}}(\mathscr{I}_\lambda) \subset \mathscr{F}$ is a submodule of finite type for every $\lambda \in \Lambda$.*
(ii) *There exists $\lambda \in \Lambda$ such that $\mathscr{F}_\mu \subset \mathscr{F}_\lambda$ for every $\mu \in \Lambda$.*

Proof. We easily reduce to the case where X is affine, say $X = \mathrm{Spec}\, A$ with A finitely presented over K^+, and for each $\lambda \in \Lambda$ the ideal $I_\lambda := \Gamma(X, \mathscr{I}_\lambda) \subset A$ is finitely generated.

(i): For given $\lambda \in \Lambda$, let $f_1, \dots, f_k$ be a finite system of generators of I_λ; then $\mathscr{F}_\lambda$ is the kernel of the map $\varphi : \mathscr{F} \to \mathscr{F}^{\oplus k}$ defined by the rule : $m \mapsto (f_1 m, \dots, f_k m)$, which is finitely generated because A is coherent.

(ii): By theorem 11.4.37 we can find primary submodules $\mathscr{G}_1, \dots, \mathscr{G}_n \subset \mathscr{F}$ such that $\mathscr{G}_1 \cap \cdots \cap \mathscr{G}_n = 0$; for every $i \leq n$ and $\lambda \in \Lambda$ set $\mathscr{H}_i := \mathscr{F}/\mathscr{G}_i$ and $\mathscr{H}_{i,\lambda} := \mathrm{Ann}_{\mathscr{H}_i}(\mathscr{I}_\lambda)$. Since the natural map $\mathscr{F} \to \oplus_{i=1}^n \mathscr{H}_i$ is injective, we have :

$$\mathscr{F}_\lambda = \mathscr{F} \cap (\mathscr{H}_{1,\lambda} \oplus \cdots \oplus \mathscr{H}_{n,\lambda}) \qquad \text{for every } \lambda \in \Lambda.$$

It suffices therefore to prove that, for every $i \leq n$, there exists $\lambda \in \Lambda$ such that $\mathscr{H}_{i,\mu} = \mathscr{H}_{i,\lambda}$ for every $\mu \in \Lambda$. Say that $\mathscr{H}_i$ is $\mathfrak{p}$-primary, for some prime ideal $\mathfrak{p} \subset A$; suppose now that there exists λ such that $I_\lambda \subset \mathfrak{p}$; since I_λ is finitely generated, we deduce that $I^n_\lambda \mathscr{H}_i = 0$ for $n \in \mathbb{N}$ large enough; from (b) we see that $\mathscr{I}^n_\lambda \in \underline{\mathscr{I}}$, so (ii) holds in this case. In case $I_\lambda \not\subset \mathfrak{p}$ for every $\lambda \in \Lambda$, we have $\mathrm{Ann}_{\mathscr{H}_i}(\mathscr{I}_\lambda) = 0$ for every $\lambda \in \Lambda$, so (ii) holds in this case as well. □

Corollary 11.4.42. *Suppose A is a valuation ring with value group Γ_A, and $\varphi : K^+ \to A$ is an essentially finitely presented local homomorphism from a valuation ring K^+ of rank one. Then:*

(i) *If the valuation of K is not discrete, φ induces an isomorphism $\Gamma \xrightarrow{\sim} \Gamma_A$.*
(ii) *If $\Gamma \simeq \mathbb{Z}$ and φ is flat, φ induces an inclusion $\Gamma \subset \Gamma_A$, and $(\Gamma_A : \Gamma)$ is finite.*

Proof. Suppose first that $\Gamma \simeq \mathbb{Z}$; then A is noetherian, hence Γ_A is discrete of rank one as well, and the assertion follows easily. In case Γ is not discrete, we claim that A has rank ≤ 1. Indeed, suppose by way of contradiction, that the rank of A is higher than one, and let $\mathfrak{m}_A \subset A$ be the maximal ideal; then we can find $a, b \in \mathfrak{m}_A \setminus \{0\}$ such that $a^{-i}b \in A$ for every $i \in \mathbb{N}$. Let us consider the A-module $M := A/bA$; we notice that $\mathrm{Ann}_M(a^i) = a^{-i}bA$ form a strictly increasing sequence of ideals, contradicting corollary 11.4.41(ii). Next we claim that φ is flat. Indeed, suppose this is not the case; then A is a κ-algebra. Let $\mathfrak{p} \subset A$ be the maximal ideal; by lemma 9.1.32, we may assume that $A/\mathfrak{p}$ is a finite extension of κ. Now, choose any finitely presented quotient $\overline{A}$ of A supported at $\mathfrak{p}$; it follows easily that $\overline{A}$ is integral over K^+, hence it is a finitely presented K^+-module by proposition 9.1.34(i), and its annihilator contains $\mathfrak{m}_K$, which is absurd in view of [75, Lemma 6.1.14]. Next, since Γ is not discrete, one sees easily that $\mathfrak{m}_K A$ is a prime ideal, and in light of the foregoing, it must then be the maximal ideal. Now the assertion follows from proposition 9.1.34(ii). □

11.4.43. Let K be a valued field of rank one, B a finitely presented K^+-algebra, and consider a pair $\underline{A} := (A, \mathrm{Fil}_\bullet A)$ consisting of a B-algebra and a B-algebra filtration on A; let also M be a finitely generated A-module, and $\mathrm{Fil}_\bullet M$ a good $\underline{A}$-filtration on M (see definition 7.9.4). By definition, this means that the Rees module $\mathsf{R}(\underline{M})_\bullet$ of the filtered $\underline{A}$-module $\underline{M} := (M, \mathrm{Fil}_\bullet M)$ is finitely generated over the graded Rees B-algebra $\mathsf{R}(\underline{A})_\bullet$. We have :

Proposition 11.4.44. *In the situation of* (11.4.43)*, suppose that A is a finitely presented B-algebra, M is a finitely presented A-module, and* $\mathrm{Fil}_\bullet A$ *is a good filtration. Then :*

(i) $\mathsf{R}(\underline{A})_\bullet$ *is a finitely presented B-algebra, and* $\mathsf{R}(\underline{M})_\bullet$ *is a finitely presented* $\mathsf{R}(\underline{A})_\bullet$*-module.*

(ii) *If furthermore,* $\mathrm{Fil}_\bullet A$ *is a positive filtration (see definition* 7.9.4(ii)*), then* $\mathrm{Fil}_i M$ *is a finitely presented B-module, for every $i \in \mathbb{Z}$.*

Proof. By lemma 7.9.8, there exists a finite system of generators $\mathbf{m} := (m_1, \dots, m_n)$ of M, and a sequence of integers $\mathbf{k} := (k_1, \dots, k_n)$ such that $\mathrm{Fil}_\bullet M$ is of the form (7.9.7).

(i): Consider first the case where A is a free B-algebra of finite type, say $A = B[t_1, \dots, t_p]$, such that $\mathrm{Fil}_\bullet A$ is the good filtration associated with the system of generators $\mathbf{t} := (t_1, \dots, t_p)$ and the sequence of integers $\mathbf{r} := (r_1, \dots, r_p)$, and moreover M is a free A-module with basis $\mathbf{m}$. Then $\mathsf{R}(\underline{A})_\bullet$ is also a free B-algebra of finite type (see example 7.9.5). Moreover, for every $j \leq n$, let $M_j \subset M$ be the A-submodule generated by m_j, and denote by $\mathrm{Fil}_\bullet M_j$ the good $\underline{A}$-filtration associated with the pair $(\{m_j\}, \{k_j\})$; clearly $\mathrm{Fil}_\bullet M = \mathrm{Fil}_\bullet M_1 \oplus \cdots \oplus \mathrm{Fil}_\bullet M_n$, therefore $\mathsf{R}(\underline{M})_\bullet = \mathsf{R}(\underline{M}_1)_\bullet \oplus \cdots \oplus \mathsf{R}(\underline{M}_n)_\bullet$, where $\underline{M}_j := (M_j, \mathrm{Fil}_\bullet M_j)$ for every $j \leq n$. Obviously each $\mathsf{R}(\underline{A})_\bullet$-module $\mathsf{R}(\underline{M}_j)_\bullet$ is free of rank one, so the proposition follows in this case.

Next, suppose that A is a free B-algebra of finite type, and M is arbitrary. Let F be a free A-module of rank n, $\mathbf{e} := (e_1, \dots, e_n)$ a basis of F, and define an A-linear surjection $\varphi : F \to M$ by the rule $e_j \mapsto m_j$ for every $j \leq n$. Then φ is even a map of filtered $\underline{A}$-modules, provided we endow F with the good $\underline{A}$-filtration $\mathrm{Fil}_\bullet F$ associated with the pair $(\mathbf{e}, \mathbf{k})$. More precisely, let $N := \mathrm{Ker}\,\varphi$; then the filtration $\mathrm{Fil}_\bullet M$ is induced by $\mathrm{Fil}_\bullet F$, meaning that $\mathrm{Fil}_i M := (N + \mathrm{Fil}_i F)/N$ for every $i \in \mathbb{Z}$. Obviously the natural map $\mathsf{R}(\underline{F})_\bullet \to \mathsf{R}(\underline{M})_\bullet$ is surjective, and its kernel is the Rees module $\mathsf{R}(\underline{N})_\bullet$ corresponding to the filtered $\underline{A}$-module $\underline{N} := (N, \mathrm{Fil}_\bullet N)$ with $\mathrm{Fil}_i N := N \cap \mathrm{Fil}_i F$ for every $i \in \mathbb{Z}$. Let $\mathbf{x} := (x_1, \dots, x_s)$ be a finite system of generators of N, and choose a sequence of integers $\mathbf{j} := (j_1, \dots, j_s)$ such that $x_i \in \mathrm{Fil}_{j_i} N$ for every $i \leq s$; denote by $\underline{L}$ the filtered $\underline{A}$-module associated as in (7.9.6), with the A-module N and the pair $(\mathbf{x}, \mathbf{j})$. Thus, $\mathsf{R}(\underline{L})_\bullet$ is a finitely generated graded $\mathsf{R}(\underline{A})_\bullet$-submodule of $\mathsf{R}(\underline{N})_\bullet$. To ease notation, set $\overline{N} := \mathsf{R}(\underline{N})_\bullet/\mathsf{R}(\underline{L})_\bullet$ and $\overline{F} := \mathsf{R}(\underline{F})_\bullet/\mathsf{R}(\underline{L})_\bullet$. Recall (definition 7.9.1(iii)) that $\mathsf{R}(\underline{A})_\bullet$ is a graded B-subalgebra of $A[U, U^{-1}]$; then we have :

Claim 11.4.45. $\overline{N} = \bigcup_{n \in \mathbb{N}} \mathrm{Ann}_{\overline{F}}(U^n)$.

Proof of the claim. It suffices to consider a homogeneous element $y := U^i z \in \mathsf{R}(\underline{F})_i$, for some some $z \in \mathrm{Fil}_i F$. Suppose that $U^n y \in \mathsf{R}(\underline{L})_{i+n}$; especially, $U^{i+n} z \in \mathsf{R}(\underline{N})_{i+n}$, so $z \in N$, hence $y \in \mathsf{R}(\underline{N})_i$. Conversely, suppose that $y \in \mathsf{R}(\underline{N})_i$; write $z = x_1 a_1 + \cdots + x_s a_s$ for some $a_1, \dots, a_s \in A$. Say that $a_r \in \mathrm{Fil}_{b_r} A$ for every $r \leq s$, and set $l := \max(i, b_1 + j_1, \dots, b_s + j_s)$. Then $U^{l-i} y = U^l z \in \mathsf{R}(\underline{L})_l$. ◊

By the foregoing, $\mathsf{R}(\underline{F})_\bullet$ is finitely presented over $\mathsf{R}(\underline{A})_\bullet$. It follows that $\overline{N}$ is finitely generated (corollary 11.4.41), hence $\mathsf{R}(\underline{N})_\bullet$ is a finitely generated $\mathsf{R}(\underline{A})_\bullet$-module, so finally $\mathsf{R}(\underline{M})_\bullet$ is finitely presented over $\mathsf{R}(\underline{A})_\bullet$.

Lastly, consider the case of an arbitrary finitely presented B-algebra A, endowed with the good B-algebra filtration associated with a system of generators $\mathbf{x} := (x_1, \dots, x_p)$ and the sequence of integers $\mathbf{r}$. We map the free B-algebra $P := B[t_1, \dots, t_p]$ onto A, by the rule: $t_j \mapsto x_j$ for every $j \leq p$. This is even a map of filtered algebras, provided we endow P with the good filtration $\mathrm{Fil}_\bullet P$ associated with the pair $(\mathbf{t}, \mathbf{r})$; more precisely, $\mathrm{Fil}_\bullet P$ induces the filtration $\mathrm{Fil}_\bullet A$ on A, hence $\mathrm{Fil}_\bullet A$ is a good $\underline{P}$-filtration. By the foregoing case, we then deduce that $\mathsf{R}(\underline{A})_\bullet$ is a finitely presented $\mathsf{R}(\underline{P})_\bullet$-module (where $\underline{P} := (P, \mathrm{Fil}_\bullet P)$), hence also a finitely presented K^+-algebra. Moreover, M is a finitely presented P-module, and clearly $\mathrm{Fil}_\bullet M$ is good when regarded as a $\underline{P}$-filtration, hence $\mathsf{R}(\underline{M})_\bullet$ is a finitely presented $\mathsf{R}(\underline{P})_\bullet$-module, so also a finitely presented $\mathsf{R}(\underline{A})_\bullet$-module.

(ii): The positivity condition implies that $\mathsf{R}(\underline{A})_0 = B$; then, taking into account (i), the assertion is just a special case of proposition 7.6.11(iii). □

Theorem 11.4.46 (Artin-Rees lemma). *Let K be a valued field of rank one, A an essentially finitely presented K^+-algebra, $I \subset A$ an ideal of finite type, M a finitely presented A-module, $N \subset M$ a finitely generated submodule. Then there exists an integer $c \in \mathbb{N}$ such that :*

$$I^n M \cap N = I^{n-c}(I^c M \cap N) \qquad \text{for every } n \geq c.$$

Proof. We reduce easily to the case where A is a finitely presented K^+-algebra; then, with all the work done so far, we only have to repeat the argument familiar from the classical noetherian case. Indeed, let us define a filtered K^+-algebra $\underline{A} := (A, \mathrm{Fil}_\bullet A)$ by the rule : $\mathrm{Fil}_n A := I^{-n}$ if $n \leq 0$, and $\mathrm{Fil}_n A := A$ otherwise; also let $\underline{M} := (M, \mathrm{Fil}_\bullet M)$ be the filtered $\underline{A}$-module such that $\mathrm{Fil}_i M := M \cdot \mathrm{Fil}_i A$ for every $i \in \mathbb{Z}$. We endow $Q := M/N$ with the filtration $\mathrm{Fil}_\bullet Q$ induced from $\mathrm{Fil}_\bullet M$, so that the natural projection $M \to M/N$ yields a map of filtered $\underline{A}$-modules $\underline{M} \to \underline{Q} := (Q, \mathrm{Fil}_\bullet Q)$. By proposition 11.4.44, there follows a surjective map of finitely presented graded $\mathsf{R}(\underline{A})_\bullet$-modules : $\pi_\bullet : \mathsf{R}(\underline{M})_\bullet \to \mathsf{R}(\underline{Q})_\bullet$, whose kernel in degree $k \leq 0$ is the A-module $I^{-k} M \cap N$. Hence we can find a finite (non-empty) system $f_1, \dots, f_r$ of generators for the $\mathsf{R}(\underline{A})_\bullet$-module $\mathrm{Ker}\, \pi_\bullet$; clearly we may assume that each f_i is homogeneous, say of degree d_i; we may also suppose that $d_i \leq 0$ for every $i \leq r$, since $\mathrm{Ker}\, \pi_0$ generates $\mathrm{Ker}\, \pi_n$, for every $n \geq 0$. Let $d := \min(d_i \mid i = 1, \dots, r)$; by inspecting the definitions, one verifies easily that

$$\mathsf{R}(\underline{A})_i \cdot \mathrm{Ker}\, \pi_j \subset \mathsf{R}(\underline{A})_{i+j-d} \cdot \mathrm{Ker}\, \pi_d \qquad \text{whenever } i + j \leq d \text{ and } 0 \geq j \geq d.$$

Therefore $\mathrm{Ker}\, \pi_{d+k} = \mathsf{R}(\underline{A})_k \cdot \mathrm{Ker}\, \pi_d$ for every $k \leq 0$, so the assertion holds with $c := -d$. □

11.4.47. In the situation of theorem 11.4.46, let M be any finitely presented A-module; we denote by $M^\wedge$ the I-adic completion of M, which is an $A^\wedge$-module.

Corollary 11.4.48. *With the notation of* (11.4.47)*, the following holds :*

(i) *The functor $M \mapsto M^\wedge$ is exact on the category of finitely presented A-modules.*
(ii) *For every finitely presented A-module M, the natural map $M \otimes_A A^\wedge \to M^\wedge$ is an isomorphism.*
(iii) *$A^\wedge$ is flat over A. If additionally $I \subset \mathrm{rad}\, A$ (the Jacobson radical of A), then $A^\wedge$ is faithfully flat over A.*

Proof. (i): Let $N \subset M$ be any injection of finitely presented A-modules; by theorem 11.4.46, the I-adic topology on M induces the I-adic topology on N. Then the assertion follows from proposition 8.2.13(i,v).

(ii): Choose a presentation $A^{\oplus p} \to A^{\oplus q} \to M \to 0$. By (i) we deduce a commutative diagram with exact rows :

$$\begin{array}{ccccccc} A^{\oplus p} \otimes_A A^\wedge & \longrightarrow & A^{\oplus q} \otimes_A A^\wedge & \longrightarrow & M \otimes_A A^\wedge & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ (A^{\oplus p})^\wedge & \longrightarrow & (A^{\oplus q})^\wedge & \longrightarrow & M^\wedge & \longrightarrow & 0 \end{array}$$

and clearly the two left-most vertical arrows are isomorphisms. The claim follows.

(iii): The first assertion means that the functor $M \mapsto M \otimes_A A^\wedge$ is exact; this follows from (i) and (ii), via a standard reduction to the case where M is finitely presented. Suppose next that $I \subset \mathrm{rad}\, A$; to conclude, it suffices to show that the image of the natural map $\mathrm{Spec}\, A^\wedge \to \mathrm{Spec}\, A$ contains the maximal spectrum $\mathrm{Max}\, A$ ([126, Th.7.3]). Since the natural map $\mathrm{Max}\, A/I \to \mathrm{Max}\, A$ is a bijection, the latter assertion follows from the :

Claim 11.4.49. For every finitely presented A-module M, the natural map $i_M : M \to M^\wedge$ induces an isomorphism $M/IM \stackrel{\sim}{\to} M^\wedge/IM^\wedge$.

Proof of the claim. From (i) we get a natural identification : $M^\wedge/(IM)^\wedge \stackrel{\sim}{\to} (M/IM)^\wedge \stackrel{\sim}{\to} M/IM$, whose inverse is the map induced by i_M. However, by (ii) the image of $(IM)^\wedge$ in $M^\wedge$ is the same as the image of $IM \otimes_A A^\wedge$, which is the same as the image of $IM^\wedge$. □

Theorem 11.4.50. *Let K be a valued field of rank one, $B \to A$ a map of finitely presented K^+-algebras. Then :*

(i) *If M is an ω-coherent A-module, M is ω-coherent as a B-module.*

(ii) *If J is a coh-injective B-module, and $I \subset B$ is a finitely generated ideal, we have :*
  (a) *$\mathrm{Hom}_B(A, J)$ is a coh-injective A-module.*
  (b) *$\bigcup_{n \in \mathbb{N}} \mathrm{Ann}_J(I^n)$ is a coh-injective B-module.*

(iii) *If $(J_n, \varphi_n \mid n \in \mathbb{N})$ is a direct system consisting of coh-injective B/I^n-modules J_n and B-linear maps $\varphi_n : J_n \to J_{n+1}$ (for every $n \in \mathbb{N}$), then $\mathrm{colim}_{n \in \mathbb{N}} J_n$ is a coh-injective B-module.*

Proof. (See (7.11.25) for the generalities on coh-injective and ω-coherent modules.)

(i): We reduce easily to the case where M is finitely presented over A. Let $x_1, \dots, x_p$ be a system of generators for the B-algebra A, and $m_1, \dots, m_n$ a system of generators for the A-module M. We let $\underline{A} := (A, \mathrm{Fil}_\bullet A)$, where $\mathrm{Fil}_\bullet A$ is the good B-algebra filtration associated with the pair $\mathbf{x} := (x_1, \dots, x_p)$ and $\mathbf{r} := (1, \dots, 1)$; likewise, let $\underline{M}$ be the filtered $\underline{A}$-module defined by the good $\underline{A}$-filtration on M associated with $\mathbf{m} := (m_1, \dots, m_n)$ and $\mathbf{k} := (0, \dots, 0)$ (see definition 7.9.4). Then claim follows easily after applying proposition 11.4.44(ii) to the filtered B-algebra $\underline{A}$ and the filtered $\underline{A}$-module $\underline{M}$.

(ii.a): Let $N \subset M$ be two coherent A-modules, and $\varphi : N \to \mathrm{Hom}_B(A, J)$ an A-linear map. According to claim 10.1.17, φ corresponds by adjunction to a unique B-linear map $\overline{\varphi} : N \to J$; on the other hand by (i), M, and M/N are ω-coherent B-modules, hence $\overline{\varphi}$ extends to a B-linear map $\overline{\psi} : M \to J$ (lemma 7.11.27). Under the adjunction, $\overline{\psi}$ corresponds to an A-linear extension $\psi : M \to \mathrm{Hom}_B(A, J)$ of φ.

(iii): Let $M \subset N$ be two finitely presented B-modules, $f : M \to J := \mathrm{colim}_{n \in \mathbb{N}} J_n$ a B-linear map. Since M is finitely generated, f factors through a map $f_n : M \to J_n$ and the natural map $J_n \to J$, provided n is large enough ([75, Prop.2.3.16(ii)]). By theorem 11.4.46 there exists $c \in \mathbb{N}$ such that $I^{n+c}N \cap M \subset I^n M$. Hence

$$f_{n+c} := \varphi_{n+c-1} \circ \cdots \circ \varphi_n \circ f_n : M \to J_{n+c}$$

factors through a unique B/I^{n+c}-linear map $\overline{f}_{n+c} : \overline{M} := M/(I^{n+c}N \cap M) \to J_{n+c}$; since I^{n+c} is finitely generated, $\overline{M}$ is a coherent submodule of the coherent B-module $\overline{N} := N/I^{n+c}N$, therefore $\overline{f}_{n+c}$ extends to a map $\overline{g} : \overline{N} \to J_{n+c}$. The composition of $\overline{g} : \overline{N} \to J_{n+c}$ with the projection $N \to \overline{N}$ and the natural map $J_{n+c} \to J$, is the sought extension of f.

(ii.b): Letting $A := B/I^n$ in (ii.a), we deduce that $J_n := \mathrm{Ann}_J(I^n)$ is a coh-injective B/I^n-module, for every $n \in \mathbb{N}$. Then the assertion follows from (iii). □

11.5. **Local duality.** Throughout this section we let $(K, |\cdot|)$ be a valued field of rank one. We shall continue to use the general notation of (11.4).

11.5.1. Let A be finitely presented K^+-algebra, $I \subset A$ an ideal generated by a finite system $\mathbf{f} := (f_i \mid i = 1, \dots, r)$, and denote by $i : Z := V(I) \to X := \mathrm{Spec}\, A$ the natural closed immersion. Let also $\mathscr{I} \subset \mathscr{O}_X$ be the ideal arising from I. For every $n \geq 0$ there is a natural epimorphism

$$i_* i^{-1}\mathscr{O}_X \to \mathscr{O}_X/\mathscr{I}^n$$

whence natural morphisms in $\mathsf{D}(A\text{-}\mathbf{Mod})$:

$$R\mathrm{Hom}^\bullet_{\mathscr{O}_X}(\mathscr{O}_X/\mathscr{I}^n, \mathscr{F}^\bullet) \to R\mathrm{Hom}^\bullet_{\mathscr{O}_X}(i_* i^{-1}\mathscr{O}_X, \mathscr{F}^\bullet) \xrightarrow{\sim} R\Gamma_Z \mathscr{F}^\bullet \qquad (11.5.2)$$

for any bounded below complex $\mathscr{F}^\bullet$ of $\mathscr{O}_X$-modules.

Theorem 11.5.3. *In the situation of* (11.5.1)*, let $M^\bullet$ be any object of* $\mathsf{D}^+(A\text{-}\mathbf{Mod})$*. Then* (11.5.2) *induces natural isomorphisms :*

$$\operatorname*{colim}_{n\in\mathbb{N}} \mathrm{Ext}^i_A(A/I^n, M^\bullet) \xrightarrow{\sim} R^i\Gamma_Z M^{\bullet\sim} \qquad \textit{for every } i \in \mathbb{N}.$$

Proof. For $\mathscr{F}^\bullet := M^{\bullet\sim}$, trivial duality (theorem 10.1.18) identifies the source of (11.5.2) with $R\mathrm{Hom}^\bullet_A(A/I^n, M^\bullet)$; then one takes cohomology in degree i and forms the colimit over n to define the sought map. Next, by usual spectral sequence arguments, we may reduce to the case where $M^\bullet$ is a single A-module M sitting in degree zero (see *e.g.* the proof of proposition 10.3.21(i)). By inspecting the definitions, one verifies easily that the morphism thus defined is the composition of the isomorphism of proposition 10.4.32(iii) and the map (7.8.41) (see the proof of proposition 10.4.32(iii)); then it suffices to show that the inverse system $(H_i\mathbf{K}_\bullet(\mathbf{f}^n) \mid n \in \mathbb{N})$ is essentially zero when $i > 0$ (lemma 7.8.42). By lemma 7.8.44, this will in turn follow, provided the following holds. For every finitely presented quotient B of A, and every $b \in B$, there exists $p \in \mathbb{N}$ such that $\mathrm{Ann}_B(b^q) = \mathrm{Ann}_B(b^p)$ for every $q \geq p$. This latter assertion is a special case of corollary 11.4.41. □

Corollary 11.5.4. *In the situation of theorem* 11.5.3*, we have natural isomorphisms :*

$$\operatorname*{colim}_{n\in\mathbb{N}} \mathrm{Ext}^i_A(I^n, M^\bullet) \xrightarrow{\sim} H^i(X\setminus Z, M^{\bullet\sim}) \qquad \textit{for every } i \in \mathbb{N}.$$

Proof. Set $U := X \setminus Z$. We may assume that $M^\bullet$ is a complex of injective A-modules, in which case the sought map is obtained by taking colimits over the direct system of composed morphisms :

$$\mathrm{Hom}_A(I^n, M^\bullet) \xrightarrow{\beta_n} \Gamma(U, M^\bullet) \to R\Gamma(U, M^{\bullet\sim})$$

where β_n is induced by the identification $(I^n)^\sim_{|U} = \mathscr{O}_U$ and the natural isomorphism :

$$\mathrm{Hom}_{\mathscr{O}_U}(\mathscr{O}_U, M^{\bullet\sim}) \simeq \Gamma(U, M^{\bullet\sim}).$$

The complex $R\Gamma(U, M^{\bullet\sim})$ is computed by a Cartan-Eilenberg injective resolution $M^{\bullet\sim} \xrightarrow{\sim} \mathscr{M}^\bullet$ of $\mathscr{O}_X$-modules, and then the usual arguments allow to reduce to the case where $M^\bullet$ consists of a single injective A-module M placed in degree zero. Finally, from the short exact sequence $0 \to I^n \to A \to A/I^n \to 0$ we deduce a commutative ladder with exact rows :

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}_A(A/I^n, M) & \longrightarrow & M & \longrightarrow & \mathrm{Hom}_A(I^n, M) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha_n} & & \| & & \downarrow{\scriptstyle \beta_n} & & \downarrow \\ 0 & \longrightarrow & \Gamma_Z M^\sim & \longrightarrow & M & \longrightarrow & \Gamma(U, M^\sim) & \longrightarrow & R^1\Gamma_Z M^\sim \longrightarrow 0 \end{array}$$

where α_n is induced from (11.5.2). From theorem 11.5.3 it follows that $\mathrm{colim}_{n\in\mathbb{N}}\,\alpha_n$ is an isomorphism, and $R^i\Gamma_Z M^\sim$ vanishes for all $i > 0$; hence $\mathrm{colim}_{n\in\mathbb{N}}\,\beta_n$ is an isomorphism, and the contention follows. □

Proposition 11.5.5. *Let $X \xrightarrow{f} Y \xrightarrow{g} S$ be two finitely presented morphisms of schemes.*

(i) *If g is smooth, $\mathscr{O}_Y[0]$ is a dualizing complex on Y.*
(ii) *If g is smooth and f is a closed immersion, then X admits a dualizing complex $\omega^\bullet_X$.*
(iii) *If X and Y are affine and $\omega^\bullet_Y$ is a dualizing complex on Y, then $f^!\omega^\bullet_Y$ is dualizing on X (notation of* (11.1.15)*).*
(iv) *Every finitely presented quasi-separated S-scheme admits a dualizing complex.*

Proof. (i) is an immediate consequence of proposition 11.4.2(i).

(ii): As in the proof of corollary 11.3.65, this follows directly from (i) and lemma 11.3.27(i).

(iii): Let us choose a factorization $f = p_Y \circ i$ where $i : X \to \mathbb{A}^n_Y$ is a finitely presented closed immersion, and $p_Y : \mathbb{A}^n_Y \to Y$ the smooth projection onto Y. In view of lemma 11.3.27(i) and (11.1.15), it suffices to prove the assertion for the morphism p_Y. To this aim, we pick a closed finitely presented immersion $h : Y \to \mathbb{A}^m_S$ and consider the fibre diagram :

$$\begin{array}{ccc} \mathbb{A}^n_Y & \xrightarrow{h'} & \mathbb{A}^{n+m}_S \\ {\scriptstyle p_Y}\downarrow & & \downarrow{\scriptstyle p_S} \\ Y & \xrightarrow{h} & \mathbb{A}^m_S. \end{array}$$

By (i) we know that the scheme $\mathbb{A}^m_S$ admits a dualizing complex $\omega^\bullet$, and then lemma 11.3.27(i) says that $h^!\omega^\bullet$ is dualizing as well. By proposition 11.3.25 we deduce that $\omega^\bullet_Y \simeq \mathscr{L}[\sigma] \otimes_{\mathscr{O}_Y} h^!\omega^\bullet$ for some invertible $\mathscr{O}_Y$-module $\mathscr{L}$ and some continuous function $\sigma : |Y| \to \mathbb{Z}$. Since p_Y is smooth, we can compute: $p^!_Y\omega^\bullet_Y \simeq (p^!_Y \circ h^!\omega^\bullet) \otimes_{\mathscr{O}_{\mathbb{A}^n_Y}} p^*_Y\mathscr{L}[\sigma]$, hence $p^!_Y\omega^\bullet_Y$ is dualizing if and only if the same holds for $p^!_Y \circ h^!\omega^\bullet$. By proposition 11.1.7(iv), the latter complex is isomorphic to $h'^\flat \circ p^!_S\omega^\bullet$ and again using lemma 11.3.27(i) we reduce to checking that $p^!_S\omega^\bullet$ is dualizing, which is clear from (i).

(iv): Let $f : X \to S$ be a finitely presented morphism, with X quasi-separated. If X is affine, (iii) says that $f^!\mathscr{O}_S[0]$ is dualizing on X. In the general case, let $(U_i \mid i = 1, \dots, n)$ be a finite covering of X consisting of affine open subsets; for each $i, j, k = 1, \dots, n$, denote by $f_i : U_i \to S$ the restriction of f, set $U_{ij} := U_i \cap U_j$, $U_{ijk} := U_{ij} \cap U_k$, and let $g_{ij} : U_{ij} \to U_i$ be the inclusion map. We know that $f^!_i\mathscr{O}_S[0]$ is dualizing on U_i, for every $i = 1, \dots, n$; moreover, for every $i, j = 1, \dots, n$ there exists a natural isomorphism

$$\psi_{ij} : g^*_{ij}f^!_i\mathscr{O}_S[0] \xrightarrow{\sim} g^*_{ji}f^!_j\mathscr{O}_S[0]$$

fulfilling the cocycle condition

$$\psi_{jk|U_{ijk}} \circ \psi_{ij|U_{ijk}} = \psi_{ik|U_{ijk}} \qquad \text{for every } i, j, k = 1, \dots, n$$

(lemma 11.1.16). In other words, $((U_i, f^!_i\mathscr{O}_S[0]); \psi_{ij} \mid i, j = 1, \dots, n)$ is a descent datum for the fibration (11.3.55). Then the assertion follows from proposition 11.3.56. □

Example 11.5.6. (i) For given $b \in \mathfrak{m}_K$, let $i_b : S_{/b} \to S$ be the closed immersion. If $b \neq 0$, a simple computation yields a natural isomorphism in $\mathsf{D}(\mathscr{O}_{S_{/b}}\text{-}\mathbf{Mod})$:

$$i_b^! \mathscr{O}_S \xrightarrow{\sim} \mathscr{O}_{S_{/b}}[-1].$$

By proposition 11.5.5(i,iii), we deduce that $\mathscr{O}_{S_{/b}}[0]$ is a dualizing complex on $S_{/b}$, for every $b \in \mathfrak{m}_K$.

(ii) Next, let $f : X \to S_{/b}$ be an affine finitely presented Cohen-Macaulay morphism, of constant fibre dimension n. Then $\omega_X^\bullet := f^! \mathscr{O}_{S_{/b}}[0]$ is a dualizing complex on X, by (i) and proposition 11.5.5(iii). Moreover, $\omega_X^\bullet$ is concentrated in degree $-n$, and $H^{-n}\omega_X^\bullet$ is a finitely presented f-Cohen-Macaulay $\mathscr{O}_X$-module. Indeed, let $i : X \to Y := \mathbb{A}^m_{S_{/b}}$ be a closed immersion of $S_{/b}$-schemes, and denote by $g : Y \to S_{/b}$ the projection. Fix any $x \in X$, let $y := i(x)$ and set

$$d_x := \dim \mathscr{O}_{f^{-1}(fx),x} \qquad d_y := \dim \mathscr{O}_{g^{-1}(gy),y}.$$

We have a natural isomorphism in $\mathsf{D}(\mathscr{O}_X\text{-}\mathbf{Mod})$

$$\omega_X^\bullet \xrightarrow{\sim} i^! \mathscr{O}_Y[m] \xrightarrow{\sim} i^* R\mathscr{H}om^\bullet_{\mathscr{O}_Y}(i_*\mathscr{O}_X, \mathscr{O}_Y[m])$$

(lemma 11.1.16(i)), and by assumption $i_*\mathscr{O}_X$ is a g-Cohen-Macaulay $\mathscr{O}_Y$-module; by theorem 11.3.67(ii), it follows that $\omega^\bullet_{X,x}$ is concentrated in degree $d_y - d_x - m$, and its cohomology in that degree is f-Cohen-Macaulay, as asserted. Lastly, since the fibres of f and g are equidimensional ([64, Ch.IV, Prop.5.2.1] and lemma 11.3.62(ii)), it is easily seen that $d_y - d_x = m - n$ (details left to the reader), whence the claim.

11.5.7. For any finitely presented morphism $f : X \to S$ we consider the map :

$$d : |X| \to \mathbb{Z} \qquad x \mapsto \mathrm{tr.\,deg}(\kappa(x)/\kappa(f(x))) + \dim \overline{\{f(x)\}}.$$

Lemma 11.5.8. *With the notation of* (11.5.7)*, let $x, y \in X$, and suppose that x is a specialization of y. We have :*

(i) *x is an immediate specialization of y if and only if $d(y) = d(x) + 1$.*
(ii) *If X is irreducible, $d(x) - d(y) = \dim X(y) - \dim X(x)$.*
(iii) *X is catenary and of finite Krull dimension.*
(iv) *If f is flat, then $\dim \mathscr{O}_{X,x} = \dim \mathscr{O}_{f^{-1}(fx),x} + \dim \mathscr{O}_{S,f(x)}$.*
(v) *If f is Cohen-Macaulay at the point x, then $\mathscr{O}_{X,x}$ is equidimensional.*

Proof. (i): In case $f(x) = f(y)$, the assertion follows from [64, Ch.IV, Prop.5.2.1]. Hence, suppose that $f(x) = s$, $f(y) = \eta$; let Z be the topological closure of $\{y\}$ in X, and endow Z with its reduced subscheme structure; notice that Z is an S-scheme of finite type. By assumption, X is quasi-compact and quasi-separated, hence $\{y\}$ is a pro-constructible subset of X, and therefore Z is the set of all specializations of y in X ([63, Ch.IV, Th.1.10.1]). Especially, y is the unique maximal point of $Z_\eta := Z \cap f^{-1}(\eta)$; also, if x is an immediate specialization of y, then x is a maximal point of $Z_s := Z \cap f^{-1}(s)$, and by lemma 11.4.1(i) and [64, Ch.IV, Prop.5.2.1], the latter implies that $d(y) = d(x) + 1$.

Conversely, suppose that $d(y) = d(x) + 1$; from lemma 11.4.1(i) and [64, Ch.IV, Prop.5.2.1] we deduce that x is a maximal point of Z_s; then x is an immediate specialization of y in X, since otherwise x would be an immediate specialization in X of a proper specialization y' of y in Z_η, and in this case, by the foregoing we would get $d(y') = d(x) + 1$, *i.e.* $d(y) = d(y')$, contradicting [64, Ch.IV, Prop.5.2.1].

(iii) It is easily seen that X has finite Krull dimension. Now, consider any sequence $y_0, \dots, y_n$ of points of X, with $y_0 := y$, $y_n := x$, and such that y_{i+1} is an immediate specialization of y_i, for $i = 0, \dots, n-1$; from (i) we deduce that $n = d(y) - d(x)$, especially n is independent of the chosen chain of specializations, so X is catenary.

(ii): We reduce easily to the case where y is the maximal point of X, in which case we may argue as in the proof of (iii) (details left to the reader).

(iv): If $f(x) = \eta$, the identity is [64, Ch.IV, Cor.6.1.2], hence we may assume that $f(x) = s$, and we need to check that

$$\dim X(x) = \dim \mathscr{O}_{f^{-1}(s),x} + 1.$$

Now, notice that – by the flatness assumption – every irreducible component of X intersect $f^{-1}(\eta)$, and it is therefore itself a flat S-scheme (with its reduced subscheme structure). Since $f^{-1}(\eta)$ is a noetherian scheme, it also follows that the set of irreducible components of X is finite; thus, let $X_1, \dots, X_n$ be the reduced irreducible components of X containing x, and $f_i : X_i \to S$ ($i = 1, \dots, n$) the corresponding restrictions of f. Then

$$\dim \mathscr{O}_{X,x} = \max_{1 \leq i \leq n} \dim \mathscr{O}_{X_i,x} \qquad \text{and} \qquad \dim \mathscr{O}_{f^{-1}(s),x} = \max_{1 \leq i \leq n} \dim \mathscr{O}_{f_i^{-1}(s),x}$$

so that it suffices to show the stated identity for each of the morphisms $f_1, \dots, f_n$. Hence, we may assume that X is irreducible, and we let z be the maximal point of X, and y any maximal generization of x in $f^{-1}(s)$. By lemma 11.4.1(i), $f^{-1}(s)$ is equidimensional, and we have

$$\dim \mathscr{O}_{f^{-1}(s),x} = \dim f^{-1}(s) - d(x) \qquad \text{and} \qquad \dim f^{-1}(s) = d(y)$$

([64, Ch.IV, Cor.5.2.3]). Arguing as in the proof of (i), we see that z is an immediate generization of y, hence $d(z) = d(y) + 1$. So, we come down to showing that $\dim X(x) = d(z) - d(x)$; the latter follows from (ii), since $\dim X(z) = 0$.

(v): By lemma 11.3.62(ii), the morphism f is equidimensional at the point x. Especially, $f(u) = \eta$ for every maximal point u of $\operatorname{Spec} \mathscr{O}_{X,x}$, and there exist an open neighborhood U of x in X and an integer $e \in \mathbb{N}$, such that $f^{-1}f(y)$ is equidimensional of dimension e, for every $y \in U$. Since f is finitely presented, it follows that

$$\operatorname{tr.deg}(\kappa(y)/\kappa(f(y))) = e$$

for every $y \in U$ such that y is maximal in $f^{-1}f(y)$ ([126, Th.5.6]). Hence, $d(u) = e + 1$ for every maximal point u of $\operatorname{Spec} \mathscr{O}_{X,x}$, and especially, for every such u, the difference $d(x) - d(u)$ is independent of u; in light of (i), the assertion follows. □

Lemma 11.5.9. (i) *The K^+-module K/K^+ is coh-injective.*

(ii) *Let A be a finitely presented K^+-algebra, $I \subset A$ a finitely generated ideal, J a coh-injective K^+-module. Then the A-module*

$$J_A := \operatorname*{colim}_{n \in \mathbb{N}} \operatorname{Hom}_{K^+}(A/I^n, J)$$

is coh-injective.

Proof. (i): It suffices to show that $\operatorname{Ext}^1_{K^+}(M, K/K^+) = 0$ whenever M is a finitely presented K^+-module. This is clear when $M = K^+$, and then [75, Ch.6, Lemma 6.1.14] reduces to the case where $M = K^+/aK^+$ for some $a \in \mathfrak{m}_K \setminus \{0\}$; in that case we can compute using the free resolution $K^+ \xrightarrow{a} K^+ \to M$, and the claim follows easily.

(ii): According to theorem 11.4.50(ii.a), the A-module $J'_A := \operatorname{Hom}_{K^+}(A, J)$ is coh-injective. However, $J_A = \bigcup_{n \in \mathbb{N}} \operatorname{Ann}_{J'_A}(I^n)$, so the assertion follows from theorem 11.4.50(ii.b). □

Theorem 11.5.10. *Let $f : X \to S$ be a finitely presented affine morphism. Then for every point $x \in X$, the $\mathscr{O}_{X,x}$-module*

$$J(x) := R^{1-d(x)}\Gamma_{\{x\}}(f^! \mathscr{O}_S[0])_{|X(x)}$$

is coh-injective, and we have a natural isomorphism in $\mathsf{D}(\mathscr{O}_{X,x}\text{-}\mathbf{Mod})$:

$$R\Gamma_{\{x\}}(f^! \mathscr{O}_S[0])_{|X(x)} \simeq J(x)[d(x) - 1].$$

Proof. Fix $x \in X$ and set $d := d(x)$.

Claim 11.5.11. The theorem holds in case $f(x) = \eta$.

Proof of the claim. Indeed, let $f_\eta : f^{-1}(\eta) \to S(\eta) := \mathrm{Spec}\, K$ be the restriction of f; according to lemma 11.1.16(i), $f^!_\eta \mathscr{O}_{S(\eta)}[0]$ is the restriction of $f^!\mathscr{O}_S[0]$, so the claim follows immediately from example 11.3.51. ◊

Hence, suppose that $f(x) = s$, the closed point of S, and say that $X = \mathrm{Spec}\, A$.

Claim 11.5.12. We may assume that x is closed in X, hence that $\kappa(x)$ is finite over $\kappa(s)$.

Proof of the claim. Arguing as in the proof of lemma 9.1.32, we can find a factorization of the morphism f as a composition $X \xrightarrow{g} Y := \mathbb{A}^d_S \xrightarrow{h} S$, such that $\xi := g(x)$ is the generic point of $h^{-1}(s) \subset Y$, the morphism g is finitely presented, and the stalk $\mathscr{O}_{Y,\xi}$ is a valuation ring. Moreover, let $g_y := g \times_Y \mathbf{1}_{Y(y)} : X(y) := X \times_Y Y(y) \to Y(y)$; we have a natural isomorphism

$$(f^!\mathscr{O}_S[0])_{|X(y)} \simeq g^!_y \mathscr{O}_{Y(y)}[d].$$

Then we may replace f by g_y, and K^+ by $\mathscr{O}_{Y,\xi}$, whence the claim. ◊

Hence, suppose now that x is closed in X, let $\mathfrak{p} \subset A$ be the maximal ideal corresponding to x, and $\overline{\mathfrak{p}}$ the image of $\mathfrak{p}$ in $\overline{A} := A \otimes_{K^+} \kappa$; we choose a finite system of elements $b_1, \dots, b_r \in \mathscr{O}_X(X)$ whose images in $\overline{A}$ generate $\overline{\mathfrak{p}}$. Pick any non-zero $a \in \mathfrak{m}_K$, and let $I \subset A$ be the ideal generated by the system $(a, b_1, \dots, b_r)$, and $\mathscr{I} \subset \mathscr{O}_X$ the corresponding coherent ideal; clearly $V(I) = \{x\}$, hence theorem 11.5.3 yields a natural isomorphism :

$$\text{(11.5.13)} \qquad \operatorname*{colim}_{n\in\mathbb{N}} R^i\mathrm{Hom}^\bullet_{\mathscr{O}_X}(\mathscr{O}_X/\mathscr{I}^n, f^!\mathscr{O}_S[0]) \xrightarrow{\sim} R^i\Gamma_{\{x\}} f^!\mathscr{O}_S[0] \qquad \text{for every } i \in \mathbb{Z}$$

where the transition maps in the colimit are induced by the natural maps $\mathscr{O}_X/\mathscr{I}^n \to \mathscr{O}_X/\mathscr{I}^m$, for every $n \geq m$. However, from lemmata 11.1.6(ii), 11.1.16(i) and corollary 10.3.2(i) we deduce as well natural isomorphisms :

$$R\mathrm{Hom}^\bullet_{\mathscr{O}_X}(\mathscr{O}_X/\mathscr{I}^n, f^!\mathscr{O}_S[0]) \xrightarrow{\sim} R\mathrm{Hom}^\bullet_{\mathscr{O}_S}(f_*\mathscr{O}_X/\mathscr{I}^n, \mathscr{O}_S) \xrightarrow{\sim} R\mathrm{Hom}^\bullet_{K^+}(A/I^n, K^+).$$

We wish to compute these Ext groups by means of lemma 7.11.27(iii); to this aim, let us remark first that A/I^n is an integral K^+-algebra for every $n \in \mathbb{N}$, hence it is a finitely presented torsion K^+-module, according to proposition 9.1.34(i). We may then use the coh-injective resolution $K^+[0] \to (0 \to K \to K/K^+ \to 0)$ (lemmata 11.5.9(i), 7.11.27(iii)) to compute :

$$\mathrm{Ext}^i_{K^+}(A/I^n, K^+) = \begin{cases} J_n := \mathrm{Hom}_{K^+}(A/I^n, K/K^+) & \text{if } i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Now the theorem follows from lemma 11.5.9(i) and theorem 11.4.50(ii.a),(iii).

We could also appeal directly to lemma 11.5.9(ii), provided we already knew that the foregoing natural identifications transform the direct system whose colimit appears in (11.5.13), into the direct system $(J_n \mid n \in \mathbb{N})$ whose transition maps are induced by the natural maps $A/I^n \to A/I^m$, for every $n \geq m$. For the sake of completeness, we check this latter assertion.

For every $n \geq m$, let $j_n : X_n := \mathrm{Spec}\, A/I^n \to X$ and $j_{mn} : X_m \to X_n$ be the natural closed immersions. We have a diagram of functors :

$$\begin{array}{ccccc}
j_{m*} \circ j^!_{mn} \circ (f \circ j_n)^! & \xrightarrow{\zeta_1} & j_{n*} \circ (j_{mn*} \circ j^!_{mn}) \circ j^!_n \circ f^! & \xleftarrow{\xi_2} & j_{m*} \circ j^!_m \circ f^! \\
{\scriptstyle \xi_1} \uparrow & \searrow^{\varepsilon_1} & & \searrow^{\varepsilon_2} & \downarrow {\scriptstyle \alpha_{mn}} \\
j_{m*} \circ (f \circ j_m)^! & \xrightarrow{\beta_{mn}} & j_{n*} \circ (f \circ j_n)^! & \xrightarrow{\zeta_2} & j_{n*} \circ j^!_n \circ f^!
\end{array}$$

where :

- ζ_1 and ζ_2 are induced by the natural isomorphism $\psi_{f,j_n} : (f \circ j_n)^! \xrightarrow{\sim} j^!_n \circ f^!$ of lemma 11.1.16(i).
- ξ_1 (resp. ξ_2) is induced by the isomorphism $\psi_{f\circ j_n, j_{mn}}$ (resp. $\psi_{j_n, j_{mn}}$).

- ε_1 and ε_2 are induced by the counit of adjunction $j_{mn*} \circ j^!_{mn} \to \mathbf{1}_{\mathsf{D}^+(\mathscr{O}_{X_n}\text{-}\mathbf{Mod})}$.
- β_{mn} (resp. α_{mn}) is induced by the natural map $(f \circ j_n)_*\mathscr{O}_{X_n} \to (f \circ j_m)_*\mathscr{O}_{X_m}$ (resp. by the map $j_{n*}\mathscr{O}_{X_n} \to j_{m*}\mathscr{O}_{X_m}$).

It follows from lemma 11.1.22 that the two triangular subdiagrams commute, and it is also clear that the same holds for the inner quadrangular subdiagram. Moreover, lemma 11.1.16(ii) yields a commutative diagram

$$
\begin{array}{ccc}
(f \circ j_m)^! & \xrightarrow{\psi_{f,j_m}} & j^!_m \circ f^! \\
{\scriptstyle \psi_{f\circ j_n, j_{mn}}}\downarrow & & \downarrow{\scriptstyle \psi_{j_n,j_{mn}} \circ f^!} \\
j^!_{mn} \circ (f \circ j_n)^! & \xrightarrow{j^!_{mn}(\psi_{f,j_n})} & j^!_{mn} \circ j^!_n \circ f^!
\end{array}
\tag{11.5.14}
$$

such that j_{m*}(11.5.14) is the diagram :

$$
\begin{array}{ccc}
j_{m*} \circ (f \circ j_m)^! & \xrightarrow{j_{m*}(\psi_{f,j_m})} & j_{m*} \circ j^!_m \circ f^! \\
{\scriptstyle \xi_1}\downarrow & & \downarrow{\scriptstyle \xi_2} \\
j_{m*} \circ j^!_{mn} \circ (f \circ j_n)^! & \xrightarrow{\zeta_1} & j_{m*} \circ j^!_{mn} \circ j^!_n \circ f^!.
\end{array}
$$

We then arrive at the following commutative diagram :

$$
\begin{array}{ccc}
R\mathrm{Hom}^\bullet_{\mathscr{O}_S}(\mathscr{O}_X/\mathscr{I}^m, \mathscr{O}_S) & \xrightarrow{R\Gamma(\psi_{f,j_m})} & R\mathrm{Hom}^\bullet_{\mathscr{O}_X}(\mathscr{O}_X/\mathscr{I}^m, f^!\mathscr{O}_S[0]) \\
{\scriptstyle R\Gamma(\beta_{mn})}\downarrow & & \downarrow{\scriptstyle R\Gamma(\alpha_{mn})} \\
R\mathrm{Hom}^\bullet_{\mathscr{O}_S}(\mathscr{O}_X/\mathscr{I}^n, \mathscr{O}_S) & \xrightarrow{R\Gamma(\psi_{f,j_n})} & R\mathrm{Hom}^\bullet_{\mathscr{O}_X}(\mathscr{O}_X/\mathscr{I}^n, f^!\mathscr{O}_S[0]).
\end{array}
$$

Now, the maps $R\Gamma(\alpha_{mn})$ are the transition morphisms of the inductive system whose colimit appears in (11.5.13), hence we may replace the latter by the inductive system formed by the maps $R\Gamma(\beta_{mn})$. Combining with corollary 10.3.2(i), we finally deduce natural A-linear isomorphisms

$$\operatorname*{colim}_{n\in\mathbb{N}} \mathrm{Ext}^i_{K^+}(A/I^n, K^+) \xrightarrow{\sim} R^i\Gamma_{\{x\}}f^!\mathscr{O}_S[0]$$

where the transition maps in the colimit are induced by the natural maps $A/I^n \to A/I^m$, for every $n \geq m$. Our assertion is an immediate consequence. $\square$

11.5.15. Let A be a local ring, set $X := \operatorname{Spec} A$ let $x \in X$ be the closed point, and suppose that

(a) either, A is noetherian and X admits a dualizing complex
(b) or else, A is essentially of finite presentation over K^+.

Notice that X admits a dualizing complex in case (b) as well. Indeed, in that case one can find a finitely presented affine S-scheme Y and a point $y \in Y$ such that $X \simeq \operatorname{Spec} \mathscr{O}_{Y,y}$; by proposition 11.5.5(iv), Y admits a dualizing complex $\omega^\bullet_Y$, and since every coherent $\mathscr{O}_X$-module extends to a coherent $\mathscr{O}_Y$-module, one verifies easily that the restriction of $\omega^\bullet_Y$ is dualizing for X (cp. the proof of proposition 11.3.37). Hence, let $\omega^\bullet$ be a dualizing complex for X. In case (a) (resp. in case (b)), it follows easily from corollary 11.3.46 (resp. from theorem 11.5.10 and propositions 11.5.5(iii) and 11.3.25), that there exists a unique $c \in \mathbb{Z}$ such that

$$J(x) := R^c\Gamma_{\{x\}}\omega^\bullet \neq 0.$$

Moreover, $J(x)$ is a coh-injective A-module, hence by lemma 7.11.27(iii), we obtain a well defined functor

$$D : \mathsf{D}^b(A\text{-}\mathbf{Mod}) \to \mathsf{D}^b(A\text{-}\mathbf{Mod})^o \quad : \quad C^\bullet \mapsto \mathrm{Hom}^\bullet_A(C^\bullet, J(x)). \tag{11.5.16}$$

Furthermore, let $\mathsf{D}^b_{\{x\}}(A\text{-}\mathbf{Mod}_{\mathrm{coh}})$ be the full subcategory of $\mathsf{D}^b(A\text{-}\mathbf{Mod}_{\mathrm{coh}})$ consisting of all complexes $C^\bullet$ such that $\mathrm{Supp}\, H^\bullet C^\bullet \subset \{x\}$. We have the following :

Corollary 11.5.17 (Local duality)**.** *In the situation of* (11.5.15)*, the following holds :*

(i) *The functor* (11.5.16) *restricts to an equivalence of categories :*

$$D_x : \mathsf{D}^b_{\{x\}}(A\text{-}\mathbf{Mod}_{\mathrm{coh}}) \stackrel{\sim}{\to} \mathsf{D}^b_{\{x\}}(A\text{-}\mathbf{Mod}_{\mathrm{coh}})^o$$

and the natural transformation $C_\bullet \to D_x \circ D_x(C_\bullet)$ is an isomorphism of functors.

(ii) *For every $i \in \mathbb{Z}$ there exists a natural isomorphism of functors :*

$$R^i\Gamma_{\{x\}} \circ \mathscr{D} \stackrel{\sim}{\to} D \circ R^{c-i}\Gamma \quad : \quad \mathsf{D}^b(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}} \to A\text{-}\mathbf{Mod}^o$$

where $\mathscr{D}$ is the duality functor corresponding to $\omega^\bullet$ (see definition 11.3.16*).*

(iii) *Let c and $\mathscr{D}$ be as in* (ii)*, $I \subset A$ a finitely generated ideal such that $V(I) = \{x\}$, and denote by $A^\wedge$ the I-adic completion of A. Then for every $i \in \mathbb{Z}$ there exists a natural isomorphism of functors :*

$$D \circ R^i\Gamma_{\{x\}} \circ \mathscr{D} \stackrel{\sim}{\to} A^\wedge \otimes_A R^{c-i}\Gamma \quad : \quad \mathsf{D}^b(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}} \to A^\wedge\text{-}\mathbf{Mod}^o_{\mathrm{coh}}.$$

Proof. Let $C_\bullet$ be any object of $\mathsf{D}^b_{\{x\}}(A\text{-}\mathbf{Mod}_{\mathrm{coh}})$; we denote by $i : \{x\} \to X$ the natural closed immersion. Obviously $C^\sim_\bullet = i_* i^{-1} C^\sim_\bullet$ (notation of (10.3)), therefore :

$$\begin{aligned}
\mathrm{Hom}^\bullet_A(C_\bullet, J(x)) &\stackrel{\sim}{\to} R\mathrm{Hom}^\bullet_A(C_\bullet, J(x)) && \text{by lemma 7.11.27(iii)}\\
&\stackrel{\sim}{\to} R\mathrm{Hom}^\bullet_{\mathscr{O}_X}(C^\sim_\bullet, R\underline{\Gamma}_{\{x\}}\omega^\bullet[c]) && \text{by theorem 10.1.18}\\
&\stackrel{\sim}{\to} R\mathrm{Hom}^\bullet_{\mathscr{O}_X}(i_* i^{-1} C^\sim_\bullet, \omega^\bullet[c]) && \text{by lemma 10.4.13(i.b)}\\
&\stackrel{\sim}{\to} R\mathrm{Hom}^\bullet_{\mathscr{O}_X}(C^\sim_\bullet, \omega^\bullet[c])
\end{aligned}$$

which easily implies (i). Next, we compute, for any object $\mathscr{F}^\bullet$ of $\mathsf{D}^b(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}}$:

$$\begin{aligned}
R^i\Gamma_{\{x\}}\mathscr{D}(\mathscr{F}^\bullet) &\stackrel{\sim}{\to} R^i\mathrm{Hom}^\bullet_{\mathscr{O}_X}(\mathscr{F}^\bullet, R\Gamma_{\{x\}}\omega^\bullet) && \text{by lemma 10.4.13(iii)}\\
&\stackrel{\sim}{\to} R^i\mathrm{Hom}^\bullet_{\mathscr{O}_X}(\mathscr{F}^\bullet, J(x)[-c])\\
&\stackrel{\sim}{\to} R^i\mathrm{Hom}^\bullet_A(R\Gamma\mathscr{F}^\bullet, J(x)[-c]) && \text{by theorem 10.1.18}\\
&\stackrel{\sim}{\to} \mathrm{Hom}_A(R^{c-i}\Gamma\mathscr{F}^\bullet, J(x)) && \text{by lemma 7.11.27(iii)}
\end{aligned}$$

whence (ii). Finally, let $\mathscr{F}^\bullet$ be any object of $\mathsf{D}^b(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}}$; if A is as in case (b) of (11.5.15), we compute :

$$\begin{aligned}
A^\wedge \otimes_A R^{c-i}\Gamma\mathscr{F}^\bullet &\stackrel{\sim}{\to} \lim_{n\in\mathbb{N}} (A/I^n) \otimes_A R^{c-i}\Gamma\mathscr{F}^\bullet && \text{by corollary 11.4.48(ii)}\\
&\stackrel{\sim}{\to} \lim_{n\in\mathbb{N}} D \circ D((A/I^n) \otimes_A R^{c-i}\Gamma\mathscr{F}^\bullet) && \text{by (i)}\\
&\stackrel{\sim}{\to} \lim_{n\in\mathbb{N}} D(\mathrm{Hom}_A(A/I^n, R^i\Gamma_{\{x\}}\mathscr{D}(\mathscr{F}^\bullet))) && \text{by (ii)}\\
&\stackrel{\sim}{\to} D(\operatorname*{colim}_{n\in\mathbb{N}} \mathrm{Hom}_A(A/I^n, R^i\Gamma_{\{x\}}\mathscr{D}(\mathscr{F}^\bullet)))\\
&\stackrel{\sim}{\to} D(R^i\Gamma_{\{x\}}\mathscr{D}(\mathscr{F}^\bullet))
\end{aligned}$$

so (iii) holds as well. If A is as in case (a) of (11.5.15), the same proof works : instead of corollary 11.4.48(ii), one has just to appeal to [126, Th.8.7 and Th.8.8]. ☐

Corollary 11.5.18. *Let $(A, \mathfrak{m})$ be a local noetherian ring, complete and separated for its $\mathfrak{m}$-adic topology, $\omega^\bullet$ a dualizing complex for $X := \mathrm{Spec}\, A$, and $c \in \mathbb{Z}$ the unique integer such that $J := R^c\Gamma_{\{\mathfrak{m}\}}\omega^\bullet \neq 0$. Let also $M^\bullet$ be a bounded complex of A-modules, and $M^{\bullet\sim}$ the induced complex of quasi-coherent $\mathscr{O}_X$-modules. Then we have a natural isomorphism*

$$\mathrm{Ext}^{c-i}_A(M^\bullet, \omega^\bullet) \stackrel{\sim}{\to} \mathrm{Hom}_A(R^i\Gamma_{\{\mathfrak{m}\}}M^{\bullet\sim}, J) \qquad \textit{for every } i \in \mathbb{Z}.$$

Proof. We may find a filtered system $(M^\bullet_\lambda \mid \lambda \in \Lambda)$ of bounded complexes of A-modules, such that M^n_λ is a submodule of finite type of M^n, for every $n \in \mathbb{Z}$, and such that $M^\bullet = \mathrm{colim}_{\lambda\in\Lambda} M^\bullet_\lambda$ in $\mathsf{C}^b(A\text{-}\mathbf{Mod})$ (details left to the reader). There follows a natural isomorphism of A-modules :

$$\mathrm{Hom}_A(R^i\Gamma_{\{\mathfrak{m}\}}M^{\bullet\sim}, J) \xrightarrow{\sim} \mathrm{Hom}_A(\operatorname*{colim}_{\lambda\in\Lambda} R^i\Gamma_{\{\mathfrak{m}\}}M^{\bullet\sim}_\lambda, J) \xrightarrow{\sim} \lim_{\lambda\in\Lambda} \mathrm{Hom}_A(R^i\Gamma_{\{\mathfrak{m}\}}M^{\bullet\sim}_\lambda, J).$$

Taking into account corollary 8.3.53, we are then reduced to proving the assertion for each $M^\bullet_\lambda$; but in this case, the sought isomorphism follows immediately from corollary 11.5.17(iii). ☐

Definition 11.5.19. Let A and $J(x)$ be as in (11.5.15), and E an A-module. We say that E is *finitely copresented* if there exist a finitely presented A-module M and an A-linear isomorphism $E \xrightarrow{\sim} \mathrm{Hom}_A(M, J(x))$.

Lemma 11.5.20. *Let A be as in* (11.5.15)*, and $I \subset A$ a finitely generated ideal, whose radical is the maximal ideal of A. For every A-module E, the following conditions are equivalent :*

(a) *E is finitely copresented and finitely generated.*
(b) *E is finitely copresented and $I^kE = 0$ for some $k \in \mathbb{N}$.*
(c) *E is finitely presented and $I^kE = 0$ for some $k \in \mathbb{N}$.*

Proof. To ease notation, for every ideal $L \subset A$ and every A-module M set

$$D(M) := \mathrm{Hom}_A(M, J(x)) \quad \text{and} \quad M[L] := \mathrm{Ann}_M(L).$$

Claim 11.5.21. Let E be any finitely copresented A-module. Then $E[I^k]$ is a finitely presented A-module for every $k \in \mathbb{N}$, and $E = \bigcup_{k\in\mathbb{N}} E[I^k]$.

Proof of the claim. We may assume that $E = D(M)$ for some finitely presented A-module M; then clearly every element of E is annihilated by some power of I, since the same holds for $J(x)$. Moreover, we have a natural identification

$$E[I^n] \xrightarrow{\sim} D(M/I^nM) \qquad \text{for every } n \in \mathbb{N}.$$

But M/I^kM is finitely presented and supported at $\{x\}$, so $D(M/I^kM)$ is finitely presented as well, by corollary 11.5.17(i). ♢

It follows immediately from claim 11.5.21 that (a)⇒(b)⇒(c).

(c)⇒(a): If E is finitely presented and $I^kE = 0$, then the same holds for $D(E)$, by corollary 11.5.17(i), and moreover $D \circ D(E) = E$, by the same token, whence (a). ☐

Proposition 11.5.22. *Let A be as in* (11.5.15)*, and $f : E \to N$ a homomorphism of A-modules, with E finitely copresented and N finitely presented. Then we have :*

(i) $\mathrm{Ker}\, f$ *is a finitely copresented A-module.*
(ii) $\mathrm{Im}\, f$ *and* $\mathrm{Coker}\, f$ *are finitely presented A-modules.*

Proof. Clearly, it suffices to show the assertions concerning $\mathrm{Ker}\, f$ and $\mathrm{Im}\, f$. To this aim, pick any finitely generated ideal $I \subset A$ such that $\mathrm{Supp}\, A/I = \{x\}$ (where x denotes the closed point of $\mathrm{Spec}\, A$); by corollary 11.4.41(ii) there exists $n \in \mathbb{N}$ such that $N[I^n] = N[I^m]$ for every integer $m \geq n$. In view of claim 11.5.21, we see that there exists $n \in \mathbb{N}$ such that $\mathrm{Im}\, f = (\mathrm{Im}\, f)[I^n]$. Taking into account corollary 11.4.41(i), we may then replace N by $N[I^n]$, and assume from start that $I^nN = 0$, so N is also finitely copresented, by lemma 11.5.20.

Now, more generally, let E and F be two finitely copresented A-modules, and say that $E = D(M)$, $F = D(P)$, where D is defined as in the proof of lemma 11.5.20; in view of claim

11.5.21 and corollary 11.5.17(i) we get natural identifications :

$$\begin{aligned}\mathrm{Hom}_A(E,F) &\stackrel{\sim}{\to} \lim_{n\in\mathbb{N}} \mathrm{Hom}_A(E[I^n],F[I^n]) \\ &\stackrel{\sim}{\to} \lim_{n\in\mathbb{N}} \mathrm{Hom}_A(P/I^nP,M/I^nM) \\ &\stackrel{\sim}{\to} \lim_{n\in\mathbb{N}} \mathrm{Hom}_A(P,M/I^nM) \\ &\stackrel{\sim}{\to} \lim_{n\in\mathbb{N}} \mathrm{Hom}_A(P,M^\wedge)\end{aligned}$$

where $M^\wedge$ denotes the I-adic completion of M. Recall that the natural map $M \to M^\wedge$ is injective (corollary 11.4.48(ii,iii)), so the same holds for the induced map

$$\mathrm{Hom}_A(P,M) \to \mathrm{Hom}_A(E,F) \qquad f \mapsto D(f). \tag{11.5.23}$$

Claim 11.5.24. With the foregoing notation, we have :

(i) Let $g : E \to F$ be any morphism. If g lies in the image of (11.5.23), then $\mathrm{Ker}\, g$, $\mathrm{Coker}\, g$ and $\mathrm{Im}\, g$ are all finitely copresented A-modules.

(ii) If either $I^kM = 0$ or $I^kP = 0$ for some $k \in \mathbb{N}$, then (11.5.23) is an isomorphism.

Proof of the claim. (i): Indeed, say that g is the image of $h : P \to M$ under (11.5.23); since $J(x)$ is coh-injective, we get natural identifications

$$\mathrm{Ker}\, g \stackrel{\sim}{\to} D(\mathrm{Coker}\, h) \qquad \mathrm{Coker}\, g \stackrel{\sim}{\to} D(\mathrm{Ker}\, h) \qquad \mathrm{Im}\, g \stackrel{\sim}{\to} D(\mathrm{Im}\, h)$$

whence the contention.

(ii): If $I^kM = 0$ for some $k \in \mathbb{N}$, then the natural map $M \to M^\wedge$ is an isomorphism, and the assertion follows. If $I^kP = 0$, then any morphism $P \to M^\wedge$ factors through $M^\wedge[I^k]$ (notation of the proof of lemma 11.5.20); however, corollaries 11.4.41(i) and 11.4.48(i) yield natural identifications

$$M[I^n] \stackrel{\sim}{\to} (M[I^n])^\wedge \stackrel{\sim}{\to} M^\wedge[I^n]$$

whence the contention (details left to the reader). ◊

Now, letting $F := N$ in claim 11.5.24, we deduce that both $\mathrm{Ker}\, f$ and $\mathrm{Im}\, f$ are finitely copresented. Lastly, since $I^n \cdot \mathrm{Im}\, f = 0$, lemma 11.5.20 says that $\mathrm{Im}\, f$ is finitely presented, as sought. □

Theorem 11.5.25. *In the situation of* (11.5.15)*, let $I \subset A$ be any finitely generated ideal, $C^\bullet$ any object of $\mathsf{D}^b(\mathscr{O}_X\text{-}\mathbf{Mod}_{\mathrm{coh}})$, and set $U := X\backslash\{x\}$. For every $i \in \mathbb{Z}$, the following conditions are equivalent :*

(a) *$I \cdot H^i(U, C^\bullet)$ is a finitely presented A-module.*

(b) *$I \cdot H^i(U, C^\bullet)$ is a finitely generated A-module.*

(c) *$I \cdot R^{i+1}\Gamma_{\{x\}}C^\bullet$ is a finitely presented A-module.*

(d) *$I \cdot R^{i+1}\Gamma_{\{x\}}C^\bullet$ is a finitely generated A-module.*

(e) *The radical of $\mathrm{Ann}_A(I \cdot R^{i+1}\Gamma_{\{x\}}C^\bullet)$ contains the maximal ideal of A.*

(f) *$I \cdot R^i\Gamma_{\{y\}}C^\bullet_{|X(y)} = 0$ for every closed point $y \in U$.*

Proof. Let $\omega^\bullet$ be a dualizing complex for X and $y \in X$ any point; then $\omega^\bullet_{|X(y)}$ is a dualizing complex for $X(y)$, and according to (11.5.15), there is a unique integer $c(y)$ such that $J(y) := R^{c(y)}\Gamma_{\{y\}}\omega^\bullet_{|X(y)} \neq 0$. If A is as in case (b) (resp. as in case (a)) of (11.5.15), we may invoke theorem 11.5.10 and lemma 11.5.8(i) (resp. lemma 11.3.50) to see that

$$c(y) + 1 = c := c(x) \quad \Leftrightarrow \quad y \text{ is a closed point of } U. \tag{11.5.26}$$

By corollary 11.5.17(i,ii), the rule

$$M \mapsto D_y(M) := \mathrm{Hom}_A(M, J(y)) \qquad \text{for every } \mathscr{O}_{X,y}\text{-module } M$$

restricts to an equivalence $\mathscr{O}_{X,y}\text{-}\mathbf{Mod}_{\mathrm{coh}} \xrightarrow{\sim} \mathscr{O}_{X,y}\text{-}\mathbf{Mod}^o_{\mathrm{coh}}$, and there is a natural isomorphism

$$(11.5.27)\qquad R^i\Gamma_{\{y\}}C^\bullet_{|X(y)} \xrightarrow{\sim} D_y(H^{c(y)-i}(X(y),(\mathscr{D}C^\bullet)_{|X(y)}))\qquad \text{in } \mathscr{O}_{X,y}\text{-}\mathbf{Mod}$$

where $\mathscr{D}$ is the duality functor corresponding to $\omega^\bullet$. Notice that $a\cdot\mathbf{1}_{D_yM} = D_y(a\cdot\mathbf{1}_M)$ for every $a\in\mathscr{O}_{X,y}$, so (11.5.27) and (11.5.26) imply that (f) holds if and only if

$$(11.5.28)\qquad I\cdot H^{c-i-1}(X(y),(\mathscr{D}C^\bullet)_{|X(y)}) = 0 \qquad \text{for every closed point } y \text{ of } U.$$

However, if A is as in case (b) (resp. as in case (a)) of (11.5.15), lemma 11.5.8(iii) (resp. theorem 11.3.52) says that every point of U specializes to a closed point, so (11.5.28) holds if and only if $I\cdot H^{c-i-1}(\mathscr{D}C^\bullet)_{|U}=0$. Since $\mathscr{D}C^\bullet$ is a complex of coherent $\mathscr{O}_X$-module, the latter condition is equivalent to saying that there exists an ideal I' whose radical is the maximal ideal of A, and such that $I'I\cdot H^{c-i-1}(\mathscr{D}C^\bullet)=0$. By invoking (11.5.27) with $y:=x$, we see that this holds if and only if $I'I\cdot R^{i+1}_{\{x\}}C^\bullet=0$. Summing up, we have shown that (e)⇔(f).

Claim 11.5.29. The A-module $I\cdot R^i\Gamma_{\{x\}}C^\bullet$ is finitely copresented, for every $i\in\mathbb{Z}$.

Proof of the claim. Notice first that, due to (11.5.27), the A-module $E := R^i\Gamma_{\{x\}}C^\bullet$ is finitely copresented, so the same holds for $E^{\oplus n}$, for every $n\in\mathbb{N}$. Since I is finitely generated, it is the image of a morphism $f: A^{\oplus n}\to A$, for some $n\in\mathbb{N}$, and then IE is naturally identified with the image of $f\otimes_A\mathbf{1}_E$. Say that $E=D_x(M)$ for a finitely presented A-module M; then clearly $f\otimes\mathbf{1}_E = D_x(f^\vee\otimes_A\mathbf{1}_M)$, where $f^\vee: A\to A^{\oplus n}$ is the transpose of f. Now the assertion follows from claim 11.5.24(i). ◊

Taking into account claim 11.5.29 and lemma 11.5.20, we deduce that (c)⇔(d)⇔(e).

Obviously, (a)⇒(b). Next, consider the exact sequence

$$H^i(X,C^\bullet)\xrightarrow{\alpha_i}H^i(U,C^\bullet)\xrightarrow{\beta_i}R^{i+1}\Gamma_{\{x\}}C^\bullet\xrightarrow{\gamma_{i+1}}H^{i+1}(X,C^\bullet)$$

(see (10.4.1)). Assume (b), which implies that $I\cdot\operatorname{Im}\beta_i$ is a finitely generated A-module, so there exists an ideal $I'\subset A$ whose radical is the maximal ideal, and such that $I'I\cdot\operatorname{Im}\beta_i=0$. On the other hand, by claim 11.5.29 and proposition 11.5.22 we know that $\operatorname{Im}\gamma_{i+1}$ is a finitely presented A-module, hence there exists an ideal $I''\subset A$ whose radical is the maximal ideal, and such that $I''\cdot\operatorname{Im}\gamma_{i+1}=0$. We conclude that $I''I'I\cdot R^{i+1}\Gamma_{\{x\}}C^\bullet=0$, whence (b)⇒(e).

Lastly, we suppose that (e) holds, and we show that (a) follows. To this aim, set $M := I\cdot H^i(U,C^\bullet)\cap\operatorname{Im}\alpha_i$, and consider the induced ladder with exact rows

$$(11.5.30)\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & I\cdot H^i(U,C^\bullet) & \longrightarrow & I\cdot\operatorname{Ker}\gamma_{i+1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \operatorname{Coker}\gamma_i & \xrightarrow{\overline{\alpha}_i} & H^i(U,C^\bullet) & \longrightarrow & \operatorname{Ker}\gamma_{i+1} & \longrightarrow & 0. \end{array}$$

From (11.5.27) we know that $\operatorname{Ker}\gamma_{i+1}$ is finitely copresented and arguing as in the proof of claim 11.5.29, we conclude that the same holds for $I\cdot\operatorname{Ker}\gamma_{i+1}$. But (e) implies that the latter A-module is annihilated by a finitely generated ideal whose radical is the maximal ideal of A, so it is finitely presented, by lemma 11.5.20. We are thus reduced to checking that M is a finitely presented A-module. Now, set $N := \operatorname{Ker}(\overline{\alpha}_i\otimes_A\mathbf{1}_{A/I})$ and notice the short exact sequence

$$0\to I\cdot\operatorname{Coker}\gamma_i\to M\to N\to 0.$$

By proposition 11.5.22(ii) and (11.5.27), the A-module $\operatorname{Coker}\gamma_i$ is finitely presented, so the same holds for $I\cdot\operatorname{Coker}\gamma_i$, because A is coherent and I is finitely generated. So, we are further reduced to showing that N is finitely presented. However, N is naturally identified with the image of the map

$$\operatorname{Tor}_1^A(\operatorname{Ker}\gamma_{i+1},A/I)\to(\operatorname{Coker}\gamma_i)\otimes_A A/I$$

induced by the bottom row of (11.5.30). On the other hand, we have already noticed that $\mathrm{Coker}\,\gamma_i$ is finitely presented, so the same holds for $(\mathrm{Coker}\,\gamma_i) \otimes_A A/I$; also, proposition 11.5.22(i) and (11.5.27) say that $\mathrm{Ker}\,\gamma_{i+1}$ is finitely copresented. Taking into account proposition 11.5.22(ii) it finally suffices to show :

Claim 11.5.31. If Q is any finitely copresented A-module and P any finitely presented A-module, the A-module $\mathrm{Tor}_i^A(Q, P)$ is finitely copresented as well, for every $i \in \mathbb{N}$.

Proof of the claim. Since I is finitely generated and A is coherent, we may find a resolution $F_\bullet \xrightarrow{\sim} P[0]$ consisting of free A-modules of finite type (details left to the reader), and then $\mathrm{Tor}_i^A(Q, P) \simeq H_i(F_\bullet \otimes_A Q)$, so the assertion follows easily from claim 11.5.24(i) (details left to the reader). □

Proposition 11.5.32. *In the situation of* (11.5.15)*, let $d \in \mathbb{N}$ be any integer, and suppose additionally that :*

- (a') *If A is as in case* (a)*, then A is regular of dimension $d+1$.*
- (b') *If A is as in case* (b)*, the structure morphism $f : X \to S$ is flat, $f(x) = s$ and $f^{-1}(s)$ is a regular scheme of dimension d.*

Let $j : U := X\setminus\{x\} \to X$ be the open immersion, $\mathscr{F}$ a flat quasi-coherent $\mathscr{O}_U$-module. Then the following two conditions are equivalent :

- (c) *$\Gamma(U, \mathscr{F})$ is a flat A-module.*
- (d) *$\delta(x, j_*\mathscr{F}) \geq d+1$.*

Proof. If $d = 0$, then A is a valuation ring (proposition 9.1.34(ii)), and U is the spectrum of the field of fractions of A, in which case both (c) and (d) hold trivially. Hence we may assume that $d \geq 1$. Suppose now that (c) holds, and set $F := \Gamma(U, \mathscr{F})$. In view of (10.4.3), we need to show that $H^i(U, \mathscr{F}) = 0$ whenever $1 \leq i \leq d-1$. If $F^\sim$ denotes the $\mathscr{O}_X$-module determined by F, then $F^\sim_{|U} = \mathscr{F}$. Moreover, by [120, Ch.I, Th.1.2], F is the colimit of a filtered family $(L_i \mid i \in I)$ of free A-modules of finite rank, hence $H^i(U, \mathscr{F}) = \mathrm{colim}_{i\in I} H^i(U, L_i^\sim)$ by proposition 10.1.10(ii), so we are reduced to the case where $\mathscr{F} = \mathscr{O}_U$, and therefore $j_*\mathscr{F} = \mathscr{O}_X$ (corollary 11.3.9). In case (a'), assertion (d) follows already, by virtue of [126, Th.17.8]. So, suppose that (b') holds; since $f^{-1}(s)$ is regular, we have $\delta(x, \mathscr{O}_{f^{-1}(s)}) = d$; then, since the topological space underlying X is noetherian, lemma 10.4.20(ii) and corollary 10.4.47 imply that $\delta(x, \mathscr{O}_X) \geq d+1$, which is (d).

Conversely, suppose that (d) holds; we consider first the case (b'), and we shall derive (c) by induction on d, the case $d = 0$ having already been dealt with. Let $\mathfrak{m}_A$ (resp. $\overline{\mathfrak{m}}_A$) be the maximal ideal of A (resp. of $\overline{A} := A/\mathfrak{m}_K A$). Suppose then that $d \geq 1$ and that the assertion is known whenever $\dim f^{-1}(s) < d$. Pick $\overline{t} \in \overline{\mathfrak{m}}_A \setminus \overline{\mathfrak{m}}_A^2$, and let $t \in \mathfrak{m}_A$ be any lifting of $\overline{t}$. Since $f^{-1}(s)$ is regular, $\overline{t}$ is a regular element of $\overline{A}$, so that t is regular in A and the induced morphism $g : X' := \mathrm{Spec}\,A/tA \to S$ is flat ([65, Ch.IV, Th.11.3.8]). Let $j' : U' := U \cap X' \to X'$ be the restriction of j; our choice of $\overline{t}$ ensures that $g^{-1}(s)$ is a regular scheme, so the pair $(X', \mathscr{F}/t\mathscr{F}_{|U'})$ fulfills the conditions of the proposition, and $\dim g^{-1}(s) = d-1$. Furthermore, since $\mathscr{F}$ is a flat $\mathscr{O}_U$-module, the sequence $0 \to \mathscr{F} \xrightarrow{t} \mathscr{F} \to \mathscr{F}/t\mathscr{F} \to 0$ is short exact. Assumption (d) means that

$$H^i(U, \mathscr{F}) = 0 \qquad \text{whenever } 1 \leq i \leq d-1. \tag{11.5.33}$$

Therefore, from the long exact cohomology sequence we deduce that $H^i(U, \mathscr{F}/t\mathscr{F}) = 0$ for $1 \leq i \leq d-2$, *i.e.*

$$\delta(x, j'_*\mathscr{F}/t\mathscr{F}) \geq d. \tag{11.5.34}$$

The same sequence also yields a left exact sequence :

$$0 \to H^0(U, \mathscr{F}) \otimes_A A/tA \xrightarrow{\alpha} H^0(U, \mathscr{F}/t\mathscr{F}) \to H^1(U, \mathscr{F}). \tag{11.5.35}$$

Claim 11.5.36. $H^0(U, \mathscr{F}) \otimes_A A/tA$ is a flat A/tA-module.

Proof of the claim. By (11.5.34) and our inductive assumption, $H^0(U, \mathscr{F}/t\mathscr{F})$ is a flat A/tA-module. In case $d = 1$, proposition 9.1.34(ii) shows that A/tA is a valuation ring, and then the claim follows from (11.5.35), and [34, Ch.VI, §3, n.6, Lemma 1]. If $d > 1$, (11.5.33) implies that α is an isomorphism, so the claim holds also in such case. ◊

Set $V := \mathrm{Spec}\, A[t^{-1}] = X \setminus V(t) \subset U$; since $\mathscr{F}_{|V}$ is a flat $\mathscr{O}_V$-module, the $A[t^{-1}]$-module $H^0(U, \mathscr{F}) \otimes_A A[t^{-1}] \simeq H^0(V, \mathscr{F}_{|V})$ is flat. Moreover, since t is regular on both A and $H^0(U, \mathscr{F})$, an easy calculation shows that $\mathrm{Tor}_i^A(A/tA, H^0(U, \mathscr{F})) = 0$ for $i > 0$. Then the contention follows from claim 11.5.36 and [75, Lemma 5.2.1].

Lastly, in case (a') holds, one picks any element $t \in \mathfrak{m}_A \setminus \mathfrak{m}_A^2$ and argues in the same way (with some simplifications : details left to the reader). □

Remark 11.5.37. The special case of proposition 11.5.32 where A is regular local ring and $\mathscr{F}$ is a locally free $\mathscr{O}_U$-module of finite rank has been studied in detail in [97].

Lemma 11.5.38. *Let A be a noetherian ring, M an A-module endowed with a finite filtration $M_0 := 0 \subset M_1 \subset \cdots \subset M_k := M$, and denote by $\mathrm{gr}^\bullet M$ the associated graded A-module. Suppose that, for every $i = 1, \dots, k$, there exists an ideal $I_i \subset A$ such that $I_i \cdot \mathrm{gr}^i M$ is an A-module of finite type. Then $\prod_{i=1}^k I_i M$ is an A-module of finite type.*

Proof. A simple induction on the length of the filtration reduces the assertion to the following

Claim 11.5.39. Let A be a noetherian ring, $I, J \subset A$ two ideals, and

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

a short exact sequence of A-modules, such that IM_1 and JM_3 are finitely generated. Then IJM_2 is finitely generated.

Proof of the claim. Indeed, the induced map $N := JM_2/(M_1 \cap JM_2) \to JM_3$ is injective, so N is finitely generated, and then the same holds for $I \otimes_A N$; but the latter maps surjectively onto $IJM_2/(IM_1 \cap IJM_2) \simeq (IJM_2 + IM_1)/IM_1$, so the latter is also of finite type, and therefore the same holds for $IJM_2 + IM_1$, whence the lemma. □

11.5.40. Let A be a noetherian ring, $\omega^\bullet$ a dualizing complex on $X := \mathrm{Spec}\, A$, and $I \subset A$ any ideal. Let also $U \subset X$ be any open subset, and $K^\bullet$ any object of $\mathsf{D}^b(\mathscr{O}_U\text{-}\mathbf{Mod})_{\mathrm{coh}}$. We denote :

- $\overline{U}$ the topological closure of U in X
- $Z := X \setminus U$ and $\partial U := \overline{U} \setminus U$, endowed with the reduced closed subscheme structures
- $U_1 := \{x \in U \mid$ there exists $y \in \partial U$ which is an immediate specialization of x in $\overline{U}\}$
- $\mathscr{D}_U : \mathsf{D}^b(\mathscr{O}_U\text{-}\mathbf{Mod})^o_{\mathrm{coh}} \to \mathsf{D}^b(\mathscr{O}_U\text{-}\mathbf{Mod})_{\mathrm{coh}}$ the duality functor associated with $\omega^\bullet_{|U}$
- $n(K^\bullet)$ the cardinality of $\{i \in \mathbb{Z} \mid H^i(\mathscr{D}_U K^\bullet) \neq 0\}$
- $d(K^\bullet) := \min(n(K^\bullet), 1 + \dim \partial U)$
- $\partial U(x) := \partial U \cap X(x)$ and $\delta(x) := \dim \partial U(x)$ for every $x \in X$
- $e(x) := \min(1 + \delta(y) \mid y$ is a specialization of x in $\overline{U})$ for every $x \in U_1$.

Here we use the convention that $\dim \partial U = -1$ if $\partial U = \varnothing$, and likewise we define the dimension of $\partial U(y)$, for any $y \in X$ such that this set is empty. On the other hand, the dimension of ∂U can be infinite, in the current setting. Notice that the integer $n(K^\bullet)$ depends only on $K^\bullet$, and does not depend on the choice of $\omega^\bullet$.

Lemma 11.5.41. *For every $x \in \partial U$, every closed point of $U \cap X(x)$ is an immediate generization of x in $X(x)$. Especially, every such point lies in U_1.*

Proof. Fix any $x \in \partial U$, set $A := \mathscr{O}_{X,x}$, and for any closed point z of $U \cap X(x)$, denote by $\mathfrak{p}_z \subset A$ the prime ideal corresponding to z, so that $\mathfrak{p}_z$ is strictly contained in the maximal ideal $\mathfrak{m}_A$ of A. Let also $I \subset A$ be any ideal such that $\mathrm{Spec}\, A/I = Z(x)$, so $I \not\subset \mathfrak{p}_z$ by construction. Set $B := A/\mathfrak{p}_z$, and denote by $\mathfrak{m}_B, \overline{I} \subset B$ the images of respectively $\mathfrak{m}_A$ and I. It suffices to show that $\dim B = 1$; suppose by contradiction that the latter fails, and pick any $f \in \overline{I} \setminus \{0\}$.

Claim 11.5.42. If $\dim B > 1$, there exists a prime ideal $\overline{\mathfrak{q}} \subset B$ of height 1, such that $f \notin \overline{\mathfrak{q}}$.

Proof of the claim. Clearly, every prime ideal of height one containing f is a maximal point of $\mathrm{Supp}\, B/fB$, hence it lies in $\mathrm{Ass}\, B/fB$ (lemma 10.5.3(i)); therefore, the set Σ of prime ideals of B of height one containing f is finite ([126, Th.6.5(i)]). If Σ is empty, Krull's *Hauptidealsatz* ([126, Th.13.5]) implies that f must be invertible, in which case the claim is obvious. If Σ contains a single prime ideal $\overline{\mathfrak{q}}_1$, notice that $\overline{\mathfrak{q}}_1 \neq \mathfrak{m}_B$, since $\dim B > 1$, and pick any $g \in \mathfrak{m}_B \setminus \mathfrak{q}_1$; again by Krull's theorem we know that there exists a prime ideal $\overline{\mathfrak{q}} \subset B$ of height one containing g, and by construction $\overline{\mathfrak{q}} \neq \overline{\mathfrak{q}}_1$, whence the claim, in this case. Otherwise, say that $\Sigma = \{\overline{\mathfrak{q}}_1, \cdots, \overline{\mathfrak{q}}_n\}$ for some $n \geq 2$. For every $i, j \leq n$ with $i \neq j$ we may find an element $g_{ij} \in \mathfrak{q}_i \setminus \mathfrak{q}_j$, and we set $g := \sum_{j=1}^n \prod_{i \neq j} g_{ij}$. It is easily seen that $g \in \mathfrak{m}_B \setminus (\overline{\mathfrak{q}}_1 \cup \cdots \cup \overline{\mathfrak{q}}_n)$, so we may apply again Krull's theorem to exhibit a prime ideal $\overline{\mathfrak{q}}$ which cannot lie in Σ, since it contains g. ◊

Let $\overline{\mathfrak{q}}$ be as in claim 11.5.42, and denote by $\mathfrak{q} \subset A$ the preimage of $\overline{\mathfrak{q}}$; then $\mathfrak{q}$ corresponds to a point $y \in U \cap X(x)$ which is different from x, and is a proper specialization of z in $X(x)$, which is impossible, since by assumption z is closed in U. □

Theorem 11.5.43. *In the situation of* (11.5.40)*, let $i \in \mathbb{Z}$ be any integer, and consider the following conditions :*

(a) $I \cdot R^i\Gamma_{\{x\}}K^\bullet_{|U(x)} = 0$ *for every* $x \in U_1$.
(b) $I^{d(K^\bullet)} \cdot H^i(U, K^\bullet)$ *is an A-module of finite type.*
(c) $I \cdot H^j(U, K^\bullet)$ *is an A-module of finite type, for every* $j \leq i$.
(d) $I^{e(x)} \cdot R^j\Gamma_{\{x\}}K^\bullet_{|U(x)} = 0$ *for every* $x \in U_1$ *and every* $j \leq i$.

Then (a)⇒(b) *and* (c)⇒(d).

Proof. (a)⇒(b): By corollary 10.3.39, we may find an object $L^\bullet$ of $\mathsf{D}^b(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}}$ with an isomorphism $L^\bullet_{|U} \xrightarrow{\sim} K^\bullet$ in $\mathsf{D}^b(\mathscr{O}_U\text{-}\mathbf{Mod})_{\mathrm{coh}}$, whence an exact sequence

$$H^i(X, L^\bullet) \to H^i(U, K^\bullet) \to R^{i+1}\Gamma_Z L^\bullet \to H^{i+1}(X, L^\bullet)$$

(see (10.4.1)) whose first and last terms are finitely generated A-modules. By an easy diagram chase, it follows that condition (b) is equivalent to

(e) $I^{d(K^\bullet)} \cdot R^{i+1}\underline{\Gamma}_Z L^\bullet$ is a coherent $\mathscr{O}_X$-module.

It will therefore suffice to show that (a)⇒(e). However, from lemma 10.4.13(iii) we get a natural isomorphism

$$R\underline{\Gamma}_Z L^\bullet \xrightarrow{\sim} R\mathscr{H}om^\bullet_Z(\mathscr{D}_X L^\bullet, \omega^\bullet)$$

where $\mathscr{D}_X$ is the dualizing functor on $\mathsf{D}^b(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}}$ defined by $\omega^\bullet$. By choosing an injective resolution for $\omega^\bullet$, we obtain a spectral sequence

$$E_2^{pq} := R^p\mathscr{H}om^\bullet_Z(H^{-q}(\mathscr{D}_X L^\bullet)[0], \omega^\bullet) \Rightarrow R^{p+q}\mathscr{H}om^\bullet_Z(\mathscr{D}_X L^\bullet, \omega^\bullet).$$

Now, for every $q \in \mathbb{Z}$, define the $\mathscr{O}_X$-module $\overline{H}^{-q}$ by the short exact sequence

$$0 \to \underline{\Gamma}_Z H^{-q}(\mathscr{D}_X L^\bullet) \to H^{-q}(\mathscr{D}_X L^\bullet) \to \overline{H}^{-q} \to 0.$$

There results an exact sequence

(11.5.44)
$$F^{pq} := R^p\mathscr{H}om^\bullet_Z(\overline{H}^{-q}[0], \omega^\bullet) \to E_2^{pq} \to G^{pq} := R^p\mathscr{H}om^\bullet_Z(\underline{\Gamma}_Z H^{-q}(\mathscr{D}_X L^\bullet)[0], \omega^\bullet)$$

Claim 11.5.45. For every $p, q \in \mathbb{Z}$ we have :

(i) E_2^{pq} and F^{pq} are quasi-coherent $\mathscr{O}_X$-modules.

(ii) G^{pq} and $\overline{H}^{-q}$ are coherent $\mathscr{O}_X$-modules.

Proof of the claim. The assertions about E_2^{pq} and F^{pq} follow immediately from lemmata 10.4.17(i) and 10.4.13(iii) and corollary 10.3.2(ii), and the assertion for $\overline{H}^{-q}$ follows from lemma 10.4.17(i). Next, notice that the natural map

$$\underline{\Gamma}_Z H^{-q}(\mathscr{D}_X L^\bullet) \to R\underline{\Gamma}_Z \circ \underline{\Gamma}_Z H^{-q}(\mathscr{D}_X L^\bullet)$$

is an isomorphism in $\mathsf{D}(\mathscr{O}_X\text{-}\mathbf{Mod})$, whence an isomorphism of $\mathscr{O}_X$-modules

$$G^{pq} \xrightarrow{\sim} R^p\mathscr{H}om^\bullet_{\mathscr{O}_X}(\underline{\Gamma}_Z H^{-q}(\mathscr{D}_X L^\bullet)[0], \omega^\bullet) \tag{11.5.46}$$

(lemma 10.4.13(vi)). But since X is noetherian, $\underline{\Gamma}_Z H^{-q}(\mathscr{D}_X L^\bullet)$ is a coherent $\mathscr{O}_X$-module, so the same holds for the target of (11.5.46) (proposition 10.3.3(i)), and the claim follows. ◊

Claim 11.5.47. In order to show (e), it suffices to check that :

(i) $I \cdot F^{pq} = 0$ for every $p, q \in \mathbb{Z}$ such that $p + q = i + 1$.

(ii) $F_x^{pq} = 0$ for every $x \in X$ and every $p \in \mathbb{Z}$ such that $p \notin [c_X(x) - \dim \partial U(x), c_X(x)]$.

Proof of the claim. Say that $\mathscr{D}_X L^\bullet \in \mathsf{D}^{[a,b]}(\mathscr{O}_X\text{-}\mathbf{Mod})$ for some $a, b \in \mathbb{Z}$ with $a \leq b$; to begin with, we show that – under assumptions (i) and (ii) of the claim – there exist for every $q = a, \dots, b$ and every $p = i+1-b, \dots, i+1-a$ and $x \in X$, integers $\nu(q), \mu(p, x) \in \mathbb{N}$ such that $I^{\nu(q)} F^{pq} = 0$ and $I^{\mu(p,x)} F_x^{pq} = 0$, and moreover

$$\sum_{q=a}^{b} \nu(q) \leq n(K^\bullet) \qquad \sum_{p=i+1-b}^{i+1-a} \mu(p, x) \leq 1 + \delta(x) \quad \text{for every } x \in X. \tag{11.5.48}$$

Indeed, if $q \in [a, b]$ is an index such that $H^{-q}(\mathscr{D}_U K^\bullet) = 0$, it follows that $H^{-q}(\mathscr{D}_X L^\bullet)_{|U} = 0$, so $\overline{H}^{-q} = 0$, therefore $F^{pq} = 0$ for every $p \in \mathbb{Z}$, and we may take $\nu(q) = 0$ in this case. But assumption (i) says that in any case we may take $\nu(q) \leq 1$ for every $q = a, \dots, b$. This already suffices to achieve the sought upper bound for the sum of the values $\nu(q)$. The other upper bound is treated similarly : we may in any case take $\mu(p, x) \leq 1$, due to condition (i), and moreover condition (ii) says that we may take $\mu(p, x) = 0$ for every $p \in \mathbb{Z}$ such that $p \notin [c_X(x) - \dim \partial U(x), c_X(x)]$, whence the contention.

Next, recall that $M := R^{i+1}\underline{\Gamma}_Z L^\bullet$ admits a finite filtration $\mathrm{Fil}^\bullet M$ whose associated graded $\mathscr{O}_X$-module is $\bigoplus_{p+q=i+1} E_\infty^{pq}$, and by construction, each term E_∞^{pq} is a quasi-coherent subquotient of E_2^{pq} (claim 11.5.45(i)), so $\mathrm{Fil}^q M$ is a quasi-coherent $\mathscr{O}_X$-module as well, for every $q = a, \dots, b$. Using claims 11.5.39 and 11.5.45(i,ii) we deduce that $I^{\nu(q)} E_2^{pq}$ is a coherent $\mathscr{O}_X$-module for every $q = a, \dots, b$, so the same holds for $I^{\nu(q)} E_\infty^{pq}$, and then (11.5.48) and lemma 11.5.38 imply that $I^{n(K^\bullet)} M$ is a coherent $\mathscr{O}_X$-module.

Set $\delta := \dim \partial U \in \mathbb{N} \cup \{\infty\}$; if $\delta \geq n(K^\bullet) - 1$, the proof is complete; otherwise $\delta \in \mathbb{N}$, and it remains only to check that $I^{1+\delta} M$ is also a coherent $\mathscr{O}_X$-module. To this aim, notice that (11.5.44) induces a short exact sequence

$$0 \to F_\infty^{pq} \to E_\infty^{pq} \to G_\infty^{pq} \to 0$$

where F_∞^{pq} (resp. G_∞^{pq}) is a certain quasi-coherent subquotient of F^{pq} (resp. of G^{pq}); from claim 11.5.45(ii) it follows that G_∞^{pq} is a coherent $\mathscr{O}_X$-module, and the foregoing implies that $I^{\mu(p,x)} F_{\infty,x}^{pq} = 0$ for every $x \in X$ and every $p = i+1-b, \dots, i+1-a$. We may then find, for every $q = a, \dots, b$ a coherent $\mathscr{O}_X$-submodule $N^q \subset \mathrm{Fil}^q M$ such that the induced morphism $\mathrm{Fil}^q M \to G_\infty^{pq}$ restricts to an epimorphism $N^q \to G_\infty^{pq}$. Set $N := \sum_{q=a}^{b} N^q \subset M$, and $P := M/N$, endow P with the filtration $\mathrm{Fil}^\bullet P$ induced by the filtration of M, and denote by $\mathrm{gr}^\bullet P$ the associated graded $\mathscr{O}_X$-module. By construction, we get an epimorphism $\psi_q : E_\infty^{pq} \to \mathrm{gr}^q P$ for every $q = a, \dots, b$, whose kernel contains the image of N^q in E_∞^{pq}. Consequently, the restriction of ψ_q to F_∞^{pq} is still an epimorphism, so that $I^{\mu(p,x)}\mathrm{gr}^{i+1-p}P_x = 0$ for every $x \in X$ and every $p = i+1-b, \dots, i+1-a$. In view of (11.5.48) we conclude that $I^{1+\delta(x)}P_x = 0$ for every $x \in X$, and since clearly $\delta(x) \leq \delta$ for every $x \in X$, it follows that $I^{1+\delta}P = 0$. Lastly, since N is a coherent $\mathscr{O}_X$-module, it suffices to invoke claim 11.5.39 to conclude the proof. ♢

Let $\mathscr{R}^\bullet$ be the residual complex arising from $\omega^\bullet$, introduced in (11.3.53). With this notation, F^{pq} is a subquotient of the A-module $\mathscr{H}om_{\mathscr{O}_X}(\overline{H}^{-q}, \underline{\Gamma}_Z\mathscr{R}^p)$. From lemma 10.4.4(iii.b), we get

$$\underline{\Gamma}_Z\mathscr{R}^p = \bigoplus_{x \in X_p \cap Z} E(x)^\sim. \qquad \text{with } X_p := \{x \in X \mid c_X(x) = p\} \text{ for every } p \in \mathbb{Z}$$

where $c_X : |X| \to \mathbb{Z}$ is the weak codimension function associated with $\omega^\bullet$ as in (11.3.49) and $E(x)$ is an injective hull of the A-module $\kappa(x)$, for every $x \in X$ (notation of (10.3)). Hence, denote by X_p^* the set of all specializations in X of the points of X_p; we deduce

$$\mathrm{Supp}\,\underline{\Gamma}_Z\mathscr{R}^p = Z \cap X_p^* \tag{11.5.49}$$

(we caution the reader that X_p is usually not a pro-constructible subset of X, so X_p^* is usually strictly contained in the topological closure of X_p in X). Moreover, since $\overline{H}^{-q}$ is a coherent $\mathscr{O}_X$-module (claim 11.5.45(ii)) and $\underline{\Gamma}_Z\mathscr{R}^p$ is quasi-coherent (lemma 10.4.17(i)), we also see that

$$\mathscr{H}om_{\mathscr{O}_X}(\overline{H}^{-q}, \underline{\Gamma}_Z\mathscr{R}^p)_x \simeq \mathrm{Hom}_{\mathscr{O}_{X,x}}(\overline{H}_x^{-q}, (\underline{\Gamma}_Z\mathscr{R}^p)_x) \qquad \text{for every } x \in X \tag{11.5.50}$$

([75, Lemma 2.4.29(i.a)]). Combining (11.5.49) and (11.5.50) we conclude that

$$\mathrm{Supp}\,F^{pq} \subset Z \cap X_p^* \cap \mathrm{Supp}\,\overline{H}^{-q}.$$

From this, we can already show condition (ii) of claim 11.5.47 : indeed, say that $F_x^{pq} \neq 0$; then $x \in X_p^*$, so $c_X(x) \geq p$, and on the other hand, by construction we have $\underline{\Gamma}_Z\overline{H}^{-q} = 0$, so $\mathrm{Supp}\,H^{-q}$ is the topological closure in X of $U \cap \mathrm{Supp}\,\overline{H}^{-q}$, therefore $x \in \partial U$ and furthermore there exists a generization z of x in X with $c_X(z) = p$, so $z \in X_p \cap \partial U(x)$, which forces the bound $c_X(x) \leq p + \dim \partial U(x)$, as required.

Lastly, to get condition (i) of claim 11.5.47 it suffices to check assertions (i) and (ii) of the following :

Claim 11.5.51. (i) $\mathrm{Supp}\,(I \cdot F^{pq}) \subset \mathrm{Supp}\,\mathscr{H}om_{\mathscr{O}_X}(I \cdot \overline{H}^{-q}, \underline{\Gamma}_Z\mathscr{R}^p)$ for every $p, q \in \mathbb{Z}$.
(ii) $\mathscr{H}om_{\mathscr{O}_X}(I \cdot \overline{H}^{-q}, \underline{\Gamma}_Z\mathscr{R}^p) = 0$ whenever $p + q = i + 1$.
(iii) $Z \cap X_p \cap \mathrm{Supp}\,(I \cdot \overline{H}^{-q}) = \varnothing$ whenever $p + q = i + 1$.

Proof of the claim. (i): Consider arbitrary $x \in X$, $a \in I$ and $f \in F_x^{pq}$; by virtue of (11.5.50), the element f is represented by an $\mathscr{O}_{X,x}$-linear map $\varphi : \overline{H}_x^{-q} \to \underline{\Gamma}_Z\mathscr{R}_x^p$, and therefore af is represented by $a\varphi$; the latter obviously factors through the submodule $I \cdot \overline{H}_x^{-q}$, so the contention follows from the natural identification

$$\mathscr{H}om_{\mathscr{O}_X}(I \cdot \overline{H}^{-q}, \underline{\Gamma}_Z\mathscr{R}^p)_x \simeq \mathrm{Hom}_{\mathscr{O}_{X,x}}(I \cdot \overline{H}_x^{-q}, (\underline{\Gamma}_Z\mathscr{R}^p)_x) \qquad \text{for every } x \in X \tag{11.5.52}$$

which is proven as (11.5.50).

(ii): Let $x \in X$ be any point, and $\varphi : I \cdot \overline{H}_x^{-q} \to (\underline{\Gamma}_Z\mathscr{R}^p)_x$ any $\mathscr{O}_{X,x}$-linear map; taking into account (11.5.52), we are reduced to showing that $\varphi = 0$. Now, notice that

$$(\underline{\Gamma}_Z\mathscr{R}^p)_x = \bigoplus_{y \in Z(x)_p} E(y) \qquad \text{where } Z(x)_p := X_p \cap Z \cap X(x).$$

If $Z(x)_p = \varnothing$, we are done; otherwise, for every $y \in Z(x)_p$ denote by $p_y : (\underline{\Gamma}_Z\mathscr{R}^p)_x \to E(y)$ the natural projection; it suffices to check that $p_y \circ \varphi = 0$ for every such y. However, $p_y \circ \varphi$ factors through an $\mathscr{O}_{X,y}$-linear map $I \cdot \overline{H}_y^{-q} \to (\underline{\Gamma}_Z\mathscr{R}^p)_y$, so we may replace x by y, and assume from start that $x \in X_p \cap Z$, in which case the contention will follow from assertion (iii) of the claim.

(iii): Fix any point $y \in X_p \cap Z$. We have already noticed that $\mathrm{Supp}\,\overline{H}^{-q}$ is the topological closure in X of $U \cap \mathrm{Supp}\,\overline{H}^{-q}$; we are then reduced to showing that $I \cdot \overline{H}^{-q}_z = 0$ for every closed point z of $X(y) \cap U$. But for any such z, we have an isomorphism

$$H^{-q}(\mathscr{D}_U K^\bullet)_z \xrightarrow{\sim} \overline{H}^{-q}_z \tag{11.5.53}$$

of $\mathscr{O}_{X,z}$-modules. On the other hand, corollary 11.5.17(iii) yields an isomorphism of $\mathscr{O}_{X,z}$-modules

$$\mathscr{O}^\wedge_{X,z} \otimes_{\mathscr{O}_{X,z}} H^{-q}(\mathscr{D}_U K^\bullet)_z \xrightarrow{\sim} D_z \circ R^{q+c_X(z)}\Gamma_{\{z\}} K^\bullet_{|U(z)} \tag{11.5.54}$$

where $\mathscr{O}^\wedge_{X,z}$ denotes the completion of the ring $\mathscr{O}_{X,z}$ and D_z denotes the local duality functor corresponding to the point z. We remark now that

$$c_X(z) = c_X(y) - 1 = p - 1. \tag{11.5.55}$$

Indeed, in view of lemma 11.3.50, identity (11.5.55) asserts that y is an immediate specialization of z in $X(y)$, and this is already known from lemma 11.5.41.

Finally, let $a \in I$ be any element; taking into account (11.5.53), we are reduced to checking that scalar multiplication by a is the zero endomorphism of $H^{-q}(\mathscr{D}_U K^\bullet)_z$, which – by virtue of (11.5.54) and (11.5.55) – is equivalent to showing that scalar multiplication by a is the zero endomorphism on $R^i\Gamma_{\{z\}} K^\bullet_{|U(z)}$, and the latter is ensured by our condition (a). ◊

(c)⇒(d): Let $x \in U_1$ be any point, pick any immediate specialization $y \in \partial U$ of x in X such that $1 + \dim \partial U(y) = e(x)$, let $U(y) := U \cap X(y)$, and notice that the natural map

$$H^j(U, K^\bullet) \otimes_A \mathscr{O}_{X,y} \to H^j(U(y), K^\bullet_{|U(y)})$$

is an isomorphism for every $j \in \mathbb{Z}$, so we may replace X by $X(y)$, U by $U(y)$ and $K^\bullet$ by $K^\bullet_{|U(y)}$, and assume from start that A is a local ring, and y is the closed point of X. In this situation, we need to show that $I^e R^j \Gamma_{\{x\}} K^\bullet_{|U(x)} = 0$, with $e := 1 + \dim \partial U$ and every $j \leq i$. Then, let $V := X \setminus \{y\}$, denote $j : U \to V$ the induced open immersion, and set

$$L^\bullet := \tau^{\leq i} Rj_* K^\bullet$$

(where $\tau^{\leq i}$ denotes the usual truncation functor), so that $L \in \mathrm{Ob}(\mathsf{D}^b(\mathscr{O}_V\text{-}\mathbf{Mod})_{\mathrm{qcoh}})$ and $L^\bullet_{|U} = \tau^{\leq i} K^\bullet$. Notice also that the natural morphism of $\mathscr{O}_X$-modules

$$H^j(U, K^\bullet)^\sim \to H^j L^\bullet$$

is an isomorphism for every $j \leq i$. Taking into account proposition 10.3.34(ii), it follows that we may find a morphism $\varphi^\bullet : C^\bullet \to L^\bullet$ in $\mathsf{D}^b(\mathscr{O}_X\text{-}\mathbf{Mod})$ such that

- $C^\bullet \in \mathrm{Ob}(\mathsf{D}^b(\mathscr{O}_X\text{-}\mathbf{Mod})_{\mathrm{coh}})$.
- The induced morphism $C^\bullet_{|U} \to \tau^{\leq i} K^\bullet_{|U}$ is an isomorphism in $\mathsf{D}^b(\mathscr{O}_U\text{-}\mathbf{Mod})_{\mathrm{coh}}$.
- $H^j\varphi^\bullet : H^j C^\bullet \to H^j L^\bullet$ is a monomorphism for every $j \in \mathbb{Z}$, and its image contains the $\mathscr{O}_X$-submodule $I \cdot H^j(U, K^\bullet)^\sim$, for every $j \leq i$.

Set $D^\bullet := \mathrm{Cone}\,\varphi$; by construction, for every $j \in \mathbb{Z}$ we have

$$\mathrm{Supp}\, H^j L^\bullet \subset \overline{U} \cap V \quad \mathrm{Supp}\, H^j D^\bullet \subset Z \cap \mathrm{Supp}\, H^i L^\bullet \subset \partial U \cap V \quad I \cdot H^j D^\bullet = 0. \tag{11.5.56}$$

Now, consider the spectral sequence

$$E_2^{pq} := H^p(V, H^q D^\bullet) \Rightarrow H^{p+q}(V, D^\bullet).$$

It follows from (11.5.56) that $E_2^{pq} = 0$ for every $p > \dim \partial U \cap V = e - 2$, and $I \cdot E_2^{pq} = 0$ for every $p, q \in \mathbb{Z}$, whence

$$I^{e-1} H^j(V, D^\bullet) = 0 \qquad \text{for every } j \in \mathbb{Z}.$$

On the other hand, we have an exact sequence of A-modules

$$H^{j-1}(V, D^\bullet) \to H^j(V, C^\bullet) \to H^j(V, L^\bullet) = H^j(U, K^\bullet) \qquad \text{for every } j \leq i.$$

In view of lemma 11.5.38, we conclude that $I^e H^j(V, C^\bullet)$ is an A-module of finite type, whence $I^e \cdot R^j\Gamma_{\{x\}} C^\bullet_{|U(x)} = 0$ for every $j \leq i$ (theorem 11.5.25). To conclude the proof, it now suffices to observe that the natural map $R^j\Gamma_{\{x\}} C^\bullet_{|U(x)} \to R^j\Gamma_{\{x\}} K^\bullet_{|U(x)}$ is an isomorphism, for every $j \leq i$. □

The following corollary recovers the results of [85, Exp.VIII, §2].

Corollary 11.5.57. *In the situation of* (11.5.40)*, for every $i \in \mathbb{Z}$ we have :*
(i) *If $R^i\Gamma_{\{x\}} K^\bullet = 0$ for every $x \in U_1$, then $H^i(U, K^\bullet)$ is an A-module of finite type.*
(ii) *The following conditions are equivalent :*
  (a) *$R^j\Gamma_{\{x\}} K^\bullet = 0$ for every $x \in U_1$ and every $j \leq i$.*
  (b) *$H^j(U, K^\bullet)$ is an A-module of finite type for every $j \leq i$.*

Proof. It is the special case of theorem 11.5.43 with $I := A$. □

11.6. **Hochster's theorem and Stanley's theorem.** The two main results of this section are theorems 11.6.35 and 11.6.43, which were proved originally respectively by Hochster in [91], and by Stanley in [155]. Our proofs follow in the main the posterior methods presented by Bruns and Herzog in [43] (which in turns, partially rely on some ideas of Danilov). These results shall be used in section 12.5, in order to show that regular log schemes are Cohen-Macaulay. We begin with some preliminaries from algebraic topology, which we develop only to the extent that is required for our special purposes: a much more general theory exists, and is well known to experts (see *e.g.* [124, Ch.IX]).

For any topological space X, and any subset $T \subset X$, we denote by $\overline{T}$ the topological closure of T in X, endowed with the topology induced from X. We let $H_\bullet(X)$ (resp. $H_\bullet(X, T)$) be the singular homology groups of X (resp. of the pair (X, T)). Also, for every $n \in \mathbb{N}$ fix a Banach norm $\|\cdot\|$ on $\mathbb{R}^n$, and for every real number $\rho > 0$, let $\mathbb{B}^n(\rho) := \{v \in \mathbb{R}^n \mid \|v\| < \rho\}$, and set $\mathbb{S}^{n-1} := \overline{\mathbb{B}}^n(1) \setminus \mathbb{B}^n(1)$ (so $\mathbb{S}^{-1} = \varnothing$).

Definition 11.6.1. (i) A *finite regular cell complex* (or briefly : a *cell complex*) is the datum of a topological space X, together with a finite filtration

$$X^{-1} = \varnothing \subset X^0 \subset X^1 \subset X^2 \subset \cdots \subset X^k = X$$

consisting of closed subspaces, such that for every $i = 0, \dots, k$ we have a decomposition

$$X^i \setminus X^{i-1} = \bigcup_{\lambda \in \Lambda_i} e^i_\lambda$$

where :

(a) Λ_i is a finite set, and for every $\lambda \in \Lambda_i$ there exists a homeomorphism

$$f_\lambda : \overline{\mathbb{B}}^i(1) \xrightarrow{\sim} \overline{e}^i_\lambda \qquad \text{such that } f_\lambda^{-1}(e^i_\lambda) = \mathbb{B}^i(1).$$

(b) $e^i_\lambda \cap e^i_\mu = \varnothing$ for any two distinct indices $\lambda, \mu \in \Lambda_i$.
(c) $\overline{e}^i_\lambda \setminus e^i_\lambda \subset X^{i-1}$ for every $\lambda \in \Lambda_i$.

(ii) The smallest $k \in \mathbb{N}$ such that $X^k = X$ is called the *dimension* of $X^\bullet$, and is denoted $\dim X^\bullet$. For every $i = 0, \dots, k$, the subsets e^i_λ are called the *i-dimensional cells of $X^\bullet$*.

(iii) A *subcomplex* of the cell complex $X^\bullet$ is a closed subset $Y \subset X$ which is a union of cells of X. Then the filtration such that $Y^i := Y \cap X^i$ for every $i \geq -1$ defines a cell complex structure $Y^\bullet$ on Y.

Remark 11.6.2. With the notation of definition 11.6.1(i), notice that X^0 is a (finite) discrete topological space. Moreover, X is a Hausdorff space : indeed, more generally, the quotient of a compact Hausdorff space by a closed equivalence relation is Hausdorff, so in particular a space is Hausdorff, if it can be covered by finitely many closed compact Hausdorff subspaces.

Lemma 11.6.3. *With the notation of definition* 11.6.1*, we have :*

(i) *For every $i = 0, \dots, \dim X$ and $q \in \mathbb{Z}$, we have natural isomorphisms of abelian groups :*

$$H_q(X^i, X^{i-1}) \xrightarrow{\sim} \begin{cases} \mathbb{Z}^{\Lambda_i} & \text{if } q = i \\ 0 & \text{otherwise}. \end{cases}$$

(ii) *More precisely, let $(b_\lambda \mid \lambda \in \Lambda_i)$ be the canonical basis of the free $\mathbb{Z}$-module $\mathbb{Z}^{\Lambda_i}$; for every $\lambda \in \Lambda_i$, the isomorphism of* (i) *maps the image of the induced map*

$$H_i f_\lambda : H_i(\overline{\mathbb{B}}^i(1), \mathbb{S}^{i-1}) \to H_i(X^i, X^{i-1})$$

isomorphically onto the direct summand generated by b_λ.

Proof. (i): This is obvious for $i = 0$. For $i = 1, \dots, k := \dim X^\bullet$ and every $\lambda \in \Lambda_i$, set

$$Y^i_\lambda := f_\lambda(\overline{\mathbb{B}}^i(1/2)) \qquad Y^i := \bigcup_{\lambda \in \Lambda_i} Y^i_\lambda \qquad A^i := \bigcup_{\lambda \in \Lambda_i} \{f_\lambda(0)\}$$

and for any $q \in \mathbb{Z}$, consider the natural group homomorphisms

$$\begin{array}{ccccc} H_q(\overline{\mathbb{B}}^i(1/2), \overline{\mathbb{B}}^i(1/2) \setminus \{0\}) & \xrightarrow{\alpha'} & H_q(\overline{\mathbb{B}}^i(1), \overline{\mathbb{B}}^i(1) \setminus \{0\}) & \xleftarrow{\beta'} & H_q(\overline{\mathbb{B}}^i(1), \mathbb{S}^{i-1}) \\ \downarrow{\scriptstyle\gamma} & & \downarrow & & \downarrow \\ H_q(Y^i, Y^i \setminus A^i) & \xrightarrow{\alpha} & H_q(X^i, X^i \setminus A^i) & \xleftarrow{\beta} & H_q(X^i, X^{i-1}). \end{array} \tag{11.6.4}$$

The map α is an isomorphism, by excision; the same holds for β, since X^{i-1} is a deformation retract of $X^i \setminus A^i$. However, for every $\lambda \in \Lambda_i$ we have natural isomorphisms

$$H_q(Y^i_\lambda, Y^i_\lambda \setminus A) \xrightarrow{\sim} \begin{cases} \mathbb{Z} & \text{if } q = i \\ 0 & \text{otherwise.} \end{cases}$$

whence the contention.

(ii): Take $q := i$ in (11.6.4); then clearly γ is a monomorphism whose image is the direct summand $H_i(Y^i_\lambda, Y^i_\lambda \setminus A)$; on the other hand, arguing as in the foregoing, we see that both α' and β' are isomorphisms. The assertion follows easily. □

Remark 11.6.5. (i) In the situation of lemma 11.6.3, notice that there is a natural bijection from Λ_i to the set of connected components $\pi_0(X^i \setminus X^{i-1})$ of $X^i \setminus X^{i-1}$, for every $i = 0, \dots, \dim X$.

(ii) The direct sum decomposition of $H_i(X^i, X^{i-1})$ provided by lemma 11.6.3(i) depends only on the filtration $X^\bullet$ (and not on the choice of homeomorphisms f_λ). Indeed, this is clear, since the map $H_i f_\lambda$ factors as a composition :

$$H_i(\overline{\mathbb{B}}^i(1), \mathbb{S}^{i-1}) \xrightarrow{\sim} H_i(\overline{e}^i_\lambda, \overline{e}^i_\lambda \setminus e^i_\lambda) \xrightarrow{l_\lambda} H_i(X^i, X^{i-1})$$

where l_λ is the homomorphism induced by the obvious map of pairs $(\overline{e}^i_\lambda, \overline{e}^i_\lambda \setminus e^i_\lambda) \to (X^i, X^{i-1})$.

(iii) In view of (ii), the maps l_λ admit natural left inverse homomorphisms

$$p_\lambda : H_i(X^i, X^{i-1}) \to H_i(\overline{e}^i_\lambda, \overline{e}^i_\lambda \setminus e^i_\lambda)$$

such that

$$p_\mu \circ l_\lambda = 0 \qquad \text{for every } \lambda, \mu \in \Lambda_i \text{ with } \lambda \neq \mu. \tag{11.6.6}$$

These maps can be described as follows. For any $\lambda \in \Lambda_i$, the natural map of pairs $(X^i, X^{i-1}) \to (X^i, X^i \setminus e^i_\lambda)$ induces a homomorphism

$$m_\lambda : H_i(X^i, X^{i-1}) \to H_i(X^i, X^i \setminus e^i_\lambda).$$

Arguing by excision as in the proof of lemma 11.6.3(ii), it is easily seen that $q_\lambda := m_\lambda \circ l_\lambda$ is an isomorphism (details left to the reader), and we set $p_\lambda := q_\lambda^{-1} \circ m_\lambda$. Obviously p_λ is left inverse to l_λ, and in oder to check (11.6.6) it suffices to show that $m_\mu \circ l_\lambda = 0$ for every $\mu \neq \lambda$. However, $m_\mu \circ l_\lambda$ is the homomorphism arising from the map of pairs $(\overline{e}^i_\lambda, \overline{e}^i_\lambda \setminus e^i_\lambda) \to (X^i, X^i \setminus e^i_\mu)$, so the assertion follows, after remarking that $\overline{e}^i_\lambda \subset X^i \setminus e^i_\mu$.

11.6.7. We attach to any cell complex $X^\bullet$ a complex of abelian groups $\mathscr{C}_\bullet(X^\bullet)$, as follows.

- For $i < 0$ and for $i > k := \dim X^\bullet$, we set $\mathscr{C}_i(X^\bullet) := 0$, and for $i = 1, \dots, k$ we let

$$\mathscr{C}_i(X^\bullet) := H_i(X^i, X^{i-1}).$$

Sometimes we write just $\mathscr{C}_i(X)$, unless it is useful to stress which filtration $X^\bullet$ on X we are considering. For every $i > 0$, the differential $d_i : \mathscr{C}_i(X^\bullet) \to \mathscr{C}_{i-1}(X^\bullet)$ is the composition

$$H_i(X^i, X^{i-1}) \xrightarrow{\partial_i} H_{i-1}(X^{i-1}) \xrightarrow{j_{i-1}} H_{i-1}(X^{i-1}, X^{i-2})$$

where ∂_i is the boundary operator of the long exact homology sequence associated with the pair (X^i, X^{i-1}), and j_{i-1} is the homomorphism induced by the obvious map of pairs $(X^{i-1}, \varnothing) \to (X^{i-1}, X^{i-2})$. In order to check that $d_i \circ d_{i+1} = 0$ for every $i \in \mathbb{Z}$, recall that every element of $H_{i+1}(X^{i+1}, X^i)$ is the class $\overline{c}$ of a singular $(i+1)$-chain c of X^{i+1}, whose boundary $\partial_{i+1}c$ is a singular i-chain of X^i; then $d_{i+1}(\overline{c})$ is the class of $\partial_{i+1}c$ in $H_i(X^i, X^{i-1})$, and $d_i \circ d_{i+1}(\overline{c})$ is the class of $\partial_i \circ \partial_{i+1} c$ in $H_{i-1}(X^{i-1}, X^{i-2})$, so it vanishes.

- It is also useful to consider an augmented version of the above complex; namely, let us set

$$\overline{\mathscr{C}}_{-1}(X^\bullet) := \mathbb{Z} \qquad \text{and} \qquad \overline{\mathscr{C}}_i(X^\bullet) := \mathscr{C}_i(X^\bullet) \qquad \text{for every } i \neq -1$$

with differential d_0 given by the rule : $d_0(b^0_\lambda) = 1$ for every $\lambda \in \Lambda_0$.

Proposition 11.6.8. *With the notation of* (11.6.7)*, there exist natural isomorphisms of abelian groups*

$$H_q \mathscr{C}_\bullet(X^\bullet) \xrightarrow{\sim} H_q(X) \qquad \textit{for every } q \in \mathbb{N}.$$

Proof. For every topological space T, let $C_\bullet(T)$ denote the complex of singular chains of T. The filtration $X^\bullet$ induces a finite filtration of complexes

$$C_\bullet(X^0) \subset C_\bullet(X^1) \subset C_\bullet(X^2) \subset \cdots \subset C_\bullet(X)$$

whence a convergent spectral sequence

$$E^1_{pq} := H_{p+q}(X^p, X^{p-1}) \Rightarrow H_{p+q}(X)$$

(see [163, Th.5.5.1]). By direct inspection, it is easily seen that the differential $d^1_{p,0} : E^1_{p,0} \to E^1_{p-1,0}$ agrees with the differential d_p of $\mathscr{C}_\bullet(X^\bullet)$, for every $p \in \mathbb{N}$. On the other hand, lemma 11.6.3 shows that this spectral sequence degenerates, whence the contention. □

11.6.9. Keep the notation of definition 11.6.1, and set $\Lambda_0 := X^0$. For every $i = 0, \dots, \dim X$ and every $\lambda \in \Lambda_i$, the abelian group $H_{i,\lambda} := H_i(\overline{e}^i_\lambda, \overline{e}^i_\lambda \setminus e^i_\lambda)$ is free of rank one; if $i > 0$, we fix one of the two generators of this group, and we denote by b^i_λ its image in $\mathscr{C}_i(X^\bullet)$ (see remark 11.6.5(ii)). For $i = 0$, each e^0_λ is a point, hence $H_{0,\lambda}$ admits a canonical identification with $\mathbb{Z}$, and we let b^0_λ be the image of 1, under the resulting map $\mathbb{Z} \xrightarrow{\sim} H_{0,\lambda} \to \mathscr{C}_0(X^\bullet)$. The system of classes $(b^i_\lambda \mid 0 \leq i \leq \dim X, \lambda \in \Lambda_i)$ is called an *orientation* for $X^\bullet$. We may write

$$d_i(b^i_\lambda) = \sum_{\mu \in \Lambda_{i-1}} [b^i_\lambda : b^{i-1}_\mu] b^{i-1}_\mu \qquad \text{for every } i \leq \dim X^\bullet \text{ and every } \lambda \in \Lambda_i$$

for a system of uniquely determined integers $[b^i_\lambda : b^{i-1}_\mu]$, called the *incidence numbers* of the cells e^i_λ and e^{i-1}_μ (relative to the chosen orientation of $X^\bullet$). With this notation, we may state :

Lemma 11.6.10. *The incidence numbers of a cell complex $X^\bullet$ fulfill the following conditions :*

(i) $\sum_{\mu\in\Lambda_{i-1}}[b^i_\lambda : b^{i-1}_\mu]\cdot[b^{i-1}_\mu : b^{i-2}_\nu] = 0$ *for every $i \geq 2$, every $\lambda \in \Lambda_i$ and every $\nu \in \Lambda_{i-2}$.*

(ii) *Let $\lambda \in \Lambda_1$ be any index, and say that $\overline{e}^1_\lambda \setminus e^1_\lambda = e^0_\mu \cup e^0_\rho$. Then $[b^1_\lambda : b^0_\mu] + [b^1_\lambda : b^0_\rho] = 0$.*

Proof. Condition (i) translates the identity $d_{i-1} \circ d_i = 0$ for the differential $d_\bullet$ of the complex $\mathscr{C}_\bullet(X^\bullet)$. The identity of (ii) follows by a simple inspection of the definition of d_1 : details left to the reader. □

11.6.11. The generalities of the previous paragraphs shall be applied to the following situation. Let (V,σ) be a strictly convex polyhedral cone such that $\langle\sigma\rangle = V$, and set $d := \dim_{\mathbb{R}} V$. We attach to σ a cell complex $C^\bullet_\sigma$ as follows. Fix $u_0 \in \sigma^\vee$ such that $\sigma \cap \mathrm{Ker}\, u_0 = \{0\}$ (corollary 6.3.14), let σ° be the topological interior of σ (in V) and set

$$C_\sigma := \sigma \cap u_0^{-1}(1) \qquad C^\circ_\sigma := \sigma^\circ \cap u_0^{-1}(1).$$

Let $v_1, \dots, v_k$ be a minimal system of generators for σ; we may assume that $u_0(v_i) = 1$ for $i = 1, \dots, k$, in which case

$$C_\sigma = \left\{\sum_{i=1}^k \lambda_i v_i \mid \lambda_1, \dots, \lambda_k \geq 0 \quad \text{and} \quad \sum_{i=1}^k \lambda_i = 1\right\} \tag{11.6.12}$$

which shows that C_σ is a compact topological space (with the topology inherited from V).

Lemma 11.6.13. *There exists a homeomorphism $\overline{\mathbb{B}}^{d-1}(1) \xrightarrow{\sim} C_\sigma$ that maps $\mathbb{B}^{d-1}(1)$ onto C°_σ.*

Proof. Set $v_0 := k^{-1}\cdot(v_1 + \cdots + v_k)$ and $W := \mathrm{Ker}\, u_0$. It suffices to show that there exists a homeomorphism $\overline{\mathbb{B}}^{d-1}(1) \xrightarrow{\sim} D_\sigma := C_\sigma - v_0 \subset W$ that maps $\mathbb{B}^{d-1}(1)$ onto $D^\circ_\sigma := C^\circ_\sigma - v_0$. However, pick a minimal system $u_1, \dots, u_t$ of generators of $\sigma^\vee$ (corollary 6.3.12(i)); since σ spans V, for every $i = 1, \dots, t$ there exists $j \leq k$ such that $u_i(v_j) > 0$. Hence – after replacing the u_i by suitable scalar multiples – we may assume that $u_i(v_0) = 1$ for every $i = 1, \dots, t$, and then lemma 6.3.2 implies that

$$D_\sigma = \{v \in W \mid u_i(v) \geq -1 \quad \text{for every } i = 1, \dots, t\}.$$

Also, proposition 6.3.11(i) implies that

$$D^\circ_\sigma = \{v \in W \mid u_i(v) > -1 \quad \text{for every } i = 1, \dots, t\}.$$

For every $v \in W$, set $\mu(v) := \min(u_i(v)\,|i = 1, \dots, t)$. It is easily seen that $\mu(v) < 0$ for every $v \in W \setminus \{0\}$; indeed, if $\mu(w) \geq 0$, then the subset $\{\lambda w \mid \lambda \in \mathbb{R}_+\}$ lies in D_σ, and since the latter is compact, it follows that $w = 0$. Moreover we have

$$(\mathbb{R}_+ w) \cap D_\sigma = \{\lambda w \mid \lambda \in [0, -1/\mu(w)]\} \qquad \text{for every } w \in W \setminus \{0\}. \tag{11.6.14}$$

Fix a Banach norm $\|\cdot\|_V$ on V, and consider the mapping

$$\varphi : D_\sigma \to W \qquad \text{such that} \qquad \varphi(0) := 0 \quad \text{and} \quad \varphi(v) := -\frac{\mu(v)}{\|v\|}\cdot v \quad \text{for } v \neq 0.$$

It is easily seen that φ is injective; since μ is a continuous mapping, the same follows for φ, and since D_σ is compact, we conclude that φ induces a homeomorphism $D_\sigma \xrightarrow{\sim} \varphi(D_\sigma)$. However, from (11.6.14) it follows that

$$\varphi(D_\sigma) = \{v \in W \mid \|v\| \leq 1\} \qquad \text{and} \qquad \varphi(D^\circ_\sigma) = \{v \in W \mid \|v\| < 1\}$$

whence the claim. □

11.6.15. Keep the notation of (11.6.11); we consider the finite filtration $C_\sigma^\bullet$ of C_σ, defined as follows. For every $i = -1, \dots, d-1$, we let $C_\sigma^i \subset C_\sigma$ be the union of the subsets $\tau \cap u_0^{-1}(1)$, where τ ranges over the (finite) set of all faces of σ of dimension $i+1$. Clearly $C_\sigma^{-1} = \varnothing$, $C_\sigma^{d-1} = C_\sigma$, and C_σ^i is a closed subset of C_σ, for every $i = 0, \dots, d-1$. Moreover, it follows easily from lemma 11.6.13 and proposition 6.3.11(i) that the datum of C_σ and its filtration $C_\sigma^\bullet$ is a cell complex. With the notation of definition 11.6.1, the indexing set Λ_i can be taken to be the set of all $(i+1)$-dimensional faces of σ, for every $i = 0, \dots, d-1$: indeed, if τ is such a face, denote by τ° the relative interior of τ (see example 6.3.16(iii)); then it is clear that $e_\tau^i := \tau^\circ \cap C_\sigma \neq \varnothing$ and $\overline{e}_\tau^i = \tau \cap C_\sigma$.

Remark 11.6.16. (i) Notice that the cell complex $C_\sigma^\bullet$ is independent – up to homeomorphism – of the choice of u_0. Indeed, say that $u'_0 \in \sigma^\vee$ is any other linear form such that $\sigma \cap \mathrm{Ker}\, u'_0 = \{0\}$, and let $C'_\sigma := \sigma \cap u_0'^{-1}(1)$. We define a homeomorphism $\psi : C'_\sigma \stackrel{\sim}{\to} C_\sigma$, by the rule :

$$v \mapsto u_0(v)^{-1} \cdot v \qquad \text{for every } v \in C'_\sigma.$$

It is easily seen that ψ restricts to homeomorphisms $C_\sigma'^i \stackrel{\sim}{\to} C_\sigma^i$ for every $i = 0, \dots, d-1$.

(ii) For any integer $i = 0, \dots, d-3$, and any cells e_τ^i, e_λ^{i+2} of $C_\sigma^\bullet$ such that $e_\tau^i \subset \overline{e}_\lambda^{i+2}$, there exist exactly two $(i+1)$-dimensional cells e_μ^{i+1}, e_ρ^{i+1} such that $e_\tau^i \subset \overline{e}_\mu^{i+2} \cap \overline{e}_\rho^{i+1}$ and $e_\mu^{i+1} \cup e_\rho^{i+1} \subset \overline{e}_\lambda^{i+2}$: indeed, this assertion is a direct translation of claim 6.3.9(ii).

Proposition 11.6.17. *With the notation of* (11.6.9) *and* (11.6.15)*, the following holds for every* $i = 0, \dots, d-2$ *:*

(i) *If* $\tau \in \Lambda_{i+1}$ *and* $\mu \in \Lambda_i$ *is not a facet of* τ*, we have* $[b_\tau^{i+1} : b_\mu^i] = 0$.
(ii) *If* $\tau \in \Lambda_{i+1}$ *and* $\mu \in \Lambda_i$ *is a facet of* τ*, then* $[b_\tau^{i+1} : b_\mu^i] \in \{1, -1\}$.
(iii) *In the situation of remark* 11.6.16(ii)*, we have :*

$$[b_\lambda^{i+2} : b_\mu^{i+1}] \cdot [b_\mu^{i+1} : b_\tau^i] + [b_\lambda^{i+2} : b_\rho^{i+1}] \cdot [b_\rho^{i+1} : b_\tau^i] = 0.$$

Proof. (i): Consider the commutative diagram

$$(11.6.18)\qquad \begin{array}{ccccc} H_{i+1}(\overline{e}_\tau^{i+1}, \overline{e}_\tau^{i+1} \setminus e_\tau^{i+1}) & \xrightarrow{\partial} & H_i(\overline{e}_\tau^{i+1} \setminus e_\tau^{i+1}) & \xrightarrow{j_\tau} & H_i(C_\sigma^i, C_\sigma^i \setminus e_\mu^i) \\ \downarrow{\scriptstyle l_\tau} & & \downarrow{\scriptstyle g} & & \uparrow{\scriptstyle m_\mu} \\ H_{i+1}(C_\sigma^{i+1}, C_\sigma^i) & \xrightarrow{\partial'} & H_i(C_\sigma^i) & \xrightarrow{j_i} & H_i(C_\sigma^i, C_\sigma^{i-1}) \end{array}$$

where l_τ and m_μ are defined as in remark 11.6.5(ii,iii) and g is induced by the inclusion map $\overline{e}_\tau^{i+1} \setminus e_\tau^{i+1} \to C_\sigma^i$, the maps ∂ and ∂' are the boundary operators of the long exact sequences attached to the pairs $(\overline{e}_\tau^{i+1}, \overline{e}_\tau^{i+1} \setminus e_\tau^{i+1})$ and $(C_\sigma^{i+1}, C_\sigma^i)$, and j_i (resp. j_τ) is deduced from the obvious map of pairs $(\overline{e}_\tau^{i+1} \setminus e_\tau^{i+1}, \varnothing) \to (C_\sigma^i, C_\sigma^i \setminus e_\mu^i)$ (resp. $(C_\sigma^i, \varnothing) \to (C_\sigma^i, C_\sigma^{i-1})$). In light of remark 11.6.5(iii), we need to check that $m_\mu \circ j_{i-1} \circ \partial' \circ l_\tau = 0$, and to this aim, it suffices to show that $j_\tau = 0$. However, the assumption implies that $\mu \cap \tau$ is a (proper) face of both μ and τ; this translates as the identity $(\overline{e}_\tau^{i+1} \setminus e_\tau^{i+1}) \cap e_\mu^i = \varnothing$, whence the claim.

(ii): Clearly $C_\tau^\bullet$ is a cell subcomplex of $C_\sigma^\bullet$, and e_τ^{i+1} and e_μ^i are cells of this subcomplex. Moreover, any orientation for $C_\sigma^\bullet$ restricts to an orientation for $C_\tau^\bullet$, and the resulting incidence number of e_τ^{i+1} and e_μ^i is the same for either cell complex. Thus, we may replace σ by τ, in which case the map g of (11.6.18) is the identity.

Claim 11.6.19. If $\sigma = \tau$ and $i > 0$, both ∂ and $k_\mu := m_\mu \circ j_i$ in (11.6.18) are isomorphisms.

Proof of the claim. For ∂, we use the long exact homology sequence of the pair $(\overline{e}_\tau^{i+1}, \overline{e}_\tau^{i+1} \setminus e_\tau^{i+1})$: since $i \geq 0$, lemma 11.6.13 implies that $H_{i+1}(\overline{e}_\tau^{i+1}) = 0$, so ∂ is injective; since $i > 0$, the same argument shows that ∂ is surjective. Next, k_μ is the homomorphism deduced from the long exact homology sequence of the pair $(C_\tau^i, C_\tau^i \setminus e_\mu^i)$, and we remark that $C_\tau^i \setminus e_\mu^i$ is contractible :

indeed, e^i_μ is homeomorphic to $\mathbb{B}^i(1)$ (lemma 11.6.13), hence $C^i_\tau \setminus e^i_\mu$ is a retraction of $C^i_\tau \setminus \{x\}$, for any $x \in e^i_\mu$, and the latter is homeomorphic to $\mathbb{R}^i$ (again by lemma 11.6.13). If $i > 1$, we deduce already that k_μ is an isomorphism. For $i = 1$, the same argument shows that k_μ is injective, so its cokernel is a cyclic torsion group that injects into $H_0(C^1_\tau \setminus e^1_\mu) \simeq \mathbb{Z}$, hence it must vanish as well. ◊

Since (11.6.18) commutes, claim 11.6.19 yields the assertion, in case $i > 0$. If $i = 0$, C^1_τ is isomorphic to $\overline{\mathbb{B}}^1(1)$, and e^0_μ is one of the two points of $\overline{\mathbb{B}}^1(1) \setminus \mathbb{B}^1(1)$, so the assertion can be checked easily, by inspecting the definitions.

(iii) follows from (i), lemma 11.6.10(i) and remark 11.6.16(ii). □

The following result says that the properties of proposition 11.6.17 completely characterize the incidence numbers of $C^\bullet_\sigma$.

Proposition 11.6.20. *Keep the notation of* (11.6.15)*, and consider a system of integers*

$$(\beta^i_{\lambda\mu} \mid i = 1, \dots, d-1,\ \lambda \in \Lambda_i, \mu \in \Lambda_{i-1})$$

such that :

(a) *If $\lambda \in \Lambda_i$ and $\mu \in \Lambda_{i-1}$ is not a facet of λ, then $\beta^i_{\lambda\mu} = 0$.*

(b) *If $\lambda \in \Lambda_i$ and $\mu \in \Lambda_{i-1}$ is a facet of λ, then $\beta^i_{\lambda\mu} \in \{1, -1\}$.*

(c) *If μ and τ are the two 1-dimensional facets of the 2-dimensional face λ of σ, then*

$$\beta^1_{\lambda\mu} + \beta^1_{\lambda\tau} = 0.$$

(d) *Let $\lambda \in \Lambda_{i+1}$ and $\mu \in \Lambda_{i-1}$ be two faces, such that μ is a face of λ, and denote by τ and ρ the two facets of λ that contain μ. Then*

$$\beta^{i+1}_{\lambda\tau}\beta^i_{\tau\mu} + \beta^{i+1}_{\lambda\rho}\beta^i_{\rho\mu} = 0.$$

Then there exists a unique orientation of $C^\bullet_\sigma$

$$(b^i_\lambda \mid i = 0, \dots, d-1,\ \lambda \in \Lambda_i)$$

such that for every $i = 1, \dots, d-1$ we have :

(11.6.21) $$[b^i_\lambda : b^{i-1}_\mu] = \beta^i_{\lambda\mu} \qquad \text{for every } \lambda \in \Lambda_i,\ \mu \in \Lambda_{i-1}.$$

Proof. We construct the orientation classes b^i_λ fulfilling condition (11.6.21), by induction on i. For $i = 0$, the condition is empty, and the classes b^0_λ are prescribed by (11.6.9). For $i = 1$, and a given $\lambda \in \Lambda_1$, denote by τ and ρ the two faces of λ; clearly the condition $[b^1_\lambda : b^0_\tau] = \beta^1_{\lambda\tau}$ (and assumption (b)) determines b^1_λ univocally, and in view of (c) and lemma 11.6.10(ii), for this choice of orientation of e^1_λ we have as well $[b^1_\lambda : b^0_\rho] = \beta^1_{\lambda\rho}$. Moreover, if $\mu \in \Lambda_0 \setminus \{\tau, \rho\}$, then (11.6.21) is verified as well, by virtue of (a) and proposition 11.6.17(i).

Now, suppose that $i > 1$ and that we have already constructed orientation classes b^j_μ as sought, for every $j < i$ and every $\mu \in \Lambda_j$. Let $\lambda \in \Lambda_i$ be any face, and fix a facet τ of λ; again, the identity

(11.6.22) $$[b^i_\lambda : b^i_\tau] = \beta^i_{\lambda\tau}$$

determines b^i_λ, and it remains to check that – with this choice of b^i_λ – condition (11.6.21) holds for every $\mu \in \Lambda_{i-1} \setminus \{\tau\}$, and by virtue of (a) and proposition 11.6.17(i), it suffices to consider the facets μ of λ. However, notice that the system of orientation classes $(b^j_\mu \mid j = 0, \dots, i-1,\ \mu \subset \lambda)$ amounts to an orientation of C^{i-1}_λ (regarded as a cell subcomplex of $C^\bullet_\sigma$), and the incidence numbers for the complex $(C^{i-1}_\lambda)^\bullet$ relative to these classes agree with the incidence numbers of $C^\bullet_\sigma$, relative to the same classes. Especially, the sums

$$c := \sum_{\mu \in \Lambda_{i-1}} \beta^i_{\lambda\mu} b^{i-1}_\mu \qquad c' := \sum_{\mu \in \Lambda_{i-1}} [b^i_\lambda : b^{i-1}_\mu] b^{i-1}_\mu$$

are well defined elements of $\mathscr{C}_{i-1}(C_\lambda^{i-1})$, and in fact :

$$\begin{aligned} d_{i-1}(c) &= \sum_{\mu\in\Lambda_{i-1}} \beta^i_{\lambda\mu}\cdot d_{i-1}(b^{i-1}_\mu) \\ &= \sum_{\mu\in\Lambda_{i-1}} \beta^i_{\lambda\mu}\cdot \sum_{\rho\in\Lambda_{i-2}} [b^{i-1}_\mu : b^{i-2}_\rho]b^{i-2}_\rho \\ &= \sum_{\mu\in\Lambda_{i-1}}\sum_{\rho\in\Lambda_{i-2}} \beta^i_{\lambda\mu}\beta^{i-1}_{\mu\rho}b^{i-2}_\rho && \text{(by inductive assumption)} \\ &= 0 && \text{(by (d))} \end{aligned}$$

and a similar calculation yields $d_{i-1}(c')=0$ as well. However, C_λ^{i-1} is homeomorphic to $\mathbb{S}^{i-1}$ (lemma 11.6.13), hence

$$\mathrm{Ker}(d_{i-1}:\mathscr{C}_{i-1}(C_\lambda^{i-1})\to\mathscr{C}_{i-2}(C_\lambda^{i-1}))\simeq\mathbb{Z}$$

which, in view of (b), implies that $c=\pm c'$. Taking into account (11.6.22), we see that actually $c=c'$, whence the claim. The uniqueness of the orientation fulfilling condition (11.6.21) can be checked easily by induction on i : the details shall be left to the reader. □

11.6.23. Let now L be a free abelian group of finite rank d, and $(L_\mathbb{R},\sigma)$ a strictly convex L-rational polyhedral cone (see (6.3.20)), such that $\langle\sigma\rangle=L_\mathbb{R}$; set

$$P:=L\cap\sigma \qquad F_\lambda:=L\cap\lambda \qquad P_\lambda:=F_\lambda^{-1}P \qquad \text{for every face } \lambda \text{ of } \sigma$$

so P and its localization P_λ are fine and saturated monoids (proposition 6.3.22(i)), and F_λ is a face of P, for every such λ. Let R be any ring, and if $\lambda,\mu\subset\sigma$ are any two faces with $\lambda\subset\mu$, denote by

$$j_{\lambda\mu}:R[P_\lambda]\to R[P_\mu]$$

the natural localization map; notice that if $0\subset\sigma$ is the unique 0-dimensional face, then $P_0=P$, and $j_{0\mu}$ is the localization map $R[P]\to R[P_\mu]$. We attach to P the complex $\overline{\mathscr{C}}{}^\bullet_P$ of $R[P]$-modules such that :

$$\overline{\mathscr{C}}{}^0_P:=R[P] \qquad\text{and}\qquad \overline{\mathscr{C}}{}^i_P:=\bigoplus_{\lambda\in\Lambda_{i-1}}R[P_\lambda] \qquad \text{for every } i=1,\dots,\dim P$$

with differentials given by the rule :

$$d^i(x_\lambda):=\sum_{\substack{\mu\in\Lambda_i\\ \lambda\subset\mu}}[b^i_\mu:b^{i-1}_\lambda]\cdot j_{\lambda\mu}(x_\lambda) \qquad \text{for every } i\geq 1,\ \lambda\in\Lambda_{i-1} \text{ and } x_\lambda\in R[P_\lambda]$$

and

$$d^0(x):=\sum_{\mu\in\Lambda_0}j_{0\mu}(x) \qquad \text{for every } x\in R[P]$$

where $(b^j_\tau\mid j=0,\dots\dim\sigma-1,\ \tau\in\Lambda_j)$ is a chosen orientation for $C^\bullet_\sigma$. Taking into account lemma 11.6.10(ii) and proposition 11.6.17(iii), it is easily seen that $d^{i+1}\circ d^i=0$ for every $i\geq 0$ (essentially, the differential d^i is the transpose of the differential d_{i-1} of $\overline{\mathscr{C}}_\bullet(C^\bullet_\sigma)$).

Set as well $X:=\mathrm{Spec}\,R[P]$, and notice that $Z:=\mathrm{Spec}\,R\langle P/\mathfrak{m}_P\rangle\simeq\mathrm{Spec}\,R$ is a closed subset of X. For any $R[P]$-module M, we denote as usual by $M^\sim$ the quasi-coherent $\mathscr{O}_X$-module arising from M.

Theorem 11.6.24. *With the notation of* (11.6.23)*, for every $R[P]$-module M we have a natural isomorphism*

$$R\Gamma_Z M^\sim \xrightarrow{\sim} M\otimes_{R[P]}\overline{\mathscr{C}}{}^\bullet_P \qquad \textit{in } \mathsf{D}^+(R[P]\text{-}\mathbf{Mod}).$$

Proof. For every face λ of σ, set $U_\lambda := \mathrm{Spec}\, R[P_\lambda]$, and denote by $g_\lambda : U_\lambda \to X$ the open immersion. We consider the chain complex $\mathscr{R}_\bullet$ of $\mathbb{Z}_X$-modules such that :

$$\mathscr{R}_0 := \mathbb{Z}_X \qquad \mathscr{R}_i := \bigoplus_{\lambda\in\Lambda_{i-1}} g_{\lambda!}\mathbb{Z}_{U_\lambda} \qquad \text{for every } i = 1, \dots, \dim P.$$

The differential $d_1 : \mathscr{R}_1 \to \mathscr{R}_0$ is just the sum of the natural morphisms $g_{\lambda!}\mathbb{Z}_{U_\lambda} \to \mathbb{Z}_X$, for λ ranging over the one-dimensional faces of σ. For $i > 1$, the differential d_i is the sum of the maps

$$d_\lambda := \sum_{\substack{\mu\in\Lambda_i\\ \mu\subset\lambda}} [b^i_\lambda : b^{i-1}_\mu] \cdot d_{\lambda\mu!} : g_{\lambda!}\mathbb{Z}_{U_\lambda} \to \mathscr{R}_{n-1} \qquad \text{for every } \lambda \in \Lambda_{i-1}$$

where $d_{\lambda\mu} : g_{\lambda!}\mathbb{Z}_{U_\lambda} \to g_{\mu!}\mathbb{Z}_{U_\mu} \subset \mathscr{R}_{n-1}$ is induced by the inclusion $U_\lambda \subset U_\mu$. With this notation, a simple inspection of the definitions yields a natural identification

$$M \otimes_{R[P]} \overline{\mathscr{C}}^\bullet_P \xrightarrow{\sim} \mathrm{Hom}^\bullet_{\mathbb{Z}}(\mathscr{R}_\bullet, M^\sim[0]) \qquad \text{in } \mathsf{C}(R[P]\text{-}\mathbf{Mod}). \tag{11.6.25}$$

On the other hand, let $\lambda_1, \dots, \lambda_n$ be the one-dimensional faces of σ; then λ_i is L-rational (see (6.3.20)), and F_{λ_i} is a fine, sharp and saturated monoid of dimension one (propositions 6.3.22(i) and 6.4.9(ii)), so $F_{\lambda_i} \simeq \mathbb{N}$ for $i = 1, \dots, n$ (theorem 6.4.18(ii)). For each $i = 1, \dots, n$, let y_i be the unique generator of F_{λ_i}, and denote by $I \subset P$ the ideal generated by $y_1, \dots, y_n$; we remark

Claim 11.6.26. The radical of I is $\mathfrak{m}_P$.

Proof of the claim. For any $x \in \mathfrak{m}_P$, pick a subset $S \subset \{1, \dots, n\}$ such that $x = \sum_{i\in S} a_i y_i$, with $a_i > 0$ for every $i \in S$. Let $N \in \mathbb{N}$ be large enough such that $Na_i \geq 1$ for every $i \in S$, and pick $b_i \in \mathbb{N}$ such that $Na_i \geq b_i \geq 1$ for every $i \in S$. It follows that $Nx - \sum_{i\in S} b_i y_i \in P$, and the claim follows. ◊

Now, let $i : Z \to X$ be the closed immersion; we have :

Claim 11.6.27. The natural map $\mathbb{Z}_X \to i_*\mathbb{Z}_Z$ induces an isomorphism

$$\mathscr{R}_\bullet \xrightarrow{\sim} i_*\mathbb{Z}_Z[0] \qquad \text{in } \mathsf{D}^+(\mathbb{Z}_X\text{-}\mathbf{Mod}). \tag{11.6.28}$$

Proof of the claim. The assertion can be checked on the stalks at the points of X, hence let $x \in X$ be any such point; if $x \in Z$, claim 11.6.26 easily implies that $(\mathscr{R}_\bullet)_x$ is concentrated in degree zero, and then $(11.6.28)_x$ is clearly an isomorphism of complexes of $\mathbb{Z}$-modules. If $x \notin Z$, let $\mathfrak{p} \subset R[P]$ be the prime ideal corresponding to x, and $\lambda \subset \sigma$ the unique face such that $P \setminus F_\lambda = \mathfrak{p} \cap P$; a simple inspection of the construction shows that

$$(\mathscr{R}_\bullet)_x \xrightarrow{\sim} \overline{\mathscr{C}}_\bullet(C^\bullet_\lambda)$$

(notation of (11.6.7)). But proposition 11.6.8 and lemma 11.6.13 imply that $\overline{\mathscr{C}}_\bullet(C^\bullet_\lambda)$ is acyclic, whence the claim. ◊

In view of (11.6.25) and claim 11.6.27, we are reduced to showing that the natural map

$$\mathrm{Hom}^\bullet_{\mathbb{Z}}(\mathscr{R}_\bullet, M^\sim[0]) \to R\mathrm{Hom}^\bullet_{\mathbb{Z}}(\mathscr{R}_\bullet, M^\sim[0])$$

is an isomorphism in $\mathsf{D}^+(R[P]\text{-}\mathbf{Mod})$. This can be done by a spectral sequence argument, along the lines of proposition 10.4.18 (which indeed includes a special case of the situation we are considering here, namely the case where P is a free monoid : the details shall be left to the reader). □

11.6.29. Notice that $\overline{\mathscr{C}}^\bullet_P$ is a complex of L-graded R-modules, hence its cohomology is L-graded as well, and we wish next to compute the graded terms $\mathrm{gr}_\bullet H^\bullet(\overline{\mathscr{C}}^\bullet_P)$. To this aim, we make the following :

Definition 11.6.30. Let V be a finite dimensional $\mathbb{R}$-vector space, $X \subset V$ any subset, and $z \in V$ any point. Then :

(a) We say that a point $x \in X$ *is visibile from* z, if $\{tz + (1-t)x \mid 0 \leq t \leq 1\} \cap X = \{x\}$.
(b) We say that a subset $S \subset X$ *is visibile from* z, if every point of S is visible from z.

Lemma 11.6.31. *In the situation of* (11.6.15)*, let* $z \in V \setminus C_\sigma$ *be any point, and denote by* S *the set of points of* C_σ *that are visible from* z*. Then :*

(i) *S is a subcomplex of $C^\bullet_\sigma$.*
(ii) *S (with the topology induced from V) is homeomorphic to $\overline{\mathbb{B}}^e(1)$, for $e \in \{d-2, d-1\}$.*

Proof. Pick a system of generators $\rho_1, \dots, \rho_n$ for $\sigma^\vee$, and choose $\rho_0 \in \sigma^\vee$ so that $C_\sigma = \sigma \cap \rho_0^{-1}(1)$; we may assume that, for some integer $k \leq n$ we have

$$\rho_i(z) < 0 \qquad \text{if and only if} \qquad 1 \leq i \leq k.$$

Say that $y \in C_\sigma$, so that $\rho_i(y) \geq 0$ for every $i = 1, \dots, n$, and set $y_t := tz + (1-t)y$ for every $t \in [0,1]$; then clearly $\rho_i(y_t) \geq 0$ for every $i = k+1, \dots, n$. Now, if $\rho_0(z) \neq 1$, we get $\rho_0(y_t) \neq 1$ for every $t \neq 0$, therefore the whole of C_σ is visible from z, in which case (i) is trivial, and (ii) follows from 11.6.13. Hence we may assume that $\rho_0(z) = 1$, so $\rho_0(y_t) = 1$ for every $t \in [0,1]$. Suppose now that $\rho_i(y) > 0$ for some $i \leq k$; then there exists a unique $t_i \in]0,1[$ such that $\rho_i(y_{t_i}) = 0$. Hence, if $s := \min(\rho_i(y) \mid i = 1, \dots, k) > 0$, let $t := \min(t_i \mid i = 1, \dots, k)$; it follows that $\rho_i(y_t) \geq 0$ for every $i = 1, \dots, n$, so $y_t \in C_\sigma$, which says that y is not visible from z. Conversely, if $s = 0$, then it follows easily that y is visible from z. We conclude that

$$S = C_\sigma \cap \bigcup_{i=1}^k \mathrm{Ker}\, \rho_i$$

which shows (i). Next, set $W := \mathrm{Ker}\, \rho_0$, and denote by $\tau_z : \rho_0^{-1}(1) \xrightarrow{\sim} W$ the translation map given by the rule : $x \mapsto x - z$ for every $x \in \rho_0^{-1}(1)$. To conclude, it suffices to check that $S' := \tau_z(S)$ is homeomorphic to $\overline{\mathbb{B}}^{d-2}(1)$. To this aim, denote by λ the convex cone in W generated by $\tau_z(C_\sigma)$. Explicitly, if $v_1, \dots, v_k \in \sigma$ have been chosen so that (11.6.12) holds, then λ is the cone generated by $v_1 - z, \dots, v_k - z$; especially, λ is a polyhedral cone, and it is easily seen that $\langle \lambda \rangle = W$. Moreover, λ is strictly convex; indeed, otherwise there exist real numbers $a_1, \dots, a_k \geq 0$, with $a_i > 0$ for at least an index $i \leq k$, such that $\sum_{i=1}^k a_i \cdot (v_i - z) = 0$, *i.e.* $\sum_{i=1}^k a_i v_i = (\sum_{i=1}^k a_i) \cdot z$, which is absurd, since $z \notin \sigma$. Pick $u \in \lambda^\vee$ such that $\lambda \cap \mathrm{Ker}\, u = 0$, and set $C_\lambda := \lambda \cap u^{-1}(1)$; by lemma 11.6.13, the subset C_λ is homeomorphic to $\overline{\mathbb{B}}^{d-2}(1)$. Lastly, let $\pi : W \setminus \mathrm{Ker}\, u \to u^{-1}(1)$ be the radial projection (so $\pi(w)$ is the intersection point of $\mathbb{R}w$ with $u^{-1}(1)$, for every $w \in W \setminus \mathrm{Ker}\, u$). It is easily seen that π maps S' bijectively onto C_λ, so the restriction of π is a homeomorphism $S' \xrightarrow{\sim} C_\lambda$, as required. □

Lemma 11.6.32. *In the situation of* (11.6.23)*, let* $\lambda \subset \sigma$ *be any face. For every* $l \in L_\mathbb{R}$ *we have:*

(i) *The set of points of σ that are visible from l is a union of faces of σ.*
(ii) *Suppose $l \in L$, and endow $R[P_\lambda]$ with its natural L-grading. Then*

$$\mathrm{gr}_l R[P_\lambda] = \begin{cases} Rl & \textit{if } \lambda \subset \sigma \textit{ is not visible from } l \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Let $\rho_1, \dots, \rho_n$ be a system of generators of $\sigma^\vee$; we may assume that, for some integer $k \leq n$ we have

$$\rho_i(l) < 0 \qquad \text{if and only if } 1 \leq i \leq k.$$

Arguing as in the proof of lemma 11.6.31, we check easily that the set of points of σ visible from l equals $S := \sigma \cap \bigcup_{i=1}^k \mathrm{Ker}\, \rho_i$, which already shows (i).

(ii): Suppose first that $\lambda \subset \sigma$ is not visible from l; then there exists $x \in \lambda \setminus S$, and since F_λ generates λ (see (6.3.20)), we may assume that $x \in F_\lambda$ (details left to the reader). A simple inspection then shows that there exists a sufficiently large $N \in \mathbb{N}$ such that $l + Nx \in \sigma$, whence $l \in R[P_\lambda]$, and so $\mathrm{gr}_l R[P_\lambda] = Rl$. Conversely, if the latter identity holds, then there exists $x \in F_\lambda$ such that $l + x \in P$, whence $\rho_i(x) > 0$ for $i = 1, \dots, k$, so x is not visible from l. □

11.6.33. In the situation of (11.6.23), let us fix a linear form $u_0 \in \sigma^\vee$ such that $\sigma \cap \mathrm{Ker}\, u_0 = 0$, and define σ° and C_σ as in (11.6.11). For any $l \in L_\mathbb{R}$, denote by S_l the set of points of σ that are visible from l, so S_l is a union of faces of σ, by lemma 11.6.32(i), and therefore $C_l := S_l \cap C_\sigma$ is a subcomplex of $C_\sigma^\bullet$.

Proposition 11.6.34. *With the notation of* (11.6.33), *suppose $l \in L$. Then the following holds :*

(i) *If $-l \in \sigma^\circ$, the complex of R-modules $\mathrm{gr}_l \overline{\mathscr{C}}{}^\bullet_P$ is isomorphic to $R[-d]$.*

(ii) *If $-l \notin \sigma^\circ$, there is a natural isomorphism of complexes of R-modules :*

$$\mathrm{gr}_l \overline{\mathscr{C}}{}^\bullet_P \xrightarrow{\sim} \mathrm{Hom}^\bullet_\mathbb{Z}(\overline{\mathscr{C}}_\bullet(C_\sigma^\bullet)/\overline{\mathscr{C}}_\bullet(C_l^\bullet), R[-1]).$$

Moreover, in this case, both $\overline{\mathscr{C}}_\bullet(C_\sigma^\bullet)$ and $\overline{\mathscr{C}}_\bullet(C_l^\bullet)$ are acyclic complexes.

Proof. (i): If $-l \in \sigma^\circ$, then $l \in P_\lambda$ if and only if $\lambda = \sigma$, so the assertion is clear.

(ii): The sought identification of complexes follows from lemma 11.6.32(ii), by a direct inspection of the constructions (and indeed, this holds even if $-l \in \sigma^\circ$: details left to the reader). Moreover, it is already known from proposition 11.6.8 and lemma 11.6.13 that $\overline{\mathscr{C}}_\bullet(C_\sigma^\bullet)$ is acyclic.

• Next, if $l \in P$, then clearly $S_l = \varnothing$, so the assertion for $\overline{\mathscr{C}}_\bullet(C_l^\bullet)$ is trivial in this case.

• Thus, suppose that $l \notin P$; if furthermore $-l \notin P$, then the convex cone τ generated by P and l is still strictly convex, so we may find $u_1 \in \tau^\vee$ such that $u_1(l) = 1$ and $\sigma \cap \mathrm{Ker}\, u_1 = 0$. Set $C'_\sigma := \sigma \cap u_1^{-1}(1)$ and $C'_l := S_l \cap C'_\sigma$. In remark 11.6.16(i) we have exhibited a homeomorphism $C'^\bullet_\sigma \xrightarrow{\sim} C^\bullet_\sigma$ that preserves the respective cell complex structures, and a simple inspection shows that this homeomorphism maps C'_l onto C_l. On the other hand, it is easily seen that C'_l is also the set of points of C'_σ that are visible from l (indeed, the segment that joins any point of C'_σ to l lies in $u_1^{-1}(1)$, so its intersection with σ equals its intersection with C'_σ : details left to the reader). By virtue of lemma 11.6.31(ii) (and proposition 11.6.8), it follows that $\overline{\mathscr{C}}_\bullet(C'^\bullet_l)$ is acyclic, so the same holds for $\overline{\mathscr{C}}_\bullet(C^\bullet_l)$.

• Lastly, suppose $-l \in P \setminus \sigma^\circ$; we let $\rho_1, \dots, \rho_n$ be a system of generators of $\sigma^\vee$, and $k \leq n$ an integer such that $\rho_i(l) < 0$ if and only if $1 \leq i \leq k$. Denote by $\tau^\vee$ (resp. $\mu^\vee$) the convex cone in $L^\vee_\mathbb{R}$ generated by $(\rho_{k+1}, \dots, \rho_n)$ (resp. by $(-\rho_1, \dots, -\rho_k)$), and let τ (resp. μ) be the dual of $\tau^\vee$ (resp. of $\mu^\vee$) in $L_\mathbb{R}$; with this notation, we have $l \in \tau \cap \mu^\circ$. Especially, $\tau \cap \mu^\circ \neq \varnothing$, and since τ° is dense in τ (by proposition 6.3.11(i)), we deduce that $\tau^\circ \cap \mu^\circ \neq \varnothing$ as well. Pick $z \in \tau^\circ \cap \mu^\circ$; by inspecting the proof of lemma 11.6.32(i), it is easily seen that $S_l = S_z$. But by construction, $-z \notin \sigma$, therefore – arguing as in the previous case – we conclude that $\overline{\mathscr{C}}_\bullet(C^\bullet_z) = \overline{\mathscr{C}}_\bullet(C^\bullet_l)$ is acyclic. □

Theorem 11.6.35 (Hochster)**.** *Let R be a Cohen-Macaulay noetherian ring, P a fine, sharp and saturated monoid. Then $R[P]$ is a Cohen-Macaulay ring.*

Proof. In view of [126, p.181, Cor.] we may assume that R is a field. We argue by induction on $d := \dim P$. If $d = 0$, we have $P = 0$, and there is nothing to show. Suppose that $d > 0$

and that the assertion is already known for every Cohen-Macaulay ring R and every monoid as above, of dimension $< d$. Set $L := P^{\mathrm{gp}}$, and let $\sigma \subset L_{\mathbb{R}}$ be the unique convex polyhedral cone such that $P = L \cap \sigma$. The ideal $\mathfrak{n} := R[\mathfrak{m}_P]$ is maximal in $R[P]$, and proposition 11.6.34 and theorem 11.6.24 show that

$$\text{(11.6.36)} \qquad \operatorname{depth} R[P]_{\mathfrak{n}} = \dim P.$$

On the other hand, we have the more general :

Claim 11.6.37. Let F be a field, P a fine monoid, A an integral domain which is an F-algebra of finite type, and F' the field of fractions of A. Then we have :

(i) For every maximal ideal $\mathfrak{m} \subset A$, the Krull dimension of $A_{\mathfrak{m}}$ equals the transcendence degree of F' over F.

(ii) For every maximal ideal $\mathfrak{m} \subset F[P]$, the Krull dimension of $F[P]_{\mathfrak{m}}$ equals $\mathrm{rk}_{\mathbb{Z}} P^{\mathrm{gp}}$.

Proof of the claim. (i): This is a straightforward consequence of [126, Th.5.6].

(ii): Choose an isomorphism : $P^{\mathrm{gp}} \xrightarrow{\sim} L \oplus T$, where T is the torsion subgroup of P^{gp}, and L is a free abelian group of finite rank; there follows an induced isomorphism (4.8.52) :

$$F[P^{\mathrm{gp}}] \xrightarrow{\sim} F[L] \otimes_F F[T].$$

Let B be the maximal reduced quotient of $F[P^{\mathrm{gp}}]$ (so the kernel of the projection $F[P^{\mathrm{gp}}] \to B$ is the nilradical); we deduce that B is a direct product of the type $\prod_{i=1}^n F'_i[L]$, where each F'_i is a finite field extension of F. By (i), the Krull dimension of $F'_i[L]_{\mathfrak{m}}$ equals $r := \mathrm{rk}_{\mathbb{Z}} P^{\mathrm{gp}}$ for every maximal ideal $\mathfrak{m} \subset F'_i[L]$, hence every irreducible component of $\operatorname{Spec} B$ has dimension r. Let also C be the maximal reduced quotient of $F[P]$; the natural map $C \to B$ is an injective localization, obtained by inverting a finite system of generators of P, hence the induced morphism $\operatorname{Spec} B \to \operatorname{Spec} C$ is an open immersion with dense image. Let Z be any (reduced) irreducible component of $\operatorname{Spec} C$; again by (i) it follows that every non-empty open subset of Z has dimension equal to $\dim Z$, so necessarily the latter equals r. ◊

From (11.6.36), corollary 6.4.12(i) and claim 11.6.37(ii) we see already that $R[P]_{\mathfrak{n}}$ is a Cohen-Macaulay ring. Next, let $\lambda_1, \dots, \lambda_k$ be the one-dimensional faces of σ, and define P_{λ_i} as in (11.6.23), for every $i = 1, \dots, k$. In light of claim 11.6.26, we have

$$\operatorname{Spec} R[P] \setminus \{\mathfrak{n}\} = \bigcup_{i=1}^{k} \operatorname{Spec} R[P_{\lambda_i}]$$

so it remains to check that $R[P_{\lambda_i}]$ is Cohen-Macaulay for every $i = 1, \dots, k$. However, we may find a decomposition $P_{\lambda_i} \xrightarrow{\sim} Q_i \times G_i$, where G_i is a free abelian group of finite type, and Q_i is a fine, sharp and saturated monoid of dimension $d - 1$ (lemma 6.2.10). Set $S_i := R[G_i]$; then $R[P_{\lambda_i}] = S_i[Q_i]$, and S_i is a Cohen-Macaulay ring ([126, Th.17.7]), so the claim follows by inductive assumption. □

11.6.38. In the situation of (11.6.23), we endow $R[P]$ with its natural L-grading, and denote by $\underline{A} := (R[P], \mathrm{gr}_\bullet R[P])$ the resulting L-graded R-algebra. Let $(M, \mathrm{gr}_\bullet M)$ be an L-graded $\underline{A}$-module (see definition 7.6.1(ii)); we set

$$M^\dagger := \bigoplus_{l \in L} \mathrm{Hom}_R(\mathrm{gr}_l M, R)$$

which is naturally an R-submodule of $M^* := \mathrm{Hom}_R(M, R)$: indeed, any R-linear map $\mathrm{gr}_l M \to R$ yields a linear form $M \to R$, after composition with the projection $M \to \mathrm{gr}_l M$. Notice that M^* is naturally an $R[P]$-module; namely for any linear form $f : M \to R$ and any

$x \in P$, one defines $x \cdot f : M \to R$ by the rule : $x \cdot f(m) := f(xm)$ for every $m \in M$. Then, it is easily seen that $M^\dagger$ is an $R[P]$-submodule of M^*; more precisely, we have

$$x \cdot \mathrm{Hom}_R(\mathrm{gr}_l M, R) \subset \mathrm{Hom}_R(\mathrm{gr}_{l-x} M, R) \qquad \text{for every } x \in P. \tag{11.6.39}$$

In light of (11.6.39), it is convenient to define an L-grading on $M^\dagger$ by the rule :

$$\mathrm{gr}_l M^\dagger := \mathrm{Hom}_R(\mathrm{gr}_{-l} M, R) \qquad \text{for every } l \in L$$

and then $(M^\dagger, \mathrm{gr}_\bullet M^\dagger)$ is naturally an L-graded $\underline{A}$-module.

Remark 11.6.40. (i) Clearly, the rule $M \mapsto M^\dagger$ yields a functor from the category $\underline{A}$-$\mathbf{Mod}$ of L-graded $\underline{A}$-modules to $\underline{A}$-$\mathbf{Mod}^o$.

(ii) Let $\underline{A}$-$\mathbf{Mod}_{\mathrm{rflx}}$ denote the full subcategory of $\underline{A}$-$\mathbf{Mod}$ whose objects are all the L-graded $\underline{A}$-modules $(M, \mathrm{gr}_\bullet M)$ such that $\mathrm{gr}_l M$ is a projective R-module of finite type, for every $l \in L$. A simple inspection of the definition shows that the functor $(-)^\dagger$ of (i) restricts to an equivalence of categories :

$$\underline{A}\text{-}\mathbf{Mod}_{\mathrm{rflx}} \to (\underline{A}\text{-}\mathbf{Mod}_{\mathrm{rflx}})^o.$$

Example 11.6.41. If $S \subset L$ is any P-submodule, notice that $L \setminus (-S)$ is also a P-submodule of L, and set

$$S^\dagger := L/(L \setminus (-S))$$

where the quotient is a pointed P-module, as in remark 4.8.17(iii). Explicitly, $S^\dagger$ is the set $(-S) \cup \{0_{S^\dagger}\}$ (where the zero element $0_{S^\dagger}$ should not be confused with the neutral element 0 of the abelian group L); the P-module structure on $S^\dagger$ is determined by the rule :

$$x \cdot s := \begin{cases} x + s & \text{if } x + s \in -S \\ 0_{S^\dagger} & \text{otherwise} \end{cases} \qquad \text{for every } s \in -S.$$

Then it is easily seen that there exists a natural isomorphism of L-graded $\underline{A}$-modules :

$$R[S]^\dagger \xrightarrow{\sim} R\langle S^\dagger \rangle$$

(notation of (6.1.31)). On the other hand, the natural map $R[S] \to (R[S]^*)^*$ induces an isomorphism $R[S] \xrightarrow{\sim} (R[S]^\dagger)^\dagger$ whence an isomorphism of L-graded $\underline{A}$-modules

$$R[S] \xrightarrow{\sim} R\langle S^\dagger \rangle^\dagger. \tag{11.6.42}$$

Theorem 11.6.43 (Stanley, Danilov). *In the situation of* (11.6.23), *set $P^\circ := L \cap \sigma^\circ$. We have :*

(i) *There exists a natural isomorphism :*

$$R\mathrm{Hom}^\bullet_{R[P]}(R, R[P^\circ]) \xrightarrow{\sim} R[-d] \qquad \textit{in } \mathsf{D}(R[P]\text{-}\mathbf{Mod})$$

(here R is regarded as an $R[P]$-module, via the augmentation map $R[P] \to R$).

(ii) *If R is a Gorenstein noetherian ring, the complex of coherent $\mathscr{O}_X$-modules $R[P^\circ]^\sim[0]$ is dualizing on X.*

Proof. (i): From proposition 11.6.34 we deduce a map of complexes of L-graded $\underline{A}$-modules

$$\varphi^\bullet : \overline{\mathscr{C}}{}^\bullet_P \to R\langle (P^\circ)^\dagger \rangle[-d]$$

(notation of example 11.6.41) with $\mathrm{gr}_l \varphi^\bullet$ a quasi-isomorphism of complexes of R-modules, for every $l \in L$. Since $\mathrm{gr}_l \overline{\mathscr{C}}{}^\bullet_P$ is a finite dimensional R-vector space for every $l \in L$, there follows – in view of (11.6.42) – a quasi-isomorphism of complexes of L-graded $\underline{A}$-modules

$$R[P^\circ][0] \xrightarrow{\sim} (\overline{\mathscr{C}}{}^\bullet_P)^\dagger[-d]. \tag{11.6.44}$$

whence a convergent spectral sequence

$$E_1^{pq} := \mathrm{Ext}^p_{R[P]}(R, (\overline{\mathscr{C}}{}^{d-q}_P)^\dagger) \Rightarrow \mathrm{Ext}^{p+q}_{R[P]}(R, R[P^\circ]).$$

Hence, the assertion follows immediately from the :

Claim 11.6.45. With the foregoing notation, we have :

(i) $E_1^{pq} = 0$ whenever $q < d$.

(ii) $E_1^{0d} \simeq R$, and $E_1^{pd} = 0$ for every $p > 0$.

Proof of the claim. (i): Let $\lambda \subset \sigma$ be any face with $\dim \lambda > 0$; it suffices to check that $E^i := \mathrm{Ext}^i_{R(P]}(R, R[P_\lambda]^\dagger) = 0$ for every $i \in \mathbb{N}$. To this aim, pick any $x \in F_\lambda \setminus \{0\}$; since $R[P_\lambda]^\dagger = R\langle P_\lambda^\dagger \rangle$ (example 11.6.41), we see that scalar multiplication by x is an automorphism on $R[P_\lambda]^\dagger$, hence also on E^i. On the other hand, scalar multiplication by x is the zero endomorphism on R, hence also on E^i, and the claim follows.

(ii): Pick a resolution $F_\bullet$ of the P-graded $\mathbb{Z}[P]$-module $\mathbb{Z} := \mathbb{Z}[P]/\mathbb{Z}[\mathfrak{m}_P]$ as in remark 7.6.18(iii), so that F_n is a free $\mathbb{Z}[P]$-module of finite type for every $n \in \mathbb{N}$, and the differential $d_n : F_n \to F_{n-1}$ is a morphism of P-graded $\mathbb{Z}[P]$-modules, for every $n > 0$. Since the augmented complex $F_\bullet \to \mathbb{Z}$ is flat as a complex of $\mathbb{Z}$-modules, the complex of $R[P]$-modules $R \otimes_\mathbb{Z} F_\bullet$ is still a free resolution of the $R[P]$-module R, and we get natural isomorphisms

$$E_1^{pd} \xrightarrow{\sim} H^p\mathrm{Hom}^\bullet_{R[P]}(R \otimes_\mathbb{Z} F_\bullet, R[P]^\dagger[0]) \xrightarrow{\sim} H^p((R \otimes_\mathbb{Z} F_\bullet)^\dagger) \qquad \text{for every } p \in \mathbb{N}.$$

(where the differentials of the complex $(R\otimes_\mathbb{Z} F_\bullet)^\dagger$ are the maps $d_\bullet^\dagger$). However, remark 11.6.40(ii) implies that the complex $(R \otimes_\mathbb{Z} F_\bullet \to R)^\dagger$ is still acyclic, whence the contention. ◊

(ii): To ease notation, set $S := \mathrm{Spec}\, R$ and $\omega_P := R[P^\circ]^\sim$; in view of proposition 11.3.37, it suffices to check that the complex of coherent $\mathscr{O}_{X(x)}$-modules $\omega_P(x)[0]$ is dualizing on $X(x)$, for every $x \in X$ (notation of definition 4.9.17(iii)). Hence, fix any such point x, and let $\mathfrak{p} \subset R[P]$ be the prime ideal corresponding to x; after replacing R by its localization at the prime ideal $\mathfrak{p} \cap R$, we may assume that R is a local ring, and the structure morphism $X \to S$ maps x to the closed point of S, corresponding to the maximal ideal $\mathfrak{m}_R$ of R. Denote by $\lambda \subset \sigma$ the unique face such that $P \setminus F_\lambda = \mathfrak{p} \cap P$, so that $x \in U_\lambda := \mathrm{Spec}\, R[P_\lambda]$. We may find a decomposition $P \xrightarrow{\sim} F_\lambda^{\mathrm{gp}} \times Q$, where Q is also a fine, sharp and saturated monoid (lemma 6.2.10), whence an isomorphism of S-schemes

$$U_\lambda \xrightarrow{\sim} \mathrm{Spec}\, R[F_\lambda^{\mathrm{gp}}] \times_S Y \qquad \text{where } Y := \mathrm{Spec}\, R[Q]$$

and by construction, $F^{-1}(\mathfrak{p} \cap P)$ is the maximal ideal of P_λ, so the induced projection $p : U_\lambda \to Y$ maps x to the maximal ideal $R[\mathfrak{m}_Q] + \mathfrak{m}_R[Q] \subset R[Q]$. Let $\tau \subset Q^{\mathrm{gp}}_\mathbb{R}$ be the unique polyhedral cone such that $Q = \tau \cap Q^{\mathrm{gp}}$, set $Q^\circ := Q \cap \tau^\circ$, and define the coherent $\mathscr{O}_Y$-module $\omega_Q := R[Q^\circ]^\sim$. It is easily seen that there is a natural identification

$$\omega_{P|U_\lambda} \xrightarrow{\sim} p^*\omega_Q.$$

In view of proposition 11.3.34 and remark 11.3.32(i), it then suffices to check that $\omega_Q(p(x))$ is dualizing on $Y(p(x))$. Thus, we may replace P by Q, and assume from start that $\mathfrak{p} = R[\mathfrak{m}_P] + \mathfrak{m}_R[P] \subset R[P]$. In this case, set $S_0 := \mathrm{Spec}\, R/\mathfrak{m}_R$, and denote by $i_1 : S_0 \to S$ and $i_2 : S \to X(S)$ the closed immersions (where i_2 corresponds to the ring homomorphism $R(P]_\mathfrak{p} \to R$ deduced from augmentation map); according to (i) and corollary 10.3.2(ii), we have a natural identification

$$i_2^!\omega_P(x)[0] \simeq \mathscr{O}_S[-d].$$

On the other hand, since R is Gorenstein, $\mathscr{O}_S[-d]$ is a dualizing complex on S; from lemma 11.3.27(i) and proposition 11.1.7(i) it follows that $i_1^!\mathscr{O}_S[-d] \xrightarrow{\sim} (i_2 \circ i_1)^!\omega_P(x)[0]$ is dualizing on S_0, and therefore it is isomorphic to $\mathscr{O}_{S_0}[c]$ for some $c \in \mathbb{Z}$. Lastly, proposition 11.3.28(i) and corollary 10.3.2(ii) now say that $\omega_P(x)[0]$ is dualizing on $X(x)$. □

Remark 11.6.46. Suppose that R is a field; then example 7.11.33 shows that $(\overline{\mathscr{C}}^0_P)^\dagger$ is the injective hull of the $R[P]$-module R. In this case, claim 11.6.45(ii) is therefore a special case of claim 7.11.24. For a general R, the $R[P]$-module $(\overline{\mathscr{C}}^0_P)^\dagger$ shall not be injective, but it can be viewed as a graded variant of the injective hull construction.

12. LOGARITHMIC GEOMETRY

12.1. **Log topoi.** Henceforth, *all topoi under consideration will be locally ringed and with enough points, and all morphisms of topoi will be morphisms of locally ringed topoi* (see (4.9.13)). The purpose of this restriction is to insure that we obtain the right notions, when we specialize to the case of schemes.

Definition 12.1.1. Let $T := (T, \mathscr{O}_T)$ be a locally ringed topos.

(i) A *pre-log structure* on T is the datum of a pair

$$(\underline{M}, \alpha)$$

where $\underline{M}$ is a T-monoid, and $\alpha : \underline{M} \to \mathscr{O}_T$ is a morphism of T-monoids, called the *structure map* of $\underline{M}$, and where the monoid structure on $\mathscr{O}_T$ is induced by the multiplication law (hence, by the multiplication in the ring $\mathscr{O}_T(U)$, for every object U of T).

(ii) A morphism $(\underline{M}, \alpha) \to (\underline{N}, \beta)$ of pre-log structures on T, is a map $\gamma : \underline{M} \to \underline{N}$ of T-monoids, such that $\beta \circ \gamma = \alpha$. We denote by

$$\mathbf{pre\text{-}log}_T$$

the category of pre-log structures on T.

(iii) A pre-log structure $(\underline{M}, \alpha)$ on T is a *log structure* if α restricts to an isomorphism:

$$\alpha^{-1}\mathscr{O}_T^\times \xrightarrow{\sim} \underline{M}^\times \xrightarrow{\sim} \mathscr{O}_T^\times .$$

The datum of a locally ringed topos $(T, \mathscr{O}_T)$ and a log structure on T is also called, for short, a *log topos*. We denote by

$$\mathbf{log}_T$$

the full subcategory of $\mathbf{pre\text{-}log}_T$ consisting of all log structures on T. When there is no danger of ambiguity, we shall often omit mentioning explicitly the map α, and therefore only write $\underline{M}$ to denote a pre-log or a log structure.

12.1.2. The category $\mathbf{log}_T$ admits an initial object, namely the log structure $(\mathscr{O}_T^\times, j)$, where j is the natural inclusion; this is called the *trivial log structure*. $\mathbf{log}_T$ admits a final object as well: this is $(\mathscr{O}_T, \mathbf{1}_{\mathscr{O}_T})$. A morphism of locally ringed topoi $f : T \to S$ induces a pair of adjoint functors :

(12.1.3) $$f^* : \mathbf{pre\text{-}log}_S \to \mathbf{pre\text{-}log}_T \qquad f_* : \mathbf{pre\text{-}log}_T \to \mathbf{pre\text{-}log}_S .$$

Namely, let $(\underline{M}, \alpha)$ (resp. $(\underline{N}, \beta)$) be a pre-log structure on S (resp. on T) and

$$f^\flat : \mathscr{O}_S \to f_*\mathscr{O}_T \qquad f^\sharp : f^{-1}\mathscr{O}_S \to \mathscr{O}_T$$

the natural morphisms (corresponding to each other under the adjunction (f^{-1}, f_*) that defines the morphism f); then

$$f^{-1}\underline{M} \xrightarrow{f^{-1}\alpha} f^{-1}\mathscr{O}_S \xrightarrow{f^\sharp} \mathscr{O}_T$$

defines $f^*(\underline{M}, \alpha)$, and $f_*(\underline{N}, \beta)$ is the pair $(\underline{P}, \gamma)$, where $\underline{P}$ is the fibre product in the cartesian diagram

$$\begin{array}{ccc} \underline{P} & \xrightarrow{\gamma} & \mathscr{O}_S \\ \downarrow & & \downarrow{\scriptstyle f^\flat} \\ f_*\underline{N} & \xrightarrow{f_*\beta} & f_*\mathscr{O}_T . \end{array}$$

Lemma 12.1.4. *Let T be a topos with enough points, $\varphi : \mathscr{M} \to \mathscr{N}$ a morphism of integral T-monoids inducing isomorphisms $\mathscr{M}^\times \xrightarrow{\sim} \mathscr{N}^\times$ and $\mathscr{M}^\sharp \xrightarrow{\sim} \mathscr{N}^\sharp$. Then φ is an isomorphism.*

Proof. This can be checked on the stalks, hence we are reduced to the corresponding assertions for a morphism $M \to N$ of monoids. Moreover, $M^\sharp$ is just the set-theoretic quotient of M by the translation action of $M^\times$ (lemma 4.8.31(iii)), so the assertion is straightforward, and shall be left as an exercise for the reader. □

Definition 12.1.5. Let $\gamma : (\underline{M}, \alpha) \to (\underline{N}, \beta)$ be a morphism of pre-log structures on the locally ringed topos T, and ξ a T-point.

(i) We say that $(\underline{M}, \alpha)$ is *integral* (resp. *saturated*) if $\underline{M}$ is an integral (resp. integral and saturated) T-monoid.
(ii) We say that γ is *flat* (resp. *saturated*) *at the point* ξ, if $\gamma_\xi : \underline{M}_\xi \to \underline{N}_\xi$ is a flat morphism of monoids (resp. a saturated morphism of integral monoids) (see remark 4.8.23(vi)).
(iii) We say that γ is *flat* (resp. *saturated*), if γ is a flat morphism of T-monoids (resp. a saturated morphism of integral T-monoids) (see definition 6.2.28). In view of proposition 4.8.26 (resp. corollary 6.2.29), this is the same as saying that γ is flat (resp. saturated) at every T-point.

12.1.6. The forgetful functor :

$$\mathbf{log}_T \to \mathbf{pre\text{-}log}_T \quad : \quad \underline{M} \mapsto \underline{M}^{\text{pre-log}}$$

admits a left adjoint :

$$\mathbf{pre\text{-}log}_T \to \mathbf{log}_T \quad : \quad (\underline{M}, \alpha) \mapsto (\underline{M}, \alpha)^{\log}$$

such that the resulting diagram :

$$(12.1.7)\qquad \begin{array}{ccc} \alpha^{-1}(\mathscr{O}_T^\times) & \longrightarrow & \underline{M} \\ \downarrow & & \downarrow \\ \mathscr{O}_T^\times & \longrightarrow & \underline{M}^{\log} \end{array}$$

is cocartesian in the category of pre-log structures. One calls $\underline{M}^{\log}$ the *log structure associated* to $\underline{M}$. From this description, it is easily seen that the counit of adjunction

$$(12.1.8)\qquad (\underline{M}^{\text{pre-log}})^{\log} \to \underline{M}$$

is an isomorphism for every log structure $\underline{M}$ on T.

Composing with the adjunction (12.1.3), we deduce a pair of adjoint functors :

$$f^* : \mathbf{log}_S \to \mathbf{log}_T \qquad f_* : \mathbf{log}_T \to \mathbf{log}_S$$

for any map of locally ringed topoi $f : T \to S$. Explicitly, if $\underline{N}$ is any log structure on S, then $f^*\underline{N}$ is the push-out in the cocartesian diagram of T-monoids :

$$\begin{array}{ccc} f^*\mathscr{O}_S^\times & \longrightarrow & f^*(\underline{N}^{\text{pre-log}}) \\ \downarrow & & \downarrow \\ \mathscr{O}_T^\times & \longrightarrow & f^*\underline{N} \end{array}$$

and on the other hand, if $\underline{M}$ is any log structure on T, then it is easily seen that the pre-log structure $f_*(\underline{M}^{\text{pre-log}})$ is actually a log structure (this can be checked on the stalks over the points of T). It follows easily that the induced map of T-monoids :

$$(12.1.9)\qquad f^*(\underline{N}^\sharp) \to (f^*\underline{N})^\sharp$$

is an isomorphism. As a consequence, for every point ξ of T, the natural map $\underline{N}_{f(\xi)} \to (f^*\underline{N})_\xi$ is a local morphism of monoids.

Remark 12.1.10. The category $\mathbf{log}_T$ admits arbitrary colimits : indeed, since the counit (12.1.8) is an isomorphism, it suffices to construct such colimits in the category of pre-log structures, and then apply the functor $(-) \mapsto (-)^{\log}$ which preserves colimits, since it is a left adjoint.

12.1.11. Let $(\underline{M}, \alpha)$ be a pre-log structure on T. The morphism α extends to a unique morphism of pointed T-monoids $\alpha_\circ : \underline{M}_\circ \to \mathscr{O}_T$, whence a new pre-log structure

$$(\underline{M}, \alpha)_\circ := (\underline{M}_\circ, \alpha_\circ).$$

Clearly, $(\underline{M}, \alpha)$ is a log structure if and only if the same holds for $(\underline{M}, \alpha)_\circ$. More precisely, for any pre-log structure $\underline{M}$ there is a natural isomorphism of log structures :

$$(\underline{M}_\circ)^{\log} \xrightarrow{\sim} (\underline{M}^{\log})_\circ.$$

Furthermore, for any morphism $f : T \to S$ of topoi, we have natural isomorphisms of pre-log (resp. log) structures

$$f^*(\underline{N}_\circ) \xrightarrow{\sim} (f^*\underline{N})_\circ \qquad f_*(\underline{M}_\circ) \xrightarrow{\sim} (f_*\underline{M})_\circ$$

for every pre-log (resp. log) structure $\underline{N}$ on S and $\underline{M}$ on T (details left to the reader).

Example 12.1.12. (i) Let $T \to S$ be a morphism of topoi. Since the initial object of a category is the empty coproduct, it follows formally that the inverse image $f^*(\mathscr{O}_S^\times, j)$ of the trivial log structure on S, is the trivial log structure on T.

(ii) Let T be a topos, and $j_U : T/U \to T$ an open subtopos (see example 4.7.8(i)). Consider the subsheaf of monoids $\underline{M} \subset \mathscr{O}_T$ such that :

$$\underline{M}(V) := \{s \in \mathscr{O}_T(V) \mid s_{|U\times V} \in \mathscr{O}_T^\times(U \times V)\} \qquad \text{for every object } V \text{ of } T.$$

Then it is easily seen that the natural map $\underline{M} \to \mathscr{O}_T$ is a log structure on T. This log structure is (naturally isomorphic to) the extension $j_{U*}\mathscr{O}_U^\times$ of the trivial log structure on T/U.

(iii) Let U be any object of the topos T, and $\underline{M}$ a log structure on T. Since $\mathscr{O}_{T/U} = (\mathscr{O}_T)_{|U}$, it is easily seen that the natural morphism of pre-log structures

$$(\underline{M}^{\text{pre-log}})_{|U} \to (\underline{M}_{|U})^{\text{pre-log}}$$

is an isomorphism.

(iv) Let $\beta : \underline{M} \to \mathscr{O}_T$ be a pre-log structure on a topos T. Then $\beta^{-1}(0) \subset \underline{M}$ is an ideal, and β factors uniquely through the natural map $\underline{M} \to \underline{M}/\beta^{-1}(0)$, and a pre-log structure

$$(\underline{M}, \beta)_{\text{red}} := (\underline{M}/\beta^{-1}(0), \overline{\beta})$$

called the *reduced pre-log structure* associated to $\underline{M}$. As usual, we shall often write just $\underline{M}_{\text{red}}$ instead of $(\underline{M}, \beta)_{\text{red}}$. We say that β is *reduced* if the induced morphism of pre-log structures $\underline{M} \to \underline{M}_{\text{red}}$ is an isomorphism.

Suppose now that $\underline{M}$ is a log structure; then it is easily seen that the same holds for $\underline{M}_{\text{red}}$. More precisely, since the tensor product is right exact (see (4.8.19)), for any pre-log structure $\underline{M}$ the natural morphism of log structures

$$(\underline{M}_{\text{red}})^{\log} \to (\underline{M}^{\log})_{\text{red}}$$

is an isomorphism.

Lemma 12.1.13. *Let $\gamma : (\underline{M}, \alpha) \to (\underline{N}, \beta)$ be a morphism of pre-log structures on T. We have:*

(i) *If $\underline{M}$ is integral (resp. saturated), then the same holds for $\underline{M}^{\log}$.*

(ii) *The unit of adjunction $\underline{M} \to \underline{M}^{\log}$ is a flat morphism.*

(iii) *If γ is flat (resp. saturated) at a T-point ξ, the same holds for the induced morphism $\gamma^{\log} : \underline{M}^{\log} \to \underline{N}^{\log}$ of log structures.*

(iv) *Especially, if γ is flat (resp. saturated), the same holds for $\gamma^{\log}$.*

Proof. In view of lemma 4.8.46(ii) and proposition 4.8.26, both (i) and (ii) can be checked on stalks. Taking into account lemma 4.8.46(i), we are reduced to showing the following. Let P be a monoid, A a ring, $\beta : P \to (A, \cdot)$ a morphism of monoids; then the natural map $P \to P' := P \otimes_{\beta^{-1}A^\times} A^\times$ is flat, and if P is integral (resp. saturated), the same holds for P'. The first assertion follows easily from example 6.1.23(vi), and the second follows from remark 6.2.5(i) (resp. from corollary 6.2.25(ix) and proposition 6.2.26).

(iii): The map $\gamma^{\log}$ can be factored as the composition of

$$\gamma' := \gamma \otimes_{\beta^{-1}\mathscr{O}_T^\times} \mathscr{O}_T^\times : \underline{N}^{\log} \to \underline{P} := \underline{M} \otimes_{\beta^{-1}\mathscr{O}_T^\times} \mathscr{O}_T^\times$$

and the natural unit of adjunction $\gamma'' : \underline{P} \to \underline{P}^{\log} = \underline{M}^{\log}$. If γ_ξ is flat, the same clearly holds for γ'_ξ, and (ii) says that γ'' is flat, hence $\gamma_\xi^{\log}$ is flat in this case. Lastly, suppose that γ_ξ is saturated, and we wish to show that $\gamma_\xi^{\log}$ is saturated. Set $P := \alpha_\xi^{-1}\mathscr{O}_{T,\xi}^\times$ and $Q := \beta_\xi^{-1}\mathscr{O}_{T,\xi}^\times$. Then the induced map $(P^{-1}\underline{M}_\xi)^\sharp \to (Q^{-1}\underline{N}_\xi)^\sharp$ is saturated (lemma 6.2.12(ii,iii)). But the latter is the same as the morphism $(\gamma_\xi^{\log})^\sharp$, and then also γ_ξ is saturated, again by lemma 6.2.12(iii). □

Lemma 12.1.14. *Let $f : T' \to T$ be a morphism of topoi, ξ a T'-point, and $\gamma : (\underline{M}, \alpha) \to (\underline{N}, \beta)$ a morphism of integral log structures on T. The following conditions are equivalent :*

(a) *γ is flat (resp. saturated) at the T-point $f(\xi)$.*
(b) *$f^*\gamma$ is flat (resp. saturated) at the T'-point ξ.*
(c) *$\gamma_\xi^\sharp$ is a flat (resp. saturated) morphism of monoids.*

Proof. The equivalence of (a) and (c) follows from corollary 6.1.49 (resp. lemma 6.2.12(ii)). By the same token, (b) holds if and only if $(f^*\gamma)_\xi^\sharp$ is flat (resp. saturated); in light of the isomorphism (12.1.9), the latter condition is equivalent to (c). □

12.1.15. For any locally ringed topos T, let us write the objects of $\mathbf{Mnd}/\Gamma(T, \mathscr{O}_T)$ in the form (M, φ), where M is any monoid, and $\varphi : M \to \Gamma(T, \mathscr{O}_T)$ a morphism of monoids. There is an obvious global sections functor :

$$\Gamma(T, -) : \mathbf{pre\text{-}log}_T \to \mathbf{Mnd}/\Gamma(T, \mathscr{O}_T) \quad : \quad (\underline{N}, \alpha) \mapsto (\Gamma(T, \underline{N}), \Gamma(T, \alpha))$$

which admits a left adjoint :

$$\mathbf{Mnd}/\Gamma(T, \mathscr{O}_T) \to \mathbf{pre\text{-}log}_T \quad : \quad (M, \varphi) \mapsto (M, \varphi)_T := (M_T, \varphi_T).$$

Indeed, M_T is the constant sheaf on (T, C_T) with value M, and φ_T is the composition of the map of constant sheaves $M_T \to \Gamma(T, \mathscr{O}_T)_T$ induced by φ, with the natural map $\Gamma(T, \mathscr{O}_T)_T \to \mathscr{O}_T$. Again, we shall often just write M_T to denote this pre-log structure.

After taking associated log structures, we deduce a left adjoint :

$$\mathbf{Mnd}/\Gamma(T, \mathscr{O}_T) \to \mathbf{log}_T \quad : \quad (M, \varphi) \mapsto M_T^{\log} := (M, \varphi)_T^{\log} \tag{12.1.16}$$

to the global sections functor. $M_T^{\log}$ is called the *constant log structure* associated to (M, φ).

Definition 12.1.17. Let T be a locally ringed topos, $(\underline{M}, \alpha)$ a log structure on T.

(i) A *chart for $\underline{M}$* is an object (P, β) of $\mathbf{Mnd}/\Gamma(T, \mathscr{O}_T)$, together with a map of pre-log structures $\omega_P : (P, \beta)_T \to \underline{M}$, inducing an isomorphism on the associated log structures. (Notation of (12.1.15).)
(ii) We say that a chart (P, β) is *finite* (resp. *integral*, resp. *fine*, resp. *saturated*) if P is a finitely generated (resp. integral, resp. fine, resp. integral and saturated) monoid.

(iii) Let $\varphi : \underline{M} \to \underline{N}$ be a morphism of log structures on T. A *chart for* φ is the datum of charts :

$$\omega_P : (P, \beta)_T \to \underline{M} \quad \text{and} \quad \omega_Q : (Q, \gamma)_T \to \underline{N}$$

for $\underline{M}$, respectively $\underline{N}$, and a morphism of monoids $\vartheta : Q \to P$, fitting into a commutative diagram :

$$\begin{array}{ccc} Q_T & \xrightarrow{\vartheta_T^{\log}} & P_T^{\log} \\ {\scriptstyle \omega_Q}\downarrow & & \downarrow{\scriptstyle \omega_P} \\ \underline{N} & \xrightarrow{\varphi} & \underline{M}. \end{array}$$

We say that such a chart is *finite* (resp. *integral*, resp. *fine*, resp. *saturated*) if the monoids P and Q are finitely generated (resp. integral, resp. fine, resp. integral and saturated). We say that the chart is *flat* (resp. *saturated*), if ϑ is a flat morphism of monoids (resp. a saturated morphism of integral monoids).

(iv) We say that $\underline{M}$ is *quasi-coherent* (resp. *coherent*) if there exist a covering family $(U_\lambda \to 1_T \mid \lambda \in \Lambda)$ of the final object 1_T in (T, C_T), and for every $\lambda \in \Lambda$, a chart (resp. a finite chart) $(P_\lambda, \beta_\lambda)$ for $\underline{M}_{|U_\lambda}$.

(v) We say that $(\underline{M}, \alpha)$ is *quasi-fine* (resp. *fine*) if it is integral and quasi-coherent (resp. and coherent).

(vi) Let ξ be any T-point. We say that a chart (P, β) is *local* (resp. *sharp*) *at the point* ξ, if the morphism $\beta_\xi : P \to \mathscr{O}_{T,\xi}$ is local (resp. if P is sharp and β_ξ is local).

Lemma 12.1.18. *Let $f : T \to S$ be a morphism of locally ringed topoi, $\underline{Q}$ a log structure on S, and ξ any point of S. The following holds :*

(i) *If $\underline{Q}$ is quasi-coherent (resp. coherent, resp. integral, resp. saturated, resp. quasi-fine, resp. fine), then the same holds for $f^*\underline{Q}$.*

(ii) *Suppose that $\underline{Q}$ is an integral log structure. Then $\underline{Q}^\sharp$ is an integral S-monoid, and $\underline{Q}$ is saturated if and only if $\underline{Q}^\sharp$ is a saturated S-monoid.*

(iii) *Suppose that $\underline{Q}$ is quasi-coherent. Then $\underline{Q}$ is integral (resp. integral and saturated, resp. fine, resp. fine and saturated) if and only if there exist a covering family $(U_\lambda \to 1_S \mid \lambda \in \Lambda)$ of the final object of S, and for every $\lambda \in \Lambda$, an integral (resp. integral and saturated, resp. fine, resp. fine and saturated) chart $(P_\lambda)_{U_\lambda} \to \underline{Q}_{|U_\lambda}$.*

(iv) *If $\underline{P}$ is any coherent log structure on S, and $\omega : \underline{P}_\xi \to \underline{Q}_\xi$ is a map of monoids, then :*

  (a) *There exist a neighborhood U of ξ and a morphism $\vartheta : \underline{P}_{|U} \xrightarrow{\sim} \underline{Q}_{|U}$ of log structures, such that $\vartheta_\xi = \omega$.*

  (b) *Moreover, for any two morphisms $\vartheta, \vartheta' : \underline{P}_{|U} \xrightarrow{\sim} \underline{Q}_{|U}$ with the property of* (a), *we may find a smaller neighborhood $V \to U$ of ξ such that $\vartheta_{|V} = \vartheta'_{|V}$.*

  (c) *Especially, if $\underline{Q}$ is also coherent, and ω is an isomorphism, we may find ϑ and a small enough U as in* (a), *such that ϑ is an isomorphism.*

(v) *If M is a finitely generated monoid, and $\omega : M \to \mathscr{O}_{S,\xi}$ a morphism of monoids, then we may find a neighborhood U of ξ and a morphism $\vartheta : M_U \to \mathscr{O}_U$ of S/U-monoids, such that $\vartheta_\xi = \omega$.*

Proof. (i): If $\underline{Q}$ is the constant log structure associated to a map of monoids $\alpha : Q \to \Gamma(S, \mathscr{O}_S)$, then $f^*\underline{Q}$ is the constant log structure associated to $\Gamma(S, f^\natural) \circ \alpha$ (where $f^\natural : \mathscr{O}_S \to f_*\mathscr{O}_T$ is the natural map). The assertions concerning quasi-coherent or coherent log structures are a straightforward consequence. Next, suppose that $\underline{Q}$ is integral (resp. saturated); we wish to show that $f^*\underline{Q}$ is integral (resp. saturated). To this aim, let $\underline{M} := f^*(\underline{Q}^{\text{pre-log}})$; by lemma

4.8.45(i), $\underline{M}$ is an integral (resp. saturated) T-monoid; then the assertion follows from lemma 12.1.13(i).

(ii): By lemma 4.8.46(ii) the assertion can be checked on stalks. Hence, suppose that $\underline{Q}$ is integral; then $\underline{Q}_\xi$ is integral by *loc.cit.*, consequently, the same holds for $\underline{Q}_\xi/\underline{Q}_\xi^\times$ (lemma 4.8.38). The second assertion follows from lemma 6.2.9(ii).

(iii): Suppose first that $\underline{Q}$ is quasi-coherent and integral. Hence, there is a covering family $(U_\lambda \to 1_S \mid \lambda \in \Lambda)$, and for every $\lambda \in \Lambda$ a monoid M_λ, a pre-log structure $\alpha_\lambda : (M_\lambda)_{U_\lambda} \to \mathscr{O}_{U_\lambda}$, and an isomorphism $((M_\lambda)_{U_\lambda}, \alpha_\lambda)^{\log} \xrightarrow{\sim} \underline{Q}_{|U_\lambda}$; whence a cocartesian diagram of S-monoids, as in (12.1.7) :

$$\begin{array}{ccc} \underline{N} := \alpha_\lambda^{-1}(\mathscr{O}_{U_\lambda}^\times) & \longrightarrow & (M_\lambda)_{U_\lambda} \\ \downarrow & & \downarrow \\ \mathscr{O}_{U_\lambda}^\times & \longrightarrow & \underline{Q}_{|U_\lambda}. \end{array} \tag{12.1.19}$$

The induced diagram $(12.1.19)^{\mathrm{int}}$ of integral S-monoids is still cocartesian. Moreover, since $\underline{Q}_{|U_\lambda}$ is integral, α_λ factors through a unique map $\beta_\lambda : ((M_\lambda)_{U_\lambda})^{\mathrm{int}} \to \underline{Q}_{|U_\lambda} \to \mathscr{O}_{U_\lambda}$, and the morphism in S underlying the induced morphism of S-monoids $\underline{N}^{\mathrm{int}} \to \underline{N}' := \beta_\lambda^{-1}(\mathscr{O}_{U_\lambda}^\times)$ is an epimorphism (this can be checked easily on the stalks). Furthermore, $((M_\lambda)_{U_\lambda})^{\mathrm{int}} \simeq (M_\lambda^{\mathrm{int}})_{U_\lambda}$ (see (4.8.49)). It follows that the natural map

$$\underline{Q}_{|U_\lambda} \to \mathscr{O}_{U_\lambda}^\times \amalg_{\underline{N}'} (M_\lambda^{\mathrm{int}})_{U_\lambda} \simeq (M_\lambda^{\mathrm{int}})_{U_\lambda}^{\log}$$

is an isomorphism, so the claim holds with $P_\lambda := M_\lambda^{\mathrm{int}}$. If Q is fine, we can find M_λ as above which is also finitely generated, in which case the resulting P_λ shall be fine.

Suppose additionally, that $\underline{Q}$ is saturated. By the previous case, we may then find a covering family $(U_\lambda \to e_S \mid \lambda \in \Lambda)$, and for every $\lambda \in \Lambda$ an integral monoid M_λ, a pre-log structure $\alpha_\lambda : (M_\lambda)_{U_\lambda} \to \mathscr{O}_{U_\lambda}$, and an isomorphism $((M_\lambda)_{U_\lambda}, \alpha_\lambda)^{\log} \xrightarrow{\sim} \underline{Q}_{|U_\lambda}$; whence a cocartesian diagram (12.1.19) consisting of integral S-monoids. The induced diagram $(12.1.19)^{\mathrm{sat}}$ is still cocartesian; one may then argue as in the foregoing, to obtain a natural isomorphism : $\underline{Q}_{|U_\lambda} \xrightarrow{\sim} (M_\lambda^{\mathrm{sat}})_{U_\lambda}^{\log}$. Furthermore, if M_λ is finitely generated, the same holds for M_λ^{sat} (corollary 6.4.1(ii)), hence the chart thus obtained shall be fine and saturated, in this case.

Conversely, if a family $(P_\lambda \mid \lambda \in \Lambda)$ of integral (resp. saturated) monoids can be found fulfilling the condition of (iii), then $(P_\lambda)_{U_\lambda}$ is an integral (resp. saturated) pre-log structure on T/U_λ (example 4.8.47(ii)), hence the same holds for its associated log structure $\underline{Q}_{|U_\lambda}$ (lemma 12.1.13(i)), and thus also for $\underline{Q}$ (lemma 4.8.46(ii), example 4.7.12, and example 12.1.12(iii)). Moreover, if each P_λ is fine, then $\underline{Q}$ is fine as well.

(iv.a): We may assume that $\underline{P}$ admits a finite chart $\alpha : M_S \to \underline{P}$, for some finitely generated monoid M, denote by $\beta : \underline{Q} \to \mathscr{O}_S$ the structure map of $\underline{Q}$, and set $\omega' := \omega \circ \alpha_\xi : M \to \underline{Q}_\xi$. According to lemma 6.1.7(ii), the morphism ω' factors through a map $\omega'' : M \to \Gamma(U', \underline{Q})$, for some neighborhood U' of ξ. By adjunction, ω'' determines a morphism of S/U'-monoids $\psi : M_{U'} \to \underline{Q}_{|U'}$ whence a morphism of pre-log structures

$$(M_{U'}, \beta_{|U'} \circ \psi) \to (\underline{Q}_{|U'}, \beta_{|U'}). \tag{12.1.20}$$

Let us make the following general observation :

Claim 12.1.21. Let N be a finitely generated monoid, F any S-monoid, $f, g : N_S \to F$ two morphisms of S-monoids, such that $f_\xi = g_\xi$. Then there exists a neighborhood U of ξ in S such that $f_{|U} = g_{|U}$.

Proof of the claim. By adjunction, the morphisms f and g correspond to unique maps of monoids $\Gamma(f), \Gamma(g) : N \to \Gamma(S, F)$; since N is finitely generated and $f_\xi = g_\xi$, we may find a neighborhood U of ξ such that the maps $N \to F(U)$ induced by $\Gamma(f)$ and $\Gamma(g)$ coincide. Again by adjunction, we deduce a unique morphism of S/U-monoids $N_U \to F_{|U}$, which by construction is just the restriction of both f and g. ◊

Let $\gamma : \underline{P} \to \mathscr{O}_S$ be the structure map of $\underline{P}$; we apply claim 12.1.21 with S replaced by S/U', to deduce that there exists a small enough neighborhood $U \to U'$ of ξ such that the restriction $(\gamma \circ \alpha)_{|U} : M_{|U} \to \mathscr{O}_{S|U}$ agrees with $\beta_{|U} \circ \psi_{|U}$. Then it is clear that the morphism of log structure associated to (12.1.20)$_{|U}$ yields the sought extension ϑ of ω.

(iv.b): By assumption we have the identity : $\vartheta_\xi = \vartheta'_\xi$; however, any morphism of log structures $\underline{P}_{|U} \to \underline{Q}_{|U}$ is already determined by its restriction to the image of any finite local chart $M_U \to \underline{P}_{|U}$. Hence the assertion follows from claim 12.1.21.

(iv.c): We apply (iv.a) to ω^{-1} to deduce the existence of a morphism $\sigma : \underline{Q}_{|U} \to \underline{P}_{|U}$ such that $\sigma_\xi = \omega^{-1}$ on some neighborhood U of ξ. Hence, $(\vartheta \circ \sigma)_\xi = \mathbf{1}_{\underline{Q}_\xi}$ and $(\sigma \circ \vartheta)_\xi = \mathbf{1}_{\underline{P}_\xi}$. By (iv.b), these identities persist on some smaller neighborhood.

(v): The proof is similar to that of (iv.a), though simpler : we leave it as an exercise for the reader. □

Definition 12.1.22. (i) A *morphism* $(T, \underline{M}) \to (S, \underline{N})$ *of topoi with pre-log* (resp. *log*) *structures*, is a pair $f := (f, \log f)$ consisting of a morphism of locally ringed topoi $f : T \to S$, and a morphism

$$\log f : f^* \underline{N} \to \underline{M}$$

of pre-log structures (resp. log structures) on T. We say that f is *log flat* (resp. *saturated*) if $\log f$ is a flat (resp. saturated) morphism of pre-log structures.

(ii) Let $(f, \log f)$ as in (i) be a morphism of log topoi, ξ a T-point; we say that f is *strict at the point* ξ, if $\log f_\xi$ is an isomorphism. We say that f is *strict*, if it is strict at every T-point.

(iii) A *chart* for φ is the datum of charts

$$\omega_P : (P, \beta)_T \to \underline{M} \quad \text{and} \quad \omega_Q : (Q, \gamma)_S \to \underline{N}$$

for $\underline{M}$, respectively $\underline{N}$, and a morphism of monoids $\vartheta : Q \to P$, such that $(f^*\omega_Q, \omega_P, \vartheta)$ is a chart for the morphism $\log f$. We say that such a chart $(\omega_Q, \omega_P, \vartheta)$ is *finite* (resp. *fine*, resp. *flat*, resp. *saturated*) if the same holds for the corresponding chart $(f^*\omega_Q, \omega_P, \vartheta)$ of $\log f$.

Remark 12.1.23. (i) Let $f : (T, \underline{M}) \to (S, \underline{N})$ be a morphism of log topoi, $g : T' \to T$ a morphism of topoi, and $f' : (T', g^*\underline{M}) \to (S, \underline{N})$ the composition of f and the natural morphism of log topoi $(T', g^*\underline{M}) \to (T, \underline{M})$; let also ξ be a T'-point. Then f' is strict at the point ξ if and only if f is strict at the point $g(\xi)$. Indeed, f' is strict at ξ if and only if $(\log f')^\sharp_\xi$ is an isomorphism (lemma 12.1.4), if and only if $(\log f)^\sharp_{g(\xi)}$ is an isomorphism (by (12.1.9)), if and only if $\log f_{g(\xi)}$ is an isomorphism (again by lemma 12.1.4).

(ii) For any log topos $(T, \underline{M})$, let us set $(T, \underline{M})_\circ := (T, \underline{M}_\circ)$. Then $(T, \underline{M})_\circ$ is a log topos (see (12.1.11)), and clearly, every morphism $f : (T, \underline{M}) \to (S, \underline{N})$ of log topoi extends naturally to a morphism $f_\circ : (T, \underline{M})_\circ \to (S, \underline{N})_\circ$ of log topoi.

12.1.24. The 2-category of log topoi admits arbitrary 2-limits indexed by usual categories. Indeed, if $\mathscr{T} := ((T_\lambda, \underline{M}_\lambda) \mid \lambda \in \Lambda)$ is any pseudo-functor (from a small category Λ, to the category of log topoi), the 2-limit of $\mathscr{T}$ is the pair $(T, \underline{M})$, where T is the 2-limit of the system of ringed topoi $(T_\lambda \mid \lambda \in \Lambda)$, and $\underline{M}$ is the colimit of the induced system of log structures $(p^*_\lambda \underline{M}_\lambda \mid \lambda \in \Lambda)$ on T, where $p_\lambda : T \to T_\lambda$ denotes the natural projection, for every $\lambda \in \Lambda$ (see remark 12.1.10).

12.1.25. Consider a 2-cartesian diagram of log topoi :

$$\begin{array}{ccc} (T',\underline{M}') & \xrightarrow{g} & (T,\underline{M}) \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ (S',\underline{N}') & \xrightarrow{h} & (S,\underline{N}). \end{array}$$

The following result yields a relative variant of the isomorphism (12.1.9) :

Lemma 12.1.26. *In the situation of* (12.1.25)*, the morphism*

$$g^*\mathrm{Coker}(\log f) \to \mathrm{Coker}(\log f')$$

induced by $\log g$ *is an isomorphism of* T'*-monoids.*

Proof. Indeed, denote by $\beta : \underline{M} \to \mathscr{O}_T$ and $\gamma : \underline{N}' \to \mathscr{O}_{S'}$ the log structures of T and S'. Fix any T'-point ξ, let $\xi := g(\xi')$, and set

$$P := \underline{M}_\xi \otimes_{\underline{N}_{f(\xi)}} \underline{N}'_{f'(\xi')} \qquad \text{and} \qquad \rho := \beta_\xi \otimes \gamma_{f'(\xi')} : P \to \mathscr{O}_{T'}.$$

Then $\underline{M}'_{\xi'} = P \otimes_{\rho^{-1}\mathscr{O}_{T',\xi'}} \mathscr{O}^\times_{T',\xi'}$, and it is easily seen that $\rho^{-1}\mathscr{O}_{T',\xi'} = \mathscr{O}^\times_{T,\xi} \otimes_{\mathscr{O}^\times_{S,f(\xi)}} \mathscr{O}^\times_{S',f'(\xi')}$. Therefore $(\underline{M}'_{\xi'})^\sharp = P/\rho^{-1}\mathscr{O}_{T',\xi'} = \underline{M}^\sharp_\xi \otimes_{\underline{N}^\sharp_{f(\xi)}} \underline{N}'^\sharp_{f'(\xi')}$, and

$$\mathrm{Coker}(\log f'_{\xi'}) = \mathrm{Coker}(\underline{N}'^\sharp_{f'(\xi')} \to \underline{M}^\sharp_\xi \otimes_{\underline{N}^\sharp_{f(\xi)}} \underline{N}'^\sharp_{f'(\xi')}) = \mathrm{Coker}(\underline{N}^\sharp_{f(\xi)} \to \underline{M}^\sharp_\xi)$$

whence the contention. ◻

12.1.27. We consider now a special situation that we will encounter in proposition 12.1.30. Namely, let Q be a monoid, and $H \subset Q^\times$ a subgroup. Let also G be an abelian group, $\rho : G \to Q^{\mathrm{gp}}$ a group homomorphism, and set :

$$H_\rho := G \times_{Q^{\mathrm{gp}}} H \qquad Q_\rho := G \times_{Q^{\mathrm{gp}}} Q.$$

The natural inclusion $H \to Q$ and the projection $Q_\rho \to Q$ determine a unique morphism :

$$Q_\rho \otimes_{H_\rho} H \to Q. \tag{12.1.28}$$

Lemma 12.1.29. *In the situation of* (12.1.27)*, suppose furthermore that the composition :*

$$G \xrightarrow{\rho} Q^{\mathrm{gp}} \to (Q/H)^{\mathrm{gp}}$$

is surjective. Then (12.1.28) *is an isomorphism.*

Proof. Set $G' := G \oplus H$, and let $\rho' : G' \to Q^{\mathrm{gp}}$ be the unique group homomorphism that extends ρ and the natural map $H \to Q \to Q^{\mathrm{gp}}$. Under the standing assumptions, ρ' is clearly surjective. Define $Q_{\rho'}$ and $H_{\rho'}$ as in (12.1.27); there is a natural isomorphism of monoids : $Q_{\rho'} \xrightarrow{\sim} Q_\rho \oplus H$, inducing an isomorphism $H_{\rho'} \xrightarrow{\sim} H_\rho \oplus H$, and defined as follows. To every $g \in G$, $h \in H$, $q \in Q$ such that $[(g,h),q] \in Q_{\rho'}$, we assign the element $[(g,h^{-1}q),a] \in Q_\rho \oplus H$ (details left to the reader). Under this isomorphism, the projection $H_{\rho'} \to H$ is identified with the map $H_\rho \oplus H \to H$ given by the rule : $(h_1,h_2) \mapsto \pi_H(h_1)\cdot h_2$, where $\pi_H : H_\rho \to H$ is the projection. It then follows that the natural map :

$$Q_\rho \otimes_{H_\rho} H \to Q_{\rho'} \otimes_{H_{\rho'}} H$$

is an isomorphism. Thus, we may replace G and ρ by G' and ρ', which allows to assume from start that ρ is surjective. However, we have natural isomorphisms :

$$\mathrm{Ker}(Q_\rho \xrightarrow{\pi_Q} Q) \simeq \mathrm{Ker}\,\rho \simeq \mathrm{Ker}(H_\rho \xrightarrow{\pi_H} H).$$

Moreover, the set underlying Q (resp. H) is the set-theoretic quotient of the set Q_ρ (resp. H_ρ) by the translation action of $\mathrm{Ker}\,\rho$; hence the natural maps $Q_\rho/\mathrm{Ker}\,\pi_Q \to Q$ and $H_\rho/\mathrm{Ker}\,\pi_H \to H$ are isomorphisms (lemma 4.8.31(ii)). We can then compute :

$$Q_\rho \otimes_{H_\rho} H \simeq (Q_\rho/\mathrm{Ker}\,\pi_Q) \otimes_{H_\rho/\mathrm{Ker}\,\pi_H} H \simeq Q \otimes_H H \simeq Q$$

as stated. □

Proposition 12.1.30. *Let T be a locally ringed topos, ξ any T-point, and $\underline{M}$ a coherent (resp. fine) log structure on T. Suppose that G is a finitely generated abelian group with a group homomorphism $G \to \underline{M}^{\mathrm{gp}}_\xi$ such that the induced map $G \to (\underline{M}^\sharp)^{\mathrm{gp}}_\xi$ is surjective. Set*

$$P := G \times_{\underline{M}^{\mathrm{gp}}_\xi} \underline{M}_\xi.$$

Then the induced morphism $P \to \underline{M}_\xi$ extends to a finite (resp. fine) chart $P_U \to \underline{M}_{|U}$ on some neighborhood U of ξ.

Proof. We begin with the following :

Claim 12.1.31. Let Y be any locally ringed topos, ξ a Y-point, and $\alpha : Q_Y \to \mathscr{O}_Y$ the constant pre-log structure associated to a map of monoids $\vartheta : Q \to \Gamma(Y, \mathscr{O}_Y)$, where Q is finitely generated. Set $S := \alpha_\xi^{-1}\mathscr{O}^\times_{Y,\xi} \subset Q_{Y,\xi} = Q$. Then :

(i) S and $S^{-1}Q$ are finitely generated monoids.
(ii) There exists a neighborhood U of ξ such that $\alpha_{|U}$ factors as the composition of the natural map of sheaves of monoids $j_U : Q_U \to (S^{-1}Q)_U$, and a (necessarily unique) pre-log structure $\alpha_S : (S^{-1}Q)_U \to \mathscr{O}_U$.
(iii) The induced map of log structures $j_U^{\log} : Q_U^{\log} \to (S^{-1}Q)_U^{\log}$ is an isomorphism.
(iv) $\alpha^{-1}_{S,\xi}(\mathscr{O}^\times_{U,\xi}) = (S^{-1}Q)^\times$ is a finitely generated group.

Proof of the claim. (i) follows from lemma 6.1.20(iv).

(ii): Since $\mathscr{O}^\times_{Y,\xi}$ is the filtered colimit of the groups $\Gamma(U, \mathscr{O}^\times_U)$, where U ranges over the neighborhoods of ξ, lemma 6.1.7(ii) and (i) imply that the induced map $S \to \mathscr{O}^\times_{Y,\xi}$ factors through $\Gamma(U, \mathscr{O}^\times_Y)$ for some neighborhood U of ξ. Then the composition of ϑ and the natural map $\Gamma(Y, \mathscr{O}_Y) \to \Gamma(U, \mathscr{O}_U)$ extends to a unique map $S^{-1}Q \to \Gamma(U, \mathscr{O}_U)$, whence the sought pre-log structure α_S on U.

(iii): Let $\underline{N}$ be any log structure on U; it is clear that every morphism of pre-log structures $Q_U \to \underline{N}$ factors uniquely through $(S^{-1}Q)_U$, whence the contention.

(iv): Indeed, by construction we have : $\alpha^{-1}_{S,\xi}(\mathscr{O}^\times_{U,\xi}) = S^{\mathrm{gp}}$. ◊

Let Y be a neighborhood of ξ such that $\underline{M}_{|Y}$ admits a finite local chart $\alpha : Q_Y \to \mathscr{O}_Y$; we lift ξ to some Y-point, ξ_Y, and choose a neighborhood $U \in \mathrm{Ob}(T/Y)$ of ξ_Y, as provided by claim 12.1.31. We may then replace T by U, ξ by ξ_Y and α by the chart α_S of claim 12.1.31(ii), which allows to assume from start that $S := \alpha_\xi^{-1}(\mathscr{O}^\times_{T,\xi})$ is a finitely generated group. Moreover, let $H := \mathrm{Ker}(S \to \mathscr{O}^\times_{T,\xi})$; clearly $\alpha_\xi : Q \to \mathscr{O}_{T,\xi}$ factors through the quotient $Q' := Q/H$, hence we may find a neighborhood U of ξ such that $\alpha_{|U}$ factors through a (necessarily unique) map of pre-log structures $\alpha_H : Q'_U \to \mathscr{O}_U$. Furthermore, if $\underline{N}$ is any log structure on U, every map of pre-log structures $Q_U \to \underline{N}$ factors through Q'_U, so that α_H is a chart for $\underline{M}_{|U}$. We may therefore replace T by U and α by α_H, which allows to assume additionally, that α_ξ is injective on the subgroup S. Now, for any finitely generated subgroup $H \subset \mathscr{O}^\times_{T,\xi}$ with $S \subset H$, we set $\underline{M}_{\xi,H} := H \amalg_S Q$; clearly, the monoids $\underline{M}_{\xi,H}$ are finitely generated, and moreover :

$$\underline{M}_\xi = \operatorname*{colim}_{S \subset H \subset \mathscr{O}^\times_{T,\xi}} \underline{M}_{\xi,H}.$$

Furthermore, we deduce a natural sequence of maps of monoids :

$$\{1\} \to \underline{M}_{\xi,H} \xrightarrow{\varphi_H} \underline{M}_\xi \xrightarrow{\psi_H} \mathscr{O}^\times_{T,\xi}/H \to \{1\}. \qquad (12.1.32)$$

Claim 12.1.33. For every subgroup H as above, the sequence (12.1.32) is exact, *i.e.* $\underline{M}_{\xi,H}$ is the kernel of ψ_H, and $\mathscr{O}^\times_{T,\xi}/H$ is the cokernel of φ_H.

Proof of the claim. By lemma 4.8.29(iii), the assertion concerning $\operatorname{Ker}\psi_H$ can be verified on the underlying map of sets; however, lemma 4.8.31(ii) says that the set $\underline{M}_\xi$ is the set-theoretic quotient $(Q \times \mathscr{O}^\times_{T,\xi})/S$, for the natural translation action of S, and a similar description holds for $\underline{M}_{\xi,H}$, therefore $\operatorname{Ker}\varphi_H$ is the set-theoretic quotient $(H \times Q)/S$, as required.

The assertion concerning $\operatorname{Coker}\varphi_H$ holds by general categorical nonsense. ◊

Let $\varepsilon : \underline{M}_\xi \to \underline{M}^{\mathrm{gp}}_\xi$ be the natural map; claim 12.1.33 implies that the sequence of abelian groups (12.1.32)$^{\mathrm{gp}}$ is right exact, and then a little diagram chase shows that :

$$\varepsilon^{-1}(\operatorname{Im}\varphi^{\mathrm{gp}}_H) = \underline{M}_{\xi,H} \qquad \text{whenever } S \subset H \subset \mathscr{O}^\times_{T,\xi}. \tag{12.1.34}$$

Since G is finitely generated, we may find H as above, large enough, so that $\underline{M}^{\mathrm{gp}}_{\xi,H}$ contains the image of G. In view of (12.1.34), we deduce that the natural map

$$G \times_{\underline{M}^{\mathrm{gp}}_{\xi,H}} \underline{M}_{\xi,H} \to P$$

is an isomorphism, so P is finitely generated, by corollary 6.4.2; moreover P is integral whenever $\underline{M}_\xi$ is. Then, lemma 12.1.18(iv.a) implies that the natural map $P \to \underline{M}_\xi$ extends to a morphism of log structures $\vartheta : P^{\log}_U \to \underline{M}_{|U}$ on some neighborhood U of ξ. It remains to show that ϑ restricts to a chart for $\underline{M}_{|V}$, on some smaller neighborhood V of ξ. To this aim, it suffices to show that the map of stalks ϑ_ξ is an isomorphism (lemma 12.1.18(iv.c)). The latter assertion follows from lemma 12.1.29. □

Proposition 12.1.30 is the basis of several frequently used tricks that allow to construct “good” charts for a given coherent log structure (and for a morphism of such structures), or to “improve” given charts. We conclude this section with a selection of these tricks.

Corollary 12.1.35. *Let T be a topos, ξ a T-point, $\underline{M}$ a fine log structure on T. Then there exist a neighborhood U of ξ in T, and a chart $P_U \to \underline{M}_{|U}$ such that :*

(a) *P^{gp} is a free abelian group of finite rank.*
(b) *The induced morphism of monoids $P \to \mathscr{O}_{T,\xi}$ is local.*

Proof. Choose a group homomorphism $G := \mathbb{Z}^{\oplus n} \to \underline{M}^{\mathrm{gp}}_\xi$ (for some integer $n \geq 0$), such that the induced map $G \to (\underline{M}^\sharp)^{\mathrm{gp}}_\xi$ is surjective, and set $P := G \times_{\underline{M}^{\mathrm{gp}}_\xi} \underline{M}_\xi$. By proposition 12.1.30, the induced map $P \to \underline{M}_\xi$ extends to a chart $P_U \to \underline{M}_{|V}$, for some neighborhood U of ξ. According to example 4.8.36(v), P^{gp} is a subgroup of G, whence (a). Next, claim 12.1.31 implies that, after replacing P by some localization (which does not alter P^{gp}), and U by a smaller neighborhood of ξ, we may achieve (b) as well. □

Corollary 12.1.36. *Let T be a topos, ξ a T-point, $\underline{M}$ a coherent log structure on T, and suppose that $\underline{M}_\xi$ is integral and saturated. Then we have :*

(i) *There exist a neighborhood U of ξ in T, and a fine and saturated chart $P_U \to \underline{M}_{|U}$ which is sharp at the point ξ.*
(ii) *Especially, there exists a neighborhood U of ξ in T, such that $\underline{M}_{|U}$ is a fine and saturated log structure.*

Proof. (i): By lemma 6.2.10, we may find a decomposition $\underline{M}_\xi = P \times \underline{M}^\times_\xi$, for a sharp submonoid $P \subset \underline{M}_\xi$. Set $G := P^{\mathrm{gp}}$; we deduce an isomorphism $G \xrightarrow{\sim} \underline{M}^{\mathrm{gp}}_\xi/\underline{M}^\times_\xi$, and clearly $P = G \times_{\underline{M}^{\mathrm{gp}}_\xi} \underline{M}_\xi$. By proposition 12.1.30, it follows that the induced map $P \to \underline{M}_\xi$ extends to a chart $\beta : P_U \to \underline{M}_{|U}$ on a neighborhood U of ξ. By construction, β is sharp at the T-point ξ; moreover, since $\underline{M}_\xi$ is saturated, it is easily seen that the same holds for P. Finally, P is finitely generated, by corollary 6.4.2.

(ii): This follows immediately from (i) and lemma 12.1.18(iii). □

Theorem 12.1.37. *Let T be a locally ringed topos, ξ a T-point, $f : \underline{M} \to \underline{N}$ a morphism of coherent (resp. fine) log structures on T. Then :*

(i) *There exists a neighborhood U of ξ, such that $f_{|U}$ admits a finite (resp. fine) chart.*
(ii) *More precisely, given a finite (resp. fine) chart $\omega_P : P_T \to \underline{M}$, we may find a neighborhood U of ξ, a finite (resp. fine) monoid Q, and a chart of $f_{|U}$ of the form*

$$(\omega_{P|U}, \omega_Q : Q_U \to \underline{N}_{|U}, \vartheta : P \to Q).$$

(iii) *Moreover, if f is a flat (resp. saturated) morphism of fine log structures and $(\omega_P, \omega_Q, \vartheta)$ is a fine chart for f, then we may find a neighborhood U of ξ, a localization map $j : Q \to Q'$, and a flat (resp. saturated) and fine chart for $f_{|U}$ of the form $(\omega_{P|U}, \omega_{Q'}, j \circ \vartheta)$, such that*
  (a) $\omega_{Q|U} = \omega_{Q'} \circ j_U$.
  (b) *The chart $\omega_{Q'}$ is local at the point ξ.*

Proof. Up to replacing T by T/U'_i for a covering $(U'_i \to 1_T \mid i \in I)$ of the final object, we may assume that we have finite (resp. fine) charts $\omega_P : P_T \to \underline{M}$ and $Q'_T \to \underline{N}$ (lemma 12.1.18(iii)), whence a morphism of pre-log structures :

$$\omega : P_T \xrightarrow{\omega_P} \underline{M} \xrightarrow{f} \underline{N}.$$

Let ξ be any T-point; there follow maps of monoids $\varphi : P \to \underline{M}_\xi \to \underline{N}_\xi$ and $\psi : Q' \to \underline{N}_\xi$. Set $G := (P \oplus Q')^{\mathrm{gp}}$, and apply proposition 12.1.30 to the group homomorphism $G \to \underline{N}_\xi^{\mathrm{gp}}$ induced by φ and ψ; for $Q := G \times_{\underline{N}_\xi^{\mathrm{gp}}} \underline{N}_\xi$, we obtain a finite (resp. fine) local chart $Q_U \to \underline{N}_{|U}$ on some neighborhood U of ξ. Then, φ and the natural map $P \to G$ determine a unique map $P \to Q$, whence a morphism $\omega' : P_U \to Q_U \to \underline{N}_{|U}$ of pre-log structures; by construction, $\omega'_\xi : P = P_{U,\xi} \to \underline{N}_\xi$ is none else than φ. By lemma 12.1.18(iv.b), we may then find a smaller neighborhood $V \to U$ of ξ such that $\omega'_{|V} = \omega_{|V}$. This proves (i) and (ii).

Next, we suppose that f is flat (resp. saturated) and both $\underline{M}$, $\underline{N}$ are fine, and we wish to show (iii). In view of claim 12.1.31, we may find – after replacing T by a neighborhood of ξ – a fine chart for f of the form $(\omega_{P'}, \omega_{Q'}, \vartheta')$, such that :

- P' and Q' are localizations of P and Q, and ϑ' is induced by ϑ;
- $\omega_P = \omega_{P'} \circ (j_P)_T$ and $\omega_Q = \omega_{Q'} \circ (j_Q)_T$, where $j_P : P \to P'$ and $j_Q : Q \to Q'$ are the localization maps;
- the induced maps $Q'^\sharp \to \underline{M}_\xi^\sharp$ and $P'^\sharp \to \underline{N}_\xi^\sharp$ are isomorphisms.

Now, by proposition 4.8.26 (resp. corollary 6.2.29), the map f_ξ is flat (resp. saturated), hence the same holds for the induced map $\underline{M}_\xi^\sharp \to \underline{N}_\xi^\sharp$, by corollary 6.1.49(i) (resp. by lemma 6.2.12(iii)). Then the map $P'^\sharp \to Q'^\sharp$ induced by ϑ' is flat (resp. saturated) as well, so the same holds for ϑ', by corollary 6.1.49(ii) (resp. again by lemma 6.2.12(iii)). Finally, $j_P \circ \vartheta' : P \to Q'$ is flat (resp. saturated), by example 6.1.23(iii) (resp. by lemma 6.2.12(i)), and $(\omega_P, \omega_{Q'}, j_P \circ \vartheta')$ is a chart for f with the sought properties. □

Corollary 12.1.38. *Let T be a topos, ξ a T-point, and $\varphi : \underline{M} \to \underline{N}$ a saturated morphism of fine and saturated log structures on T. We have :*

(i) *There exist a neighborhood U of ξ, and a fine and saturate chart $(\omega_P, \omega_Q, \vartheta : P \to Q)$ of $\varphi_{|U}$, such that ω_P and ω_Q are sharp at the point ξ.*
(ii) *More precisely, suppose that $(\omega_P, \omega_Q, \vartheta : P \to Q)$ is a fine and saturated chart for φ, such that $\underline{M}$ is sharp at the point ξ, and ω_Q is local at ξ. Then there exists a section $\sigma : Q^\sharp \to Q$ of the projection $Q \to Q^\sharp$, such that $(\omega_P, \omega_Q \circ \sigma_T, \vartheta^\sharp)$ is a chart for φ.*

Proof. After replacing T by some neighborhood of ξ, we may assume that $\underline{M}$ admits a chart $\omega_P : P_T \to \underline{M}$ which is fine, saturated, and sharp at the point ξ (corollary 12.1.36(i)). Then, by theorem 12.1.37(iii), we may find a neighborhood U of ξ, and a fine and saturated chart $(\omega_{P|U}, \omega_Q, \vartheta : P \to Q)$ for $\varphi_{|U}$, such that ω_Q is local at ξ, so that ϑ is also a local morphism. Hence, it suffices to show assertion (ii).

(ii): We notice the following :

Claim 12.1.39. Let $\vartheta : P \to Q$ a local and saturated morphism of fine and saturated monoids, and suppose that P is sharp. Then there exists a section $\sigma : Q^\sharp \to Q$ of the projection $Q \to Q^\sharp$, such that $\vartheta(P)$ lies in the image of σ.

Proof of the claim. Pick an isomorphism $\beta : Q \xrightarrow{\sim} Q^\sharp \times Q^\times$ as in lemma 6.2.10, and denote by $\psi : P \to Q^\times$ the composition of ϑ with the induced projection $Q \to Q^\times$. The morphism $\vartheta^\sharp$ is still local and saturated (lemma 6.2.12(iii)), hence corollary 6.2.32(ii) implies that $\vartheta^{\sharp\mathrm{gp}}$ extends to an isomorphism $P^{\mathrm{gp}} \oplus L \xrightarrow{\sim} Q^{\sharp\mathrm{gp}}$, where L is a free abelian group of finite rank. Thus, we may extend ψ^{gp} to a group homomorphism $\psi' : Q^{\sharp\mathrm{gp}} \to Q^\times$. Define an automorphism α of $Q^\sharp \times Q^\times$, by the rule : $(x, g) \mapsto (x, g \cdot \psi'(x)^{-1})$. The restriction $\sigma : Q^\sharp \to Q$ of $(\alpha \circ \beta)^{-1}$ will do. ◊

With the notation of claim 12.1.39 it is easily seen that $\omega_Q \circ \sigma_T$ is still a chart for $\underline{N}$, hence $(\omega_P, \omega_Q \circ \sigma_T, \vartheta^\sharp)$ is a chart for φ as required. □

Corollary 12.1.40. *Let $f : (T, \underline{M}) \to (S, \underline{N})$ a morphism of log topoi with coherent (resp. fine) log structures, and suppose that $\underline{N}$ admits a finite (resp. fine) chart $\omega_Q : Q_S \to \underline{N}$. Let also ξ be any T-point; we have :*

- (i) *There exist a neighborhood U of ξ, and a finite (resp. fine) chart $(\omega_{Q|U}, \omega_P, \vartheta : Q \to P)$ for the morphism $f_{|U}$.*
- (ii) *Moreover, if $\underline{M}$ and $\underline{N}$ are fine and f is log flat (resp. saturated) then, on some neighborhood U of ξ, we may also find a chart $(\omega_{Q|U}, \omega_P, \vartheta)$ which is flat (resp. saturated) and fine.*

Proof. This is an immediate consequence of theorem 12.1.37. □

12.2. **Log schemes.** We specialize now to the case of a scheme X. Whereas in [75, §6.4] we considered only pre-log structures on the Zariski site of a scheme, hereafter we shall treat uniformly the categories of log structures on the topoi $X^\sim_{\text{ét}}$ and $X^\sim_{\mathrm{Zar}}$ (notation of (4.9.13)).

12.2.1. Henceforth, we choose $\tau \in \{\text{ét}, \mathrm{Zar}\}$ (see (4.9.19)), and whenever we mention a topology on a scheme X, it will be implicitly meant that this is the topology X_τ (unless explicitly stated otherwise). Let X be a scheme; a *pre-log structure* (resp. *a log structure*) on X is a pre-log structure (resp. a log structure) on the topos $X^\sim_\tau$. The datum of a scheme X and a log structure on X is called briefly a *log scheme*. It is known that a morphism of schemes $X \to Y$ is the same as a morphism of locally ringed topoi $X^\sim_\tau \to Y^\sim_\tau$, hence we may define a morphism of log schemes $(X, \underline{M}) \to (Y, \underline{N})$ as a morphism of log topoi $(X^\sim_\tau, \underline{M}) \to (Y^\sim_\tau, \underline{N})$ (and likewise for morphisms of schemes with pre-log structures). We denote by $\mathbf{pre\text{-}log}_\tau$ (resp. $\mathbf{log}_\tau$) the category of schemes with pre-log structures (resp. of log schemes) on the chosen topology τ. We denote by :

$$\mathbf{int.log}_\tau \qquad \mathbf{sat.log}_\tau \qquad \mathbf{qcoh.log}_\tau \qquad \mathbf{coh.log}_\tau$$

the full subcategory of the category $\mathbf{log}_\tau$, consisting of all log schemes with integral (resp. integral and saturated, resp. quasi-coherent, resp. coherent) log structures.

A scheme with a quasi-fine (resp. fine, resp. quasi-fine and saturated, resp. fine and saturated) log structure is called, briefly, a *quasi-fine log scheme* (resp. a *fine log scheme*, resp. a *qfs log* *scheme*, resp. a *fs log scheme*), and we denote by

$$\mathbf{qf.log}_\tau \qquad \mathbf{f.log}_\tau \qquad \mathbf{qfs.log}_\tau \qquad \mathbf{fs.log}_\tau$$

the full subcategory of $\mathbf{log}_\tau$ consisting of all quasi-fine (resp. fine, resp. qfs, resp. fs) log schemes on the topology τ. In case it is clear (or indifferent) which topology we are dealing with, we will usually omit the subscript τ. There is an obvious (forgetful) functor :

$$F : \mathbf{log} \to \mathsf{Sch}$$

to the category of schemes, and it is easily seen that this functor is a fibration. For every scheme X, we denote by $\mathbf{log}_X$ the fibre category $F^{-1}(X)$ *i.e.* the category of all log structures on X (or $\mathbf{log}_{X_\tau}$, if we need to specify the topology τ). The same notation shall be used also for the various subcategories : so for instance we shall write $\mathbf{int.log}_X$ for the full subcategory of all integral log structures on X. Moreover, we shall say that the log scheme $(X, \underline{M})$ is *locally noetherian* if the underlying scheme X is locally noetherian.

12.2.2. Most of the forthcoming assertions hold in both the étale and Zariski topoi, with the same proof. However, it may occasionally happen that the proof of some assertion concerning $X_\tau^\sim$ (for $\tau \in \{\text{ét}, \text{Zar}\}$), is easier for one choice or the other of these two topologies; in such cases, it is convenient to be able to change the underlying topology, to suit the problem at hand. This is sometimes possible, thanks to the following general considerations.

The morphism of locally ringed topoi $\widetilde{u}$ of (4.9.15) induces a pair of adjoint functors :

$$\widetilde{u}^* : \mathbf{log}_{\text{Zar}} \to \mathbf{log}_{\text{ét}} \qquad \widetilde{u}_* : \mathbf{log}_{\text{ét}} \to \mathbf{log}_{\text{Zar}}$$

as well as analogous adjoint pairs for the corresponding categories of sheaves of monoids (resp. of pre-log structures) on the two sites. It follows formally that $\widetilde{u}^*$ sends constant log structures to constant log structures, *i.e.* for every scheme X, and every object $M := (M, \varphi)$ of $\mathbf{Mnd}/\Gamma(X, \mathscr{O}_X)$ we have a natural isomorphism :

$$\widetilde{u}^*(X_{\text{Zar}}, M^{\log}_{X_{\text{Zar}}}) \simeq (X_{\text{ét}}, M^{\log}_{X_{\text{ét}}}).$$

More generally, lemma 12.1.18(i) shows that $\widetilde{u}^*$ preserves the subcategories of quasi-coherent (resp. coherent, resp. integral, resp. fine, resp. fine and saturated) log structures.

Proposition 12.2.3. (i) *The functor $\widetilde{u}^*$ on log structures is faithful and conservative.*

(ii) *The functor $\widetilde{u}^*$ restricts to a fully faithful functor :*

$$\widetilde{u}^* : \mathbf{int.log}_{\text{Zar}} \to \mathbf{int.log}_{\text{ét}}.$$

(iii) *Let $(X_{\text{ét}}, \underline{M})$ be any log scheme. Then the counit of adjunction $\widetilde{u}^*\widetilde{u}_*(X_{\text{ét}}, \underline{M}) \to (X_{\text{ét}}, \underline{M})$ is an isomorphism if and only if the same holds for the counit of adjunction $\widetilde{u}^*\widetilde{u}_*(\underline{M}^\sharp) \to \underline{M}^\sharp$.*

Proof. (i): Let $(X_{\text{Zar}}, \underline{M})$ be any log structure; set $(X_{\text{ét}}, \underline{M}_{\text{ét}}) := \widetilde{u}^*(X, \underline{M})$, and denote by $\widetilde{u}^\natural_X : \widetilde{u}^*\mathscr{O}_{X_{\text{Zar}}} \to \mathscr{O}_{X_{\text{ét}}}$ the natural map of structure rings. Since $\widetilde{u}$ is a morphism of locally ringed topoi, we have :

$$(\widetilde{u}^\natural_X)^{-1}\mathscr{O}^\times_{X_{\text{ét}}} = \widetilde{u}^*(\mathscr{O}^\times_{X_{\text{Zar}}}).$$

It follows easily that $\underline{M}_{\text{ét}}$ is the push-out in the cocartesian diagram :

$$\begin{array}{ccc} \widetilde{u}^*\underline{M}^\times & \xrightarrow{\alpha} & \mathscr{O}^\times_{X_{\text{ét}}} \\ \downarrow & & \downarrow \\ \widetilde{u}^*\underline{M} & \xrightarrow{\beta} & \underline{M}_{\text{ét}}. \end{array}$$

However, for every geometric point ξ of X, the natural map $\mathscr{O}_{X,|\xi|} \to \mathscr{O}_{X,\xi}$ is faithfully flat, hence α_ξ is injective, and therefore also β_ξ, in light of lemma 4.8.31(ii). The faithfulness of $\widetilde{u}^*$

is an easy consequence. Next, let $f : \underline{M} \to \underline{N}$ be a morphism of log structures on X_{Zar}, set $(X_{\text{ét}}, \underline{N}_{\text{ét}}) := \widetilde{u}^*(X, \underline{N})$, and suppose that $\widetilde{u}^* f : \underline{M}_{\text{ét}} \to \underline{N}_{\text{ét}}$ is an isomorphism; we wish to show that f is an isomorphism. However, β induces an isomorphism of monoids :

$$\widetilde{u}^*(\underline{M}^\sharp) \xrightarrow{\sim} \underline{M}^\sharp_{\text{ét}} \tag{12.2.4}$$

and likewise for $\underline{N}$; it follows already that f induces an isomorphism $\underline{M}^\sharp \xrightarrow{\sim} \underline{N}^\sharp$. To conclude, it suffices to invoke lemma 12.1.4.

(ii): Let us suppose that $\underline{M}$ is integral. According to [28, Prop.3.4.1], it suffices to show that the unit of adjunction $(X, \underline{M}) \to \widetilde{u}_*(X_{\text{ét}}, \underline{M}_{\text{ét}})$ is an isomorphism. Now, from the isomorphism (12.2.4) we deduce a commutative diagram :

$$\begin{array}{ccc} \underline{M}^\sharp & \longrightarrow & (\widetilde{u}_*\underline{M}_{\text{ét}})^\sharp \\ \downarrow & & \downarrow \\ \widetilde{u}_*\widetilde{u}^*(\underline{M}^\sharp) & \longrightarrow & \widetilde{u}_*(\underline{M}^\sharp_{\text{ét}}) \end{array}$$

whose bottom arrow is an isomorphism, and whose left vertical arrow is an isomorphism as well, by lemma 4.9.27(iii) (and again [28, Prop.3.4.1]). We claim that also the right vertical arrow is an isomorphism. Indeed, since $\widetilde{u}_*$ is left exact, the latter arrow is a monomorphism, hence it suffices to show it is an epimorphism; however, since $\underline{M}_{\text{ét}}$ is an integral log structure (lemma 12.1.18(i)), it is easily seen that the projection $\underline{M}_{\text{ét}} \to \underline{M}^\sharp_{\text{ét}}$ is a $\mathscr{O}^\times_{X_{\text{ét}}}$-torsor (in the topos $X^\sim_{\text{ét}}/\underline{M}^\sharp_{\text{ét}}$). Then the contention follows from the exact sequence of pointed sheaves (4.9.12), and the vanishing result of lemma 4.9.27(iv).

Summing up, we conclude that the top horizontal arrow in the above diagram is an isomorphism, so the assertion follows from lemma 12.1.4.

(iii): Set $(X_{\text{ét}}, (\widetilde{u}_*\underline{M})_{\text{ét}}) := \widetilde{u}^*\widetilde{u}_*(X_{\text{ét}}, \underline{M})$. To begin with, lemma 4.9.27(iv) easily implies that the natural morphism $(\widetilde{u}_*\underline{M})^\sharp \to \widetilde{u}_*(\underline{M}^\sharp)$ is an isomorphism; together with the general isomorphism (12.1.9), this yields a short exact sequence of $X_{\text{ét}}$-monoids :

$$0 \to \mathscr{O}^\times_{X_{\text{ét}}} \to (\widetilde{u}_*\underline{M})_{\text{ét}} \to \widetilde{u}^*\widetilde{u}_*(\underline{M}^\sharp) \to 0$$

which easily implies the assertion : the details shall be left to the reader. □

We shall prove later on some more results in the same vein (see corollary 12.2.24).

12.2.5. Arguing as in (12.1.24), we see easily that all finite limits are representable in the category of log schemes. The rule $X \mapsto (X, \mathscr{O}^\times_X)$ defines a fully faithful inclusion of the category of schemes into the category of log schemes. Hence, we shall regard a scheme as a log scheme with trivial log structure. Especially, if $(X, \underline{M})$ is any log scheme, and $Y \to X$ is any morphism of schemes, we shall often use the notation :

$$Y \times_X (X, \underline{M}) := (Y, \mathscr{O}^\times_Y) \times_{(X, \mathscr{O}^\times_X)} (X, \underline{M}). \tag{12.2.6}$$

Especially, if ξ is any τ-point of X, the *localization* of $(X, \underline{M})$ at ξ is the log scheme

$$(X(\xi), \underline{M}(\xi)) := X(\xi) \times_X (X, \underline{M})$$

(see definition 4.9.17(ii,iii)). If $\tau = \text{ét}$, this operation is also called the *strict henselization* of X at ξ.

Definition 12.2.7. (i) For every integer $n \in \mathbb{N}$, we have the subset :

$$(X, \underline{M})_n := \{x \in X \mid \dim \underline{M}_\xi \leq n \text{ for every } \tau\text{-point } \xi \to X \text{ localized at } x\}.$$

By remark 6.1.11(i), $(X, \underline{M})_0$ consists of all $x \in X$ such that $\underline{M}_\xi = \mathscr{O}^\times_{X,\xi}$ for every τ-point ξ of X localized at x; this subset is called the *trivial locus* of $(X, \underline{M})$, and is also denoted $(X, \underline{M})_{\mathrm{tr}}$.

(ii) If $f : (X, \underline{M}) \to (Y, \underline{N})$ is a morphism of log schemes, we denote by $\mathrm{Str}(f) \subset X$ the *strict locus* of f, which is the subset consisting of all points $x \in X$ such that f is strict at every τ-point localized at x (see definition 12.1.22(ii)).

Remark 12.2.8. Let $f : (X, \underline{M}) \to (Y, \underline{N})$ be any morphism of log schemes.

(i) Since $\log f$ induces local morphisms on stalks, it is easily seen that f restricts to a map

$$f_{\mathrm{tr}} : (X, \underline{M})_{\mathrm{tr}} \to (Y, \underline{N})_{\mathrm{tr}}.$$

(ii) Especially, we have $(X, \underline{M})_{\mathrm{tr}} \subset \mathrm{Str}(f)$.

12.2.9. Let $\underline{X} := (X_i \mid i \in I)$ be a cofiltered family of quasi-separated schemes, with affine transition morphisms $f_\varphi : X_j \to X_i$, for every morphism $\varphi : j \to i$ in I. Denote by X the limit of $\underline{X}$, and for each $i \in I$ let $\pi_i : X \to X_i$ be the natural projection.

Lemma 12.2.10. *In the situation of* (12.2.9)*, let* $\mathscr{X} := ((X_i, \underline{M}_i) \mid i \in I)$ *be a cofiltered system of log schemes, with transition morphisms* $(f_\varphi, \log f_\varphi) : (X_i, \underline{M}_i) \to (X_j, \underline{M}_j)$ *for every morphism* $\varphi : i \to j$ *in* I*. We have :*

(i) *The limit of the system* $\mathscr{X}$ *exists in the category* **log**.

(ii) *Let* $(X, \underline{M})$ *denote the limit of the system* $\mathscr{X}$*. If* X_i *is quasi-compact and quasi-separated for every* $i \in I$*, then the natural map*

$$\operatorname*{colim}_{i \in I} \Gamma(X_i, \underline{N}_i) \to \Gamma(X, \underline{N})$$

is an isomorphism.

Proof. (i): Let X be the limit of the system of schemes $\underline{X}$, and endow X with the sheaf of monoids $\underline{M} := \mathrm{colim}_{i \in I}\, \pi_i^* \underline{M}_i$, where π^* is the pull-back functor for sheaves of monoids (see (4.8.43)), and the transition maps $\pi_j^* \underline{M}_j \to \pi_i^* \underline{M}_i$ are induced by the morphisms $\log f_\varphi : f_\varphi^* \underline{M}_j \to \underline{M}_i$, for every $\varphi : i \to j$ in I. Then the structure maps of the log structures $\underline{M}_i$ induce a well defined morphism of X-monoids $\underline{M} \to \mathscr{O}_X$, and we claim that the resulting scheme with pre-log structure $(X, \underline{M})$ is actually a log scheme. Indeed, the assertion can be checked on the stalks, and notice that, for every τ-point ξ of X we have a natural identification

$$\mathscr{O}_{X,\xi} \xrightarrow{\sim} \operatorname*{colim}_{i \in I} \mathscr{O}_{X_i, \pi_i(\xi_i)}.$$

(This is clear for $\tau = \mathrm{Zar}$, and for $\tau = \text{ét}$ one uses [66, Ch.IV, Prop.18.8.18(ii)]); it then suffices to invoke lemma 4.8.46(i). Lastly, it is easily seen that $(X, \underline{M})$ is a limit of the system $\mathscr{X}$: the details shall be left to the reader.

(ii): In view of the explicit construction in (i), the assertion for $\tau = \mathrm{Zar}$ follows immediately from proposition 10.1.10, and for $\tau = \text{ét}$ it follows from [9, Exp.VII, Rem.5.14]. □

Example 12.2.11. Let X be a scheme. For the following example we choose to work with the étale topology $X_{\text{ét}}$ on X. A *divisor* on X is a closed subscheme $D \subset X$ which is regularly embedded in X and of codimension 1 ([66, Ch.IV, Déf.19.1.3, §21.2.12]). Suppose moreover that X is noetherian, let $D \subset X$ be a divisor, and denote by $(D_i \mid i \in I)$ the irreducible reduced components of D. We say that D is a *strict normal crossings* divisor, if :

- $\mathscr{O}_{X,x}$ is a regular ring, for every $x \in D$.
- D is a reduced subscheme.
- For every subset $J \subset I$, the (scheme theoretic) intersection $\bigcap_{j \in J} D_j$ is regular of pure codimension $\sharp J$ in X.

A closed subscheme D of a noetherian scheme X is a *normal crossings divisor* if, for every $x \in X$ there exists an étale neighborhood $f : U \to X$ of x such that $f^{-1}D$ is a strict normal crossings divisor in U.

Suppose that D is a normal crossings divisor of a noetherian scheme X, and let $j : U := X\setminus D \to X$ be the natural open immersion. We claim that the log structure $j_*\mathscr{O}_U^\times$ is fine (this is the direct image of the trivial log structure on $U_{\text{ét}}$: see example 12.1.12(ii)). To see this, let ξ be any geometric point of X localized at a point of D; up to replacing X by an étale neighborhood of ξ, we may assume that D is a strict normal crossings divisor; we can assume as well that X is affine and small enough, so that the irreducible components $(D_\lambda \mid \lambda \in \Lambda)$ are of the form $V(I_\lambda)$, where $I_\lambda \subset A := \Gamma(X, \mathscr{O}_X)$ is a principal divisor, say generated by an element $x_\lambda \in A$, for every $\lambda \in \Lambda$. We claim that $j_*\mathscr{O}_U^\times$ is the constant log structure associated to the pre-log structure :

$$\alpha : \mathbb{N}_X^{(\Lambda)} \to \mathscr{O}_X \quad : \quad e_\lambda \mapsto x_\lambda$$

where $(e_\lambda \mid \lambda \in \Lambda)$ is the standard basis of $\mathbb{N}^{(\Lambda)}$. Indeed, let $S \subset \Lambda$ be the largest subset such that the image of ξ lies in $D_S := \bigcap_{\lambda\in S} D_\lambda$, we have $x_\lambda \in \mathscr{O}_{X,x}^\times$ for all $\lambda \notin S$, so that the push-out of the induced diagram of stalks $\mathscr{O}_{X,\xi}^\times \leftarrow \alpha^{-1}\mathscr{O}_{X,\xi}^\times \to \mathbb{N}^{(\Lambda)}$ is the same as the push-out P_S of the analogous diagram $\mathscr{O}_{X,\xi}^\times \leftarrow \alpha_S^{-1}\mathscr{O}_{X,\xi}^\times \to \mathbb{N}^{(S)}$, where $\alpha_S : \mathbb{N}_X^{(S)} \to \mathscr{O}_X$ is the restriction of α. Suppose that $a \in \mathscr{O}_{X,\xi}$ and a is invertible on $X(\xi)\setminus D_S$; the minimal associated primes of $A/(a)$ are all of height one, and they must therefore be found among the prime ideals Ax_λ, with $\lambda \in S$. It follows easily that a is of the form $u \cdot \prod_{\lambda\in S} x_\lambda^{k_\lambda}$ for certain $k_\lambda \in \mathbb{N}$ and $u \in \mathscr{O}_{X,\xi}^\times$. Therefore, the natural map $\beta_\xi : P_S \to (j_*\mathscr{O}_U^\times)_\xi$ is surjective. Moreover, the family $(x_\lambda \mid \lambda \in S)$ is a regular system of parameters of $\mathscr{O}_{X,\xi}$ ([63, Ch.0, Prop.17.1.7]), therefore the natural map $\mathrm{Sym}^n_{\kappa(\xi)}(\mathfrak{m}_\xi/\mathfrak{m}_\xi^2) \to \mathfrak{m}_\xi^n/\mathfrak{m}_\xi^{n+1}$ is an isomorphism for every $n \in \mathbb{N}$ (here $\mathfrak{m}_\xi \subset \mathscr{O}_{X,\xi}$ is the maximal ideal); it follows easily that β_ξ is also injective.

Example 12.2.12. Suppose that X is a regular noetherian scheme, and $D \subset X$ a divisor on X; let $U := X\setminus D$. If D is not a normal crossings divisor, the log structure $\underline{M} := j_*\mathscr{O}_U^\times$ on $X_{\text{ét}}$ is not necessarily fine. For a counterexample, let K be an algebraically closed field, $C \subset \mathbb{A}^2_K$ a nodal cubic; take $D \subset X := \mathbb{A}^3_K$ to be the (reduced, affine) cone over the cubic C, with vertex $x_0 \in \mathbb{A}^3$, and pick a geometric point ξ localized at x_0. It is easily seen that, away from the vertex, D is a normal crossings divisor, hence $\underline{M}_{|X\setminus\{x_0\}}$ is a fine log structure on $X\setminus\{x_0\}$, by example 12.2.11. More precisely, let $y_0 \in C$ be the unique singular point, $L \subset D$ the line spanned by x_0 and y_0, and η a geometric point localized at the generic point of L. By inspecting the argument in example 12.2.11, we find that :

$$\underline{M}_\eta \simeq \mathbb{N}^{\oplus 2} \oplus \mathscr{O}_{X,\eta}^\times$$

(indeed, an isomorphism is obtained by choosing $a, b \in \mathscr{O}_{C,y_0}$ such that $V(a)$ and $V(b)$ are the two branches of the cubic C in an étale neighborhood of y_0). On the other hand, let $\mathfrak{p} \subset A := K[T_1, T_2, T_3]$ be the prime ideal corresponding to x_0, and $I \subset A_\mathfrak{p}$ the ideal defining the closed subscheme $X(x_0)\cap D$ in $X(x_0)$; we claim that $I\cdot\mathscr{O}_{X,\xi}$ is still a prime ideal, necessarily of height one. Indeed, let $B := A_\mathfrak{p}/I$, and denote by $A^\wedge$ (resp. $B^\wedge$) the $\mathfrak{p}$-adic completion of $A_\mathfrak{p}$ (resp. of B); then $B^\wedge$ is also the completion of the reduced ring $\mathscr{O}_{X,\xi}/I$, hence it suffices to show that $\mathrm{Spec}\, B^\wedge$ is irreducible. However, we may assume that $C \subset \mathrm{Spec}\, K[T_1, T_2]$ is the affine cubic defined by the ideal $J \subset K[T_1, T_2]$ generated by $T_1^3 - T_2^2 + T_1T_2$. Then I is generated by the element $f := T_1^3 - T_2^2T_3 + T_1T_2T_3$; also, $\mathfrak{p} = (T_1, T_2, T_3)$, so that $A^\wedge \simeq K[[T_1, T_2, T_3]]$ and $B^\wedge \simeq A^\wedge/(f)$. Suppose $\mathrm{Spec}\, B^\wedge$ is not irreducible. This means that $V(f) \subset \mathrm{Spec}\, A^\wedge$ is a union $V(f) = Z_1 \cup \cdots \cup Z_n$ of $n \geq 2$ irreducible components Z_i. Since $A^\wedge$ is a local regular ring, each such irreducible component Z_i is a divisor, defined by some principal prime ideal $\mathfrak{q}_i$ in $A^\wedge$. Let a_i be a generator for $\mathfrak{q}_i$; then f admits factorizations of the form $f = a_ib_i$ for some non-invertible $b_i \in A^\wedge$. Fix some i, and set $a := a_i$, $b := b_i$; since f is homogeneous of degree 3, we must have $a \in \mathfrak{p}^k\setminus\mathfrak{p}^{k+1}$ for either $k = 1$ or $k = 2$ ($k \neq 0$ since a is not a unit, and $k \neq 3$, since b is not a unit); then $b \in \mathfrak{p}^{3-k}\setminus\mathfrak{p}^{4-k}$. Write $a = a' + a''$ and $b = b' + b''$, where $a'' \in \mathfrak{p}^{k+1}$,

$b'' \in \mathfrak{p}^{4-k}$ and a' (resp. b') is homogeneous of degree k (resp. $3-k$). Then $f = ab = a'b' + c$, where $c \in \mathfrak{p}^4$ and $a'b'$ is homogeneous of degree 3. This means that $f = a'b'$ is a factorization of f in A. However, f is irreducible in A, a contradiction. (Instead of this elementary argument, one can appeal to [133, Th.43.20], which runs as follows. If R is a local domain, then there is a natural bijection between the set of minimal prime ideals of the henselization R^{h} of R and the set of maximal ideals of the normalization of R in its ring of fractions. In our case, the normalization D^ν of D is the cone over the normalization of C, hence the only point of D^ν lying over x_0 is the vertex of D^ν.)

It follows that any choice of a generator of I yields an isomorphism :

$$\underline{M}_\xi \simeq \mathbb{N} \oplus \mathscr{O}^\times_{X,\xi}.$$

Suppose now that – in an étale neighborhood V of ξ – the log structure $\underline{M}$ is associated to a pre-log structure $\alpha : P_V \to \mathscr{O}_V$, for some monoid P; hence $\underline{M}_{|V}$ is the push-out of the diagram $\mathscr{O}^\times_V \leftarrow \alpha^{-1}(\mathscr{O}^\times_V) \to P_V$, whence isomorphisms :

$$P/\alpha^{-1}_\xi(\mathscr{O}^\times_{X,\xi}) \simeq \underline{M}_\xi/\mathscr{O}^\times_{X,\xi} \simeq \mathbb{N} \qquad P/\alpha^{-1}_\eta(\mathscr{O}^\times_{X,\eta}) \simeq \underline{M}_\eta/\mathscr{O}^\times_{X,\eta} \simeq \mathbb{N}^{\oplus 2}.$$

But clearly $\alpha^{-1}_\xi(\mathscr{O}^\times_{X,\xi}) \subset \alpha^{-1}_\eta(\mathscr{O}^\times_{X,\eta})$, so we would have a surjection of monoids $\mathbb{N} \to \mathbb{N}^{\oplus 2}$, which is absurd.

On the other hand, we remark that the log structure $j_*\mathscr{O}^\times_U$ *on the Zariski site* X_{Zar} is fine : indeed, one has a global chart $\mathbb{N}_{X_{\mathrm{Zar}}} \to j_*\mathscr{O}^\times_U$, provided by the equation defining the divisor D.

12.2.13. Let R be a ring, M a monoid, and set $S := \operatorname{Spec} R[M]$. The unit of adjunction $\varepsilon_M : M \to R[M]$ can be regarded as an object (M, ε_M) of $\mathbf{Mnd}/\Gamma(S, \mathscr{O}_S)$, whence a constant log structure $M^{\log}_S$ on S (see (12.1.15)). The rule

$$M \mapsto \operatorname{Spec}(R, M) := (S, M^{\log}_S)$$

is clearly functorial in M. Namely, to any morphism $\lambda : M \to N$ of monoids, we attach the morphism of log schemes

$$\operatorname{Spec}(R, \lambda) := (\operatorname{Spec} R[\lambda], \lambda^{\log}_{\operatorname{Spec} R[N]}) : \operatorname{Spec}(R, N) \to \operatorname{Spec}(R, M).$$

Likewise, if P is a pointed monoid, $\operatorname{Spec} R\langle P\rangle$ is a closed subscheme of $\operatorname{Spec} R[P]$, and we may define

$$\operatorname{Spec}\langle R, P\rangle := \operatorname{Spec}(R, P) \times_{\operatorname{Spec} R[P]} \operatorname{Spec} R\langle P\rangle.$$

Lastly, if M is a non-pointed monoid, notice the natural isomorphism of log schemes

$$\operatorname{Spec}\langle R, M_\circ\rangle \xrightarrow{\sim} \operatorname{Spec}(R, M)_\circ.$$

(Notation of remark 12.1.23(ii) : the details shall be left to the reader.)

Lemma 12.2.14. *With the notation of* (12.2.13)*, let* $a \in M$ *be any element, and set* $M_a := S_a^{-1}M$*, where* $S_a := \{a^n \mid n \in \mathbb{N}\}$*. Then* $U_a := \operatorname{Spec} R[M_a]$ *is an open subscheme of* S*, and the induced morphism of log schemes :*

$$\operatorname{Spec}(R, M_a) \to \operatorname{Spec}(R, M) \times_S U_a$$

is an isomorphism.

Proof. Let $\beta_S : M_S \to \mathscr{O}_S$ and $\beta_{U_a} : (M_a)_{U_a} \to \mathscr{O}_{U_a}$ be the natural charts, and denote by $\varphi : M \to M_a$ the localization map. For every τ-point ξ of U_a, we have the identity :

$$\beta_{S,\xi} = \beta_{U_a,\xi} \circ \varphi : M \to \mathscr{O}_{S,\xi}.$$

Let $Q := \beta^{-1}_{U_a,\xi}\mathscr{O}^\times_{U_a,\xi}$. The assertion is a straightforward consequence of the following :

Claim 12.2.15. The induced commutative diagram of monoids :

$$\begin{array}{ccc} \varphi^{-1}Q & \longrightarrow & M \\ \downarrow & & \downarrow \\ Q & \longrightarrow & M_a \end{array}$$

is cocartesian.

Proof of the claim. Let $b \in M$, and suppose that $\beta_{U_a,\xi}(a^{-1}b) \in \mathscr{O}^\times_{U_a,\xi}$. Since $\beta_{U_a,\xi}(a) \in \mathscr{O}^\times_{U_a,\xi}$, we deduce that the same holds for $\beta_{U_a,\xi}(b)$, *i.e.* $b \in \varphi^{-1}Q$. Let $Q' := \varphi(\varphi^{-1}Q)$; we conclude that $Q = S_a^{-1}Q'$, the submonoid of M_a generated by Q' and a^{-1}. The claim follows easily. □

12.2.16. In the same vein, let X be a R-scheme, and (M, φ) any object of $\mathbf{Mnd}/\Gamma(X, \mathscr{O}_X)$. The map φ induces, via the adjunction of (4.8.50), a homomorphism of R-algebras $R[M] \to \Gamma(X, \mathscr{O}_X)$, whence a map of schemes $f : X \to S := \mathrm{Spec}\, R[M]$, inducing a morphism

$$X \times_S \mathrm{Spec}(R, M) \to (X, (M, \varphi)^{\log}_X) \qquad (12.2.17)$$

of log schemes.

Lemma 12.2.18. *In the situation of* (12.2.16)*, we have :*

(i) *The map* (12.2.17) *is an isomorphism.*
(ii) *The log scheme* $\mathrm{Spec}(R, M)$ *represents the functor*

$$\mathbf{log} \to \mathbf{Set} \quad : \quad (Y, \underline{N}) \mapsto \mathrm{Hom}_{\mathbb{Z}\text{-}\mathsf{Alg}}(R, \Gamma(Y, N)) \times \mathrm{Hom}_{\mathbf{Mnd}}(P, \Gamma(Y, \underline{N})).$$

Proof. (i): The log structure $f^*(M_S^{\log})$ of $\mathrm{Spec}(R, M) \times_S X$ represents the functor :

$$F : \mathbf{log}_X \to \mathbf{Set} \quad : \quad \underline{N} \mapsto \mathrm{Hom}_{\mathbf{Mnd}/\Gamma(S,\mathscr{O}_S)}((M, \varepsilon_M), \Gamma(S, f_*\underline{N})).$$

However, if $\underline{N}$ is any log structure on X, the pre-log structure $(f_*\underline{N})^{\text{pre-log}}$ is the same as $f_*(\underline{N}^{\text{pre-log}})$ (see [75, (6.4.8)]). From the explicit construction of direct images for pre-log structures, and since the global sections functor is left exact (because it is a right adjoint), we deduce a cartesian diagram of monoids :

$$\begin{array}{ccc} \Gamma(S, f_*\underline{N}) & \longrightarrow & \Gamma(S, \mathscr{O}_S) \\ \downarrow & & \downarrow \\ \Gamma(X, \underline{N}) & \longrightarrow & \Gamma(X, \mathscr{O}_X). \end{array}$$

It follows easily that F is naturally isomorphic to the functor given by the rule :

$$\underline{N} \mapsto \mathrm{Hom}_{\mathbf{Mnd}/\Gamma(X,\mathscr{O}_X)}((M, \varphi), \Gamma(X, \underline{N})).$$

The latter is of course the functor represented by $(M, \varphi)^{\log}_X$.

(ii) can now be deduced formally from (i), or proved directly by inspecting the definitions. □

12.2.19. From (12.2.13) it is also clear that the rule $(R, M) \mapsto \mathrm{Spec}(R, M)$ defines a functor

$$\mathbb{Z}\text{-}\mathsf{Alg}^o \times \mathbf{Mnd}^o \to \mathbf{log}$$

which commutes with fibre products; namely, say that

$$(R', M') \leftarrow (R, M) \to (R'', M'')$$

are two morphisms of $\mathbb{Z}\text{-}\mathsf{Alg} \times \mathbf{Mnd}$; then there is a natural isomorphism of log schemes :

$$\mathrm{Spec}(R' \otimes_R R'', M' \otimes_M M'') \xrightarrow{\sim} \mathrm{Spec}(R', M') \times_{\mathrm{Spec}(R,M)} \mathrm{Spec}(R'', M'').$$

For the proof, one compares the universal properties characterizing these log schemes, using lemma 12.2.18(ii) : details left to the reader.

12.2.20. Let X be a scheme, $\underline{M}$ a sheaf of monoids on X_τ. We say that $\underline{M}$ is *locally constant* if there exist a covering family $(U_\lambda \to X \mid \lambda \in \Lambda)$ and for every $\lambda \in \Lambda$, a monoid P_λ and an isomorphism of sheaves of monoids $\underline{M}_{|U_\lambda} \simeq (P_\lambda)_{U_\lambda}$. We say that $\underline{M}$ is *weakly constructible* if, for every affine open subset $U \subset X$ we can find finitely many locally closed constructible subsets $Z_1, \dots, Z_n \subset U$ such that :

- $U = Z_1 \cup \dots \cup Z_n$
- $\underline{M}_{|Z_i}$ is a locally constant sheaf of monoids, for every $i = 1, \dots, n$.

If moreover, $\underline{M}_\xi$ is a finitely generated monoid for every τ-point ξ of X, we say that $\underline{M}$ is *constructible*. If $\varphi : \underline{M} \to \underline{N}$ is a morphism of constructible sheaves of monoids on X_τ, then it is easily seen that $\mathrm{Ker}\,\varphi$, $\mathrm{Coker}\,\varphi$ and $\mathrm{Im}\,\varphi$ are constructible (corollary 6.4.3(i)). Moreover, constructibility is preserved under coequalizers; if $\underline{M}$ is weakly constructible with integral stalks $\underline{M}_\xi$ at every τ-point ξ of X, then a fibre product $\underline{M}' \times_{\underline{M}} \underline{M}''$ is constructible if the same holds for both $\underline{M}'$ and $\underline{M}''$; furthermore, under the same assumptions for $\underline{M}$, an equalizer $\mathrm{Equal}(\varphi, \varphi' : \underline{N} \to \underline{M})$ is constructible if the same holds for $\underline{N}$ (corollaries 6.4.2 and 6.4.3(ii)).

If X is locally noetherian and both $\underline{M}$ and $\underline{N}$ are weakly constructible, then the same holds for $\mathrm{Ker}\,\varphi$, $\mathrm{Coker}\,\varphi$ and $\mathrm{Im}\,\varphi$.

Suppose that $\underline{M}$ is a sheaf of monoids on X_τ, and for every $x \in X$ choose a τ-point $\overline{x}$ localized at x; the *rank* of $\underline{M}$ is the function

$$\mathrm{rk}_{\underline{M}} : X \to \mathbb{N} \cup \{\infty\} \qquad x \mapsto \dim_{\mathbb{Q}} \underline{M}^{\mathrm{gp}}_{\overline{x}} \otimes_{\mathbb{Z}} \mathbb{Q}.$$

It is clear from the definitions that the rank function of a constructible sheaf of monoid is constructible on X.

Lemma 12.2.21. *Let X be a scheme, $\varphi : \underline{Q} \to \underline{Q}'$ a morphism of coherent log structures on X_τ. Then :*

(i) *The sheaves $\underline{Q}^\sharp$ and* $\mathrm{Coker}\,\varphi$ *are constructible.*
(ii) *$(X, \underline{Q})_n$ is an open subset of X, for every integer $n \geq 0$. (See definition* 12.2.7(i)*.)*
(iii) *The rank functions of $\underline{Q}^\sharp$ and* $\mathrm{Coker}\,\varphi$ *are (constructible and) upper semicontinuous.*

Proof. (i): Suppose that $\underline{Q}$ admits a finite chart $\alpha : M_X \to \underline{Q}$. Pick a finite system of generators $\Sigma \subset M$, and for every $S \subset \Sigma$, set :

$$Z_S := \bigcap_{s \in S} D(s) \cap \bigcap_{t \in \Sigma \setminus S} V(t)$$

where, as usual $D(s)$ (resp. $V(s)$) is the open (resp. closed) subset of the points $x \in X$ such that the image $s(x) \in \kappa(x)$ of s is invertible (resp. vanishes). Clearly each Z_S is a constructible subset of X, and their union equals X. Moreover, for every $S \subset \Sigma$, and every τ-point ξ supported on Z_S, the submonoid $N_\xi := \alpha_\xi^{-1}(\mathscr{O}^\times_{X,\xi}) \subset M$ is a face of M (lemma 6.1.20(i)), hence it is the submonoid $\langle S \rangle$ generated by $\Sigma \cap N_\xi = S$ (lemma 6.1.20(ii)). It follows easily that $\underline{Q}^\sharp_{|Z_S} \simeq (M/\langle S \rangle)_{|Z_S}$.

More generally, suppose that $\underline{Q}$ is coherent. We may assume that X is affine. Then, by the foregoing, we may find a finite set Λ and a covering family $(f_\lambda : U_\lambda \to X \mid \lambda \in \Lambda)$ of X_τ, such that U_λ is affine and $\underline{Q}^\sharp_{|U_\lambda}$ is a constructible sheaf of monoids on $(U_\lambda)_\tau$, for every $\lambda \in \Lambda$. Since f_λ is finitely presented, it maps constructible subsets to constructible subsets ([66, Ch.IV, Th.1.8.4]); if $\tau = \mathrm{Zar}$, it follows easily that the restriction of $\underline{Q}^\sharp$ to $f_\lambda(U_\lambda)_\tau$ is constructible, therefore $\underline{Q}^\sharp$ is constructible. For $\tau = \text{ét}$ we remark :

Claim 12.2.22. Let $f : Y \to X$ a surjective morphism of schemes, locally of finite presentation, and $\mathscr{F}$ a sheaf of monoids on $X_{\text{ét}}$. Then $\mathscr{F}$ is constructible if and only if the same holds for $f^*_{\text{ét}}\mathscr{F}$.

Proof of the claim. This can be proved as [10, Exp.IX, Prop.2.8]. ◊

Hence, also for $\tau = \text{ét}$ we see that $\underline{Q}^\sharp$ is constructible. Next, notice that $\operatorname{Coker}\varphi = \operatorname{Coker}\varphi^\sharp$; by the foregoing, $\underline{Q}'^\sharp$ is constructible as well, so the same holds for $\operatorname{Coker}\varphi$.

(ii,iii): Let $x \in X$ be any point, and ξ a τ-point of X localized at x; by theorem 12.1.37(i) we may find a neighborhood $f : U \to X$ of ξ in X_τ and a finite chart $(\omega_P, \omega_{P'}, \vartheta)$ for $\varphi_{|U}$. By claim 12.1.31, we may assume, after replacing U by a smaller neighborhood of ξ, that ω_P and $\omega_{P'}$ are local at the point ξ, in which case $\underline{Q}^\sharp_\xi = P^\sharp$ and $\operatorname{Coker}\varphi_\xi = \operatorname{Coker}\vartheta$. Let $r : X \to \mathbb{N}$ (resp. $r' : U \to \mathbb{N}$) denote the rank function of $\underline{Q}^\sharp$ (resp. of $\underline{Q}^\sharp_{|U}$); then it is clear that $r' = r \circ f$. On the other hand, for every $y \in U$ and every τ-point η of U localized at y, the stalk $\underline{Q}^\sharp_\eta$ is a quotient of $P^\sharp$, hence $r'(y) \leq r(x)$ and $\dim \underline{Q}_\eta \leq \dim \underline{Q}_\xi$. Since f is an open mapping, this shows (ii), and also that the rank function of $\underline{Q}^\sharp$ is upper semicontinuous. Likewise, let s (resp. s') denote the rank function of $\operatorname{Coker}\varphi$ (resp. $\operatorname{Coker}\varphi_{|U}$); then $s' = s \circ f$, and $\operatorname{Coker}\varphi_\eta$ is a quotient of $\operatorname{Coker}\vartheta$ for every τ-point η of U; the latter implies that $s'(y) \leq s(x)$ for every $y \in U$, which shows that s is upper semicontinuous. □

Remark 12.2.23. By lemma 12.2.21(ii), if $\underline{Q}$ is any coherent log structure on a scheme X, then $(X, \underline{Q})_n$ is naturally an open subscheme of X for every $n \in \mathbb{N}$, and even naturally a log scheme, with the log structure inherited via the open immersion in X. Henceforth, for every such n and $\underline{Q}$, the notation $(X, \underline{Q})_n$ will refer to either the underlying open subscheme of X, or to the corresponding open log subscheme of $(X, \underline{Q})$, depending on the context.

Corollary 12.2.24. *Let X be a scheme, $\underline{M}$ a log structure on X_{Zar}, and set*

$$(X_{\text{ét}}, \underline{M}_{\text{ét}}) := \widetilde{u}^*(X_{\text{Zar}}, \underline{M}).$$

Then $\underline{M}$ is integral (resp. coherent, resp. fine, resp. fine and saturated) if and only if the same holds for $\underline{M}_{\text{ét}}$.

Proof. It has already been remarked that $(X_{\text{ét}}, \underline{M}_{\text{ét}})$ is coherent (resp. fine, resp. fine and saturated) whenever the same holds for $(X_{\text{Zar}}, \underline{M})$; furthermore, the proof of proposition 12.2.3(i) shows that the natural map on stalks $\underline{M}_{|\xi|} \to \underline{M}_{\text{ét},\xi}$ is injective for every geometric point ξ of X, therefore $\underline{M}$ is integral whenever the same holds for $\underline{M}_{\text{ét}}$.

Next, we suppose that $\underline{M}_{\text{ét}}$ is coherent, and we wish to show that $\underline{M}$ is coherent.

Let $x \in X$ be any point, ξ a geometric point localized at x. By assumption there exists a finitely generated monoid P', with a morphism $\alpha : P' \to \underline{M}_{\text{ét},\xi}$ inducing an isomorphism :

$$P' \otimes_{\beta^{-1}\underline{M}^\times_{\text{ét},\xi}} \underline{M}^\times_{\text{ét},\xi} \to \underline{M}_{\text{ét},\xi} \simeq \underline{M}_x \otimes_{\underline{M}^\times_x} \mathscr{O}^\times_{X,\xi}.$$

It follows easily that we may find a finitely generated submonoid $Q \subset \underline{M}_x$, such that the image of α lies in $(Q \cdot \underline{M}^\times_x) \otimes_{\underline{M}^\times_x} \mathscr{O}^\times_{X,\xi}$, and therefore the natural map :

$$(Q \cdot \underline{M}^\times_x) \otimes_{\underline{M}^\times_x} \mathscr{O}^\times_{X,\xi} \to \underline{M}_{\text{ét},\xi}$$

is surjective. Then lemma 4.8.31(ii) implies that $Q \cdot \underline{M}^\times_x = \underline{M}_x$, in other words, the induced map $Q \to \underline{M}^\sharp_x = \underline{M}^\sharp_{\text{ét},\xi}$ is surjective. Set

$$P := Q^{\text{gp}} \times_{\underline{M}^{\text{gp}}_{\text{ét},\xi}} \underline{M}_{\text{ét},\xi}.$$

In this situation, proposition 12.1.30 tells us that the induced map $\beta_\xi : P \to \underline{M}_{\text{ét},\xi}$ extends to an isomorphism of log structures $\beta^{\log} : P^{\log}_{U_{\text{ét}}} \to \underline{M}_{\text{ét}|U_{\text{ét}}}$ on some étale neighborhood $U \to X$ of

ξ. On the other hand, it is easily seen that the diagram of monoids :

$$\begin{array}{ccc} \underline{M}_x & \longrightarrow & \underline{M}_x^{\mathrm{gp}} \\ \downarrow & & \downarrow \\ \underline{M}_{\text{ét},\xi} & \longrightarrow & \underline{M}^{\mathrm{gp}}_{\text{ét},\xi} \end{array}$$

is cartesian, therefore β_ξ factors uniquely through a morphism $\beta'_x : P \to \underline{M}_x$. The latter extends to a morphism of log structures $\beta'^{\log} : P^{\log}_{U'_{\mathrm{Zar}}} \to \underline{M}_{|U'_{\mathrm{Zar}}}$ on some (Zariski) open neighborhood U' of x in X (lemma 12.1.18(iv.a),(v)). By inspecting the construction, we find a commutative diagram of monoids :

$$\begin{array}{ccc} (\widetilde{u}^* P^{\log}_{U'_{\mathrm{Zar}}})_\xi & \longrightarrow & P^{\log}_{U_{\text{ét}},\xi} \\ {\scriptstyle (\widetilde{u}^*\beta'^{\log})_\xi}\downarrow & & \downarrow{\scriptstyle \beta^{\log}_\xi} \\ (\widetilde{u}^*\underline{M})_\xi & = & \underline{M}_{\text{ét},\xi} \end{array}$$

(where $\widetilde{u}^*$ is the functor on log structures of (12.2.2)) whose horizontal arrows and right vertical arrow are isomorphisms; it follows that $(\widetilde{u}^*\beta'^{\log})_\xi$ is an isomorphism as well, therefore $\widetilde{u}^*\beta'^{\log}$ restricts to an isomorphism on some smaller étale neighborhood $f : U'' \to U'$ of ξ (lemma 12.1.18(iv.c)). Since f is an open morphism, we deduce that the restriction of $(\widetilde{u}^*\beta'^{\log})_\xi$ is already an isomorphism on $f(U'')_{\text{ét}}$, and $f(U'')$ is a (Zariski) open neighborhood of x in X. Finally, in light of proposition 12.2.3(i), we conclude that the restriction of $\beta'^{\log}$ is an isomorphism on $f(U'')_{\mathrm{Zar}}$, so $\underline{M}$ is coherent, as stated.

Lastly, we suppose that $\underline{M}_{\text{ét}}$ is fine and saturated, and we wish to show that $\underline{M}$ is saturated. However, the assertion can be checked on the stalks, hence let ξ be any geometric point of X; the proof of proposition 12.2.3 shows that the natural map $\underline{M}^\sharp_{|\xi|} \to \underline{M}^\sharp_{\text{ét},\xi}$ is bijective, so the assertion follows from lemma 6.2.9(ii). □

Corollary 12.2.25. *The category* **coh.log** *admits all finite limits. More precisely, the limit (in the category of log structures) of a finite system of coherent log structures, is coherent.*

Proof. Let Λ be a finite category, $\mathscr{X} := ((X_\lambda, \underline{M}_\lambda) \mid \lambda \in \Lambda)$ an inverse system of schemes with coherent log structures indexed by Λ. Denote by Y the limit of the system $(X_\lambda \mid \lambda \in \Lambda)$ of underlying schemes, and by $\pi_\lambda : Y \to X_\lambda$ the natural morphism, for every $\lambda \in \Lambda$. It is easily seen that the limit of $\mathscr{X}$ is naturally isomorphic to the limit of the induced system $\mathscr{Y} := ((Y, \pi^*_\lambda \underline{M}_\lambda) \mid \lambda \in \Lambda)$, and in view of lemma 12.1.18(i), we may therefore replace $\mathscr{X}$ by $\mathscr{Y}$, and assume that $\mathscr{X}$ is a finite inverse system in the category $\mathbf{log}_X$, for some scheme X (especially, the underlying maps of schemes are $\mathbf{1}_X$, for every morphism $\lambda \to \mu$ in Λ).

It suffices then to show that the push-out of two morphisms $g : \underline{N} \to \underline{M}$, $h : \underline{N} \to \underline{M}'$ of coherent log structures on X, is coherent. However, notice that the assertion is local on the site X_τ, hence we may assume that $\underline{N}$ admits a finite chart $Q_X \to \underline{N}$. Then, thanks to theorem 12.1.37(ii), we may further assume that both f and g admit finite charts of the form $Q_X \to P_X$ and respectively $Q_X \to P'_X$, for some finitely generated monoids P and P'. We deduce a natural map $(P \amalg_Q P')_X \xrightarrow{\sim} P_X \amalg_{Q_X} P'_X \to \underline{M} \amalg_{\underline{N}} \underline{M}'$, which is the sought finite chart. □

Lemma 12.2.26. *In the situation of* (12.2.9)*, suppose that X_i is quasi-compact and quasi-separated for every $i \in I$, and that there exist $i \in I$, and a coherent log structure $\underline{N}_i$ on X_i, such that the log structure $\underline{N} := \pi^*_i \underline{N}_i$ on X_τ has a finite chart $\beta : Q_X \to \underline{N}$. Then there exist a morphism $\varphi : j \to i$ in I and a chart $\beta_j : Q_{X_j} \to \underline{N}_j := f^*_\varphi \underline{N}_i$ for $\underline{N}_j$, such that $\pi^*_j \beta_j = \beta$.*

Proof. After replacing I by I/i, we may assume that $\underline{N}_j$ is well defined for every $j \in I$, and i is the final object of I. In this case, notice that $(X, \underline{N})$ is the limit of the cofiltered system of log schemes $((X_j, \underline{N}_j) \mid j \in I)$. We begin with the following :

Claim 12.2.27. In order to prove the lemma, it suffices to show that, for every τ-point ξ of X there exist $j(\xi) \in I$, a neighborhood $U_\xi \to X_{j(\xi)}$ of $\pi_{j(\xi)}(\xi)$ in $X_{j(\xi),\tau}$, and a chart $\beta_{j(\xi)} : Q_{U_\xi} \to \underline{N}_{j(\xi)|U_\xi}$ such that $(\mathbf{1}_{U_\xi} \times_{X_{j(\xi)}} \pi_{j(\xi)})^* \beta_{j(\xi)} = \beta$.

Proof of the claim. Clearly we may assume that U_ξ is an affine scheme, for every τ-point ξ of X. Under the stated assumptions, X is quasi-compact, hence we may find a finite set $\{\xi_1, \dots, \xi_n\}$ of τ-points of X, such that $(U_{\xi_k} \times_{X_{j(\xi_k)}} X \to X \mid k = 1, \dots, n)$ is covering in X_τ. Since I is cofiltered, we may then find $j \in I$ and morphisms $\varphi_k : j \to j(\xi_k)$ for every $k = 1, \dots, n$. After replacing each $\beta_{j(\xi_k)}$ by $f^*_{\varphi_k} \beta_{j(\xi_k)}$, we may assume that $j(\xi_k) = j$ for every $k \leq n$.

In this case, set $\beta_k := \beta_{j(\xi_k)}$, and $U_k := U_{\xi_k}$ for every $k \leq n$; let also $U_{kl} := U_k \times_{X_j} U_l$, $U^\sim_{kl} := U_{kl} \times_{X_j} X$ for every $k, l \leq n$, and denote by $\pi_{kl} : U^\sim_{kl} \to U_{kl}$ the natural projection. Hence, for every $k, l \leq n$ we have a morphism of U_{kl}-monoids :

$$\beta_{kl} := \beta_{k|U_{kl}} : Q_{U_{kl}} \to \underline{N}_{j|U_{kl}}$$

and by construction we have $\pi^*_{kl} \beta_{kl} = \pi^*_{lk} \beta_{lk}$ under the natural identification : $U^\sim_{kl} \xrightarrow{\sim} U^\sim_{lk}$. Notice now that U_{kl} is quasi-compact and quasi-separated for every $k, l \leq n$, hence the natural map

$$\operatorname*{colim}_{i \in I} \Gamma(U_{kl} \times_{X_j} X_i, \underline{N}_i) \to \Gamma(U^\sim_{kl}, \underline{N})$$

is an isomorphism (lemma 12.2.10(ii)). On the other hand, β_{kl} is given by a morphism of monoids $b_{kl} : Q \to \Gamma(U_{kl}, \underline{N}_j)$, and likewise $\pi^*_{kl} \beta_{kl}$ is given by the morphism $Q \to \Gamma(U^\sim_{kl}, \underline{N})$ obtained by composiiton of b_{kl} and the natural map $\Gamma(U_{kl}, \underline{N}_j) \to \Gamma(U^\sim_{kl}, \underline{N})$. By lemma 6.1.7(ii), we may then find a morphism $j' \to j$ in I such that the following holds. Set

$$V_k := U_k \times_{X_j} X_{j'} \qquad V_{kl} := V_k \times_{X_{j'}} V_l \qquad \text{for every } k, l \leq n$$

and let $p_k : V_k \to U_k$ be the projection for every $k \leq n$; let also $\beta'_k := p^*_k \beta_k$, which is a chart $Q_{V_k} \to \underline{N}_{j'|V_k}$ for the restriction of $\underline{N}_{j'}$. Then

$$\beta'_{k|V_{kl}} = \beta'_{l|V_{kl}} \qquad \text{for every } k, l \leq n \tag{12.2.28}$$

under the natural identification $V_{kl} \xrightarrow{\sim} V_{lk}$. By construction, the system of morphisms $(V_k \times_{X_{j'}} X \to X \mid k = 1, \dots, n)$ is covering in X_τ; after replacing j' by a larger index, we may then assume that the system of morphisms $(V_k \to X_{j'} \mid k = 1, \dots, n)$ is covering in $X_{j',\tau}$ ([65, Ch.IV, Th.8.10.5]). In this case, (12.2.28) implies that the local charts β'_k glue to a well defined chart $\beta_{j'} : Q_{X_{j'}} \to \underline{N}_{j'}$, and a direct inspection shows that we have indeed $\pi^*_{j'} \beta_{j'} = \beta$. ◊

Now, denote by $\alpha : Q_X \to \mathscr{O}_X$ the composition of β and the structure map of $\underline{N}$, and let ξ be a τ-point of X; we have a natural isomorphism

$$\underline{N}_\xi = \underline{N}_{i,\pi_i(\xi)} \otimes_{\underline{N}^\times_{i,\pi_i(\xi)}} \mathscr{O}^\times_{X,\xi} \xrightarrow{\sim} \operatorname*{colim}_{j \in I} \underline{N}_{i,\pi_i(\xi)} \otimes_{\underline{N}^\times_{i,\pi_i(\xi)}} \mathscr{O}^\times_{X_j,\pi_j(\xi_j)} \xrightarrow{\sim} \operatorname*{colim}_{j \in I} \underline{N}_{j,\pi_j(\xi)}$$

([66, Ch.IV, Prop.18.8.18(ii)]). It follows that β_ξ and α_ξ factor through morphisms of monoids $\beta_{\xi,j} : Q \to \underline{N}_{j,\pi_j(\xi)}$ and $\alpha_{\xi,j} : Q \to \mathscr{O}_{X_j,\pi_j(\xi)}$ for some $j \in I$ (lemma 6.1.7(ii)), and again we may replace I by I/j, after which we may assume that $\beta_{\xi,j}$ and $\alpha_{\xi,j}$ are defined for every $j \in I$. Then $\alpha_{\xi,j}$ extends to a pre-log structure $\alpha_j : Q_{U_j} \to \mathscr{O}_{U_j}$ on some neighborhood $U_j \to X_j$ of $\pi_j(\xi)$ in $X_{j,\tau}$ (lemma 12.1.18(v)), and we may also assume that $\beta_{\xi,j}$ extends to a morphism of pre-log structures $\beta_j : (Q_{U_j}, \alpha_j) \to \underline{N}_{j|U_j}$ (lemma 12.1.18(iv.a)). Notice as well that

$$\underline{N}^\sharp_{j,\xi} \simeq \underline{N}^\sharp_\xi \quad \text{and} \quad \beta^{-1}_{\xi,j} \underline{N}^\times_{j,\pi_j(\xi)} = \beta^{-1}_\xi \underline{N}^\times_\xi \qquad \text{for every } j \in I$$

and since β_ξ induces an isomorphism $Q/\beta^{-1}_\xi \underline{N}^\times_\xi \xrightarrow{\sim} \underline{N}^\sharp_\xi$, we deduce that $\beta_{j,\xi}$ (which is the same as $\beta_{\xi,j}$) induces an isomorphism $Q/\alpha^{-1}_{j,\xi} \mathscr{O}^\times_{X_j,\pi_j(\xi)} \xrightarrow{\sim} \underline{N}^\sharp_{j,\xi}$ for every $j \in I$. In turn, it then follows from lemma 12.1.4 that $\beta_{j,\xi}$ induces an isomorphism $(Q_{U_j}, \alpha_j)^{\log}_{\pi_j(\xi)} \xrightarrow{\sim} \underline{N}_{j,\pi_j(\xi)}$.

Next, by lemma 12.1.18(iv.b,c) we may find, for every $j \in I$, a neighborhood $U'_j \to U_j$ of ξ in X_τ, such that the restriction $(Q_{U'_j}, \alpha_{j|U'_j}) \to \underline{N}_{j|U'_j}$ of β_j is a chart for $\underline{N}_{j|U'_j}$.

By construction, the morphism $((\mathbf{1}_{U_j} \times_{X_j} \pi_j)^*\beta_j)_\xi : Q \to \underline{N}_\xi$ is the same as β_ξ, so by lemma 12.1.18(iv.b) we may find a neighborhood $V_j \to U'_j \times_{X_j} X$ of ξ in X_τ, such that

$$((\mathbf{1}_{U_j} \times_{X_j} \pi_j)^*\beta_j)_{|V_j} = \beta_{|V_j} \qquad \text{and} \qquad ((\mathbf{1}_{U_j} \times_{X_j} \pi_j)^*\alpha_j)_{|V_j} = \alpha_{|V_j}.$$

Claim 12.2.29. In the situation of (12.2.9), let $Y \to X$ be an object of the site X_τ, with Y quasi-compact and quasi-separated. We have :

- (i) There exist $i \in I$, an object $Y_i \to X_i$ of $X_{i,\tau}$, and an isomorphism of X-schemes $Y \xrightarrow{\sim} Y_i \times_{X_i} X$.
- (ii) Moreover, if $Y \to X$ is covering in X_τ, then we may find $i \in I$ and $Y_i \to X_i$ as in (i), which is covering in $X_{i,\tau}$.

Proof of the claim. (i) is obtained by combining [65, Ch.IV, Th.8.8.2(ii)] and [66, Ch.IV, Prop.17.7.8(ii)] (and [65, Ch.IV, Cor.8.6.4] if τ = Zar). Assertion (ii) follows from (i) and [65, Ch.IV, Th.8.10.5]. ◊

By claim 12.2.29(i), we may assume that $V_j = U''_j \times_{X_j} X$ for some neighborhood $U''_j \to U'_j$ of $\pi_j(\xi)$ in $X_{j,\tau}$. To conclude, it suffices to invoke claim 12.2.27. □

Proposition 12.2.30. *In the situation of* (12.2.9)*, suppose that X_i is quasi-compact and quasi-separated for every $i \in I$. Then the natural functor :*

$$\text{2-}\operatorname*{colim}_{I} \mathbf{coh.log}_{X_i} \to \mathbf{coh.log}_X \tag{12.2.31}$$

is an equivalence.

Proof. To begin with, we show :

Claim 12.2.32. The functor (12.2.31) is faithful. Namely, for a given $i \in I$, let $\underline{M}_i$ and $\underline{N}_i$ be two coherent log structures on X_i, and $f_i, g_i : \underline{M}_i \to \underline{N}_i$ two morphisms, such that $\pi_i^* f_i$ agrees with $\pi_i^* g_i$. Then there exists a morphism $\psi : j \to i$ in I such that $f_\psi^* f_i = f_\psi^* g_i$.

Proof of the claim. Set $\underline{M} := \pi_i^*\underline{M}_i$, and define likewise the coherent log structure $\underline{N}$ on X. For any τ-point ξ of X, pick a neighborhood $U_\xi \to X_i$ of $\pi_i(\xi)$ in X_i, and a finite chart $\beta : P_{U_\xi} \to \underline{M}_{i|U_\xi}$ for the restriction of $\underline{M}_i$. The morphisms $f_{i|U_\xi}$ and $g_{i|U_\xi}$ are determined by the induced maps $\varphi := \Gamma(U_\xi, f_i) \circ \beta$ and $\gamma := \Gamma(U_\xi, g_i) \circ \beta$, and the assumption means that the composition of φ and the natural map $\Gamma(U_\xi, \underline{N}_i) \to \Gamma(U_\xi \times_{X_i} X, \underline{N})$ equals the composition of γ with the same map.

It then follows from lemmata 12.2.10(ii) and 6.1.7(ii) that there exists a morphism $\psi : i(\xi) \to i$ in I such that the composition of φ with the natural map $\Gamma(U_\xi, \underline{N}_i) \to \Gamma(U_\xi \times_{X_i} X_{i(\xi)}, f_\psi^*\underline{N})$ equals the composition of γ with the same map; in other words, if we set $U'_\xi := U_\xi \times_{X_i} X_{i(\xi)}$, we have $f_\psi^* f_{i|U'_\xi} = f_\psi^* g_{i|U'_\xi}$. Next, since X is quasi-compact, we may find finitely many τ-points $\xi_1, \dots, \xi_n$ such that the family $(U'_{\xi_k} \times_{X_{i(\xi_k)}} X \to X)$ is covering in X_τ. Then, by [65, Ch.IV, Th.8.10.5] we may find $j \in I$ and morphisms $\psi_k : j \to i(\xi_k)$ in I, for $k = 1, \dots, n$, such that the induced family $(U''_k := U'_{\xi_k} \times_{X_{i(\xi_k)}} X_j \to X_j)$ is covering in $X_{j,\tau}$. By construction we have

$$f^*_{\psi_k \circ \psi} f_{i|U''_k} = f^*_{\psi_k \circ \psi} g_{i|U''_k} \qquad \text{for } k = 1, \dots, n$$

therefore $f^*_{\psi_k \circ \psi} f_i = f^*_{\psi_k \circ \psi} g_i$, as required. ◊

Claim 12.2.33. The functor (12.2.31) is full. Namely, let $i \in I$ and $\underline{M}_i$, $\underline{N}_i$ as in claim 12.2.32, and suppose that $f : \pi_i^*\underline{M}_i \to \pi_i^*\underline{N}_i$ is a given morphism of log structures; then there exist a morphism $\psi : j \to i$ in I, and a morphism of log structures $f_j : f_\psi^*\underline{M}_i \to f_\psi^*\underline{N}_i$ such that $\pi_j^* f_j = f$.

Proof of the claim. Indeed, by theorem 12.1.37(i) we may find a covering family $(U_\lambda \to X \mid \lambda \in \Lambda)$ in X_τ, and for each $\lambda \in \Lambda$ a finite chart

$$\beta_\lambda : P_{\lambda,U_\lambda} \to \underline{M}_{|U_\lambda} \quad \gamma_\lambda : Q_{\lambda,U_\lambda} \to \underline{N}_{|U_\lambda} \quad \vartheta_\lambda : P_\lambda \to Q_\lambda$$

for the restriction $f_{|U_\lambda}$. Clearly we may assume that each U_λ is affine, and since X is quasi-compact, we may assume as well that Λ is a finite set; in this case, claim 12.2.29 implies that there exist a morphism $j \to i$ in I, a covering family $(U_{j,\lambda} \to X_j \mid \lambda \in \Lambda)$, and isomorphism of X-schemes $U_\lambda \stackrel{\sim}{\to} U_{j,\lambda} \times_{X_j} X$ for every $\lambda \in \Lambda$. Next, lemma 12.2.26 says that, after replacing j by some larger index, we may assume that for every $\lambda \in \Lambda$ there exist charts

$$\beta_{j,\lambda} : P_{\lambda,U_{j,\lambda}} \to \underline{M}_j := f_\psi^* \underline{M}_i \qquad \text{and} \qquad \gamma_{j,\lambda} : Q_{\lambda,U_{j,\lambda}} \to \underline{N}_j := f_\psi^* \underline{N}_i$$

with $\pi_j^* \beta_{j,\lambda} = \beta_\lambda$ and $\pi_j^* \gamma_{j,\lambda} = \gamma_\lambda$. Set $f_{j,\lambda} := \vartheta^{\log}_{\lambda,U_{j,\lambda}} : \underline{M}_j \to \underline{N}_j$; by construction we have :

$$(\mathbf{1}_{U_{j,\lambda}} \times_{X_j} \pi_j)^* f_{j,\lambda} = f_{|U_\lambda} \qquad \text{for every } \lambda \in \Lambda.$$

Next, for every $\lambda, \mu \in \Lambda$, let $U_{j,\lambda\mu} := U_{j,\lambda} \times_{X_j} U_{j,\mu}$; we deduce, for every $\lambda, \mu \in \Lambda$, two morphisms of log structures $f_{j,\lambda|U_{j,\lambda\mu}}, f_{j,\mu|U_{\lambda\mu}} : \underline{M}_{j|U_{\lambda\mu}} \to \underline{N}_{j|U_{\lambda\mu}}$, which agree after pull back to $U_{j,\lambda\mu} \times_{X_j} X$. By applying claim 12.2.32 to the cofiltered system of schemes $(U_{j,\lambda\mu} \times_{X_j} X_{j'} \mid j' \in I/j)$, we may then achieve – after replacing j by some larger index – that $f_{j,\lambda|U_{j,\lambda\mu}} = f_{j,\mu|U_{\lambda\mu}}$, in which case the system $(f_{j,\lambda} \mid \lambda \in \Lambda)$ glues to a well defined morphism f_j as sought. ♢

Finally, let us show that (12.2.31) is essentially surjective. Indeed, let $\underline{M}$ be a coherent log structure on X; let us pick a covering family $\mathscr{U} := (U_\lambda \to X \mid \lambda \in \Lambda)$ and finite charts $\beta_\lambda : P_{\lambda,U_\lambda} \to \underline{M}_{|U_\lambda}$ for every $\lambda \in \Lambda$. As in the foregoing, we may assume that each U_λ is affine, and Λ is a finite set, in which case, according to claim 12.2.29(ii) we may find $i \in I$ and a covering family $(U_{i,\lambda} \to X_i \mid \lambda \in \Lambda)$ with isomorphisms of X-schemes $U_{i,\lambda} \times_{X_i} X \stackrel{\sim}{\to} U_\lambda$ for every $\lambda \in \Lambda$; for every $j \in I/i$ and every $\lambda \in \Lambda$, let us set $U_{j,\lambda} := U_{i,\lambda} \times_{X_i} X_j$. The composition $\alpha_\lambda : P_{\lambda,U_\lambda} \to \mathscr{O}_{U_\lambda}$ of β_λ and the structure map of $\underline{M}_{|U_\lambda}$ is determined by a morphism of monoids

$$P_\lambda \to \Gamma(U_\lambda, \mathscr{O}_{U_\lambda}) = \operatorname*{colim}_{i \in I/i} \Gamma(U_{j,\lambda}, \mathscr{O}_{U_{j,\lambda}}).$$

Then, as usual, lemma 6.1.7(ii) implies that there exists $j \in I/i$ such that α_λ descends to a pre-log structure $P_{\lambda,U_{j,\lambda}} \to \mathscr{O}_{U_{j,\lambda}}$ on $U_{j,\lambda}$, whose associated log structure we denote by $\underline{M}_{j,\lambda}$. For every $\lambda, \mu \in \Lambda$, let $U_{j,\lambda\mu} := U_{j,\lambda} \times_{X_j} U_{j,\mu}$; by construction we have isomorphisms

$$(12.2.34) \qquad (\mathbf{1}_{U_{j,\lambda\mu}} \times_{X_j} \pi_j)^* \underline{M}_{j,\lambda|U_{j,\lambda\mu}} \stackrel{\sim}{\to} (\mathbf{1}_{U_{j,\mu\lambda}} \times_{X_j} \pi_j)^* \underline{M}_{j,\mu|U_{j,\mu\lambda}} \qquad \text{for every } \lambda, \mu \in \Lambda.$$

By applying claim 12.2.33 to the cofiltered system $(U_{j,\lambda\mu} \times_{X_j} X_{j'} \mid j' \in I/j)$, we can then obtain – after replacing j by a larger index – isomorphisms $\omega_{\lambda\mu} : \underline{M}_{j,\lambda|U_{j,\lambda\mu}} \stackrel{\sim}{\to} M_{j,\mu|U_{j,\mu\lambda}}$ for every $\lambda, \mu \in \Lambda$, whose pull-back to $U_{j,\lambda\mu} \times_{X_j} X$ are the isomorphisms (12.2.34). Lastly, in view of claim 12.2.32, we may achieve – after further replacement of j by a larger index – that the system $\mathscr{D} := (\underline{M}_{j,\lambda}, \omega_{\lambda\mu} \mid \lambda, \mu \in \Lambda)$ is a descent datum for the fibration F of (12.2.1), whose pull-back to X is isomorphic to the natural descent datum for $\underline{M}$, associated to the covering family $\mathscr{U}$. Then $\mathscr{D}$ glues to a log structure $\underline{M}_j$, such that $\pi_j^* \underline{M}_j \simeq \underline{M}$. □

Corollary 12.2.35. *In the situation of* (10.1.8)*, suppose that* X_λ *is quasi-compact for every* $\lambda \in \mathrm{Ob}(\Lambda)$*, and that we have a morphism of log schemes with coherent log structures*

$$(g, \log g) : (Y, \underline{N}) \to (X, \underline{M}).$$

Then there exist $\lambda \in \Lambda$*, and a morphism of log schemes with coherent log structures*

$$(g_\lambda, \log g_\lambda) : (Y_\lambda, \underline{N}_\lambda) \to (X_\lambda, \underline{M}_\lambda) \qquad \textit{such that} \qquad \log g = \psi_\lambda^* \log g_\lambda.$$

Proof. The assertion is an immediate consequence of proposition 12.2.30. □

Corollary 12.2.36. *In the situation of* (10.1.8)*, let* $0 \in \mathrm{Ob}(\Lambda)$ *be an index such that* $\underline{N}_0$ *and* $\underline{M}_0$ *are coherent log structures on* Y_0*, and respectively* X_0*, and* $(g_0, \log g_0) : (Y_0, \underline{N}_0) \to (X_0, \underline{M}_0)$ *a morphism of log schemes. Suppose also that* X_λ *is quasi-compact (as well as quasi-separated) for every* $\lambda \in \mathrm{Ob}(\Lambda)$*; then we have :*

(i) *If the morphism of log schemes*

$$(g, \psi_0^* \log g_0) : Y \times_{Y_0} (Y_0, \underline{N}_0) \to X \times_{X_0} (X_0, \underline{M}_0)$$

admits a chart $(\omega, \omega', \vartheta : P \to Q)$*, there exists a morphism* $u : \lambda \to 0$ *in* Λ *such that the morphism of log schemes*

$$(g_\lambda, \psi_u^* \log g_0) : Y_\lambda \times_{Y_0} (Y_0, \underline{N}_0) \to X_\lambda \times_{X_0} (X_0, \underline{M}_0)$$

admits a chart $(\omega_\lambda, \omega'_\lambda, \vartheta)$.

(ii) *If* $(g, \psi_0^* \log g_0)$ *is a log flat (resp. saturated) morphism of fine log schemes, there exists a morphism* $u : \lambda \to 0$ *in* Λ *such that* $(g_\lambda, \psi_u^* \log g_0)$ *is a log flat (resp. saturated) morphism of fine log schemes.*

Proof. (i): Under the current assumptions, X and Y are quasi-compact. In view of lemma 12.2.26, we may then find a morphism $v : \mu \to 0$, such that ω and ω' descend to charts $\omega_v : P_{X_\mu} \to \varphi_v^* \underline{M}_0$ and $\omega'_v : Q_{Y_\mu} \to \psi_v^* \underline{N}_0$. We deduce two morphisms of pre-log structures $P_{Y_\mu} \to \psi_v^* \underline{N}_0$, namely

$$\beta_1 := \psi_v^*(\log g_0) \circ g_\mu^* \omega_v \qquad \text{and} \qquad \beta_2 := \omega'_v \circ \vartheta_{Y_\mu}$$

and by construction we have $\psi_\mu^* \beta_1^{\log} = \psi_\mu^* \beta_2^{\log}$. By proposition 12.2.30 it follows that there exists $w : \lambda \to \mu$ such that $\psi_w^* \beta_1^{\log} = \psi_w^* \beta_2^{\log}$. The latter means that $(\varphi_w^* \omega_v, \psi_w^* \omega'_v, \vartheta)$ is a chart for the morphism of log schemes $(g_\lambda, \psi_{v \circ w}^* \log g_0) : (Y_\lambda, \psi_{v \circ w}^* \underline{N}_0) \to (X_\lambda, \varphi_{v \circ w}^* \underline{M}_0)$, *i.e.* the claim holds with $u := v \circ w$.

(ii): Suppose therefore that $(g, \psi_0^* \log g_0)$ is a log flat (resp.saturated) morphism, and let $\mathscr{U} := (U_i \to X \mid i \in I)$ be a covering family for X_τ, such that $U_i \times_{X_0} (X_0, \underline{M}_0)$ admits a finite (resp. fine) chart, and U_i is affine for every $i \in I$. Since X is quasi-compact, we may assume that I is a finite set, and then there exists $\lambda \in \Lambda$ such that $\mathscr{U}$ descends to a covering family $\mathscr{U}_\lambda := (U_{\lambda,i} \to X_\lambda \mid i \in I)$ for $X_{\lambda,\tau}$ (claim 12.2.29(ii)). After replacing Λ by Λ/λ, we may assume that $\lambda = 0$, in which case we set $U_{\lambda,i} := U_{0,i} \times_{X_0} X_\lambda$ for every object λ of Λ, and every $i \in I$. Clearly, it suffices to show that there exists $u : \lambda \to 0$ such that $U_{\lambda,i} \times_{X_\lambda} (g_\lambda, \psi_u^* \log g_0)$ is flat (resp. saturated) for every $i \in I$. Set $Y'_{\lambda,i} := Y_\lambda \times_{X_\lambda} U_{\lambda,i}$ for every $\lambda \in \Lambda$; we may then replace the cofiltered system $\underline{X}$ and $\underline{Y}$, by respectively $(U_{\lambda,i} \mid \lambda \in \Lambda)$ and $(Y'_{\lambda,i} \mid \lambda \in \Lambda)$, which allows to assume from start, that $X \times_{X_0} (X_0, \underline{M}_0)$ admits a finite (resp. fine) chart. In this case, lemma 12.2.26 allows to further reduce to the case where $\underline{M}_0$ admits a finite (resp. fine) chart.

Then, by theorem 12.1.37(iii), we may find a covering family $\mathscr{V} := (V_j \to Y \mid j \in J)$ for Y_τ, consisting of finitely many affine schemes V_j, and for every $j \in J$, a flat (resp. saturated) and fine chart for the induced morphism $V_j \times_{Y_0} (Y_0, \underline{N}_0) \to X \times_{X_0} (X_0, \underline{M}_0)$. As in the foregoing, after replacing Λ by some category Λ/λ, we may assume that $\mathscr{V}$ descends to a covering family $\mathscr{V}_0 := (V_{0,j} \to Y_0 \mid j \in J)$ for $Y_{0,\tau}$, in which case we set $V_{\lambda,j} := V_{0,j} \times_{Y_0} Y_\lambda$ for every $\lambda \in \Lambda$. Clearly, it suffices to show that there exists $\lambda \in \Lambda$ such that the induced morphism $V_{\lambda,j} \times_{Y_0} (Y_0, \underline{N}_0) \to X_\lambda \times_{X_0} (X_0, \underline{M}_0)$ is log flat (resp. saturated). Thus, we may replace $\underline{Y}$ by the cofiltered system $(V_{\lambda,j} \mid \lambda \in \Lambda)$, which allows to assume that $(g, \psi_0^* \log g_0)$ admits a flat (resp. saturated) and fine chart. In this case, the assertion follows from (i). □

Proposition 12.2.37. *The inclusion functors :*

$$\mathbf{qf.log} \to \mathbf{qcoh.log} \qquad \mathbf{qfs.log} \to \mathbf{qf.log}$$

admit right adjoints :

$$\mathbf{qcoh}.\mathbf{log} \to \mathbf{qf}.\mathbf{log} \ : \ (X,\underline{M}) \mapsto (X,\underline{M})^{\mathrm{qf}} \qquad \mathbf{qf}.\mathbf{log} \to \mathbf{qfs}.\mathbf{log} \ : \ (X,\underline{M}) \mapsto (X,\underline{M})^{\mathrm{qfs}}.$$

Proof. Let $(X,\underline{M})$ be a scheme with quasi-coherent (resp. quasi-fine) log structure. We need to construct a morphism of log schemes

$$\varphi : (X,\underline{M})^{\mathrm{qf}} \to (X,\underline{M}) \qquad (\text{resp. } \varphi : (X,\underline{M})^{\mathrm{qfs}} \to (X,\underline{M}))$$

such that $(X,\underline{M})^{\mathrm{qf}}$ (resp. $(X,\underline{M})^{\mathrm{qfs}}$) is a quasi-fine (resp. qfs) log scheme, and the following holds. Every morphism of log schemes $\psi : (Y,\underline{N}) \to (X,\underline{M})$ with $(Y,\underline{N})$ quasi-fine (resp. qfs), factors uniquely through φ. To this aim, suppose first that $\underline{M}$ admits a chart (resp. a quasi-fine chart) $\alpha : P_X \to \underline{M}$. By lemma 12.2.18(i), α determines an isomorphism

$$(X,\underline{M}) \stackrel{\sim}{\to} \mathrm{Spec}(\mathbb{Z},P) \times_{\mathrm{Spec}\,\mathbb{Z}[P]} X.$$

Since $\underline{N}$ is integral (resp. and saturated), the morphism $(Y,\underline{N}) \to \mathrm{Spec}(\mathbb{Z},P)$ induced by ψ factors uniquely through $\mathrm{Spec}(\mathbb{Z},P^{\mathrm{int}})$ (resp. $\mathrm{Spec}(\mathbb{Z},P^{\mathrm{sat}})$) (lemma 12.2.18(ii)). Taking into account lemma 12.1.18(iii), it follows easily that we may take

$$(X,\underline{M})^{\mathrm{qf}} := \mathrm{Spec}(\mathbb{Z},P^{\mathrm{int}}) \times_{\mathbb{Z}[P]} X \qquad (X,\underline{M})^{\mathrm{qfs}} := \mathrm{Spec}(\mathbb{Z},P^{\mathrm{sat}}) \times_{\mathbb{Z}[P]} X$$

Next, notice that the universal property of $(X',\underline{M}') := (X,\underline{M})^{\mathrm{qf}}$ (resp. of $(X',\underline{M}') := (X,\underline{M})^{\mathrm{qfs}}$) is local on X_τ : namely, suppose that $(X',\underline{M}')$ has already been found, and let $U \to X$ be an object of X_τ, with a morphism $(Y,\underline{N}) \to (U,\underline{M}_{|U})$ from a quasi-fine (resp. qfs) log scheme; there follows a unique morphism

$$(Y,\underline{N}) \to (U,\underline{M}_{|U}) \times_{(X,\underline{M})} (X',\underline{M}') \stackrel{\sim}{\to} U \times_X (X',\underline{M}')$$

(notation of (12.2.6)). Thus $(U,\underline{M}_{|U})^{\mathrm{qf}} \simeq U \times_X (X,\underline{M})^{\mathrm{qf}}$ (resp. $(U,\underline{M}_{|U})^{\mathrm{qfs}} \simeq U \times_X (X,\underline{M})^{\mathrm{qfs}}$,) since the latter is a quasi-fine (resp. qfs) log scheme. Therefore, for a general quasi-coherent (resp. quasi-fine) log structure $\underline{M}$, choose a covering family $(U_\lambda \to X \mid \lambda \in \Lambda)$ such that $(U_\lambda,\underline{M}_{|U_\lambda})$ admits a chart for every $\lambda \in \Lambda$; it follows that the family $\mathscr{U} := ((U_\lambda,\underline{M}_{|U_\lambda})^{\mathrm{qf}} \mid \lambda \in \Lambda)$, together with the natural isomorphisms :

$$U_\mu \times_X (U_\lambda,\underline{M}_{|U_\lambda})^{\mathrm{qf}} \simeq U_\lambda \times_X (U_\mu,\underline{M}_{|U_\mu})^{\mathrm{qf}} \qquad \text{for every } \lambda,\mu \in \Lambda$$

is a descent datum for the fibred category over X_τ, whose fibre over any object $U \to X$ is the category of affine U-schemes endowed with a log structure (resp. likewise for the family $\mathscr{U} := ((U_\lambda,\underline{M}_{|U_\lambda})^{\mathrm{qfs}} \mid \lambda \in \Lambda)$). Using faithfully flat descent ([82, Exp.VIII, Th.2.1]), one sees that $\mathscr{U}$ comes from a quasi-fine (resp. qfs) log scheme which enjoys the sought universal property. □

Remark 12.2.38. (i) By inspecting the proof of proposition 12.2.37, we see that the quasi-fine log scheme associated to a coherent log scheme, is actually fine, and the qfs log scheme associated to a fine log scheme, is a fs log scheme (one applies corollary 6.4.1(i)). Hence we obtain functors

$$\mathbf{coh}.\mathbf{log} \to \mathbf{f}.\mathbf{log} \ : \ (X,\underline{M}) \mapsto (X,\underline{M})^{\mathrm{f}} \qquad \mathbf{f}.\mathbf{log} \to \mathbf{fs}.\mathbf{log} \ : \ (X,\underline{M}) \mapsto (X,\underline{M})^{\mathrm{fs}}$$

which are right adjoint to the inclusion functors $\mathbf{f}.\mathbf{log} \to \mathbf{coh}.\mathbf{log}$ and $\mathbf{fs}.\mathbf{log} \to \mathbf{f}.\mathbf{log}$.

(ii) Notice as well that, for every log scheme $(X,\underline{M})$ with quasi-coherent (resp. fine) log structure, the morphism of schemes underlying the counit of adjunction $(X,\underline{M})^{\mathrm{qf}} \to (X,\underline{M})$ (resp. $(X,\underline{M})^{\mathrm{fs}} \to (X,\underline{M})$) is a closed immersion (resp. is finite).

(iii) Furthermore, let $f : Y \to X$ be any morphism of schemes; the proof of proposition 12.2.37 also yields a natural isomorphism of $(Y,f^*\underline{M})$-schemes :

$$(Y,f^*\underline{M})^{\mathrm{qf}} \stackrel{\sim}{\to} Y \times_X (X,\underline{M})^{\mathrm{qf}} \qquad (\text{resp. } (Y,f^*\underline{M})^{\mathrm{qfs}} \stackrel{\sim}{\to} Y \times_X (X,\underline{M})^{\mathrm{qfs}}).$$

(iv) Let $(X, \underline{M})$ be a quasi-fine log scheme, and suppose that X is a normal, irreducible scheme, and $(X, \underline{M})_{\rm tr}$ is a dense subset of X. Denote by $X^{\rm qfs}$ the scheme underlying $(X, \underline{M})^{\rm qfs}$; then we claim that the projection $X^{\rm qfs} \to X$ (underlying the counit of adjunction) admits a natural section :

$$\sigma_X : X \to X^{\rm qfs}.$$

The naturality means that if $f : (X, \underline{M}) \to (Y, \underline{N})$ is any morphism of quasi-fine log schemes, where Y is also normal and irreducible, and $(Y, \underline{N})_{\rm tr}$ is dense in Y, then the induced diagram of schemes :

$$\begin{array}{ccc} X & \xrightarrow{\sigma_X} & X^{\rm qfs} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^{\rm qfs}} \\ Y & \xrightarrow{\sigma_Y} & Y^{\rm qfs} \end{array} \qquad (12.2.39)$$

commutes, and therefore it is cartesian, by virtue of (iii). Indeed, suppose first that X is affine, say $X = \operatorname{Spec} A$ for some normal domain A, and $\underline{M}$ admits an integral chart, given by a morphism $\beta : P \to A$, for some integral monoid P; we have to exhibit a ring homomorphism $P^{\rm sat} \otimes_P A \to A$, whose composition with the natural map $A \to P^{\rm sat} \otimes_P A$ is the identity of A. The latter is the same as the datum of a morphism of monoids $P^{\rm sat} \to A$ whose restriction to P agrees with β. However, since the trivial locus of $\underline{M}$ is dense in X, the image of P in A does not contain 0, hence β extends to a group homomorphism $\beta^{\rm gp} : P^{\rm gp} \to \operatorname{Frac}(A)^\times$; since A is integrally closed in $\operatorname{Frac}(A)$, we have $\beta^{\rm gp}(P^{\rm sat}) \subset A$, as required. Next, suppose that $U \to X$ is an object of X_τ, with U also affine and irreducible; then U is normal and the trivial locus of $(U, \underline{M}_{|U})$ is dense in U. Thus, the foregoing applies to $(U, \underline{M}_{|U})$ as well, and by inspecting the constructions we deduce a natural identification :

$$\sigma_U = \mathbf{1}_U \times_X \sigma_X.$$

Lastly, for a general $(X, \underline{M})$, we can find a covering family $(U_\lambda \to X \mid \lambda \to \Lambda)$ in X_τ, such that U_λ is affine, and $(U_\lambda, \underline{M}_{|U_\lambda})$ admits an integral chart for every $\lambda \in \Lambda$; proceeding as above, we obtain a system of morphisms $(\sigma_\lambda : U_\lambda \to U^{\rm qfs} \mid \lambda \in \Lambda)$, as well as natural identifications :

$$\mathbf{1}_{U_\mu} \times_X \sigma_\lambda = \mathbf{1}_{U_\lambda} \times_X \sigma_\mu \qquad \text{for every } \lambda, \mu \in \Lambda.$$

In other words, we have defined a descent datum for the category fibred over X_τ, whose fibre over any object $U \to X$ is the category of all morphisms of schemes $U \to U^{\rm qfs}$. By invoking faithfully flat descent ([82, Exp.VIII, Th.2.1]), we see that this descent datum yields a morphism $\sigma_X : X \to X^{\rm qfs}$ such that $\mathbf{1}_{U_\lambda} \times_X \sigma_X = \sigma_\lambda$ for every $\lambda \in \Lambda$. The verification that σ_X is a section of the projection $X^{\rm qfs} \to X$, and that (12.2.39) commutes, can be carried out locally on X_τ, in which case we can assume that $\underline{M}$ admits a chart as above, and one can check explicitly these assertions, by inspecting the constructions.

12.3. **Logarithmic differentials and smooth morphisms.** In this section we introduce the logarithmic version of the usual sheaves of relative differentials, and we study some special classes of morphisms of log schemes.

Definition 12.3.1. Let $(X, \underline{M} \stackrel{\alpha}{\to} \mathscr{O}_X)$ and $(Y, \underline{N} \stackrel{\beta}{\to} \mathscr{O}_Y)$ be two schemes with pre-log structures, and $f : (X, \underline{M}) \to (Y, \underline{N})$ a morphism of schemes with pre-log structures. Let also $\mathscr{F}$ be an $\mathscr{O}_X$-module. An *f-linear derivation of $\underline{M}$ with values in $\mathscr{F}$* is a pair $(\partial, \log \partial)$ consisting of maps of sheaves :

$$\partial : \mathscr{O}_X \to \mathscr{F} \qquad \log \partial : \log \underline{M} \to \mathscr{F}$$

such that :

- ∂ is a derivation (in the usual sense).
- $\log \partial$ is a morphism of sheaves of (additive) monoids on X_τ.

- $\partial \circ f^\natural = 0$ and $\log \partial \circ \log f = 0$.
- $\partial \circ \alpha(m) = \alpha(m) \cdot \log \partial(m)$ for every object U of X_τ, and every $m \in \underline{M}(U)$.

The set of all f-linear derivations with values in f shall be denoted by :

$$\mathrm{Der}_{(Y,\underline{N})}((X, \underline{M}), \mathscr{F}).$$

The f-linear derivations shall also be called $(Y, \underline{N})$-*linear derivations*, when there is no danger of ambiguity.

12.3.2. The set $\mathrm{Der}_{(Y,\underline{N})}((X, \underline{M}), \mathscr{F})$ is clearly functorial in $\mathscr{F}$, and moreover, for any object U of X_τ, any $s \in \mathscr{O}_X(U)$, and any f-linear derivation $(\partial, \log \partial)$, the restriction $(s\cdot\partial_{|U}, s\cdot\log \partial_{|U})$ is an element of $\mathrm{Der}_{(Y,\underline{N})}((U, \underline{M}_{|U}), \mathscr{F})$, hence the rule $U \mapsto \mathrm{Der}_{(Y,\underline{N})}((U, \underline{M}_{|U}), \mathscr{F})$ defines an $\mathscr{O}_X$-module :

$$\mathscr{D}er_{(Y,\underline{N})}((X, \underline{M}), \mathscr{F}).$$

In case $\underline{M} = \mathscr{O}_X^\times$ and $\underline{N} = \mathscr{O}_Y^\times$ are the trivial log structures, the f-linear derivations of $\underline{M}$ are the same as the usual f-linear derivations, *i.e.* the natural map

$$\mathscr{D}er_{(Y,\mathscr{O}_Y^\times)}((X, \mathscr{O}_X^\times), \mathscr{F}) \to \mathscr{D}er_Y(X, \mathscr{F}) \tag{12.3.3}$$

is an isomorphism. In the category of usual schemes, the functor of derivations is represented by the module of relative differentials. This construction extends to the category of schemes with pre-log structures. Namely, let us make the following :

Definition 12.3.4. Let $(X, \underline{M} \xrightarrow{\alpha} \mathscr{O}_X)$ and $(Y, \underline{N} \xrightarrow{\beta} \mathscr{O}_Y)$ be two schemes with pre-log structures, and $f : (X, \underline{M}) \to (Y, \underline{N})$ a morphism of schemes with pre-log structures. The *sheaf of logarithmic differentials* of f is the $\mathscr{O}_X$-module :

$$\Omega^1_{X/Y}(\log \underline{M}/\underline{N}) := (\Omega^1_{X/Y} \oplus (\mathscr{O}_X \otimes_{\mathbb{Z}} \log \underline{M}^{\mathrm{gp}}))/R$$

where R is the $\mathscr{O}_X$-submodule generated locally on X_τ by local sections of the form :

- $(d\alpha(a), -\alpha(a) \otimes \log a)$ with $a \in \underline{M}(U)$
- $(0, 1 \otimes \log a)$ with $a \in \mathrm{Im}((f^{-1}\underline{N})(U) \to \underline{M}(U))$

where U ranges over all the objects of X_τ (here we use the notation of (6.1)). For arbitrary $a \in \underline{M}(U)$, the class of $(0, 1 \otimes \log a)$ in $\Omega^1_{X/Y}(\log \underline{M}/\underline{N})$ shall be denoted by $d \log a$.

12.3.5. It is easy to verify that $\Omega^1_{X/Y}(\log \underline{M}/\underline{N})$ represents the functor

$$\mathscr{F} \mapsto \mathrm{Der}_{(Y,\underline{N})}((X, \underline{M}), \mathscr{F})$$

on $\mathscr{O}_X$-modules. Consequently, (12.3.3) translates as a natural isomorphism of $\mathscr{O}_X$-modules :

$$\Omega^1_{X/Y} \xrightarrow{\sim} \Omega^1_{X/Y}(\log \mathscr{O}_X^\times/\mathscr{O}_Y^\times). \tag{12.3.6}$$

Furthermore, let us fix a scheme with pre-log structure $(S, \underline{N})$, and define the category :

$$\mathbf{pre\text{-}log}/(S, \underline{N})$$

as in (1.1.1). Also, let $\mathbf{Mod.pre\text{-}log}/(S, \underline{N})$ be the category whose objects are all the pairs $((X, \underline{M}), \mathscr{F})$, where $(X, \underline{M})$ is a $(S, \underline{N})$-scheme, and $\mathscr{F}$ is an $\mathscr{O}_X$-module. The morphisms

$$((X, \underline{M}), \mathscr{F}) \to ((Y, \underline{N}), \mathscr{G})$$

in $\mathbf{Mod.pre\text{-}log}/(S, \underline{N})$ are the pairs (f, φ) consisting of a morphism $f : (X, \underline{M}) \to (Y, \underline{N})$ of $(S, \underline{N})$-schemes, and a morphism $\varphi : f^*\mathscr{G} \to \mathscr{F}$ of $\mathscr{O}_X$-modules. With this notation, we claim that the rule :

$$(X, \underline{M}) \mapsto ((X, \underline{M}), \Omega^1_{X/S}(\log \underline{M}/\underline{N})) \tag{12.3.7}$$

defines a functor $\mathbf{pre\text{-}log}/(S,\underline{N}) \to \mathbf{Mod}.\mathbf{pre\text{-}log}/(S,\underline{N})$. Indeed, slightly more generally, consider a commutative diagram of schemes with pre-log structures :

$$\begin{array}{ccc} (X,\underline{M}) & \xrightarrow{g} & (X',\underline{M}') \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ (S,\underline{N}) & \xrightarrow{h} & (S',\underline{N}'). \end{array} \tag{12.3.8}$$

An $\mathscr{O}_X$-linear map :

$$g^*\Omega^1_{X'/S'}(\log \underline{M}'/\underline{N}') \xrightarrow{dg} \Omega_{X/S}(\log \underline{M}/\underline{N}) \tag{12.3.9}$$

is the same as a natural transformation of functors :

$$\mathscr{D}er_{(S,\underline{N})}((X,\underline{M}),\mathscr{F}) \to \mathscr{D}er_{(S',\underline{N}')}((X',\underline{M}'),g_*\mathscr{F}) \tag{12.3.10}$$

on all $\mathscr{O}_X$-modules $\mathscr{F}$. The latter can be defined as follows. Let $(\partial,\log\partial)$ be an $(S,\underline{N})$-linear derivation of $\underline{M}$ with values in $\mathscr{F}$; then we deduce morphisms :

$$\partial' : \mathscr{O}_{X'} \xrightarrow{g^\natural} g_*\mathscr{O}_X \xrightarrow{g_*\partial} g_*\mathscr{F} \qquad \log\partial' : \underline{M}' \xrightarrow{\log g} g_*\underline{M} \xrightarrow{g_*\log\partial} g_*\mathscr{F}$$

and it is easily seen that $(\partial',\log\partial')$ is a $(S',\underline{N}')$-linear derivation of $\underline{M}'$ with values in $g_*\mathscr{F}$.

12.3.11. Consider two morphisms $(X,\underline{M}) \xrightarrow{f} (Y,\underline{N}) \xrightarrow{g} (Z,\underline{P})$ of schemes with pre-log structures. A direct inspection of the definitions shows that :

$$\mathscr{D}er_{(Y,\underline{N})}((X,\underline{M}),\mathscr{F}) = \mathrm{Ker}(\mathscr{D}er_{(Z,\underline{P})}((X,\underline{M}),\mathscr{F}) \to \mathscr{D}er_{(Z,\underline{P})}((Y,\underline{N}),f_*\mathscr{F}))$$

for every $\mathscr{O}_X$-module $\mathscr{F}$, whence a right exact sequence of $\mathscr{O}_X$-modules :

$$f^*\Omega^1_{Y/Z}(\log \underline{N}/\underline{P}) \xrightarrow{df} \Omega^1_{X/Z}(\log \underline{M}/\underline{P}) \to \Omega^1_{X/Y}(\log \underline{M}/\underline{N}) \to 0 \tag{12.3.12}$$

extending the standard right exact sequence for the usual sheaves of relative differentials.

Proposition 12.3.13. *Suppose that the diagram* (12.3.8) *is cartesian. Then the map* (12.3.9) *is an isomorphism.*

Proof. If (12.3.8) is cartesian, X is the scheme $X' \times_{S'} S$, and $\underline{M}$ is the push-out of the diagram:

$$f^{-1}\underline{N} \leftarrow (f' \circ g)^{-1}\underline{N}' \xrightarrow{\varphi} g^*\underline{M}'.$$

Suppose now that $\log\partial : \underline{M}' \to g_*\mathscr{F}$ and $\partial : \mathscr{O}_{X'} \to g_*\underline{\mathscr{F}}$ define a f'-linear derivation of $\underline{M}'$. By adjunction, we deduce morphisms $\alpha : g^{-1}\mathscr{O}_{X'} \to \mathscr{F}$ and $\beta : g^{-1}\underline{M}' \to \mathscr{F}$. By construction, we have $\beta \circ \varphi = 0$, hence β extends uniquely to a morphism $\log\partial' : \underline{M} \to \mathscr{F}$ such that $\log\partial' \circ \log f = 0$. Likewise, α extends by linearity to a unique f-linear derivation $\partial : \mathscr{O}_X \to \mathscr{F}$. One checks easily that $(\partial',\log\partial')$ is a f-linear derivation of $\underline{M}$, and that every f-linear derivation of $\underline{M}$ with values in $\mathscr{F}$ is obtained in this fashion. ◻

12.3.14. The functor (12.3.7) admits a left adjoint. Indeed, let $((X,\underline{M} \xrightarrow{\alpha} \mathscr{O}_X),\mathscr{F})$ be any object of $\mathbf{Mod}.\mathbf{pre\text{-}log}/(S,\underline{N})$; we define an $(S,\underline{N})$-scheme $(X \oplus \mathscr{F},\underline{M} \oplus \mathscr{F})$ as follows. $X \oplus \mathscr{F}$ is the spectrum of the $\mathscr{O}_X$-algebra $\mathscr{O}_X \oplus \mathscr{F}$, whose multiplication law is given by the rule :

$$(s,t)\cdot(s',t') := (ss',st'+s't) \qquad \text{for every local section } s \text{ of } \mathscr{O}_X \text{ and } t \text{ of } \mathscr{F}.$$

Likewise, we define a composition law on the sheaf $\underline{M} \oplus \mathscr{F}$, by the rule :

$$(m,t)\cdot(m',t') := (mm',\alpha(m)\cdot t' + \alpha(m')\cdot t) \qquad \text{for every local section } m \text{ of } \underline{M} \text{ and } t \text{ of } \mathscr{F}.$$

Then $\underline{M} \oplus \mathscr{F}$ is a sheaf of monoids, and α extends to a pre-log structure $\alpha \oplus \mathbf{1}_\mathscr{F} : \underline{M} \oplus \mathscr{F} \to \mathscr{O}_X \oplus \mathscr{F}$. The natural map $\mathscr{O}_X \to \mathscr{O}_X \oplus \mathscr{F}$ is a morphism of algebras, whence a natural map of schemes $\pi : X \oplus \mathscr{F} \to X$, which extends to a morphism of schemes with pre-log structures $(X \oplus \mathscr{F}, \underline{M} \oplus \mathscr{F}) \to (X, \underline{M})$, by letting $\log \pi : \pi^* \underline{M} \to \underline{M} \oplus \mathscr{F}$ be the map induced by the natural monomorphism (notice that π^* induces an equivalence of sites $X_\tau \xrightarrow{\sim} (X \oplus \mathscr{F})_\tau$).

Now, let $(Y, \underline{P})$ be any $(S, \underline{N})$-scheme, and :

$$\varphi : \mathscr{F} := ((X, \underline{M}), \mathscr{F}) \to \Omega := ((Y, \underline{P}), \Omega^1_{Y/S}(\log \underline{P}/\underline{N}))$$

a morphism in $\mathbf{Mod.pre\text{-}log}/(S, \underline{N})$. By definition, φ consists of a morphism $f : (X, \underline{M}) \to (Y, \underline{P})$ and an $\mathscr{O}_X$-linear map $f^*\Omega^1_{Y/S}(\log \underline{P}/\underline{N}) \to \mathscr{F}$, which is the same as a $(S, \underline{N})$-linear derivation :

$$\partial : \mathscr{O}_Y \to f_*\mathscr{F} \qquad \log \partial : \underline{P} \to f_*\mathscr{F}.$$

In turns, the latter yields a morphism of $(S, \underline{N})$-schemes :

$$(Y \oplus f_*\mathscr{F}, \underline{P} \oplus f_*\mathscr{F}) \to (Y, \underline{P}) \tag{12.3.15}$$

determined by the map of algebras :

$$\mathscr{O}_Y \to \mathscr{O}_Y \oplus f_*\mathscr{F} \qquad s \mapsto (s, \partial s) \quad \text{for every local section } s \text{ of } \mathscr{O}_Y$$

and the map of monoids :

$$\underline{P} \mapsto \underline{P} \oplus f_*\mathscr{F} \qquad p \mapsto (p, \log \partial p) \quad \text{for every local section } p \text{ of } \underline{P}.$$

Finally, we compose (12.3.15) with the natural morphism

$$(X \oplus \mathscr{F}, \underline{M} \oplus \mathscr{F}) \to (Y \oplus f_*\mathscr{F}, \underline{P} \oplus f_*\mathscr{F})$$

that extends f, to obtain a morphism $g_\varphi : (X \oplus \mathscr{F}, \underline{M} \oplus \mathscr{F}) \to (Y, \underline{P})$. We leave to the reader the verification that the rule $\varphi \mapsto g_\varphi$ establishes a natural bijection :

$$\mathrm{Hom}_{\mathbf{Mod.pre\text{-}log}/(S,\underline{N})}(\mathscr{F}, \Omega) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{pre\text{-}log}/(S,\underline{N})}((X \oplus \mathscr{F}, \underline{M} \oplus \mathscr{F}), (Y, \underline{P})).$$

12.3.16. In the situation of definition 12.3.4, let $(\partial, \log \partial)$ be an f-linear derivation of $\underline{M}$ with values in an $\mathscr{O}_X$-module $\mathscr{F}$. Consider the map :

$$\partial' : \mathscr{O}_X^\times \to \mathscr{F} \quad : \quad u \mapsto u^{-1} \cdot \partial u \quad \text{for all local sections } u \text{ of } \mathscr{O}_X^\times.$$

By definition, $\partial' \circ \alpha : \alpha^{-1}\underline{M} \to \mathscr{F}$ agrees with the restriction of $\log \partial$; in view of the cocartesian diagram (12.1.7), we deduce that $\log \partial$ extends uniquely to an f-linear derivation $\log \partial^{\log}$ of $\underline{M}^{\log}$. Let $f^{\log} : (X, \underline{M}^{\log}) \to (Y, \underline{N}^{\log})$ be the map deduced from f; a similar direct verification shows that $\log \partial^{\log}$ is a $f^{\log}$-linear derivation. There follow natural identifications :

$$\Omega^1_{X/Y}(\log(\underline{M}/\underline{N})) = \Omega^1_{X/Y}(\log(\underline{M}^{\log}/\underline{N})) = \Omega^1_{X/Y}(\log(\underline{M}^{\log}/\underline{N}^{\log})). \tag{12.3.17}$$

Moreover, if $\underline{M}$ and $\underline{N}$ are log structures, the natural map :

$$\mathscr{O}_X \otimes_\mathbb{Z} \log \underline{M}^{\mathrm{gp}} \to \Omega^1_{X/Y}(\log(\underline{M}/\underline{N})) \qquad a \otimes b \mapsto a \cdot d\log(b) \tag{12.3.18}$$

is an epimorphism. Indeed, we have $da = d(a+1)$ for every local section $a \in \mathscr{O}_X(U)$ (for any étale X-scheme U), and locally on X_τ, either a or $1 + a$ is invertible in $\mathscr{O}_X$ (this holds certainly on the stalks, hence on appropriate small neighborhoods $U' \to U$); hence da lies in the image of (12.3.18).

Example 12.3.19. Let R be a ring, $\varphi : N \to M$ be any map of monoids, and set :

$$S := \mathrm{Spec}\, R \qquad S[M] := \mathrm{Spec}\, R[M] \qquad S[N] := \mathrm{Spec}\, R[N].$$

Also, let $f : \mathrm{Spec}(R, M) \to \mathrm{Spec}(R, N)$ be the morphism of log schemes induced by φ (see (12.2.13)). With this notation, we claim that (12.3.18) induces an isomorphism :

$$\mathscr{O}_{S[M]} \otimes_\mathbb{Z} \mathrm{Coker}\, \varphi^{\mathrm{gp}} \xrightarrow{\sim} \Omega^1_{S[M]/S[N]}(\log M^{\log}_{S[M]}/N^{\log}_{S[N]})$$

To see this, we may use (12.3.12) to reduce to the case where $N = \{1\}$. Next, notice that the functor

$$\mathbf{Mnd}^o \to \mathbf{pre\text{-}log}/(S, \mathscr{O}_S^\times) \quad : \quad M \mapsto \mathrm{Spec}(R, M)$$

commutes with limits, and the same holds for the functor (12.3.7), since the latter is a right adjoint. Hence, we may assume that M is finitely generated; then lemma 6.1.7(i) further reduces to the case where $M = \mathbb{N}^{\oplus n}$ for some integer $n \geq 0$, and even to the case where $n = 1$. Set $X := S[\mathbb{N}]$; it is easy to see that a S-linear derivation of $\mathbb{N}_X^{\log}$ with values in an $\mathscr{O}_X$-module $\mathscr{F}$, is completely determined by a map of additive monoids $\mathbb{N} \to \Gamma(X, \mathscr{F})$, and the latter is the same as an $\mathscr{O}_X$-linear map $\mathscr{O}_X \to \mathscr{F}$, whence the contention.

Lemma 12.3.20. *Let $f : (X, \underline{M}) \to (Y, \underline{N})$ be a morphism of schemes with quasi-coherent log structures. Then :*

(i) *The $\mathscr{O}_X$-module $\Omega^1_{X/Y}(\log \underline{M}/\underline{N})$ is quasi-coherent.*
(ii) *If $\underline{M}$ is coherent, X is noetherian, and $f : X \to Y$ is locally of finite type, then $\Omega^1_{X/Y}(\log \underline{M}/\underline{N})$ is a coherent $\mathscr{O}_X$-module.*
(iii) *If both $\underline{M}$ and $\underline{N}$ are coherent, and $f : X \to Y$ is locally of finite presentation, then $\Omega^1_{X/Y}(\log \underline{M}/\underline{N})$ is a quasi-coherent $\mathscr{O}_X$-module of finite presentation.*

Proof. Applying the right exact sequence (12.3.12) to the sequence $(X, \underline{M}) \xrightarrow{f} (Y, \underline{N}) \to (Y, \mathscr{O}_Y^\times)$, we may easily reduce to the case where $\underline{N} = \mathscr{O}_Y^\times$ is the trivial log structure on Y. In this case, $\underline{N}$ admits the chart given by the unique map of monoids : $\{1\} \to \Gamma(Y, \mathscr{O}_Y)$, and f admits the chart $\{1\} \to P$, whenever $\underline{M}$ admits a chart $P \to \Gamma(X, \mathscr{O}_X)$.

Hence, everything follows from the following assertion, whose proof shall be left to the reader. Suppose that $\varphi : A \to B$ is a ring homomorphism, $\underline{M}$ (resp. $\underline{N}$) is the constant log structure on $X := \mathrm{Spec}\, B$ (resp. on $Y := \mathrm{Spec}\, A$) associated to a map of monoids $\alpha : P \to B$ (resp. $\beta : Q \to A$), and $f : (X, \underline{M}) \to (Y, \underline{N})$ is defined by φ admits a chart $\vartheta : Q \to P$. Then $\Omega^1_{X/Y}(\log \underline{M}/\underline{N})$ is the quasi-coherent $\mathscr{O}_X$-module $L^\sim$, associated to the B-module $L := (\Omega^1_{B/A} \oplus (B \otimes_\mathbb{Z} P^{\mathrm{gp}}))/R$, where R is the submodule generated by the elements of the form $(0, 1 \otimes \log \vartheta(q))$ for all $q \in Q$, and those of the form $(d\alpha(m), -\alpha(m) \otimes \log m)$, for all $m \in M$. $\square$

12.3.21. Let us fix a log scheme $(Y, \underline{N})$. To any pair of $(Y, \underline{N})$-schemes $X := (X, \underline{M})$, $X' := (X', \underline{M}')$, we attach a contravariant functor

$$\mathscr{H}_Y(X', X) : (X'_\tau)^o \to \mathbf{Set}$$

by assigning, to every object U of X'_τ, the set of all morphisms $(U, \underline{M}'_{|U}) \to (X, \underline{M})$ of $(Y, \underline{N})$-schemes. It is easily seen that $\mathscr{H}_Y(X', X)$ is a sheaf on X'_τ. Any morphism $\varphi : (X'', \underline{M}'') \to (X', \underline{M}')$ (resp. $\psi : (X, \underline{M}) \to (X'', \underline{M})$) of $(Y, \underline{N})$-schemes induces a map of sheaves :

$$\varphi^* : \varphi^* \mathscr{H}_Y(X', X) \to \mathscr{H}_Y(X'', X) \qquad (\text{resp. } \psi_* : \mathscr{H}_Y(X', X) \to \mathscr{H}_Y(X', X''))$$

in the obvious way.

Definition 12.3.22. With the notation of (12.3.21) :

(i) We say that a morphism $i : (T', \underline{L}') \to (T, \underline{L})$ of log schemes is a *closed immersion* (resp. an *exact closed immersion*) if the underlying morphism of schemes $T' \to T$ is a closed immersion, and $\log i : i^*\underline{L} \to \underline{L}'$ is an epimorphism (resp. an isomorphism) of T'_τ-monoids.
(ii) We say that a morphism $i : (T', \underline{L}') \to (T, \underline{L})$ of log schemes is an *exact nilpotent immersion* if i is an exact closed immersion, and the ideal $\mathscr{I} := \mathrm{Ker}(\mathscr{O}_T \to i_*\mathscr{O}_{T'})$ is nilpotent.

(iii) We say that a morphism $f : (X, \underline{M}) \to (Y, \underline{N})$ of log schemes is *formally smooth* (resp. *formally unramified*, resp. *formally étale*) if, for every exact nilpotent immersion $i : T' \to T$ of fine $(Y, \underline{N})$-schemes, the induced map of sheaves $i^* : i^*\mathscr{H}_Y(T, X) \to \mathscr{H}_Y(T', X)$ is an epimorphism (resp. a monomorphism, resp. an isomorphism).

(iv) We say that a morphism $f : (X, \underline{M}) \to (Y, \underline{N})$ of log schemes is *smooth* (resp. *unramified*, resp. *étale*) if the underlying morphism $X \to Y$ is locally of finite presentation, and f is formally smooth (resp. formally unramified, resp. formally étale).

Example 12.3.23. Let $(S, \underline{P})$ be a log scheme, $(f, \log f) : (X, \underline{M}) \to (Y, \underline{N})$ be a morphism of S-schemes, such that Y is a separated S-scheme. The pair $(\mathbf{1}_{(X,\underline{M})}, (f, \log f))$ induces a morphism

$$\Gamma_f : (X, \underline{M}) \to (X', \underline{M}') := (X, \underline{M}) \times_S (Y, \underline{N})$$

the *graph* of f. Then it is easily seen that Γ_f is a closed immersion of log schemes. Indeed, the morphism of schemes underlying Γ_f is a closed immersion ([59, Ch.I, 5.4.3]) and it is easily seen that the morphism $\log \Gamma_f : \Gamma_f^*\underline{M}' \to \underline{M}$ is an epimorphism on the underlying sheaves of sets, so it is *a fortiori* an epimorphism of X_τ-monoids.

Proposition 12.3.24. *Let $f : (X, \underline{M}) \to (Y, \underline{N})$, $g : (Y, \underline{N}) \to (Z, \underline{P})$, and $h : (Y', \underline{N}') \to (Y, \underline{N})$ be morphisms of log schemes. Denote by $\mathbf{P}$ either one of the properties : "formally smooth", "formally unramified", "formally étale", "smooth", "unramified", "étale". The following holds :*

(i) *If f and g enjoy the property $\mathbf{P}$, then the same holds for $g \circ f$.*

(ii) *If $(f, \log f)$ enjoys the property $\mathbf{P}$, then the same holds for*

$$(f, \log f) \times_{(Y,\underline{N})} (Y', \underline{N}') : (X, \underline{M}) \times_{(Y,\underline{N})} (Y', \underline{N}') \to (Y', \underline{N}').$$

(iii) *Let $(j_\lambda : U_\lambda \to X \mid \lambda \in \Lambda)$ be a covering family in X_τ; endow U_λ with the log structure $(\underline{M})_{|U_\lambda}$ and suppose that $f_\lambda := (f \circ j_\lambda, (\log f)_{|U_\lambda}) : (U_\lambda, (\underline{M})_{|U_\lambda}) \to (Y, \underline{N})$ enjoys the property $\mathbf{P}$, for every $\lambda \in \Lambda$. Then f enjoys the property $\mathbf{P}$ as well.*

(iv) *An open immersion of log schemes is étale.*

(v) *A closed immersion of log schemes is formally unramified.*

Proof. (i), (ii) and (iv) are left to the reader.

(iii): To begin with, if each f_λ is locally of finite presentation, the same holds for f, by [66, Ch.IV, lemme 17.7.5], hence we may assume that $\mathbf{P}$ is either "formally smooth" or "formally unramified". (Clearly, the case where $\mathbf{P}$ is "formally étale" will follow.)

Now, let $i : T' \to T$ be an exact nilpotent immersion of $(Y, \underline{N})$-schemes, and ξ a τ-point of T'. By inspecting the definitions, it is easily seen that the stalk $\mathscr{H}_Y(T, X)_\xi$ is the union of the images of the stalks $\mathscr{H}_Y(T, U_\lambda)_\xi$, for every $\lambda \in \Lambda$, and likewise for $\mathscr{H}_Y(T', X)_\xi$. It readily follows that f is formally smooth whenever all the f_λ are formally smooth.

Lastly, suppose that all the f_λ are formally unramified, and let $s_\xi, t_\xi \in \mathscr{H}_Y(T, X)_{i(\xi)}$ be two sections whose images in $\mathscr{H}_Y(T', X)_\xi$ agree; after replacing T by a neighborhood of $i(\xi)$ in T_τ, we may assume that s_ξ, t_ξ are represented by two $(Y, \underline{N})$-morphisms $s, t : T \to (X, \underline{M})$ such that $s \circ i = t \circ i$. Choose $\lambda \in \Lambda$ such that U_λ is a neighborhood of $s \circ i(\xi) = t \circ i(\xi)$; this means that there exist a neighborhood $p' : U' \to T'$ of ξ, and a morphism $s_{U'} : U' \to U_\lambda$ lifiting $s \circ i$ (and thus, also $t \circ i$). Then we may find a neighborhood $p : U \to T$ of $i(\xi)$ which identifies p' with $p \times_T \mathbf{1}_{T'}$ ([66, Ch.IV, Th.18.1.2]), and since j_λ is étale, we may furthermore find morphisms $s_U, t_U : U \to U_\lambda$ such that $j_\lambda \circ s_U = s \circ i = t \circ i = j_\lambda \circ t_U$. Set $i_U := i \times_T \mathbf{1}_U : U' \to U$; by construction, s_U and t_U yield two sections of $i_U^*\mathscr{H}_Y(U, U_\lambda)_\xi$, whose images in $\mathscr{H}_Y(U', U_\lambda)_\xi$ coincide. Since f_λ is formally unramified, it follows that – up to replacing U by a neighborhood of $i(\xi)$ in U_τ – we must have $s_U = t_U$, so $s_\xi = t_\xi$, and we conclude that f is formally unramified.

(v): Consider a commutative diagram of log schemes :

$$\begin{array}{ccc} (T',\underline{L}') & \xrightarrow{h'} & (X,\underline{M}) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\ (T,\underline{L}) & \xrightarrow{h} & (Y,\underline{N}) \end{array}$$

where f is a closed immersion, and i is an exact closed immersion. We are easily reduced to showing that there exists at most a morphism $(g, \log) : (T, \underline{L}) \to (X, \underline{M})$ such that $f \circ g = h$ and $h' = g \circ i$. Since $f : X \to Y$ is a closed immersion of schemes, there exists at most one morphism of schemes $g : T \to X$ lifting h and extending h' ([66, Ch.IV, Prop.17.1.3(i)]). Hence we may assume that such a g is given, and we need to check that there exists at most one morphism $\log g : g^*\underline{M} \to \underline{L}$ whose composition with $g^*(\log f) : h^*\underline{N} \to g^*\underline{M}$ equals $\log h$. However, by assumption $\log f$ is an epimorphism, hence the same holds for $g^*(\log f)$ (proposition 1.3.25(iv)), whence the contention. □

Corollary 12.3.25. *Let f and g be as proposition* 12.3.24. *We have :*

(i) *If $g \circ f$ is formally unramified, the same holds for f.*
(ii) *If $g \circ f$ is formally smooth (resp. formally étale) and g is formally unramified, then f is formally smooth (resp. formally étale).*
(iii) *Suppose that g is formally étale. Then f is formally smooth (resp. formally unramified, resp. formally étale) if and only if the same holds for $g \circ f$.*

Proof. (i): Let $Y = \bigcup_{\lambda\in\Lambda}$ be an affine open covering of Y, and for each $\lambda \in \Lambda$, let $f_\lambda : U_\lambda \times_Y (X, \underline{M}) \to (U_\lambda, \underline{N}_{|U_\lambda})$ be the restriction of f; in light of proposition 12.3.24(iii) it suffices to show that each f_λ is formally unramified, and on the other hand, the restriction $g \circ f_\lambda : U_\lambda \times_Y (X, \underline{M}) \to (Z, \underline{P})$ of $g \circ f$ is formally unramified, by proposition 12.3.24(i),(iv). It follows that we may replace f and g respectively by f_λ and g_λ, which allows to assume that Y is affine, especially separated, so that g is a separated morphism of schemes. In such situation, one may – in view of example 12.3.23 – argue as in the proof of [66, Ch.IV, Prop.17.1.3] : the details shall be left to the reader.

(ii) is a formal consequence of the definitions (cp. the proof of [66, Ch.IV, Prop.17.1.4]), and (iii) follows from (ii) and proposition 12.3.24(i). □

Proposition 12.3.26. *Let $f : (X, \underline{M}) \to (Y, \underline{N})$ be a morphism of log schemes, and i an exact closed immersion of $(Y, \underline{N})$-schemes, defined by an ideal $\mathscr{I} := \mathrm{Ker}(\mathscr{O}_T \to i_*\mathscr{O}_{T'})$ with $\mathscr{I}^2 = 0$. For any global section $s : T' \to \mathscr{H}_Y(T', X)$, denote by $\mathscr{T}_s$ the morphism :*

$$i^*\mathscr{H}_Y(T, X) \times_{\mathscr{H}_Y(T',X)} T' \to T'$$

deduced from $i^* : i^*\mathscr{H}_Y(T, X) \to \mathscr{H}_Y(T', X)$. Let also $U \subset T'$ be the image *of $\mathscr{T}_s$* (i.e. *the subset of all $t' \in T'$ such that $\mathscr{T}_{s,\xi} \neq \varnothing$ for every τ-point ξ localized at t'), and suppose that $U \neq \varnothing$. We have :*

(i) *U is an open subset of T', and $\mathscr{T}_{s|U}$ is a torsor for the abelian sheaf*

$$\mathscr{G} := \mathscr{H}om_{\mathscr{O}_{T'}}(s^*\Omega^1_{X/Y}(\log(\underline{M}/\underline{N})), \mathscr{I})_{|U}.$$

(ii) *If f is a smooth morphism of log schemes with coherent log structures, we have :*
 (a) *The $\mathscr{O}_X$-module $\Omega^1_{X/Y}(\log(\underline{M}/\underline{N}))$ is locally free of finite type.*
 (b) *If T' is affine, $\mathscr{T}_s$ is a trivial $\mathscr{G}$-torsor.*

Proof. (i): To any τ-point ξ of U, and any two given local sections h and g of $\mathscr{T}_{s,\xi}$, we assign the f-linear derivation of $\underline{M}_{s(\xi)}$ with values in $s_*\mathscr{I}_\xi$ given by the rule :

$$\begin{aligned} a &\mapsto h^*(a) - g^*(a) && \text{for every } a \in \mathscr{O}_{X,s(\xi)} \\ d\log m &\mapsto \log u(m) := u(m) - 1 && \text{for every } m \in \underline{M}_{s(\xi)} \end{aligned}$$

where $u(m)$ is the unique local section of $\mathrm{Ker}(\mathscr{O}_{T,\xi} \to i_*\mathscr{O}_{T',\xi})$ such that

$$\log(g(m) \cdot u(m)) = \log h(m).$$

We leave to the reader the laborious, but straightforward verification that the above map is well-defined, and yields the sought bijection between $\mathscr{T}_{s,\xi}$ and $\mathscr{G}_\xi$.

(ii.b) is standard : first, since f is smooth, we have $U = T'$; by lemma 12.3.20(iii), the $\mathscr{O}_X$-module $\Omega^1_{X/Y}(\log(\underline{M}/\underline{N}))$ is quasi-coherent and finitely presented, hence $\mathscr{G}$ is quasi-coherent; however, the obstruction to gluing local sections of a $\mathscr{G}$-torsor lies in $H^1(T_\tau, \mathscr{G})$ (see (4.9.11)); the latter vanishes whenever T is affine.

(ii.a) We may assume that X is affine, say $X = \mathrm{Spec}\, A$; then $\Omega^1_{X/Y}(\log(\underline{M}/\underline{N}))$ is the quasi-coherent $\mathscr{O}_X$-module arising from a finitely presented A-module Ω (lemma 12.3.20(iii)). Let I be any A-module, and set :

$$T_I := X \oplus I^\sim \qquad \underline{L}_I := \underline{M} \oplus I^\sim$$

(notation of (12.3.14)). Let $i : X \to T_I$ (resp. $\pi : T_I \to X$) be the natural closed immersion (resp. the natural projection); then $(T_I, \underline{L}_I)$ is a fine log scheme, and π (resp. i) extends to a morphism of log schemes $(\pi, \log \pi) : (T_I, \underline{L}_I) \to (X, \underline{M})$ (resp. $(i, \log i) : (X, \underline{M}) \to (T_I, \underline{L}_I)$) with $\log \pi : \pi^*\underline{M} \to \underline{L}_I$ (resp. $\log i : i^*\underline{L}_I \to \underline{M}$) induced by the obvious inclusion (resp. projection) map. Notice that $(i, \log i)$ is an exact immersion and $(I^\sim)^2 = 0$, hence (i) says that the set $\mathscr{T}(T_I)$ of all morphisms $g : (T_I, \underline{L}_I) \to (X, \underline{M})$ such that

$$f \circ g = f \circ \pi \qquad g \circ i = \mathbf{1}_{(X,\underline{M})}$$

is in bijection with $\mathrm{Hom}_A(\Omega, I)$. Moreover, any map $\varphi : I \to J$ of A-modules induces a morphism $i_\varphi : (T_J, \underline{L}_J) \to (T_I, \underline{L}_I)$ of log schemes, and if φ is surjective, i_φ is an exact nilpotent immersion. Furthermore, we have a commutative diagram of sets :

$$\begin{array}{ccc} \mathscr{T}(T_I) & \xrightarrow{\ i^*_\varphi\ } & \mathscr{T}(T_J) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_A(\Omega, I) & \xrightarrow{\ \varphi_*\ } & \mathrm{Hom}_A(\Omega, J) \end{array}$$

whose vertical arrows are bijections, and where i^*_φ (resp. φ_*) is given by the rule $g \mapsto g \circ i_\varphi$, (resp. $\psi \mapsto \varphi \circ \psi$). Assertion (ii.b) implies that ι^*_φ is surjective when f is smooth and φ is surjective, hence the same holds for φ_*, *i.e.* Ω is a projective A-module, as stated. □

Corollary 12.3.27. *Let $(f, \log f) : (X, \underline{M}) \to (Y, \underline{N})$ be a morphism of log schemes. We have:*

(i) *If $\underline{M}$ and $\underline{N}$ are fine log structures, and f is strict (see definition* 12.1.22(ii)*), then $(f, \log f)$ is smooth (resp. étale) if and only if the underlying morphism of schemes $f : X \to Y$ is smooth (resp. étale).*

(ii) *Suppose that $(f, \log f)$ is a smooth (resp. étale) morphism of log schemes on the Zariski sites of X and Y, and either :*

  (a) *$\underline{M}$ and $\underline{N}$ are both integral log structures,*

  (b) *or else $\underline{N}$ is coherent (on Y_{Zar}) and $\underline{M}$ is fine (on X_{Zar}).*

  Then the induced morphism of log schemes on étale sites :

$$\widetilde{u}^*(f, \log f) := (f, \widetilde{u}^*_X \log f) : \widetilde{u}^*_X(X, \underline{M}) \to \widetilde{u}^*_Y(Y, \underline{N})$$

is smooth (resp. étale). (Notation of (12.2.2).*)*

(iii) *Suppose that* $(f, \log f)$ *is a morphism of log schemes on the Zariski sites of* X *and* Y*, and* $\underline{M}$ *is an integral log structure on* X_{Zar}*. Suppose also that* $\widetilde{u}^*(f, \log f)$ *is smooth (resp. étale). Then the same holds for* $(f, \log f)$*.*

Proof. (i): Suppose first that $(f, \log f)$ is smooth (resp. étale). Let $i : T' \to T$ be a nilpotent immersion of affine schemes, defined by an ideal $\mathscr{I} \subset \mathscr{O}_T$ such that $\mathscr{I}^2 = 0$; let $s : T' \to X$ and $t : T \to Y$ be morphisms of schemes such that $f \circ s = t \circ i$. By lemma 12.1.18(i), the log structures $\underline{L} := t^*\underline{N}$ and $\underline{L}' := s^*\underline{M}$ are fine, and by choosing the obvious maps $\log s$ and $\log t$, we deduce a commutative diagram :

$$\begin{array}{ccc} (T', \underline{L}') & \xrightarrow{s} & (X, \underline{M}) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\ (T, \underline{L}) & \xrightarrow{t} & (Y, \underline{N}). \end{array} \tag{12.3.28}$$

Then proposition 12.3.26(ii.b) says that there exists a morphism (resp. a unique morphism) of schemes $g : T \to X$ such that $g \circ i = s$ and $f \circ g = t$, *i.e.* f is smooth (resp. étale).

The converse is easy, and shall be left as an exercise for the reader.

(ii): Suppose first that $(f, \log f)$ is smooth, $\underline{N}$ is coherent and $\underline{M}$ is fine. By proposition 12.3.24(iii), the assertion to prove is local on $X_{\text{ét}}$, hence we may assume that f admits a chart, given by a morphism of finitely generated monoids $P \to P'$ and commutative diagrams :

$$P'_{X_{\mathrm{Zar}}} \to \underline{M} \qquad \omega : P_{Y_{\mathrm{Zar}}} \to \underline{N}$$

(theorem 12.1.37(i)). Now, consider a commutative diagram of log schemes on étale sites :

$$\begin{array}{ccc} (T', \underline{L}') & \xrightarrow{s} & \widetilde{u}^*_X(X, \underline{M}) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \widetilde{u}^* f} \\ (T, \underline{L}) & \xrightarrow{t} & \widetilde{u}^*_Y(Y, \underline{N}) \end{array} \tag{12.3.29}$$

where i is an exact nilpotent immersion of fine log schemes, defined by an ideal $\mathscr{I} \subset \mathscr{O}_T$ such that $\mathscr{I}^2 = 0$. Since the assertion to prove is local on $T_{\text{ét}}$, we may assume that T is affine and – in view of theorem 12.1.37(ii) – that t admits a chart, given by a morphism of finitely generated monoids $\varphi : P \to Q$, and commutative diagrams :

$$\begin{array}{ccc} P_{T_{\text{ét}}} = t^*P_{Y_{\text{ét}}} & \xrightarrow{t^*u^*_Y\omega} & t^*\widetilde{u}^*_Y\underline{N} \\ {\scriptstyle \varphi_T}\downarrow & & \downarrow{\scriptstyle \log t} \\ Q_{T_{\text{ét}}} & \xrightarrow{\beta} & \underline{L} \end{array} \qquad \begin{array}{ccc} P & \longrightarrow & \Gamma(Y, \mathscr{O}_Y) \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle t^\natural} \\ Q & \xrightarrow{\psi} & \Gamma(T, \mathscr{O}_T) \end{array} \tag{12.3.30}$$

Since i is an exact closed immersion, it follows that the morphism :

$$Q_{T'_{\text{ét}}} \xrightarrow{i^*\beta} i^*\underline{L} \xrightarrow{\log i} \underline{L}'$$

is a chart for $\underline{L}'$. Especially, $(T', \underline{L}')$ is isomorphic to $(T'_{\text{ét}}, Q^{\log}_{T'})$, the constant log structure deduced from the morphism $i^\natural \circ \psi : Q \to \Gamma(T', \mathscr{O}_{T'})$. The latter is also the log scheme $\widetilde{u}^*_{T'}(T_{\mathrm{Zar}}, Q^{\log}_{T'})$. From proposition 12.2.3(ii) we deduce that there exists a unique morphism $s_{\mathrm{Zar}} : (T_{\mathrm{Zar}}, Q^{\log}_{T'}) \to (X, \underline{M})$, such that $\widetilde{u}^* s_{\mathrm{Zar}} = s$. On the other hand, by inspecting (12.3.30) we find that there exists a unique morphism $t_{\mathrm{Zar}} : (T_{\mathrm{Zar}}, Q^{\log}_T) \to (Y, \underline{N})$ such that $\widetilde{u}^* t_{\mathrm{Zar}} = t$.

These morphisms can be assembled into a diagram :

$$(12.3.31)\qquad \begin{array}{ccc} (T'_{\mathrm{Zar}}, Q^{\log}_{T'}) & \xrightarrow{s_{\mathrm{Zar}}} & (X, \underline{M}) \\ {\scriptstyle i_{\mathrm{Zar}}}\downarrow & & \downarrow{\scriptstyle f} \\ (T_{\mathrm{Zar}}, Q^{\log}_{T}) & \xrightarrow{t_{\mathrm{Zar}}} & (Y, \underline{N}) \end{array}$$

where i_{Zar} is an exact closed immersion. By construction, the diagram $\widetilde{u}^*$(12.3.31) is naturally isomorphic to (12.3.29); especially, (12.3.31) commutes (proposition 12.2.3(i)). Now, since f is smooth, proposition 12.3.26(ii.b) implies that there exists a morphism $v : (T_{\mathrm{Zar}}, Q^{\log}_T) \to (X, \underline{M})$ such that $f \circ v = t_{\mathrm{Zar}}$ and $v \circ i_{\mathrm{Zar}} = s_{\mathrm{Zar}}$; then $\widetilde{u}^* v$ provides an appropriate lifting of t, which allows to conclude that $\widetilde{u}^*(f, \log f)$ is smooth.

Next, suppose that $(f, \log f)$ is étale (and we are still in case (b) of the corollary); then there exists a unique morphism v with the properties stated above. However, from proposition 12.2.3(i),(ii) we deduce easily that the natural map :

$$\mathscr{H}_Y((T_{\mathrm{Zar}}, Q^{\log}_T), (X, \underline{M}))(T) \to \mathscr{H}_{\widetilde{u}^*(Y,\underline{N})}((T, \underline{L}), \widetilde{u}^*(X, \underline{M}))(T)$$

is a bijection, and the same holds for the analogous map for T'. In view of the foregoing, this shows that the map $\mathscr{H}_{\widetilde{u}^*(Y,\underline{N})}((T, \underline{L}), \widetilde{u}^*(X, \underline{M}))(T) \to \mathscr{H}_{\widetilde{u}^*(Y,\underline{N})}((T', \underline{L}'), \widetilde{u}^*(X, \underline{M}))(T')$ is bijective, whenever t admits a chart and T is affine. We easily conclude that $\widetilde{u}^*(f, \log f)$ is étale.

The case where both $\underline{N}$ and $\underline{M}$ are both integral, is similar, though easier : we may assume that $\underline{L}$ admits a chart, in which case $(T, \underline{L})$ is of the form $\widetilde{u}^*(T_{\mathrm{Zar}}, Q^{\log}_T)$ for some finite integral monoid Q; then $(T', \underline{L}')$ admits a similar description, and again, by appealing to proposition 12.2.3(ii) we deduce that (12.3.29) is of the form $\widetilde{u}^*$(12.3.31), in which case we conclude as in the foregoing.

Conversely, if $\widetilde{u}^*(f, \log f)$ is smooth, consider again a commutative diagram (12.3.28) of log schemes on Zariski sites, with i an exact closed immersion, defined by a sheaf of ideals $\mathscr{I} \subset \mathscr{O}_T$ such that $\mathscr{I}^2 = 0$. By applying everywhere the pull-back functors from Zariski to étale sites, we deduce a commutative diagram $\widetilde{u}^*$(12.3.28) of log schemes on étale sites, and it is easy to see that $\widetilde{u}^*(i, \log i)$ is again an exact nilpotent immersion. According to proposition 12.3.26(ii.b), after replacing T by any affine open subset, we may find a morphism of log schemes $g : \widetilde{u}^*_T(T, \underline{L}) \to \widetilde{u}^*_X(X, \underline{M})$, such that $g \circ \widetilde{u}^* i = \widetilde{u}^* s$ and $\widetilde{u}^* f \circ g = t$. By proposition 12.2.3(i),(ii) there exists a unique morphism $g' : (T, \underline{L}) \to (X, \underline{M})$ such that $\widetilde{u}^* g' = g$, and necessarily $g' \circ i = s$ and $f \circ g' = t$. We conclude that $(f, \log f)$ is smooth.

Finally, if $\widetilde{u}^*(f, \log f)$ is étale, the morphism g exhibited above is unique, and therefore the same holds for g', so $(f, \log f)$ is étale as well. □

Proposition 12.3.32. *In the situation of* (12.3.11)*, suppose that the log structures* $\underline{M}, \underline{N}, \underline{P}$ *are coherent, and consider the following conditions :*

(a) *f is smooth (resp. étale).*
(b) *df is a locally split monomorphism (resp. an isomorphism).*

Then (a)⇒(b)*, and if $g \circ f$ is smooth, then* (b)⇒(a).

Proof. We may assume that the schemes under consideration are affine, say $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, $Z = \operatorname{Spec} C$; then (12.3.12) amounts to an exact sequence of A-modules :

$$A \otimes_B \Omega(g) \xrightarrow{df} \Omega(g \circ f) \xrightarrow{\omega} \Omega(f) \to 0.$$

On the other hand, let $i : (T'\underline{L}') \to (T, \underline{L})$ be an exact closed immersion of $(Y, \underline{N})$-schemes, defined by an ideal $\mathscr{I} \subset \mathscr{O}_T$ with $\mathscr{I}^2 = 0$. Suppose that $s : T' \to \mathscr{H}_Y(T', X)$ is a global section, *i.e.* a given morphism of $(Y,\underline{N})$-schemes $(T'\underline{L}') \to (X,\underline{M})$. In this situation, we have a natural sequence of morphisms :

$$\mathscr{H}_1 := \mathscr{H}_Y(T,X) \xrightarrow{\alpha} \mathscr{H}_2 := \mathscr{H}_Z(T,X) \xrightarrow{f_*} \mathscr{H}_3 := \mathscr{H}_Z(T,Y)$$

where α is the obvious monomorphism. Let us consider the pull-back of these sheaves along the global section s :

$$\mathscr{T} := T' \times_{\mathscr{H}_Y(T',X)} i^*\mathscr{H}_1 \qquad \mathscr{T}' := T' \times_{\mathscr{H}_Z(T',X)} i^*\mathscr{H}_2 \qquad \mathscr{T}'' := T' \times_{\mathscr{H}_Z(T',Y)} i^*\mathscr{H}_3.$$

By proposition 12.3.26(i), any choice of a global section of $\mathscr{T}(T)$ determines a commutative diagram of sets :

$$\begin{array}{ccccc} \mathrm{Hom}_A(\Omega(f),\mathscr{I}(T)) & \xrightarrow{\omega^*} & \mathrm{Hom}_A(\Omega(g\circ f),\mathscr{I}(T)) & \xrightarrow{df^*} & \mathrm{Hom}_B(\Omega(g),\mathscr{I}(T)) \\ \downarrow & & \downarrow & & \downarrow \\ \mathscr{T}(T) & \longrightarrow & \mathscr{T}'(T) & \xrightarrow{f_*} & \mathscr{T}''(T) \end{array} \tag{12.3.33}$$

whose vertical arrows are bijections.

In order to show that df is a locally split monomorphism (resp. an isomorphism), it suffices to show that the map $df^* := \mathrm{Hom}_A(df, I)$ is surjective for every A-module I. To this aim, we take $(T,L) := (T_I,\underline{L}_I)$, $(T',\underline{L}') := (X,\underline{M})$, and let i be the nilpotent immersion defined by the ideal $I \subset A \oplus I$, as in the proof of proposition 12.3.26(ii.a). If $\mathscr{I} \subset \mathscr{O}_T$ is the corresponding sheaf of ideals, then $\mathscr{I}(T) = I$; moreover, the inclusions $A \to A \oplus I$ and $\underline{M} \subset \underline{M} \oplus \mathscr{I}$ determine a morphism $h : (T_I,\underline{L}_I) \to (X,\underline{M})$, which is a global section of $\mathscr{T}(T)$. Having made these choices, consider the resulting diagram (12.3.33) : if f is étale, f_* is an isomorphism, and when f is smooth, f_* is surjective (proposition 12.3.26(ii.b)), whence the contention.

For the converse, we may suppose that df is a split monomorphism, hence df^* in (12.3.33) is a split surjection. We have to show that $\mathscr{T}(T)$ is not empty, and by assumption (and proposition 12.3.26(ii.b)) we know that $\mathscr{T}'(T) \neq \varnothing$. Choose any $\widetilde{h} \in \mathscr{T}'(T)$; then t and $f_*\widetilde{h}$ are two elements of $\mathscr{T}''(T)$, so we may find $\varphi \in \mathrm{Hom}_B(\Omega(g),\mathscr{I}(T))$ such that $\varphi + f_*\widetilde{h} = t$ (where the sum denotes the action of $\mathrm{Hom}_B(\Omega(g),\mathscr{I}(T))$ on its torsor $\mathscr{T}''$). Then we may write $\varphi = \psi \circ df$ for some $\psi : \Omega(g\circ f) \to \mathscr{I}(Y)$, and it follows easily that $\psi + \widetilde{h}$ lies in $\mathscr{T}(T)$. Finally, if df is an isomorphism, we have $\Omega(f) = 0$, hence $\mathscr{T}(T)$ contains exactly one element. □

Proposition 12.3.34. *Let R be a ring, $\varphi : P \to Q$ a morphism of finitely generated monoids, such that* $\mathrm{Ker}\,\varphi^{\mathrm{gp}}$ *and the torsion subgroup of* $\mathrm{Coker}\,\varphi^{\mathrm{gp}}$ *(resp.* $\mathrm{Ker}\,\varphi^{\mathrm{gp}}$ *and* $\mathrm{Coker}\,\varphi^{\mathrm{gp}}$*) are finite groups whose orders are invertible in R. Then, the induced morphism*

$$\mathrm{Spec}(R,\varphi) : \mathrm{Spec}(R,Q) \to \mathrm{Spec}(R,P)$$

is smooth (resp. étale).

Proof. To ease notation, set

$$f := \mathrm{Spec}(R,\varphi) \qquad (X,\underline{M}) := \mathrm{Spec}(R,Q) \qquad (Y,\underline{N}) := \mathrm{Spec}(R,P).$$

Clearly f is finitely presented. We have to show that, for every commutative diagram like (12.3.28) with i an exact nilpotent immersion of fine log schemes, there is, locally on T_τ at least one morphism (resp. a unique morphism) $h : (T,\underline{L}) \to (X,\underline{M})$ such that $f \circ h = h \circ i$.

Let $\mathscr{I} := \mathrm{Ker}(\mathscr{O}_T \to i_*\mathscr{O}_{T'})$; by considering the $\mathscr{I}$-adic filtration on $\mathscr{O}_T$, we reduce easily to the case where $\mathscr{I}^2 = 0$, and then we may embed $\mathscr{I}$ in $\underline{L}$ via the morphism :

$$\mathscr{I} \to \mathscr{O}_T^\times \subset \underline{L} \qquad x \mapsto 1 + x.$$

(Here $\mathscr{I}$ is regarded as a sheaf of abelian groups, via its addition law.) Since i is exact, the natural morphism $\underline{L}/\mathscr{I} \to i_*\underline{L}'$ is an isomorphism, whence a commutative diagram :

$$\begin{array}{ccc} \underline{L} & \xrightarrow{\log i} & i_*\underline{L}' \\ \downarrow & & \downarrow \\ \underline{L}^{\mathrm{gp}} & \xrightarrow{(\log i)^{\mathrm{gp}}} & i_*(\underline{L}')^{\mathrm{gp}} \simeq \underline{L}^{\mathrm{gp}}/\mathscr{I} \end{array} \tag{12.3.35}$$

and since $\underline{L}$ is integral, one sees easily that (12.3.35) is cartesian (it suffices to consider the stalks over the τ-points).

• On the other hand, suppose first that both $\mathrm{Ker}\,\varphi^{\mathrm{gp}}$ and $\mathrm{Coker}\,\varphi^{\mathrm{gp}}$ are finite groups whose order is invertible in R, hence in $\mathscr{I}$; then a standard diagram chase shows that we may find a unique map $g : P_T^{\mathrm{gp}} \to L^{\mathrm{gp}}$ of abelian sheaves that fits into a commutative diagram :

$$\begin{array}{ccccc} Q_T^{\mathrm{gp}} & \longrightarrow & t^*\underline{N}^{\mathrm{gp}} & \xrightarrow{(\log t)^{\mathrm{gp}}} & \underline{L}^{\mathrm{gp}} \\ {\scriptstyle \varphi_T^{\mathrm{gp}}}\downarrow & & {\scriptstyle g}\nearrow & & \downarrow{\scriptstyle (\log i)^{\mathrm{gp}}} \\ P_T^{\mathrm{gp}} & \longrightarrow & i_*s^*\underline{M}^{\mathrm{gp}} & \xrightarrow{i_*(\log s)^{\mathrm{gp}}} & i_*(\underline{L}')^{\mathrm{gp}}. \end{array} \tag{12.3.36}$$

• More generally, we may write : $\mathrm{Coker}\,\varphi^{\mathrm{gp}} \simeq G \oplus H$, where H is a finite group with order invertible in R, and G is a free abelian group of finite rank. The direct summand G lifts to a direct summand $G' \subset P^{\mathrm{gp}}$. Extend φ^{gp} to a map $\psi : Q^{\mathrm{gp}} \oplus G' \to P^{\mathrm{gp}}$, by the rule : $(x, g) \mapsto \varphi^{\mathrm{gp}}(x) \cdot g$. Given any τ-point ξ of T, we may extend the morphism $Q^{\mathrm{gp}} = Q^{\mathrm{gp}}_{T,\xi} \to \underline{L}^{\mathrm{gp}}_\xi$ in $(12.3.36)_\xi$, to a map $\omega : Q^{\mathrm{gp}} \oplus G' \to \underline{L}^{\mathrm{gp}}_\xi$ whose composition with $(\log i)^{\mathrm{gp}}_\xi$ agrees with the composition of ψ and the bottom map $P^{\mathrm{gp}} = P^{\mathrm{gp}}_{T,\xi} \to i_*(\underline{L}')^{\mathrm{gp}}_\xi$ of $(12.3.36)_\xi$. By the usual arguments, ω extends to a map of abelian sheaves $\vartheta : (Q^{\mathrm{gp}} \oplus G')_U \to (\underline{L}^{\mathrm{gp}})_{|U}$ on some neighborhood $U \to T$ of ξ, and if U is small enough, the composition $(\log i)^{\mathrm{gp}}_{|U} \circ \vartheta$ agrees with the composition of ψ_U and the bottom map $\beta : P_U^{\mathrm{gp}} \to i_*(\underline{L}'^{\mathrm{gp}})_{|U}$ of $(12.3.36)_{|U}$. We may then replace T by U, and since $\mathrm{Ker}\,\psi = \mathrm{Ker}\,\varphi^{\mathrm{gp}}$, and $\mathrm{Coker}\,\psi = H$, the same diagram chase as in the foregoing shows that we may again find a morphism $g : P_T^{\mathrm{gp}} \to \underline{L}^{\mathrm{gp}}$ fitting into a commutative diagram :

$$\begin{array}{ccc} (Q^{\mathrm{gp}} \oplus G')_T & \xrightarrow{\vartheta} & \underline{L}^{\mathrm{gp}} \\ {\scriptstyle \psi_T}\downarrow & {\scriptstyle g}\nearrow & \downarrow{\scriptstyle (\log i)^{\mathrm{gp}}} \\ P_T^{\mathrm{gp}} & \xrightarrow{\beta} & i_*\underline{L}'^{\mathrm{gp}} \end{array}$$

In either case, in view of (12.3.35), the morphisms

$$P_T \to P_T^{\mathrm{gp}} \xrightarrow{g} \underline{L}^{\mathrm{gp}} \qquad \text{and} \qquad P_T \to i_*s^*\underline{M} \xrightarrow{i_* \log s} i_*\underline{L}'$$

determine a unique morphism $P_T \to \underline{L}$, which induces a morphism of log schemes $(T, \underline{L}) \to (X, \underline{M})$ with the sought property. □

Theorem 12.3.37. *Let $f : (X, \underline{M}) \to (Y, \underline{N})$ be a morphism of fine log schemes. Assume we are given a fine chart $\beta : Q_Y \to \underline{N}$ of $\underline{N}$. Then the following conditions are equivalent :*

(a) *f is smooth (resp. étale).*

(b) *There exist a covering family $(g_\lambda : U_\lambda \to X \mid \lambda \in \Lambda)$ in $X_{\text{ét}}$, and for every $\lambda \in \Lambda$, a fine chart $(\beta, (P_\lambda)_{U_\lambda} \to g_\lambda^*\underline{M}, \varphi_\lambda : Q \to P_\lambda)$ of the induced morphism of log schemes*

$$f_{|U_\lambda} := (f \circ g_\lambda, g_\lambda^* \log f) : (U_\lambda, g_\lambda^*\underline{M}) \to (Y, \underline{N})$$

such that :

(i) $\operatorname{Ker}\varphi_\lambda^{\mathrm{gp}}$ *and the torsion subgroup of* $\operatorname{Coker}\varphi_\lambda^{\mathrm{gp}}$ *(resp.* $\operatorname{Ker}\varphi_\lambda^{\mathrm{gp}}$ *and* $\operatorname{Coker}\varphi_\lambda^{\mathrm{gp}}$*) are finite groups of orders invertible in* $\mathscr{O}_{U_\lambda}$.
(ii) *The natural morphism of* Y*-schemes* $p_\lambda : U_\lambda \to Y \times_{\operatorname{Spec}\mathbb{Z}[Q]} \operatorname{Spec}\mathbb{Z}[P_\lambda]$ *is étale.*

Proof. Suppose first that $\tau = \text{ét}$, so $\underline{M}$ and $\underline{N}$ are log structures on étale sites. Then the log structure $g_\lambda^*\underline{M}$ on $(U_\lambda)_{\text{ét}}$ is just the restriction of $\underline{M}$, and $g_\lambda^* \log f$ is the restriction of $\log f$ to $(U_\lambda)_{\text{ét}}$. In case $\tau = \text{Zar}$, the log structure $g_\lambda^*\underline{M}$ can be described as follows. Form the log scheme $(X, \underline{M}') := \widetilde{u}_X^*(X, \underline{M})$ (notation of (12.2.2)), take the restriction $\underline{M}'_{|U_\lambda}$ of $\underline{M}'$ to $(U_\lambda)_{\text{ét}}$, and push forward to the Zariski site to obtain $\widetilde{u}_{X*}(\underline{M}'_{|U_\lambda}) = g_\lambda^*\underline{M}$ (by proposition 12.2.3(ii)). Since f is smooth (resp. étale) if and only if $\widetilde{u}^*f$ is (corollary 12.3.27(ii),(iii)), we conclude that the assertion concerning f holds if and only if the corresponding assertion for $\widetilde{u}^*f$ does. Hence, it suffices to consider the case of log structures on étale sites. Set $S := \operatorname{Spec}\mathbb{Z}[Q]$.

(b)⇒(a): Taking into account lemma 12.2.18(i), we deduce a commutative diagram of log schemes :

$$\begin{array}{ccccc}
(U_\lambda, \underline{M}_{|U_\lambda}) & \xrightarrow{\sim} & \operatorname{Spec}(\mathbb{Z}, P_\lambda) \times_{\operatorname{Spec}\mathbb{Z}[P_\lambda]} U_\lambda & \xrightarrow{p_\lambda} & \operatorname{Spec}(\mathbb{Z}, P_\lambda) \times_{\operatorname{Spec}\mathbb{Z}[Q]} Y \\
{\scriptstyle g_\lambda}\downarrow & & & & \downarrow{\scriptstyle \pi_\lambda} \\
(X, \underline{M}) & \xrightarrow{f} & (Y, \underline{N}) & \xrightarrow{\sim} & \operatorname{Spec}(\mathbb{Z}, Q) \times_{\operatorname{Spec}\mathbb{Z}[Q]} Y.
\end{array}$$

It follows by corollary 12.3.27(i) (resp. by propositions 12.3.24(ii) and 12.3.34) that p_λ (resp. π_λ) is smooth. Hence $f \circ g_\lambda$ is smooth, by proposition 12.3.24(i). Finally, f is smooth, by proposition 12.3.24(iii).

(a)⇒(b): Suppose that f is smooth, and fix a geometric point ξ of X. Since $(12.3.18)_\xi$ is a surjection, we may find elements $t_1, \dots, t_r \in \underline{M}_\xi$ such that $(d\log t_i \mid i = 1, \dots, r)$ is a basis of the free $\mathscr{O}_{X,\xi}$-module $\Omega^1_{X/Y}(\log \underline{M}/\underline{N})_\xi$ (proposition 12.3.26(ii.a)). Moreover, the kernel of $(12.3.18)_\xi$ is generated by sections of the form :

- $1 \otimes \log a$ where $a \in N' := \operatorname{Im}(\log f_\xi : \underline{N}_{f(\xi)} \to \underline{M}_\xi)$
- $\sum_{j=1}^s \alpha(m_i) \otimes \log m_i$ where $m_1, \dots, m_s \in \underline{M}_\xi$ and $\sum_{j=1}^s d\alpha(m_j) = 0$ in $\Omega^1_{X/Y}$

whence a well-defined $\mathscr{O}_{X,\xi}$-linear map :

$$\Omega^1_{X/Y}(\log \underline{M}/\underline{N})_\xi \to \kappa(\xi) \otimes_\mathbb{Z} (\underline{M}_\xi^{\mathrm{gp}}/(\underline{M}_\xi^\times \cdot N')) \quad : \quad d\log a \mapsto 1 \otimes a. \tag{12.3.38}$$

Consider the map of monoids :

$$\varphi : P_1 := \mathbb{N}^{\oplus r} \oplus Q \to \underline{M}_\xi$$

which is given by the rule : $e_i \mapsto t_i$ on the canonical basis $e_1, \dots, e_r$ of $\mathbb{N}^{\oplus r}$, and on the summand Q it is given by the map $Q \xrightarrow{\beta_\xi} \underline{N}_{f(\xi)} \xrightarrow{\log f_\xi} \underline{M}_\xi$. Since (12.3.38) is a surjection, we see that the same holds for the induced map

$$\kappa(\xi) \otimes_\mathbb{Z} P_1^{\mathrm{gp}} \xrightarrow{\mathbf{1}_{\kappa(\xi)} \otimes_\mathbb{Z} \varphi^{\mathrm{gp}}} \kappa(\xi) \otimes_\mathbb{Z} \underline{M}_\xi^{\mathrm{gp}} \to \kappa(\xi) \otimes_\mathbb{Z} (\underline{M}_\xi^{\mathrm{gp}}/\underline{M}_\xi^\times).$$

It follows that the cokernel of the map $\overline{\varphi} : P_1^{\mathrm{gp}} \to \underline{M}_\xi^{\mathrm{gp}}/\underline{M}_\xi^\times$ induced by φ^{gp}, is a finite group (lemma 12.2.21(i)) annihilated by an integer n which is invertible in $\mathscr{O}_{X,\xi}$. Let $m_1, \dots, m_s \in \underline{M}_\xi^{\mathrm{gp}}$ be finitely many elements, whose images in $\underline{M}_\xi^{\mathrm{gp}}/\underline{M}_\xi^\times$ generate $\operatorname{Coker}\overline{\varphi}$; therefore we may find $u_1, \dots, u_s \in \underline{M}_\xi^\times$, and $x_1, \dots, x_s \in P_1^{\mathrm{gp}}$, such that $m_i^n \cdot u_i = \overline{\varphi}(x_i)$ for every $i \leq s$. However, since $\mathscr{O}_{X,\xi}$ is strictly henselian, $\underline{M}_\xi^\times \simeq \mathscr{O}_{X,\xi}^\times$ is n-divisible, hence we may find $v_1, \dots, v_s$ in $\underline{M}_\xi^\times$ such that $u_i = v_i^n$ for $i = 1, \dots, n$. Define group homomorphisms :

$$\mathbb{Z}^{\oplus s} \xrightarrow{\gamma} \mathbb{Z}^{\oplus s} \oplus P_1^{\mathrm{gp}} \xrightarrow{\delta} \underline{M}_\xi^{\mathrm{gp}}$$

by the rules : $\gamma(e_i) = (-ne_i, x_i)$ and $\delta(e_i, y) = m_i v_i \cdot \varphi(y)$ for every $i = 1, \dots, s$ and every $y \in P_1^{\mathrm{gp}}$. It is easily seen that δ factors through a group homomorphism $h : G := \operatorname{Coker}\gamma \to$

$\underline{M}_\xi$; moreover the natural map $P_1^{\mathrm{gp}} \to G$ is injective, and its cokernel is annihilated by n; furthermore, the induced map $G \to \underline{M}_\xi^\sharp$ is surjective. Let $P := h^{-1}\underline{M}_\xi$. Then, the natural map $Q^{\mathrm{gp}} \to P^{\mathrm{gp}}$ is injective, and the torsion subgroup of $P^{\mathrm{gp}}/Q^{\mathrm{gp}}$ is annihilated by n. We deduce that the rule : $x \mapsto d\log h(x)$ for every $x \in P^{\mathrm{gp}}$, induces an isomorphism :

$$\mathscr{O}_{X,\xi} \otimes_{\mathbb{Z}} (P^{\mathrm{gp}}/Q^{\mathrm{gp}}) \xrightarrow{\sim} \Omega^1_{X/Y}(\log \underline{M}/\underline{N})_\xi. \tag{12.3.39}$$

It follows that we may find an étale neighborhood $U \to X$ of ξ, such that (12.3.39) extends to an isomorphism of $\mathscr{O}_U$-modules :

$$\mathscr{O}_U \otimes_{\mathbb{Z}} (P^{\mathrm{gp}}/Q^{\mathrm{gp}}) \xrightarrow{\sim} \Omega^1_{U/Y}(\log \underline{M}_{|U}/\underline{N}). \tag{12.3.40}$$

Next, proposition 12.1.30 says that, after replacing U by a smaller étale neighborhood of ξ, the restriction $h_{|P} : P \to \underline{M}_\xi$ extends to a chart $P_U \to \underline{M}_{|U}$, whence a strict morphism

$$p : (U, \underline{M}_{|U}) \to (Y', P^{\log}_{Y'}) := (Y, \underline{N}) \times_{\mathrm{Spec}(\mathbb{Z},Q)} \mathrm{Spec}(\mathbb{Z}, P).$$

as sought. Taking into account corollary 12.3.27(i), it remains only to show :

Claim 12.3.41. p is étale.

Proof of the claim. By proposition 12.3.13 and example 12.3.19, we have natural isomorphisms

$$\Omega^1_{Y'/Y}(\log P^{\log}_{Y'}/\underline{N}) \xrightarrow{\sim} \mathscr{O}_{Y'} \otimes_{\mathbb{Z}} (P^{\mathrm{gp}}/Q^{\mathrm{gp}}).$$

In view of (12.3.40), it follows that the map

$$dp : p^*\Omega^1_{Y'/Y}(\log P^{\log}_{Y'}/\underline{N}) \to \Omega^1_{U/Y}(\log \underline{M}_{|U}/\underline{N})$$

is an isomorphism, and then the claim follows from proposition 12.3.32. □

Corollary 12.3.42. *Keep the notation of theorem* 12.3.37. *Suppose that f is a smooth morphism of fs log schemes, and that Q is fine, sharp and saturated. Then there exist a covering family $(g_\lambda : U_\lambda \to X)$ in $X_{\mathrm{ét}}$, and fine and saturated charts $(\beta, (P_\lambda)_{U_\lambda} \to g_\lambda^*\underline{M}, \varphi_\lambda : Q \to P_\lambda)$ of $f_{|U_\lambda}$ fulfilling conditions* (b.i) *and* (b.ii) *of the theorem, and such that moreover φ_λ is injective, and $P_\lambda^\times$ is a torsion-free abelian group, for every $\lambda \in \Lambda$.*

Proof. Notice that, under the stated assumptions, Q^{gp} is a torsion-free abelian group; hence theorem 12.3.37 already implies the existence of a covering family $(g_\lambda : U_\lambda \to X)$ in $X_{\mathrm{ét}}$, and of fine charts $(\beta, \omega'_\lambda : (P'_\lambda)_{U_\lambda} \to g_\lambda^*\underline{M}, \varphi'_\lambda : Q \to P'_\lambda)$ such that φ'_λ is injective and $\mathrm{Coker}(\varphi'_\lambda)^{\mathrm{gp}}$ is a finite group of order invertible in $\mathscr{O}_{U_\lambda}$, for every $\lambda \in \Lambda$. Since $g_\lambda^*\underline{M}$ is saturated (lemma 12.1.18(i)), it is clear that ω'_λ factors through a morphism $(P'_\lambda)^{\mathrm{sat}}_{U_\lambda} \to g_\lambda^*\underline{M}$ of pre-log structures, which is again a chart, so we may assume that P'_λ is fine and saturated, for every $\lambda \in \Lambda$ (corollary 6.4.1(ii)), in which case we may find an isomorphism of monoids $P'_\lambda = P_\lambda \times G$, where $P_\lambda^\times$ is torsion-free, and G is a finite group (lemma 6.2.10). Let d be the order of $\mathrm{Coker}(\varphi'_\lambda)^{\mathrm{gp}}$, and denote by φ_λ the composition of φ'_λ and the projection $P'_\lambda \to P_\lambda$; since Q^{gp} is a torsion-free abelian group, we deduce a short exact sequence :

$$0 \to G \to \mathrm{Coker}(\varphi'_\lambda)^{\mathrm{gp}} \to \mathrm{Coker}\,\varphi_\lambda^{\mathrm{gp}} \to 0 \tag{12.3.43}$$

and notice that φ_λ is also injective. We may assume that U_λ and Y are affine, say $U_\lambda = \mathrm{Spec}\, B_\lambda$ and $Y = \mathrm{Spec}\, A$, and since d is invertible in $\mathscr{O}_{U_\lambda}$, we reduce easily – via base change by a finite morphism $Y' \to Y$ – to the case where A contains the subgroup $\boldsymbol{\mu}_d \subset \overline{\mathbb{Q}}^\times$ of d-th power roots of 1. The chart ω'_λ determines a morphism of monoids $P'_\lambda \to B_\lambda$, and the map $f^\sharp : A \to B_\lambda$ factors through the natural ring homomorphism

$$A \to A \otimes_{R[Q]} R[P'_\lambda] \xrightarrow{\sim} A \otimes_{R[Q]} (R[P_\lambda] \otimes_R R[G]) \qquad \text{where} \qquad R := \mathbb{Z}[d^{-1}, \boldsymbol{\mu}_d].$$

On the other hand, let $\Gamma := \mathrm{Hom}_{\mathbb{Z}}(G, \boldsymbol{\mu}_d)$; then we have a natural decomposition

$$R[G] \simeq \prod_{\chi\in\Gamma} e_\chi R[G]$$

where e_χ is the idempotent of $R[G]$ defined as in (14.6.11) (cp. the proof of theorem 14.6.23(i)); each factor is a ring isomorphic to R, whence a corresponding decomposition of U_λ as a disjoint union of Y-schemes $U_\lambda = \coprod_{\chi\in\Gamma} U_{\lambda,\chi}$. We are then further reduced to the case where $U_\lambda = U_{\lambda,\chi}$ for some character χ of G. In view of (12.3.43), it is easily seen that the composition

$$Q^{\mathrm{gp}} \xrightarrow{(\varphi'_\lambda)^{\mathrm{gp}}} (P'_\lambda)^{\mathrm{gp}} \to G \xrightarrow{\chi} \boldsymbol{\mu}_d$$

extends to a well defined group homomorphism $\overline{\chi} : P_\lambda^{\mathrm{gp}} \to \boldsymbol{\mu}_d$, whence a map $\overline{\chi}_{U_\lambda} : P_{\lambda,U_\lambda} \to \mu_{d,U_\lambda} \subset g_\lambda^*\underline{M}$ of sheaves on $U_{\lambda,\tau}$. Define $\omega_\lambda : P_{\lambda,U_\lambda} \to g_\lambda^*\underline{M}$ by the rule : $s \mapsto \omega'_\lambda(s)\cdot\overline{\chi}_{U_\lambda}(s)$ for every local section s of P_{λ,U_λ}. It is easily seen that ω_λ is again a chart for $g_\lambda^*\underline{M}$ (*e.g.* one may apply lemma 12.1.4). Lastly, a direct inspection shows that $(\beta, \omega_\lambda, \varphi_\lambda)$ is a chart of $f_{|U_\lambda}$ with the sought properties. □

12.3.44. Let $(Y_i \mid i \in I)$ be a cofiltered system of quasi-compact and quasi-separated schemes, with affine transition morphisms, and suppose that 0 is an initial object of the indexing category I. Suppose also that $g_0 : X_0 \to Y_0$ is a finitely presented morphism of schemes. Let $X_i := X_0 \times_{Y_0} Y_i$ for every $i \in I$, and denote $g_i : X_i \to Y_i$ the induced morphism. Let also $g : X \to Y$ be the limit of the family of morphisms $(g_i \mid i \in I)$.

Corollary 12.3.45. *In the situation of* (12.3.44)*, suppose that* $(g, \log g) : (X, \underline{M}) \to (Y, \underline{N})$ *is a smooth morphism of fine log schemes. Then there exist* $i \in I$*, and a smooth morphism* $(g_i, \log g_i) : (X_i, \underline{M}_i) \to (Y_i, \underline{N}_i)$ *of fine log schemes, such that* $\log g = \pi_i^* \log g_i$.

Proof. First, using corollary 12.2.35, we may find $i \in I$ such that $(g, \log g)$ descends to a morphism $(g_i, \log g_i)$ of log schemes with coherent log structures. After replacing I by I/i, we may then suppose that $i = 0$, in which case we set $(X_i, \underline{M}_i) := X_i \times (X_0, \underline{M}_0)$, and define likewise $(Y_i, \underline{N}_i)$ for every $i \in I$.

Next, arguing as in the proof of corollary 12.2.36(ii), we may assume that $\underline{N}_0$ admits a fine chart. In this case, also $\underline{N}$ admits a fine chart, and then we my find a covering family $\mathscr{U} := (U_\lambda \to X \mid \lambda \in \Lambda)$ for X_τ, such that the induced morphism $(U_\lambda, \underline{M}_{|U_\lambda}) \to (Y, \underline{N})$ admits a chart fulfilling conditions (i) and (ii) of theorem 12.3.37. Moreover, under the current assumptions, X is quasi-compact, hence we may assume that Λ is a finite set, in which case there exists $i \in I$ such that $\mathscr{U}$ descends to a covering family $(U_{i,\lambda} \to X_i \mid \lambda \in \Lambda)$ for $X_{i,\tau}$ (claim 12.2.29(ii)), and as usual, we may then reduce to the case where $i = 0$. It then suffices to show that there exists $i \in I$ such that the induced morphism $U_{0,\lambda} \times_{X_0} (X_i, \underline{M}_i) \to (Y_i, \underline{N}_i)$ is smooth (proposition 12.3.24(iii)). Thus, we may replace the system $(X_i \mid i \in I)$ by the system of schemes $(U_{0,\lambda} \times_{X_0} X_i \mid i \in I)$, and assume from start that $(g, \log g)$ admits a fine chart fulfilling conditions (i) and (ii) of theorem 12.3.37. In this case, corollary 12.2.36(i) and [65, Ch.IV, Prop.17.7.8(ii)] imply more precisely that there exists $i \in I$ such that the morphism $(X_i, \underline{M}_i) \to (Y_i, \underline{N}_i)$ fulfills conditions (i) and (ii) of theorem 12.3.37, whence the contention. □

12.4. **Logarithmic blow up of a coherent ideal.** This section introduces the logarithmic version of the scheme-theoretic blow up of a coherent ideal, which will be exhibited as the logarithmic homogeneous spectrum of a certain graded algebra, naturally attached to any sheaf of ideals in a log structure.

12.4.1. We shall consider first the logarithmic counterparts of the notions introduced in section 10.6. To begin with, let X be any scheme, N a monoid, and $\underline{P}$ a (commutative) N-graded monoid of the topos X_τ (see definition 4.8.8). Notice that $\underline{P}^\times = \coprod_{n\in N}(\underline{P}^\times \cap \underline{P}_n)$, hence the sheaf of invertible sections of a N-graded monoid on X is a N-graded abelian sheaf.

Definition 12.4.2. Let X be a scheme, and $\beta : \underline{M} \to \mathscr{O}_X$ a log structure on X_τ.

(i) A $\mathbb{N}$-*graded* $\mathscr{O}_{(X,\underline{M})}$-*algebra* is a datum $(\mathscr{A}, \underline{P}, \alpha, \beta_\mathscr{A})$ consisting of a $\mathbb{N}$-graded $\mathscr{O}_X$-algebra $\mathscr{A} := \oplus_{n\in\mathbb{N}}\mathscr{A}_n$ (on the site X_τ), a graded monoid $\underline{P}$ on X_τ, and a commutative diagram :

$$\begin{CD} \underline{M} @>\alpha>> \underline{P} \\ @V\beta VV @VV\beta_\mathscr{A}V \\ \mathscr{O}_X @>>> \mathscr{A} \end{CD}$$

where $\beta_\mathscr{A}$ restricts to a morphism of graded monoids $\underline{P} \to \coprod_{n\in\mathbb{N}} \mathscr{A}_n$ (and the composition law on the target is induced by the multiplication law of $\mathscr{A}$), α is a morphism of monoids $\underline{M} \to (\underline{P})_0$, and the bottom map is the natural morphism $\mathscr{O}_X \to \mathscr{A}_0$. We say that the $\mathbb{N}$-graded $\mathscr{O}_{(X,\underline{M})}$-algebra $(\mathscr{A}, \underline{P}, \alpha, \beta_\mathscr{A})$ is *quasi-coherent*, if $\mathscr{A}$ is a quasi-coherent $\mathscr{O}_X$-algebra.

(ii) A *morphism* $(g, \log g) : (\mathscr{A}, \underline{P}, \alpha, \beta_\mathscr{A}) \to (\mathscr{A}', \underline{P}', \alpha', \beta'_\mathscr{A})$ *of* $\mathbb{N}$-*graded* $\mathscr{O}_{(X,\underline{M})}$-*algebras* is a commutative diagram :

$$\begin{CD} \underline{P} @>\log g>> \underline{P}' \\ @V\beta_\mathscr{A} VV @VV\beta'_\mathscr{A}V \\ \mathscr{A} @>g>> \mathscr{A}' \end{CD}$$

where $\log g$ is a morphism of $\mathbb{N}$-graded monoid such that $\log g \circ \alpha = \alpha'$, and g is a morphism of $\mathscr{O}_Z$-algebras.

12.4.3. Let M be a monoid, S an M-module; we say that an element $s \in S$ is *invertible*, if the translation map $M \to S : m \mapsto m \cdot s$ (for all $m \in M$) is an isomorphism. It is easily seen that the subset $S^\times$ consisting of all invertible elements of S, is naturally an $M^\times$-module.

Let $\underline{M}$ be a sheaf of monoids on X_τ, and $\mathscr{N}$ an $\underline{M}$-module; by restriction of scalars, $\mathscr{N}$ is naturally an $\underline{M}^\times$-module. We define the $\underline{M}^\times$-submodule $\mathscr{N}^\times \subset \mathscr{N}$, by the rule :

$$\mathscr{N}^\times(U) := \mathscr{N}(U)^\times \qquad \text{for every object } U \text{ of } X_\tau.$$

Conversely, for any $\underline{M}^\times$-module $\mathscr{P}$, the extension of scalars $\mathscr{N} \otimes_{\underline{M}^\times} \underline{M}$ defines a functor $\underline{M}^\times\text{-}\mathbf{Mod} \to \underline{M}\text{-}\mathbf{Mod}$. It is easily seen that the latter restricts to an equivalence from the full subcategory $\underline{M}^\times\text{-}\mathbf{Inv}$ of $\underline{M}^\times$-torsors to the subcategory $(\underline{M}\text{-}\mathbf{Inv})^\times$ of $\underline{M}\text{-}\mathbf{Mod}$ whose objects are all invertible $\underline{M}$-modules, and whose morphisms are the isomorphisms of $\underline{M}$-modules (see definition 4.8.6(iv)); the functor $\mathscr{N} \mapsto \mathscr{N}^\times$ provides a quasi-inverse $(\underline{M}\text{-}\mathbf{Inv})^\times \to \underline{M}^\times\text{-}\mathbf{Inv}$.

Especially, if $(X, \underline{M})$ is a log scheme, we see that the category of invertible $\underline{M}$-modules is equivalent to that of invertible $\mathscr{O}_X^\times$-modules, hence also to that of invertible $\mathscr{O}_X$-modules.

If $\varphi : (Z, \underline{N}) \to (X, \underline{M})$ is a morphism of log schemes, and $\mathscr{N}$ a $\underline{M}$-module, we let :

$$\varphi^*\mathscr{N} := \varphi^{-1}\mathscr{N} \otimes_{\varphi^{-1}\underline{M}} \underline{N}. \tag{12.4.4}$$

Clearly, $\varphi^*\mathscr{N}$ is an invertible $\underline{N}$-module, whenever $\mathscr{N}$ is an invertible $\underline{M}$-module.

12.4.5. Keep the notation of definition 12.4.2(i); by faithfully flat descent, the restriction of $\mathscr{A}$ to the Zariski site of X is again a quasi-coherent $\mathscr{O}_X$-algebra, which we denote again by $\mathscr{A}$. We may then set $Y := \mathrm{Proj}\,\mathscr{A}$, and let $\pi : Y \to X$ be the natural morphism. The composition of $\pi^{-1}\beta_{\mathscr{A}}$ and the morphism (10.6.24), yields a map on the site Y_τ :

$$\pi^{-1}\underline{P}_n \to \mathscr{O}_Y(n) \qquad \text{for every } n \in \mathbb{N}. \tag{12.4.6}$$

Set $\mathscr{O}_Y(\bullet) := \coprod_{n\in\mathbb{Z}} \mathscr{O}_Y(n)$; it is easily seen that the morphisms (10.6.21) induce a natural $\mathbb{Z}$-graded monoid structure on $\mathscr{O}_Y(\bullet)$, and the coproduct of the maps (12.4.6) amounts to a morphism of graded monoids :

$$\omega_\bullet : \pi^{-1}\underline{P} \to \mathscr{O}_Y(\bullet).$$

We let $\underline{Q}$ be the push-out in the cocartesian diagram :

$$\begin{array}{ccc} \omega_\bullet^{-1}(\mathscr{O}_Y(\bullet)^\times) & \longrightarrow & \pi^{-1}\underline{P} \\ \downarrow & & \downarrow \\ \mathscr{O}_Y(\bullet)^\times & \longrightarrow & \underline{Q}. \end{array} \tag{12.4.7}$$

Clearly $\underline{Q}$ is naturally a $\mathbb{Z}$-graded monoid, in such a way that all the arrows in (12.4.7) are morphisms of $\mathbb{Z}$-graded monoids. For every $n \in \mathbb{Z}$, let $\underline{Q}_n$ be the degree n subsheaf of $\underline{Q}$; the map $\omega_\bullet$ and the natural inclusion $\mathscr{O}_Y(\bullet)^\times \to \mathscr{O}_Y(\bullet)$ determine a unique morphism $\underline{Q} \to \mathscr{O}_Y(\bullet)$, whose restriction in degree zero is a pre-log structure :

$$\beta^\sim_{\mathscr{A}} : \underline{Q}_0 \to \mathscr{O}_Y.$$

Clearly α induces a unique morphism $\alpha^\sim : \pi^{-1}\underline{M} \to \underline{Q}_0$, such that the diagram of monoids :

$$\begin{array}{ccc} \pi^{-1}\underline{M} & \xrightarrow{\alpha^\sim} & \underline{Q}_0 \\ {\scriptstyle \pi^{-1}\beta}\downarrow & & \downarrow{\scriptstyle \beta^\sim_{\mathscr{A}}} \\ \pi^{-1}\underline{\mathscr{O}}_X & \xrightarrow{\pi^\natural} & \mathscr{O}_Y \end{array}$$

commutes. Denote by $\underline{P}^\sim$ the log structure associated to $\beta^\sim_{\mathscr{A}}$; the *homogeneous spectrum* of the quasi-coherent $\mathbb{N}$-graded algebra $(\mathscr{A}, \underline{P}, \alpha, \beta_{\mathscr{A}})$ is defined as the $(X, \underline{M})$-scheme :

$$\mathrm{Proj}(\mathscr{A}, \underline{P}) := (Y, \underline{P}^\sim).$$

We also let :

$$U_1(\mathscr{A}, \underline{P}) := U_1(\mathscr{A}) \times_Y \mathrm{Proj}(\mathscr{A}, \underline{P}).$$

Furthermore, for every $n \in \mathbb{Z}$ we have a natural morphism of $\underline{Q}_0$-monoids :

$$\underline{Q}_0 \otimes_{\mathscr{O}_Y^\times} \mathscr{O}_Y(n)^\times \to \underline{Q}_n \tag{12.4.8}$$

and it is easily seen that $(12.4.8)_{|U_1(\mathscr{A})}$ is an isomorphism. We set :

$$\underline{P}^\sim(n) := (\underline{P}^\sim \otimes_{\underline{Q}_0} \underline{Q}_n)_{|U_1(\mathscr{A})} \qquad \text{for every } n \in \mathbb{Z}.$$

Hence (12.4.8) induces a natural isomorphism :

$$(\underline{P}^\sim \otimes_{\mathscr{O}_Y^\times} \mathscr{O}_Y(n)^\times)_{|U_1(\mathscr{A})} \xrightarrow{\sim} \underline{P}^\sim(n). \tag{12.4.9}$$

Especially, $\underline{P}^\sim(n)$ is an invertible $\underline{P}^\sim_{|U_1(\mathscr{A})}$-module, for every $n \in \mathbb{Z}$. From (12.4.9), we also deduce natural isomorphisms of $\underline{P}^\sim_{|U_1(\mathscr{A})}$-modules :

$$\underline{P}^\sim(n) \otimes_{\underline{P}^\sim} \underline{P}^\sim(m) \xrightarrow{\sim} \underline{P}^\sim(n+m) \qquad \text{for every } n, m \in \mathbb{Z} \tag{12.4.10}$$

and of $\mathscr{O}_{U_1(\mathscr{A})}$-modules :

$$\underline{P}^\sim(n) \otimes_{\underline{P}^\sim} \mathscr{O}_{U_1(\mathscr{A})} \xrightarrow{\sim} \mathscr{O}_Y(n)_{|U_1(\mathscr{A})} \qquad \text{for every } n \in \mathbb{Z}. \tag{12.4.11}$$

Additionally, the morphism $\pi^{-1}\underline{P}_n \to \underline{Q}_n$ deduced from (12.4.7), yields a natural map of $\underline{P}^\sim_{|U_1(\mathscr{A})}$-modules :

(12.4.12) $$\lambda_n : (\pi^*\underline{P}_n)_{|U_1(\mathscr{A})} \to \underline{P}^\sim(n) \qquad \text{for every } n \in \mathbb{N}.$$

Example 12.4.13. Let $(Z, \gamma : \underline{N} \to \mathscr{O}_Z)$ be a log scheme, $\mathscr{L}$ an invertible $\underline{N}$-module, and set :

$$\beta_{\mathscr{A}(\mathscr{L})} := \mathrm{Sym}^\bullet_{\underline{N}}\mathscr{L} \otimes_{\underline{N}} \gamma : \mathrm{Sym}^\bullet_{\underline{N}}\mathscr{L} \to \mathscr{A}(\mathscr{L}) := \mathrm{Sym}^\bullet_{\mathscr{O}_Z}(\mathscr{L} \otimes_{\underline{N}} \mathscr{O}_Z)$$

which is a morphisms of $\mathbb{N}$-graded monoids (notation of example 4.8.10). Clearly $\mathscr{A}(\mathscr{L})$ is also a $\mathbb{N}$-graded quasi-coherent $\mathscr{O}_Z$-algebra. Denote also :

$$\alpha_{\mathscr{L}} : \underline{N} \to \mathrm{Sym}^\bullet_{\underline{N}}\mathscr{L}$$

the natural morphism that identifies $\underline{N}$ to $\mathrm{Sym}^0_{\underline{N}}\mathscr{L}$; then the datum

$$(\mathscr{A}(\mathscr{L}), \mathrm{Sym}^\bullet_{\underline{N}}\mathscr{L}, \alpha_{\mathscr{L}}, \beta_{\mathscr{A}(\mathscr{L})})$$

is a quasi-coherent $\mathscr{O}_{(Z,\underline{N})}$-algebra, and a direct inspection of the definitions shows that the induced morphism of log schemes :

(12.4.14) $$\pi_{(Z,\underline{N})} : \mathbb{P}(\mathscr{L}) := \mathrm{Proj}(\mathscr{A}(\mathscr{L}), \mathrm{Sym}^\bullet_{\underline{N}}\mathscr{L}) \to (Z, \underline{N})$$

is an isomorphism. Furthermore, we have natural isomorphisms as in (10.6.34) :

$$\pi^*_{(Z,\underline{N})}(\mathscr{L}^{\otimes n} \otimes_{\underline{N}} \mathscr{O}_Z) \xrightarrow{\sim} \mathscr{O}_{\mathbb{P}(\mathscr{L})}(n) \qquad \text{for every } n \in \mathbb{Z}.$$

Let $\underline{P}^\sim_{\mathscr{L}} \to \mathscr{O}_{\mathbb{P}(\mathscr{L})}$ be the log structure of $\mathbb{P}(\mathscr{L})$; there follows a natural identification :

(12.4.15) $$\pi^*_{(Z,\underline{N})}\mathscr{L}^{\otimes n} \xrightarrow{\sim} \underline{P}^\sim_{\mathscr{L}} \otimes_{\mathscr{O}^\times_{\mathbb{P}(\mathscr{L})}} \mathscr{O}_{\mathbb{P}(\mathscr{L})}(n)^\times \xrightarrow{\sim} \underline{P}^\sim_{\mathscr{L}}(n) \qquad \text{for every } n \in \mathbb{Z}$$

where the last isomorphism is (12.4.9), in view of the fact that $U_1(\mathscr{A}(\mathscr{L})) = \mathrm{Proj}\,\mathscr{A}(\mathscr{L})$.

Example 12.4.16. (i) Let $(Z, \underline{N})$ be a log scheme, and $n \in \mathbb{N}$ any integer. We define an $\mathbb{N}$-grading on $\mathbb{N}^{\oplus n}$, by setting

$$\mathrm{gr}^k\mathbb{N}^{\oplus n} := \{a_\bullet := (a_1, \dots, a_n) \mid a_1 + \cdots + a_n = k\} \qquad \text{for every } k \in \mathbb{N}.$$

We then define the $\mathbb{N}$-graded monoid

$$\mathrm{Sym}^\bullet_{\underline{N}}\underline{N}^{\oplus n} := \mathbb{N}^{\oplus n}_Z \times \underline{N}$$

whose $\mathbb{N}$-grading is deduced in the obvious way from the foregoing grading of $\mathbb{N}^{\oplus n}$. The log structure $\gamma : \underline{N} \to \mathscr{O}_Z$ extends naturally to a map of $\mathbb{N}$-graded Z-monoids :

$$\mathrm{Sym}^\bullet_{\underline{N}}\gamma^{\oplus n} : \mathrm{Sym}^\bullet_{\underline{N}}\underline{N}^{\oplus n} \to \mathrm{Sym}^\bullet_{\mathscr{O}_Z}\mathscr{O}_Z^{\oplus n}.$$

Namely, if $e_1, \dots, e_n$ is the canonical basis of the free $\mathscr{O}_Z$-module $\mathscr{O}_Z^{\oplus n}$, then $\mathrm{Sym}^k_{\mathscr{O}_Z}\mathscr{O}_Z^{\oplus n}$ is a free $\mathscr{O}_Z$-module with basis

$$\{e_{a_\bullet} := e_1^{a_1} \cdots e_n^{a_n} \mid a_\bullet \in \mathrm{gr}^k\mathbb{N}^{\oplus n}\}$$

and $\mathrm{Sym}^\bullet_{\underline{N}}\gamma^{\oplus n}$ is given by the rule : $(a_\bullet, x) \mapsto \gamma(x)\cdot e_{a_\bullet}$ for every $a_\bullet \in \mathbb{N}^{\oplus n}$, every τ-open subset of Z, and every section $x \in \underline{N}(U)$. Clearly $\mathrm{Sym}^\bullet_{\underline{N}}\gamma^{\oplus n}$ defines an $\mathbb{N}$-graded $\mathscr{O}_{(Z,\underline{N})}$-algebra

$$\mathrm{Sym}^\bullet_{(\mathscr{O}_Z,\underline{N})}(\mathscr{O}_Z, \underline{N})^{\oplus n} := (\mathrm{Sym}^\bullet_{\mathscr{O}_Z}\mathscr{O}_Z^{\oplus n}, \mathrm{Sym}^\bullet_{\underline{N}}\underline{N}^{\oplus n}) \qquad \text{for every } n \in \mathbb{N}.$$

We set :

$$\mathbb{P}^n_{(Z,\underline{N})} := \mathrm{Proj}\,\mathrm{Sym}^\bullet_{(\mathscr{O}_Z,\underline{N})}(\mathscr{O}_Z, \underline{N})^{\oplus n+1}$$

and we call it the *projective n-dimensional space* over $(Z, \underline{N})$.

(ii) Denote by $\underline{N}^\sim$ the log structure of $\mathbb{P}^n_{(Z,\underline{N})}$, and by $\underline{N}^\sim(k)$ the $\underline{N}^\sim$-modules defined as in (12.4.9), for every $k \in \mathbb{Z}$. By simple inspection we get a commutative diagram of monoids :

$$\begin{array}{ccc} \Gamma(Z, \mathrm{Sym}^k_{\underline{N}} \underline{N}^{\oplus n+1}) & \xrightarrow{\Gamma(Z,\mathrm{Sym}^k_{\underline{N}}\gamma^{\oplus n+1})} & \Gamma(Z, \mathrm{Sym}^k_{\mathscr{O}_Z} \mathscr{O}_Z^{\oplus n+1}) \\ \downarrow & & \downarrow \\ \Gamma(\mathbb{P}^n_{(Z,\underline{N})}, \underline{N}^\sim(k)) & \longrightarrow & \Gamma(\mathbb{P}^n_{(Z,\underline{N})}, \mathscr{O}_{\mathbb{P}^n_{(Z,\underline{N})}}(k)) \end{array} \qquad \text{for every } k \in \mathbb{N}$$

whose right vertical arrow is an isomorphism. Especially, the natural basis of the free $\underline{N}(Z)$-module $\Gamma(Z, \mathrm{Sym}^1_{\underline{N}} \underline{N}^{\oplus n+1}) = \underline{N}(Z)^{\oplus n+1}$ yields a distinguished system of $n+1$ elements

$$\varepsilon_0, \dots, \varepsilon_n \in \Gamma(\mathbb{P}^n_{(Z,\underline{N})}, \underline{N}^\sim(1)).$$

On the other hand, we have as well the distinguished system of global sections

$$T_0, \dots, T_n \in \Gamma(\mathbb{P}^n_{(Z,\underline{N})}, \mathscr{O}_{\mathbb{P}^n_{(Z,\underline{N})}}(1))$$

corresponding to the natural basis of $\Gamma(Z, \mathrm{Sym}^1_{\mathscr{O}_Z} \mathscr{O}_Z^{\oplus n+1}) = \mathscr{O}_Z(Z)^{\oplus n+1}$. For each $i = 0, \dots, n$, the largest open subset $U_i \subset \mathbb{P}^n_{(Z,\underline{N})}$ such that $T_i \in \Gamma(U_i, \mathscr{O}_{\mathbb{P}^n_{(Z,\underline{N})}}(1)^\times)$ is the complement of the hyperplane where T_i vanishes. Moreover, notice that

$$T_i^{-1} T_j \in \underline{N}^\sim(U_i) \qquad \text{for every } i, j = 0, \dots, n.$$

With this notation, the isomorphism (12.4.9) yields the identification :

$$\varepsilon_j = T_i^{-1} T_j \otimes T_i \qquad \text{on } U_i \qquad \text{for every } i, j = 0, \dots, n$$

from which we also see that, for every $i = 0, \dots, n$, the open subset U_i is the largest such that $\varepsilon_i \in \Gamma(U_i, \underline{N}^\sim(1)^\times)$. By the same token, we obtain :

(12.4.17) $$(\mathbb{P}^n_{(Z,\underline{N})}, \underline{N}^\sim)_{\mathrm{tr}} = U_0 \cap \dots \cap U_n \simeq \mathbb{G}^n_{m,Z}.$$

12.4.18. Let $(Z, \underline{N})$ be a log scheme, $(\varphi, \log \varphi) : (\mathscr{A}, \underline{P}) \to (\mathscr{A}', \underline{P}')$ a morphism of quasi-coherent $\mathbb{N}$-graded $\mathscr{O}_{(Z,\underline{N})}$-algebras. We let (notation of (10.6.26)) :

$$G(\varphi, \log \varphi) := G(\varphi) \times_{\mathrm{Proj}\,\mathscr{A}'} \mathrm{Proj}(\mathscr{A}', \underline{P}').$$

Denote also by $\pi : Y := \mathrm{Proj}\,\mathscr{A} \to Z$ and $\pi' : Y' := \mathrm{Proj}\,\mathscr{A}' \to Z'$ the natural projections; there follows, on the one hand, a morphism of $\mathbb{N}$-graded monoids :

(12.4.19) $$\pi'^{-1}(\log \varphi) : (\mathrm{Proj}\,\varphi)^{-1}(\pi^{-1}\underline{P}) \to (\pi'^{-1}\underline{P}')_{|G(\varphi)}$$

and on the other hand, a morphism of $\mathbb{Z}$-graded monoids :

(12.4.20) $$(\mathrm{Proj}\,\varphi)^{-1}\mathscr{O}_Y(\bullet)^\times \to \mathscr{O}_{Y'}(\bullet)^\times_{|G(\varphi)}$$

deduced from (10.6.28). Define the $\mathbb{Z}$-graded monoid $\underline{Q}$ on Y_τ as in (12.4.7), and the analogous $\mathbb{Z}$-graded monoid $\underline{Q}'$ on Y'_τ. Then (12.4.19) and (12.4.20) determine a unique morphism of $\mathbb{Z}$-graded monoids :

$$\vartheta : (\mathrm{Proj}\,\varphi)^{-1}\underline{Q} \to \underline{Q}'_{|G(\varphi)}$$

and by construction, the restriction of ϑ in degree zero is a morphism of pre-log structures :

$$\vartheta_0 : (\mathrm{Proj}\,\varphi)^*((\underline{Q})_0, \beta^\sim_{\mathscr{A}}) \to ((\underline{Q}')_0, \beta^\sim_{\mathscr{A}'})$$

whence a morphism of $(Z, \underline{N})$-schemes :

$$\mathrm{Proj}(\varphi, \log \varphi) : G(\varphi, \log \varphi) \to \mathrm{Proj}(\mathscr{A}, \underline{P}).$$

Moreover, on the one hand, (10.6.28) induces an isomorphism of $\mathscr{O}^\times_{Y'}$-modules :

$$\nu^\times_{\mathscr{A}(n)} : (\mathscr{O}^\times_{Y'} \otimes_{(\mathrm{Proj}\,\varphi)^{-1}\mathscr{O}^\times_Y} (\mathrm{Proj}\,\varphi)^{-1}\mathscr{O}_Y(n)^\times)_{|G_1(\varphi)} \xrightarrow{\sim} \mathscr{O}_{Y'}(n)^\times_{|G_1(\varphi)} \qquad \text{for every } n \in \mathbb{Z}$$

(notation of (10.6.27)). On the other hand, for every $n \in \mathbb{Z}$, the morphism of $(\mathrm{Proj}\,\varphi)^{-1}(\underline{Q})_0$-modules ϑ_n determines a morphism of $\underline{P}'^{\sim}$-modules :

$$\vartheta_n^{\sim} : \mathrm{Proj}(\varphi, \log\,\varphi)^* \underline{P}^{\sim}(n)_{|G_1(\varphi)} \to \underline{P}'^{\sim}(n)_{|G_1(\varphi)} \tag{12.4.21}$$

and by inspecting the construction, it is easily seen that the isomorphism (12.4.9) (and the corresponding one for $\underline{P}'^{\sim}(n)$) identifies $\vartheta_n^{\sim}$ with $\nu^{\times}_{\mathscr{A}(n)} \otimes_{\mathscr{O}^{\times}_{Y'}} \underline{P}'^{\sim}$; especially, $\vartheta_n^{\sim}$ is an isomorphism.

12.4.22. Let $\psi : (Z', \underline{N}') \to (Z, \underline{N})$ be a morphism of log schemes, and $(\mathscr{A}, \underline{P}, \alpha, \beta_{\mathscr{A}})$ a $\mathbb{N}$-graded quasi-coherent $\mathscr{O}_{(Z,\underline{N})}$-algebra. We may view $\underline{P}$ as a $\underline{N}$-module, via the morphism α, hence we may form the $\underline{N}'$-module $\psi^*\underline{P}$, as in (12.4.4). Moreover, by remark (6.1.25)(i), $\psi^*\underline{P}$ is a $\mathbb{N}$-graded sheaf of monoids on Z'_τ, such that

$$\psi^*(\mathscr{A}, \underline{P}) := (\psi^*\mathscr{A}, \psi^*\underline{P}, \psi^*\alpha, \psi^*\beta_{\mathscr{A}}) \tag{12.4.23}$$

is a $\mathbb{N}$-graded quasi-coherent $\mathscr{O}_{(Z',\underline{N}')}$-algebra, and in view of the isomorphism (6.1.26), we obtain a natural isomorphism of $(Z', \underline{N}')$-schemes:

$$\mathrm{Proj}\,\psi^*(\mathscr{A}, \underline{P}) \xrightarrow{\sim} (Z', \underline{N}') \times_{(Z,\underline{N})} \mathrm{Proj}(\mathscr{A}, \underline{P}). \tag{12.4.24}$$

Furthermore, denote by $\pi_{(\mathscr{A},\underline{P})} : U_1(\psi^*(\mathscr{A}, \underline{P})) \to U_1(\mathscr{A}, \underline{P})$ the morphism deduced from (12.4.24), and by $\pi_Y : Y' := \mathrm{Proj}\,\psi^*\mathscr{A} \to Y := \mathrm{Proj}\,\mathscr{A}$ the underlying morphism of schemes. From (10.6.31) we obtain natural isomorphisms :

$$(\mathscr{O}^{\times}_{Y'} \otimes_{\pi_Y^{-1}\mathscr{O}^{\times}_Y} \pi_Y^{-1}\mathscr{O}_Y(n)^{\times})_{|U_1(\psi^*\mathscr{A})} \xrightarrow{\sim} \mathscr{O}_{Y'}(n)^{\times}_{|U_1(\psi^*\mathscr{A})} \qquad \text{for every } n \in \mathbb{Z}$$

and the latter induce natural identifications :

$$\pi^*_{(\mathscr{A},\underline{P})}\underline{P}^{\sim}(n) \xrightarrow{\sim} (\psi^*\underline{P})^{\sim}(n) \qquad \text{for every } n \in \mathbb{Z}. \tag{12.4.25}$$

12.4.26. Keep the notation of (12.4.5), and let $\log\mathscr{C}_{(X,\underline{M})}$ be the category whose objects are the pairs $((Z, \underline{N}), \mathscr{L})$, where $(Z, \underline{N})$ is a $(X, \underline{M})$-scheme, and $\mathscr{L}$ is an invertible $\underline{N}$-module. The morphisms $((Z, \underline{N}), \mathscr{L}) \to ((Z', \underline{N}'), \mathscr{L}')$ are the pairs (φ, h), where $\varphi : (Z, \underline{N}) \to (Z', \underline{N}')$ is a morphism of $(X, \underline{M})$-schemes, and $h : \varphi^*\mathscr{L}' \xrightarrow{\sim} \mathscr{L}$ is an isomorphism of $\underline{N}$-modules (with composition of morphisms defined in the obvious way). There is an obvious forgetful functor :

$$\mathsf{p} : \log\mathscr{C}_{(X,\underline{M})} \to \mathscr{C}_X \quad : \quad ((Z, \underline{N}), \mathscr{L}) \mapsto (Z, \mathscr{L} \otimes_{\underline{N}} \mathscr{O}_Z)$$

and the functor $F_{\mathscr{A}}$ can be lifted to a functor :

$$F_{(\mathscr{A},\underline{P})} : \log\mathscr{C}_{(X,\underline{M})} \to \mathbf{Set}$$

which assigns to any object $((Z, \gamma : \underline{N} \to \mathscr{O}_Z), \mathscr{L})$ the set consisting of all morphisms of $\mathbb{N}$-graded quasi-coherent $\mathscr{O}_{(Z,\underline{N})}$-algebras :

$$\begin{array}{ccc} \psi^*\underline{P} & \xrightarrow{\log g} & \mathrm{Sym}^{\bullet}_{\underline{N}}\mathscr{L} \\ {\scriptstyle \psi^*\beta_{\mathscr{A}}}\downarrow & & \downarrow{\scriptstyle \beta_{\mathscr{A}(\mathscr{L})}} \\ \psi^*\mathscr{A} & \xrightarrow{g} & \mathscr{A}(\mathscr{L}) \end{array}$$

where $\psi : (Z, \underline{N}) \to (X, \underline{M})$ is the structural morphism, and g is an epimorphism on the underlying $\mathscr{O}_Z$-modules.

Proposition 12.4.27. $(U_1(\mathscr{A}, \underline{P}), \underline{P}^{\sim}(1)) \in \mathrm{Ob}(\log\mathscr{C}_{(X,\underline{M})})$ *represents the functor* $F_{(\mathscr{A},\underline{P})}$.

Proof. Given $((Z, \underline{N}) \xrightarrow{\psi} (X, \underline{M}), \mathscr{L}) \in \mathrm{Ob}(\log\mathscr{C}_{(X,\underline{M})})$ and $(g, \log g) \in F_{(\mathscr{A},\underline{P})}((Z, \underline{N}), \mathscr{L})$, define $\mathbb{P}(\mathscr{L})$ as in (12.4.14); there follows a morphism of $(Z, \underline{N})$-schemes :

$$\mathrm{Proj}(g, \log g) : \mathbb{P}(\mathscr{L}) \to \mathrm{Proj}\,\psi^*(\mathscr{A}, \underline{P}).$$

In view of (12.4.14) and (12.4.24), this is the same as a morphism of $(X,\underline{M})$-schemes :

$$\mathbb{P}(g,\log g):(Z,\underline{N})\to \mathrm{Proj}(\mathscr{A},\underline{P})$$

and arguing as in the proof of lemma 10.6.33, we see that the image of $\mathbb{P}(g,\log g)$ lands in $U_1(\mathscr{A},\underline{P})$. Next, combining (12.4.15), (12.4.21) and (12.4.25), we deduce a natural isomorphism :

$$\begin{aligned}\pi^*_{(Z,\underline{N})}\circ\mathbb{P}(g,\log g)^*\underline{P}^\sim(1) &\xrightarrow{\sim} \mathrm{Proj}(g,\log g)^*\circ\pi^*_{(\mathscr{A},\underline{P})}\underline{P}^\sim(1)\\ &\xrightarrow{\sim} \mathrm{Proj}(g,\log g)^*(\psi^*\underline{P})^\sim(1)\\ &\xrightarrow{\sim} \underline{P}^\sim_{\mathscr{L}}(1)\\ &\xrightarrow{\sim} \pi^*_{(Z,\underline{N})}\mathscr{L}\end{aligned}$$

whence an isomorphism $h(g,\log g):\mathbb{P}(g,\log g)^*\underline{P}^\sim(1)\xrightarrow{\sim}\mathscr{L}$, and the datum :

$$(\mathbb{P}(g,\log g),h(g,\log g)):((Z,\underline{N}),\mathscr{L})\to(U_1(\mathscr{A},\underline{P}),\underline{P}^\sim(1))$$

is a well defined morphism of $\log\mathscr{C}_{(X,\underline{M})}$.

Conversely, let $\varphi:=(\beta,\log\beta):(Z,\underline{N})\to U_1(\mathscr{A},\underline{P})$ be a morphism of $(X,\underline{M})$-schemes, and $h:\varphi^*\underline{P}^\sim(1)\xrightarrow{\sim}\mathscr{L}$ an isomorphism of $\underline{N}$-modules. In view of (12.4.10), we deduce an isomorphism :

$$h^{\otimes n}:\varphi^*\underline{P}^\sim(n)\xrightarrow{\sim}\mathscr{L}^{\otimes n}\qquad\text{for every } n\in\mathbb{Z}.$$

Combining with (12.4.12), we may define the map of $\underline{N}$-modules :

$$\log\widehat{g}(\varphi,h):=\bigoplus_{n\in\mathbb{N}}h^{\otimes n}\circ\beta^*(\lambda_n):\psi^*\underline{P}\to\mathrm{Sym}^\bullet_{\underline{N}}\mathscr{L}.$$

On the other hand, in view of (12.4.11), we have an isomorphism of $\mathscr{O}_Z$-modules :

$$h\otimes_{\underline{N}}\mathscr{O}_Z:\beta^*\mathscr{O}_Y(1)_{|U_1(\mathscr{A})}\xrightarrow{\sim}\varphi^*\underline{P}^\sim(1)\otimes_{\underline{N}}\mathscr{O}_Z\xrightarrow{\sim}\mathscr{L}\otimes_{\underline{N}}\mathscr{O}_Z$$

(where, as usual, $Y:=\mathrm{Proj}\,\mathscr{A}$). We let (notation of (10.6.36)) :

$$\widehat{g}(\varphi,h):=g(\beta,h\otimes_{\underline{N}}\mathscr{O}_Z)$$

and notice that the pair $(\widehat{g}(\varphi,h),\log\widehat{g}(\varphi,h))$ is an element of $F_{(\mathscr{A},\underline{P})}((Z,\underline{N}),\mathscr{L})$. Summing up, we have exhibited two natural transformations :

$$\vartheta:F_{(\mathscr{A},\underline{P})}\Rightarrow\mathrm{Hom}_{\log\mathscr{C}_{(X,\underline{M})}}(-,(U_1(\mathscr{A},\underline{P}),\underline{P}^\sim(1)))\quad(g,\log g)\mapsto\mathbb{P}(g,\log g)$$
$$\sigma:\mathrm{Hom}_{\log\mathscr{C}_{(X,\underline{M})}}(-,(U_1(\mathscr{A},\underline{P}),\underline{P}^\sim(1)))\Rightarrow F_{(\mathscr{A},\underline{P})}\qquad(\varphi,h)\mapsto(\widehat{g}(\varphi,h),\log\widehat{g}(\varphi,h))$$

and it remains to show that these transformations are isomorphisms of functors. However, the latter fit into an essentially commutative diagram of natural transformations :

$$\begin{array}{ccccc}F_{(\mathscr{A},\underline{P})} & \xRightarrow{\vartheta} & \mathrm{Hom}_{\log\mathscr{C}_{(X,\underline{M})}}(-,(U_1(\mathscr{A},\underline{P}),\underline{P}^\sim(1))) & \xRightarrow{\sigma} & F_{(\mathscr{A},\underline{P})}\\ \Downarrow & & \Downarrow & & \Downarrow\\ F_{\mathscr{A}}\circ\mathsf{p} & \Longrightarrow & \mathrm{Hom}_{\mathscr{C}_X}(\mathsf{p}(-),(U_1(\mathscr{A}),\mathscr{O}_Y(1)_{|U_1(\mathscr{A})})) & \Longrightarrow & F_{\mathscr{A}}\circ\mathsf{p}\end{array}$$

whose bottom line is given by the natural transformations (10.6.37). Moreover, given an isomorphism h of invertible $\underline{N}$-modules, the discussion in (12.4.3) leads to the identity :

$$h=(h\otimes_{\underline{N}}\mathscr{O}_Z)^\times\otimes_{\mathscr{O}_{Z^\times}}\underline{N}. \qquad (12.4.28)$$

Likewise, we have natural identifications :

$$\mathrm{Sym}^n_{\underline{N}}\mathscr{L}=(\mathrm{Sym}^n_{\mathscr{O}_Z}\mathscr{L}\otimes_{\underline{N}}\mathscr{O}_Z)^\times\otimes_{\mathscr{O}_Z^\times}\underline{N}\qquad\text{for every } n\in\mathbb{Z}$$

which show that $\beta_{\mathscr{A}}$ and g determine uniquely $\log g$. This – and an inspection of the proof of lemma 10.6.33 – already implies that $\sigma\circ\vartheta$ is the identity automorphism of the functor

$F_{(\mathscr{A},\underline{P})}$. Finally, let $\varphi = (\beta, \log \beta)$ and h as in the foregoing, so that (φ, h) is a morphism in $\log \mathscr{C}_{(X,\underline{M})}$; to conclude we have to show that $(\varphi', h') := \vartheta \circ \sigma(\varphi, h)$ equals (φ, h). Say that $\varphi' := (\beta', \log \beta')$; by the above (and by the proof of lemma 10.6.33) we already know that $\beta = \beta'$, and (12.4.28) implies that $h = h'$. Hence, it remains only to show that $\log \beta = \log \beta'$, which can be checked directly on the stalks over the τ-points of Z : we leave the details to the reader. $\square$

Example 12.4.29. (i) Let $\psi : (Z', \underline{N}') \to (Z, \underline{N})$ be a morphism of log schemes, $n \in \mathbb{N}$ any integer, and $\mathbb{P}^n_{(Z,\underline{N})}$ the n-dimensional projective space over $(Z, \underline{N})$, defined as in example 12.4.16. A simple inspection of the definitions yields a natural isomorphism of $\mathscr{O}_{(Z',\underline{N}')}$-algebras

$$\mathrm{Sym}^\bullet_{(\mathscr{O}_{Z'},\underline{N}')}(\mathscr{O}_{Z'}, \underline{N}')^{\oplus n} \xrightarrow{\sim} \psi^*\mathrm{Sym}^\bullet_{(\mathscr{O}_Z,\underline{N})}(\mathscr{O}_Z, \underline{N})^{\oplus n} \qquad \text{for every } n \in \mathbb{N}$$

whence a natural isomorphism of $(Z', \underline{N}')$-schemes :

$$\mathbb{P}^n_{(Z',\underline{N}')} \xrightarrow{\sim} (Z', \underline{N}') \times_{(Z,\underline{N})} \mathbb{P}^n_{(Z,\underline{N})} \qquad \text{for every } n \in \mathbb{N}.$$

(ii) Let $\mathscr{L}$ be any invertible $\underline{N}$-module; notice that a morphism of $\mathbb{N}$-graded monoids

$$\mathrm{Sym}^\bullet_{\underline{N}} \underline{N}^{\oplus n} \to \mathrm{Sym}^\bullet_{\underline{N}} \mathscr{L}$$

which is the identity map in degree zero, is the same as the datum of a sequence

$$(\beta_i : \underline{N} \to \mathscr{L} \mid i = 1, , \dots, n)$$

of morphisms of $\underline{N}$-modules, and the latter is the same as a sequence $(b_1, \dots, b_n)$ of global sections of $\mathscr{L}$. Since $\mathrm{Sym}^1_{\mathscr{O}_Z} \mathscr{O}_Z^{\oplus n+1}$ generates $\mathrm{Sym}^\bullet_{\mathscr{O}_Z} \mathscr{O}_Z^{n+1}$, proposition 12.4.27 and (i) imply that $\mathbb{P}^n_{(Z,\underline{N})}$ represents the functor $\log \mathscr{C}_{(Z,\underline{N})} \to \mathbf{Set}$ that assigns to any pair $((X, \underline{M}), \mathscr{L})$ the set of all sequences $(b_0, \dots, b_n)$ of global sections of $\mathscr{L}$. The bijection

$$\text{(12.4.30)} \qquad \mathrm{Hom}_{\log \mathscr{C}_{(Z,\underline{N})}}(((X, \underline{M}), \mathscr{L}), (\mathbb{P}^n_{(Z,\underline{N})}, \underline{N}^\sim(1))) \xrightarrow{\sim} \Gamma(X, \mathscr{L})^{n+1}$$

can be explicited as follows. Let $(\varphi, h) : ((X, \underline{M}), \mathscr{L}) \to (\mathbb{P}^n_{(Z,\underline{N})}, \underline{N}^\sim(1))$ be a given morphism in $\log \mathscr{C}_{(Z,\underline{N})}$; then $h : \varphi^*\underline{N}^\sim(1) \to \mathscr{L}$ is an isomorphism of $\underline{M}$-modules, which induces a map on global sections

$$\Gamma(h) : \Gamma(\mathbb{P}^n_{(Z,\underline{N})}, \underline{N}^\sim(1))) \to \Gamma(X, \mathscr{L}).$$

Now, $\underline{N}^\sim(1)$ admits a distiguished system of global sections $\varepsilon_0, \dots, \varepsilon_n$ (example 12.4.16(ii)), and the bijection (12.4.30) assigns to (φ, h) the sequence $(\Gamma(h)(\varepsilon_0), \dots, \Gamma(h)(\varepsilon_n))$.

(iii) Given a sequence $b_\bullet := (b_0, \dots, b_n)$ as in (ii), denote by

$$f_{b_\bullet} : (X, \underline{M}) \to \mathbb{P}^n_{(Z,\underline{N})}$$

the corresponding morphism. A direct inspection of the definitions shows that the formation of $f_{b_\bullet}$ is compatible with arbitrary base changes $h : (Z', \underline{N}') \to (Z, \underline{N})$. Namely, set $(X', \underline{M}') := (Z', \underline{N}') \times_{(Z,\underline{N})} (X, \underline{M})$, let $g : (X', \underline{M}') \to (X, \underline{M})$ be the induced morphism, $\mathscr{L}' := g^*\mathscr{L}$, and suppose that $\mathscr{L}'$ is also invertible (which always holds, if $\underline{M}'$ is an integral log structure); the sequence $b_\bullet$ pulls back to a corresponding sequence $b'_\bullet := (b'_0, \dots, b_n)$ of global sections of $\mathscr{L}'$, and there follows a cartesian diagram of log schemes :

$$\begin{array}{ccc} (X', \underline{M}') & \xrightarrow{f_{b'_\bullet}} & \mathbb{P}^n_{(Z',\underline{N}')} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \mathbb{P}^n_h} \\ (X, \underline{M}) & \xrightarrow{f_{b_\bullet}} & \mathbb{P}^n_{(Z,\underline{N})}. \end{array}$$

The details shall be left to the reader.

Definition 12.4.31. Let $(X, \underline{M})$ be a log scheme, and $\mathscr{I} \subset \underline{M}$ an ideal of $\underline{M}$ (see (3.7.22)).

(i) Let $f : (Y, \underline{N}) \to (X, \underline{M})$ be a morphism of log schemes; then $f^{-1}\mathscr{I}$ is an ideal in the sheaf of monoids $f^{-1}\underline{M}$; we let :

$$\mathscr{I}\underline{N} := \log f(f^{-1}\mathscr{I}) \cdot \underline{N}$$

which is the smallest ideal of $\underline{N}$ containing the image of $f^{-1}\mathscr{I}$.

(ii) A *logarithmic blow up* of the ideal $\mathscr{I}$ is a morphism of log schemes

$$\varphi : (X', \underline{M}') \to (X, \underline{M})$$

which enjoys the following universal property. The ideal $\mathscr{I}\underline{M}'$ is invertible, and every morphism of log schemes $(Y, \underline{N}) \to (X, \underline{M})$ such that $\mathscr{I}\underline{N} \subset \underline{N}$ is invertible, factors uniquely through φ.

Remark 12.4.32. (i) Keep the notation of definition 12.4.31. By the usual general nonsense, it is clear that the blow up $(X', \underline{M}')$ is determined up to unique isomorphism of $(X, \underline{M})$-schemes.

(ii) Moreover, let $f : Y \to X$ be a morphism of schemes. Then we claim that the natural projection :

$$(Y', \underline{M}'_Y) := Y \times_X (X', \underline{M}') \to (Y, \underline{M}_Y) := Y \times_X (X, \underline{M})$$

is a logarithmic blow up of $\mathscr{I}\underline{M}_Y$, provided $\mathscr{I}\underline{M}'_Y$ is an invertible ideal; especially, this holds whenever $\underline{M}'$ is an integral log structure (lemma 12.1.18(i)). The proof is left as an exercise for the reader.

12.4.33. Let $(X, \underline{M})$ be a log scheme, $\mathscr{I} \subset \underline{M}$ an ideal; we shall show the existence of the logarithmic blow up of $\mathscr{I}$, under fairly general conditions. To this aim, we introduce the graded *blow up $\mathscr{O}_X$-algebra* :

$$\mathsf{B}(X, \underline{M}, \mathscr{J}) := \bigoplus_{n\in\mathbb{N}} \mathscr{I}^n \otimes_{\underline{M}} \mathscr{O}_X$$

where $\mathscr{I}^n \otimes_{\underline{M}} \mathscr{O}_X$ is the sheaf associated to the presheaf $U \mapsto \mathscr{I}^n(U) \otimes_{\underline{M}(U)} \mathscr{O}_X(U)$ on X_τ. Here $\mathscr{I}^0 := \underline{M}$, and for every $n > 0$ we let $\mathscr{I}^n$ be the sheaf associated to the presheaf $U \mapsto \mathscr{I}(U)^n$ on X_τ. The graded multiplication law of the blow up $\mathscr{O}_X$-algebra is induced by the multiplication $\mathscr{I}^n \times \mathscr{I}^m \to \mathscr{I}^{n+m}$, for every $n, m \in \mathbb{N}$.

The natural map $\mathscr{I}^n \to \mathscr{I}^n \otimes_{\underline{M}} \mathscr{O}_X$ induces a morphism of sheaves of monoids :

$$\coprod_{n\in\mathbb{N}} \mathscr{I}^n \to \mathsf{B}(X, \underline{P}, \mathscr{J}).$$

The latter defines a $\mathbb{N}$-graded $\mathscr{O}_{(X,\underline{M})}$-algebra, which we denote $\mathscr{B}(X, \underline{M}, \mathscr{I})$.

12.4.34. Suppose first that $\mathscr{I}$ is invertible; then it is easily seen that, locally on X_τ, $\mathscr{I}$ is generated by a regular local section (see example 4.8.36(i)), hence the same holds for the power $\mathscr{I}^n$, for every $n \in \mathbb{N}$. Therefore $\mathscr{I}^n$ is a locally free $\underline{M}$-module of rank one, and we have a natural isomorphism :

$$\mathscr{B}(X, \underline{M}, \mathscr{I}) \stackrel{\sim}{\to} (\mathscr{A}(\mathscr{I}), \mathrm{Sym}^\bullet_{\underline{M}}\mathscr{I})$$

(notation of example (12.4.13)). It follows that in this case, the natural projection :

$$\pi_{(X,\underline{M},\mathscr{I})} : \mathrm{Proj}\, \mathscr{B}(X, \underline{M}, \mathscr{I}) \to (X, \underline{M})$$

is an isomorphism of log schemes.

12.4.35. The formation of $\mathscr{B}(X, \underline{M}, \mathscr{I})$ is obviously functorial with respect to morphisms of log schemes; more precisely, such a morphism $f : (Y, \underline{N}) \to (X, \underline{M})$ induces a morphism of $\mathbb{N}$-graded $\mathscr{O}_{(Y,\underline{N})}$-algebras:

$$\mathscr{B}(f, \mathscr{I}) : f^*\mathscr{B}(X, \underline{M}, \mathscr{I}) \to \mathscr{B}(Y, \underline{N}, \mathscr{I}\underline{N})$$

(notation of (12.4.23)) which is an epimorphism on the underlying $\mathscr{O}_Y$-modules. Moreover, if $g : (Z, \underline{Q}) \to (Y, \underline{N})$ is another morphism, we have the identity :

$$\mathscr{B}(f \circ g, \mathscr{I}) = \mathscr{B}(g, \mathscr{I}\underline{N}) \circ g^*\mathscr{B}(f, \mathscr{I}). \tag{12.4.36}$$

Furthermore, the construction of the blow up algebra is local for the topology of X_τ : if $U \to X$ is any object of X_τ, we have a natural identification

$$\mathscr{B}(U, \underline{M}_{|U}, \mathscr{I}_{|U}) \xrightarrow{\sim} \mathscr{B}(X, \underline{M}, \mathscr{I})_{|U}.$$

In the presence of charts for the log structure $\underline{M}$, we can give a handier description for the blow up algebra; namely, we have the following :

Lemma 12.4.37. *Let X be a scheme, $\alpha : \underline{P} \to \mathscr{O}_X$ a pre-log structure, $\beta : \underline{P} \to \underline{P}^{\log}$ the natural morphism of pre-log structures. Let also $\mathscr{I} \subset \underline{P}$ be an ideal, and set $\mathscr{I}\underline{P}^{\log} := \beta(\mathscr{I}) \cdot \underline{P}^{\log}$ (which is the ideal of $\underline{P}^{\log}$ generated by the image of $\mathscr{I}$). Then :*

(i) *There is a natural isomorphism of graded $\mathscr{O}_X$-algebras :*

$$\bigoplus_{n\in\mathbb{N}} \mathscr{I}^n \otimes_{\underline{P}} \mathscr{O}_X \xrightarrow{\sim} \mathsf{B}(X, \underline{P}^{\log}, \mathscr{I}\underline{P}^{\log}).$$

(ii) *Especially, suppose $(X, \underline{M})$ is a log scheme that admits a chart $\beta : P_X \to \underline{M}$, let $I \subset P$ be an ideal, and set $I\underline{M} := \beta(I_X) \cdot \underline{M}$. Then $\mathscr{B}(X, \underline{M}, I\underline{M})$ is a $\mathbb{N}$-graded quasi-coherent $\mathscr{O}_{(X,\underline{M})}$-algebra.*

(iii) *In the situation of* (ii)*, set $(S, P_S^{\log}) := \mathrm{Spec}(\mathbb{Z}, P)$ (see* (12.2.13)*), and denote by $f : (X, \underline{M}) \to (S, P_S^{\log})$ the natural morphism. Then the map :*

$$\mathscr{B}(f, IP_S^{\log}) : f^*\mathscr{B}(S, P_S^{\log}, IP_S^{\log}) \to \mathscr{B}(X, \underline{M}, I\underline{M})$$

is an isomorphism of $\mathbb{N}$-graded $\mathscr{O}_{(X,\underline{M})}$-algebras.

Proof. (i): Since the functor (4.8.51) commutes with all colimits, we have a natural isomorphism of sheaves of rings on X_τ :

$$\mathbb{Z}[\underline{P}^{\log}] \xrightarrow{\sim} \mathbb{Z}[\underline{P}] \otimes_{\mathbb{Z}[\alpha^{-1}\mathscr{O}_X^\times]} \mathbb{Z}[\mathscr{O}_X^\times].$$

We are therefore reduced to showing that the natural morphism

$$\mathbb{Z}[\mathscr{I}^n] \otimes_{\mathbb{Z}[\alpha^{-1}\mathscr{O}_X^\times]} \mathbb{Z}[\mathscr{O}_X^\times] \to \mathscr{I}^n\mathbb{Z}[\underline{P}^{\log}]$$

is an isomorphism for every $n \in \mathbb{N}$. The latter assertion can be checked on the stalks over the τ-points of X; to this aim, we invoke the more general :

Claim 12.4.38. Let G be an abelian group, $\varphi : H \to P'$ and $\psi : H \to G$ two morphisms of monoids, $I \subset P'$ an ideal. Then the natural map

$$\mathbb{Z}[I] \otimes_{\mathbb{Z}[H]} \mathbb{Z}[G] \to I\mathbb{Z}[P' \otimes_H G] \tag{12.4.39}$$

is an isomorphism.

Proof of the claim. Recall that $\mathbb{Z}[I] = I\mathbb{Z}[P']$, and the map (12.4.39) is induced by the natural identification: $\mathbb{Z}[P'] \otimes_{\mathbb{Z}[H]} \mathbb{Z}[G] \xrightarrow{\sim} \mathbb{Z}[P' \otimes_H G]$ (see (4.8.52)). Especially, (12.4.39) is clearly surjective, and it remains to show that it is also injective. To this aim, notice first that ψ factors through the unit of adjunction $\eta : H \to H^{\mathrm{gp}}$; the morphism $\mathbb{Z}[\eta] : \mathbb{Z}[H] \to \mathbb{Z}[H^{\mathrm{gp}}] = H^{-1}\mathbb{Z}[H]$ is a localization map (see (4.8.53)), especially it is flat, hence (12.4.39) is injective when $G =$

$H^{\rm gp}$ and $\psi = \eta$. It follows easily that we may replace H by $H^{\rm gp}$, P' by $P' \otimes_H H^{\rm gp}$, I by $I \cdot (P' \otimes_H H^{\rm gp})$, and therefore assume that H is a group. Let $L := \psi(H)$; arguing as in the foregoing, we may consider separately the case where ψ is the surjection $H \to L$ and the case where ψ is the injection $i : L \to G$. However, one sees easily that $\mathbb{Z}[i] : \mathbb{Z}[L] \to \mathbb{Z}[G]$ is a flat morphism, hence it suffices to consider the case where ψ is a surjective group homomorphism. Set $K := \operatorname{Ker}\psi$; we have a natural identification : $P' \otimes_H G \simeq P' \otimes_K \{1\}$, hence we may further reduce to the case where $G = \{1\}$. Then the contention is that the augmentation map $\mathbb{Z}[K] \to \mathbb{Z}$ induces an isomorphism $\omega : \mathbb{Z}[I] \otimes_{\mathbb{Z}[K]} \mathbb{Z} \xrightarrow{\sim} I\mathbb{Z}[P'/K]$. From lemma 4.8.31(ii), we derive easily that $I\mathbb{Z}[P'/K] = \mathbb{Z}[I/K]$, where I/K is the set-theoretic quotient of I for the K-action deduced from φ. However, any set-theoretic section $I/K \to I$ of the natural projection $I \to I/K$ yields a well-defined surjection $\mathbb{Z}[I/K] \to \mathbb{Z}[I] \otimes_{\mathbb{Z}[K]} \mathbb{Z}$ whose composition with ω is the identity map. The claim follows. ◊

(ii): Let U be an affine object of X_τ, say $U = \operatorname{Spec} A$; from (i), we see that $\mathscr{B}(X, \underline{M}, I\underline{M})_{|U}$ is the quasi-coherent $\mathscr{O}_U$-algebra associated to the A-algebra $\oplus_{n\in\mathbb{N}}\mathbb{Z}[I^n] \otimes_{\mathbb{Z}[P]} A$.

(iii): In view of (i) we know already that $\mathscr{B}(f, IP_S^{\log})$ induces an isomorphism on the underlying $\mathscr{O}_X$-algebras; hence, by lemma 12.2.18(i), it remains to show that $\mathscr{B}(f, IP_S^{\log})$ induces an isomorphism :

$$f^*(I^n P_S^{\log}) \xrightarrow{\sim} I^n f^*(P_S^{\log}) \qquad \text{for every } n \in \mathbb{N}.$$

Let $\gamma : f^{-1}P_S^{\log} \to f^{-1}\mathscr{O}_S \to \mathscr{O}_X$ be the natural map; after replacing I by I^n, we come down to showing that the natural map :

$$f^{-1}(IP_S^{\log}) \otimes_{\gamma^{-1}\mathscr{O}_X^\times} \mathscr{O}_X^\times \to I \cdot (f^{-1}P_S^{\log} \otimes_{\gamma^{-1}\mathscr{O}_X^\times} \mathscr{O}_X^\times)$$

is an isomorphism. This assertion can be checked on the stalks over the τ-points of X; if x is such a point, let $G := \mathscr{O}_{X,x}^\times$, $H := \gamma_x^{-1}G$ and $P' := P_{S,f(x)}^{\log}$. The map under consideration is the natural morphism of P'-modules :

$$\omega : (IP') \otimes_H G \to I(P' \otimes_H G)$$

and it suffices to show that $\mathbb{Z}[\omega]$ is an isomorphism; however, the latter is none else than (12.4.39), so we may appeal to claim 12.4.38 to conclude. □

12.4.40. We wish to generalize lemma 12.4.37(ii) to log structures that do not necessarily admit global charts. Namely, suppose now that $\underline{M}$ is a quasi-coherent log structure on X, and $\mathscr{I} \subset \underline{M}$ is a coherent ideal (see definition 4.8.6(v)). For every τ-point ξ of X, we may then find a neighborhood U of ξ in X_τ, a chart $\beta : P_U \to \underline{M}_{|U}$, and local sections $s_1, \dots, s_n \in \mathscr{I}(U)$ which form a system of generators for $\mathscr{I}_{|U}$. We may then write $s_{i,\xi} = u_i \cdot \beta(x_i)$ for certain $x_1, \dots, x_n \in P$ and $u_1, \dots, u_n \in \mathscr{O}_{X,\xi}^\times$. Up to shrinking U, we may assume that $u_1, \dots, u_n \in \mathscr{O}_X^\times(U)$, and it follows that $\mathscr{I}_{|U} = I \cdot \underline{M}_{|U}$, where $I \subset P$ is the ideal generated by $x_1, \dots, x_n$. In other words, locally on X_τ, the datum $(X, \underline{M}, \mathscr{I})$ is of the type considered in lemma 12.4.37(ii); therefore the blow up $\mathscr{O}_X$-algebra $\mathscr{B}(X, \underline{M}, \mathscr{I})$ is quasi-coherent. We may then consider the natural projection :

$$\pi_{(X,\underline{M},\mathscr{I})} : \mathrm{Bl}_{\mathscr{I}}(X, \underline{M}) := \operatorname{Proj} \mathscr{B}(X, \underline{M}, \mathscr{I}) \to (X, \underline{M}). \tag{12.4.41}$$

Next, let $f : (Y, \underline{N}) \to (X, \underline{M})$ be a morphism of log schemes; it is easily seen that $\mathscr{I}\underline{N}$ is a coherent ideal of $\underline{N}$, hence $\mathscr{B}(Y, \underline{N}, \mathscr{I}\underline{N})$ is quasi-coherent as well, and since the map $\mathscr{B}(f, \mathscr{I})$ of (12.4.35) is an epimorphism on the underlying $\mathscr{O}_Y$-modules, we have :

$$G(\mathscr{B}(f, \mathscr{I})) = \mathrm{Bl}_{\mathscr{I}\underline{N}}(Y, \underline{N})$$

(notation of (12.4.18)) whence a closed immersion of $(Y, \underline{N})$-schemes :

$$\operatorname{Proj} \mathscr{B}(f, \mathscr{I}) : \mathrm{Bl}_{\mathscr{I}\underline{N}}(Y, \underline{N}) \to (Y, \underline{N}) \times_{(X,\underline{M})} \mathrm{Bl}_{\mathscr{I}}(X, \underline{M})$$

([60, Ch.II, Prop.3.6.2(i)] and (12.4.24)), which is the same as a morphism of $(X, \underline{M})$-schemes :

$$\varphi(f, \mathscr{I}) : \mathrm{Bl}_{\mathscr{I}\underline{N}}(Y, \underline{N}) \to \mathrm{Bl}_{\mathscr{I}}(X, \underline{M}).$$

Moreover, if $g : (Z, \underline{Q}) \to (Y, \underline{N})$ is another morphism, (12.4.36) induces the identity :

$$\varphi(f, \mathscr{I}) \circ \varphi(g, \mathscr{I}\underline{N}) = \varphi(f \circ g, \mathscr{I}). \tag{12.4.42}$$

Example 12.4.43. In the situation of lemma 12.4.37(iii), notice that $\mathscr{O}_S$ is a flat $\mathbb{Z}[P_S]$-algebra, and therefore :

$$\mathsf{B}(S, P_S^{\log}, IP_S^{\log}) = \bigoplus_{n \in \mathbb{N}} I^n \mathscr{O}_S.$$

Moreover, (12.4.24) specializes to a natural isomorphism of $(X, \underline{M})$-schemes :

$$\mathrm{Bl}_{I\underline{M}}(X, \underline{M}) \stackrel{\sim}{\to} X \times_S \mathrm{Bl}_{IP_S^{\log}}(S, P_S^{\log}). \tag{12.4.44}$$

We wish to give a more explicit description of the log structure of $\mathrm{Bl}_{IP_S^{\log}}(S, P_S^{\log})$. To begin with, recall that $S' := \mathrm{Proj}\, \mathsf{B}(S, P_S^{\log}, IP_S^{\log})$ admits a distinguished covering by (Zariski) affine open subsets : namely, for every $a \in I$, consider the localization

$$P_a := T_a^{-1} P \qquad \text{where} \qquad T_a := \{a^n \mid n \in \mathbb{N}\}$$

and let $Q_a \subset P_a$ be the submonoid generated by the image of P and the subset $\{a^{-1}b \mid b \in I\}$; then

$$S' = \bigcup_{a \in I} \mathrm{Spec}\, \mathbb{Z}[Q_a].$$

Hence, let us set

$$U_a := \mathrm{Spec}(\mathbb{Z}, Q_a) \qquad \text{for every } a \in I.$$

We claim that these locally defined log structures glue to a well defined log structure $\underline{Q}$ on the whole of S'_τ. Indeed, let $a, b \in I$; we have

$$U_{a,b} := \mathrm{Spec}\, \mathbb{Z}[Q_a] \cap \mathrm{Spec}\, \mathbb{Z}[Q_b] = \mathrm{Spec}\, \mathbb{Z}[Q_a \otimes_P Q_b]$$

and it is easily seen that $Q_a \otimes_P Q_b = Q_a[b^{-1}a]$, *i.e.* the localization of Q_a obtained by inverting its element $a^{-1}b$, and this is of course the same as $Q_b[a^{-1}b]$. Then lemma 12.2.14 implies that the log structures of U_a and U_b agree on $U_{a,b}$, whence the contention. It is easy to check that the resulting log scheme is precisely $\mathrm{Bl}_{IP_S^{\log}}(S, P_S^{\log})$: the details shall be left to the reader.

Proposition 12.4.45. *Let $(X, \underline{M})$ be a log scheme with quasi-coherent log structure, and $\mathscr{I} \subset \underline{M}$ a coherent ideal. Then, the morphism* (12.4.41) *is a logarithmic blow up of $\mathscr{I}$.*

Proof. Let $f : (Y, \underline{N}) \to (X, \underline{M})$ be a morphism of log schemes, and suppose that $\mathscr{I}\underline{N}$ is an invertible ideal of $\underline{N}$; in this case, we have already remarked (see (12.4.34)) that the projection $\pi_{(Y, \underline{N}, \mathscr{I}\underline{N})} : \mathrm{Bl}_{\mathscr{I}\underline{N}}(Y, \underline{N}) \to (Y, \underline{N})$ is an isomorphism. We deduce a morphism :

$$\varphi(f, \mathscr{I}) \circ \pi^{-1}_{(Y, \underline{N}, \mathscr{I}\underline{N})} : (Y, \underline{N}) \to \mathrm{Bl}_{\mathscr{I}}(X, \underline{M}). \tag{12.4.46}$$

To conclude, it remains to show that (12.4.46) is the only morphism of log schemes whose composition with $\pi_{(X, \underline{M}, \mathscr{I})}$ equals f. The latter assertion can be checked locally on X_τ, hence we may assume that $\underline{M}$ admits a chart $P_X \to \underline{M}$, such that $\mathscr{I} = I\underline{M}$ for some finitely generated ideal $I \subset P$. In this case, in view of (12.4.44), the set of morphisms of $(X, \underline{M})$-schemes $(Y, \underline{N}) \to \mathrm{Bl}_{\mathscr{I}}(X, \underline{M})$ is in natural bijection with the set of $(S, P_S^{\log})$-morphisms $(Y, \underline{N}) \to B := \mathrm{Bl}_{IP_S^{\log}}(S, P_S^{\log})$ (notation of example 12.4.43). In other words, we may assume that $(X, \underline{M}) = (S, P_S^{\log})$, and $\mathscr{I} = IP_S^{\log}$. Then f is determined by $\log f$, *i.e.* by a map $\beta : P \to$

$\underline{N}(Y)$. Let $a_1, \dots, a_k$ be a system of generators for I; for each $i = 1, \dots, k$, we let $U_i \subset Y$ be the subset of all $y \in Y$ for which there exists a τ-point ξ of Y with $|\xi| = y$ and

$$a_i \underline{N}_\xi = I \underline{N}_\xi . \tag{12.4.47}$$

Notice that if $y \in U_i$, then (12.4.47) holds for every τ-point ξ of Y localized at y (details left to the reader).

Claim 12.4.48. The subset U_i is open in Y_i for every $i = 1, \dots, k$, and $Y = \bigcup_{i=1}^k U_i$.

Proof of the claim. Say that $y \in U_i$, and let ξ be a τ-point of Y localized at y, such that (12.4.47) holds. This means that, for every $j = 1, \dots, k$, there exists $u_j \in \underline{N}_\xi$ such that $a_j = u_j a_i$. Then, we may find a τ-neighborhood $h : U' \to$ of ξ such that this identity persists in $\underline{N}(U')$; thus, $h(U') \subset U_i$. Since h is an open map, this shows that U_i is an open subset.

Next, let ξ be any τ-point of Y; by assumption, we have $I\underline{N}_\xi = b\underline{N}_\xi$ for some $b \in \underline{N}_\xi$; this means that for every $i = 1, \dots, k$ there exists $u_i \in \underline{N}_\xi$ such that $a_i = u_i b$. Since $a_1, \dots, a_k$ generate I, we must have $u_i \in \underline{N}_\xi^\times$ for at least an index $i \leq k$, in which case $|\xi| \in U_i$, and this shows that the U_i cover the whole of Y, as claimed. ◊

It is easily seen that, for every $i = 1, \dots, k$, any morphism $(U_i, \underline{N}_{|U_i}) \to B$ of $(S, P_S^{\log})$-schemes factors through the open immersion $\mathrm{Spec}\,(Z, Q_{a_i}) \to B$ (where Q_a, for an element $a \in P$, is defined as in example 12.4.43). Conversely, by construction β extends to a unique morphism of monoids $Q_{a_i} \to \underline{N}(U_i)$. Summing up, there exists at most one morphism of $(S, P_S^{\log})$-schemes $(U_i, \underline{N}_{|U_i}) \to B$. In light of claim 12.4.48, the proposition follows. □

12.4.49. Keep the notation of proposition 12.4.45; by inspecting the construction, it is easily seen that the log structure $\underline{M}'$ of $\mathrm{Bl}_{\mathscr{I}}(X, \underline{M})$ is quasi-coherent, and if $\underline{M}$ is coherent (resp. quasi-fine, resp. fine), then the same holds for $\underline{M}'$. However, simple examples show that $\underline{M}'$ may fail to be saturated, even in cases where $(X, \underline{M})$ is a fs log scheme. Due to the prominent role played by fs log schemes, it is convenient to introduce the special notation :

$$\mathrm{sat.Bl}_{\mathscr{I}}(X, \underline{M}) := (\mathrm{Bl}_{\mathscr{I}}(X, \underline{M}))^{\mathrm{qfs}}$$

for the *saturated logarithmic blow up* of a coherent ideal $\mathscr{I}$ in a quasi-fine log structure $\underline{M}$ (notation of proposition 12.2.37). Clearly the projection $\mathrm{sat.Bl}_{\mathscr{I}}(X, \underline{M}) \to (X, \underline{M})$ is a final object of the category of saturated $(X, \underline{M})$-schemes in which the preimage of $\mathscr{I}$ is invertible. If $(X, \underline{M})$ is a fine log scheme, $\mathrm{sat.Bl}_{\mathscr{I}}(X, \underline{M})$ is a fs log scheme. Moreover, for any morphism of schemes $f : Y \to X$, let $\underline{M}_Y := f^*\underline{M}$; from remarks (12.4.32)(ii) and 12.2.38(iii), we deduce a natural isomorphism of $(Y, \underline{M}_Y)$-schemes :

$$\mathrm{sat.Bl}_{\mathscr{I}\underline{M}_Y}(Y, \underline{M}_Y) \xrightarrow{\sim} Y \times_X \mathrm{sat.Bl}_{\mathscr{I}}(X, \underline{M}). \tag{12.4.50}$$

Theorem 12.4.51. *Let $(X, \underline{M})$ be a quasi-fine log scheme with saturated log structure, $\mathscr{I} \subset \underline{M}$ an ideal, ξ a τ-point of X. Suppose that, in a neighborhood of ξ, the ideal $\mathscr{I}$ is generated by at most two sections, and denote :*

$$\varphi : \mathrm{Bl}_{\mathscr{I}}(X, \underline{M}) \to (X, \underline{M}) \qquad \textit{(resp. } \varphi_{\mathrm{sat}} : \mathrm{sat.Bl}_{\mathscr{I}}(X, \underline{M}) \to (X, \underline{M})\textit{)}$$

the logarithmic (resp. saturated logarithmic) blow up of $\mathscr{I}$. Then :

(i) *If $\mathscr{I}_\xi$ is an invertible ideal of $\underline{M}_\xi$, the natural morphisms*

$$\varphi^{-1}(\xi) \to \mathrm{Spec}\,\kappa(\xi) \qquad \varphi_{\mathrm{sat}}^{-1}(\xi) \to \mathrm{Spec}\,\kappa(\xi)$$

are isomorphisms.

(ii) *Otherwise, $\varphi^{-1}(\xi)$ is a $\kappa(\xi)$-scheme isomorphic to $\mathbb{P}^1_{\kappa(\xi)}$; furthermore, the same holds for the reduced fibre $\varphi_{\mathrm{sat}}^{-1}(\xi)_{\mathrm{red}}$, provided $\underline{M}$ is fine.*

Proof. After replacing X by a τ-neighborhood of ξ, we reduce to the case where $\underline{M}$ admits an integral and saturated chart $\alpha : P_X \to \underline{M}$ (lemma 12.1.18(iii)), and if $\underline{M}$ is a fs log structure, we may also assume that the chart α is fine and sharp at ξ (corollary 12.1.36(i)). Furthermore, we may assume that $\mathscr{I}$ is generated by at most two elements of $\underline{M}(X)$, and if $\mathscr{I}_\xi$ is principal, we may assume that the same holds for $\mathscr{I}$. In the latter case, since $\underline{M}$ is integral, $\mathscr{I}$ is invertible, hence φ and φ_{sat} are isomorphisms, so (i) follows already.

Now, suppose that $\mathscr{I}_\xi$ is not invertible, and let $a', b' \in \underline{M}(X)$ be a system of generators for $\mathscr{I}$; we can write $a' = \alpha(a) \cdot u$, $b' = \alpha(b) \cdot v$ for some $a, b \in P$ and $u, v \in \kappa^\times$. Set $t := a^{-1}b$, and let $J \subset P$ be the ideal generated by a and b; clearly $J\underline{M} = \mathscr{I}$, and $Pa, Pb \neq J$.

Consider first the case where $X = \operatorname{Spec} \kappa$, where κ is a field (resp. a separably closed field, in case $\tau =$ ét). In this situation, a pre-log structure on X_τ is the same as a morphism of monoids $\beta : P \to \kappa$, the associated log structure is the induced map of monoids

$$\beta^{\log} : P \otimes_{P_0} \kappa^\times \to \kappa \qquad \text{where} \qquad P_0 := \beta^{-1}\kappa^\times$$

and α is the natural map $P \to P \otimes_{P_0} \kappa^\times$. After replacing P by its localization $P_0^{-1}P$, we may also assume that $P_0 = P^\times$. Let

$$(S, P_S^{\log}) := \operatorname{Spec}(\kappa, P) \qquad J^\sim := JP_S^{\log} \qquad (Y, \underline{N}) := \mathrm{Bl}_{J^\sim}(S, P_S^{\log}).$$

Denote by $\varepsilon : \kappa[P] \to \kappa$ the homomorphism of κ-algebras induced by β via the adjunction (4.8.50), and set $I := \operatorname{Ker} \varepsilon$. In view of lemma 12.4.37(i) and (12.4.50), we have natural cartesian diagrams of κ-schemes :

(12.4.52)
$$\begin{array}{ccc} \varphi^{-1}(\xi) & \longrightarrow & (Y, \underline{N}) \\ \downarrow & & \downarrow \\ |\xi| & \xrightarrow{\operatorname{Spec} \varepsilon} & S \end{array} \qquad \begin{array}{ccc} \varphi_{\mathrm{sat}}^{-1}(\xi) & \longrightarrow & (Y, \underline{N})^{\mathrm{sat}} \\ \downarrow & & \downarrow \\ |\xi| & \xrightarrow{\operatorname{Spec} \varepsilon} & S. \end{array}$$

On the other hand, (a, b) can be regarded as a pair of global sections of $J^\sim\underline{N}$, so the universal property of example 12.4.29(ii) yields a morphism of $(S, P_S^{\log})$-schemes :

$$f_{(a,b)} : (Y, \underline{N}) \to \mathbb{P}^1_{(S, P_S^{\log})}.$$

In light of example 12.4.29(i), the assertion concerning $\varphi^{-1}(\xi)$ will then follow from the :

Claim 12.4.53. The morphism $|\xi| \times_S f_{(a,b)}$ is an isomorphism of κ-schemes.

Proof of the claim. Let $Q_a \subset P^{\mathrm{gp}}$ (resp. $Q_b \subset P^{\mathrm{gp}}$) be the submonoid generated by P and t (resp. by P and t^{-1}); by inspecting example 12.4.43, we see that Y is covered by two affine open subsets :

$$U_a := \operatorname{Spec} \kappa[Q_a] \qquad U_b := \operatorname{Spec} \kappa[Q_b]$$

and $U_a \cap U_b = \operatorname{Spec} \kappa[Q_a \otimes_P Q_b]$. On the other hand, $\mathbb{P}^1_{(S, P_S^{\log})}$ is covered as well by two affine open subsets U'_0 and U'_∞, both isomorphic to $\operatorname{Spec} \kappa[P \oplus \mathbb{N}]$, and such that $U'_0 \cap U'_\infty = \operatorname{Spec} \kappa[P \oplus \mathbb{Z}]$, as usual. Moreover, a direct inspection shows that $f_{(a,b)}$ restricts to morphisms

$$U_a \to U'_0 \qquad U_b \to U'_\infty$$

induced respectively by the maps of κ-algebras

$$\omega_0 : \kappa[P \oplus \mathbb{N}] \to \kappa[Q_a] \qquad \omega_\infty : \kappa[P \oplus \mathbb{N}] \to \kappa[Q_b]$$

such that $\omega_0(x, n) = x \cdot t^n$ and $\omega_\infty(x, n) = x \cdot t^{-n}$ for every $(x, n) \in P \oplus \mathbb{N}$. We show that $\overline{\omega}_0 := \omega_0 \otimes_{\kappa[P]} \kappa[P]/I$ is an isomorphism; the same argument will apply also to ω_∞, so the claim shall follow.

Indeed, clearly the $\kappa[P]$-algebra $\kappa[Q_a]$ is generated by t, hence ω_0 is surjective, and then the same holds for $\overline{\omega}_0$. Next, set $H_a := I \cdot \kappa[Q_a]$; it is easily seen that H_a consists of all sums of the form $\sum_{j=0}^n c_j t^j$, for arbitrary $n \in \mathbb{N}$, with $c_j \in I$ for every $j = 0, \dots, n$. Clearly, an element $p(T) \in \kappa[\mathbb{N}] = \kappa[T]$ lies in $\mathrm{Ker}\, \overline{\omega}_0$ if and only if $p(t) \in H_a$, so we come down to the following assertion. Let $c_0, \dots, c_n \in \kappa[P]$ such that

$$\sum_{j=0}^{n} c_j t^j = 0 \tag{12.4.54}$$

in $\kappa[Q_a]$; then c_j lies in the ideal $\kappa[\beta^{-1}(0)]$ of $\kappa[P]$, for every $j = 0, \dots, n$. Since P is integral, (12.4.54) is equivalent to the identity : $\sum_{j=0}^n c_j a^{n-j} b^j = 0$ in $\kappa[P]$. For every $x \in P$, denote by $\pi_x : \kappa[P] \to \kappa$ the κ-linear projection such that $\pi_x(x) = 1$, and $\pi_x(y) = 0$ for every $y \in P \backslash \{x\}$.

Suppose, by way of contradiction, that $c_i \notin \kappa[\beta^{-1}(0)]$ for some $i \leq n$, hence $\pi_x(c_i) \neq 0$ for some $x \in P_0$; since $P_0 = P^\times$, we may replace c_j by $x^{-1}c_j$, for every $j \leq n$, and assume that $\pi_1(c_i) \neq 0$, hence $\pi_{a^{n-i}b^i}(c_i a^{n-i} b^i) \neq 0$ (again, using the assumption that P is integral). Thus, there exists $j \neq i$ with $j \leq n$, such that $\pi_{a^{n-i}b^i}(c_j a^{n-j} b^j) \neq 0$, and we may then find an element $c \in P$ such that $\pi_{a^{n-i}b^i}(c a^{n-j} b^j) = 1$, *i.e.* $c a^{n-j} b^j = a^{n-i} b^i$; up to swapping the roles of a and b, we may assume that $i > j$, in which case we may write $t^{i-j} = c$; since P is saturated, it follows that $t \in P$, hence J is generated by a, which is excluded. ◊

Next, we assume that $\underline{M}$ is a fs log structure (and X is still $\mathrm{Spec}\, \kappa$), and we consider the morphism φ_{sat}. As already remarked, we may assume that α is sharp at ξ, and P fine and saturated; the sharpness condition amounts to saying that $\beta(x) = 0$ for every $x \in P \setminus \{1\}$, therefore I is the augmentation ideal of the graded κ-algebra $\kappa[P]$. By inspecting the proof of proposition 12.2.37, we see that $(Y, \underline{N})^{\mathrm{sat}}$ is covered by two affine open subsets

$$U_a^{\mathrm{fs}} := \mathrm{Spec}\, \kappa[Q_a^{\mathrm{sat}}] \qquad U_b^{\mathrm{fs}} := \mathrm{Spec}\, \kappa[Q_b^{\mathrm{sat}}]$$

and $U_a^{\mathrm{fs}} \cap U_b^{\mathrm{fs}} = \mathrm{Spec}\, \kappa[Q_a^{\mathrm{sat}} \otimes_P Q_b^{\mathrm{sat}}]$. Since J is not principal, we have $t \notin P$, and since P is saturated, we deduce that t is not a torsion element of P^{gp}; as the latter is a finitely generated abelian group, it follows that we may find a *unimodular* element $u \in P^{\mathrm{gp}}$ such that t lies in the submonoid $\mathbb{N}u \subset P^{\mathrm{gp}}$ generated by u; this condition means that $t = u^k$ for some $k \in \mathbb{N}$, and $\mathbb{N}u$ is not properly contained in another rank one free submonoid of P^{gp}. Write $u = a'^{-1}b'$ for some $a', b' \in P$, let $J' \subset P$ be the ideal generated by a' and b', and $R_{a'}$ (resp. $R_{b'}$) the submonoid of P^{gp} generated by P and u (resp. by P and u^{-1}); clearly $R_{a'}^{\mathrm{sat}} = Q_a^{\mathrm{sat}}$, and $R_{b'}^{\mathrm{sat}} = Q_b^{\mathrm{sat}}$. Denote by $\underline{N}'$ the log structure of $(Y, \underline{N})^{\mathrm{sat}}$, and $\mathscr{J}' := J'\underline{N}'$; it is easily seen that

$$\mathscr{J}'_{|U_a^{\mathrm{fs}}} = a'\underline{N}'_{|U_a^{\mathrm{fs}}} \qquad \mathscr{J}'_{|U_b^{\mathrm{fs}}} = b'\underline{N}'_{|U_b^{\mathrm{fs}}}$$

hence $\mathscr{J}'$ is invertible, and example 12.4.29(ii) yields a morphism of $(S, P_S^{\log})$-schemes

$$f_{(a',b')} : (Y, \underline{N})^{\mathrm{sat}} \to \mathbb{P}^1_{(S, P_S^{\log})}.$$

In light of example 12.4.29(i), the assertion concerning $\varphi_{\mathrm{sat}}^{-1}(\xi)$ will then follow from the :

Claim 12.4.55. $(|\xi| \times_S f_{(a',b')})_{\mathrm{red}} : \varphi_{\mathrm{sat}}^{-1}(\xi)_{\mathrm{red}} \to \mathbb{P}^1_\kappa$ is an isomorphism of κ-schemes.

Proof of the claim. As in the proof of claim 12.4.53, the morphism $f_{(a',b')}$ restricts to morphisms $U_a^{\mathrm{fs}} \to U'_0$ and $U_b^{\mathrm{fs}} \to U'_\infty$ induced by maps of κ-algebras :

$$\omega_0 : \kappa[P \oplus \mathbb{N}] \to \kappa[Q_a^{\mathrm{sat}}] \qquad \omega_\infty : \kappa[P \oplus \mathbb{N}] \to \kappa[Q_b^{\mathrm{sat}}]$$

such that $\omega_0(x, n) = x \cdot u^n$ and $\omega_\infty(x, n) = x \cdot u^{-n}$ for every $(x, n) \in P \oplus \mathbb{N}$, and again, it suffices to show that

$$(\omega_0 \otimes_{\kappa[P]} \kappa[P]/I)_{\mathrm{red}} : \kappa[T] \to (\kappa[R_a^{\mathrm{sat}}]/I\kappa[R_a^{\mathrm{sat}}])_{\mathrm{red}}$$

is an isomorphism (where for any ring A, we denote A_{red} be the maximal reduced quotient of A, *i.e.* $A_{\mathrm{red}} := A/\mathrm{nil}(A)$, where $\mathrm{nil}(A)$ is the nilradical of A). We break the latter verification in two steps : first, let us check that the map

$$\overline{\omega}'_0 : \kappa[T] \to \kappa[R_a]/I\kappa[R_a] \qquad p(T) \mapsto p(u) \pmod{I\kappa[R_a]}$$

is an isomorphism. Indeed, $\overline{\omega}'_0$ is induced by the map of monoids $\varphi : \mathbb{N} \to R_a$ such that $n \mapsto u^n$ for every $n \in \mathbb{N}$; if $t = u^k$, the map φ fits into the cocartesian diagram :

$$\begin{array}{ccc} \mathbb{N} & \xrightarrow{\psi} & Q_a \\ {\scriptstyle \boldsymbol{k}_{\mathbb{N}}}\downarrow & & \downarrow{\scriptstyle j} \\ \mathbb{N} & \xrightarrow{\varphi} & R_a \end{array}$$

where $\boldsymbol{k}_{\mathbb{N}}$ is the k-Frobenius map, ψ is given by the rule : $n \mapsto t^n$ for every $n \in \mathbb{N}$, and j is the natural inclusion. Hence :

$$\overline{\omega}'_0 = (\kappa[\psi] \otimes_{\kappa[P]} \kappa[P]/I) \otimes_{\kappa[T^k]} \kappa[T].$$

However, the proof of claim 12.4.53 shows that $\kappa[\psi] \otimes_{\kappa[P]} \kappa[P]/I$ is an isomorphism, whence the contention. Lastly, let show that the natural map

$$\overline{\omega}''_0 : \kappa[R_a]/I\kappa[R_a] \to (\kappa[R_a^{\mathrm{sat}}]/I\kappa[R_a^{\mathrm{sat}}])_{\mathrm{red}}$$

is an isomorphism. Indeed, it is clear that the natural map $\omega''_0 : \kappa[R_a] \to \kappa[R_a^{\mathrm{sat}}]$ is integral and injective, hence $\mathrm{Spec}\,\omega''_0$ is surjective; therefore $\mathrm{Spec}\,\overline{\omega}''_0$ is still surjective and integral. However, the foregoing shows that $\kappa[R_a]/I\kappa[R_a]$ is reduced, so we deduce that $\overline{\omega}''_0$ is injective. To show that $\overline{\omega}''_0$ is surjective, it suffices to show that the classes of the generating system $R_a^{\mathrm{sat}} \subset \kappa[R_a^{\mathrm{sat}}]$ lie in the image of $\overline{\omega}''_0$. Hence, let $x \in R_a^{\mathrm{gp}}$, with $x^m \in R_a$ for some $m > 0$, so that $x^m = y \cdot u^n$ for some $n \in \mathbb{N}$ and $y \in P$. If $y \neq 1$, we have $y \in I$, hence the image of x^m vanishes in $\kappa[R_a^{\mathrm{sat}}]/I\kappa[R_a^{\mathrm{sat}}]$, and the image of x vanishes in the reduced quotient; finally, if $y = 1$, the identity $x^m = u^n$ implies that m divides n, since u is unimodular; hence $x = u^{n/m}$ and the image of x agrees with $\overline{\omega}''_0(u^{n/m})$. $\diamondsuit$

Finally, let us return to a general quasi-fine log scheme $(X, \underline{M})$; the theorem will follow from the more precise :

Claim 12.4.56. In the situation of the theorem, suppose moreover that

(a) $\underline{M}$ admits a saturated chart $\alpha : P_X \to \underline{M}$
(b) $\mathscr{I} = J\underline{M}$, where $J \subset P$ is an ideal generated by two elements $a, b \in P$
(c) $\mathscr{I}_\xi$ is not invertible.

Then we have :

(i) There exists a morphism of $(X, \underline{M})$-schemes : $\mathrm{Bl}_{\mathscr{I}}(X, \underline{M}) \to \mathbb{P}^1_{(X,\underline{M})}$ inducing an isomorphism of $\kappa(\xi)$-schemes $\varphi^{-1}(\xi) \xrightarrow{\sim} \mathbb{P}^1_{\kappa(\xi)}$.
(ii) If furthermore, P is fine (and saturated) and α is sharp at ξ, then there exists a morphism of $(X, \underline{M})$-schemes : $\mathrm{sat}.\mathrm{Bl}_{\mathscr{I}}(X, \underline{M}) \to \mathbb{P}^1_{(X,\underline{M})}$ inducing an isomorphism of $\kappa(\xi)$-schemes $\varphi_{\mathrm{sat}}^{-1}(\xi)_{\mathrm{red}} \xrightarrow{\sim} \mathbb{P}^1_{\kappa(\xi)}$.

Proof of the claim. (i): Denote by $\underline{N}$ the log structure of $\mathrm{Bl}_{\mathscr{I}}(X, \underline{M})$; the elements a, b define global sections of the invertible $\underline{N}$-module $\mathscr{I}\underline{N}$, and we claim that the corresponding morphism of $(X, \underline{M})$-schemes $f_{(a,b)} : \mathrm{Bl}_{\mathscr{I}}(X, \underline{M}) \to \mathbb{P}^1_{(X,\underline{M})}$ will do. Indeed, set $(|\xi|, \underline{M}_\xi) := |\xi| \times_X (X, \underline{M})$, and recall that there exists natural isomorphisms

$$|\xi| \times_X \mathbb{P}^1_{(X,\underline{M})} \xrightarrow{\sim} \mathbb{P}^1_{(|\xi|,\underline{M}_\xi)} \qquad |\xi| \times_X \mathrm{Bl}_{\mathscr{I}}(X, \underline{M}) \xrightarrow{\sim} \mathrm{Bl}_{\mathscr{I}\underline{M}_\xi}(|\xi|, \underline{M}_\xi)$$

(example 12.4.29(i) and remark 12.4.32(ii)). Denote by $\underline{N}_\xi$ the log structure of $\mathrm{Bl}_{\mathscr{I}\underline{M}_\xi}(|\xi|, \underline{M}_\xi)$; by example 12.4.29(iii), the base change

$$|\xi| \times_X f : \mathrm{Bl}_{\mathscr{I}\underline{M}_\xi}(|\xi|, \underline{M}_\xi) \to \mathbb{P}^1_{(|\xi|, \underline{M}_\xi)}$$

is the unique morphism $f_{(\overline{a},\overline{b})}$ of $(|\xi|, \underline{M}_\xi)$-schemes corresponding to the pair $(\overline{a}, \overline{b})$ of global sections of $\mathscr{I}\underline{N}_\xi$ obtained by pulling back the pair (a, b). Therefore, in order to check that $|\xi| \times_X f$ is an isomorphism, we may replace from start $(X, \underline{M})$ by $(|\xi|, \underline{M}_\xi)$ (whose log structure is still quasi-fine, by lemma 12.1.18(i), and assume that $X = \mathrm{Spec}\, \kappa$, where κ is a field (resp. a separably closed field, in case $\tau = \text{ét}$), in which case the assertion is just claim 12.4.53.

(ii): Denote by $\underline{N}'$ the log structure of $\mathrm{sat.Bl}_{\mathscr{I}}(X, \underline{M})$, define a', b' and J' as in the foregoing, and set again $\mathscr{J}' := J'\underline{N}'$. Again, it is easily seen that $\mathscr{J}'$ is an invertible $\underline{N}'$-module, and the pair (a', b') yields a morphism $f_{(a',b')} : \mathrm{sat.Bl}_{\mathscr{I}}(X, \underline{M}) \to \mathbb{P}^1_{(X,\underline{M})}$ which fulfills the sought condition. Indeed, denote by $\underline{N}'_\xi$ the log structure of $\mathrm{sat.Bl}_{\mathscr{I}\underline{M}_\xi}(|\xi|, \underline{M}_\xi)$; in light of (12.4.50) and example 12.4.29(iii), the base change

$$|\xi| \times_X f_{(a',b')} : \mathrm{sat.Bl}_{\mathscr{I}\underline{M}_\xi}(|\xi|, \underline{M}_\xi) \to \mathbb{P}^1_{(|\xi|, \underline{M}_\xi)}$$

is the unique morphism $f_{(\overline{a}',\overline{b}')}$ of $(|\xi|, \underline{M}_\xi)$-schemes corresponding to the pair $(\overline{a}', \overline{b}')$ of global sections of $\mathscr{J}'\underline{N}'_\xi$ obtained by pulling back the pair (a', b'). Thus, the assertion is just claim 12.4.55. □

12.5. **Regular log schemes.** In this section we introduce the logarithmic version of the classical regularity condition for locally noetherian schemes. This theory is essentially due to K.Kato ([112]), and we mainly follow his exposition, except in a few places where his original arguments are slightly flawed, in which cases we supply the necessary corrections.

Lemma 12.5.1. *Let A be a noetherian local ring, N an A-module of finite type, $\alpha : P \to A$ and $\beta : Q \to A$ two morphisms of pointed monoids, with $P^\sharp$ and $Q^\sharp$ both fine. Suppose that α and β induce the same constant log structure on* $\mathrm{Spec}\, A$ *(see* (12.1.15)*). Then N is α-flat if and only if it is β-flat.*

Proof. Let ξ be a τ-point localized at the closed point of $X := \mathrm{Spec}\, A$, set $B := \mathscr{O}_{X,\xi}$ and let $\varphi : A \to B$ be the natural map. Let also M be the push-out of the diagram of monoids $P \leftarrow (\varphi \circ \alpha)^{-1}B^\times \to B^\times$ deduced from α; then $M \simeq P^{\log}_{X,\xi}$, the stalk at the point ξ of the constant log structure on X_τ associated to α. Since φ is faithfully flat, it is easily seen that N is α-flat if and only if $N \otimes_A B$ is $\varphi \circ \alpha$-flat.

Hence we may replace A by B, α by $\varphi \circ \alpha$, N by $N \otimes_A B$, and Q by M, after which we may assume that $Q = P \amalg_{\alpha^{-1}(A^\times)} A^\times$; especially, there exists a morphism of monoids $\gamma : P \to Q$ such that $\alpha = \beta \circ \gamma$, and moreover β is a local morphism.

Next, set $S := \alpha^{-1}(A^\times)$; clearly γ extends to a morphism of monoids $\gamma' : S^{-1}P \to Q$, and α and $\beta \circ \gamma'$ induce the same constant log structure on X_τ. Arguing as in the proof of proposition 6.4.31, we see that N is P-flat if and only if it is $S^{-1}P$-flat. Hence, we may replace P by $S^{-1}P$, which allows to assume that γ induces an isomorphism $P \amalg_{P^\times} A^\times \xrightarrow{\sim} Q$, therefore also an isomorphism $P^\sharp \xrightarrow{\sim} Q^\sharp$. The latter implies that $\mathfrak{m}_Q = \mathfrak{m}_P Q$; moreover, notice that the morphism of monoids $P^\times \to A^\times$ is faithfully flat, so the natural map :

$$\mathrm{Tor}_1^{\mathbb{Z}\langle P\rangle}(N, \mathbb{Z}\langle P/\mathfrak{m}_P\rangle) \to \mathrm{Tor}_1^{\mathbb{Z}\langle Q\rangle}(N, \mathbb{Z}\langle (P/\mathfrak{m}_P) \otimes_P Q\rangle) \to \mathrm{Tor}_1^{\mathbb{Z}\langle Q\rangle}(N, \mathbb{Z}\langle Q/\mathfrak{m}_Q\rangle)$$

is an isomorphism. The assertion follows. □

Lemma 12.5.2. *Let M be an integral monoid, A a ring, $\varphi : M \to A$ a morphism of monoids, and set $S :=$ $\mathrm{Spec}\, A$. Suppose that A is φ-flat. Then the log structure $(M, \varphi)^{\log}_S$ on S_τ is the subsheaf of monoids of $\mathscr{O}_S$ generated by $\mathscr{O}_S^\times$ and the image of M.*

Proof. To ease notation, set $\underline{M} := (M,\varphi)^{\log}_S$; let ξ be any τ-point of S. Then the stalk $\underline{M}_\xi$ is the push-out of the diagram : $\mathscr{O}^\times_{S,\xi} \leftarrow \varphi^{-1}_\xi(\mathscr{O}^\times_{S,\xi}) \to M$ where $\varphi_\xi : M \to \mathscr{O}_{S,\xi}$ is deduced from φ. Hence $\underline{M}_\xi$ is generated by $\mathscr{O}^\times_{S,\xi}$ and the image of M, and it remains only to show that the structure map $\underline{M}_\xi \to \mathscr{O}_{S,\xi}$ is injective. Therefore, let $a, b \in M$ and $u, v \in \mathscr{O}^\times_{S,\xi}$ such that :

(12.5.3) $$\varphi_\xi(a)\cdot u = \varphi_\xi(b)\cdot v.$$

We come down to showing :

Claim 12.5.4. There exist $c, d \in M$ such that :

$$\varphi_\xi(c), \varphi_\xi(d) \in \mathscr{O}^\times_{S,\xi} \qquad ac = bd \qquad \varphi_\xi(c)\cdot v = \varphi_\xi(d)\cdot u.$$

Proof of the claim. Let $\mathfrak{m}_\xi \subset \mathscr{O}_{S,\xi}$ be the maximal ideal, and set $\mathfrak{p} := \varphi^{-1}_\xi(\mathfrak{m}_\xi)$, so that φ_ξ extends to a local morphism $\varphi_\mathfrak{p} : M_\mathfrak{p} \to \mathscr{O}_{S,\xi}$. Since $\mathscr{O}_{S,\xi}$ is a flat A-algebra, $\mathscr{O}_{S,\xi}$ is φ-flat, and consequently it is faithfully $\varphi_\mathfrak{p}$-flat (lemma 6.1.36). Then, assumption (12.5.3) leads to the identity:

$$aM_\mathfrak{p} \otimes_{M_\mathfrak{p}} \mathscr{O}_{S,\xi} = \varphi_\xi(a)\cdot\mathscr{O}_{S,\xi} = \varphi_\xi(b)\cdot\mathscr{O}_{S,\xi} = bM_\mathfrak{p} \otimes_{M_\mathfrak{p}} \mathscr{O}_{S,\xi}$$

whence $aM_\mathfrak{p} = bM_\mathfrak{p}$, by faithful $\varphi_\mathfrak{p}$-flatness. It follows that there exist $x, y \in M_\mathfrak{p}$ such that $ax = b$ and $by = a$, hence $axy = a$, which implies that $xy = 1$, since $M_\mathfrak{p}$ is an integral module. The latter means that there exist $c, d \in M\backslash\mathfrak{p}$ such that $ac = bd$ in M. We deduce easily that

$$\varphi_\xi(a)\cdot\varphi_\xi(c)\cdot v = \varphi_\xi(a)\cdot\varphi_\xi(d)\cdot u.$$

Thus, to complete the proof, it suffices to show that $\varphi_\xi(a)$ is regular in $\mathscr{O}_{S,\xi}$. However, the morphism of M-modules $\mu_a : M \to M : m \mapsto am$ (for all $m \in M$) is injective, hence the same holds for the map $\mu_a \otimes_M \mathscr{O}_{S,\xi} : \mathscr{O}_{S,\xi} \to \mathscr{O}_{S,\xi}$, which is just multiplication by $\varphi(a)$. ☐

12.5.5. Let $(X, \underline{M})$ be a locally noetherian log scheme, with coherent log structure (on the site X_τ, see (12.2.1)), and let ξ be any τ-point of X. We denote by $I(\xi, \underline{M}) \subset \mathscr{O}_{X,\xi}$ the ideal generated by the image of the maximal ideal of $\underline{M}_\xi$, and we set :

$$d(\xi, \underline{M}) := \dim \mathscr{O}_{X,\xi}/I(\xi, \underline{M}) + \dim \underline{M}_\xi.$$

Lemma 12.5.6. *In the situation of* (12.5.5)*, suppose furthermore that* $(X, \underline{M})$ *is a fs log scheme. Then we have the inequality :*

$$\dim \mathscr{O}_{X,\xi} \le d(\xi, \underline{M}).$$

Proof. According to corollary 12.1.36(i), there exist a neighborhood $U \to X$ of ξ in X_τ, and a fine and saturated chart $\alpha : P_U \to \underline{M}_{|U}$, which is sharp at the point ξ. Especially, $P \simeq \underline{M}^\sharp_\xi$, therefore $\dim P = \dim \underline{M}_\xi$ (corollary 6.4.12(ii)). Notice that $\mathscr{O}_{X,\xi}$ is a noetherian local ring (this is obvious for $\tau =$ Zar, and follows from [66, Ch.IV, Prop.18.8.8(iv)] for $\tau =$ ét), hence to conclude it suffices to apply corollary 6.4.27(i) to the induced map of monoids $P \to \mathscr{O}_{X,\xi}$. ☐

Definition 12.5.7. Let $(X, \underline{M})$ be a locally noetherian fs log scheme, ξ a τ-point of X.

(i) We say that $(X, \underline{M})$ is *regular at the point* ξ, if the following holds :
  (a) the inequality of lemma (12.5.6) is actually an equality, and
  (b) the local ring $\mathscr{O}_{X,\xi}/I(\xi, \underline{M})$ is regular.
(ii) We denote by $(X, \underline{M})_{\mathrm{reg}} \subset X$ the set of points x such that $(X, \underline{M})$ is regular at any (hence all) τ-points of X localized at x.
(iii) We say that $(X, \underline{M})$ is *regular*, if $(X, \underline{M})_{\mathrm{reg}} = X$.
(iv) Let K be a field, and X a K-scheme. We say that the K-log scheme $(X, \underline{M})$ is *geometrically regular*, if $E \times_K (X, \underline{M})$ is regular, for every finite field extension E of K.

Remark 12.5.8. (i) Certain constructions produce log structures $\underline{M} \to \mathscr{O}_X$ that are morphisms of pointed monoids. It is then useful to extend the notion of regularity to such log structures. We shall say that $(X, \underline{M})$ is a *pointed regular* log scheme, if there exists a log structure $\underline{N}$ on X, such that $\underline{M} = \underline{N}_\circ$ (notation of (12.1.11)), and $(X, \underline{N})$ is a regular log scheme.

(ii) Likewise, if K is a field, X a K-scheme, and $\underline{M}$ a log structure on X, we shall say that the K-log scheme $(X, \underline{M})$ is *geometrically pointed regular*, if $\underline{M} = \underline{N}_\circ$ for some log structure $\underline{N}$ on X, such that $(X, \underline{N})$ is geometrically regular.

12.5.9. Let $(X, \underline{M})$ be a locally noetherian fs log scheme, ξ a τ-point of X, and $\mathscr{O}^\wedge_{X,\xi}$ the completion of $\mathscr{O}_{X,\xi}$. The next result is the logarithmic version of the classical characterization of complete regular local rings ([126, Th.29.7 and Th.29.8]).

Theorem 12.5.10. *With the notation of* (12.5.9)*, the log scheme* $(X, \underline{M})$ *is regular at the point* ξ *if and only if there exist :*

(a) *a complete regular local ring* $(R, \mathfrak{m}_R)$*, and a local ring homomorphism* $R \to \mathscr{O}^\wedge_{X,\xi}$*;*
(b) *a fine and saturated chart* $P_{X(\xi)} \to \underline{M}(\xi)$ *which is sharp at the closed point* ξ *of* $X(\xi)$*, such that the induced continuous ring homomorphism*

$$R[[P]] \to \mathscr{O}^\wedge_{X,\xi}$$

is an isomorphism if $\mathscr{O}_{X,\xi}$ *contains a field, and otherwise it is a surjection, with kernel generated by an element* $\vartheta \in R[[P]]$ *whose constant term lies in* $\mathfrak{m}_R \backslash \mathfrak{m}_R^2$.

Proof. Suppose first that (a) and (b) hold. If R contains a field, then it follows that

$$\dim \mathscr{O}_{X,\xi} = \dim \mathscr{O}^\wedge_{X,\xi} = \dim R + \dim P$$

by corollary 6.4.27(ii); moreover, in this case $I(\xi, \underline{M}) = \mathfrak{m}_P \mathscr{O}_{X,\xi}$, hence $\mathscr{O}_{X,\xi}/I(\xi, \underline{M}) = \mathscr{O}_{X,\xi}/\mathfrak{m}_P \mathscr{O}_{X,\xi}$, whose completion is $\mathscr{O}^\wedge_{X,\xi}/\mathfrak{m}_P \mathscr{O}^\wedge_{X,\xi} \simeq R$ so that $\mathscr{O}_{X,\xi}/I(\xi, \underline{M})$ is regular ([63, Ch.0, Prop.17.3.3(i)]). Furthermore,

$$\dim R = \dim \mathscr{O}^\wedge_{X,\xi}/\mathfrak{m}_P \mathscr{O}^\wedge_{X,\xi} = \dim \mathscr{O}_{X,\xi}/I(\xi, \underline{M})$$

([126, Th.15.1]). Hence $(X, \underline{M})$ is regular at the point ξ. If R does not contain a field, we obtain $\dim \mathscr{O}_{X,\xi} = \dim R + \dim P - 1$. On the other hand, let ϑ_0 be the image of ϑ in $\mathfrak{m}_R$; then we have $\mathscr{O}^\wedge_{X,\xi}/\mathfrak{m}_P \mathscr{O}^\wedge_{X,\xi} \simeq R/\vartheta_0 R$, which is regular of dimension $\dim\, R - 1$, and again we invoke [63, Ch.0, Prop.17.3.3(i)] to see that $(X, \underline{M})$ is regular at ξ.

Conversely, suppose that $(X, \underline{M})$ is regular at ξ. Suppose first that $\mathscr{O}_{X,\xi}$ contains a field; then we may find a field $k \subset \mathscr{O}^\wedge_{X,\xi}$ mapping isomorphically to the residue field of $\mathscr{O}^\wedge_{X,\xi}$ ([126, Th.28.3]). Pick a sequence $(t_1, \dots, t_r)$ of elements of $\mathscr{O}_{X,\xi}$ whose image in the regular local ring $\mathscr{O}_{X,\xi}/I(\xi, \underline{M})$ forms a regular system of parameters. Let also P a fine saturated monoid for which there exist a neighborhood $U \to X$ of ξ and a chart $P_U \to \underline{M}_{|U}$, sharp at the point ξ. There follows a map of monoids $\alpha : P \to \mathscr{O}_{X,\xi}$, and necessarily the image of $\mathfrak{m}_P$ lies in the maximal ideal of $\mathscr{O}_{X,\xi}$, and generates $I(\xi, \underline{M})$. We deduce a continuous ring homomorphism

$$k[[P \times \mathbb{N}^{\oplus r}]] \to \mathscr{O}^\wedge_{X,\xi}$$

which extends α, and which maps the generators $T_1, \dots, T_r$ of $\mathbb{N}^{\oplus r}$ onto respectively $t_1, \dots, t_r$. This map is clearly surjective, and by comparing dimensions (using corollary 6.4.27(ii)) one sees that it is an isomorphism. Then the theorem holds in this case, with $R := k[[\mathbb{N}^{\oplus r}]]$.

Next, if $\mathscr{O}_{X,\xi}$ does not contain a field, then its residue characteristic is a positive integer p, and we may find a complete discrete valuation ring $V \subset \mathscr{O}^\wedge_{X,\xi}$ whose maximal ideal is pV, and such that V/pV maps isomorphically onto the residue field of $\mathscr{O}^\wedge_{X,\xi}$ ([126, Th.29.3]). Again, we choose a morphism of monoids $\alpha : P \to \mathscr{O}_{X,\xi}$ as in the foregoing, and a sequence $(t_1, \dots, t_r)$

of elements of $\mathscr{O}_{X,\xi}$ lifting a regular system of parameters for $\mathscr{O}_{X,\xi}/I(\xi,\underline{M})$, by means of which we define a continuous ring homomorphism

$$\varphi : V[[P \times \mathbb{N}^{\oplus r}]] \to \mathscr{O}^{\wedge}_{X,\xi}$$

as in the previous case. Again, it is clear that φ is surjective. The image of the ideal J generated by the maximal ideal of $P \times \mathbb{N}^{\oplus r}$ is the maximal ideal of $\mathscr{O}^{\wedge}_{X,\xi}$; in particular, there exists $x \in J$ such that $\vartheta := p - x$ lies in $\mathrm{Ker}\,\varphi$. If we let $R := V[[\mathbb{N}^{\oplus r}]]$, it is clear that $\vartheta \in \mathfrak{m}_R \setminus \mathfrak{m}_R^2$.

Claim 12.5.11. Let A be a ring, π a regular element of A such that $A/\pi A$ is an integral domain, P a fine and sharp monoid. Let also ϑ be an element of $A[[P]]$ whose constant term is π (see (6.4.25)). Then $A[[P]]/(\vartheta)$ is an integral domain.

Proof of the claim. To ease notation, set $A_0 := A/\pi A$, $B := A[[P]]$ and $C := B/\vartheta B$. Choose a decomposition (6.4.26), and set $\mathrm{Fil}^\gamma_n B := \prod_{i\geq n} A[[\mathrm{gr}^\gamma_i P]]$ for every $n \in \mathbb{N}$. $\mathrm{Fil}^\gamma_\bullet B$ is a separated filtration by ideals of B, and we may consider the induced filtration $\mathrm{Fil}^\gamma_\bullet C$ on C. First, we remark that $\mathrm{Fil}^\gamma_\bullet C$ is also separated. This comes down to checking that

$$\bigcap_{n\geq 0} \vartheta B + \mathrm{Fil}^\gamma_n B = \vartheta B.$$

To this aim, suppose that, for a given $x \in B$ we have identities of the type $x = \vartheta y_n + z_n$, with $y_n \in B$ and $z_n \in \mathrm{Fil}^\gamma_n B$ for every $n \in \mathbb{N}$. Then, since π is regular, an easy induction shows that $\mathrm{gr}^\gamma_i(y_n) = \mathrm{gr}^\gamma_i(y_m)$ whenever $n, m > i$, and moreover $x = \vartheta \cdot \sum_{i\in\mathbb{N}} \mathrm{gr}^\gamma_i(y_{i+1})$, which shows the contention. It follows that, in order to show that C is a domain, it suffices to show that the same holds for the graded ring $\mathrm{gr}^\gamma_\bullet C$ associated to $\mathrm{Fil}^\gamma_\bullet C$. However, notice that :

$$\mathrm{Fil}^\gamma_n B \cap \vartheta B = \vartheta \cdot \mathrm{Fil}^\gamma_n B \qquad \text{for every } n \in \mathbb{N}.$$

(Indeed, this follows easily from the fact that π is a regular element : the verification shall be left to the reader). Hence, we may compute :

$$\mathrm{gr}^\gamma_n C = \frac{\mathrm{Fil}^\gamma_n B + \vartheta B}{\mathrm{Fil}^\gamma_{n+1} + \vartheta B} \simeq \frac{\mathrm{Fil}^\gamma_n B}{\mathrm{Fil}^\gamma_{n+1} + \vartheta \mathrm{Fil}^\gamma_n} \simeq A[\mathrm{gr}^\gamma_n P]/\vartheta A[\mathrm{gr}^\gamma_n P] \simeq A_0[\mathrm{gr}^\gamma_n P].$$

Thus, $\mathrm{gr}^\gamma_\bullet A \simeq A_0[P]$, which is a domain, since by assumption A_0 is a domain. ◊

From claim 12.5.11(ii) we deduce that $R[[P]]/(\vartheta)$ is an integral domain. Then, again by comparing dimensions, we see that φ factors through an isomorphism $R[[P]]/(\vartheta) \xrightarrow{\sim} \mathscr{O}^{\wedge}_{X,\xi}$. □

Remark 12.5.12. Resume the notation of (12.5.9), and suppose that $\mathscr{O}_{X,\xi}/I(\xi,\underline{M})$ is a regular local ring. Let $P_{X(\xi)} \to \underline{M}(\xi)$ be a chart as in theorem 12.5.10(b), and $\alpha : P \to \mathscr{O}_{X,\xi}$ the corresponding morphism of monoids. Moreover, if $\mathscr{O}_{X,\xi}$ contains a field, let V denote a coefficient field of $\mathscr{O}^{\wedge}_{X,\xi}$, and otherwise, let V be a complete discrete valuation ring whose maximal ideal is generated by p, the residue characteristic of $\mathscr{O}_{X,\xi}$. In either case, pick a ring homomorphism $V \to \mathscr{O}^{\wedge}_{X,\xi}$ inducing an isomorphism of V/pV onto the residue field of $\mathscr{O}_{X,\xi}$. Let as well $t_1, \dots, t_r \in \mathscr{O}_{X,\xi}$ be any sequence of elements whose image in $\mathscr{O}_{X,\xi}/I(\xi,\underline{M})$ forms a regular system of parameters, and extend the map α to a morphism of monoids $P \times \mathbb{N}^{\oplus r} \to \mathscr{O}_{X,\xi}$, by the rule : $e_i \mapsto t_i$, where $e_1, \dots, e_r$ is the natural basis of $\mathbb{N}^{\oplus r}$. Then by inspecting the proof of theorem 12.5.10, we see that the induced continuous ring homomorphism :

$$V[[P \times \mathbb{N}^{\oplus r}]] \to \mathscr{O}^{\wedge}_{X,\xi}$$

is always surjective, and if V is not a field, its kernel contains an element ϑ whose constant term in V is $\vartheta_0 = p$. If V is a field (resp. if V is a discrete valuation ring) then $(X, \underline{M})$ is regular at the point ξ, if and only if this map is an isomorphism (resp. if and only if the kernel of this map is generated by ϑ).

Corollary 12.5.13. *Let $(X, \underline{M})$ be a regular log scheme. Then the scheme X is normal and Cohen-Macaulay.*

Proof. Let $x \in X$ be any point, and ξ a τ-point localized at x; we have to show that $\mathscr{O}_{X,x}$ is Cohen-Macaulay; in light of [66, Ch.IV, Cor.18.8.13(a)] (when $\tau = \text{ét}$), it suffices to show that the same holds for $\mathscr{O}_{X,\xi}$. Then [126, Th.17.5] further reduces to showing that the completion $\mathscr{O}^{\wedge}_{X,\xi}$ is Cohen-Macaulay; the latter follows easily from theorems 12.5.10 and 11.6.35(i). Next, in order to prove that X is normal, it suffices to show that $\mathscr{O}_{X,x}$ is regular, whenever x has codimension one in X ([126, Th.23.8]). Again, by [66, Ch.IV, Cor.18.8.13(c)] (when $\tau = \text{ét}$) and [63, Ch.0, Prop.17.1.5], we reduce to showing that $\mathscr{O}^{\wedge}_{X,\xi}$ is regular for such a point x. However, for a point of codimension one we have $r := \dim \underline{M}_\xi \leq 1$. If $r = 0$, then $I(\xi, \underline{M}) = \{0\}$, hence $\mathscr{O}_{X,\xi}$ is regular. Lastly, if $r = 1$, we see that $\underline{M}^\sharp_\xi \simeq \mathbb{N}$ (theorem 6.4.18(iii)); consequently, there exist a regular local ring R and an isomorphism $\mathscr{O}^{\wedge}_{X,\xi} \simeq R[[\mathbb{N}]]/(\vartheta)$, where $\vartheta = 0$ if $\mathscr{O}_{X,\xi}$ contains a field, and otherwise the constant term of ϑ lies in $\mathfrak{m}_R \setminus \mathfrak{m}_R^2$. Since $R[[\mathbb{N}]]$ is again regular, the assertion follows in either case. □

Corollary 12.5.14. *Let $i : (X', \underline{M}') \to (X, \underline{M})$ be an exact closed immersion of regular log schemes (see definition* 12.3.22(i)*). Then the underlying morphism of schemes $X' \to X$ is a regular closed immersion.*

Proof. Let ξ be a τ-point of X, and denote by J the kernel of $i^\natural_\xi : \mathscr{O}_{X,\xi} \to \mathscr{O}_{X',\xi}$. To ease notation, let as well $A := \mathscr{O}_{X,\xi}/I(\xi, \underline{M})$ and $A' := \mathscr{O}_{X',\xi}/I(\xi, \underline{M}')$ Since $\log i : i^*\underline{M} \to \underline{M}'$ is an isomorphism, $i^\natural_\xi$ induces an isomorphism :

$$\mathscr{O}_{X,\xi}/(J + I(\xi, \underline{M})) \xrightarrow{\sim} A'.$$

Since A and A' are regular, there exists a sequence of elements $t_1, \dots, t_k \in J$, whose image in A forms the beginning of a regular system of parameters and generate the kernel of the induced map $A \to A'$ ([63, Ch.0, Cor.17.1.9]). Extend this sequence by suitable elements of $\mathscr{O}_{X,\xi}$, to obtain a sequence $(t_1, \dots, t_r)$ whose image in A is a regular system of parameters. We deduce a surjection $\varphi : V[[P \times \mathbb{N}^{\oplus r}]] \to \mathscr{O}^{\wedge}_{X,\xi}$ as in remark 12.5.12, where V is either a field or a complete discrete valuation ring. Denote by $(e_1, \dots, e_r)$ the natural basis of $\mathbb{N}^{\oplus r}$. Now, suppose first that V is a complete discrete valuation ring; then $\mathrm{Ker}\, \varphi$ is generated by an element $\vartheta \in V[[P \times \mathbb{N}^{\oplus r}]]$, whose constant term is a uniformizer in V; we deduce easily from claim 12.5.11 that $(e_1, \dots, e_r, \vartheta)$ is a regular sequence in $V[[P \times \mathbb{N}^{\oplus r}]]$; hence the same holds for the sequence $(\vartheta, e_1, \dots, e_r)$, in view of [126, p.127, Cor.]. This implies that $(t_1, \dots, t_r)$ is a regular sequence in $\mathscr{O}^{\wedge}_{X,\xi}$, hence $(t_1, \dots, t_k)$ is a regular sequence in $\mathscr{O}_{X,\xi}$, which is the contention. The case where V is a field is analogous, though simpler : the details shall be left to the reader. □

Remark 12.5.15. In the situation of remark 12.5.12, suppose that P is a free monoid of finite rank d, let $\{f_1, \dots, f_d\} \subset \mathscr{O}_{X,\xi}$ be the image of the (unique) basis of P, and for every $i \leq d$ let $Z_i \subset X$ be the zero locus of f_i; then $\bigcup_{i=1}^d Z_i$ is a strict normal crossings divisor in the sense of example 12.2.11. The proof is analogous to that of corollary 12.5.14 : the sequence $(f_1, \dots, f_d, \vartheta)$ is regular in $V[[P \oplus \mathbb{N}^{\oplus r}]]$, hence also its permutation $(\vartheta, f_1, \dots, f_d)$ is regular, whence the claim (the details shall be left to the reader).

Proposition 12.5.16. *Resume the situation of* (12.5.9). *We have :*

- (i) *The log scheme $(X, \underline{M})$ is regular at the τ-point ξ if and only if the following two conditions hold :*
  - (a) *The ring $\mathscr{O}_{X,\xi}/I(\xi, \underline{M})$ is regular.*
  - (b) *There exists a morphism of monoids $P \to \mathscr{O}_{X,\xi}$ from a fine monoid P, whose associated constant log structure on $X(\xi)$ is the same as $\underline{M}(\xi)$, and such that $\mathscr{O}_{X,\xi}$ is P-flat.*

(ii) *Moreover, if the equivalent conditions of* (i) *hold, and* $Q_X \to \underline{M}(\xi)$ *is any fine chart, then* $\mathscr{O}_{X,\xi}$ *is Q-flat, for the induced map of monoids* $Q \to \mathscr{O}_{X,\xi}$.

Proof. (i): Suppose first that $(X, \underline{M})$ is regular at the point ξ; then by definition (a) holds. Next, we may find a fine monoid P and a local morphism $\alpha : P \to A := \mathscr{O}_{X,\xi}$ whose associated constant log structure on $X(\xi)$ is the same as $\underline{M}(\xi)$, and a regular local ring R, with a ring homomorphism $R \to A$ such that the induced continuous map $R[[P]] \to A^\wedge$ fulfills condition (b) of theorem 12.5.10 (where $A^\wedge$ is the completion of A). Since the completion map $\varphi : A \to A^\wedge$ is faithfully flat, A is α-flat if and only if $A^\wedge$ is $\varphi \circ \alpha$-flat; in light of proposition 6.1.39(i), it then suffices to show that, for every ideal $I \subset P$, the natural map

$$A^\wedge \overset{\mathbf{L}}{\otimes}_P P/I \to A^\wedge \otimes_P P/I$$

is an isomorphism in $\mathsf{D}^-(A^\wedge\text{-}\mathbf{Mod})$ (notation of (6.1.33)). To this aim we remark :

Claim 12.5.17. For any ideal $I \subset P$, the natural morphism

$$R[P] \overset{\mathbf{L}}{\otimes}_P P/I \to R\langle P/I \rangle$$

is an isomorphism in $\mathsf{D}^-(R[P]\text{-}\mathbf{Mod})$.

Proof of the claim. We consider the change of rings spectral sequence for Tor :

$$E^2_{ij} := \mathrm{Tor}_i^{\mathbb{Z}[P]}(\mathrm{Tor}_j^{\mathbb{Z}}(R, \mathbb{Z}[P]), \mathbb{Z}\langle P/I \rangle) \Rightarrow \mathrm{Tor}^{\mathbb{Z}}_{i+j}(R, \mathbb{Z}\langle P/I \rangle)$$

([163, Th.5.6.6]). Clearly $E^2_{ij} = 0$ whenever $j > 0$, whence isomorphisms :

$$\mathrm{Tor}_i^{\mathbb{Z}\langle P \rangle}(R\langle P \rangle, \mathbb{Z}\langle P/I \rangle) \xrightarrow{\sim} \mathrm{Tor}_i^{\mathbb{Z}}(R, \mathbb{Z}\langle P/I \rangle)$$

for every $i \in \mathbb{N}$. The claim follows easily. ◊

In light of claim 12.5.17 we deduce natural isomorphisms :

$$A^\wedge \overset{\mathbf{L}}{\otimes}_P P/I \xrightarrow{\sim} A^\wedge \overset{\mathbf{L}}{\otimes}_{R[P]} (R[P] \overset{\mathbf{L}}{\otimes}_P P/I) \xrightarrow{\sim} A^\wedge \overset{\mathbf{L}}{\otimes}_{R[P]} R\langle P/I \rangle$$

in $\mathsf{D}^-(A^\wedge\text{-}\mathbf{Mod})$. Now, if A contains a field, we have $A^\wedge \simeq R[[P]]$, which is a flat $R[P]$-algebra ([126, Th.8.8]), and the contention follows. If A does not contain a field, the complex

$$0 \to R[[P]] \xrightarrow{\vartheta} R[[P]] \to A^\wedge \to 0$$

is a $R[P]$-flat resolution of $A^\wedge$. Since $R[[P]]/IR[[P]] \simeq \lim_{n \in \mathbb{N}} R\langle P/(I \cup \mathfrak{m}_P^n) \rangle$, we come down to the following :

Claim 12.5.18. Let $\vartheta \in R[P]$ be any element whose constant term ϑ_0 is a regular element in R. Then, for every ideal $I \subset P$, the image of ϑ in $R\langle P/I \rangle$ is a regular element.

Proof of the claim. In view of (6.4.22), P can be regarded as a graded monoid $P = \coprod_{n \in \mathbb{N}} P_n$, so $R[P]$ is a graded algebra, and $I = \coprod_{n \in \mathbb{N}} (I \cap P_n)$ is a graded ideal. Thus, $\langle P/I \rangle$ is a graded R-algebra as well, and the claim follows easily (details left to the reader). ◊

Conversely, suppose that conditions (a) and (b) hold. By virtue of lemma 12.5.1, we may assume that P is fine, sharp and saturated, and that the map $P \to \mathscr{O}_{X,\xi}$ is local. According to remark 12.5.12, we may find a regular local ring R of dimension $\leq 1 + \dim A/\mathfrak{m}_P A$, and a surjective ring homomorphism :

$$\varphi : R[[P]] \to A^\wedge$$

Suppose first that $\dim R = 1 + \dim A/\mathfrak{m}_P A$; then $\mathrm{Ker}\,\varphi$ contains an element ϑ whose constant term ϑ_0 lies in $\mathfrak{m}_R \setminus \mathfrak{m}_R^2$, and φ induces an isomorphism $R_0 := R/\vartheta_0 R \xrightarrow{\sim} A^\wedge/\mathfrak{m}_P A^\wedge$. We have to show that φ induces an isomorphism $\overline{\varphi} : R[[P]]/(\vartheta) \xrightarrow{\sim} A^\wedge$. To this aim, we consider the $\mathfrak{m}_P$-adic filtrations on these rings; for the associated graded rings we have :

$$\mathrm{gr}_n R[[P]]/(\vartheta) \simeq R_0\langle \mathfrak{m}_P^n/\mathfrak{m}_P^{n+1}\rangle \qquad \mathrm{gr}_n A^\wedge = \mathfrak{m}_P^n A^\wedge/\mathfrak{m}_P^{n+1}A^\wedge.$$

Hence $\mathrm{gr}_n\overline{\varphi}$ is the natural map $A^\wedge/\mathfrak{m}_P A^\wedge \otimes_P (\mathfrak{m}_P^n/\mathfrak{m}_P^{n+1}) \to \mathfrak{m}_P^n A^\wedge/\mathfrak{m}_P^{n+1}A^\wedge$, and the latter is an isomorphism, since $A^\wedge$ is P-flat. The assertion follows in this case. The remaining case where $\dim R = \dim A/\mathfrak{m}_P A$ is similar, but easier, so shall be left to the reader, as an exercise.

(ii) follows immediately from (i) and lemma 12.5.1. □

Corollary 12.5.19. *Let $(X, \underline{M})$ be a log scheme, ξ a τ-point of X, and suppose that $(X, \underline{M})$ is regular at ξ. Then the following conditions are equivalent :*

(a) *$\mathscr{O}_{X,\xi}$ is a regular local ring.*

(b) *$\underline{M}_\xi^\sharp$ is a free monoid of finite rank.*

Proof. (b)⇒(a) : Indeed, it suffices to show that $\mathscr{O}_{X,\xi}^\wedge$ is regular ([63, Ch.0, Prop.17.3.3(i)]); by theorem 12.5.10, the latter is isomorphic to either $R[[P]]$ or $R[[P]]/\vartheta R[[P]]$, where R is a regular local ring, $P := \underline{M}_\xi$, and the constant term of ϑ lies in $\mathfrak{m}_R \setminus \mathfrak{m}_R^2$. Since, by assumption, P is a free monoid of finite rank, it is easily seen that rings of the latter kind are regular ([63, Ch.0, Cor.17.1.8]).

(a)⇒(b) : By proposition 12.5.16(ii), $R := \mathscr{O}_{X,\xi}$ is P-flat, for a morphism $P := \underline{M}_\xi^\sharp \to R$ that induces the log structure $\underline{M}(\xi)$ on $X(\xi)$. It follows that

$$\mathfrak{m}_P R/\mathfrak{m}_P^2 R = (\mathfrak{m}_P/\mathfrak{m}_P^2) \otimes_P R = R^{\oplus r}$$

where $r := \mathrm{rk}^\circ_{P^\times}\mathfrak{m}_P/\mathfrak{m}_P^2$ is the cardinality of $\mathfrak{m}_P \setminus \mathfrak{m}_P^2$. In view of [66, Ch.IV, Prop.16.9.3, Cor.16.9.4, Cor.19.1.2], we deduce that the image of $\mathfrak{m}_P \setminus \mathfrak{m}_P^2$ is a regular sequence of R of length r, hence $\dim P = \dim R - \dim R/\mathfrak{m}_P R = r$, since $(X, \underline{M})$ is regular at ξ. Then the assertion follows from proposition 6.4.23. □

Corollary 12.5.20. *Let $(X, \underline{M})$ be a regular log scheme, ξ any τ-point of X, and $\mathfrak{p} \subset \underline{M}_\xi$ an ideal. We have :*

(i) *If $\mathfrak{p}$ is a prime ideal, $\mathfrak{p}\mathscr{O}_{X,\xi}$ is a prime ideal of $\mathscr{O}_{X,\xi}$, and* $\mathrm{ht}\,\mathfrak{p} = \mathrm{ht}\,\mathfrak{p}\mathscr{O}_{X,\xi}$.

(ii) *If $\mathfrak{p}$ is a radical ideal, $\mathfrak{p}\mathscr{O}_{X,\xi}$ is a radical ideal of $\mathscr{O}_{X,\xi}$.*

Proof. To ease notation, set $A := \mathscr{O}_{X,\xi}$. Pick a chart $\beta : P_{X(\xi)} \to \underline{M}(\xi)$ with P fine, sharp and saturated (corollary 12.1.36(i)); by proposition 12.5.16(ii), the ring A is P-flat for the resulting map $P \to A$. Let $\mathfrak{q} := \beta^{-1}\mathfrak{p} \subset P$; then $\mathfrak{p}A = \mathfrak{q}A$.

(i): Since $\mathfrak{p}$ is a prime ideal, $\mathfrak{q}$ is a prime ideal of P, and in order to prove that $\mathfrak{p}A$ is a prime ideal, it suffices therefore to show that the completion $(A/\mathfrak{q}A)^\wedge$ of $A/\mathfrak{q}A$ is an integral domain. However, remark 12.5.12 implies that $(A/\mathfrak{q}A)^\wedge \simeq B/\vartheta B$, where $B := R[[P\backslash\mathfrak{q}]]$, with $(R, \mathfrak{m}_R)$ a regular local ring, and ϑ is either zero, or else it is an element whose constant term $\vartheta_0 \in R$ lies in $\mathfrak{m}_R\backslash\mathfrak{m}_R^2$. The assertion is obvious when $\vartheta = 0$, and otherwise it follows from claim 12.5.11. Next, by going down (corollary 6.4.34), we see that :

$$\mathrm{ht}\,\mathfrak{p}A \geq \mathrm{ht}\,\mathfrak{p}.$$

On the other hand, notice that we have a natural identification $\mathrm{Spec}\,\underline{M}_\xi \xrightarrow{\sim} \mathrm{Spec}\,P$, especially $\mathrm{ht}\,\mathfrak{p} = \mathrm{ht}\,\mathfrak{q}$, hence $\dim A/\mathfrak{p}A = \dim(A/\mathfrak{q}A)^\wedge = \dim R + \dim(P\backslash\mathfrak{q}) - \varepsilon$, where ε is either 0 or 1 depending on whether R does or does not contain a field (corollary 6.4.27(ii)). Thus :

$$\mathrm{ht}\,\mathfrak{p}A \leq \dim A - \dim A/\mathfrak{p}A = \dim P - \dim(P\backslash\mathfrak{q}) = \mathrm{ht}\,\mathfrak{p}$$

(corollary 6.4.12(iii)), which completes the proof.

(ii): In this case, $\mathfrak{q}$ is a radical ideal of P, so it can be written as a finite intersection of prime ideals of P (lemmata 6.1.16 and 6.1.20(iii)); then the assertion follows from (i) and lemma 6.1.37 (details left to the reader). □

Lemma 12.5.21. *Let X be a scheme, $\underline{M}$ a fs log structure on X_{Zar}, and ξ a geometric point of X. Then $(X, \underline{M})$ is regular at the point $|\xi|$ if and only if $\widetilde{u}_X^*(X, \underline{M})$ is regular at ξ.*

Proof. Set $(Y, \underline{N}) := \widetilde{u}_X^*(X, \underline{M})$ (notation of (12.2.2) : of course $Y = X$, but the sheaf $\mathscr{O}_Y$ is defined on the site $X_{\text{ét}}$, hence $B := \mathscr{O}_{Y,\xi}$ is the strict henselization of $A := \mathscr{O}_{X,|\xi|}$), and let $\alpha : \underline{M}_{|\xi|} \to B$ be the induced morphism of monoids; since α is local, it is easily seen that $\underline{N}_\xi$ is isomorphic to the push-out of the diagram $\underline{M}_{|\xi|} \leftarrow \underline{M}^\times_{|\xi|} \to B^\times$, especially $\dim \underline{M}_{|\xi|} = \dim \underline{N}_\xi$. Moreover, $I(\xi, \underline{N}) = I(|\xi|, \underline{M})B$, so that $B_0 := B/I(\xi, \underline{N})$ is the strict henselization of $A_0 := A/I(|\xi|, \underline{M})$ ([66, Ch.IV, Prop.18.6.8]); hence A_0 is regular if and only if the same holds for B_0 ([66, Ch.IV, Cor.18.8.13]). Finally, $\dim A = \dim B$ and $\dim A_0 = \dim B_0$ ([126, Th.15.1]). The lemma follows. ☐

12.5.22. Let $(Y, \underline{N})$ be a log scheme, $\overline{y}$ a τ-point of Y, and $\alpha : Q_Y^{\log} \to \underline{N}$ a fine and saturated chart which is sharp at $\overline{y}$. Let also $\varphi : Q \to P$ be an injective morphism of monoids, with P fine and saturated, and such that $P^\times$ is a torsion-free abelian group. Define $(X, \underline{M})$ as the fibre product in the cartesian diagram of log schemes :

$$(12.5.23)\qquad \begin{array}{ccc} (X, \underline{M}) & \xrightarrow{\ f\ } & (Y, \underline{N}) \\ \downarrow & & \downarrow{\scriptstyle h} \\ \mathrm{Spec}(\mathbb{Z}, P) & \xrightarrow{\mathrm{Spec}(\mathbb{Z}, \varphi)} & \mathrm{Spec}(\mathbb{Z}, Q) \end{array}$$

where h is induced by α; especially, h is strict.

Lemma 12.5.24. *In the situation of* (12.5.22)*, let $\overline{x}$ be any τ-point of X such that $f(\overline{x}) = \overline{y}$, and suppose that $(Y, \underline{N})$ is regular at $\overline{y}$. Then $(X, \underline{M})$ is regular at $\overline{x}$.*

Proof. To begin with, we show :

Claim 12.5.25. The natural map :

$$\mathscr{O}_{X,\overline{x}} \overset{\mathbf{L}}{\otimes}_P P/I \to \mathscr{O}_{X,\overline{x}} \otimes_P P/I$$

is an isomorphism in $\mathsf{D}^-(\mathscr{O}_{X,\overline{x}}\text{-}\mathbf{Mod})$, for every ideal $I \subset P$ (notation of (6.1.33)).

Proof of the claim. It suffices to show that this map is an isomorphism in $\mathsf{D}^-(\mathscr{O}_{Y,\overline{y}}\text{-}\mathbf{Mod})$, and in the latter category we have a commutative diagram :

$$(12.5.26)\qquad \begin{array}{ccccc} (\mathscr{O}_{Y,\overline{y}} \overset{\mathbf{L}}{\otimes}_Q P) \overset{\mathbf{L}}{\otimes}_P P/I & \longrightarrow & & & \mathscr{O}_{Y,\overline{y}} \overset{\mathbf{L}}{\otimes}_Q P/I \\ \downarrow & & & & \downarrow \\ (\mathscr{O}_{Y,\overline{y}} \otimes_Q P) \overset{\mathbf{L}}{\otimes}_P P/I & \longrightarrow & (\mathscr{O}_{Y,\overline{y}} \otimes_Q P) \otimes_P P/I & \xrightarrow{\sim} & \mathscr{O}_{Y,\overline{y}} \otimes_Q P/I. \end{array}$$

Since $\mathscr{O}_{X,\overline{x}}$ is a localization of $\mathscr{O}_{Y,\overline{y}} \otimes_Q P$, we are reduced to showing that the bottom arrow of (12.5.26) is an isomorphism. However, the top arrow of (12.5.26) is always an isomorphism. Moreover, on the one hand, since $(Y, \underline{N})$ is regular at $\overline{y}$, the ring $\mathscr{O}_{Y,\overline{y}}$ is Q-flat (proposition 12.5.16(ii)), and on the other hand, since φ is injective, P/I is an integral Q-module, so the two vertical arrows are isomorphisms as well, and the claim follows. ◊

Claim 12.5.27. $\mathscr{O}_{X,\overline{x}}/I(\overline{x}, \underline{M})$ is a regular ring.

Proof of the claim. Let $\beta : P \to A := \mathscr{O}_{X,\overline{x}}$ be the morphism deduced from h, and set :

$$S := \beta^{-1}(\mathscr{O}^\times_{X,\overline{x}}) \qquad I(\overline{x}, P) := P \backslash S.$$

Then the $\mathbb{Z}[P]$-algebra A is a localization of the $\mathbb{Z}[P]$-algebra

$$B := S^{-1}(\mathscr{O}_{Y,\overline{y}} \otimes_Q P)$$

and it suffices to show that $B/I(\overline{x}, \underline{M})$ is regular.

It is easily seen that $I(\overline{x}, \underline{M}) = I(\overline{x}, P)A$ and $\varphi(\mathfrak{m}_Q) \subset I(\overline{x}, P)$; on the other hand, since α is sharp at the point $\overline{y}$, we have $I(\overline{y}, \underline{N}) = \mathfrak{m}_Q \mathscr{O}_{Y,\overline{y}}$, and $Q \setminus \mathfrak{m}_Q = \{1\}$. Let p be the residue characteristic of $\mathscr{O}_{Y,\overline{y}}$; there follow isomorphisms of $\mathbb{Z}_{(p)}$-algebras :

$$B/I(\overline{x}, \underline{M})B \simeq S^{-1}\mathscr{O}_{Y,\overline{y}} \otimes_Q P/I(\overline{x}, P) \simeq \mathscr{O}_{Y,\overline{y}}/I(\overline{y}, \underline{N}) \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_{(p)}[S^{\mathrm{gp}}].$$

By assumption, $\mathscr{O}_{Y,\overline{y}}/I(\overline{y}, \underline{N})$ is regular, hence we are reduced to showing that $\mathbb{Z}_{(p)}[S^{\mathrm{gp}}]$ is a smooth $\mathbb{Z}_{(p)}$-algebra ([66, Ch.IV, Prop.17.5.8(iii)]). However, under the current assumptions P^{gp} is a free abelian group of finite rank, hence the same holds for S^{gp}, and the contention follows easily. ◊

In light of proposition 6.1.40(i), claims 12.5.25 and 12.5.27 assert that conditions (a) and (b) of proposition 12.5.16(i) are satisfied for the τ-point $\overline{x}$ of $(X, \underline{M})$, so the latter is regular at $\overline{x}$, as stated. □

Theorem 12.5.28. *Let $f : (X, \underline{M}) \to (Y, \underline{N})$ be a smooth morphism of locally noetherian fs log schemes, ξ a τ-point of X, and suppose that $(Y, \underline{N})$ is regular at the point $f(\xi)$. Then $(X, \underline{M})$ is regular at the point ξ.*

Proof. In case $\tau = \mathrm{Zar}$, lemma 12.5.21 and corollary 12.3.27(ii) reduce the assertion to the corresponding one for $\widetilde{u}^* f$. Hence, we may assume that $\tau = \text{ét}$. Next, since the assertion is local on X_τ, we may assume that $\underline{N}$ admits a fine and saturated chart $\alpha : Q_Y^{\log} \to \underline{N}$ which is sharp at $f(\xi)$ (corollary 12.1.36(i)); then, by corollary 12.3.42, we may further assume that there exist an injective morphism of fine and saturated monoids $\varphi : Q \to P$, such that $P^\times$ is torsion-free, and a cartesian diagram as in (12.5.23). Then the assertion follows from lemma 12.5.24. □

Corollary 12.5.29. *Let $f : (X, \underline{M}) \to (Y, \underline{N})$ be a smooth morphism of locally noetherian fine log schemes. Then, for every $y \in (Y, \underline{N})_{\mathrm{tr}}$, the log scheme $\mathrm{Spec}\, \kappa(y) \times_Y (X, \underline{M})$ is geometrically regular.*

Proof. The trivial log structure on $\mathrm{Spec}\, \kappa(y)$ is obviously saturated, so the same holds for the log structure of $\mathrm{Spec}\, \kappa(y) \times_Y (X, \underline{M})$. Hence, the assertion is an immediate consequence of theorem 12.5.28. □

In the same vein we have :

Proposition 12.5.30. *Let $f : (X, \underline{M}) \to (Y, \underline{N})$ be a strict morphism of locally noetherian fs log schemes, ξ a τ-point of X, and suppose that the morphism of schemes $f_\xi : X(\xi) \to Y(f(\xi))$ is flat. The following holds :*

(i) *If $(X, \underline{M})$ is regular at the point ξ, then $(Y, \underline{N})$ is regular at the point $f(\xi)$.*
(ii) *Conversely, if $(Y, \underline{N})$ is regular at the point $f(\xi)$, and the fibre $f_\xi^{-1}(f(\xi))$ is a regular scheme (notation of (4.9.20)), then $(X, \underline{M})$ is regular at the point ξ.*

Proof. Set $A := \mathscr{O}_{Y,f(\xi)}$ and $B := \mathscr{O}_{X,\xi}$, let $g : A \to B$ be the induced ring homomorphism, pick a fine chart $\beta : P \to A$ for the log structure $\underline{N}(f(\xi))$, and denote by $I \subset A$ the ideal generated by the image of the maximal ideal of $\underline{N}_{f(\xi)}$.

(i): By proposition 12.5.16(ii), the local ring B/IB is regular, $g \circ \beta : P \to B$ is a fine chart for the log structure $\underline{M}(\xi)$, and B is $(g \circ \beta)$-flat. Since the induced map $A/I \to B/IB$ is flat and local, also A/I is regular ([63, Ch.0, Prop.17.3.3(i)]), and since g is faithfully flat, A is β-flat (remark 6.1.35(iii)). Then the assertion follows from proposition 12.5.16(i).

(ii): By proposition 12.5.16(ii), the local ring A/I is regular, and A is β-flat. Since g is flat, the same holds for the induced map $A/I \to B/IB$; it follows that also B/IB is regular ([63, Ch.0, Prop.17.3.3(ii)]), the induced map $P \to B$ is a chart for $\underline{M}(\xi)$ and B is $(g \circ \beta)$-flat, whence the contention, again by proposition 12.5.16(i). □

Theorem 12.5.31. *Let $(X, \underline{M})$ be a locally noetherian fs log scheme. Then the subset $(X, \underline{M})_{\mathrm{reg}}$ is closed under generization.*

Proof. Let ξ be a τ-point of X, with support $x \in (X, \underline{M})_{\mathrm{reg}}$, and let η be a generization of ξ, whose support is a strict generization y of x. We have to show that $(X, \underline{M})$ is regular at the point η. Since the assertion is local on X, we may assume that $(X, \underline{M})$ admits a fine and saturated chart $\beta : P_X \to \underline{M}$, sharp at the point ξ (corollary 12.1.36(i)). Let $\alpha : P \to \mathscr{O}_{X,\eta}$ be the morphism deduced from β_η, and set $\mathfrak{p} := \alpha^{-1}\mathfrak{m}_\eta$, where $\mathfrak{m}_\eta \subset \mathscr{O}_{X,\eta}$ is the maximal ideal. We consider the cartesian diagram of log schemes :

$$\begin{array}{ccc} (X', \underline{M}')_\circ & \longrightarrow & (X, \underline{M}) \\ \big\downarrow & & \big\downarrow{\scriptstyle g} \\ \mathrm{Spec}\langle \mathbb{Z}, P/\mathfrak{p}\rangle & \longrightarrow & \mathrm{Spec}(\mathbb{Z}, P) \end{array}$$

where g is induced by β (notation of (12.2.13)). Clearly ξ and η induce τ-points on X', which we denote by the same names. We have natural identifications:

$$\mathscr{O}_{X,\xi}/I(\xi, \underline{M}) \xrightarrow{\sim} \mathscr{O}_{X',\xi}/I(\xi, \underline{M}') \qquad \underline{M}'_\eta = \mathscr{O}^\times_{X',\eta} \qquad \mathscr{O}_{X,\eta}/I(\eta, \underline{M}) \xrightarrow{\sim} \mathscr{O}_{X',\eta}$$

([66, Ch.IV, Prop.18.6.8]). Moreover, from proposition 12.5.16(ii) we know that $\mathscr{O}_{X,\xi}$ is P-flat; then corollary 6.4.34 yields the inequality :

$$\dim \mathscr{O}_{X',\xi} = \dim \mathscr{O}_{X,\xi} \otimes_P P/\mathfrak{p} \geq \dim \mathscr{O}_{X,\xi} - \mathrm{ht}(\mathfrak{p}) = \dim \mathscr{O}_{X,\xi}/I(\xi, \underline{M}) + \dim P/\mathfrak{p}$$

in other words : $\dim \mathscr{O}_{X',\xi} = d(\xi, \underline{M}')$ (notation of definition 12.5.7), so $(X', \underline{M}')$ is regular at ξ. By the same token, $\mathscr{O}_{X,\eta}$ is $P_\mathfrak{p}$-flat; from proposition 12.5.16(i), it follows that $(X, \underline{M})$ is regular at η if and only if the same holds for $(X', \underline{M}')$.

Hence we may replace $(X, \underline{M})$ by $(X', \underline{M}')$, and P by $P \setminus \mathfrak{p}$, after which we may assume that $y \in (X, \underline{M})_{\mathrm{tr}}$. In this case :

$$\alpha(P) \subset \mathscr{O}^\times_{X,\eta} \tag{12.5.32}$$

and we have to show that $\mathscr{O}_{X,\eta}$ is a regular local ring, or equivalently, that $\mathscr{O}_{X,y}$ is regular ([66, Ch.IV, Cor.18.8.13(c)]). Now, if $P = 0$, then $\mathscr{O}_{X,x}$ is regular, and the assertion follows from [126, Th.19.3]; thus, we may assume that $P \neq 0$.

Denote by Y the topological closure of y in X, endowed with its reduced subscheme structure. According to [60, Ch.II, Prop.7.1.7] (and [66, Ch.IV, Prop.18.8.8(iv)] in case $\tau = \text{ét}$), we may find a local injective ring homomorphism $j : \mathscr{O}_{Y,\xi} \to V$, where V is a discrete valuation ring. Let also $\overline{\beta} : P \to \mathscr{O}_{Y,\xi}$ be the morphism deduced from β. Then (12.5.32) implies that $j \circ \overline{\beta}(P) \subset V \setminus \{0\}$, hence $j \circ \overline{\beta}$ extends to a homomorphism of groups $P^{\mathrm{gp}} \to K^\times$, where $K^\times := (V \setminus \{0\})^{\mathrm{gp}}$ is the multiplicative group of the field of fractions of V; after composition with the valuation $K^\times \to \mathbb{Z}$ of V, there follows a group homomorphism

$$\varphi : P^{\mathrm{gp}} \to \mathbb{Z}.$$

Notice also that $j \circ \overline{\beta}$ is a local morphism, since the same holds for j and $\overline{\beta}$; consequently, $\varphi(P^{\mathrm{gp}}) \neq \{0\}$. Set $Q := \varphi^{-1}\mathbb{N}$; then $\varphi(Q)$ is a non-trivial submonoid of $\mathbb{N}$, and $\dim Q = \dim Q/\mathrm{Ker}\,\varphi = \dim \varphi(Q) = 1$. Set

$$T := \mathrm{Spec}\, V \qquad (S, \underline{Q}) := \mathrm{Spec}(\mathbb{Z}, Q).$$

Notice that Q is saturated and fine (corollary 6.4.2), so there exists an isomorphism :

$$\mathbb{Z}^{\oplus r} \times \mathbb{N} \xrightarrow{\sim} Q \qquad \text{for some } r \in \mathbb{N}$$

(theorem 6.4.18(iii)); the latter determines a chart :

$$\mathbb{N}_S \to \underline{Q} \tag{12.5.33}$$

which is sharp at every τ-point of S localized outside the trivial locus $\mathrm{Spec}(\mathbb{Z}, Q)_{\mathrm{tr}}$. We consider the cartesian diagram of log schemes :

$$\begin{array}{ccc} (X', g'^*\underline{Q}) & \xrightarrow{g'} & \mathrm{Spec}(\mathbb{Z}, \underline{Q}) \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ (X, \underline{M}) & \xrightarrow{g} & \mathrm{Spec}(\mathbb{Z}, P) \end{array}$$

where f is the morphism of log schemes induced by the inclusion map $\psi : P \to Q$. Notice that f is a smooth morphism (proposition 12.3.34); moreover, the restriction

$$f_{\mathrm{tr}} : \mathrm{Spec}(\mathbb{Z}, Q)_{\mathrm{tr}} \to \mathrm{Spec}(\mathbb{Z}, P)_{\mathrm{tr}}$$

of f, is just the morphism $\mathrm{Spec}\,\mathbb{Z}[\psi^{\mathrm{gp}}] : \mathrm{Spec}\,\mathbb{Z}[Q^{\mathrm{gp}}] \to \mathrm{Spec}\,\mathbb{Z}[P^{\mathrm{gp}}]$, hence it is an isomorphism of schemes. It follows that f' is a smooth morphism (proposition 12.3.24(ii)), and its restriction f'_{tr} to the trivial loci, is an isomorphism.

The homomorphism j induces a morphism $h : T \to X$, such that the closed point of T maps to x and the generic point maps to y. By construction $g \circ h$ lifts to a morphism of schemes $h' : T \to S$, and the pair (h, h') determines a morphism $T \to X'$. Let $x', y' \in X'$ be the images of respectively the closed point and the generic point of T, and choose τ-points ξ' and η' localized at x' and respectively y'; then the image of x' in X is the point x, therefore $(X', g'^*\underline{Q})$ is regular at the point ξ', by theorem 12.5.28. Furthermore, $g'(\xi')$ lies outside $\mathrm{Spec}(\mathbb{Z}, Q)_{\mathrm{tr}}$, hence (12.5.33) induces a chart $\mathbb{N}_{X'} \to g'^*\underline{Q}$, which is sharp at ξ'. In light of corollary 12.5.19, we deduce that $\mathscr{O}_{X',x'}$ is a regular ring, and then the same holds also for $\mathscr{O}_{X',y'}$ ([63, Ch.0, Cor.17.3.2]). However, y' lies in the trivial locus of $(X', g'^*\underline{Q})$, and its image in X is y, so by the foregoing the natural map $\mathscr{O}_{X,y} \to \mathscr{O}_{X',y'}$ is an isomorphism, and the contention follows. $\square$

Remark 12.5.34. Notice that the proof of [60, Chap.II, Prop.7.1.7] (that is invoked in the proof of theorem 12.5.31) is slightly incorrect : indeed, with the notation of the proof of *loc.cit.*, it is implicitly assumed that $B/\mathfrak{m}B \neq 0$, which may fail, *e.g.* take $A := k[[x_2]]$ (for any field k), whose maximal ideal is generated by $x_1 := x_2^2$ and x_2, so that $B = k((x_2))$ in this case. The proof can be amended by considering the scheme $X := \mathrm{Spec}\,A$ and the blow-up $\pi : E \to X$ of the coherent ideal $\mathfrak{m}\mathscr{O}_X$; since π is proper and surjective, we see that there exists $i \in \{1, \dots, n\}$ such the image of the restriction $\mathrm{Spec}\,A[x_1/x_i, \dots, x_n/x_i] \to X$ of π contains $\mathfrak{m}$, and the argument then carries through with $B := A[x_1/x_i, \dots, x_n/x_i]$. See also [102, Th.6.4.3].

12.5.35. Let $(X, \underline{M})$ be any log scheme, ξ a τ-point of X. There follows a continuous map

$$\psi_\xi : X(\xi) \to T_\xi := \mathrm{Spec}\,\underline{M}_\xi$$

that sends the closed point of $X(\xi)$ to the closed point of T_ξ. For every point $\mathfrak{p} \in T_\xi$, let $\overline{\{\mathfrak{p}\}}$ be the topological closure of $\{\mathfrak{p}\}$ in T_ξ; then

$$X(\mathfrak{p}) := \psi_\xi^{-1}\overline{\{\mathfrak{p}\}}$$

is a closed subset of $X(\xi)$, which we endow with its reduced subscheme structure, and we set

$$(X(\mathfrak{p}), \underline{M}(\mathfrak{p})) := ((X(\xi), \underline{M}(\xi)) \times_X X(\mathfrak{p}))_{\mathrm{red}}$$

(notation of example 12.1.12(iv)). Notice that $U_\mathfrak{p} := \psi_\xi^{-1}(\mathfrak{p})$ is an open subset of $X(\mathfrak{p})$, for every $\mathfrak{p} \in T_\xi$. We call the family

$$(U_\mathfrak{p} \mid \mathfrak{p} \in T_\xi)$$

of locally closed subschemes of $X(\xi)$, the *logarithmic stratification* of $(X(\xi), \underline{M}(\xi))$. For instance, $U_\varnothing = (X(\overline{x}), \underline{M}(\overline{x}))_{\mathrm{tr}}$. More generally, it is clear from the definition that

$$U_\mathfrak{p} = (X(\mathfrak{p}), \underline{M}(\mathfrak{p}))_{\mathrm{tr}} \qquad \text{for every } \mathfrak{p} \in T_\xi. \tag{12.5.36}$$

Corollary 12.5.37. *In the situation of* (12.5.35)*, suppose that* $(X, \underline{M})$ *is regular at* ξ*. Then :*

(i) *The log scheme* $(X(\mathfrak{p}), \underline{M}(\mathfrak{p}))$ *is pointed regular, for every* $\mathfrak{p} \in T_\xi$*.*

(ii) *The scheme* $X(\mathfrak{p})$ *is irreducible, and its codimension in* X *equals the height of* $\mathfrak{p}$ *in* T_ξ*, for every* $\mathfrak{p} \in T_\xi$*.*

(iii) *The scheme* $U_\mathfrak{p}$ *is regular and irreducible, for every* $\mathfrak{p} \in T_\xi$*.*

Proof. (i): By theorem 12.5.31, it suffices to show that $(X(\mathfrak{p}), \underline{M}(\mathfrak{p}))$ is pointed regular at ξ, for every $\mathfrak{p} \in T_\xi$. However, say that $X(\xi) = \mathrm{Spec}\, A$, and let $P \to \underline{M}(\xi)$ be a fine and saturated chart, sharp at the τ-point ξ. Then $X(\mathfrak{p}) = \mathrm{Spec}\, A_0$, where $A_0 := A/\mathfrak{p}A$, and $\underline{M}(\mathfrak{p}) = \underline{N}_\circ$, where $\underline{N}$ is the fs log structure deduced from the induced map $\beta : P\backslash\mathfrak{p} \to A_0$. By proposition 12.5.16(ii), the ring A is P-flat; then A_0 is β-flat, and the assertion follows from proposition 12.5.16(i).

Next, (ii) is a rephrasing of corollary 12.5.20(i), and (iii) follows from (i),(ii) and (12.5.36), by virtue of corollary 12.5.19. □

Proposition 12.5.38. *Let* $(X, \beta : \underline{M} \to \mathscr{O}_X)$ *be a regular log scheme, set* $U := (X, \underline{M})_{\mathrm{tr}}$*, and denote by* $j : U \to X$ *the open immersion. Then the morphism* β *induces identifications:*

$$\underline{M} \xrightarrow{\sim} j_*\mathscr{O}_U^\times \cap \mathscr{O}_X \qquad \underline{M}^{\mathrm{gp}} \xrightarrow{\sim} j_*\mathscr{O}_U^\times.$$

Proof. Notice that the scheme X is normal (corollary 12.5.13), hence both $j_*\mathscr{O}_U^\times$ and $\mathscr{O}_X$ are subsheaves of the sheaf $i_*\mathscr{O}_{X_0}$, where X_0 is the subscheme of maximal points of X, and $i : X_0 \to X$ is the natural morphism; so we may intersect these two sheaves inside the latter.

In view of lemma 12.5.2 and proposition 12.5.16(ii), we know already that β is injective, and clearly the image of β lands in $j_*\mathscr{O}_U^\times$, so it remains only to show that β (resp. β^{gp}) induces an epimorphism onto $j_*\mathscr{O}_U^\times \cap \mathscr{O}_X$ (resp. onto $j_*\mathscr{O}_U^\times$). The assertions can be checked on the stalks, hence let ξ be any τ-point of X; to begin with, we show :

Claim 12.5.39. The induced map $\beta_\xi^{\mathrm{gp}} : \underline{M}_\xi^{\mathrm{gp}} \to (j_*\mathscr{O}_U^\times)_\xi$ is a surjection.

Proof of the claim. Set $A := \mathscr{O}_{X,\xi}$; notice that $(X(\xi), \underline{M}(\xi))_{\mathrm{tr}}$ is the complement of the union of the finitely many closed subsets of the form $Z_\mathfrak{p} := \mathrm{Spec}\, A/\mathfrak{p}A$, where $\mathfrak{p} \subset \underline{M}_\xi$ runs over the prime ideals of height one. Due to corollary 12.5.20(i), each $Z_\mathfrak{p}$ is an irreducible divisor in $\mathrm{Spec}\, A$. Now, let $s \in (j_*\mathscr{O}_U^\times)_\xi$; then the divisor of s is of the form $\sum_{\mathrm{ht}\mathfrak{p}=1} n_\mathfrak{p}[\mathfrak{p}]$, for some $n_\mathfrak{p} \in \mathbb{Z}$. By lemmata 12.1.18(ii) and 12.2.21(i), $\underline{M}_\xi^\sharp$ is a fine and saturated monoid; then lemma 6.4.39 and proposition 6.4.50 say that the fractional ideal $I := \bigcap_{\mathrm{ht}\,\mathfrak{p}=1} \mathfrak{m}_\mathfrak{p}^{n_\mathfrak{p}}$ of $\underline{M}_\xi$ is reflexive, and therefore the fractional ideal IA of A is reflexive as well (lemma 6.4.45(ii)). Since A is normal, by considering the localizations of $I\mathscr{O}_{X,\xi}$ at the prime ideals of height one of A, we deduce easily that $IA = sA$ ([34, Ch.VII, §4, n.2, Cor. du Th.2]). Thus, we may write $s = \sum_{i=1}^n \beta_\xi^{\mathrm{gp}}(x_i)a_i$ for certain $a_i \in A$ and $x_i \in I$ (for $i = 1, \dots, n$). Then we must have $\beta_\xi^{\mathrm{gp}}(x_i)a_i \notin s \cdot \mathfrak{m}_\xi$ for at least one index $i \leq n$ (where $\mathfrak{m}_\xi \subset A$ is the maximal ideal); for such i, it follows that $s^{-1} \cdot \beta_\xi^{\mathrm{gp}}(x_i) \in A^\times$, whence $s \in \underline{M}_\xi^{\mathrm{gp}}$, as required. ◊

Now, let $s \in (j_*\mathscr{O}_U^\times)_\xi \cap A$; by claim 12.5.39, we may find $x \in \underline{M}_\xi^{\mathrm{gp}}$ such that $\beta_\xi^{\mathrm{gp}}(x) = s$. To conclude, it suffices to show that $x \in \underline{M}_\xi$. By theorem 6.4.18(i), we are reduced to showing that $x \in (\underline{M}_\xi)_\mathfrak{p}$ for every prime ideal $\mathfrak{p} \subset \underline{M}_\xi$ of height one. Set $\mathfrak{q} := \mathfrak{p}A$; then $\mathfrak{q}$ is a prime ideal of height one (corollary 12.5.20(i)), and β_ξ extends to a well defined map of monoids $\beta_\mathfrak{p} : (\underline{M}_\xi)_\mathfrak{p} \to A_\mathfrak{q}$. Furthermore, $A_\mathfrak{q}$ is a discrete valuation ring (corollary 12.5.13), whose valuation we denote $v : A_\mathfrak{q} \setminus \{0\} \to \mathbb{N}$. In view of theorem 6.4.18(ii), we see that $x \in (\underline{M}_\xi)_\mathfrak{p}$ if and only if $(v \circ \beta_\mathfrak{p})^{\mathrm{gp}}(x) \geq 0$. However, clearly $v(s) \geq 0$, whence the contention. □

Remark 12.5.40. Let $\mathbf{reg.log}_\tau$ denote the full subcategory of $\mathbf{log}_\tau$ whose objects are the regular log schemes. As an immediate consequence of proposition 12.5.38 (and of remark 12.2.8(i)) we see that the forgetful functor F of (12.2.1) restricts to a fully faithful functor

$$\mathbf{reg.log}_\tau \to \mathbf{Open} \qquad (X, \underline{M}) \mapsto ((X, \underline{M})_{\mathrm{tr}} \to X)$$

where $\mathbf{Open}$ is full subcategory of $\mathsf{Morph}(\mathsf{Sch})$ whose objects are the open immersions.

12.5.41. Let $(X, \underline{M})$ be a quasi-compact and regular log scheme. Set $Z := X \setminus (X, \underline{M})_{\mathrm{tr}}$; then Z is a closed subset of X (lemma 12.2.21(ii)), and we endow it with its reduced subscheme structure. Denote by $\omega_X \subset \mathscr{O}_X$ the sheaf of ideals corresponding to Z. We have :

Theorem 12.5.42. *In the situation of* (12.5.41)*, the complex* $\omega_X[0]$ *is dualizing on* X*. (See definition* 11.3.16.*)*

Proof. In light of proposition 11.3.37, we may assume that X is local, say $X = \operatorname{Spec} A$, for a local noetherian ring A. Next, pick any τ-point ξ localized at the closed point of X; by virtue of proposition 11.3.34 and [66, Ch.IV, Prop.18.8.8(iv) and Prop.18.8.12(ii)], we may further reduce to the case where $(X, \underline{M}) = (X(\xi), \underline{M}(\xi))$. Let $X^\wedge := \operatorname{Spec} A^\wedge$, where $A^\wedge$ denotes the completion of the local ring A, and denote by $f : X^\wedge \to X$ the natural morphism; by applying again proposition 11.3.34, we reduce to checking that $f^*\omega_X[0]$ is dualizing on $X^\wedge$. However, theorem 12.5.10 says that there exist a local ring homomorphism $R \to A^\wedge$, with R a complete regular local ring, and a fine and saturated chart $P_X \to \underline{M}$, sharp at the closed point of X, such that the induced continuous ring homomorphism $\varphi : R[[P]] \to A^\wedge$ is an isomorphism if A contains a field, and otherwise it is a surjection, whose kernel is generated by a regular element $\vartheta \in R[[P]]$. With this notation, a simple inspection shows that Z is the union of the closed subsets $X(\mathfrak{p}) \subset X$, where $\mathfrak{p} \subset P$ ranges over all prime ideals $\mathfrak{p} \neq \varnothing$ (notation of (12.5.35)). Then, corollary 12.5.20(i) says that ω_X is the intersection of the ideals of $\mathscr{O}_X$ of the form $\mathfrak{p}\mathscr{O}_X$, where $\mathfrak{p} \subset P$ is an arbitrary non-empty prime ideal. Denote by $P^\circ \subset P$ the intersection of all the (finitely many) non-empty prime ideals of P; lemma 6.1.37 and proposition 12.5.16(ii) imply that $\omega_X = P^\circ \cdot \mathscr{O}_X$ (details left to the reader), and since f is a flat morphism, we deduce that $f^*\omega_X = P^\circ \cdot \mathscr{O}_{X^\wedge}$. The ideal P° can also be described as follows. Set $V := P^{\mathrm{gp}}_\mathbb{R}$, denote by σ the unique strictly convex polyhedral cone such that $P = \sigma \cap P^{\mathrm{gp}}$, and let σ° be the topological interior of σ in V; then it is easily seen that $P^\circ = P \cap \sigma^\circ$.

Set $Y := \operatorname{Spec} R[P]$, $\omega_Y := P^\circ \cdot \mathscr{O}_Y$, and let $y \in Y$ be the image of the closed point of $Y^\wedge := \operatorname{Spec} R[[P]]$, under the natural morphism $g : Y^\wedge \to Y$. Theorem 11.6.43(ii) says that $\omega_Y[0]$ is dualizing on Y; then, since g is flat, propositions 11.3.34 and 11.3.37 imply that $g^*\omega_Y[0]$ is dualizing on $Y^\wedge$.

Now, suppose that A contains a field; then φ induces an isomorphism $X^\wedge \xrightarrow{\sim} Y^\wedge$ which identifies $f^*\omega_X$ with $g^*\omega_Y$, whence the contention.

Lastly, if A does not contain a field, φ induces a regular closed immersion $i : X^\wedge \to Y^\wedge$, so $i^!(g^*\omega_Y[0])$ is dualizing on $X^\wedge$ (lemma 11.3.27(i)). The latter is the complex of $\mathscr{O}_{X^\wedge}$-modules arising from the complex of $A^\wedge$-modules $L^\bullet := R\mathrm{Hom}^\bullet_{R[[P]]}(A^\wedge[0], R[[P^\circ]])$ (corollary 10.3.2(ii); here $R[[P^\circ]] := P^\circ \cdot R[[P]]$). However, $A^\wedge[0]$ is naturally isomorphic (in $\mathsf{D}(R[[P]]\text{-}\mathbf{Mod})$) to the Koszul complex $\mathbf{K}_\bullet(\vartheta)$ (notation of remark 7.8.1(ii) : this is a bounded complex of free $R[[P]]$-modules), so we see that

$$L^\bullet \simeq (R[[P^\circ]] \otimes_{R[[P]]} A^\wedge)[-1].$$

However, we have natural $A^\wedge$-linear isomorphisms :

$$\begin{aligned} R[[P^\circ]] \otimes_{R[[P]]} A^\wedge &\stackrel{\sim}{\to} R[P^\circ] \otimes_{R[P]} A^\wedge && \text{(since } g \text{ is flat)} \\ &\stackrel{\sim}{\to} \mathbb{Z}[P^\circ] \otimes_{\mathbb{Z}[P]} A \otimes_A A^\wedge \\ &\stackrel{\sim}{\to} (P^\circ \cdot A) \otimes_A A^\wedge && \text{(by proposition 12.5.16(ii))} \end{aligned}$$

which shows that $i^!(g^*\omega_Y[0]) \simeq f^*\omega_X[-1]$, and concludes the proof of the theorem. □

12.6. **Resolution of singularities of regular log schemes.** Most of this section concerns results that are special to the class of log schemes over the Zariski topology; these are then applied to étale fs log structures, after we have shown that every such log structure admits a logarithmic blow up which descends to the Zariski topology (proposition 12.6.52). The same statement – with an unnecessary restriction to regular log schemes – can be found in the article [135] by W.Niziol : see theorem 5.6 of *loc.cit.* We mainly follow her treatment, except for fleshing out some details, and correcting some inaccuracies.

Therefore, *we let here* $\tau = \mathrm{Zar}$, *and all log structures considered in this section until* (12.6.48) *are defined on the Zariski sites of their underlying schemes.*

12.6.1. Let $f : (Y, \underline{N}) \to (X, \underline{M})$ be any morphism of log schemes; we remark that f is a morphism of monoidal spaces, *i.e.* for every $y \in Y$, the map $\underline{M}_{f(y)} \to \underline{N}_y$ induced by $\log f$, is local. Indeed, it has been remarked in (12.1.6) that the natural map $\underline{M}_{f(y)} \to (f^*\underline{M})_y$ is local, and on the other hand, a section $s \in (f^*\underline{M})_y$ is invertible if and only if its image in $\mathscr{O}_{Y,y}$ is invertible, if and only if $\log f_y(s)$ is invertible in $\underline{N}_y$, whence the contention. We shall denote

$$f^\sharp : (Y, \underline{N})^\sharp \to (X, \underline{M})^\sharp$$

the morphism of sharp monoidal spaces induced by f in the obvious way (*i.e.* the underlying continuous map is the same as the continuous map underlying f, and $\log f^\sharp : f^*\underline{M}^\sharp \to \underline{N}^\sharp$ is $(\log f)^\sharp$: see definition 6.5.1).

12.6.2. We let $\mathscr{K}$ be the category whose objects are all data of the form $\underline{X} := ((X, \underline{M}), F, \psi)$, where $(X, \underline{M})$ is a log scheme, F is a fan, and $\psi : (X, \underline{M})^\sharp \to F$ is a morphism of sharp monoidal spaces, such that $\log \psi : \psi^*\mathscr{O}_F \to \underline{M}^\sharp$ is an isomorphism. The morphisms :

$$((Y, \underline{N}), F', \psi') \to \underline{X}$$

in $\mathscr{K}$ are all the pairs (f, φ), where $f : (Y, \underline{N}) \to (X, \underline{M})$ is a morphism of log schemes, and $\varphi : F' \to F$ is a morphism of fans, such that the diagram

$$\begin{array}{ccc} (Y, \underline{N})^\sharp & \xrightarrow{f^\sharp} & (X, \underline{M})^\sharp \\ {\scriptstyle \psi'}\downarrow & & \downarrow{\scriptstyle \psi} \\ F' & \xrightarrow{\varphi} & F \end{array}$$

commutes. Especially, notice that lemma 12.1.4 implies the identity :

$$\mathrm{Str}(f) = \psi'^{-1}\mathrm{Str}(\varphi). \qquad (12.6.3)$$

(notation of definitions 6.5.1(ii) and 12.2.7(ii)). We shall say that an object $\underline{X}$ is *locally noetherian*, if the same holds for the scheme X. Likewise, a morphism (f, φ) in $\mathscr{K}$ is *quasi-compact*, (resp. *quasi-separated*, resp. *separated*, resp. *locally of finite type*, resp. *of finite type*) if the same holds for the morphism of schemes underlying f. We say that f is *étale*, if the same holds for the morphism of log schemes underlying f. Also, the *trivial locus* of $\underline{X}$ is defined as the subset $(X, \underline{M})_{\mathrm{tr}}$ of X.

12.6.4. There is an obvious (forgetful) functor :

$$F : \mathscr{K} \to \mathsf{Sch} \qquad ((X, \underline{M}), F, \psi) \mapsto X$$

which is a fibration. More precisely, every morphism of schemes $S' \to S$ induces a *base change* functor (notation of (1.1.27)) :

$$F\mathscr{K}/S \to F\mathscr{K}/S' \qquad \underline{X} \mapsto S' \times_S \underline{X}$$

unique up to natural isomorphism of functors. Namely, for $\underline{X} := ((X, \underline{M}), F, \psi)$ one lets

$$S' \times_S \underline{X} := (S' \times_S (X, \underline{M}), F, \psi')$$

where $\psi' := \psi \circ \pi^\sharp$, and $\pi : S' \times_S (X, \underline{M}) \to (X, \underline{M})$ is the natural projection.

Example 12.6.5. (i) Let P be a monoid, R a ring, and set $(S, P_S^{\log}) := \mathrm{Spec}(R, P)$ (see (12.2.13)). The unit of adjunction $\varepsilon_P : P \to R[P]$ determines a unique morphism of sharp monoidal spaces

$$\psi_P : \mathrm{Spec}(R, P)^\sharp \to T_P := (\mathrm{Spec}\, P)^\sharp$$

(proposition 6.5.6), and we claim that $\underline{S} := (\mathrm{Spec}(R, P), T_P, \psi_P)$ is an object of $\mathscr{K}$.

The assertion can be checked on the stalks, hence let ξ be a point of S; then ξ corresponds to a prime ideal $\mathfrak{p}_\xi \subset R[P]$, and by inspecting the definitions, we see that $\psi_P(\xi) = \varepsilon_P^{-1}(\mathfrak{p}_\xi) = \mathfrak{q}_\xi := P \cap \mathfrak{p}_\xi \in \mathrm{Spec}\, P$. Again, a direct inspection shows that the morphism

$$(\log \psi_P)_\xi : \mathscr{O}_{T_P, \mathfrak{q}_\xi} \to (P_S^{\log})^\sharp_\xi$$

is none else than the natural isomorphism $P^\sharp_{\mathfrak{q}_\xi} \xrightarrow{\sim} P^{\log}_{S,\xi}/\mathscr{O}^\times_{S,\xi}$ deduced from ε_P and the natural identifications

$$P^{\log}_{S,\xi} = \mathscr{O}^\times_{S,\xi} \otimes_{P \setminus \mathfrak{q}_\xi} P = \mathscr{O}^\times_{S,\xi} \otimes_{P^\times_{\mathfrak{q}_\xi}} P_{\mathfrak{q}_\xi}.$$

(ii) The construction of $\underline{S}$ is clearly functorial in P. Namely, say that $\lambda : P \to Q$ is a morphism of monoids, and set $S' := \mathrm{Spec}\, R[Q]$, $T_Q := (\mathrm{Spec}\, Q)^\sharp$. There follows a morphism

$$\underline{S}' := (\mathrm{Spec}(R, Q), T_Q, \psi_Q) \to \underline{S} \qquad (12.6.6)$$

in $\mathscr{K}$, whose underlying morphism of log schemes is $\mathrm{Spec}(R, \lambda)$, and whose underlying morphism of fans is just $(\mathrm{Spec}\, \lambda)^\sharp$.

12.6.7. Example 12.6.5 can be globalized to more general log schemes, at least under some additional assumptions. Namely, let $(X, \underline{M})$ be a regular log scheme. For every point x of X, let $\mathfrak{m}_x \subset \mathscr{O}_{X,x}$ be the maximal ideal; we set :

$$F(X) := \{x \in X \mid I(x, \underline{M}) = \mathfrak{m}_x\}$$

(notation of (12.5.5)), and we endow $F(X)$ with the topology induced from X. The *fan of* $(X, \underline{M})$ is the sharp monoidal space

$$F(X, \underline{M}) := (F(X), \underline{M}_{|F(X)})^\sharp.$$

We wish to show that $F(X, \underline{M})$ is indeed a fan. Let $U \subset X$ be any open subset; to begin with, it is clear that $F(U, \underline{M}_{|U})$ is naturally an open subset of $F(X, \underline{M})$; hence the contention is local on X, so we may assume that X is affine, say $X = \mathrm{Spec}\, A$ for a noetherian ring A, and that $\underline{M}$ admits a finite chart $P_X \to \underline{M}$; denote by $\beta : P \to A$ the induced morphism of monoids. Let now ξ be a point of X, and $\mathfrak{p}_\xi \subset A$ the corresponding prime ideal; set $\mathfrak{q}_\xi := \beta^{-1}\mathfrak{p}_\xi \in \mathrm{Spec}\, P$. By inspecting the definitions, it is easily seen that $I(\xi, \underline{M}) = \mathfrak{q}_\xi A_{\mathfrak{p}_\xi}$. Therefore, for any prime ideal $\mathfrak{q} \subset P$, let $V(\mathfrak{q})_{\max}$ be the finite set consisting of all the maximal points of the closed subset $V(\mathfrak{q}) := \mathrm{Spec}\, A/\mathfrak{q}A$ (*i.e.* the minimal prime ideals of $A/\mathfrak{q}A$); in light of corollary 12.5.20(i), it follows easily that

$$F(X) = \bigcup_{\mathfrak{q}\in\mathrm{Spec}\, P} V(\mathfrak{q})_{\max} \tag{12.6.8}$$

especially, $F(X)$ is a finite set (lemma 6.1.20(iii)). Let $t \in F(X)$ be any element, and denote by $U(t) \subset F(X)$ the subset of all generizations of t in $F(X)$; as a corollary, we see that $U(t)$ is an open subset of $F(X)$. Moreover, (12.6.8) also implies that :

$$U(t) = \bigcup_{\mathfrak{q}\in\mathrm{Spec}\, P} (\mathrm{Spec}\, \mathscr{O}_{X,t}/\mathfrak{q}\mathscr{O}_{X,t})_{\max}.$$

However, corollary 12.5.20(i) says that $\mathrm{Spec}\, \mathscr{O}_{X,t}/\mathfrak{q}\mathscr{O}_{X,t}$ is irreducible for every $\mathfrak{q} \in \mathrm{Spec}\, P$, therefore the set $U(t)$ is naturally identified with a subset of $\mathrm{Spec}\, P$. Furthermore, arguing as in example 12.6.5(i) we find a natural isomorphism $P^\sharp_{\mathfrak{q}_t} \xrightarrow{\sim} \mathscr{O}_{F(X,\underline{M}),t}$. Moreover, if $t' \in U(t)$ is any other point, the specialization map $\mathscr{O}_{F(X,\underline{M}),t} \to \mathscr{O}_{F(X,\underline{M}),t'}$ corresponds – under the above isomorphism – to the natural morphism $P^\sharp_{\mathfrak{q}_t} \to P^\sharp_{\mathfrak{q}'_t}$ induced by the localization map $P_{\mathfrak{q}_t} \to P_{\mathfrak{q}'_t}$. This shows that the open monoidal subspace $(U(t), \underline{M}_{|U(t)})$ is naturally isomorphic to $(\mathrm{Spec}\, P_{\mathfrak{q}_t})^\sharp$, hence $F(X, \underline{M})$ is a fan, as stated.

Remark 12.6.9. (i) The discussion in (12.6.7) shows more precisely that if :

- (a) $(U, \underline{M})$ is a log scheme with $U = \mathrm{Spec}\, A$ affine,
- (b) there exists a morphism $\beta : P \to A$ from a finitely generated monoid P, inducing the log structure $\underline{M}$ on U, and
- (c) $\beta(\mathfrak{q})A$ is a prime ideal for every $\mathfrak{q} \in \mathrm{Spec}\, P$

then $F(U, \underline{M})$ is naturally identified with $(\mathrm{Spec}\, P)^\sharp$. In this situation, denote by

$$f_\beta : (U, \underline{M})^\sharp \to T_P := (\mathrm{Spec}\, P)^\sharp$$

the morphism of sharp monoidal spaces deduced from β (proposition 6.5.6); by inspecting the definitions, we see that the associated map $\log f_\beta : f^*_\beta \mathscr{O}_{T_P} \to \underline{M}^\sharp$ is an isomorphism. Via the foregoing natural identification, there results a morphism $\pi_U : (U, \underline{M})^\sharp \to F(U, \underline{M})$ which can be described without reference to P. Indeed, let $x \in U$ be any point, and $\mathfrak{p} \in \mathrm{Spec}\, A$ the corresponding prime ideal; by inspecting the definitions we find that

$$\pi_U(x) = I(x, \underline{M}) \ \text{ which is the largest prime ideal in } \mathrm{Spec}\, A_\mathfrak{p} \cap F(U, \underline{M}).$$

Especially, $\pi_U(x)$ is a generization of x, and the inverse of $(\log \pi_U)_x : \mathscr{O}_{F(U,\underline{M}),\pi_U(x)} \xrightarrow{\sim} \underline{M}^\sharp_x$ is induced by the specialization map $\underline{M}_x \to \underline{M}_{\pi_U(x)}$ (which induces an isomorphism on the associated sharp quotient monoids).

(ii) Finally, it follows easily from corollary 12.5.20(i) and [65, Ch.IV, Cor.8.4.3], that every regular log scheme $(X, \underline{M})$ admits an affine open covering $X = \bigcup_{i\in I} U_i$, such that each $(U_i, \underline{M}_{|U_i})$ fulfills conditions (a)–(c) above, and the intrinsic description in (i) shows that the morphisms π_{U_i} glue to a well defined morphism $\pi_X : (X, \underline{M})^\sharp \to F(X, \underline{M})$ of sharp monoidal spaces, such that the datum

$$\mathscr{K}(X, \underline{M}) := ((X, \underline{M}), F(X, \underline{M}), \pi_X)$$

is an object of $\mathscr{K}$.

(iii) Notice that $\pi_X^{-1}(t)$ is an irreducible locally closed subset of X of codimension equal to the height of t, for every $t \in F(X, \underline{M})$; also $\pi_X^{-1}(t)$ is a regular scheme, for its reduced subscheme structure. Indeed, we have already observed that t is the unique maximal point of $\pi_X^{-1}(t)$, and then the assertion follows immediately from corollary 12.5.37(ii,iii). Furthermore, the inclusion map $j : F(X, \underline{M}) \to X$ is a continuous section of π_X, and notice that $j \circ \pi_X(U) =$

$j^{-1}U$ for every open subset $U \subset X$, since j maps every $t \in F(X, \underline{M})$ to the unique maximal point of $\pi_X^{-1}(t)$. It follows easily that π_X is an open map.

12.6.10. Denote by $\mathscr{K}_{\rm int}$ the full subcategory of $\mathscr{K}$ whose objects are the data $((X, \underline{M}), F, \psi)$ such that $\underline{M}$ is an integral log structure, and F is an integral fan. There is an obvious functor

$$\mathscr{K}_{\rm int} \to \mathbf{int.Fan} \quad : \quad ((X, \underline{M}), F, \psi) \mapsto F$$

to the category of integral fans, which shall be used to construct useful morphisms of log schemes, starting from given morphisms of fans. This technique rests on the following three results :

Lemma 12.6.11. *Let $((X, \underline{M}), F, \psi)$ be an object of $\mathscr{K}_{\rm int}$, with F locally fine. Then, for each point $x \in X$ there exist an open neighborhood $U \subset X$ of x, a fine chart $Q_U \to \underline{M}_{|U}$ (for some fine monoid Q depending on x), and an isomorphism of monoids $Q^\sharp \stackrel{\sim}{\to} \mathscr{O}_{F,\psi(x)}$.*

Proof. The assertion is local on X, hence we may assume that $F = (\mathrm{Spec}\, P)^\sharp$ for a fine monoid P. In this case, ψ is determined by the corresponding map

$$\overline{\beta} : P_X \to \underline{M}^\sharp.$$

Indeed, for any $x \in X$, the point $\psi(x)$ is the prime ideal $\overline{\beta}_x^{-1}(\mathfrak{m}_x) \subset P$, where $\mathfrak{m}_x \subset \underline{M}_x^\sharp$ is the maximal ideal; moreover :

$$\mathscr{O}_{F,\psi(x)} = P/\overline{S}_x \qquad \text{where} \qquad \overline{S}_x := \overline{\beta}_x^{-1}(1)$$

and – under this identification – the isomorphism $\log \psi_x : \mathscr{O}_{F,\psi(x)} \stackrel{\sim}{\to} \underline{M}_x^\sharp$ is deduced from $\overline{\beta}_x$ in the obvious way.

Now, let $x \in X$ be any point; after replacing X by the open subset $\psi^{-1}U(\psi(x))$, we may assume that $P = \mathscr{O}_{F,\psi(x)}$ (notation of (6.5.16)), and by assumption $\log \psi_x : P \to \underline{M}_x^\sharp$ is an isomorphism. Pick a surjection $\alpha : \mathbb{Z}^{\oplus r} \to P^{\rm gp}$, and let Q be the pull-back in the cartesian diagram :

$$\begin{array}{ccc} Q & \longrightarrow & \mathbb{Z}^{\oplus r} \\ \downarrow & & \downarrow {\scriptstyle (\log \psi_x)^{\rm gp} \circ \alpha} \\ \underline{M}_x^\sharp & \longrightarrow & \underline{M}_x^{\rm gp}/\underline{M}_x^\times. \end{array}$$

By construction, α restricts to a morphism of monoids $\vartheta : Q \to P$, inducing an isomorphism $Q^\sharp \stackrel{\sim}{\to} P$, whence an isomorphisms of fans $(\mathrm{Spec}\, P)^\sharp \stackrel{\sim}{\to} (\mathrm{Spec}\, Q)^\sharp$, and Q is fine, by corollary 6.4.2. Moreover, the choice of a lifting $\mathbb{Z}^{\oplus r} \to \underline{M}_x^{\rm gp}$ of $(\log \psi_x)^{\rm gp} \circ \alpha$ determines a morphism

$$\beta_x : Q \to \underline{M}_x^\sharp \times_{\underline{M}_x^{\rm gp}/\underline{M}_x^\times} \underline{M}_x^{\rm gp} = \underline{M}_x$$

which lifts $\log \psi_x \circ \vartheta$ (here it is needed that $\underline{M}$ is integral). Next, after replacing X by an open neighborhood of x, we may assume that β_x extends to a map of pre-log structures $\beta : Q_X \to \underline{M}$ lifting $\overline{\beta}$ (lemma 12.1.18(iv.b),(v)). It remains only to show that β is a chart for $\underline{M}$, which can be checked on the stalks. Thus, let $y \in X$ be any point, and set $S_y := \beta_y^{-1}\underline{M}_y^\times$; with the foregoing notation, the stalk $Q_{X,y}^{\log}$ of the induced log structure is naturally isomorphic to $(S_y^{-1}Q \times \mathscr{O}_{X,y}^\times)/S_y^{\rm gp}$, therefore

$$(Q_{X,y}^{\log})^\sharp \simeq Q/S_y \simeq P/\overline{S}_y$$

and the induced map $P/\overline{S}_y \to \underline{M}_y^\sharp$ is again deduced from $\overline{\beta}_y$, so it is an isomorphism; then the same holds for $\beta_y^{\log} : Q_{X,y}^{\log} \to \underline{M}_y$ (lemma 12.1.4). □

Lemma 12.6.12. *Let $\mu : P \to P'$ be a morphism of integral monoids such that μ^{gp} is surjective, $\varphi : (\operatorname{Spec} P')^\sharp \to (\operatorname{Spec} P)^\sharp$ the induced morphism of affine fans, and denote by*

$$\lambda : P \to Q := P^{\mathrm{gp}} \times_{P'^{\mathrm{gp}}} P'$$

the map of monoids determined by μ and the unit of adjunction $P \to P^{\mathrm{gp}}$. Then :

(i) *The natural projection $Q \to P'$ induces an isomorphism $\omega : (\operatorname{Spec} P')^\sharp \to (\operatorname{Spec} Q)^\sharp$ of affine fans, such that $(\operatorname{Spec} \lambda)^\sharp \circ \omega = \varphi$.*
(ii) *The induced morphism (12.6.6) in $\mathscr{K}_{\mathrm{int}}$ (with $R := \mathbb{Z}$) is $\mathbf{int}.\mathbf{Fan}$-cartesian.*
(iii) *If P and P' are fine, the morphism (12.6.6) is étale.*

Proof. (See (3.1) for generalities concerning inverse images and cartesian morphisms relative to a functor.) Notice first that, since μ^{gp} is surjective, the projection $Q \to P'$ induces an isomorphism $Q^\sharp \xrightarrow{\sim} P'^\sharp$, whence (i).

(ii): Notice that the log structure of $\operatorname{Spec}(\mathbb{Z}, Q)$ is integral, by lemma 12.1.18(iii). Next, set

$$F' := (\operatorname{Spec} P')^\sharp.$$

Define $\underline{S}$, $\underline{S}'$ as in example 12.6.5(ii) (with $R := \mathbb{Z}$), let $g : ((Y, \underline{N}), F'', \psi_Y) \to \underline{S}$ be any morphism of $\mathscr{K}_{\mathrm{int}}$, and $\varphi' : F'' \to F'$ a morphism of integral fans, such that the image of g in $\mathbf{int}.\mathbf{Fan}$ equals $\varphi \circ \varphi'$; we must show that g factors through a unique morphism $h : ((Y, \underline{N}), F'', \psi_Y) \to \underline{S}'$, whose image in $\mathbf{int}.\mathbf{Fan}$ equals φ'. As usual, we may reduce to the case where $Y = \operatorname{Spec} A$ is affine, $F'' = \operatorname{Spec} P''$ is an affine fan for a sharp integral monoid P'', and ψ_Y is given by a map of sheaves $P''_Y \to \underline{N}^\sharp$. In such situation, the morphism $(Y, \underline{N}) \to \operatorname{Spec}(\mathbb{Z}, P)$ underlying g is determined by a morphism of monoids $P \to \underline{N}(Y)$, or which is the same, a map of sheaves $\gamma_Y : P_Y \to \underline{N}$; likewise, φ' is given by a morphism of monoids $P' \to P''$, and composing with ψ_Y, we get a map of sheaves $\alpha_Y : P'_Y \to \underline{N}^\sharp$. Finally, the condition that g lies over $\varphi \circ \varphi'$ translates as the commutativity of the following diagram of sheaves :

$$\begin{array}{ccccc} P_Y^{\mathrm{gp}} & \longrightarrow & P_Y'^{\mathrm{gp}} & \longleftarrow & P'_Y \\ {\scriptstyle \gamma_Y^{\mathrm{gp}}}\downarrow & & \downarrow & & \downarrow{\scriptstyle \alpha_Y} \\ \underline{N}^{\mathrm{gp}} & \longrightarrow & \underline{N}^{\mathrm{gp}}/\underline{N}^\times & \longleftarrow & \underline{N}^\sharp. \end{array}$$

There follows a unique morphism of sheaves :

$$Q_Y \to \underline{N}^{\mathrm{gp}} \times_{\underline{N}^{\mathrm{gp}}/\underline{N}^\times} \underline{N}^\sharp = \underline{N} \tag{12.6.13}$$

(here it is needed that $\underline{N}$ is integral) such that the diagram :

$$\begin{array}{ccccc} P_Y & \longrightarrow & Q_Y & \longrightarrow & P'_Y \\ {\scriptstyle \gamma_Y}\downarrow & & \downarrow & & \downarrow{\scriptstyle \alpha_Y} \\ \underline{N} & = & \underline{N} & \longrightarrow & \underline{N}^\sharp \end{array}$$

commutes. Then (12.6.13) determines a morphism of log schemes $h_Y : (Y, \underline{N}) \to \operatorname{Spec}(\mathbb{Z}, Q)$, such that the pair (h_Y, φ') is the unique morphism h in $\mathscr{K}_{\mathrm{int}}$ with the sought properties.

(iii): If both P and P' are fine, so is Q (corollary 6.4.2), and by construction, λ^{gp} is an isomorphism. Then the assertion follows from theorem 12.3.37. □

Proposition 12.6.14. *Let $\underline{X} := ((X, \underline{M}), F, \psi)$ be an object of $\mathscr{K}_{\mathrm{int}}$. Let also $\varphi : F' \to F$ be an integral partial subdivision, with F locally fine and F' integral. We have :*

(i) *$(X, \underline{M})$ is a fine log scheme.*
(ii) *If F is saturated, $(X, \underline{M})$ is a fs log scheme.*
(iii) *$\underline{X}$ admits an inverse image over φ (relative to the functor of (12.6.10)).*
(iv) *If φ is finite, then the cartesian morphism $(f, \varphi) : \varphi^* \underline{X} \to \underline{X}$, is quasi-compact.*

(v) *If F' is locally fine, the morphism (f, φ) is quasi-separated and étale.*

Proof. (i): This is just a restatement of lemma 12.6.11.

(ii): In light of (i), we only have to show that $\underline{M}_x$ is a saturated monoid, for every $x \in X$. Since by assumption $\underline{M}^\sharp_x$ is saturated, the assertion follows from lemma 6.2.9(ii).

Claim 12.6.15. In order to show (iii)–(v), we may assume that :

(a) $F = (\operatorname{Spec} P)^\sharp$ for a fine monoid P, and X is an affine scheme.
(b) The map of global sections $P \to \Gamma(X, \underline{M}^\sharp)$ determined by ψ, comes from a morphism of pre-log structures $\beta : P_X \to \underline{M}$ which is a fine chart for $\underline{M}$.

Proof of the claim. To begin with, suppose that $((X', \underline{M}'), F', \psi')$ is the sought preimage of $\underline{X}$; let $U \subset X$ be any open subset, and $V \subset F$ an open subset such that $\psi(U) \subset V$. Then it is easily seen that the object

$$\varphi^* \underline{X} \times_{\underline{X}} (U, V) := ((f^{-1}U, \underline{M}'_{|f^{-1}U}), \varphi^{-1}V, \psi'_{|f^{-1}U})$$

is a preimage of $((U, \underline{M}_{|U}), V, \psi_{|U})$ over the restriction $\varphi^{-1}V \to V$ of φ. Now, suppose that we have found an affine open covering $X = \bigcup_{i \in I} U_i$, and for every $i \in I$ an affine open subset $V_i \subset F$ with $\psi(U_i) \subset V_i$, such that the object $\underline{U}_i := ((U_i, \underline{M}_{|U_i}), V_i, \psi_{|U_i})$ admits a preimage over the restriction $\varphi_i : \varphi^{-1}V_i \to V_i$ of φ; then the foregoing implies that there are natural isomorphisms:

$$\varphi_i^* \underline{U}_i \times_{\underline{U}_i} (U_{ij}, V_{ij}) \xrightarrow{\sim} \varphi_j^* \underline{U}_j \times_{\underline{U}_j} (U_{ij}, V_{ij})$$

for every $i, j \in I$, where $U_{ij} := U_i \cap U_j$ and $V_{ij} := V_i \cap V_j$. Thus, we may glue all these inverse images along these isomorphisms, to obtain the sought inverse image of $\underline{X}$.

Moreover, if φ is finite, the same will hold for the restrictions φ_i, and if each cartesian morphism $\varphi_i^* \underline{U}_i \to \underline{U}_i$ is quasi-compact, the same will hold also for (f, φ) ([58, Ch.I, §6.1]). Likewise, if each morphism $\varphi_i^* \underline{U}_i \to \underline{U}_i$ is étale and quasi-separated, then the same will hold for (f, φ) ([58, Ch.I, Prop.6.1.11] and proposition 12.3.24(iii)).

Therefore, we may replace X by any U_i, and F by the corresponding V_i, which reduces the proof of (iii)–(v) to the case where condition (a) is fulfilled. Lastly, in light of lemma 12.6.11 we may suppose that the open subsets U_i are small enough, so that also condition (b) is fulfilled. $\Diamond$

In view of claim 12.6.15, we shall assume henceforth that conditions (a) and (b) are fulfilled. Let $\underline{S} := (\operatorname{Spec}(\mathbb{Z}, P), T_P, \psi_P)$ be the object of $\mathscr{K}_{\mathrm{int}}$ considered in example 12.6.5(i) (with $R := \mathbb{Z}$); in this situation, β determines a morphism of schemes $f_\beta : X \to S$, and in view of lemma 12.2.14 we have $\underline{X} \simeq X \times_S \underline{S}$ (notation of (12.6.4)). Therefore, if we find an inverse image $\underline{S}'$ for $\underline{S}$ over φ, the object $X \times_S \underline{S}'$ will provide the sought inverse image of $\underline{X}$. Thus, in order to show (iii)–(v), we may further reduce to the case where $\underline{X} = \underline{S}$ ([58, Ch.I, Prop.6.1.5(iii), Prop.6.1.9(iii)] and proposition 12.3.24(ii)).

Claim 12.6.16. In order to prove (iii)–(v), we may assume that F' is affine.

Proof of the claim. Indeed, say that $F' = \bigcup_{i \in I} V_i$ is an open covering, and let $\varphi_i : V_i \to T_P$ be the restriction of φ; suppose that we have found, for each $i \in I$, an inverse image $\varphi_i^* \underline{S}$ of $\underline{S}$ over φ_i; again, it follows easily that an inverse image for $\underline{S}$ over φ can be constructed by gluing the objects $\varphi_i^* \underline{S}$. This already implies that, in order to show (iii), we may assume that F' is affine.

Now, suppose that φ is finite, so we may find an open covering as above, such that furthermore each V_i is affine, and I is a finite set. Suppose that each of the corresponding morphisms $\varphi_i^* \underline{S} \to \underline{S}$ is quasi-compact; by the foregoing, the schemes underlying the objects $\varphi_i^* \underline{S}$ give a finite open covering of the scheme underlying $\varphi^* \underline{S}$, and then it is clear that (iv) holds.

Next, suppose furthermore that each morphism $\varphi_i^* \underline{S} \to \underline{S}$ is étale. It follows that the morphism $\varphi^* \underline{S} \to \underline{S}$ is also étale (proposition 12.3.24(ii)); then, since S is noetherian, the scheme underlying $\varphi^*\underline{S}$ is locally noetherian ([58, Ch.I, Prop.6.2.2]), and therefore the morphism $\varphi^*\underline{S} \to \underline{S}$ is quasi-separated ([58, Ch.I, Cor.6.1.13]). ◊

In view of claim 12.6.16, we shall assume that $F' = (\operatorname{Spec} P')^\sharp$ is affine as well, for a sharp and integral P', and that φ is given by a morphism $\lambda : P \to P'$ inducing a surjection on the associated groups (details left to the reader). Then assertions (iii) and (iv) are now straightforward consequences of lemma 12.6.12(i,ii), and (v) follows from lemma 12.6.12(iii), after one remarks that, when F' is fine, one may choose for P' a fine monoid. □

Remark 12.6.17. Keep the assumptions of proposition 12.6.14, and let $t \in F_0$ a point of F of height zero (see (6.5.16)). Notice that the inclusion map $j_t : \{t\} \to F$ is an open immersion, hence the fibre $X_t := \psi^{-1}(t) \subset X$ is open; indeed, it is clear from the definitions, that $\psi^{-1}(F_0)$ is precisely the trivial locus of $\underline{X}$. Moreover, let $t' \in \varphi^{-1}(t)$ since the group homomorphism $\mathscr{O}^{\mathrm{gp}}_{F,t} \to \mathscr{O}^{\mathrm{gp}}_{F',t'}$ is surjective, we see that t' is of height zero in F', and φ restricts to an isomorphism of fans $(\{t'\}, \mathscr{O}_{F,t'}) \xrightarrow{\sim} (\{t\}, \mathscr{O}_{F,t})$. Set $\underline{X}_t := j_t^*\underline{X}$ (whose underlying scheme is X_t), and define likewise $\varphi^*\underline{X}_{t'} := (\varphi \circ j_{t'})^*\underline{X}$ (whose underlying scheme is an open subset of the trivial locus of $\varphi^*\underline{X}$). We deduce that :

- The trivial locus of $\varphi^*\underline{X}$ is the preimage of $(X, \underline{M})_{\mathrm{tr}}$.
- For every $t' \in F'_0$, the restriction of $(f, \varphi) : \varphi^*\underline{X}_{t'} \to \underline{X}_t$ is an isomorphism.

Corollary 12.6.18. *In the situation of proposition* 12.6.14*, let* $((X', \underline{M}'), F', \psi') := \varphi^*\underline{X}$*. The following holds :*

(i) *If* $F' = F^{\mathrm{sat}}$*, and* $\varphi : F^{\mathrm{sat}} \to F$ *is the counit of adjunction, then* $(X', \underline{M}') = (X, \underline{M})^{\mathrm{fs}}$*, and* f *is the counit of adjunction.*

(ii) *If* φ *is the blow up of a coherent ideal* $\mathscr{I}$ *of* $\mathscr{O}_F$*, then* f *is the blow up of* $\mathscr{I}\underline{M}$*, the unique ideal of* $\underline{M}$ *whose image in* $\underline{M}^\sharp$ *equals* $\psi^{-1}\mathscr{I}$*.*

(iii) *Suppose moreover, that* $(X, \underline{M})$ *is regular,* F' *is saturated , and* $\underline{X} = \mathscr{K}(X, \underline{M})$ *(notation of remark* 12.6.9(iii)*). Then* $(X', \underline{M}')$ *is regular, and* $\varphi^*\underline{X}$ *is isomorphic to* $\mathscr{K}(X', \underline{M}')$*.*

Proof. To start with, we remark that the assertions are local on X. Indeed, this is clear for (i), and for (iii) it follows easily from remark 12.6.9(i,,ii); concerning (ii), suppose that $X = \bigcup_{i\in I} U_i$ is an open covering, such that $\varphi^*(U_i, \underline{M}_{|U_i})$ is the blow up of the ideal $\mathscr{I}\underline{M}_{|U_i}$, for every $i \in I$. For every $i, j \in I$ set $U_{ij} := U_i \cap U_j$; by the universal property of the blow up, there are unique isomorphisms of $(U_{ij}, \underline{M}_{|U_{ij}})$-schemes :

$$U_{ij} \times_{U_i} \varphi^*(U_i, \underline{M}_{|U_i}) \xrightarrow{\sim} U_{ij} \times_{U_j} \varphi^*(U_j, \underline{M}_{|U_j}).$$

Then both $\varphi^*(X, \underline{M})$ and the blow up of $\mathscr{I}\underline{M}$ are necessarily obtained by gluing along these isomorphisms, so they are isomorphic.

Thus, we may assume that $F = (\operatorname{Spec} P)^\sharp$ for some fine monoid P, and $(X, \underline{M}) = X \times_S (S, P^{\log}_S)$ for some morphism of schemes $X \to S$, where as usual $(S, P^{\log}_S, F, \psi_P)$ is defined as in example 12.6.5(i). In this case, (i) follows from remarks 12.2.38(iii), 6.5.8(ii) and lemma 12.6.12(ii).

Likewise, remark 12.4.32(ii) allows to reduce (ii) to the case where $\underline{X} = (S, P^{\log}_S, F, \psi_P)$, and $\mathscr{I} = I^\sim$ for some ideal $I \subset P$; in which case we conclude by inspecting the explicit description in example 12.4.43.

(iii): From proposition 12.6.14(ii,v) and theorem 12.5.28 we already see that $(X', \underline{M}')$ is regular. Next, we are easily reduced to the case where X is affine, say $X = \operatorname{Spec} A$, and $F' = (\operatorname{Spec} Q)^\sharp$, for some saturated monoid Q, and by lemma 12.6.12, we may assume that φ is induced by a morphism of monoids $\lambda : P \to Q$ such that λ^{gp} is an isomorphism, and $(X', \underline{M}') = \varphi^*\underline{X} = X \times_S \underline{S}'$, where $\underline{S}' := (\operatorname{Spec}(\mathbb{Z}, Q), F', \psi_Q)$ is defined as in example

12.6.5(ii). Let $\mathfrak{q} \subset Q$ be any prime ideal, and set $\mathfrak{p} := \mathfrak{q} \cap P$; in light of remark 12.6.9(i), it then suffices to show that $Q/\mathfrak{q} \otimes_P A = Q/\mathfrak{q} \otimes_{P/\mathfrak{p}} A/\mathfrak{p}A$ is an integral domain. To this aim, let us remark :

Claim 12.6.19. Let $\lambda : P \to Q$ be a morphism of fine monoids, such that λ^{gp} is an isomorphism, $F \subset Q$ any face, and denote $\lambda_F : F \cap P \to P$ the inclusion map. Then :

(i) The natural map $\mathrm{Coker}\, \lambda_F^{\mathrm{gp}} \to P^{\mathrm{gp}}/(F \cap P)^{\mathrm{gp}}$ is injective.

(ii) If moreover, P is saturated, then $\mathrm{Coker}\, \lambda_F^{\mathrm{gp}}$ is a free abelian group of finite rank.

Proof of the claim. The map of (i) is obtained via the snake lemma, applied to the ladder of abelian groups :

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (F \cap P)^{\mathrm{gp}} & \longrightarrow & P^{\mathrm{gp}} & \longrightarrow & P^{\mathrm{gp}}/(F \cap P)^{\mathrm{gp}} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \lambda_F^{\mathrm{gp}}} & & \downarrow{\scriptstyle \lambda^{\mathrm{gp}}} & & \downarrow & & \\ 0 & \longrightarrow & F^{\mathrm{gp}} & \longrightarrow & Q^{\mathrm{gp}} & \longrightarrow & Q^{\mathrm{gp}}/F^{\mathrm{gp}} & \longrightarrow & 0. \end{array}$$

taking into account that both $\mathrm{Ker}\, \lambda^{\mathrm{gp}}$ and $\mathrm{Coker}\, \lambda^{\mathrm{gp}}$ vanish. Then (i) is obvious, and (ii) comes down to checking that $P^{\mathrm{gp}}/(F \cap P)^{\mathrm{gp}}$ is torsion-free, in case P is fine and saturated. But since $F \cap P$ is a face of P, the latter assertion is an easy consequence of proposition 6.4.9. ◊

Now, set $F := Q \setminus \mathfrak{q}$; we come down to checking that $B := F \otimes_{F \cap P} A/\mathfrak{p}A$ is an integral domain, and remark 12.6.9(i) tells us that $A/\mathfrak{p}A$ is a domain. However, $A/\mathfrak{p}A$ is $(F \cap P)$-flat (proposition 12.5.16(ii)), hence the natural map

$$B \to C := F^{\mathrm{gp}} \otimes_{(F \cap P)^{\mathrm{gp}}} \mathrm{Frac}(A/\mathfrak{p}A)$$

is injective; on the other hand, claim 12.6.19(ii) implies that $C = A[\mathrm{Coker}\, \lambda_F^{\mathrm{gp}}]$ is a free (polynomial) A-algebra, whence the contention. □

Example 12.6.20. In the situation of example 6.5.27, take $T := \mathrm{Spec}\, P$ for a fine monoid P, and define $\underline{S}$ as in example 12.6.5(i). The k-Frobenius map $\boldsymbol{k}_P$ (example 6.5.10(i)) induces an endomorphism $\boldsymbol{k}_{\underline{S}} := (\mathrm{Spec}(R, \boldsymbol{k}_P), \boldsymbol{k}_T)$ of $\underline{S}$ in $\mathscr{K}$. By proposition 12.6.14(iii) and example 6.5.27, there exists a unique morphism $\underline{g}$ fitting into a commutative diagram of $\mathscr{K}_{\mathrm{int}}$:

(12.6.21)
$$\begin{array}{ccc} \varphi^*\underline{S} & \longrightarrow & \underline{S} \\ \downarrow{\scriptstyle \underline{g}} & & \downarrow{\scriptstyle \boldsymbol{k}_{\underline{S}}} \\ \varphi^*\underline{S} & \longrightarrow & \underline{S} \end{array}$$

(where the horizontal arrows are the cartesian morphisms). Say that $\varphi^*\underline{S} = ((Y, \underline{N}), F, \psi)$; then we have $\underline{g} = (g, \boldsymbol{k}_F)$ for a unique endomorphism $g := (g, \log g)$ of the log scheme $(Y, \underline{N})$. Let $U := \mathrm{Spec}\, P' \subset F$ be any open affine subset ; since $\boldsymbol{k}_F$ is the identity on the underlying topological spaces, g restricts to an endomorphism $g_{|\psi^{-1}U}$ of $\psi^{-1}U \times_Y (Y, \underline{N})$. In view of lemma 12.6.12, the latter log scheme is of the form $\underline{S}'$ as in example 12.6.5(ii), with $Q := P^{\mathrm{gp}} \times_{P'^{\mathrm{gp}}} P'$. Then $g_{|\psi^{-1}U}$ is induced by an endomorphism ν of Q, fitting into a commutative diagram :

$$\begin{array}{ccc} P & \longrightarrow & Q \\ \downarrow{\scriptstyle \boldsymbol{k}_P} & & \downarrow{\scriptstyle \nu} \\ P & \longrightarrow & Q \end{array}$$

whose horizontal arrows are the natural injections. Since $Q \subset P^{\mathrm{gp}}$, it is clear that $\nu = \boldsymbol{k}_Q$. Especially, $g : Y \to Y$ is a finite morphism of schemes. Furthermore, for every point $y \in Y$, we have a commutative diagram of monoids :

$$\begin{array}{ccccc} \underline{N}_{g(y)} & \longrightarrow & \underline{N}^\sharp_{g(y)} & \xleftarrow{\sim} & \mathscr{O}_{F,\psi(y)} \\ \downarrow{\scriptstyle \log g_y} & & \downarrow{\scriptstyle \log g^\sharp_y} & & \downarrow{\scriptstyle \boldsymbol{k}_{\psi(y)}:=(\log \boldsymbol{k}_F)_{\psi(y)}} \\ \underline{N}_y & \longrightarrow & \underline{N}^\sharp_y & \xleftarrow{\sim} & \mathscr{O}_{F,\psi(y)}. \end{array} \qquad (12.6.22)$$

12.6.23. Let $(K, |\cdot|)$ be a valued field, Γ and K^+ respectively the value group and the valuation ring of $|\cdot|$; denote by $1 \in \Gamma$ the neutral element, and by $\Gamma_+ \subset \Gamma$ the submonoid consisting of all elements ≤ 1. Set $S := \operatorname{Spec} K^+$; we consider the log scheme $(S, \mathscr{O}^*_S)$, where $\mathscr{O}^*_S \subset \mathscr{O}_S$ is the subsheaf such that $\mathscr{O}^*_S(U) := \mathscr{O}_S(U)\backslash\{0\}$ for every open subset $U \subset S$. To this log scheme we associate the object of $\mathscr{K}_{\mathrm{int}}$:

$$\mathsf{f}(K, |\cdot|) := ((S, \mathscr{O}^*_S), \operatorname{Spec} \Gamma_+, \psi_\Gamma)$$

where ψ_Γ is the morphism of monoidal spaces arising from the isomorphism

$$\Gamma_+ \xrightarrow{\sim} (K^+\backslash\{0\})/(K^+)^\times$$

deduced from the valuation $|\cdot|$. It is well known that ψ_Γ is a homeomorphism (see *e.g.* [75, §6.1.26]). We shall use objects of this kind to state some separation and properness criteria for log schemes whose existence is established via proposition 12.6.14. To this aim, we need to digress a little, to prove the following auxiliary results, which are refinements of the standard valuative criteria for morphisms of schemes.

12.6.24. Let $f : X \to Y$ be a morphism of schemes, R an integral ring, and K the field of fractions of R. Let us denote by $X(R)_{\max} \subset X(R)$ the set of morphisms $\operatorname{Spec} R \to X$ which map $\operatorname{Spec} K$ to a maximal point of X. There follows a commutative diagram of sets :

$$\begin{array}{ccc} X(R)_{\max} & \longrightarrow & X(K)_{\max} \\ \downarrow & & \downarrow \\ Y(R) & \longrightarrow & Y(K). \end{array} \qquad (12.6.25)$$

Proposition 12.6.26. *Let $f : X \to Y$ be a morphism of schemes. The following conditions are equivalent :*

(a) *f is separated.*

(b) *f is quasi-separated, and for every valued field $(K, |\cdot|)$, the map of sets :*

$$X(K^+)_{\max} \to X(K)_{\max} \times_{Y(K)} X(K^+)_{\max} \qquad (12.6.27)$$

deduced from diagram (12.6.25) *(with $R := K^+$, the valuation ring of $|\cdot|$), is injective.*

Proof. (a) ⇒ (b) by the valuative criterion of separation ([58, Ch.I, Prop.5.5.4]). Conversely, we shall show that if (b) holds, then the assumptions of the criterion of *loc.cit.* are fulfilled. Indeed, let $(L, |\cdot|_L)$ be any valued field, with valuation ring L^+, and suppose we have two morphisms $\sigma_1, \sigma_2 : \operatorname{Spec} L^+ \to X$ whose restrictions to $\operatorname{Spec} L$ agree, and such that $f \circ \sigma_1 = f \circ \sigma_2$.

Let $s, \eta \in \operatorname{Spec} L^+$ be respectively the closed point and the generic point, set $x := \sigma_1(\eta) = \sigma_2(\eta) \in X$, and denote by $\varphi : \mathscr{O}_{X,x} \to L$ the ring homomorphism corresponding to the restriction of σ_1 (and σ_2); then x admits two specializations $x_i := \sigma_i(s) \in X$ (for $i = 1, 2$) such that φ sends the image $A_i \subset \mathscr{O}_{X,x}$ of the specialization map $\mathscr{O}_{X,x_i} \to \mathscr{O}_{X,x}$, into L^+, and the maximal ideal of A_i into the maximal ideal $\mathfrak{m}_L$ of L^+, for both $i = 1, 2$. Moreover, $y := f(x_1) = f(x_2)$.

Denote by $B \subset \mathscr{O}_{X,x}$ the smallest subring containing A_1 and A_2, and set $\mathfrak{p} := B \cap \varphi^{-1}\mathfrak{m}_L$. Then $\varphi(B_\mathfrak{p}) \subset L^+$ as well, and $B_\mathfrak{p}$ dominates both A_1 and A_2. Now, let $t \in X$ be a maximal point which specializes to x; by [58, Ch.I, Prop.5.5.2] we may find a valued field $(E, |\cdot|_E)$

with a local ring homomorphism $\mathscr{O}_{X,t} \to E$, such that the valuation ring E^+ of $|\cdot|_E$ dominates the image of the specialization map $\mathscr{O}_{X,x} \to \mathscr{O}_{X,t}$. Let $\kappa(E)$ be the residue field of E^+, and $\overline{B}_\mathfrak{p} \subset \kappa(E)$ the image of $B_\mathfrak{p}$. By the same token, we may find a valuation ring $V \subset \kappa(E)$ with fraction field $\kappa(E)$, and dominating $\overline{B}_\mathfrak{p}$; then it is easily seen that the preimage $K^+ \subset E^+$ of V is a valuation ring with field of fractions $K = E$, which dominates the image of $B_\mathfrak{p}$ in $\mathscr{O}_{X,t}$ ([126, Th.10.1(iv)]). Hence, K^+ dominates the images of $\mathscr{O}_{X,x_i}$, for both $i = 1, 2$, and therefore, also the image of $\mathscr{O}_{Y,y}$; in other words, in this way we obtain two elements in $X(K^+)_{\max}$ whose images agree in $Y(K^+)$, and whose restrictions $\operatorname{Spec} K \to X$ coincide. By assumption, these two K^+-points must then coincide, especially $x_1 = x_2$, and therefore $\sigma_1 = \sigma_2$, as required. □

Proposition 12.6.28. *Let $f : X \to Y$ be a quasi-compact morphism of schemes. The following conditions are equivalent :*

(a) *f is universally closed.*
(b) *For every valued field $(K, |\cdot|)$, the corresponding map* (12.6.27) *is surjective.*

Proof. (a) ⇒ (b) by the valuative criterion of [58, Ch.I, Prop.5.5.8]. Conversely, we will show that (b) implies that the conditions of *loc.cit.* are fulfilled. Hence, let $(L, |\cdot|_L)$ be any valued field, with valuation ring L^+, and suppose that we have a morphism $\sigma : \operatorname{Spec} L \to X$, whose composition with f extends to a morphism $\operatorname{Spec} L^+ \to Y$; we have to show that σ extends to a morphism $\operatorname{Spec} L^+ \to X$. Denote by $\eta, s \in \operatorname{Spec} L^+$ respectively the generic and the closed point, let $x \in X$ be the image of η, and $y \in Y$ the image of s. Then σ corresponds to a ring homomorphism $\sigma^\natural : \mathscr{O}_{X,x} \to L$, and L^+ dominates the image of the map $\mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ determined by f. Moreover, $\sigma^\natural$ factors through the residue field $\kappa(x)$ of $\mathscr{O}_{X,x}$. Then $\kappa(x)^+ := \kappa(x) \cap L^+$ is a valuation ring with fraction field $\kappa(x)$, and we are reduced to showing that there exists a specialization $x' \in X$ of x, such that $f(x') = y$, and such that $\kappa(x)^+$ dominates the image of the specialization map $\mathscr{O}_{X,x'} \to \mathscr{O}_{X,x}$.

Let now $t \in X$ be a maximal point which specializes to x; by [58, Ch.I, Prop.5.5.8] we may find a valued field $(E, |\cdot|_E)$ with a local ring homomorphism $\mathscr{O}_{X,t} \to E$, whose valuation ring E^+ dominates the image of the specialization map $\mathscr{O}_{X,x} \to \mathscr{O}_{X,t}$. Let $\kappa(E)$ be the residue field of E^+; the induced map $\mathscr{O}_{X,x} \to \kappa(E)$ factors through $\kappa(x)$, and by [126, Th.10.2] we may find a valuation ring $V \subset \kappa(E)$ with field of fractions $\kappa(E)$, which dominates $\kappa(x)^+$. The preimage $K^+ \subset E^+$ of V is a valuation ring with field of fractions $K = E$ ([126, Th.10.1]). By construction, K^+ dominates the image of $\mathscr{O}_{X,y}$, in which case assumption (b) says that there exists a specialization x' of x such that $f(x') = y$, and such that K^+ dominates the image of the specialization map $\mathscr{O}_{X,x'} \to \mathscr{O}_{X,t}$. A simple inspection then shows that $\kappa(x)^+$ dominates the image of $\mathscr{O}_{X,x'}$ in $\mathscr{O}_{X,x}$, as required. □

Corollary 12.6.29. *Let $f : X \to Y$ be a morphism of schemes which is quasi-separated and of finite type. Then the following conditions are equivalent :*

(a) *f is proper.*
(b) *For every valued field $(K, |\cdot|)$, the corresponding diagram* (12.6.25) *(with $R := K^+$ the valuation ring of $|\cdot|$) is cartesian.*

Proof. It is immediate from propositions 12.6.26 and 12.6.28. □

12.6.30. We are now ready to return to log schemes. Resume the situation of (12.6.23), let $\underline{X} := ((X, \underline{M}), F, \psi)$ be any object of $\mathscr{K}$, and denote by $\alpha : \underline{M} \to \mathscr{O}_X$ the structure map of $\underline{M}$.

Suppose we are given a morphism $\sigma : S \to X$ of schemes, and we ask whether there exists a morphism of log structures $\beta : \sigma^*\underline{M} \to \mathscr{O}_S^*$, such that the pair (σ, β) is a morphism of log schemes $(S, \mathscr{O}_S^*) \to (X, \underline{M})$. By definition, this holds if and only if the composition

$$\overline{\beta} : \sigma^*\underline{M} \xrightarrow{\sigma^*\alpha} \sigma^*\mathscr{O}_X \xrightarrow{\sigma^\natural} \mathscr{O}_S$$

factors through $\mathscr{O}_S^*$. Moreover, in this case the factorization is unique, so that σ determines β uniquely. Let $\eta \in S$ be the generic point; we claim that the stated condition is fulfilled, if and only if $t := \sigma(\eta)$ lies in $(X, \underline{M})_{\mathrm{tr}}$ (see definition 12.2.7(i)). Indeed, if t lies in the trivial locus, $\underline{M}_t = \mathscr{O}_{X,t}^\times$, so certainly the image of $\overline{\beta}_t : \underline{M}_t \to \mathscr{O}_{S,\eta}$ lies in $\mathscr{O}_{S,\eta}^* = K^\times$. Now, if $s \in S$ is any other point, the composition

$$\underline{M}_{\sigma(s)} \xrightarrow{\overline{\beta}_{\sigma(s)}} \mathscr{O}_{S,s} \to \mathscr{O}_{S,\eta} = K$$

factors through $\underline{M}_t$, hence its image lies in $K^\times \cap \mathscr{O}_{S,s} = \mathscr{O}_{S,s}^*$, whence the contention. Conversely, if $\overline{\beta}$ factors through $\mathscr{O}_S^*$, it follows especially that the image of the stalk of the structure map $\underline{M}_t \to \mathscr{O}_{X,t}$ lies in the preimage of $\mathscr{O}_{S,\eta}^* = \mathscr{O}_{S,\eta}^\times$, and the latter is just $\mathscr{O}_{X,t}^\times$, so t lies in the trivial locus.

Next, let $f : (S, \mathscr{O}_S^*) \to (X, \underline{M})$ be any morphism of log schemes. We claim that there exists a unique morphism of fans $\varphi := (\varphi, \log\varphi) : \mathrm{Spec}\,\Gamma_+ \to F$, such that the pair (f, φ) is a morphism $\mathsf{f}(K, |\cdot|) \to \underline{X}$ in $\mathscr{K}$. Indeed, since ψ_Γ is a homeomorphism, there exists a unique continuous map φ on the underlying topological spaces, such that $\varphi \circ \psi_\Gamma = \psi \circ f$, and for the same reason, the map $\log f^\sharp : f^*\underline{M}/\mathscr{O}_S^\times \to \mathscr{O}_S^*/\mathscr{O}_S^\times$ is of the form $\psi_\Gamma^*(\log\varphi)$ for a unique morphism of sheaves $\log\varphi$ as sought. Then $\log\varphi$ will be a local morphism, since the same holds for $\log f$ (see (12.6.1)).

Summing up, we have shown that the natural map

$$\mathrm{Hom}_{\mathscr{K}}(\mathsf{f}(K, |\cdot|), \underline{X}) \to X(K^+) \quad : \quad (\sigma, \varphi) \mapsto \sigma$$

is injective, and its image is the set of all the morphisms $S \to X$ which map η into $(X, \underline{M})_{\mathrm{tr}}$.

Proposition 12.6.31. *Let $\underline{X} := ((X, \underline{M}), F, \psi)$ be an object of $\mathscr{K}_{\mathrm{int}}$, and $\varphi : F' \to F$ an integral partial subdivision, with F and F' locally fine. We have :*

(i) *If the induced map $F'(\mathbb{N}) \to F(\mathbb{N})$ is injective, the cartesian morphism $\varphi^*\underline{X} \to \underline{X}$ is separated.*

(ii) *If φ is a proper subdivision, the cartesian morphism $\varphi^*\underline{X} \to \underline{X}$ is proper, and induces an isomorphism of schemes $(\varphi^*\underline{X})_{\mathrm{tr}} \xrightarrow{\sim} (X, \underline{M})_{\mathrm{tr}}$.*

Proof. The assertions are local on X (cp. the proof of claim 12.6.15), so we may assume – by lemma 12.6.11 – that $\underline{M}$ admits a chart $P_X \to \underline{M}$, and $F = (\mathrm{Spec}\,P)^\sharp$. In this case, let $S := \mathrm{Spec}\,\mathbb{Z}[P]$, and denote by $\underline{S}$ the object of $\mathscr{K}_{\mathrm{int}}$ attached to P, as in example 12.6.5(i) (with $R := \mathbb{Z}$); then $(X, \underline{M})$ is isomorphic to $X \times_S (S, P_S^{\log})$, and if $f_S : \varphi^*\underline{S} \to \underline{S}$ is the cartesian morphism over φ, then the cartesian morphism $f : \varphi^*\underline{X} \to \underline{X}$ is given by the pair $(\mathbf{1}_X \times_S f_S, \varphi)$ (see the proof of proposition 12.6.14(iii)). Thus, we may replace $\underline{X}$ by $\underline{S}$ ([58, Ch.I, Prop.5.3.1(iv)]), in which case lemma 12.6.12 shows that the log scheme $(X', \underline{M}')$ underlying $\varphi^*\underline{S}$ admits an open covering consisting of affine log schemes of the form $\mathrm{Spec}(\mathbb{Z}, Q)$, for a fine monoid Q. Notice that, for such Q we have :

$$\mathrm{Spec}(\mathbb{Z}, Q)_{\mathrm{tr}} = \mathbb{Z}[Q^{\mathrm{gp}}]$$

which is a dense open subset of $\mathrm{Spec}\,\mathbb{Z}[Q]$.

(i): According to proposition 12.6.14(v), the morphism f is quasi-separated, so we may apply the criterion of proposition 12.6.26. Indeed, let $(K, |\cdot|)$ be any valued field, and suppose that $\sigma_i \in X'(K^+)_{\max}$ (for $i = 1, 2$) are two sections, whose images in $X'(K)_{\max} \times_{S(K)} S(K^+)$ coincide; we have to show that $\sigma_1 = \sigma_2$. However, we have just seen that the maximal points of X' lie in $(X', \underline{M}')_{\mathrm{tr}}$, so the discussion of (12.6.30) shows that both σ_i extend uniquely to morphisms $\sigma'_i : \mathsf{f}(K, |\cdot|) \to \varphi^*\underline{S}$ in $\mathscr{K}$, and it suffices to show that $\sigma'_1 = \sigma'_2$. By definition, the datum of σ'_i is equivalent to the datum of a morphism $\sigma''_i : \mathsf{f}(K, |\cdot|) \to \underline{S}$, and a morphism of fans $\varphi'_i : \mathrm{Spec}\,\Gamma_+ \to F'$. Again by (12.6.30), the morphisms σ''_1 and σ''_2 agree if and only if they induce the same morphisms of schemes; the latter holds by assumption, since σ_1 and σ_2 yield the same element of $S(K^+)$. On the other hand, in view of (b) and proposition 6.5.24, the elements $\varphi_1, \varphi_2 \in F'(\Gamma_+)$ coincide if and only if their images in $F(\Gamma_+)$ coincide; but again, this last condition holds since the images of both σ_i agree in $S(K^+)$.

(ii): In view of (i) and proposition 12.6.14(iv),(v) we know already that f is separated and of finite type, so we may apply the criterion of corollary 12.6.29. Hence, let $\sigma \in S(K^+)$ be a section, and $x \in X'(K)_{\max}$ a K-rational point such that $f(x)$ is the image of σ in $S(K)$; in view of (i), it suffices to show that σ lifts to a section $\widetilde{\sigma} \in X'(K^+)_{\max}$, whose image in $X'(K)$ is x. Since $x \in (X', \underline{M}')_{\mathrm{tr}}$, remark 12.6.17 implies that $f(x) \in (S, P_S^{\log})_{\mathrm{tr}}$, and then the discussion in (12.6.30) says that σ underlies a unique morphism $(\sigma', \beta) : \mathsf{f}(K, |\cdot|) \to \underline{S}$. By proposition 6.5.26, the element $\beta \in F(\Gamma_+)$ lifts to an element $\beta' \in F'(\Gamma_+)$, and finally the pair $((\sigma, \beta), \beta')$ determines a unique morphism $\mathsf{f}(K, |\cdot|) \to \varphi^*\underline{S}$, whose underlying morphism of schemes is the sought $\widetilde{\sigma}$. Lastly, notice that the map $F'(\{1\}) \to F(\{1\})$ induced by φ, is bijective by propositions 6.5.24 and 6.5.26 (where $\{1\}$ is the monoid with one element), which means that φ restricts to a bijection on the points of height zero; then remark 12.6.17 implies the second assertion of (ii). ☐

Theorem 12.6.32. *Let $(X, \underline{M})$ be a regular log scheme. Then there exists a smooth morphism of log schemes $f : (X', \underline{M}') \to (X, \underline{M})$, whose underlying morphism of schemes is proper and birational, and such that X' is a regular scheme. More precisely, f restricts to an isomorphism of schemes $f^{-1}X_{\mathrm{reg}} \to X_{\mathrm{reg}}$ on the preimage of the open locus of regular points of X.*

Proof. We use the object $\underline{X} := ((X, \underline{M}), F(X, \underline{M}), \pi_X)$ attached to $(X, \underline{M})$ as in remark 12.6.9(ii). Indeed, it is clear that $F(X, \underline{M})$ is locally fine and saturated, hence theorem 6.6.31 yields an integral, proper, simplicial subdivision $\varphi : F' \to F(X, \underline{M})$ which restricts to an isomorphism $\varphi^{-1}F(X, \underline{M})_{\mathrm{sim}} \xrightarrow{\sim} F(X, \underline{M})_{\mathrm{sim}}$. Take $(f, \varphi) : \varphi^*\underline{X} \to \underline{X}$ to be the cartesian morphism over φ, and denote by $(X', \underline{M}')$ the log scheme underlying $\varphi^*\underline{X}$; it follows already from proposition 12.6.31(ii) that f is proper on the underlying schemes. Next, corollary 12.5.19 shows that X_{reg} is $\pi^{-1}F(X, \underline{M})_{\mathrm{sim}}$, so f restricts to an isomorphism $f^{-1}X_{\mathrm{reg}} \xrightarrow{\sim} X_{\mathrm{reg}}$. Furthermore, f is étale, by proposition 12.6.14(v), hence the log scheme $(X', \underline{M}')$ is regular (theorem 12.5.28). Finally, again by corollary 12.5.12 we see that X' is regular. ☐

12.6.33. Let now $(Y, \underline{N})$ be a regular log scheme, such that $F(Y, \underline{N})$ is affine (notation of remark 12.6.9(ii)), say isomorphic to $(\mathrm{Spec}\, P)^\sharp$, for some fine, sharp and saturated monoid P. Let $I \subset P$ be an ideal generated by two elements $a, b \in P$, and denote by $f : (Y', \underline{N}') \to (Y, \underline{N})$ the saturated blow up of the ideal $I\underline{N}$ of $\underline{N}$ (see (12.4.49)). Set $U' := (Y', \underline{N}')_{\mathrm{tr}}$, $U := (Y, \underline{N})_{\mathrm{tr}}$, and denote $j : U \to Y$, $j' : U' \to Y'$ the open immersions. In this situation we have :

Lemma 12.6.34. (i) $H^1(Y', \underline{N}'^{\sharp\mathrm{gp}}) = 0$.

(ii) *Suppose moreover, that $R^1 j'_* \mathscr{O}_{U'}^\times = 0$. Then $R^1 j_* \mathscr{O}_U^\times = 0$.*

Proof. (i): Since $\underline{N}'^{\sharp\mathrm{gp}} = \pi_X^* \mathscr{O}^{\mathrm{gp}}_{F(Y', \underline{N}')}$, claim 12.6.39(ii) and remark 12.6.9(iii) reduce to showing

$$H^1(F(Y', \underline{N}'), \mathscr{O}^{\mathrm{gp}}_{F(Y', \underline{N}')}) = 0. \tag{12.6.35}$$

However, by corollary 12.6.18(iii), the fan $F(Y', \underline{N}')$ is the saturated blow up of the ideal $I\mathscr{O}_{F(Y,\underline{N})}$ of $\mathscr{O}_{F(Y,\underline{N})}$, hence it admits the affine covering

$$F(Y', \underline{N}') = (\mathrm{Spec}\, P[a^{-1}b]^{\mathrm{sat}})^\sharp \cup (\mathrm{Spec}\, P[b^{-1}a]^{\mathrm{sat}})^\sharp.$$

Notice now that every affine fan is a local topological space, hence the left hand-side of (12.6.35) is computed by the Čech cohomology of $\mathscr{O}^{\mathrm{gp}}_{F(Y', \underline{N}')}$ relative to this covering (theorem 10.2.24(ii)).

However, the intersection of the two open subsets is $(\mathrm{Spec}\, P[a^{-1}b, b^{-1}a])^\sharp$, and clearly the restriction map

$$H^0(\mathrm{Spec}\, P[a^{-1}b]^{\mathrm{sat}}, \mathscr{O}^{\mathrm{gp}}_{F(Y',\underline{N}')}) \to H^0(\mathrm{Spec}\, P[a^{-1}b, b^{-1}a], \mathscr{O}^{\mathrm{gp}}_{F(Y',\underline{N}')})$$

is surjective. The assertion is an immediate consequence.

(ii): Let $y \in Y$ be any point; according to (12.4.50), the morphism

$$f \times_Y Y(y) : (Y', \underline{N}') \times_Y Y(y) \to (Y(y), \underline{N}(y))$$

is the saturated blow up of the ideal $I\underline{N}(y)$; on the other hand, let $U(y) := U \times_Y Y(y)$, $U'(y) := U' \times_Y Y(y)$ and denote by $j_y : U(y) \to Y(y)$ and $j'_y : U'(y) \to Y' \times_Y Y(y)$ the open immersions; in light of proposition 10.1.10(ii), it suffices to show that $R^1 j_{y*}\mathscr{O}_{U(y)} = 0$, and the assumption implies that $R^1 j'_{y*}\mathscr{O}_{U'(y)} = 0$. Summing up, we may replace Y be $Y(y)$, and assume from start that Y is local, and y is its closed point. From the assumption we get :

$$H^1(Y', j'_*\mathscr{O}^\times_{U'}) = H^1(U', \mathscr{O}^\times_{Y'}) = \mathrm{Pic}\, U'.$$

On the other hand, recall that $\underline{N}'^{\sharp\mathrm{gp}} = j'_*\mathscr{O}^\times_{U'}/\mathscr{O}^\times_{Y'}$ (proposition 12.5.38); combining with (i), we deduce that the natural map

$$\mathrm{Pic}\, Y' \to \mathrm{Pic}\, U'$$

is surjective. Set $Y'_0 := f^{-1}(y) \subset Y'$, endow Y'_0 with its reduced subscheme structure, and let $i : Y'_0 \to Y'$ be the closed immersion. If I is an invertible ideal of P, then f is an isomorphism, in which case the assertion is obvious. We may then assume that I is not invertible, in which case claim 12.4.56(ii) says that there exists a morphism of $(Y, \underline{N})$-schemes $h : (Y', \underline{N}') \to \mathbb{P}^1_{(Y,\underline{N})}$ inducing an isomorphism of $\kappa(y)$-schemes

(12.6.36) $$(h \times_Y \mathrm{Spec}\, \kappa(y))_{\mathrm{red}} : Y'_0 \xrightarrow{\sim} \mathbb{P}^1_{\kappa(y)}.$$

Let us remark :

Claim 12.6.37. Let S be a noetherian local scheme, s the closed point of S, and $f : X \to S$ a proper morphism of schemes. Suppose that $\dim X(s) \leq 1$, and $H^1(X(s), \mathscr{O}_{X(s)}) = 0$. Then the natural map

$$\mathrm{Pic}\, X \to \mathrm{Pic}\, X(s)$$

is injective.

Proof of the claim. Say that $S = \mathrm{Spec}\, A$ for a local ring A, and denote by $\mathfrak{m}_A \subset A$ the maximal ideal. For every $k \in \mathbb{N}$, set $S_n := \mathrm{Spec}\, A/\mathfrak{m}_A^{k+1}$, and let $i_n : X_n := X \times_S S_n \to X$ be the closed immersion. Let $\mathscr{L}$ be any invertible $\mathscr{O}_X$-module, and suppose that $i_0^*\mathscr{L} \simeq \mathscr{O}_{X_0}$; we have to show that $\mathscr{L} \simeq \mathscr{O}_X$. We notice that, for every $k \in \mathbb{N}$, the natural map

$$H^0(X_{k+1}, i_{k+1}^*\mathscr{L}) \to H^0(X_k, i_k^*\mathscr{L})$$

is surjective : indeed, its cokernel is an A-submodule of $H^1(X_0, \mathfrak{m}_A^k\mathscr{L}/\mathfrak{m}_A^{k+1}\mathscr{L})$, and since $\dim X_0 \leq 1$, the natural map

$$(\mathfrak{m}_A^k/\mathfrak{m}_A^{k+1}) \otimes_{\kappa(s)} H^1(X_0, i_0^*\mathscr{L}) \xrightarrow{\sim} H^1(X_0, (\mathfrak{m}_A^k/\mathfrak{m}_A^{k+1}) \otimes_{\kappa(s)} \mathscr{L}) \to H^1(X_0, \mathfrak{m}_A^k\mathscr{L}/\mathfrak{m}_A^{k+1}\mathscr{L})$$

is surjective; on the other hand, our assumptions imply that $H^1(X_0, i_0^*\mathscr{L}) = 0$, whence the contention. Let $A^\wedge$ be the $\mathfrak{m}_A$-adic completion of A; taking into account [61, Ch.III, Th.4.1.5], we deduce that the natural map

$$H^0(X, \mathscr{L}) \otimes_A A^\wedge \to H := H^0(X_0, i^*\mathscr{L})$$

is a continuous surjection, for the $\mathfrak{m}_A$-adic topologies. Since H is a discrete space for this topology, and since the image of $H^0(X, \mathscr{L})$ is dense in the $\mathfrak{m}_A$-adic topology of $H^0(X, \mathscr{L}) \otimes_A A^\wedge$, we conclude that the restriction map $H^0(X, \mathscr{L}) \to H$ is surjective as well. Let $\overline{s} \in H$ be a global section of $i_0^*\mathscr{L}$ whose image in $i_0^*\mathscr{L}_x$ is a generator of the latter A_0-module, for every

$x \in X_0$, and pick $s \in H^0(X, \mathscr{L})$ whose image in H equals $\overline{s}$. It remains only to check that, for every $x \in X$, the A-module $\mathscr{L}_x$ is generated by the image s_x of s. However, since X is proper, every $x \in X$ specializes to a point of X_0, hence we may assume that $x \in X_0$, in which case one concludes easily, by appealing to Nakayama's lemma (details left to the reader). $\diamondsuit$

Combining claim 12.6.37 and [122, Prop.11.1(i)], we see that the induced map

$$i^* : \mathrm{Pic}\, Y' \to \mathrm{Pic}\, Y'_0$$

is injective. Let now $\mathscr{L}$ be any invertible $\mathscr{O}_U$-module; we have to show that $\mathscr{L}$ extends to an invertible $\mathscr{O}_Y$-module (which is then isomorphic to $\mathscr{O}_Y$). However, notice that f restricts to an isomorphism $g : U' \xrightarrow{\sim} U$, hence $g^*\mathscr{L}$ is an invertible $\mathscr{O}_{U'}$-module, and by the foregoing there exists an invertible $\mathscr{O}_{Y'}$-module $\mathscr{L}'$ such that $\mathscr{L}'_{|U'} \simeq g^*\mathscr{L}$. In light of the isomorphism (12.6.36), there exists an invertible $\mathscr{O}_{\mathbb{P}^1_Y}$-module $\mathscr{L}''$ such that $i^*\mathscr{L}' \simeq i^*h^*\mathscr{L}''$. Therefore

$$\mathscr{L}' \simeq h^*\mathscr{L}''.$$

Now, on the one hand, claim 12.6.37 implies that $\mathscr{L}'' = \mathscr{O}_{\mathbb{P}^1_Y}(n)$ for some $n \in \mathbb{N}$; on the other hand, (12.4.17) implies that h restricts to a morphism of schemes $h_{\mathrm{tr}} : U' \to \mathbb{G}_{m,Y}$, and $\mathscr{L}''_{|\mathbb{G}_{m,Y}} = \mathscr{O}_{\mathbb{P}^1_Y}(n)_{|\mathbb{G}_{m,Y}} = \mathscr{O}_{\mathbb{G}_{m,Y}}$, so finally $g^*\mathscr{L} = \mathscr{O}_{U'}$, hence $\mathscr{L} = \mathscr{O}_U$, whence the contention. $\square$

The following result complements proposition 12.5.38.

Theorem 12.6.38. *Let $(X, \underline{M})$ be any regular log scheme, set $U := (X, \underline{M})_{\mathrm{tr}}$ and denote by $j : U \to X$ the open immersion. Then we have :*

$$R^1 j_* \mathscr{O}_U^\times = 0.$$

Proof. We begin with the following general :

Claim 12.6.39. Let $\pi : T_1 \to T_2$ be a continuous open and surjective map of topological spaces, such that $\pi^{-1}(t)$ is an irreducible topological space (with the subspace topology) for every $t \in T_2$. Then, we have :

(i) For every sheaf S on T_2, the natural map $S \to \pi_*\pi^*S$ is an isomorphism.
(ii) For every abelian sheaf S on T_2, the natural map $S[0] \to R\pi_*\pi^*S$ is an isomorphism in $\mathsf{D}(\mathbb{Z}_{T_2}\text{-}\mathbf{Mod})$.

Proof of the claim. (i): Since π is open, π^*S is the sheaf associated to the presheaf : $U \mapsto S(\pi U)$, for every open subset U of T_1. We show, more precisely, that this presheaf is already a sheaf; since π is surjective, the claim shall follow immediately. Now, let $U \subset T_1$ be an open subset, and $(U_i \mid i \in I)$ a family of open subsets of X covering U; for every $i, j \in I$, set $U_{ij} := U_i \cap U_j$. It suffices to show that $S(\pi U)$ is the equalizer of the two maps :

$$\prod_{i \in I} S(\pi U_i) \rightrightarrows \prod_{i,j \in I} S(\pi U_{ij}).$$

Since S is a sheaf, the latter will hold, provided we know that $\pi U_i \cap \pi U_j = \pi U_{ij}$ for every $i, j \in I$. Hence, let $t \in \pi U_i \cap \pi U_j$; this means that $\pi^{-1}(t) \cap U_i \neq \varnothing$ and $\pi^{-1}(t) \cap U_j \neq \varnothing$. Since $\pi^{-1}(t)$ is irreducible, we deduce that $\pi^{-1}(t) \cap U_{ij} \neq \varnothing$, as required.

(ii): The proof of (i) also shows that, for every flabby abelian sheaf J on T_2, the abelian sheaf π^*J is flabby on T_1. Hence, if S is any abelian sheaf on T_2, we obtain a flabby resolution of π^*S of the form $\pi^*J_\bullet$, by taking a flabby resolution $S \to J_\bullet$ of S on T_2. According to remark 7.3.31(vi), there is a natural isomorphism

$$\pi_*\pi^*J \xrightarrow{\sim} R\pi_*\pi^*S$$

in $\mathsf{D}(\mathbb{Z}_{T_2}\text{-}\mathbf{Mod})$. Then the assertion follows from (i). $\diamondsuit$

After these preliminaries, let us return to the log scheme $(X, \underline{M})$, and its associated object $\underline{X} := ((X, \underline{M}), F(X, \underline{M}), \pi_X)$. The assertion to prove is local on X, hence we may assume that $F(X, \underline{M}) = (\operatorname{Spec} P)^\sharp$, for some sharp, fine and saturated monoid P. Next, by theorem 12.6.32 (and its proof) there exists an integral proper simplicial subdivision $\varphi : F' \to F(X, \underline{M})$, such that the log scheme $(X', \underline{M}')$ underlying $\varphi^*\underline{X}$ is regular, X' is regular, and the morphism $X' \to X$ restricts to an isomorphism $U' := (X', \underline{M}')_{\mathrm{tr}} \to U$. In this situation, we may find a further subdivision $\varphi' : F'' \to F'$ such that both φ' and $\varphi \circ \varphi'$ are compositions of saturated blow up of ideals generated by at most two elements of P (example 6.6.15(iii)). Say that $\varphi \circ \varphi' = \varphi_r \circ \varphi_{r-1} \cdots \circ \varphi_1$, where each φ_i is a saturated blow up of the above type. By proposition 12.6.14(iii), we deduce a sequence of morphisms of log schemes

$$(X_1, \underline{M}_1) \xrightarrow{g_1} \cdots \to (X_{r-1}, \underline{M}_{r-1}) \xrightarrow{g_{r-1}} (X_r, \underline{M}_r) \xrightarrow{g_r} (X_{r+1}, \underline{M}_{r+1}) := (X, \underline{M})$$

each of which is the blow up of a corresponding ideal, and by the same token, φ' induces a morphism $g : (X_1, \underline{M}_1) \to (X', \underline{M}')$ of $(X, \underline{M})$-schemes. For every $i = 1, \cdots, r+1$, set $U_i := (X_i, \underline{M}_i)_{\mathrm{tr}}$, and let $j_i : U_i \to X_i$ be the open immersion; especially, $U_{r+1} = U$. We shall show, by induction on i, that

$$R^1 j_{i*}\mathscr{O}^\times_{U_i} = 0 \qquad \text{for } i = 1, \dots, r+1. \tag{12.6.40}$$

Notice first that the stated vanishing translates the following assertion. For every $x \in X_i$ and every invertible $\mathscr{O}_{U_i}$-module $\mathscr{L}$, there exists an open neighborhood U_x of x in X_i such that $\mathscr{L}_{|U_x \cap U_i}$ extends to an invertible $\mathscr{O}_{U_x}$-module. However, it follows immediately from propositions 11.3.8 and 11.3.15, that every invertible $\mathscr{O}_{U'}$-module extends to an invertible $\mathscr{O}_{X'}$-module. Since g restricts to an isomorphism $U_1 = g^{-1}U' \xrightarrow{\sim} U'$, we easily deduce that every invertible $\mathscr{O}_{U_1}$-module extends to an invertible $\mathscr{O}_{X_1}$-module (namely : if $\mathscr{L}$ is such a module, extend $g_{|U_1*}\mathscr{L}$ to an invertible $\mathscr{O}_{X'}$-module, and pull the extension back to X_1, via g^*). Summing up, we see that (12.6.40) holds for $i = 1$.

Next, suppose that (12.6.40) has already been shown to hold for a given $i \leq r$; by lemma 12.6.34(ii), it follows that (12.6.40) holds for $i + 1$, so we are done. □

12.6.41. Let X be a scheme, $\underline{M}$ a fine log structure on the Zariski site of X, x a point of X, and notice that every fractional ideal of $\underline{M}_x$ is finitely generated (lemma 6.4.39(iv)). Say that $X(x) = \operatorname{Spec} A$; in view of lemma 6.4.45(ii) there is a natural map of abelian groups

$$\operatorname{Div}(\underline{M}_x) \to \operatorname{Div}(A).$$

Composing with the map $I \mapsto I^\sim$ as in (11.3.12), we get, by virtue of *loc.cit.*, a map

$$\operatorname{Div}(\underline{M}_x) \to \operatorname{Div} X(x). \tag{12.6.42}$$

Corollary 12.6.43. *With the notation of* (12.6.41)*, suppose furthermore that* $(X, \underline{M})$ *is regular at the point* x*. Then* (12.6.42) *induces an isomorphism*

$$\overline{\operatorname{Div}}(\underline{M}_x) \xrightarrow{\sim} \overline{\operatorname{Div}}(X(x)).$$

Proof. By virtue of proposition 6.4.55(ii) (and remark 11.3.11(i)), the map under investigation is already known to be injective. To show surjectivity, let K be the field of fractions of A, and $L \subset K$ any reflexive fractional ideal of A; we may then regard $\mathscr{L} := L^\sim$ as a coherent $\mathscr{O}_X$-submodule of $i_*\mathscr{O}_{X_0}$, where $i : X_0 \to X$ is the inclusion map of the subscheme $X_0 := \operatorname{Spec} K$. By proposition 11.3.15, the $\mathscr{O}_U$-module $\mathscr{L}_{|U}$ is invertible, hence it extends to an invertible $\mathscr{O}_{X(x)}$-module $\mathscr{L}'$, by virtue of theorem 12.6.38. Since $X(x)$ is local, $\mathscr{L}'$ is a free $\mathscr{O}_{X(x)}$-module, and therefore $\mathscr{L}_{|U} \simeq \mathscr{O}_U$. Thus, pick $a \in \mathscr{L}(U) \subset K$ which generates $\mathscr{L}_{|U}$; after replacing L by $a^{-1}L$, we may assume that $\mathscr{L}_{|U} = \mathscr{O}_U$ as subsheaves of $i_*\mathscr{O}_{X_0}$. Let Σ be the set of points of height one of $X(x)$ contained in $X(x) \setminus U$; for each $y \in \Sigma$, the maximal ideal $\mathfrak{m}_y$ of $\mathscr{O}_{X(x),y}$ is generated by a single element a_y, and there exists $k_y \in \mathbb{Z}$ such that $a_y^{k_y}$ generates the $\mathscr{O}_{X(x),y}$-submodule $\mathscr{L}_y$ of K. To ease notation, let $P := \underline{M}_x$, and denote $\psi : X(x) \to \operatorname{Spec} P$ the natural continuous map; also, let $\mathfrak{m}_{\psi(y)}$ be the maximal ideal of the localization $P_{\psi(y)}$ for every $y \in X(x)$, and set

$$I := \bigcap_{y\in\Sigma} \mathfrak{m}_{\psi(y)}^{k_y}.$$

In light of lemma 6.4.39(i,ii) and proposition 6.4.50, it is easily seen that $I/P^\times$ is a reflexive fractional ideal of the fine and saturated monoid $P^\sharp$, so I is a reflexive fractional ideal of P. Then $IA \subset K$ is a reflexive fractional ideal of A. Set $\mathscr{L}'' := (IA)^\sim \subset i_*\mathscr{O}_{X_0}$; then $\mathscr{L}''_{|U} = \mathscr{L}_{|U}$ and $\mathscr{L}''_y = \mathscr{L}_y$ for every $y \in \Sigma$. It follows that, for every $y \in \Sigma$, there exists an open neighborhood U_y of y in $X(x)$ such that $\mathscr{L}''_{|U_y} = \mathscr{L}_{|U_y}$. Let $U' := U \cup \bigcup_{y\in\Sigma} U_y$, and denote by $j' : U' \to X(x)$ the open immersion. Notice that $\delta'(y, \mathscr{O}_X) \geq 2$ for every $y \in X \setminus U'$ (corollary 12.5.13). In light of proposition 11.3.8(ii) (and remark 11.3.11(i)), we deduce

$$\mathscr{L} = j'_*\mathscr{L}_{|U'} = j'_*\mathscr{L}''_{|U'} = \mathscr{L}''$$

whence the contention. □

12.6.44. Let $\underline{M}$ be a log structure on the Zariski site of a local scheme X, such that $(X, \underline{M})$ is a regular log scheme. Let $x \in X$ be the closed point, say that $X = \operatorname{Spec} B$ for some local ring B, and let $\beta : P \to B$ a chart for $\underline{M}$ which is sharp at x. As usual, if M is any B-module, we denote $M^\sim$ the quasi-coherent $\mathscr{O}_X$-module arising from M.

Theorem 12.6.45. *In the situation of* (12.6.44)*, suppose as well that* $\dim P = 2$ *and* $\dim X = 2$*. Then every indecomposable reflexive* $\mathscr{O}_X$*-module is isomorphic to* $(IB)^\sim$*, for some reflexive fractional ideal* I *of* P*. (Notation of* (6.4.44)*.)*

Proof. Set $Q := \operatorname{Div}_+(P)$ and define $\varphi : P \to Q$ as in example 6.4.64. The chart β defines a morphism $\psi : (X, \underline{M}) \to \operatorname{Spec}(\mathbb{Z}, P)$, and we let $(X', \underline{M}')$ be the fibre product in the cartesian diagram

$$\begin{array}{ccc} (X', \underline{M}') & \longrightarrow & \operatorname{Spec}(\mathbb{Z}, Q) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \operatorname{Spec}(\mathbb{Z},\varphi)} \\ (X, \underline{M}) & \xrightarrow{\psi} & \operatorname{Spec}(\mathbb{Z}, P). \end{array}$$

Arguing as in the proof of claim 13.3.37, it is easily seen that X' is a local scheme and $(X', \underline{M}')$ is regular. Moreover, X' is a regular scheme, since Q is a free monoid (corollary 12.5.19), and f is a finite morphism of Kummer type (lemma 13.3.6).

Claim 12.6.46. The $\mathscr{O}_X$-module $f_*\mathscr{O}_{X'}$ is isomorphic to a finite direct sum $(I_1B \oplus \cdots \oplus I_kB)^\sim$, where $I_1, \dots, I_k$ are reflexive fractional ideals of P, and $\mathscr{O}_X$ is a direct summand of $f_*\mathscr{O}_{X'}$.

Proof of the claim. In view of example 6.4.64, we see that $f_*\mathscr{O}_{X'} = (Q \otimes_P B)^\sim$ is the direct sum of the $\mathscr{O}_X$-modules $(\operatorname{gr}_\gamma Q \otimes_P B)^\sim$, where γ ranges over the elements of $Q^{\mathrm{gp}}/P^{\mathrm{gp}}$, and $\operatorname{gr}_\bullet Q$ denotes the φ-grading of Q. But for each such γ, the natural map $\operatorname{gr}_\gamma Q \otimes_P B \to \operatorname{gr}_\gamma Q \cdot B$ is an isomorphism, since B is P-flat (proposition 12.5.16(ii)). This shows the first assertion, and the second is clear as well, since $\operatorname{gr}_0 Q = P$. ◊

Denote by x' the closed point of X'; from corollary 12.5.20(i) we see that $U := (X, \underline{M})_1$ is the complement of $\{x\}$ and $U' := (X', \underline{M}')_1$ is the complement of $\{x'\}$ (notation of definition 12.2.7(i)). Let also $j : U \to X$ and $j' : U' \to X'$ be the open immersions.

Claim 12.6.47. (i) The restriction $g : U' \to U$ of f is a flat morphism of schemes.

(ii) The functor $j^* : \mathscr{O}_X\text{-}\mathbf{Rflx} \to \mathscr{O}_U\text{-}\mathbf{Rflx}$ is an equivalence (see definition 11.3.1(iii)).

Proof of the claim. (i): It suffices to check that the restriction $\mathrm{Spec}(\mathbb{Z}, Q)_1 \to \mathrm{Spec}(\mathbb{Z}, P)_1$ of $\mathrm{Spec}(\mathbb{Z}, \varphi)$ is flat. However, we have the affine open covering

$$\mathrm{Spec}(\mathbb{Z}, P)_1 = \mathrm{Spec}\,\mathbb{Z}[P_{\mathfrak{p}_1}] \cup \mathrm{Spec}\,\mathbb{Z}[P_{\mathfrak{p}_2}]$$

where $\mathfrak{p}_1, \mathfrak{p}_2 \subset P$ are the two prime ideals of height one (see example 6.4.19(i)). Hence, we are reduced to showing that the morphism of log schemes underlying

$$\mathrm{Spec}(\mathbb{Z}, \varphi_{\mathfrak{p}_i}) : \mathrm{Spec}(\mathbb{Z}, Q_{\mathfrak{p}_i}) \to \mathrm{Spec}(\mathbb{Z}, P_{\mathfrak{p}_i})$$

is flat for $i = 1, 2$. However, it is clear that $Q_{\mathfrak{p}_i}$ is an integral $P_{\mathfrak{p}_i}$-module, hence it suffices to check that $Q_{\mathfrak{p}_i}$ is a flat $P_{\mathfrak{p}_i}$-module, for $i = 1, 2$ (proposition 6.1.52), or equivalently, that $Q^\sharp_{\mathfrak{p}_i}$ is a flat $P^\sharp_{\mathfrak{p}_i}$-module (corollary 6.1.49(ii)). The latter assertion follows immediately from the discussion of (6.4.61).

(ii): From proposition 11.3.8 we see that j^* is full and essentially surjective. Moreover, it follows from remark 11.3.5 that every reflexive $\mathscr{O}_X$-module is S_1, so j^* is also faithful (details left to the reader). ♢

In light of claim 12.6.47(ii), it suffices to show that every indecomposable reflexive $\mathscr{O}_U$-module $\mathscr{F}$ is isomorphic to $(IB)^\sim_{|U}$, for some reflexive fractional ideal I of P. However, for such $\mathscr{F}$, claim 12.6.47(i) and lemma 11.3.7(i) imply that $g^*\mathscr{F}$ is a reflexive $\mathscr{O}_{U'}$-module. From proposition 11.3.8 and corollary 10.4.23 we deduce that $\delta'(x', j'_*g^*\mathscr{F}) \geq 2$, so $j'_*g^*\mathscr{F}$ is a free $\mathscr{O}_{X'}$-module of finite rank ([63, Ch.0, Prop.17.3.4]), and finally, $g^*\mathscr{F}$ is a free $\mathscr{O}_{U'}$-module of finite rank. Taking into account claim 12.6.46, it follows that $g_*g^*\mathscr{F} = \mathscr{F} \otimes_{\mathscr{O}_U} g_*\mathscr{O}_{U'}$ is a direct sum of $\mathscr{O}_U$-modules of the type $(IB)^\sim_{|U}$, for various $I \in \mathrm{Div}(P)$; moreover, $\mathscr{F}$ is a direct summand of $g_*g^*\mathscr{F}$. Then, we may find a decomposition $g_*g^*\mathscr{F} = \mathscr{F}_1 \oplus \cdots \oplus \mathscr{F}_t$ such that $\mathscr{F}_i$ is an indecomposable $\mathscr{O}_U$-module for $i = 1, \dots, t$, and $\mathscr{F}_1 = \mathscr{F}$ (details left to the reader : notice that – since reflexive $\mathscr{O}_U$-modules are S_1 – the length t of such a decomposition is bounded by the dimension of the $\kappa(\eta)$-vector space $(g_*g^*\mathscr{F})_\eta$, where η is the maximal point of X). On the other hand, notice that

$$\begin{aligned}
\mathrm{End}_{\mathscr{O}_U}((IB)^\sim_{|U}) &= \mathrm{End}_B(IB) && \text{(claim 12.6.47(ii))}\\
&= (IB : IB) && \text{(by (6.4.53))}\\
&= (I : I)B && \text{(lemma 6.4.45(i))}\\
&= B && \text{(proposition 6.4.55(i))}
\end{aligned}$$

for every $I \in \mathrm{Div}(P)$. Now the contention follows from theorem 7.11.3. □

12.6.48. Let now $(X_{\text{ét}}, \underline{M})$ be a quasi-coherent log scheme on the étale site of X. Pick a covering family $(U_\lambda \mid \lambda \in \Lambda)$ of X in $X_{\text{ét}}$, and for every $\lambda \in \Lambda$, a chart $P_{\lambda,U_\lambda} \to \underline{M}_{|U_\lambda}$. The latter induce isomorphisms of log schemes $(U_\lambda, \underline{M}_{|U_\lambda}) \xrightarrow{\sim} \widetilde{u}^*(U_{\lambda,\mathrm{Zar}}, P^{\log}_{U_{\lambda,\mathrm{Zar}}})$ (see (12.2.2)); in other words, every quasi-coherent log structure on $X_{\text{ét}}$ descends, locally on $X_{\text{ét}}$, to a log structure on the Zariski site. However, $(X_{\text{ét}}, \underline{M})$ may fail to descend to a unique log structure on the whole of X_{Zar}. Our present aim is to show that, at least under a few more assumptions on $\underline{M}$, we may find a blow up $(X'_{\text{ét}}, \underline{M}') \to (X_{\text{ét}}, \underline{M})$ such that $(X'_{\text{ét}}, \underline{M}')$ descends to a log structure on X'_{Zar}. To begin with, for every $\lambda \in \Lambda$ let

$$T_\lambda := (\mathrm{Spec}\, P_\lambda)^\sharp \qquad \text{and} \qquad S_\lambda := \mathrm{Spec}\,\mathbb{Z}[P_\lambda].$$

Also, let $\underline{S}_\lambda := (\mathrm{Spec}(\mathbb{Z}, P_\lambda), T_\lambda, \psi_{P_\lambda})$ be the object of $\mathscr{K}$ attached to P_λ, as in example 12.6.5(i). From the isomorphism

$$(U_{\lambda,\mathrm{Zar}}, \widetilde{u}_*\underline{M}_{|U_\lambda}) \xrightarrow{\sim} U_\lambda \times_{S_\lambda} \mathrm{Spec}(\mathbb{Z}, P_\lambda)$$

we deduce an object

$$\underline{U}_\lambda := U_\lambda \times_{S_\lambda} \underline{S}_\lambda = ((U_{\lambda,\mathrm{Zar}}, \widetilde{u}_*\underline{M}_{|U_\lambda}), T_\lambda, \psi_\lambda).$$

Suppose now that $\underline{M}$ is a fs log structure; then we may choose for each P_λ a fine and saturated monoid (lemma 12.1.18(iii)). Next, suppose that X is quasi-compact; in this case we may assume that Λ is a finite set, hence $\mathscr{S} := \{P_\lambda \mid \lambda \in \Lambda\}$ is a finite set of monoids, and consequently we may choose a finite sequence of integers $\underline{c}(\mathscr{S})$ fulfilling the conditions of (6.6.25) relative to the category $\mathscr{S}$-**Fan**. Then, for every $\lambda \in \Lambda$ we have a well defined integral roof $\rho_\lambda : T_\lambda(\mathbb{Q}_+) \to \mathbb{Q}_+$, and we denote by $f_\lambda : T(\rho_\lambda) \to T_\lambda$ the corresponding subdivision. There follows a cartesian morphism

$$f_\lambda^* \underline{U}_\lambda \to \underline{U}_\lambda$$

whose underlying morphism of log schemes is a saturated blow up of the ideal $\mathscr{I}_{\rho_\lambda} \widetilde{u}_*\underline{M}_{|U_\lambda}$ (proposition 6.6.13 and corollary 12.6.18). Next, for $\lambda, \mu \in \Lambda$, set $U_{\lambda\mu} := U_\lambda \times_X U_\mu$ and let

$$\underline{U}_{\lambda\mu} := ((U_{\lambda\mu}, \widetilde{u}_*\underline{M}_{|U_{\lambda\mu}}), T_\lambda, \psi_{\lambda\mu})$$

where $\psi_{\lambda\mu}$ is the composition of ψ_λ and the projection $U_{\lambda\mu} \to U_\lambda$. Denote by $(U_\lambda^\sim, \underline{M}_\lambda^\sim)$ (resp. $(U_{\lambda\mu}^\sim, \underline{M}_{\lambda\mu}^\sim)$) the log scheme underlying $f_\lambda^* \underline{U}_\lambda$ (resp. $f_\lambda^* \underline{U}_{\lambda\mu}$). Also, for any $\lambda, \mu, \gamma \in \Lambda$, let $\pi_{\lambda\mu\gamma} : U_{\lambda\mu} \times_X U_\gamma \to U_{\lambda\mu}$ be the natural projection.

Lemma 12.6.49. *In the situation of* (12.6.48)*, we have :*

(i) *There exists a unique isomorphism of log schemes* $g_{\lambda\mu} : (U_{\lambda\mu}^\sim, \underline{M}_{\lambda\mu}^\sim) \xrightarrow{\sim} (U_{\mu\lambda}^\sim, \underline{M}_{\mu\lambda}^\sim)$ *fitting into a commutative diagram :*

$$\begin{array}{ccc} (U_{\lambda\mu}^\sim, \underline{M}_{\lambda\mu}^\sim) & \xrightarrow{g_{\lambda\mu}} & (U_{\mu\lambda}^\sim, \underline{M}_{\mu\lambda}^\sim) \\ \downarrow & & \downarrow \\ U_{\lambda\mu} & = & U_{\mu\lambda} \end{array}$$

whose vertical arrows are the saturated blow up morphisms.

(ii) *There exist natural isomorphisms of* $\mathscr{O}_{U_{\lambda\mu}}$*-modules*

$$\omega_{\lambda\mu} : g_{\lambda\mu}^* \mathscr{O}_{U_{\mu\lambda}}(1) \xrightarrow{\sim} \mathscr{O}_{U_{\lambda\mu}}(1)$$

such that $(\pi_{\mu\gamma\lambda}^* \omega_{\mu\lambda}) \circ (\pi_{\lambda\mu\gamma}^* \omega_{\lambda\mu}) = \pi_{\lambda\gamma\mu}^* \omega_{\lambda\gamma}$ *for every* $\lambda, \mu, \gamma \in \Lambda$.

Proof. (i): By the universal property of the saturated blow up, it suffices to show that :

$$\mathscr{I}_{\rho_\lambda} \widetilde{u}_*\underline{M}_{|U_{\lambda\mu}} = \mathscr{I}_{\rho_\mu} \widetilde{u}_*\underline{M}_{|U_{\mu\lambda}} \qquad \text{on } U_{\lambda\mu} = U_{\mu\lambda}. \tag{12.6.50}$$

The assertion is local on $U_{\lambda\mu}$, hence let $x \in U_{\lambda\mu}$ be any point; we get an isomorphism of $\widetilde{u}_*\underline{M}_x$-monoids :

$$\mathscr{O}_{T_\lambda, \psi_{\lambda\mu}(x)} \xrightarrow{\sim} \mathscr{O}_{T_\mu, \psi_{\mu\lambda}(x)}$$

whence an isomorphism of fans $U(\psi_{\lambda\mu}(x)) \xrightarrow{\sim} U(\psi_{\mu\lambda}(x))$ (notation of (6.5.16)). Therefore, the subset $U(x) := \psi_{\lambda\mu}^{-1} U(\psi_{\lambda\mu}(x)) \cap \psi_{\mu\lambda}^{-1} U(\psi_{\mu\lambda}(x))$ is an open neighborhood of x in $U_{\lambda\mu}$, and both $\psi_{\lambda\mu}$ and $\psi_{\mu\lambda}$ factor through the same morphism of monoidal spaces :

$$\psi(x) : (U(x), (\widetilde{u}_*\underline{M}^\sharp)_{|U(x)}) \to F(x) := (\operatorname{Spec} \widetilde{u}_*\underline{M}_x)^\sharp$$

and open immersions $F(x) \to T_\lambda$ and $F(x) \to T_\mu$. It then follows from (6.6.25) that the preimage of $\mathscr{I}_{\rho_\lambda}$ on $F(x)$ agrees with the preimage of $\mathscr{I}_{\rho_\mu}$, whence the contention.

(ii): By inspecting the definitions, it is easily seen that the epimorphism (10.6.24) identifies naturally $\mathscr{O}_{U_{\lambda\mu}^\sim}(1)$ to the ideal $\mathscr{I}_{\rho_{\lambda\mu}} \mathscr{O}_{U_{\lambda\mu}^\sim}$ of $\mathscr{O}_{U_{\lambda\mu}^\sim}$ generated by the image of $\mathscr{I}_{\rho_\lambda} \widetilde{u}_*\underline{M}_{|U_{\lambda\mu}}$. Hence the assertion follows again from (12.6.50). □

12.6.51. Lemma 12.6.49 implies that

$$((U^\sim_\lambda, \mathscr{O}_{U^\sim_\lambda}(1)), (g_{\lambda\mu}, \omega_{\lambda\mu}) \mid \lambda, \mu \in \Lambda)$$

is a descent datum – relative to the faithfully flat and quasi-compact morphism $\coprod_{\lambda\in\Lambda} U_\lambda \to X$ – for the fibred category of schemes endowed with an ample invertible sheaf. According to [82, Exp.VIII, Prop.7.8], such a datum is effective, hence it yields a projective morphism $\pi : X^\sim \to X$ together with an ample invertible sheaf $\mathscr{O}_{X^\sim}(1)$ on $X^\sim$, with isomorphisms $g_\lambda : X^\sim \times_X U_\lambda \stackrel{\sim}{\to} U^\sim_\lambda$ of U_λ-schemes and $\pi^*_\lambda \mathscr{O}_{X^\sim}(1) \stackrel{\sim}{\to} g^*_\lambda \mathscr{O}_{U^\sim_\lambda}(1)$ of invertible modules.

Then the datum $(\widetilde{u}^* \underline{M}^\sim_\lambda, \widetilde{u}^* \log g_{\lambda\mu} \mid \lambda, \mu \in \Lambda)$ likewise determines a unique sheaf of monoids $\underline{M}^\sim$ on $X^\sim_{\text{ét}}$, and the structure maps of the log structures $\underline{M}_\lambda$ glue to a well defined morphism of sheaves of monoids $\underline{M}^\sim \to \mathscr{O}_{X^\sim_{\text{ét}}}$, so that $(X^\sim_{\text{ét}}, \underline{M}^\sim)$ is a log scheme, and the projection π extends to a morphism of log schemes $(\pi, \log \pi) : (X^\sim_{\text{ét}}, \underline{M}^\sim) \to (X_{\text{ét}}, \underline{M})$.

Proposition 12.6.52. *In the situation of* (12.6.51)*, the counit of adjunction*

$$\widetilde{u}^* \widetilde{u}_*(X^\sim_{\text{ét}}, \underline{M}^\sim) \to (X^\sim_{\text{ét}}, \underline{M}^\sim)$$

is an isomorphism.

Proof. (This is the counit of the adjoint pair $(\widetilde{u}^*, \widetilde{u}_*)$ of (12.1.6), relating the categories of log structures on $X^\sim_{\text{Zar}}$ and $X^\sim_{\text{ét}}$.) Recall that there exist natural epimorphisms $(\mathbb{Q}^{\oplus d}_+)_{T(\rho_\lambda)} \to \mathscr{O}_{T(\rho_\lambda),\mathbb{Q}}$ (see (6.6.27)), which induce epimorphisms of $U^\sim_{\lambda,\text{Zar}}$-monoids

$$\vartheta_\lambda : (\mathbb{Q}^{\oplus d}_+)_{U^\sim_{\lambda,\text{Zar}}} \to (\underline{M}^\sim_\lambda)^\sharp_\mathbb{Q} \qquad \text{for every } \lambda \in \Lambda.$$

The compatibility with open immersions expressed by (6.6.28) implies that the system of maps $(\widetilde{u}^* \vartheta_\lambda \mid \lambda \in \Lambda)$ glues to a well defined epimorphism of $X^\sim_{\text{ét}}$-monoids :

$$\vartheta : (\mathbb{Q}^{\oplus d}_+)_{X^\sim_{\text{ét}}} \to (\underline{M}^\sim)^\sharp_\mathbb{Q}.$$

In view of lemma 4.9.27(ii), it follows that the counit of adjunction :

$$\widetilde{u}^* \widetilde{u}_* (\underline{M}^\sim)^\sharp_\mathbb{Q} \to (\underline{M}^\sim)^\sharp_\mathbb{Q}$$

is an isomorphism. By applying again lemma 4.9.27(ii) to the monomorphism $(\underline{M}^\sim)^\sharp \to (\underline{M}^\sim)^\sharp_\mathbb{Q}$, we deduce that also the counit

$$\widetilde{u}^* \widetilde{u}_* (\underline{M}^\sim)^\sharp \to (\underline{M}^\sim)^\sharp$$

is an isomorphism. Then the assertion follows from proposition 12.2.3(iii). □

Corollary 12.6.53. *Let $(X_{\text{ét}}, \underline{M})$ be a quasi-compact regular log scheme. Then there exists a smooth morphism of log schemes $(X'_{\text{ét}}, \underline{M}') \to (X_{\text{ét}}, \underline{M})$ whose underlying morphism of schemes is proper and birational, and such that X' is a regular scheme. More precisely, f restricts to an isomorphism $(X'_{\text{ét}}, \underline{M}')_{\text{tr}} \to (X_{\text{ét}}, \underline{M})_{\text{tr}}$ on the trivial loci.*

Proof. Given such $(X_{\text{ét}}, \underline{M})$, we construct first the morphism $\pi : (X^\sim_{\text{ét}}, \underline{M}^\sim) \to (X_{\text{ét}}, \underline{M})$ as in (12.6.51). Since $\pi \times_X \mathbf{1}_{U_\lambda}$ is proper for every $\lambda \in \Lambda$, if follows that π is proper ([64, Ch.IV, Prop.2.7.1]). Likewise, notice that each morphism $U^\sim_\lambda \to U_\lambda$ induces an isomorphism $(U^\sim_\lambda, \underline{M}^\sim_\lambda)_{\text{tr}} \stackrel{\sim}{\to} (U_\lambda, \widetilde{u}_* \underline{M}_{|U_\lambda})_{\text{tr}}$ (remark 12.6.17). It follows easily that π restricts an isomorphism on the trivial loci. Hence, we may replace $(X_{\text{ét}}, \underline{M})$ by $(X^\sim_{\text{ét}}, \underline{M}^\sim_{\text{ét}})$. Then, by corollary 12.3.27(ii) and proposition 12.6.52 we are further reduced to showing that there exists a proper morphism $\pi' : (X'_{\text{Zar}}, \underline{M}') \to \widetilde{u}_*(X^\sim_{\text{ét}}, \underline{M}^\sim)$ with X' regular, restricting to an isomorphism on the trivial loci. However, in light of lemma 12.5.21 (and again, proposition 12.6.52), the sought π' is provided by the more precise theorem 12.6.32. □

12.7. **Local properties of the fibres of a smooth morphism.** Resume the situation of example 12.6.5(ii), and to ease notation set $\varphi := (\mathrm{Spec}\,\lambda)^\sharp$, and $(f, \log f) := \mathrm{Spec}(R, \lambda)$. Suppose now, that $\lambda : P \to Q$ is an integral, local and injective morphism of fine monoids. Then $f : S' \to S$ is flat and finitely presented (theorem 6.2.3). Moreover :

Lemma 12.7.1. *In the situation of* (12.7)*, for every point $s \in S$, the fibre $f^{-1}(s)$ is either empty, or else it is pure-dimensional, of dimension*

$$\dim f^{-1}(s) = \dim Q - \dim P = \mathrm{rk}_{\mathbb{Z}}\,\mathrm{Coker}\,\lambda^{\mathrm{gp}}.$$

Proof. To begin with, notice that $\lambda^{-1}Q^\times = P^\times$, whence the second stated identity, in view of corollary 6.4.12(i). To prove the first stated identity, we easily reduce to the case where R is a field. Notice that the image of f is an open subset $U \subset S$ ([64, Ch.IV, Th.2.4.6]), especially U (resp. S') is pure-dimensional of dimension $\mathrm{rk}_{\mathbb{Z}} P^{\mathrm{gp}}$ (resp. $\mathrm{rk}_{\mathbb{Z}} Q^{\mathrm{gp}}$) by claim 11.6.37(ii). Hence, fix any closed point $s \in S$, and set $X := f^{-1}(s)$. From [64, Ch.IV, Cor.6.1.2] we deduce that, for every closed point $s' \in X$, the Krull dimension of $\mathscr{O}_{X,s'}$ equals $r := \mathrm{rk}_{\mathbb{Z}}\mathrm{Coker}\,\lambda^{\mathrm{gp}}$. More precisely, say that $Z \subset X$ is any irreducible component; we may find a closed point $s' \in Z$ which does not lie on any other irreducible component of X, and then the foregoing implies that the dimension of Z equals r, as stated.

Next, let $s \in U$ be any point, and denote K the residue field of $\mathscr{O}_{U,s}$, and $\pi : R[P] \to K$ the natural map. Let $y \in \mathrm{Spec}\,K[P]$ be the K-rational closed point such that $a(y) = \pi(a)$ for every $a \in P$; then the image of y in S equals s, and if we let $f_K := \mathrm{Spec}\,K[\lambda]$, we have an isomorphism of K-schemes $f_K^{-1}(y) \stackrel{\sim}{\to} f^{-1}(s)$. The foregoing shows that $f_K^{-1}(y)$ is pure-dimensional of dimension r, hence the same holds for $f^{-1}(s)$. □

Now, let us fix $s \in S$, such that $f^{-1}(s) \neq \varnothing$, and suppose that $\psi_P(s) = \mathfrak{m}_P$ is the closed point of T_P. For every $\mathfrak{q} \in \varphi^{-1}(\mathfrak{m}_P)$, the closure $\overline{\{\mathfrak{q}\}}$ of $\{\mathfrak{q}\}$ in T_Q is the image of the natural map $\mathrm{Spec}\,Q/\mathfrak{q} \to T_Q$. We deduce natural isomorphisms of schemes :

$$S_0 := \psi_P^{-1}\{\mathfrak{m}_\mathfrak{p}\} \stackrel{\sim}{\to} \mathrm{Spec}\,R\langle P/\mathfrak{m}_P\rangle \qquad S'_\mathfrak{q} := \psi_Q^{-1}\overline{\{\mathfrak{q}\}} \stackrel{\sim}{\to} \mathrm{Spec}\,R\langle Q/\mathfrak{q}\rangle$$

under which, the restriction $f_\mathfrak{q} : S'_\mathfrak{q} \to S_0$ of f is identified with $\mathrm{Spec}\,R\langle\lambda_\mathfrak{q}\rangle$, where $\lambda_\mathfrak{q} : P/\mathfrak{m}_P \to Q/\mathfrak{q}$ is induced by λ. The latter is an integral and injective morphism as well (corollary 6.1.51). Explicitly, set $F := Q\setminus\mathfrak{q}$, and let $\lambda_F : P^\times \to F$ be the restriction of λ; then $\lambda_\mathfrak{q} = (\lambda_F)_\circ$, and lemma 12.7.1 yields the identity :

$$\dim f_\mathfrak{q}^{-1}(s) = \dim Q/\mathfrak{q} = \dim Q - \mathrm{ht}\,\mathfrak{q}. \tag{12.7.2}$$

Also, notice that T_Q is a finite set under the current assumptions (lemma 6.1.20(iii)), and clearly

$$f^{-1}(s) = \bigcup_{\mathfrak{q}\in\mathrm{Max}(\varphi^{-1}\mathfrak{m}_P)} f_\mathfrak{q}^{-1}(s)$$

where, for a topological space T, we denote by $\mathrm{Max}(T)$ the set of maximal points of T. Therefore, for every irreducible component Z of $f^{-1}(s)$ there must exist $\mathfrak{q} \in \mathrm{Max}(\varphi^{-1}\mathfrak{m}_P)$ such that $Z \subset f_\mathfrak{q}^{-1}(s)$, and especially,

$$\dim f^{-1}(s) = \dim f_\mathfrak{q}^{-1}(s). \tag{12.7.3}$$

Conversely, we claim that (12.7.3) holds for every $\mathfrak{q} \in \mathrm{Max}(\varphi^{-1}\mathfrak{m}_P)$. Indeed, suppose that $\dim f_\mathfrak{q}^{-1}(s) < \dim f^{-1}(s)$ for one such $\mathfrak{q}$, and let Z be an irreducible component of $f_\mathfrak{q}^{-1}(s)$; let also Z' be an irreducible component of $f^{-1}(s)$ containing Z. By the foregoing, there exists $\mathfrak{q}' \in \mathrm{Max}(\varphi^{-1}\mathfrak{m}_P)$ with $Z' \subset f_{\mathfrak{q}'}^{-1}(s)$. Set $\mathfrak{q}'' := \mathfrak{q} \cup \mathfrak{q}'$; then clearly $\mathfrak{q}'' \in \varphi^{-1}(\mathfrak{m}_P)$, and

$$\overline{\{\mathfrak{q}''\}} = \overline{\{\mathfrak{q}\}} \cap \overline{\{\mathfrak{q}'\}}.$$

Especially, $Z \subset f_{\mathfrak{q}''}^{-1}(s)$; however, it follows from (12.7.2) that $\dim f_{\mathfrak{q}''}^{-1}(s) < \dim f_{\mathfrak{q}}^{-1}(s)$ (since $\mathrm{ht}\,\mathfrak{q} < \mathrm{ht}\,\mathfrak{q}''$); but this is absurd, since $\dim Z = \dim f_{\mathfrak{q}}^{-1}(s)$ (lemma 12.7.1). The same counting argument also shows that every maximal point of $f^{-1}(s)$ gets mapped necessarily to a maximal point of $\varphi^{-1}(\mathfrak{m}_P)$; in other words, we have shown that ψ_Q restricts to a surjective map :

$$\mathrm{Max}(f^{-1}s) \to \mathrm{Max}(\varphi^{-1}\mathfrak{m}_P).$$

More precisely, let $s' \in f^{-1}(s)$ be a point such that $\psi_Q(s') = \mathfrak{m}_Q$, the closed point of T_Q. Then clearly $s' \in f_{\mathfrak{q}}^{-1}(s)$ for every $\mathfrak{q} \in \mathrm{Max}(\varphi^{-1}\mathfrak{m}_P)$, and it follows that the foregoing surjection restricts further to a surjective map :

$$f_{s'}^{-1}(s) \to \mathrm{Max}(\varphi^{-1}\mathfrak{m}_P). \qquad (12.7.4)$$

On the other hand, since $\log \psi_Q$ is an isomorphism, we have

$$(Q_{S'}^{\log})^\sharp_{\overline{s}'} = \mathscr{O}_{T_Q,\mathfrak{q}} = Q^\sharp_{\mathfrak{q}} \qquad \text{for every } s' \in \psi_Q^{-1}(\mathfrak{q})$$

(where $\overline{s}'$ denotes any τ-point of S' localized at s'); explicitly, if $F = Q \setminus \mathfrak{q}$, then $Q^\sharp_{\mathfrak{q}} = Q/F$; likewise, we get $(P_S^{\log})^\sharp_{\overline{s}} = P/\varphi^{-1}F$. Taking into account (12.7.2), we deduce :

$$\dim f^{-1}(f(s')) - \dim f_{\mathfrak{q}}^{-1}(f(s')) = \mathrm{rk}_{\mathbb{Z}}\,\mathrm{Coker}\,(\log f)^{\mathrm{gp}}_{\overline{s}'} \qquad \text{for every } s' \in \psi_Q^{-1}(\mathfrak{q}).$$

Thus, for every $n \in \mathbb{N}$, let

$$U_n := \{s' \in S' \mid \mathrm{rk}_{\mathbb{Z}}\,\mathrm{Coker}\,(\log f)^{\mathrm{gp}}_{\overline{s}'} = n\}.$$

The foregoing implies that for every $s \in S$, the subset $U_0 \cap f^{-1}(s)$ is open and dense in $f^{-1}(s)$, and $U_n \cap f^{-1}(s)$ is either empty, or else it is a subset of pure codimension n in $f^{-1}(s)$.

12.7.5. In the situation of (12.7), suppose moreover that $\log f$ is saturated; notice that this condition holds if and only if $\log\varphi$ is saturated, if and only if the same holds for λ (lemma 6.2.12(iii)). Then, corollary 6.2.32(ii) says that $\mathrm{Coker}(\log f^\sharp)^{\mathrm{gp}}_{\overline{s}'}$ is torsion-free for every $s' \in S'$; especially, $\mathrm{Coker}(\log f^\sharp)^{\mathrm{gp}}_{\overline{s}'}$ vanishes for every $s' \in U_0$. Then corollary 6.2.32(i) implies that $\log f^\sharp_{\overline{s}'}$ is an isomorphism for every $s' \in U_0$, in which case the same holds for $\log f_{\overline{s}'}$ (lemma 12.1.4); in other words, U_0 is the strict locus of f (see definition 12.2.7(ii)).

Lemma 12.7.6. *Let K be an algebraically closed field of characteristic p, and $\chi : P \to (K, \cdot)$ a local morphism of monoids. Let also $\lambda : P \to Q$ be as in* (12.7.5). *We have :*

(i) *The K-algebra $Q \otimes_P K$ is Cohen-Macaulay.*
(ii) *If moreover, either $p = 0$, or else the order of the torsion subgroup of $\Gamma := \mathrm{Coker}\,\lambda^{\mathrm{gp}}$ is not divisible by p, then $Q \otimes_P K$ is reduced (*i.e. *its nilradical is trivial).*

Proof. (i): Since χ is a local morphism, we have $\chi^{-1}(0) = \mathfrak{m}_P$, and χ is determined by its restriction $P^\times \to K^\times$, which is a homomorphism of abelian groups. Notice that $K^\times$ is divisible, hence it is injective in the category of abelian groups; since the unit of adjunction $P \to P^{\mathrm{sat}}$ is local (lemma 6.2.9(iii)), it follows that χ extends to a local morphism $\chi' : P^{\mathrm{sat}} \to K$. Notice that $Q \otimes_P K = Q^{\mathrm{sat}} \otimes_{P^{\mathrm{sat}}} K$ (lemma 6.2.12(iv)), hence we may replace P by P^{sat} and Q by Q^{sat}, which allows to assume from start that Q is saturated. In this case, by theorem 11.6.35(i), both $K[P]$ and $K[Q]$ are Cohen-Macaulay; since $K[\lambda]$ is flat, theorem 10.4.38 and [64, Ch.IV, Cor.6.1.2] imply that $Q \otimes_P K$ is Cohen-Macaulay as well.

(ii): Let $Q = \bigoplus_{\gamma\in\Gamma} Q_\gamma$ be the λ-grading of Q (see remark 6.2.5(iii)). Under the current assumptions, λ is exact (lemma 6.2.30(ii)), consequently $Q_0 = P$ (remark 6.2.5(v)). Moreover, Q_γ is a finitely generated P-module, for every $\gamma \in \Gamma$ (corollary 6.4.5), hence either $Q_\gamma = \varnothing$, or else Q_γ is a free P-module of rank one, say generated by an element u_γ (remark 6.2.5(iv)). Furthermore, $Q_\gamma^k = Q_{k\gamma}$ for every integer $k > 0$ and every $\gamma \in \Gamma$ (proposition 6.2.31). Thus, whenever $Q_\gamma \neq \varnothing$, the element $u_\gamma \otimes 1$ is a basis of the K-vector space $Q_\gamma \otimes_P K$, and $(u_\gamma \otimes 1)^k$ is a basis of $Q_{k\gamma} \otimes_P K$, for every $k > 0$; especially, $u_\gamma \otimes 1$ is not nilpotent. In view of proposition 7.6.28(ii), the contention follows. □

12.7.7. Let $f : (X, \underline{M}) \to (Y, \underline{N})$ be a smooth and log flat morphism of fine log schemes. For every $n \in \mathbb{N}$, let $U(f, n) \subset X$ be the subset of all $x \in X$ such that $\mathrm{rk}_{\mathbb{Z}}\mathrm{Coker}\,(\log f^\sharp_{\overline{x}})^{\mathrm{gp}} = n$, for any τ-point $\overline{x}$ localized at x. By lemma 12.2.21(iii), $U(f, n)$ is a locally closed subset (resp. an open subset) of X, for every $n > 0$ (resp. for $n = 0$).

Theorem 12.7.8. *In the situation of* (12.7.7)*, we have :*

(i) *f is a flat morphism of schemes.*
(ii) *For all $y \in Y$, every connected component of $f^{-1}(y)$, is pure dimensional, and $f^{-1}(y) \cap U_n$ is either empty, or else it has pure codimension n in $f^{-1}(y)$, for every $n \in \mathbb{N}$.*
(iii) *Moreover, if f is saturated, we have :*
  (a) *The strict locus* $\mathrm{Str}(f)$ *of f is open in X, and* $\mathrm{Str}(f) \cap f^{-1}(y)$ *is a dense subset of $f^{-1}(y)$, for every $y \in Y$.*
  (b) *$f^{-1}(y)$ is geometrically reduced and Cohen-Macaulay, for every $y \in Y$.*
  (c) $\mathrm{Coker}(\log f^\sharp_{\overline{x}})^{\mathrm{gp}}$ *is a free abelian group of finite rank, for every τ-point $\overline{x}$ of X.*

Proof. Let ξ be any τ-point of X; according to corollary 12.1.35, there exist a neighborhood V of ξ, and a fine chart $P_V \to \underline{N}_{|V}$, such that P^{gp} is a free abelian group of finite rank, and the induced morphism of monoids $P \to \mathscr{O}_{Y,\xi}$ is local.

By lemma 12.1.14, [64, Ch.IV, Th.2.4.6, Prop.2.5.1] and [66, Ch.IV, Prop.17.5.7], it suffices to prove the theorem for the morphism $f \times_Y V : (X, \underline{M}) \times_Y V \to (V, \underline{N}_{|V})$. Hence we may assume from start that $\underline{N}$ admits a fine chart $\beta : P_Y \to \underline{N}$, such that P^{gp} is torsion-free and the induced map $P \to \mathscr{O}_{Y,\xi}$ is local.

By theorem 12.3.37, remark 12.1.23(i), [64, Ch.IV, Th.2.4.6] and [66, Ch.IV, Prop.17.5.7] (and again lemma 12.1.14) we may further assume that f admits a fine chart $(\beta, \omega_Q : Q_X \to \underline{M}, \lambda)$, such that λ is injective, the torsion subgroup of $\mathrm{Coker}\,\lambda^{\mathrm{gp}}$ is a finite group whose order is invertible in $\mathscr{O}_X$, and the induced morphism of schemes $X \to Y \times_{\mathrm{Spec}\,\mathbb{Z}[Q]} \mathrm{Spec}\,\mathbb{Z}[P]$ is étale. Moreover, by theorem 12.1.37(iii), we may assume – after replacing Q by a localization, and X by a neighborhood of ξ in X_τ – that the morphism $\lambda : P \to Q$ is integral (resp. saturated, if f is saturated), and the morphism $Q \to \mathscr{O}_{X,\xi}$ induced by $\omega_{P,\xi}$ is local, so λ is local as well.

In this case, the same sort of reduction as in the foregoing shows that, in order to prove (i) and (ii), it suffices to consider a morphism f as in (12.7), for which these assertions have already been established. Likewise, in order to show (iii), it suffices to consider a morphism f as in (12.7.5). For such a morphism, assertions (iii.a) and (iii.c) are already known, and (iii.b) is an immediate consequence of lemma 12.7.6. □

Theorem 12.7.8(iii) admits the following partial converse :

Proposition 12.7.9. *Let $f : (X, \underline{M}) \to (Y, \underline{N})$ be a smooth and log flat morphism of fs log schemes, such that $f^{-1}(y)$ is geometrically reduced, for every $y \in Y$. Then f is saturated.*

Proof. Fix a τ-point ξ of X; the assertion can be checked on stalks, hence we may assume that $\underline{N}$ admits a fine and saturated chart $\beta : P_Y \to \underline{N}$ (lemma 12.1.18(iii)), such that the induced morphism $\alpha_P : P \to \mathscr{O}_{Y,f(\xi)}$ is local (claim 12.1.31). Then, by corollary 12.3.42, we may find an étale morphism $g : U \to X$ and a τ-point ξ' of U with $g(\xi') = \xi$, such that the induced morphism of log schemes $f_U : (U, g^*\underline{M}) \to (Y, \underline{N})$ admits a fine and saturated chart $(\beta, \omega_Q : Q_U \to g^*\underline{M}, \lambda)$, where λ is injective, and the induced ring homomorphism

(12.7.10) $$Q \otimes_P \mathscr{O}_{Y,f(\xi)} \to \mathscr{O}_{U,\xi'}$$

is étale. By [66, Ch.IV, Prop.17.5.7], the fibres of f_U are still geometrically reduced, hence we are reduced to the case where $U = X$ and $\xi = \xi'$. Furthermore, after replacing Q by a localization, and X by a neighborhood of ξ, we may assume that the map $\alpha_Q : Q \to \mathscr{O}_{X,\xi}$ induced by $\omega_{Q,\xi}$ is local and λ is integral (theorem 12.1.37(iii) and lemma 6.2.9(i)). Lastly, let K be the residue field of $\mathscr{O}_{Y,f(\xi)}$; our assumption on $f^{-1}(y)$ means that the ring $A := \mathscr{O}_{X,\xi} \otimes_{\mathscr{O}_{Y,f(\xi)}} K$ is reduced.

We shall apply the criterion of proposition 6.2.31. Thus, let $Q = \bigoplus_{\gamma \in \Gamma} Q_\gamma$ be the λ-grading of Q and notice as well that λ is a local morphism (since the same holds for α_P and α_Q), therefore it is exact (lemma 6.2.30(ii)); consequently $Q_0 = P$ (remark 6.2.5(v)). Moreover, Q_γ is a finitely generated P-module, for every $\gamma \in \Gamma$ (corollary 6.4.5), hence either $Q_\gamma = \varnothing$, or else Q_γ is a free P-module of rank one.

We have to prove that $Q_\gamma^k = Q_{k\gamma}$ for every integer $k > 0$ and every $\gamma \in \Gamma$. In case $Q_\gamma = \varnothing$, this is the assertion that $Q_{k\gamma} = \varnothing$ as well. However, since Q is saturated, the same holds for $\Gamma' := (\lambda P)^{-1} Q / (\lambda P)^{\mathrm{gp}}$ (lemma 6.2.9(i,ii)), so it suffices to remark that $\Gamma' \subset \Gamma$ is precisely the submonoid consisting of all those $\gamma \in \Gamma$ such that $Q_\gamma \neq \varnothing$.

Therefore, fix a generator u_γ for every P-module $Q_\gamma \neq \varnothing$, and by way of contradiction, suppose that there exist $\gamma \in \Gamma'$ and $k > 0$ such that u_γ^k does not generate the P-module $Q_{k\gamma}$; this means that there exists $a \in \mathfrak{m}_P$ such that $u_\gamma^k = a \cdot u_{k\gamma}$. Now, notice that the induced morphism of monoids $\beta : P \to K$ is local, especially $\beta(a) = 0$, therefore $(u_\gamma \otimes 1)^k = 0$ in the K-algebra $Q \otimes_P K$. Denote by $I \subset Q \otimes_P K$ the annihilator of $u_\gamma \otimes 1$, and notice that, since (12.7.10) is flat, IA is the annihilator of the image u' of $u_\gamma \otimes 1$ in A.

On the other hand, it is easily seen that I is the graded ideal generated by $(u_\mu \otimes 1 \mid \mu \in \Gamma'')$ where $\Gamma'' \subset \Gamma'$ is the subset of all μ such that $u_\gamma \cdot u_\mu$ is not a generator of the P-module $Q_{\gamma+\mu}$. Clearly $u_\mu \notin Q^\times$ for every $\mu \in \Gamma''$, therefore the image of $u_\mu \otimes 1$ in A lies in the maximal ideal. Therefore $IA \neq A$, *i.e.* u' is a non-zero nilpotent element, a contradiction. □

12.7.11. Let $(X, \underline{M})$ be any log scheme, and $\overline{x}$ any geometric point, localized at a point $x \in X$. Suppose that $y \in X$ is a generization of x, and $\overline{y}$ a geometric point localized at y, and assume first that the log structure $\underline{M}$ is defined on the étale site $X_{\mathrm{\acute{e}t}}$; then, arguing as in (4.9.23), we may extend uniquely any strict specialization morphism $X(\overline{y}) \to X(\overline{x})$ to a morphism of log schemes $(X(\overline{y}), \underline{M}(\overline{y})) \to (X(\overline{x}), \underline{M}(\overline{x}))$ fitting into a commutative diagram

$$\begin{array}{ccc} (X(\overline{y}), \underline{M}(\overline{y})) & \longrightarrow & (X(y), \underline{M}(y)) \\ \downarrow & & \downarrow \\ (X(\overline{x}), \underline{M}(\overline{x})) & \longrightarrow & (X(x), \underline{M}(x)) \end{array}$$

whose right vertical arrow is induced by the natural isomorphism

$$(X(y), \underline{M}(y)) \xrightarrow{\sim} (X(x), \underline{M}(x)) \times_{X(x)} X(y).$$

A simple inspection shows that the induced morphism

$$\Gamma(X(\overline{x}), \underline{M}(\overline{x}))^\sharp \to \Gamma(X(\overline{y}), \underline{M}(\overline{y}))^\sharp$$

is naturally identified with the morphism $\underline{M}^\sharp_{\overline{x}} \to \underline{M}^\sharp_{\overline{y}}$ obtained from the specialization map $\underline{M}_{\overline{x}} \to \underline{M}_{\overline{y}}$. In case $\underline{M}$ is defined on the Zariski site X_{Zar}, we may apply the foregoing construction to the log scheme $\widetilde{u}^*(X, \underline{M})$, to obtain again the strictly local log schemes $(X(\overline{x}), \underline{M}(\overline{x}))$ and $(X(\overline{y}), \underline{M}(\overline{y}))$ a corresponding commutative diagram, as in the foregoing case.

12.7.12. Let $g : (X, \underline{M}) \to (Y, \underline{N})$ be a morphism of log schemes, $\overline{x}$ any geometric point of X, and set $\overline{y} := g(\overline{x})$. The log structures $\underline{M} \to \mathscr{O}_X$ and $\underline{N} \to \mathscr{O}_Y$, and the map $\log g_{\overline{x}}$ induce a commutative diagram of continuous maps

$$\begin{array}{ccc} X(\overline{x}) & \xrightarrow{g_{\overline{x}}} & Y(\overline{y}) \\ {\scriptstyle\psi_{\overline{x}}}\downarrow & & \downarrow{\scriptstyle\psi_{\overline{y}}} \\ \mathrm{Spec}\,\underline{M}_{\overline{x}} & \xrightarrow{\varphi_{\overline{x}}} & \mathrm{Spec}\,\underline{N}_{\overline{y}} \end{array} \tag{12.7.13}$$

(notation of 4.9.20), and notice that $\psi_{\overline{x}}$ (resp. $\psi_{\overline{y}}$) maps the closed point of $X(\overline{x})$ (resp. of $Y(\overline{y})$) to the closed point $t_{\overline{x}} \in \mathrm{Spec}\,\underline{M}_{\overline{x}}$ (resp. $t_{\overline{y}} \in \mathrm{Spec}\,\underline{N}_{\overline{y}}$).

Proposition 12.7.14. *In the situation of* (12.7.12)*, suppose that g is a smooth morphism of fine log schemes, and moreover :*

(a) *either g is a saturated morphism*
(b) *or $(X, \underline{M})$ is a fs log scheme.*

Then the following holds :

(i) *The map $\psi_{\overline{x}}$ restricts to a bijection :*
$$\mathrm{Max}(g_{\overline{x}}^{-1}(\overline{y})) \xrightarrow{\sim} \mathrm{Max}(\varphi_{\overline{x}}^{-1}(t_{\overline{y}})).$$
(ii) *For every irreducible component Z of $g_{\overline{x}}^{-1}(\overline{y})$, set*
$$(Z, \underline{M}(Z)) := (X(\overline{x}), \underline{M}(\overline{x})) \times_{X(\overline{x})} Z.$$
Then the $\kappa(\overline{y})$-log scheme $(Z, \underline{M}(Z)_{\mathrm{red}})$ is geometrically pointed regular. (Notation and terminology of example 12.1.12(iv) *and remark* 12.5.8(ii)*.)*

Proof. By corollary 12.1.35, there exist a neighborhood $U \to Y$ of $\overline{y}$, and a fine chart $\beta : P_U \to \underline{N}_{|U}$ such that $\beta_{\overline{y}}$ is a local morphism, and P^{gp} is torsion-free. Now, let
$$g_{\overline{x}} := (g_{\overline{x}}, \log g_{\overline{x}}) : (X(\overline{x}), \underline{M}(\overline{x})) \to (Y(\overline{y}), \underline{N}(\overline{y}))$$
be the morphism of log schemes induced by g, as in 12.7.11; by theorem 12.3.37, we may find a fine chart for $g_{\overline{x}}$ of the type $(i_{\overline{y}}^*\beta, \omega, \lambda : P \to Q)$, where λ is injective, and the order of the torsion subgroup of $\mathrm{Coker}\,\lambda^{\mathrm{gp}}$ is invertible in $\mathscr{O}_{X,x}$. Moreover, set $R := \mathscr{O}_{Y(\overline{y}),\overline{y}}$; then the induced map $X(\overline{x}) \to \mathrm{Spec}\,Q \otimes_P R$ is pro-étale, and – after replacing Q by some localization – we may assume that $\omega_{\overline{x}} : Q \to \mathscr{O}_{X(\overline{x}),\overline{x}}$ is local (claim 12.1.31), hence the same holds for λ. Furthermore, under assumption (a) (resp. (b)), we may also suppose that Q is saturated, by lemmata 6.2.9(ii) and 12.1.18(ii) (resp. that λ is saturated, by theorem 12.1.37(iii)).

Let us now define $f : S' \to S$ as in (12.7); notice that $\omega_{\overline{x}}$ induces a closed immersion $Y(\overline{y}) \to S$, and we have a natural identification of $Y(\overline{y})$-schemes :
$$\mathrm{Spec}\,Q \otimes_P R = Y(\overline{y}) \times_S S'.$$
Denote by $\overline{s}$ the image of $\overline{y}$ in S, and $\overline{s}'$ the image of $\overline{x}$ in $Y(\overline{y}) \times_S S' \subset S'$; there follows an isomorphism of $Y(\overline{y})$-schemes :
$$X(\overline{x}) \xrightarrow{\sim} Y(\overline{y}) \times_{S(\overline{s})} S'(\overline{s}')$$
([66, Ch.IV, Prop.18.8.10]). Moreover, our chart induces isomorphisms :
$$\mathrm{Spec}\,\underline{M}_{\overline{x}} \xrightarrow{\sim} T_Q \qquad \mathrm{Spec}\,\underline{N}_{\overline{y}} \xrightarrow{\sim} T_P$$
which identify $\varphi_{\overline{x}}$ to the map $\varphi : T_Q \to T_P$ of (12.7). In view of these identifications, we see that (12.7.13) is the restriction to the closed subset $X(\overline{x})$, of the analogous diagram :

$$\begin{array}{ccc} S'(\overline{s}') & \xrightarrow{f_{\overline{s}'}} & S(\overline{s}) \\ {\scriptstyle\psi_{\overline{s}'}}\downarrow & & \downarrow{\scriptstyle\psi_{\overline{s}}} \\ T_Q & \xrightarrow{\varphi} & T_P. \end{array}$$

We may thus assume that $X = S'$, $Y = S$ and $g = f$. Moreover, let s (resp. s') be the support of $\overline{s}$ (resp. of $\overline{s}'$); the morphism $\pi : S'(\overline{s}') \to S'(s')$ is flat, hence it restricts to a surjection

$$\mathrm{Max}(f_{\overline{s}'}^{-1}\overline{s}) \to \mathrm{Max}(f_{s'}^{-1}s).$$

In order to prove (i), it suffices then to show that the map $\mathrm{Max}(f_{\overline{s}'}^{-1}\overline{s}) \to \mathrm{Max}(\varphi^{-1}\mathfrak{m}_P)$, defined as the composition of the foregoing map and the surjection (12.7.4), is injective. This boils down to the assertion that, for every $\mathfrak{q} \in \mathrm{Max}(\varphi^{-1}\mathfrak{m}_P)$ the point s' lies on a unique irreducible component of the fibre $\pi^{-1}(f_{\mathfrak{q},s'}^{-1}s)$. However, let $\overline{\beta} : P \to \kappa(s)$ be the composition of the chart $P \to R$ and the projection $R \to \kappa(s)$; then $f_{\mathfrak{q}}^{-1}(s) = \mathrm{Spec}\, Q/\mathfrak{q} \otimes_P \kappa(s)$. Since π is pro-étale, the assertion will follow from [66, Ch.IV, Prop.17.5.7] and corollary 12.5.13, together with :

Claim 12.7.15. The log scheme $W_{\mathfrak{q}} := \mathrm{Spec}\langle\mathbb{Z}, Q/\mathfrak{q}\rangle \times_{\mathrm{Spec}\,\mathbb{Z}[P]} \mathrm{Spec}\,\kappa(s)$ is geometrically pointed regular.

Proof of the claim. To ease notation, set $F := Q \setminus \mathfrak{q}$; by assumption $\lambda^{-1}F = P^\times$, so that

$$W_{\mathfrak{q}} = (W'_{\mathfrak{q}})_\circ \qquad \text{where} \qquad W'_{\mathfrak{q}} := \mathrm{Spec}(\mathbb{Z}, F) \times_{\mathrm{Spec}(\mathbb{Z},P^\times)} \mathrm{Spec}\,\kappa(s)$$

(notation of (12.2.13); notice that the log structures of $\mathrm{Spec}(\mathbb{Z}, P^\times)$ and $\mathrm{Spec}\,\kappa(s)$ are trivial). Moreover, let $\lambda_F : P^\times \to F$ be the restriction of λ; then λ_F^{gp} is injective, and

$$\mathrm{Coker}\,\lambda_F^{\mathrm{gp}} \subset \mathrm{Coker}\,\lambda^{\mathrm{gp}}$$

(corollary 6.2.33(i)), hence the order of the torsion subgroup of $\mathrm{Coker}\lambda_F^{\mathrm{gp}}$ is invertible in $\mathscr{O}_{S,s}$. Moreover, if λ is saturated, then the same holds for λ_F (corollary 6.2.33(ii)), and it is easily seen that if Q is saturated, the same holds for F. Consequently, the morphism $\mathrm{Spec}(\mathbb{Z}, \lambda_F)$ is smooth (proposition 12.3.34). The same then holds for the morphism $W'_{\mathfrak{q}} \to \mathrm{Spec}\,\kappa(s)$ obtained after base change of $\mathrm{Spec}(\mathbb{Z}, \lambda_F)$ along the morphism $h : \mathrm{Spec}(\mathbb{Z}, P^\times) \to \mathrm{Spec}\,\kappa(s)$ induced by $\overline{\beta}$ (proposition 12.3.24(ii)).

Now we notice that under either of the assumptions (a) or (b), $W'_{\mathfrak{q}}$ is a fs log scheme. Indeed, under assumption (b), this follows by remarking that $\mathrm{Spec}\,\kappa(s)$ is trivially a fs log scheme, and $\mathrm{Spec}(\mathbb{Z}, \lambda_F)$ is saturated. Under assumption (a), $\mathrm{Spec}(\mathbb{Z}, F)$ is a fs log scheme, and it suffices to observe that h is a strict morphism. Lastly, since the morphism $W'_{\mathfrak{q}} \to \mathrm{Spec}\,\kappa(s)$ is obviously saturated, we apply corollary 12.5.29 to conclude. ◊

(ii): In light of the foregoing, we see that, for any irreducible component Z of $g_{\overline{x}}^{-1}(\overline{y})$, there exists a unique $\mathfrak{q}(Z) \in \mathrm{Max}(\varphi^{-1}\mathfrak{m}_P)$ such that Z is isomorphic to the strict henselization of $f_{\mathfrak{q}(Z)}^{-1}(s)$, at the geometric point $\overline{s}'$. Notice now that the log structure of $W_{\mathfrak{q}(Z)}$ is reduced, by virtue of claim 12.7.15 and proposition 12.5.38; then, a simple inspection of the definitions shows that $(Z, \underline{M}(Z)_{\mathrm{red}})$ is isomorphic to the strict henselization $W_{\mathfrak{q}(Z)}(\overline{s}')$. Invoking again claim 12.7.15, we deduce the contention. □

12.7.16. In the situation of (12.7.12), suppose that Y is a normal scheme, and $(Y, \underline{N})_{\mathrm{tr}}$ is a dense subset of Y. Let $\overline{\eta}$ be a geometric point of $Y(\overline{y})$, localized at the generic point η, and

$$(U_{\mathfrak{q}} \mid \mathfrak{q} \in \mathrm{Spec}\,\underline{M}_{\overline{x}})$$

the logarithmic stratification of $(X(\overline{x}), \underline{M}(\overline{x}))$ (see (12.5.35)). Notice that $\psi_{\overline{x}}(g^{-1}(\eta))$ lies in the preimage $\Sigma \subset \mathrm{Spec}\,\underline{M}_{\overline{x}}$ of the maximal point $\varnothing$ of $\mathrm{Spec}\,\underline{N}_{\overline{y}}$; therefore, $g_{\overline{x}}^{-1}(\eta)$ is the union of the subsets $U_{\mathfrak{q}} \times_Y |\eta|$, for all $\mathfrak{q} \in \Sigma$.

Proposition 12.7.17. *In the situation of* (12.7.16)*, suppose that g is a smooth morphism of fine log schemes. Then the following holds :*

(i) *X is a normal scheme.*
(ii) *The scheme $g_{\overline{x}}^{-1}(\overline{\eta})$ is normal and irreducible.*

(iii) *For every* $\mathfrak{q} \in \Sigma$, *the* $\kappa(\eta)$*-scheme* $U_\mathfrak{q} \times_Y |\eta|$ *is non-empty, geometrically normal and geometrically irreducible, of pure dimension* $\dim g_{\overline{x}}^{-1}(\eta) - \mathrm{ht}\,\mathfrak{q}$.

(iv) *Especially, set* $W := (X(\overline{x})\backslash U_\varnothing) \times_Y |\overline{\eta}|$*; then* $\psi_{\overline{x}}$ *induces a bijection :*

$$\mathrm{Max}(W) \xrightarrow{\sim} \{\mathfrak{q} \in \Sigma \mid \mathrm{ht}\mathfrak{q} = 1\}.$$

(v) *For every* $w \in \mathrm{Max}(W)$, *the stalk* $\mathscr{O}_{g_{\overline{x}}^{-1}(\overline{\eta}),w}$ *is a discrete valuation ring.*

Proof. Set $R := \mathscr{O}_{Y(\overline{y}),\overline{y}}$; arguing as in the proof of proposition 12.7.14, we may find :

- a local, flat and saturated morphism $\lambda : P \to Q$ of fine monoids, such that the order of the torsion subgroup of $\mathrm{Coker}\,\lambda^{\mathrm{gp}}$ is invertible in R;
- local morphisms of monoids $P \to R$, $Q \to \mathscr{O}_{X(\overline{x}),\overline{x}}$ which are charts for the log structures deduced from $\underline{N}$ and respectively $\underline{M}$, and such that the induced morphism of $Y(\overline{y})$-schemes $X(\overline{x}) \to \mathrm{Spec}\, Q \otimes_P R$ is pro-étale.

By [66, Ch.IV, Prop.17.5.7], we may then assume that $(X, \underline{M})$ (resp. $(Y, \underline{N})$) is the scheme $\mathrm{Spec}\, Q \otimes_P R$ (resp. $\mathrm{Spec}\, R$), endowed with the log structure deduced from the natural map $Q \to Q \otimes_P R$ (resp. the chart $P \to R$), and g is the natural projection. Suppose first that R is excellent, and let R' be the normalization of R in a finite extension K' of $\mathrm{Frac}(R)$; then R' is also strictly local and noetherian, and if y' denotes the closed point of $Y' := \mathrm{Spec}\, R'$, then the residue field extension $\kappa(y) \subset \kappa(y')$ is purely inseparable. Set

$$(X', \underline{M}') := (X, \underline{M}) \times_Y Y' \qquad (Y', \underline{N}') := (Y, \underline{N}) \times_Y Y'$$

and let $g' : (X', \underline{M}') \to (Y', \underline{N}')$ be the induced morphism of log schemes; it follows especially that the restriction $g'^{-1}(y') \to g^{-1}(y)$ is a homeomorphism on the underlying topological spaces. Hence, there is a geometric point $\overline{x}'$ of X', unique up to isomorphism, whose image in X agrees with $\overline{x}$, and we easily deduce an isomorphism of Y-schemes ([66, Ch.IV, Prop.18.8.10])

$$X'(\overline{x}') \xrightarrow{\sim} X(\overline{x}) \times_Y Y'. \tag{12.7.18}$$

Let η' be the generic point of Y'; by assumption, $\underline{N}'$ is trivial in a Zariski neighborhood of η', hence $(X', \underline{N}') \times_{Y'} |\eta'|$ is a fs log schemes (since g is saturated), and then the same log scheme is also regular (theorem 12.5.28), therefore $g'^{-1}(\eta')$ is a normal scheme (corollary 12.5.13). On the other hand, R' is a Krull domain ([133, Th.33.10]), and g' is flat with reduced fibres (theorem 12.7.8(iii.b)), so X' is a noetherian normal scheme (lemma 9.8.2), consequently the same holds for $X'(\overline{x}')$ ([66, Ch.IV, Prop.18.8.12(i)]); especially, the latter is irreducible, so the same holds for $X'(\overline{x}') \times_{Y'} |\eta'|$. In view of (12.7.18), it follows that $X(\overline{x}) \times_Y |\eta'|$ is also normal and irreducible. Since K' is arbitrary, this completes the proof of (i) and (ii), in this case.

Next, if R is any normal ring, we may write R as the union of a filtered family $(R_i \mid i \in I)$ of excellent normal local subrings ([64, Ch.IV, (7.8.3)(ii,vi)]). For each $i \in I$, denote by $\overline{y}_i$ the image of $\overline{y}$ in $\mathrm{Spec}\, R_i$; then the strict henselization R_i^{sh} of R_i at $\overline{y}_i$ is also a subring of R, so we may replace R_i by R_i^{sh}, which allows to assume that each R_i is strictly local, normal and noetherian ([66, Ch.IV, Prop.18.8.8(iv), Prop.18.8.12(i)]). Up to replacing I by a cofinal subset, we may assume that the image of P lies in R_i, for every $i \in I$. For each $i \in I$, set $X_i := \mathrm{Spec}\, Q \otimes_P R_i$, $Y_i := \mathrm{Spec}\, R_i$, and endow X_i (resp. Y_i) with the log structure $\underline{M}_i$ (resp. $\underline{N}_i$) deduced from the natural map $Q \to Q \otimes_P R_i$ (resp. $P \to R_i$). There follows a system of morphisms of log schemes $g_i : (X_i, \underline{M}_i) \to (Y_i, \underline{N}_i)$ for every $i \in I$, whose limit is the morphism g. Moreover, since $(Y, \underline{N})_{\mathrm{tr}}$ is dense in Y, the image of P lies in $R\backslash\{0\}$, hence it lies in $R_i \backslash \{0\}$ for every $i \in I$, and the latter means that $(Y_i, \underline{N}_i)_{\mathrm{tr}}$ is dense in Y_i, for every $i \in I$. Let $\overline{\eta}_i$ (resp. $\overline{x}_i$) be the image of $\overline{\eta}$ (resp. of $\overline{x}$) in Y_i (resp. in X_i); moreover, for each $i \in I$, let $x_i \in X_i$ be the support of $\overline{x}_i$. By the previous case, we know that $g_{i,\overline{x}_i}^{-1}(\overline{\eta}_i)$ is normal and irreducible. However, X (resp. $g_{\overline{x}}^{-1}(\overline{\eta})$) is the limit of the system of schemes $(X_i \mid i \in I)$ (resp.

$(g^{-1}_{i,\overline{x}_i}(\overline{\eta}_i) \mid i \in I))$, so (i) and (ii) follow. (Notice that the colimit of a filtered system of integral (resp. normal) domains, is an integral (resp. normal) domain : exercise for the reader.)

(iii): For every $\mathfrak{q} \in \operatorname{Spec} Q = \operatorname{Spec} \underline{M}_{\overline{x}}$, set $X_\mathfrak{q} := \operatorname{Spec} Q/\mathfrak{q} \otimes_P R$; since the chart $Q \to \mathscr{O}_{X(\overline{x}),\overline{x}}$ is local, it is clear that $x \in X_\mathfrak{q}$ for every such $\mathfrak{q}$, and :

$$X_\mathfrak{q}(\overline{x}) = \bigcup_{\mathfrak{p} \in \operatorname{Spec} Q/\mathfrak{q}} U_\mathfrak{p}.$$

If $\varphi(\mathfrak{q}) = \varnothing$, the induced map $P \to Q/\mathfrak{q}$ is still flat (corollary 6.1.51), hence the projection $X_\mathfrak{q}(\overline{x}) \to Y$ is a flat morphism of schemes, especially $g^{-1}_{\overline{x}}(\eta) \cap X_\mathfrak{q}(\overline{x}) \neq \varnothing$. Furthermore, the subset $X_\mathfrak{q}(\overline{x})$ is pure-dimensional, of codimension $\operatorname{ht} \mathfrak{q}$ in $g^{-1}_{\overline{x}}(\eta)$, by (12.7.2) and [64, Ch.IV, Cor.6.1.4]. It follows that $U_\mathfrak{q}$ is a dense open subset of $X_\mathfrak{q}(\overline{x})$. To conclude, it remains only to show that each $X_\mathfrak{q}(\overline{x})$ is geometrically normal and geometrically irreducible; however, let $j_\mathfrak{q} : X_\mathfrak{q} \to X$ be the closed immersion; the induced morphism of log schemes $g_\mathfrak{q} : (X_\mathfrak{q}, j^*_\mathfrak{q}\underline{M}) \to (Y, \underline{N})$ is also smooth, hence the assertion follows from (ii).

(iv) is a straightforward consequence of (iii).

(v): Notice that $A := \mathscr{O}_{g^{-1}_{\overline{x}}(\overline{\eta}),w}$ is ind-étale over the noetherian ring $Q \otimes_P \kappa(\overline{\eta})$, hence its strict henselization is noetherian, and then A itself is noetherian ([66, Ch.IV, Prop.18.8.8(iv)]). Since X is normal, and w is a point of height one in $g^{-1}_{\overline{x}}(\overline{\eta})$, we conclude that A is a discrete valuation ring. □

13. Étale coverings of schemes and log schemes

13.1. **Acyclic morphisms of schemes.** For any scheme X, we shall denote by :

$$\mathsf{Cov}(X)$$

the category whose objects are the finite étale morphisms $E \to X$; the morphisms $(E \to X) \to (E' \to X)$ are the X-morphisms of schemes $E \to E'$. By faithfully flat descent, $\mathsf{Cov}(X)$ is naturally equivalent to the subcategory of $X^\sim_{\text{ét}}$ consisting of all locally constant constructible sheaves. If $f : X \to Y$ is any morphism of schemes, and $\varphi : E \to Y$ is an object $\mathsf{Cov}(Y)$, then $f^*\varphi := \varphi \times_Y X : E \times_Y X \to X$ is an object of $\mathsf{Cov}(X)$; more precisely, we have a fibration :

$$\mathsf{Cov} \to \mathsf{Sch} \qquad (13.1.1)$$

over the category of schemes, whose fibre, over any scheme X, is the category $\mathsf{Cov}(X)$.

Lemma 13.1.2. *Let f be a morphism of schemes, and suppose that either one of the following conditions holds :*

(a) *f is integral and surjective.*
(b) *f is faithfully flat.*
(c) *f is proper and surjective.*

Then f is of universal 2-descent for the fibred category (13.1.1).

Proof. This is [9, Exp.VIII, Th.9.4]. □

Lemma 13.1.3. *In the situation of definition* 10.5.42*, let $U \subset X$ be any open subset with $Y \subset U$. If $\operatorname{Lef}(X, Y)$ holds, the closed immersion $j : Y \to U$ induces a fully faithful functor*

$$j^* : \mathsf{Cov}(U) \to \mathsf{Cov}(Y).$$

Proof. Let $\mathsf{Cov}(\mathfrak{X})$ be the full subcategory of $\mathscr{O}_\mathfrak{X}$-$\mathsf{Alg}_{\text{lfft}}$ consisting of all finite étale $\mathscr{O}_\mathfrak{X}$-algebras (notation of lemma 10.5.43(ii)); the category $\mathsf{Cov}(U)$ is a full subcategory of $\mathscr{O}_U$-$\mathsf{Alg}_{\text{lfft}}$, so lemma 10.5.43(ii) already implies that the functor $\mathsf{Cov}(U) \to \mathsf{Cov}(\mathfrak{X})$ is fully faithful, hence we are reduced to showing that the functor :

$$\mathsf{Cov}(\mathfrak{X}) \to \mathsf{Cov}(Y) \qquad \mathscr{A} \mapsto \operatorname{Spec} \mathscr{A} \otimes_{\mathscr{O}_\mathfrak{X}} \mathscr{O}_Y$$

is fully faithful. To this aim, let $\mathscr{I} \subset \mathscr{O}_X$ be the ideal defining the closed immersion $Y \subset X$; consider the direct system of schemes

$$(Y_n := \mathrm{Spec}\,\mathscr{O}_X/\mathscr{I}^{n+1} \mid n \in \mathbb{N})$$

and let $\mathsf{Cov}(Y_\bullet)$ be the category consisting of all direct systems $(E_n \mid n \in \mathbb{N})$ of schemes, such that E_n is finite étale over Y_n, and such that the transition maps $E_n \to E_{n+1}$ induce isomorphisms $E_n \stackrel{\sim}{\to} E_{n+1} \times_{Y_{n+1}} Y_n$ for every $n \in \mathbb{N}$. The morphisms in $\mathsf{Cov}(Y_\bullet)$ are the morphisms of direct systems of schemes. We have a natural fully faithful functor :

$$\mathsf{Cov}(\mathfrak{X}) \to \mathsf{Cov}(Y_\bullet) \qquad \mathscr{A} \mapsto (\mathrm{Spec}\,\mathscr{A}/\mathscr{I}^{n+1}\mathscr{A} \mid n \in \mathbb{N})$$

([59, Ch.I, Cor.10.6.10(ii)]). Finally, the functor $\mathsf{Cov}(Y_\bullet) \to \mathsf{Cov}(Y)$ given by the rule : $(E_n \mid n \in \mathbb{N}) \to E_0$ is an equivalence, by [66, Ch.IV, Th.18.1.2]. The claim follows. □

13.1.4. Consider a cofiltered system $X_\bullet := (X_\lambda \mid \lambda \in \Lambda)$ of quasi-compact and quasi-separated schemes, with affine transition morphisms; let X be the limit of $X_\bullet$, and for every $\lambda \in \Lambda$, let $p_\lambda : X \to X_\lambda$ be the natural morphism. The functors $p^*_\lambda : \mathsf{Cov}(X_\lambda) \to \mathsf{Cov}(X)$ define a pseudo-cocone in the 2-category $\mathbf{Cat}$, whence a functor :

$$\text{(13.1.5)} \qquad 2\text{-}\underset{\lambda\in\Lambda}{\mathrm{colim}}\,\mathsf{Cov}(X_\lambda) \to \mathsf{Cov}(X).$$

Lemma 13.1.6. *In the situation of* (13.1.4)*, the functor* (13.1.5) *is an equivalence.*

Proof. It is a rephrasing of [65, Ch.IV, Th.8.8.2, Th.8.10.5] and [66, Ch.IV, Prop.17.7.8(ii)]. □

Lemma 13.1.7. *Let X be a scheme, $j : U \to X$ an open immersion with dense image, and $f : X' \to X$ an integral surjective and radicial morphism. The following holds :*

(i) *The morphism f induces an equivalence of sites*

$$f^* : X'_{\text{ét}} \to X_{\text{ét}}$$

and the functor $f^* : \mathsf{Cov}(X) \to \mathsf{Cov}(X')$ is an equivalence.

(ii) *The functor $j^* : \mathsf{Cov}(X) \to \mathsf{Cov}(U)$ is faithful.*

(iii) *Suppose that X is reduced and normal, and moreover :*
   (a) *either X has finitely many maximal points*
   (b) *or else, j is a quasi-compact open immersion.*

Then j^* is fully faithful, and its essential image consists of all the objects φ of $\mathsf{Cov}(U)$ such that $\varphi \times_X X(\overline{x})$ lies in the essential image of the pull-back functor :

$$\mathsf{Cov}(X(\overline{x})) \to \mathsf{Cov}(X(\overline{x}) \times_X U)$$

for every geometric point $\overline{x}$ of X. (Notation of definition 4.9.17(ii).*)*

(iv) *Furthermore, if X is locally noetherian and regular, and $X\setminus U$ has codimension ≥ 2 in X, then j^* is an equivalence.*

Proof. (i) follows from [9, Exp.VIII, Th.1.1].

(ii): Indeed, let $\varphi : E \to X$ and $\varphi' : E' \to X$ be any two finite étale morphisms, and $f, g : E \to E'$ two morphisms of X-schemes such that $f \times_X U = g \times_X U$; let $\Delta_{E'} \to E' \times_X E'$ be the open and closed diagonal immersion, $(f, g) : E \to E' \times_X E'$ the morphism deduced from f and g, and set $D := (f, g)^{-1}\Delta_{E'}$. Then D is the largest open subset of E' such that $f_{|D} = g_{|D}$; on the other hand, by assumption $\varphi^{-1}U \subset D$, and since φ is an open map, $\varphi^{-1}U$ is dense in E. Lastly, D is also a closed subset of E, so $D = E$, whence the claim.

(iii): Choose a covering $X = \bigcup_{i\in I} V_i$ consisting of affine open subsets, let

$$X' := \coprod_{i\in I} V_i \qquad X'' := X' \times_X X'$$

and denote by $g : X' \to X$ the induced morphism; set also $j'_i := j \times_X V_i$ for every $i \in I$. By lemma 13.1.2, g is of universal 2-descent for the fibred category Cov. On the other hand, the induced open immersion $j'' : U \times_X X'' \to X''$ has dense image, hence the corresponding functor j''^* is faithful, by (ii). By corollary 3.5.30(ii), the full faithfulness of j^* follows from the full faithfulness of the pull-back functor j'^* corresponding to the open immersion $j' := j \times_X X'$. The latter holds if and only if the same holds for all the pull-back functors $j_i'^*$. Hence, we may replace X by V_i, and assume from start that X is affine, say $X = \operatorname{Spec} A$. Let $E \to X$ and $E' \to X$ be two objects of $\mathsf{Cov}(X)$, and $h : E \times_X U \to E' \times_X U$ a morphism, and write $E = \operatorname{Spec} B$, $E' = \operatorname{Spec} B'$ for finite étale A-algebras B and B'; we have to check that h extends to a morphism $E \to E'$.

• To this aim, consider first the case where X has finitely many maximal points $\eta_1, \dots, \eta_s$. Under the current assumptions, A is the product of s domains, and its total ring of fractions $\operatorname{Frac} A$ is the product of fields $\kappa(\eta_1) \times \cdots \times \kappa(\eta_s)$. The restrictions $h_{\eta_i} := h \times_U X(\eta_i) : E(\eta_i) \to E'(\eta_i)$ induce a map of $\operatorname{Frac} A$-algebras

$$h^\natural_\eta := \prod_{i=1}^s h^\natural_{\eta_i} : B' \otimes_A \operatorname{Frac} A \to B \otimes_A \operatorname{Frac} A.$$

On the other hand, by proposition 9.8.3, B (resp. B') is the normalization of A in $B \otimes_A \operatorname{Frac} A$ (resp. in $B' \otimes_A \operatorname{Frac} A$). It follows that $h^\natural_\eta$ restricts to a map $B' \to B$, and the corresponding morphism $E \to E'$ is necessarily an extension of h. This shows that j^* is fully faithful in this case.

• Next, suppose that assumption (b) holds, and set $\mathscr{E} := B^\sim$, $\mathscr{E}' := B'^\sim$; then $\mathscr{E}$ and $\mathscr{E}'$ are étale $\mathscr{O}_X$-algebras and locally free $\mathscr{O}_X$-modules of finite type, and h corresponds to a morphism $h^\sharp : \mathscr{E} \to \mathscr{E}'$. Under the current assumptions, $j_*\mathscr{O}_U$ is a quasi-coherent $\mathscr{O}_X$-algebra, and $\mathscr{O}_{X,x}$ is integrally closed in $(j_*\mathscr{O}_U)_x$, for every $x \in X$ ([75, Prop.8.2.31(i)]). Likewise, $j_*j^*\mathscr{E} = \mathscr{E} \otimes_{\mathscr{O}_X} j_*\mathscr{O}_U$, so $\mathscr{E}'_x$ is the integral closure of $\mathscr{O}_{X,x}$ in $(j_*j^*\mathscr{E})_x$, for every $x \in X$. Lastly, $\mathscr{E}_x$ is integral over $\mathscr{O}_{X,x}$, so the map $(j_*j^*h^\sharp)_x : (j_*j^*\mathscr{E})_x \to (j_*j^*\mathscr{E}')_x$ restricts to a map $\mathscr{E}_x \to \mathscr{E}'_x$, and the assertion follows.

To proceed, we make the following general remark.

Claim 13.1.8. Let Z be a scheme, $V_0 \subset Z$ an open subset, and $\varphi : E \to V_0$ an object of $\mathsf{Cov}(V_0)$. Suppose that, for every open subset $V \subset Z$ containing V_0, the pull-back functor $j_V^* : \mathsf{Cov}(V) \to \mathsf{Cov}(V_0)$ is fully faithful. Let $\mathscr{F}$ be the family consisting of all the data (V, ψ, α) where $V \subset X$ is any open subset with $V_0 \subset V$, $\psi : E_V \to V$ is a finite étale morphism, and $\alpha : \psi^{-1}V_0 \xrightarrow{\sim} E$ is an isomorphism of V_0-schemes. $\mathscr{F}$ is preordered (see example 1.1.6(iii)) by the relation such that $(V, \psi, \alpha) \geq (V', \psi', \alpha')$ if and only if $V' \subset V$ and there is an isomorphism $\beta : \psi^{-1}V' \xrightarrow{\sim} E_{V'}$ of V'-schemes, such that $\alpha' \circ j_{V'}^*(\beta) = \alpha$. Then the partially ordered quotient $\mathscr{F}'$ of $\mathscr{F}$ admits a supremum (see example 1.1.16). (Notice that $\mathscr{F}$ is a proper class, but it can be replaced by a subset that meets every isomorphism class.)

Proof of the claim. Using Zorn's lemma, it is easily seen that every element of $\mathscr{F}'$ can be dominated by a maximal element, and it remains to show that any two maximal elements (V, ψ, α) and (V', α', ψ) of $\mathscr{F}$ are isomorphic; to see this, set $V'' := V \cap V'$: by assumption, the isomorphism $\alpha'^{-1} \circ \alpha : \psi^{-1}V_0 \xrightarrow{\sim} \psi'^{-1}V_0$ extends to an isomorphism of V''-schemes $\psi^{-1}V'' \xrightarrow{\sim} \psi'^{-1}V''$, using which one can glue E_V and $E_{V'}$ to obtain a datum $(V \cup V', \psi'', \alpha'')$ which is larger than both our maximal elements, hence it is isomorphic to both. ◇

Claim 13.1.9. In the situation of claim 13.1.8, suppose that Z is reduced and normal, V_0 is dense in Z, and either one of the assumptions (iii.a) or (iii.b) hold for Z and the open immersion $V_0 \to Z$. Let $(V_{\max}, \psi, \alpha)$ be a supremum for $\mathscr{F}$, and $\overline{z}$ a geometric point of Z, such that $\varphi \times_Z Z(\overline{z})$ extends to a finite étale covering of $Z(\overline{z})$; then the support of $\overline{z}$ lies in $V_{\max}$.

Proof of the claim. By lemma 13.1.6, there exist an étale neighborhood $g : Y \to Z$ of $\overline{z}$, with Y affine, a finite étale morphism $\varphi_Y : E_Y \to Y$, and an isomorphism $h : \varphi \times_{V_0} Y \simeq \varphi_Y \times_Z V_0$. We have a natural essentially commutative diagram :

$$\begin{array}{ccccc} \mathsf{Cov}(gY) & \xrightarrow{\alpha} & \mathsf{Desc}(\mathsf{Cov}, g) & \longrightarrow & \mathsf{Cov}(Y) \\ \downarrow & & \downarrow & & \downarrow \\ \mathsf{Cov}(V_0 \cap gY) & \xrightarrow{\beta} & \mathsf{Desc}(\mathsf{Cov}, g \times_Z V_0) & \longrightarrow & \mathsf{Cov}(Y \times_Z V_0) \end{array}$$

where α and β are equivalences, by lemma 13.1.2. Moreover, let $Y' := Y \times_Z Y$ and $Y'' := Y' \times_Z Y$; it is easily seen that if assumption (iii.a) (resp. (iii.b)) holds for Z and V_0, the same holds for Y' and $Y' \times_Z V_0$, and also for Y'' and $Y'' \times_Z V_0$. By the foregoing, it follows that the functors

$$\mathsf{Cov}(Y') \to \mathsf{Cov}(Y' \times_Z V_0) \qquad \text{and} \qquad \mathsf{Cov}(Y'') \to \mathsf{Cov}(Y'' \times_Z V_0)$$

are fully faithful, hence the right square subdiagram is 2-cartesian (corollary 3.5.30(iii)). Thus, the datum $(\varphi \times_Z gY, \varphi_Y, h)$ determines an object φ' of $\mathsf{Cov}(gY)$, together with an isomorphism $\varphi' \times_Z V_0 \simeq \varphi \times_Z gY$, which we may use to glue φ and φ' to a single object φ'' of $\mathsf{Cov}(V_0 \cup gY)$. The claim follows. ◊

The foregoing shows that the assumptions of claim 13.1.8 are fulfilled, with $Z := X$, $V_0 := U$ and any object φ of $\mathsf{Cov}(U)$, hence there exists a largest open subset $U_{\max} \subset X$ over which φ extends. However, claim 13.1.9 shows that $U_{\max} = X$, so the proof of (iii) is complete.

(iv): In view of (iii) and [66, Ch.IV, Cor.18.8.13], we are reduced to the case where X is a regular local scheme, and it suffices to show that j^* is essentially surjective. We argue by induction on the dimension n of $X \setminus U$. If $n = 0$, then $X \setminus U$ is the closed point, in which case it suffices to invoke the Zariski-Nagata purity theorem ([85, Exp.X, Th.3.4(i)]). Suppose $n > 0$ and that the assertion is already known for smaller dimensions. Let φ be a given finite étale covering of U, and x a maximal point of $X \setminus U$; then $X(x) \setminus U = \{x\}$, so $\varphi_{|U \cap X(x)}$ extends to a finite étale morphism φ_x over $X(x)$. In turns, φ_x extends to an affine open neighborhood $V \subset X$ of X, and up to shrinking V, this extension φ' agrees with φ on $U \cap V$, by lemma 13.1.6. Hence we can glue φ and φ', and replace U by $U \cup V$. Repeating the procedure for every maximal point of $X \setminus U$, we reduce the dimension of $X \setminus U$; then we conclude by the inductive assumption. □

Definition 13.1.10. Let $f : X \to S$ be a morphism of schemes, $\overline{x}$ a geometric point of X localized at $x \in X$; set $s := f(x)$, $\overline{s} := f(\overline{x})$, and denote by $f_{\overline{x}} : X(\overline{x}) \to S(\overline{s})$ the morphism of strictly local schemes induced by f. Let also $\mathbb{L} \subset \mathbb{N}$ be any non-empty set of prime numbers.

(i) Let also $\mathbf{P}$ be any property of schemes (such as : connected, irreducible, integral, normal, . . .); we shall say that *the Milnor fibres of f at the point x have property* $\mathbf{P}$, if the scheme $f_{\overline{x}}^{-1}(\xi)$ has property $\mathbf{P}$, for every strict geometric point ξ of $S(\overline{s})$ (see definition 4.9.17(i)).

(ii) We say that f is *locally (-1)-acyclic at the point x*, if the Milnor fibres of f at the point x are non-empty.

(iii) We say that f is *locally 0-acyclic at the point x*, if the Milnor fibres of f at the point x are connected (and in particular, non-empty).

(iv) We say that a group G is a *finite $\mathbb{L}$-group* if G is finite and all the primes dividing the order of G lie in $\mathbb{L}$. We say that G is an *$\mathbb{L}$-group* if it is a filtered union of finite $\mathbb{L}$-groups.

(v) We say that f is *locally 1-aspherical for $\mathbb{L}$ at the point x*, if we have :

$$H^1(f_{\overline{x}}^{-1}(\xi)_{\text{ét}}, G) = \{1\}$$

for every strict geometric point ξ of $S(\overline{s})$, and every $\mathbb{L}$-group G (where 1 denotes the trivial G-torsor).

(vi) We say that f is *locally* (-1)*-acyclic* (resp. *locally* 0*-acyclic*, resp. *locally* 1*-aspherical for* $\mathbb{L}$), if f is locally (-1)-acyclic (resp. locally 0-acyclic, resp. locally 1-aspherical for $\mathbb{L}$) at every point of X.

(vii) We say that f is (-1)*-acyclic* (resp. 0*-acyclic*) if the unit of adjunction : $\mathscr{F} \to f_*f^*\mathscr{F}$ is a monomorphism (resp. an isomorphism) for every sheaf $\mathscr{F}$ on $S_{\text{ét}}$.

See section 4.1 for generalities about G-torsors for a group object G on a topos T. Here we shall be mainly concerned with the case where T is the étale topos $X^\sim_{\text{ét}}$ of a scheme X, and G is representable by a group scheme, finite and étale over X. In this case, using faithfully flat descent one can show that any G-torsor is representable by a *principal G-homogeneous space*, *i.e.* a finite, surjective, étale morphism $E \to X$ with a G-action $G \to \mathrm{Aut}_X(E)$ such that the induced morphism of X-schemes

$$G \times E \to E \times_X E$$

is an isomorphism.

13.1.11. If G_X is the constant $X^\sim_{\text{ét}}$-group arising from a finite group G and X is non-empty and connected, right G_X-torsors are also understood as G-valued characters of the étale fundamental group of X. Indeed, let ξ be a geometric point of X; recall ([82, Exp.V, §7]) that $\pi := \pi_1(X_{\text{ét}}, \xi)$ is defined as the automorphism group of the fibre functor

$$F_\xi : \mathsf{Cov}(X) \to \mathbf{Set} \qquad (E \xrightarrow{f} X) \mapsto f^{-1}\xi.$$

We endow π with its natural profinite topology, as in (3.6.7), so that F_ξ can be viewed as an equivalence of categories

$$F_\xi : \mathsf{Cov}(X) \to \pi\text{-}\mathbf{Set}.$$

Lemma 13.1.12. *In the situation of* (13.1.11)*, there exists a natural bijection of pointed sets :*

$$H^1(X_{\text{ét}}, G_X) \xrightarrow{\sim} H^1_{\text{cont}}(\pi, G)$$

from the pointed set of right G_X-torsors, to the first non-abelian continuous cohomology group of P with coefficients in G (see (3.6.1)*).*

Proof. Let $f : E \to X$ be a right G_X-torsor, and fix a geometric point $s \in F_\xi(E)$; given any $\sigma \in \pi$, there exists a unique $g_{s,\sigma} \in G$ such that

$$s \cdot g_{s,\sigma} = \sigma_E(s).$$

Any $g \in G$ determines a X-automorphism $g_E : E \to E$, and by definition, the automorphism $F_\xi(g_E)$ on $F_\xi(E)$ commutes with the left action of any element $\tau \in \pi$; however $F_\xi(g_E)$ is just the right action of g on $F_\xi(E)$, hence we may compute :

$$s \cdot g_{s,\tau} \cdot g_{s,\sigma} = (\tau_E(s)) \cdot g_{s,\sigma} = \tau_E(s \cdot g_{s,\tau}) = \tau_E \cdot \sigma_E(s)$$

so the rule $\sigma \mapsto g_{s,\sigma}$ defines a group homomorphism $\rho_{s,f} : \pi \to G$ which is clearly continuous. We claim that the conjugacy class of $\rho_{s,f}$ does not depend on the choice of s. Indeed, if $s' \in F_\xi(E)$ is another choice, there exists a (unique) element $h \in G$ such that $h(s) = s'$; arguing as in the foregoing we see that σ_E commutes with the right action of h on $F_\xi(E)$. In other words, $\sigma_E(s) = h^{-1} \circ \sigma_E(s')$, so that $g_{s',\sigma} = h \circ g_{s,\sigma} \circ h^{-1}$.

Therefore, denote by ρ_f the conjugacy class of $\rho_{s,f}$; we claim that ρ_f depends only on the isomorphism class of the G_X-torsor E. Indeed, any isomorphism $t : E \xrightarrow{\sim} E'$ of right G_X-torsors induces a bijection $F_\xi(t) : F_\xi(E) \xrightarrow{\sim} F_\xi(E')$, equivariant for the action of G, and for any $\sigma \in \pi$ we have $F_\xi(t) \circ \sigma_E = \sigma_{E'} \circ F_\xi(t)$, whence the assertion.

Conversely, given a continuous group homomorphism $\rho : \pi \to G$, let us endow G with the induced left π-action, and right G-action. Then G is an object of π-$\mathbf{Set}$, to which there corresponds a finite étale morphism $E_\rho \to X$, with an isomorphism $E_\rho \times_X |\xi| \stackrel{\sim}{\to} G$ of sets with left π-action. Since the right action of G is π-equivariant, we have a corresponding G-action by X-automorphisms on E_ρ, so E_ρ is G-torsor, and its image under the map of the lemma is clearly the conjugacy class of ρ.

Finally, in order to show that the map of the lemma is injective, it suffices to prove that, for any right G_X-torsor $(f : E \to X, \varphi : E \times G \to E)$ and any $s \in F_\xi(E)$, there exists an isomorphism of right G_X-torsors $E_{\rho_{s,f}} \stackrel{\sim}{\to} E$. However, s and $\rho_{s,f}$ determine an identification of sets with left π-action :

$$G \stackrel{\sim}{\to} F_\xi(E) \tag{13.1.13}$$

whence an isomorphism $t : E_{\rho_{s,f}} \stackrel{\sim}{\to} E$ in $\mathsf{Cov}(X)$. Moreover, (13.1.13) also identifies the right G-action on $F_\xi(E)$ to the natural right G-action on G; the latter is π-equivariant, hence it induces a right G-action $\varphi' : E \times G \to E$, such that t is G-equivariant. To conclude, it suffices to show that $\varphi = \varphi'$. In view of [66, Ch.IV, Cor.17.4.8], the latter assertion can be checked on the stalks over the geometric point ξ, where it holds by construction. ☐

Remark 13.1.14. (i) In the situation of (13.1.11), let $E \to X$ be any right G_X-torsor, and $\rho_E : \pi \to G$ the corresponding representation. Then the proof of lemma 13.1.12 shows that the left π-action on $F_\xi(E)$ is isomorphic to the left π-action on G induced by ρ_E; especially, ρ_E is surjective if and only if π acts transitively on $F_\xi(E)$, if and only if the scheme E is connected (since a decomposition of E into connected components corresponds to a decomposition of $F_\xi(E)$ into orbits for the π-action).

(ii) Let $\varphi : G' \to G$ be a homomorphism of finite groups, and

$$H^1_{\mathrm{cont}}(\pi, \varphi) : H^1_{\mathrm{cont}}(\pi, G') \to H^1_{\mathrm{cont}}(\pi, G)$$

the induced map. Denote by r (resp. l) the right (resp. left) translation action of G on itself. Let also $E' \to X$ be a principal G'-homogeneous space, given by a map $\rho : G' \to \mathrm{Aut}_X(E')$, and denote by $c' \in H^1_{\mathrm{cont}}(\pi, G')$ the class of E'. Then the class $c := H^1_{\mathrm{cont}}(\pi, \varphi)(c)$ can be described geometrically as follows. The scheme $E' \times G$ admits an obvious right G-action, induced by r. Moreover, it admits as well a right G'-action : namely, to any element $g \in G'$, we assign the X-automorphism $\rho_g \times l_{\varphi(g^{-1})} : E' \times G \stackrel{\sim}{\to} E' \times G$. Set $E := (E \times G)/G'$; it is easily seen that E is a principal G-homogeneous space, and its class is precisely c (the detailed verification shall be left as an exercise for the reader).

(iii) Consider a commutative diagram of schemes

$$\begin{array}{ccc} E' & \xrightarrow{\ g\ } & E \\ & {\scriptstyle f'}\searrow \quad \swarrow{\scriptstyle f} & \\ & X & \end{array}$$

where f (resp. f') is a G-torsor (resp. a G'-torsor) for a given finite group G (resp. G'). Denote by $\rho : G \to \mathrm{Aut}_X(E)$ and $\rho' : G' \to \mathrm{Aut}_X(E')$ the respective actions. We have :

(a) Suppose that there exists a group homomorphism $\varphi : G' \to G$, such that $\rho \circ \varphi = g \circ \rho'$. Then g induces a G'-equivariant map

$$F_\xi(E') \to \mathrm{Res}(\varphi) F_\xi(E)$$

whence an isomorphism $(E' \times G)/G' \stackrel{\sim}{\to} G$ of G-torsors. Hence, let c (resp. c') denote any representative of the equivalence class of E (resp. E') in $H^1_{\mathrm{cont}}(\pi, G)$ (resp.

$H^1_{\mathrm{cont}}(\pi, G'))$; in view of (ii), it follows that

$$H^1(\pi, \varphi)(c') = c.$$

In other words, the induced diagram of continuous group homomorphisms

$$\begin{array}{ccc} & \pi_1(X_{\text{ét}}, \xi) & \\ {}^{c'}\swarrow & & \searrow^{c} \\ G' & \xrightarrow{\ \varphi\ } & G \end{array}$$

commutes, up to composition with an inner automorphism of G. (Details left to the reader.)

(b) If E is connected, a group homomorphism $\varphi : G \to G'$ fulfilling the condition of (a) exists and is unique up to composition with an inner automorphism of G. Indeed, fix any $e' \in F_\xi(E')$ and let $e := f(e')$; if $g' \in G'$, define $\varphi(g')$ as the unique $g \in G$ such that $f(e' \cdot g') = e \cdot g$; also, in view of (i) we may pick $\sigma_{g'} \in \pi_1(X, \xi)$ such that $\sigma_{g'} \cdot e' = e' \cdot g'$, and notice that $\sigma_{g'} \cdot e = e \cdot \varphi(g')$ for every $g' \in G'$. Now, if $h' \in G'$ is any other element, we may compute :

$$f(e' \cdot g'h') = f(\sigma_{g'} \cdot e' \cdot h') = \sigma_{g'} \cdot f(e' \cdot h') = \sigma_{g'} \cdot e \cdot \varphi(h') = e \cdot \varphi(g') \cdot \varphi(h')$$

whence $\varphi(g'h') = \varphi(g') \cdot \varphi(h')$, as required.

Lemma 13.1.15. *Let $f : X \to Y$ be a morphism of schemes, $\mathscr{F}$, $\mathscr{G}$ two sheaves on $Y_{\text{ét}}$. Then :*

(i) *If $\mathscr{F}$ is locally constant and constructible, the natural map :*

$$\vartheta_f : f^*\mathscr{H}om_{Y^\sim_{\text{ét}}}(\mathscr{F}, \mathscr{G}) \to \mathscr{H}om_{X^\sim_{\text{ét}}}(f^*\mathscr{F}, f^*\mathscr{G})$$

is an isomorphism.

(ii) *If f is 0-acyclic, it induces a fully faithful functor*

$$f^* : \mathsf{Cov}(Y) \to \mathsf{Cov}(X) \quad : \quad (E \to Y) \mapsto (E \times_Y X \to X).$$

Proof. (i): Suppose we have a cartesian diagram of schemes :

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y. \end{array}$$

Then, according to (4.8.2), we have a natural isomorphism :

$$\vartheta_{g'} \circ g'^*\vartheta_f \Rightarrow \vartheta_{f'} \circ f'^*\vartheta_g$$

(an invertible 2-cell, in the terminology of (2.1)). Now, if g – and therefore g' – is a covering morphism, ϑ_g and $\vartheta_{g'}$ are isomorphisms, and $g'^*\vartheta_f$ is an isomorphism if and only if the same holds for ϑ_f. Summing up, in this case ϑ_f is an isomorphism if and only if the same holds for $\vartheta_{f'}$. Thus, we may choose g such that g^* is a constant sheaf, and after replacing f by f', we may assume that $\mathscr{F} = S_Y$ is the constant sheaf associated with a finite set S. Since the functors

$$\mathscr{H}om_{Y^\sim_{\text{ét}}}(-, \mathscr{G}) : (Y^\sim_{\text{ét}})^o \to Y^\sim_{\text{ét}} \quad \text{and} \quad f^* : Y^\sim_{\text{ét}} \to X^\sim_{\text{ét}}$$

are left exact, we may further reduce to the case where $S = \{1\}$ is the set with one element, in which case $\mathscr{F} = 1_Y$ is the final object of $Y^\sim_{\text{ét}}$, and $f^*\mathscr{F} = 1_X$ is the final object of $X^\sim_{\text{ét}}$. Moreover, we have a natural identification :

$$\mathscr{H}om_{Y^\sim_{\text{ét}}}(1_Y, \mathscr{G}) \xrightarrow{\sim} \mathscr{G} \quad : \quad \sigma \mapsto \sigma(1)$$

and likewise for $\mathscr{H}om_{X^\sim_{\text{ét}}}(1_Y, f^*\mathscr{G})$. Using the foregoing characterization, it is easily checked that, under these identifications, ϑ_f is the identity map of $f^*\mathscr{G}$, whence the claim.

(ii): It has already been remarked that $\mathsf{Cov}(Y)$ is equivalent to the category of locally constant constructible sheaves on $Y_{\text{ét}}$, and likewise for $\mathsf{Cov}(X)$. Let E and F be two objects of $\mathsf{Cov}(Y)$; we have natural bijections :

$$\begin{aligned}\mathrm{Hom}_{\mathsf{Cov}(X)}(f^*E, f^*F) &\xrightarrow{\sim} \Gamma(X, \mathscr{H}om_{X^{\sim}_{\text{ét}}}(f^*E, f^*F)) \\ &\xrightarrow{\sim} \Gamma(Y, f_*f^*\mathscr{H}om_{Y^{\sim}_{\text{ét}}}(E, F)) && \text{by (i)} \\ &\xrightarrow{\sim} \Gamma(Y, \mathscr{H}om_{Y^{\sim}_{\text{ét}}}(E, F)) && \text{since } f \text{ is 0-acyclic} \\ &\xrightarrow{\sim} \mathrm{Hom}_{\mathsf{Cov}(Y)}(E, F)\end{aligned}$$

as stated. □

Lemma 13.1.16. *Let $f : X \to S$ be a quasi-compact morphism of schemes, and suppose that :*

(a) *f is locally (-1)-acyclic.*
(b) *For every strict geometric point ξ of S, the induced morphism $f_\xi : f^{-1}(\xi) \to |\xi|$ is 0-acyclic (*i.e. *f has non-empty geometrically connected fibres).*

Then f is 0-acyclic.

Proof. Let $\mathscr{F}$ be a sheaf on $S_{\text{ét}}$. For every strict geometric point ξ of S, we have a commutative diagram :

$$\begin{array}{ccc}\mathscr{F}_\xi & \xrightarrow{\varepsilon_\xi} & (f_*f^*\mathscr{F})_\xi \\ {\scriptstyle\alpha}\downarrow & & \downarrow \\ \Gamma(|\xi|, \xi^*\mathscr{F}) & \xrightarrow{f^*_\xi} & \Gamma(f^{-1}(\xi), f^*_\xi \circ \xi^*\mathscr{F})\end{array} \tag{13.1.17}$$

where $\varepsilon : \mathscr{F} \to f_*f^*\mathscr{F}$ is the unit of adjunction. The map α is an isomorphism, and the same holds for f^*_ξ, since f_ξ is 0-acyclic. Hence ε_ξ is injective, which shows already that f is (-1)-acyclic. It remains to show that ε_ξ is surjective. Hence, let $t \in (f_*f^*\mathscr{F})_\xi$ be any section. From (13.1.17) we see that there exists a section $t' \in \mathscr{F}_\xi$ such that the images of t and $\varepsilon_\xi(t')$ agree on $\Gamma(f^{-1}(\xi), f^*_\xi \circ \xi^*\mathscr{F})$. We may find an étale neighborhood $g : U \to S$ of ξ, such that t' (resp. t) extends to a section $t'_U \in \mathscr{F}(U)$ (resp. $t_U \in \Gamma(X \times_S U, f^*\mathscr{F})$). Let $X_U := X \times_S U$, $f_U := f \times_S U : X_U \to U$, and for every geometric point $\overline{x}$ of X_U, denote by $f^*_{\overline{x}} : \mathscr{F}_{f_U(\overline{x})} \to f^*_U\mathscr{F}_{\overline{x}}$ the natural isomorphism (4.9.22). We set

$$V := \{x \in X_U \mid t_{U,\overline{x}} = f^*_{\overline{x}}(t'_{U,f(\overline{x})})\}$$

where $\overline{x}$ is any geometric point of X localized at x, and $t_{U,\overline{x}} \in f^*\mathscr{F}_{\overline{x}}$ (resp. $t'_{U,f(\overline{x})} \in \mathscr{F}_{f_U(\overline{x})}$) denotes the image of t_U (resp. of t'_U). Clearly V is an open subset of X_U, and we have :

Claim 13.1.18. (i) $V = f_U^{-1}f_U(V)$.
(ii) $f_U(V) \subset U$ is an open subset.

Proof of the claim. (i): Given a point $u \in U$, choose a strict geometric point $\overline{u}$ localized at u, and set $\overline{s} := g(\overline{u})^{\text{st}}$; by assumption, the morphism $f_{\overline{s}} : f^{-1}(\overline{s}) \to |\overline{s}|$ is 0-acyclic, hence the image of t_U in $\Gamma(f^{-1}(\overline{s}), f^*_{\overline{s}} \circ \overline{s}^*\mathscr{F})$ is of the form $f^*_{\overline{s}}t''$, for some $t'' \in \mathscr{F}_{\overline{s}}$. It follows that $V \cap f_U^{-1}(u)$ is either the whole of $f_U^{-1}(u)$ or the empty set, according to whether t'' agrees or not with the image of t'_U in $\mathscr{F}_{\overline{s}} = g^*\mathscr{F}_{\overline{u}}$.

(ii): The subset $X \setminus V$ is closed, especially pro-constructible; since f is quasi-compact, we deduce that $f_U(X \setminus V)$ is a pro-constructible subset of U (corollary 8.1.50). It then follows from (i) that $f_U(V)$ is ind-constructible, hence we are reduced to showing that $f_U(V)$ is closed under generizations (proposition 8.1.47(ii)). To this aim, since V is open, it suffices to show that f_U is *generizing*, *i.e.* that the induced maps $X_U(x) \to U(u)$ are surjective, for every $u \in U$ and every $x \in f_U^{-1}(u)$. However, choose a geometric point $\overline{x}$ localized at x, and let $\overline{u} := f_U(\overline{x})$; since the natural maps $X_U(\overline{x}) \to X_U(x)$ and $U(\overline{u}) \to U(u)$ are surjective, it suffices to show that the same holds for the map $f_{U,\overline{x}} : X_U(\overline{x}) \to U(\overline{u})$. The image of $\overline{x}$ (resp. $\overline{U}$) in X (resp. in S) is a geometric point which we denote by the same name; since the natural maps $X_U(\overline{x}) \to X(\overline{x})$ and $U(\overline{u}) \to S(\overline{u})$ are isomorphisms, we are reduced to showing that $f_{\overline{x}} : X(\overline{x}) \to S(\overline{u})$ is surjective, which holds, since f is locally (-1)-acyclic. ♢

Set $W := f_U(V)$; in view of claim 13.1.18, W is an étale neighborhood of ξ, and the natural map $\mathscr{F}(W) \to f^*\mathscr{F}(U)$ sends the restriction $t'_{U|W}$ of t'_U to the restriction $t_{U|V}$ of t_U, whence the claim. □

13.1.19. Consider now a cartesian diagram of schemes :

(13.1.20)
$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

where g is a local morphism of strictly local schemes, and denote by s (resp. by s') the closed point of S (resp. of S'). Let $x' \in f'^{-1}(s')$ be any point, $\overline{x}'$ a geometric point of X' localized at x', and set $x := g'(x')$, $\overline{x} := g'(\overline{x}')$. Then g' induces a morphism of S'-schemes :

(13.1.21) $$X'(\overline{x}') \to X(\overline{x}) \times_S S'.$$

Lemma 13.1.22. *In the situation of* (13.1.19)*, suppose that g is an integral morphism. Then :*

(i) *The induced morphism $f'^{-1}(s') \to f^{-1}(s)$ induces a homeomorphism on the underlying topological spaces.*
(ii) (13.1.21) *is an isomorphism.*

Proof. If g is integral, $\kappa(s')$ is a purely inseparable algebraic extension of $\kappa(s)$, hence the morphism $T' := \operatorname{Spec}\kappa(s') \to T := \operatorname{Spec}\kappa(s)$ is radicial, and the same holds for the induced morphisms :

$$f'^{-1}(s') \xrightarrow{\sim} f^{-1}(s) \times_T T' \to f^{-1}(s) \qquad f_x^{-1}(s) \times_T T' \to f_x^{-1}(s)$$

([59, Ch.I, Prop.3.5.7(ii)]). Especially (i) holds, and therefore the natural map $X'(x') \to X(x) \times_S S'$ is an isomorphism; we see as well that $f_x^{-1}(s) \times_T T'$ is a local scheme. Then the assertion follows from [66, Ch.IV, Rem.18.8.11]. □

Proposition 13.1.23. *Let $f : X \to S$ and $g : S' \to S$ be morphisms of schemes, with g quasi-finite, and set $X' := X \times_S S'$. Suppose that f is locally (-1)-acyclic (resp. locally 0-acyclic); then the same holds for $f' := f \times_S S' : X' \to S'$.*

Proof. Let $\overline{s}'$ be any geometric point of S', and set $\overline{s} := g(\overline{s}')$. Denote by $s \in S$ (resp. $s' \in S'$) the support of $\overline{s}$ (resp. $\overline{s}'$); then f is locally (-1)-acyclic at the points of $f^{-1}(s)$, if and only if $f_{\overline{s}} := f \times_S S(\overline{s}) : X \times_S S(\overline{s}) \to S(\overline{s})$ enjoys the same property at the points of $f_{\overline{s}}^{-1}(\overline{s})$. Likewise, f' is locally (-1)-acyclic (resp. locally 0-acyclic) at the points of $f'^{-1}(s')$, if and only if $f'_{\overline{s}'} : X' \times_{S'} S'(\overline{s}') \to S'(\overline{s}')$ enjoys the same property at the points of $f_{\overline{s}}^{-1}(\overline{s})$. Hence, we may replace g by $g_{\overline{s}'} : S'(\overline{s}') \to S(\overline{s})$, and f by the induced morphism $X \times_S S(\overline{s}) \to S(\overline{s})$, which allows to assume that g is finite ([66, Ch.IV, Th.18.5.11]), hence integral. Let ξ' (resp. $\overline{x}'$) be any strict geometric point of S' (resp. of $f'^{-1}(s')$), and let $\xi := g(\xi')^{\mathrm{st}}$, (resp. let $\overline{x}$ be the image of $\overline{x}'$ in X); we have natural morphisms :

$$f'^{-1}_{\overline{x}'}(\xi') \xrightarrow{\alpha} X(\overline{x}) \times_S \xi' \xrightarrow{\beta} f_{\overline{x}}^{-1}(\xi).$$

However, α is an isomorphism, by lemma 13.1.22(ii), and β is a radicial morphism, since the field extension $\kappa(\xi) \subset \kappa(\xi')$ is purely inseparable ([59, Ch.I, Prop.3.5.7(ii)]). The claim follows. □

Lemma 13.1.24. (i) *Let S be a strictly local scheme, $s \in S$ the closed point, $f : X \to S$ a morphism of schemes, $\overline{x}$ (resp. ξ) a strict geometric point of $f^{-1}(s)$ (resp. of S). We may find :*

(a) *A cartesian diagram* (13.1.20)*, with S' strictly local, irreducible and normal.*
(b) *A strict geometric point $\overline{x}'$ of $f'^{-1}(s')$ with $g'(\overline{x}')^{\mathrm{st}} = \overline{x}$.*
(c) *A strict geometric point ξ' of S' localized at the generic point η' of S', with $g(\xi')^{\mathrm{st}} = \xi$, and such that* (13.1.21) *induces an isomorphism :*

$$f'^{-1}_{\overline{x}'}(\xi') \stackrel{\sim}{\to} f^{-1}_{\overline{x}}(\xi). \qquad (13.1.25)$$

(ii) *Moreover, if S is noetherian, we may find S' as in* (i)*, such that $\mathscr{O}_{S'}(S')$ is a Krull domain.*
(iii) *Alternatively, we may find S' as in* (i)*, with separably closed residue field $\kappa(\eta')$.*

Proof. (i): Denote by $Z \subset S$ the closure of the image of ξ, endow Z with its reduced subscheme structure, set $Y := X \times_S Z \subset X$, and let $h_{\overline{x}} : Y(\overline{x}) \to Z$ be the natural morphism. Then Z is a strictly local scheme ([66, Ch.IV, Prop.18.5.6(i)]). Moreover, the closed immersion $Y \to X$ induces an isomorphism $Y(\overline{x}) \stackrel{\sim}{\to} X(\overline{x}) \times_S Z$ of Z-schemes (lemma 13.1.22(ii)). By construction, ξ factors through a strict geometric point ξ' of Z, and we deduce an isomorphism $h^{-1}_{\overline{x}}(\xi') \stackrel{\sim}{\to} f^{-1}_{\overline{x}}(\xi)$ of Z-schemes. Thus, we may replace (S, X, ξ) by (Z, Y, ξ'), and assume that S is the spectrum of a strictly local domain, and ξ is localized at the generic point of S.

Claim 13.1.26. Let A be a strictly local domain, and A^ν the normalization of A in an algebraic extension K of $F := \mathrm{Frac}\, A$. Then A^ν is strictly local.

Proof of the claim. A^ν is the union of a filtered family $(A_\lambda \mid \lambda \in \Lambda)$ of finite A-subalgebras of F; since A is henselian, each A_λ is a product of henselian local rings, hence it is a local henselian ring, so the same holds for A^ν. Moreover, the residue field of A^ν is separably closed, since it is an algebraic extension of the residue field of A, which is separably closed by assumption. ◊

Say that $S = \mathrm{Spec}\, A$, and denote by A^ν the normalization of A in its field of fractions F. By claim 13.1.26, A^ν is strictly henselian, so we may fulfill condition (a) by taking $S' := \mathrm{Spec}\, A^\nu$. Condition (b) holds as well, due to lemma 13.1.22(i). Finally, it is clear that ξ lifts to a unique strict geometric point ξ' of S', and it follows from lemma 13.1.22(ii) that (13.1.25) is an isomorphism, as required.

(ii): A direct inspection of the proof of (i) reveals that if S is noetherian, the scheme S' exhibited is the spectrum of the normalization of a noetherian domain; the assertion then follows from [133, Th.33.10].

(iii): Let A^ν_K be the normalization of A in a separable closure K of F; the proof of (i) applies as well with $S' := \mathrm{Spec}\, A^\nu_K$, whence the assertion. □

13.1.27. Consider now a cofiltered family $\mathscr{S} := (S_\lambda \mid \lambda \in \Lambda)$ of affine schemes. Denote by S the limit of $\mathscr{S}$, and suppose moreover that Λ admits a final element $0 \in \Lambda$. Let $f_0 : X_0 \to S_0$ be a finitely presented morphism of schemes, and set :

$$X_\lambda := X_0 \times_{S_0} S_\lambda \qquad f_\lambda := f_0 \times_{S_0} S_\lambda : X_\lambda \to S_\lambda \qquad \text{for every } \lambda \in \Lambda.$$

Set as well $X := X_0 \times_{S_0} S$ and $f := f_0 \times_{S_0} S : X \to S$. Let $x \in X$ be any point, and $s := f(x)$. Let also ξ be a strict geometric point of S. We deduce a compatible system of points $x_\lambda := p'_\lambda(x) \in X_\lambda$, whence a cofiltered system of local schemes

$$\mathscr{X} := (X_\lambda(x_\lambda) \mid \lambda \in \Lambda).$$

Moreover, we get a compatible system of strict geometric points $(\xi_\lambda := p_\lambda(\xi)^{\mathrm{st}} \mid \lambda \in \Lambda)$, with :

$$\xi \stackrel{\sim}{\to} \lim_{\lambda \in \Lambda} \xi_\lambda.$$

Choose a geometric point $\overline{x}$ of X localized at x, and set likewise $\overline{x}_\lambda := p'_\lambda(\overline{x})$; then $\mathscr{X}$ lifts to a system $\mathscr{X}^{\mathrm{sh}} := (X_\lambda(\overline{x}_\lambda) \mid \lambda \in \Lambda)$, whose limit is naturally isomorphic to $X(\overline{x})$ ([66, Ch.IV, Prop.18.8.18(ii)]). Furthermore, $\mathscr{X}^{\mathrm{sh}}$ induces a natural isomorphism of $\kappa(\xi)$-schemes :

$$f_{\overline{x}}^{-1}(\xi) \xrightarrow{\sim} \lim_{\lambda \in \Lambda} f_{\lambda,\overline{x}_\lambda}^{-1}(\xi_\lambda) \tag{13.1.28}$$

where, as usual, $f_{\overline{x}} : X(\overline{x}) \to S$ (resp. $f_{\lambda,\overline{x}_\lambda} : X_\lambda(\overline{x}_\lambda) \to S_\lambda$) is deduced from f (resp. from f_λ). These remarks, together with the following lemma 13.1.29, and the previous lemma 13.1.24, will allow in many cases, to reduce the study of the fibres of $f_{\overline{x}}$, to the case where the base S is strictly local, excellent and normal.

Lemma 13.1.29. *Let S be a strictly local normal scheme. Then there exists a cofiltered family $\mathscr{S} := (S_\lambda \mid \lambda \in \Lambda)$ consisting of strictly local normal excellent schemes, such that :*

(a) *S is isomorphic to the limit of $\mathscr{S}$.*

(b) *The natural morphism $S \to S_\lambda$ is dominant for every $\lambda \in \Lambda$.*

Proof. Say that $S = \operatorname{Spec} A$, and write A as the union of a filtered family $\mathscr{A} := (A_\lambda \mid \lambda \in \Lambda)$ of excellent noetherian local subrings, which we may assume to be normal, by [64, Ch.IV, (7.8.3)(ii),(vi)]. Proceeding as in (13.1.27), we choose a compatible family of geometric points $\overline{s}_\lambda$ localized at the closed points of $\operatorname{Spec} A_\lambda$, for every $\lambda \in \Lambda$; using these geometric points, we lift $\mathscr{A}$ to a filtered family $(A_\lambda^{\mathrm{sh}} \mid \lambda \in \Lambda)$ of strict henselizations, whose colimit is naturally isomorphic to A. Moreover, each A_λ^{sh} is noetherian, normal and excellent ([66, Ch.IV, Prop.18.8.8(iv), Prop.18.8.12(i)] and proposition 9.7.19(ii)). Let η be the generic point of S, $h_\lambda : S \to S_\lambda := \operatorname{Spec} A_\lambda^{\mathrm{sh}}$ the natural morphism, and $\eta_\lambda^{\mathrm{sh}} := h_\lambda(\eta)$ for every $\lambda \in \Lambda$. The cofiltered system $(S_\lambda \mid \lambda \in \Lambda)$ fulfills condition (a). Moreover, by construction, the image of $\eta_\lambda^{\mathrm{sh}}$ in $\operatorname{Spec} A_\lambda$ is the generic point η_λ; then $\eta_\lambda^{\mathrm{sh}}$ is the generic point of S_λ, since the latter is the only point of S_λ lying over η_λ. Hence (b) holds as well. □

Part (ii) of the following proposition improves upon [10, Exp.XV, Th.4.1].

Proposition 13.1.30. *Let $f : X \to S$ be a flat morphism of schemes. We have :*

(i) *f is locally (-1)-acyclic.*

(ii) *Suppose moreover, that f has geometrically reduced fibres, and :*

  (a) *either f is locally finitely presented,*

  (b) *or else, S is locally noetherian.*

  Then f is locally 0-acyclic.

(iii) *Furthermore, if f is locally finitely presented and has geometrically normal fibres, then the Milnor fibres of f at every point of X are geometrically normal and irreducible.*

Proof. Let $x \in X$ be any point, set $s := f(x)$, choose a geometric point $\overline{x}$ of X localized at x, set $\overline{s} := f(\overline{x})$, and let ξ be any strict geometric point of $S(\overline{s})$.

(i): If f is flat, the induced morphism $f_{\overline{x}} : X(\overline{x}) \to S(\overline{s})$ is faithfully flat; especially, $f_{\overline{x}}$ is surjective ([126, Th.7.3(i)]).

(ii),(iii): Set $X' := X \times_S S(\overline{s})$. The natural morphism $X(\overline{x}) \to X$ factors uniquely through a morphism of $S(\overline{s})$-schemes $j : X(\overline{x}) \to X'$, and if we denote by $\overline{x}'$ the image in X' of $\overline{x}$, then j induces an isomorphism of $S(\overline{s})$-schemes : $X(\overline{x}) \xrightarrow{\sim} X'(\overline{x}')$. Hence we may replace S by $S(\overline{s})$, and assume that S is strictly local, when (a) holds, and even strictly local and noetherian, when (b) holds. By lemma 13.1.24, we are further reduced to the case where $S = S(\overline{s}) = \operatorname{Spec} A$ is strictly local and normal, ξ is localized at the generic point of S, and :

(a') either f is finitely presented,

(b') or else, A is a (not necessarily noetherian) Krull domain.

Claim 13.1.31. (i) In case (b') holds, $f_{\overline{x}}^{-1}(\xi)$ is connected.

(ii) If (a') holds and f has geometrically normal fibres, then $f_{\overline{x}}^{-1}(\xi)$ is normal and irreducible.

Proof of the claim. Let F be the field of fractions of A; the field $\kappa(\xi)$ is algebraic over F, hence, in order to show (i), it suffices to check that $X(\overline{x}) \times_S \operatorname{Spec} K$ is connected for every finite field extension $F \subset K$ ([65, Ch.IV, Prop.8.4.1(ii)]). Likewise, in order to show (ii), it suffices to check that $X(\overline{x}) \times_S \operatorname{Spec} K$ is normal and irreducible for every such K. Let A_K be the normalization of A in K, set $S_K := \operatorname{Spec} A_K$, $X_K := X \times_S S_K$, let $f_K : X_K \to S_K$ be the induced morphism, and pick any geometric point $\overline{x}_K$ of X_K whose image in X agrees with $\overline{x}$. By claim 13.1.26, S_K is strictly local, and if (b') holds, then A_K is again a Krull domain ([34, Ch.VII, §1, n.8, Prop.12]). Moreover, lemma 13.1.22(ii) yields a natural identification :

$$X_K(\overline{x}_K) \times_{S_K} \operatorname{Spec} K \xrightarrow{\sim} X(\overline{x}) \times_S \operatorname{Spec} K.$$

Clearly, if f has geometrically reduced (resp. geometrically normal) fibres, the same holds for f_K, hence we may replace f by f_K, and reduce to checking the claim for $K = F$.

Now, $B := \mathscr{O}_{X,\overline{x}}$ is clearly a flat A-algebra, and notice that the geometric fibres $f_{\overline{x}}$ are cofiltered limits of schemes that are étale over the fibres of f; if the fibres of f are geometrically reduced (resp. geometrically normal), it follows that the same holds for the fibres of $f_{\overline{x}}$. In case A is a Krull domain, B is integrally closed in $B \otimes_A F$ (lemma 9.8.2); especially these two rings have the same idempotents, whence (i).

Lastly, if the assumptions of (ii) hold, then we know already that the fibres of $f_{\overline{x}}$ are geometrically normal, so that $X(\overline{x})$ is normal, by [65, Ch.IV, Prop.11.3.13(ii)], and therefore $B \otimes_A F$ is a normal domain, whence (ii). ◊

Claim 13.1.31 yields already (iii), and also shows that (ii) holds when (b') holds.

Finally, let us check that (ii) holds when (a') holds. By lemma 13.1.29, the scheme S is the limit of a cofiltered family $(S_\lambda \mid \lambda \in \Lambda)$ of strictly local excellent and normal schemes, such that the natural morphisms $p_\lambda : S \to S_\lambda$ are dominant. By claim 9.8.4, there exists $\lambda \in \Lambda$ and a flat morphism of schemes $f_\lambda : X_\lambda \to S_\lambda$ with geometrically reduced fibres, with an isomorphism of S-schemes $S \times_{S_\lambda} X_\lambda \xrightarrow{\sim} X$; then, for every $\mu \in \Lambda$ with $\mu \geq \lambda$, set $X_\mu := S_\mu \times_{S_\lambda} X_\lambda$ and $f_\mu := S_\mu \times_{S_\lambda} f_\lambda : X_\mu \to S_\mu$. After replacing Λ by a coinitial subset, we may assume that f_μ is defined for every $\lambda \in \Lambda$. For every such λ, let $x_\lambda \in X_\lambda$ be the image of x. Arguing as in (13.1.27), we obtain a compatible system of strict geometric points ξ_λ of S_λ (resp. $\overline{x}_\lambda$ of X_λ), such that $p_\lambda(\xi)$ factors through ξ_λ; whence an isomorphism (13.1.28). Thus, $f_{\overline{x}}^{-1}(\xi)$ is reduced if and only if $f_{\lambda,\overline{x}_\lambda}^{-1}(\xi_\lambda)$ is reduced for every sufficiently large $\lambda \in \Lambda$ ([65, Ch.IV, Prop.8.7.2]). Furthermore, since p_λ is dominant, ξ_λ is localized at the generic point of S_λ, for every $\lambda \in \Lambda$. Thus, we are reduced to the case where $S = S(s)$ is the spectrum of a strictly local noetherian normal domain A, and ξ is localized at the generic point of S; since such A is a Krull domain ([126, Th.12.4(i)]), this is covered by claim 13.1.31. □

Remark 13.1.32. In the statement and proof of proposition 13.1.30(iii) we use a definition of *geometrically normal scheme* (over a given field) which is more general than that introduced originally in [64, Ch.IV, Def.6.7.6] (which includes some unnecessary noetherian assumptions), and agrees *e.g.* with the definition in [107, Tag 0380, Tag 038M].

Example 13.1.33. (i) Let A be an excellent local ring, and $A^\wedge$ the completion of A. Then the natural morphism :

$$f : \operatorname{Spec} A^\wedge \to \operatorname{Spec} A$$

is locally 0-acyclic. Indeed, this follows from proposition 13.1.30(ii) (and from the excellence assumption, which includes the geometric regularity of the formal fibres of A).

(ii) Suppose additionally, that A is strictly local. Then f is 0-acyclic. To see this, we apply the criterion of lemma 13.1.16 : indeed, since f is flat, it is (-1)-acyclic (proposition 13.1.30(i)); it remains to show that f has geometrically connected fibres, and since $A^\wedge$ is strictly local ([66, Ch.IV, Prop.18.5.14]), this is the same as showing that f is locally 0-acyclic at the closed point of $\operatorname{Spec} A^\wedge$, which has already been remarked in (i).

(iii) More generally, f is 0-acyclic whenever A is excellent and henselian. Indeed, in this case the argument of (ii) again reduces to showing that f has geometrically connected fibres. However, consider the natural commutative diagram :

$$\begin{array}{ccc} \mathrm{Spec}\,(A^\wedge)^{\mathrm{sh}} & \longrightarrow & \mathrm{Spec}\,A^\wedge \\ {\scriptstyle f^{\mathrm{sh}}}\downarrow & & \downarrow{\scriptstyle f} \\ \mathrm{Spec}\,A^{\mathrm{sh}} & \longrightarrow & \mathrm{Spec}\,A. \end{array} \tag{13.1.34}$$

Since A is henselian, A^{sh} is the colimit of a filtered family of finite étale and local A-algebras. Since A and $A^\wedge$ have the same residue field, it follows easily that $A^\wedge \otimes_A A^{\mathrm{sh}}$ is the colimit of a filtered family of finite étale and local $A^\wedge$-algebras, hence it is strictly henselian, and therefore (13.1.34) is cartesian, especially the geometric fibres of f are connected if and only if the same holds for the geometric fibres of f^{sh}, and the latter are reduced (even regular), since A is excellent. Hence, we come down to showing that f^{sh} is locally 0-acyclic at the closed point of $\mathrm{Spec}\,(A^\wedge)^{\mathrm{sh}}$, which holds again by proposition 13.1.30(ii).

Proposition 13.1.35. *Let $f : X \to S$ be a morphism of schemes locally of finite type, and suppose that either one of the following conditions holds :*

(a) *For some $d \in \mathbb{N}$, the fibres of f are all of pure dimension d.*
(b) *f is* quasi-flat*, in the sense of* [64, Ch.IV, (2.3.3)] i.e. *there exists a quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ of finite type which is S-flat and with* $\mathrm{Supp}\,\mathscr{F} = X$ *(notation of definition* 10.1.1(iv)*).*

Then the Milnor fibres of f at every point of X have finitely many irreducible components.

Proof. Suppose first that (a) holds. We may assume that $X = \mathrm{Spec}\,B$ is affine, and by lemma 13.1.24(iii), we may also assume that $S = \mathrm{Spec}\,A$ is strictly local and normal, and the field of fractions F of A is separably closed. In this situation, we need to check that $X(\overline{x}) \times_S \mathrm{Spec}\,F$ has finitely many irreducible components, for every geometric point $\overline{x}$ of X whose image in S is localized at the maximal ideal $\mathfrak{m} \subset A$. By assumption, $B/\mathfrak{m}B$ is of pure dimension d; by the Nother normalization lemma, we may then find a finite map $\kappa(\mathfrak{m})[T_1, \dots, T_d] \to B/\mathfrak{m}B$ of $\kappa(\mathfrak{m})$-algebras, and lift it to a map of A-algebras $\varphi : A[T_1, \dots, T_d] \to B$ which will be quasi-finite at the support $x \in \mathrm{Spec}\,B$ of $\overline{x}$. After replacing X by an affine open neighborhood of x, we may therefore assume that the morphism $g : X \to \mathbb{A}^d_S$ induced by φ is quasi-finite ([65, Ch.IV, Cor.13.1.4]). Then, since $X_F := X \times_S \mathrm{Spec}\,F$ is of pure dimension d, it follows easily that the restriction $g_F : X_F \to \mathbb{A}^d_F$ of g is maximizing (see definition 8.1.46(ii)). Set $\overline{y} := g(\overline{x})$, let $h : X(\overline{x}) \to \mathbb{A}^d_S(\overline{y})$ be the induced morphism, and consider the commutative diagram of schemes :

$$\begin{array}{ccc} X(\overline{x}) \times_S \mathrm{Spec}\,F & \longrightarrow & X_F \\ {\scriptstyle h \times_S \mathrm{Spec}\,K}\downarrow & & \downarrow{\scriptstyle g_F} \\ \mathbb{A}^d_S(\overline{y}) \times_S \mathrm{Spec}\,F & \longrightarrow & \mathbb{A}^d_F. \end{array}$$

Notice that both horizontal arrows are flat morphisms, hence they are generizing, and the fibres of the bottom horizontal arrows are 0-dimensional; since g_F is maximizing, it then follows that the same holds for $h \times_S \mathrm{Spec}\,K$. On the other hand, h is a finite morphism, by Zariski's main theorem, so it has finite fibres, and $\mathbb{A}^d_S(\overline{y}) \times_S \mathrm{Spec}\,K$ is an integral scheme, by proposition 13.1.30(iii), whence the assertion.

Next, suppose that (b) holds; again, we fix a geometric point $\overline{x}$ of X, and we are easily reduced to the case where S is strictly local, X is affine, and the image $\overline{s}$ of $\overline{x}$ in S is localized at the closed point. Then we may find a closed immersion $i : X \to Y := \mathbb{A}^n_S$, for some $n \in \mathbb{N}$; according to [86, Part I, Th.3.4.1], the $\mathscr{O}_Y$-module $i_*\mathscr{F}$ is finitely presented in an affine open neighborhood U of the support of $\overline{y} := i(\overline{x})$, hence we may replace Y by U and X by $X \times_Y U$, and assume that $i_*\mathscr{F}$ is a finitely presented Y-scheme. Let $\mathscr{I} \subset \mathscr{O}_Y$ be the 0-th Fitting ideal of $i_*\mathscr{F}$, set $X' := \mathrm{Spec}\,\mathscr{O}_Y/\mathscr{I}$, and let $g : X' \to S$ be the induced projection; then X' is a closed subscheme of Y, and $i(X) = \mathrm{Supp}_Y(i_*\mathscr{F}) = X'$, hence the maximal reduced subschemes of X and X' are S-isomorphic. It follows easily that the $S(\overline{s})$-schemes $X(\overline{x})$ and $X'(\overline{y})$ have isomorphic maximal reduced subschemes. Hence, for every strict geometric point ξ of $S(\overline{s})$, the fibres $f_{\overline{x}}^{-1}(\xi)$ and $g_{\overline{x}}^{-1}(\xi)$ are homeomorphic. Thus, we may replace $f : X \to S$ by $g : X' \to S$, and assume from start that X is a finitely presented S-scheme.

Claim 13.1.36. There exist finitely many closed subschemes $Z_1, \dots, Z_k \subset X$ such that $X = Z_1 \cup \cdots \cup Z_k$, and integers $d_1, \dots, d_k \in \mathbb{N}$ such that for every $i = 1, \dots, k$, all the fibres of the restriction $Z_i \to S$ of f are of pure dimension d_i.

Proof of the claim. Say that $S = \mathrm{Spec}\,A$, $X = \mathrm{Spec}\,B$ and $\mathscr{F} = M^\sim$ for a finitely presented A-algebra B and an A-flat B-module M of finite presentation. Then we may find a noetherian subring $A_0 \subset A$, an A_0-algebra B_0 of finite type, and a B_0-module M_0 with an isomorphism of A-algebras $A \otimes_{A_0} B_0 \xrightarrow{\sim} B$ and of B-modules $B \otimes_{B_0} M_0 \xrightarrow{\sim} M$. Moreover, we may assume that M_0 is a flat A_0-module ([65, Ch.IV, Cor.11.2.6.1]), and that $\mathrm{Supp}_{B_0} M_0 = \mathrm{Spec}\,B_0$ ([65, Ch.IV, Cor.8.3.3]). In view of [64, Ch.IV, Cor.4.1.4], we are easily reduced to checking the assertion for the corresponding morphism $\mathrm{Spec}\,B_0 \to \mathrm{Spec}\,A_0$. The latter is provided by [65, Ch.IV, Prop.12.1.1.5]. ◊

To a decomposition $X = Z_1 \cup \cdots \cup Z_k$ as in claim 13.1.36, there corresponds a decomposition $X(\overline{x}) = Z_1(\overline{x}) \cup \cdots \cup Z_k(\overline{x})$ for every geometric point $\overline{x}$ of X, hence it suffices to check the assertion for each of the restrictions $Z_i \to S$ of f, so we may assume that condition (a) holds, in which case the assertion is already known. □

For future use, we point out the following

Proposition 13.1.37. *Let $g : X \to Y$ be a flat morphism of excellent noetherian schemes, with X strictly local, and Y normal. Let $U \subset X$ be an open subset, and $Z \subset Y$ a closed subscheme. Suppose that :*

(i) *$g^{-1}(z) \subset U$ for every maximal point z of Z.*
(ii) *$U \cap g^{-1}(z)$ is a dense open subset of $g^{-1}(z)$, for every $z \in Z$.*
(iii) *The fibres $g^{-1}(z)$ are reduced, for every $z \in Z$.*

Then the induced functor $\mathsf{Cov}(U) \to \mathsf{Cov}(U \times_Y Z)$ is fully faithful.

Proof. Indeed, say that $X = \mathrm{Spec}\,B$, $Y = \mathrm{Spec}\,A$, $Z = V(I)$ for some ideal $I \subset A$, and denote by $B^\wedge$ the $\mathfrak{m}_B$-adic completion of the local ring B (where $\mathfrak{m}_B \subset B$ denotes the maximal ideal). Let also $f : \mathrm{Spec}\,B^\wedge \to Y$ be the induced morphism, and $U^\wedge \subset \mathrm{Spec}\,B^\wedge$ the preimage of U. In light of example 13.1.33(ii) and lemma 13.1.15(ii), it suffices to show that the induced functor $\mathsf{Cov}(U^\wedge) \to \mathsf{Cov}(U^\wedge \times_Y Z)$ is fully faithful. In view of lemma 13.1.3, we are further reduced to checking that conditions (a)–(c) of proposition 10.5.46 hold for the induced ring homomorphism $\varphi : A \to B^\wedge$, the open subset $U^\wedge$, and the ideal I. However, by example 10.5.41, we have $\mathrm{Ass}_A(I, A) = \mathrm{Max}(Z)$, hence (c) follows trivially from our assumption (i). Next, since $B^\wedge$ is a faithfully flat B-algebra, assumption (ii) implies that $U^\wedge \cap f^{-1}(z)$ is a dense open subset of $f^{-1}(z)$, for every $z \in Z$. Moreover, since B is excellent, the natural morphism $\mathrm{Spec}\,B^\wedge \to X$ is regular, so the same holds for the induced morphism $f^{-1}(z) \to g^{-1}(z)$, and then our assumption (iii) implies – together with [126, Th.32.3(i)] – that $f^{-1}(z)$ is reduced, for every $z \in Z$, whence condition (b). Lastly, we check condition (a), *i.e.* we show that $B^\wedge$ is I-adically complete. Indeed, let C be the I-adic completion of $B^\wedge$; the natural map $B^\wedge \to C$ is injective, and it admits a left inverse, constructed as follows. Let $\underline{a} := (a_n \mid n \in \mathbb{N})$ be a given sequence of elements of $B^\wedge$, which is Cauchy for the I-adic topology; then $\underline{a}$ is also Cauchy for the $\mathfrak{m}_B$-adic topology, and it is easily seen that the limit l of $\underline{a}$ in the $\mathfrak{m}_B$-adic topology depends only on the class $[\underline{a}]$ of $\underline{a}$ in C, so we get a well defined ring homomorphism $\lambda : C \to B^\wedge$ by the rule : $[\underline{a}] \mapsto l$, and clearly λ is the sought left inverse. It remains to check that λ is injective; thus, suppose that $l = 0$, and that $[\underline{a}] \neq 0$; this means that there exists $N \in \mathbb{N}$ such that $a_n \notin I^N$, for every $n \in \mathbb{N}$. Now, the induced sequence $(\overline{a}_n \mid n \in \mathbb{N})$ of elements of $B^\wedge/I^N$ is stationary, and on the other hand, it converges $\mathfrak{m}_B$-adically to 0; therefore $\overline{a}_n = 0$ for every sufficiently large $n \in \mathbb{N}$, a contradiction. □

13.1.38. Let A be a noetherian normal ring, and endow the A-algebra $A[[t]]$ with its t-adic topology. Let

$$\varphi : \mathfrak{X} := \operatorname{Spf} A[[t]] \to X := \operatorname{Spec} A[[t]] \qquad \pi : X \to S := \operatorname{Spec} A \qquad i : S \to X$$

be respectively the natural morphism of locally ringed spaces, the natural projection, and the closed immersion determined by the ring homomorphism $A[[t]] \to A$ given by the rule : $f(t) \mapsto f(0)$, for every $f(t) \in A[[t]]$. Let also $U_0 \subset \operatorname{Spec} A$ be an open subset, $U := \pi^{-1}U_0$ and $\mathfrak{U} := \varphi^{-1}U$. Finally, denote by $\mathscr{E}$ a locally free $\mathscr{O}_U$-module of finite rank, and set $\mathscr{E}^\wedge := \varphi^*_{|\mathfrak{U}}\mathscr{E}$, which is a locally free $\mathscr{O}_{\mathfrak{U}}$-module of finite rank.

Lemma 13.1.39. *In the situation of* (13.1.38)*, suppose that $S \setminus U_0$ has codimension ≥ 2 in S. Then:*

(i) *The natural map*
$$\Gamma(U, \mathscr{E}) \to \Gamma(\mathfrak{U}, \mathscr{E}^\wedge)$$
is an isomorphism of $A[[t]]$-modules.

(ii) *The restriction $i_{|U_0} : U_0 \to U$ of i induces an equivalence :*
$$i^*_{|U_0} : \mathsf{Cov}(U) \to \mathsf{Cov}(U_0) \quad : \quad (E \to U) \mapsto (E \times_U U_0 \to U_0).$$

Proof. (i): To begin with, set $Z := S \setminus U_0$; since the morphism π is flat, hence generizing ([126, Th.9.5]), the closed subset $X \setminus U = \pi^{-1}Z$ has codimension ≥ 2 in X. Since A and $A[[t]]$ are both normal, we deduce :

$$\operatorname{depth}_{X \setminus V}\mathscr{O}_X \geq 2 \qquad \operatorname{depth}_Z \mathscr{O}_S \geq 2 \tag{13.1.40}$$

(theorem 10.4.21 and [126, Th.23.8]); therefore (corollary 10.4.23) :

$$\Gamma(U, \mathscr{O}_X) = \Gamma(X, \mathscr{O}_X) = A[[t]]. \tag{13.1.41}$$

Next, the short exact sequences of $\mathscr{O}_X$-modules :

$$0 \to i_*\mathscr{O}_S \to \mathscr{O}_X/t^{n+1}\mathscr{O}_X \to \mathscr{O}_X/t^n\mathscr{O}_X \to 0 \qquad \text{for every } n \in \mathbb{N}$$

induce exact sequences

$$R^j\Gamma_Z i_*\mathscr{O}_S \to R^j\Gamma_Z \mathscr{O}_X/t^{n+1}\mathscr{O}_X \to R^j\Gamma_Z \mathscr{O}_X/t^n\mathscr{O}_X \qquad \text{for every } n, j \in \mathbb{N}. \tag{13.1.42}$$

Then (13.1.40) and (13.1.42) yield inductively :

$$\operatorname{depth}_Z \mathscr{O}_X/t^n\mathscr{O}_X \geq 2 \qquad \text{for every } n \in \mathbb{N}$$

and again corollary 10.4.23 implies :

$$\Gamma(U, \mathscr{O}_X/t^n\mathscr{O}_X) = A[t]/t^nA[t] \qquad \text{for every } n \in \mathbb{N}. \tag{13.1.43}$$

Since U is quasi-compact, we may find a left exact sequence $P := (0 \to \mathscr{E} \to \mathscr{O}_U^{\oplus m} \to \mathscr{O}_U^{\oplus n})$ of $\mathscr{O}_U$-modules (corollary 10.3.28). Since φ is a flat morphism of locally ringed spaces, the sequence φ^*P is still left exact. Since the global section functors are left exact, we are then reduced to the case where $\mathscr{E} = \mathscr{O}_U$. Then we may write :

$$\mathscr{E}^\wedge = \mathscr{O}_{\mathfrak{U}} = \lim_{n \in \mathbb{N}} \mathscr{O}_U/t^n\mathscr{O}_U$$

where, for each $n \in \mathbb{N}$, we regard $\mathscr{O}_U/t^n\mathscr{O}_U$ as a sheaf of (pseudo-discrete) rings on $\mathfrak{U} = V(t) \subset U$. The functor $\Gamma(\mathfrak{U}, -)$ is a right adjoint, hence commutes with limits, and we deduce an isomorphism :

$$\Gamma(\mathfrak{U}, \mathscr{E}^\wedge) \stackrel{\sim}{\to} \lim_{n\in\mathbb{N}} \Gamma(U, \mathscr{O}_U/t^n\mathscr{O}_U).$$

(This is even a homeomorphism, provided we view the target as a limit of rings with the discrete topology.) Taking (13.1.43) into account, we obtain $\Gamma(\mathfrak{U}, \mathscr{E}^\wedge) = A[[t]]$ which, together with (13.1.41), implies the contention.

(ii): Notice that (i) and lemma 10.5.44 imply that $\mathrm{Lef}(U, i(U_0))$ holds (see definition 10.5.42). Since the pull-back functor $\pi^*_{|U} : \mathsf{Cov}(U_0) \to \mathsf{Cov}(U)$ is a right quasi-inverse to $i^*_{|U_0}$, the latter is essentially surjective. The full faithfulness is a special case of lemma 13.1.3. □

13.2. **Local asphericity of smooth morphisms of schemes.** Let S be a strictly local scheme, $s \in S$ the closed point, $f : X \to S$ a smooth morphism, $\overline{x}$ any geometric point of $f^{-1}(s)$, and denote by $f_{\overline{x}} : X(\overline{x}) \to S$ the induced morphism of strictly local schemes. For any open subset $U \subset S$ we have a base change functor :

(13.2.1) $$f^*_{\overline{x}} : \mathsf{Cov}(U) \to \mathsf{Cov}(f^{-1}_{\overline{x}}U) \qquad (E \to U) \mapsto (E \times_U f^{-1}_{\overline{x}}U).$$

Theorem 13.2.2. *In the situation of* (13.2)*, we have :*

(i) *The functor* (13.2.1) *is fully faithful.*
(ii) *Suppose moreover that S is excellent and normal, and that $S\backslash U$ has codimension ≥ 2 in S. Then* (13.2.1) *is an equivalence of categories.*

Proof. (i): In view of lemma 13.1.15(ii) it suffices to show that $f_{\overline{x}}$ is 0-acyclic (since in that case, the same will obviously hold also for its restriction $f^{-1}_{\overline{x}}U \to U$). To begin with, $f_{\overline{x}}$ is locally (-1)-acyclic, by proposition 13.1.30(i), hence it remains only to show that f is locally 0-acyclic at the point x (lemma 13.1.16). The latter assertion follows from proposition 13.1.30(ii) and [66, Ch.IV, Th.17.5.1].

(ii): In light of (i), it suffices to show that (13.2.1) is essentially surjective, under the assumptions of (ii). We argue by induction on the relative dimension n of f. Let $x \in X$ be the support of $\overline{x}$. We may find an open neighborhood $U \subset X$ of x, and an étale morphism of S-schemes $\varphi : U \to \mathbb{A}^n_S$ ([66, Ch.IV, Cor.17.11.4]). Let $\overline{x}' := \varphi(\overline{x})$; there follows an isomorphism of S-schemes : $X(\overline{x}) \stackrel{\sim}{\to} \mathbb{A}^n_S(\overline{x}')$, hence we may assume from start that $X = \mathbb{A}^n_S$, and f is the natural projection. Especially, the theorem holds for $n = 0$. Suppose then, that $n > 0$, and that the theorem is already known when the relative dimension is $< n$. Write $f = h \circ g$, where

$$g : X \simeq \mathbb{A}^{n-1}_S \times_S \mathbb{A}^1_S \to \mathbb{A}^1_S \qquad \text{and} \qquad h : \mathbb{A}^1_S \to S$$

are the natural projections; set $\overline{x}_1 := g(\overline{x})$, and $U_1 := h^{-1}_{\overline{x}_1}U$, (where $h_{\overline{x}_1} : S_1 := \mathbb{A}^1_S(\overline{x}_1) \to S$ is the morphism induced by h). We have $S_1\backslash U_1 = h^{-1}_{\overline{x}_1}(S\backslash U)$, and since flat maps are generizing ([126, Th.9.5]) we easily see that the codimension of $S_1\backslash U_1$ in S_1 equals the codimension of $S\backslash U$ in S. From our inductive assumption, we deduce that the base change functor $\mathsf{Cov}(U_1) \to \mathsf{Cov}(f^{-1}_{\overline{x}}U)$ is essentially surjective, and hence it suffices to show that the same holds for the functor $\mathsf{Cov}(U) \to \mathsf{Cov}(U_1)$. Thus, we are reduced to the case where $X = \mathbb{A}^1_S$. Suppose now, that $E \to f^{-1}_{\overline{x}}U$ is a finite étale morphism; we can write $f_{\overline{x}}$ as the limit of a cofiltered family of smooth morphisms $(f_\lambda : Y_\lambda \to S \mid \lambda \in \Lambda)$, where each Y_λ is an affine étale $\mathbb{A}^1_S$-scheme. Then $f^{-1}_{\overline{x}}U$ is the limit of the family $(Y_\lambda \times_S U \mid \lambda \in \Lambda)$. By [65, Ch.IV, Th.8.8.2(ii), Th.8.10.5] and [66, Ch.IV, Prop.17.7.8], we may find a $\lambda \in \Lambda$, a finite étale morphism $E_\lambda \to Y_\lambda \times_S U$ and an isomorphism of $f^{-1}_{\overline{x}}U$-schemes : $E_\lambda \times_{Y_\lambda} \mathbb{A}^1_S(\overline{x}) \stackrel{\sim}{\to} E$. Denote by $y \in Y_\lambda$ the image of the closed point of $\mathbb{A}^1_S(\overline{x})$, and by $\overline{y}$ the geometric point of Y_λ obtained as the image of $\overline{x}$ (the latter is viewed naturally as a geometric point of $\mathbb{A}^1_S(\overline{x})$); by construction, y lies in the closed fibre $Y_0 := Y_\lambda \times_S \mathrm{Spec}\,\kappa(s)$, which is an étale $\mathbb{A}^1_{\kappa(s)}$-scheme, and we may therefore find a specialization $z \in Y_0$ of y, with z a closed point. Pick a geometric point $\overline{z}$ of Y_0 localized at z, and a strict specialization map $Y_\lambda(\overline{z}) \to Y_\lambda(\overline{y})$ as in (4.9.23); there follows a commutative diagram :

$$\begin{array}{ccc} \mathbb{A}^1_S(\overline{x}) \simeq Y_\lambda(\overline{y}) & \longrightarrow & Y_\lambda(\overline{z}) \\ \downarrow & & \downarrow \\ Y_\lambda(y) & \longrightarrow & Y_\lambda(z). \end{array}$$

The finite étale covering $E_\lambda \times_{Y_\lambda} Y_\lambda(y) \to Y_\lambda(y) \times_S U$ lies in the essential image of the functor

$$\mathsf{Cov}(Y_\lambda(z) \times_S U) \to \mathsf{Cov}(Y_\lambda(y) \times_S U) \qquad C \mapsto C \times_{Y_\lambda(z)} Y_\lambda(y).$$

It follows that $E \to f_{\overline{x}}^{-1}U$ lies in the essential image of the functor

$$\mathsf{Cov}(f^{-1}_{\lambda,\overline{z}}U) \to \mathsf{Cov}(f^{-1}_{\overline{x}}U) \qquad C \mapsto C \times_{Y(\overline{z})} Y(\overline{y}) \simeq C \times_{Y(\overline{z})} \mathbb{A}^1_S(\overline{x})$$

and therefore it suffices to show that the pull-back functor $\mathsf{Cov}(U) \to \mathsf{Cov}(f_{\overline{z}}^{-1}U)$ is essentially surjective. In other words, we may replace x by z, and assume throughout that x is a closed point of $\mathbb{A}^1_S$.

Claim 13.2.3. Under the current assumptions, we may find a strictly local normal scheme T, with closed point t, a finite surjective morphism $g : T \to S$, and a finite morphism of $\mathrm{Spec}\,\kappa(s)$-schemes :

$$\mathrm{Spec}\,\kappa(t) \to \mathrm{Spec}\,\kappa(x).$$

Proof of the claim. Since $\kappa(x)$ is a finite extension of $\kappa(s)$, it is generated by finitely many algebraic elements $u_1, \dots u_n$, and an easy induction allows to assume that $n = 1$. In this case, one constructs first a scheme T' by taking any lifting of the minimal polynomial of u_1 : for the details, see e.g. [61, Ch.0, (10.3.1.2.)], which shows that the resulting T is local, finite and flat over S, so T' maps surjectively onto S. Next, we may replace T' by its maximal reduced subscheme, which is still strictly local and finite over S. Next, since S is excellent, the normalization $(T')^\nu$ of T' is finite over S ([64, Ch.IV, Scholie 7.8.3(vi)]); let T be any irreducible component of $(T')^\nu$; by [66, Ch.IV, Prop.18.8.10], T fulfills all the sought conditions. ◊

Choose $g : T \to S$ as in claim 13.2.3; since the residue field extension $\kappa(s) \to \kappa(t)$ is algebraic and purely inseparable, there exists a unique point $x' \in \mathbb{A}^1_S(\overline{x}) \times_S T$ lying over t, and we may find a unique strict geometric point $\overline{x}'$ of $\mathbb{A}^1_S(\overline{x}) \times_S T$ localized at x', and lying over $\overline{x}$. In view of [66, Ch.IV, Prop.18.8.10], there follows a natural isomorphism of T-schemes :

$$\mathbb{A}^1_S(\overline{x}) \times_S T \xrightarrow{\sim} \mathbb{A}^1_T(\overline{x}').$$

Denote by $f_{\overline{x}'} := f_{\overline{x}} \times_S T : \mathbb{A}^1_T(\overline{x}') \to T$ the natural projection, and set $U_T := g^{-1}U$; since the morphism $g : T \to S$ is generizing ([126, Th.9.4(ii)]), it is easily seen that $T \setminus U_T$ has codimension ≥ 2 in T.

Let $F : \mathsf{Cov} \to \mathbf{Sch}$ be the fibred category (13.1.1). We have a natural essentially commutative diagram of categories :

$$\begin{array}{ccc} \mathsf{Cov}(U) & \longrightarrow & \mathsf{Desc}(F, g \times_S U) \\ \downarrow & & \downarrow \delta \\ \mathsf{Cov}(f^{-1}_{\overline{x}}U) & \longrightarrow & \mathsf{Desc}(F, g \times_S f^{-1}_{\overline{x}}U) \end{array} \tag{13.2.4}$$

where, for any morphism of schemes h, we have denoted by $\mathsf{Desc}(F, h)$ the category of descent data for the fibred category F, relative to the morphism h.

According to lemma 13.1.2, the morphism g is of universal 2-descent for the fibred category F, so the horizontal arrows in (13.2.4) are equivalences. Hence, the theorem will follow, once we know that δ is essentially surjective. However, we have :

Claim 13.2.5. (i) Set $U'_T := U_T \times_S T$ and $U''_T := U'_T \times_S T$. The pull-back functors :

$$\mathsf{Cov}(U'_T) \to \mathsf{Cov}(\mathbb{A}^1_T(\overline{x}') \times_T U'_T) \qquad \mathsf{Cov}(U''_T) \to \mathsf{Cov}(\mathbb{A}^1_T(\overline{x}') \times_T U''_T)$$

are fully faithful.

(ii) Suppose that the pull-back functor

$$\mathsf{Cov}(U_T) \to \mathsf{Cov}(f^{-1}_{\overline{x}'} U_T)$$

is essentially surjective. Then the same holds for the functor δ.

Proof of the claim. (i): Let $\overline{z}''$ be any geometric point of $X'' := \mathbb{A}^1_T \times_S T \times_S T$ whose strict image in $\mathbb{A}^1_T$ is $\overline{x}'$, and let $\overline{z}'$ be the image of $\overline{z}''$ in $X' := \mathbb{A}^1_T \times_S T$; by lemma 13.1.22(ii), the natural morphisms :

$$X'(\overline{z}') \to \mathbb{A}^1_T(\overline{x}') \times_S T \qquad X''(\overline{z}'') \to \mathbb{A}^1(\overline{x}') \times_S T \times_S T$$

are isomorphisms (notice that $T \times_S T$ is also strictly local). Then the claim follows from assertion (i) of the theorem, applied to the projections $X' \to T \times_S T$ and $X'' \to T \times_S T \times_S T$.

(ii): Recall that an object of $\mathsf{Desc}(F, g \times_S f^{-1}_{\overline{x}'} U)$ consists of a finite étale morphism $E'_T \to f^{-1}_{\overline{x}'} U_T$ and a X'-isomorphism $\beta' : E'_T \times_S T \xrightarrow{\sim} T \times_S E'_T$ fulfilling a cocycle condition on $E \times_S T \times_S T$. By assumption, E_T descends to a finite étale morphism $E_T \to U_T$; then (i) implies that β' descends to a U'_T-isomorphism $\beta : E_T \times_S T \xrightarrow{\sim} T \times_S E_T$, and the cocycle identity for β' descends to a cocycle identity for β. ◊

In view of claim 13.2.5, we may replace (S, U, x) by (T, U_T, x'), and therefore assume that x is a $\kappa(s)$-rational point of $\mathbb{A}^1_{\kappa(s)}$. In this case, any choice of coordinate t on $\mathbb{A}^1_S$ yields a section $\sigma_{\overline{x}} : S \to \mathbb{A}^1_S(\overline{x})$ of the natural projection, such that $\sigma_{\overline{x}}(s) = x$. To conclude the proof of the theorem, it suffices to show that the pull-back functor :

$$\mathsf{Cov}(f^{-1}_{\overline{x}} U) \xrightarrow{\sigma^*_{\overline{x}}} \mathsf{Cov}(U)$$

is fully faithful.

Say that $S = \operatorname{Spec} A$; then the scheme $\mathbb{A}^1_S(\overline{x})$ is the spectrum of $A\{t\}$, the henselization of $A[t]$ along the ideal $\mathfrak{m}\{t\}$ generated by t and the maximal ideal $\mathfrak{m}$ of A. Let $A^\wedge$ (resp. $A\{t\}^\wedge$) be the $\mathfrak{m}$-adic (resp. $\mathfrak{m}\{t\}$-adic) completion of A (resp. of $A\{t\}$), and notice the natural isomorphism:

$$A\{t\}^\wedge / tA\{t\}^\wedge \xrightarrow{\sim} A^\wedge$$

(indeed, it is easy to check that $A\{t\}^\wedge \simeq A^\wedge[[t]]$), whence a natural diagram of schemes :

$$\begin{array}{ccc} X^\wedge := \operatorname{Spec} A\{t\}^\wedge & \xrightarrow{g'} & \operatorname{Spec} A\{t\} \\ {\scriptstyle \pi} \downarrow \uparrow {\scriptstyle \sigma} & & {\scriptstyle f_{\overline{x}}} \downarrow \uparrow {\scriptstyle \sigma_{\overline{x}}} \\ S^\wedge := \operatorname{Spec} A^\wedge & \xrightarrow{g} & \operatorname{Spec} A \end{array}$$

(where π is the natural projection) whose horizontal arrows commute with both the downward arrows and the upward ones. Set $U^\wedge := g^{-1} U$; by example 13.1.33(ii) and lemma 13.1.15(ii), the pull-back functors

$$g^* : \mathsf{Cov}(U) \to \mathsf{Cov}(U^\wedge) \qquad g'^* : \mathsf{Cov}(f^{-1}_{\overline{x}} U) \to \mathsf{Cov}(\pi^{-1} U^\wedge)$$

are fully faithful. Consequently, we are easily reduced to showing that the pull-back functor : $\mathsf{Cov}(\pi^{-1} U^\wedge) \xrightarrow{\sigma^*} \mathsf{Cov}(U^\wedge)$ is an equivalence. The latter holds by lemma 13.1.39(ii). □

Example 13.2.6. As an application of theorem 13.2.2, suppose that $K \subset E$ is an extension of separably closed fields, V_K a geometrically normal and strictly local K-scheme, $U \subset V_K$ an open subset, and ξ a geometric point of $V_E := V_K \times_K E$, whose image in V_K is supported on the closed point. Then the induced functor

$$\mathsf{Cov}(U) \to \mathsf{Cov}(U \times_{V_K} V_E(\xi))$$

is fully faithful, and it is an equivalence in case V_K is excellent and $V_K \setminus U$ has codimension ≥ 2 in V_K. Indeed, let K^{a} (resp. E^{a}) be an algebraic closure of K (resp. E), and choose a homomorphism $K^{\mathrm{a}} \to E^{\mathrm{a}}$ extending the inclusion of K into E. Then both $V_{K^{\mathrm{a}}} := V_K \times_K K^{\mathrm{a}}$ and $V_{E^{\mathrm{a}}}(\xi) := V_E(\xi) \times_E E^{\mathrm{a}}$ are still normal and strictly local (lemma 13.1.22(ii)), and the induced functors

$$\mathsf{Cov}(U) \to \mathsf{Cov}(U \times_K K^{\mathrm{a}}) \qquad \mathsf{Cov}(U \times_{V_K} V_E(\xi)) \to \mathsf{Cov}(U \times_{V_K} V_{E^{\mathrm{a}}}(\xi))$$

are equivalences (lemma 13.1.7(i)). It then suffices to show that the induced functor

$$\mathsf{Cov}(U \times_K K^{\mathrm{a}}) \to \mathsf{Cov}(U \times_{V_K} V_{E^{\mathrm{a}}}(\xi))$$

has the asserted properties. Hence, we may replace K by K^{a} and E by E^{a}, and assume from start that $K \subset E$ is an extension of algebraically closed fields. In this case, E can be written as the colimit of a filtered family $(R_\lambda \mid \lambda \in \Lambda)$ of smooth K-algebras; correspondingly, V_E is the limit of a cofiltered system $(V_\lambda \mid \lambda \in \Lambda)$ of smooth V_K-schemes, and – by lemma 13.1.6 – $\mathsf{Cov}(U \times_{V_K} V_E(\xi))$ is the 2-colimit of the system of categories

$$\mathsf{Cov}(U \times_{V_K} V_\lambda(\xi_\lambda)) \qquad (\lambda \in \Lambda)$$

(where, for each $\lambda \in \Lambda$, we denote by ξ_λ the image of ξ in V_λ). Now the contention follows directly from theorem 13.2.2.

Theorem 13.2.7. *Let $f : X \to S$ is a smooth morphism of schemes, $\mathbb{L} \subset \mathbb{N}$ be a set of primes, and suppose that all the elements of $\mathbb{L}$ are invertible in $\mathscr{O}_S$. Then f is 1-aspherical for $\mathbb{L}$.*

Proof. Let $\overline{x}$ be any geometric point of X, $\overline{s} := f(\overline{x})$, and $\overline{\eta}$ a strict geometric point of $S(\overline{s})$. To ease notation, set $T := X(\overline{x})$, let $f_{\overline{x}} : T \to S(\overline{s})$ be the natural map, and $T_{\overline{\eta}} := f_{\overline{x}}^{-1}(\overline{\eta})$; we have to show that $H^1(T_{\overline{\eta},\text{ét}}, G) = \{1\}$ for every $\mathbb{L}$-group G. Arguing as in the proof of proposition 13.1.30, we reduce to the case where $S = S(\overline{s})$. Then, by lemma 13.1.24, we can further assume that S is normal and $\overline{\eta}$ is localized at the generic point η of S. By lemma 13.1.29, S is the limit of a cofiltered system $(S_\lambda \mid \lambda \in \Lambda)$ of strictly local, normal and excellent schemes, with dominant transition maps, and as usual, after replacing Λ by a coinitial subset, we may assume that f (resp. $\overline{\eta}$) descends to a compatible system of morphisms $(f_\lambda : X_\lambda \to S_\lambda \mid \lambda \in \Lambda)$, (resp. of strict geometric points $\overline{\eta}_\lambda$ localized at the generic point of S_λ). By [66, Ch.IV, Prop.17.7.8(ii)], there exists $\lambda \in \Lambda$ such that f_μ is smooth for every $\mu \geq \lambda$. Then, in view of [9, Exp.VII, Rem.5.14] and the isomorphism (13.1.28), we may replace f by f_λ, and $\overline{\eta}$ by $\overline{\eta}_\lambda$, and assume from start that S is strictly local, normal and excellent, and G is a finite $\mathbb{L}$-group.

Let $(\varphi : E_{\overline{\eta}} \to T_{\overline{\eta}}, \rho : G \to \mathrm{Aut}_{T_{\overline{\eta}}}(E_{\overline{\eta}}))$ be a principal G-homogeneous space; we come down to showing that $E_{\overline{\eta}}$ has a section $T_{\overline{\eta}} \to E_{\overline{\eta}}$. By [66, Ch.IV, Prop.17.7.8(ii)] and [65, Ch.IV, Th.8.8.2(ii), Th.8.10.5], we may find a finite separable extension $\kappa(\eta) \subset L$, and a principal G-homogeneous space

$$\varphi_L : E_L \to T_L := T \times_S \operatorname{Spec} L \qquad \rho_L : G \to \mathrm{Aut}_{T_L}(E_L)$$

such that

$$\varphi = \varphi_L \times_{\operatorname{Spec} L} \operatorname{Spec} \kappa(\overline{\eta}) \qquad \rho = \rho_L \times_{\operatorname{Spec} L} \operatorname{Spec} \kappa(\overline{\eta}).$$

Say that $S = \operatorname{Spec} A$, denote by A_L the normalization of A in L, and set $S_L := \operatorname{Spec} A_L$. Then S_L is again normal and excellent ([64, Ch.IV, (7.8.3)(ii,vi)]), and the residue field of A_L is an algebraic extension of the residue field of A, hence it is separably closed, so S_L is strictly local as well. Thus, we may replace S by S_L, and assume that $E_{\overline{\eta}}$ descends to a principal G-homogeneous space $E_\eta \to T_\eta := f_{\overline{x}}^{-1}(\eta)$ on T_η. Next, we may write η as the limit of the filtered system of affine open subsets of S, so that – by the same arguments – we find an affine open subset $U \subset S$ and a principal G-homogeneous space $E_U \to T_U := f_{\overline{x}}^{-1}U$, with a G-equivariant isomorphism of T_η-schemes : $E_U \times_{T_U} T_\eta \xrightarrow{\sim} E_\eta$. Denote by $D_1, \dots, D_n$ the irreducible components of $S \backslash U$ which have codimension one in S, and for every $i \leq n$, endow D_i with its reduced closed subscheme structure, and set $D'_i := D_i \times_S T$. Let also η_T be the generic point of T.

Claim 13.2.8. For given $i \leq n$, let y be the generic point of D_i. We have:

(i) T and E_U are noetherian normal schemes, and $T(y) := T \times_S S(y)$ is regular.
(ii) D'_i is an integral scheme of pure codimension one in T, and $D'(y) := D'_i \times_{D_i} D_i(y)$ is regular (and irreducible).
(iii) Let also $z \in D'_i$ be the generic point, and $\mathfrak{m}_y$ (resp. $\mathfrak{m}_z$) the maximal ideal of $\mathscr{O}_{S,y}$ (resp. of $\mathscr{O}_{T,z}$); then $\mathscr{O}_{S,y}$ and $\mathscr{O}_{T,z}$ are discrete valuation rings, and $\mathfrak{m}_y \cdot \mathscr{O}_{T,z} = \mathfrak{m}_z$.
(iv) Let $t \in A$ be any element such that $t \cdot \mathscr{O}_{S,y} = \mathfrak{m}_y$. Then there exist an integer $m > 0$ such that $(m, \mathrm{char}\,\kappa(s)) = 1$, a finite étale covering

$$E_y \to T(y)[t^{1/m}] := T(y) \times_S \mathrm{Spec}\,A[T]/(T^m - t)$$

and an isomorphism of $T(y)[t^{1/m}] \times_T T_U$-schemes :

$$E_y \times_T T_U \xrightarrow{\sim} E_U \times_T T(y)[t^{1/m}].$$

Proof of the claim. (i): Since S is normal by assumption, the assertion for T and E_U follows from [66, Ch.IV, Prop.17.5.7, Prop.18.8.12(i), Prop.18.8.8(iv)]. Let $u \in T(y)$ be any point; denote by $w \in X$ the image of u. Since $\mathscr{O}_{S,y}$ is a discrete valuation ring, $\mathscr{O}_{X,w}$ is a regular local ring ([66, Ch.IV, Prop.17.5.8(iii)]); moreover, the natural map $\mathscr{O}_{X,w} \to \mathscr{O}_{T(y),u}$ induces isomorphisms on strict henselizations, therefore $\mathscr{O}_{T(y),u}$ is regular ([66, Ch.IV, Prop.18.8.13]).

(ii): By virtue of proposition 13.1.30(ii), the geometric generic fibre of the induced morphism $D'_i \to D_i$ is connected, hence the same holds for its generic fibre. But arguing as in the proof of (i), we see that this generic fibre is noetherian and regular, so it is irreducible and reduced. Since moreover the morphism $D'_i \to D_i$ is flat, it follows easily that D'_i is irreducible and reduced as well. Lastly, D'_i is of pure codimension one in T, by virtue of [64, Ch.IV, Cor.6.1.4].

(iii): From (i) and (ii) we see that $\mathscr{O}_{S,y}$ and $\mathscr{O}_{T,z}$ are discrete valuation rings; since moreover D'_i is reduced, the same holds for $\mathscr{O}_{T,z}/\mathfrak{m}_y\mathscr{O}_{T,z}$, so $\mathfrak{m}_y \cdot \mathscr{O}_{T,z} = \mathfrak{m}_z$.

(iv): By (iii) we have $t \cdot \mathscr{O}_{T,z} = \mathfrak{m}_z$. Notice that $T(z) \times_T T_U = T(\eta_T)$, and $E_{\eta_T} := E_U \times_T T(z)$ is a disjoint union of spectra of finite separable extensions $L_1, \dots, L_k$ of $\kappa(\eta_T)$. Moreover, E_{η_T} is a principal G-homogeneous space over $T(\eta_T)$, *i.e.* every L_j is a Galois extension of $\kappa(\eta_T)$, with Galois group $G_j := \mathrm{Gal}(L_j/\kappa(\eta_T)) \subset G$. Since G is an $\mathbb{L}$-group, the same holds for G_j, hence $E_U \times_T T(y)$ is tamely ramified along the divisor $\overline{\{z\}}$ (the topological closure of $\{z\}$ in $T(y)$), and the assertion follows from Abhyankar's lemma [82, Exp.XIII, Prop.5.2]. ◊

Claim 13.2.9. There exist :

(a) a finite dominant morphism $S' \to S$, such that both S' and $T' := T \times_S S'$ are strictly local and normal;
(b) an open subset $U' \subset S'$, such that $S' \backslash U'$ has codimension ≥ 2 in S';
(c) a finite étale morphism $E' \to T'_{U'} := T \times_S U'$, with an isomorphism of $T'_{U'}$-schemes :

$$E' \times_T T_\eta \simeq E_\eta \times_T T'_{U'}.$$

Proof of the claim. For every $i \leq n$, let y_i be the maximal point of D_i, and choose $t_i \in A$ whose image in $\mathscr{O}_{S,y_i}$ generates the maximal ideal. Choose also $m_i \in \mathbb{N}$ with $(m_i, \mathrm{char}\,\kappa(s)) = 1$

and such that there exists a finite étale covering $E_i \to T(y_i)[t_i^{1/m_i}]$ extending the étale covering $E_U \times_T T(y_i)[t_i^{1/m_i}]$ (claim 13.2.8(iv)). Fix an algebraic closure K of $\mathrm{Frac}\, A$, pick for each $i = 1, \dots, n$ an m_i-th root τ_i of t_i in K, and let S' be the normalization of $\mathrm{Spec}\, A[\tau_1, \dots, \tau_n]$. Then S' is finite over S, hence it is excellent ([64, Ch.IV, (7.8.3)(ii,vi)]), and strictly local (cp. the proof of lemma 13.1.24). Set $E'_\eta := E_\eta \times_S S'$, $T' := T \times_S S'$; since the geometric fibres of $f_{\overline{x}}$ are connected (proposition 13.1.30(ii)), the same holds for the geometric fibres of the induced morphism $T' \to S'$, therefore T' is connected, and then it is also strictly local, by the usual arguments. Notice also that T' is the limit of a cofiltered family of smooth S'-schemes, hence it is reduced and normal ([66, Ch.IV, Prop.17.5.7]). Say that $E'_\eta = \mathrm{Spec}\, C$, $T = \mathrm{Spec}\, B$, $T' = \mathrm{Spec}\, B'$, and let C' be the integral closure of B' in C. Notice that $C \otimes_B \kappa(\eta_T)$ is a finite product of finite separable extensions of the field $B' \otimes_B \kappa(\eta_T)$, and consequently the natural morphism $\varphi' : E_{T'} := \mathrm{Spec}\, C' \to T'$ is finite ([126, §33, Lemma 1]). Define :

$$U' := \{y \in S' \mid \varphi' \times_{S'} S'(y) : E_{T'}(y) \to T'(y) \text{ is étale}\}.$$

Let now $y \in U'$ any point; then $S'(y)$ is the limit of the cofiltered family $(U_\lambda \mid \lambda \in \Lambda)$ of affine open neighborhoods of y in S', and $\varphi' \times_{S'} S'(y)$ the limit of the system of morphisms $(\varphi'_\lambda := \varphi' \times_{S'} U_\lambda \mid \lambda \in \Lambda)$; we may then find $\lambda \in \Lambda$ such that φ'_λ is étale ([66, Ch.IV, Prop.17.7.8(ii)]), hence $U_\lambda \subset U'$, which shows that U' is open. Furthermore, from [66, Ch.IV, Prop.17.5.7] it follows that $E_U \times_T T'$ is normal, whence an isomorphism of T'-schemes :

$$E_{T'} \times_S U \simeq E_U \times_T T'$$

(cp. the proof of lemma 13.1.7(iii)) especially, $U \times_S S' \subset U'$. Likewise, by construction we have natural morphisms : $T'(y_i) \to T(y_i)[t_i^{1/m_i}]$, and using the fact that all the schemes in view are normal we deduce isomorphisms of $T'(y_i)$-schemes :

$$E_{T'}(y_i) \xrightarrow{\sim} E_i \times_{T(y_i)[t_i^{1/m_i}]} T'(y_i).$$

Thus, U' contains all the points of S' of codimension ≤ 1, since the image in S of any such point lies in $U \cup \{y_1, \dots, y_n\}$. The morphism $E' := E_{T'} \times_{S'} U' \to T'_{U'}$ fulfills conditions (a)-(c). $\diamondsuit$

Now, choose $S' \to S$, $U' \subset S'$, and $E' \to T'_{U'}$ as in claim 13.2.9; since the corresponding T' is local, there exists a unique point $x' \in X' := X \times_S S'$ lying over x; pick a geometric point $\overline{x}'$ of X' localized at x', and lying over $\overline{x}$; it then follows from [66, Ch.IV, Prop.18.8.10] that the natural morphism $X'(\overline{x}') \to T'$ is an isomorphism. In such situation, theorem 13.2.2 says that there exists a finite étale covering $E \to U'$ with an isomorphism of $T'_{U'}$-schemes : $E \times_{U'} T'_{U'} \xrightarrow{\sim} E'$, whence an isomorphism of $T_{\overline{\eta}}$-schemes :

$$E_{\overline{\eta}} \simeq E(\overline{\eta}) \times_{\mathrm{Spec}\, \kappa(\overline{\eta})} T_{\overline{\eta}}.$$

Since $\kappa(\overline{\eta})$ is separably closed, the étale morphism $E(\overline{\eta}) \to \mathrm{Spec}\, \kappa(\overline{\eta})$ admits a section, hence the same holds for φ, as claimed. $\square$

13.2.10. Let $f : X \to S$ be a morphism of schemes, and $j : U \subset X$ an open immersion, and set $U_\eta := U \cap f^{-1}(\eta)$ for every $\eta \in S_{\max}$, where $S_{\max} \subset S$ denotes the subset of all maximal points of S. We deduce a natural essentially commutative diagram of functors :

$$\mathscr{D}(S,f,U) \quad : \quad \begin{array}{ccc} \mathsf{Cov}(X) & \xrightarrow{\quad j^* \quad} & \mathsf{Cov}(U) \\ {\scriptstyle \prod_\eta \iota^*_\eta}\downarrow & & \downarrow{\scriptstyle \prod_\eta \iota^*_{\eta|U}} \\ \prod_{\eta \in S_{\max}} \mathsf{Cov}(f^{-1}\eta) & \xrightarrow{\prod_\eta j^*_\eta} & \prod_{\eta \in S_{\max}} \mathsf{Cov}(U_\eta) \end{array}$$

where $j_\eta : U_\eta \to f^{-1}(\eta)$ is the restriction of j and $\iota_\eta : f^{-1}(\eta) \to X$ is the natural immersion. Let us say that $U \subset X$ is *fibrewise dense*, if $f^{-1}(s) \cap U$ is dense in $f^{-1}(s)$, for every $s \in S$. Then we have :

Theorem 13.2.11. *In the situation of* (13.2.10)*, suppose that f is smooth, and U is fibrewise dense. The following holds :*

(i) *The restriction functor j^* is fully faithful.*
(ii) *The diagram $\mathscr{D}(S, f, U)$ is 2-cartesian.*
(iii) *If furthermore, $f^{-1}S_{\max} \subset U$, then j^* is an equivalence.*

Proof. Assertion (ii) means that the functors j^* and ι_η^* induce an equivalence $(j, \iota_\bullet)^*$ from $\mathsf{Cov}(X)$ to the category $\mathscr{C}(X, U)$ of data

(13.2.12) $$\underline{E} := (\varphi, (\psi_\eta, \alpha_\eta \mid \eta \in S_{\max}))$$

where φ (resp. ψ_η) is an object of $\mathsf{Cov}(U)$ (resp. of $\mathsf{Cov}(f^{-1}\eta)$, for every $\eta \in S_{\max}$), and $\alpha_\eta : \varphi \times_U \operatorname{Spec} \kappa(\eta) \xrightarrow{\sim} \psi_\eta \times_{f^{-1}\eta} U_\eta$ is an isomorphism of U-schemes, for every $\eta \in S_{\max}$ (see example 3.3.12(ii)). On the basis of this description, it is easily seen that (i),(ii)⇒(iii). Furthermore, we remark :

Claim 13.2.13. (i) If j^* is fully faithful, then the same holds for $(j, \iota_\bullet)^*$.

(ii) For every open subset $U' \subset X$ containing U, suppose that :
 (a) The pull-back functor $\mathsf{Cov}(U') \to \mathsf{Cov}(U)$ is fully faithful.
 (b) If $f^{-1}S_{\max} \subset U'$, the pull-back functor $\mathsf{Cov}(X) \to \mathsf{Cov}(U')$ is an equivalence.
Then assertion (ii) holds.

Proof of the claim. (i): Since $f^{-1}\eta$ is a normal (even regular) scheme ([66, Ch.IV, Prop.17.5.7]), the pull-back functors ι_η^* are fully faithful (lemma 13.1.7(iii)); the assertion is an immediate consequence.

(ii): In light of (i), it remains only to check that $(j, \iota_\bullet)^*$ is essentially surjective. Thus, let $\varphi : E \to U$ be a finite étale morphism, such that $i_\eta^* \varphi$ extends to a finite étale morphism $\varphi'_\eta : E'_\eta \to f^{-1}(\eta)$, for every maximal point $\eta \in S$. By claim 13.1.8, there is a largest open subset $U_{\max}$ containing U, over which φ extends to a finite étale morphism $\varphi_{\max}$. To conclude, we have to show that $U_{\max} = X$. However, for any maximal point η, let $i_\eta : f^{-1}(\eta) \to X(\eta)$ be the natural closed immersion. By lemma 13.1.7(i), i_η^* is an equivalence, hence we may find a finite étale morphism $\varphi'_{(\eta)} : E'(\eta) \to X(\eta)$ such that $i_\eta^* \varphi'_{(\eta)} \simeq \varphi'_\eta$. By the same token, we also see that $E'(\eta) \times_{X(\eta)} U(\eta)$ is $U(\eta)$-isomorphic to $E \times_U U(\eta)$.

Next, $S(\eta)$ is the limit of the filtered system $\mathscr{V}$ of all open subsets $V \subset S$ with $\eta \in V$, hence lemma 13.1.6 ensures that we may find $V \in \mathscr{V}$ and an object $\varphi'_V : E'_V \to f^{-1}V$ of $\mathsf{Cov}(f^{-1}V)$ such that $\varphi'_V \times_V S(\eta) \simeq \varphi'_{(\eta)}$, and after shrinking V, we may also assume (again by lemma 13.1.6) that $E'_V \times_X U$ is U-isomorphic to $E \times_S V$. Hence we may glue E' and E along the common intersection, to deduce a finite étale morphism $E' \to U' := U \cup f^{-1}V$ that extends φ. It follows that $f^{-1}(\eta) \subset f^{-1}V \subset U_{\max}$. Since η is arbitrary, (b) implies that the pull-back functor $\mathsf{Cov}(X) \to \mathsf{Cov}(U_{\max})$ is an equivalence, especially φ lies in the essential image of j^*, as claimed. ◊

Claim 13.2.14. (i) Suppose that S is noetherian and normal, and X is separated. Then (i) holds.

(ii) If furthermore, S is also excellent, then (ii) holds as well.

Proof of the claim. (i): Under the assumptions of the claim, X is normal and noetherian ([66, Ch.IV, Prop.17.5.7]), so (i) follows from lemma 13.1.7(iii), which also says – more generally – that assumption (a) of claim 13.2.13(ii) holds in this case, hence in order to show (ii) it suffices to check that assumption (b) of claim 13.2.13(ii) holds whenever $U \cup f^{-1}S_{\max} \subset U' \subset X$, especially $X\setminus U'$ has codimension ≥ 2 in X. Suppose first that S is regular; then the same holds for X ([66, Ch.IV, Prop.17.5.8]), and the contention follows from lemma 13.1.7(iv).

In the general case, let $S_{\mathrm{reg}} \subset S$ be the regular locus, which is open since S is excellent, and contains all the points of codimension ≤ 1, by Serre's normality criterion ([64, Ch.IV, Th.5.8.6]). Consider the restriction $f^{-1}S_{\mathrm{reg}} \to S_{\mathrm{reg}}$ of f, and the fibrewise dense open immersion $j_{\mathrm{reg}} : U' \cap f^{-1}S_{\mathrm{reg}} \subset f^{-1}S_{\mathrm{reg}}$; by the foregoing, the functor j^*_{reg} is an equivalence, hence we are easily reduced to showing that the functor $\mathsf{Cov}(X) \to \mathsf{Cov}(U' \cup f^{-1}S_{\mathrm{reg}})$ is an equivalence, *i.e.* we may assume that $V := f^{-1}S_{\mathrm{reg}} \subset U'$. Moreover, since the full faithfulness of j^* is already known, we only need to show that any finite étale morphism $\varphi : E \to U'$ extends to an object of $\mathsf{Cov}(X)$. To this aim, by lemma 13.1.7(iii), it suffices to prove that $\varphi \times_{U'} (X(\overline{x}) \times_X U')$ extends to an object of $\mathsf{Cov}(X(\overline{x}))$, for every geometric point $\overline{x}$ of X. Let $\overline{s} := f(\overline{x})$, and denote by $s \in S$ the support of $\overline{s}$; by assumption, we may find a geometric point ξ of $f^{-1}(s)$, whose support lies in $U' \cap f^{-1}(s)$, and a strict specialization morphism $X(\xi) \to X(\overline{x})$. There follows an essentially commutative diagram :

$$\begin{array}{ccccc} \mathsf{Cov}(X(\overline{x})) & \xrightarrow{\ \rho\ } & \mathsf{Cov}(X(\overline{x}) \times_X V) & \xleftarrow{\ \alpha\ } & \mathsf{Cov}(S(\overline{s}) \times_S S_{\mathrm{reg}}) \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \gamma} & \swarrow{\scriptstyle \beta} & \\ \mathsf{Cov}(X(\xi)) & \xrightarrow{\ \tau\ } & \mathsf{Cov}(X(\xi) \times_X V) & & \end{array}$$

where α and β are both equivalences, by theorem 13.2.2(ii); hence γ is an equivalence as well. Moreover, both $\mathsf{Cov}(X(\overline{x}))$ and $\mathsf{Cov}(X(\xi))$ are equivalent to the category of finite sets, and δ is obviously an equivalence. By construction, $\gamma(\varphi \times_{U'} (X(\xi) \times_X V))$ lies in the essential image of τ, hence $\varphi \times_{U'} (X(\xi) \times_X V)$ lies in the essential image of ρ, so say it is isomorphic to $\rho(\varphi')$ for some object φ' of $\mathsf{Cov}(X(\overline{x}))$. Using (i) (and [66, Ch.IV, Prop.18.8.12]) one checks easily that $\varphi' \times_X U' \simeq \varphi \times_{U'} (X(\xi) \times_X U')$, whence the contention. $\Diamond$

Claim 13.2.15. Let $m \in \mathbb{N}$ be any integer. Assertions (i) and (ii) hold if S and X are affine schemes of finite type over $\mathrm{Spec}\,\mathbb{Z}$, the fibres of f have pure dimension m, and furthermore :

$$\dim f^{-1}(s)\setminus U < m \qquad \text{for every } s \in S. \tag{13.2.16}$$

Proof of the claim. Indeed, in this situation, S admits finitely many maximal points, hence the normalization morphism $S^\nu \to S$ is integral and surjective. Set :

$$S_2 := S^\nu \times_S S^\nu \qquad U_1 := U \times_S S^\nu \qquad U_2 := U \times_S S_2.$$

Let $\beta : X_1 := X \times_S S^\nu \to X$, $f_1 : X_1 \to S^\nu$ and $j_2 : U_2 \to X_2 := X \times_S S_2$ be the induced morphisms; clearly $f_1^{-1}(s')$ has pure dimension m for every $s' \in S^\nu$, and from (13.2.16) we deduce that $\dim f_1^{-1}(s') \setminus U_1 < m$, especially, U_1 is dense in every fibre of f_1; by the same token, U_2 is dense in X_2. Then j_2^* is faithful (lemma 13.1.7(ii)), and lemma 13.1.2 and corollary 3.5.30(ii) imply that j^* is fully faithful, provided the same holds for the functor $j_1^* : \mathsf{Cov}(X_1) \to \mathsf{Cov}(U_1)$. In other words, in order to prove assertion (i), we may replace (f, U) by (f_1, U_1), which allows to assume that S is an affine normal scheme, and then it suffices to invoke claim 13.2.14, to conclude.

Concerning assertion (ii) : by the foregoing, we already know that j^* is fully faithful, so the same holds for $(j, \iota_\bullet)^*$ (claim 13.2.13(i)). To show that $(j, \iota_\bullet)^*$ is essentially surjective, let $\underline{E}$ be an object as in (13.2.12) of the category $\mathscr{C}(X, U)$; the normalization morphism induces a bijection $S^\nu_{\max} \xrightarrow{\sim} S_{\max} : \eta^\nu \mapsto \eta$, and clearly $\kappa(\eta^\nu) = \kappa(\eta)$ for every $\eta \in S_{\max}$, whence a datum

$$\underline{E}^\nu := (\varphi_1 := \varphi \times_U U_1, (\psi_\eta, \alpha_\eta \mid \eta^\nu \in S^\nu_{\max}))$$

of the analogous category $\mathscr{C}(X_1, U_1)$; by claim 13.2.14(ii), we may find $\varphi'_1 \in \mathrm{Ob}(\mathsf{Cov}(X_1))$ and an isomorphism $\alpha : \varphi'_1 \times_{X_1} U_1 \xrightarrow{\sim} \varphi_1$. Let $U_3 := U_2 \times_U U_1$, and denote by $j_3 : U_3 \to$

$X_3 := X_2 \times_X X_1$ the natural open immersion; by the foregoing, we know already that both j_2^* and j_3^* are fully faithful; then corollary 3.5.30(iii) says that the natural essentially commutative diagram :

$$\begin{array}{ccc} \mathsf{Desc}(\mathsf{Cov}, \beta) & \longrightarrow & \mathsf{Desc}(\mathsf{Cov}, \beta \times_X U) \\ \downarrow & & \downarrow \\ \mathsf{Cov}(X_1) & \longrightarrow & \mathsf{Cov}(U_1) \end{array}$$

is 2-cartesian. Thus, let $\rho : \mathsf{Cov}(U) \to \mathsf{Desc}(\mathsf{Cov}, \beta \times_X U)$ be the functor defined in (3.5.22); it follows that the datum $(\varphi'_1, \rho(\varphi), \alpha)$ comes from a descent datum (φ'_1, ω) in $\mathsf{Desc}(\mathsf{Cov}, \beta)$. By lemma 13.1.2, the latter descends to an object φ' of $\mathsf{Cov}(X)$, and by construction we have $(j, \iota_\bullet)^*\varphi' = \underline{E}$, as required. ◊

Next, we consider assertions (i) and (ii) in case where both X and S are affine. We may find an affine open covering $X = V_0 \cup \cdots \cup V_n$ such that the fibres of $f_{|V_i} : V_i \to fV_i$ are of pure dimension i, for every $i = 0, \dots, n$ ([66, Ch.IV, Prop.17.10.2]). For $i = 0, \dots, n$, let $j_i : V_i \cap U \to V_i$ be the induced open immersion; we have natural equivalences of categories :

$$\mathsf{Cov}(X) \xrightarrow{\sim} \prod_{i=0}^{n} \mathsf{Cov}(V_i) \qquad \mathsf{Cov}(U) \xrightarrow{\sim} \prod_{i=0}^{n} \mathsf{Cov}(V_i \cap U)$$

which induce a natural identification : $j^* = j_0^* \times \cdots \times j_n^*$. It follows j^* is fully faithful if and only if the same holds for every j_i^*, and moreover $\mathscr{D}(S, f, U)$ decomposes as a product of n diagrams $\mathscr{D}(S, f_{|V_i}, U \cap V_i)$. Hence we may replace f by $f_{|V_m}$, for any $m \leq n$, after which we may also assume that all the fibres f have the same pure dimension m. In that case, notice that the assumption on U is equivalent to (13.2.16). Next, say that $X = \operatorname{Spec} A$, and let $I \subset A$ be an ideal such that $V(I) = X \backslash U$; we may write I as the union of the filtered family $(I_\lambda \mid \lambda \in \Lambda)$ of its finitely generated subideals. Set $U_\lambda := X \backslash V(I_\lambda)$ for every $\lambda \in \Lambda$; it follows that $U = \bigcup_{\lambda \in \Lambda} U_\lambda$. For every $\lambda \in \Lambda$, set

$$Z_\lambda := \{s \in S \mid \dim f^{-1}(s) \backslash U_\lambda < m\}.$$

Claim 13.2.17. (i) Z_λ is a constructible subset of S, for every $\lambda \in \Lambda$.

(ii) We have $Z_\lambda \subset Z_\mu$ whenever $\mu \geq \lambda$, and moreover $S = \bigcup_{\lambda \in \Lambda} Z_\lambda$.

Proof of the claim. (i): Let $V_\lambda := \{x \in X \mid \dim_x f^{-1}(f(x)) \setminus U_\lambda = m\}$; according to [65, Ch.IV, Prop.9.9.1], every V_λ is a constructible subset of X, hence $f(V_\lambda)$ is a constructible subset of S ([63, Ch.IV, Th.1.8.4]), so the same holds for $Z_\lambda = S \backslash f(V_\lambda)$.

(ii): Let $\mu, \lambda \in \Lambda$, such that $\mu \geq \lambda$; then it is clear that $f^{-1}(s) \backslash U_\lambda \subset f^{-1}(s) \backslash U_\mu$ for every $s \in S$; using (13.2.16), the claim follows easily. ◊

Claim 13.2.17 and [63, Ch.IV, Cor.1.9.9] imply that $Z_\lambda = S$ for every sufficiently large $\lambda \in \Lambda$. Hence, after replacing Λ by a cofinal subset, we may assume that all the open subsets U_λ are fibrewise dense. We have a natural essentially commutative diagram :

$$\begin{array}{ccc} \mathsf{Cov}(U) & \longrightarrow & 2\text{-}\lim\limits_{\lambda \in \Lambda} \mathsf{Cov}(U_\lambda) \\ \downarrow & & \downarrow \\ \prod_{\eta \in S_{\max}} \mathsf{Cov}(U_\eta) & \longrightarrow & \prod_{\eta \in S_{\max}} 2\text{-}\lim\limits_{\lambda \in \Lambda} \mathsf{Cov}(f^{-1}(\eta) \cap U_\lambda) \end{array}$$

whose horizontal arrows are equivalences (notation of definition 2.5.1(i)); it follows formally that j^* is fully faithful, provided the same holds for all the pull-back functors $\mathsf{Cov}(X) \to \mathsf{Cov}(U_\lambda)$, and likewise, $\mathscr{D}(S, f, U)$ is 2-cartesian, provided the same holds for all the diagrams

$\mathscr{D}(S, f, U_\lambda)$. Hence, we may replace U by U_λ, and assume that U is constructible, and (13.2.16) still holds.

Next, we may write S as the limit of a cofiltered family $(S_\lambda \mid \lambda \in \Lambda)$ of affine schemes of finite type over $\operatorname{Spec}\mathbb{Z}$, and f as the limit of a cofiltered family $f_\bullet := (f_\lambda : X_\lambda \to S_\lambda \in \Lambda)$ of affine finitely presented morphisms, such that :

- The natural morphism $g_\lambda : S \to S_\lambda$ is dominant for every $\lambda \in \Lambda$.
- f_λ is smooth for every $\lambda \in \Lambda$ ([66, Ch.IV, Prop.17.7.8(ii)]), and $f_\mu = f_\lambda \times_{S_\lambda} S_\mu$ whenever $\mu \geq \lambda$.

Furthermore, we may find $\lambda \in \Lambda$ such that $U = U_\lambda \times_{S_\lambda} S$ ([65, Ch.IV, Cor.8.2.11]), so that j is the limit of the cofiltered system of open immersions $(j_\mu : U_\mu := U_\lambda \times_{S_\lambda} S_\mu \to X_\lambda \mid \mu \geq \lambda)$, and after replacing Λ by a cofinal subset, we may assume that j_μ is defined for every $\mu \in \Lambda$. For every $\lambda \in \Lambda$ and $n \in \mathbb{N}$, let $X_{\lambda,n} \subset X_\lambda$ be the open and closed subset consisting of all $x \in X_\lambda$ such that $\dim_x f^{-1}f(x) = n$; clearly $f_\bullet$ restricts to a cofiltered family $f_{\bullet,m} := (f_{\lambda|X_{\lambda,m}} : X_{\lambda,m} \to S_\lambda \mid \lambda \in \Lambda)$, whose limit is again f. Hence we may replace X_λ by $X_{\lambda,m}$, and assume that the fibres of f_λ have pure dimension m, for every $\lambda \in \Lambda$. For every $\lambda \in \Lambda$, let :

$$Z'_\lambda := \{s \in S_\lambda \mid \dim f_\lambda^{-1}(s)\backslash U_\lambda = m\}.$$

and endow Z'_λ with its constructible topology $\mathscr{T}_\lambda$; since Z'_λ is a constructible subset of S_λ ([65, Ch.IV, Prop.9.9.1]), $(Z'_\lambda, \mathscr{T}_\lambda)$ is a compact topological space, and due to (13.2.16), we have :

$$\lim_{\lambda\in\Lambda} Z'_\lambda = \varnothing.$$

Then [41, Ch.I, §9, n.6, Prop.8] implies that $Z'_\lambda = \varnothing$ for every sufficiently large $\lambda \in \Lambda$. Set :

$$\mathsf{Cov}(X_\bullet) := 2\text{-}\underset{\mu\geq\lambda}{\operatorname{colim}}\, \mathsf{Cov}(X_\mu) \qquad \mathsf{Cov}(U_\bullet) := 2\text{-}\underset{\mu\geq\lambda}{\operatorname{colim}}\, \mathsf{Cov}(U_\mu).$$

(See definition 2.5.1(ii).) There follows an essentially commutative diagram of categories:

$$(13.2.18)\qquad \begin{array}{ccc} \mathsf{Cov}(X) & \longrightarrow & \mathsf{Cov}(X_\bullet) \\ {\scriptstyle j^*}\downarrow & & \downarrow{\scriptstyle j^*_\bullet} \\ \mathsf{Cov}(U) & \longrightarrow & \mathsf{Cov}(U_\bullet) \end{array}$$

where $j^*_\bullet$ is the 2-colimit of the system of pull-back functors $j^*_\mu : \mathsf{Cov}(X_\mu) \to \mathsf{Cov}(U_\mu)$. In light of lemma 13.1.6, the horizontal arrows of (13.2.18) are equivalences, so j^* will be fully faithful, provided the same holds for the functors j^*_μ, for every large enough $\mu \in \Lambda$.

Hence, in order to prove assertion (i) when X and S are affine, we may assume that S is of finite type over $\operatorname{Spec}\mathbb{Z}$, the fibres of f have pure dimension m, and (13.2.16) holds, which is the case covered by claim 13.2.15.

Concerning assertion (ii), since the morphism g_μ is dominant, for every $\eta' \in (S_\mu)_{\max}$ we may find $\eta \in S_{\max}$ such that $g_\mu(\eta) = \eta'$. Denote by :

$$h : f^{-1}\eta \to f_\mu^{-1}\eta' \qquad \text{and} \qquad j'_\mu : (U_\mu)_{\eta'} := U_\mu \cap f_\mu^{-1}\eta' \to f_\mu^{-1}\eta'$$

the natural morphisms. With this notation, we have the following :

Claim 13.2.19. The induced essentially commutative diagram :

$$\begin{array}{ccc} \mathsf{Cov}(f_\mu^{-1}\eta') & \xrightarrow{j'^*_\mu} & \mathsf{Cov}((U_\mu)_{\eta'}) \\ {\scriptstyle h^*}\downarrow & & \downarrow{\scriptstyle h^*_U} \\ \mathsf{Cov}(f^{-1}\eta) & \xrightarrow{j^*_\eta} & \mathsf{Cov}(U_\eta) \end{array}$$

is 2-cartesian.

Proof of the claim. The pair (h^*, j'^*_μ) induces a functor $(h, j'_\mu)^*$ from $\mathsf{Cov}(f_\mu^{-1}\eta')$ to the category of data of the form $(\varphi, \varphi', \alpha)$, where φ' (resp φ) is a finite étale covering of $(U_\mu)_{\eta'}$ (resp. of $f^{-1}\eta$) and $\alpha : \varphi \times_{f^{-1}\eta} U_\eta \xrightarrow{\sim} \varphi' \times_{\eta'} \eta$ is an isomorphism in $\mathsf{Cov}(U_\eta)$, and the contention is that $(h, j'_\mu)^*$ is an equivalence. The full faithfulness of the functors j'^*_μ and j^*_η (lemma 13.1.7(iii)) easily implies the full faithfulness of $(h, j'_\mu)^*$. To prove that $(h, j'_\mu)^*$ is essentially surjective, amounts to showing that if $\varphi' : E' \to (U_\mu)_{\eta'}$ is a finite étale morphism and

$$\varphi'' := \varphi' \times_{\eta'} \eta : E'' := E' \times_{\eta'} \eta \to U_\eta$$

extends to a finite étale morphism $\varphi : E \to f^{-1}\eta$, then φ' extends to a finite étale covering of $f_\mu^{-1}\eta'$. Now, let L be the maximal purely inseparable extension of $\kappa(\eta')$ contained in $\kappa(\eta)$. Since the induced morphism $\eta'' := \mathrm{Spec}\, L \to \mathrm{Spec}\, \kappa(\eta')$ is radicial, the base change functors

$$\mathsf{Cov}(f_\mu^{-1}\eta') \to \mathsf{Cov}((f_\mu^{-1}\eta') \times_{\eta'} \eta'') \qquad \mathsf{Cov}((U_\mu)_{\eta'}) \to \mathsf{Cov}((U_\mu)_{\eta'} \times_{\eta'} \eta'')$$

are equivalences (lemma 13.1.7(i)). Thus, we may replace η' by η'', and assume that the field extension $\kappa(\eta') \subset \kappa(\eta)$ is separable, hence the induced morphism $\mathrm{Spec}\, \kappa(\eta) \to \mathrm{Spec}\, \kappa(\eta')$ is regular ([34, Ch.VIII, §7, no.3, Cor.1]), and then the same holds for the morphism h ([64, Ch.IV, Prop.6.8.3(iii)]). Given φ' as above, set $\mathscr{A} := (j'_\mu \circ \varphi')_* \mathscr{O}_{E'}$; then $\mathscr{A}$ is a quasi-coherent $\mathscr{O}_{f_\mu^{-1}\eta'}$-algebra, and we may define the quasi-coherent $\mathscr{O}_{f_\mu^{-1}\eta'}$-algebra $\mathscr{B}$ as the integral closure of $\mathscr{O}_{f_\mu^{-1}\eta'}$ in $\mathscr{A}$. By [64, Ch.IV, Prop.6.14.1], $h^*\mathscr{B}$ is the integral closure of $\mathscr{O}_{f^{-1}\eta}$ in $h^*\mathscr{A} = j_{\eta *} \circ h^*_U(\varphi''_* \mathscr{O}_{E''}) = j_{\eta *}(\varphi_* \mathscr{O}_E)$. By proposition 9.8.3, it then follows that $h^*\mathscr{B} = \varphi^* \mathscr{O}_E$, therefore $\mathscr{B}$ is a finite étale $\mathscr{O}_{f_\mu^{-1}\eta'}$-algebra ([66, Ch.IV, Prop.17.7.3(ii)] and [64, Ch.IV, Prop.2.7.1]). The claim follows. ◊

By the foregoing, we already know that j^* is fully faithful, hence the same holds for $(j, \iota_\bullet)^*$ (claim 13.2.13(i)). To show that $(j, \iota_\bullet)^*$ is essentially surjective, consider any $\underline{E}$ as in (13.2.12); we may find $\mu \in \Lambda$, and a finite étale morphism $\varphi' : E_\mu \to U_\mu$ such that $\varphi = \varphi' \times_{U_\mu} U$, whence objects $\varphi'_{\eta'} := \varphi' \times_{U_\mu} (U_\mu)_{\eta'}$ in $\mathsf{Cov}((U_\mu)_{\eta'})$, for every $\eta' \in (S_\mu)_{\max}$. By construction, we have $\varphi'_{\eta'} \times_{\eta'} \eta \simeq \psi_\eta \times_{f^{-1}\eta} U_\eta$ for every $\eta' \in (S_\mu)_{\max}$ and every $\eta \in S_{\max}$ such that $g_\mu(\eta) = \eta'$. Then claim 13.2.19 shows that, for every $\eta' \in (S_\mu)_{\max}$ there exists an object $\psi'_{\eta'}$ of $\mathsf{Cov}(f_\mu^{-1}\eta')$ with isomorphisms :

$$\psi'_{\eta'} \times_{f_\mu^{-1}\eta'} f^{-1}\eta \simeq \psi_\eta \qquad \alpha_{\eta'} : \psi'_{\eta'} \times_{f_\mu^{-1}\eta'} (U_\mu)_{\eta'} \xrightarrow{\sim} \varphi'_{\eta'}.$$

Therefore, the datum $\underline{E}_\mu := (\varphi', (\psi'_{\eta'}, \alpha'_{\eta'} \mid \eta' \in (S_\mu)_{\max}))$ is an object of the 2-limit of the diagram of categories

$$\mathsf{Cov}(U_\mu) \xleftarrow{j^*_\mu} \mathsf{Cov}(X_\mu) \xrightarrow{\prod_{\eta'} \iota^*_{\eta'}} \prod_{\eta' \in (S_\mu)_{\max}} \mathsf{Cov}(f_\mu^{-1}\eta')$$

(where $\iota_{\eta'} : f_\mu^{-1}\eta' \to X_\mu$ is the natural immersion, for every $\eta' \in (S_\mu)_{\max}$). By claim 13.2.15, the datum $\underline{E}_\mu$ comes from an object φ_μ of $\mathsf{Cov}(X_\mu)$. Let φ'' be the image of φ'_μ in $\mathsf{Cov}(X)$; by construction we have $(j, \iota_\bullet)^*\varphi'' = \underline{E}$, as required.

This conclude the proof of (i) and (ii), in case X and S are affine. To deal with the general case, let $X = \bigcup_{i \in I} V_i$ be a covering consisting of affine open subschemes, and for every $i \in I$, let $fV_i = \bigcup_{\lambda \in \Lambda_i} S_{i\lambda}$ be an affine open covering of the open subscheme $fV_i \subset S$; set also $V_{i\lambda} := V_i \cap f^{-1}S_{i\lambda}$ for every $i \in I$ and $\lambda \in \Lambda_i$. The restrictions $f_{|V_{i\lambda}} : V_{i\lambda} \to S_{i\lambda}$ are smooth morphisms; moreover, the image of the open immersion

$$j_{i\lambda} := j_{|U \cap V_{i\lambda}} : U \cap V_{i\lambda} \to V_{i\lambda}$$

is dense in every fibre of $f_{|V_i}$. The induced morphism

$$\beta : X' := \coprod_{i \in I} \coprod_{\lambda \in \Lambda_i} V_{i\lambda} \to X \qquad (13.2.20)$$

is faithfully flat, hence of universal 2-descent for (13.1.1); moreover, it is easily seen that $X'' := X' \times_X X'$ is separated, and $j'' := j \times_X X''$ is a dense open immersion, hence j''^* is faithful (lemma 13.1.7(ii)). Then, by corollary 3.5.30(ii), j^* is fully faithful, provided the pull-back functor $\mathsf{Cov}(X') \to \mathsf{Cov}(X' \times_X U)$ is fully faithful, *i.e.* provided the same holds for the functors $j^*_{i\lambda} : \mathsf{Cov}(V_{i\lambda}) \to \mathsf{Cov}(V_{i\lambda} \cap U)$. However, each $V_{i\lambda}$ is affine ([59, Ch.I, Prop.5.5.10]), hence assertion (i) is already known for the morphisms $f_{|V_{i\lambda}}$ and the open subsets $U \cap V_{i\lambda}$; this concludes the proof of (i).

To show (ii), we use the criterion of claim 13.2.13(ii) : indeed, assumption (a) is already known, hence we are reduced to showing that assertion (iii) holds. To this aim, we consider again the morphism β of (13.2.20), and denote by $f'' : X'' \to S$ the induced morphism. Clearly f'' is smooth, and j'' is an open immersion, such that $f''^{-1}(s) \times_X U$ is dense in $f''^{-1}(s)$, for every $s \in S$; then assertion (i) implies that j''^* is fully faithful. Moreover it is easily seen that $X''' := X'' \times_X X'$ is separated, and $j \times_X X'''$ is a dense open immersion, so j'''^* is faithful (lemma 13.1.7(ii)), and therefore corollary 3.5.30(ii) reduces to showing that the pull-back functor $\mathsf{Cov}(X') \to \mathsf{Cov}(X' \times_X U)$ is an equivalence, or – what is the same – that this holds for the pull-back functors $j^*_{i\lambda}$, which is already known. □

13.2.21. We consider now the local counterpart of theorem 13.2.11. Namely, suppose that $f : X \to S$ is a smooth morphism, let $\overline{x}$ be any geometric point of X, set $\overline{s} := f(\overline{x})$, and let $s \in S$ be the support of $\overline{s}$. Then f induces the morphism $f_{\overline{x}} : X(\overline{x}) \to S(\overline{s})$, and the latter has geometrically irreducible fibres by proposition 13.1.30(iii). For every open immersion $j : U \to X(\overline{x})$, we may consider the diagram $\mathscr{D}(S(\overline{x}), f_{\overline{x}}, U)$ as in (13.2.10).

Theorem 13.2.22. *In the situation of* (13.2.21)*, suppose that U contains the generic point of $f_{\overline{x}}^{-1}(s)$. Then :*

(i) $j^* : \mathsf{Cov}(X(\overline{x})) \to \mathsf{Cov}(U)$ *is fully faithful.*
(ii) *The diagram $\mathscr{D}(S(\overline{x}), f_{\overline{x}}, U)$ is 2-cartesian.*

Proof. (i): To begin with, since $f_{\overline{x}}$ is generizing ([126, Th.9.5]), and S is local, every fibre of $f_{\overline{x}}$ has a point that specializes to the generic point η_s of $f_{\overline{x}}^{-1}(s)$; since the fibres are irreducible, it follows that the generic point of every fibre specializes to η_s. Therefore U is fibrewise dense in $X(\overline{x})$, and moreover it is connected. Now, the category $\mathsf{Cov}(X)$ is equivalent to the category of finite sets, hence every object in the essential image of j^* is (isomorphic to) a finite disjoint union of copies of U; since U is connected, the morphisms of U-schemes between two such objects E and E' are in natural bijection with the set-theoretic mappings $\pi_0(E) \to \pi_0(E')$ of their sets of connected components, whence the assertion.

(ii): Let us write U as the union of a filtered family $(U_\lambda \mid \lambda \in \Lambda)$ of constructible open subsets of $X(\overline{x})$; up to replacing Λ by a cofinal subset, we may assume that $\eta_s \in U_\lambda$ for every $\lambda \in \Lambda$. Arguing as in the proof of theorem 13.2.11, we see that $\mathscr{D}(S(\overline{x}), f_{\overline{x}}, U)$ is the 2-limit of the system of diagrams $\mathscr{D}(S(\overline{x}), f_{\overline{x}}, U_\lambda)$, hence it suffices to show the assertion with $U = U_\lambda$, for every $\lambda \in \Lambda$, which allows to assume that U is quasi-compact. Next, arguing as in the proof of proposition 13.1.30, we are reduced to the case where $S = S(\overline{s})$. We may write $X(\overline{x})$ as the limit of a cofiltered system $(X_\lambda \mid \lambda \in \Lambda)$ of affine schemes, étale over X, and for $\lambda \in \Lambda$ large enough, we may find an open subset $U_\lambda \subset X_\lambda$ such that $U = U_\lambda \times_{X_\lambda} X(\overline{x})$ ([65, Ch.IV, Cor.8.2.11]). For every $\mu \geq \lambda$, set $U_\mu := U_\lambda \times_{X_\lambda} X_\mu$, and denote by $f_\mu : X_\mu \to S$ the natural morphism. Suppose first that S is irreducible; then, from lemma 13.1.6 it is easily seen that $\mathscr{D}(S, f_{\overline{x}}, U)$ is the 2-colimit of the system of diagrams $\mathscr{D}(S, f_\mu, U_\mu)$, so the assertion follows from theorem 13.2.11(ii) (more generally, this argument works whenever $S_{\max}$ is a finite set, since filtered 2-colimits of categories commute with finite products).

In the general case, let $\varphi : E \to U$ be a finite étale morphism, and suppose that $\varphi_\eta := \varphi \times_{X(\overline{x})} f_{\overline{x}}^{-1}(\eta)$ extends to an object ψ_η of $\mathsf{Cov}(f_{\overline{x}}^{-1}\eta)$, for every $\eta \in S_{\max}$. The assertion boils down to showing that φ extends to an object φ' of $\mathsf{Cov}(X(\overline{x}))$. To this aim, for every $\eta \in S_{\max}$, let $Z_\eta \to S$ be the closed immersion of the topological closure of η in S (which we endow with its reduced scheme structure); set also $Y_\eta := X \times_S Z_\eta$. Then Z_η is a strictly local scheme ([66, Ch.IV, Prop.18.5.6(i)]), and $\overline{x}$ factors through the closed immersion $Y \to X$, which induces an isomorphism of Z-schemes :

$$Y_\eta(\overline{x}) \xrightarrow{\sim} X(\overline{x}) \times_S Z_\eta$$

(lemma 13.1.22(ii)). By the foregoing case, $\varphi \times_S Z_\eta$ extends to an object $\overline{\psi}_\eta$ of $\mathsf{Cov}(Y_\eta(\overline{x}))$. However, Z_η is the limit of the cofiltered system $(Z_{\eta,i} \mid i \in I(\eta))$ consisting of the constructible closed subschemes of S that contain Z_η. By lemma 13.1.6, it follows that we may find $i \in I(\eta)$ and an object $\overline{\psi}_{\eta,i}$ of $\mathsf{Cov}(X(\overline{x}) \times_S Z_{\eta,i})$ whose image in $\mathsf{Cov}(Y_\eta(\overline{x}))$ is isomorphic to $\overline{\psi}_\eta$, and if i is large enough, $\overline{\psi}_{\eta,i} \times_{X(\overline{x})} U$ agrees with $\varphi \times_S Z_{\eta,i}$ in $\mathsf{Cov}(U \times_S Z_{\eta,i})$. For each $\eta, \eta' \in S_{\max}$, choose $i \in I(\eta)$, $i' \in I(\eta')$ with these properties, and to ease notation, set :

$$X'_\eta := X \times_S Z_{\eta,i} \qquad X''_{\eta\eta'} := X'_\eta \times_S Z_{\eta',i'} \qquad \varphi'_\eta := \overline{\psi}_{\eta,i}$$

and denote by $\alpha_\eta : \varphi'_\eta \times_{X(\overline{x})} U \xrightarrow{\sim} \varphi \times_S Z_{\eta,i}$ the given isomorphism. As in the foregoing, we notice that $\overline{x}$ factors through X'_η, and the closed immersion $X'_\eta \to X$ induces an isomorphism $X'_\eta(\overline{x}) \xrightarrow{\sim} X(\overline{x}) \times_S Z_{\eta,i}$ of $Z_{\eta,i}$-schemes. According to [63, Ch.IV, Cor.1.9.9], we may then find a finite subset $T \subset S_{\max}$ such that the induced morphism :

$$\beta : X_1 := \coprod_{\eta \in T} X'_\eta(\overline{x}) \to X(\overline{x})$$

is surjective. Set $X_2 := X_1 \times_{X(\overline{x})} X_1$ and $X_3 := X_2 \times_{X(\overline{x})} X_1$; notice that X_2 is the disjoint union of schemes of the form $X(\overline{x}) \times_S Z_{\eta,i} \times_S Z_{\eta',i'}$, for $\eta, \eta' \in T$, and again, the latter is naturally isomorphic to $X''_{\eta\eta'}(\overline{x})$, for a unique lifting of the geometric point $\overline{x}$ to a geometric point of $X''_{\eta\eta'}$. Similar considerations can be repeated for X_3, and in light of (i), we deduce that the pull-back functors :

$$\mathsf{Cov}(X_i \times_{X(\overline{x})} U) \to \mathsf{Cov}(X_i) \qquad i = 1, 2, 3$$

are fully faithful, in which case corollary 3.5.30(iii) says that the essentially commutative diagram of categories :

$$\begin{array}{ccc} \mathsf{Desc}(\mathsf{Cov}, \beta) & \longrightarrow & \mathsf{Desc}(\mathsf{Cov}, \beta \times_{X(\overline{x})} U) \\ \downarrow & & \downarrow \\ \prod_{\eta \in T} \mathsf{Cov}(X'_\eta(\overline{x})) & \longrightarrow & \prod_{\eta \in T} \mathsf{Cov}(X'_\eta(\overline{x}) \times_{X(\overline{x})} U) \end{array}$$

is 2-cartesian. Let $\rho : \mathsf{Cov}(U) \to \mathsf{Desc}(\mathsf{Cov}, \beta \times_{X(\overline{x})} U)$ be the functor defined in (3.5.22); it follows that the datum $((\varphi'_\eta, \alpha_\eta \mid \eta \in T), \rho(\varphi))$ comes from a descent datum (φ'_1, ω) in $\mathsf{Desc}(\mathsf{Cov}, \beta)$. By lemma 13.1.2, the latter descends to an object φ' of $\mathsf{Cov}(X(\overline{x}))$, and by construction we have $j^*\varphi' = \varphi$, as required. □

13.3. **Étale coverings of log schemes.** We resume the general notation of (12.2.1), especially, we choose implicitly τ to be either the Zariski or étale topology, and we shall omit further mention of this choice, unless the omission might be a source of ambiguities.

13.3.1. Let $\underline{Y} := ((Y, \underline{N}), T, \psi)$ be an object of $\mathscr{K}_{\mathrm{int}}$ (see (12.6.10) : especially $\tau = \mathrm{Zar}$ here), $\varphi : T' \to T$ an integral proper subdivision of the fan T (definition 6.5.22(ii),(iii)), and suppose that both T and T' are locally fine and saturated. Set $((Y', \underline{N}'), T', \psi') := \varphi^*\underline{Y}$ (proposition 12.6.14(iii)), and let $f : Y' \to Y$ be the morphism of schemes underlying the cartesian morphism $\varphi^*\underline{Y} \to \underline{Y}$.

Proposition 13.3.2. *In the situation of* (13.3.1)*, the following holds :*

(i) *For every geometric point ξ of Y, the fibre $f^{-1}(\xi)$ is connected (and hence, non-empty).*
(ii) *The functor $f^* : \mathsf{Cov}(Y) \to \mathsf{Cov}(Y')$ is an equivalence.*

Proof. (i): The assertion is local on Y, hence we may assume that $T = (\mathrm{Spec}\, P)^\sharp$ for a fine, sharp and saturated monoid P. In this case, we may find a subdivision $\varphi' : T'' \to T'$ such that both φ' and $\varphi \circ \varphi'$ are compositions of saturated blow up of ideals generated by at most two elements of P (example 6.6.15(iii)). We are reduced to showing the assertion for the morphisms φ' and $\varphi \circ \varphi'$, after which, we may further assume that φ is the saturated blow up of an ideal generated by two elements of $\Gamma(T, \mathscr{O}_T)$, in which case the assertion follows from the more precise theorem 12.4.51.

(ii): First, we claim that the assertion is local on the Zariski topology of Y. Indeed, let $Y = \bigcup_{i \in I} U_i$ be a Zariski open covering, and set $U_{ij} := U_i \cap U_j$ for every $i, j \in I$; according to corollary 3.5.30(ii) and lemma 13.1.2, it suffices to prove the contention for the objects $(U_i \times_Y (Y, \underline{N}), T, \psi_{|U_i})$ and $(U_{ij} \times_Y (Y, \underline{N}), T, \psi_{|U_{ij}})$, and the morphism φ. Hence, we may again suppose that $T = \mathrm{Spec}\, P$, for a monoid P as in (i).

Arguing as in the proof of (i), we may next reduce to the case where φ is the saturated blow up of an ideal generated by two elements of $\Gamma(T, \mathscr{O}_T)$. Now, the morphism of schemes f induces a morphism of topoi $Y'^{\sim}_{\text{ét}} \to Y^{\sim}_{\text{ét}}$ which we denote again by f; then, assertion (i) and the proper base change theorem [10, Exp.XII, Th.5.1(i)] imply that the unit of adjunction $\mathscr{F} \to f_* f^* \mathscr{F}$ is an isomorphism for every sheaf $\mathscr{F}$ on Y, so f^* is fully faithful (proposition 1.1.20(iii)). Next, if $\varphi : E' \to Y'$ is a finite étale covering, Y' decomposes into a disjoint union of open and closed subsets $Y' = Y'_0 \cup \cdots \cup Y'_k$, such that $(\varphi_* \mathscr{O}_{E'})_{|Y'_r}$ is locally free of rank r, for every $r = 0, \dots, k$. Set $Y_r := f(Y'_r)$ for $r = 0, \dots, k$; since the fibres of f are connected, we see that $Y'_r = f^{-1} Y_r$ for every such r. Moreover, since f is a closed map and the topology of Y is induced from that of Y' via f, we see that Y_r is a closed subset of Y for $r = 0, \dots, k$, so $Y = Y_0 \cup \cdots \cup Y_k$ is a partition of Y by open and closed subsets. After replacing Y by Y_r, we may then assume that $\varphi_* \mathscr{O}_E$ is locally free of rank r, and notice that the isomorphism classes of étale coverings of this type are classified by the pointed set $H^1(Y'_{\text{ét}}, S_{r,Y'})$, where S_r is the symmetric group on r elements. Thus, it suffices to show that the induced map :

$$H^1(Y_{\text{ét}}, S_{r,Y}) \to H^1(Y'_{\text{ét}}, S_{r,Y'}) \tag{13.3.3}$$

is a bijection. However, theorem 4.9.9 yields the exact sequence of pointed sets :

$$\{1\} \to H^1(Y_{\text{ét}}, f_* S_{r,Y'}) \xrightarrow{u} H^1(Y'_{\text{ét}}, S_{r,Y'}) \to H^0(Y_{\text{ét}}, R^1 f_* S_{r,Y'}).$$

On the other hand, assertion (i) and the proper base change theorem [10, Exp.XII, Th.5.1(i)] imply that the unit of adjunction $S_{r,Y} \to f_* f^* S_{r,Y} = f_* S_{r,Y'}$ is an isomorphism, hence u is naturally identified to (13.3.3), and we are reduced to showing that the natural morphism

$$\tau_{f, S_{r,Y'}} : 1_{Y^{\sim}_{\text{ét}}} \to R^1 f_* S_{r,Y'}$$

is an isomorphism (notation of (4.9.3)). The latter can be checked on the stalks, and in view of [10, Exp.XII, Cor.5.2(ii)] we are reduced to showing that $H^1(f^{-1}(\xi)_{\text{ét}}, S_r) = \{1\}$ for every geometric point ξ of Y. However, according to theorem 12.4.51, the reduced geometric fibre $f^{-1}(\xi)_{\text{red}}$ is either isomorphic to $|\xi|$ (in which case the contention is trivial), or else it is isomorphic to the projective line $\mathbb{P}^1_{\kappa(\xi)}$, in which case – in view of lemma 13.1.7(i) – it suffices to show that every finite étale morphism $E \to \mathbb{P}^1_{\kappa(\xi)}$ admits a section, which is well known. $\square$

The class of étale morphisms of log schemes was introduced in section 12.3 : its definition and its main properties parallel those of the corresponding notion for schemes, found in [63, Ch.IV, §17]. In the present section this theme is further advanced : we will consider the logarithmic analogue of the classical notion of *étale covering* of a scheme. To begin with, we make the following :

Definition 13.3.4. (i) Let $\varphi : T \to S$ be a morphism of fans. We say that φ is *of Kummer type*, if the map $(\log \varphi)_t : \mathscr{O}_{S,\varphi(t)} \to \mathscr{O}_{T,t}$ is of Kummer type, for every $t \in T$ (see definition 6.4.58).

(ii) Let $f : (Y, \underline{N}) \to (X, \underline{M})$ be a morphism of log schemes.

(a) We say that f is *of Kummer type*, if for every τ-point ξ of Y, the morphism of monoids $(\log f)_\xi : f^*\underline{M}_\xi \to \underline{N}_\xi$ is of Kummer type.

(b) A *Kummer chart* for f is the datum of charts

$$\omega_P : P_Y \to \underline{N} \qquad \omega_Q : Q_X \to \underline{M}$$

and a morphism of monoids $\vartheta : Q \to P$ such that $(\omega_P, \omega_Q, \vartheta)$ is a chart for f (see definition 12.1.17(iii)), and ϑ is of Kummer type.

Remark 13.3.5. (i) Let $f : P \to Q$ be a morphism of monoids of Kummer type, with P integral and saturated. It follows easily from lemma 6.4.59(iii,v), that the induced morphism of fans $(\mathrm{Spec}\, f)^\sharp : (\mathrm{Spec}\, Q)^\sharp \to (\mathrm{Spec}\, P)^\sharp$ is of Kummer type.

(ii) In the situation of definition 13.3.4(ii), it is easily seen that f is of Kummer type, if and only if the morphism of Y-monoids $f^*\underline{M}^\sharp_\xi \to \underline{N}^\sharp_\xi$ deduced from $\log f$, is of Kummer type for every τ-point ξ of Y.

(iii) In the situation of definition 13.3.4(ii.b), suppose that the chart $(\omega_P, \omega_Q, \vartheta)$ is of Kummer type, and Q is integral and saturated. It then follows easily from (i), (ii) and example 12.6.5(ii), that f is of Kummer type. In the same vein, we have the following :

Lemma 13.3.6. *Let $f : (Y, \underline{N}) \to (X, \underline{M})$ be a morphism of log schemes with coherent log structures, ξ a τ-point of Y, and suppose that :*

(a) *$\underline{M}_{f(\xi)}$ is fine and saturated.*

(b) *The morphism $(\log f)_\xi : \underline{M}_{f(\xi)} \to \underline{N}_\xi$ is of Kummer type.*

Then there exists a (Zariski) open neighborhood U of $|\xi|$ in Y, such that the restriction $f_{|U} : (U, \underline{N}_{|U}) \to (X, \underline{M})$ of f is of Kummer type.

Proof. By corollary 12.1.36(i) and theorem 12.1.37(ii), we may find a neighborhood $U' \to Y$ of ξ in Y_τ, and a finite chart $(\omega_P, \omega_Q, \vartheta)$ for the restriction $f_{|U'}$, with Q fine and saturated. Set

$$S_P := \omega^{-1}_{P,\xi}\underline{N}^\times_\xi \quad S_Q := \omega^{-1}_{Q,f(\xi)}\underline{M}^\times_{f(\xi)} \quad P' := S_P^{-1}P \quad Q' := S_Q^{-1}Q.$$

According to claim 12.1.31(iii) we may find neighborhoods $U'' \to U'$ of ξ in Y_τ, and $V \to X$ of $f(\xi)$ in X_τ, such that the charts $\omega_{P|U''}$ and $\omega_{Q|V}$ extend to charts

$$\omega_{P'} : P'_{U''} \to \underline{N}_{|U''} \qquad \omega_{Q'} : Q'_V \to \underline{M}_{|V}.$$

Clearly ϑ extends as well to a unique morphism $\vartheta' : Q' \to P'$, and after shrinking U'' we may assume that the restriction $f_{|U''} : (U'', \underline{N}_{|U''}) \to (X, \underline{M})$ factors through a morphism $f' : (U'', \underline{N}_{|U''}) \to (V, \underline{M}_{|V})$, in which case it is easily seen that the datum $(\omega_{P'}, \omega_{Q'}, \vartheta')$ is a chart for f'. Notice as well that Q' is still fine and saturated (lemma 6.2.9(i)), and by claim 12.1.31(iv) the maps $\omega_{P'}$ and $\omega_{Q'}$ induce isomorphisms :

$$P'^\sharp \xrightarrow{\sim} \underline{N}^\sharp_\xi \qquad Q'^\sharp \xrightarrow{\sim} \underline{M}^\sharp_{f(\xi)}.$$

Our assumption (b) then implies that the map $Q'^\sharp \to P'^\sharp$ deduced from ϑ' is of Kummer type, and then the morphism of fans $\mathrm{Spec}\, \vartheta' : \mathrm{Spec}\, P' \to \mathrm{Spec}\, Q'$ is of Kummer type as well (remark 13.3.5(i)). However, let

$$\overline{\omega}_{P'} : (U'', \underline{N}^\sharp_{|U''}) \to (\mathrm{Spec}\, P')^\sharp \qquad \overline{\omega}_{Q'} : (V, \underline{M}^\sharp_{|V}) \to (\mathrm{Spec}\, Q')^\sharp$$

be the morphisms of monoidal spaces deduced from $\omega_{P'}$ and $\omega_{Q'}$; we obtain a morphism

$$(f', \mathrm{Spec}\, \vartheta') : (U'', \underline{N}_{|U''}, (\mathrm{Spec}\, P')^\sharp, \overline{\omega}_{P'}) \to (V, \underline{M}_{|V}, (\mathrm{Spec}\, Q')^\sharp, \overline{\omega}_{Q'})$$

in the category $\mathscr{K}$ of (12.6.2), which – in view of remark 13.3.5(ii) – shows that f' is of Kummer type. This already concludes the proof in case $\tau = \mathrm{Zar}$, and for $\tau = \text{ét}$ it suffices to remark that the image of U'' in Y is a Zariski open neighborhood U of ξ, such that the restriction $f_{|U}$ is of Kummer type. □

We shall use the following criterion :

Proposition 13.3.7. *Let k be a field, $f : P \to Q$ an injective morphism of fine monoids, with P sharp. Suppose that the scheme $\operatorname{Spec} k\langle Q/\mathfrak{m}_P Q\rangle$ admits an irreducible component of Krull dimension 0. We have :*

(i) *f is of Kummer type, and $k\langle Q/\mathfrak{m}_P Q\rangle$ is a finite k-algebra.*
(ii) *If moreover, $Q^\times$ is a torsion-free abelian group, then Q is sharp, and $k\langle Q/\mathfrak{m}_P Q\rangle$ is a local k-algebra with k as residue field.*

Proof. Notice that the assumption means especially that $Q/\mathfrak{m}_P Q \neq \{1\}$, so f is a local morphism. Set $I := \mathrm{rad}(\mathfrak{m}_P Q)$ (notation of definition 6.1.8(ii)), and let $\mathfrak{p}_1, \dots, \mathfrak{p}_n \subset Q$ be the minimal prime ideals containing I; then $I = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_n$, by lemma 6.1.16. Clearly the natural closed immersion $\operatorname{Spec} k\langle Q/I\rangle \to \operatorname{Spec} k\langle Q/\mathfrak{m}_P Q\rangle$ is a homeomorphism; on the other hand, say that $\mathfrak{q} \subset k\langle Q\rangle$ is a prime ideal containing I; then $\mathfrak{q} \cap Q$ is a prime ideal of Q containing I, hence $\mathfrak{p}_i \subset \mathfrak{q}$ for some $i \leq n$, *i.e.* $\operatorname{Spec} k\langle Q/I\rangle$ is the union of its closed subsets $\operatorname{Spec} k\langle Q/\mathfrak{p}_i Q\rangle$, for $i = 1, \dots, n$. The Krull dimension of each irreducible component of $\operatorname{Spec} k\langle Q/\mathfrak{p}_i\rangle = \operatorname{Spec} k[Q\backslash\mathfrak{p}_i]$ equals

$$\mathrm{rk}_{\mathbb{Z}}(Q\backslash\mathfrak{p})^\times + \dim Q\backslash\mathfrak{p} = \mathrm{rk}_{\mathbb{Z}} Q^\times + d - \mathrm{ht}(\mathfrak{p}_i) \qquad \text{where } d := \dim Q$$

(claim 11.6.37(ii) and corollary 6.4.12(i,ii)). Our assumption then implies that $Q^\times$ is a finite group, and $\mathrm{ht}(\mathfrak{p}_i) = d$, *i.e.* $\mathfrak{p}_i = \mathfrak{m}_Q$, for at least an index $i \leq n$, and therefore $I = \mathfrak{m}_Q$. Furthermore, since $\mathfrak{m}_Q$ is finitely generated, we have $\mathfrak{m}_Q^n \subset \mathfrak{m}_P Q$ for a sufficiently large integer $n > 0$, and then it follows easily that $k\langle Q/\mathfrak{m}_P Q\rangle$ is a finite k-algebra. If $Q^\times$ is torsion-free, then we also deduce that Q is sharp, and moreover the maximal ideal of $k\langle Q/\mathfrak{m}_P Q\rangle$ generated by $\mathfrak{m}_Q$ is nilpotent, hence the latter k-algebra is local.

Let now $\mathfrak{q} \subset Q$ be any prime ideal, and pick $x \in \mathfrak{m}_Q \backslash \mathfrak{q}$; the foregoing implies that there exists an integer $r > 0$ such that $x^r = f(p_\mathfrak{q})q$ for some $p_\mathfrak{q} \in \mathfrak{m}_P$ and $q \in Q$, hence $f(p_\mathfrak{q}) \notin \mathfrak{q}$, and $f(p_\mathfrak{q})$ is not invertible in Q, since f is local. If now $\mathfrak{q}$ has height $d-1$, it follows that $f(p_\mathfrak{q})$ is a generator of $(Q\backslash\mathfrak{q})_\mathbb{R}$, which is an extremal ray of the polyhedral cone $Q_\mathbb{R}$, and every such extremal ray is of this form (proposition 6.4.9); furthermore, the latter cone is strictly convex, since $Q^\times$ is finite. Hence the set $S := (f(p_\mathfrak{q}) \mid \mathrm{ht}(\mathfrak{q}) = d-1)$ is a system of generators of the polyhedral cone $Q_\mathbb{R}$ (see (6.3.15)). Let $Q' \subset Q$ be the submonoid generated by S; then $S_\mathbb{Q} = S_\mathbb{R} \cap Q_\mathbb{Q} = Q_\mathbb{Q}$ (proposition 6.3.22(iii)), *i.e.* f is of Kummer type. □

13.3.8. Let $f : (Y_{\mathrm{Zar}}, \underline{N}) \to (X_{\mathrm{Zar}}, \underline{M})$ be a morphism of log schemes with Zariski log structures; it follows easily from the isomorphism (12.1.9) and remark 13.3.5(ii) that f is of Kummer type if and only if $\widetilde{u}^* f : (X_{\text{ét}}, \widetilde{u}^*\underline{M}) \to (Y_{\text{ét}}, \widetilde{u}^*\underline{N})$ is a morphism of Kummer type between schemes with étale log structures (notation of (12.2.2)). Suppose now that $\underline{M}$ is an integral and saturated log structure on X_{Zar}, and denote by $\mathbf{s.Kum}(X_{\mathrm{Zar}}, \underline{M})$ (resp. $\mathbf{s.Kum}(X_{\text{ét}}, \widetilde{u}^*\underline{M})$) the full subcategory of $\mathbf{sat.log}/(X_{\mathrm{Zar}}, \underline{M})$ (resp. of $\mathbf{sat.log}/(X_{\text{ét}}, \widetilde{u}^*\underline{M})$ whose objects are all the morphisms of Kummer type. In view of the foregoing (and of lemma 12.1.18(i)), we see that $\widetilde{u}^*$ restricts to a functor :

$$\mathbf{s.Kum}(X_{\mathrm{Zar}}, \underline{M}) \to \mathbf{s.Kum}(X_{\text{ét}}, \widetilde{u}^*\underline{M}). \tag{13.3.9}$$

Lemma 13.3.10. *The functor* (13.3.9) *is an equivalence.*

Proof. By virtue of proposition 12.2.3(ii) we know already that (13.3.9) is fully faithful, hence we only need to show its essential surjectivity. Thus, let $f : (Y_{\text{ét}}, \underline{N}) \to (X_{\text{ét}}, \widetilde{u}^*\underline{M})$ be a morphism of Kummer type. By remark 13.3.5(ii), we know that $\log f$ induces an isomorphism

$$\widetilde{u}^* f^* \underline{M}^\sharp_{\mathbb{Q}} \xrightarrow{\sim} f^* \widetilde{u}^* \underline{M}^\sharp_{\mathbb{Q}} \xrightarrow{\sim} \underline{N}^\sharp_{\mathbb{Q}}$$

(notation of (6.3.20)). Since $\underline{N}$ is integral and saturated, the natural map $\underline{N}^\sharp \to \underline{N}^\sharp_{\mathbb{Q}}$ is a monomorphism, so that the counit of adjunction $\widetilde{u}^*\widetilde{u}_*\underline{N}^\sharp \to \underline{N}^\sharp$ is an isomorphism (lemma 4.9.27(ii)), and then the same holds for the counit of adjunction $\widetilde{u}^*\widetilde{u}_*(Y, \underline{N}) \to (Y, \underline{N})$ (proposition 12.2.3(iii)). □

Proposition 13.3.11. *Let $f : (Y, \underline{N}) \to (X, \underline{M})$ be a morphism of fs log schemes. The following conditions are equivalent:*

(a) *Every geometric point of X admits an étale neighborhood $U \to X$ such that $Y_U := U \times_X Y$ decomposes as a disjoint union $Y_U = \bigcup_{i=1}^n Y_i$ of open and closed subschemes (for some $n \in \mathbb{N}$), and we have :*
   (i) *Each restriction $Y_i \times_Y (Y, \underline{N}) \to U \times_X (X, \underline{M})$ of $f \times_X \mathbf{1}_U$ admits a fine, saturated Kummer chart $(\omega_{P_i}, \omega_{Q_i}, \vartheta_i)$ such that P_i and Q_i are sharp, and the order of $\mathrm{Coker}\,\vartheta_i^{\mathrm{gp}}$ is invertible in $\mathscr{O}_U$.*
   (ii) *The induced morphism of U-schemes $Y_i \to U \times_{\mathrm{Spec}\,\mathbb{Z}[P_i]} \mathrm{Spec}\,\mathbb{Z}[Q_i]$ is an isomorphism, for every $i = 1, \dots, n$.*

(b) *f is étale, and the morphism of schemes underlying f is finite.*

Proof. (a) ⇒ (b): Indeed, it is easily seen that the morphism $\mathrm{Spec}\,\mathbb{Z}[\vartheta_i]$ is finite, hence the same holds for the restriction $Y_i \to U$ of f, in view of (a.ii), and then the same holds for $f \times_X \mathbf{1}_U$, so finally f is finite on the underlying schemes ([64, Ch.IV, Prop.2.7.1]), and it is étale by the criterion of theorem 12.3.37.

(b) ⇒ (a): Arguing as in the proof of theorem 12.3.37, we may reduce to the case where $\tau = \text{ét}$. Suppose first that both X and Y are strictly local. In this case, $\underline{M}$ admits a fine and saturated chart $\omega_P : P_X \to \underline{M}$, sharp at the closed point (corollary 12.1.36(i)). Moreover, f admits a chart $(\omega_P, \omega_Q : Q_Y \to \underline{N}, \vartheta : P \to Q)$, for some fine monoid Q such that $Q^\times$ is torsion-free; also ϑ is injective, the induced morphism of X-schemes

$$g : Y \to X' := X \times_{\mathrm{Spec}\,\mathbb{Z}[P]} \mathrm{Spec}\,\mathbb{Z}[Q]$$

is étale, and the order of $\mathrm{Coker}\,\vartheta^{\mathrm{gp}}$ is invertible in $\mathscr{O}_X$ (corollary 12.3.42). Since f is finite, g is also closed ([60, Ch.II, Prop.6.1.10]), hence its image Z is an open and closed local subscheme, finite over X. Let k be the residue field of the closed point of X; it follows that $X' \times_X \mathrm{Spec}\,k \simeq \mathrm{Spec}\,k\langle Q/\mathfrak{m}_P Q\rangle$ admits an irreducible component of Krull dimension zero (namely, the intersection of Z with the fibre of X' over the closed point of X), in which case the criterion of proposition 13.3.7(ii) ensures that ϑ is of Kummer type and Q is sharp, hence X' is finite over X, and moreover $X' \times_X \mathrm{Spec}\,k$ is a local scheme with k as residue field. Since X is strictly henselian, it follows that X' itself is strictly local, and therefore g is an isomorphism, so the proposition is proved in this case.

Let now X be a general scheme, and ξ a geometric point of X; denote by $X(\xi)$ the strict henselization of X at the point ξ, and set $Y(\xi) := X(\xi) \times_X Y$. Since f is finite, $Y(\xi)$ decomposes as the disjoint union of finitely many open and closed strictly local subschemes $Y_1(\xi), \dots, Y_n(\xi)$. Then we may find an étale neighborhood $U \to X$ of ξ, and open and closed subschemes $Y_1, \dots, Y_n$ of $Y \times_X U$, with isomorphisms of $X(\xi)$-schemes $Y_i(\xi) \xrightarrow{\sim} Y_i \times_U X(\xi)$, for every $i = 1, \dots, n$ ([65, Ch.IV, Cor.8.3.12]). We may then replace X by U, and we reduce to proving the proposition for each of the restrictions $Y_i \to U$ of $f \times_X \mathbf{1}_U$; hence we may assume that $Y(\xi)$ is strictly local. By the foregoing case, we may find a chart

(13.3.12) $$P_{X(\xi)} \to \underline{M}(\xi) \qquad Q_{Y(\xi)} \to \underline{N}(\xi) \qquad \vartheta : P \to Q$$

of $f \times_X \mathbf{1}_{X(\xi)}$, with ϑ of Kummer type, such that P and Q are sharp, the order d of $\mathrm{Coker}\,\vartheta^{\mathrm{gp}}$ is invertible in $\mathscr{O}_{X(\xi)}$, and the induced morphism of $X(\xi)$-schemes $Y(\xi) \to X(\xi) \times_{\mathrm{Spec}\,\mathbb{Z}[P]} \mathrm{Spec}\,\mathbb{Z}[Q]$ is an isomorphism. By corollary 12.2.36 we may find an étale neighborhood $U \to X$ of ξ such that (13.3.12) extends to a chart for $f \times_X \mathbf{1}_U$. After shrinking U, we may assume that d is invertible in $\mathscr{O}_U$. Lastly, after further shrinking of U, we may ensure that the induced morphism of U-schemes $U \times_X Y \to U \times_{\mathrm{Spec}\,\mathbb{Z}[P]} \mathrm{Spec}\,\mathbb{Z}[Q]$ is an isomorphism ([65, Ch.IV, Cor.8.8.2.4]). $\square$

Definition 13.3.13. (i) Let $f : (Y, \underline{N}) \to (X, \underline{M})$ be a morphism of fs log schemes. We say that f is an *étale covering* of $(X, \underline{M})$, if f fulfills the equivalent conditions (a) and (b) of proposition 13.3.11. We denote by

$$\mathsf{Cov}(X, \underline{M})$$

the full subcategory of $\mathbf{log}/(X, \underline{M})$, whose objects are the étale coverings of $(X, \underline{M})$.

(ii) Let K be a field, E a finite separable field extension of K, and w a discrete valuation of E; set $v := w_{|K}$, and denote by Γ_v and $\kappa(v)$ (Γ_w and $\kappa(w)$) the value group of v (resp. of w) and the residue field of the valuation ring K^+ of v (resp. of the valuation ring E^+ of w). We say that the ring extension $K^+ \subset E^+$ *is tamely ramified* if the *ramification index*

$$e(w/v) := (\Gamma_w : \Gamma_v)$$

(which is known to be finite) is invertible in $\kappa(v)$, and the residue field extension $\kappa(v) \to \kappa(w)$ is separable.

Remark 13.3.14. (i) Notice that all morphisms in $\mathsf{Cov}(X, \underline{M})$ are étale coverings, in light of corollary 12.3.25(ii).

(ii) Moreover, $\mathsf{Cov}(X, \underline{M})$ is a Galois category (see [82, Exp.V, Déf.5.1]), and if ξ is any geometric point of $(X, \underline{M})_{\mathrm{tr}}$, we obtain a fibre functor for this category, by the rule : $f \mapsto f^{-1}(\xi)$, for every étale covering f of $(X, \underline{M})$. (Details left to the reader.) We shall denote by

$$\pi_1((X, \underline{M})_{\text{ét}}, \xi)$$

the corresponding fundamental group.

Example 13.3.15. Let $(f, \log f) : (Y, \underline{N}) \to (X, \underline{M})$ be an étale covering of a regular log scheme $(X, \underline{M})$, and suppose that X is strictly local of dimension 1 and Y is connected (hence strictly local as well). Let $x \in X$ (resp. $y \in Y$) be the closed point; it follows that $\mathscr{O}_{X,x}$ is a strictly henselian discrete valuation ring (corollary 12.5.13). The same holds for $\mathscr{O}_{Y,y}$ in view of theorem 12.5.28 and [66, Ch.IV, Prop.18.5.10]. In case $\dim \underline{M}_x = 0$, the log structures $\underline{M}$ and $\underline{N}$ are trivial, so $f : Y \to X$ is an étale morphism of schemes. Otherwise we have $\dim \underline{M}_x = 1 = \dim \underline{N}_y$ (lemma 6.4.59(i)), and then $\underline{M}^\sharp_x \simeq \mathbb{N} \simeq \underline{N}^\sharp_y$ (theorem 6.4.18(ii)); also, the choice of a chart for f as in proposition 13.3.11, induces an isomorphism

$$\mathscr{O}_{X,x} \otimes_{\underline{M}^\sharp_x} \underline{N}^\sharp_y \xrightarrow{\sim} \mathscr{O}_{Y,y}$$

where the map $\underline{M}^\sharp_x \to \underline{N}^\sharp_y$ is the N-Frobenius map of $\mathbb{N}$, where $N > 0$ is an integer invertible in $\mathscr{O}_{X,x}$. Moreover, notice that the image of the maximal ideal of $\underline{M}_x$ generates the maximal ideal of $\mathscr{O}_{X,x}$ (and likewise for the image of $\underline{N}_y$ in $\mathscr{O}_{Y,y}$), so the structure map of $\underline{M}$ induces an isomorphism $\underline{M}^\sharp_x \xrightarrow{\sim} \Gamma_+$, onto the submonoid of the value group $(\Gamma, \leq)$ of $\mathscr{O}_{X,x}$ consisting of all elements ≤ 1 (and likewise for $\underline{N}^\sharp_y$). In other words, $\mathscr{O}_{Y,y}$ is obtained from $\mathscr{O}_{X,x}$ by adding the N-th root of a uniformizer. It then follows that the ring homomorphism $\mathscr{O}_{X,x} \to \mathscr{O}_{Y,y}$ is a finite tamely ramified extension of discrete valuation rings.

Definition 13.3.16. Let X be a normal scheme, and $Z \to X$ a closed immersion such that :

(a) Z is constructible in X.
(b) Z is a union of irreducible closed subsets of codimension one in X.

(c) For every maximal point $z \in Z$, the stalk $\mathscr{O}_{X,z}$ is a discrete valuation ring.

Set $U := X \setminus Z$, let $f : U' \to U$ be an étale covering, and $g : X' \to X$ the normalization of X in U'. Notice that $g^{-1}Z$ is a union of irreducible closed subsets of codimension one in X' (by the going down theorem [126, Th.9.4(ii)]). Notice also that for every $z \in \mathrm{Max}\, Z$, the induced map $A := \mathscr{O}_{X,z} \to B := g_*\mathscr{O}_{X',z}$ is a finite ring homomorphism, since A is a noetherian domain, and since $\mathrm{Frac}(A) \otimes_A B$ is an étale $\mathrm{Frac}(A)$-algebra ([126, §33, Lemma 1]). It follows easily that, for every geometric point ξ of X' localized at a maximal point of $g^{-1}Z$, the induced extension of strictly henselian discrete valuation rings :

$$\mathscr{O}_{X,g(\xi)} \to \mathscr{O}_{X',\xi} \tag{13.3.17}$$

is finite (notation of definition 4.9.17(iii).

(i) We say that f is *tamely ramified along* Z if, for every geometric point ξ of X' localized at a maximal point of $g^{-1}Z$, the corresponding extension (13.3.17) is tamely ramified. Hence, let $\Gamma_{g(\xi)} \to \Gamma_\xi$ be the induced map of value groups; then the *ramification index of* f *at* ξ

$$e_\xi(f) := (\Gamma_\xi : \Gamma_{g(\xi)})$$

is invertible in the residue field $\kappa(\xi)$, and g induces an isomorphism $\kappa(g(\xi)) \xrightarrow{\sim} \kappa(\xi)$.

(ii) Suppose that f is tamely ramified along Z, and that the set of maximal points of Z is finite. Then the *ramification index of* f *along* Z is the least common multiple $e_Z(f)$ of the ramification indices $e_\xi(f)$, where ξ ranges over the set of all geometric points of X' supported at a maximal point of $g^{-1}Z$.

(iii) We denote by $\mathbf{Tame}(X, U)$ the full subcategory of $\mathsf{Cov}(U)$ whose objects are the étale coverings of U that are tamely ramified along Z.

Lemma 13.3.18. *Let K, K', L be subfields of a given field, and $V_{L'}$ a discrete valuation ring of $L' := K'L$; for every subfield $E \subset L'$ set $V_E := E \cap V_{L'}$, and for every pair $E \subset E'$ of such subfields, let $e(E'/E)$ be the ramification index of the extension $V_E \subset V_{E'}$. We have :*

(i) *Suppose moreover that :*

(a) $V_K \neq K$.

(b) *K' is a finite separable extension of K, and the extension $V_K \subset V_{K'}$ is tamely ramified.*

Then the extension $V_L \subset V_{L'}$ is tamely ramified, and $e(L'/L)$ divides $e(K'/K)$.

(ii) *Suppose that :*

(a) *The extension $V_L \subset V_{L'}$ is tamely ramified, and both V_K and V_L are strictly henselian.*

(b) *$[K' : K] = [L' : L] < +\infty$ and $e(L'/K') = 1$.*

Then the extension $V_K \subset V_{K'}$ is tamely ramified, $e(K'/K) = e(L'/L)$, and $e(L/K) = 1$.

Proof. Let κ_K be the residue field of V_K, and define likewise the residue fields $\kappa_{K'}, \kappa_L$ and $\kappa_{L'}$.

(i): To begin with, let us remark :

Claim 13.3.19. In order to show (i), we may assume that V_K and V_L are strictly henselian.

Proof of the claim. Let us fix a separable closure Ω of $\kappa_{L'}$, and let $V_K^{\mathrm{sh}}, V_L^{+\mathrm{sh}}, V_{K'}^{+\mathrm{sh}}, V_{L'}^{+\mathrm{sh}}$ be the strict henselizations of $V_K, V_L, V_{K'}$ and $V_{L'}$ relative to the corresponding maps to Ω. According to [75, Lemma 6.2.5], all these are discrete valuation rings, and we denote by $K^{\mathrm{sh}}, L^{\mathrm{sh}}, K'^{\mathrm{sh}}$ and L'^{sh} the respective fields of fractions; also, the ramification indices $e(K^{\mathrm{sh}}/K)$, $e(L^{\mathrm{sh}}/L)$, $e(K'^{\mathrm{sh}}/K')$ and $e(L'^{\mathrm{sh}}/L')$ of the extensions $V_K \subset V_K^{\mathrm{sh}}$, $V_L \subset V_L^{\mathrm{sh}}$, $V_{K'} \subset V_{K'}^{\mathrm{sh}}$ and $V_{L'} \subset V_{L'}^{\mathrm{sh}}$ all equal one. Moreover, using the explicit construction of strict henselizations of *loc.cit.*, it is easily seen that $K'^{\mathrm{sh}} = K' \cdot K^{\mathrm{sh}}$ and $L'^{\mathrm{sh}} = L' \cdot L^{\mathrm{sh}}$ (details left to the reader), and furthermore the residue field of V_K^{sh} is a separable closure of κ_K, and likewise for the residue fields of $V_L^{\mathrm{sh}}, V_{K'}^{\mathrm{sh}}$ and $V_{L'}^{\mathrm{sh}}$. Then it follows easily that if the extension $V_L^{\mathrm{sh}} \subset V_{L'}^{\mathrm{sh}}$ is tamely ramified, the same holds for the extension $V_L \subset V_{L'}$, and we have both $e(L'^{\mathrm{sh}}/L^{\mathrm{sh}}) = e(L'/L)$ and

$e(K'^{\mathrm{sh}}/K^{\mathrm{sh}}) = e(K'/K)$. Thus, it suffices to prove assertion (i) for the discrete valuation ring $V^{\mathrm{sh}}_{L'}$ and the fields $K^{\mathrm{sh}}, K'^{\mathrm{sh}}, L^{\mathrm{sh}}, L'^{\mathrm{sh}}$. ◊

Henceforth, we thus assume that V_K and V_L are strictly henselian; since K' is separable over K, it is clear that L' is separable over L, and $V_{K'}$ (resp. $V_{L'}$) is strictly henselian as well, and is the integral closure of V_K in K' (resp. of V_L in L'); then $V_{K'}$ is a finite V_K-module ([126, §33, Lemma 1]). Next, let π_K (resp. $\pi_{K'}$, resp. π_L) be a uniformizer for V_K (resp. for $V_{K'}$, resp. for V_L); the assumption means that $\pi_K = \pi^e_{K'} \cdot a$ for some $a \in V^\times_{K'}$, and $e := e(K'/K)$ is invertible in the residue field of V_K. Since $V_{K'}$ is strictly henselian, we have $a = b^e$ for some $b \in V_{K'}$, and after replacing $\pi_{K'}$ by $b \cdot \pi_{K'}$, we may assume that $\pi_K = \pi^e_{K'}$; since $\kappa_K = \kappa_{K'}$, we have :

$$V_{K'} = V_K + \pi_{K'} V_{K'} = V_K + \pi_{K'} V_K + \pi^2_{K'} V_{K'} = \cdots = V_K + \pi_{K'} V_K + \cdots + \pi^{e-1}_{K'} V_K + \pi_K V_{K'}$$

whence $V_{K'} = V_K[\pi_{K'}]$, by Nakayama's lemma; therefore $K' = K(\pi_{K'})$, and K' is the splitting field of $X^e - \pi_K$ over K. It follows that $L' = L(\pi_{K'})$ is the splitting field of the same polynomial over L, and especially, it is a finite Galois extension of L. Moreover, the injective map

$$\mathrm{Gal}(L'/L) \to L^\times \qquad \sigma \mapsto \sigma(\pi_{K'})/\pi_{K'}$$

is a character with values in the e-torsion subgroup of $L^\times$; hence $[L' : L]$ divides e. Recalling that $[L' : L] = e(L'/L) \cdot [\kappa_{L'} : \kappa_L]$ ([34, Ch.VI, §8, no.5, Cor.1]), the assertion follows.

(ii): The assumptions imply that $\kappa_L = \kappa_{L'}$; then, [34, Ch.VI, §8, no.5, Cor.1] yields :

$$\begin{aligned} e(K'/K) \cdot [\kappa_{K'} : \kappa_K] \cdot e(L/K) = [K' : K] \cdot e(L/K) &= [L' : L] \cdot e(L/K) \\ &= e(L' : L) \cdot e(L/K) \\ &= e(L' : K) \\ &= e(L'/K') \cdot e(K'/K) = e(K'/K) \end{aligned}$$

whence $[\kappa_{K'} : \kappa_K] = e(L/K) = 1$ and $e(K'/K) = e(L'/L)$. □

Proposition 13.3.20. *Let $\varphi : X' \to X$ be a maximizing morphism of normal schemes (see definition* 8.1.46(ii)*), $Z \subset X$ a closed subset, and set $Z' := \varphi^{-1}Z$. Suppose that :*

- (a) *The closed immersions $Z \to X$ and $Z' \to X'$ satisfy conditions* (a),(b),(c) *of definition* 13.3.16.
- (b) *φ restricts to a maximizing map $Z' \to Z$.*

Set moreover $U := X \setminus Z$ and $U' := X' \setminus Z'$. We have :

- (i) *$\varphi^* : \mathsf{Cov}(U) \to \mathsf{Cov}(U')$ restricts to a functor*
$$\mathbf{Tame}(X, U) \to \mathbf{Tame}(X', U').$$
- (ii) *Suppose that $\mathrm{Max}\, Z$ and $\mathrm{Max}\, Z'$ are finite sets. Then, for any object $f : V \to U$ of $\mathbf{Tame}(X, U)$, the index $e_{Z'}(\varphi^* f)$ divides $e_Z(f)$.*
- (iii) *Suppose that X is regular and noetherian, and let $j : U \to X$ be the open immersion. Then the essential image of the pull-back functor $j^* : \mathsf{Cov}(X) \to \mathbf{Tame}(X, U)$ consists of the objects $f : V \to U$ such that $e_Z(f) = 1$.*

Proof. (i,ii): Let $V \to U$ be an object of $\mathbf{Tame}(X, U)$, and denote by $g : W \to X$ (resp. $g' : W' \to X'$) the normalization of X in V (resp. of X' in $V \times_U U'$). Let w' be a geometric point of W' localized at a maximal point of $g'^{-1}Z'$, and denote by w (resp. x', resp. x) the image of w' in W (resp. in X', resp. in X); in order to check (i) and (ii), we have to show that the ramification index of g' at the geometric point w' is invertible in $\kappa(w')$, and the residue field extension $\kappa(x') \subset \kappa(w')$ is trivial. To this aim, in view of proposition 9.8.3, we may replace X' by $X'(x')$, X by $X(x)$, W by $W(w)$, and assume from start that X, X' and W are strictly local, say $X = \mathrm{Spec}\, A$, $X' = \mathrm{Spec}\, B$ and $W = \mathrm{Spec}\, C$ for some strictly henselian discrete valuation rings A, B, C. Then, by construction, we have :

$$\mathrm{Frac}\, \mathscr{O}_{W',w'} = \mathrm{Frac}(B) \cdot \mathrm{Frac}(C).$$

hence the assertion follows from lemma 13.3.18.

(iii): Let $f : V \to U$ be an object of $\mathbf{Tame}(X, U)$. By claim 13.1.8, there exists a largest open subset $U_{\max} \subset X$ containing U, and such that f is the restriction of an étale covering $f_{\max} : V' \to U_{\max}$; now, if $e_Z(f) = 1$, every point of codimension one of $X\backslash U$ lies in $U_{\max}$, by claim 13.1.9. Hence $X\backslash U$ has codimension ≥ 2 in X, in which case lemma 13.1.7(iv) implies that $X = U_{\max}$, as required. □

Remark 13.3.21. Let X be a normal scheme, and $Z \to X$ a closed immersion verifying conditions (a),(b),(c) of definition 13.3.16. Then the assumptions (a) and (b) of proposition 13.3.20 are fulfilled, notably, when φ is finite and maximizing, or when φ is flat and X' is locally noetherian ([126, Th.9.4(ii), Th.9.5]).

13.3.22. Let now $\underline{X} := (X_i \mid i \in I)$ be a cofiltered system of quasi-compact and quasi-separated normal schemes, such that, for every morphism $\varphi : i \to j$ in the indexing category I, the corresponding transition morphism $f_\varphi : X_i \to X_j$ is maximizing and affine. Suppose also, that for every $i \in I$, there exists a closed immersion $Z_i \to X_i$, fulfilling conditions (a),(b),(c) of definition 13.3.16, such that $\mathrm{Max}\, Z_i$ is a finite set, and for every morphism $\varphi : i \to j$ of I, we have :

- $Z_i = f_\varphi^{-1} Z_j$.
- The corresponding morphism f_φ restricts to a maximizing map $Z_i \to Z_j$.

Let also X be the limit of $\underline{X}$, and Z the limit of the system $(Z_i \mid i \in I)$, and suppose furthermore that the closed immersion $Z \to X$ fulfills as well conditions (a),(b),(c) of definition 13.3.16.

Proposition 13.3.23. *In the situation of* (13.3.22)*, we have :*

(i) *The induced morphism $\pi_i : Z \to Z_i$ is maximizing for every $i \in I$.*
(ii) *The universal cone $(X \to X_i \mid i \in I)$ induces an equivalence of categories :*

$$2\text{-}\operatorname*{colim}_I \mathbf{Tame}(X_i, X_i\backslash Z_i) \to \mathbf{Tame}(X, X\backslash Z).$$

Proof. (i): More precisely, we shall prove that there is a natural homeomorphism :

$$\mathrm{Max}\, Z \xrightarrow{\sim} \lim_{i \in I} \mathrm{Max}\, Z_i.$$

(Notice the each $\mathrm{Max}\, Z_i$ is a discrete finite set, hence this will show that $\mathrm{Max}\, Z$ is a profinite topological space.) Indeed, suppose that $z := (z_i \mid i \in I)$ is a maximal point of Z. For every $i \in I$, let $T_i \subset \mathrm{Max}\, Z_i$ be the subset of maximal generizations of z_i in Z_i. It is easily seen that $f_\varphi T_i \subset T_j$ for every morphism $\varphi : i \to j$ in I. Clearly T_i is a finite non-empty set for every $i \in I$, hence the limit T of the cofiltered system $(T_i \mid i \in I)$ is non-empty. However, any point of T is a generization of z in Z, hence it must coincide with z. The assertion follows easily.

(ii): To begin with, proposition 13.3.20(i) shows that, for every morphism $\varphi : i \to j$ in I, the transition morphism f_φ induces a pull-back functor $\mathbf{Tame}(X_j, X_j\backslash Z_j) \to \mathbf{Tame}(X_i, X_i\backslash Z_i)$, so the 2-colimit in (ii) is well-defined, and combining (i) with proposition 13.3.20(i) we obtain indeed a well-defined functor from this 2-colimit to $\mathbf{Tame}(X, X\backslash Z)$.

The full faithfulness of the functor of (ii) follows from lemma 13.1.6. Next, let $g : V \to X\backslash Z$ be an object of $\mathbf{Tame}(X, X\backslash Z)$; invoking again lemma 13.1.6, we may descend g to an étale covering $g_j : V_j \to X_j\backslash Z_j$, for some $j \in I$; after replacing I by I/j, we may assume that j is the final object of I and we may define $g_i := f_\varphi^*(g_j)$ for every $\varphi : i \to j$ in I. To conclude the proof, it suffices to show that there exists $i \in I$ such that g_i is tamely ramified along Z_i.

Now, let $\overline{g}_i : \overline{V}_i \to X_i$ be the normalization of X_i in $V_i \times_{X_j} X_i$, for every $i \in I$, and $\overline{g} : \overline{V} \to X$ the normalization of X in V. Given a geometric point $\overline{v}$ localized at a maximal point of $\overline{g}^{-1}Z$, let $\overline{v}_i$ (resp. $\overline{z}_i$) be the image of $\overline{v}$ in $\overline{V}_i$ (resp. in Z_i), and $z_i \in Z_i$ the support of $\overline{z}_i$, for every $i \in I$. Let also $\overline{z}$ be the image of $\overline{v}$ in Z, and $z \in Z$ the support of $\overline{z}$. Then

$$\mathscr{O}_{X,\overline{z}} = \operatorname*{colim}_{i\in I} \mathscr{O}_{X_i,\overline{z}_i} \qquad \mathscr{O}_{\overline{V},\overline{v}} = \operatorname*{colim}_{i\in I} \mathscr{O}_{\overline{V}_i,\overline{v}_i}$$

and $\operatorname{Frac}\mathscr{O}_{\overline{V},\overline{v}} = \operatorname{Frac}(\mathscr{O}_{X,\overline{z}}) \cdot \operatorname{Frac}(\mathscr{O}_{\overline{V}_i,\overline{v}_i})$ for every $i \in I$ ([66, Ch.IV, Prop.18.8.18(ii)]), and it is easily seen that there exists $i \in I$ such that the ramification index of the extension $\mathscr{O}_{\overline{V}_i,\overline{v}_i} \subset \mathscr{O}_{\overline{V},\overline{v}}$ equals 1, and such that $[\operatorname{Frac}\mathscr{O}_{\overline{V}_i,\overline{v}_i} : \operatorname{Frac}\mathscr{O}_{X_i,\overline{z}_i}] = [\operatorname{Frac}\mathscr{O}_{\overline{V},\overline{v}} : \operatorname{Frac}\mathscr{O}_{X,\overline{z}}]$. Moreover, for every sufficiently large i, the induced map $\overline{g}^{-1}(z) \to \overline{g}_i^{-1}(z_i)$ is a bijection. Then, by lemma 13.3.18(ii), for every such index i, the finite extension $\mathscr{O}_{X_i,\overline{z}_i} \to \mathscr{O}_{\overline{V}_i,\overline{v}_i}$ is already tamely ramified, and since I is cofiltered, it follows that there exists $i \in I$ such that the induced morphism $\overline{V}_i \times_{X_i} X_i(\overline{z}_i) \to X_i(\overline{z}_i)$ is already tamely ramified. However, notice that $\pi_i^{-1}(z_i)$ is open in $\operatorname{Max} Z$, for every $i \in I$, and every $z_i \in \operatorname{Max} Z_i$. Therefore, we may find a finite subset $I_0 \subset I$, and for every $i \in I_0$ a subset $T_i \subset \operatorname{Max} Z_i$, such that :

- $\operatorname{Max} Z = \bigcup_{i\in I_0} \pi_i^{-1}(T_i)$.
- For every geometric point $\overline{z}_i$ localized in T_i, the morphism $\overline{V}_i \times_{X_i} X_i(\overline{z}_i) \to X_i(\overline{z}_i)$ is tamely ramified.

Since I is cofiltered, we may find $k \in I$ with morphisms $\varphi_i : k \to i$ for every $i \in I_0$; after replacing T_i by $f_{\varphi_i}^{-1}(T_i)$ for every $i \in I_0$, we may then assume that $I_0 = \{k\}$, so that $\operatorname{Max} Z = \pi_k^{-1}(T_k)$. It follows that $\operatorname{Max} Z_i = f_\varphi^{-1} T_k$ for some $i \in I$ and some $\varphi : i \to k$. Then it is clear that g_i is tamely ramified along Z_i, as required. □

13.3.24. Let X be a scheme, $U \subset X$ a connected open subset, ξ any geometric point of U, and $N > 0$ an integer which is invertible in $\mathscr{O}_U$; then the N-Frobenius map $\boldsymbol{N}$ of $\mathscr{O}_U^\times$ gives a *Kummer exact sequence* of abelian sheaves on $X_{\text{ét}}$:

$$0 \to \boldsymbol{\mu}_{N,U} \to \mathscr{O}_U^\times \xrightarrow{\boldsymbol{N}} \mathscr{O}_U^\times \to 0$$

(where $\boldsymbol{\mu}_{N,U}$ is the N-torsion subsheaf of $\mathscr{O}_{U_{\text{ét}}}^\times$: see [10, Exp.IX, §3.2]). Suppose now that $\boldsymbol{\mu}_{N,U}$ is a constant sheaf on $U_{\text{ét}}$; this means especially that $\boldsymbol{\mu}_N := (\boldsymbol{\mu}_{N,U})_\xi$ is a cyclic group of order N. Indeed, $\boldsymbol{\mu}_N$ certainly contains such a subgroup (since N is invertible in $\mathscr{O}_U$), so denote by ζ one of its generators; then every $u \in \boldsymbol{\mu}_N$ satisfies the identity $0 = u^N - 1 = \prod_{i=1}^N (u - \zeta^i)$, and each factor u_ζ^i of this decomposition vanishes on a closed subset U_i of $U(\xi)$; clearly $U_i \cap U_j = \varnothing$ for $i \neq j$, so we get a decomposition of $U(\xi)$ as a disjoint union of open and closed subsets $U(\xi) = U_1 \cup \cdots \cup U_N$ such that $u = \zeta^i$ on U_i, for every $i \leq N$; since $U(\xi)$ is connected, it follows that $U(\xi) = U_i$ for some i, *i.e.* ζ generates $\boldsymbol{\mu}_N$. There follows a natural map :

$$\partial_N : \Gamma(U_{\text{ét}}, \mathscr{O}_U^\times) \to H^1(U_{\text{ét}}, \boldsymbol{\mu}_{N,U}) \xrightarrow{\sim} \operatorname{Hom}_{\text{cont}}(\pi_1(U_{\text{ét}}, \xi), \boldsymbol{\mu}_N)$$

(lemma 13.1.12). Recall the geometric interpretation of ∂_N : a given $u \in \mathscr{O}_U^\times(U)$ is viewed as a morphism of schemes $u : U \to \mathbb{G}_m$, where $\mathbb{G}_m := \operatorname{Spec}\mathbb{Z}[\mathbb{Z}]$ is the standard multiplicative group scheme; we let U' be the fibre product in the cartesian diagram of schemes :

$$\begin{array}{ccc} U' & \longrightarrow & \mathbb{G}_m \\ {\scriptstyle \varphi_u}\downarrow & & \downarrow{\scriptstyle \operatorname{Spec}\mathbb{Z}[\boldsymbol{N}_\mathbb{Z}]} \\ U & \xrightarrow{u} & \mathbb{G}_m. \end{array} \tag{13.3.25}$$

Then φ_u is a torsor under the $U_{\text{ét}}$-group $\boldsymbol{\mu}_{N,U}$, and to such torsor, lemma 13.1.12 attaches a well defined continuous group homomorphism as required.

If $M > 0$ is any integer dividing N, a simple inspection yields a commutative diagram :

$$\begin{array}{ccc} \Gamma(U_{\text{ét}}, \mathscr{O}_X^\times) & \xrightarrow{\partial_N} & \mathrm{Hom}_{\mathrm{cont}}(\pi_1(U_{\text{ét}}, \xi), \boldsymbol{\mu}_N) \\ & \searrow^{\partial_M} & \downarrow{\scriptstyle \pi_{N,M}} \\ & & \mathrm{Hom}_{\mathrm{cont}}(\pi_1(U_{\text{ét}}, \xi), \boldsymbol{\mu}_M) \end{array} \tag{13.3.26}$$

where $\pi_{N,M}$ is induced by the map $\boldsymbol{\mu}_N \to \boldsymbol{\mu}_M$ given by the rule : $x \mapsto x^{N/M}$ for every $x \in \boldsymbol{\mu}_N$.

13.3.27. Let now X be a strictly local scheme, x the closed point of X, and denote by p the characteristic exponent of the residue field $\kappa(x)$ (so p is either 1 or a positive prime integer). Let also $\beta : \underline{M} \to \mathscr{O}_X$ be a log structure on $X_{\text{ét}}$, take $U := (X, \underline{M})_{\mathrm{tr}}$, suppose that $U \neq \varnothing$, and fix a geometric point ξ of U; for every integer $N > 0$ with $(N, p) = 1$, we get a morphism of monoids :

$$\underline{M}_x \xrightarrow{\beta_x} \Gamma(U_{\text{ét}}, \mathscr{O}_X^\times) \xrightarrow{\partial_N} \mathrm{Hom}_{\mathrm{cont}}(\pi_1(U_{\text{ét}}, \xi), \boldsymbol{\mu}_N) \tag{13.3.28}$$

whose kernel contains $\underline{M}_x^N$, the image of the N-Frobenius endomorphism of $\underline{M}_x$. Notice that the N-Frobenius map of $\mathscr{O}_{X,x}^\times$ is surjective : indeed, since the residue field $\kappa(x)$ is separably closed, and $(N, p) = 1$, all polynomials in $\kappa(x)[T]$ of the form $T^N - u$ (for $u \neq 0$) split as a product of distinct monic polynomials of degree 1; since $\mathscr{O}_{X,x}$ is henselian, the same holds for all polynomials in $\mathscr{O}_{X,x}[T]$ of the form $T^N - u$, with $u \in \mathscr{O}_{X,x}^\times$. It follows that (13.3.28) factors through a natural map:

$$\underline{M}_x^{\sharp\mathrm{gp}} \to \mathrm{Hom}_{\mathrm{cont}}(\pi_1(U_{\text{ét}}, \xi), \boldsymbol{\mu}_N)$$

which is the same as a group homomorphism :

$$\underline{M}_x^{\sharp\mathrm{gp}} \times \pi_1(U_{\text{ét}}, \xi) \to \boldsymbol{\mu}_N. \tag{13.3.29}$$

In view of the commutative diagram (13.3.26), it is easily seen that the pairings (13.3.29), for N ranging over all the positive integers with $(N, p) = 1$, assemble into a single pairing :

$$\underline{M}_x^{\mathrm{gp}} \times \pi_1((X, \underline{M})_{\mathrm{tr},\text{ét}}, \xi) \to \prod_{\ell \neq p} \mathbb{Z}_\ell(1)$$

(where ℓ ranges over the prime numbers different from p, and $\mathbb{Z}_\ell(1) := \lim_{n \in \mathbb{N}} \boldsymbol{\mu}_{\ell^n}$). The latter is the same as a group homomorphism :

$$\pi_1((X, \underline{M})_{\mathrm{tr},\text{ét}}, \xi) \to \underline{M}_x^{\mathrm{gp}\vee} \otimes_{\mathbb{Z}} \prod_{\ell \neq p} \mathbb{Z}_\ell(1). \tag{13.3.30}$$

13.3.31. Let $f : (X, \underline{M}) \to (Y, \underline{N})$ be a morphism of log schemes, such that both X and Y are strictly local, and f maps the closed point x of X to the closed point y of Y. Then f restricts to a morphism of schemes $f_{\mathrm{tr}} : (X, \underline{M})_{\mathrm{tr}} \to (Y, \underline{N})_{\mathrm{tr}}$ (remark 12.2.8(i)). Fix again a geometric point ξ of $(X, \underline{M})_{\mathrm{tr}}$; we get a diagram of group homomorphisms :

$$\begin{array}{ccc} \pi_1((X, \underline{M})_{\mathrm{tr},\text{ét}}, \xi) & \xrightarrow{\pi_1(f_{\mathrm{tr}}, \xi)} & \pi_1((Y, \underline{N})_{\mathrm{tr},\text{ét}}, f(\xi)) \\ \downarrow & & \downarrow \\ \underline{N}_y^{\mathrm{gp}\vee} \otimes_{\mathbb{Z}} \prod_{\ell \neq p} \mathbb{Z}_\ell(1) & \longrightarrow & \underline{M}_x^{\mathrm{gp}\vee} \otimes_{\mathbb{Z}} \prod_{\ell \neq p} \mathbb{Z}_\ell(1) \end{array}$$

whose vertical arrows (resp. bottom horizontal arrow) are the maps (13.3.30) (resp. is induced by $(\log f_x)^{\mathrm{gp}\vee}$), and by inspecting the constructions, it is easily seen that this diagram commutes.

13.3.32. Suppose now that $(X, \underline{M})$ as in (13.3.27) is a fs log scheme, so that $P := \underline{M}^\sharp_x$ is fine and saturated; in this case, we wish to give a second construction of the pairing (13.3.29).

• Namely, for every integer $N > 0$ set $S_P := \operatorname{Spec} \mathbb{Z}[P]$ and consider the finite morphism of schemes $g_N : S_P \to S_P$ induced by the N-Frobenius endomorphism of P. Notice that S_P contains the dense open subset $U_P := \operatorname{Spec} \mathbb{Z}[1/N, P^{\mathrm{gp}}]$, and it is easily seen that the restriction $g_{N|U_P} : U_P \to U_P$ of g_N is an étale morphism. Let $\tau \in U_P$ be any geometric point; then τ corresponds to a ring homomorphism $\mathbb{Z}[1/N, P^{\mathrm{gp}}] \to \kappa := \kappa(\tau)$, which is the same as a character $\chi_\tau : P^{\mathrm{gp}} \to \kappa^\times$. Likewise, any $\tau' \in g_N^{-1}(\tau)$ is determined by a character $\chi_{\tau'} : P^{\mathrm{gp}} \to \kappa^\times$ extending χ_τ, *i.e.* such that $\chi_{\tau'}(x^N) = \chi_\tau(x)$ for every $x \in P^{\mathrm{gp}}$. Notice that every character χ_τ as above admits at least one such extension $\chi_{\tau'}$, since κ is separably closed, and N is invertible in κ. Let C_N be the cokernel of the N-Frobenius endomorphism of P^{gp}; there follows a short exact sequence of finite abelian groups :

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(C_N, \kappa^\times) \to \mathrm{Hom}_{\mathbb{Z}}(P^{\mathrm{gp}}, \kappa^\times) \xrightarrow{N} \mathrm{Hom}_{\mathbb{Z}}(P^{\mathrm{gp}}, \kappa^\times) \to 0.$$

Especially, we see that the fibre $g_N^{-1}(\tau)$ is naturally a torsor under the group $\mathrm{Hom}_{\mathbb{Z}}(C_N, \kappa^\times) = \mathrm{Hom}_{\mathbb{Z}}(P^{\mathrm{gp}}, \boldsymbol{\mu}_N(\kappa))$, where $\boldsymbol{\mu}_N(\kappa) \subset \kappa^\times$ is the N-torsion subgroup. Hence $g_{N|U_P}$ is an étale Galois covering of U_P, whose Galois group is naturally isomorphic to $\mathrm{Hom}_{\mathbb{Z}}(P^{\mathrm{gp}}, \boldsymbol{\mu}_N(\kappa))$; therefore, $g_{N|U_P}$ is classified by a well defined continuous representation :

$$\pi_1(U_{P,\text{ét}}, \tau) \to \mathrm{Hom}_{\mathbb{Z}}(P^{\mathrm{gp}}, \boldsymbol{\mu}_N(\kappa)) \qquad (13.3.33)$$

(lemma 13.1.12). The corresponding action of $\mathrm{Hom}_{\mathbb{Z}}(P^{\mathrm{gp}}, \boldsymbol{\mu}_N(\kappa))$ on U_P can be extracted from the construction : namely, let $\chi : P^{\mathrm{gp}} \to \boldsymbol{\mu}_N(\kappa)$ be a given character; by definition, the action of χ sends the geometric point τ to the geometric point τ' whose character $\chi_{\tau'} : P^{\mathrm{gp}} \to \kappa^\times$ is given by the rule : $x \mapsto \chi(x) \cdot \chi_\tau$ for every $x \in P^{\mathrm{gp}}$. Consider the automorphism ρ_χ of the $\mathbb{Z}$-algebra $\mathbb{Z}[P^{\mathrm{gp}}]$ given by the rule $x \mapsto \chi(x) \cdot x$ for every $x \in P^{\mathrm{gp}}$, and notice that $\rho(\tau) = \tau'$; since the fibre functors are faithful on étale coverings, we conclude that χ acts as $\operatorname{Spec} \rho_\chi$ on U_P.

• Moreover, if $\lambda : Q \to P$ is any morphism of fine and saturated monoids, clearly we have a commutative diagram

$$\begin{array}{ccc} U_P & \xrightarrow{\operatorname{Spec} \mathbb{Z}[\lambda]_{|U_P}} & U_Q \\ {\scriptstyle g_{N|U_P}} \downarrow & & \downarrow {\scriptstyle g_{N|U_Q}} \\ U_P & \xrightarrow{\operatorname{Spec} \mathbb{Z}[\lambda]_{|U_P}} & U_Q \end{array}$$

whence, by remark 13.1.14(iii.a), a well defined group homomorphism

$$\mathrm{Hom}_{\mathbb{Z}}(P^{\mathrm{gp}}, \boldsymbol{\mu}_N) \to \mathrm{Hom}_{\mathbb{Z}}(Q^{\mathrm{gp}}, \boldsymbol{\mu}_N) \qquad (13.3.34)$$

that makes commute the induced diagram :

$$\begin{array}{ccc} \pi_1(U_{P,\text{ét}}, \xi) & \longrightarrow & \pi_1(U_Q, \xi') \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathbb{Z}}(P^{\mathrm{gp}}, \boldsymbol{\mu}_N) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(Q^{\mathrm{gp}}, \boldsymbol{\mu}_N) \end{array}$$

whose vertical arrows are the maps (13.3.33). By inspecting the constructions, it is easily seen that (13.3.34) is just the map $\mathrm{Hom}_{\mathbb{Z}}(\lambda^{\mathrm{gp}}, \boldsymbol{\mu}_N)$.

• Next, by corollary 12.1.36(i) the projection $\underline{M}_x \to P$ admits a splitting

$$\alpha : P \to \underline{M}_x$$

(which defines a sharp chart for $\underline{M}$); if N is invertible in $\mathscr{O}_X$, this splitting induces a morphism of schemes $X \to S_P$, which restricts to a morphism $U \to U_P$. If we let τ be the image of ξ, we deduce a continuous group homomorphism $\pi_1(U_{\text{ét}}, \xi) \to \pi_1(U_{P,\text{ét}}, \tau)$ ([82, Exp.V, §7]), whose composition with (13.3.33) yields a continuous map :

(13.3.35) $$\pi_1(U_{\text{ét}}, \xi) \to \mathrm{Hom}_{\mathbb{Z}}(P^{\mathrm{gp}}, \boldsymbol{\mu}_N(\kappa)).$$

We claim that the pairing $P^{\mathrm{gp}} \times \pi_1(U_{\text{ét}}, \xi) \to \boldsymbol{\mu}_N(\kappa)$ arising from (13.3.35) agrees with (13.3.29), under the natural identification $\boldsymbol{\mu}_N(\kappa) \xrightarrow{\sim} \boldsymbol{\mu}_N$. Indeed, for given $s \in P$, let $j_s : \mathbb{Z} \to P^{\mathrm{gp}}$ be the map such that $j_s(n) := s^n$ for every $n \in \mathbb{Z}$; by composing (13.3.35) with $\mathrm{Hom}_{\mathbb{Z}}(j_s, \boldsymbol{\mu}_N(\kappa))$, we obtain a map $\pi_1(U_{\text{ét}}, \xi) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \boldsymbol{\mu}_N(\kappa)) \xrightarrow{\sim} \boldsymbol{\mu}_N(\kappa)$; in view of the foregoing, it is easily seen that the corresponding $\boldsymbol{\mu}_N(\kappa)$-torsor is precisely $\varphi_{\alpha(s)}$, in the notation of (13.3.25), whence the contention.

Proposition 13.3.36. *In the situation of* (13.3.27)*, suppose that* $(X, \underline{M})$ *is a regular log scheme. Then* (13.3.30) *is a surjection.*

Proof. From the discussion of (13.3.32), we are reduced to showing that the morphism (13.3.35) is surjective, for every $N > 0$ such that $(N, p) = 1$. The latter comes down to showing that the corresponding torsor $g_{N|U_P} \times_{U_P} U : U' \to U$ is connected (remark 13.1.14(i)). However, the map $\alpha : P \to \underline{M}_x$ induces a morphism of log schemes $\psi : (X, \underline{M}) \to \mathrm{Spec}(\mathbb{Z}, P)$ (see (12.2.13)); we remark the following :

Claim 13.3.37. Slightly more generally, for any Kummer morphism $\nu : P \to Q$ of monoids, with Q fine, sharp and saturated, define $(X_\nu, \underline{M}_\nu)$ as the fibre product in the cartesian diagram:

(13.3.38) $$\begin{array}{ccc} (X_\nu, \underline{M}_\nu) & \longrightarrow & \mathrm{Spec}(\mathbb{Z}, Q) \\ {\scriptstyle f_\nu}\downarrow & & \downarrow{\scriptstyle \mathrm{Spec}(\mathbb{Z},\nu)} \\ (X, \underline{M}) & \xrightarrow{\psi} & \mathrm{Spec}(\mathbb{Z}, P) \end{array}$$

Then $(X_\nu, \underline{M}_\nu)$ is regular, and X_ν is strictly local.

Proof of the claim. By lemma 12.5.24, $(X_\nu, \underline{M}_\nu)$ is regular at every point of the closed fibre $X_\nu \times_X \mathrm{Spec}\, \kappa(x)$, and since f_ν is a finite morphism, every point of X_ν specializes to a point of the closed fibre, therefore $(X_\nu, \underline{M}_\nu)$ is regular (theorem 12.5.31). Moreover, X_ν is connected, provided the same holds for the closed fibre; in turn, this follows immediately from proposition 13.3.7(ii). Then X_ν is strictly local, by [66, Ch.IV, Prop.18.5.10]. ♢

From claim 13.3.37 we see that X_ν is normal (corollary 12.5.13) for every such ν, especially this holds for the N-Frobenius endomorphism of P, in which case U' is an open subset of X_ν; since the latter is connected, the same then holds for U'. □

Example 13.3.39. (i) In the situation of example 12.6.20, let $X \to S$ be any morphism of schemes, set $\underline{X} := X \times_S \underline{S}$, denote by $(X, \underline{M})$ the log scheme underlying $\underline{X}$, and define $(X_{(k)}, \underline{M}_{(k)})$ as the fibre product in the cartesian diagram of log schemes :

$$\begin{array}{ccc} (X_{(k)}, \underline{M}_{(k)}) & \xrightarrow{\pi_{(k)}} & \mathrm{Spec}(R, P) \\ {\scriptstyle \boldsymbol{k}_X}\downarrow & & \downarrow{\scriptstyle \mathrm{Spec}(R,\boldsymbol{k}_P)} \\ (X, \underline{M}) & \xrightarrow{\pi} & \mathrm{Spec}(R, P) \end{array}$$

where π is the natural projection. Set as well

$$\underline{X}_{(k)} := ((X_{(k)}, \underline{M}_{(k)}), T_P, \psi_P \circ \pi^\sharp_{(k)})$$

which is an object of $\mathscr{K}_{\mathrm{int}}$; by proposition 12.6.14(iii), diagram (6.5.28) (with $T := T_P$) underlies a commutative diagram in $\mathscr{K}_{\mathrm{int}}$:

$$\begin{array}{ccc} \varphi^*\underline{X}_{(k)} & \longrightarrow & \underline{X}_{(k)} \\ {\scriptstyle \underline{g}_X}\downarrow & & \downarrow{\scriptstyle (\boldsymbol{k}_X,\boldsymbol{k}_{T_P})} \\ \varphi^*\underline{X} & \longrightarrow & \underline{X} \end{array}$$

whose horizontal arrows are cartesian. Clearly this diagram is isomorphic to $X \times_S$ (12.6.21); we deduce that the morphism g_X of log schemes underlying $\underline{g}_X$ is finite, and of Kummer type; by proposition 13.3.11, g_X is even an étale covering, if k is invertible in $\mathscr{O}_X$.

(ii) Suppose furthermore, that $(X, \underline{M})$ is regular; by claim 13.3.37 it follows that $(X_{(k)}, \underline{M}_{(k)})$ is regular as well, and then the same holds for the log schemes $(X_\varphi, \underline{M}_\varphi)$ and $(Y, \underline{N})$ underlying respectively $\varphi^*\underline{X}$ and $\varphi^*\underline{X}_{(k)}$ (proposition 12.6.14(v) and theorem 12.5.28). Let now $y \in Y$ be a point of height one in Y, lying in the closed subset $Y \setminus (Y, \underline{N})_{\mathrm{tr}}$, and set $x := g_X(y)$; arguing as in example 13.3.15, we see that $\underline{N}^\sharp_y$ and $\underline{M}^\sharp_{\varphi,x}$ are both isomorphic to $\mathbb{N}$, and the induced map $\mathscr{O}_{X_\varphi,x} \to \mathscr{O}_{Y,y}$ is an extension of discrete valuation rings, whose corresponding extension of valued fields is finite. Denote by $\Gamma_x \to \Gamma_y$ the associated extension of value groups; in view of (12.6.22) we see that the ramification index $(\Gamma_y : \Gamma_x)$ equals k.

13.3.40. Resume the situation of (13.3.32). Every (finite) discrete quotient map

$$\rho : \underline{M}^{\mathrm{gp}\vee}_x \otimes_{\mathbb{Z}} \prod_{\ell \neq p} \mathbb{Z}_\ell(1) \to G$$

corresponds, via composition with (13.3.30), to a G_U-torsor, which can be explicitly constructed as follows. Set $P := \underline{M}^\sharp_x$, pick a splitting α as in (13.3.32), choose an integer $N > 0$ large enough, so that $(N, p) = 1$, and ρ factors through a group homomorphism

$$\overline{\rho} : \mathrm{Hom}_{\mathbb{Z}}(P^{\mathrm{gp}}, \boldsymbol{\mu}_N(\kappa)) \to G$$

and set $H := \mathrm{Ker}\,\overline{\rho}$. Now, via (13.3.30), the quotient $G' := \mathrm{Hom}_{\mathbb{Z}}(P^{\mathrm{gp}}, \boldsymbol{\mu}_N(\kappa))$ corresponds to the G'_U-torsor on U obtained from $g_N : U_P \to U_P$ via pull-back along the morphism $U \to U_P$ given by the chart α (notation of (13.3.32)). According to remark 13.1.14(ii), the sought G_U is therefore isomorphic to the one obtained from the quotient $\overline{g}_N : U_P/H \to U_P$, via pull-back along the same morphism. To exhibit such quotient, consider the exact sequence:

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(G, \kappa^\times) \to P^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{Z}/N\mathbb{Z} \to \mathrm{Hom}_{\mathbb{Z}}(H, \kappa^\times) \to 0.$$

Define $Q^{\mathrm{gp}} \subset P^{\mathrm{gp}}$ as the kernel of the induced map $P^{\mathrm{gp}} \to \mathrm{Hom}_{\mathbb{Z}}(H, \kappa^\times)$, and set $Q := Q^{\mathrm{gp}} \cap P$. By construction, the N-Frobenius endomorphism of P factors through an injective map $\nu : P \to Q$ and the inclusion map $j : Q \to P$, so g_N factors as a composition :

$$U_P \to U_Q \xrightarrow{h} U_P.$$

The maps on geometric points $U_P(\kappa) \to U_Q(\kappa) \to U_P(\kappa)$ induced by g_N and h correspond to $j^{\mathrm{gp}*}$ and respectively $\nu^{\mathrm{gp}*}$ in the resulting commutative diagram

$$
\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
0 & \to & H & \to & \mathrm{Hom}_{\mathbb{Z}}(P^{\mathrm{gp}}, \boldsymbol{\mu}_N(\kappa)) & \to & G & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & H & \to & \mathrm{Hom}_{\mathbb{Z}}(P^{\mathrm{gp}}, \kappa^\times) & \xrightarrow{j^{\mathrm{gp}*}} & \mathrm{Hom}_{\mathbb{Z}}(Q^{\mathrm{gp}}, \kappa^\times) & \to & 0 \\
 & & & & & & \downarrow{\scriptstyle \nu^{\mathrm{gp}*}} & & \\
 & & & & & & \mathrm{Hom}_{\mathbb{Z}}(P^{\mathrm{gp}}, \kappa^\times) & & \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}
$$

whose rows and column are exact. Hence, let τ and $\chi_\tau : P^{\mathrm{gp}} \to \kappa^\times$ be as in (13.3.32); the fibre $h^{-1}(\tau)$ corresponds to the set of all characters $\chi_{\tau'} : Q^{\mathrm{gp}} \to \kappa^\times$ whose restriction $\chi_{\tau'} \circ \nu^{\mathrm{gp}}$ to P^{gp} agrees with χ_τ. Then, with the notation of (13.3.38), we conclude that the restriction $(f_\nu)_{\mathrm{tr}} : (X_\nu, \underline{M}_\nu)_{\mathrm{tr}} \to U$ is the sought G_U-torsor. Notice as well that f_ν is an étale covering of $(X, \underline{M})$.

• Here is another handier description of the same submonoid Q of P. Notice that ρ is the same as a group homomorphism

$$\rho^\dagger : P^{\mathrm{gp}\vee} \to \mathrm{Hom}_{\mathbb{Z}}(\textstyle\prod_{\ell \neq p} \mathbb{Z}_\ell(1), G)$$

and let $L := \mathrm{Ker}\,\rho^\dagger$. Fix a generator ζ_N of $\boldsymbol{\mu}_N(\kappa)$; by definition

$$
\begin{aligned}
Q^{\mathrm{gp}} &= \{x \in P^{\mathrm{gp}} \mid t(x) \otimes \xi = 0 \text{ for all } t \in P^{\mathrm{gp}\vee} \text{ and } \xi \in \boldsymbol{\mu}_N \text{ such that } \overline{\rho}(t \otimes \xi) = 0\} \\
&= \{x \in P^{\mathrm{gp}} \mid t(x) \otimes \zeta_N = 0 \text{ for all } t \in P^{\mathrm{gp}\vee} \text{ such that } \overline{\rho}(t \otimes \zeta_N) = 0\} \\
&= \{x \in P^{\mathrm{gp}} \mid t(x) \in N\mathbb{Z} \text{ for all } t \in L\}.
\end{aligned}
$$

On the other hand, notice that the N-Frobenius of $P^{\mathrm{gp}\vee}$ factors through a map $\beta : P^{\mathrm{gp}\vee} \to L$ and the inclusion map $i : L \to P^{\mathrm{gp}\vee}$, and $\beta \circ i$ is the N-Frobenius map of L. Let $\omega : P^{\mathrm{gp}} \xrightarrow{\sim} (P^{\mathrm{gp}\vee})^\vee$ be the natural isomorphism. We may then write

$$
\begin{aligned}
Q^{\mathrm{gp}} &= \{x \in P^{\mathrm{gp}} \mid \omega(x)(t) \in N\mathbb{Z} \text{ for all } t \in L\} \\
&= \{x \in P^{\mathrm{gp}} \mid \omega(x) \circ i \in N \cdot L^\vee = i^\vee \circ \beta^\vee(L^\vee)\} \\
&= \omega^{-1}(\mathrm{Im}\,\beta^\vee).
\end{aligned}
$$

In other words, ω induces a natural isomorphism $Q^{\mathrm{gp}} \xrightarrow{\sim} L^\vee$, that identifies $\beta^\vee$ with the map $j^{\mathrm{gp}} : Q^{\mathrm{gp}} \to P^{\mathrm{gp}}$, and then necessarily also the map $i^\vee$ with ν. Now, the image of Q inside $L^\vee$ can be recovered just as $L^\vee \cap P^{\vee\vee}_{\mathbb{Q}}$ (the intersection here takes place in $L^\vee_{\mathbb{Q}}$, which contains $(P^{\mathrm{gp}\vee})^\vee$, via the injective map $i^\vee$: details left to the reader).

13.3.41. Our chief supply of tamely ramified coverings comes from the following source. Let $f : (Y, \underline{N}) \to (X, \underline{M})$ be an étale morphism of log schemes, whose underlying morphism of schemes is finite, and suppose that $(X, \underline{M})$ is regular. Then $(Y, \underline{N})$ is regular, and both X and Y are normal schemes (theorem 12.5.28 and corollary 12.5.13). Moreover, lemma 6.4.59(ii) and proposition 13.3.11 imply that $(Y, \underline{N})_{\mathrm{tr}} = f^{-1}(X, \underline{M})_{\mathrm{tr}}$, hence the restriction of f

$$f_{\mathrm{tr}} : (Y, \underline{N})_{\mathrm{tr}} \to (X, \underline{M})_{\mathrm{tr}}$$

is a finite étale morphism of schemes (corollary 12.3.27(i)). Furthermore, it follows easily from corollary 12.5.20(i) that the closed subset $X\setminus(X,\underline{M})_{\mathrm{tr}}$ is a union of irreducible closed subsets of codimension 1 in X (and the union is locally finite on the Zariski topology of X); the same holds also for $Y\setminus(Y,\underline{N})_{\mathrm{tr}}$, especially Y is the normalization of $(Y,\underline{N})_{\mathrm{tr}}$ over X. Finally, example 13.3.15 implies that f_{tr} is tamely ramified along $X\setminus(X,\underline{M})_{\mathrm{tr}}$. It is then clear that the rule $f\mapsto f_{\mathrm{tr}}$ defines a functor :

$$F_{(X,\underline{M})}:\mathsf{Cov}(X,\underline{M})\to\mathbf{Tame}(X,(X,\underline{M})_{\mathrm{tr}}).$$

It follows easily from remark 12.5.40 that $F_{(X,\underline{M})}$ is fully faithful; therefore, any choice of a geometric point ξ of $(X,\underline{M})_{\mathrm{tr}}$ determines a surjective group homomorphism :

$$\pi_1((X,\underline{M})_{\mathrm{tr,\acute{e}t}},\xi)\to\pi_1((X,\underline{M})_{\mathrm{\acute{e}t}},\xi) \tag{13.3.42}$$

([82, Exp.V, Prop.6.9]).

Proposition 13.3.43. *In the situation of* (13.3.27)*, suppose that* $(X,\underline{M})$ *is a regular log scheme. Then the map* (13.3.30) *factors through* (13.3.42)*, and induces an isomorphism :*

$$\pi_1((X,\underline{M})_{\mathrm{\acute{e}t}},\xi)\xrightarrow{\sim}\underline{M}^{\mathrm{gp}\vee}_{\overline{x}}\otimes_{\mathbb{Z}}\prod_{\ell\neq p}\mathbb{Z}_\ell(1). \tag{13.3.44}$$

Proof. The discussion of (13.3.40) shows that (13.3.30) factors through (13.3.42), and proposition 13.3.36 implies that (13.3.44) is surjective. Next, let $f:(Y,\underline{N})\to(X,\underline{M})$ be an étale covering; according to proposition 13.3.11, f admits a Kummer chart $(\omega_P,\omega_Q,\vartheta)$ with Q fine, sharp and saturated, such that the order k of $\mathrm{Coker}\,\vartheta^{\mathrm{gp}}$ is invertible in $\mathscr{O}_Y$. Moreover, the induced morphism of X-schemes $Y\to X\times_{\mathrm{Spec}\,\mathbb{Z}[P]}\mathrm{Spec}\,\mathbb{Z}[Q]$ is an isomorphism. With this notation, denote by G the cokernel of the induced group homomorphism $\mathrm{Hom}_{\mathbb{Z}}(\vartheta^{\mathrm{gp}},\boldsymbol{k}_{P^{\mathrm{gp}}})$; there follows a map $\rho:P^{\mathrm{gp}}\otimes_{\mathbb{Z}}\prod_{\ell\neq p}\mathbb{Z}_\ell(1)\to G$, and by inspecting the construction, one may check that the corresponding G_U-torsor, constructed as in (13.3.40), is isomorphic to $f\times_X\mathbf{1}_U$ (details left to the reader). This shows the injectivity of (13.3.44), and completes the proof of the proposition. □

The following result is the logarithmic version of the classical Abhyankar's lemma.

Theorem 13.3.45. *The functor* $F_{(X,\underline{M})}$ *is an equivalence.*

Proof. First we show how to reduce to the case where $\tau=\mathrm{\acute{e}t}$.

Claim 13.3.46. Let $(X_{\mathrm{Zar}},\underline{M})$ be a regular log structure on the Zariski site of X, and suppose that the theorem holds for $\widetilde{u}^*(X_{\mathrm{Zar}},\underline{M})$. Then the theorem holds for $(X_{\mathrm{Zar}},\underline{M})$ as well.

Proof of the claim. Let $g:V\to(X_{\mathrm{Zar}},\underline{M})_{\mathrm{tr}}$ be an étale covering whose normalization over X is tamely ramified along $X\setminus(X_{\mathrm{Zar}},\underline{M})_{\mathrm{tr}}$. By assumption, there exists an étale covering $f:(Y_{\mathrm{\acute{e}t}},\underline{N})\to\widetilde{u}^*(X_{\mathrm{Zar}},\underline{M})$ such that $f_{\mathrm{tr}}=g$; by lemma 13.3.10, we may then find a morphism of log schemes of Kummer type $f_{\mathrm{Zar}}:\widetilde{u}_*(Y_{\mathrm{\acute{e}t}},\underline{N})\to(X_{\mathrm{Zar}},\underline{M})$ such that $\widetilde{u}^*f_{\mathrm{Zar}}=f$, therefore f_{Zar} is an étale covering of $(X_{\mathrm{Zar}},\underline{M})$ (corollary 12.3.27(iii)), and clearly $(f_{\mathrm{Zar}})_{\mathrm{tr}}=g$, so the functor $F_{(X_{\mathrm{Zar}},\underline{M})}$ is essentially surjective. Full faithfulness for the same functor is derived formally from the full faithfulness of the functor $F_{\widetilde{u}^*(X_{\mathrm{Zar}},\underline{M})}$, and that of the functor (13.3.9) : details left to the reader. ◊

Henceforth we assume that $\tau=\mathrm{\acute{e}t}$.

Claim 13.3.47. Let $g:V\to(X,\underline{M})_{\mathrm{tr}}$ be an object of $\mathbf{Tame}(X,(X,\underline{M})_{\mathrm{tr}})$. Then g lies in the essential image of $F_{(X,\underline{M})}$ if (and only if) there exists an étale covering family $(U_\lambda\to X\mid\lambda\in\Lambda)$ of X, such that $g\times_X U_\lambda$ lies in the essential image of $F_{(U_\lambda,\underline{M}_{|U_\lambda})}$, for every $\lambda\in\Lambda$.

Proof of the claim. We have to exhibit a finite étale covering $f : (Y, \underline{N}) \to (X, \underline{M})$ such that $f_{\rm tr} = g$. However, given such f, theorem 12.5.28 and corollary 12.5.13 imply that Y is the normalization of V over X, and proposition 12.5.38 says that $\underline{N} = j_*\mathscr{O}_V^\times \cap \mathscr{O}_Y$, where $j : V \to Y$ is the open immersion; then $\log f : f^*\underline{M} \to \underline{N}$ is completely determined by f, hence by g. Thus, we come down to showing that :

(i) $\underline{N} := j_*\mathscr{O}_V^\times \cap \mathscr{O}_Y$ is a regular log structure on the normalization Y of V over X.

(ii) The unique morphism $f : (Y, \underline{N}) \to (X, \underline{M})$ is an étale covering.

However, [126, §33, Lemma 1] implies that Y is finite over X. To show (ii), it then suffices to prove that each restriction $f_\lambda := f \times_X \mathbf{1}_{U_\lambda}$ is étale (proposition 12.3.24(iii)). Likewise, (i) holds, provided the restriction $\underline{N}_{|Y_\lambda}$ is a regular log structure on $Y_\lambda := Y \times_X U_\lambda$, for every $\lambda \in \Lambda$. Let $j_\lambda : V_\lambda := V \times_X U_\lambda \to Y_\lambda$ be the induced open immersion. It is clear that $\underline{N}_{|Y_\lambda} = j_{\lambda *}\mathscr{O}_{V_\lambda}^\times \cap \mathscr{O}_{Y_\lambda}$, and Y_λ is the normalization of V_λ over U_λ (proposition 9.8.3); moreover, $(U_\lambda, \underline{M}_{|U_\lambda})$ is again regular, so our assumption implies that $(Y_\lambda, \underline{N}_{|Y_\lambda})$ is the unique regular log structure on Y_λ whose trivial locus is V_λ, and that f_λ is indeed étale. ◊

Claim 13.3.48. If $F_{(X(\xi), \underline{M}(\xi))}$ is essentially surjective for every geometric point ξ of X, the same holds for $F_{(X,\underline{M})}$.

Proof of the claim. Indeed, let $g : V \to (X, \underline{M})_{\rm tr}$ be an object of $\mathbf{Tame}(X, (X, \underline{M})_{\rm tr})$, and ξ any geometric point of X. By claim 13.3.47 it suffices to find an étale neighborhood $U \to X$ of ξ, such that $g \times_X \mathbf{1}_U$ lies in the essential image of $F_{(U, \underline{M}_{|U})}$. Denote by Y the normalization of X in V, which is a finite X-scheme ([126, §33, Lemma 1]). The scheme $Y(\xi) := Y \times_X X(\xi)$ decomposes as a finite disjoint union of strictly local open and closed subschemes $Y_1(\xi), \dots, Y_n(\xi)$, and we may find an étale neighborhood $U \to X$ of ξ, and a decomposition of $Y \times_X U$ by open and closed subschemes $Y_1, \dots, Y_n$, with isomorphisms of $X(\xi)$-schemes $Y_i(\xi) \xrightarrow{\sim} Y_i \times_X X(\xi)$ for every $i = 1, \dots, n$ ([65, Ch.IV, Cor.8.3.12]). We are then reduced to showing that all the restrictions $Y_i \to U$ lie in the essential image of $F_{(U, \underline{M}_{|U})}$. Thus, we may replace X by U, and Y by any Y_i, after which we may assume that $Y(\xi)$ is strictly local. Proposition 12.5.16 and theorem 12.5.31 imply that $(X(\xi), \underline{M}(\xi))$ is a regular log scheme. Clearly $g_\xi := g \times_X \mathbf{1}_{X(\xi)}$ is an object of $\mathbf{Tame}(X(\xi), (X(\xi), \underline{M}(\xi))_{\rm tr})$, so by assumption there exists a finite étale covering $h : (Z, \underline{N}) \to (X(\xi), \underline{M}(\xi))$ such that $h_{\rm tr} = g_\xi$. By theorem 12.5.28 and corollary 12.5.13 we know that Z is a normal scheme, and we deduce that $Z = Y(\xi)$ ([66, Ch.IV, Prop.17.5.7]).

According to proposition 13.3.11, the morphism h admits a fine and saturated Kummer chart $(\omega_P, \omega_Q, \vartheta : P \to Q)$, such that the order d of $\operatorname{Coker} \vartheta^{\rm gp}$ is invertible in $\mathscr{O}_{X,\xi}$, and such that the induced map $Y(\xi) \to X(\xi) \times_{\operatorname{Spec} P} \operatorname{Spec} Q$ is an isomorphism. By proposition 12.2.30, there exist an étale neighborhood $U \to X$ of ξ, and a coherent log structure $\underline{N}'$ on $Y' := Y \times_X U$ with an isomorphism $Y(\xi) \times_{Y'} (Y', \underline{N}') \xrightarrow{\sim} (Y(\xi), \underline{N})$. Moreover, let $h' : Y' \to U$ be the projection; after shrinking U, the map $\log h$ descends to a morphism of log structures $h'^*\underline{M}_{|U} \to \underline{N}'$, whence a morphism $(h', \log h') : (Y', \underline{N}') \to (U, \underline{M}_{|U})$ of log schemes, such that $h' \times_U \mathbf{1}_{X(\xi)} = h$. After further shrinking U, we may also assume that h' admits a Kummer chart $(\omega'_P, \omega'_Q, \vartheta)$ (corollary 12.2.36), that d is invertible in $\mathscr{O}_U$, and that the induced morphism $Y' \to U \times_{\operatorname{Spec} P} \operatorname{Spec} Q$ is an isomorphism ([65, Ch.IV, Cor.8.8.24]). Then h' is an étale covering, by proposition 13.3.11, and by construction, $F_{(U, \underline{M}_{|U})}(h') = g \times_X \mathbf{1}_U$, as desired. ◊

Claim 13.3.49. Assume that X is strictly local, denote by x the closed point of X, set $U := (X, \underline{M})_{\rm tr}$, and $P := \underline{M}_x^\sharp$. Let $h : V \to U$ be any connected non-empty étale covering, tamely ramified along $X \setminus U$; then h lies in the essential image of $F_{(X,\underline{M})}$, provided there exist a fine, sharp and saturated monoid Q, and a morphism $\nu : P \to Q$ of Kummer type, such that

$$h_\nu := h \times_X X_\nu : V \times_X X_\nu \to U \times_X X_\nu$$

admits a section (notation of (13.3.38)).

Proof of the claim. Suppose first that the order of $\mathrm{Coker}\,\nu^{\mathrm{gp}}$ is invertible in $\mathscr{O}_X$; in this case, f_ν is an étale covering of log schemes (proposition 13.3.11), hence $(f_\nu)_{\mathrm{tr}}$ is an étale covering of the scheme U. Then, by composing a section of h_ν with the projection $V \times_X X_\nu \to V$ we deduce a morphism $U \times_X X_\nu \to V$ of étale coverings of U. Such morphism shall be open and closed, hence surjective, since V is connected; hence the $\pi_1(U,\xi)$-set $h^{-1}(\xi)$ will be a quotient of $f_\nu^{-1}(\xi)$, on which $\pi_1(U,\xi)$ acts through (13.3.30), as required. Next, for a general morphism ν of Kummer type, let $L \subset Q^{\mathrm{gp}}$ be the largest subgroup such that $\nu^{\mathrm{gp}}(P^{\mathrm{gp}}) \subset L$, and $(L : P^{\mathrm{gp}})$ is invertible in $\mathscr{O}_X$. Set $Q' := L \cap Q$, and notice that $Q'^{\mathrm{gp}} = L$. Indeed, every element of L can be written in the form $x = b^{-1}a$, for some $a, b \in Q$; then choose $n > 0$ such that $b^n \in \nu P$, write $x = b^{-n} \cdot (b^{n-1}a)$ and remark that $b^{-n}, b^{n-1}a \in Q'$. The morphism ν factors as the composition of $\nu' : P \to Q'$ and $\psi : Q' \to Q$, and therefore f_ν factors through a morphism $f_\psi : X_\nu \to X_{\nu'}$. In view of the previous case, we are reduced to showing that the morphism $h_{\nu'}$ already admits a section, hence we may replace $(X, \underline{M})$ by $(X_{\nu'}, \underline{M}_{\nu'})$ (which is still regular and strictly local, by claim 13.3.37), h by $h_{\nu'}$, and ν by ψ, after which we may assume that the order of $\mathrm{Coker}\,\nu^{\mathrm{gp}}$ is p^m for some integer $m > 0$, where $p > 0$ is the characteristic of the residue field $\kappa(x)$, and then we need to show that h already admits a section.

Next, using again claim 13.3.37 and an easy induction, we may likewise reduce to the case where $m = 1$. Say that $X = \mathrm{Spec}\,A$, and suppose first that A is a $\mathbb{F}_p$-algebra (where $\mathbb{F}_p$ is the finite field with p elements), so that $X_\nu = \mathrm{Spec}\,A \otimes_{\mathbb{F}_p[P]} \mathbb{F}_p[Q]$; it is easily seen that the ring homomorphism $\mathbb{F}_p[\nu] : \mathbb{F}_p[P] \to \mathbb{F}_p[Q]$ is invertible up to Φ, in the sense of [75, Def.3.5.8(i)]. Especially, $\mathrm{Spec}\,\mathbb{F}_p[\nu]$ is integral, surjective and radicial, hence the same holds for f_ν, and therefore the morphism of sites :

$$f_\nu^* : X_{\text{ét}} \to X_{\nu,\text{ét}}$$

is an equivalence of categories (lemma 13.1.7(i)). It follows that in this case, h admits a section if and only if the same holds for h_ν.

Next, suppose that the field of fractions K of A has characteristic zero; we may write $X_\nu \times_X \mathrm{Spec}\,K = \mathrm{Spec}\,K_\nu$ and $V \times_X \mathrm{Spec}\,K = \mathrm{Spec}\,E$ for two field extensions K_ν and E of K, such that $[K_\nu : K] = p$, and the section of h_ν yields a map $E \to K_\nu$ of K-algebras. Therefore we have either $E = K$ (in which case $V = U$, and then we are done), or else $E = K_\nu$, in which case $V = (X_\nu, \underline{M}_\nu)_{\mathrm{tr}}$, since both these schemes are normal and finite over U.

Hence, let us assume that $V = (X_\nu, \underline{M}_\nu)$, and pick any point $\eta \in X$ of codimension one, whose residue field $\kappa(\eta)$ has characteristic p. Then $X_\nu \times_X X(\eta) = \mathrm{Spec}\,B$, where $B := \mathscr{O}_{X,\eta} \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q]$, and $\overline{B} := B \otimes_A \kappa(\eta) = \kappa(\eta) \otimes_{\mathbb{F}_p[P]} \mathbb{F}_p[Q]$. Since the map $\mathbb{F}_p[P] \to \mathbb{F}_p[Q]$ is invertible up to Φ, we easily deduce that $\overline{B}$ is local, and its residue field is a purely inseparable extension of $\kappa(\eta)$, say of degree d. Therefore $f_\nu^{-1}\eta$ consists of a single point η', and $\mathscr{O}_{X_\nu,\eta'} \simeq B$ is a normal and finite $\mathscr{O}_{X,\eta}$-algebra, hence it is a discrete valuation ring. Let e denote the ramification index of the extension $\mathscr{O}_{X,\eta} \to \mathscr{O}_{X_\nu,\eta'}$; then $ed = p$ ([34, Ch.VI, §8, n.5, Cor.1]). Suppose that $e = p$; in view of [75, Lemma 6.2.5], the ramification index of the induced extension of strict henselizations $\mathscr{O}^{\mathrm{sh}}_{X,\eta} \to \mathscr{O}^{\mathrm{sh}}_{X_\nu,\eta'}$ is still equal to p, which is impossible, since h is tamely ramified along $X \setminus U$. In case $e = 1$, we must have $d = p$, and since the residue field $\kappa(\eta)^{\mathrm{s}}$ of $\mathscr{O}^{\mathrm{sh}}_{X,\eta}$ is a separable closure of $\kappa(\eta)$, it follows easily that the residue field of $\mathscr{O}^{\mathrm{sh}}_{X_\nu,\eta'}$ must be a purely inseparable extension of $\kappa(\eta)^{\mathrm{s}}$ of degree p, which again contradicts the tameness of h. ◊

Claim 13.3.50. If $(X, \underline{M}) = (X, \underline{M})_2$, then $F_{(X,\underline{M})}$ is essentially surjective (notation of definition 12.2.7(i)).

Proof of the claim. By claim 13.3.48, we may assume that X is strictly henselian. In this case, let U, P and $h : V \to U$ be as in claim 13.3.49. By assumption $d := \dim P \leq 2$, and we have to show that $h \times_X X_\nu$ admits a section, for a suitable Kummer morphism $\nu : P \to Q$ (notation of (13.3.38)). If $d = 0$, then $P = \{1\}$, in which case $U = X$ is strictly local, so its fundamental group is trivial, and there is nothing to prove. In case $d = 1$, then $P \simeq \mathbb{N}$ (theorem 6.4.18(ii)), in which case X is a regular scheme (corollary 12.5.19), and $X \backslash U$ is a regular divisor (remark 12.5.15) and then the assertion follows from the classical Abhyankar's lemma ([82, Exp.XIII, Prop.5.2]). For $d = 2$, we may find $e_1, e_2 \in P$, and an integer $N > 0$ such that

$$\mathbb{N}e_1 \oplus \mathbb{N}e_2 \subset P \subset P' := \mathbb{N}\frac{e_1}{N} \oplus \mathbb{N}\frac{e_2}{N}$$

(see example 6.4.19(i)). Especially, the inclusion $\nu : P \to P'$ is a morphism of Kummer type. Since a composition of morphisms of Kummer type is obviously of Kummer type, claims 13.3.49 and 13.3.37 imply that we may replace $(X, \underline{M})$ by $(X_\nu, \underline{M}_\nu)$ and h by h_ν (notation of (13.3.38)), after which we may assume that P is isomorphic to $\mathbb{N}^{\oplus 2}$. In this case, X is again regular and $X \backslash U$ is a strict normal crossings divisor, so the assertion follows again from [82, Exp.XIII, Prop.5.2]. $\diamondsuit$

Claim 13.3.51. In the situation of (13.3.1), suppose that $(Y, \underline{N})$ is regular. Then the functor $f^*_{\mathrm{tr}} : \mathsf{Cov}(Y, \underline{N})_{\mathrm{tr}} \to \mathsf{Cov}(Y', \underline{N}')_{\mathrm{tr}}$ restricts to an equivalence :

$$\mathbf{Tame}(Y, (Y, \underline{N})_{\mathrm{tr}}) \xrightarrow{\sim} \mathbf{Tame}(Y', (Y', \underline{N}')_{\mathrm{tr}}). \tag{13.3.52}$$

Proof of the claim. Arguing as in the proof of proposition 13.3.2, we may reduce to the case where $\varphi : T' \to T$ is the saturated blow up of an ideal generated by two elements of $\Gamma(T, \mathscr{O}_T)$.

Notice that $(f, \log f)$ is an étale morphism (proposition 12.6.14(v)), and it restricts to an isomorphism $f_{\mathrm{tr}} : (Y', \underline{N}')_{\mathrm{tr}} \xrightarrow{\sim} (Y, \underline{N})_{\mathrm{tr}}$ (remark 12.6.17). Especially, $(Y', \underline{N}')$ is regular, and f^*_{tr} is trivially an equivalence from the étale coverings of $(Y, \underline{N})_{\mathrm{tr}}$ to those of $(Y', \underline{N}')_{\mathrm{tr}}$.

Let $g : V \to (Y, \underline{N})_{\mathrm{tr}}$ be an object of $\mathbf{Tame}(Y, (Y, \underline{N})_{\mathrm{tr}})$. By claim 13.3.50 and lemma 12.5.21, the functor $F_{\widetilde{u}^*(Y,\underline{N})_2}$ is essentially surjective, and then the same holds for $F_{(Y,\underline{N})_2}$, in view of claim 13.3.46; hence we may find an étale covering $(W, \underline{Q}) \to (Y, \underline{N})_2$, and an isomorphism $V \xrightarrow{\sim} (Y, \underline{N})_{\mathrm{tr}} \times_Y W$ of $(Y, \underline{N})_{\mathrm{tr}}$-schemes. On the other hand, example 6.5.56(ii) shows that φ restricts to a morphism $T'_1 \to T_2$ (notation of (6.5.16)), consequently f restricts to a morphism $(Y', \underline{N}')_1 \to (Y, \underline{N})_2$. There follows a well defined étale covering :

$$(Y', \underline{N}')_1 \times_{(Y,\underline{N})_2} (W, \underline{Q}) \to (Y', \underline{N}')_1.$$

whose image under $F_{(Y',\underline{N}')_1}$ is $f^*_{\mathrm{tr}}(g)$. However, $(Y', \underline{N}')_1$ contains all the points of height one of $Y' \backslash (Y', \underline{N}')_{\mathrm{tr}}$ (corollary 12.5.20(i)), hence $f^*_{\mathrm{tr}}(g)$ is tamely ramified along $Y' \backslash (Y', \underline{N}')_{\mathrm{tr}}$ (see (13.3.41)). This shows that (13.3.52) is well defined, and clearly this functor is fully faithful, since the same holds for f^*_{tr}. Lastly, let $y \in Y \setminus (Y, \underline{N})_{\mathrm{tr}}$ be any point of height one; by proposition 12.6.31(ii) and corollary 12.6.29, there exists a point $y' \in Y' \setminus (Y', \underline{N}')_{\mathrm{tr}}$ with $f(y') = y$, and since f restricts to an isomorphism f_{tr}, the corresponding map $\beta : \mathscr{O}_{Y,y} \to \mathscr{O}_{Y',y'}$ is local, injective, and induces an isomorphism on the respective fields of fractions. Since $\mathscr{O}_{Y,y}$ is a discrete valuation ring, it follows easily that β is an isomorphism, and therefore y' is a point of height one in Y'. Consequently, the equivalence $(f^{-1}_{\mathrm{tr}})^* : \mathsf{Cov}(Y', \underline{N}')_{\mathrm{tr}} \xrightarrow{\sim} \mathsf{Cov}(Y, \underline{N})_{\mathrm{tr}}$ induced by the isomorphism of schemes f^{-1}_{tr} restricts to a well defined functor $\mathbf{Tame}(Y', (Y', \underline{N}')_{\mathrm{tr}}) \to \mathbf{Tame}(Y, (Y, \underline{N})_{\mathrm{tr}})$ which yields a quasi-inverse for (13.3.52). $\diamondsuit$

Suppose again that $(X, \underline{M})$ is strictly henselian, define P, U and $h : V \to U$ as in claim 13.3.49, and set $T_P := (\mathrm{Spec}\, P)^\sharp$; it is easily seen that the counit of adjunction

$$\widetilde{u}^* \widetilde{u}_*(X, \underline{M}) \to (X, \underline{M})$$

is an isomorphism (cp. (12.6.48)), hence $\widetilde{u}_*(X, \underline{M})$ is regular (lemma 12.5.21). Let

$$\underline{X} := (\widetilde{u}_*(X, \underline{M}), T_P, \pi_X)$$

be the object of $\mathscr{K}$ arising from $\widetilde{u}_*(X, \underline{M})$, as in (12.6.7); by theorem 12.6.32 (and its proof), we may find an integral proper simplicial subdivision $\varphi : F \to T_P$, such that the morphism of schemes underlying $(f, \varphi) : \varphi^*\underline{X} \to \underline{X}$ is a resolution of singularities for X. For every integer $k \geq 1$ we define $(X_{(k)}, \underline{M}_{(k)})$, $(X_\varphi, \underline{M}_\varphi)$, $(Y, \underline{N})$, g_X and $\boldsymbol{k}_X$ as in example 13.3.39 : this makes sense, since, in the current situation, $\underline{X}$ is isomorphic to $X \times_S \underline{S}$ (where $\underline{S}$ is defined as in example 12.6.5(i)). Moreover, by the same token, the morphism of schemes $Y \to X_{(k)}$ is a resolution of singularities.

Set as well $U_\varphi := (X_\varphi, \underline{M}_\varphi)_{\mathrm{tr}}$ and $U_{(k)} := (X_{(k)}, \underline{M}_{(k)})_{\mathrm{tr}}$; by claim 13.3.51 and proposition 13.3.20(i), we have an essentially commutative diagram of functors :

$$\begin{array}{ccc} \mathbf{Tame}(X, U) & \xrightarrow{f^*_{\mathrm{tr}}} & \mathbf{Tame}(X_\varphi, U_\varphi) \\ {\scriptstyle \boldsymbol{k}^*_X}\downarrow & & \downarrow{\scriptstyle g^*_X} \\ \mathbf{Tame}(X_{(k)}, U_{(k)}) & \longrightarrow & \mathbf{Tame}(Y, (Y, \underline{N})_{\mathrm{tr}}) \end{array}$$

whose horizontal arrows are equivalences. Let k be the ramification index of h_φ along $X_\varphi \backslash U_\varphi$; in light of claim 13.3.49, we are reduced to showing that $\boldsymbol{k}^*_X(h)$ admits a section. Set $V_\varphi := V \times_U U_\varphi$ and $h_\varphi := f^*_{\mathrm{tr}}(h) : V_\varphi \to U_\varphi$; then it suffices to show that $g^*_X(h_\varphi)$ has a section.

Let $\overline{V}_\varphi \to X_\varphi$ be the normalization of h_φ over X_φ, and $\overline{W} \to Y$ the normalization of $\overline{V}_\varphi \times_{X_\varphi} Y$ in $V_\varphi \times_{X_\varphi} Y$. Choose a geometric point w of $\overline{W}$ whose support has height one; denote by y (resp. v, resp. x) the image of w in Y (resp. in $\overline{V}_\varphi$, resp. in X_φ), and suppose that the support of x lies in $X_\varphi \setminus U_\varphi$. Denote by $K(x)$ the field of fractions of the strictly henselian local ring $\mathscr{O}_{X,x}$, and define likewise $K(w)$, $K(y)$ and $K(v)$. There follows a commutative diagram of inclusions of valued fields :

$$\begin{array}{ccc} K(x) & \longrightarrow & K(y) \\ \downarrow & & \downarrow \\ K(v) & \longrightarrow & K(w) \end{array}$$

such that $K(w)$ is the compositum of $K(y)$ and $K(v)$: cp. the proof of proposition 13.3.20(i).

Example 13.3.39(ii) shows that the ramification index of $\mathscr{O}_{Y,y}$ over $\mathscr{O}_{X_\varphi,x}$ is k, which is a multiple of the ramification index of $\mathscr{O}_{\overline{V}_\varphi,v}$ over $\mathscr{O}_{X_\varphi,x}$. It then follows (*e.g.* from [75, Claim 6.2.15]) that $K(v) \subset K(y)$, and therefore $K(w) = K(y)$; this shows that the ramification index of $g^*_X(h_\varphi)$ along $Y \backslash (Y, \underline{N})_{\mathrm{tr}}$ equals 1, therefore $g^*_X(h_\varphi)$ is the restriction of an étale covering $\overline{h}$ of Y (proposition 13.3.20(iii)). By proposition 13.3.2(ii), $\overline{h}$ is the pull-back of an étale covering of $X_{(k)}$. Since $X_{(k)}$ is strictly local (claim 13.3.37), it follows that $\overline{h}$ admits a section, hence the same holds for $g^*_X(h_\varphi)$, as required. □

Remark 13.3.53. A proof of theorem 13.3.45 similar to the one given here can be found in [130, §2.3].

13.3.54. As an application of theorem 13.3.45, we shall determine the fundamental group of the "punctured" scheme obtained by removing the closed point from a strictly local regular log scheme $(X, \underline{M})$ (with non-trivial log structure) of dimension ≥ 2. Indeed, let $x \in X$ be the closed point, and set $r := \dim \underline{M}_x$. Also, let $\overline{y}$ be any geometric point of X, localized at a point $y \in U := X \setminus \{x\}$, and ξ a geometric point of X localized at the maximal point. Let p (resp. p') denote the characteristic exponent of $\kappa(x)$ (resp. of $\kappa(y)$). Pick any lifting of ξ to a geometric point ξ_y of $X(\overline{y})$; since $(X(\overline{y}), \underline{M}(\overline{y}))$ is regular (theorem 12.5.31), according to proposition 13.3.43, the vertical arrows of the commutative diagram in (13.3.31) factor through the surjections (13.3.42), and we get a commutative diagram of group homomorphisms :

$$\begin{array}{ccc} \pi_1((X(\overline{y}),\underline{M}(\overline{y}))_{\text{ét}},\xi_y) & \xrightarrow{\varphi_y} & \pi_1((X,\underline{M})_{\text{ét}},\xi) \\ \downarrow & & \downarrow \\ \underline{M}^{\text{gp}\vee}_{\overline{y}} \otimes_{\mathbb{Z}} \prod_{\ell\neq p'} \mathbb{Z}_\ell(1) & \longrightarrow & \underline{M}^{\text{gp}\vee}_x \otimes_{\mathbb{Z}} \prod_{\ell\neq p} \mathbb{Z}_\ell(1) \end{array}$$

whose vertical arrows are the isomorphisms (13.3.44), and whose top (resp. bottom) horizontal arrow is induced by the natural morphism $X(\overline{y}) \to X$ (resp. by the specialization map $\underline{M}_x \to \underline{M}_{\overline{y}}$). Now, by corollary 12.5.13, the scheme U is connected and normal, hence the restriction functor $\mathsf{Cov}(U) \to \mathbf{Tame}(X,(X,\underline{M})_{\text{tr}})$ is fully faithful (lemma 13.1.7(iii)); by [82, Exp.V, Prop.6.9] and theorem 13.3.45, it follows that the induced group homomorphism

(13.3.55) $$\pi_1((X,\underline{M})_{\text{ét}},\xi) \to \pi_1(U_{\text{ét}},\xi)$$

is surjective (see (13.3.41)). Clearly, the image of φ_y lies in the kernel of (13.3.55). Especially:

$$(U,\underline{M}_{|U})_r \neq \varnothing \Rightarrow \pi_1(U_{\text{ét}},\xi) = \{1\}$$

since, for any geometric point $\overline{y}$ of $(X,\underline{M})_r$, the specialization map $\underline{M}_x \to \underline{M}_{\overline{y}}$ is an isomorphism (notation of definition 12.2.7(i)). Thus, suppose that $(X,\underline{M})_r = \{x\}$, set $P := \underline{M}^\sharp_x$, and denote by $\psi : X \to \operatorname{Spec} P$ the natural continuous map; we have $\underline{M}^\sharp_{\overline{y}} = P_{\psi(y)}$, and if we let $F_y := P \setminus \psi(y)$, we get a short exact sequence of free abelian groups of finite rank :

(13.3.56) $$0 \to F^{\text{gp}}_y \to \underline{M}^{\sharp\text{gp}}_x \to \underline{M}^{\sharp\text{gp}}_{\overline{y}} \to 0.$$

It is easily seen that ψ is surjective; hence, for any $\mathfrak{p} \in \operatorname{Spec} P$ of height $r-1$, pick $y \in \psi^{-1}(\mathfrak{p})$. With this choice, we have $F^{\text{gp}}_y = \mathbb{Z}$; considering $(13.3.56)^\vee$, we deduce that $\pi_1(U_{\text{ét}},\xi)$ is a quotient of $\widehat{\mathbb{Z}}'(1) := \prod_{\ell\neq p} \mathbb{Z}_\ell(1)$. More precisely, let $(\operatorname{Spec} P)_{r-1}$ be the set of prime ideals of P of height $r-1$; then we have :

Theorem 13.3.57. *If $(X,\underline{M})_r = \{x\}$, we have a natural identification :*

(13.3.58) $$\frac{P^\vee}{\sum_{\mathfrak{p}\in(\operatorname{Spec} P)_{r-1}} P^\vee_{\mathfrak{p}}} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}'(1) \xrightarrow{\sim} \pi_1(U_{\text{ét}},\xi)$$

and these two abelian groups are cyclic of finite order prime to p.

Proof. The foregoing discussion already yields a natural surjective map as stated; it remains only to check the injectivity, and to verify that the source is a cyclic finite group.

However, since $d \geq 2$, and $(X,\underline{M})_r = \{x\}$, corollary 12.5.20(i) implies that $r \geq 2$, in which case $(\operatorname{Spec} P)_{r-1}$ must contain at least two distinct elements, say $\mathfrak{p}$ and $\mathfrak{q}$. Let $F := P \setminus \mathfrak{p}$; the image H of $(P_{\mathfrak{q}})^{\sharp\text{gp}\vee}$ in $F^{\text{gp}\vee}$ is clearly a non-trivial subgroup, hence $F^{\text{gp}\vee}/H$ is a cyclic finite group, and then the same holds for the source of (13.3.58).

Next, it follows easily from lemma 13.1.7(iii) that (13.3.55) identifies $\pi_1(U_{\text{ét}},\xi)$ with the quotient of $\pi_1((X,\underline{M})_{\text{ét}},\xi)$ by the sum of the images of the maps φ_y, for y ranging over all the points of U. However, it is clear that the same sum is already spanned by the sum of the images of the φ_y such that $y \in (X,\underline{M})_{r-1}$; whence the theorem. ☐

Example 13.3.59. Let $P \subset \mathbb{N}^{\oplus 2}$ be the submonoid of all pairs (a,b) such that $a+b \in 2\mathbb{N}$. Clearly P is fine and saturated of dimension 2. Let K be any field; the K-scheme $X := \operatorname{Spec} K[P]$ is the singular quadric in $\mathbb{A}^3_K$ cut by the equation $XY - Z^2 = 0$. It is easily seen that $\operatorname{Spec}(K,P)_2$ consists of a single point x (the vertex of the cone); let $\overline{x}$ be a geometric point of X localized at x, and $U \subset X(\overline{x})$ the complement of the closed point. Then U is a normal K-scheme of dimension 1, and we have a natural isomorphism

$$\pi_1(U,\xi) \xrightarrow{\sim} \mathbb{Z}/2\mathbb{Z}.$$

Indeed, a simple inspection shows that P admits exactly two prime ideals of height one, namely $\mathfrak{p} := P \setminus (2\mathbb{N} \oplus \{0\})$ and $\mathfrak{q} := P \setminus (\{0\} \oplus 2\mathbb{N})$. Then, $P_\mathfrak{p} = \{(a,b) \in \mathbb{Z} \oplus \mathbb{N} \mid a+b \in 2\mathbb{N}\}$, and similarly $P_\mathfrak{q}$ is a submonoid of $\mathbb{N} \oplus \mathbb{Z}$. The quotients $P^\sharp_\mathfrak{p}$ and $P^\sharp_\mathfrak{q}$ are both isomorphic to $\mathbb{N}$, and are both generated by the class of $(1,1)$. Let $\varphi : P^\sharp_\mathfrak{p} \to \mathbb{Z}$ be a map of monoids; then the image of φ in $P^{\mathrm{gp}\vee}$ is the unique map of monoids $P \to \mathbb{Z}$ given by the rule : $(2,0) \mapsto 0$, $(1,1) \mapsto \varphi(1,1)$, $(0,2) \mapsto 2\varphi(1,1)$. Likewise, a map $\psi : P^\sharp_\mathfrak{q} \to \mathbb{Z}$ gets sent to the morphism $P \to \mathbb{Z}$ such that $(2,0) \mapsto 2\psi(1,1)$, $(1,1) \mapsto \psi(1,1)$, and $(0,2) \mapsto 0$. We see therefore that $(P_\mathfrak{p})^{\sharp\mathrm{gp}\vee} + (P_\mathfrak{q})^{\sharp\mathrm{gp}\vee}$ is a subgroup of index two in $P^{\mathrm{gp}\vee}$, and the contention follows from theorem 13.3.57.

13.4. **Local acyclicity of smooth morphisms of log schemes.** In this section we consider a smooth and saturated morphism $f : (X, \underline{M}) \to (Y, \underline{N})$ of fine log schemes. We fix a geometric point $\overline{x}$ of X, localized at a point x, and let $\overline{y} := f(\overline{x})$. We shall suppose as well that Y is strictly local and normal, that $(Y, \underline{N})_{\mathrm{tr}}$ is a dense open subset of Y, and that $\overline{y}$ is localized at the closed point y of Y. Let $\overline{\eta}$ be a strict geometric point of Y, localized at the generic point η; to ease notation, set :

$$U := f^{-1}_{\overline{x}}(\eta) \qquad U_{\mathrm{tr}} := (X(\overline{x}), \underline{M}(\overline{x}))_{\mathrm{tr}} \cap U \qquad \overline{U} := U \times_{|\eta|} |\overline{\eta}| \qquad \overline{U}_{\mathrm{tr}} := U_{\mathrm{tr}} \times_{|\eta|} |\overline{\eta}|$$

and notice that U_{tr} is a dense open subset of U, by virtue of proposition 12.7.17(ii,iv). Choose a geometric point ξ of $\overline{U}_{\mathrm{tr}}$, and let ξ' be the image of ξ in U_{tr}. There follows a short exact sequence of topological groups :

$$1 \to \pi_1(\overline{U}_{\mathrm{tr,ét}}, \xi) \to \pi_1(U_{\mathrm{tr,ét}}, \xi') \xrightarrow{\alpha} \pi_1(|\eta|_{\mathrm{ét}}, \overline{\eta}) \to 1.$$

On the other hand, let p be the characteristic exponent of $\kappa := \kappa(\overline{y})$; for every integer e such that $(e,p) = 1$, the discussion of (13.3.27) yields a commutative diagram :

(13.4.1)
$$\begin{array}{ccccc}
\underline{N}_{\overline{y}} & \longrightarrow & \kappa(\eta)^\times & \longrightarrow & \mathrm{Hom}_{\mathrm{cont}}(\pi_1(|\eta|_{\mathrm{ét}}, \overline{\eta}), \boldsymbol{\mu}_e(\kappa)) \\
{\scriptstyle \log f_{\overline{x}}}\downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{Hom}_{\mathrm{cont}}(\alpha, \boldsymbol{\mu}_e(\kappa))} \\
\underline{M}_{\overline{x}} & \longrightarrow & \Gamma(U_{\mathrm{tr}}, \mathscr{O}^\times_U) & \longrightarrow & \mathrm{Hom}_{\mathrm{cont}}(\pi_1(U_{\mathrm{tr,ét}}, \xi'), \boldsymbol{\mu}_e(\kappa))
\end{array}$$

such that the composition of the two top (rep. bottom) horizontal arrows factors through $\underline{N}^\sharp_{\overline{y}}$ (resp. $\underline{M}^\sharp_{\overline{x}}$). There follows a system of natural group homomorphisms :

(13.4.2)
$$\mathrm{Coker}\,(\log f)^{\mathrm{gp}}_{\overline{x}} \to \mathrm{Hom}_{\mathrm{cont}}(\pi_1(\overline{U}_{\mathrm{tr,ét}}, \xi), \boldsymbol{\mu}_e(\kappa)) \qquad \text{where } (e,p) = 1$$

which assemble into a group homomorphism :

(13.4.3)
$$\pi_1(\overline{U}_{\mathrm{tr,ét}}, \xi) \to \mathrm{Coker}\,(\log f^{\mathrm{gp}}_{\overline{x}})^\vee \otimes_\mathbb{Z} \prod_{\ell \neq p} \mathbb{Z}_\ell(1).$$

13.4.4. We wish to give a second description of the map (13.4.3), analogous to the discussion in (13.3.32). To this aim, let R be a ring; for any monoid P, and any integer $e > 0$, denote by $\boldsymbol{e}_P : P \to P$ the e-Frobenius map of P. Let $\lambda : P \to Q$ be a local morphism of finitely generated monoids. For any integer $e > 0$, we get a commutative diagram of monoids :

(13.4.5)
$$\begin{array}{ccccc}
P & \xrightarrow{\boldsymbol{e}_P} & P & & \\
{\scriptstyle\lambda}\downarrow & & \downarrow{\scriptstyle\lambda_e} & \searrow^{\lambda} & \\
Q & \xrightarrow{\mu'} & Q' & \xrightarrow{\boldsymbol{e}_{Q|P}} & Q
\end{array}$$

whose square subdiagram is cocartesian, and such that $\boldsymbol{e}_{Q|P} \circ \mu' = \boldsymbol{e}_Q$. The latter induces a commutative diagram of log schemes :

$$\begin{array}{ccccc} \mathrm{Spec}(R,Q) & \xrightarrow{g_{Q|P}} & \mathrm{Spec}(R,Q') & \xrightarrow{g'} & \mathrm{Spec}(R,Q) \\ & \searrow^{g} & \downarrow{\scriptstyle g_e} & & \downarrow{\scriptstyle g} \\ & & \mathrm{Spec}(R,P) & \xrightarrow{g_P} & \mathrm{Spec}(R,P) \end{array}$$

whose square subdiagram is cartesian, and such that $g_Q := g' \circ g_{Q|P}$ is the morphism induced by $\boldsymbol{e}_Q$. Also, by example 12.6.5, we have a commutative diagram of monoidal spaces :

$$\begin{array}{ccccc} \mathrm{Spec}(R,Q) & \xrightarrow{g_{Q|P}} & \mathrm{Spec}(R,Q') & \xrightarrow{g_e} & \mathrm{Spec}(R,P) \\ \downarrow{\scriptstyle \psi_Q} & & \downarrow{\scriptstyle \psi_{Q'}} & & \downarrow{\scriptstyle \psi_P} \\ T_Q & \xrightarrow{\varphi} & T_{Q'} & \longrightarrow & T_P \end{array}$$

(where $\varphi := (\mathrm{Spec}\, \boldsymbol{e}_{Q|P})^\sharp$) which determines morphisms in the category $\mathscr{K}$:

$$(\mathrm{Spec}(R,Q), T_Q, \psi_Q) \to (\mathrm{Spec}(R,Q'), T_{Q'}, \psi_{Q'}) \to (\mathrm{Spec}(R,P), T_P, \psi_P).$$

13.4.6. Now, suppose that the closed point $\mathfrak{m}_Q$ of T_Q lies in the strict locus of $(\mathrm{Spec}\, \lambda)^\sharp$ (which just means that $\lambda^\sharp$ is an isomorphism). Notice that the functor $M \mapsto M^\sharp$ commutes with colimits (since it is a left adjoint); taking into account lemma 6.1.13, we deduce that the square subdiagram of $(13.4.5)^\sharp$ is still cocartesian, and therefore $\boldsymbol{e}^\sharp_{Q|P}$ is an isomorphism, so the same holds for φ.

More generally, let $\mathfrak{q} \in \mathrm{Spec}\, Q$ be any prime ideal in the strict locus of $(\mathrm{Spec}\, \lambda)^\sharp$; set $\mathfrak{p} := \lambda^{-1}\mathfrak{q}$, $\mathfrak{r} := \varphi(\mathfrak{q})$, and recall that $T_{Q_\mathfrak{q}} := \mathrm{Spec}\, Q_\mathfrak{q}$ is naturally an open subset of T_Q (see (6.5.16)). A simple inspection shows that the restriction $T_{Q_\mathfrak{q}} \to T_{Q'_\mathfrak{r}}$ of φ is naturally identified with the morphism of affine fans $(\mathrm{Spec}\, \boldsymbol{e}_{Q_\mathfrak{q}|P_\mathfrak{p}})^\sharp$. Since $\mathfrak{q}$ is the closed point of $T_{Q_\mathfrak{q}}$, the foregoing shows that $T_{Q_\mathfrak{q}}$ lies in the strict locus of φ; in other words, $\mathrm{Str}(\varphi)$ is an open subset of T_Q, and we have

$$\mathrm{Str}((\mathrm{Spec}\, \lambda)^\sharp) \subset \mathrm{Str}(\varphi). \tag{13.4.7}$$

Moreover, recall that $\mathrm{Spec}\, \boldsymbol{e}_Q : T_Q \to T_Q$ is the identity on the underlying topological space (see example 6.5.10(i)), and by construction it factors through φ, so the latter is injective on the underlying topological spaces. Especially, $\mathrm{Str}(\varphi) = \varphi^{-1}\varphi(\mathrm{Str}(\varphi))$, and therefore

$$\mathrm{Str}(g_{Q|P}) = \psi_Q^{-1}(\mathrm{Str}(\varphi)) = g^{-1}_{Q|P}(\psi^{-1}_{Q'}\varphi(\mathrm{Str}(\varphi))). \tag{13.4.8}$$

Hence, set $g := \mathrm{Spec}(R, \lambda)$; from (13.4.7) and (12.6.3) we obtain :

$$\mathrm{Str}(g) \subset \mathrm{Str}(g_{Q|P})$$

and together with (13.4.8) we deduce that :

- $\mathrm{Str}(g_{Q|P})$ is an open subset of $\mathrm{Spec}\, R[Q]$.
- $\psi^{-1}_{Q'}\varphi(\mathrm{Str}(\varphi))$ is a locally closed subscheme of $\mathrm{Spec}\, R[Q']$.
- The restriction $\mathrm{Str}(g) \to \psi^{-1}_{Q'}\varphi(\mathrm{Str}(\varphi))$ of $g_{Q|P}$ is a finite morphism.

Lastly, suppose that e is invertible in R; in this case, the morphisms g_P and g_Q are étale (proposition 12.3.34), so the same holds for $g_{Q|P}$ (corollary 12.3.25(iii)). From corollary 12.3.27(i) we deduce that the restriction $\mathrm{Str}(g) \to \psi^{-1}_{Q'}\varphi(\mathrm{Str}(\varphi))$ of $g_{Q|P}$ is a (finite) étale covering.

13.4.9. In the situation of (13.4.4), suppose additionally, that R is a $\mathbb{Z}[1/e, \boldsymbol{\mu}_e]$-algebra (where $\boldsymbol{\mu}_e$ is the e-torsion subgroup of $\mathbb{C}^\times$), P and Q are fine monoids, and λ is integral, so that Q' is also fine. Set

$$G_P := \mathrm{Hom}_{\mathbb{Z}}(P^{\mathrm{gp}}, \boldsymbol{\mu}_e) \qquad G_Q := \mathrm{Hom}_{\mathbb{Z}}(Q^{\mathrm{gp}}, \boldsymbol{\mu}_e) \qquad G_{Q|P} := \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Coker}\, \lambda^{\mathrm{gp}}, \boldsymbol{\mu}_e).$$

Notice that the trivial locus $\mathrm{Spec}(R, P)_{\mathrm{tr}}$ is the open subset $\mathrm{Spec}\, R[P^{\mathrm{gp}}]$, and likewise for $\mathrm{Spec}(R, Q)_{\mathrm{tr}}$ and $\mathrm{Spec}(R, Q')_{\mathrm{tr}}$; therefore, (13.4.5)$^{\mathrm{gp}}$ induces a cartesian diagram of schemes

$$\begin{array}{ccc} \mathrm{Spec}(R, Q')_{\mathrm{tr}} & \xrightarrow{g'_{\mathrm{tr}}} & \mathrm{Spec}(R, Q)_{\mathrm{tr}} \\ {\scriptstyle g_{e,\mathrm{tr}}}\downarrow & & \downarrow{\scriptstyle g_{\mathrm{tr}}} \\ \mathrm{Spec}(R, P)_{\mathrm{tr}} & \xrightarrow{g_{P,\mathrm{tr}}} & \mathrm{Spec}(R, P)_{\mathrm{tr}}. \end{array}$$

Fix a geometric point τ'_Q of $\mathrm{Spec}(R, Q')_{\mathrm{tr}}$, and let $\tau_Q := g'(\tau'_Q)$, $\tau_P := g(\tau_Q)$, $\tau'_P := g_e(\tau'_Q)$. It was shown in (13.3.32) that $g_P^{-1}(\tau_P)$ is a G_P-torsor, so $g_{P,\mathrm{tr}}$ is a Galois étale covering, corresponding to a continuous representation (13.3.33) into G_P. Hence $g'^{-1}(\tau_Q)$ is a G_P-torsor, and g'_{tr} is a Galois étale covering, whose corresponding representation of $\pi_1(\mathrm{Spec}(R, Q)_{\mathrm{tr,ét}}, \tau_Q)$ is obtained by composing (13.3.33) with the natural continuous group homomorphism

$$\pi_1(g_{\mathrm{tr}}, \tau_Q) : \pi_1(\mathrm{Spec}(R, Q)_{\mathrm{tr,ét}}, \tau_Q) \to \pi_1(\mathrm{Spec}(R, P)_{\mathrm{tr,ét}}, \tau_P).$$

By the same token, $g_Q^{-1}(\tau_Q)$ is a G_Q-torsor, so also $g_{Q,\mathrm{tr}}$ is a Galois étale covering, and a simple inspection shows that the surjection

$$g_Q^{-1}(\tau_Q) \to g'^{-1}(\tau_Q)$$

induced by $g_{Q|P}$ is G_Q-equivariant, for the G_Q-action on the target obtained from the map

$$\mathrm{Hom}_{\mathbb{Z}}(\lambda^{\mathrm{gp}}, \boldsymbol{\mu}_e) : G_Q \to G_P.$$

The situation is summarized by the commutative diagram of continuous group homomorphisms

$$\begin{array}{ccccc} \pi_1(\mathrm{Spec}(R, Q')_{\mathrm{tr,ét}}, \tau'_Q) & \longrightarrow & \pi_1(\mathrm{Spec}(R, Q)_{\mathrm{tr,ét}}, \tau_Q) & \longrightarrow & G_Q \\ {\scriptstyle \pi_1(g_{e,\mathrm{tr}}, \tau'_Q)}\downarrow & & {\scriptstyle \pi_1(g_{\mathrm{tr}}, \tau_Q)}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}_{\mathbb{Z}}(\lambda^{\mathrm{gp}}, \boldsymbol{\mu}_e)} \\ \pi_1(\mathrm{Spec}(R, P)_{\mathrm{tr,ét}}, \tau'_P) & \longrightarrow & \pi_1(\mathrm{Spec}(R, P)_{\mathrm{tr,ét}}, \tau_P) & \longrightarrow & G_P \end{array}$$

whose horizontal right-most arrows are the maps (13.3.33). Consequently, $g_{Q|P}^{-1}(\tau'_Q)$ is a $G_{Q|P}$-torsor, and $g_{Q|P,\mathrm{tr}}$ is a Galois étale covering, classified by a continuous group homomorphism

$$\mathrm{Ker}\, \pi_1(g_{e,\mathrm{tr}}, \tau'_Q) \to \mathrm{Ker}\, \pi_1(g_{\mathrm{tr}}, \tau_Q) \to G_{Q|P}. \tag{13.4.10}$$

13.4.11. Let us return to the situation of (13.4), and assume additionally, that both $(X, \underline{M})$ and $(Y, \underline{N})$ are fs log schemes. Take $R := \mathscr{O}_{Y,\overline{y}}$ in (13.4.4); by corollary 12.3.42 and theorem 12.1.37(iii), we may assume that there exist

- a local and saturated morphism $\lambda : P \to Q$ of fine and saturated monoids, such that P is sharp, $Q^\times$ is a free abelian group of finite type, say of rank r, and $\mathrm{Ass}_{\mathbb{Z}}\, \mathrm{Coker}\, \lambda^{\mathrm{gp}}$ does not contain the characteristic exponent of $\kappa(\overline{y})$;
- a morphism of schemes $\pi : Y \to \mathrm{Spec}\, R[P]$, which is a section of the projection $\mathrm{Spec}\, R[P] \to Y$, such that

  $$(Y, \underline{N}) = Y \times_{\mathrm{Spec}\, R[P]} \mathrm{Spec}(R, P) \qquad (X, \underline{M}) = Y \times_{\mathrm{Spec}\, R[P]} \mathrm{Spec}(R, Q)$$

  and f is obtained by base change from the morphism $g := \mathrm{Spec}(R, \lambda)$. Moreover, the induced chart $Q_X \to \underline{M}$ shall be local at the geometric point $\overline{x}$.

By claim 12.1.39, we may further assume that the projection $Q \to Q^\sharp$ admits a section $\sigma : Q^\sharp \to Q$, such that $\lambda(P)$ lies in the image of σ. In this case, g factors through the morphism $\mathrm{Spec}(R,P) \to \mathrm{Spec}(R,Q^\sharp)$ induced by $\lambda^\sharp$, and σ induces an isomorphism of log schemes :

$$\mathrm{Spec}(R,Q) = \mathbb{G}_{m,Y}^{\oplus r} \times_Y \mathrm{Spec}(R,Q^\sharp)$$

(where $\mathbb{G}_{m,Y}^{\oplus r}$ denotes the standard torus of rank r over Y). In this situation, X is smooth over $Y \times_{\mathrm{Spec}\,R[P]} \mathrm{Spec}\,R[Q^\sharp]$, and more precisely

$$(X,\underline{M}) = \mathbb{G}_{m,Y}^{\oplus r} \times_{\mathrm{Spec}\,R[P]} \mathrm{Spec}(R,Q^\sharp). \tag{13.4.12}$$

Summing up, after replacing Q by $Q^\sharp$, we may assume that Q is also sharp, and (13.4.12) holds with $\mathrm{Spec}(R,Q)$ instead of $\mathrm{Spec}(R,Q^\sharp)$. Moreover, the image of x in T_Q is the closed point $\mathfrak{m}_Q$.

Let $e > 0$ be an integer which is invertible in R; by inspecting the definition, it is easily seen that there exists a finite separable extension K_e of $\kappa(\eta) = \mathrm{Frac}(R)$, such that the normalization Y_e of Y in $\mathrm{Spec}\,K_e$ fits into a commutative diagram of schemes :

$$\begin{array}{ccc} Y_e & \longrightarrow & Y \\ {\scriptstyle\pi_e}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ \mathrm{Spec}\,R[P] & \xrightarrow{g_P} & \mathrm{Spec}\,R[P] \end{array}$$

(whose top horizontal arrow is the obvious morphism); namely, π is defined by some morphism of monoids $\beta : P \to R$, and one takes for K_e any subfield of $\kappa(\overline{\eta})$ containing $\kappa(\eta)$ and the e-th roots of the elements of $\beta(P)$. Set $(Y_e,\underline{N}_e) := Y_e \times_{\mathrm{Spec}\,R[P]} \mathrm{Spec}(R,P)$, and define log schemes $(X_e,\underline{M}_e)$, $(X'_e,\underline{M}'_e)$ so that the two square subdiagrams of the diagram of log schemes

$$\begin{array}{ccccc} (X'_e,\underline{M}'_e) & \xrightarrow{h_e} & (X_e,\underline{M}_e) & \longrightarrow & \mathbb{G}_{m,Y}^{\oplus r} \times_Y (Y_e,\underline{N}_e) \\ \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle\pi'_e} \\ \mathrm{Spec}(R,Q) & \xrightarrow{g_{Q|P}} & \mathrm{Spec}(R,Q') & \xrightarrow{g_e} & \mathrm{Spec}(R,P) \end{array}$$

are cartesian (here π'_e is the composition of π_e and the projection $\mathbb{G}_{m,Y}^{\oplus r} \times_Y Y_e \to Y_e$), whence a commutative diagram :

$$\begin{array}{ccc} (X'_e,\underline{M}'_e) & \xrightarrow{h_e} & (X_e,\underline{M}_e) \\ & {\scriptstyle f'_e}\searrow & \big\downarrow{\scriptstyle f_e} \\ & & (Y_e,\underline{N}_e). \end{array} \tag{13.4.13}$$

Notice that both f_e and f'_e are smooth and saturated morphisms of fine log schemes. Also, by construction Y_e is strictly local, and $(Y_e,\underline{N}_e)_{\mathrm{tr}}$ is a dense subset of Y_e. In other words, f_e and f'_e are still of the type considered in (13.4). Moreover, h_e is étale, since the same holds for $g_{Q|P}$, and the discussion in (13.4.6) shows that the restriction

$$\mathrm{Str}(f'_e) \to X_e$$

is an étale morphism of schemes. Furthermore, the discussion in (13.4.9) shows that

$$h_{e,\mathrm{tr}} : (X'_e,\underline{M}'_e)_{\mathrm{tr}} \to (X_e,\underline{M}_e)_{\mathrm{tr}}$$

is a Galois étale covering.

13.4.14. More precisely, notice that $\lambda^\sharp = \log f^\sharp_{\overline{x}}$; combining with (13.4.10), we deduce a continuous group homomorphism :

$$\mathrm{Ker}\, \pi_1(f_{e,\mathrm{tr}}, \xi'_e) \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Coker}\,(\log f)^{\mathrm{gp}}_{\overline{x}}, \boldsymbol{\mu}_e(\kappa)) \qquad (13.4.15)$$

where ξ'_e is the image in X_e of the geometric point ξ. The geometric point $\overline{y}$ lifts uniquely to a geometric point $\overline{y}_e$ of Y_e, localized at the closed point y_e, and the pair $(\overline{x}, \overline{y}_e)$ determines a unique geometric point $\overline{x}_e$ such that $f_e(\overline{x}_e) = \overline{y}_e$. Also, since the field extension $\kappa(y) \to \kappa(y_e)$ is purely inseparable, it is easily seen that the induced map $f_e^{-1}(y_e) \to f^{-1}(y)$ is a homeomorphism; there follows an isomorphism of $X(\overline{x})$-schemes ([66, Ch.IV, Prop.18.8.10]) :

$$X_e(\overline{x}_e) \stackrel{\sim}{\to} X(\overline{x}) \times_Y Y_e.$$

Let η_e be the generic point of Y_e; by construction, $\overline{\eta}$ lifts to a geometric point $\overline{\eta}_e$ of Y_e, localized at η_e, and we have continuous group homomorphisms :

$$\pi_1(\overline{U}_{\mathrm{tr}}, \xi) \stackrel{\sim}{\to} \mathrm{Ker}\,(\pi_1(U_{e,\mathrm{tr}}, \xi'_e) \to \pi_1(\eta_e, |\overline{\eta}_e|)) \to \mathrm{Ker}\, \pi_1(f_{e,\mathrm{tr}}, \xi'_e). \qquad (13.4.16)$$

The composition of (13.4.15) and (13.4.16) is a continuous group homomorphism

$$\pi_1(\overline{U}_{\mathrm{tr}}, \xi) \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Coker}\,(\log f)^{\mathrm{gp}}_{\overline{x}}, \boldsymbol{\mu}_e(\kappa)) \qquad (13.4.17)$$

whence, finally, a pairing

$$\mathrm{Coker}\,(\log f)^{\mathrm{gp}}_{\overline{x}} \times \pi_1(\overline{U}_{\mathrm{tr,ét}}, \xi) \to \boldsymbol{\mu}_e(\kappa).$$

We claim that this pairing agrees with the one deduced from (13.4.2). Indeed, by tracing back through the constructions, we see that (13.4.17) is the homomorphism arising from the Galois covering of $\overline{U}_{\mathrm{tr}}$, which is obtained from $g_{Q|P}$, after base change along the composition

$$\overline{U}_{\mathrm{tr}} \to X_e \to \mathrm{Spec}\, R[Q'].$$

On the other hand, the discussion of (13.3.32) shows that the homomorphism $\pi_1(U_{\mathrm{tr,ét}}, \xi') \to G_Q$ arising from the bottom row of (13.4.1), classifies the Galois covering $C \to \overline{U}_{\mathrm{tr}}$ obtained from g_Q, by base change along the same map. By the same token, the top row of (13.4.1) corresponds to the G_P-Galois covering $C' \to |\eta|$ obtained by base change of g_P along the composition $|\eta| \to Y \to S_P$. The map $\log f_{\overline{x}}$ induces a morphism of schemes $C \to C' \times_{|\eta|} U_{\mathrm{tr}}$, and (13.4.2) corresponds to the $G_{Q|P}$-torsor obtained from a fibre of this morphism. Evidently, this torsor is isomorphic to $g^{-1}_{Q|P}(\tau'_Q)$, whence the contention.

13.4.18. In the situation of (13.4), recall that there is a natural bijection between the set of maximal points of $f^{-1}_{\overline{x}}(\overline{y})$, and the set Σ of maximal points of the closed fibre of the induced map

$$\mathrm{Spec}\, \underline{M}_{\overline{x}} \to \mathrm{Spec}\, \underline{N}_{\overline{y}} \qquad (13.4.19)$$

(proposition 12.7.14). For every $\mathfrak{q} \in \Sigma$, denote by $\eta_{\mathfrak{q}}$ the corresponding maximal point of $f^{-1}_{\overline{x}}(\overline{y})$, choose a geometric point $\overline{\eta}_{\mathfrak{q}}$ localized at $\eta_{\mathfrak{q}}$, and let $X(\overline{\eta}_{\mathfrak{q}})$ be the strict henselization of $X(\overline{x})$ at $\overline{\eta}_{\mathfrak{q}}$. Also, set

$$U_{\mathfrak{q}} := U \times_{X(\overline{x})} X(\overline{\eta}_{\mathfrak{q}}) \qquad \overline{U}_{\mathfrak{q}} := U_{\mathfrak{q}} \times_{|\eta|} |\overline{\eta}|$$

and notice that $\overline{U}_{\mathfrak{q}}$ is an irreducible normal scheme. Notice as well that f induces a strict morphism $(X(\overline{\eta}_{\mathfrak{q}}), \underline{M}(\overline{\eta}_{\mathfrak{q}})) \to (Y, \underline{N})$ (theorem 12.7.8(iii.a)), and therefore the log structure of $U_{\mathfrak{q}} \times_{X(\overline{\eta}_{\mathfrak{q}})} (X(\overline{\eta}_{\mathfrak{q}}), \underline{M}(\overline{\eta}_{\mathfrak{q}}))$ is trivial.

Recall that $Z := \overline{U} \setminus \overline{U}_{\mathrm{tr}}$ is a finite union of irreducible closed subsets of codimension one in $\overline{U}$, and $\mathscr{O}_{\overline{U},z}$ is a discrete valuation ring, for each maximal point $z \in Z$ (proposition

12.7.17(iv,v)); especially, the category $\mathbf{Tame}(\overline{U}, \overline{U}_{\mathrm{tr}})$ is well defined (definition 13.3.16(iii)). We denote

$$\mathbf{Tame}(f, \overline{x})$$

the full subcategory of $\mathbf{Tame}(\overline{U}, \overline{U}_{\mathrm{tr}})$ consisting of all the coverings $C \to \overline{U}_{\mathrm{tr}}$ such that, for every $\mathfrak{q} \in \Sigma$, the induced covering

$$C \times_{\overline{U}_{\mathrm{tr}}} \overline{U}_{\mathfrak{q}} \to \overline{U}_{\mathfrak{q}}$$

is trivial (*i.e.* $C \times_{\overline{U}_{\mathrm{tr}}} \overline{U}_{\mathfrak{q}}$ is a disjoint union of copies of $\overline{U}_{\mathfrak{q}}$). It is easily seen that $\mathbf{Tame}(f, \overline{x})$ is a Galois category (see [82, Exp.V, Déf.5.1]), and we obtain a fibre functor for this category, by restriction of the usual fibre functor $\varphi \mapsto \varphi^{-1}(\xi)$ defined on all étale coverings φ of $\overline{U}_{\mathrm{tr}}$; we denote by $\pi_1(\overline{U}_{\mathrm{tr}}/Y_{\mathrm{\acute{e}t}}, \xi)$ the corresponding fundamental group. According to [82, Exp.V, Prop.6.9], the fully faithful inclusion $\mathbf{Tame}(f, \overline{x}) \to \mathsf{Cov}(\overline{U}_{\mathrm{tr}})$ induces a continuous surjective group homomorphism

$$\pi_1(\overline{U}_{\mathrm{tr,\acute{e}t}}, \xi) \to \pi_1(\overline{U}_{\mathrm{tr}}/Y_{\mathrm{\acute{e}t}}, \xi). \tag{13.4.20}$$

Proposition 13.4.21. *The map* (13.4.3) *factors through* (13.4.20)*, and the induced group homomorphism :*

$$\pi_1(\overline{U}_{\mathrm{tr}}/Y_{\mathrm{\acute{e}t}}, \xi) \to \mathrm{Coker}\,(\log f^{\mathrm{gp}}_{\overline{x}})^\vee \otimes_{\mathbb{Z}} \prod_{\ell \neq p} \mathbb{Z}_\ell(1). \tag{13.4.22}$$

is surjective.

Proof. Let $\sigma_Y : Y \to Y^{\mathrm{qfs}}$ be the natural morphism of schemes exhibited in remark 12.2.38(iv), and set

$$(Y, \underline{N}') := Y \times_{Y^{\mathrm{qfs}}} (Y, \underline{N})^{\mathrm{qfs}} \qquad (X, \underline{M}') := (Y, \underline{N}') \times_{(Y, \underline{N})} (X, \underline{M}).$$

Since f is saturated, both $(X, \underline{M}')$ and $(Y, \underline{N}')$ are fs log schemes (see remark 12.2.38(i)); also, the morphism of schemes underlying the induced morphism of log schemes $f' : (X, \underline{M}') \to (Y, \underline{N}')$, agrees with that underlying f. Moreover, by construction we have $\underline{N}'^{\sharp}_{\overline{z}} = (\underline{N}^{\sharp}_{\overline{z}})^{\mathrm{sat}}$ for every geometric point $\overline{z}$ of Y, especially $(Y, \underline{N}')_{\mathrm{tr}} = (Y, \underline{N})_{\mathrm{tr}}$. Likewise, $\underline{M}'^{\sharp}_{\overline{z}} = (\underline{M}^{\sharp}_{\overline{z}})^{\mathrm{sat}}$ (lemma 6.2.12(iii,iv)), therefore $\mathrm{Str}(f') = \mathrm{Str}(f)$, and especially, $(X, \underline{M}')_{\mathrm{tr}} = (X, \underline{M})_{\mathrm{tr}}$. Furthermore, notice that the the natural map

$$\mathrm{Coker}\,(\log f)^{\mathrm{gp}}_{\overline{x}} \to \mathrm{Coker}\,(\log f')^{\mathrm{gp}}_{\overline{x}} \tag{13.4.23}$$

is surjective, and its kernel is a quotient of $(\underline{M}^{\mathrm{sat}}_{\overline{x}})^\times / \underline{M}^\times_{\overline{x}}$, especially it is a torsion subgroup. However, the cokernel of $(\log f)^{\mathrm{gp}}_{\overline{x}}$ equals the cokernel of $(\log f^\sharp_{\overline{x}})^{\mathrm{gp}}$, hence it is torsion-free (corollary 6.2.32(ii)), so (13.4.23) is an isomorphism. Thus, we may replace $\underline{N}$ (resp. $\underline{M}$) by $\underline{N}'$ (resp. $\underline{M}'$), and assume from start that f is a smooth, saturated morphism of fs log schemes.

In this case, in light of the discussion of (13.4.14), it suffices to prove that (13.4.17) is a surjection, and that it factors through $\pi_1(\overline{U}_{\mathrm{tr}}/Y_{\mathrm{\acute{e}t}}, \xi)$. To prove the surjectivity comes down to showing that $\overline{U}_{\mathrm{tr}} \times_{X_e} X'_e$ is a connected scheme. However, let x_e be the support of $\overline{x}_e$, and notice that $\psi_P \circ \pi_e(y_e) = \mathfrak{m}_P$, the closed point of T_P. Since x maps to the closed point of T_Q, we deduce easily that the image of x_e in $T_{Q'}$ is the closed point $\mathfrak{m}_{Q'}$, *i.e.* x_e lies in the closed subscheme $X_e \times_{S'_Q} \mathrm{Spec}\,\kappa\langle Q'/\mathfrak{m}_{Q'}\rangle$. Notice as well that $\boldsymbol{e}_{Q|P}$ is of Kummer type (see definition 6.4.58); by proposition 13.3.7(ii), it follows that there exists a unique geometric point $\overline{x}'_e$ of X'_e lying over $\overline{x}_e$, whence an isomorphism of $X_e(\overline{x}_e)$-schemes ([66, Ch.IV, Prop.18.8.10])

$$X'_e(\overline{x}'_e) \xrightarrow{\sim} X'_e \times_{X_e} X_e(\overline{x}_e).$$

Hence $\overline{U}_{\mathrm{tr}} \times_{X_e} X'_e$ is an open subset of $|\overline{\eta}_e| \times_{Y_e} X'_e(\overline{x}_e)$, and the latter is an irreducible scheme, by proposition 12.7.17(ii). We also deduce that the induced morphism

$$h_{e,\overline{x}} : (X'_e(\overline{x}'_e), \underline{M}'_e(\overline{x}'_e)) \to (X_e(\overline{x}_e), \underline{M}_e(\overline{x}_e))$$

is a finite étale covering of log schemes. From the discussion in (13.4.11), we see that the restriction of $h_{e,\overline{x}}$

$$\mathrm{Str}(h_{e,\overline{x}}) \to X_e(\overline{x}_e)$$

is an étale morphism, and $\mathrm{Str}(h_{e,\overline{x}})$ contains the strict locus of the induced morphism

$$f'_{e,\overline{x}} : (X'_e(\overline{x}'_e), \underline{M}'_e(\overline{x}'_e)) \to (Y_e, \underline{N}_e).$$

Since the field extension $\kappa(\overline{y}) \to \kappa(\overline{y}_e)$ is purely inseparable, $\overline{\eta}_{\mathfrak{q}}$ lifts uniquely to a geometric point $\eta_{e,\mathfrak{q}} \in X_e(\overline{x}_e)$, and as usual we deduce that the strict henselization $X_e(\overline{\eta}_{\mathfrak{q}})$ of $X_e(\overline{x}_e)$ at $\overline{\eta}_{e,\mathfrak{q}}$ is isomorphic, as an $X_e(\overline{x}_e)$-scheme, to $X_e(\overline{x}_e) \times_{X(\overline{x})} X(\overline{\eta}_{\mathfrak{q}})$. Moreover, if $\eta_{e,\mathfrak{q}}$ is the support of $\overline{\eta}_{e,\mathfrak{q}}$, a simple inspection shows that the fibre $h^{-1}_{e,\overline{x}}(\eta_{e,\mathfrak{q}})$ consists of maximal points of $f'^{-1}_{e,\overline{x}}(y_e)$. By theorem 12.7.8(iii.a), every point of $h^{-1}_{e,\overline{x}}(\eta_{e,\mathfrak{q}})$ lies in $\mathrm{Str}(f'_{e,\overline{x}})$, therefore the induced morphism $X'_e \times_{X_e} X_e(\overline{\eta}_{\mathfrak{q}}) \to X_e(\overline{\eta}_{\mathfrak{q}})$ is finite and étale. Taking into account (13.3.41), we conclude that the étale covering $X'_e \times_{X_e} \overline{U}_{\mathrm{tr}} \to \overline{U}_{\mathrm{tr}}$ is an object of $\mathbf{Tame}(f, \overline{x})$, whence the proposition. □

13.4.24. Say that $Y = \mathrm{Spec}\, R$; for every algebraic field extension K of $\kappa(\eta) = \mathrm{Frac}(R)$, let R_K be the normalization of R in K, set $|\eta_K| := \mathrm{Spec}\, K$ and

$$Y_K := \mathrm{Spec}\, R_K \quad (Y_K, \underline{N}_K) := Y_K \times_Y (Y, \underline{N}) \quad (X_K, \underline{M}_K) := Y_K \times_Y (X, \underline{M}).$$

Moreover, let $f_K : (X_K, \underline{M}_K) \to (Y_K, \underline{N}_K)$ be the induced morphism, and $\overline{y}_K$ any geometric point localized at the closed point y_K of the strictly local scheme Y_K; since the extension $\kappa(\overline{y}) \to \kappa(\overline{y}_K)$ is purely inseparable, there exists a unique geometric point $\overline{x}_K$ of X_K lifting $\overline{x}$, and we have an isomorphism of $(X(\overline{x}), \underline{M}(\overline{x}))$-schemes :

$$(X_K(\overline{x}_K), \underline{M}_K(\overline{x}_K)) \xrightarrow{\sim} X_K \times_X (X(\overline{x}), \underline{M}(\overline{x})) = Y_K \times_Y (X(\overline{x}), \underline{M}(\overline{x})).$$

Clearly the morphism f_K is again of the type considered in (13.4.18); especially, the maximal points of $f^{-1}_{K,\overline{x}_K}(\overline{y}_K)$ are in natural bijection with the elements of Σ, and it is natural to denote

$$U_K := U \times_Y |\eta_K| \qquad U_{K,\mathrm{tr}} := U_{\mathrm{tr}} \times_Y |\eta_K| \qquad U_{K,\mathfrak{q}} := U_{K,\mathrm{tr}} \times_{X(\overline{x})} X(\overline{\eta}_{\mathfrak{q}})$$

for every $\mathfrak{q} \in \Sigma$. Then, let $\overline{\eta}_{K,\mathfrak{q}}$ be the unique geometric point of $f^{-1}_{K,\overline{x}_K}(\overline{y}_K)$ lying over $\overline{\eta}_{\mathfrak{q}}$, and $\eta_{K,\mathfrak{q}}$ the support of $\overline{\eta}_{K,\mathfrak{q}}$; as usual, we have

$$X_K(\overline{\eta}_{K,\mathfrak{q}}) = X(\overline{\eta}_{\mathfrak{q}}) \times_Y Y_K \tag{13.4.25}$$

hence the above notation is consistent with the one introduced for the original morphism f. Furthermore, if z is any maximal point of $\overline{U} \backslash \overline{U}_{\mathrm{tr}}$, the image z_K of z in U_K is a maximal point of $U_K \backslash U_{K,\mathrm{tr}}$, and since the induced map $\mathscr{O}_{U_K, z_K} \to \mathscr{O}_{\overline{U}, z}$ is faithfully flat, proposition 12.7.17(iv,v) easily implies that $\mathscr{O}_{U_K, z_K}$ is a discrete valuation ring. We may then denote

$$\mathbf{Tame}(f, \overline{x}, K)$$

the full subcategory of $\mathbf{Tame}(U_K, U_{K,\mathrm{tr}})$, consisting of those objects $C \to U_{K,\mathrm{tr}}$, such that the induced covering $C \times_{U_{K,\mathrm{tr}}} U_{K,\mathfrak{q}} \to U_{K,\mathfrak{q}}$ is trivial, for every $\mathfrak{q} \in \Sigma$. We have a natural functor

$$2\text{-}\operatorname*{colim}_{K} \mathbf{Tame}(f, \overline{x}, K) \to \mathbf{Tame}(f, \overline{x}) \tag{13.4.26}$$

where the 2-colimit ranges over the filtered family of all finite separable extensions K of $\kappa(\eta)$.

Lemma 13.4.27. *The functor* (13.4.26) *is an equivalence.*

Proof. Let $\overline{h} : \overline{C} \to \overline{U}_{\mathrm{tr}}$ be an object of the category $\mathbf{Tame}(f, \overline{x})$. According to [66, Ch.IV, Prop.17.7.8(ii)] and [65, Ch.IV, Prop.8.10.5], we may find a finite extension K of $\kappa(\eta)$, such that $\overline{h}$ descends to a finite étale morphism

$$h_K : C_K \to U_{K,\mathrm{tr}}.$$

Let C'_K (resp. $\overline{C}'$) denote the normalization of C_K (resp. of $\overline{C}$) over U_K (resp. over $\overline{U}$). Since the morphism $|\overline{\eta}| \to |\eta_K|$ is pro-étale, we have $\overline{C}' = C'_K \times_{|\eta_K|} |\overline{\eta}|$ (proposition 9.8.3), and it follows easily that C_K is tamely ramified along the divisor $U_K \setminus U_{K,\mathrm{tr}}$. From (13.4.25) we get a natural isomorphism :

$$\overline{U}_{\mathfrak{q}} \stackrel{\sim}{\to} U_{K,\mathfrak{q}} \times_{|\eta_K|} |\overline{\eta}|.$$

Thus, after replacing K by a larger finite separable extension of $\kappa(\eta)$, we may assume that the induced morphism $C_K \times_{U_{K,\mathrm{tr}}} U_{K,\mathfrak{q}} \to U_{K,\mathfrak{q}}$ is a trivial étale covering, for every $\mathfrak{q} \in \Sigma$.

This shows that (13.4.26) is essentially surjective; likewise one shows the full faithfulness : the details shall be left to the reader. □

13.4.28. In the situation of (13.4.24), let K be an algebraic extension of $\kappa(\eta)$, and

$$h : C \to U_{K,\mathrm{tr}}$$

any object of $\mathbf{Tame}(U_K, U_{K,\mathrm{tr}})$; denote by C' the normalization of $X_K(\overline{x}_K)$ in C, and let $h' : C' \to X_K(\overline{x}_K)$ be the induced morphism of schemes. We claim that there exists a largest non-empty open subset

$$E(h) \subset X_K(\overline{x}_K)$$

such that the restriction $h'^{-1}E(h) \to E(h)$ of h' is étale. Indeed, in any case, h' restricts to an étale morphism on a dense open subset containing $U_{K,\mathrm{tr}}$, and there exists a largest open subset $E' \subset C'$ such that $h'_{|E'}$ is étale (claim 13.1.8 and lemma 13.1.7(iii)). Then it is easily seen that $E(h) := X_K(\overline{x}_K) \setminus h'(C' \setminus E')$ will do.

Lemma 13.4.29. *With the notation of* (13.4.28)*, the category* $\mathbf{Tame}(f, \overline{x}, K)$ *is the full subcategory of* $\mathbf{Tame}(U_K, U_{K,\mathrm{tr}})$ *consisting of those objects* $h : C \to U_{K,\mathrm{tr}}$ *such that* $E(h)$ *contains the maximal points of* $X_K(\overline{x}_K) \times_{Y_K} |y_K|$.

Proof. In view of claim 13.1.9, this characterization is a rephrasing of the definition of the category $\mathbf{Tame}(f, \overline{x}, K)$. □

13.4.30. Keep the notation of (13.4.28), and suppose that h is an object of $\mathbf{Tame}(f, \overline{x}, K)$. Fix $\mathfrak{q} \in \Sigma$; then lemma 13.4.29 says that $\eta_{K,\mathfrak{q}} \in E(h)$. Thus, we obtain a functor

$$\mathbf{Tame}(f, \overline{x}, K) \to \mathsf{Cov}(|\eta_{K,\mathfrak{q}}|) \qquad : \qquad C \mapsto C' \times_{X_K(\overline{x}_K)} |\eta_{K,\mathfrak{q}}|.$$

However, the natural morphism $|\eta_{K,\mathfrak{q}}| \to |\eta_{\mathfrak{q}}|$ is radicial, hence it induces an equivalence

$$\mathsf{Cov}(|\eta_{\mathfrak{q}}|) \stackrel{\sim}{\to} \mathsf{Cov}(|\eta_{K,\mathfrak{q}}|)$$

(lemma 13.1.7(i)). Combining these two functors in the special special case where $K := \kappa(\overline{\eta})$, we get a functor

$$\mathbf{Tame}(f, \overline{x}) \to \mathsf{Cov}(|\eta_{\mathfrak{q}}|) \quad : \quad (C \to \overline{U}_{\mathrm{tr}}) \mapsto (C_{|\eta_{\mathfrak{q}}} \to |\eta_{\mathfrak{q}}|). \tag{13.4.31}$$

Now, the rule $\varphi \mapsto \varphi^{-1}(\overline{\eta}_{\mathfrak{q}})$ yields a fibre functor for the Galois category $\mathsf{Cov}(|\eta_{\mathfrak{q}}|)$; by composition with (13.4.31), we deduce a fibre functor for $\mathbf{Tame}(f, \overline{x})$, whose group of automorphisms we denote $\pi_1(\overline{U}_{\mathrm{tr}}/Y_{\mathrm{\acute{e}t}}, \overline{\eta}_{\mathfrak{q}})$. Also, set $F(\mathfrak{q}) := \underline{M}_{\overline{x}} \setminus \mathfrak{q}$, and notice that the structure map $\underline{M}_{\overline{x}} \to \mathscr{O}_{X(\overline{x}),\overline{x}}$ induces a group homomorphism

$$F(\mathfrak{q})^{\mathrm{gp}} \to \kappa(\eta_{\mathfrak{q}})^{\times}. \tag{13.4.32}$$

Lemma 13.4.33. *With the notation of* (13.4.30)*, we have :*

(i) *The natural map* $\mathrm{Coker}\,(\log f_{\overline{x}}) \to F(\mathfrak{q})$ *induces a commutative diagram of groups*

$$\begin{array}{ccccc}
\pi_1(|\eta_{\mathfrak{q}}|_{\text{ét}}, \overline{\eta}_{\mathfrak{q}}) & \longrightarrow & \pi_1(\overline{U}_{\mathrm{tr}}/Y_{\text{ét}}, \overline{\eta}_{\mathfrak{q}}) & \longrightarrow & \pi_1(\overline{U}_{\mathrm{tr}}/Y_{\text{ét}}, \xi) \\
\downarrow{\scriptstyle\alpha} & & & & \downarrow{\scriptstyle\beta} \\
F(\mathfrak{q})^{\mathrm{gp}\vee} \otimes_{\mathbb{Z}} \prod_{\ell\neq p} \mathbb{Z}_\ell(1) & & \xrightarrow{\quad\gamma\quad} & & \mathrm{Coker}\,(\log f^{\mathrm{gp}}_{\overline{x}})^\vee \otimes_{\mathbb{Z}} \prod_{\ell\neq p} \mathbb{Z}_\ell(1)
\end{array}$$

where β *is* (13.4.22)*, and* α *is deduced from* (13.4.32)*, as in the discussion of* (13.3.27).

(ii) α *is surjective, and* γ *is an isomorphism.*

Proof. (i): The proof amounts to unwinding the definitions, and shall be left as an exercice for the reader. Notice that the second arrow on the top row is only well-defined up to inner automorphisms, but since the groups on the bottom row are abelian, the ambiguity does not affect the statement.

(ii): Notice that $\log f_{\overline{x}}$ restricts to a map of monoids $\mathscr{O}^\times_{Y,\overline{y}} \to F(\mathfrak{q})$, which induces an isomorphism $\mathrm{Coker}(\log f)^\sharp \stackrel{\sim}{\to} F(\mathfrak{q})^\sharp$; we deduce that γ is an isomorphism. Next, let Z be the topological closure of $\eta_{\mathfrak{q}}$ in $X(\overline{x})$, and endow Z with its reduced subscheme structure; set also $(Z, \underline{M}(Z)) := Z \times_{X(\overline{x})} (X, \underline{M})$. The map α factors as a composition

$$\pi_1(|\eta_{\mathfrak{q}}|_{\text{ét}}, \overline{\eta}_{\mathfrak{q}}) \to \pi_1((Z, \underline{M}(Z))_{\mathrm{tr,ét}}, \overline{\eta}_{\mathfrak{q}}) \to F(\mathfrak{q})^{\mathrm{gp}\vee} \otimes_{\mathbb{Z}} \prod_{\ell\neq p} \mathbb{Z}_\ell(1)$$

where the first map is surjective, by lemma 13.1.7(ii). Lastly, notice that $\underline{M}(Z)_{\mathrm{red},\overline{x}} = F(\mathfrak{q})_\circ$; by propositions 13.3.43 and 12.7.14(ii), it follows that the second map is surjective as well, so the proof of (ii) is complete. □

13.4.34. In the situation of (12.3.44), suppose that Y_i is a strictly local normal scheme for every $i \in I$, and the transition morphisms $Y_j \to Y_i$ are local and dominant, for every morphism $i \to j$ in I. Let $\overline{x}$ be a geometric point of X, and denote by $\overline{x}_i$ the image of $\overline{x}$ in X_i, for every $i \in I$. Suppose that the image $\overline{y}$ of $\overline{x}$ in Y is localized at the closed point. Also, let $\overline{\eta}$ be a strict geometric point of Y, localized at the generic point η_i, and denote by $\overline{\eta}_i$ (resp. $\overline{y}_i$) the strict image of $\overline{\eta}$ (resp. $\overline{y}$) in Y_i (see definition 4.9.17(v)).

Lemma 13.4.35. *In the situation of* (13.4.34)*, suppose that* $(g, \log g) : (X, \underline{M}) \to (Y, \underline{N})$ *is a smooth and saturated morphism of fine log schemes. Then there exist* $i \in I$*, and a smooth and saturated morphism* $(g_i, \log g_i) : (X_i, \underline{M}_i) \to (Y_i, \underline{N}_i)$ *of fine log schemes, such that* $\log g = \pi_i^* \log g_i$.

Proof. By corollary 12.3.45, we can descend $(g, \log g)$ to a smooth morphism $(g_i, \log g_i)$ of fine log schemes, and after replacing I by I/i, we may assume that $i = 0$. Then the contention follows from corollary 12.2.36(ii). □

13.4.36. Keep the situation of (13.4.34), and suppose that $(g_0, \log g_0) : (X_0, \underline{M}_0) \to (Y_0, \underline{N}_0)$ is a smooth and saturated morphism of fine log schemes; set

$$(X_i, \underline{M}_i) := X_i \times_{X_0} (X_0, \underline{M}_0) \qquad (Y_i, \underline{N}_i) := Y_i \times_{Y_0} (Y_0, \underline{Y}_0)$$

and denote $(g_i, \log g_i) : (X_i, \underline{M}_i) \to (Y_i, \underline{N}_i)$ the induced morphism of log schemes, for every $i \in I$. Also, let $(g, \log g) : (X, \underline{M}) \to (Y, \underline{N})$ be the limit of the system of morphisms $((g_i, \log g_i) \mid i \in I)$. These are morphisms of the type considered in (13.4), so we may define $U_i := g^{-1}_{i,\overline{x}_i}(\eta_i)$, and introduce likewise the schemes $U_{i,\mathrm{tr}}$, $\overline{U}_i$ and $\overline{U}_{i,\mathrm{tr}}$ as in (13.4). Moreover, set $Z_i := \overline{U}_i \backslash \overline{U}_{i,\mathrm{tr}}$ for every $i \in I$; clearly $Z_i = Z_j \times_{X_j(\overline{x}_j)} X_i(\overline{x}_i)$ for every morphism $i \to j$ in I. Also, each Z_i is a finite union of irreducible subsets of codimension one, and for every $i \to j$ in I, the transition morphisms $X_i \to X_j$ restrict to maps

$$\mathrm{Max}\, Z_i \to \mathrm{Max}\, Z_j \qquad \mathrm{Max}\, X_i(\overline{x}_i) \times_{Y_i} |\overline{y}_i| \to \mathrm{Max}\, X_j(\overline{x}_j) \times_{Y_j} |\overline{y}_j|$$

Combining proposition 13.3.23(ii) and lemma 13.4.29, we deduce a fully faithful functor

$$(13.4.37) \qquad 2\text{-}\underset{i\in I}{\mathrm{colim}}\,\mathbf{Tame}(g_i,\overline{x}_i) \to \mathbf{Tame}(g,\overline{x}).$$

Lemma 13.4.38. *The functor* (13.4.37) *is an equivalence.*

Proof. It remains only to show the essential surjectivity. Hence, let h be a given object of $\mathbf{Tame}(g,\overline{x})$; by proposition 13.3.23(ii), we know that there exists $j \in I$ such that h descends to an étale covering $h_j : V_j \to \overline{U}_{j,\mathrm{tr}}$, tamely ramified along Z_j, and after replacing I by I/i, we may assume that j is the final object of I, and define $h_i := \overline{U}_{i,\mathrm{tr}} \times_{\overline{U}_{j,\mathrm{tr}}} h_j$ for every $i \to j$ in I. Now, let $E' \subset E(h)$ be a constructible open subset containing the maximal points of $X(\overline{x}) \times_Y |\overline{y}|$. For every $i \in I$, let $\overline{Y}_i$ be the normalization of Y_i in $\mathrm{Spec}\,\kappa(\overline{\eta}_i)$, and set $\overline{X}_i := X_i(\overline{x}_i) \times_{Y_i} \overline{Y}_i$; according to [65, Ch.IV, Th.8.3.11], there exists $i \in I$ such that E' descends to a constructible open subset $E'_i \subset \overline{X}_i$, and then necessarily E'_i contains all the maximal points of $X_i(\overline{x}_i) \times_{Y_i} |\overline{y}_i|$. As usual, we may assume that i is the final object, so E'_i is defined for every $i \in I$. Lastly, since h extends to an étale covering on E', we see that h_i extends to an étale covering of E'_k, for some $k \in I$ ([66, Ch.IV, Prop.17.7.8(i)]). In view of lemma 13.4.29, the contention follows. □

Theorem 13.4.39. *The map* (13.4.22) *is an isomorphism.*

Proof. Arguing as in the proof of proposition 13.4.21, we may assume that both $(X,\underline{M})$ and $(Y,\underline{N})$ are fs log schemes, and in view of proposition 13.4.21, we need only show that (13.4.22) is injective. This comes down to the following assertion. For every object $\overline{h} : \overline{C} \to \overline{U}_{\mathrm{tr}}$ of the category $\mathbf{Tame}(f,\overline{x})$, the induced action of $\pi_1(\overline{U}_{\mathrm{tr}},\xi)$ on $\overline{h}^{-1}(\xi)$ factors through the quotient $\mathrm{Coker}\,(\log f^{\mathrm{gp}}_{\overline{x}})^\vee \otimes_{\mathbb{Z}} \prod_{\ell\neq p}\mathbb{Z}_\ell(1)$.

• By lemma 13.4.27, there exist a finite separable extension K of $\kappa(\eta)$, and an object $h : C_K \to U_{K,\mathrm{tr}}$ of $\mathbf{Tame}(f,\overline{x},K)$, with an isomorphism $C_K \times_{U_{K,\mathrm{tr}}} \overline{U}_{\mathrm{tr}} \xrightarrow{\sim} \overline{C}$ of $\overline{U}_{\mathrm{tr}}$-schemes. Since $\log f_{\overline{x}} = \log f_{K,\overline{x}_K}$, the theorem will hold for the morphism f and the point $\overline{x}$, if and only if it holds for f_K and the point $\overline{x}_K$.

Claim 13.4.40. The theorem holds if Y is noetherian of dimension one.

Proof of the claim. In this case, Y is the spectrum of a strictly henselian discrete valuation ring R, and the same then holds for Y_K. Hence, we may replace throughout f by f_K, and assume from start that there exists an object $h : C \to U_{\mathrm{tr}}$ of $\mathbf{Tame}(f,\overline{x},\kappa(\eta))$, with an isomorphism $\overline{C} \xrightarrow{\sim} C \times_{U_{\mathrm{tr}}} \overline{U}_{\mathrm{tr}}$ of $\overline{U}_{\mathrm{tr}}$-schemes. Endow Y with the fine log structure $\underline{N}'$ such that $\Gamma(Y,\underline{N}') = R\setminus\{0\}$; since $(Y,\underline{N})_{\mathrm{tr}}$ is dense in Y, we have a well defined morphism of log schemes $\pi : (Y,\underline{N}') \to (Y,\underline{N})$, which is the identity on the underlying schemes. Set $(X,\underline{M}') := (Y,\underline{N}') \times_{(Y,\underline{N})} (X,\underline{M})$. Then $(Y,\underline{N}')$ is a regular log scheme, and consequently the same holds for $(X,\underline{M}')$, by theorem 12.5.28.

Furthermore, π trivially restricts to a strict morphism on the open subset $|\eta|$, hence the induced morphism $(X,\underline{M}') \times_Y |\eta| \to (X,\underline{M}) \times_Y |\eta|$ is an isomorphism, especially U_{tr} is the trivial locus of $(X(\overline{x}),\underline{M}'(\overline{x})) \times_Y |\eta|$. However, it is easily seen that $(X(\overline{x}),\underline{M}'(\overline{x}))_{\mathrm{tr}}$ does not intersect the closed fibre $f^{-1}_{\overline{x}}(\overline{y})$, so finally $U_{\mathrm{tr}} = (X(\overline{x}),\underline{M}'(\overline{x}))_{\mathrm{tr}}$.

From theorem 13.3.45, we deduce that h extends to an étale covering of $(X(\overline{x}),\underline{M}'(\overline{x}))$. Then, arguing as in (13.4) we get a commutative diagram of groups :

$$\begin{array}{ccc} \pi_1(\overline{U}_{\mathrm{tr,\acute{e}t}},\xi) & \longrightarrow & \mathrm{Coker}\,(\log f^{\mathrm{gp}}_{\overline{x}})^\vee \otimes_{\mathbb{Z}} \prod_{\ell\neq p}\mathbb{Z}_\ell(1) \\ \downarrow & & \downarrow \\ \pi_1(U_{\mathrm{tr,\acute{e}t}},\xi') & \xrightarrow{\ \alpha\ } & \underline{M}^{\mathrm{gp}\vee}_{\overline{x}} \otimes_{\mathbb{Z}} \prod_{\ell\neq p}\mathbb{Z}_\ell(1) \end{array}$$

whose top horizontal arrow is (13.4.3), and whose right vertical arrow is deduced from the projection $\underline{M}^{\mathrm{gp}}_{\overline{x}} \to \mathrm{Coker}\,(\log f^{\mathrm{gp}}_{\overline{x}})$. Lastly, proposition 13.3.43 shows that the natural action of $\pi_1(U_{\mathrm{tr,\acute{e}t}}, \xi')$ on $h^{-1}(\xi')$ factors through α, so the claim follows. $\diamondsuit$

• Next, suppose that Y is an arbitrary normal, strictly local scheme. The discussion in (13.4.14) implies that, in order to prove the theorem, it suffices to find an integer $e > 0$, a $(X, \underline{M})$-scheme $(X'_e, \underline{M}'_e)$ as in (13.4.11), and a geometric point $\overline{x}'_e$ of X'_e lying over $\overline{x}$, such that $h \times_{X(\overline{x})} X'_e(\overline{x}'_e)$ is a trivial covering. To this aim, we write Y as the limit of a cofiltered system $(Y_i \mid i \in I)$ of strictly local excellent and normal schemes (lemma 13.1.29), and we denote by η_i the generic point of Y_i, for every $i \in I$. By lemma 13.4.35, we may then descend $(f, \log f)$ to a smooth and saturated morphism $(f_i, \log f_i) : (X_i, \underline{M}_i) \to (Y_i, \underline{N}_i)$, for some $i \in I$, and as usual, we may assume that i is the final object of I. Let $\overline{x}_i$ be the image of $\overline{x}$ in X_i; by lemmata 13.4.38 and 13.4.27, the object h of $\mathbf{Tame}(f, \overline{x}, \kappa(\eta))$ descends to an object h_i of $\mathbf{Tame}(f_i, \overline{x}_i, K)$, for some $i \in I$, and some finite separable extension K of $\kappa(\eta_i)$. It suffices therefore to find $e > 0$, a $(X_i, \underline{M}_i)$-scheme $(X'_{i,e}, \underline{M}'_{i,e})$, and a geometric point $\overline{x}'_{i,e}$ of $X'_{i,e}$ lying over $\overline{x}_i$, such that $h_i \times_{X_i(\overline{x}_i)} X'_{i,e}(\overline{x}'_{i,e})$ is a trivial covering. In other words, we may replace throughout Y by $Y_{i,K}$, and assume from start that Y is excellent, and $\overline{h}$ descends to an object $h : C \to U_{\mathrm{tr}}$ of $\mathbf{Tame}(f, \overline{x}, \kappa(\eta))$.

• By [60, Ch.II, Prop.7.1.7] (see also remark 12.5.34) we may find a discrete valuation ring V and a local injective morphism $R \to \mathsf{V}$ inducing an isomorphism on the respective fields of fractions. Let V^{sh} be the strict henselization of V (at a geometric point whose support is the closed point), and set

$$\mathsf{Y} := \mathrm{Spec}\,\mathsf{V}^{\mathrm{sh}} \quad (\mathsf{Y}, \underline{\mathsf{N}}) := \mathsf{Y} \times_Y (Y, \underline{N}) \quad (\mathsf{X}, \underline{\mathsf{M}}) := \mathsf{Y} \times_Y (X, \underline{M}).$$

Also, let $\mathsf{f} : (\mathsf{X}, \underline{\mathsf{M}}) \to (\mathsf{Y}, \underline{\mathsf{N}})$ be the induced morphism. Denote by $\overline{\mathsf{y}}$ a geometric point localized at the closed point y of Y; also, pick any geometric point $\overline{\mathsf{x}}$ of X, whose image in X is $\overline{x}$; the induced morphism

(13.4.41) $$(\mathsf{X}(\overline{\mathsf{x}}), \underline{\mathsf{M}}(\overline{\mathsf{x}})) \to (X(\overline{x}), \underline{M}(\overline{x}))$$

restricts to a flat morphism $\mathsf{f}^{-1}_{\overline{\mathsf{x}}}(\overline{\mathsf{y}}) \to f^{-1}_{\overline{x}}(\overline{y})$ and from proposition 12.7.14, we see that the latter induces a bijection between the sets of maximal points of the two fibres. On the other hand, let $\overline{\eta}_{\mathsf{V}}$ denote a geometric point localized at the generic point η_{V} of Y; then (13.4.41) restricts to an ind-étale morphism $\mathsf{f}^{-1}_{\overline{\mathsf{x}}}(\eta_{\mathsf{V}}) \to f^{-1}_{\overline{x}}(\eta)$. Hence, set

$$\mathsf{U}_{\mathrm{tr}} := (\mathsf{X}(\overline{\mathsf{x}}), \underline{\mathsf{M}}(\overline{\mathsf{x}}))_{\mathrm{tr}} \times_{\mathsf{Y}} |\eta_{\mathsf{V}}|.$$

From lemma 13.4.29, it follows easily that the covering $C \times_{U_{\mathrm{tr}}} \mathsf{U}_{\mathrm{tr}} \to \mathsf{U}_{\mathrm{tr}}$ is an object of $\mathbf{Tame}(\mathsf{f}, \overline{\mathsf{x}}, \kappa(\eta_{\mathsf{V}}))$.

For any integer $e > 0$ invertible in R, pick a $(Y, \underline{N})$-scheme $(Y_e, \underline{N}_e)$ as in (13.4.11), so that we may define the étale morphism $(X'_e, \underline{M}'_e) \to (X_e, \underline{M}_e)$ of $(Y_e, \underline{N}_e)$-schemes as in (13.4.13). Notice that the morphism $(X'_e, \underline{M}'_e) \to (Y_e, \underline{N}_e)$ is again of the type considered in (13.4), and there exists, up to isomorphism, a unique geometric point $\overline{x}'_e$ of X'_e lifting $\overline{x}$; moreover, for any geometric point $\overline{y}_e$ supported at y_e, the induced map

$$\mathrm{Spec}\,\underline{M}'_{e,\overline{x}'_e} \to \mathrm{Spec}\,\underline{N}'_{e,\overline{y}_e}$$

is naturally identified with (13.4.19). Likewise, pick a $(\mathsf{Y}, \underline{\mathsf{N}})$-scheme $(\mathsf{Y}_e, \underline{\mathsf{N}}_e)$ in the same fashion, and denote by η_e (resp. $\eta_{\mathsf{V},e}$) the generic point of Y_e (resp. of Y_e), and by $\mathsf{y}_e \in \mathsf{Y}_e$ (resp. $y_e \in Y_e$) the closed point. We may choose Y_e so that $\kappa(\eta_{\mathsf{V},e})$ contains $\kappa(\eta_e)$, in which case we have a strict morphism

$$(\mathsf{Y}_e, \underline{\mathsf{N}}_e) \to (Y_e, \underline{N}_e)$$

of log schemes, and we may set $(\mathsf{X}'_e, \underline{\mathsf{M}}'_e) := \mathsf{Y}_e \times_{Y_e} (X'_e, \underline{M}'_e)$. Again, there exists, up to isomorphism, a unique geometric point $\overline{\mathsf{x}}'_e$ of X'_e lifting $\overline{\mathsf{x}}$, and by claim 13.4.40, we may assume that both e and $\kappa(\eta_{\mathsf{V},e})$ have been chosen large enough, so that the base change

$$h \times_{X(\overline{x})} \mathsf{X}'_e(\overline{\mathsf{x}}'_e)$$

shall be a trivial étale covering. Hence, we may replace Y by Y_e, Y by Y_e, X by X'_e, and h by $h \times_{X(\overline{x})} X'_e(\overline{x}'_e)$, and assume from start that $C \times_{U_{\mathrm{tr}}} \mathsf{U}_{\mathrm{tr}}$ is a trivial covering of U_{tr}. The theorem will follow, once we show that – in this case – h is a trivial étale covering.

Let C' (resp. C') be the normalization of $X(\overline{x})$ in C (resp. of $\mathsf{X}(\overline{\mathsf{x}})$ in $C \times_{U_{\mathrm{tr}}} \mathsf{U}_{\mathrm{tr}}$), and define $E(h)$ as in (13.4.28). Then C' is a trivial étale covering of $\mathsf{X}(\overline{\mathsf{x}})$, and since $X(\overline{x})$ is excellent, the induced morphism $h' : C' \to X(\overline{x})$ is finite.

Claim 13.4.42. For every maximal point η_{q} of $f_{\overline{x}}^{-1}(\overline{y})$, the induced covering $C_{|\eta_{\mathsf{q}}} \to |\eta_{\mathsf{q}}|$ is trivial (notation of (13.4.31)).

Proof of the claim. Let Z_{q} denote the topological closure of $\{\eta_{\mathsf{q}}\}$ in $X(\overline{x})$, and endow Z_{q} with its reduced subscheme structure; then $E_{\mathsf{q}} := E(h) \cap Z_{\mathsf{q}}$ is non-empty (lemma 13.4.29), and geometrically normal (proposition 12.7.14(ii) and corollary 12.5.13). Also, (13.4.41) induces an isomorphism of $\kappa(\mathsf{y})$-schemes

$$E_{\mathsf{q}} \times_{X(\overline{x})} \mathsf{X}(\overline{x}) \xrightarrow{\sim} E_{\mathsf{q}} \times_{\kappa(y)} \kappa(\mathsf{y}).$$

Moreover, Z_{q} is strictly local, and $Z_{\mathsf{q}} \times_{X(\overline{x})} \mathsf{X}(\overline{\mathsf{x}})$ is the strict henselization of $Z_{\mathsf{q}} \times_{\kappa(y)} \kappa(\mathsf{y})$ at the point x. Furthermore, the morphism $h'' := h' \times_{X(\overline{x})} E_{\mathsf{q}}$ is an étale covering of E_{q}, and $h'' \times_{\kappa(y)} \kappa(\mathsf{y})$ is naturally identified with the restriction of C' to the subscheme $E_{\mathsf{q}} \times_{X(\overline{x})} \mathsf{X}(\overline{\mathsf{x}})$ ([66, Ch.IV, Prop.17.5.7]), hence it is a trivial covering. By example 13.2.6, it follows that h'' is trivial as well. Since $C_{|\eta_{\mathsf{q}}}$ is the restriction of h'' to $|\eta_{\mathsf{q}}|$, the claim follows. ◊

Clearly (13.4.41) maps each stratum U_{q} of the logarithmic stratification of $\mathsf{X}(\overline{\mathsf{x}})$, to the corresponding stratum U_{q} of the logarithmic stratification of $X(\overline{x})$ (see (12.5.35)). More precisely, since (13.4.41) $\times_Y |\eta|$ is ind-étale, proposition 12.7.17(iii) implies that the generic point of $\mathsf{U}_{\mathsf{q}} \times_{\mathsf{Y}} |\eta_{\mathsf{V}}|$ gets mapped to the generic point of $U_{\mathsf{q}} \times_Y |\eta|$. We conclude that $E(h)$ contains the generic point of every stratum $U_{\mathsf{q}} \times_Y |\eta|$.

Claim 13.4.43. $X(\overline{x}) \times_Y |\eta| \subset E(h)$.

Proof of the claim. Notice first that $(X, \underline{M}) \times_Y |\eta|$ is a regular log scheme (corollary 12.5.29).

For any geometric point ξ of $X(\overline{x})$, denote by $(X(\xi), \underline{M}(\xi))$ the strict henselization of $(X(\overline{x}), \underline{M}(\overline{x}))$ at ξ, and set $C'(\xi) := C' \times_{X(\overline{x})} X(\xi)$. Now, suppose that the support of ξ lies in the stratum $U_{\mathsf{q}} \times_Y |\eta|$, and let ξ_{q} be a geometric point localized at the generic point of U_{q}. By assumption, $C'(\xi)$ is a finite $X(\xi)$-scheme, tamely ramified along the non-trivial locus of $(X(\xi), \underline{M}(\xi))$. Likewise, $C'(\xi_{\mathsf{q}})$ is tamely ramified along the non-trivial locus of $(X(\xi_{\mathsf{q}}), \underline{M}(\xi_{\mathsf{q}}))$. Pick any strict specialization map $(X(\xi_{\mathsf{q}}), \underline{M}(\xi_{\mathsf{q}})) \to (X(\xi), \underline{M}(\xi))$ (see (12.7.11)); it induces a functor

$$\mathsf{Cov}(X(\xi), \underline{M}(\xi)) \to \mathsf{Cov}(X(\xi_{\mathsf{q}}), \underline{M}(\xi_{\mathsf{q}})) \tag{13.4.44}$$

and theorem 13.3.45 implies that $C'(\xi)$ is an object of the source of (13.4.44), which is mapped, under this functor, to the object $C'(\xi_{\mathsf{q}})$. By proposition 13.3.43, for any geometric point ξ of $X(\overline{x}) \times_Y |\eta|$, the category $\mathsf{Cov}(X(\xi), \underline{M}(\xi))$ is equivalent to the category of finite sets with a continuous action of $(\underline{M}_\xi)^{\mathrm{gp}\vee} \otimes_{\mathbb{Z}} \prod_{\ell \neq p} \mathbb{Z}_\ell(1)$. On the other hand, clearly $\underline{M}(\overline{x})^\sharp$ restricts to a constant sheaf of monoids on $(U_{\mathsf{q}})_\tau$. In view of (13.3.54), we deduce that (13.4.44) is an equivalence; lastly, we have seen that the induced morphism $C'(\xi_{\mathsf{q}}) \to X(\xi_{\mathsf{q}})$ is étale, *i.e.* is a trivial covering, therefore the same holds for the morphism $C'(\xi) \to X(\xi)$, and consequently the support of ξ lies in $E(h)$ (claim 13.1.9). Since ξ is arbitrary, the assertion follows. ◊

Claim 13.4.45. There exists a non-empty open subset $U_Y \subset Y$ such that $X(\overline{x}) \times_Y U_Y \subset E(h)$.

Proof of the claim. Since $X(\overline{x})$ is a noetherian scheme, $E(h)$ is a constructible open subset, hence $Z := X(\overline{x}) \setminus E(h)$ is a constructible closed subset of $X(\overline{x})$. The subset $f_{\overline{x}}(Z)$ is pro-constructible ([63, Ch.IV, Prop.1.9.5(vii)]) and does not contain η (by claim 13.4.43), hence neither does its topological closure W ([63, Ch.IV, Th.1.10.1]). It is easily seen that $U_Y := Y \setminus W$ will do. ◊

Claim 13.4.46. Let U_Y be as in claim 13.4.45. We have :

(i) There exists an irreducible closed subset Z of Y of dimension one, such that $Z \cap (Y, \underline{N})_{\mathrm{tr}} \cap U_Y \neq \varnothing$.

(ii) For any Z as in (i), the induced functor $\mathsf{Cov}(E(h)) \to \mathsf{Cov}(Z \times_Y E(h))$ is fully faithful.

Proof of the claim. (i): More generally, let $(A, \mathfrak{m})$ be any local noetherian domain of Krull dimension $d \geq 1$, and $W \subset \mathrm{Spec}\, A$ a proper closed subset; we show that there exists an irreducible closed subset $Z \subset \mathrm{Spec}\, A$ of dimension one, not contained in W. To this aim, we may assume that $W = \mathrm{Spec}\, A/fA$ for some $f \in \mathfrak{m} \setminus \{0\}$; let $\mathfrak{n}$ be any maximal ideal of $A[f^{-1}]$, and $\mathfrak{p} := A \cap \mathfrak{n}$. Since $A/\mathfrak{p}[f^{-1}]$ is a field, [63, Ch.0, Cor.16.3.3] implies that $Z := \mathrm{Spec}\, A/\mathfrak{p}$ will do.

(ii): It suffices to check that conditions (i)–(iii) of proposition 13.1.37 hold for Z and the open subset $E(h)$. However, condition (i) is immediate, since the generic point η_Z of Z lies in U_Y. Likewise, condition (ii) holds trivially for the fibre over the point η_Z, so it suffices to consider the fibre over the closed point y of Z, in which case the assertion is just lemma 13.4.29. Lastly, condition (iii) follows directly from theorem 12.7.8(iii.b) and [66, Ch.IV, Prop.18.8.10, 18.8.12(i)]. ◊

Let Z be as in claim 13.4.46(i), and endow Z with its reduced subscheme structure. Let also Z' be the normalization of Z; then both Z and Z' are strictly local, and the morphism $Z' \to Z$ is radicial and surjective, hence the induced functor

$$\mathsf{Cov}(Z \times_Y E(h)) \to \mathsf{Cov}(Z' \times_Y E(h))$$

is an equivalence (lemma 13.1.7(i)). Taking into account claim 13.4.46, we are thus reduced to showing that the morphism

$$(Z' \times_Y E(h)) \times_{X(\overline{x})} C' \to Z' \times_Y E(h)$$

is a trivial étale covering. However, let $\eta_{Z'}$ be the generic point of Z', and set

$$(Z', \underline{N}') := Z' \times_Y (Y, \underline{N}) \qquad (X', \underline{M}') := Z' \times_Y (X, \underline{M}).$$

The open subset $(Z', \underline{N}')_{\mathrm{tr}}$ is dense in Z', by virtue of claim 13.4.46(i), so the induced morphism $f' : (X', \underline{M}') \to (Z', \underline{N}')$ is still of the type considered in (13.4), the geometric point $\overline{x}$ lifts uniquely (up to isomorphism) to a geometric point $\overline{x}'$ of X', and $h \times_Y Z'$ is an object of $\mathbf{Tame}(f', \overline{x}', \kappa(\eta_{Z'}))$ (proposition 13.3.20(i)). Hence, we may replace from start $(X, \underline{M})$ by $(X', \underline{M}')$, $(Y, \underline{N})$ by $(Z', \underline{N}')$, h by $h \times_Y Z'$, after which, we may assume that Y is noetherian and of dimension one. Moreover, taking into account claim 13.4.42, we may assume that the induced covering $C_{|\eta_\mathfrak{q}} \to |\eta_\mathfrak{q}|$ is trivial, for every maximal point $\eta_\mathfrak{q}$ of $f_{\overline{x}}^{-1}(\overline{y})$, and it remains to show that h is trivial under these assumptions.

To this aim, we look at the corresponding commutative diagram of groups, provided by lemma 13.4.33(i) : with the notation of *loc.cit.*, we see that in the current situation, β is an isomorphism as well, by claim 13.4.40, therefore lemma 13.4.33(ii) says that the group homomorphism $\pi_1(|\eta_\mathfrak{q}|_{\acute{e}t}, \overline{\eta}_\mathfrak{q}) \to \pi_1(\overline{U}_{\mathrm{tr}}/Y_{\acute{e}t}, \xi)$ is surjective, for any maximal point $\eta_\mathfrak{q}$ of $f_{\overline{x}}^{-1}(\overline{y})$. From this, we deduce that $\overline{h}$ is a trivial covering, and therefore there exists an étale covering $C_Y \to |\eta|$ with an isomorphism $C \overset{\sim}{\to} C_Y \times_{|\eta|} U_{\mathrm{tr}}$ ([82, Exp.IX, Th.6.1]). Denote by C_Y^ν the normalization of Y in C_Y. Also, set $E' := E(h) \subset \mathrm{Str}(f_{\overline{x}})$. In light of theorem 12.7.8(iii.a) and lemma 13.4.29, it is easily seen that the restriction $E' \to Y$ of $f_{\overline{x}}$ is surjective; the latter is also a smooth morphism of schemes (corollary 12.3.27(i)). It follows that $C^{\nu}_Y \times_Y E'$ is the normalization of E' in C ([66, Ch.IV, Prop.17.5.7]), especially, it is an étale covering of E'. We then deduce that C^{ν}_Y is already a (trivial) étale covering of Y ([66, Ch.IV, Prop.17.7.1(ii)]), and then clearly h must be a trivial covering as well. □

Remark 13.4.47. Theorem 13.4.39 is the local acyclicity result that gives the name to this section. However, the title is admittedly not self-explanatory, and its full justification would require the introduction of a more advanced theory of the *log-étale site*, that lies beyond the bounds of this treatise. In rough terms, we can try to describe the situation as follows. In lieu of the standard strict henselization, one should consider a suitable notion of strict *log henselization* for points of the *log-étale topoi* associated with log schemes. Then, for $f : X \to Y$ as in (13.4) with saturated log structures on both X and Y, and log-étale points $\widetilde{x}$ of X with image $\widetilde{y}$ in Y, one should look, not at our $f_{\overline{x}}$, but rather at the induced morphism $f_{\widetilde{x}}$ of strict log henselizations (of X at $\widetilde{x}$ and of Y at $\widetilde{y}$). The (suitably defined) *log geometric fibres* of $f_{\widetilde{x}}$ will be the *log Milnor fibres* of f at the log-étale point $\widetilde{x}$, and one can state for such fibres an acyclicity result : namely, the prime-to-p quotients of their (again, suitably defined) *log fundamental groups* vanish. The proof proceeds by reduction to our theorem 13.4.39, which, with hindsight, is seen to supply the essential geometric information encoded in the more sophisticated log-étale language.

14. THE ALMOST PURITY TOOLBOX

The sections of this rather heterogeneous chapter are each devoted to a different subject, and are linked to each other only very loosely, if they are at all. They have been lumped here together, because they each contribute a distinct self-contained little theory, that will find application in one step or other of the proof of the almost purity theorem or of its applications in chapter 17. The exception is section 14.8 : it studies a class of rings more general than the measurable algebras introduced in section 14.5; the results of section 14.8 will not be used elsewhere in this treatise, but they might be interesting for other purposes.

Section 14.4 develops the yoga of almost pure pairs (see definition 14.4.1(i)); the relevance to the almost purity theorem is clear, since the latter establishes the almost purity of certain pairs (X, Z) consisting of a scheme X and a closed subscheme $Z \subset X$. This section provides the means to perform various kinds of reductions in the proof of the almost purity theorem, allowing to replace the given pair (X, Z) by more tractable ones.

Section 14.5 introduces measurable (and more generally, ind-measurable) K^+-algebras, for K^+ a fixed valuation ring of rank one : see definitions 14.5.3(ii) and 14.5.64. For modules over a measurable algebra, one can define a well-behaved real-valued normalized length. This length function is non-negative, and additive for short exact sequences of modules. Moreover, the length of an almost zero module vanishes (for the standard almost structure associated with K^+). Conversely, under some suitable assumptions, a module of normalized length zero will be almost zero.

Lastly, section 14.6 studies some questions concerning the formation of quotients of affine almost schemes under a finite group action.

14.1. **Non-flat almost structures.**

This section contains some material that complements the generalities of [75, §2.4, §2.5] : indeed, whereas many of the preliminaries in *loc.cit.* make no assumptions on the basic setup $(V, \mathfrak{m})$ that underlies the whole discussion, for the more advanced results it is usually required that $\widetilde{\mathfrak{m}} := \mathfrak{m} \otimes_V \mathfrak{m}$ is a flat V-module. We shall show how, with more work, one can remove this condition (or at least, weaken it significantly) and still recover most of the useful almost ring theory of [75]. This extension shall be applied in a later section, in order to state and prove the most general case of almost purity.

14.1.1. We begin with a simple observation, that completes the discussion of [75, §2.1.6].

Proposition 14.1.2. *Let V be a reduced ring, and consider ideals $\mathfrak{m}, J_1, \dots, J_k \subset V$ such that:*

(a) $\mathfrak{m}$ *is a filtered union of principal ideals.*

(b) $J_1 \otimes_V \cdots \otimes_V J_k$ *is a flat V-module.*

Then the following holds :

(i) *The natural map $\mathfrak{m} \otimes_V \mathfrak{m} \to \mathfrak{m}^2$ is an isomorphism.*

(ii) *If the set $\mathrm{Min}\, V$ of minimal prime ideals of V is a quasi-compact subset of $\mathrm{Spec}\, V$, then the natural map $J_1 \otimes_V \cdots \otimes_V J_k \to J_1 \cdots J_k$ is an isomorphism.*

Proof. (i): We are easily reduced to checking the following. For every $x \in V$, the natural map $\mu : Vx \otimes_V Vx \to Vx^2$ is an isomorphism. Now, $I_n := \mathrm{Ann}_V(x^n)$ for $n = 1, 2$, and notice that $I_1 \subset I_2$; the map μ is naturally identified with the map

$$V/I_1 \otimes_V V/I_1 \to V/I_2 \qquad \overline{a} \otimes \overline{b} \mapsto \overline{ab}.$$

The latter is the composition of the natural isomorphism $V/I_1 \otimes_V V/I_1 \stackrel{\sim}{\to} V/I_1$ and the projection $V/I_1 \to V/I_2$, so we are reduced to checking that $I_1 = I_2$. But if $x^2 y = 0$ for some $y \in V$, then $(xy)^2 = 0$, hence $xy = 0$, since V is reduced, and so $y \in I_1$, whence the contention.

(ii): Set $L := J_1 \otimes_V \cdots \otimes_V J_k$ and $J := J_1 \cdots J_k$; the natural map $\varphi : L \to J$ is surjective, so it remains to check that it is injective, under the assumptions of (ii). However, notice that, for every $\mathfrak{p} \in Y := \mathrm{Min}\, V$, the localization $V_\mathfrak{p}$ is a field, since V is reduced; it follows easily that $\varphi_\mathfrak{p} : L_\mathfrak{p} \to J_\mathfrak{p}$ is an isomorphism for every such $\mathfrak{p}$. We are then reduced to checking that the natural map $\psi_L : L \to \prod_{\mathfrak{p} \in \mathrm{Min}\, V} L_\mathfrak{p}$ is injective. However, endow Y with the topology induced by $X := \mathrm{Spec}\, V$ via the inclusion map $i : Y \to X$, and denote by $L^\sim$ the quasi-coherent $\mathscr{O}_X$-module arising from L. Then ψ_L is a composition of maps

$$L \xrightarrow{\psi'_L} \Gamma(Y, i^* L^\sim) \xrightarrow{\psi''_L} \prod_{\mathfrak{p} \in Y} L^\sim_\mathfrak{p} = \prod_{\mathfrak{p} \in Y} L_\mathfrak{p}$$

and ψ''_L is obviously injective. Thus, it remains to show that ψ'_L is injective. But, by Lazard's theorem [120, Ch.I,Th.1.2], L is the colimit of a filtered system $(F_\lambda \mid \lambda \in \Lambda)$ of free V-modules of finite rank, so $L^\sim$ is the colimit of the induced filtered system $(F^\sim_\lambda \mid \lambda \in \Lambda)$ of quasi-coherent $\mathscr{O}_X$-modules. By lemma 10.1.7(i), we are then reduced to checking that the natural map $F_\lambda \to \Gamma(Y, i^* F^\sim_\lambda)$ is injective for every $\lambda \in \Lambda$. For the latter, it suffices to check that the natural map $\psi_V : V \to \Gamma(Y, i^* \mathscr{O}_X)$ is injective; but the composition of ψ_V with the natural map $\Gamma(Y, i^* \mathscr{O}_X) \to \prod_{\mathfrak{p} \in Y} \mathscr{O}_{X,\mathfrak{p}} = \prod_{\mathfrak{p} \in Y} V_\mathfrak{p}$ is injective, since V is reduced, whence the claim. □

Remark 14.1.3. As a corollary, we see that if $(V, \mathfrak{m})$ is a basic setup (in the sense of [75, §2.1.1]), such that V is a reduced ring, then the natural map $\widetilde{\mathfrak{m}} \to \mathfrak{m}$ is an isomorphism, provided that either (a) $\mathfrak{m}$ is a filtered union of principal ideal, or else (b) $\widetilde{\mathfrak{m}}$ is a flat V-module, and $\mathrm{Min}\, V$ is a quasi-compact subset of $\mathrm{Spec}\, V$.

14.1.4. Let now $(V, \mathfrak{m})$ be any *basic setup* in the sense of [75, §2.1.1], and R any V-algebra. For every interval $I \subset \mathbb{N}$, we have a localization functor

$$\mathsf{C}^I(R\text{-}\mathbf{Mod}) \to \mathsf{C}^I(R^a\text{-}\mathbf{Mod}) \qquad K^\bullet \mapsto K^{\bullet a} \tag{14.1.5}$$

from complexes of R-modules, to complexes of R^a-modules, which is obviously exact, hence it induces a derived localization functor :

$$\mathsf{D}^I(R\text{-}\mathbf{Mod}) \to \mathsf{D}^I(R^a\text{-}\mathbf{Mod}). \tag{14.1.6}$$

The functor (14.1.5) admits a left (resp. right) adjoint

$$\mathsf{C}^I(R^a\text{-}\mathbf{Mod}) \to \mathsf{C}^I(R\text{-}\mathbf{Mod}) \qquad K^\bullet \mapsto K^\bullet_! \qquad (\text{ resp. } K^\bullet \mapsto K^\bullet_*) \tag{14.1.7}$$

defined by applying termwise to $K^\bullet$ the functor $M \mapsto M_!$ (resp. $M \mapsto M_*$) for R^a-modules given by [75, §2.2.10, §2.2.21]. However, if $\widetilde{\mathfrak{m}}$ is not flat, the functor $M \mapsto M_!$ is obviously not exact, and the localization functor $R\text{-}\mathbf{Mod} \to R^a\text{-}\mathbf{Mod}$ does not send injectives to injectives (cp. [75, Cor.2.2.24]). This makes it trickier to deal with constructions in the derived category; for instance, if $\widetilde{\mathfrak{m}}$ is flat, we get a left adjoint to (14.1.6), simply by deriving trivially the exact functor $M \mapsto M_!$. This fails in the general case, but we shall see later that a suitable derived version of the construction of $M_!$ is still available.

14.1.8. For any interval $I \subset \mathbb{N}$, let Σ_I be the multiplicative set of morphisms φ in $\mathsf{D}^I(R\text{-}\mathbf{Mod})$ such that φ^a is an isomorphism in $\mathsf{D}^I(R^a\text{-}\mathbf{Mod})$; arguing as in the proof of [163, Prop.10.4.4], one sees that Σ_I is right locally small, hence the localized category $\Sigma_I^{-1}\mathsf{D}^I(R\text{-}\mathbf{Mod})$ has small Hom-sets (proposition 1.6.16(ii)) which are independent (up to natural isomorphism) of the choice of universe. Obviously the derived localization functor factors through a natural functor

$$\Sigma_I^{-1}\mathsf{D}^I(R\text{-}\mathbf{Mod}) \to \mathsf{D}^I(R^a\text{-}\mathbf{Mod}). \tag{14.1.9}$$

Lemma 14.1.10. (i) *For every interval I, the functor* (14.1.9) *is an equivalence.*

(ii) *For every $K^\bullet, L^\bullet \in \mathrm{Ob}(\mathsf{D}^I(R\text{-}\mathbf{Mod}))$ the induced R^a-linear morphism*

$$\mathrm{Hom}_{\mathsf{D}^I(R\text{-}\mathbf{Mod})}(K^\bullet, L^\bullet)^a \to \mathrm{Hom}_{\mathsf{D}^I(R^a\text{-}\mathbf{Mod})}(K^{\bullet a}, L^{\bullet a})^a$$

is an isomorphism.

Proof. (i): A proof is sketched in [75, §2.4.9], in case $\widetilde{\mathfrak{m}}$ is flat, but in fact this assumption is superfluous. Indeed, since the unit of adjunction $M \to M_!^a$ is an isomorphism ([75, Prop.2.2.23(ii)]), it is clear that the functor $K^\bullet \mapsto K_!^\bullet$ of (14.1.7) descends to a functor

$$\mathsf{D}^I(R^a\text{-}\mathbf{Mod}) \to \Sigma_I^{-1}\mathsf{D}^I(R\text{-}\mathbf{Mod}) \tag{14.1.11}$$

such that the composition (14.1.9) ∘ (14.1.11) is naturally isomorphic to the identity automorphism of $\mathsf{D}^I(R^a\text{-}\mathbf{Mod})$. Likewise, a simple inspection shows that (14.1.11) ∘ (14.1.9) is naturally isomorphic to the identity of $\Sigma_I^{-1}\mathsf{D}^I(R\text{-}\mathbf{Mod})$, whence the contention.

(ii): The R-module $\mathrm{Hom}_{\Sigma_I^{-1}\mathsf{D}^I(R\text{-}\mathbf{Mod})}(K^\bullet, L^\bullet)$ is calculated as the colimit of the system of R-modules $(\mathrm{Hom}_{\mathsf{D}^I(R\text{-}\mathbf{Mod})}(K'^\bullet, L^\bullet) \mid \varphi^\bullet : K'^\bullet \to K^\bullet)$, where $\varphi^\bullet$ ranges over the elements of Σ_I with target $K^\bullet$. For such a morphism $\varphi^\bullet$, we have $\mathfrak{m} \cdot H^i(\mathrm{Cone}\,\varphi^\bullet) = 0$ for every $i \in \mathbb{Z}$.

Claim 14.1.12. Let $C^\bullet$ be any object of $\mathsf{D}^I(R\text{-}\mathbf{Mod})$ such that $(H^iC^\bullet)^a = 0$ for every $i \in \mathbb{Z}$. Then $\mathrm{Hom}_{\mathsf{D}^I(R\text{-}\mathbf{Mod})}(C^\bullet, L^\bullet)^a = 0$ for every $L^\bullet \in \mathrm{Ob}(\mathsf{D}^I(R\text{-}\mathbf{Mod}))$.

Proof of the claim. Set $H_{n,k} := \mathrm{Hom}_{\mathsf{D}^I(R\text{-}\mathbf{Mod})}(C^\bullet, (\tau^{\geq -n}L^\bullet)[k])$ for every $n, k \in \mathbb{N}$; the natural morphism $L^\bullet \to \lim_{n\in\mathbb{N}} \tau^{\geq -n}L^\bullet$ is an isomorphism in $\mathsf{C}(R\text{-}\mathbf{Mod})$, so we have a short exact sequence

$$0 \to \lim_{n\in\mathbb{N}}{}^1 H_{n,-1} \to \mathrm{Hom}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(C^\bullet, L^\bullet) \to \lim_{n\in\mathbb{N}} H_{n,0} \to 0$$

(proposition 7.3.45(ii)). Hence, it suffices to check that $H_{n,0}^a = H_{n,-1}^a = 0$ for every $n \in \mathbb{N}$, and we are reduced to the case where $L^\bullet$ is bounded below.

Likewise, set $H'_n := \mathrm{Hom}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(\tau^{\leq n}C^\bullet, L^\bullet)$ for every $n, k \in \mathbb{Z}$; since the natural morphism $\mathrm{colim}_{n\in\mathbb{N}}\, \tau^{\leq n}C^\bullet \to C^\bullet$ is an isomorphism in $\mathsf{C}(R\text{-}\mathbf{Mod})$, we have an isomorphism

$$\mathrm{Hom}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(C^\bullet, L^\bullet) \xrightarrow{\sim} \lim_{n\in\mathbb{N}} H'_n$$

(proposition 7.3.45(i)), so we may assume that $C^\bullet$ is a bounded above complex. In this situation, say that $L^\bullet \in \mathrm{Ob}(\mathsf{D}^{\geq a}(R\text{-}\mathbf{Mod}))$ and $C^\bullet \in \mathrm{Ob}(\mathsf{D}^{\leq b}(R\text{-}\mathbf{Mod}))$ for some $a, b \in \mathbb{Z}$; we may assume that $a \leq b$, and we show, by induction on n, that $(H'_n)^a = 0$ for every $n \geq a - 1$. The claim will follow for $n = b$. Indeed, notice that

$$H'_r = \mathrm{Hom}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(\tau^{\leq r}C^\bullet, \tau^{\leq r}L^\bullet) \qquad \text{for every } r \in \mathbb{Z}$$

by virtue of remark 7.3.14(iv), so the assertion is clear for $n = a - 1$. Next, suppose that the assertion is already known for every integer $< n$; from the short exact of complexes $0 \to \tau^{\leq n-1}C^\bullet \to \tau^{\leq n}C^\bullet \to (H^n C^\bullet)[-n] \to 0$ we deduce the exact sequence

$$\mathrm{Hom}_{\mathsf{D}(R\text{-}\mathbf{Mod})}((H^n C^\bullet)[-n], L^\bullet) \to H'_n \to H'_{n-1}$$

(remark 7.3.35). But since $\mathfrak{m} \cdot H^n C^\bullet = 0$, the first term in this exact sequence is annihilated by $\mathfrak{m}$, and the same holds for the third term, by inductive assumption; then it holds for the middle term as well, as required. ◊

By claim 14.1.12, we have $\mathrm{Hom}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(\mathrm{Cone}\,\varphi^\bullet, L^\bullet)^a = 0$; by considering the long exact $\mathrm{Hom}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(-, L^\bullet)$-sequence associated with the distinguished triangle $K'^\bullet \to K^\bullet \to \mathrm{Cone}\,\varphi$, we deduce an R^a-linear isomorphism

$$\mathrm{Hom}_{\mathsf{D}^I(R\text{-}\mathbf{Mod})}(\varphi^\bullet, L^\bullet)^a : \mathrm{Hom}_{\mathsf{D}^I(R\text{-}\mathbf{Mod})}(K'^\bullet, L^\bullet)^a \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{D}^I(R\text{-}\mathbf{Mod})}(K^\bullet, L^\bullet)^a$$

(see remark 7.3.35). In light of (i), the assertion follows straightforwardly. □

14.1.13. We sketch a few generalities on derived tensor products, that will be applied to construct useful objects in various derived categories. Recall first that the tensor product $-\otimes_R -$ on R-modules descends to a bifunctor $-\otimes_{R^a} -$ ([75, §2.2.6, §2.2.12]), and if M is a flat R-module, then M^a is a flat R^a-module ([75, Lemma 2.4.7]). It follows that the category R^a-$\mathbf{Mod}$ has enough flat objects, so every bounded above complex of R^a-modules admits a bounded above flat resolution. Now, given bounded above complexes $K^\bullet, L^\bullet$ of R^a-modules, set

$$K^\bullet \overset{\mathbf{L}}{\otimes}_{R^a} L^\bullet := (K^\bullet_! \overset{\mathbf{L}}{\otimes}_R L^\bullet_!)^a$$

which is a well defined object of $\mathsf{D}(R^a\text{-}\mathbf{Mod})$.

- We claim that this rule yields a well defined functor

$$- \overset{\mathbf{L}}{\otimes}_{R^a} - : \mathsf{D}^-(R^a\text{-}\mathbf{Mod}) \times \mathsf{D}^-(R^a\text{-}\mathbf{Mod}) \to \mathsf{D}^-(R^a\text{-}\mathbf{Mod}).$$

Indeed, suppose $\varphi^\bullet : K^\bullet_1 \to K^\bullet_2$ is a morphism in $\mathsf{C}(R^a\text{-}\mathbf{Mod})$, inducing an isomorphism in $\mathsf{D}(R^a\text{-}\mathbf{Mod})$, and set $C^\bullet := \mathrm{Cone}\,(\varphi^\bullet_!)$; clearly, $C^{\bullet a} = 0$ in $\mathsf{D}(R^a\text{-}\mathbf{Mod})$. Now, pick any flat bounded above resolution $P^\bullet \to L^\bullet_!$, so that $K^\bullet_{i!} \otimes_R P^\bullet$ computes $K^\bullet_{i!} \overset{\mathbf{L}}{\otimes}_R L^\bullet_!$, for $i = 1, 2$. We get natural isomorphism :

$$\mathrm{Cone}(\varphi^\bullet \overset{\mathbf{L}}{\otimes}_{R^a} L^\bullet) \xrightarrow{\sim} (C^\bullet \otimes_R P^\bullet)^a \xrightarrow{\sim} C^{\bullet a} \otimes_{R^a} P^{\bullet a} = 0.$$

so the derived tensor product depends only on the image of $K^\bullet$ in $\mathsf{D}^-(R^a\text{-}\mathbf{Mod})$; likewise for the argument $L^\bullet$, whence the contention.

- Next, suppose that $P^\bullet \to K^\bullet$ is a bounded above resolution, with $P^\bullet$ a complex of flat R^a-modules; we claim that there is a natural isomorphism in $\mathsf{D}(R^a\text{-}\mathbf{Mod})$

$$K^\bullet \overset{\mathbf{L}}{\otimes}_{R^a} L^\bullet \xrightarrow{\sim} P^\bullet \otimes_{R^a} L^\bullet.$$

Indeed, pick any bounded above flat resolution $Q^\bullet \to L^\bullet_!$; we have natural isomorphisms

$$K^\bullet \overset{\mathbf{L}}{\otimes}_{R^a} L^\bullet \xrightarrow{\sim} (K^\bullet_! \otimes_R Q^\bullet)^a \xrightarrow{\sim} K^\bullet \otimes_{R^a} Q^{\bullet a} \xrightarrow{\sim} P^\bullet \otimes_{R^a} Q^{\bullet a}$$

in $\mathsf{D}(R^a\text{-}\mathbf{Mod})$, where the last holds, since $Q^{\bullet a}$ is a complex of flat R^a-modules. Finally, the induced map

$$P^\bullet \otimes_{R^a} Q^{\bullet a} \to P^\bullet \otimes_{R^a} L^\bullet$$

is also an isomorphism in $\mathsf{D}(R^a\text{-}\mathbf{Mod})$, since $P^\bullet$ is a complex of flat R^a-modules, so the claim follows.

• Notice that, for any $K^\bullet \in \mathrm{Ob}(\mathsf{D}^-(R\text{-}\mathbf{Mod}))$ and any flat resolution $P^\bullet \to K^\bullet$, the induced morphism $P^{\bullet a} \to K^{\bullet a}$ is a flat resolution; it follows that, for any $L^\bullet \in \mathrm{Ob}(\mathsf{D}(R\text{-}\mathbf{Mod}))$ we get a natural isomorphism

$$(K^\bullet \overset{\mathbf{L}}{\otimes}_R L^\bullet)^a \xrightarrow{\sim} K^{\bullet a} \overset{\mathbf{L}}{\otimes}_{R^a} L^{\bullet a} \qquad \text{in } \mathsf{D}(R^a\text{-}\mathbf{Mod}). \tag{14.1.14}$$

Remark 14.1.15. Clearly, for the derived tensor products of R^a-modules, one has the same commutativity and associativity isomorphisms as the ones detailed in remark 7.3.39(i) for usual modules, as well as the vanishing properties of lemma 7.3.42(ii).

We are now ready to return to the question of the existence of adjoints to derived localization. The key point is the following :

Lemma 14.1.16. *In the situation of* (14.1.4)*, let $K^\bullet$ be any complex of R-modules, $i \in \mathbb{Z}$ any integer and suppose that :*
   (a) $K^\bullet \in \mathrm{Ob}(\mathsf{D}^{\leq i}(R\text{-}\mathbf{Mod}))$
   (b) $K^{\bullet a} \in \mathrm{Ob}(\mathsf{D}^{\leq i-1}(R^a\text{-}\mathbf{Mod}))$.

Then $\mathfrak{m} \overset{\mathbf{L}}{\otimes}_V K^\bullet \in \mathrm{Ob}(\mathsf{D}^{\leq i-1}(R\text{-}\mathbf{Mod}))$.

Proof. We apply the standard spectral sequence

$$E^2_{pq} := \mathrm{Tor}^V_p(\mathfrak{m}, H^q K^\bullet) \Rightarrow H^{q-p}(\mathfrak{m} \overset{\mathbf{L}}{\otimes}_V K^\bullet).$$

Indeed, (a) says that $H^q K^\bullet = 0$ for every $q > i$, and (b) says that $(H^i K^\bullet)^a = 0$, and therefore $\mathfrak{m}_V \otimes_V H^i K^\bullet = 0$ ([75, Rem.2.1.4(i)]). In either case, we conclude that $E^2_{pq} = 0$ whenever $q - p \geq i$, whence the lemma. □

14.1.17. Now, let us define inductively :

$$\mathfrak{M}^\bullet_0 := V[0] \quad \text{and} \quad \mathfrak{M}^\bullet_{i+1} := \mathfrak{m} \overset{\mathbf{L}}{\otimes}_V \mathfrak{M}^\bullet_i \quad \text{for every } i \in \mathbb{N}.$$

A simple induction shows that $\mathfrak{M}^\bullet_i \in \mathsf{D}^{\leq 0}(V\text{-}\mathbf{Mod})$ for every $i \in \mathbb{N}$, so all these derived tensor product are well defined in $\mathsf{D}^{\leq 0}(V\text{-}\mathbf{Mod})$. Moreover, from the short exact sequence of V-modules

$$\Sigma \quad : \quad 0 \to \mathfrak{m} \to V \to V/\mathfrak{m} \to 0$$

we obtain a distiguished triangle

$$\mathfrak{M}^\bullet_i \overset{\mathbf{L}}{\otimes}_V \Sigma \quad : \quad \mathfrak{M}^\bullet_{i+1} \to \mathfrak{M}^\bullet_i \to \mathfrak{M}^\bullet_i \overset{\mathbf{L}}{\otimes}_V (V/\mathfrak{m}) \to \mathfrak{M}^\bullet_{i+1}[1] \qquad \text{for every } i \in \mathbb{N}.$$

Especially, we get an inverse system of morphisms in $\mathsf{D}^{\leq 0}(V\text{-}\mathbf{Mod})$:

$$\cdots \to \mathfrak{M}^\bullet_{i+1} \xrightarrow{\pi^\bullet_i} \mathfrak{M}^\bullet_i \xrightarrow{\pi^\bullet_{i-1}} \mathfrak{M}^\bullet_{i-1} \to \cdots \to \mathfrak{M}^\bullet_0 := V[0].$$

Also, from (14.1.14) we deduce natural isomorphisms in $\mathsf{D}^{\leq 0}(V^a\text{-}\mathbf{Mod})$:

$$\mathfrak{M}^{\bullet a}_i \xrightarrow{\sim} V^a[0] \qquad \text{for every } i \in \mathbb{N} \tag{14.1.18}$$

and under these identifications, the morphism $\pi^{\bullet a}_i$ corresponds to the identity automorphism of $V^a[0]$ (details left to the reader). Furthermore, we deduce the following derived version of [75, Rem.2.1.4(i)] :

Proposition 14.1.19. *Let $a, b \in \mathbb{Z}$ be any integers with $a \leq b$. We have :*
   (i) *For every $K^\bullet \in \mathrm{Ob}(\mathsf{D}^{[a,b]}(R\text{-}\mathbf{Mod}))$, the following conditions are equivalent :*
      (a) $K^{\bullet a} \simeq 0$ *in* $\mathsf{D}^{[a,b]}(R^a\text{-}\mathbf{Mod})$.
      (b) $\tau^{\geq a}(\mathfrak{M}^\bullet_{b-a+1} \overset{\mathbf{L}}{\otimes}_V K^\bullet) \simeq 0$ *in* $\mathsf{D}^{[a,b]}(R\text{-}\mathbf{Mod})$.

(ii) *For every morphism* $\varphi^\bullet : K^\bullet \to L^\bullet$ *in* $\mathsf{D}^{\leq b}(R\text{-}\mathbf{Mod})$*, the following conditions are equivalent :*

(a) $\varphi^{\bullet a}$ *is an isomorphism in* $\mathsf{D}^{[a,b]}(R^a\text{-}\mathbf{Mod})$.

(b) $\tau^{\geq a}(\mathfrak{M}^\bullet_{b-a+2} \overset{\mathbf{L}}{\otimes}_V \varphi^\bullet)$ *is an isomorphism in* $\mathsf{D}^{[a,b]}(R\text{-}\mathbf{Mod})$.

Proof. (i): From (14.1.18), it is easily seen that (b)⇒(a). The other direction follows straightforwardly from lemma 14.1.16, via an easy descending induction on b.

(ii): Again, the direction (b)⇒(a) is immediate from (14.1.18). For the other direction, denote by $C^\bullet$ the cone of $\varphi^\bullet$; then $C^\bullet \in \mathrm{Ob}(\mathsf{D}^{[a-1,b]}(R\text{-}\mathbf{Mod}))$, and $C^{\bullet a} \simeq 0$ in $\mathsf{D}^{[a-1,b]}(R^a\text{-}\mathbf{Mod})$. From (i) we deduce that $\tau^{\geq a-1}(\mathfrak{M}^\bullet_{b-a+2} \overset{\mathbf{L}}{\otimes}_V C^\bullet) \simeq 0$ in $\mathsf{D}^{[a-1,b]}(R\text{-}\mathbf{Mod})$. Since the derived tensor product is a triangulated functor, the assertion follows easily : details left to the reader. □

Corollary 14.1.20. *With the notation of* (14.1.17)*, the morphism*

$$\tau^{\geq 2-i}\pi^\bullet_i : \tau^{\geq 2-i}\mathfrak{M}^\bullet_{i+1} \to \tau^{\geq 2-i}\mathfrak{M}^\bullet_i$$

is an isomorphism in $\mathsf{D}^{[2-i,0]}(V\text{-}\mathbf{Mod})$ *for every integer* $i \geq 2$.

Proof. By construction, we have a natural isomorphism :

$$\mathrm{Cone}(\tau^{\geq 2-i}\pi^\bullet_i) \xrightarrow{\sim} \tau^{\geq 1-i}(\mathfrak{M}^\bullet_i \overset{\mathbf{L}}{\otimes}_V (V/\mathfrak{m})) \qquad \text{in } \mathsf{D}^{[1-i,0]}(V\text{-}\mathbf{Mod})$$

in light of which, the assertion is an immediate consequence of proposition 14.1.19(i). □

Proposition 14.1.21. *In the situation of* (14.1.4)*, we have :*

(i) *The localization functor* $\mathsf{D}^+(R\text{-}\mathbf{Mod}) \to \mathsf{D}^+(R^a\text{-}\mathbf{Mod})$ *admits the right adjoint :*

$$\mathsf{D}^+(R^a\text{-}\mathbf{Mod}) \to \mathsf{D}^+(R\text{-}\mathbf{Mod}) \quad : \quad K^\bullet \mapsto K^\bullet_{[*]} := R\mathrm{Hom}^\bullet_{R^a}(R^a[0], K^\bullet). \tag{14.1.22}$$

(ii) *Let* $a, b, i \in \mathbb{Z}$ *be any three integers with* $b \geq a$ *and* $i \geq b - a + 2$*. The localization functor* $\mathsf{D}^{[a,b]}(R\text{-}\mathbf{Mod}) \to \mathsf{D}^{[a,b]}(R^a\text{-}\mathbf{Mod})$ *admits the left adjoint :*

$$\mathsf{D}^{[a,b]}(R^a\text{-}\mathbf{Mod}) \to \mathsf{D}^{[a,b]}(R\text{-}\mathbf{Mod}) \quad : \quad K^\bullet \mapsto K^\bullet_{[!]} := \tau^{\geq a}(\mathfrak{M}^\bullet_i \overset{\mathbf{L}}{\otimes}_V K^\bullet_{[*]}).$$

and the right adjoint

$$\mathsf{D}^{[a,b]}(R^a\text{-}\mathbf{Mod}) \to \mathsf{D}^{[a,b]}(R\text{-}\mathbf{Mod}) \quad : \quad K^\bullet \mapsto \tau^{\leq b}K^\bullet_{[*]}.$$

(iii) *For every* $K^\bullet \in \mathrm{Ob}(\mathsf{D}^+(R^a\text{-}\mathbf{Mod}))$ *(resp.* $L^\bullet \in \mathrm{Ob}(\mathsf{D}^{[a,b]}(R^a\text{-}\mathbf{Mod}))$*) the counit of adjunction is an isomorphism*

$$(K^\bullet_{[*]})^a \xrightarrow{\sim} K^\bullet \qquad (\textit{ resp. } (\tau^{\leq b}L^\bullet_{[*]})^a \xrightarrow{\sim} L^\bullet \ ).$$

(iv) *For every* $L^\bullet \in \mathrm{Ob}(\mathsf{D}^{[a,b]}(R^a\text{-}\mathbf{Mod}))$*, the unit of adjunction is an isomorphism*

$$L^\bullet \to (L^\bullet_{[!]})^a.$$

Proof. (i): This is analogous to lemma 10.1.16(iii). Recall the construction : we know that the category R^a-$\mathbf{Mod}$ admits enough injectives ([75, 2.2.18]), hence (14.1.22) can be represented by $I^\bullet_*$, where $K^\bullet \xrightarrow{\sim} I^\bullet$ is any injective resolution of $K^\bullet$, and $I^\bullet_*$ is obtained by applying termwise to $I^\bullet$ the functor $M \mapsto M_*$ of [75, §2.2.10]. Indeed, taking into account [75, Cor.2.2.19] we get natural isomorphisms :

$$\begin{aligned}
\mathrm{Hom}_{\mathsf{D}^+(R\text{-}\mathbf{Mod})}(L^\bullet, I^\bullet_*) &\xrightarrow{\sim} H^0\mathrm{Hom}^\bullet_R(L^\bullet, I^\bullet_*) \\
&\xrightarrow{\sim} H^0\mathrm{Hom}^\bullet_{R^a}(L^{\bullet a}, I^\bullet) \\
&\xrightarrow{\sim} \mathrm{Hom}_{\mathsf{D}(R^a\text{-}\mathbf{Mod})}(L^{\bullet a}, I^\bullet) \\
&\xrightarrow{\sim} \mathrm{Hom}_{\mathsf{D}(R^a\text{-}\mathbf{Mod})}(L^{\bullet a}, K^\bullet)
\end{aligned}$$

for every bounded below complex $L^\bullet$ of R-modules.

(iii) follows by direct inspection of the definitions, taking into account that, for every R^a-module M, the counit of adjunction $(M_*)^a \to M$ is an isomorphism ([75, Prop.2.2.14(iii)]) : details left to the reader.

(ii): We consider first the assertion concerning the right adjoint : let $K^\bullet \in \mathsf{D}^{[a,b]}(R^a\text{-}\mathbf{Mod})$; we may find an injective resolution $K^\bullet \stackrel{\sim}{\to} I^\bullet$ such that $I^j = 0$ for every $j < a$, in which case $I^\bullet_* \in \mathsf{D}^{\geq a}(R\text{-}\mathbf{Mod})$, and $\tau^{\leq b} I^\bullet_*$ represents $\tau^{\leq b} K^\bullet_{[*]}$ in $\mathsf{D}^{[a,b]}(R\text{-}\mathbf{Mod})$. Then, in view of (i), the assertion is reduced to remark 7.3.14(iv). For the left adjoint, let us set

$$\omega L^\bullet := \tau^{\geq a}(\mathfrak{M}^\bullet_i \overset{\mathbf{L}}{\otimes}_V L^\bullet) \qquad \text{for every } L^\bullet \in \mathrm{Ob}(\mathsf{D}^{[a,b]}(R\text{-}\mathbf{Mod})).$$

Then $\omega L^\bullet$ is naturally an object of $\mathsf{D}^{[a,b]}(R\text{-}\mathbf{Mod})$, as explained in remark 7.3.39(ii), and likewise for $K^\bullet_{[!]}$, if $K^\bullet$ is any object of $\mathsf{D}^{[a,b]}(R^a\text{-}\mathbf{Mod})$. We begin with the following :

Claim 14.1.23. Let $K^\bullet, L^\bullet \in \mathrm{Ob}(\mathsf{D}^{[a,b]}(R\text{-}\mathbf{Mod}))$ be any two objects. Then the natural map

$$\mathrm{Hom}_{\mathsf{D}^{[a,b]}(R\text{-}\mathbf{Mod})}(\omega K^\bullet, L^\bullet) \to \mathrm{Hom}_{\mathsf{D}^{[a,b]}(R^a\text{-}\mathbf{Mod})}(K^{\bullet a}, L^{\bullet a})$$

is an isomorphism.

Proof of the claim. Arguing as in [75, §2.2.2], and taking into account lemma 14.1.10(i), we reduce to showing that for any $K^\bullet \in \mathsf{D}^{[a,b]}(R\text{-}\mathbf{Mod})$, the natural morphism

$$\varphi_{K^\bullet} : \omega K^\bullet \to K^\bullet$$

is initial in the full subcategory of $\mathsf{D}^{[a,b]}(R\text{-}\mathbf{Mod})/K^\bullet$ whose objects are the morphisms $\psi : L^\bullet \to K^\bullet$ that lie in $\Sigma_{[a,b]}$. However, for any such ψ, we have a commutative diagram in $\mathsf{D}^{[a,b]}(R\text{-}\mathbf{Mod})$:

$$\begin{array}{ccc} \omega L^\bullet & \xrightarrow{\varphi_{L^\bullet}} & L^\bullet \\ \downarrow & & \downarrow \psi \\ \omega K^\bullet & \xrightarrow{\varphi_{K^\bullet}} & K^\bullet \end{array}$$

whose left vertical arrow is an isomorphism, by proposition 14.1.19(ii). There follows a morphism $\varphi_{K^\bullet} \to \psi$, and we have to check that this is the unique morphism from $\varphi_{K^\bullet}$ to ψ. However, say that $\alpha, \beta : \varphi_{K^\bullet} \to \psi$ are two such morphisms; then their difference is a morphism $\gamma := \alpha - \beta : \omega K^\bullet \to L^\bullet$ such that $\psi \circ \gamma = 0$, so γ factors through a morphism $\overline{\gamma} : \omega K^\bullet \to \mathrm{Cone}\,\psi[-1]$. Set $C^\bullet := \tau^{\leq b}\mathrm{Cone}\,\psi[-1]$; then $C^\bullet \in \mathsf{D}^{[a,b]}(R\text{-}\mathbf{Mod})$, and according to remark 7.3.14(iv), $\overline{\gamma}$ lifts uniquely to a morphism $\omega K^\bullet \to C^\bullet$ that we denote again $\overline{\gamma}$. We deduce a commutative diagram

$$\begin{array}{ccc} \omega \circ \omega K^\bullet & \longrightarrow & \omega C^\bullet \\ {\scriptstyle \varphi_{\omega_{K^\bullet}}} \downarrow & & \downarrow {\scriptstyle \varphi_{C^\bullet}} \\ \omega K^\bullet & \xrightarrow{\overline{\gamma}} & C^\bullet. \end{array}$$

Now, by construction $C^{\bullet a} = 0$, therefore $\omega C^\bullet = 0$ (proposition 14.1.19(i)); on the other hand, $\varphi_{\omega_{K^\bullet}}$ is an isomorphism, by proposition 14.1.19(ii). We conclude that $\overline{\gamma} = 0$, whence $\alpha = \beta$, as sought. ◊

Assertion (ii.a) is an immediate consequence of (iii) and claim 14.1.23; from this, also (iv) is immediate : details left to the reader. □

Remark 14.1.24. Let $a, b, i \in \mathbb{Z}$ be any three integers such that $a \leq b$ and $i \geq b - a + 2$. From propositions 14.1.21(ii.a,iii) and 14.1.19(ii) we deduce a natural isomorphism

$$\tau^{\geq a}(\mathfrak{M}^\bullet_i \overset{\mathbf{L}}{\otimes}_V K^\bullet) \stackrel{\sim}{\to} (K^{\bullet a})_{[!]} \qquad \text{for every } K^\bullet \in \mathrm{Ob}(\mathsf{D}^{[a,b]}(R\text{-}\mathbf{Mod}))$$

(details left to the reader), which allows to compute $(K^{\bullet a})_{[!]}$ purely in terms of $K^\bullet$ and operations within $\mathsf{D}(R\text{-}\mathbf{Mod})$. It turns out that an analogous isomorphism is available also for $K^\bullet_{[*]}$: this is contained in the following

Lemma 14.1.25. *Let $a, b, i \in \mathbb{N}$ be any integers such that $a \leq b$ and $i \geq b - a + 2$. For every $K^\bullet \in \mathrm{Ob}(\mathsf{D}^{[a,b]}(R\text{-}\mathbf{Mod}))$, we have a natural isomorphism :*

$$\tau^{\leq b} R\mathrm{Hom}^\bullet_V(\mathfrak{M}^\bullet_i, K^\bullet) \xrightarrow{\sim} \tau^{\leq b} K^{\bullet a}_{[*]}.$$

Proof. Theorem 10.1.18(ii) yields a natural isomorphism

$$R\mathrm{Hom}^\bullet_V(\mathfrak{M}^\bullet_i, K^\bullet) \xrightarrow{\sim} R\mathrm{Hom}^\bullet_R(\mathfrak{M}^\bullet_i \overset{\mathbf{L}}{\otimes}_V R[0], K^\bullet).$$

To compute the right-hand side, we may fix an injective resolution $K^\bullet \xrightarrow{\sim} I^\bullet$; the complex $I^{\bullet a}$ is not necessarily injective, but we can find an injective resolution $\varphi : I^{\bullet a} \xrightarrow{\sim} J^\bullet$ (in the category of bounded below complexes of R^a-modules). In view of (14.1.14) and (14.1.18), the morphism φ induces a natural transformation

$$\psi : R\mathrm{Hom}^\bullet_R(\mathfrak{M}^\bullet_i \overset{\mathbf{L}}{\otimes}_V R[0], K^\bullet) \to R\mathrm{Hom}^\bullet_{R^a}(R^a[0], K^{\bullet a}) = (K^{\bullet a})_{[*]}$$

and it suffices to show that, for every $j \leq b$, the map

$$H^j\psi : \mathrm{Hom}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(\mathfrak{M}^\bullet_i \overset{\mathbf{L}}{\otimes}_V R[0], K^\bullet[j]) \to \mathrm{Hom}_{\mathsf{D}(R^a\text{-}\mathbf{Mod})}(R^a[0], K^{\bullet a}[j])$$

is an isomorphism. However, by remark 7.3.14(iv), the latter is the same as a map

$$\mathrm{Hom}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(\mathfrak{M}^\bullet_i \overset{\mathbf{L}}{\otimes}_V R[0], \tau^{\leq 0}K^\bullet[j]) \to \mathrm{Hom}_{\mathsf{D}(R^a\text{-}\mathbf{Mod})}(R^a[0], \tau^{\leq 0}K^{\bullet a}[j]) \tag{14.1.26}$$

and a direct inspection shows that (14.1.26) agrees with the map arising in claim 14.1.23, for every $j \leq b$. Especially, for every such j, the map (14.1.26) is an isomorphism, as sought. □

Proposition 14.1.27. *Let $a, b, c \in \mathbb{Z}$ be any three integers such that $a \leq b$, and $K^\bullet, L^\bullet$ any two objects of $\mathsf{D}^{[a,b]}(R\text{-}\mathbf{Mod})$). Suppose that*

(a) $\mathrm{Hom}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(K^\bullet, X[-j]) = 0$ *for all $j \in [c, b]$ and all R-modules X with $X^a = 0$.*
(b) $\mathrm{Hom}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(Y[-j], L^\bullet) = 0$ *for all $j \in [a, c]$ and all R-modules Y with $Y^a = 0$.*

Then the natural map

$$\mathrm{Hom}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(K^\bullet, L^\bullet) \to \mathrm{Hom}_{\mathsf{D}(R^a\text{-}\mathbf{Mod})}(K^{\bullet a}, L^{\bullet a}) \tag{14.1.28}$$

is an isomorphism.

Proof. We start out with the following observation :

Claim 14.1.29. Consider the following conditions :

(a') $\mathrm{Hom}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(K^\bullet, X^\bullet) = 0$ for every $X^\bullet \in \mathrm{Ob}(\mathsf{D}^{[c,b]}(R\text{-}\mathbf{Mod}))$ such that $X^{\bullet a} = 0$.
(b') $\mathrm{Hom}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(Y^\bullet, L^\bullet) = 0$ for every $Y^\bullet \in \mathrm{Ob}(\mathsf{D}^{[a,c]}(R\text{-}\mathbf{Mod}))$ such that $Y^{\bullet a} = 0$.

Then (a)⇔(a') and (b)⇔(b').

Proof of the claim. Obviously (a')⇒(a). For the converse, one argues by decreasing induction on $c \leq b$. Indeed, the case $c = b$ is immediate. Then, suppose that the sought equivalence has already been established for some $d \leq b$; if $X^\bullet \in \mathsf{D}^{[d-1,b]}$ and $X^{\bullet a} = 0$, and if we know that (a) holds with $c := d - 1$, we set $H^\bullet := H^cX^\bullet[-c]$, and consider the distinguished triangle

$$H^\bullet \to X^\bullet \to \tau^{\geq d}X^\bullet \to H^\bullet[1].$$

By inductive assumption, we have $\mathrm{Hom}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(K^\bullet, \tau^{\geq d}X^\bullet) = 0$, and (a) says that

$$\mathrm{Hom}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(K^\bullet, H^\bullet) = 0.$$

It then follows that $\mathrm{Hom}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(K^\bullet, X^\bullet) = 0$, which shows that the equivalence holds for c.

The proof of the equivalence (b)⇔(b') is wholly analogous. ◊

Fix an integer $i \geq b - a + 2$, and set

$$C^\bullet := \mathrm{Cone}(\pi_0^\bullet \circ \cdots \circ \pi_i^\bullet : \mathfrak{M}_i^\bullet \to V[0])$$

(notation of (14.1.17)); notice that $C^\bullet \in \mathsf{D}^{\leq 1}(R\text{-}\mathbf{Mod})$, and $C^{\bullet a} = 0$. By virtue of condition (b), claim 14.1.29 and remark 7.3.14(iv), it follows that :

$$R^j\mathrm{Hom}^\bullet_R(C^\bullet, L^\bullet) = \mathrm{Hom}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(\tau^{\geq a}(C^\bullet[-j]), L^\bullet) = 0 \qquad \text{for every } j < c$$

In other words, $D^\bullet := R\mathrm{Hom}^\bullet_R(C^\bullet, L^\bullet) \in \mathsf{D}^{\geq c}(R\text{-}\mathbf{Mod})$, and clearly $D^{\bullet a} = 0$; also notice the induced distinguished triangle

$$\Sigma \quad : \quad D^\bullet \to L^\bullet \to R\mathrm{Hom}^\bullet_R(\mathfrak{M}_i^\bullet, L^\bullet) \to D^\bullet[-1].$$

Now, condition (a), claim 14.1.29 and remark 7.3.14(iv) imply that

$$\mathrm{Hom}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(K^\bullet, D^\bullet[j]) = \mathrm{Hom}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(K^\bullet, \tau^{\leq b}D^\bullet[j]) = 0 \qquad \text{for every } j \leq 0$$

whence, by considering the distinguished triangle $R\mathrm{Hom}^\bullet_R(K^\bullet, \Sigma)$, natural isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(K^\bullet, L^\bullet) &\xrightarrow{\sim} \mathrm{Hom}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(K^\bullet, R\mathrm{Hom}^\bullet_R(\mathfrak{M}_i^\bullet, L)) \\
&\xrightarrow{\sim} \mathrm{Hom}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(\mathfrak{M}_i^\bullet \overset{\mathbf{L}}{\otimes}_V K^\bullet, L^\bullet) && \text{(by [163, Th.10.8.7])} \\
&\xrightarrow{\sim} \mathrm{Hom}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(\tau^{\geq a}\mathfrak{M}_i^\bullet \overset{\mathbf{L}}{\otimes}_V K^\bullet, L^\bullet) && \text{(by remark 7.3.14(iv))} \\
&\xrightarrow{\sim} \mathrm{Hom}_{\mathsf{D}(R^a\text{-}\mathbf{Mod})}(K^{\bullet a}, L^{\bullet a}) && \text{(by claim 14.1.23)}
\end{aligned}$$

whose composition, after a simple inspection, is seen to agree with the map (14.1.28). □

Remark 14.1.30. (i) For every interval $I \subset \mathbb{N}$, denote by

$$\Phi_I : \mathsf{D}^I(R/\mathfrak{m}R\text{-}\mathbf{Mod}) \to \mathsf{D}^I(R\text{-}\mathbf{Mod})$$

the forgetful functor. It follows easily from lemma 10.1.16(iii) and remark 7.3.14(iv), that, for every interval $[a, b]$, the functor $\Phi_{[a,b]}$ admits the right adjoint

$$\mathsf{D}^{[a,b]}(R\text{-}\mathbf{Mod}) \to \mathsf{D}^{[a,b]}(R/\mathfrak{m}R\text{-}\mathbf{Mod}) \qquad K^\bullet \mapsto \Psi^r_{[a,b]}K^\bullet := \tau^{\leq b}R\mathrm{Hom}^\bullet_R(R/\mathfrak{m}R[0], K^\bullet).$$

(ii) Likewise, theorem 10.1.18 and remark 7.3.14(iv) imply that $\Phi_{[a,b]}$ admits the left adjoint

$$\mathsf{D}^{[a,b]}(R\text{-}\mathbf{Mod}) \to \mathsf{D}^{[a,b]}(R/\mathfrak{m}R\text{-}\mathbf{Mod}) \qquad K^\bullet \mapsto \Psi^l_{[a,b]}K^\bullet := \tau^{\geq a}(K^\bullet \otimes_R R/\mathfrak{m}R[0]).$$

(iii) Consider as well the following two conditions :

(a”) $\mathrm{Hom}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(K^\bullet, \Phi_{[c,b]}X^\bullet) = 0$ for every $X^\bullet \in \mathrm{Ob}(\mathsf{D}^{[c,b]}(R/\mathfrak{m}R\text{-}\mathbf{Mod}))$.

(b”) $\mathrm{Hom}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(\Phi_{[a,c]}Y^\bullet, L^\bullet) = 0$ for every $Y^\bullet \in \mathrm{Ob}(\mathsf{D}^{[a,c]}(R/\mathfrak{m}R\text{-}\mathbf{Mod}))$.

Then, arguing as in the proof of claim 14.1.29 it is easily seen that condition (a) of proposition 14.1.27 is equivalent to (a”), and condition (b) is equivalent to (b”).

Proposition 14.1.31. *Let $a, b \in \mathbb{Z}$ be any two integers such that $a \leq b$. For every object $K^\bullet$ of $\mathsf{D}^{[a,b]}(R\text{-}\mathbf{Mod})$, the following conditions are equivalent :*

(a) *$K^\bullet$ lies in the essential image of the left adjoint functor $X^\bullet \mapsto X^\bullet_{[!]}$.*

(b) $K^\bullet \overset{\mathbf{L}}{\otimes}_R R/\mathfrak{m}R[0] \in \mathrm{Ob}(\mathsf{D}^{<a-1}(R/\mathfrak{m}R\text{-}\mathbf{Mod}))$.

Proof. We start out with the following :

Claim 14.1.32. We may assume that $V = R$.

Proof of the claim. Clearly condition (b) does not depend on the underlying ring V. It suffices then to remark that condition (a) depends only on the basic setup $(R, \mathfrak{m}R)$ (as opposed to the original basic setup $(V, \mathfrak{m})$). Indeed, notice that there is a natural equivalence

$$\Omega : R^a\text{-}\mathbf{Mod} \xrightarrow{\sim} (R, \mathfrak{m}R)^a\text{-}\mathbf{Mod}$$

(where R^a denotes, as in the foregoing, the image of R in the category of $(V, \mathfrak{m})$-algebras), and the induced equivalence of the respective derived categories fits into an essentially commutative diagram

$$\begin{array}{ccc} & \mathsf{D}(R\text{-}\mathbf{Mod}) & \\ \swarrow & & \searrow \\ \mathsf{D}(R^a\text{-}\mathbf{Mod}) & \xrightarrow{\mathsf{D}(\Omega)} & \mathsf{D}((R, \mathfrak{m}R)^a\text{-}\mathbf{Mod}) \end{array}$$

whose downward arrows are the forgetful functors. Especially, the left (resp. right) adjoints of these two forgetful functors share the same essential images. ◊

Henceforth, we assume that $V = R$ (and therefore, $\mathfrak{m} = \mathfrak{m}R$). Fix an integer $i \geq b - a + 2$, set $L^\bullet := \tau^{\geq a-1}(\mathfrak{M}_i^\bullet \overset{\mathbf{L}}{\otimes}_V K^\bullet)$, and notice first that, taking into account proposition 14.1.21(iv) and remark 14.1.24, condition (a) is equivalent to :

(c) The morphism $\pi_0^\bullet \circ \cdots \circ \pi_{i-1}^\bullet : \mathfrak{M}_i \to V[0]$ induces an isomorphism $\tau^{\geq a} L^\bullet \to K^\bullet$.

(c)⇒(b): Indeed, set $H := H^{a-1} L^\bullet$; if (c) holds, we have a distinguished triangle

$$H[a-1] \to L^\bullet \to K^\bullet \to H[a]$$

whence a distinguished triangle in $\mathsf{D}(V/\mathfrak{m}\text{-}\mathbf{Mod})$

$$H[a-1] \overset{\mathbf{L}}{\otimes}_V V/\mathfrak{m}[0] \to L^\bullet \overset{\mathbf{L}}{\otimes}_V V/\mathfrak{m}[0] \to K^\bullet \overset{\mathbf{L}}{\otimes}_V V/\mathfrak{m}[0] \to H[a] \overset{\mathbf{L}}{\otimes}_V V/\mathfrak{m}[0]. \tag{14.1.33}$$

However, we have natural isomorphisms

$$\begin{aligned} \tau^{\geq a-1}(L^\bullet \overset{\mathbf{L}}{\otimes}_V V/\mathfrak{m}[0]) &\xrightarrow{\sim} \tau^{\geq a-1}((\mathfrak{M}_i^\bullet \overset{\mathbf{L}}{\otimes}_V K^\bullet) \overset{\mathbf{L}}{\otimes}_V V/\mathfrak{m}[0]) && \text{(by lemma 7.3.42(ii))} \\ &\xrightarrow{\sim} \tau^{\geq a-1}(\mathfrak{M}_i^\bullet \overset{\mathbf{L}}{\otimes}_V (K^\bullet \overset{\mathbf{L}}{\otimes}_V V/\mathfrak{m}[0])) && \text{(by remark 7.3.39(i))} \\ &\xrightarrow{\sim} \tau^{\geq a-1}(\mathfrak{M}_i^\bullet \overset{\mathbf{L}}{\otimes}_V \tau^{\geq a-1}(K^\bullet \overset{\mathbf{L}}{\otimes}_V V/\mathfrak{m}[0])) && \text{(by lemma 7.3.42(ii))} \\ &\xrightarrow{\sim} 0 && \text{(by proposition 14.1.19(i))} \end{aligned}$$

whence (b), after considering the distinguished triangle $\tau^{\geq a-1}$(14.1.33).

(b)⇒(a): We remark

Claim 14.1.34. Condition (b) is equivalent to condition (a”) of remark 14.1.30(iii), with $c := a - 1$.

Proof of the claim. Indeed, (b) holds if and only if $\Psi^l_{[a-1,b]} K^\bullet = 0$ in $\mathsf{D}^{[a-1,b]}(R/\mathfrak{m}R\text{-}\mathbf{Mod})$ (notation of remark 14.1.30(ii)). If the latter condition holds, then clearly (a”) holds with $c := a - 1$. Conversely, if (a”) holds for this value of c, then $\mathrm{Hom}_{\mathsf{D}(R/\mathfrak{m}R\text{-}\mathbf{Mod})}(\Psi^l_{[a-1,b]} K^\bullet, X^\bullet) = 0$ for every $X^\bullet \in \mathsf{D}^{[a-1,b]}(R/\mathfrak{m}R\text{-}\mathbf{Mod})$; especially, the identity automorphism of $\Psi^l_{[a-1,b]} K^\bullet$ factors through 0, so $\Psi^l_{[a-1,b]} K^\bullet$ vanishes. ◊

From claim 14.1.34 and remark 14.1.30(iii) we deduce that if (b) holds, condition (a) of proposition 14.1.27 holds for $c := a - 1$, and condition (b) of the same proposition holds trivially for this value of c, for every $L^\bullet \in \mathsf{D}^{[a;b]}(R\text{-}\mathbf{Mod})$. We conclude that the natural map

$$\mathrm{Hom}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(K^\bullet, L^\bullet) \to \mathrm{Hom}_{\mathsf{D}(R^a\text{-}\mathbf{Mod})}(K^{\bullet a}, L^{\bullet a}) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(K^{\bullet a}_{[!]}, L^\bullet)$$

is an isomorphism, for every $L^\bullet \in \mathsf{D}^{[a;b]}(R\text{-}\mathbf{Mod})$, whence (a). □

Proposition 14.1.35. *Let $a, b \in \mathbb{Z}$ be any two integers such that $a \leq b$. For every $L^\bullet \in \mathrm{Ob}(\mathsf{D}^{[a,b]}(R\text{-}\mathbf{Mod}))$, the following conditions are equivalent :*

(a) *$L^\bullet$ lies in the essential image of the right adjoint functor $X^\bullet \mapsto \tau^{\leq b} X^\bullet_{[*]}$.*

(b) $R\mathrm{Hom}^\bullet_R(R/\mathfrak{m}R[0], L^\bullet) \in \mathrm{Ob}(\mathsf{D}^{>b+1}(R/\mathfrak{m}R\text{-}\mathbf{Mod}))$.

Proof. By the same argument as in the proof of claim 14.1.32, we reduce to the case where $V = R$. Next, fix $i \in \mathbb{N}$ such that $i \geq b - a + 2$, define

$$K^\bullet := R\mathrm{Hom}^\bullet_V(\mathfrak{M}^\bullet_i, L^\bullet) \qquad \mathfrak{P}_i := \mathfrak{M}_i \overset{\mathbf{L}}{\otimes}_V V/\mathfrak{m}[0]$$

and notice that

$$\tau^{\geq a-b-1}\mathfrak{P}_i = 0 \qquad \text{in } \mathsf{D}^{[a-b-1,0]}(V/\mathfrak{m}\text{-}\mathbf{Mod}) \tag{14.1.36}$$

due to proposition 14.1.19(i). Moreover, in view of proposition 14.1.21(iii) and lemma 14.1.25, condition (a) is equivalent to :

(c) The morphism $\pi^\bullet_0 \circ \cdots \circ \pi^\bullet_{i-1} : \mathfrak{M}_i \to V[0]$ induces an isomorphism $L^\bullet \xrightarrow{\sim} \tau^{\leq b} K^\bullet$.

(c)⇒(b): We argue as in the proof of proposition 14.1.31; namely, set $H := H^{b+1}K^\bullet$; if (c) holds, we obtain a distinguished triangle

$$H[-b-2] \to L^\bullet \to \tau^{\leq b+1} K^\bullet \to H[-b-1] \tag{14.1.37}$$

and by considering the induced distinguished triangle $\tau^{\leq b+1} R\mathrm{Hom}^\bullet_V(V/\mathfrak{m}[0], (14.1.37))$, we reduce to observing that

$$\begin{aligned}
\tau^{\leq b+1} R\mathrm{Hom}^\bullet_V(V/\mathfrak{m}[0], \tau^{\leq b+1} K^\bullet) &\xrightarrow{\sim} \tau^{\leq b+1} R\mathrm{Hom}^\bullet_V(V/\mathfrak{m}[0], K^\bullet) && \text{(by lemma 7.3.42)} \\
&\xrightarrow{\sim} \tau^{\leq b+1} R\mathrm{Hom}^\bullet_V(\mathfrak{P}^\bullet_i, L^\bullet) && \text{(by [163, Th.10.8.7])} \\
&\xrightarrow{\sim} \tau^{\leq b+1} R\mathrm{Hom}^\bullet_V(\tau^{\geq a-b-1}\mathfrak{P}^\bullet_i, L^\bullet) && \text{(by lemma 7.3.42)} \\
&\xrightarrow{\sim} 0 && \text{(by (14.1.36)).}
\end{aligned}$$

(b)⇒(a): Again, we proceed as in the proof of the corresponding assertion in proposition 14.1.31; namely, arguing as in the proof of claim 14.1.34, we see that condition (b) is equivalent to condition (b”) of remark 14.1.30(iii), with $c := b + 1$. Hence, if (b) holds, condition (b) of proposition 14.1.27 holds for $c := b + 1$, and notice that condition (a) of *loc.cit.* holds trivially for this value of c, for every $K^\bullet \in \mathsf{D}^{[a;b]}(R\text{-}\mathbf{Mod})$. We conclude that the natural map

$$\mathrm{Hom}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(K^\bullet, L^\bullet) \to \mathrm{Hom}_{\mathsf{D}(R^a\text{-}\mathbf{Mod})}(K^{\bullet a}, L^{\bullet a}) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(K^\bullet, \tau^{\leq b} L^{\bullet a}_{[*]})$$

is an isomorphism, for every $K^\bullet \in \mathsf{D}^{[a;b]}(R\text{-}\mathbf{Mod})$, whence (a). □

14.1.38. Let A be any V^a-algebra; recall that the localization functor V-Alg $\to V^a$-Alg admits a left adjoint $R \mapsto R_{!!}$, whose restriction to the subcategory of $A_{!!}$-algebras yields a left adjoint for the localization functor $A_{!!}$-Alg $\to A$-Alg ([75, Prop.2.2.29]). In [75], we have studied the deformation theory of A-algebras by means of this left adjoint, under the assumption that $\widetilde{\mathfrak{m}}$ is V-flat; here we wish to show that the same can be repeated in the current setting, if one appeals instead to the results of the foregoing paragraphs. To begin with, we remark :

Proposition 14.1.39. *Let $A \to B$ be a morphism of V^a-algebras, N a $B_{!!}$-module. We have:*

(i) *If the unit of adjunction $N \to N^a_*$ is injective, the natural map*

$$\mathrm{Exal}_{A_{!!}}(B_{!!}, N) \to \mathrm{Exal}_A(B, N^a) \tag{14.1.40}$$

is a bijection (notation of [75, §2.5.7]*).*

(ii) *If $N^a = 0$, then* $\mathrm{Exal}_{A_{!!}}(B_{!!}, N) = 0$.

Proof. (i): Consider any square-zero extension of A-algebras

$$\Sigma \quad : \quad 0 \to N^a \to E \xrightarrow{\varphi} B \to 0.$$

There follows a square-zero extension of $A_{!!}$-algebras

$$\Sigma_{!!} \quad : \quad 0 \to N_!^a/\mathrm{Ker}\,\varphi_{!!} \to E_{!!} \to B_{!!} \to 0$$

Under the stated assumption, the counit of adjunction $N_!^a \to N$ factors uniquely through a B-linear map $g_\varphi : N_!^a/\mathrm{Ker}\,\varphi_{!!} \to N$; then $g_\varphi * \Sigma_{!!}$ (defined as in [75, §2.5.5]) yields an element of $\mathrm{Exal}_{A_{!!}}(B_{!!}, N)$ whose image under (14.1.40) equals the class of Σ. Conversely, if

$$\Omega \quad : \quad 0 \to N \to F \xrightarrow{\psi} B_{!!} \to 0$$

is a square-zero extension of $A_{!!}$-algebras, then by adjunction we get a natural map

$$\Omega^a_{!!} \to \Omega$$

which in turns, by simple inspection, induces an isomorphism $g_{\psi^a} * \Omega^a_{!!} \xrightarrow{\sim} \Omega$ in the category of square-zero $A_{!!}$-algebra extensions of $B_{!!}$ (details left to the reader). The assertion follows.

(ii): Suppose $N^a = 0$, and let Ω be as in the foregoing; it follows that $\psi^a : F^a \to B$ is an isomorphism of A-algebras. By adjunction, the morphism $(\psi^a)^{-1}$ corresponds to a map of $A_{!!}$-algebras $\varphi : B_{!!} \to E$, and it is easily seen that $\psi \circ \varphi$ is the identity automorphism of $B_{!!}$, whence the assertion. □

Definition 14.1.41. Let $f : A \to B$ be a morphism of V^a-algebras. We set

$$\mathbb{L}^a_{B/A} := (\mathbb{L}_{B_{!!}/A_{!!}})^a$$

which is a simplicial complex of B-modules that we call the *almost cotangent complex* of f.

Remark 14.1.42. (i) In case $\widetilde{\mathfrak{m}}$ is a flat V-module, we have introduced in [75, Def.2.5.20] a simplicial $B_{!!}$-module $\mathbb{L}_{B/A}$. Notice that the notation of *loc.cit.* agrees with the current one: indeed, [75, Prop.8.1.7(ii)] shows that complex $(\mathbb{L}_{B/A})^a$ obtained by applying the derived localization functor to $\mathbb{L}_{B/A}$ is naturally isomorphic (in $\mathsf{D}(s.B\text{-}\mathbf{Mod})$) to the complex of definition 14.1.41.

(ii) Depending on the context, we will want to regard $\mathbb{L}_{B/A}$ either as a simplicial object, or as a cochain complex, via the Dold-Kan isomorphism ([163, Th.8.4.1]). The resulting slight notational ambiguity should not be a source of confusion.

Theorem 14.1.43. *In the situation of definition* 14.1.41*, let N be any B-module. Then there are natural isomorphisms*

$$\mathrm{Der}_A(B, N) \xrightarrow{\sim} \mathrm{Ext}^0_B(\mathbb{L}^a_{B/A}, N)$$
$$\mathrm{Exal}_A(B, N) \xrightarrow{\sim} \mathrm{Ext}^1_B(\mathbb{L}^a_{B/A}, N).$$

(Notation of [75, Def.2.5.22(i)]*; so, here we view $\mathbb{L}^a_{B/A}$ as an object of $\mathsf{D}^{\leq 0}(B\text{-}\mathbf{Mod})$.)*

Proof. The first isomorphism follows easily from [103, II.1.2.4.2] and the natural isomorphism

$$\Omega_{B_{!!}/A_{!!}} \xrightarrow{\sim} (\Omega_{B/A})_! \qquad (14.1.44)$$

proved in [75, Lemma 2.5.29] (the proof in *loc.cit.* does not use the assumption that $\widetilde{\mathfrak{m}}$ is V-flat). Clearly we have

(14.1.45) $\quad \mathrm{Hom}_{\mathsf{D}(B_{!!}\text{-}\mathbf{Mod})}(Y[0], N_*[0]) = 0 \qquad$ for every $B_{!!}$-module Y such that $Y^a = 0$.

On the other hand, we have :

Claim 14.1.46. $\mathrm{Hom}_{\mathsf{D}(B_{!!}\text{-}\mathbf{Mod})}(\mathbb{L}_{B_{!!}/A_{!!}}, X^\bullet) = 0$ for every $X^\bullet \in \mathrm{Ob}(\mathsf{D}^{[0,1]}(B_{!!}\text{-}\mathbf{Mod}))$ such that $X^{\bullet a} = 0$.

Proof of the claim. For every $X^\bullet \in \mathrm{Ob}(\mathsf{D}^{[0,1]}(B_{!!}\text{-}\mathbf{Mod}))$ we have a distinguished triangle

$$H^0X^\bullet[0] \to X^\bullet \to H^1X^\bullet[-1] \to (H^0X^\bullet)[1]$$

which reduces to considering the cases where $X^\bullet = M[j]$ for some almost zero $B_{!!}$-module M, and $j = 0, -1$. The case where $j = -1$ follows from [103, III.1.2.3] and proposition 14.1.39(ii). The case where $j = 0$ follows easily from [103, II.1.2.4.2] and (14.1.44) : details left to the reader. ♢

Now, (14.1.45) says that $L^\bullet := N_*[0]$ fulfills condition (b) of proposition 14.1.27, and claim 14.1.46 says that $K^\bullet := \mathbb{L}_{B_{!!}/A_{!!}}$ fulfills condition (a), so the natural map

$$\mathrm{Ext}^1_{B_{!!}}(\mathbb{L}_{B_{!!}/A_{!!}}, N_*) \to \mathrm{Ext}^1_B(\mathbb{L}^a_{B/A}, N)$$

is an isomorphism. Taking into account proposition 14.1.39(i) and [103, III.1.2.3], the theorem follows. □

14.1.47. For the further study the almost cotangent complex, we shall need some preliminaries concerning the derived functors of certain non-additive functors. This material generalizes the results of [75, §8.1], that were obtained under the assumption that $\widetilde{\mathfrak{m}}$ is V-flat.

Lemma 14.1.48. *Let $(V, \mathfrak{m})$ be any basic setup, R a simplicial V-algebra, $n \in \mathbb{N}$ an integer, M and N two R-modules such that $H_iM = H_iN = 0$ for every $i \geq n$. The following holds :*

(i) *If $M^a = 0$ in $\mathsf{D}(R^a\text{-}\mathbf{Mod})$, then $a \cdot \mathbf{1}_M = 0$ in $\mathsf{D}(R\text{-}\mathbf{Mod})$, for every $a \in \mathfrak{m}$.*

(ii) *If $\varphi : M \to N$ is a morphism of R-modules such that φ^a in an isomorphism in $\mathsf{D}(R^a\text{-}\mathbf{Mod})$, then for every $a \in \mathfrak{m}$ we may find a morphism $\psi : N \to M$ in $\mathsf{D}(R\text{-}\mathbf{Mod})$, such that $\psi \circ \varphi = a \cdot \mathbf{1}_M$ and $\varphi \circ \psi = a \cdot \mathbf{1}_N$ in $\mathsf{D}(R\text{-}\mathbf{Mod})$.*

Proof. (i): For every R-module X, set (notation of remark 7.10.31(iii))

$$\tau^{\leq -1}X := \sigma \circ \omega X$$

According to [103, I.3.2.1.9(ii)], there exists a natural sequence of morphisms

$$(14.1.49) \qquad \tau^{\leq -1}X \to X \to s.H_0(X) \to \sigma(\tau^{\leq -1}X) \qquad \text{in } \mathsf{D}(R\text{-}\mathbf{Mod})$$

whose induced sequence of normalized complexes is a distinguished triangle in $\mathsf{D}(V\text{-}\mathbf{Mod})$ (*i.e.* a distinguished triangle of $\mathsf{D}(R\text{-}\mathbf{Mod})$, in the terminology of [103, I.3.2.2.4], and in view of [103, I.3.2.2.5]). Let now n and M be as in the lemma; we argue by induction on n. The case where $n = 0$ is trivial, so suppose that $n > 0$, and that the assertion has already been proven for all almost zero R-modules N such that $H_iN = 0$ for every $i \geq n-1$. Especially, for $N := \tau^{\leq 1}X$ and $P := s.H_0(M)$ we have $a \cdot \mathbf{1}_{\omega N} = 0$ and $a \cdot \mathbf{1}_P = 0$ in $\mathsf{D}(R\text{-}\mathbf{Mod})$, for every $a \in \mathfrak{m}$. Since the adjunction $\sigma \circ \omega N \to N$ is an isomorphism in $\mathsf{D}(R\text{-}\mathbf{Mod})$ ([103, I.3.2.1.10]), we deduce that

$$\mathrm{Hom}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(M, N)^a = 0 = \mathrm{Hom}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(M, P)^a$$

whence $\mathrm{End}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(M)^a = 0$, by virtue of (14.1.49) (with $X := M$) and [103, I.3.2.2.10]. The assertion follows.

(ii): Set $C := \mathrm{Cone}\,\varphi$; according to [103, I.3.2.2], we have a distinguished triangle

$$(14.1.50) \qquad M \xrightarrow{\varphi} N \to C \to \sigma M \qquad \text{in } \mathsf{D}(R\text{-}\mathbf{Mod})$$

whence – by [103, I.3.2.2.10] – an exact sequence of V-modules

$$\mathrm{Hom}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(N, M) \xrightarrow{\alpha} \mathrm{End}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(N) \xrightarrow{\beta} \mathrm{Hom}_{\mathsf{D}(R\text{-}\mathbf{Mod})}(N, C).$$

Now, let us write $a = \sum_{i=1}^n a_ib_i$ for some $a_1, b_1, \dots, a_n, b_n \in \mathfrak{m}$; the assumption on φ implies that $C^a = 0$ in $\mathsf{D}(R^a\text{-}\mathbf{Mod})$, therefore (i) yields $\beta(a_i \cdot \mathbf{1}_N) = a_i \cdot \beta(\mathbf{1}_N) = 0$, so there exists a morphism $\psi_i : N \to M$ in $\mathsf{D}(R\text{-}\mathbf{Mod})$ such that $\alpha(\psi_i) = a_i \cdot \mathbf{1}_N$, *i.e.* $\varphi \circ \psi_i = a_i \cdot \mathbf{1}_N$ for every $i = 1, \dots, n$. Likewise, by considering the long exact sequence $\mathbb{E}\mathsf{xt}^\bullet_R((14.1.50), M)$

provided by [103, I.3.2.2.10] we find, for every $i = 1, \dots, n$, a morphism $\psi'_i : N \to M$ such that $\psi'_i \circ \varphi = b_i \cdot \mathbf{1}_M$. Thus,

$$a_i \cdot \psi'_i = \psi'_i \circ \varphi \circ \psi_i = b_i \cdot \psi_i \qquad \text{for every } i = 1, \dots, n$$

and a simple computation shows that $\psi := \sum_{i=1}^n b_i \cdot \psi_i$ will do. □

Remark 14.1.51. Before considering non-additive functors, let us see how to define derived tensor products in $\mathsf{D}(R^a\text{-}\mathbf{Mod})$, for any simplicial V-algebra R. We proceed as in (14.1.13) : for given R^a-modules M, N, set

$$M \overset{\ell}{\otimes}_{R^a} N := (M_! \overset{\ell}{\otimes}_R N_!)^a.$$

(i) We claim that this rule yields a well defined functor

$$- \overset{\ell}{\otimes}_{R^a} - : \mathsf{D}(R^a\text{-}\mathbf{Mod}) \times \mathsf{D}(R^a\text{-}\mathbf{Mod}) \to \mathsf{D}(R^a\text{-}\mathbf{Mod}).$$

Indeed, say that $\varphi : M \to M'$ is a quasi-isomorphism of R^a-modules, and set $C := \mathrm{Cone}(\varphi_!)$. We need to check that $(\varphi_! \overset{\ell}{\otimes}_R N)^a$ is a quasi-isomorphism, for any R-module N; in light of remark 7.10.31(iv), it then suffices to show that $(C \overset{\ell}{\otimes}_R N)^a = 0$; but the latter R^a-module may be computed as

$$(C \otimes_R \perp^R_\bullet N)^a = C^a \otimes_{R^a} (\perp^R_\bullet N)^a$$

whence the claim, since $C^a = 0$ in $R^a\text{-}\mathbf{Mod}$.

(ii) Next, suppose that M is a flat R^a-module; then we claim that the natural morphism of R^a-modules

$$M \overset{\ell}{\otimes}_{R^a} N \to M \otimes_{R^a} N$$

is a quasi-isomorphism, for every R^a-module N. Indeed, by Eilenberg-Zilber's theorem 7.4.50, the assertion comes down to checking that the augmented simplicial R^a-module

$$(M_! \otimes_R \perp^R_\bullet N_!)^a \to M \otimes_{R^a} N$$

is aspherical. But for every $k \in \mathbb{N}$, the k-th column of the latter is isomorphic to the augmented R^a-module

$$M[k] \otimes_{R^a[k]} (\perp^{R[k]}_\bullet N_![k])^a \to M[k] \otimes_{R^a[k]} N[k]$$

which is aspherical, since $M[k]$ is a flat $R^a[k]$-module, whence the assertion.

(iii) Just as in (14.1.13), the foregoing immediately implies yields a natural isomorphism

$$(M \overset{\ell}{\otimes}_R N)^a \xrightarrow{\sim} M^a \overset{\ell}{\otimes}_{R^a} N^a \qquad \text{in } \mathsf{D}(R^a\text{-}\mathbf{Mod})$$

for any two R-modules M and N (details left to the reader).

(iv) Furthermore, we get suspension and loop functors σ and ω for R^a-modules, by the rule :

$$\sigma M := (\sigma M_!)^a \quad \text{and} \quad \omega M := (\omega M_*)^a \qquad \text{for every } R^a\text{-module } M$$

from which it follows that σ is left adjoint to ω. Then, it is clear that the assertions of remark 7.10.31(iii) hold as well for these functors.

(v) Likewise, we define the cone of a morphism $\varphi : M \to N$ of R^a-modules, by the rule

$$\mathrm{Cone}\,\varphi := (\mathrm{Cone}\,\varphi_!)^a$$

and then the assertion of remark 7.10.31(iv) holds also for morphisms of R^a-modules.

(vi) In view of (iv) and (v), it is then easy to check that also lemma 7.10.32 holds *verbatim* for $A := V$, and any two R^a-modules X, Y.

Remark 14.1.52. (i) In the same vein, we may define derived tensor products of R^a-algebras, for any simplicial V-algebra R. Namely, if S and S'' are any two R^a-algebras, we set

$$S \overset{\ell}{\otimes}_{R^a} S' := (S_{!!} \overset{\ell}{\otimes}_R S'_{!!})^a$$

(see (14.1.38)), where $\overset{\ell}{\otimes}_R$ denotes the derived tensor product for R-algebras, defined in example 7.10.25. In view of remarks 7.10.31(ii) and 14.1.51(i), it is easily seen that this rule defines a functor

$$- \overset{\ell}{\otimes}_{R^a} - : \mathsf{D}(R^a\text{-}\mathsf{Alg}) \times \mathsf{D}(R^a\text{-}\mathsf{Alg}) \to \mathsf{D}(R^a\text{-}\mathsf{Alg})$$

and moreover, the formation of these tensor products commutes with the forgetful functor $\mathsf{D}(R^a\text{-}\mathsf{Alg}) \to \mathsf{D}(R^a\text{-}\mathbf{Mod})$.

(ii) Moreover, they are computed by arbitrary flat resolutions : if S (or S') is a flat R^a-algebras, then the natural morphism

$$S \overset{\ell}{\otimes}_{R^a} S' \to S \otimes_{R^a} S'$$

is an isomorphism in $\mathsf{D}(R^a\text{-}\mathsf{Alg})$.

(iii) Furthermore, if $R^a \to S$ is a given morphism of simplicial V^a-algebras, we obtain a well defined functor

$$\mathsf{D}(R^a\text{-}\mathsf{Alg}) \to \mathsf{D}(S\text{-}\mathsf{Alg}) \qquad S' \mapsto S \overset{\ell}{\otimes}_{R^a} S'.$$

Namely, given an R^a-algebra S', we pick a resolution $P \to S'$ with P a flat R^a-algebra, and endow $S \otimes_{R^a} P$ with its natural S-algebra structure, which is independent, up to natural isomorphism, of the choice of P. All the verifications are exercises for the reader.

Definition 14.1.53. Let V be a ring, A a V-algebra, $d \in \mathbb{N}$ and $T : A\text{-}\mathbf{Mod} \to A\text{-}\mathbf{Mod}$ a functor. We say that T is *V-homogeneous of degree d* if we have

$$T(a \cdot \mathbf{1}_M) = a^d \cdot \mathbf{1}_{TM} \qquad \text{for every } A\text{-module } M \text{ and every } a \in V.$$

Remark 14.1.54. Let $(V, \mathfrak{m})$ be a basic setup, A a V-algebra, $d \in \mathbb{N}$, and $T : A\text{-}\mathbf{Mod} \to A\text{-}\mathbf{Mod}$ a V-homogeneous functor of degree d. Suppose that either $d \leq 1$ or else $\mathfrak{m}$ fulfills condition $(\mathbf{B})$ of [75, §2.1.6].

(i) Let also $\varphi : M \to N$ be a homomorphism of A-modules such that $\varphi^a : M^a \to N^a$ is an isomorphism (for the almost structure given by $(V, \mathfrak{m})$); arguing as in the proof of lemma 14.1.48(ii), it is easily seen that, for every $a \in \mathfrak{m}$ there exists a B-linear map $\psi : N \to M$ such that $\varphi \circ \psi = a \cdot \mathbf{1}_N$ and $\psi \circ \varphi = a \cdot \mathbf{1}_M$. Then, the V-homogeneity property of T implies that $(T\varphi)^a$ is an isomorphism as well (details left to the reader).

(ii) Hence, T induces a well defined functor

$$T^a : A^a\text{-}\mathbf{Mod} \to A^a\text{-}\mathbf{Mod} \qquad M^a \mapsto (TM)^a \qquad (\varphi^a : M^a \to N^a) \mapsto (T\varphi)^a.$$

For every cardinality c, let $\mathscr{M}_c(A^a)$ be the set of isomorphism classes of A^a-modules which admit a set of generators of cardinality $\leq c$. We endow $\mathscr{M}_c(A^a)$ with the uniform structure as in [75, Def.2.3.1(i)]. Let ω be an infinite cardinality, such that $\mathfrak{m}$ is generated by at most ω elements; then there exists a cardinality $\omega' \geq \omega$ such that the isomorphism class of $T^a M$ lies in $\mathscr{M}_{\omega'}(A^a)$, for every A^a-module M whose isomorphism class lies in $\mathscr{M}_\omega(A^a)$; thus, T^a induces a map

$$\mathscr{M}_{\omega,\omega'}(T^a) : \mathscr{M}_\omega(A^a) \to \mathscr{M}_{\omega'}(A^a) \qquad N \mapsto T^a N.$$

Lemma 14.1.55. *Let $(V, \mathfrak{m})$ be a basic setup, A a V-algebra, $I, J \subset A$ two ideals, and $\varphi : M' \to M$ a morphism of A^a-modules with $I \cdot \mathrm{Ker}\,\varphi = J \cdot \mathrm{Coker}\,\varphi = 0$. Then there exists an A^a-linear morphism $\lambda : I \otimes_A J \otimes_A M \to M'$ with*

$$\varphi \circ \lambda = \mu_{I,J} \otimes_A \mathbf{1}_M \qquad \textit{and} \qquad \lambda \circ (I \otimes_A J \otimes_A \varphi) = \mu_{I,J} \otimes_A \mathbf{1}_{M'}$$

where $\mu_{I,J} : I \otimes_A J \to A$ is the multiplication law : $a \otimes b \mapsto ab$ for every $a \in I$ and $b \in J$.

Proof. Let $M' \xrightarrow{\overline{\varphi}} M_0 := \mathrm{Im}\,\varphi \xrightarrow{i} M$ be the natural factorization of φ; then for every $a \in I$ and $b \in J$ we have A^a-linear morphisms

$$\psi_a : M_0 \to M' \quad \text{and} \quad \mu_b : M \to M_0 \qquad \text{such that} \qquad \psi_a \circ \overline{\varphi} = a \cdot \mathbf{1}_{M'} \quad \text{and} \quad i \circ \mu_b = b \cdot \mathbf{1}_M.$$

Let us check that $\overline{\varphi} \circ \psi_a = a \cdot \mathbf{1}_{M_0}$; since $\overline{\varphi}$ is an epimorphism, it suffices to show that $\overline{\varphi} \circ \psi_a \circ \overline{\varphi} = (a \cdot \mathbf{1}_{M_0}) \circ \overline{\varphi}$, which is clear. Likewise, let us check that $\mu_b \circ i = b \cdot \mathbf{1}_{M_0}$; since i is a monomorphism, it suffices to show that $i \circ \mu_b \circ i = i \circ b \cdot \mathbf{1}_{M_0}$, which is clear. For every $a \in I$ and $b \in J$ set $\lambda_{a,b} := \psi_a \circ \mu_b : M \to M'$; we deduce that

$$\varphi \circ \lambda_{a,b} = i \circ \overline{\varphi} \circ \psi_a \circ \mu_b = i \circ (a \cdot \mathbf{1}_{M_0}) \circ \mu_b = i \circ \mu_b \circ (a \cdot \mathbf{1}_M) = ab \cdot \mathbf{1}_M$$

and a similar computation yields : $\lambda_{a,b} \circ \varphi = ab \cdot \mathbf{1}_{M'}$. Thus, we obtain a map

$$I \times J \times M \to M' \qquad (a, b, x) \mapsto \lambda_{a,b}(x).$$

A simple inspection shows that this map is A-trilinear, hence it factors uniquely through an A-linear map $\lambda : I \otimes_A J \otimes_A M \to M'$, and the foregoing easily implies that λ fulfills the required identities. □

Proposition 14.1.56. *In the situation of remark* 14.1.54 *the following holds :*

(i) *If T commutes with filtered colimits, and for every free A-module L of finite rank, $(TL)^a$ is a flat A^a-module (resp. is the zero A^a-module), then for every flat A^a-module M, the A^a-module $T^a M$ is flat (resp. is the zero A^a-module).*

(ii) *If for every free A-module L of finite rank (resp. for every free A-module L), $(TL)^a$ is an almost projective A^a-module, then for every almost projective and almost finitely generated A^a-module M (resp. for every almost projective A^a-module M), the A^a-module $T^a M$ is almost projective.*

(iii) *The map $\mathscr{M}_{\omega,\omega'}(T^a)$ of remark* 14.1.54(ii) *is uniformly continuous.*

Proof. (i): Say that $M = N^a$ for an A-module N, and let $\mathscr{F}$ be the category whose objects are the pairs (C, φ), where C is a finitely presented A-module and $\varphi : C \to N$ is an A-linear map; the morphisms $\psi : (C, \varphi) \to (C', \varphi')$ in $\mathscr{F}$ are the A-linear maps $\psi : C \to C'$ such that $\varphi' \circ \psi = \varphi$, with the obvious composition law. We have a functor

$$F : \mathscr{F} \to A\text{-}\mathbf{Mod} \qquad (C, \varphi) \mapsto C \qquad ((C, \varphi) \xrightarrow{\psi} (C', \varphi')) \mapsto (C \xrightarrow{\psi} C')$$

as well as a natural co-cone $\beta : F \Rightarrow c_N$ given by the rule : $(C, \varphi) \mapsto \varphi$. It is easily seen that $\mathscr{F}$ is filtered, and β is a universal co-cone : the details are left to the reader. Since T commutes with filtered colimits, TN is isomorphic to the colimit of the functor $T \circ F : \mathscr{F} \to A\text{-}\mathbf{Mod}$, and $T * \beta : T \circ F \Rightarrow c_{TN}$ is again a universal co-cone. We need to show that $\mathfrak{m}\mathrm{Tor}_1^A(TN, X) = 0$ for every A-module X (resp. that $\mathfrak{m}TN = 0$), and by the foregoing it then suffices to check that for every $(C, \varphi) \in \mathrm{Ob}(\mathscr{F})$, the induced map

$$\lambda := \mathrm{Tor}_1^A(T\varphi, X) : \mathrm{Tor}_1^A(TC, X) \to \mathrm{Tor}_1^A(TN, X)$$

is almost zero (resp. that $T\varphi$ is almost zero). However, since N^a is a flat A^a-module, [75, Lemma 2.4.17] implies that for every $a \in \mathfrak{m}$ there exists a free A-module L of finite rank and A-linear maps $\varphi' : C \to L$, $\varphi'' : L \to M$ with $a \cdot \varphi = \varphi'' \circ \varphi'$. Hence $a^d \cdot T\varphi = T(\varphi'') \circ T(\varphi'')$, and therefore $a^d \cdot \lambda^a = 0$, since by assumption $(TL)^a$ is a flat A^a-module (resp. and therefore $a^d \cdot (T\varphi)^a = 0$, since $(TL)^a = 0$); the contention follows, since we have either $d \leq 1$, or else $\mathfrak{m}$ fulfills condition $(\mathbf{B})$.

(ii): Suppose first that for every free A-module L of finite rank, $(TL)^a$ is almost projective, and let M be an almost projective and almost finitely generated A-module. According to [75, 2.4.15], for every $a \in \mathfrak{m}$ there exists a free A-module L of finite rank and A^a-linear morphisms $\varphi' : M \to L^a$, $\varphi'' : L^a \to M$ such that $a \cdot \mathbf{1}_M = \varphi'' \circ \varphi'$. We deduce that

$$a^d \cdot \mathrm{Ext}^1_A(\mathbf{1}_{TM}, X)^a = \mathrm{Ext}^1_A(T\varphi', X)^a \circ \mathrm{Ext}^1_A(T\varphi'', X)^a = 0$$

for every A-module X, since $\mathrm{Ext}^1_A(TL, X)^a = 0$ by assumption (and by [75, Rem.2.4.12(i)]). The contention follows, again because either $d \leq 1$ or else $\mathfrak{m}$ fulfills condition $(\mathbf{B})$. One argues similarly in case M is almost projective and $(TL)^a$ is almost projective for every free A-module L : the details shall be left to the reader.

(iii): Let $\mathfrak{m}_0 \subset \mathfrak{m}$ be a subideal of finite type, and $\varphi : M' \to M$ a morphism of A^a-module with $\mathfrak{m}_0 \mathrm{Ker}\,\varphi = \mathfrak{m}_0 \mathrm{Coker}\,\varphi = 0$. Denote by $\mathfrak{m}_0^{(2d)} \subset \mathfrak{m}_0$ the subideal generated by the system $(a^d \mid a \in \mathfrak{m}_0^2)$. By lemma 14.1.55, for every $a \in \mathfrak{m}_0^2$ there exists a morphism $\lambda : M \to M'$ of A^a-modules with $\varphi \circ \lambda = a \cdot \mathbf{1}_M$ and $\lambda \circ \varphi = a \cdot \mathbf{1}_{M'}$; therefore

$$T^a(\varphi) \circ T^a(\lambda) = a^d \cdot \mathbf{1}_{T^aM} \qquad \text{and} \qquad T^a(\lambda_a) \circ T^a(\varphi) = a^d \cdot \mathbf{1}_{T^aM'}$$

which implies that $\mathfrak{m}_0^{(2d)} \cdot \mathrm{Ker}\,T^a(\varphi) = \mathfrak{m}_0^{(2d)} \cdot \mathrm{Coker}\,T^a(\varphi) = 0$, whence the assertion, since we either have $d \leq 1$ or $\mathfrak{m}$ fulfills condition $(\mathbf{B})$. ☐

Example 14.1.57. Let $(V, \mathfrak{m})$ be a basic setup such that $\mathfrak{m}$ satisfies condition $(\mathbf{B})$; let A be a V-algebra, and P any A-module. According to [37, §9, n.3], for every integer $n > 0$ we have a complex of A-modules

$$\Sigma^\bullet_A(P) \qquad : \qquad 0 \to \Lambda^n_A P \to \cdots \to (\Lambda^j_A P) \otimes_A (\mathrm{Sym}^{n-j}_A P) \to \cdots \to \mathrm{Sym}^n_A P \to 0$$

which is functorial for morphisms of A-modules, and is acyclic if P is a flat A-module ([37, §9, n.3, Prop.3]). Clearly, for every $j \in \mathbb{N}$ the functor

$$T^j : A\text{-}\mathbf{Mod} \to A\text{-}\mathbf{Mod} \qquad P \mapsto H^j(\Sigma^\bullet_A(P))$$

fulfills the conditions of (14.1.58) (with $d := n$) and $T^jP = 0$ if P is a flat A-module. By proposition 14.1.56(i), we deduce that the induced complex of A^a-modules $\Sigma^\bullet_A(P)^a$ is acyclic, if P^a is a flat A^a-module. With this observation, all the results of [75, §4.3, §4.4] extend now *verbatim* to the case where $\mathfrak{m}$ fulfills condition $(\mathbf{B})$ (whereas the proofs in *loc.cit.* used the stronger condition that $\mathfrak{m}$ is a flat V-module : the details shall be left to the reader).

14.1.58. Let $\pi : V\text{-}\mathsf{Alg.Mod} \to V\text{-}\mathsf{Alg}$ be the functor such that $\pi(A, M) := A$ and $\pi(f, \varphi) := f$ for every object (A, M) and every morphism (f, φ) of V-$\mathsf{Alg.Mod}$. We consider a functor

$$T : V\text{-}\mathsf{Alg.Mod} \to V\text{-}\mathsf{Alg.Mod} \qquad \text{such that } \pi \circ T = \pi.$$

Let also be R a simplicial V-algebra, $d \in \mathbb{N}$ and suppose that :

- T is *V-homogeneous of degree d, i.e.* for every V-algebra A, the restriction of T

$$T_A : A\text{-}\mathbf{Mod} \to A\text{-}\mathbf{Mod}$$

  is V-homogeneous of degree d.
- T_A commutes with filtered colimits (cp. (7.10.26)) for every V-algebra A.
- Either $d \leq 1$ or else the ideal $\mathfrak{m} \cdot H_0(R)$ of $H_0(R)$ satisfies condition $(\mathbf{B})$ of [75, §2.1.6].

Remark 14.1.59. (i) As detailed in (7.10.26), the functor T induces a functor

$$T_R : R\text{-}\mathbf{Mod} \to R\text{-}\mathbf{Mod} \qquad (M[n] \mid n \in \mathbb{N}) \mapsto (T_{R[n]}M[n] \mid n \in \mathbb{N}).$$

(ii) Let T and T' be two functors as in (14.1.58), V-homogeneous of degrees respectively d and d'; then we get the functor V-homogeneous of degree $d + d'$

$$T \otimes T' : V\text{-}\mathsf{Alg.Mod} \to V\text{-}\mathsf{Alg.Mod} \qquad (A, M) \mapsto (A, T_AM \otimes_A T'_AM)$$

and it is easily seen that $(T \otimes T')_A$ commutes with filtered colimits, for every V-algebra A.

(iii) Likewise, if $f : T \to T'$ is a natural transformation of functors fulfilling the conditions of (14.1.58), for the same degree d; then the same holds for the functors $\mathrm{Ker}\, f$ and $\mathrm{Coker}\, f$.

(iv) In the situation of (14.1.58), let $f : H_0(R) \to B$ be any morphism of V-algebras. If condition $(\mathbf{B})$ holds for $\mathfrak{m} \cdot H_0(R)$, it holds also also for $\mathfrak{m}B$; then remark 14.1.54(ii) implies that T_B induces a functor $T_{B^a} := T_B^a : B^a\text{-}\mathbf{Mod} \to B^a\text{-}\mathbf{Mod}$. Especially, the above holds with $B := R[p]$, for any $p \in \mathbb{N}$, and f the natural map given by the degeneracies of the simplicial V-algebra R. It is then clear that T_R induces a well defined functor

$$T_{R^a} : R^a\text{-}\mathbf{Mod} \to R^a\text{-}\mathbf{Mod}.$$

The following result shows that, in this situation, the construction of left derived functors descends likewise to R^a-modules.

Theorem 14.1.60. *In the situation of* (14.1.58)*, the following holds :*

(i) *Let $\varphi : M \to N$ be a morphism of R-modules, and $n \in \mathbb{N}$ an integer such that $H_i(\varphi)^a : (H_iM)^a \to (H_iN)^a$ is an isomorphism of V^a-modules, for every $i \leq n$. Then*

$$H_i(LT\varphi)^a : H_i(LTM)^a \to H_i(LTN)^a$$

is an isomorphism of V^a-modules, for every $i \leq n$.

(ii) *Especially, the functor T_R induces a well defined left derived functor*

$$LT_{R^a} : \mathsf{D}(R^a\text{-}\mathbf{Mod}) \to \mathsf{D}(R^a\text{-}\mathbf{Mod}).$$

Proof. Clearly (ii) follows from (i).

(i): In light of corollary 7.10.34(i), we may replace M and N by respectively $\mathrm{cosk}_n M$ and $\mathrm{cosk}_n N$, after which, we may assume that $H_iM = H_iN = 0$ for every $i > n$, so φ^a is an isomorphism in $\mathsf{D}(R^a\text{-}\mathbf{Mod})$. Next, by proposition 7.10.28, we may replace M and N by their respective standard free resolutions, in which case we are reduced to checking that the induced map $(T_R\varphi)^a$ is an isomorphism in $\mathsf{D}(R^a\text{-}\mathbf{Mod})$. Since T is homogeneous and $(\mathbf{B})$ holds for $\mathfrak{m} \cdot H_0(R)$, the latter assertion follows straightforwardly from lemma 14.1.48(ii). ☐

14.1.61. Next, we wish to decide how much of the foregoing theory can be salvaged, when we drop condition $(\mathbf{B})$. We will concentrate on the functors that are relevant to the later study of the cotangent complex. Thus, henceforth, for every integer $d \in \mathbb{N}$, we shall denote by T^d one of the three standard functors

$$\mathrm{Sym}^d, \Lambda^d, \Gamma^d : V\text{-}\mathsf{Alg.Mod} \to V\text{-}\mathsf{Alg.Mod}$$

(namely, the symmetric and exterior d-th power functors, and the d-th divided power functor). Notice that the functor T^d fulfills the conditions of (14.1.58), for every $d \in \mathbb{N}$. Moreover, in all three cases we have natural identifications :

$$T^1 \xrightarrow{\sim} \mathbf{1}_{V\text{-}\mathsf{Alg.Mod}}. \tag{14.1.62}$$

In general, we can no longer expect that T^d descends to almost modules; indeed, consider the following :

Example 14.1.63. Let $(V, \mathfrak{m})$ be as in (14.1.61), and $p \in \mathbb{N}$ any prime integer. The Frobenius map $\Phi_R : R/pR \to R/pR$ for V-algebras R, can be seen as a homogeneous polynomial law of degree p on the V-module V/pV. Denote by $\mathfrak{m}^{(p)} \subset V$ the ideal generated by $(x^p \mid x \in \mathfrak{m})$. Clearly Φ descends to a polynomial law

$$\overline{\Phi} : V/(pV + \mathfrak{m}) \to W_p := V/(pV + \mathfrak{m}^{(p)})$$

which is still homogeneous of degree p, so it factors through a unique V-linear map

$$\Gamma^p_A(V/(pV + \mathfrak{m})) \to W_p.$$

The latter is surjective, since its image contains the class of the unit element of V. Now, the proof of [75, Prop.2.1.7(ii)] shows that – if $(\mathbf{B})$ does not hold for $\mathfrak{m}$ – there exists some prime p such that $pV + \mathfrak{m}^{(p)}$ does not contain $\mathfrak{m}$, therefore $W_p^a \neq 0$. This shows that the functor

$$V\text{-}\mathbf{Mod} \to V^a\text{-}\mathbf{Mod} \qquad M \mapsto (\Gamma_A^p M)^a$$

does not factor through V^a-$\mathbf{Mod}$.

However, we will see that example 14.1.63 is, in a sense, the worst that can happen.

Lemma 14.1.64. *In the situation of* (14.1.61)*, we have :*

(i) *There are natural transformations*

$$T^{i+j} \to T^i \otimes T^j \to T^{i+j} \qquad \textit{for every } i, j \in \mathbb{N}$$

whose composition equals $\binom{i+j}{i} \cdot \mathbf{1}_{T^{i+j}}$. *(Notation of remark* 14.1.59(ii).*)*

(ii) *For every simplicial V-algebra R, the maps of* (i) *induce natural transformations*

$$LT_R^{i+j} \to LT_R^i \overset{\ell}{\otimes}_R LT_R^j \to LT_R^{i+j} \qquad \textit{for every } i, j \in \mathbb{N}$$

whose composition equals $\binom{i+j}{i} \cdot \mathbf{1}_{LT_R^{i+j}}$.

Proof. (i): For $T = \Lambda$, the sought morphism $\Lambda^{i+j} \to \Lambda^i \otimes \Lambda^j$ is the one denoted $\Delta_{i,j}$ in [75, §4.3.20], and the morphism $\Lambda^i \otimes \Lambda^j \to \Lambda^{i+j}$ is given by the rule : $x \otimes y \mapsto x \wedge y$, for every $(A, M) \in \mathrm{Ob}(V\text{-}\mathsf{Alg.Mod})$, every $x \in \Lambda_A^i M$ and every $y \in \Lambda_A^j M$. The sought identity follows easily from the explicit formula given in [75, (4.3.21)] : details left to the reader.

For $T = \mathrm{Sym}$, the natural transformation $\mathrm{Sym}^{i+j} \to \mathrm{Sym}^i \otimes \mathrm{Sym}^j$ is given by a similar formula; namely, for every object (A, M) of V-$\mathsf{Alg.Mod}$, every sequence of elements $x_1, \dots, x_{i+j} \in M$, and every subset $I \subset \{1, \dots, i+j\}$, set $x_I := \prod_{k \in I} x_k \in \mathrm{Sym}_A^k M$ (where the multiplication is formed in the graded ring $\mathrm{Sym}_A^\bullet M$); one checks easily that the rule

$$x_1 \cdots x_{i+j} \mapsto \sum_{I,J} x_I \otimes x_J$$

defines a well defined A-linear map on $\mathrm{Sym}_A^{i+j} M$ (where the sum ranges over all the partitions (I, J) of $\{1, \dots, i+j\}$ such that the cardinality of I equals i). Then one defines a map $\mathrm{Sym}_A^i M \otimes_A \mathrm{Sym}_A^j M \to \mathrm{Sym}_A^{i+j} M$ by the rule : $u \otimes v \mapsto u \cdot v$ for every $u \in \mathrm{Sym}_A^i M$ and $v \in \mathrm{Sym}_A^j M$. Again, the sought identity is verified by direct computation.

If $T = \Gamma$, then for every object (A, M) of V-$\mathsf{Alg.Mod}$ we define a homogeneous polynomial law $M \rightsquigarrow \Gamma_A^i M \otimes_A \Gamma_A^j M$ of degree $i + j$, by the rule : $x \mapsto x^{[i]} \otimes x^{[j]}$ for every A-algebra B, and every $x \in B \otimes_A M$. This law yields a transformation $\Gamma^{i+j} \to \Gamma^i \otimes \Gamma^j$ as sought. Next, the multiplication of the graded ring functor $\Gamma^\bullet$ gives a transformation $\Gamma^i \otimes \Gamma^j \to \Gamma^{i+j}$. The composition of these two transformations is characterized as the unique transformation ψ of Γ^{i+j} such that $\psi_{(A,M)}(x^{[i+j]}) = x^{[i]} \cdot x^{[j]}$ for every object (A, M) of V-$\mathsf{Alg.Mod}$, and every $x \in M$. Then, by corollary 9.5.56 we must have $\psi_{(A,M)} = \binom{i+j}{i} \mathbf{1}_M$, as required.

(ii): Recall that the three functors Λ, Γ and Sym transform free A-modules into free A-modules, for every V-algebra A. Therefore, the sought natural transformation is none else than the map

$$T_R^{i+j}(\perp_\bullet^R M) \to T_R^i(\perp_\bullet^R M) \otimes_R T_R^j(\perp_\bullet^R M) \to T_R^{i+j}(\perp_\bullet^R M)$$

given by (i), for any R-module M. □

Proposition 14.1.65. *In the situation of* (14.1.61)*, let $p \in \mathbb{N}$ be a prime integer, R a simplicial $V \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$-algebra, M an R-module such that $M^a = 0$ in* $\mathsf{D}(R^a\text{-}\mathbf{Mod})$. *Then*

$$(LT_R^d M)^a = 0 \qquad \textit{in } \mathsf{D}(R^a\text{-}\mathbf{Mod}), \textit{for every } d \in \mathbb{N} \textit{ such that } (p, d) = 1.$$

Proof. In light of corollary 7.10.34(i), we may replace M by $\mathrm{cosk}_i M$, in which case lemma 14.1.48(i) implies that $a \cdot \mathbf{1}_M = 0$ for every $a \in \mathfrak{m}$. Now, we apply lemma 14.1.64(ii) with $i = 1$ and $j = d-1$ (notice that $j \geq 0$, since $d > 0$); in view of (14.1.62), there result natural transformations

$$LT^d_R M \to M \otimes_R LT^{d-1}_R M \to LT^d_R M$$

whose composition is $d \cdot \mathbf{1}_{LT^d_R M}$. Since the image of d is invertible in R, it follows easily that $a \cdot \mathbf{1}_{LT^d_R M} = 0$ for every $a \in \mathfrak{m}$, whence the assertion. □

Theorem 14.1.66. *Let $d, n, p \in \mathbb{N}$ be any three integers, with $d > 0$ and p a prime. Let also R be a simplicial $V \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$-algebra, M an R-module, and suppose that*

(a) $H_i M = 0$ *for every $i < n$.*
(b) $M^a = 0$ *in* $\mathsf{D}(R^a\text{-}\mathbf{Mod})$.

Then we have :

(i) $H_i(L\Gamma^d_R M)^a = 0$ *for every $i < n$.*
(ii) $H_i(L\Lambda^d_R M)^a = 0$ *for every $i < n + p - 1$.*
(iii) $H_i(L\mathrm{Sym}^d_R M)^a = 0$ *for every $i < n + 2(p-1)$.*

Proof. (i) is just a special case of corollary 7.10.34(ii), and for $d < p$, assertions (ii) and (iii) follow from the more general proposition 14.1.65, hence we may assume that $d \geq p$. For every $j = 0, \dots, d$, set

$$F^j := \Gamma^j \otimes \Lambda^{d-j} \qquad G^j := \Lambda^j \otimes \mathrm{Sym}^{d-j}.$$

By remark 14.1.59(ii), these functors are homogeneous of degree d, and fulfill the conditions of (14.1.58). Moreover, since Γ, Λ and Sym transform free modules into free modules, we have natural isomorphisms of functors :

$$(14.1.67) \quad LF^j_R \xrightarrow{\sim} L\Gamma^j_R \overset{\ell}{\otimes}_R L\Lambda^{d-j}_R \quad LG^j_R \xrightarrow{\sim} L\Lambda^j_R \overset{\ell}{\otimes}_R L\mathrm{Sym}^{d-j}_R \qquad \text{for every } j = 0, \dots, d$$

(details left to the reader).

(ii): According to [103, I.4.3.1.7], there is a natural complex of functors :

$$(14.1.68) \qquad 0 \to F^d_R \xrightarrow{\partial_{d-1}} F^{d-1}_R \to \cdots \to F^1_R \xrightarrow{\partial_0} F^0_R \to 0$$

which is exact on flat R-modules. Set $Z_p := \mathrm{Ker}\,\partial_{p-1}$; due to remark 14.1.59(iii) we may consider the derived functor LZ_p, and from corollary 7.10.34(ii) and assumption (a), we get

$$(14.1.69) \qquad H_i(LZ_p M) = 0 \qquad \text{for every } i < n.$$

The evaluation of (14.1.68) on $(\perp^R_\bullet M)^\Delta$ gives an exact sequence of R-modules :

$$0 \to LZ_p M \to LF^{p-1}_R M \to \cdots \to LF^1_R M \to LF^0_R M \to 0.$$

Notice as well, that

$$(14.1.70) \qquad (LF^j_R M)^a = 0 \qquad \text{in } \mathsf{D}(R^a\text{-}\mathbf{Mod}) \text{ for } j = 1, \dots, p-1$$

in view of (14.1.67), remark 14.1.51(iii), and proposition 14.1.65. The assertion now follows from (14.1.69) and claim 7.10.50.

(iii): According to [103, I.4.3.1.7], there is a natural complex of functors :

$$\Sigma \quad : \quad 0 \to G^d_R \xrightarrow{\partial_{d-1}} G^{d-1}_R \to \cdots \to G^1_R \xrightarrow{\partial_0} G^0_R \to 0.$$

Set $Z'_p := \mathrm{Ker}\,\partial_{p-1}$; due to remark 14.1.59(iii) we may consider the derived functor LZ'_p, and since Σ is exact on flat R-modules, we obtain two exact sequences of R-modules :

$$\Sigma' \quad : \quad 0 \to LG^d_R M \to LG^{d-1}_R M \to \cdots \to LG^p_R M \to LZ'_p M \to 0$$
$$\Sigma'' \quad : \quad 0 \to LZ'_p M \to LG^{p-1}_R M \to \cdots \to LG^1_R M \to LG^0_R M \to 0.$$

On the one hand, in light of (ii), remark 14.1.51(iii,vi) and (14.1.67), we see that

$$\mathrm{cosk}_{n+p-1}(LG^j_R M)^a = 0 \qquad \text{in } \mathsf{D}(R^a\text{-}\mathbf{Mod}) \text{ for every } j = 1, \dots, d.$$

By applying claim 7.10.50 to the exact sequence $\mathrm{cosk}_{n+p-1}\Sigma'^a$, we deduce that $H_i(LZ'_p M)^a = 0$ for every $i < n+p-1$. On the other hand, (14.1.67) and proposition 14.1.65 imply that

$$(LG^j_R M)^a = 0 \qquad \text{in } \mathsf{D}(R^a\text{-}\mathbf{Mod}) \text{ for } j = 1, \dots, p-1$$

so the assertion follows, after applying claim 7.10.50 to the exact sequence Σ''. □

14.1.71. Let now $\varphi : R \to S$ be any morphism of simplicial V^a-algebras; pick any resolution $\rho : P \to S$ with P a flat R-algebra, and consider the composition

$$S \otimes_R P \xrightarrow{S \otimes_R \rho} S \otimes_R S \xrightarrow{\mu_S} S \tag{14.1.72}$$

where μ_S is the multiplication law of S. Then (14.1.72) represents a morphism

$$\Delta(\varphi) : S \overset{\ell}{\otimes}_R S \to S \qquad \text{in } \mathsf{D}(R\text{-}\mathsf{Alg})$$

which is independent (up to unique isomorphism) of the choice of P. Notice that if φ is an isomorphism in $\mathsf{D}(s.V^a\text{-}\mathsf{Alg})$, then the same holds for $\varphi \overset{\ell}{\otimes}_R \varphi : R \to S \overset{\ell}{\otimes}_R S$; also, we have

$$\varphi = \Delta(\varphi) \circ (\varphi \overset{\ell}{\otimes}_R \varphi) \qquad \text{in } \mathsf{D}(s.V^a\text{-}\mathsf{Alg}).$$

More generally, set $C := \mathrm{Cone}\,\varphi$; then, by considering the section $S \overset{\ell}{\otimes}_R \varphi$ of $\Delta(\varphi)$, we get a natural decomposition

$$S \overset{\ell}{\otimes}_R S \xrightarrow{\sim} S \oplus (S \overset{\ell}{\otimes}_R C) \qquad \text{in } \mathsf{D}(R\text{-}\mathbf{Mod})$$

which identifies $\Delta(\varphi)$ with the natural projection, whence a natural isomorphism

$$\mathrm{Cone}\,\Delta(\varphi) \xrightarrow{\sim} \sigma S \overset{\ell}{\otimes}_R C. \tag{14.1.73}$$

Thus, say that $C = \sigma^k C'$ in $\mathsf{D}(R\text{-}\mathbf{Mod})$, for some R-module C' with $H_0 C' \neq 0$; there follows a distinguished triangle in $\mathsf{D}(R\text{-}\mathbf{Mod})$

$$\sigma^k C' \to \sigma^k S \overset{\ell}{\otimes}_R C' \to \sigma^{2k} C' \overset{\ell}{\otimes}_R C' \to \sigma^{k+1} C'.$$

Especially, if $k \geq 2$, then $H_k C = H_k(S \overset{\ell}{\otimes}_R C)$, and we see that, in this case, $\mathrm{Cone}\,\Delta(\varphi) = \sigma^{k+1} C''$ in $\mathsf{D}(R\text{-}\mathbf{Mod})$, for some C'' such that $H_0 C'' \neq 0$. However, it is clear from (14.1.73) that $\mathrm{Cone}\,\Delta^2(\varphi) = \sigma^2 C''$ for some object C'' of $\mathsf{D}(R\text{-}\mathbf{Mod})$. We conclude that if $\Delta^n(\varphi)$ is an isomorphism in $\mathsf{D}(R\text{-}\mathbf{Mod})$ for some $n \geq 2$, then the same holds already for $\Delta^2(\varphi)$.

Definition 14.1.74. In the situation of (14.1.71), we say that φ is a *weakly étale morphism*, if $\Delta^2(\varphi)$ is an isomorphism in $\mathsf{D}(R\text{-}\mathsf{Alg})$.

14.1.75. Let $\varphi : R' \to R$ be any morphism of simplicial V^a-algebras, and S, T two R-algebras. Since the standard resolution $F_R(S) := F_{R_{!!}}(S_{!!})^a$ of example 7.10.25 is clearly functorial in both R and S, we have a natural map

$$F_{R'}(S)^\Delta \to F_R(S)^\Delta$$

of simplicial R'-algebras, whence a natural morphism

$$S \overset{\ell}{\otimes}_{R'} T \to F_R(S)^\Delta \otimes_{R'} T \to S \overset{\ell}{\otimes}_R T. \tag{14.1.76}$$

Lemma 14.1.77. *In the situation of* (14.1.75)*, suppose that φ is a quasi-isomorphism. Then the same holds for* (14.1.76)*.*

Proof. Let $\psi : F_{R'}(S)^\Delta \otimes_{R'} R \to F_R(S)^\Delta$ be the natural morphism; by construction, (14.1.76) factors as the composition of $\psi \otimes_R \mathbf{1}_T$ and the natural isomorphism

$$F_{R'}(S)^\Delta \otimes_{R'} T \xrightarrow{\sim} (F_{R'}(S)^\Delta \otimes_{R'} R) \otimes_R T.$$

Since both $F_{R'}(S)^\Delta \otimes_{R'} R$ and $F_R(S)^\Delta$ are flat R-algebras, remark 14.1.52(ii) then reduces to checking that ψ is a quasi-isomorphism. Since the natural maps $\beta : F_R(S)^\Delta \to S$ and $\alpha : F_{R'}(S)^\Delta \to F_{R'}(\Delta) \otimes_{R'} R$ are quasi-isomorphisms, it suffices to show that the composition $\beta \circ \psi \circ \alpha$ is a quasi-isomorphism. But the latter is none else than the standard resolution $F_{R'}(S) \to S$, whence the lemma. □

Proposition 14.1.78. *Let $\varphi : R \to S$ and $\varphi' : R' \to S'$ be two morphisms of simplicial V-algebras, and suppose we have a commutative diagram in $\mathsf{D}(R\text{-}\mathsf{Alg})$*

(14.1.79)
$$\begin{array}{ccc} R & \xrightarrow{\varphi} & S \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\psi'} \\ R' & \xrightarrow{\varphi'} & S' \end{array}$$

where ψ and ψ' are isomorphisms (in $\mathsf{D}(R\text{-}\mathsf{Alg})$). Then φ^a is weakly étale if and only if the same holds for φ'^a.

Proof. Denote by $\mathsf{Hot}(V\text{-}\mathsf{Alg})$ the homotopy category of simplicial V-algebras, and recall that the multiplicative system of quasi-isomorphisms in $\mathsf{Hot}(V\text{-}\mathsf{Alg})$ admits a right calculus of fractions ([103, I.3.1.8(ii)]). We begin with the following special case :

Claim 14.1.80. Suppose that ψ and ψ' are also morphisms of simplicial V-algebras, and that (14.1.79) commutes in the category $\mathsf{Hot}(V\text{-}\mathsf{Alg})$. Then the proposition holds.

Proof of the claim. Indeed, in this situation it follows easily from lemma 14.1.77 that φ^a (resp. φ'^a) is weakly étale if and only if the same holds for $\psi'^a \circ \varphi^a$ (resp. for $\varphi'^a \circ \psi^a$), so we are reduced to the case where $R = R'$, $S = S'$, and $\varphi, \varphi' : R \to S$ are two homotopic morphisms of simplicial V-algebras. In this case, there exist morphisms of simplicial V-algebras $\varphi'' : R \to S''$ and $d_0, d_1 : S'' \to S$, such that d_0 and d_1 are quasi-isomorphisms, and $\varphi = d_0 \circ \varphi''$, $\varphi' = d_1 \circ \varphi''$ (see [103, I.2.3.2]). Then the claim follows by applying repeatedly lemma 14.1.77. ◊

Next, there exist a simplicial V-algebra S'' and morphisms $\beta : S'' \to S$ and $\gamma : S'' \to S'$ of simplicial V-algebras, such that both β and γ are quasi-isomorphisms, and $\psi' = \gamma \circ \beta^{-1}$ in $\mathsf{D}(s.V\text{-}\mathsf{Alg})$. Moreover, there exist morphisms $\varphi'' : R'' \to S''$ and $\psi'' : R'' \to R$ such that ψ'' is a quasi-isomorphism, and the resulting diagram

$$\begin{array}{ccc} R'' & \xrightarrow{\varphi''} & S'' \\ {\scriptstyle\psi''}\downarrow & & \downarrow{\scriptstyle\beta} \\ R & \xrightarrow{\varphi} & S \end{array}$$

commutes in $\mathsf{Hot}(V\text{-}\mathsf{Alg})$. By claim 14.1.80, we may then replace φ by φ'' and ψ' by β, after which we may assume that ψ' is a morphism of simplicial V-algebras.

Likewise, we may find morphisms $\beta' : R'' \to R$ and $\gamma' : R'' \to R'$ of simplicial V-algebras that are quasi-isomorphisms, and such that $\psi = \gamma' \circ \beta'^{-1}$ in $\mathsf{D}(s.V\text{-}\mathsf{Alg})$; again by lemma 14.1.77, we see that φ^a is weakly étale if and only if the same holds for $\varphi^a \circ \beta^a$, so we may replace φ by $\varphi \circ \beta$, ψ by $\psi \circ \gamma$, and assume from start that also ψ is a map of simplicial V-algebras.

In this situation, by applying once again lemma 14.1.77 we see that φ^a (resp. φ'^a) is weakly étale if and only if the same holds for $\psi'^a \circ \varphi^a$ (resp. for $\varphi'^a \circ \psi^a$). Therefore, we may assume from start that $R = R'$, $S = S'$, and $\varphi, \varphi' : R \to S$ are two maps of simplicial V-algebras that represent the same morphism in $\mathsf{D}(s.V\text{-}\mathsf{Alg})$. In this case, we may find a morphism $\vartheta : R''' \to R$ of simplicial V-algebras, such that ϑ is a quasi-isomorphism, and $\varphi \circ \vartheta, \varphi' \circ \vartheta : R''' \to S$ are homotopic maps; then the assertion follows from claim 14.1.80 (and again, from lemma 14.1.77: details left to the reader). □

Corollary 14.1.81. *Let $\varphi : R \to S$ and $\psi : S \to T$ be any two morphisms of simplicial V^a-algebras. We have :*

(i) *If φ and ψ are weakly étale, the same holds for $\psi \circ \varphi$.*
(ii) *If φ and $\psi \circ \varphi$ are weakly étale, the same holds for ψ.*
(iii) *If φ is weakly étale, and $R \to R'$ is any morphism of simplicial V^a-algebras, then $\varphi' := R' \overset{\ell}{\otimes}_R \varphi$ is weakly étale.*
(iv) *φ is weakly étale if and only if the same holds for $\Delta(\varphi)$.*

Proof. (iv) is immediate from the discussion of (14.1.71).

(iii): Endow $S' := R' \overset{\ell}{\otimes}_R S$ with the R'-algebra structure deduced from its left tensor factor; then the assertion is an immediate consequence of the following more general

Claim 14.1.82. There exists a commutative diagram in $\mathsf{D}(R'\text{-}\mathsf{Alg})$

$$\begin{array}{ccc} S' \overset{\ell}{\otimes}_{R'} S' & \xrightarrow{\qquad\qquad} & R' \overset{\ell}{\otimes}_R (S \overset{\ell}{\otimes}_R S) \\ & {\scriptstyle \Delta(\varphi')} \searrow \quad \swarrow {\scriptstyle R' \overset{\ell}{\otimes}_R \Delta(\varphi)} & \\ & S' & \end{array}$$

whose horizontal arrow is an isomorphism.

Proof of the claim. Pick any resolution $\rho : P \to S$ with P a flat R-algebra; then $R' \otimes_R P$ represents $R' \overset{\ell}{\otimes}_R S$, and we have a natural isomorphism of R'-algebras

$$(R' \otimes_R P) \otimes_{R'} (R' \otimes_R P) \xrightarrow{\sim} R' \otimes_R (P \otimes_R P).$$

In view of remark 7.10.31(i), the source of this map represents $S' \overset{\ell}{\otimes}_{R'} S'$, and the target represents $R' \overset{\ell}{\otimes}_R (S \overset{\ell}{\otimes}_R S)$. Under these identifications, the morphism $R' \overset{\ell}{\otimes}_R \Delta(\varphi)$ becomes the map $\mathbf{1}_{R'} \otimes_R (\mu_S \circ (\rho \otimes_R \rho))$ (notation of (14.1.71)), whereas $\Delta(\varphi')$ is represented by the multiplication map $\mu_{R' \otimes_R P} = \mathbf{1}_{R'} \otimes_R \mu_P$ of $R' \otimes_R P$. The claim follows straightforwardly. ◊

(i): Pick a resolution $P \to S_{!!}$ with P a flat $R_{!!}$-algebra, and a resolution $Q \to T_{!!}$ with Q a flat P-algebra; in light of proposition 14.1.78 it suffices to show that the resulting morphism $R \to Q^a$ is weakly étale, so we may replace S by P^a and T by Q^a, and assume from start that both φ and ψ are flat morphisms. Due to (iv), it then suffices to check that $\Delta(\psi \circ \varphi) : T \otimes_R T \to T$ is weakly étale; however, the latter can be factored as the composition

$$T \otimes_R T \xrightarrow{\sim} T \otimes_S (S \otimes_R S) \otimes_S T \xrightarrow{T \otimes_S \Delta(\varphi) \otimes_S T} T \otimes_S T \xrightarrow{\Delta(\psi)} T \qquad (14.1.83)$$

and by (iv) the maps $\Delta(\psi)$ and $\Delta(\varphi)$ are weakly étale, so the same holds for $T \otimes_S \Delta(\varphi) \otimes_S T$, in view of (iii). We may therefore replace ψ by $\Delta(\psi)$ and φ by $T \otimes_S \Delta(\varphi) \otimes_S T$, after which, we may assume that $\Delta(\psi)$ is an isomorphism in $\mathsf{D}(s.A\text{-}\mathsf{Alg})$. In this case, the factorization (14.1.83) makes it clear that $\Delta(\psi \circ \varphi)$ is weakly étale, as stated.

(ii): Set $S' := S \overset{\ell}{\otimes}_R S$ and $T' := S \overset{\ell}{\otimes}_R T$; we may factor ψ as the composition

$$S \xrightarrow{S \overset{\ell}{\otimes}_R (\psi \circ \varphi)} T' \xrightarrow{T' \overset{\ell}{\otimes}_{S'} \Delta(\varphi)} T$$

so the assertion follows from (i), (iii) and (iv). □

Theorem 14.1.84. *Let $\varphi : R \to S$ be a morphism of simplicial V-algebras, $c, p \in \mathbb{N}$ two integers, with p a prime, and suppose that φ^a is weakly étale. We have :*

(i) *If the ideal $\mathfrak{m} \cdot H_0(S)$ of $H_0(S)$ satisfies condition $(\mathbf{B})$ of* [75, §2.1.6]*, then*

$$\mathbb{L}^a_{S/R} = 0 \qquad \textit{in } \mathsf{D}(S^a\text{-}\mathbf{Mod}).$$

(ii) *If S is a $\mathbb{Z}_{(p)}$-algebra and $H_i\mathbb{L}_{S/R} = 0$ for every $i < c$, then*

$$H_i\mathbb{L}^a_{S/R} = 0 \qquad \textit{for every } i < c + 2p - 1.$$

Proof. We begin with the following :

Claim 14.1.85. We may assume that both $\Delta(\varphi^a)$ and $H_0(\varphi)$ are isomorphisms.

Proof of the claim. To ease notation, set $S' := S \overset{\ell}{\otimes}_R S$, and consider the sequence of morphisms

$$S \xrightarrow{\mathbf{1}_S \overset{\ell}{\otimes}_R \varphi} S' \xrightarrow{\mu_B} S \tag{14.1.86}$$

where μ_S is the composition of the multiplication map $S \otimes_R S \to S$ and the natural map $S' \to S \otimes_R S$; the transitivity triangle ([103, III.2.1.2]) relative to (14.1.86) yields a natural isomorphism

$$\mathbb{L}_{S'/S} \xrightarrow{\sim} \sigma S \otimes_{S'} \mathbb{L}_{S'/S} \xrightarrow{\sim} \sigma S \otimes_{S'} (S' \otimes_S \mathbb{L}_{S/R}) \qquad \text{in } \mathsf{D}(S\text{-}\mathbf{Mod})$$

where the last isomorphism follows from the base change theorem of [103, III.2.2.1]. Thus, $\mathbb{L}_{S'/S} \simeq \sigma\mathbb{L}_{S/R}$ in $\mathsf{D}(S\text{-}\mathbf{Mod})$, so assertion (i) (resp. (ii)) holds for φ, provided it holds for $\Delta(\varphi)$, and then the claim follows from corollary 14.1.81(iv). ◊

Henceforth, we assume that both $\Delta(\varphi^a)$ and $H_0(\varphi)$ are isomorphisms. Let $P \to S$ be a resolution, with P an R-algebra, such that $P[n]$ is a free $R[n]$-algebra for every $n \in \mathbb{N}$; set $P' := S \otimes_R P$, and recall that there are natural isomorphisms

$$\mathbb{L}_{S/R} \xrightarrow{\sim} S \otimes_P \Omega^1_{P/R} \xrightarrow{\sim} S \otimes_{P'} \Omega^1_{P'/S}$$

where $\Omega^1_{P/R}$ denotes the flat simplicial P-module such that $\Omega^1_{P/R}[n] := \Omega_{P[n]/R[n]}$ (and likewise for $\Omega^1_{P'/S}$), with faces and degeneracies deduced from those of P and R (resp. those of P' and S), in the obvious way. In other words, set

$$J := \mathrm{Ker}(\mu_S : P' \to S)$$

(with μ_S as in the proof of claim 14.1.85); then $\mathbb{L}_{S/R} \simeq J/J^2$ in $\mathsf{D}(S\text{-}\mathbf{Mod})$, and notice that

$$J^a = 0 \qquad \text{in } \mathsf{D}(S^a\text{-}\mathbf{Mod}) \tag{14.1.87}$$

since $\Delta(\varphi^a)$ is an isomorphism.

Claim 14.1.88. J is a quasi-regular ideal, and $H_0 J = 0$.

Proof of the claim. Since $H_0(\varphi)$ is an isomorphism, it is clear that $H_0 J = 0$. Next, for every $n \in \mathbb{N}$, the $S[n]$-algebra $P'[n]$ is free, hence we reduce to showing the following. Let B be any ring, $C := B[X_i \mid i \in I]$ any free B-algebra (for any set I), and $f : C \to B$ any morphism of B-algebras; then $\mathrm{Ker}\, f$ is a quasi-regular ideal of C. However, for every $i \in I$, set $b_i := f(X_i)$, and let $g : C \xrightarrow{\sim} C$ be the isomorphism of B-algebras such that $g(X_i) = X_i - a_i$ for every $i \in I$; clearly $\mathrm{Ker}\, f$ is a quasi-regular ideal if and only if the same holds for $g^{-1}\mathrm{Ker}\, f$. But the latter is the ideal $(X_i \mid i \in I)$, so the assertion follows from remark 7.10.44(iii). ◊

(i): We need to show that $H_n(J/J^2)^a = 0$ for every $n \in \mathbb{N}$, and we shall argue by induction on n. The assertion for $n = 0$ is clear from (14.1.87), in view of the exact sequence

$$H_n J \to H_n(J/J^2) \to H_{n-1} J^2 \qquad \text{for every } n \in \mathbb{N}.$$

Hence, suppose that $n > 0$, and the assertion is already known for every degree $< n$. The same exact sequence reduces to checking that $H_{n-1}(J^2)^a = 0$. Now, from claim 14.1.88 and theorem 7.10.48, we know that $H_{n-1}J^n = 0$. Therefore, by an easy induction, we are further reduced to showing that $H_{n-1}(J^i/J^{i+1})^a = 0$ for every $i = 2, \dots, n-1$. However, on the one hand, proposition 7.10.45(ii.a) says that the natural map

$$L\mathrm{Sym}^i_S(J/J^2) \to J^i/J^{i+1}$$

is an isomorphism in $\mathsf{D}(S\text{-}\mathbf{Mod})$, for every $i \in \mathbb{N}$; on the other hand, the inductive assumption and theorem 14.1.60(i) imply that $H_j(L\mathrm{Sym}^i J/J^2)^a = 0$ for every $j \leq n$, whence the contention.

(ii): We show, by induction on n, that $H_n(J/J^2)^a = 0$ for every $n < c + 2p - 1$. Arguing as in the foregoing, we may assume that $c + 2p - 1 > n > 0$, and the sought vanishing is already known in degrees $< n$, in which case we reduce to checking that $H_{n-1}(L\mathrm{Sym}^i_S J/J^2)^a = 0$ for every $i = 2, \dots, n-1$. Set $M := \mathrm{cosk}_n(s.\mathrm{trunc}_n J/J^2)$; then $H_j M = 0$ for every $j < c$, and the inductive assumption implies that $M^a = 0$ in $\mathsf{D}(S^a\text{-}\mathbf{Mod})$, so $H_{n-1}(L\mathrm{Sym}^i_S M)^a = 0$, by theorem 14.1.66(iii). Lastly, by corollary 7.10.34(i), we see that

$$H_{n-1}(L\mathrm{Sym}^i_S M) = H_{n-1}(L\mathrm{Sym}^i_S J/J^2) \qquad \text{for every } i \in \mathbb{N}$$

whence the contention. □

Remark 14.1.89. Let $A \to B$ be any morphism of simplicial rings such that the multiplication map $\mu_B : B \otimes_A B \to B$ induces an isomorphism $H_0(\mu_B) : H_0(B) \otimes_{H_0(A)} H_0(B) \xrightarrow{\sim} H_0(B)$. Pick a resolution $\varphi : P \to B$ such that $P[n]$ is a free $A[n]$-algebra for every $n \in \mathbb{N}$, and set $P' := B \otimes_A P$, $J := \mathrm{Ker}\,(\mu_B \circ (B \otimes_A \varphi) : P' \to B)$. The assumption on B implies that $H_0 J = 0$. On the other hand, arguing as in the proof of claim 14.1.88 we see that J is a quasi-regular ideal. Taking into account proposition 7.10.45(ii) the spectral sequence associated with the J-adic filtration of P' can be written as

$$E^2_{pq} := H_{p+q}(L\mathrm{Sym}^q_B(\mathbb{L}_{B/A})) \Rightarrow \mathrm{Tor}^A_{p+q}(B, B).$$

Though the J-adic filtration is not finite, using proposition 7.2.18(i) and theorem 7.10.48 one sees that this spectral sequence is convergent, and the filtration on its abutment is finite. We will not use this result.

Corollary 14.1.90. *Let $\varphi : A \to B$ be a morphism of V^a-algebras, $p \in \mathbb{N}$ a prime integer, and suppose that*

(a) *V is a $\mathbb{Z}_{(p)}$-algebra*
(b) *the induced morphism $s.\varphi : s.A \to s.B$ of simplicial V^a-algebras is weakly étale.*

Then $H_i \mathbb{L}^a_{B/A} = 0$ for every $i \leq 2p$.

Proof. Notice that condition (a) implies that $B_{!!}$ is a $\mathbb{Z}_{(p)}$-algebra; letting $c := 0$ in theorem 14.1.84(ii), we deduce that $H_i \mathbb{L}^a_{B/A} = 0$ for $i = 0, 1$. But in view of claim 14.1.46, the latter implies that $H_i \mathbb{L}_{B_{!!}/A_{!!}} = 0$ for $i = 0, 1$. So, actually the morphism φ fulfills the conditions of theorem 14.1.84(ii), with $c = 2$, whence the assertion. □

Remark 14.1.91. As an application we can now generalize the deformation theory for almost rings in [75, §3.2] to the case of an arbitrary basic setup $(V, \mathfrak{m})$. Indeed, [75, Lemmata 3.2.1 and 3.26, Prop.3.2.9 and Cor.3.2.11] hold *verbatim* for any such basic setup, with the same proof : the only difference is that instead of invoking [75, 3.2.1] we must appeal to our theorem 14.1.43 in the explanation of [75, §3.2.7]. Next, let $A \to B$ be any weakly étale morphism of V^a-algebras; for every prime integer p, let $V_{(p)} := \mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} V$, and denote

$$j_p : (V, \mathfrak{m})^a\text{-}\mathsf{Alg} \to (V_{(p)}, \mathfrak{m}V_{(p)})^a\text{-}\mathsf{Alg} \qquad C \mapsto \mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} C$$

the natural functor; with this notation, it is easily seen that

$$\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} B_{!!} = (j_p B)_{!!}$$

(details left to the reader), and in view of [103, II,2.2.1] we deduce a natural isomorphism

$$\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} \mathbb{L}_{B_{!!}/A_{!!}} \xrightarrow{\sim} \mathbb{L}_{(j_p B)_{!!}/(j_p A)_{!!}}.$$

Corollary 14.1.90 then easily implies that $H_i \mathbb{L}^a_{B/A} = 0$ for every $i \leq 4$. Combining with the results just quoted from [75], we see that the whole of [75, Th.3.2.18] still holds under the current assumptions. In the same vein, we may complement as follows the descent results of [75, §3.4] :

Corollary 14.1.92. *Consider a cartesian diagram of V^a-algebras*

$$\mathscr{D} \quad : \quad \begin{array}{ccc} A_0 & \longrightarrow & A_1 \\ \downarrow & & \downarrow \\ A_2 & \longrightarrow & A_3 \end{array}$$

such that the morphism $A_1 \to A_3$ is an epimorphism on the underlying V^a-modules. Then the resulting morphism $A_0 \to A_1 \times A_2$ is of universal effective descent for the fibred categories $\mathbf{w}.\mathbf{\acute{E}t}$ *of weakly étale morphisms and* $\mathbf{\acute{E}t}$ *of étale morphisms.*

Proof. Arguing as in the proof of [75, Prop.3.4.33], and using [75, Cor.3.4.22], we see that the morphism $A_0 \to A_1 \times A_2$ is of effective descent for the fibred categories $\mathbf{w}.\mathbf{\acute{E}t}$ and $\mathbf{\acute{E}t}$. Next, consider any morphism $A_0 \to B_0$ of V^a-algebras, and set $B_i := B_0 \otimes_{A_0} A_i$ for $i = 1, 2, 3$. The resulting diagram $B_0 \otimes_{A_0} \mathscr{D}$ is not necessarily cartesian; however, we notice :

Claim 14.1.93. The induced morphism $f : B_0 \to B'_0 := B_1 \times_{B_3} B_2$ is an epimorphism on the underlying V^a-modules, with nilpotent kernel.

Proof of the claim. Indeed, we have a short exact sequence of A_0-modules

$$0 \to A_0 \to A_1 \times A_2 \to A_3 \to 0$$

inducing a right exact sequence $B_0 \to B_1 \times B_2 \to B_3 \to 0$, which shows that f is an epimorphism. Newt, for any B-module M we show more generally that

$$\mathrm{Ann}_B(M \otimes_{A_0} A_1) \cdot \mathrm{Ann}_B(M \otimes_{A_0} A_2) \subset \mathrm{Ann}_B(M).$$

Letting $M := B$, it will then follow easily that $(\mathrm{Ker}\, f)^2 = 0$. Now, let $x, y \in B_*$ be two almost elements such that $x \cdot (M \otimes_{A_0} A_1) = 0$ and $y \cdot (M \otimes_{A_0} A_2) = 0$, and let also $I := \mathrm{Ker}\,(A_1 \to A_3) = \mathrm{Ker}\,(A_0 \to A_2)$; since $M \otimes_{A_0} A_2 = M/IM$, we deduce that $yM \subset IM$, and on the other hand the natural morphism of A_0-modules $M \otimes_{A_0} A_1 \otimes_{A_1} I \xrightarrow{\sim} M \otimes_{A_0} I \to IM$ is an epimorphism, so that $x \cdot IM = 0$, and the assertion follows. ◊

Now, by the foregoing we known already that the induced morphism $B'_0 \to B_1 \times B_2$ is of effective descent for $\mathbf{w}.\mathbf{\acute{E}t}$ and $\mathbf{\acute{E}t}$. But claim 14.1.93 together with remark 14.1.91 implies that the base change functors

$$B_0\text{-}\mathbf{w}.\mathbf{\acute{E}t} \to B'_0\text{-}\mathbf{w}.\mathbf{\acute{E}t} \qquad \text{and} \qquad B_0\text{-}\mathbf{\acute{E}t} \to B'_0\text{-}\mathbf{\acute{E}t}$$

are equivalences of categories; thus, also the induced morphism $B_0 \to B_1 \times B_2$ is of effective descent, as required. □

14.1.94. We conclude this section with a few words on the cohomology of sheaves of almost modules. Namely, let $(V, \mathfrak{m})$ again be an arbitrary basic setup, which we view as a basic setup relative to the one-point topos $\{\mathrm{pt}\}$, in the sense of [75, §3.3]. Let $(X, \mathscr{O}_X)$ be a ringed topos, $\mathfrak{m}_X \subset \mathscr{O}_X$ an ideal, such that $(\mathscr{O}_X, \mathfrak{m}_X)$ is a basic setup relative to the topos X, and suppose also that we are given a morphism of ringed topoi :

$$\pi : (X, \mathscr{O}_X) \to (\{\mathrm{pt}\}, V) \quad \text{such that} \quad (\pi^*\mathfrak{m}) \cdot \mathscr{O}_X \subset \mathfrak{m}_X.$$

We deduce a natural morphism :

$$R\pi_* : \mathsf{D}^+((\mathscr{O}_X, \mathfrak{m}_X)^a\text{-}\mathbf{Mod}) \to \mathsf{D}^+((V, \mathfrak{m})^a\text{-}\mathbf{Mod})$$

which can be constructed as usual, by taking injective resolutions. In case $\widetilde{\mathfrak{m}}$ is a flat V-module, this is the same as setting

$$R\pi_*(K^\bullet) := (R\pi_* K^\bullet_!)^a.$$

Using [75, Cor.2.2.24] one may verify that the two definitions coincide : the details shall be left to the reader. In many cases, both statements and proofs of results concerning the cohomology of $\mathscr{O}_X$-modules carry over *verbatim* to $\mathscr{O}_X^a$-modules. One sets, as customary :

$$H^\bullet(X, K^\bullet) := H^\bullet R\pi_* K^\bullet \qquad \text{for every object } K^\bullet \text{ of } \mathsf{D}^+((\mathscr{O}_X, \mathfrak{m}_X)^a\text{-}\mathbf{Mod}).$$

14.1.95. As an illustration, we consider the following situation. Suppose that :

- $f : A \to A'$ is a map of V-algebras, and set :
$$\varphi := \mathrm{Spec}\, f : X' := \mathrm{Spec}\, A' \to X := \mathrm{Spec}\, A.$$
- $t \in A$ is an element which is regular both in A and in A', and such that the induced map $A/tA \to A'/tA'$ is an isomorphism.
- $U \subset X$ is any open subset containing $D(t) := \{\mathfrak{p} \in X \mid t \notin \mathfrak{p}\}$, and set $U' := \varphi^{-1}U$.
- $\mathscr{F}$ is a quasi-coherent $\mathscr{O}_U^a$-module, and set $\mathscr{F}' := \varphi^*_{|U'}\mathscr{F}$, which is a quasi-coherent $\mathscr{O}_{U'}^a$-module.

Then we have natural morphisms of A^a-modules :

$$H^q(U, \mathscr{F}) \to H^q(U', \mathscr{F}') \qquad \text{for every } q \in \mathbb{N}. \tag{14.1.96}$$

Lemma 14.1.97. *In the situation of* (14.1.95)*, suppose moreover that t is $\mathscr{F}$-regular. Then* (14.1.96) *is an isomorphism for every $q > 0$, and induces an isomorphism of A'^a-modules :*

$$H^0(U, \mathscr{F}) \otimes_A A' \xrightarrow{\sim} H^0(U', \mathscr{F}').$$

Proof. To ease notation, we shall write φ instead of $\varphi_{|U'}$. To start out, we remark :

Claim 14.1.98. (i) $\mathscr{T}\!or_1^{\mathscr{O}_U^a}(\mathscr{F}, \mathscr{O}_U^a/t\mathscr{O}_U^a) = 0$ and $\mathrm{Tor}_1^{A^a}(H^0(U, \mathscr{F}), A/tA) = 0$.

(ii) The $\mathscr{O}_U^a$-module $\varphi_*\mathscr{F}' = \mathscr{F} \otimes_{\mathscr{O}_U^a} \varphi_*\mathscr{O}_{U'}^a$ (resp. the A'^a-module $H^0(U, \mathscr{F}) \otimes_{A^a} A'^a$) is t-torsion-free.

Proof of the claim. (i): Since t is regular on $\mathscr{O}_U$, we have a short exact sequence : $\mathscr{E} := (0 \to \mathscr{O}_U^a \to \mathscr{O}_U^a \to \mathscr{O}_U^a/t\mathscr{O}_U^a \to 0)$, and since t is regular on $\mathscr{F}$, the sequence $\mathscr{E} \otimes_{\mathscr{O}_U^a} \mathscr{F}$ is still exact; the vanishing of $\mathscr{T}\!or_1^{\mathscr{O}_U^a}(\mathscr{F}, \mathscr{O}_U^a/t\mathscr{O}_U^a)$ is an easy consequence. For the second stated vanishing one argues similarly, using the exact sequence $0 \to A \to A \to A/tA \to 0$.

(ii): Under the current assumptions, the natural map $\mathscr{O}_U/t\mathscr{O}_U \to \varphi_*(\mathscr{O}_{U'}/t\mathscr{O}_{U'})$ is an isomorphism, whence a short exact sequence : $\mathscr{E}' := (0 \to \varphi_*\mathscr{O}_{U'}^a \to \varphi_*\mathscr{O}_{U'}^a \to \mathscr{O}_U^a/t\mathscr{O}_U^a \to 0)$. In view of (i), the sequence $\mathscr{E}' \otimes_{\mathscr{O}_U^a} \mathscr{F}$ is still exact, so $\mathscr{F} \otimes_{\mathscr{O}_U^a} \varphi_*\mathscr{O}_{U'}^a$ is t-torsion-free. An analogous argument works as well for $H^0(U, \mathscr{F}) \otimes_{A^a} A'^a$. ◊

Claim 14.1.99. For every $n > 0$, the map $A/t^nA \to A'/t^nA'$ induced by f is an isomorphism.

Proof of the claim. Indeed, since t is regular on both A and A', and $A/tA \xrightarrow{\sim} A'/tA'$, the map of graded rings $\oplus_{n\in\mathbb{N}} t^n A/t^{n+1}A \to \oplus_{n\in\mathbb{N}} t^n A'/t^{n+1}A'$ is bijective. Then the claim follows from [34, Ch.III, §2, n.8, Cor.3]. $\Diamond$

Let $j : D(t) \to U$ be the natural open immersion, and set :

$$\mathscr{F}[t^{-1}] := j_* j^* \mathscr{F} \simeq \mathscr{F} \otimes_{\mathscr{O}^a_U} j_* \mathscr{O}^a_{D(t)}.$$

Notice the natural isomorphisms :

$$j_* j^* \varphi_* \mathscr{F}' \simeq (\mathscr{F} \otimes_{\mathscr{O}^a_U} \varphi_* \mathscr{O}^a_{U'}) \otimes_{\mathscr{O}^a_U} j_* \mathscr{O}^a_{D(t)} \simeq \mathscr{F}[t^{-1}] \otimes_{\mathscr{O}^a_U} \varphi_* \mathscr{O}^a_{U'}. \tag{14.1.100}$$

Since t is $\mathscr{F}$-regular, the natural map $\mathscr{F} \to \mathscr{F}[t^{-1}]$ is a monomorphism, and the same holds for the corresponding map $\varphi_* \mathscr{F}' \to j_* j^* \varphi_* \mathscr{F}'$, in view of claim 14.1.98(ii). Moreover, $\mathscr{G} := \mathscr{F}[t^{-1}]/\mathscr{F}$ can be written as the increasing union of its subsheaves $\mathrm{Ann}_{\mathscr{G}}(t^n)$ (for all $n \in \mathbb{N}$); hence claim 14.1.99 implies that the natural map $\mathscr{G} \to \mathscr{G} \otimes_{\mathscr{O}^a_U} \varphi_* \mathscr{O}^a_{U'}$ is an isomorphism. There follows a ladder of short exact sequences :

$$\begin{array}{ccccccccc} 0 & \to & \mathscr{F} & \to & \mathscr{F}[t^{-1}] & \to & \mathscr{G} & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & \varphi_* \mathscr{F}' & \to & \mathscr{F}[t^{-1}] \otimes_{\mathscr{O}^a_U} \varphi_* \mathscr{O}^a_{U'} & \to & \mathscr{G} & \to & 0. \end{array} \tag{14.1.101}$$

On the other hand, since j is an affine morphism, we may compute :

$$H^q(U, \mathscr{F}[t^{-1}]) \simeq H^q(U, Rj_* j^* \mathscr{F}) \simeq H^q(D(t), j^* \mathscr{F}) \simeq 0 \qquad \text{for every } q > 0.$$

Likewise, from (14.1.100) we get :

$$H^q(U, \mathscr{F}[t^{-1}] \otimes_{\mathscr{O}^a_U} \varphi_* \mathscr{O}^a_{U'}) \simeq 0 \qquad \text{for every } q > 0.$$

Thus, in the commutative diagram :

$$\begin{array}{ccc} H^{q-1}(U, \mathscr{G}) & \xrightarrow{\partial} & H^q(U, \mathscr{F}) \\ {\scriptstyle \partial'} \downarrow & & \downarrow \\ H^q(U, \varphi_* \mathscr{F}) & \xrightarrow{\sim} & H^q(U', \mathscr{F}') \end{array}$$

the boundary maps ∂ and ∂' are isomorphisms whenever $q > 1$, and the right vertical arrow is (14.1.96), so the assertion follows already for every $q > 1$. To deal with the remaining cases with $q = 0$ or 1, we look at the ladder of exact cohomology sequences:

$$\begin{array}{ccccccccccc} 0 & \to & H^0(U, \mathscr{F}) & \xrightarrow{\alpha} & H^0(D(t), \mathscr{F}) & \xrightarrow{\beta} & H^0(U, \mathscr{G}) & \to & H^1(U, \mathscr{F}) & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \downarrow & & \\ 0 & \to & H^0(U', \mathscr{F}') & \to & H^0(D(t), \mathscr{F}) \otimes_{A^a} A'^a & \xrightarrow{\beta'} & H^0(U, \mathscr{G}) & \to & H^1(U', \mathscr{F}') & \to & 0 \end{array}$$

deduced from (14.1.101). On the one hand, we remark that the natural inclusion $\mathrm{Im}\,\beta \subset \mathrm{Im}\,\beta'$ factors as a composition

$$\mathrm{Im}\,\beta \xrightarrow{\iota} M := A'^a \otimes_{A^a} \mathrm{Im}\,\beta \xrightarrow{\pi} \mathrm{Im}\,\beta'$$

where ι is given by the rule : $x \mapsto 1 \otimes x$, for all $x \in A^a_*$, and π is an epimorphism.

On the other hand, the image of β is isomorphic to $N := \mathrm{Coker}\,\alpha$, and the latter is the increasing union of its submodules $\mathrm{Ann}_N(t^n)$ (for all $n \in \mathbb{N}$). Again from claim 14.1.99 we deduce that $M = \mathrm{Im}\,\beta$, hence $\mathrm{Im}\,\beta' = \mathrm{Im}\,\beta$, which shows that (14.1.96) is an isomorphism also for $q = 1$. By the same token, $\mathrm{Ker}(\alpha \otimes_{A^a} \mathbf{1}_{A'^a})$ is the increasing union of its t^n-torsion submodules (for all $n \in \mathbb{N}$), since it is a quotient of $\mathrm{Tor}_1^{A^a}(\mathrm{Im}\,\beta, A'^a)$; hence $\mathrm{Ker}(\alpha \otimes_{A^a} \mathbf{1}_{A'^a}) = 0$, in view of claim 14.1.98(ii). This easily implies the last assertion for $q = 0$. $\square$

14.2. **Inverse systems of almost modules.** The considerations of this section expand upon [75, §2.3.14, §2.4.1, Lemmata 2.3.15, 2.4.2 and 2.4.13].

14.2.1. Let V be a ring, $(\mathfrak{m}_\lambda \mid \lambda \in \Lambda)$ a filtered system of ideals of V such that $\mathfrak{m}_\lambda^2 = \mathfrak{m}_\lambda$ for every λ, and set $\mathfrak{m} := \bigcup_{\lambda\in\Lambda} \mathfrak{m}_\lambda$. Then $(V, \mathfrak{m})$ and $(V, \mathfrak{m}_\lambda)$ for every $\lambda \in \Lambda$ are basic setups, and we have an obvious compatible system of localization functors :

$$\begin{array}{ccc} (V,\mathfrak{m})^a\text{-}\mathbf{Mod} & \xrightarrow{\pi_\lambda} & (V,\mathfrak{m}_\lambda)^a\text{-}\mathbf{Mod} \\ & \searrow^{\pi_\mu} & \downarrow{\scriptstyle \pi_{\lambda\mu}} \\ & & (V,\mathfrak{m}_\mu)^a\text{-}\mathbf{Mod} \end{array} \qquad \text{for every } \lambda \geq \mu \text{ in } \Lambda. \tag{14.2.2}$$

Proposition 14.2.3. *(i) The compatible system* (14.2.2) *induces an equivalence of categories*

$$\omega : (V, \mathfrak{m})^a\text{-}\mathbf{Mod} \xrightarrow{\sim} 2\text{-}\lim_{\lambda\in\Lambda} (V, \mathfrak{m}_\lambda)^a\text{-}\mathbf{Mod}.$$

(ii) If $\mathfrak{m}_\lambda$ fulfills condition $(\mathbf{B})$ *of* [75, §2.1.6] *for every $\lambda \in \Lambda$, then the same holds for $\mathfrak{m}$.*

Proof. (ii) is obvious. To show (i), we construct a quasi-inverse τ for ω as follows. Recall that an object of the above 2-limit is a datum $(M_\bullet, f_{\bullet\bullet}) := ((M_\lambda \mid \lambda \in \Lambda), (f_{\lambda\mu} : \pi_{\lambda\mu}(M_\lambda) \xrightarrow{\sim} M_\mu \mid \lambda \geq \mu))$ consisting of $(V, \mathfrak{m}_\lambda)^a$-modules M_λ for every $\lambda \in \Lambda$, and isomorphisms $f_{\lambda\mu}$ of $(V, \mathfrak{m}_\mu)^a$-modules, for every $\lambda \geq \mu$ in Λ. Such a datum is required moreover to satisfy the compatibility condition :

$$f_{\mu\nu} \circ a_{\mu\nu}(f_{\lambda\mu}) = f_{\lambda\nu} \qquad \text{whenever } \lambda \geq \mu \geq \nu.$$

However, the discussion of [75, §2.2.2] allows to describe such a datum more concretely as follows. Choose, for every $\lambda \in \Lambda$ a V-module M'_λ whose image in $(V, \mathfrak{m}_\lambda)^a$-$\mathbf{Mod}$ represents M_λ; the isomorphism $f_{\lambda\mu}$ corresponds then to a unique V-linear isomorphism

$$f'_{\lambda\mu} : \widetilde{\mathfrak{m}}_\mu \otimes_V M'_\lambda \xrightarrow{\sim} \widetilde{\mathfrak{m}}_\mu \otimes_V M'_\mu \qquad \text{whenever } \lambda \geq \mu$$

(with $\widetilde{\mathfrak{m}}_\lambda := \mathfrak{m}_\lambda \otimes_V \mathfrak{m}_\lambda$ for every $\lambda \in \Lambda$) such that

$$f'_{\mu\nu} \circ (\widetilde{\mathfrak{m}}_\nu \otimes_V f'_{\lambda\mu}) = \widetilde{\mathfrak{m}}_\nu \otimes_V f'_{\lambda\nu} \qquad \text{whenever } \lambda \geq \mu \geq \nu. \tag{14.2.4}$$

Set $M_{\lambda!} := \widetilde{\mathfrak{m}}_\lambda \otimes_V M'_\lambda$; by composing the inverse of $f'_{\lambda\mu}$ with the natural map $\widetilde{\mathfrak{m}}_\mu \otimes_V M'_\lambda \to M_{\lambda!}$, we obtain a well defined V-linear map

$$f''_{\lambda\mu} : M_{\mu!} \to M_{\lambda!} \qquad \text{whenever } \lambda \geq \mu$$

and in light of (14.2.4) we deduce easily that $f''_{\lambda\nu} = f''_{\mu\nu} \circ f''_{\lambda\mu}$ whenever $\lambda \geq \mu \geq \nu$ (details left to the reader). We now let

$$\tau(M_\bullet, f_{\bullet\bullet}) := \big(\operatorname*{colim}_{\lambda\in\Lambda} M_{\lambda!}\big)^a \in \mathrm{Ob}((V, \mathfrak{m})^a\text{-}\mathbf{Mod}).$$

If $(N_\bullet, g_{\bullet\bullet})$ is another object of the above 2-limit of categories, a morphism $(M_\bullet, f_{\bullet\bullet}) \to (N_\bullet, g_{\bullet\bullet})$ is the datum of a system of morphisms $h_\lambda : M_\lambda \to N_\lambda$ fulfilling obvious compatibility conditions with respect ot the isomorphisms $f_{\bullet\bullet}$ and $g_{\bullet\bullet}$; after choosing representative V-modules N'_λ for each N_λ, the morphism h_λ corresponds to a unique V-linear map $h'_\lambda : M_! \to N_! := \widetilde{\mathfrak{m}}_\lambda \otimes_V N'_\lambda$, for every $\lambda \in \Lambda$. It is then clear that the resulting system $h'_\bullet$ yields a natural transformation $M_{\bullet!} \to N_{\bullet!}$ of direct systems indexed by Λ, whence a well defined morphism of $(V, \mathfrak{m})^a$-modules

$$\tau(h_\bullet) := \big(\operatorname*{colim}_{\lambda\in\Lambda} h'_\lambda\big)^a : \tau(M_\bullet, f_{\bullet\bullet}) \to \tau(N_\bullet, g_{\bullet\bullet}).$$

This completes the construction of our functor τ. It is easily seen that the definition of τ is independent of all choices, up to a canonical isomorphism of functors. Especially, if $(M_\bullet, f_{\bullet\bullet}) = \omega(M^a)$ for some V-module M, we may assume that M'_λ has been chosen equal to M, and

$f'_{\lambda\mu} = \widetilde{\mathfrak{m}}_\lambda \otimes_V \mathbf{1}_M$ for every $\lambda, \mu \in \Lambda$ with $\lambda \geq \mu$. With these choices, we get a natural isomorphism $\tau \circ \omega(M^a) \stackrel{\sim}{\to} M^a$ for every $(V, \mathfrak{m})^a$-module M^a. Next, for any given datum $(M_\bullet, f_{\bullet\bullet})$ as in the foregoing, set $(P_\bullet, t_{\bullet\bullet}) := \omega \circ \tau(M_\bullet, f_{\bullet\bullet})$; for every $\mu \in \Lambda$, the subset $\Lambda(\mu) := \{\lambda \in \Lambda \mid \lambda \geq \mu\}$ is cofinal in Λ, hence P_μ is naturally isomorphic to

$$\big(\operatorname*{colim}_{\lambda \in \Lambda(\mu)} M_{\lambda !}\big)^a \stackrel{\sim}{\to} \operatorname*{colim}_{\lambda \in \Lambda(\mu)} (M_{\lambda !})^a \stackrel{\sim}{\to} \operatorname*{colim}_{\lambda \in \Lambda(\mu)} \pi_{\lambda\mu}(M_\lambda)$$

where the transition morphisms in the latter colimit are given by the isomorphisms $f_{\lambda\mu}$; *i.e.* is naturally isomorphic to M_μ. Lastly, under this identification, it is easily seen that the morphism $t_{\lambda\mu}$ corresponds to $f_{\lambda\mu}$ whenever $\lambda \geq \mu$ in Λ (details left to the reader); summing up, we have obtained a natural isomorphism $\omega \circ \tau(M_\bullet, f_{\bullet\bullet}) \stackrel{\sim}{\to} (M_\bullet, f_{\bullet\bullet})$, and the proof is concluded. □

Corollary 14.2.5. *With the notation of* (14.2.1)*, for every $(V, \mathfrak{m})^a$-module M and every $\lambda \in \Lambda$, denote by M_λ the image of M in $(V, \mathfrak{m}_\lambda)^a$-**Mod**. Let A be a $(V, \mathfrak{m})^a$-algebra, M an A-module, B an A-algebra, and $r \in \mathbb{N}$. Then the following holds :*

(i) *We have natural isomorphisms of V-modules :*

$$\operatorname*{colim}_{\lambda \in \Lambda} M_{\lambda !} \stackrel{\sim}{\to} M_! \qquad M_* \stackrel{\sim}{\to} \lim_{\lambda \in \Lambda} M_{\lambda *}.$$

(ii) *The system of localization functors A-$\mathsf{Alg} \to A_\lambda$-$\mathsf{Alg}$ induces a natural equivalence*

$$A\text{-}\mathsf{Alg} \stackrel{\sim}{\to} 2\text{-}\lim_{\lambda \in \Lambda} A_\lambda\text{-}\mathsf{Alg}.$$

(iii) *The A-module M is flat (resp. faithfully flat, resp. almost projective, resp. almost finitely generated, resp. almost finitely presented) if and only if the same holds for the A_λ-module M_λ, for every $\lambda \in \Lambda$.*

(iv) *Suppose that $\mathfrak{m}_\lambda$ fulfills condition* $(\mathbf{B})$ *for every $\lambda \in \Lambda$. Then the A-module M is almost projective of almost finite rank (resp. of finite rank $\leq r$) if and only if the same holds for the A_λ-module M_λ, for every $\lambda \in \Lambda$.*

(v) *The A-algebra B is flat (resp. faithfully flat, resp. weakly unramified, resp. weakly étale, resp. unramified, resp. étale) if and only if the same holds for the A_λ-algebra B_λ, for every $\lambda \in \Lambda$.*

Proof. (i): The assertion for $M_!$ is clear. For the assertion concerning M_*, it suffices to notice that for any two $(V, \mathfrak{m})^a$-modules M and N, proposition 14.2.3(i) implies that the natural map

$$\mathrm{Hom}_{(V,\mathfrak{m})^a\text{-}\mathbf{Mod}}(M, N) \to \lim_{\lambda \in \Lambda} \mathrm{Hom}_{(V,\mathfrak{m}_\lambda)^a\text{-}\mathbf{Mod}}(M_\lambda, N_\lambda) \qquad (14.2.6)$$

is an isomorphism of V-modules.

(ii),(iii): Clearly the functors π_λ and $\pi_{\lambda\mu}$ of (14.2.1) are all compatible with tensor products and with the alHom functors; moreover, for every $\lambda \in \Lambda$, every short exact sequence of $(V, \mathfrak{m}_\lambda)^a$-modules is isomorphic to Σ^a, for some short exact sequence Σ of V-modules (and likewise for short exact sequences of $(V, \mathfrak{m})^a$-modules : details left to the reader). In view of proposition 14.2.3(i), assertion (ii) follows straightforwardly, and taking into account the isomorphism (14.2.6), we also deduce that the A-module M is flat (resp. almost projective) if and only if the same holds for the A_λ-module M_λ, for every $\lambda \in \Lambda$.

Next, if every M_λ is a faithfully flat A_λ-module, and X is an A-module with $M \otimes_A X = 0$, it follows that $M_\lambda \otimes_{A_\lambda} X_\lambda = 0$ for every $\lambda \in \Lambda$, whence $X_\lambda = 0$ for every such λ, and hence $X = 0$, by proposition 14.2.3(i). Conversely, if M is faithfully flat and $\lambda \in \Lambda$, consider any A-module X such that $M_\lambda \otimes_{A_\lambda} X_\lambda = 0$; then $(M \otimes_A X)_\lambda = 0$, *i.e.* $\widetilde{\mathfrak{m}}_\lambda \otimes (M_* \otimes_{A_*} X_*) = 0$, so that $M \otimes_A (\widetilde{\mathfrak{m}}_\lambda \otimes_V X) = 0$. By assumption, this implies that $\widetilde{\mathfrak{m}}_\lambda \otimes_V X = 0$, and therefore $X_\lambda = 0$; this shows that M_λ is faithfully flat.

Next, say that $A = R^a$ and $M = N^a$ for some V-algebra R and some R-module N; suppose that each M_λ is almost finitely generated (resp. almost finitely presented), and let $\mathfrak{m}_0 \subset \mathfrak{m}$

be any finitely generated subideal. We then find $\lambda \in \Lambda$ such that $\mathfrak{m}_0 \subset \mathfrak{m}_\lambda$, and by applying [75, Cor.2.3.13] to the A_λ-module M_λ we find a finitely generated (resp. finitely presented) R-module N' and an R-linear map $N' \to N$ whose kernel and cokernel are annihilated by $\mathfrak{m}_0$. Then, by applying [75, Cor.2.3.13] to the A-module M, we deduce the assertion.

(iv): If every M_λ is almost projective of almost finite rank, we already know that M is almost projective, and it remains only to check that M is of almost finite rank. To this aim, let $\varepsilon \in \mathfrak{m}$ be any element, choose $\lambda \in \Lambda$ such that $\varepsilon \in \mathfrak{m}_\lambda^2$, and write $\varepsilon = \sum_{i=1}^n \varepsilon_i\varepsilon'_i$ for some $n \in \mathbb{N}$ and $\varepsilon_1, \varepsilon'_1, \dots, \varepsilon_n, \varepsilon'_n \in \mathfrak{m}_\lambda$; if M_λ is of almost finite rank, we may then find $j \in \mathbb{N}$ such that $\varepsilon_i \cdot \Lambda^j_{A_\lambda} M_\lambda = 0$ for $i = 1, \dots, n$, in which case it follows easily that $\varepsilon_i\varepsilon'_i \cdot \Lambda^j_A M = 0$ for every $i = 1, \dots, n$ (details left to the reader), whence the assertion. One argues likewise in case each M_λ has rank $\leq r$: the details shall be left to the reader.

(v) follows immediately from (i), (ii) and (iii). □

Remark 14.2.7. (i) Let V be a ring, $\mathfrak{m}, \mathfrak{m}' \subset V$ two ideals such that $(V, \mathfrak{m})$ and $(V, \mathfrak{m}')$ are both basic setups, and let $\mathfrak{m}'' := \mathfrak{m} \cdot \mathfrak{m}'$; set as usual $\widetilde{\mathfrak{m}} := \mathfrak{m} \otimes_V \mathfrak{m}$, and define likewise the V-modules $\widetilde{\mathfrak{m}}'$ and $\widetilde{\mathfrak{m}}''$. It is easily seen that $(V, \mathfrak{m}'')$ is a basic setup as well. Moreover, we claim that there exists a natural isomorphism of V-modules :

$$\widetilde{\mathfrak{m}} \otimes_V \widetilde{\mathfrak{m}}' \xrightarrow{\sim} \widetilde{\mathfrak{m}}'' \qquad x \otimes y \otimes x' \otimes y' \mapsto xx' \otimes yy'. \tag{14.2.8}$$

Indeed, let $\mu : \mathfrak{m} \otimes_V \mathfrak{m}' \to \mathfrak{m}''$ be the multiplication map; we know already that $\operatorname{Ker} \mu$ is annihilated by both $\mathfrak{m}$ and $\mathfrak{m}'$, hence $\mathfrak{m}'' \otimes_V \mu : \mathfrak{m}'' \otimes_V \mathfrak{m} \otimes_V \mathfrak{m}' \to \widetilde{\mathfrak{m}}''$ is an isomorphism, and likewise for $\mu \otimes_V \mathfrak{m} \otimes_V \mathfrak{m}' : \widetilde{\mathfrak{m}} \otimes_V \widetilde{\mathfrak{m}}' \to \mathfrak{m}'' \otimes_V \mathfrak{m} \otimes_V \mathfrak{m}'$, so the same holds for their composition, which is (14.2.8).

(ii) Let M be any V-module; denote by $(M, \mathfrak{m})^a \in \operatorname{Ob}((V, \mathfrak{m})^a\text{-}\mathbf{Mod})$ the image of M, and define likewise $(M, \mathfrak{m}')^a \in \operatorname{Ob}((V, \mathfrak{m}')^a\text{-}\mathbf{Mod})$ and $(M, \mathfrak{m}'')^a \in \operatorname{Ob}((V, \mathfrak{m}'')^a\text{-}\mathbf{Mod})$. We deduce from (i) natural V-linear identifications

$$(M, \mathfrak{m}'')^a_* \xrightarrow{\sim} ((M, \mathfrak{m})^a_*, \mathfrak{m}')^a_* \xrightarrow{\sim} ((M, \mathfrak{m}')^a_*, \mathfrak{m})^a_*$$
$$(M, \mathfrak{m}'')^a_! \xrightarrow{\sim} ((M, \mathfrak{m})^a_!, \mathfrak{m}')^a_! \xrightarrow{\sim} ((M, \mathfrak{m}')^a_!, \mathfrak{m})^a_!.$$

14.2.9. Let V be any ring; a *V-linear abelian category* is the datum of an abelian category $\mathscr{A}$ together with a V-module structure on $\operatorname{Hom}_{\mathscr{A}}(A, B)$ for every $A, B \in \operatorname{Ob}(\mathscr{A})$, such that the composition law of $\mathscr{A}$ is a V-bilinear map

$$\operatorname{Hom}_{\mathscr{A}}(A, B) \times \operatorname{Hom}_{\mathscr{A}}(B, C) \to \operatorname{Hom}_{\mathscr{A}}(A, C) \qquad \text{for every } A, B, C \in \operatorname{Ob}(\mathscr{A}).$$

We say that a functor $F : \mathscr{A} \to \mathscr{A}'$ between V-linear abelian categories is *V-linear* if the map

$$\operatorname{Hom}_{\mathscr{A}}(A, B) \to \operatorname{Hom}_{\mathscr{A}'}(FA, FB) \qquad f \mapsto Ff$$

is V-linear, for every $A, B \in \operatorname{Ob}(\mathscr{A})$.

Remark 14.2.10. Let V be a ring, and $\mathscr{A}, \mathscr{A}'$ two V-linear abelian categories.

(i) For every $x \in V$ and $A \in \operatorname{Ob}(\mathscr{A})$ we define the suboobject $xA := \operatorname{Im} x \cdot \mathbf{1}_A \subset A$. Moreover, if $J \subset V$ is an ideal generated by a finite system $x_1, \dots, x_n \in V$, we let $JA := \sum_{i=1}^n x_i A \subset A$. Let us check that the subobject JA does not depend on the choice of a finite system of generators for J : it suffices to show that for every $a \in J$ we have $aA \subset JA$; to this aim, say that $a = \sum_{i=1}^n a_i x_i$ for certain $a_1, \dots, a_n \in V$, and for every $i = 1, \dots, n$ denote by $A \xrightarrow{e_i} A^{\oplus n} \xrightarrow{p_i} A$ the inclusion into the i-the direct summand and the projection onto the i-th direct factor of $A^{\oplus n}$. Set $\varphi := \sum_{i=1}^n x_i \cdot p_i$ and $\psi := \sum_{i=1}^n a_i \cdot e_i$; it is easily seen that $\operatorname{Im} \varphi = JA$ and the V-bilinearity of the composition law implies that $\varphi \circ \psi = a \cdot \mathbf{1}_A$. Since $\operatorname{Im} \varphi \circ \psi \subset \operatorname{Im} \varphi$, the assertion follows.

(ii) Recall that the opposite $\mathscr{A}^o$ of the abelian category $\mathscr{A}$ is abelian; moreover, $\mathscr{A}^o$ inherits from $\mathscr{A}$ an obvious V-linear structure : the reader may spell out the details. Likewise, if $F : \mathscr{A} \to \mathscr{A}'$ any V-linear functor, then $F^o : \mathscr{A}^o \to \mathscr{A}'^o$ is again V-linear.

(iii) Obviously, for every V-algebra A, the category A-$\mathbf{Mod}$ is naturally a V-linear abelian category. If $(V, \mathfrak{m})$ is any basic setup, and A any $(V, \mathfrak{m})^a$-algebra, then also A-$\mathbf{Mod}$ is a V-linear abelian category.

(iv) Let $J \subset V$ be any ideal, and $A \in \mathrm{Ob}(\mathscr{A})$ any object. We say that $JA = 0$ if $xA = 0$ for every $x \in J$; notice that if J is finitely generated, this notation agrees with that of (i). We say that $\mathscr{A}$ is *J-torsion-free* if there are no non-zero objects A of $\mathscr{A}$ with $JA = 0$. For instance, if $(V, \mathfrak{m})$ is any basic setup, and A any $(V, \mathfrak{m})^a$-algebra, then A-$\mathbf{Mod}$ is $\mathfrak{m}$-torsion-free. Clearly, if $\mathscr{A}$ is J-torsion-free, the same holds for $\mathscr{A}^o$, for its natural V-linear structure.

(v) Let $f : A \to B$ be any morphism of $\mathscr{A}$, and $J, J' \subset V$ two ideals such that $J \cdot \mathrm{Ker}\, f = J' \cdot \mathrm{Coker}\, f = 0$ (notation of (iv)). Then for every $a \in J$ and $b \in J'$ there exists a morphism

$$g : B \to A \qquad \text{such that} \qquad f \circ g = ab \cdot \mathbf{1}_B \quad \text{and} \quad g \circ f = ab \cdot \mathbf{1}_A.$$

Indeed, write $f = e \circ p$, where $A \xrightarrow{p} \mathrm{Im}\, f \xrightarrow{e} B$ are the natural morphisms. Since $J \cdot \mathrm{Ker}\, p = J \cdot \mathrm{Ker}\, f = 0$, the endomorphism $a \cdot \mathbf{1}_A$ factors through p and a morphism $g_1 : \mathrm{Im}\, f \to A$. Likewise, since $J' \cdot \mathrm{Coker}\, e = J' \cdot \mathrm{Coker}\, f = 0$, the endomorphism $b \cdot \mathbf{1}_B$ factors through e and a morphism $g_2 : B \to \mathrm{Im}\, f$. Then $g := g_1 \circ g_2$ fulfills the stated conditions.

(vi) In the situation of (v), let $F : \mathscr{A} \to \mathscr{A}'$ be a V-linear functor; we get $Ff \circ Fg = ab \cdot \mathbf{1}_{FB}$ and $Fg \circ Ff = ab \cdot \mathbf{1}_{FA}$. It follows easily that $JJ' \cdot \mathrm{Ker}\, Ff = JJ' \cdot \mathrm{Coker}\, Ff = 0$.

Definition 14.2.11. Let $(V, \mathfrak{m})$ be a basic setup, $\mathscr{A}$ a V-linear abelian category, I a small category, and consider a functor

$$M_\bullet : I \to \mathscr{A} \qquad i \mapsto M_i \qquad (j \xrightarrow{\varphi} i) \mapsto (M_j \xrightarrow{M_\varphi} M_i).$$

(i) We say that $M_\bullet$ is *essentially zero*, if for every $i \in \mathrm{Ob}(I)$ there exists a morphism $\varphi : j \to i$ in I with $M_\varphi = 0$.

(ii) We say that $M_\bullet$ is *almost essentially zero*, if for every $i \in \mathrm{Ob}(I)$ and every subideal $\mathfrak{m}_0 \subset \mathfrak{m}$ of finite type there exists a morphism $\varphi : j \to i$ in I with $\mathfrak{m}_0 \cdot M_\varphi = 0$. We denote

$$\mathsf{Fun}(I, \mathscr{A})_{\mathrm{a.ess.0}}$$

the full subcategory of $\mathsf{Fun}(I, \mathscr{A})$ whose objects are the almost essentially zero functors.

(iii) We say that $M_\bullet$ is *almost essentially constant*, if there exist $L \in \mathrm{Ob}(\mathscr{A})$ and a cone $\pi_\bullet : c_L \Rightarrow M_\bullet$ inducing almost essentially zero functors

$$\mathrm{Ker}\, \pi_\bullet : I \to A\text{-}\mathbf{Mod} \qquad \text{and} \qquad \mathrm{Coker}\, \pi_\bullet : I \to A\text{-}\mathbf{Mod}.$$

(iv) We say that $M_\bullet$ has the *almost Mittag-Leffler property* if for every $i \in \mathrm{Ob}(I)$ and every subideal $\mathfrak{m}_0 \subset \mathfrak{m}$ of finite type there exists a morphism $j \xrightarrow{\varphi} i$ in I with :

$$\mathfrak{m}_0 \cdot \mathrm{Im}\, M_\varphi \subset \mathrm{Im}\, M_{\varphi \circ \psi} \qquad \text{for every morphism } \psi : k \to j \text{ in } I.$$

(v) We say that $M_\bullet$ is a *Cauchy functor*, if for every subideal $\mathfrak{m}_0 \subset \mathfrak{m}$ of finite type there exists $i \in \mathrm{Ob}(I)$ with $\mathfrak{m}_0 \cdot \mathrm{Ker}\, M_\varphi = \mathfrak{m}_0 \cdot \mathrm{Coker}\, M_\varphi = 0$ for every morphism $\varphi : j \to i$ of I.

We say that $M_\bullet$ is *null*, if it is both a Cauchy functor and almost essentially zero.

(vi) Let $\mathbf{P}(M_\bullet)$ be either one of the conditions : "$M_\bullet$ is essentially zero", "$M_\bullet$ is almost essentially zero", "$M_\bullet$ is almost essentially constant", "$M_\bullet$ has the almost Mittag-Leffler property", "$M_\bullet$ is a Cauchy functor" or "$M_\bullet$ is null". We say that $M_\bullet$ *has the dual* $\mathbf{P}$ *property* if $\mathbf{P}(M_\bullet^o)$ holds, for the opposite functor $M_\bullet^o : I^o \to \mathscr{A}^o$ (see remark 14.2.10(ii)).

Remark 14.2.12. (i) In the situation of definition 14.2.11, for every $M \in \mathrm{Ob}(\mathscr{A})$ and every subideal $\mathfrak{m}_0 \subset \mathfrak{m}$ of finite type, set $M[\mathfrak{m}_0] := \bigcap_{x \in \mathfrak{m}_0} \mathrm{Ker}\, x \cdot \mathbf{1}_M$. Notice that $M_\bullet$ is almost essentially zero if and only if the following holds. For every such $\mathfrak{m}_0$, the induced functor

$$M_\bullet / M_\bullet[\mathfrak{m}_0] : I \to \mathscr{A} \qquad i \mapsto M_i / M_i[\mathfrak{m}_0]$$

is essentially zero.

(ii) Suppose that I is cofiltered and $\mathscr{A}$ is complete, and let us set :

$$M_i^\triangle := \bigcap_{\varphi:j\to i} \mathrm{Im}\, M_\varphi \qquad \text{for every } i \in \mathrm{Ob}(I).$$

We claim that M_ψ restricts to a morphism in $\mathscr{A}$

$$M_\psi^\triangle : M_{i'}^\triangle \to M_i^\triangle \qquad \text{for every morphism } \psi : i' \to i \text{ in } I.$$

Indeed, it suffices to check that $M_\psi(M_{i'}^\triangle) \subset \mathrm{Im}\, M_\varphi$ for every morphism $\varphi : j \to i$ in I. However, since I is cofiltered, for every such φ there exist $k \in \mathrm{Ob}(I)$ and morphisms $\varphi' : k \to i'$ and $\psi' : k \to j$ such that $\varphi\circ\psi' = \psi\circ\varphi'$. It follows that $M_\psi(M_{i'}^\triangle) \subset M_\psi(\mathrm{Im}\,\varphi') = M_\varphi(\mathrm{Im}\,\psi')$ whence the contention. Next, let $\tau_\bullet : c_L \Rightarrow M_\bullet$ be any cone; notice that

$$\mathrm{Im}\,\tau_i \subset M_i^\triangle \qquad \text{for every } i \in \mathrm{Ob}(I). \tag{14.2.13}$$

On the other hand, a direct inspection of the definitions shows that

$$(M_\bullet/M_\bullet^\triangle)_i^\triangle = 0 \qquad \text{for every } i \in \mathrm{Ob}(I).$$

Hence, (14.2.13) implies that every cone $c_{L'} \Rightarrow M_\bullet/M_\bullet^\triangle$ is the zero morphism, so we must have $\lim_I M_\bullet/M_\bullet^\triangle = 0$, and since $\lim_I$ is a left exact functor (because it is a right adjoint : see (1.3)) we conclude that the inclusion $M_\bullet^\triangle \to M_\bullet$ induces an isomorphism :

$$\lim_I M_\bullet^\triangle \xrightarrow{\sim} \lim_I M_\bullet.$$

(iii) In the situation of (ii), let A be any $(V,\mathfrak{m})^a$-algebra, and take $\mathscr{A} := A\text{-}\mathbf{Mod}$ (see remark 14.2.10(iii)); notice then that $M_\bullet$ has the almost Mittag-Leffler property if and only if the following holds. For every $i \in \mathrm{Ob}(I)$, the induced system $M_{\bullet/i} := (\mathrm{Im}\, M_\varphi \mid \varphi : j \to i)$ of submodules of M_i is a Cauchy net of the uniform space $\mathscr{I}_A(M_i)$ defined as in [75, Def.2.3.1]. In this case, by [75, Lemma 2.3.5], the Cauchy sequence $M_{\bullet/i}$ admits a unique limit in $\mathscr{I}_A(M_i)$, and by direct inspection we see that this limit is $M_i^\triangle$, since for every subideal $\mathfrak{m}_0 \subset \mathfrak{m}$ of finite type there exists a morphism $\varphi : j \to i$ in I such that $\mathfrak{m}_0 \cdot \mathrm{Im}\, M_\varphi \subset M_i^\triangle$.

(iv) Notice that $M_\bullet : I \to \mathscr{A}$ is a Cauchy functor if and only if the following holds. For every subideal $\mathfrak{m}_0 \subset \mathfrak{m}$ of finite type there exists $i \in \mathrm{Ob}(I)$ such that for every pair of morphisms $k \xrightarrow{\psi} j \xrightarrow{\varphi} i$ of I, we have $\mathfrak{m}_0 \cdot \mathrm{Ker}\, M_\psi = \mathfrak{m}_0 \cdot \mathrm{Coker}\, M_\psi = 0$. Indeed, the latter condition clearly implies that $M_\bullet$ is a Cauchy functor; conversely, suppose that $M_\bullet$ is a Cauchy functor, and let $\mathfrak{m}_0 \subset \mathfrak{m}_1 \subset \mathfrak{m}$ be subideals of finite type with $\mathfrak{m}_0 \subset \mathfrak{m}_1^2$; by assumption, we may find $i \in \mathrm{Ob}(I)$ such that, for every φ and ψ as in the foregoing, $\mathfrak{m}_1$ annihilates all but the first and the fourth terms in the induced exact sequence :

$$0 \to \mathrm{Ker}\, M_\psi \to \mathrm{Ker}\, M_{\varphi\circ\psi} \to \mathrm{Ker}\, M_\varphi \to \mathrm{Coker}\, M_\psi \to \mathrm{Coker}\, M_{\varphi\circ\psi} \to \mathrm{Coker}\, M_\varphi \to 0.$$

It follows easily that $\mathfrak{m}_1 \cdot \mathrm{Ker}\, M_\psi = \mathfrak{m}_0 \cdot \mathrm{Coker}\, M_\psi$, as required. Notice also that every Cauchy functor has the almost Mittag-Leffler property : the details shall be left to the reader.

(v) Suppose again that I is cofiltered. In view of (iv), we deduce that a functor $M_\bullet : I \to \mathscr{A}$ is null if and only if the following holds. For every subideal $\mathfrak{m}_0 \subset \mathfrak{m}$ of finite type there exists $i \in \mathrm{Ob}(I)$ such that $\mathfrak{m}_0 M_j = 0$ for every morphism $\varphi : j \to i$: the details shall be left to the reader. It follows easily that the full subcategory of $\mathsf{Fun}(I, \mathscr{A})$ whose objects are the null functors, is a Serre subcategory : again, we leave the details to the reader.

Lemma 14.2.14. *In the situation of definition* 14.2.11*, the following holds :*

(i) *The category* $\mathsf{Fun}(I, \mathscr{A})_{\text{a.ess.0}}$ *is a Serre subcategory of* $\mathsf{Fun}(I, \mathscr{A})$*. Hence*

$$\mathscr{L}(I, \mathscr{A}) := \mathsf{Fun}(I, \mathscr{A})/\mathsf{Fun}(I, \mathscr{A})_{\text{a.ess.0}}.$$

is an abelian category whose objects are the functors $I \to \mathscr{A}$.

(ii) *If $\mathscr{A}$ is $\mathfrak{m}$-torsion-free (see remark* 14.2.10(iv)*), then for every almost essentially zero functor $M_\bullet : I \to A$-$\mathbf{Mod}$, we have* $\lim_I M_\bullet = 0$.

(iii) *Suppose that I is cofiltered, and $\mathscr{A}$ is complete and $\mathfrak{m}$-torsion-free. Then the functor*

$$\operatorname*{Lim}_I : \mathsf{Fun}(I, \mathscr{A}) \to \mathscr{A}$$

(see (1.3)*) factors through the localization* $\mathsf{Fun}(I, \mathscr{A}) \to \mathscr{L}(I, \mathscr{A})$ *and a functor*

$$\mathscr{L}(I, \mathscr{A}) \to \mathscr{A} \qquad M_\bullet \mapsto \lim_I M_\bullet.$$

Proof. (i): Consider a short exact sequence of functors from I to A-$\mathbf{Mod}$:

$$(14.2.15) \qquad 0 \to M'_\bullet \xrightarrow{f_\bullet} M_\bullet \xrightarrow{g_\bullet} M''_\bullet \to 0.$$

It is easily seen that if $M_\bullet$ is almost essentially zero, the same holds for $M'_\bullet$ and $M''_\bullet$. Conversely, suppose that $M'_\bullet$ and $M''_\bullet$ are almost essentially zero, and let $i \in \mathrm{Ob}(I)$, and $\mathfrak{m}_0 \subset \mathfrak{m}$ a finitely generated subideal. By assumption there exists a morphism $\varphi : j \to i$ in I such that $\mathfrak{m}_0 \cdot \operatorname{Im} M'_\varphi = 0$, and also a morphism $\psi : k \to j$ such that $\mathfrak{m}_0 \cdot \operatorname{Im} M''_\psi = 0$. It follows that there exists a morphism $h : \mathfrak{m}_0 M_k \to M'_j$ such that $f_j \circ h : \mathfrak{m}_0 M_k \to M_j$ is the restriction of M_ψ. Then $M_{\varphi\circ\psi}(\mathfrak{m}_0^2 M_k) = \mathfrak{m}_0 \cdot \operatorname{Im}(M_\varphi \circ f_j \circ h) = \mathfrak{m}_0 \cdot \operatorname{Im}(f_i \circ M'_\varphi \circ h) = 0$, which shows that $M_\bullet$ is almost essentially zero.

(ii): Suppose first that $M_\bullet$ is essentially zero. Let $X \in \mathrm{Ob}(\mathscr{A})$ be any object, and $\tau_\bullet : c_X \Rightarrow M_\bullet$ any cone; by assumption, for evey $i \in \mathrm{Ob}(I)$ there exists a morphism $\varphi : j \to i$ in I with $M_\varphi = 0$. Then $\tau_i = M_\varphi \circ \tau_j = 0$ for every such i, so that $\tau_\bullet = 0$, whence the contention.

For a general almost essentially zero functor $M_\bullet$, and every subideal $\mathfrak{m}_0 \subset \mathfrak{m}$ of finite type, we have a short exact sequence of functors

$$0 \to M_\bullet[\mathfrak{m}_0] \to M_\bullet \to M_\bullet/M_\bullet[\mathfrak{m}_0] \to 0$$

whose right-most term is essentially zero, by remark 14.2.12(i)), so that $\lim_I M_\bullet/M_\bullet[\mathfrak{m}_0] = 0$ by the previous case. On the other hand, clearly $\mathfrak{m}_0$ annihilates $\lim_I M_\bullet[\mathfrak{m}_0]$. The functor Lim_I is left exact (since it admits a left adjoint : see (1.3)); it then follows that $\mathfrak{m}_0$ annihilates as well $\lim_I M_\bullet$, and since $\mathfrak{m}_0$ is arbitrary and $\mathscr{A}$ is $\mathfrak{m}$-torsion-free, the claim follows.

(iii): As Lim_I is left exact, in the situation of (14.2.15) we see easily from (ii) that if $M''_\bullet$ is almost essentially zero, then $f_\bullet$ induces an isomorphism $\lim_I M'_\bullet \xrightarrow{\sim} \lim_I M_\bullet$. Thus, it remains only to check that if $M'_\bullet$ is almost essentially zero, then $g_\bullet$ induces an isomorphism $g : \lim_I M_\bullet \xrightarrow{\sim} \lim_I M''_\bullet$. To this aim, we suppose first that $M_\bullet$ is the constant functor of value M, and that for every subideal $\mathfrak{m}_0 \subset \mathfrak{m}$ of finite type there exists $i \in \mathrm{Ob}(I)$ such that $\mathfrak{m}_0 \cdot M'_i = 0$. Then, for every morphism $\varphi : j \to i$ in I we have as well $\mathfrak{m}_0 \cdot M'_j = 0$. Hence, for such $\mathfrak{m}_0$ and i, and for every $b \in \mathfrak{m}_0$ and every $\varphi : j \to i$, the endomorphism $b \cdot \mathbf{1}_M : M \to M$ is the composition of $g_j : M \to M''_j$ and a unique morphism $h_j : M''_j \to M$ of $\mathscr{A}$, and it is easily seen that $g_j \circ h_j = b \cdot \mathbf{1}_{M''_j}$. Moreover, the system of morphisms $(g_j \mid \varphi : j \to i)$ amounts to a natural cocone $h_\bullet : M''_\bullet \circ \mathsf{s}_i \Rightarrow M_\bullet \circ \mathsf{s}_i$, where $\mathsf{s}_i : I/i \to I$ denotes the source functor. Since s_i is coinitial (example 1.5.9(i)), we deduce a morphism of $\mathscr{A}$

$$h : L'' := \lim_I M''_\bullet \to M = \lim_I M_\bullet.$$

By construction, we have $g \circ h = b \cdot \mathbf{1}_{L''}$ and $h \circ g = b \cdot \mathbf{1}_M$. Especially, b annihilates $\operatorname{Ker} g$ and $\operatorname{Coker} g$, and since $\mathfrak{m}_0$ and b are arbitrary, the assertion follows in this case.

Lastly, let $M'_\bullet$ be an arbitrary almost essentially zero functor, and notice that

$$N_\varphi := f_i(\operatorname{Im} M'_\varphi) = M_\varphi(\operatorname{Ker} g_j) \qquad \text{for every morphism } \varphi : j \to i \text{ in } I.$$

Hence, for every such φ, the morphism M''_φ is the composition of unique morphisms of $\mathscr{A}$

$$h_\varphi : M''_j \xrightarrow{h_\varphi} Q_\varphi := M_i/N_\varphi \xrightarrow{\overline{g}_\varphi} M''_i \qquad \text{such that} \qquad h_\varphi \circ g_j = \pi_\varphi \circ M_\varphi \quad \text{and} \quad \overline{g}_\varphi \circ \pi_\varphi = g_i$$

where $\pi_\varphi : M_i \to Q_i$ is the projection. For any fixed $i \in \mathrm{Ob}(I)$, let $c_{M_i} : I/i \to \mathscr{A}$ be the constant functor of value M_i; we get a short exact sequence of functors

$$0 \to N_\bullet \to c_{M_i} \xrightarrow{\pi_\bullet} Q_\bullet \to 0$$

where $N_\bullet, Q_\bullet : I/i \to \mathscr{A}$ are given by the rules : $\varphi \mapsto N_\varphi$ and respectively $\varphi \mapsto Q_\varphi$ for every morphism $\varphi : j \to i$ in I. By construction, for every subideal $\mathfrak{m}_0 \subset \mathfrak{m}$ of finite type there exists $\varphi \in \mathrm{Ob}(I/i)$ such that $\mathfrak{m}_0 \cdot N_\varphi = 0$; by the foregoing case, it follows that $\pi_\bullet$ induces an isomorphism

$$\pi_i : M_i \xrightarrow{\sim} Q := \lim_{I/i} Q_\bullet.$$

On the other hand, the system $(h_\varphi \mid \varphi \in \mathrm{Ob}(I/i))$ amounts to a natural transformation $h_\bullet : M''_\bullet \circ \mathsf{s}_i \Rightarrow Q_\bullet$ (the details are left to the reader), whence an induced morphism of $\mathscr{A}$

$$h_i : L'' := \lim_I M''_\bullet \xrightarrow{\sim} \lim_{I/i} M''_\bullet \circ \mathsf{s}_i \to Q.$$

In turn, the system $(\pi_i^{-1} \circ h_i \mid i \in \mathrm{Ob}(I))$ amounts to a cone $c_{L''} \Rightarrow M_\bullet$, whence an induced morphism of $\mathscr{A}$

$$h : L'' \to L := \lim_I M_\bullet.$$

Claim 14.2.16. $g \circ h = \mathbf{1}_{L''}$.

Proof of the claim. Indeed, let also $\tau_\bullet : c_L \Rightarrow M_\bullet$ and $\tau''_\bullet : c_{L''} \Rightarrow M''_\bullet$ be the universal cones; it suffices to show that $\tau''_i \circ g \circ h = \tau''_i$ for every $i \in \mathrm{Ob}(I)$. But we have

$$\tau''_i \circ g \circ h = \tau''_i = g_i \circ \tau_i \circ h = g_i \circ \pi_i^{-1} \circ h_i = \overline{g}_\varphi \circ \pi_\varphi \circ \pi_i^{-1} \circ h_i$$

for every morphism $\varphi : j \to i$ in I, and on the other hand $\tau''_i = M''_\varphi \circ \tau''_j = \overline{g}_\varphi \circ h_\varphi \circ \tau''_j$ so we are reduced to checking that

$$\pi_\varphi \circ \pi_i^{-1} \circ h_i = h_\varphi \circ \tau''_j \qquad \text{for every morphism } \varphi : j \to i \text{ in } I.$$

Now, let $\tau^Q_\bullet : c_Q \Rightarrow Q_\bullet$ be the universal cone; then $\pi_\varphi \circ \pi_i^{-1} = \tau^Q_\varphi$ for every such φ, and finally, $\tau^Q_\varphi \circ h_i = h_\varphi \circ \tau''_j$, as required. ◊

From claim 14.2.16 we see that g is an epimorphism; but it is also a monomorphism, since $\lim_I M'_\bullet = 0$, and since the functor $\lim_I$ is left exact. Hence g is an isomorphism, as stated. □

Remark 14.2.17. Keep the situation of definition 14.2.11, and suppose that I is cofiltered.

(i) Consider a short exact sequence (14.2.15) of functors $I \to \mathscr{A}$, such that $M'_\bullet$ and $M''_\bullet$ are Cauchy functors. Then $M_\bullet$ is also a Cauchy functor. Indeed, let $\mathfrak{m}_0 \subset \mathfrak{m}_1 \subset \mathfrak{m}$ be two subideals of finite type with $\mathfrak{m}_0 \subset \mathfrak{m}_1^2$. Since I is cofiltered, taking into account remark 14.2.12(iv), we find $i \in \mathrm{Ob}(I)$ such that $\mathfrak{m}_1$ annihilates the kernel and cokernel of both M'_φ and M''_φ, for every morphism $\varphi : j \to i$; by the snake lemma, it follows easily that $\mathfrak{m}_0 \cdot \mathrm{Ker}\, M_\varphi = \mathfrak{m}_0 \cdot \mathrm{Coker}\, M_\varphi = 0$ for every such φ, whence the contention.

(ii) Moreover, if $X_\bullet, Y_\bullet : I \to \mathscr{A}$ are Cauchy functors, and $f_\bullet : X_\bullet \to Y_\bullet$ is any morphism of functors, then $\mathrm{Ker}\, f_\bullet$, $\mathrm{Coker}\, f_\bullet$ and $\mathrm{Im}\, f_\bullet$ are Cauchy functors. Indeed, let us check first the assertion for $Z_\bullet := \mathrm{Im}\, f_\bullet$: since the latter is a subfunctor of $Y_\bullet$, for every subideal $\mathfrak{m}_0 \subset \mathfrak{m}$ of finite type we find $i \in \mathrm{Ob}(I)$ such that for every pair of morphisms $k \xrightarrow{\psi} j \xrightarrow{\varphi} i$ we have $\mathfrak{m}_0 \cdot \mathrm{Ker}\, Z_\psi = 0$ (remark 14.2.12(iv)). Likewise, since $Z_\bullet$ is a quotient of $X_\bullet$, for every such $\mathfrak{m}_0$ there exists $i' \in \mathrm{Ob}(I)$ such that for every pair of morphisms $k' \xrightarrow{\psi'} j' \xrightarrow{\varphi'} i'$ we have $\mathfrak{m}_0 \cdot \mathrm{Coker}\, Z_{\psi'} = 0$. Since I is cofiltered, we find $i'' \in \mathrm{Ob}(I)$ with morphisms $i \leftarrow i'' \to i'$, and then $\mathfrak{m}_0$ annihilates the kernel of cokernel of Z_λ, for every morphism $\lambda : j'' \to i''$ of I.

Lastly, by considering the short exact sequences of functors :

$$0 \to \mathrm{Ker}\, f_\bullet \to X_\bullet \to Z_\bullet \to 0 \qquad 0 \to Z_\bullet \to Y_\bullet \to \mathrm{Coker}\, f_\bullet \to 0$$

and combining with (i), we deduce the assertions for $\mathrm{Ker}\, f_\bullet$ and $\mathrm{Coker}\, f_\bullet$.

Proposition 14.2.18. *With the notation of definition* 14.2.11 *and of remark* 14.2.12(ii)*, assume that I is cofiltered, and $\mathscr{A}$ is complete and $\mathfrak{m}$-torsion-free. Then we have :*

(i) *If $M_\bullet : I \to \mathscr{A}$ has the almost Mittag-Leffler property, the following holds :*

(a) *$M^\triangle_\varphi : M^\triangle_j \to M^\triangle_i$ is an epimorphism in $\mathscr{A}$, for every morphism $\varphi : j \to i$ of I.*

(b) *The inclusions $(M^\triangle_i \to M_i \mid i \in \mathrm{Ob}(I))$ define an isomorphism $M^\triangle_\bullet \xrightarrow{\sim} M_\bullet$ in $\mathscr{L}(I, \mathscr{A})$.*

(ii) *For every functor $M_\bullet : I \to \mathscr{A}$ the following conditions are equivalent :*

(a) *$M_\bullet$ is almost essentially constant.*

(b) *$M_\bullet$ is isomorphic to a constant functor, in the category $\mathscr{L}(I, \mathscr{A})$ of lemma* 14.2.14(i).

(c) *Every universal cone $c_L \Rightarrow M_\bullet$ is an isomorphism in the category $\mathscr{L}(I, \mathscr{A})$.*

(d) *$M_\bullet$ has the almost Mittag-Leffler property, and $M^\triangle_\bullet$ is a Cauchy functor.*

(iii) *Every Cauchy functor $M_\bullet : I \to \mathscr{A}$ is almost essentially constant.*

Proof. (i.a): Pick a morphism $k \xrightarrow{\psi} j$ such that $\mathfrak{m}_0 \cdot \mathrm{Im}\, M_\psi \subset M^\triangle_j$ (remark 14.2.12(iii)). Then

$$\mathfrak{m}_0 M^\triangle_i \subset \mathfrak{m}_0 \cdot \mathrm{Im}\, M_{\varphi\circ\psi} = M_\varphi(\mathfrak{m}_0 \cdot \mathrm{Im}\, M_\psi) \subset M^\triangle_\varphi(M^\triangle_j).$$

Since $\mathfrak{m}_0$ is arbitrary, the assertion follows.

(i.b): For every $i \in \mathrm{Ob}(I)$, let $\beta_i : M^\triangle_i \to M_i$ be the inclusion; we need to check that the resulting functor $\mathrm{Coker}\, \beta_\bullet : I \to \mathscr{A}$ is almost essentially zero. Indeed, let $i \in \mathrm{Ob}(I)$ and $\mathfrak{m}_0 \subset \mathfrak{m}$ any subideal of finite type; since $M_\bullet$ has the almost Mittag-Leffler property, there exists a morphism $\varphi : j \to i$ in I such that $\mathfrak{m}_0 \cdot \mathrm{Im}\, M_\varphi \subset M^\triangle_i$. Hence, $\mathfrak{m}_0$ annihilates the image of the induced morphism $\mathrm{Coker}\, \beta_j \to \mathrm{Coker}\, \beta_i$, whence the assertion.

(ii.a)⇒(ii.d): Let $L \in \mathrm{Ob}(\mathscr{A})$ with a cone $\tau_\bullet : c_L \Rightarrow M_\bullet$ such that the functors $\mathrm{Ker}\, \tau_\bullet$ and $\mathrm{Coker}\, \tau_\bullet$ are almost essentially zero. This means that for every $i \in \mathrm{Ob}(I)$ and every subideal $\mathfrak{m}_0 \subset \mathfrak{m}$ of finite type, there exists a morphism $\varphi : j \to i$ such that $\mathfrak{m}_0 \cdot \mathrm{Im}\, M_\varphi \subset \mathrm{Im}\, \tau_i$. But clearly $\mathrm{Im}\, \tau_i \subset M^\triangle_i$, so we see that $M_\bullet$ has the almost Mittag-Leffler property. By the same token, since $\mathfrak{m}_0 M^\triangle_i \subset \mathfrak{m}_0 \cdot \mathrm{Im}\, M_\varphi$, we get $\mathfrak{m}_0 M^\triangle_i \subset \mathrm{Im}\, \tau_i$, and since $\mathfrak{m}_0$ is arbitrary, we conclude that $M^\triangle_i = \mathrm{Im}\, \tau_i$ for every $i \in \mathrm{Ob}(I)$. Next, by assumption, for every $i \in \mathrm{Ob}(I)$ and every subideal $\mathfrak{m}_0 \subset \mathfrak{m}$ of finite type there exists a morphism $\varphi : j \to i$ of I such that $\mathfrak{m}_0$ annihilates the image of the induced morphism $\mathrm{Ker}\, \tau_j \to \mathrm{Ker}\, \tau_i$; but obviously the latter is a monomorphism, so the condition means that for every such $\mathfrak{m}_0$ there exists $j \in \mathrm{Ob}(I)$ such that $\mathfrak{m}_0 \cdot \mathrm{Ker}\, \tau_j = 0$. Now, let $\psi : k \to j$ be any morphism of I, and set $K := \tau_k^{-1}(\mathrm{Ker}\, M^\triangle_\psi)$; it is easily seen that $K \subset \mathrm{Ker}\, \tau_j$, hence $\mathfrak{m}_0 K = 0$, and finally $\mathfrak{m}_0 \cdot \mathrm{Ker}\, M^\triangle_\psi = 0$, since we have already established that $\tau_k : L \to M^\triangle_k$ is an epimorphism.

(ii.d)⇒(ii.b): In light of (i.b), we are reduced to checking that the functor $M^\triangle_\bullet$ is almost essentially constant. Thus, let $M^\triangle$ be the limit of the functor $M^\triangle_\bullet$, and $\tau_\bullet : c_{M^\triangle} \Rightarrow M^\triangle_\bullet$ the universal cone; let also $\mathfrak{m}_0 \subset \mathfrak{m}$ be any subideal of finite type, and pick $i \in \mathrm{Ob}(I)$ such that $\mathfrak{m}_0 \cdot \mathrm{Ker}\, M^\triangle_\varphi = 0$ for every morphism $\varphi : j \to i$ of I. Recall that the source functor $\mathsf{s}_i : I/i \to I$ is cofinal, so that $M^\triangle$ represents also the limit of $M^\triangle_\bullet \circ \mathsf{s}_i : I/i \to \mathscr{A}$. It follows easily that $\mathrm{Ker}(\tau_i : M^\triangle \to M^\triangle_i)$ represents the limit of the induced functor

$$(0 \times_{M^\triangle_i} M^\triangle_\bullet) \circ \mathsf{s}_i : I \to \mathscr{A} \qquad (j \xrightarrow{\varphi} i) \mapsto \mathrm{Ker}\, M^\triangle_\varphi = 0 \times_{M^\triangle_i} M^\triangle_j.$$

But then it is clear that $\mathfrak{m}_0 \cdot \mathrm{Ker}\, \tau_i = 0$. Especially, the functor $\mathrm{Ker}\, \tau_\bullet : I \to \mathscr{A}$ is trivially almost essentially zero. To conclude, we show that τ_i is an epimorphism for every $i \in \mathrm{Ob}(I)$. To this aim, let $\mathfrak{m}_0 \subset \mathfrak{m}$ be any subideal of finite type; in view of (i.a), we may assume that $\mathfrak{m}_0 \cdot \mathrm{Ker}\, M^\triangle_\varphi = 0$ for every morphism $\varphi : j \to i$. Hence, for every $b \in \mathfrak{m}_0$ we have a unique morphism in $\mathscr{A}$

$$f_{\varphi,b} : M_i^\triangle \to M_j^\triangle \qquad \text{such that} \qquad M_\varphi^\triangle \circ f_{\varphi,b} = b \cdot \mathbf{1}_{M_i^\triangle}.$$

By inspecting the construction, it is easily seen that $M_\psi^\triangle \circ f_{\varphi\circ\psi,b} = f_{\varphi,b}$ for every morphism $\psi : k \to j$ in I, hence the rule $\varphi \mapsto f_{\varphi,b}$ defines a cone $c_{M_i^\triangle} \Rightarrow M_\bullet^\triangle \circ \mathsf{s}_i$, whence a unique morphism of A-modules

$$h : M_i^\triangle \to M^\triangle \qquad \text{such that } f_{\varphi,b} = \tau_j \circ h \text{ for every } \varphi : j \to i.$$

Especially, $\tau_i \circ h = M_\varphi^\triangle \circ \tau_j \circ h = M_\varphi^\triangle \circ f_{\varphi,b} = b \cdot \mathbf{1}_{M_i^\triangle}$, and since $b \in \mathfrak{m}_0$ is arbitrary, the assertion follows.

(ii.b)⇒(ii.a): It suffices to prove :

Claim 14.2.19. Let $M_\bullet, M'_\bullet : I \to \mathscr{A}$ be two functors, $f_\bullet : M_\bullet \Rightarrow M'_\bullet$ a natural transformation, and suppose that $f_\bullet$ is an isomorphism in $\mathscr{L}(I, \mathscr{A})$. Then $M_\bullet$ is almost essentially constant if and only if the same holds for $M'_\bullet$.

Proof of the claim. Let L and L' be the limits of $M_\bullet$ and respectively $M'_\bullet$; denote by $\tau_\bullet : c_L \Rightarrow M_\bullet$ and $\tau'_\bullet : c_{L'} \Rightarrow M'_\bullet$ the respective universal cones. There exists a unique morphism $f : L \to L'$ in $\mathscr{A}$ that makes commute the diagram :

$$\begin{array}{ccc} c_L & \xrightarrow{c_f} & c_{L'} \\ {\scriptstyle\tau_\bullet}\downarrow & & \downarrow{\scriptstyle\tau'_\bullet} \\ M_\bullet & \xrightarrow{f_\bullet} & M'_\bullet. \end{array}$$

But lemma 14.2.14(iii) implies that f is an isomorphism; as $f_\bullet$ is an isomorphism in $\mathscr{L}(I, \mathscr{A})$, it follows that $\tau_\bullet$ is an isomorphism in $\mathscr{L}(I, \mathscr{A})$ if and only if the same holds for $\tau'_\bullet$. ◊

Lastly, clearly (c)⇒(a). Suppose then that (a) holds, and let $\tau_\bullet : c_L \Rightarrow M_\bullet$ be a universal cone; let also $\mu_\bullet : c_X \Rightarrow M_\bullet$ be another cone that is an isomorphism in $\mathscr{L}(I, \mathscr{A})$. Then there exists a unique morphism $f : X \to L$ in $\mathscr{A}$ such that $\tau_\bullet \odot c_f = \mu_\bullet$. Set $\mu := \lim_I \mu_\bullet$ and $\tau := \lim_I \tau_\bullet$, so that $\tau \circ f = \mu$. Now, τ is an isomorphism, and the same holds for $\mu_\bullet$, by lemma 14.2.14(iii); hence f is an isomorphism, so the cone $\mu_\bullet$ is universal, whence (c).

(iii): We know already that $M_\bullet$ has the almost Mittag-Leffler property (remark 14.2.12(iv)); moreover, $\operatorname{Ker} M_\varphi^\triangle \subset \operatorname{Ker} M_\varphi$ for every morphism φ of I, hence $M_\bullet$ fulfills condition (ii.c), whence the contention. □

Proposition 14.2.20. *Let $\mathscr{A}$ and $\mathscr{A}'$ be V-linear abelian categories, I a small category, $F : \mathscr{A} \to \mathscr{A}'$ a V-linear functor, and $M_\bullet, M'_\bullet : I \to \mathscr{A}$ two functors. The following holds :*

(i) *Let $f_\bullet : M_\bullet \Rightarrow M'_\bullet$ be a natural transformation. If* $\operatorname{Ker} f_\bullet$ *and* $\operatorname{Coker} f_\bullet$ *are almost essentially zero (resp. null, resp. dual almost essentially zero, resp. dual null), then the same holds for the kernel and cokernel of $Ff_\bullet : FM_\bullet \Rightarrow FM'_\bullet$.*

(ii) *If $M_\bullet$ is a Cauchy functor, the same holds for $FM_\bullet$.*

(iii) *If $M_\bullet$ is almost essentially constant (resp. dual almost essentially constant), so is $FM_\bullet$.*

(iv) *Suppose that F is right exact (resp. left exact). If $M_\bullet$ has the almost Mittag-Leffler property (resp. the dual almost Mittag-Leffler property), then the same holds for $FM_\bullet$.*

Proof. (ii): The assertion follows immediately from remark 14.2.10(vi).

(i): We can write $f_\bullet = e_\bullet \circ p_\bullet$, where $e_\bullet : I \to \mathscr{A}$ is a monomorphism of functors, and $p_\bullet : I \to \mathscr{A}$ is an epimorphism of functors, and then clearly it suffices to check the stated assertions for $e_\bullet$ and $p_\bullet$. Moreover, if the stated assertions are known for all monomorphisms of functors and all V-linear categories $\mathscr{A}$, they follow for every epimorphism as well, by considering the opposite transformation $f_\bullet^o : M'^o_\bullet \Rightarrow M^o_\bullet$ and invoking remark 14.2.10(ii).

Thus, let $f_\bullet$ be a monomorphism, $\varphi : j \to i$ any morphism in I, and $a \in V$ that annihilates the morphism $\mathrm{Coker}\, f_j \to \mathrm{Coker}\, f_i$ induced by M'_φ. The latter means that $\mathrm{Im}(aM'_\varphi) \subset \mathrm{Im}\, f_i$; it follows that there exists a morphism $g : M'_j \to M_i$ in $\mathscr{A}$ such that $f_i \circ g = aM'_\varphi$, and therefore

$$f_i \circ g \circ f_j = aM'_\varphi \circ f_j = af_i \circ M_\varphi = f_i \circ (aM_\varphi)$$

whence $g \circ f_j = aM_\varphi$, since f_i is a monomorphism. The induced identities $Ff_i \circ Fg = aFM'_\varphi$ and $Fg \circ Ff_j = aFM_\varphi$ imply that $\mathrm{Im}(aFM'_\varphi) \subset \mathrm{Im}\, Ff_i$ and $aM_\varphi(\mathrm{Ker}\, Ff_j) = 0$. Hence, if $\mathrm{Ker}\, f_\bullet$ and $\mathrm{Coker}\, f_\bullet$ are almost essentially zero (resp. dual almost essentially zero) then the same holds for $\mathrm{Ker}\, Ff_\bullet$ and $\mathrm{Coker}\, Ff_\bullet$. Combining with (ii), we deduce that if $\mathrm{Ker}\, f_\bullet$ and $\mathrm{Coker}\, f_\bullet$ are null (resp. dual null), the same follows for $\mathrm{Ker}\, Ff_\bullet$ and $\mathrm{Coker}\, Ff_\bullet$.

(iii) is an immediate consequence of (i).

(iv): As usual, the assertion in case F is left exact will follow from the assertion for the case F is right exact, by considering $M^o_\bullet$ and F^o. Hence, suppose that F is right exact and $M_\bullet$ has the almost Mittag-Leffler property. For a given subideal $\mathfrak{m}_0 \subset \mathfrak{m}$ of finite type and $i \in \mathrm{Ob}(I)$, choose a morphism $\varphi : j \to i$ in I such that for every morphism $\psi : k \to j$ of I we have $\mathfrak{m}_0 \cdot \mathrm{Im}\, M_\varphi \subset \mathrm{Im}\, M_{\varphi\circ\psi}$. Let C be the cokernel of the morphism $\overline{M}_{\varphi\circ\psi} : M_k \to \mathrm{Im}\, M_\varphi$ of $\mathscr{A}$ induced by $M_{\varphi\circ\psi}$; the assumption means that $\mathfrak{m}_0 C = 0$. Since F is right exact, FC is the cokernel of $F\overline{M}_{\varphi\circ\psi} : FM_k \to F(\mathrm{Im}\, M_\varphi)$, and obviously $\mathfrak{m}_0 FC = 0$. By the same token, $M_\varphi : M_j \to M_i$ factors through epimorphisms $FM_j \to F(\mathrm{Im}\, M_\varphi) \to \mathrm{Im}\, FM_\varphi$; we conclude that the cokernel of the morphism $FM_k \to \mathrm{Im}\, FM_\varphi$ induced by $FM_{\varphi\circ\psi}$ is a quotient of FC, and especially is annihilated by $\mathfrak{m}_0$. The assertion follows. □

Remark 14.2.21. Keep the notation of definition 14.2.11, and let $M_\bullet : I \to \mathscr{A}$ be any functor.

(i) By unwinding the definitions, we find that $M_\bullet$ has the dual almost Mittag-Leffler property if and only if the following holds. For every subideal $\mathfrak{m}_0 \subset \mathfrak{m}$ of finite type, and every $i \in \mathrm{Ob}(I)$ there exists a morphism $\varphi : i \to j$ in I such that

$$\mathfrak{m}_0 \cdot \mathrm{Ker}\, M_{\psi\circ\varphi} \subset \mathrm{Ker}\, M_\varphi \qquad \text{for every morphism } \psi : j \to k \text{ in } I.$$

(ii) The natural isomorphism $\mathsf{Fun}(I, \mathscr{A}) \xrightarrow{\sim} \mathsf{Fun}(I^o, \mathscr{A}^o)^o$ of remark 1.1.19(i) identifies $(\mathsf{Fun}(I^o, \mathscr{A}^o)_{\text{a.ess.0}})^o$ with the full subcategory of $\mathsf{Fun}(I, \mathscr{A})$ whose objects are the dual almost essentially zero functors. Since the opposite of a Serre subcategory is a Serre subcategory, lemma 14.2.14(i) implies that the quotient

$$\mathscr{C}(I, \mathscr{A}) := \mathsf{Fun}(I, \mathscr{A})/(\mathsf{Fun}(I^o, \mathscr{A}^o)_{\text{a.ess.0}})^o$$

is again an abelian category, whose objects are the functors $I \to \mathscr{A}$.

(iii) The rest of lemma 14.2.14 dualizes as well : first, part (ii) of the lemma implies that if $\mathscr{A}$ is $\mathfrak{m}$-torsion-free and $M_\bullet$ is dual almost essentially zero, then $\mathrm{colim}_I\, M_\bullet = 0$. Next, if I is filtered, and $\mathscr{A}$ is cocomplete and $\mathfrak{m}$-torsion-free, the functor Colim_I of remark 1.3.3(ii) factors through the localization $\mathsf{Fun}(I, \mathscr{A}) \to \mathscr{C}(I, \mathscr{A})$ and a functor

$$\mathscr{C}(I, \mathscr{A}) \to \mathscr{A} \qquad M_\bullet \mapsto \operatorname*{colim}_I M_\bullet.$$

(iv) Also remark 14.2.12(ii,iii) admits a valid dual : namely, suppose that I is filtered and $\mathscr{A}$ is cocomplete. We set

$$M_i^\triangledown := \bigcup_{\varphi:i\to j} \mathrm{Ker}\, M_\varphi \subset M_i \qquad \text{for every } i \in \mathrm{Ob}(I).$$

Then, arguing as in 14.2.12(ii) one easily checks that M_φ restricts to a morphism

$$M_\psi^\triangledown : M_i^\triangledown \to M_{i'}^\triangledown \qquad \text{for every morphism } \psi : i \to i' \text{ in } I$$

whence an induced functor $M_\bullet^\triangledown : I \to \mathscr{A}$, and a direct inspection of the definitions shows that

$$(M_\bullet^\triangledown)_\bullet^\triangledown = M_\bullet^\triangledown. \qquad (14.2.22)$$

Notice moreover that every cocone $\tau : M_\bullet \Rightarrow c_L$ factors (uniquely) through the projection $\pi_\bullet : M_\bullet \to M_\bullet/M_\bullet^\triangledown$. Combining with (14.2.22) we deduce that

$$\operatorname*{colim}_I M_\bullet^\triangledown = 0$$

and furthermore, $\pi_\bullet$ induces an isomorphism in $\mathscr{A}$:

$$\operatorname*{colim}_I M_\bullet \xrightarrow{\sim} \operatorname*{colim}_I M_\bullet/M_\bullet^\triangledown.$$

(v) Under the assumptions of (iv), we deduce that $M_\bullet$ has the dual almost Mittag-Leffler property if and only if for every subideal $\mathfrak{m}_0 \subset \mathfrak{m}$ of finite type and every $i \in \mathrm{Ob}(I)$ there exists a morphism $\varphi : i \to j$ such that $\mathfrak{m}_0 \cdot M_i^\triangledown \subset \operatorname{Ker} M_\varphi$. In the case where $\mathscr{A} = A$-$\mathbf{Mod}$ for a given $(V, \mathfrak{m})^a$-algebra A, it follows that for every $i \in \mathrm{Ob}(I)$ the system $(\operatorname{Ker} M_\varphi \mid \varphi : i \to j)$ is a Cauchy net in the uniform space $\mathscr{I}_A(M_i)$, and its unique limit is the A-submodule $M_i^\triangledown$.

(vi) Suppose that I is filtered, and is $\mathscr{A}$ is cocomplete and $\mathfrak{m}$-torsion-free. Then :

- By dualizing proposition 14.2.18(ii), we see that the following conditions are equivalent :
  - (a) $M_\bullet$ is dual almost essentially constant.
  - (b) $M_\bullet$ is isomorphic to a constant functor, in the category $\mathscr{C}(I, \mathscr{A})$ of remark 14.2.21(ii).
  - (c) $M_\bullet$ has the dual almost Mittag-Leffler property, and for every subideal $\mathfrak{m}_0 \subset \mathfrak{m}$ of finite type, there exists $i \in \mathrm{Ob}(I)$ such that $\mathfrak{m}_0$ annihilates the cokernel of the morphism

$$M_i/M_i^\triangledown \to M_j/M_j^\triangledown \qquad (14.2.23)$$

induced by M_φ, for every morphism $\varphi : i \to j$ in I (notation of remark 14.2.21(iv)).

- Moreover, the dual of proposition 14.2.18(i) states that if $M_\bullet$ has the dual almost Mittag-Leffler property, then (14.2.23) is a monomorphism for every morphism $\varphi : i \to j$ in I, and the projections $(M_i \to M_i/M_i^\triangledown \mid i \in \mathrm{Ob}(I))$ define an isomorphism $M_\bullet \xrightarrow{\sim} M_\bullet/M_\bullet^\triangledown$ in $\mathscr{C}(I, \mathscr{A})$.

Lemma 14.2.24. *In the situation of definition* 14.2.11*, let A be any $(V, \mathfrak{m})^a$-algebra, $M_\bullet$: $\mathbb{N}^o \to A$-$\mathbf{Mod}$ a functor whose transition morphisms $M_{m,n} : M_m \to M_n$ are epimorphisms for every $m, n \in \mathbb{N}$, and set $L := \lim_{n\in\mathbb{N}} M_n$. Then we have :*

(i) *The universal cone $\tau_\bullet : c_L \Rightarrow M_\bullet$ is an epimorphism of functors.*

(ii) $\lim^1_{n\in\mathbb{N}} M_n = 0$.

(iii) *For every almost finitely generated (resp. almost finite projective) A-module N, the cone $N \otimes_A \tau_\bullet$ induces an epimorphism (resp. an isomorphism)*

$$\varphi_N : N \otimes_A L \to \lim_{n\in\mathbb{N}} N \otimes_A M_n.$$

(iv) *More generally, if N is an A-module and $a \in A_*$ such that $a \cdot \mathbf{1}_N$ factors as a composition of A-linear morphisms $N \xrightarrow{\alpha} N_0 \xrightarrow{\beta} N$ with N_0 almost finitely generated (resp. almost finite projective), then $a \cdot \operatorname{Coker} \varphi_N = 0$ (resp. $a \cdot \operatorname{Ker} \varphi_N = 0$).*

Proof. (i): Let $M_{\bullet!} : \mathbb{N}^o \to A_*$-$\mathbf{Mod}$ be the composition of $M_\bullet$ with the left adjoint $(-)_! :$ A-$\mathbf{Mod} \to A_*$-$\mathbf{Mod}$ of the localization $(-)^a$. Set $T := \lim_{n\in\mathbb{N}} M_{n!}$, and let $\mu_\bullet : c_T \Rightarrow M_{\bullet!}$ be the universal cone; then $L = T^a$ and the natural isomorphism $M_\bullet \xrightarrow{\sim} (M_{\bullet!})^a$ identifies $\tau_\bullet$ with $\mu_\bullet^a$. Thus, it suffices to check that $\mu_\bullet$ is an epimorphism of functors. However, the transition morphisms $(M_{m,n})_! : M_{m!} \to M_{n!}$ are surjective, so the assertion is clear.

(ii): Recall that for every functor $M_\bullet : \mathbb{N}^o \to A$-$\mathbf{Mod}$ (resp. $N_\bullet : \mathbb{N}^o \to A_*$-$\mathbf{Mod}$) the right derived functor $R\lim_{\mathbb{N}^o} M_\bullet$ (resp. $R\lim_{\mathbb{N}^o} N_\bullet$) is computed by a natural two-term complex $\Pi(M_\bullet)$ (resp. $\Pi(N_\bullet)$) : see [163, §3.5]. With this notation, we have $\Pi(M_\bullet) = \Pi(M_{\bullet!})^a$, since the localization $(-)^a$ commutes with all products. However, $\lim^1_{n\in\mathbb{N}} M_{n!} = 0$, since the transition morphisms $(M_{m,n})_!$ are surjective ([163, Lemma 3.5.3]). The assertion follows.

(iii): Suppose first that N is finitely generated; hence there exists $r \in \mathbb{N}$ and a short exact sequence $0 \to K \to A^{\oplus r} \to N \to 0$ of A-modules. For every $n \in \mathbb{N}$, let $C_n \subset A^{\oplus r} \otimes_A M_n$ be the image of $K \otimes_A M_n$; we get an inverse system $(C_n \mid n \in \mathbb{N})$ whose transition morphisms are also epimorphisms (details left to the reader). By virtue of (ii), there follows an epimorphism of A-modules

$$\lim_{n\in\mathbb{N}} A^{\oplus r} \otimes_A M_n = A^{\oplus r} \otimes_A L \xrightarrow{\psi} \lim_{n\in\mathbb{N}} N \otimes_A M_n.$$

Lastly, it is easily seen that ψ factors through φ_N and the induced morphism $A^{\oplus r} \otimes_A L \to N \otimes_A L$, whence the contention in this case. Next, consider the functor

$$T : A_*\text{-}\mathbf{Mod} \to A_*\text{-}\mathbf{Mod} \qquad X \mapsto (\mathrm{Coker}\,\varphi_{X^a})_*$$

that assigns to every A-linear morphism $f : N \to N'$ the induced map $(\mathrm{Coker}\,\varphi_{N^a})_* \to (\mathrm{Coker}\,\varphi_{N'^a})_*$ of A_*-modules. It is easily seen that T is V-homogeneous of degree 1 (see definition 14.1.53), and the associated functor $T^a : A\text{-}\mathbf{Mod} \to A\text{-}\mathbf{Mod}$ is given by the rule: $N \mapsto \mathrm{Coker}\,\varphi_N$ for every A-module N (notation of remark 14.1.54(ii)). By the foregoing, $T^a N = 0$ whenever N is finitely generated; however, on the one hand, for every suitable pair of infinite cardinalities ω, ω' the associated map $\mathscr{M}_{\omega,\omega'}(T^a) : \mathscr{M}_\omega(A) \to \mathscr{M}_{\omega'}(A)$ is uniformly continuous (proposition 14.1.56(iii)), and on the other hand, it is easily seen that the subset $\{0\}$ is closed in the topology of the uniform space $\mathscr{M}(A)$. Hence, $T^a N = 0$ for every almost finitely generated A-module, as stated.

Next, suppose that N is almost finite projective; then for every $a \in \mathfrak{m}$ there exists $r \in \mathbb{N}$ and A-linear morphisms $N \xrightarrow{\alpha} A^{\oplus r} \xrightarrow{\beta} N$ whose composition is $a \cdot \mathbf{1}_N$ ([75, Lemma 2.4.15]). There follows a commutative diagram :

$$\begin{array}{ccccc} N \otimes_A L & \xrightarrow{\alpha\otimes_A L} & A^{\oplus r} \otimes_A L & \xrightarrow{\beta\otimes_A L} & N \otimes_A L \\ \downarrow{\scriptstyle \varphi_N} & & \downarrow & & \downarrow{\scriptstyle \varphi_N} \\ \lim_{n\in\mathbb{N}} N \otimes_A M_n & \longrightarrow & \lim_{n\in\mathbb{N}} A^{\oplus r} \otimes_A M_n & \longrightarrow & \lim_{n\in\mathbb{N}} N \otimes_A M_n \end{array}$$

whose central arrow is an isomorphism, and we already know that φ_N is an epimorphism. By a simple diagram chase we deduce that $a \cdot \mathrm{Ker}\,\varphi_N = 0$, and since a is arbitrary, we conclude that φ_N is an isomorphism in this case, as stated.

(iv): In light of (iii) we have : $a \cdot \mathbf{1}_{TN} = T\beta \circ T\alpha = 0$, *i.e.* $a \cdot \mathrm{Coker}\,\varphi_N = 0$. If N_0 is almost finite projective, consider the functor $T' : A^a\text{-}\mathbf{Mod} \to A^a\text{-}\mathbf{Mod}$ such that $T'X := \mathrm{Ker}\,\varphi_X$ for every A^a-module X : by (iii) we have $T'N_0 = 0$, whence $a \cdot \mathbf{1}_{T'N} = 0$. ☐

Proposition 14.2.25. *In the situation of definition* 14.2.11*, let A be any $(V,\mathfrak{m})^a$-algebra, $M_\bullet : I \to A\text{-}\mathbf{Mod}$ any functor, and $\tau_\bullet : c_L \Rightarrow M_\bullet$ and $\tau'_\bullet : M_\bullet \Rightarrow c_{L'}$ be respectively a universal cone and a universal cocone. We have :*

(i) *Suppose that I is cofiltered and consider the following conditions :*

(a) *The functor* $\mathrm{Coker}\,\tau_\bullet$ *is almost essentially zero.*

(b) *$M_\bullet$ has the almost Mittag-Leffler property.*

Then (a)⇒(b)*, and if there exists a coinitial functor* $\mathbb{N}^o \to I$*, then* (b)⇒(a)*.*

(ii) *Suppose that is I filtered. Then $M_\bullet$ has the dual almost Mittag-Leffler property if and only if the functor* $\mathrm{Ker}\,\tau'_\bullet$ *is dual almost essentially zero.*

(iii) *Suppose that $I = \mathbb{N}^o$, where $\mathbb{N}$ is endowed with its standard total ordering. If $M_\bullet$ has the almost Mittag-Leffler property, then* $\lim^1_{n\in\mathbb{N}} M_n = 0$.

Proof. (iii): Suppose first that $M_\bullet$ is almost essentially zero, and consider, for every finitely generated subideal $\mathfrak{m}_0 \subset \mathfrak{m}$, the short exact sequence of functors

$$\Sigma_\bullet \quad : \quad 0 \to M_\bullet[\mathfrak{m}_0] \to M_\bullet \to M_\bullet/M_\bullet[\mathfrak{m}_0] \to 0$$

whose right-most term is an essentially zero inverse system, by remark 14.2.12(i)), and therefore

$$\lim_{n\in\mathbb{N}} M_n/M_n[\mathfrak{m}_0] = \lim_{n\in\mathbb{N}}{}^1 M_n/M_n[\mathfrak{m}_0] = 0.$$

On the other hand, clearly $\mathfrak{m}_0$ annihilates $\lim^1_{n\in\mathbb{N}} M_n[\mathfrak{m}_0]$. From the long exact cohomology sequence associated with $\Sigma_\bullet$ it follows that $\mathfrak{m}_0$ annihilates as well $\lim^1_{n\in\mathbb{N}} M_n$. Since $\mathfrak{m}_0$ is arbitrary, the claim follows in this case.

Next, let $M_\bullet$ be an arbitrary functor with the almost Mittag-Leffler property; for every $m, n \in \mathbb{N}$ with $m > n$, we let $M_{m,n} : M_m \to M_n$ be the transition morphism. By proposition 14.2.18(i.a), the restriction $M^\triangle_{p,n} : M^\triangle_m \to M^\triangle_n$ of $M_{m,n}$ is an epimorphism for every such n and p. Let us then consider the short exact sequence of inverse systems

$$\Sigma'_\bullet \quad : \quad 0 \to M^\triangle_\bullet \to M_\bullet \to M_\bullet/M^\triangle_\bullet \to 0.$$

Proposition 14.2.18(i.b) implies that $M_\bullet/M^\triangle_\bullet$ is almost essentially zero, so $\lim^1_{n\in\mathbb{N}} M_n/M^\triangle_n = 0$, by the previous case; also, $\lim^1_{n\in\mathbb{N}} M^\triangle_n = 0$, by lemma 14.2.24(ii). The assertion follows then, by considering the long exact $R\lim$ sequence associated with $\Sigma'_\bullet$.

(i.a)⇒(i.b): By assumption, for every subideal $\mathfrak{m}_0 \subset \mathfrak{m}$ of finite type and every $i \in \mathrm{Ob}(I)$ there exists a morphism $\varphi : j \to i$ in I such that $\mathfrak{m}_0$ annihilates the image of the induced morphism $\mathrm{Coker}\,\tau_j \to \mathrm{Coker}\,\tau_i$. The latter means that $\mathfrak{m}_0 \cdot \mathrm{Im}\,M_\varphi \subset \mathrm{Im}\,\tau_i$; but clearly $\mathrm{Im}\,\tau_i \subset \mathrm{Im}\,M_{\varphi\circ\psi}$ for every morphism $\psi : k \to j$ in I, whence the contention.

Next, suppose that there exists a coinitial functor $\lambda : \mathbb{N}^o \to I$, and that $M_\bullet$ has the almost Mittag-Leffler property; notice that τ_i factors through a morphism $\mu_i : L \to M^\triangle_i$ and the inclusion $M^\triangle_i \to M_i$, for every $i \in \mathrm{Ob}(I)$. There follows a short exact sequence of functors :

$$\mathrm{Coker}\,\mu_\bullet \to \mathrm{Coker}\,\tau_\bullet \to M_\bullet/M^\triangle_\bullet \to 0.$$

By proposition 14.2.18(i.b), $M_\bullet/M^\triangle_\bullet$ is almost essentially zero, hence L also represents the limit of $M^\triangle_\bullet$, and $\mu_\bullet$ is a universal cone; by lemma 14.2.14(i) it then suffices to show that $\mathrm{Coker}\,\mu_\bullet = 0$. Now, $\mu_\bullet * \lambda : c_L \Rightarrow M^\triangle_\bullet \circ \lambda$ is still a universal cone (remark 1.5.5(ii,iii)). Since we also know that $M^\triangle_\varphi$ is an epimorphism for every morphism φ of I (proposition 14.2.18(i.a)), we are then reduced to checking that $\mu_{\lambda(n)} : L \to M^\triangle_{\lambda(n)}$ is an epimorphism for every $n \in \mathbb{N}$. The latter follows from lemma 14.2.24(i).

(ii): Suppose that $\mathrm{Ker}\,\tau'_\bullet$ is dual almost essentially zero; this means that for every subideal $\mathfrak{m}_0 \subset \mathfrak{m}$ of finite type, and every $i \in \mathrm{Ob}(I)$ there exists a morphism $\varphi : i \to j$ in I such that $M_\varphi(\mathfrak{m}_0 \cdot \mathrm{Ker}\,\tau'_i) = 0$. But we have $\mathrm{Ker}\,M_{\psi\circ\varphi} \subset \mathrm{Ker}\,\tau'_i$ for every morphism $\psi : j \to k$ in I; whence $\mathfrak{m}_0 \cdot \mathrm{Ker}\,M_{\psi\circ\varphi} \subset \mathrm{Ker}\,M_\varphi$ for every such ψ. Hence $M_\bullet$ has the dual Mittag-Leffler property, by remark 14.2.21(i). Conversely, suppose that $M_\bullet$ has the dual Mittag-Leffler property; we know that the functor $M^\triangledown_\bullet$ is almost essentially zero (remark 14.2.21(vi)) hence it suffices to check that $M^\triangledown_i = \mathrm{Ker}\,\tau'_i$ for every $i \in \mathrm{Ob}(I)$. After replacing I by i/I, we may assume that i is an initial object of I, whence an induced cone $\mu_\bullet : c_{M_i} \Rightarrow M_\bullet$, and notice that $M^\triangledown_i = \mathrm{colim}_I\,\mathrm{Ker}\,\mu_\bullet$. On the other hand, $\mathrm{colim}_I\,\mu_\bullet = \tau'_i$; since filtered colimits in A-**Mod** are exact, the assertion follows. □

14.2.26. *Pontryagin duality.* Recall that for every ring R and every R-module N, we have a natural R-module structure on

$$N^\vee := \mathrm{Hom}_\mathbb{Z}(N, \mathbb{Q}/\mathbb{Z}).$$

Namely, for every $a \in R$ and every $\mathbb{Z}$-linear map $\varphi : N \to \mathbb{Q}/\mathbb{Z}$ we set $a \cdot \varphi := \varphi \circ (a\mathbf{1}_N)$. Especially, $R^\vee$ is naturally an R-module, and there follows a natural R-linear isomorphism :

$$\mathrm{Hom}_R(N, R^\vee) \xrightarrow{\sim} N^\vee$$

assigning to every R-linear map $\varphi : N \to R^\vee$ the $\mathbb{Z}$-linear map $\varphi' : N \to \mathbb{Q}/\mathbb{Z}$ such that $\varphi'(n) := \varphi(n)(1)$ for every $n \in N$. Its inverse assigns to every $\mathbb{Z}$-linear map $\psi : N \to \mathbb{Q}/\mathbb{Z}$ the homomorphism of R-modules $\psi' : N \to R^\vee$ that sends every $n \in N$ to the $\mathbb{Z}$-linear map $\psi'(n) : R \to \mathbb{Q}/\mathbb{Z}$ such that $\psi'(n)(a) := \psi(an)$ for every $a \in R$. Since $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module, there results an exact functor

$$R\text{-}\mathbf{Mod} \to R\text{-}\mathbf{Mod}^o \qquad N \mapsto N^\vee.$$

Moreover, it is easily seen that the natural *biduality R-linear map* is an injection :

$$N \to N^{\vee\vee} \qquad \text{for every } R\text{-module } N.$$

Let now $(V, \mathfrak{m})$ be a basic setup, and A a $(V, \mathfrak{m})^a$-algebra; we consider the associated A-module

$$J_A := ((A_*)^\vee)^a$$

and from the foregoing it is easily seen that the *Pontryagin duality* functor

$$(-)^\vee : A\text{-}\mathbf{Mod} \to A\text{-}\mathbf{Mod}^o \qquad M \mapsto M^\vee := \mathrm{alHom}_A(M, J_A)$$

is again exact and V-linear, where $\mathrm{alHom}_A(-,-)$ denotes the bifunctor of *almost homomomorphisms* defined as in [75, §2.2.11]; indeed, we have a natural isomorphism of A-modules

$$M^\vee \xrightarrow{\sim} ((M_*)^\vee)^a \qquad \text{for every } A\text{-module } M$$

(details left to the reader). Moreover, it follows that the biduality morphism $M \to M^{\vee\vee}$ is a monomorphism of A-modules; especially, $(-)^\vee$ is a conservative functor.

Remark 14.2.27. (i) For every small category I, we get a natural extension of Pontryagin duality to functors $I \to A\text{-}\mathbf{Mod}$. Namely, we have a conservative exact functor :

$$\mathsf{Fun}(I, A\text{-}\mathbf{Mod})^o \to \mathsf{Fun}(I^o, A\text{-}\mathbf{Mod}) \qquad M_\bullet \mapsto M_\bullet^\vee := (-)^\vee \circ M_\bullet.$$

(ii) For every A-module M, and every A-submodule $N \subset M$, let $i_N : N \to M$ the inclusion; it is easily seen that the induced map

$$\mathscr{I}_A(M) \to \mathscr{I}_A(M^\vee) \qquad N \mapsto \mathrm{Ker}(i_N^\vee : M^\vee \to N^\vee)$$

is uniformly continuous, for the uniform structures defined as in [75, Def.2.3.1(i)].

(iii) Then, if I is cofiltered, and $M_\bullet : I \to A\text{-}\mathbf{Mod}$ is any functor with the almost Mittag-Leffler property, combining with remarks 14.2.12(iii) and 14.2.21(v) we deduce that the A-linear epimorphism $i^\vee_{M_i^\triangle} : M_i^\vee \to (M_i^\triangle)^\vee$ factors through an isomorphism of A-modules

$$M_i^\vee/(M_\bullet^\vee)_i^\triangledown \xrightarrow{\sim} (M_i^\triangle)^\vee \qquad \text{for every } i \in \mathrm{Ob}(I).$$

Lemma 14.2.28. *With the notation of remark* 14.2.27*, for every functor* $M_\bullet : I \to A\text{-}\mathbf{Mod}$ *the following holds :*

(i) *$M_\bullet$ is essentially zero (resp. almost essentially zero) if and only if $M_\bullet^\vee$ is dual essentially zero (resp. dual almost essentially zero).*

(ii) *Suppose that I is cofiltered. Then $M_\bullet$ is almost essentially constant (resp. has the almost Mittag-Leffler property) if and only if $M_\bullet^\vee$ is dual almost essentially constant (resp. has the dual almost Mittag-Leffler property).*

Proof. (i): By direct inspection we see that if $M_\bullet$ is essentially zero (resp. almost essentially zero), then $M^\vee$ is dual essentially zero (resp. dual almost essentially zero). Likewise, if $M_\bullet$ is dual essentially zero (resp. dual almost essentially zero), then $M^\vee$ is essentially zero (resp. almost essentially zero). From this, the converse assertions follow easily, since the biduality morphism $M_\bullet \to M_\bullet^{\vee\vee}$ is a monomorphism.

(ii): For a given pair of morphisms $k \xrightarrow{\psi} j \xrightarrow{\varphi} i$ of I, set $N := \mathrm{Im} M_\varphi / \mathrm{Im}\, M_{\varphi\circ\psi}$. Since Pontryagin duality is exact and conservative, for every subideal $\mathfrak{m}_0 \subset \mathfrak{m}$ we have :

$$\mathfrak{m}_0 N = 0 \Leftrightarrow \mathfrak{m}_0 N^\vee = 0 \Leftrightarrow \mathfrak{m}_0 \cdot \mathrm{Ker}(\mathrm{Im}\, M^\vee_{\varphi\circ\psi} \to M^\vee_\varphi) = 0 \Leftrightarrow \mathfrak{m}_0 \mathrm{Ker}\, M^\vee_{\varphi\circ\psi} \subset \mathrm{Ker} M^\vee_\varphi.$$

Taking into account remark 14.2.21(i), we deduce that $M_\bullet$ has the almost Mittag-Leffler property if and only if $M^\vee_\bullet$ has the dual almost Mittag-Leffler property.

Next, proposition 14.2.20(ii) implies that if $M_\bullet$ is almost essentially constant, then $M^\vee_\bullet$ is dual almost essentially constant. Conversely, if $M^\vee_\bullet$ is dual almost essentially constant, then $M^\vee_\bullet$ has the dual almost Mittag-Leffler property, and for every subideal $\mathfrak{m}_0 \subset \mathfrak{m}$ of finite type there exists $i \in \mathrm{Ob}(I)$ with

$$\mathfrak{m}_0 \cdot \mathrm{Coker}(M^\vee_i / (M^\vee_\bullet)^\nabla_i \to M^\vee_j / (M^\vee_\bullet)^\nabla_j) = 0 \qquad \text{for every morphism } i \xrightarrow{\varphi} j \tag{14.2.29}$$

(remark 14.2.21(vi)). By the foregoing, we deduce already that $M_\bullet$ has the almost Mittag-Leffler property. Moreover, by remark 14.2.27(iii), condition (14.2.29) is equivalent to :

$$\mathfrak{m}_0 \cdot \mathrm{Coker}((M^\triangle_i)^\vee \to (M^\triangle_j)^\vee) = 0 \qquad \text{for every morphism } i \xrightarrow{\varphi} j.$$

But $\mathrm{Coker}((M^\triangle_i)^\vee \to (M^\triangle_j)^\vee) = (\mathrm{Ker}\, M^\triangle_\varphi)^\vee$, and in light of proposition 14.2.18(ii), we deduce that $M_\bullet$ is almost essentially constant. □

Proposition 14.2.30. *Let I be a small cofiltered category, and*

$$0 \to M'_\bullet \to M_\bullet \to M''_\bullet \to 0$$

a short exact sequence of functors $I \to A$-$\mathbf{Mod}$. We have:

(i) *If $M'_\bullet$ and $M''_\bullet$ are almost essentially constant (resp. have the almost Mittag-Leffler property), then the same holds for $M_\bullet$.*

(ii) *If $M_\bullet$ has the almost Mittag-Leffler property, the same holds for $M''_\bullet$.*

(iii) *If $M_\bullet$ has the almost Mittag-Leffler property and $M''_\bullet$ is almost essentially constant, then $M'_\bullet$ has the almost Mittag-Leffler property.*

(iv) *If $f_\bullet : X_\bullet \to Y_\bullet$ is a natural transformation between almost essentially constant functors, then* Ker $f_\bullet$ *and* Coker $f_\bullet$ *are almost essentially constant.*

Proof. (ii) is clear from the definitions.

(iv): By proposition 14.2.18(ii), the universal cones $\tau^X_\bullet : c_X \Rightarrow X_\bullet$ and $\tau^Y_\bullet : c_X \Rightarrow Y_\bullet$ are isomorphisms in $\mathscr{L}(I, A\text{-}\mathbf{Mod})$. Thus, there exists a unique morphism $f : X \to Y$ of A-modules that makes commute the diagram

$$\begin{array}{ccc} c_X & \xrightarrow{c_f} & c_Y \\ {\scriptstyle \tau^X_\bullet}\downarrow & & \downarrow{\scriptstyle \tau^Y_\bullet} \\ X_\bullet & \xrightarrow{f_\bullet} & Y_\bullet \end{array}$$

whose vertical arrows are isomorphism in $\mathscr{L}(I, A\text{-}\mathbf{Mod})$. Set $K := \mathrm{Ker}\, f$ and $C := \mathrm{Coker}\, f$; there follow induced isomorphisms $c_K \xrightarrow{\sim} \mathrm{Ker}\, f_\bullet$ and $c_C \xrightarrow{\sim} \mathrm{Coker}\, f_\bullet$ in $\mathscr{L}(I, A\text{-}\mathbf{Mod})$, whence the contention, in light of proposition 14.2.18(ii).

(i): Let L, L' and L'' be the colimits of the functors $M^\vee_\bullet, M'^\vee_\bullet$ and respectively $M''^\vee_\bullet$; since filtered colimits are exact in the category A-$\mathbf{Mod}$, we get a commutative diagram of functors with exact rows :

$$\begin{array}{ccccccccc} 0 & \to & M''^\vee_\bullet & \to & M^\vee_\bullet & \to & M'^\vee_\bullet & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & c_{L''} & \to & c_L & \to & c_{L'} & \to & 0 \end{array}$$

whose vertical arrows are the universal cocones. Consider first the case where $M'_\bullet$ and $M''_\bullet$ are almost essentially constant; then $M'^\vee_\bullet$ and $M''^\vee_\bullet$ are dual almost essentially constant (lemma 14.2.28(ii)), and arguing as in the proof of (iv) we deduce that the first and third vertical arrows are isomorphisms in the quotient category $\mathscr{C}(I, A\text{-}\mathbf{Mod})$, hence the same holds for the middle one, by the 5-lemma. Then $M^\bullet_\bullet$ is dual almost essentially constant, by remark 14.2.21(vi), hence $M_\bullet$ is almost essentially constant, again by lemma 14.2.28(ii).

Next, if $M'_\bullet$ and $M''_\bullet$ have the almost Mittag-Leffler property, then $M'^\vee_\bullet$ and $M''^\vee_\bullet$ have the dual almost Mittag-Leffler property (lemma 14.2.28(ii)), hence the kernels of the first and third vertical arrows are dual almost essentially zero (proposition 14.2.25(ii)). Then the same holds for the kernel of the middle vertical arrow, so $M^\vee_\bullet$ has the dual almost Mittag-Leffler property, again by proposition 14.2.25(ii); finally, $M_\bullet$ has the almost Mittag-Leffler property, by lemma 14.2.28(ii).

(iii): We consider again the induced commutative ladder with exact rows. Arguing as in the proof of (ii), we see that the kernel of the middle vertical arrow and the cokernel of the left-most vertical arrow are both dual almost essentially zero, hence the same holds for the kernel of the right-most vertical arrow, by the snake lemma. Then $M'^\vee_\bullet$ has the dual almost Mittag-Leffler property, by proposition 14.2.25(ii), and finally, $M'_\bullet$ has the almost Mittag-Leffler property, by lemma 14.2.28(ii). □

14.2.31. Let us now consider a basic setup $(V, \mathfrak{m})$, a (small) cofiltered category I, and a functor

$$I \to (V, \mathfrak{m})^a\text{-}\mathsf{Alg.Mod} \qquad i \mapsto (A_i, M_i)$$

where $(V, \mathfrak{m})^a$-$\mathsf{Alg.Mod}$ is defined as in [75, Def.2.5.22(ii)]. Hence, for every $i \in \mathrm{Ob}(I)$, the pair (A_i, M_i) consists of a $(V, \mathfrak{m})^a$-algebra A_i, and an A_i-module M_i. To each morphism $\varphi : j \to i$ of I, the functor assigns a pair $(A_\varphi, g_\varphi) : (A_j, M_j) \to (A_i, M_i)$, where $A_\varphi : A_j \to A_i$ is a morphism of $(V, \mathfrak{m})^a$-algebras, and $g_\varphi : A_i \otimes_{A_j} M_j \to M_i$ is an A_i-linear morphism. We get two obvious induced functors

$$A_\bullet : I \to (V, \mathfrak{m})^a\text{-}\mathsf{Alg} \qquad M_\bullet : I \to (V, \mathfrak{m})^a\text{-}\mathbf{Mod}.$$

Namely, for every morphism φ as in the foregoing we let $M_\varphi : M_j \to M_i$ be the composition of g_φ with the natural A_j-linear morphism $M_j \to A_i \otimes_{A_j} M_j$. Set also

$$A := \lim_I A_\bullet \qquad \text{and} \qquad M := \lim_I M_\bullet.$$

Notice that $A_\bullet$ (resp. $M_\bullet$) can also be viewed as a functor with values in A-Alg (resp. A-$\mathbf{Mod}$).

Proposition 14.2.32. *With the notation of* (14.2.31)*, the following holds :*

(i) *If $A_\bullet : I \to A\text{-}\mathbf{Mod}$ has the almost Mittag-Leffler property, and g_φ is an epimorphism for every morphism φ of I, then $M_\bullet : I \to A\text{-}\mathbf{Mod}$ has the almost Mittag-Leffler property.*

(ii) *Suppose that g_φ is an isomorphism for every morphism φ of I. Then, if $A_\bullet : I \to A\text{-}\mathbf{Mod}$ is almost essentially constant (resp. is a Cauchy functor), the same holds for $M_\bullet$.*

(iii) *Moreover, under the assumptions of* (i) *(resp.* (ii)*), the natural morphism*

$$\alpha_i : A_i \otimes_A M_i^\triangle \to M_i \qquad (\textit{resp. } \beta_i : A_i \otimes_A M \to M_i\ )$$

is an epimorphism (resp. an isomorphism) for every $i \in \mathrm{Ob}(I)$.

(iv) *Lastly, under the assumptions of* (i)*, if there exists a coinitial functor $\mathbb{N}^o \to I$, then β_i is an epimorphism for every $i \in \mathrm{Ob}(I)$.*

Proof. We fix universal cones $\tau^A_\bullet : c_A \Rightarrow A_\bullet$ and $\tau^M_\bullet : c_M \Rightarrow M_\bullet$.

Claim 14.2.33. Assertions (ii) and (iii) hold if $A_\bullet$ is a Cauchy functor and g_φ is an isomorphism for every morphism φ of I.

Proof of the claim. Indeed, in this case for every subideal $\mathfrak{m}_0 \subset \mathfrak{m}$ of finite type we find $i \in \mathrm{Ob}(I)$ such that for every morphism $\varphi : j \to i$ we have $\mathfrak{m}_0 \cdot \mathrm{Ker}\, A_\varphi = \mathfrak{m}_0 \cdot \mathrm{Coker}\, A_\varphi = 0$. By virtue of remark 14.2.10(vi), it follows that $\mathfrak{m}_0^2$ annihilates the kernel and cokernel of $A_\varphi \otimes_{A_j} M_j$, for every such φ. Since g_φ is an isomorphism, it follows that $\mathfrak{m}_0^2 \cdot \mathrm{Ker}\, M_\varphi = \mathfrak{m}_0^2 \cdot \mathrm{Coker}\, M_\varphi = 0$, for every such φ. This shows that $M_\bullet$ is a Cauchy functor.

Next, notice that $\mathrm{Ker}\, \tau^A_\bullet$ and $\mathrm{Coker}\, \tau^A_\bullet$ are also Cauchy functors (remark 14.2.17(ii)), and moreover they are almost essentially zero, since $A_\bullet$ is almost essentially constant (proposition 14.2.18(ii,iii)). Hence, $\mathrm{Ker}\, \tau^A_\bullet$ and $\mathrm{Coker}\, \tau^A_\bullet$ are null. The same argument shows that $\mathrm{Ker}\, \tau^M_\bullet$ and $\mathrm{Coker}\, \tau^M_\bullet$ are null, since $M_\bullet$ is a Cauchy functor. Moreover, the kernel and cokernel of $\tau^A_\bullet \otimes_A M$ are null as well, since the functor $- \otimes_A M : A\text{-}\mathbf{Mod} \to A\text{-}\mathbf{Mod}$ is V-linear (proposition 14.2.20(i)). Since we have :

$$\beta_i \circ (\tau^A_i \otimes_A M) = \tau^M_i \qquad \text{for every } i \in \mathrm{Ob}(I)$$

and as the null functors form a full Serre subcategory of $\mathsf{Fun}(I, A\text{-}\mathbf{Mod})$ (remark 14.2.12(v)), it follows that also $\mathrm{Ker}\, \beta_\bullet$ and $\mathrm{Coker}\, \beta_\bullet$ are null functors. Now, let $\mathfrak{m}_0 \subset \mathfrak{m}$ be a subideal of finite type, and $i \in \mathrm{Ob}(I)$; since I is cofiltered, we deduce that there exists a morphism $\varphi : j \to i$ such that $\mathfrak{m}_0 \cdot \mathrm{Ker}\, \beta_j = \mathfrak{m}_0 \cdot \mathrm{Coker}\, \beta_j = 0$. Since the functor $A_i \otimes_{A_j} - : A_j\text{-}\mathbf{Mod} \to A_i\text{-}\mathbf{Mod}$ is V-linear, it follows that $\mathfrak{m}_0^2$ annihilates the kernel and cokernel of $A_i \otimes_{A_j} \beta_j : A_i \otimes_A M \to A_i \otimes_{A_j} M_j$ (remark 14.2.10(vi)). But notice that

$$g_\varphi \circ (A_i \otimes_{A_j} \beta_j) = \beta_i.$$

So finally $\mathfrak{m}_0^2 \cdot \mathrm{Ker}\, \beta_i = \mathfrak{m}_0^2 \cdot \mathrm{Coker}\, \beta_i = 0$. Since $\mathfrak{m}_0$ is arbitrary, we conclude that β_i is an isomorphism, as stated. $\Diamond$

We can now complete the proof of (ii) and (iii), for the case where $A_\bullet$ is almost essentially constant, and g_φ is still an isomorphism for every morphism φ of I. To this aim, set

$$\overline{A}_\varphi := \mathrm{Im}\, A_\varphi \quad \text{and} \quad \overline{M}_\varphi := \overline{A}_\varphi \otimes_{A_j} M_j \qquad \text{for every morphism } \varphi : j \to i \text{ of } I.$$

Let $j \xrightarrow{\varphi} i$ and $j' \xrightarrow{\varphi'} i'$ be two object of $\mathsf{Morph}(I)$, and $(\psi, \psi') : \varphi' \to \varphi$ a morphism of $\mathsf{Morph}(I)$; by definition, $\psi : j' \to j$ and $\psi' : i' \to i$ are morphisms in I such that $\psi' \circ \varphi' = \varphi \circ \psi$, and then it is clear that $A_{\psi'}$ restricts to a morphism of $(V, \mathfrak{m})^a$-algebras

$$\overline{A}_{(\psi,\psi')} : \overline{A}_{\varphi'} \to \overline{A}_\varphi.$$

Then, $M_\psi : M_{j'} \to M_j$ induces a $\overline{A}_\varphi$-linear morphism

$$\overline{g}_{(\psi,\psi')} : \overline{A}_\varphi \otimes_{\overline{A}_{\varphi'}} \overline{M}_{\varphi'} \xrightarrow{\sim} \overline{A}_\varphi \otimes_{A_{j'}} M_{j'} \to \overline{M}_\varphi$$

and it is easily seen that the rules :

$$\varphi \mapsto (\overline{A}_\varphi, \overline{M}_\varphi) \qquad \text{and} \qquad (\varphi' \xrightarrow{(\psi,\psi')} \varphi) \mapsto (\overline{A}_{(\psi,\psi')}, \overline{g}_{(\psi,\psi')})$$

for every $\varphi \in \mathrm{Ob}(\mathsf{Morph}(I))$ and every morphism (ψ, ψ') of $\mathsf{Morph}(I)$, define a functor

$$\mathsf{Morph}(I) \to (V, \mathfrak{m})^a\text{-}\mathsf{Alg.Mod}$$

whence, as in (14.2.31), two functors

$$\overline{A}_\bullet : \mathsf{Morph}(I) \to A\text{-}\mathsf{Alg} \qquad \overline{M}_\bullet : \mathsf{Morph}(I) \to A\text{-}\mathbf{Mod}$$

and notice also that :

(14.2.34) $$\overline{A}^\triangle_\varphi = A^\triangle_i \qquad \text{for every morphism } \varphi : j \to i \text{ in } I.$$

Next, let $\mathscr{I}$ be the category whose objects are the pairs $(j \xrightarrow{\varphi} i, \mathfrak{m}_0)$ where φ is a morphism of I and $\mathfrak{m}_0 \subset \mathfrak{m}$ is a subideal of finite type such that :

$$\mathfrak{m}_0 \overline{A}_\varphi \subset \overline{A}^\triangle_\varphi$$

(notation of remark 14.2.12(ii)). For every $(\varphi,\mathfrak{m}_0),(\varphi',\mathfrak{m}'_0)\in\mathrm{Ob}(\mathscr{I})$, the set of morphisms $(\varphi',\mathfrak{m}'_0)\to(\varphi,\mathfrak{m}_0)$ in $\mathscr{I}$ is empty if $\mathfrak{m}_0\not\subset\mathfrak{m}'_0$, and otherwise consists of the morphisms $\varphi'\to\varphi$ in $\mathsf{Morph}(I)$. The composition law is then the same as that of $\mathsf{Morph}(I)$, so that the projection $(\varphi,\mathfrak{m}_0)\mapsto\varphi$ and the rule : $i\mapsto\mathbf{1}_i$ for every $i\in\mathrm{Ob}(I)$ define functors

$$\mathscr{I}\xrightarrow{\pi}\mathsf{Morph}(I)\xleftarrow{\delta}I.$$

Claim 14.2.35. The functors π and δ are coinitial.

Proof of the claim. It is easily seen that both $\mathscr{I}$ and $\mathsf{Morph}(I)$ are cofiltered, so we can apply the (dual of the) criterion of lemma 1.5.7(i). Condition (a) of the lemma is trivially verified by π. To check condition (b) for π, consider any object $\varphi:j\to i$ of $\mathsf{Morph}(I)$, any object $(\varphi':j'\to i',\mathfrak{m}_0)$ of $\mathscr{I}$, and a pair of morphisms $(\psi,\psi'),(\mu,\mu'):\varphi'\to\varphi$. Since I is cofiltered, we find morphisms $\nu:j''\to j'$ and $\nu':i''\to i'$ in I such that $\psi\circ\nu=\mu\circ\nu$ and $\psi'\circ\nu'=\mu'\circ\nu'$. Then we also find $k\in\mathrm{Ob}(I)$ with morphisms $\lambda:k\to j''$ and $\lambda':k\to i''$ such that $\nu'\circ\lambda'=\varphi'\circ\nu\circ\lambda$. Set $\varphi'':=\nu'\circ\lambda'$; we deduce a morphism $(\nu\circ\lambda,\mathbf{1}_{i'}):\varphi''\to\varphi'$ in $\mathsf{Morph}(I)$ with $(\psi,\psi')\circ(\nu\circ\lambda,\mathbf{1}_{i'})=(\mu,\mu')\circ(\nu\circ\lambda,\mathbf{1}_{i'})$. From (14.2.34) we get $\overline{A}{}^{\triangle}_{\varphi''}=\overline{A}{}^{\triangle}_{\varphi'}$. By assumption we have $\mathfrak{m}_0\cdot\overline{A}_{\varphi'}\subset\overline{A}{}^{\triangle}_{\varphi'}$, and clearly $\overline{A}_{\varphi''}\subset\overline{A}_{\varphi'}$. Hence $\mathfrak{m}_0\cdot\overline{A}_{\varphi''}\subset\overline{A}{}^{\triangle}_{\varphi''}$, so $(\varphi'',\mathfrak{m}_0)\in\mathrm{Ob}(\mathscr{I})$, and $(\nu\circ\lambda,\mathbf{1}_{i'})$ is a morphism $(\varphi'',\mathfrak{m}_0)\to(\varphi',\mathfrak{m}_0)$ in $\mathscr{I}$ such that $(\psi,\psi')\circ\pi(\nu\circ\lambda,\mathbf{1}_{i'})=(\mu,\mu')\circ\pi(\nu\circ\lambda,\mathbf{1}_{i'})$, as required. The verification of conditions (a) and (b) for the functor δ is easy, and shall be left to the reader. ◊

We consider then the induced functor :

$$\overline{A}_\bullet\circ\pi:\mathscr{I}\to A\text{-}\mathbf{Mod}\qquad(j\xrightarrow{\varphi}i,\mathfrak{m}_0)\mapsto\overline{A}_\varphi.$$

Notice that $\mathrm{Im}\,\tau^A_i\subset\overline{A}_\varphi$ for every morphism $\varphi:j\to i$ of I; we deduce a cone

$$\overline{\tau}{}^A_\bullet:c_A\Rightarrow\overline{A}_\bullet\qquad(j\xrightarrow{\varphi}i)\mapsto(\tau^A_i:A\to\overline{A}_\varphi)$$

such that $\overline{\tau}{}^A_\bullet*\delta=\tau^A_\bullet$. In light of claim 14.2.35, it follows that $\overline{\tau}{}^A_\bullet$ is a universal cone, and by the same token, the same holds for the cone $\overline{\tau}{}^A_\bullet*\pi:c_A\Rightarrow\overline{A}_\bullet\circ\pi$.

Claim 14.2.36. $\overline{A}_\bullet\circ\pi$ is a Cauchy functor.

Proof of the claim. By proposition 14.2.18, the functor $A^{\triangle}_\bullet:I\to A\text{-}\mathbf{Mod}$ is Cauchy and $A_\bullet$ has the almost Mittag-Leffler property, so $A^{\triangle}_\varphi:A^{\triangle}_j\to A^{\triangle}_i$ is an epimorphism for every morphism $\varphi:j\to i$ in I. Hence, for every subideal $\mathfrak{m}_0\subset\mathfrak{m}$ of finite type there exists $i\in\mathrm{Ob}(I)$ with $\mathfrak{m}_0\cdot\mathrm{Ker}\,A^{\triangle}_\varphi=0$ for every such φ, and moreover there exists $\varphi:j\to i$ with $\mathfrak{m}_0\overline{A}_\varphi\subset A^{\triangle}_i$. For such φ, we have $(\varphi,\mathfrak{m}_0)\in\mathrm{Ob}(\mathscr{I})$; now, let $(\psi,\psi'):(\varphi':j'\to i',\mathfrak{m}'_0)\to(\varphi,\mathfrak{m}_0)$ be any morphism in $\mathscr{I}$; this means that $\mathfrak{m}_0\subset\mathfrak{m}'_0$ and $\mathfrak{m}'_0\overline{A}_{\varphi'}\subset\overline{A}{}^{\triangle}_{\varphi'}=A^{\triangle}_{i'}$. Also, $\overline{A}_{(\psi,\psi')}:\overline{A}_{\varphi'}\to\overline{A}_\varphi$ is the restriction of $A_{\psi'}:A_{i'}\to A_i$. Therefore

$$\mathfrak{m}_0\cdot\mathrm{Ker}\,\overline{A}_{(\psi,\psi')}\subset A^{\triangle}_{i'}\cap\mathrm{Ker}\,A_{\psi'}\subset\mathrm{Ker}\,A^{\triangle}_{\psi'}$$

whence $\mathfrak{m}_0^2\cdot\mathrm{Ker}\,\overline{A}_{(\psi,\psi')}=0$. Lastly, $\mathrm{Coker}\,\overline{A}_{(\psi,\psi')}$ is a quotient of $\overline{A}_\varphi/A_{\psi'}(A^{\triangle}_{i'})=\overline{A}_\varphi/A^{\triangle}_i$, which is annihilated by $\mathfrak{m}_0$, so $\mathfrak{m}_0\cdot\mathrm{Coker}\,\overline{A}_{(\psi,\psi')}=0$. ◊

Since the morphisms g_φ are isomorphisms for every morphism φ of I, a simple inspection shows that $\overline{g}_{(\psi,\psi')}$ is an isomorphism for every morphism (ψ,ψ') of $\mathsf{Morph}(I)$. Together with claims 14.2.36, and 14.2.33, we deduce that assertion (iii) holds for the functors $\overline{A}_\bullet\circ\pi$ and $\overline{M}_\bullet\circ\pi$, and moreover $\overline{M}_\bullet\circ\pi$ is a Cauchy functor.

However, let $\lambda:I\to\mathscr{I}$ be the functor given by the rules : $i\mapsto(\mathbf{1}_i,0)$ and $\varphi\mapsto(\varphi,\varphi)$ for every $i\in\mathrm{Ob}(I)$ and every morphism φ of I. It is easily seen that $\overline{A}_\bullet\circ\pi\circ\lambda=A_\bullet$ and $\overline{M}_\bullet\circ\pi\circ\lambda=M_\bullet$. Then assertion (iii) for $A_\bullet$ and $M_\bullet$ follows straightforwardly. Especially, the functor $M_\bullet$ is isomorphic to the functor $A_\bullet\otimes_A M$. Since the functor $-\otimes_A M:A\text{-}\mathbf{Mod}\to$

A-$\mathbf{Mod}$ is V-linear, proposition 14.2.20(iii) implies then that $M_\bullet$ is almost essentially constant, and the proof of (iii) is completed, under the assumptions of (ii).

Lastly, suppose that the assumptions of (i) hold; hence g_φ is an epimorphism for every morphism $\varphi : j \to i$ in I, and it follows easily that the induced morphism $A_i \otimes_{A_j} \mathrm{Im}\, M_\varphi \to M_i$ is an epimorphism as well, for every such φ. On the other hand, by assumption for every $i \in \mathrm{Ob}(I)$ and every subideal $\mathfrak{m}_0 \subset \mathfrak{m}$ of finite type there exists $\varphi : j \to i$ such that $\mathfrak{m}_0 \mathrm{Im}\, M_\varphi \subset M_i^\triangle$. Hence, $\mathfrak{m}_0 M_i \subset \mathrm{Im}\, \alpha_i$. Since $\mathfrak{m}_0$ is arbitrary, we conclude that α_i is an epimorphism.

(i): By assumption, for every $i \in \mathrm{Ob}(I)$ and every subideal $\mathfrak{m}_0 \subset \mathfrak{m}$ there exists a morphism $\varphi : j \to i$ in I such that for every morphism $\psi : k \to j$ in I we have $\mathfrak{m}_0 \cdot \mathrm{Im}\, A_{\varphi\circ\psi} \subset \mathrm{Im}\, A_\varphi$. It then follows that for every such $\mathfrak{m}_0$, φ and ψ we have

$$\mathfrak{m}_0 \cdot \mathrm{Im}(A_\varphi \otimes_{A_k} M_k) \subset \mathrm{Im}(A_{\varphi\circ\psi} \otimes_{A_k} M_k).$$

Since g_ψ is an epimorphism and $g_{\varphi\circ\psi} \circ (A_{\varphi\circ\psi} \otimes_{A_k} M_k) = M_{\varphi\circ\psi}$, we deduce :

$$\mathfrak{m}_0 \cdot \mathrm{Im}\, M_\varphi = \mathfrak{m}_0 \cdot \mathrm{Im}(M_\varphi \circ g_\psi) = \mathfrak{m}_0 \cdot \mathrm{Im}(g_{\varphi\circ\psi} \circ (A_\varphi \otimes_{A_k} M_k)) \subset \mathrm{Im}\, M_{\varphi\circ\psi}$$

which shows that $M_\bullet$ has the almost Mittag-Leffler property, as stated.

(iv): Notice that $\tau_i^M : M \to M_i$ factors through a morphism $\mu_i : M \to M_i^\triangle$ and the inclusion $M_i^\triangle \to M_i$, for every $i \in \mathrm{Ob}(I)$. Since $M_\bullet / M_\bullet^\triangle$ is almost essentially zero (proposition 14.2.18(i.b)), it follows that $\mu_\bullet : c_M \Rightarrow M_\bullet^\triangle$ is a universal cone. If $\lambda : \mathbb{N}^o \to I$ is a coinitial functor, the cone $\mu_\bullet * \lambda$ is still universal (remark 1.5.5(ii,iii)), and then $\mu_{\lambda(n)} : M \to M_{\lambda(n)}^\triangle$ is an epimorphism for every $n \in \mathbb{N}$ (lemma 14.2.24(i)). Since $M_\varphi^\triangle$ is an epimorphism for every $i \in \mathrm{Ob}(I)$ (proposition 14.2.18(i.a)), we conclude that $\mu_i : M \to M_i^\triangle$ is an epimorphism for every $i \in \mathrm{Ob}(I)$. Since we already know that α_i is an epimorphism, it follows that the same holds for β_i, for every such i. □

14.2.37. Keep the notation of (14.2.31), and consider a second functor

$$I \to (V, \mathfrak{m})^a\text{-}\mathsf{Alg.Mod} \qquad i \mapsto (A'_i, N_i).$$

as well as the corresponding induced functors

$$A'_\bullet : I \to (V, \mathfrak{m})^a\text{-}\mathsf{Alg} \qquad N_\bullet : I \to (V, \mathfrak{m})^a\text{-}\mathbf{Mod}$$

and suppose that $A_\bullet = A'_\bullet$. Let also N be the limit of $N_\bullet$. Then we get an induced functor

$$I \to (V, \mathfrak{m})^a\text{-}\mathsf{Alg.Mod} \qquad i \mapsto (A_i, M_i \otimes_{A_i} N_i)$$

and the corresponding induced functor

$$M_\bullet \otimes_{A_\bullet} N_\bullet : I \to (V, \mathfrak{m})^a\text{-}\mathbf{Mod} \qquad i \mapsto M_i \otimes_{A_i} N_i.$$

Proposition 14.2.38. *In the situation of* (14.2.37)*, suppose that $A_\bullet$, $M_\bullet$ and $N_\bullet$ are almost essentially constant functors with values in A-$\mathbf{Mod}$. Then the same holds for $M_\bullet \otimes_{A_\bullet} N_\bullet$, and $M \otimes_A N$ represents the limit of $M_\bullet \otimes_{A_\bullet} N_\bullet$.*

Proof. For every $i \in \mathrm{Ob}(I)$, the V^a-linear map

$$\varphi_i : M_i \otimes_{V^a} A_i \otimes_{V^a} N_i \to M_i \otimes_{V^a} N_i \qquad x \otimes a \otimes y \mapsto ax \otimes y - x \otimes ay$$

yields a natural identification : $\mathrm{Coker}\, \varphi_i \xrightarrow{\sim} M_i \otimes_{A_i} N_i$. In view of proposition 14.2.30(iv), we are then easily reduced to showing that the two functors :

$$I \to (V, \mathfrak{m})^a\text{-}\mathbf{Mod} \qquad i \mapsto M_i \otimes_{V^a} A_i \otimes_{V^a} N_i \qquad i \mapsto M_i \otimes_{V^a} N_i$$

are both almost essentially constant, and their limits are represented by $M \otimes_{V^a} A \otimes_{V^a} N$ and respectively $M \otimes_{V^a} N$. Moreover, if both $P_\bullet := M_\bullet \otimes_{V^a} A_\bullet$ and $P_\bullet \otimes_{V^a} N_\bullet$ are almost essentially constant with limits $P := M \otimes_{V^a} A$ and respectively $P \otimes_{V^a} N$, then $M_\bullet \otimes_{V^a} A_\bullet \otimes_{V^a} N_\bullet$ will be almost essentially constant with the sought limit. Hence, we are further reduced to showing the proposition in case $A_\bullet$ is the constant functor with $A_i = V^a$ for every $i \in \mathrm{Ob}(I)$. Now, let $\tau_\bullet : c_M \Rightarrow M_\bullet$ be the universal cone. We have right exact sequences :

$$K_i := (\mathrm{Ker}\,\tau_i) \otimes_{V^a} N_i \to \mathrm{Ker}(\tau_i \otimes_{V^a} N_i) \to T_i \to 0 \qquad \text{for every } i \in \mathrm{Ob}(I)$$

where T_i is the cokernel of the natural morphism $\mathrm{Tor}_1^{V^a}(M_i, N_i) \to \mathrm{Tor}_1^{V^a}(\mathrm{Coker}\,\tau_i, N_i)$. Since, by assumption, both $\mathrm{Ker}\,\tau_\bullet$ and $\mathrm{Coker}\,\tau_\bullet$ are almost essentially zero, it is easily seen that the same holds for the induced functors $K_\bullet$ and $T_\bullet$, and then the same follows for the functor $\mathrm{Ker}(\tau_\bullet \otimes_{V^a} N_\bullet)$, by lemma 14.2.14(i). Likewise, $\mathrm{Coker}(\tau_\bullet \otimes_{V^a} N_\bullet)$ is the functor $(\mathrm{Coker}\,\tau_\bullet) \otimes_{V^a} N_\bullet$, which is again almost essentially zero. Thus, $M_\bullet \otimes_{V^a} N_\bullet$ is isomorphic to $M \otimes_{V^a} N_\bullet$ in the category $\mathscr{L}(I, (V, \mathfrak{m})^a\text{-}\mathbf{Mod})$. Lastly, let $F : (V, \mathfrak{m})^a\text{-}\mathbf{Mod} \to (V, \mathfrak{m})^a\text{-}\mathbf{Mod}$ be the functor such that $FX := M \otimes_{V^a} X$ for every V^a-module X; since $N_\bullet$ is almost essentially constant, the same holds for $FN_\bullet = M \otimes_{V^a} N_\bullet$, and its limit is $M \otimes_{V^a} N$, by proposition 14.2.20(i,iii), whence the assertion. □

Theorem 14.2.39. *Let $(V, \mathfrak{m})$ be a basic setup, I a small cofiltered category, and consider an almost essentially constant functor $A_\bullet : I \to (V, \mathfrak{m})^a$-$\mathsf{Alg}$. Denote by A the limit of $A_\bullet$, and let $\tau_\bullet^A : c_A \Rightarrow A_\bullet$ be a universal cone; let also M be an A-module, B an A-algebra, $r \in \mathbb{N}$ and*

$$A_\bullet\text{-}\mathbf{Mod} : I \to \mathbf{Cat} \quad i \mapsto A_i\text{-}\mathbf{Mod} \qquad \textit{and} \qquad A_\bullet\text{-}\mathsf{Alg} : I \to \mathbf{Cat} \quad i \mapsto A_i\text{-}\mathsf{Alg}$$

the pseudo-functors naturally induced by $A_\bullet$. The following holds :

(i) *The cone $\tau_\bullet^A$ induces equivalences of categories*

$$A\text{-}\mathsf{Alg} \xrightarrow{\sim} 2\text{-}\lim_I A_\bullet\text{-}\mathsf{Alg} \qquad A\text{-}\mathbf{Mod} \xrightarrow{\sim} 2\text{-}\lim_I A_\bullet\text{-}\mathbf{Mod}.$$

(ii) *The A-module M is almost finitely generated (resp. almost finitely presented, resp. flat, resp. faithfully flat resp. almost projective) if and only if the same holds for the A_i-module $A_i \otimes_A M$, for every $i \in \mathrm{Ob}(I)$.*

(iii) *Suppose that $(V, \mathfrak{m})$ fulfills condition* $(\mathbf{B})$ *of* [75, §2.1.6]. *Then M is almost projective of almost finite rank (resp. of finite rank $\leq r$) if and only if the same holds for the A_i-module $A_i \otimes_A M$, for every $i \in \mathrm{Ob}(I)$.*

(iv) *The A-algebra B is weakly unramified (resp. unramified, resp. weakly étale, resp. étale) if and only if the same holds for the A_i-algebra $A_i \otimes_A B$, for every $i \in \mathrm{Ob}(I)$.*

Proof. (i): Let $\pi : (V, \mathfrak{m})^a$-$\mathsf{Alg.Mod} \to (V, \mathfrak{m})^a$-$\mathsf{Alg}$ be the natural projection functor. Then $2\text{-}\lim_I A_\bullet\text{-}\mathbf{Mod}$ is the subcategory of $\mathsf{Fun}(I, (V, \mathfrak{m})^a\text{-}\mathsf{Alg.Mod})$ whose objects are the functors

$$(A_\bullet, M_\bullet) : I \to (V, \mathfrak{m})^a\text{-}\mathsf{Alg.Mod} \qquad i \mapsto (A_i, M_i)$$

with $\pi \circ (A_\bullet, M_\bullet) = A_\bullet$, and such that, with the notation of (14.2.31), the morphism $g_\varphi : A_i \otimes_{A_j} M_j \to M_i$ is an isomorphim of A_i-modules, for every morphism $\varphi : j \to i$ in I. The morphisms $\beta_\bullet : (A_\bullet, M_\bullet) \to (A_\bullet, M'_\bullet)$ are the natural transformations such that $\pi * \beta_\bullet = \mathbf{1}_{A_\bullet}$.

Under this identification, the functor $(A_\bullet, A_\bullet \otimes_A -) : A\text{-}\mathbf{Mod} \to 2\text{-}\lim_I A_\bullet\text{-}\mathbf{Mod}$ of (i) assigns to every A-module M the functor

$$(A_\bullet, A_\bullet \otimes_A M) : I \to (V, \mathfrak{m})^a\text{-}\mathsf{Alg.Mod} \qquad i \mapsto (A_i, A_i \otimes_A M).$$

Conversely, following (14.2.31), with every functor $(A_\bullet, M_\bullet)$ as in the foregoing, we associate a functor $M_\bullet : I \to A\text{-}\mathbf{Mod}$, and let M be the limit of $M_\bullet$. From proposition 14.2.32(iii) we obtain a natural isomorphism

$$(A_\bullet, A_\bullet \otimes_A M) \xrightarrow{\sim} (A_\bullet, M_\bullet).$$

This shows that $(A_\bullet, A_\bullet \otimes_A -)$ is essentially surjective. Next, let M, N be two A-modules; according to proposition 14.2.20(i), the kernel and cokernel of $\tau_\bullet^A \otimes_A N : c_N \Rightarrow A_\bullet \otimes_A N$

are almost essentially zero, and therefore $\tau_\bullet^A \otimes_A N$ is a universal cone. There follow natural isomorphisms

$$\mathrm{Hom}_A(M,N) \stackrel{\sim}{\to} \lim_I \mathrm{Hom}_A(M, A_\bullet \otimes_A N) \stackrel{\sim}{\to} \lim_I \mathrm{Hom}_{A_i}(A_\bullet \otimes_A M, A_\bullet \otimes_A N)$$

and notice that the latter limit is naturally identified with the set of morphisms $(A_\bullet, A_\bullet \otimes_A M) \to (A_\bullet, A_\bullet \otimes_A N)$ in 2-$\lim_I A_\bullet$-**Mod**. Thus $(A_\bullet, A_\bullet \otimes_A -)$ is an equivalence.

(ii): Suppose first that $M_i := A_i \otimes_A M$ is almost finitely generated for every $i \in \mathrm{Ob}(I)$. Let $N_\bullet : \Lambda \to A$-**Mod** be any functor from a small filtered category Λ; denote by N the colimit of $N_\bullet$, and let $\tau_\bullet^N : N_\bullet \Rightarrow c_N$ be a universal cocone. We have two V-linear functors

$$F : A\text{-}\mathbf{Mod} \to A\text{-}\mathbf{Mod} \qquad P \mapsto \operatorname*{colim}_\Lambda \mathrm{alHom}_A(M, P \otimes_A N_\bullet)$$
$$G : A\text{-}\mathbf{Mod} \to A\text{-}\mathbf{Mod} \qquad P \mapsto \mathrm{alHom}_A(M, P \otimes_A N)$$

and $\tau_\bullet^N$ induces a natural transformation

$$\beta_\bullet : F \Rightarrow G \qquad P \mapsto \operatorname*{colim}_\Lambda \mathrm{alHom}_A(M, P \otimes_A \tau_\bullet^N) : FP \to GP$$

and according to [75, Prop.2.3.16(i)], it suffices to check that β_A is a monomorphism. However, we have a commutative diagram of functors :

$$\begin{array}{ccc} c_{FA} & \xrightarrow{c_{\beta_A}} & c_{GA} \\ {\scriptstyle F*\tau_\bullet^A}\downarrow & & \downarrow{\scriptstyle G*\tau_\bullet^A} \\ F \circ A_\bullet & \xrightarrow{\beta_\bullet * A_\bullet} & G \circ A_\bullet \end{array}$$

and by propositions 14.2.18(ii) and 14.2.20(i), the vertical arrows of the diagram are both isomorphisms in the category $\mathscr{L}(I, A\text{-}\mathbf{Mod})$. We claim that $\beta_{A_i} : FA_i \to GA_i$ is a monomorphism for every $i \in \mathrm{Ob}(I)$. Indeed :

$$FA_i = \operatorname*{colim}_\Lambda \mathrm{alHom}_{A_i}(M_i, A_i \otimes_A N_\bullet) \qquad \text{and} \qquad GA_i = \mathrm{alHom}_{A_i}(M_i, A_i \otimes_A N)$$

and since by assumption the A_i-module M_i is almost finitely generated, the assertion follows from [75, Prop.2.3.16(i)]. We deduce that c_{β_A} is also a monomorphism in $\mathscr{L}(I, A\text{-}\mathbf{Mod})$, *i.e.* β_A is a monomorphism, as required.

One argues likewise, in case M_i is almost finitely presented for every $i \in \mathrm{Ob}(I)$, invoking [75, Prop.2.3.16(ii)] to see that in this case β_{A_i} is an isomorphism for every such i, whence β_A is an isomorphism, so that M is almost finitely presented, again by [75, Prop.2.3.16(ii)].

Next, suppose that M_i is an almost projective A_i-module for every $i \in \mathrm{Ob}(I)$, and let $f : N' \to N$ be an epimorphism of A-modules. We attach to the pair (M, N) the V-linear functor

$$H_{M,N} : A\text{-}\mathbf{Mod} \to A\text{-}\mathbf{Mod} \qquad P \mapsto \mathrm{alHom}_A(P \otimes_A M, P \otimes_A N)$$

and define likewise $H_{M,N'} : A\text{-}\mathbf{Mod} \to A\text{-}\mathbf{Mod}$. The morphism f induces a natural transformation $H_{M,f} : H_{M,N} \Rrightarrow H_{M,N'}$, whence a commutative diagram of functors :

$$\begin{array}{ccc} c_{H_{M,N}(A)} & \xrightarrow{c_{H_{M,f}(A)}} & c_{H_{M,N'}(A)} \\ {\scriptstyle H_{M,N}*\tau_\bullet^A}\downarrow & & \downarrow{\scriptstyle H_{M,N'}*\tau_\bullet^A} \\ H_{M,N} \circ A_\bullet & \xrightarrow{H_{M,f}*A_\bullet} & H_{M,N'} \circ A_\bullet \end{array}$$

whose vertical arrows are isomorphisms in $\mathscr{L}(I, A\text{-}\mathbf{Mod})$, by virtue of propositions 14.2.18(ii) and 14.2.20(i); moreover, $H_{M,N'} * \tau_i^A$ is an epimorphism for every $i \in \mathrm{Ob}(I)$, since M_i is an almost projective A_i-module. As in the previous case, we deduce that $H_{M,f}(A)$ is an epimorphism of A-modules, so M is almost projective.

Suppose next that M_i is flat for every $i \in \mathrm{Ob}(I)$, and let $f : N \to N'$ be a monomorphism of A-modules. Set $N_i := A_i \otimes_A N$ and $N'_i := A_i \otimes_A N'$ for every $i \in \mathrm{Ob}(I)$, and let $\overline{N}_i := \mathrm{Im}(A_i \otimes_A f) \subset N'_i$. Clearly the rule : $i \mapsto \overline{N}_i$ defines a subfunctor $\overline{N}_\bullet$ of $A_\bullet \otimes_A N'$. By proposition 14.2.20(i), the kernel of $A_\bullet \otimes_A f : A_\bullet \otimes_A N \Rightarrow A_\bullet \otimes_A N'$ is almost essentially zero, hence the induced natural transformation $\pi_\bullet : A_\bullet \otimes_A N \to \overline{N}_\bullet$ is an isomorphism in $\mathscr{L}(I, A\text{-}\mathbf{Mod})$. By the same token, the same holds for the natural transformation $\tau^A_\bullet \otimes_A N : c_N \Rightarrow A_\bullet \otimes_A N$, and therefore also for the composition $\mu_\bullet := \pi_\bullet \odot (\tau^A_\bullet \otimes_A N) : c_N \Rightarrow \overline{N}_\bullet$. We obtain therefore a commutative diagram of functors :

$$\begin{array}{ccc} c_{M\otimes_A N} & \xrightarrow{c_{M\otimes_A f}} & c_{M\otimes_A N'} \\ {\scriptstyle M\otimes_A \mu_\bullet}\downarrow & & \downarrow{\scriptstyle M\otimes_A \tau^A_\bullet \otimes_A N} \\ M \otimes_A \overline{N}_\bullet & \longrightarrow & M \otimes_A A_i \otimes_A N' \end{array}$$

whose vertical arrows are, as usual, isomorphisms in $\mathscr{L}(I, A\text{-}\mathbf{Mod})$. Let $j_i : \overline{N}_i \to N'_i$ be the inclusion; we have natural isomorphisms $M \otimes_A \overline{N}_i \xrightarrow{\sim} M_i \otimes_{A_i} \overline{N}_i$ and $M \otimes_A A_i \otimes_A N' \xrightarrow{\sim} M_i \otimes_{A_i} N'_i$ for every $i \in \mathrm{Ob}(I)$, that identify the bottom horizontal arrow with the morphism $M \otimes_{A_i} j_i$. The latter is a monomorphism for every $i \in \mathrm{Ob}(I)$, since M_i is a flat A_i-module; as usual it follows that $M \otimes_A f$ is a monomorphism as well, so M is flat.

Lastly, suppose that M_i is a faithfully flat A_i-module for every $i \in \mathrm{Ob}(I)$; by the foregoing, we know already that M is a flat A-module. Then, let X be any A-module such that $M \otimes_A X = 0$; it follows that $M_i \otimes_{A_i} (A_i \otimes_A X) = 0$ for every $i \in \mathrm{Ob}(I)$, whence $A_i \otimes_A X = 0$ for every such i. But by the foregoing, X is the limit of the functor $A_\bullet \otimes_A X$, whence $X = 0$; this shows that M is faithfully flat.

(iii): Suppose that M_i is an almost projective A_i-module of almost finite rank, for every $i \in \mathrm{Ob}(I)$. By the foregoing, we know already that M is an almost projective A-module, and it remains to check that it is of almost finite rank. To this aim, define the category $\mathscr{I}$ and the functors $\pi : \mathscr{I} \to \mathsf{Morph}(I)$ and $\overline{A}_\bullet : \mathsf{Morph}(I) \to A\text{-}\mathsf{Alg}$ as in the proof of proposition 14.2.32; recall that π is coinitial, and $\overline{A}_\bullet \circ \pi$ is a Cauchy functor (claims 14.2.35 and 14.2.36). Moreover, the cone $\overline{\tau}^A_\bullet : c_A \Rightarrow \overline{A}_\bullet$ deduced from $\tau^A_\bullet$, is universal, so the same holds for $\overline{\tau}^A_\bullet * \pi$. It follows that $\mathrm{Ker}\,(\overline{\tau}^A_\bullet * \pi)$ and $\mathrm{Coker}\,(\overline{\tau}^A_\bullet * \pi)$ are null functors (remark 14.2.17(ii)). Now, let $\mathfrak{m}_0 \subset \mathfrak{m}$ be a subideal of finite type; by the foregoing there exists $(\varphi : j \to i, \mathfrak{m}_1) \in \mathrm{Ob}(\mathscr{I})$ such that the kernel and cokernel of $\overline{\tau}^A_\varphi : A \to \overline{A}_\varphi := \mathrm{Im}\, A_\varphi$ are both annihilated by $\mathfrak{m}_0$. On the other hand, since M_j has almost finite rank, there exists $n \in \mathbb{N}$ such that $\mathfrak{m}_0 \cdot \Lambda^n_{A_j} M_j = 0$, and then we have as well $\mathfrak{m}_0 \cdot (\overline{A}_\varphi \otimes_{A_j} \Lambda^n_{A_j} M_j) = 0$. But by remark 14.2.10(vi), the kernel of

$$\overline{\tau}^A_\varphi \otimes_A \Lambda^n_A M : \Lambda^n_A M \to \overline{A}_\varphi \otimes_A \Lambda^n_A M \xrightarrow{\sim} \overline{A}_\varphi \otimes_{A_j} \Lambda^n_{A_j} M_j$$

is annihilated by $\mathfrak{m}_0^2$. We conclude that $\mathfrak{m}_0^3 \cdot \Lambda^n_A M = 0$, whence the contention. Lastly, if M_i has rank $\leq r$ for every $i \in \mathrm{Ob}(I)$, the same argument shows that $\mathfrak{m} \cdot \mathfrak{m}_0^2 \cdot \Lambda^r_A M = 0$ for every such $\mathfrak{m}_0$, *i.e.* $\Lambda^r_A M = 0$, so M has rank $\leq r$.

(iv): Set $C := B \otimes_A B$, and let $\mu : C \to B$ be the multiplication law of the A-algebra B; the functor $C_\bullet := C \otimes_A A_\bullet : I \to (V, \mathfrak{m})^a\text{-}\mathsf{Alg}$ is still almost essentially constant, and $C \otimes_A \tau^A_\bullet : c_C \Rightarrow C_\bullet$ is an isomorphism in $\mathscr{L}(I, A\text{-}\mathbf{Mod})$ (proposition 14.2.20(i,iii)). Suppose now that $B_i := B \otimes_A A_i$ is a weakly unramified A_i-algebra, *i.e.* that the morphism $\mu \otimes_A A_i : C_i \to B_i$ is flat for every $i \in \mathrm{Ob}(I)$. By (ii), it follows that μ is flat, *i.e.* B is a weakly unramified A-algebra. One argues likewise in case B_i is an unramified (resp. weakly étale, resp. étale) A_i-algebra for every $i \in \mathrm{Ob}(I)$. □

The following notion of *flatness for almost modules, relative to a given base ring* will find application in section 17.5.

Definition 14.2.40. Let $(V, \mathfrak{m})$ be a basic setup, $f : A \to V$ a ring homomorphism, and M any V-module; we say that M^a is *flat over* A (resp. *faithfully flat over* A) if the functor

$$A\text{-}\mathbf{Mod} \to (V, \mathfrak{m})^a\text{-}\mathbf{Mod} \qquad N \mapsto M^a \otimes_A N := (M \otimes_A N)^a$$

is exact (resp. is exact and faithful).

Lemma 14.2.41. *In the situation of definition* 14.2.40*, let* $g : V \to B$ *be a ring homomorphism, and* N *a* B*-module. The following holds :*

(i) *If the* V^a*-module* B^a *is faithfully flat over* A*, the map* $g \circ f : A \to B$ *is injective.*
(ii) M^a *is faithfully flat over* A *if and only if it is flat over* A *and* $(M/\mathfrak{n}M)^a \neq 0$ *for every maximal ideal* $\mathfrak{n} \subset A$*.*
(iii) *If the* V^a*-module* B^a *is flat (resp. faithfully flat) over* A*, and* N^a *is a flat (resp. faithfully flat)* B^a*-module, then* N^a *is flat (resp. faithfully flat) over* A*.*

Proof. (i): Set $I := \mathrm{Ker}\,(g \circ f)$; by assumption, the inclusion map $I \to A$ induces a monomorphism $(I \otimes_A B)^a \to B^a$ of V^a-modules, *i.e.* the natural morphism $(I \otimes_A B)^a \to (IB)^a = 0$ is an isomorphism of V^a-modules. By assumption, the latter implies that $I = 0$.

Assertions (ii) and (iii) shall be left to the reader. □

14.2.42. In the situation of definition 14.2.40, suppose that A is *noetherian*, let $I \subset A$ be a given ideal, and set $A_n := A/I^n$, $V_n := V/I^nV$, for every $n \in \mathbb{N}$. Consider an inverse system $M_\bullet := (M_n, \varphi_n \mid n \in \mathbb{N})$, where M_n is a V_n-module such that M_n^a is flat over A_n, and $\varphi_n : M_{n+1} \to M_n$ is a V-linear map, such that the induced inverse system of V^a-modules $M_\bullet^a := (M_n^a, \varphi_n^a \mid n \in \mathbb{N})$ has the almost Mittag-Leffler property (see definition 14.2.11(iv)). Let also M be the limit of $M_\bullet$ (in the category V-$\mathbf{Mod}$). The following result refines [107, Tag 0912].

Proposition 14.2.43. *With the notation of* (14.2.42)*, the following holds :*

(i) M^a *is flat over* A*.*
(ii) *Every* A*-module* N *of finite type induces an isomorphism of* V^a*-modules*

$$(N \otimes_A M)^a \xrightarrow{\sim} \lim_{n\in\mathbb{N}} (N \otimes_A M_n)^a.$$

Proof. (ii): Let $(F_\bullet, d_\bullet) \to N$ be a resolution of N by free A-modules of finite rank (hence the differential $d_i : F_i \to F_{-1}$ is an A-linear map, for every $i > 0$, and $d_0 : F_0 \to N$ is an A-linear surjection). There follows, for every $n \in \mathbb{N}$, exact sequences of A-modules :

$$0 \to \mathrm{Tor}_1^A(N, M_n) \to \mathrm{Coker}(d_2) \otimes_A M_n \xrightarrow{d_{1,n}} F_0 \otimes_A M_n \to N \otimes_A M_n \to 0$$
$$0 \to \mathrm{Im}(d_2 \otimes_A M_n) \to F_1 \otimes_A M_n \to \mathrm{Coker}(d_2) \otimes_A M_n \to 0$$

where $d_{1,n}$ is induced by $d_1 \otimes_A M_n$, for every such n. Since M_n^a is flat over A_n, we have a natural isomorphism of V^a-modules :

$$\mathrm{Tor}_1^A(N, M_n)^a \xrightarrow{\sim} (\mathrm{Tor}_1^A(N, A_n) \otimes_A M_n)^a \qquad \text{for every } n \in \mathbb{N}.$$

Claim 14.2.44. (i) The inverse system $(\mathrm{Tor}_1^A(N, A_n) \mid n \in \mathbb{N})$ is essentially zero.

(ii) Consider an exact sequence of inverse systems of V^a-modules :

$$K_{\bullet\bullet} \quad : \quad (K_{3,n} \mid n \in \mathbb{N}) \xrightarrow{\varphi_{3,\bullet}} (K_{2,n} \mid n \in \mathbb{N}) \xrightarrow{\varphi_{2,\bullet}} (K_{1,n} \mid n \in \mathbb{N}) \xrightarrow{\varphi_{1,\bullet}} (K_{0,n} \mid n \in \mathbb{N}) \to 0$$

such that $K_{3,\bullet}$ is almost essentially zero, and $K_{2,\bullet}$ has the almost Mittag-Leffler property. Then $K_{\bullet\bullet}$ induces a short exact sequence of V^a-modules :

$$0 \to \lim_{n\in\mathbb{N}} K_{2,n} \to \lim_{n\in\mathbb{N}} K_{1,n} \to \lim_{n\in\mathbb{N}} K_{0,n} \to 0.$$

Proof of the claim. (i): Let $K := \mathrm{Ker}(d_0 : F_0 \to N)$; we have a natural identification :

$$\mathrm{Tor}_1^A(N, A_n) \xrightarrow{\sim} (K \cap I^n F)/(I^n K) \qquad \text{for every } n \in \mathbb{N}$$

and by the Artin-Rees lemma there exists $c \in \mathbb{N}$ such that $K \cap I^{n+c}F_0 \subset I^n K$ for every $n \in \mathbb{N}$, whence the assertion.

(ii): Set $J_n := \mathrm{Im}\,\varphi_{2,n}$ for every $n \in \mathbb{N}$; we get induced exact sequences of inverse systems :

$$K_{3,\bullet} \to K_{2,\bullet} \to J_\bullet \to 0 \qquad 0 \to J_\bullet \to K_{2,\bullet} \to K_{1,\bullet} \to 0$$

whence, by lemma 14.2.14(iii), an isomorphism of V^a-modules :

$$\lim_{n\in\mathbb{N}} K_{2,n} \xrightarrow{\sim} \lim_{n\in\mathbb{N}} J_n$$

and notice that $J_\bullet$ has the almost Mittag-Leffler property, as the same holds for $K_{2,\bullet}$ (proposition 14.2.30(ii)); then, proposition 14.2.25(iii) yields a short exact sequence of V^a-modules :

$$0 \to \lim_{n\in\mathbb{N}} J_n \to \lim_{n\in\mathbb{N}} K_{1,n} \to \lim_{n\in\mathbb{N}} K_{0,n} \to 0$$

whence the assertion. $\diamondsuit$

From claim 14.2.44(i) we deduce easily that the inverse system $(\mathrm{Tor}_1^A(N, M_n)^a \mid n \in \mathbb{N})$ is also essentially zero. On the other hand, since $M_\bullet^a$ has the almost Mittag-Leffler property, the same holds for the inverse system of V^a-modules $(\mathrm{Coker}\, d_2 \otimes_A M_n)^a$; Likewise, with proposition 14.2.30(ii)) one checks that the inverse system $(\mathrm{Im}(d_2 \otimes_A M_n)^a \mid n \in \mathbb{N})$ has the almost Mittag-Leffler property; then, claim 14.2.44(ii) yields short exact sequences of V^a-modules :

$$0 \to \lim_{n\in\mathbb{N}} (\mathrm{Coker}\, d_2 \otimes_A M_n)^a \to \lim_{n\in\mathbb{N}} (F_0 \otimes_A M_n)^a \to \lim_{n\in\mathbb{N}} (N \otimes_A M_n)^a \to 0$$

$$0 \to \lim_{n\in\mathbb{N}} \mathrm{Im}(d_2 \otimes_A M_n)^a \to \lim_{n\in\mathbb{N}} (F_1 \otimes_A M_n)^a \to \lim_{n\in\mathbb{N}} (\mathrm{Coker}\, d_2 \otimes_A M_n)^a \to 0.$$

Lastly, since F_i is a free A-module of finite type, clearly we have a natural identification

$$\lim_{n\in\mathbb{N}} (F_i \otimes_A M_n)^a \xrightarrow{\sim} (F_i \otimes_A M)^a \qquad \text{for every } i \in \mathbb{N}.$$

Summing up, we get a right exact sequence of V^a-modules

$$(F_1 \otimes_A M)^a \xrightarrow{(d_1 \otimes_A M)^a} (F_0 \otimes_A M)^a \to \lim_{n\in\mathbb{N}} (N \otimes_A M_n)^a \to 0$$

whence the contention.

(i): We need to check that every A-linear injection $f : N \to N'$ of A-modules of finite type induces a monomorphism $(f \otimes_A M)^a$. However, for every $n \in \mathbb{N}$ we have an exact sequence :

$$\mathrm{Tor}_1^A(\mathrm{Coker}\, f, M_n) \to N \otimes_A M_n \xrightarrow{f \otimes_A M_n} N' \otimes_A M_n \to \mathrm{Coker}\, f_n \otimes_A M_n \to 0$$

and arguing as in the proof of (i) we see that the inverse system $(\mathrm{Tor}_1^A(\mathrm{Coker}\, f, M_n)^a \mid n \in \mathbb{N})$ is essentially zero, and the inverse system $(N \otimes_A M_n \mid n \in \mathbb{N})$ has the almost Mittag-Leffler property, therefore claim 14.2.44(ii) implies that the inverse system $((f \otimes_A M_n)^a \mid n \in \mathbb{N})$ induces a monomorphism $\lim_{n\in\mathbb{N}}(N \otimes_A M_n)^a \to \lim_{n\in\mathbb{N}}(N' \otimes_A M_n)^a$. By (i), the latter is naturally identified with $(f \otimes_A M)^a$, whence the asssertion. $\square$

For future reference, let us also point out the following almost analogue of proposition 8.2.13(i,v) :

Lemma 14.2.45. *Let $(V, \mathfrak{m})$ be a basic setup, $f : M' \to M$ a morphism of V^a-modules, and $\mathscr{T}$ a linear topology on M. Endow M' (resp. $M'' := \mathrm{Coker}\, f$) with the linear topology $\mathscr{T}'$ (resp. $\mathscr{T}''$) induced by $\mathscr{T}$ via f (resp. via the projection $\pi : M \to M''$). We have :*

(i) *f induces a short exact sequence of the respective separated completions :*

$$0 \to (M', \mathscr{T}')^\wedge \xrightarrow{f^\wedge} (M, \mathscr{T})^\wedge \xrightarrow{\pi^\wedge} (M'', \mathscr{T}'')^\wedge.$$

(ii) *If $\mathscr{T}$ has a countable fundamental system of open submodules, $\pi^\wedge$ is an epimorphism.*

Proof. (i): See [75, §5.3] for generalities concerning linear topologies on V^a-modules. By definition, $\mathscr{T}$ is determined by a cofiltered family $(M_\lambda \mid \lambda \in \Lambda)$ of V^a-submodules of M, and $\mathscr{T}'$ and $\mathscr{T}''$ are likewise given by the families $(M'_\lambda := f^{-1}M_\lambda \mid \lambda \in \Lambda)$ and $(M''_\lambda := \pi(M_\lambda)) \mid \lambda \in \Lambda)$. We need to check that f induces an exact sequence of V^a-modules :

$$0 \to \lim_{\lambda\in\Lambda} M'/M'_\lambda \to \lim_{\lambda\in\Lambda} M/M_\lambda \to \lim_{\lambda\in\Lambda} M''/M''_\lambda.$$

To this aim, it suffices to show that the induced sequence of V-modules :

$$0 \to \lim_{\lambda\in\Lambda}(M'/M'_\lambda)_* \to \lim_{\lambda\in\Lambda}(M/M_\lambda)_* \to \lim_{\lambda\in\Lambda}(M''/M''_\lambda)_*$$

is exact. However, f induces a short exact sequence of V^a-modules

$$0 \to M'/M'_\lambda \to M/M_\lambda \to M''/M''_\lambda \to 0 \qquad \text{for every } \lambda \in \Lambda$$

whence a corresponding short exact sequence of V-modules

$$0 \to (M'/M'_\lambda)_* \to (M/M_\lambda)_* \to (M''/M''_\lambda)_* \qquad \text{for every } \lambda \in \Lambda.$$

The assertion follows immediately.

(ii): Under the stated assumptions, we may suppose that $\Lambda = \mathbb{N}^o$; then the assertion follows from proposition 14.2.25(iii). □

We conclude with two homological applications to usual rings : the techiques developed in this section are employed in the proof of the following propositions, but only trivial almost structures intervene.

Proposition 14.2.46. *Let $g : A \to B$ be a ring homomorphism, $I \subset A$ an ideal generated by a sequence $\mathbf{f} := (f_1, \dots, f_r)$, and $g(\mathbf{f}) := (g(f_1), \dots, g(f_r))$. Endow A and B with their I-adic topologies, and suppose that g is adically flat, and that B is complete and separated. We have :*

(i) *If A satisfies condition $(\mathrm{c})_{\mathbf{f}}$, the ring B satisfies $(\mathrm{c})_{g(\mathbf{f})}$ (notation of (7.8.21)).*

(ii) *Suppose additionally that also A is complete and separated, that g is adically faithfully flat, and that B satisfies $(\mathrm{c})_{g(\mathbf{f})}$; then A satisfies $(\mathrm{c})_{\mathbf{f}}$.*

Proof. (i): By assumption, the inverse system $(H_i(\mathbf{f}, I^n) \mid n \in \mathbb{N})$ is essentially zero for every $i > 0$, and we need to check that the same holds for the inverse system $(H_i(g(\mathbf{f}), I^nB) \mid n \in \mathbb{N})$. To this aim, we notice :

Claim 14.2.47. Let R be a ring, $J \subset R$ an ideal generated by a finite sequence $\mathbf{g}$, and consider the following conditions :

(a) R satisfies condition $(\mathrm{c})_{\mathbf{g}}$ of (7.8.21).

(b) For every $i, n \in \mathbb{N}$ there exists an integer $n' \geq n$ such that for every integer $m \geq n$ the natural map
$$H_i(\mathbf{g}, J^{n'}/J^{m'}) \to H_i(\mathbf{g}, J^n/J^m)$$
vanishes for some integer $m' \geq \max(m, n')$.

(c) For every $i, n \in \mathbb{N}$ there exists an integer $n' \geq n$ such that the natural map
$$H_i(\mathbf{g}, J^{n'}) \to H_i(\mathbf{g}, J^n/J^m)$$
vanishes for every integer $m \geq n$.

Then (a)⇒(b)⇒(c), and if R is J-adically complete and separated, we have also (c)⇒(a).

Proof of the claim. (a)⇒(b): By assumption, for every $i \in \mathbb{N}$ there exists a map $\varphi_i : \mathbb{N} \to \mathbb{N}$ with $\varphi_i(n) \geq n$, and such that the induced map $H_i(\mathbf{g}, J^{\varphi_i(n)}) \to H_i(\mathbf{g}, J^n)$ vanishes, for every $n \in \mathbb{N}$. Clearly, we may assume that $\varphi_i(n) \geq \varphi_i(m)$ whenever $n \geq m$ Set also $\varphi_{-1}(n) := n$

for every $n \in \mathbb{N}$, and define $\psi_i := \varphi_{i-1} \circ \psi_i$ for every $i \in \mathbb{N}$. For every $i, n, m \in \mathbb{N}$ with $m \geq n$ we get a commutative diagram with exact horizontal rows :

$$\begin{array}{ccccccc}
H_i(\mathbf{g}, J^{\psi_i(m)}) & \longrightarrow & H_i(\mathbf{g}, J^{\psi_i(n)}) & \longrightarrow & H_i(\mathbf{g}, J^{\psi_i(n)}/J^{\varphi_i(m)}) & \longrightarrow & H_{i-1}(\mathbf{g}, J^{\psi_i(m)}) \\
\downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_3} \\
H_i(\mathbf{g}, J^{\varphi_i(m)}) & \longrightarrow & H_i(\mathbf{g}, J^{\varphi_i(n)}) & \longrightarrow & H_i(\mathbf{g}, J^{\varphi_i(n)}/J^{\varphi_i(m)}) & \longrightarrow & H_{i-1}(\mathbf{g}, J^{\varphi_i(m)}) \\
\downarrow{\scriptstyle \beta_0} & & \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \beta_2} & & \downarrow{\scriptstyle \beta_3} \\
H_i(\mathbf{g}, J^m) & \longrightarrow & H_i(\mathbf{g}, J^n) & \longrightarrow & H_i(\mathbf{g}, J^n/J^m) & \longrightarrow & H_{i-1}(\mathbf{g}, J^m)
\end{array}$$

and by construction, the composition $\beta_k \circ \alpha_k$ vanishes for $k = 0, 1, 3$. By the snake lemma, it follows that $\beta_2 \circ \alpha_2 = 0$ as well; then $n' := \psi_i(n)$ and $m' := \psi_i(m)$ will do.

(b)⇒(c) is trivial.

(c)⇒(a): Recall that for every $n \in \mathbb{N}$, the natural map $J^n \to \lim_{k \in \mathbb{N}} J^n/J^{n+k}$ is an isomorphism (remark 8.3.3(iii,iv)); then, according to [163, Th.3.5.8], we have a short exact sequence

$$L_{i,n} := \lim_{k \in \mathbb{N}}{}^1 H_{i+1}(\mathbf{g}, J^n/J^{n+k}) \to H_i(\mathbf{g}, J^n B) \xrightarrow{\tau_{i,n}} L'_{i,n} := \lim_{k \in \mathbb{N}} H_i(\mathbf{f}, J^n/J^{n+k}) \to 0$$

for every $i, n \in \mathbb{N}$. On the other hand, for every $i, n, k \in \mathbb{N}$ we have an exact sequence :

$$H_{i+1}(\mathbf{g}, J^n) \xrightarrow{\alpha_{i,n,k}} H_{i+1}(\mathbf{g}, J^n/J^{n+k}) \xrightarrow{\beta_{i,n,k}} H_i(\mathbf{g}, J^{n+k}) \xrightarrow{\gamma_{i,n,k}} H_i(\mathbf{g}, J^n).$$

But notice that (c) implies that the inverse system $H_i(\mathbf{f}, I^{n+k} \mid k \in \mathbb{N})$ has the Mittag-Leffler property, for every $i, n \in \mathbb{N}$; then, the inverse system $(\mathrm{Im}\, \gamma_{i,n,k} \mid k \in \mathbb{N})$ is essentially constant, and consequently $(\mathrm{Ker}\, \gamma_{i,n,k} \mid k \in \mathbb{N})$ has the Mittag-Leffler property (proposition 14.2.30(iii)). The same holds for $(\mathrm{Im}\, \alpha_{i,n,k} \mid k \in \mathbb{N})$ (proposition 14.2.30(ii)), and finally we conclude that $(H_{i+1}(\mathbf{g}, J^n/J^{n+k}) \mid k \in \mathbb{N})$ has the Mittag-Leffler property, for every $i, n \in \mathbb{N}$ (proposition 14.2.30(i)). Hence, $L_{i,n} = 0$ for every $i, n \in \mathbb{N}$ (proposition 14.2.25(iii)), and $\tau_{i,n}$ is an isomorphism. Lastly, (c) implies that for every $i, n \in \mathbb{N}$ there exists an integer $n' \geq n$ such that the induced map $H_i(\mathbf{g}, J^{n'}) \to L'_{i,n}$ vanishes, so the same holds for the natural map $H_i(\mathbf{g}, J^{n'}) \to H_i(\mathbf{g}, J^n)$, whence (a). ◊

We are then reduced to checking that condition (b) of claim 14.2.47 holds for $R := B$, $\mathbf{g} := g(\mathbf{f})$, and $J := IB$. But since g is adically flat for the I-adic topologies, we have a natural identification

$$B \otimes_A H_i(\mathbf{f}, I^n/I^{n+k}) \xrightarrow{\sim} H_i(g(\mathbf{f}), I^n B/I^{n+k} B) \qquad \text{for every } i, n, k \in \mathbb{N}. \tag{14.2.48}$$

On the other hand, by assumption condition (a) of claim 14.2.47 holds for $R := A$, $\mathbf{g} := \mathbf{f}$, and $J := I$, so condition (b) of the claim holds as well for $A, \mathbf{f}$ and I, whence the assertion.

(ii): Again, we have the isomorphisms (14.2.48), and by assumption, condition (b) of claim 14.2.47 holds for $R := B$, $\mathbf{g} := g(\mathbf{f})$, and $J := IB$. Under the current assumptions, it then follows that condition (b) of claim 14.2.47 also holds for $R := A$, $\mathbf{g} := \mathbf{f}$ and $J := I$. Since A is I-adically complete, the assertion follows. □

Proposition 14.2.49. *Let $g : A \to B$ be a ring homomorphism, $\mathbf{f} := (f_1, \dots, f_r)$ a finite sequence of elements of A that generates an ideal $I \subset A$, $n \in \mathbb{N} \setminus \{0\}$, and $\mathbf{f}'$ the image of $\mathbf{f}$ in B. Suppose that B/IB is a flat A/I-algebra, and consider the following conditions :*

- (a) *The natural map $\varphi_i : B \otimes_A H_i(\mathbf{f}, A) \to H_i(\mathbf{f}', B)$ is bijective for $i = 0, \dots, n-1$ and surjective for $i = n$.*
- (b) $\mathrm{Tor}_i^A(B, M) = 0$ *for $i = 1, \dots, n$, every $k \in \mathbb{N} \setminus \{0\}$, and every A/I^k-module M.*
- (c) $\mathrm{Tor}_i^A(B, A/I) = 0$ *for $i = 1, \dots, n$.*
- (d) $\mathrm{Tor}_1^A(B, A/I) = 0$.

(e) *g is adically flat for the I-adic topologies of A and B.*

Then we have :

(i) (a)⇔(b)⇔(c) ⇒(d)⇒(e).

(ii) *If* (a) *holds for all $n \geq 0$, and A verifies $(\mathrm{c})_{\mathbf{f}}$ of* (7.8.21)*, then B verifies $(\mathrm{c})_{\mathbf{f}'}$.*

(iii) *If* (e) *holds and A satisfies $(\mathrm{c})_{\mathbf{f}}$, and B satisfies $(\mathrm{c})_{\mathbf{f}'}$, then* (a) *holds for all $n \geq 0$.*

Proof. (d)⇒(e) holds by by [126, Th.22.3], and trivially (b)⇒(c)⇒(d).

(c)⇒(b): The short exact sequence $0 \to IM \to M \to M/IM \to 0$ induces an exact sequence

$$\mathrm{Tor}_i^A(B, IM) \to \mathrm{Tor}_i^A(B, M) \to \mathrm{Tor}_i^A(B, M/IM) \qquad \text{for every } i \in \mathbb{N}.$$

Clearly IM is annihilated by I^{k-1}, and M/IM is annhilated by I; by a simple induction, we are then reduced to the case where $k = 1$. In this case, we find a set S and an A-linear surjection $h : (A/I)^{(S)} \to M$, whence an exact sequence :

$$T_i := \mathrm{Tor}_i^A(B, (A/I)^{(S)}) \to \mathrm{Tor}_i^A(B, M) \to \mathrm{Tor}_{i-1}^A(B, \mathrm{Ker}\, h) \qquad \text{for every } i > 0$$

and since the functor $\mathrm{Tor}_i^A(B, -)$ commutes with direct limits and finite sums, we have $T_i = 0$ for $i = 1, \dots, n$, due to (a). Hence, we are further reduced to checking that $\mathrm{Tor}_1^A(B, M) = 0$ for every A/I-module M, and we reduced easily first to the case where M is of finite type, say generated by k elements (since $\mathrm{Tor}_1^A(B, -)$ commutes with direct limits), and then to the case where $k = 1$, arguing by induction on k : the details are left to the reader. So, we may assume that $M = A/J$ for some ideal $J \subset A$ with $I \subset J$. In this case, notice that the short exact sequence $0 \to J/I \to A/I \to A/J \to 0$ is still exact after tensoring by B, since B/IB is a flat A/I-algebra; so the induced map $\mathrm{Tor}_1^A(B, A/I) \to \mathrm{Tor}_1^A(B, A/J)$ is surjective, whence the assertion.

(b)⇒(a): We consider the Künneth spectral sequence (see [163, Th.5.6.4])

$$E^2_{pq} := \mathrm{Tor}_p^A(B, H_q(\mathbf{f}, A)) \Rightarrow H_{p+q}(\mathbf{f}', B)$$

and recall that $H_q(\mathbf{f}, A)$ is an A/I-module for every $q \in \mathbb{N}$ (lemma 7.8.2(i)); thus, $E^2_{pq} = 0$ for every $p = 1, \dots, n$, due to (b). It follows easily that $E^\infty_{p,i-p} = 0$ for $p = 1, \dots, i$, and $E^\infty_{0,i} = E^2_{0,i}$, if $i < n$. Hence, φ_i is bijective for $i < n$. Lastly, for $i = n$ we get an exact sequence

$$E^2_{n+1,0} \xrightarrow{\partial^{n+1}_{n+1,0}} E^2_{0,n} = B \otimes_A H_n(\mathbf{f}, A) \to H_n(\mathbf{f}', B) \to 0$$

which shows that φ_n is surjective.

(a)⇒(c): We argue by induction on $n \in \mathbb{N} \setminus \{0\}$. If $n = 1$, notice that $E^\infty_{1,0} = E^2_{1,0} = \mathrm{Tor}_1^A(B, A/I)$, and we have a short exact sequence :

$$E^2_{0,1} = B \otimes_A H_1(\mathbf{f}, A) \to H_1(\mathbf{f}', B) \to E^\infty_{1,0} \to 0.$$

But (a) implies that $E^\infty_{1,0} = 0$, whence (c). Next, let $n > 1$, and suppose that it is already known that $\mathrm{Tor}_i^A(B, A/I) = 0$ for $i = 1, \dots, n-1$. By the foregoing, it follows that $\mathrm{Tor}_i^A(B, M) = 0$ for $i = 1, \dots, n-1$ and every A/I-module M. Hence, $E^2_{pq} = 0$ whenever $p = 1, \dots, n-1$, and therefore $E^\infty_{pq} = 0$ for every such p. In particular, $E^\infty_{0,n-1} = H_{n-1}(\mathbf{f}', B)$, and on the other hand, (a) implies that $H_{n-1}(\mathbf{f}', B) = E^2_{0,n-1}$. It follows that $E^r_{0,n-1} = E^2_{0,n-1}$ for every $r \geq 2$, and especially, the differential $\partial^r_{r,n-r} : E^r_{r,n-r} \to E^2_{0,n-1}$ is the zero map for every such r. But notice that $E^\infty_{n,0} = \mathrm{Ker}(\partial^n_{n,0} : E^2_{n,0} \to E^2_{0,n-1})$, so finally $E^\infty_{n,0} = E^2_{n,0}$. The subquotients of the filtration on the abutment $H_n(\mathbf{f}', B)$ vanish, except possibly for $E^\infty_{0,n}$ and $E^\infty_{n,0}$, so we arrive at the short exact sequence :

$$0 \to E^\infty_{0,n} \to H_n(\mathbf{f}', B) \to E^2_{n,0} \to 0.$$

where $E^\infty_{0,n}$ is a quotient of $E^2_{0,n} = B \otimes_A H_n(\mathbf{f}, A)$. Since φ_n is surjective by assumption, we deduce that $E^2_{n,0} = 0$, *i.e.* $\mathrm{Tor}^A_n(B, A/I) = 0$, as required.

(ii): To ease notation, set $H_i(A) := H_i(\mathbf{f}, A)$ and $H_{i,n}(A) := H_i(\mathbf{f}, A/I^n)$ for every $i, n \in \mathbb{N}$, and define likewise $H_i(B)$ and $H_{i,n}(B)$. The short exact sequences $0 \to I^n \to A \to A/I^n \to 0$ and $0 \to I^nB \to B \to B/I^nB \to 0$ yield a commutative ladder :

$$\begin{array}{ccccccccc} B\otimes_A H_{i+1}(A) & \xrightarrow{\psi_{i+1,n}} & B\otimes_A H_{i+1,n}(A) & \longrightarrow & B\otimes_A H_i(\mathbf{f}, I^n) & \longrightarrow & B\otimes_A H_i(A) & \xrightarrow{\psi_{i,n}} & B\otimes_A H_{i,n}(A) \\ {\scriptstyle\varphi_{i+1}}\downarrow & & {\scriptstyle\beta_{i+1,n}}\downarrow & & \downarrow{\scriptstyle\gamma_{i,n}} & & \downarrow{\scriptstyle\varphi_i} & & \downarrow{\scriptstyle\beta_{i,n}} \\ H_{i+1}(B) & \xrightarrow{\psi'_{i+1,n}} & H_{i+1,n}(B) & \longrightarrow & H_i(\mathbf{f}', I^nB) & \longrightarrow & H_i(B) & \xrightarrow{\psi'_{i,n}} & H_{i,n}(B) \end{array}$$

whose bottom horizontal row is exact, and the same holds for its top horizontal row, since we know already that (a) implies (e), and since $H_i(\mathbf{f}, M)$ is an A/I-module for every A-module M (lemma 7.8.2(ii)). By the same token, $\beta_{i,n}$ is bijective for every $i, n \in \mathbb{N}$, so the same holds for $\gamma_{i,n}$, by the five lemma, whence the assertion.

(iii): Under the assumptions of (iii), the horizontal rows of the commutative ladder in the proof of (ii) are still exact, and both $(B \otimes_A H_i(\mathbf{f}, I^n) \mid n \in \mathbb{N})$ and $(H_i(\mathbf{f}', I^nB) \mid n \in \mathbb{N})$ are essentially zero inverse systems, for every $i > 0$. The same also holds trivially for $i = 0$. In light of lemma 14.2.14(iii), it follows that the systems $(\psi_{i,n} \mid n \in \mathbb{N})$ and $(\psi'_{i,n} \mid n \in \mathbb{N})$ induce isomorphisms :

$$B \otimes_A H_i(A) \xrightarrow{\sim} \lim_{n\in\mathbb{N}} B \otimes_A H_{i,n}(A) \qquad H_i(B) \xrightarrow{\sim} \lim_{n\in\mathbb{N}} H_{i,n}(B).$$

Moreover, $\beta_{i,n}$ is still an isomorphism for every $n \in \mathbb{N}$, whence (a). ☐

Corollary 14.2.50. *Let $(A, \mathfrak{m})$ be a local noetherian ring, $\mathbf{f}$ a system of parameters for A, and $g : B \to B'$ a homomorphism of A-algebras. Endow B and B' with their $\mathfrak{m}$-adic topologies, and let $\mathbf{g}$ (resp. $\mathbf{g}'$) be the image of $\mathbf{f}$ in B (resp. in B'). Suppose that g is adically flat; then :*

(i) *If B satisfies condition $(\mathrm{c})_{\mathbf{g}}$ and B' satisfies condition $(\mathrm{c})_{\mathbf{g}'}$ of* (7.8.21)*, we have :*

$$\mathrm{depth}_A B' \geq \mathrm{depth}_A B.$$

(ii) *If B' is complete and separated, and if* $\mathrm{depth}_A B \geq \dim A$*, then* $\mathrm{depth}_A B' \geq \dim A$.

(iii) *If B and B' are complete and separated, g is adically faithfully flat, and* $\mathrm{depth}_A B' \geq \dim A$*, then* $\mathrm{depth}_A B \geq \dim A$.

Proof. (i) follows directly from propositions 14.2.49(iii) and 10.4.32(i), and lemma 7.8.2(v).

(ii): The assumptions trivially imply that B satisfies condition $(\mathrm{d})_{\mathbf{f}}$ of (7.8.21), so it also satisfies condition $(\mathrm{c})_{\mathbf{f}}$ (proposition 7.8.25(i)). Then B' satisfies condition $(\mathrm{c})_{\mathbf{f}}$, by proposition 14.2.46(i), and we apply (i) to conclude.

(iii): The assumptions trivially imply that B' satisfies condition $(\mathrm{d})_{\mathbf{f}}$ of (7.8.21), so it also satisfies condition $(\mathrm{c})_{\mathbf{f}}$ (proposition 7.8.25(i)). Then B satisfies condition $(\mathrm{c})_{\mathbf{f}}$, by proposition 14.2.46(i). Then, by proposition 14.2.49(iii), we have $B' \otimes_B H_i(\mathbf{g}, B) = 0$ for every $i > 0$; but since g is adically faithfully flat, and since $H_i(\mathbf{f}, B)$ is an A/I-module, we get $H_i(\mathbf{g}, B) = 0$ for every $i > 0$. Then the assertion follows from proposition 10.4.32(i), and lemma 7.8.2(v). ☐

14.3. **Quasi-coherent sheaves of almost modules and almost rings.** Throughout this section, we fix a basic setup $(V, \mathfrak{m})$ (see [75, §2.1.1]) and we set $S := \mathrm{Spec}\, V$; for every S-scheme X we may consider the sheaf $\mathscr{O}^a_X$ of almost algebras on X, and we refer to [75, §5.5] for the definition of quasi-coherent $\mathscr{O}^a_X$-modules and algebras. However, whereas in [75] it was assumed that $\widetilde{\mathfrak{m}} := \mathfrak{m} \otimes_V \mathfrak{m}$ is a flat V-module, in this section the basic setup can be arbitrary, except where it is explicitly said otherwise. This of course requires some care when quoting from [75]; however, it turns out that – thanks to the work done in section 14.1 – most of the results in [75] do extend *verbatim* to the case of a general setup. The main exception is the theory of the finite and almost finite rank of almost projective modules, that relies on the existence of a well behaved exterior power functor, which is available only if the basic setup satisfies some minimal conditions (see (14.1.61)). In any case, whenever we need to import some theorem from [75], we shall comment on its range of validity.

Definition 14.3.1. Let X be an S-scheme, $\mathscr{A}$ a quasi-coherent $\mathscr{O}_X^a$-algebra, and $\mathscr{F}$ an $\mathscr{A}$-module which is quasi-coherent as an $\mathscr{O}_X^a$-module.

(i) $\mathscr{F}$ is said to be an $\mathscr{A}$-module *of almost finite type* (resp. *of almost finite presentation*, resp *flat*, resp. *faithfully flat*) if, for every affine open subset $U \subset X$, the $\mathscr{A}(U)$-module $\mathscr{F}(U)$ is almost finitely generated (resp. almost finitely presented, resp. flat, resp. faithfully flat).

(ii) $\mathscr{F}$ is said to be an *almost coherent* $\mathscr{A}$-module if it is an $\mathscr{A}$-module of almost finite type, and for every open subset $U \subset X$, every quasi-coherent $\mathscr{A}_{|U}$-submodule of $\mathscr{F}_{|U}$ of almost finite type, is almost finitely presented.

(iii) We say that $\mathscr{F}$ is a *torsion-free* $\mathscr{O}_X^a$-module if we have $\operatorname{Ker} b \cdot \mathbf{1}_{\mathscr{F}(U)} = 0$, for every affine open subset $U \subset X$ and every regular element $b \in \mathscr{O}_X(U)$.

(iv) Suppose that $\mathfrak{m}$ satisfies condition $(\mathbf{B})$ of [75, §2.1.6]. Then we say that $\mathscr{F}$ is an $\mathscr{A}$-module *of almost finite rank* (resp. *of finite rank*) if, for every affine open subset $U \subset X$, the $\mathscr{A}(U)$-module $\mathscr{F}(U)$ is almost finitely generated projective of almost finite (resp. finite) rank.

Remark 14.3.2. In the situation of definition 14.3.1 :

(i) We denote by

$$\mathscr{E}nd_{\mathscr{A}}(\mathscr{F})$$

the $\mathscr{O}_X$-module of *$\mathscr{A}$-linear endomorphisms of $\mathscr{F}$*, defined by the rule : $U \mapsto \operatorname{End}_{\mathscr{A}(U)}(\mathscr{F}(U))$ for every open subset $U \subset X$. It is easily seen that $\mathscr{E}nd_{\mathscr{A}}(\mathscr{F})^a$ is an $\mathscr{A}$-module.

(ii) Suppose that $\mathscr{F}$ is a flat and almost finitely presented $\mathscr{A}$-module. Then there exists a *trace morphism*

$$\operatorname{tr}_{\mathscr{F}/\mathscr{A}} : \mathscr{E}nd_{\mathscr{A}}(\mathscr{F}) \to \mathscr{A}$$

that, on every affine open subset $U \subset X$, induces the trace morphism $\operatorname{tr}_{\mathscr{F}(U)\mathscr{A}(U)}$ of the almost finitely generated projective $\mathscr{A}(U)$-module $\mathscr{F}(U)$ (details left to the reader : see [75, §4.1], which does not depend on any assumption on the basic setup).

14.3.3. Let $f : Y \to X$ be any morphism of V-schemes. The usual functor f^* for quasi-coherent $\mathscr{O}_X$-module admits a variant for quasi-coherent $\mathscr{O}_X^a$-modules; namely, we define

$$f^* : \mathscr{O}_X^a\text{-}\mathbf{Mod}_{\mathrm{qcoh}} \to \mathscr{O}_Y^a\text{-}\mathbf{Mod}_{\mathrm{qcoh}} \qquad \mathscr{F} \mapsto (f^*\mathscr{F}_!)^a.$$

If f is also quasi-compact and quasi-separated, we have as well a direct image functor

$$f_* : \mathscr{O}_Y^a\text{-}\mathbf{Mod}_{\mathrm{qcoh}} \to \mathscr{O}_X^a\text{-}\mathbf{Mod}_{\mathrm{qcoh}} \qquad \mathscr{G} \mapsto (f_*\mathscr{G}_!)^a$$

([59, Ch.I, Prop.9.2.1]). Moreover, if $\mathscr{F}$ (resp. $\mathscr{G}$) is any quasi-coherent $\mathscr{O}_X$-module (resp. $\mathscr{O}_Y$-module), it is easily seen that the natural morphism

$$f^*(\mathscr{F}^a) \to (f^*\mathscr{F})^a \qquad \text{resp. } f_*(\mathscr{G}^a) \to (f_*\mathscr{G})^a$$

is an isomorphism. It follows that, if $Z \subset X$ is any constructible closed subset, there is a well defined functor of sections with support in Z :

$$\underline{\Gamma}_Z : \mathscr{O}_X\text{-}\mathbf{Mod}_{\mathrm{qcoh}} \to \mathscr{O}_X\text{-}\mathbf{Mod}_{\mathrm{qcoh}} \qquad \mathscr{F} \mapsto (\underline{\Gamma}_Z\mathscr{F}_!)^a$$

(see (10.4.16)) and again, for every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, the natural map

$$\underline{\Gamma}_Z(\mathscr{F}^a) \to (\underline{\Gamma}_Z\mathscr{F})^a$$

is an isomorphism (details left to the reader).

Lemma 14.3.4. *Suppose that* $\mathfrak{m}$ *fulfills condition* $(\mathbf{B})$*, and let* X *be a reduced* S*-scheme,* $\mathscr{F}$ *a flat quasi-coherent* $\mathscr{O}_X^a$*-module of almost finite type. We have :*

(i) *If X is reduced and $\mathscr{F}$ is an $\mathscr{O}_X^a$-module of almost finite presentation, then $\mathscr{F}$ is an $\mathscr{O}_X^a$-module of almost finite rank.*
(ii) *If X is integral, then $\mathscr{F}$ is an $\mathscr{O}_X^a$-module of finite rank.*

Proof. (i): We may assume that X is affine, say $X = \operatorname{Spec} R$ for a reduced V-algebra R, and $\mathscr{F} = P^\sim$ for a flat and almost finitely presented R^a-module P. Then P is almost projective ([75, Prop.2.4.18(ii)]), and we need to check that it has almost finite rank. Thus, let $\varepsilon \in \mathfrak{m}$ be any element and set $R' := R[\varepsilon^{-1}]$; then $P' := R' \otimes_{R^a} P$ is a projective R'-module of finite rank, so there exists $n \in \mathbb{N}$ such that $R' \otimes_R \Lambda^n_R P = \Lambda^n_{R'} P' = 0$. Since $\Lambda^n_R P$ is also almost finitely presented (see [75, §4.3.1]), we deduce easily that there exists $N \in \mathbb{N}$ such that $\varepsilon^N \cdot \Lambda^n_R P = 0$, and since moreover $\Lambda^n_R P$ is a flat R^a-module and R is reduced, we have

$$\operatorname{Ann}_{\Lambda^n_R P}(\varepsilon^N) = \operatorname{Ann}_R(\varepsilon^N) \cdot \Lambda^n_R P = \operatorname{Ann}_R(\varepsilon) \cdot \Lambda^n_R P$$

i.e. $\varepsilon \cdot \Lambda^n_R P = 0$, whence the contention.

(ii): Let η be the generic point of X; if $\kappa(\eta)^a = 0$, we have $\mathscr{O}_X^a = 0$ as well, and there is nothing to show. If $\kappa(\eta)^a \neq 0$, the almost structure on $\kappa(\eta)$ is the trivial one (the "classical limit" case of [75, Ex.2.1.2(ii)]); in this case, clearly $\mathscr{F}_\eta$ is a free $\kappa(\eta)^a$-module of finite rank, and let r be this rank. From [75, Prop.2.4.19], it follows easily that $\mathscr{F}$ is almost finitely presented. Moreover, the exterior powers of $\mathscr{F}$ are still flat $\mathscr{O}_X^a$-modules; then the r-th exterior power vanishes, since it vanishes at the generic point of X. □

Lemma 14.3.5. *Let $f : Y \to X$ a faithfully flat and quasi-compact morphism of S-schemes, $\mathscr{A}$ a quasi-coherent $\mathscr{O}_X^a$-algebra, $\mathscr{F}$ a quasi-coherent $\mathscr{A}$-module, and $r \in \mathbb{N}$. Then :*

(i) *The $\mathscr{A}$-module $\mathscr{F}$ is of almost finite type (resp. of almost finite presentation, resp. flat, resp. faithfully flat) if and only if the same holds for the $f^*\mathscr{A}$-module $f^*\mathscr{F}$.*
(ii) *Suppose that $\mathfrak{m}$ fulfills condition $(\mathbf{B})$. Then the $\mathscr{A}$-module $\mathscr{F}$ is of almost finite rank (resp. of finite rank $\leq r$) if and only if the same holds for the $f^*\mathscr{A}$-module $f^*\mathscr{F}$.*
(iii) *If the $f^*\mathscr{A}$-module $f^*\mathscr{F}$ is almost coherent, then the same holds for the $\mathscr{A}$-module $\mathscr{F}$.*

Proof. The assertions are local on X, so we may assume that X is affine; then Y admits a finite covering $(U_i \mid i \in I)$ consisting of affine open subsets, and we may further reduce to the case where Y is the disjoint union of the schemes U_i, so Y is affine as well.

(i): For the conditions "almost finite type", "almost finite presentation", and "flat", the assertion follows from [75, Rem.3.2.26(ii)], which holds for any basic setup. It remains to check that if $A \to B$ is a faithfully flat map of V-algebras, and M is any A-module, such that $(B \otimes_A M)^a$ is a faithfully flat B^a-module, then M^a is a faithfully flat A^a-module. To this aim, let X be any A-module such that $(M \otimes_A X)^a = 0$; then $(B \otimes_A X)^a \otimes_{B^a} (B \otimes_A X)^a = 0$, and therefore $(B \otimes_A X)^a = 0$, *i.e.* $\widetilde{\mathfrak{m}} \otimes_V B \otimes_A X = 0$, whence $\widetilde{\mathfrak{m}} \otimes_V X = 0$, *i.e.* $X^a = 0$. Since we know already that M^a is a flat A^a-module, the contention follows.

(ii): It suffices to apply [75, Rem.3.2.26(iii)], and recall that exterior powers commute with any base changes : the details shall be left to the reader.

(iii) follows easily from (i). □

Lemma 14.3.6. *Let X be any S-scheme.*

(i) *The full subcategory $\mathscr{O}_X^a$-$\mathbf{Mod}_{\mathrm{acoh}}$ of $\mathscr{O}_X^a$-$\mathbf{Mod}_{\mathrm{qcoh}}$ consisting of all almost coherent modules is abelian and closed under extensions. (More precisely, the embedding $\mathscr{O}_X^a$-$\mathbf{Mod}_{\mathrm{acoh}} \to \mathscr{O}_X^a$-$\mathbf{Mod}_{\mathrm{qcoh}}$ is an exact functor.)*
(ii) *If X is a coherent scheme, then every quasi-coherent $\mathscr{O}_X^a$-module of almost finite presentation is almost coherent.*

Proof. Both assertions are local on X, hence we may assume that X is affine, say $X = \operatorname{Spec} R$.

(i): Let $f : M \to N$ be a morphism of almost coherent R^a-modules. It is clear that $\operatorname{Coker} f$ is again almost coherent. Moreover, $f(M) \subset N$ is almost finitely generated, hence almost finitely presented; then by [75, Lemma 2.3.18], $\mathrm{Ker}\, f$ is almost finitely generated, and so it is almost coherent as well. Similarly, using [75, Lemma 2.3.18] one sees that $\mathscr{O}^a_X\text{-}\mathbf{Mod}_{\mathrm{acoh}}$ is closed under extensions.

(ii): Let M be an almost finitely presented R^a-module; according to [75, Cor.2.3.13], for every finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$ there exist a finitely presented R-module N and an R-linear map $\varphi : N \to M_*$ whose kernel and cokernel are annihilated by $\mathfrak{m}_0$. Likewise, if $M' \subset M$ is an almost finitely generated R^a-submodule, we may find finitely many almost elements $x_1, \dots, x_n \in M'_*$ that generate an R-module containing $\mathfrak{m}_0 M'_*$. Let $a_1, \dots, a_k$ be a finite system of generators for $\mathfrak{m}_0$; for each $i \leq k$ and $j \leq n$ we may find $y_{ij} \in N$ such that $\varphi(y_{ij}) = a_i x_j$. Denote by N' the R-submodule of N generated by $(y_{ij} \mid i \leq k,\ j \leq n)$. Then φ restricts to an R-linear map $N' \to M'_*$ whose kernel is annihilated by $\mathfrak{m}_0$ and whose cokernel is annihilated by $\mathfrak{m}_0^2$. Since R is coherent, N' is finitely presented, and since $\mathfrak{m}_0$ is arbitrary, we conclude that M' is almost finitely presented, as stated. □

Lemma 14.3.7. *Let X be an S-scheme, $\varphi : \mathscr{A} \to \mathscr{B}$ a morphism of quasi-coherent $\mathscr{O}^a_X$-algebras, and $\mathscr{B}_\bullet := (\mathscr{B}_\lambda \mid \lambda \in \Lambda)$ a filtered system of quasi-coherent and faithfully flat $\mathscr{A}$-algebras. We have :*

(i) *φ is faithfully flat if and only if φ is injective and* $\mathrm{Coker}\,\varphi$ *is a flat $\mathscr{A}$-module.*

(ii) *The colimit of $\mathscr{B}_\bullet$ is a faithfully flat $\mathscr{A}$-algebra.*

Proof. It is easily seen that (i)⇒(ii). In order to show (i), we may assume that X is affine, in which case φ is the morphism of quasi-coherent $\mathscr{O}^a_X$-algebras arising from a morphism $f : A \to B$ of $\mathscr{O}_X(X)^a$-algebras. Now, let M be any A-module; if f is injective, the short exact sequence of A-modules $0 \to A \to B \to \mathrm{Coker}\, f \to 0$ induces a long exact sequence

$$0 \to \mathrm{Tor}_1^A(B, M) \to \mathrm{Tor}_1^A(\mathrm{Coker}\, f, M) \to M \xrightarrow{f \otimes_A M} B \otimes_A M \to \mathrm{Coker}\, f \otimes_A M \to 0.$$

Then, if $\mathrm{Coker}\, f$ is a flat A-module, we deduce that $f \otimes_A M$ is injective and $\mathrm{Tor}_1^A(B, M) = 0$ for every A-module M, so B is a faithfully flat A-algebra. Conversely, if the latter holds, then f and $f \otimes_M A$ are both injective, hence the foregoing short exact sequence shows that $\mathrm{Tor}_1^A(\mathrm{Coker}\, f, M) = 0$ for every A-module M, so that $\mathrm{Coker}\, f$ is a flat A-module. □

Definition 14.3.8. Let X be an S-scheme, and $\mathscr{A}$ a quasi-coherent $\mathscr{O}^a_X$-algebra.

(i) We say that a quasi-coherent $\mathscr{A}$-algebra $\mathscr{B}$ is *almost finite* (resp. *weakly unramified*, resp. *weakly étale*, resp. *unramified*, resp. *étale*) if, for every affine open subset $U \subset X$, the $\mathscr{A}(U)$-algebra $\mathscr{B}(U)$ is almost finite (resp. weakly unramified, resp. weakly étale, resp. unramified, resp. étale) .

(ii) We define the $\mathscr{O}_X$-algebra $\mathscr{A}_{!!}$ by the short exact sequence :

$$\widetilde{\mathfrak{m}} \otimes_V \mathscr{O}_X \to \mathscr{O}_X \oplus \mathscr{A}_! \to \mathscr{A}_{!!} \to 0$$

analogous to [75, §2.2.25]. This agrees with the definition of [75, §3.3.2], corresponding to the basic setup $(\mathscr{O}_X, \mathfrak{m}\mathscr{O}_X)$ relative to the Zariski topos of X; in general, this is *not* the same as forming the algebra $\mathscr{A}_{!!}$ relative to the basic setup $(V, \mathfrak{m})$. The reason why we prefer the foregoing version of $\mathscr{A}_{!!}$, is explained by the following lemma 14.3.9.

Lemma 14.3.9. *Let X be an S-scheme, $\mathscr{A}$ a quasi-coherent $\mathscr{O}^a_X$-algebra. Then $\mathscr{A}_{!!}$ is a quasi-coherent $\mathscr{O}_X$-algebra, and if $\mathscr{A}$ is almost finite, then $\mathscr{A}_{!!}$ is an integral $\mathscr{O}_X$-algebra.*

Proof. First, $\mathscr{A}_!$ is a quasi-coherent $\mathscr{O}_X$-module ([75, §5.5.4]), hence the same holds for $\mathscr{A}_{!!}$. It remains to show that if $\mathscr{A}$ is almost finite, then $\mathscr{A}_{!!}(U) = \mathscr{A}(U)_{!!}$ is an integral $\mathscr{O}_X(U)$-algebra for every affine open subset $U \subset X$. Since $\mathscr{A}(U)_{!!} = \mathscr{O}_X(U) + \mathfrak{m}\mathscr{A}(U)_{!!}$, we need only show that every element of $\mathfrak{m}\mathscr{A}(U)_{!!}$ is integral over $\mathscr{O}_X(U)$. However, by adjunction we get a map :

$$\mathscr{A}(U)_{!!} \to A_U := \mathscr{O}^a_X(U)_* + \mathfrak{m}\mathscr{A}(U)_*$$

whose kernel is annihilated by $\mathfrak{m}$, and according to [75, Lemma 5.1.13(i)], A_U is integral over $\mathscr{O}_X^a(U)_*$. Let $a \in \mathfrak{m}\mathscr{A}(U)_{!!}$ be any element, and $\overline{a}$ its image in $\mathfrak{m}\mathscr{A}(U)_*$; we can then find $b_0, \dots, b_n \in \mathscr{O}_X(U)_*$ such that $\overline{a}^{n+1} + \sum_{i=0}^n b_i \overline{a}^i = 0$ in A_U. It follows that $(\varepsilon a)^{n+1} + \sum_{i=0}^n \varepsilon^{n+1-i} b_i (\varepsilon a)^i = 0$ in $\mathscr{A}(U)_{!!}$, for every $\varepsilon \in \mathfrak{m}$. Since $\varepsilon^{n+1-i} b_i$ lies in the image of $\mathscr{O}_X(U)$ for every $i \leq n$, the claim follows easily. □

Remark 14.3.10. With the notation of definition 14.3.8 we have :

(i) $\mathscr{B}$ is an unramified $\mathscr{A}$-algebra if and only if, for every affine open subset $U \subset X$, there exists an idempotent element $e_{\mathscr{B}(U)/\mathscr{A}(U)} \in (\mathscr{B} \otimes_{\mathscr{A}} \mathscr{B})(U)_*$, uniquely characterized by the following conditions :

(a) $\mu_{\mathscr{B}/\mathscr{A}}(e_{\mathscr{B}(U)/\mathscr{A}(U)}) = 1$, where $\mu_{\mathscr{B}/\mathscr{A}} : \mathscr{B} \otimes_{\mathscr{A}} \mathscr{B} \to \mathscr{B}$ is the multiplication morphism of the $\mathscr{A}$-algebra $\mathscr{B}$.

(b) $e_{\mathscr{B}(U)/\mathscr{A}(U)} \cdot \operatorname{Ker} \mu_{\mathscr{B}/\mathscr{A}}(U) = 0$

([75, Lemma 3.1.4]). It is easily seen that, on $U' \subset U$, the element $e_{\mathscr{B}(U)/\mathscr{A}(U)}$ restricts to $e_{\mathscr{B}(U')/\mathscr{A}(U')}$; hence, the system $(e_{\mathscr{B}(U)/\mathscr{A}(U)} \mid U \subset X)$, for U ranging over all the affine open subsets of X, determines a global section

$$e_{\mathscr{B}/\mathscr{A}} \in \Gamma(X, \mathscr{B} \otimes_{\mathscr{A}} \mathscr{B})_*$$

which we call the *diagonal idempotent* of the $\mathscr{A}$-algebra $\mathscr{B}$ (see [75, §5.5.4]).

(ii) In the situation of (i), let $\mathscr{A} \to \mathscr{A}'$ be any morphism of quasi-coherent $\mathscr{O}_X^a$-algebras, and set $\mathscr{B}' := \mathscr{A}' \otimes_{\mathscr{A}} \mathscr{B}$. Then the induced morphism $\mathscr{A}' \to \mathscr{B}'$ is unramified, and in view of the characterization provided by (i), it is easily seen that $e_{\mathscr{B}'/\mathscr{A}'}$ is the image in $\mathscr{B}' \otimes_{\mathscr{A}'} \mathscr{B}'$ of $e_{\mathscr{B}/\mathscr{A}}$.

(iii) If $\mathscr{B}$ is a flat and almost finitely presented $\mathscr{A}$-algebra, there exists a *trace form*

$$t_{\mathscr{B}/\mathscr{A}} : \mathscr{B} \otimes_{\mathscr{A}} \mathscr{B} \to \mathscr{A}$$

that, on each open affine subset $U \subset X$, induces the trace form $t_{\mathscr{B}(U)/\mathscr{A}(U)}$ of [75, §4.1.12] (details left to the reader).

(iv) Suppose that $\mathscr{B}$ is a flat, unramified and almost finitely presented $\mathscr{A}$-algebra. Then the diagonal idempotent and the trace form of $\mathscr{B}$ are related by the identity

$$t_{\mathscr{B}/\mathscr{A}}(e_{\mathscr{B}/\mathscr{A}} \cdot (1 \otimes b)) = b = t_{\mathscr{B}/\mathscr{A}}(e_{\mathscr{B}/\mathscr{A}} \cdot (b \otimes 1))$$

for every affine open subset $U \subset X$ and every $b \in \mathscr{B}(U)_*$ ([75, Rem.4.1.17]).

Definition 14.3.11. (i) Let X be an S-scheme. For a monomorphism $\mathscr{R} \subset \mathscr{S}$ of quasi-coherent $\mathscr{O}_X$-algebras, the integral closure $\mathrm{i.c.}(\mathscr{R}, \mathscr{S})$ of $\mathscr{R}$ in $\mathscr{S}$ is a quasi-coherent $\mathscr{O}_X$-algebra. For a monomorphism $\mathscr{A} \to \mathscr{B}$ of $\mathscr{O}_X^a$-algebras, the *integral closure of $\mathscr{A}$ in $\mathscr{B}$* is

$$\mathrm{i.c.}(\mathscr{A}, \mathscr{B}) := \mathrm{i.c.}(\mathscr{A}_{!!}, \mathscr{B}_{!!})^a$$

(see also [75, Lemma 8.2.28]). In view of lemma 14.3.9, this is a well defined quasi-coherent $\mathscr{O}_X^a$-algebra. It is characterized as the unique $\mathscr{A}$-subalgebra of $\mathscr{B}$ such that :

$$\mathrm{i.c.}(\mathscr{A}, \mathscr{B})(U) = \mathrm{i.c.}(\mathscr{A}(U), \mathscr{B}(U))$$

for every affine open subset $U \subset X$ (notation of [75, Def.8.2.27]). We say that $\mathscr{A}$ is *integrally closed in $\mathscr{B}$*, if $\mathscr{A} = \mathrm{i.c.}(\mathscr{A}, \mathscr{B})$. We say that $\mathscr{B}$ is an *integral $\mathscr{A}$-algebra* if $\mathrm{i.c.}(\mathscr{A}, \mathscr{B}) = \mathscr{B}$.

(ii) Likewise, the weak normalization $\mathscr{C}$ and the p-integral closure $\mathscr{D}$ of the image of $\mathscr{A}_{!!}$ in $\mathscr{B}_{!!}$ are quasi-coherent $\mathscr{O}_X$-algebras (see (9.8.27)); then we define the *weak normalization* (resp. the *p-integral closure*) of $\mathscr{A}$ in $\mathscr{B}$ as the quasi-coherent $\mathscr{O}_X^a$-algebra $\mathscr{C}^a$ (resp. $\mathscr{D}^a$).

Remark 14.3.12. (i) In the notation of definition 14.3.11, if $\mathscr{R}'$ is the integral closure of $\mathscr{R}$ in $\mathscr{S}$, then $\mathscr{R}'^a$ is the integral closure of $\mathscr{R}^a$ in $\mathscr{S}^a$. Indeed, we may assume that X is affine, in which case the assertion follows from [75, Lemma 8.2.28].

(ii) It follows that $\mathscr{B}$ is an integral $\mathscr{A}$-algebra if and only if $\mathscr{B}_{!!}$ is an integral $\mathscr{A}_{!!}$-algebra. Indeed, suppose that $\mathscr{B}$ is an integral $\mathscr{A}$-algebra, so that $\mathscr{B} = \mathrm{i.c.}(\mathscr{A}_{!!}, \mathscr{B}_{!!})^a$, whence by adjunction, a morphism of $\mathscr{A}_{!!}$-algebras $\mathscr{B}_{!!} \to \mathrm{i.c.}(\mathscr{A}_{!!}, \mathscr{B}_{!!})$, which must be the identity map.

(iii) Also, if $\mathscr{R}''$ is the weak normalization of $\mathscr{R}$ in $\mathscr{S}$, then it follows easily from proposition 9.8.29(i,ii) that $\mathscr{R}''^a$ is the weak normalization of $\mathscr{R}^a$ in $\mathscr{S}^a$: the details are left to the reader.

Proposition 14.3.13. *Suppose that the ideal* $\mathfrak{m}$ *fulfills condition* $(\mathbf{B})$ *of* [75, §2.1.6]. *Let X be a quasi-compact S-scheme, $\mathscr{A}$ a quasi-coherent $\mathscr{O}_X^a$-algebra, and $\mathscr{B}$ a flat $\mathscr{A}$-algebra that is almost finitely presented as an $\mathscr{A}$-module. Then $\mathscr{B}(X)$ is an integral $\mathscr{A}(X)$-algebra.*

Proof. For every $b \in \mathscr{B}(X)_*$ and every $i \in \mathbb{N}$, set :

$$c_i(b) := \mathrm{tr}_{\Lambda^i_{\mathscr{A}}\mathscr{B}/\mathscr{A}}(\Lambda^i_{\mathscr{A}}(b\mathbf{1}_{\mathscr{B}})) \in \mathscr{A}(X)$$

(see remark 14.3.2(ii)). In view of [75, Lemma 8.2.28], it suffices to show

Claim 14.3.14. For every $b \in \mathscr{B}(X)_*$ and $a \in \mathfrak{m}$, there exists $n \in \mathbb{N}$ such that :

(i) $c_i(ab) = 0$ for every $i > n$.
(ii) $\sum_{i=0}^n (-ab)^{i+1} \cdot c_{n-i}(ab) = 0$.

Proof of the claim. Since X admits a finite affine covering, and since the trace morphism commutes with arbitrary base changes, we are easily reduced to the case where X is affine. In this case, set $A := \mathscr{A}(X)$ and $B := \mathscr{B}(X)$; then B is an almost finitely generated projective A-algebra, and there exists $n \in \mathbb{N}$ such that $a\mathbf{1}_B$ factors through A-linear maps $u : B \to A^{\oplus n}$ and $v : A^{\oplus n} \to B$ ([75, Lemma 2.4.15]). Thus, $ab\mathbf{1}_B = v \circ (u \circ b\mathbf{1}_B)$, and we get :

$$c_i(ab) = \mathrm{tr}_{\Lambda^i_A A^{\oplus n}/A}(\Lambda^i_A(u \circ b\mathbf{1}_B \circ v)) \qquad \text{for every } i \in \mathbb{N}$$

([75, Lemma 4.1.2(i)]), from which (i) already follows. By the same token, we also obtain :

$$\chi := \sum_{i=0}^n (-u \circ b\mathbf{1}_B \circ v)^i \cdot c_{n-i}(ab) = 0$$

([75, Prop.4.4.30]). However, the left-hand side of the identity of (ii) is none else than $(-1) \cdot v \circ \chi \circ (u \circ b\mathbf{1}_B)$; whence the claim. ☐

The following lemma generalizes [75, Cor. 4.4.31].

Lemma 14.3.15. *Let A be a V^a-algebra, $B \subset A$ a V^a-subalgebra, P an almost finitely generated projective A-module, and φ an A-linear endomorphism of P. Suppose that :*

(a) *B is integrally closed in A.*
(b) *φ is integral over B_*.*

Then we have :

(i) $\mathrm{tr}_{P/A}(\varphi) \in B_*$.
(ii) *If* $\mathfrak{m}$ *fulfills condition* $(\mathbf{B})$ *of* [75, §2.1.6], *then* $\det(\mathbf{1}_P + T\varphi) \in B_*[[T]]$.

Proof. For the meaning of assumption (b), see the proof of [75, Cor. 4.4.31].

(i): For every $\varepsilon \in \mathfrak{m}$ we may find a free A-module L of finite rank, and A-linear morphisms $u : P \to L$, $v : L \to P$ such that $v \circ u = \varepsilon\mathbf{1}_P$ ([75, Lemma 2.4.15]). It follows that

$$\mathrm{tr}_{L/A}(u \circ \varphi \circ v) = \mathrm{tr}_{P/A}(\varphi \circ v \circ u) = \varepsilon \cdot \mathrm{tr}_{P/A}(\varphi)$$

([75, Lemma 4.1.2]). On the other hand, say that the polynomial $T^n + \sum_{j=0}^{n-1} b_j T^j \in B_*[T]$ annihilates φ; it is easily seen that $T^n + \sum_{j=0}^{n-1} b_j \varepsilon^{n-j} T^j$ annihilates $u \circ \varphi \circ v$, so the latter is integral over B_* as well, and therefore its trace lies in B_* ([75, Cor.4.4.31 and Rem.8.2.30(i)]). Summing up, we have shown that $\varepsilon \cdot \mathrm{tr}_{P/A}(\varphi) \in B_*$ for every $\varepsilon \in \mathfrak{m}$, whence the contention.

(ii): Recall that $\det(\mathbf{1}_P + T\varphi)$ is the power series in the variable T, whose coefficient in degree i is the trace of $\Lambda^i_A\varphi$ on $\Lambda^i_A P$: see [75, §4.3.1, §4.3.3].

Claim 14.3.16. Let Q be another almost finitely generated projective A-module, ψ be an endomorphism of P that is also integral over B_*. Then the endomorphism $\varphi \otimes_A \psi$ of $P \otimes_A Q$ is integral over B_*.

Proof of the claim. The tensor product defines a map of unital associative B_*-algebras

$$\mathrm{End}_A(P) \otimes_{B_*} \mathrm{End}_A(Q) \to \mathrm{End}_A(P \otimes_A Q).$$

By assumption, the subalgebra $B_*[\varphi] \subset \mathrm{End}_A(P)$ is finite over B_*, and similarly for $B_*[\psi] \subset \mathrm{End}_A(Q)$. Hence, the same holds for the image of $B_*[\varphi] \otimes_{B_*} B_*[\psi]$ in $\mathrm{End}_A(P \otimes_A Q)$, whence the claim. ♢

It follows easily from claim 14.3.16, that the endomorphism $\Lambda^i_A \varphi$ of $\Lambda^i_A P$ is integral over B_*. Hence, we may replace P by $\Lambda^i_A P$, and reduce to showing that the trace of φ lies in B_*, which is known, by (i). □

Lemma 14.3.17. *Let A be a V^a-algebra, $I \subset \mathrm{rad}(A)$ a tight ideal (see* [75, Def.5.1.5]*), and P a flat and almost finitely generated A-module. Let also $r \in \mathbb{N}$. We have :*

(i) *If P/IP is a faithfully flat A-module, then P is a faithfully flat A-module.*
(ii) *If $\mathfrak{m}$ fulfills condition* $(\mathbf{B})$*, and P is an almost projective A-module such that P/IP is an A/I-module of almost finite rank (resp. of finite rank $\leq r$), then P is an A-module of almost finite rank (resp. of finite rank $\leq r$).*

Proof. (i): It suffices to show that for every almost finitely generated A-module $M \neq 0$, we have $M \otimes_A P \neq 0$. To this aim, it suffices to check that $M \otimes_A P/IP \neq 0$; since $M \otimes_A P/IP = (M \otimes_A A/I) \otimes_{A/I} P/IP$, and since by assumption P/IP is a faithfully flat A/I-module, we are further reduced to showing that $M \otimes_A A/I \neq 0$; the latter follows from [75, Lemma 5.1.7].

(ii): Suppose first that P/IP is of finite rank $\leq r$, so that $\Lambda^r_{A/I}(P/IP) = 0$. Hence $(\Lambda^r_A P) \otimes_A P/IP = 0$, and then [75, Lemma 5.1.7] yields $\Lambda^r_A P = 0$, as sought. In the more general case where P/IP has almost finite rank, pick $n \in \mathbb{N}$ and a finitely generated subideal $\mathfrak{m}_0 \subset \mathfrak{m}$, such that $I^n \subset \mathfrak{m}_0$. Let also $\mathfrak{m}_1 \subset \mathfrak{m}$ be any finitely generated subideal such that $\mathfrak{m}_0 \subset \mathfrak{m}_1^{n+1}$; by assumption, there exists $i \in \mathbb{N}$ such that $\mathfrak{m}_1 \Lambda^i_{A/I}(P/IP) = 0$, *i.e.* $\mathfrak{m}_1 \Lambda^i_A P \subset I \Lambda^i_A P$. Therefore

$$\mathfrak{m}_0 \Lambda^i_A P \subset \mathfrak{m}_1^{n+1} \Lambda^i_A P \subset I^{n+1} \Lambda^i_A P \subset \mathfrak{m}_0 I \Lambda^i_A P$$

whence $\mathfrak{m}_0 \Lambda^i_A P = 0$, by [75, Lemma 5.1.7]. Since $\mathfrak{m}_0$ is arbitrary, we deduce that P has almost finite rank, as claimed. □

Lemma 14.3.18. *Let $A \to B$ be a morphism of V^a-algebras which is a monomorphism on the underlying V^a-modules, P an almost finitely generated and almost projective A-module, and $r \in \mathbb{N}$. Suppose as well that $\mathfrak{m}$ fulfills condition* $(\mathbf{B})$*; the following holds :*

(i) *P is an A-module of almost finite rank (resp. of finite rank $\leq r$) if and only if the same holds for the B-module $B \otimes_A P$.*
(ii) *Suppose that P has almost finite rank. Then P is a faithfully flat A-module if and only if the same holds for the B-module $B \otimes_A P$.*

Proof. (i): According to proposition 14.1.56(i), the A-module $\Lambda^i_A P$ is flat for every $i \in \mathbb{N}$; hence the natural A-linear morphism

$$\Lambda^i_A P \to B \otimes_A \Lambda^i_A P \stackrel{\sim}{\to} \Lambda^i_B(B \otimes_A P)$$

is a monomorphism for every $i \in \mathbb{N}$; the assertion follows straightforwardly.

(ii): According to [75, Th.4.3.28] (which holds whenever $\mathfrak{m}$ fulfills condition $(\mathbf{B})$) we have V^a-algebras A_0, A' and an isomorphism of V^a-algebras $A \stackrel{\sim}{\to} A_0 \times A'$ such that $A_0 \otimes_A P = 0$ and $A' \otimes_A P$ is a faithfully flat A'-module. Suppose now that $B \otimes_A P$ is a faithfully flat B-module; since $(B \otimes_A A_0) \otimes_B (B \otimes_A P) \simeq B \otimes_A A_0 \otimes_A P = 0$, it follows that $B \otimes_A A_0 = 0$.

On the other hand, the induced A-linear morphism $A_0 \to B \otimes_A A_0$ is a monomorphism, since A_0 is a flat A-module; thus, $A_0 = 0$, and consequently $P = A' \otimes_A P$ is faithfully flat. $\square$

Lemma 14.3.19. *Let $j : U \to X$ be an open quasi-compact immersion of S-schemes, $\mathscr{B}$ a quasi-coherent $\mathscr{O}_U^a$-algebra, $\mathscr{A}$ a quasi-coherent $\mathscr{O}_X^a$-algebra, and $j^*\mathscr{A} \to \mathscr{B}$ an étale morphism. Then the induced morphism*

$$\mathscr{A} \to j_*j^*\mathscr{A} \to j_*\mathscr{B}$$

is flat if and only if it is étale.

Proof. We may assume that $j_*\mathscr{B}$ is a flat $\mathscr{A}$-algebra, and it suffices to show that $j_*\mathscr{B}$ admits a diagonal idempotent as in remark 14.3.10(i). To this aim, let us remark more generally :

Claim 14.3.20. Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism, $\mathscr{F}$ a flat quasi-coherent $\mathscr{A}$-module, and $\mathscr{G}$ an $f^*\mathscr{A}$-module, quasi-coherent as an $\mathscr{O}_Y^a$-module. Then the natural map

$$\mathscr{F} \otimes_{\mathscr{A}} f_*\mathscr{G} \to f_*(f^*\mathscr{F} \otimes_{f^*\mathscr{A}} \mathscr{G})$$

is an isomorphism.

Proof of the claim. This is proved just as for $\mathscr{O}_X$-modules. Namely, one reduces easily to the case where X is affine, and one needs to check that the natural map

$$\mathscr{F}(X) \otimes_{\mathscr{A}(X)} \mathscr{G}(Y) \to (f^*\mathscr{F} \otimes_{f^*\mathscr{A}} \mathscr{G})(Y)$$

is an isomorphism. However, by assumption Y can be covered by finitely many affine open subsets $U_1, \dots, U_n$, and the intersection $U_{ij} := U_i \cap U_j$ can be covered by finitely many affine open subsets $U_{ij1}, \dots, U_{ijn}$, for every $i, j = 1, \dots, n$. Since $f^*\mathscr{F}$, $\mathscr{G}$ and $f^*\mathscr{A}$ are quasi-coherent $\mathscr{O}_Y^a$-modules, the natural maps

$$\mathscr{F}(X) \otimes_{\mathscr{A}(X)} \mathscr{G}(U_i) \to (f^*\mathscr{F} \otimes_{f^*\mathscr{A}} \mathscr{G})(U_i)$$
$$\mathscr{F}(X) \otimes_{\mathscr{A}(X)} \mathscr{G}(U_{ijk}) \to (f^*\mathscr{F} \otimes_{f^*\mathscr{A}} \mathscr{G})(U_{ijk})$$

are isomorphisms, for every $i, j, k = 1, \dots, n$. Then we get :

$$(f^*\mathscr{F} \otimes_{\mathscr{A}} \mathscr{G})(Y) = \operatorname{Ker}(\prod_{i=1}^n \mathscr{F}(X) \otimes_{\mathscr{A}(X)} \mathscr{G}(U_i) \rightrightarrows \prod_{i,j,k=1}^n \mathscr{F}(X) \otimes_{\mathscr{A}(X)} \mathscr{G}(U_{ijk}))$$
$$= \mathscr{F}(X) \otimes_{\mathscr{A}(X)} \operatorname{Ker}(\prod_{i=1}^n \mathscr{G}(U_i) \rightrightarrows \prod_{i,j,k=1}^n \mathscr{G}(U_{ijk}))$$

(since $\mathscr{F}(X)$ is a flat $\mathscr{A}(X)$-module) whence the claim. $\Diamond$

Now, set $\mathscr{R} := \mathscr{B} \otimes_{j^*\mathscr{A}} \mathscr{B}$; since j is quasi-compact and $j_*\mathscr{B}$ is a flat $\mathscr{A}$-algebra, claim 14.3.20 implies that the natural morphism :

$$j_*\mathscr{B} \otimes_{\mathscr{A}} j_*\mathscr{B} \to j_*\mathscr{R}$$

is an isomorphism. Especially, the diagonal idempotent of $\mathscr{B}$ extends to a global section of $j_*\mathscr{B} \otimes_{\mathscr{A}} j_*\mathscr{B}$, and the assertion follows. $\square$

The criterion of lemma 14.3.19 has limited usefulness, since it is not usually known a priori that $j_*\mathscr{B}$ is a flat $\mathscr{A}$-algebra. In several situations, one can however apply the following variant.

Proposition 14.3.21. *Let X be an S-scheme, $j : U \to X$ a quasi-compact open immersion, $\mathscr{A} \to \mathscr{B}$ a morphism of quasi-coherent $\mathscr{O}_X^a$-algebras, and suppose that :*

(a) *The units of adjunction $\mathscr{A} \to j_*j^*\mathscr{A}$ and $\mathscr{B} \to j_*j^*\mathscr{B}$ are monomorphisms.*
(b) *$\mathscr{A}$ is integrally closed in $j_*j^*\mathscr{A}$, and $\mathscr{B}$ is an integral $\mathscr{A}$-algebra.*
(c) *$j^*\mathscr{B}$ is an étale $j^*\mathscr{A}$-algebra and an almost finitely presented $j^*\mathscr{A}$-module.*
(d) *The diagonal idempotent $e_{j^*\mathscr{B}/j^*\mathscr{A}}$ is a global section of the subalgebra*

$$\operatorname{Im}(\mathscr{B} \otimes_{\mathscr{A}} \mathscr{B} \to j_*j^*(\mathscr{B} \otimes_{\mathscr{A}} \mathscr{B})).$$

Then $\mathscr{B}$ is an étale $\mathscr{A}$-algebra and an almost finitely presented $\mathscr{A}$-module.

Proof. We may assume that X is affine. Under our assumptions, the restriction map $\mathscr{A}(X)_* \to \mathscr{A}(U)_*$ is injective, and its image is integrally closed in $\mathscr{A}(U)_*$ ([75, Rem.8.2.30]); moreover, $\mathscr{B}(X)_{!!}$ is an integral $\mathscr{A}(X)_{!!}$-algebra (remark 14.3.12(ii)). In view of lemma 14.3.15(i), we deduce a commutative diagram

$$(14.3.22)\qquad \begin{array}{ccc} \mathscr{B} \otimes_{\mathscr{A}} \mathscr{B} & \xrightarrow{\quad t \quad} & \mathscr{A} \\ \downarrow & & \downarrow \\ j_*j^*(\mathscr{B} \otimes_{\mathscr{A}} \mathscr{B}) & \xrightarrow{j_*(t_{j^*\mathscr{B}/j^*\mathscr{A}})} & j_*j^*\mathscr{A} \end{array}$$

whose vertical arrows are the units of adjunctions, and where t is a uniquely determined $\mathscr{A}$-bilinear form. Then we are reduced to showing :

Claim 14.3.23. Let $j : U \to X$ be as in the proposition, and $\mathscr{A} \to \mathscr{B}$ a morphism of quasi-coherent $\mathscr{O}_X^a$-algebras fulfilling conditions (a), (c), (d) of the proposition, and such that there exists a bilinear form t making commute diagram 14.3.22. Then $\mathscr{B}$ is an étale $\mathscr{A}$-algebra and an almost finitely presented $\mathscr{A}$-module.

Proof of the claim. The proof proceeds by a familiar argument (see *e.g.* the proof of [75, Claim 3.5.33]). Namely, for a given $\varepsilon \in \mathfrak{m}$, we can write

$$\varepsilon \cdot e_{j^*\mathscr{B}/j^*\mathscr{A}} = \sum_{j=1}^{m} a_j \otimes b_j \qquad \text{where } a_j, b_j \in \mathscr{B}(X)_* \text{ for every } j = 1, \dots, m.$$

In light of remark 14.3.10(iv), we deduce

$$\varepsilon \cdot a = \sum_{j=1}^{m} t(a \otimes a_j) \cdot b_j \qquad \text{for every } a \in \mathscr{B}(X)_*.$$

Indeed, the identity holds after restriction to U, and assumption (a) implies that the restriction map $\mathscr{B}(X)_* \to \mathscr{B}(U)_*$ is injective. We may then define $\mathscr{A}$-linear maps : $\mathscr{B} \xrightarrow{\varphi} \mathscr{A}^{\oplus m} \xrightarrow{\psi} \mathscr{B}$ by the rules :

$$\varphi(a) := (t(a \otimes a_1), \dots, t(a \otimes a_m)) \qquad \psi(s_1, \dots, s_m) := \sum_{j=1}^{m} s_j b_j$$

for every open subset $U \subset X$, every $a \in \mathscr{B}(U)_*$, and every $s_1, \dots, s_m \in \mathscr{A}(U)_*$. Thus we have $\psi \circ \varphi = \varepsilon \mathbf{1}_{\mathscr{B}}$, and since ε is arbitrary, this already proves that $\mathscr{B}$ is an $\mathscr{A}$-module of almost finite type, so we may apply [75, Lemma 2.4.15 and Prop.2.4.18(i)] to deduce that $\mathscr{B}$ is a flat and almost finitely presented $\mathscr{A}$-module. Now, assumption (a) and claim 14.3.20 imply that the horizontal arrows of the induced diagram

$$\begin{array}{ccc} \mathscr{B} \otimes_{\mathscr{A}} \mathscr{B} & \longrightarrow & j_*j^*(\mathscr{B} \otimes_{\mathscr{A}} \mathscr{B}) \\ {\scriptstyle \mu_{\mathscr{B}/\mathscr{A}}} \downarrow & & \downarrow {\scriptstyle j_*j^*\mu_{\mathscr{B}/\mathscr{A}}} \\ \mathscr{B} & \longrightarrow & j_*j^*\mathscr{B} \end{array}$$

are monomorphisms. From the characterization of remark 14.3.10(i), it follows easily that the section $e_{j^*\mathscr{B}/j^*\mathscr{A}}$ is the diagonal idempotent for the morphism $\mathscr{A} \to \mathscr{B}$, so $\mathscr{B}$ is an unramified $\mathscr{A}$-algebra, and the proof is concluded. $\square$

Corollary 14.3.24. *Let X be an S-scheme, $j : U \to X$ a quasi-compact open immersion, $\mathscr{A} \to \mathscr{B}$ a morphism of quasi-coherent $\mathscr{O}_U^a$-algebras, and suppose that :*

(a) *$\mathscr{B}$ is an étale $\mathscr{A}$-algebra and an almost finitely presented $\mathscr{A}$-module.*

(b) *The diagonal idempotent* $e_{\mathscr{B}/\mathscr{A}}$ *is a global section of the subalgebra*

$$\mathrm{Im}((j_*\mathscr{B}) \otimes_{j_*\mathscr{A}} (j_*\mathscr{B}) \to j_*(\mathscr{B} \otimes_{\mathscr{A}} \mathscr{B})).$$

Then $j_*\mathscr{B}$ *is an étale* $j_*\mathscr{A}$*-algebra and an almost finitely presented* $j_*\mathscr{A}$*-module.*

Proof. Indeed, the induced morphism $j_*\mathscr{A} \to j_*\mathscr{B}$ fulfills conditions (a), (c) and (d) of proposition 14.3.21, and diagram 14.3.22 trivially commutes with $t := j_*(t_{\mathscr{B}/\mathscr{A}})$, so the corollary follow from claim 14.3.23. □

Using lemma 14.3.15 we can also relax one assumption in [75, Prop.8.2.31(i)]; namely, we have the following :

Proposition 14.3.25. *Let* $A \subset B$ *be a pair of* V^a*-algebras, such that* $A = \mathrm{i.c.}(A, B)$*. Then, for every étale almost finite projective* A*-algebra* A_1 *we have* $A_1 = \mathrm{i.c.}(A_1, A_1 \otimes_A B)$*.*

Proof. Set $B_1 := A_1 \otimes_A B$, and suppose that $x \in B_{1*}$ is integral over A_{1*}. Let $e \in (A_1 \otimes_A A_1)_*$ be the diagonal idempotent of the unramified A-algebra A_1; for given $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \mathfrak{m}$ we write $\varepsilon_1 \cdot e = \sum_{j=1}^k c_i \otimes d_i$ for some $c_i, d_i \in A_{1*}$. According to remark 14.3.10(iv) and [75, Prop.4.1.8(ii)], we have $\sum_{j=1}^k c_i \cdot \mathrm{Tr}_{B_1/B}(xd_i) = \varepsilon \cdot x$. On the other hand, $\varepsilon_2 \cdot x$ and $\varepsilon_3 \cdot d_i$ are integral over A_* (by [75, Lemma 5.1.13(i)], which holds for arbitrary basic setups); then lemma 14.3.15(i) implies that $\mathrm{Tr}_{B_1/B}(\varepsilon_2\varepsilon_3 \cdot xd_i) \in A_*$ for every $i = 1, \dots, k$. Hence $\varepsilon_1\varepsilon_2\varepsilon_3 \cdot x \in A_{1*}$, and since $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are arbitrary, the assertion follows. □

Lemma 14.3.26. *Let* X *be an* S*-scheme,* $\mathscr{A}$ *a quasi-coherent* $\mathscr{O}_X^a$*-algebra. We have :*

(i) *Let* $f : Y \to X$ *a faithfully flat and quasi-compact morphism. Then* $\mathscr{A}$ *is an étale (resp. weakly étale, resp. weakly unramified)* $\mathscr{O}_X^a$*-algebra if and only if* $f^*\mathscr{A}$ *is an étale (resp. weakly étale, resp. weakly unramified)* $\mathscr{O}_Y^a$*-algebra.*

(ii) *Suppose that* X *is integral and* $\mathscr{A}$ *is a weakly étale* $\mathscr{O}_X^a$*-algebra. Let* $\eta \in X$ *be the generic point; then* $\mathscr{A}$ *is an étale* $\mathscr{O}_X^a$*-algebra if and only if* $\mathscr{A}_\eta$ *is an étale* $\mathscr{O}_{X,\eta}^a$*-algebra.*

(iii) *Suppose that* X *is normal and irreducible, and* $\mathscr{A}$ *is integral, torsion-free and unramified over* $\mathscr{O}_X^a$*. Then* $\mathscr{A}$ *is étale over* $\mathscr{O}_X^a$*, and almost finitely presented as an* $\mathscr{O}_X^a$*-module.*

(iv) *Let* $j : X \to Y$ *be an open quasi-compact immersion of* S*-schemes, with* Y *normal and irreducible, and such that* $j_*\mathscr{O}_X = \mathscr{O}_Y$*, and suppose that* $\mathscr{A}$ *is integral and torsion-free over* $\mathscr{O}_X^a$*. Then* $j_*\mathscr{A}$ *is an integral* $\mathscr{O}_Y^a$*-algebra.*

Proof. (i): Arguing as in the proof of lemma 14.3.5, one reduces easily to the case where X and Y are affine, in which case the assertion follows from [75, §3.4.1].

(ii): Let $U \subset X$ be any affine open subset; by assumption $\mathscr{A}(U)$ is flat over the almost ring $B := \mathscr{A}(U) \otimes_{\mathscr{O}_X^a(U)} \mathscr{A}(U)$, and $\mathscr{A}(U) \otimes_{\mathscr{O}_X^a(U)} \kappa(\eta)^a$ is almost projective over $B \otimes_{\mathscr{O}_X^a(U)} \kappa(\eta)^a$. Then the assertion follows from [75, Prop.2.4.19].

(iii): Let $\eta \in X$ be the generic point. We begin with the following :

Claim 14.3.27. Suppose that $V = \mathfrak{m}$ (the "classical limit" case of [75, Ex.2.1.2(ii)]), and let $f : A \to B$ be a local homomorphism of local rings. Then :

(i) f^a is weakly étale if and only if f extends to an isomorphism of strict henselizations $A^{\mathrm{sh}} \to B^{\mathrm{sh}}$.

(ii) Especially, if A is a field and f^a is weakly étale, then B is a separable algebraic extension of A.

Proof of the claim. (i): In the classical limit case, a weakly étale morphism is the same as an "absolutely flat" map as defined in [137]; then the assertion is none else than [137, Th.5.2].

(ii): If A is a field, A^{sh} is a separable closure of A; hence (i) implies that B is a subring of A^{sh}, hence it is a subfield of A^{sh}. ◊

Claim 14.3.28. Suppose that $V = \mathfrak{m}$. Let A be a field and $f : A \to B$ a ring homomorphism such that f^a is weakly étale. Then :

(i) Every finitely generated A-subalgebra of B is finite étale over A (that is, in the usual sense of [66, Ch.IV, §17]).
(ii) f^a is étale if and only if f is étale (in the usual sense of [66]).

Proof of the claim. (i): From claim 14.3.27(ii) we see that B is reduced of Krull dimension ≤ 0, and all its residue fields are separable algebraic extensions of A. We consider first the case of a monogenic extension $C := A[b] \subset B$. For every prime ideal $\mathfrak{p} \subset B$, let $b_\mathfrak{p}$ be the image of b in $B_\mathfrak{p}$; then for every such $\mathfrak{p}$ we may find an irreducible separable monic polynomial $P_\mathfrak{p}(T) \in A[T]$ with $P_\mathfrak{p}(b_\mathfrak{p}) = 0$. This identity persists in an open neighborhood $U_\mathfrak{p} \subset \operatorname{Spec} B$ of $\mathfrak{p}$, and finitely many such $U_\mathfrak{p}$ suffice to cover $\operatorname{Spec} B$. Thus, we find finitely many $P_{\mathfrak{p}_1}(T), \dots, P_{\mathfrak{p}_k}(T)$ such that $\prod_{i=1}^k P_{\mathfrak{p}_i}(b) = 0$ holds in B, and after omitting repetitions, we may assume that all these polynomials are distinct. Since C is a quotient of the separable A-algebra $A[T]/(\prod_{i=1}^k P_{\mathfrak{p}_i}(T))$, the claim follows in this case. In the general case, we may write $C = A[b_1, \dots, b_n]$ for certain $b_1, \dots, b_n \in B$. Then $\operatorname{Spec} C$ is a reduced closed subscheme of $\operatorname{Spec} A[b_1] \times_{\operatorname{Spec} A} \cdots \times_{\operatorname{Spec} A} \operatorname{Spec} A[b_n]$; by the foregoing, the latter is étale over $\operatorname{Spec} A$, hence the same holds for $\operatorname{Spec} C$.

(ii): We may assume that f^a is étale, and we have to show that f is étale, *i.e.* that B is a finitely generated A-module. Hence, let $e_{B/A} \in B \otimes_A B$ be the diagonal idempotent (see remark 14.3.10(i)); we choose a finitely generated A-subalgebra $C \subset B$ such that $e_{B/A}$ is the image of an element $e' \in C \otimes_A C$. Notice that $1 - e'$ lies in the kernel of the multiplication map $\mu_{C/A} : C \otimes_A C \to C$. By (i), the morphism $A \to C$ is étale, hence it admits as well a diagonal idempotent $e_{C/A} \in C \otimes_A C$; on the other hand, [75, Lemma 3.1.2(v)] says that B^a is étale over C^a, especially B is a flat C-algebra, hence the natural map $C \otimes_A C \to B \otimes_A B$ is injective. Since $1 - e_{C/A}$ lies in the kernel of the multiplication map $\mu_{B/A} : B \otimes_A B \to B$, we have :

$$e_{B/A}(1 - e_{C/A}) = 0 = e_{C/A}(1 - e') = e_{C/A}(1 - e_{B/A})$$

from which it follows easily that $e_{B/A} = e_{C/A}$. Moreover, the induced morphism $\operatorname{Spec} B \to \operatorname{Spec} C$ has dense image, so it must be surjective, since $\operatorname{Spec} C$ is a discrete finite set; therefore B is even a faithfully flat C-algebra. Let $J := \operatorname{Ker}(B \otimes_A B \to B \otimes_C B)$; then J is the ideal generated by all elements of the form $1 \otimes c - c \otimes 1$, where $c \in C$; clearly this is the same as the extension of the ideal $I_{C/A} := \operatorname{Ker} \mu_{C/A}$. However, $I_{C/A}$ is generated by the idempotent $1 - e_{C/A}$ ([75, Cor.3.1.9]), consequently J is generated by $1 - e_{B/A}$, *i.e.* $J = \operatorname{Ker} \mu_{B/A}$. So finally, the multiplication map $\mu_{B/C} : B \otimes_C B \to B$ is an isomorphism, whose inverse is the map $B \to B \otimes_C B : b \mapsto b \otimes 1$. The latter is of the form $j \otimes_C \mathbf{1}_B$, where $j : C \to B$ is the natural inclusion map. By faithfully flat descent we conclude that $C = B$, whence the claim. ◊

Next, we remark that $\mathscr{A}_\eta$ is a finitely presented $\mathscr{O}^a_{X,\eta}$-module. Indeed, since $K := \mathscr{O}_{X,\eta}$ is a field, we have either $\mathfrak{m}K = 0$, in which case the category of K^a-modules is trivial and there is nothing to show, or else $\mathfrak{m}K = K$. So we may assume that we are in the "classical limit" case, and then the assertion follows from claim 14.3.28(ii).

Now, set $\mathscr{B} := \mathscr{A} \otimes_{\mathscr{O}^a_X} \mathscr{A}$, and let $j : X(\eta) \to X$ be the natural morphism; since X is normal and $\mathscr{A}$ is torsion-free, the units of adjunction $\mathscr{O}^a_X \to j_*j^*\mathscr{O}_{X(\eta)}$ and $\mathscr{A} \to j_*j^*\mathscr{A}$ are monomorphisms. Since $\mathscr{A}$ is unramified over $\mathscr{O}^a_X$, the diagonal idempotent of $j^*\mathscr{A}$ lies in the image of the restriction map $\mathscr{B}(X)_* \to \mathscr{B}_{\eta *}$. In view of these observations, an easy inspection shows that the proof of proposition 14.3.21 carries over *verbatim* to the current situation, and yields assertion (iii).

(iv): Let $\mathscr{T} \subset \mathscr{A}_{!!}$ be the maximal torsion $\mathscr{O}_X$-subsheaf, and set $\mathscr{B} := \mathscr{A}_{!!}/\mathscr{T}$. Then $\mathscr{B}$ is an integral, torsion-free $\mathscr{O}_X$-algebra, by remark 14.3.12(ii), and $\mathscr{B}^a \simeq \mathscr{A}$, hence $(j_*\mathscr{B})^a \simeq j_*\mathscr{A}$. Let $\eta \in Y$ be the generic point; in light of remark 14.3.12(ii), it then suffices to show :

Claim 14.3.29. Under the assumptions of (v), let $\mathscr{R}$ be an integral quasi-coherent and torsion-free $\mathscr{O}_X$-algebra. Then $j_*\mathscr{R}$ is an integral $\mathscr{O}_Y$-algebra.

Proof of the claim. Let us write $\mathscr{R}_\eta$ as the filtered union of the family $(R_\lambda \mid \lambda \in \Lambda)$ of its finite $\mathscr{O}_{Y,\eta}$-subalgebras, and for every $\lambda \in \Lambda$, let $\mathscr{R}_\lambda \subset \mathscr{R}$ be the quasi-coherent $\mathscr{O}_X$-subalgebra such that $\mathscr{R}_\lambda(V) = R_\lambda \cap \mathscr{R}(V)$ for every non-empty open subset $V \subset X$. Then $\mathscr{R}$ is the filtered colimit of the family $(\mathscr{R}_\lambda \mid \lambda \in \Lambda)$, and clearly it suffices to show the claim for every $\mathscr{R}_\lambda$. We may thus assume from start that $\mathscr{R}_\eta$ is a finite $\mathscr{O}_{Y,\eta}$-algebra. Let $\mathscr{R}^\nu$ be the integral closure of $\mathscr{R}$, *i.e.* the quasi-coherent $\mathscr{O}_X$-algebra such that $\mathscr{R}^\nu(V)$ is the integral closure of $\mathscr{R}(V)$ in $\mathscr{R}_\eta$, for every non empty affine open subset $V \subset X$. Clearly it suffices to show that $j_*\mathscr{R}^\nu$ is integral, hence we may replace $\mathscr{R}$ by $\mathscr{R}^\nu$ and assume from start that $\mathscr{R}$ is integrally closed. In this case, for every non-empty open subset $V \subset X$ the restriction map $\mathscr{R}(V) \to \mathscr{R}_\eta$ induces a bijection between the idempotents of $\mathscr{R}(V)$ and those of $\mathscr{R}_\eta$. Especially, $\mathscr{R}(X)$ admits finitely many idempotents, and moreover we have a natural decomposition $\mathscr{R} = \mathscr{R}_1 \times \cdots \times \mathscr{R}_k$, as a product of $\mathscr{O}_X$-algebras, such that $\mathscr{R}_{i,\eta}$ is a field for every $i \leq k$. It then suffices to show the claim for every $j_*\mathscr{R}_i$, and then we may assume throughout that $\mathscr{R}_\eta$ is a field. Up to replacing $\mathscr{R}$ by its normalization in a finite extension of $\mathscr{R}_\eta$, we may even assume that $\mathscr{R}_\eta$ is a finite normal extension of $\mathscr{O}_{Y,\eta}$. Hence, let $V \subset Y$ be any non-empty affine open subset, and $a \in \mathscr{R}(X \cap V)$ any element. Let $P(T)$ be the minimal polynomial of a over the field $\mathscr{O}_{Y,\eta}$; we have to show that the coefficients of $P(T)$ lie in $\mathscr{O}_Y(V) = \mathscr{O}_X(X \cap V)$, and since Y is normal, it suffices to prove that these coefficients are integral over $\mathscr{O}_X(W)$, for every non-empty affine open subset $W \subset X \cap V$. However, since $\mathscr{R}_\eta$ is normal over $\mathscr{O}_{Y,\eta}$, such coefficients can be written as some elementary symmetric polynomials of the conjugates of a in $\mathscr{R}_\eta$. Hence, we come down to showing that the conjugates of a are still integral over $\mathscr{O}_X(W)$. The latter assertion is clear: indeed, if $Q(T) \in \mathscr{O}_X(W)[T]$ is a monic polynomial with $Q(a) = 0$, then we have also $Q(a') = 0$ for every conjugate a' of a. □

Proposition 14.3.30. *Let X be a quasi-compact and quasi-separated S-scheme, $j : U \to X$ a quasi-compact open immersion, and $\mathscr{A}$ a quasi-coherent $\mathscr{O}_U^a$-algebra, almost finitely presented as an $\mathscr{O}_U^a$-module. Then for every finitely generated subideal $\mathfrak{m}_0 \subset \mathfrak{m}$, there exist a quasi-coherent $\mathscr{O}_X$-algebra $\mathscr{B}$, finitely presented as an $\mathscr{O}_X$-module, and a morphism $\mathscr{B}^a_{|U} \to \mathscr{A}$ of $\mathscr{O}_U^a$-algebras, whose kernel and cokernel are annihilated by $\mathfrak{m}_0$.*

Proof. Set $\mathscr{C} := \mathrm{i.c.}(\mathscr{O}_X^a, j_*\mathscr{A})$; then $\mathscr{C}$ is a quasi-coherent $\mathscr{O}_X^a$-algebra, and $\mathscr{C}_{|U} = \mathscr{A}$, by lemma 14.3.9. According to proposition 10.3.31, we may write $\mathscr{C}_{!!}$ as the colimit of a filtered family $(\mathscr{F}_i \mid i \in I)$ of finitely presented quasi-coherent $\mathscr{O}_X$-modules. Pick a finitely generated subideal $\mathfrak{m}_1 \subset \mathfrak{m}$ such that $\mathfrak{m}_0 \subset \mathfrak{m}_1^2$; for every affine open subset $U' \subset U$, we may find $i \in I$ such that $\mathfrak{m}_1 \cdot \mathscr{A}_{!!}(U') \subset \mathrm{Im}(\mathscr{F}_i(U') \to \mathscr{A}_{!!}(U'))$, and since U is quasi-compact, finitely many such open subsets cover U; hence, we may find $i \in I$ such that $\mathfrak{m}_1 \cdot \mathscr{A}_{!!} \subset \mathrm{Im}(\mathscr{F}_{i|U} \to \mathscr{A}_{!!})$; we set $\mathscr{F} := \mathscr{F}_i$, and let $\varphi : \mathscr{G} := \mathrm{Sym}^\bullet_{\mathscr{O}_X}\mathscr{F} \to \mathscr{C}_{!!}$ be the induced morphism of quasi-coherent $\mathscr{O}_X$-algebras. Notice that $\mathscr{G}$ is finitely presented as an $\mathscr{O}_X$-algebra. Using again proposition 10.3.31 (or [59, Ch.I, Cor.9.4.9]) we may write $\mathrm{Ker}\,\varphi$ as the colimit of a filtered family $(\mathscr{K}_\lambda \mid \lambda \in \Lambda)$ of quasi-coherent $\mathscr{O}_X$-submodules of finite type. Fix a finite covering $(U'_i \mid i \in I)$ of X consisting of affine open subsets, and for every $i \in I$, let $f_{i,1}, \dots, f_{i,n(i)}$ be a finite set of generators of $\mathscr{F}(U'_i)$; we may find monic polynomials $P_{i,1}(T), \dots, P_{i,n(i)}(T)$ with coefficients in $\mathscr{O}_X(U'_i)$ such that $P_{i,j}(\varphi(f_{i,j})) = 0$ in $\mathscr{C}(U'_i)_{!!}$, for every $i \in I$ and every $j \leq n(i)$. Let $\lambda \in \Lambda$ be chosen large enough, so that $P_{i,j}(f_{i,j}) \in \mathscr{K}_\lambda(U'_i)$ for every $i \in I$ and every $j \leq n(i)$. Let also $\mathscr{K}'_\lambda \subset \mathscr{G}$ be the ideal generated by $\mathscr{K}_\lambda$; then $\mathscr{G}' := \mathscr{G}/\mathscr{K}'_\lambda$ is an integral $\mathscr{O}_X$-algebra of finite presentation, hence it is finitely presented as an $\mathscr{O}_X$-module, in view of claim 9.1.35. Clearly $\varphi_{|U}$ descends to a map $\varphi' : \mathscr{G}'_{|U} \to \mathscr{A}_{!!}$, whose cokernel is annihilated by $\mathfrak{m}_1$, and by [75, Claim 2.3.12 and the proof of Cor.2.3.13] we may find, for every affine open subset $U' \subset U$, a finitely generated submodule $K_{U'} \subset \mathrm{Ker}\,(\varphi'_{U'} : \mathscr{G}'(U') \to \mathscr{A}(U')_{!!})$ such that $\mathfrak{m}_1^2 \cdot \mathrm{Ker}\,\varphi'_{U'} \subset K_{U'}$. Another invocation of proposition 10.3.31 ensures the existence of a quasi-coherent $\mathscr{O}_X$-submodule of finite type $\mathscr{J} \subset \mathrm{Ker}\,\varphi$ such that $K_{U'} \subset \mathscr{J}(U')$ for all the U' of a finite covering of U; the $\mathscr{O}_X$-algebra $\mathscr{B} := \mathscr{G}'/\mathscr{J} \cdot \mathscr{G}'$ fulfills the required conditions. □

14.3.31. Let $(X_i \mid i \in I)$ be a cofiltered system of quasi-compact and quasi-separated S-schemes, with affine transition morphisms $h_\varphi : X_j \to X_i$, for every morphism $\varphi : j \to i$ in I. Let also $U_i \subset X_i$ be a quasi-compact open subset, for every $i \in I$, such that $U_j = h_\varphi^{-1} U_i$ for every $\varphi : j \to i$ in I. Set

$$X := \lim_{i \in I} X_i \qquad U := \lim_{i \in I} U_i.$$

and denote by $h_i : U \to X_i$ the natural morphism, for every $i \in I$.

Corollary 14.3.32. *In the situation of* (14.3.31)*, let $\mathscr{A}$ be any flat almost finitely presented $\mathscr{O}_U^a$-algebra. Then, for every finitely generated subideal $\mathfrak{m}_0 \subset \mathfrak{m}$ there exist $i \in I$, a quasi-coherent $\mathscr{O}_{X_i}$-algebra $\mathscr{R}$, finitely presented as an $\mathscr{O}_{X_i}$-module, and a map of $\mathscr{O}_U^a$-algebras $f : h_i^* \mathscr{R}^a \to \mathscr{A}$ such that :*

(a) $\mathrm{Ker}\, f$ *and* $\mathrm{Coker}\, f$ *are annihilated by* $\mathfrak{m}_0$.
(b) *For every $x \in U_i$ and every $b \in \mathfrak{m}_0$, the map $b \cdot \mathbf{1}_{\mathscr{R},x} : \mathscr{R}_x \to \mathscr{R}_x$ factors through a free $\mathscr{O}_{U_i,x}$-module.*

Proof. From [65, Ch.IV, Th.8.3.11] it is easily seen that X is a quasi-compact and quasi-separated S-scheme, and U is a quasi-compact open subset of X. Pick a finitely generated subideal $\mathfrak{m}_1$ such that $\mathfrak{m}_0 \in \mathfrak{m}_1^2$; according to proposition 14.3.30, we may find a quasi-coherent $\mathscr{O}_X$-algebra $\mathscr{B}$, finitely presented as an $\mathscr{O}_X$-module, and a morphism $g : \mathscr{B}^a_{|U} \to \mathscr{A}$ of $\mathscr{O}_U^a$-algebras whose kernel and cokernel are annihilated by $\mathfrak{m}_1$. By standard arguments we deduce

$$\mathfrak{m}_1^2 \cdot \mathrm{Tor}^1_{\mathscr{O}_X(U')}(M, \mathscr{B}(U')) = 0 \qquad (14.3.33)$$

for every affine open subset $U' \subset U$ and every $\mathscr{O}_X(U')$-module M. Now, let us remark, quite generally :

Claim 14.3.34. Let A be any ring, M and N any two A-modules, $\varphi : M \to N$ any A-linear map, and $M^\vee := \mathrm{Hom}_A(M, A)$. Then φ factors through a free A-module of finite rank if and only if it lies in the image of the natural map

$$M^\vee \otimes_A N \to \mathrm{Hom}_A(M, N). \qquad (14.3.35)$$

Proof of the claim. Suppose first that φ factors as the composition of A-linear maps $\psi : M \to A^{\oplus n}$ and $\psi' : A^{\oplus n} \to N$, for some $n \in \mathbb{N}$. For every $i = 1, \dots, n$, let $p_i : A^{\oplus n} \to A$ (resp. $e_i : A \to A^{\oplus n}$) be the natural epimorphism (resp. monomorphism), and set $\psi_i := p_i \circ \psi$, $\psi'_i := \psi' \circ e_i$. Then $\varphi = \sum_{i=1}^n \psi'_i \circ \psi_i$ and each summand $\psi'_i \circ \psi_i$ lies in the image of (14.3.35). Conversely, if φ lies in the image of (14.3.35), we have $\varphi = \sum_{i=1}^n \psi'_i \circ \psi_i$, for some $\psi_1, \dots, \psi_n \in M^\vee$ and $\psi'_1, \dots, \psi'_n \in \mathrm{Hom}_A(A, N) = N$, and we may define $\psi := \sum_{i=1}^n e_i \circ \psi_i$, $\psi'_i := \sum_{i=1}^n \psi'_i \circ p_i$. Then $\varphi = \psi' \circ \psi$. ◊

Now, pick a finite covering $(U'_{i,\lambda} \mid \lambda \in \Lambda)$ of U_i, consisting of affine open subsets, and a finite system $b_1, \dots, b_k$ of generators of $\mathfrak{m}_1^2$, and set $U'_\lambda := h_i^{-1} U'_{i,\lambda}$ for every $\lambda \in \Lambda$; from claim 14.3.34, (14.3.33) and [75, Lemma 2.4.17] we deduce that for every $\lambda \in \Lambda$ and every $j = 1, \dots, k$ there exists $n(\lambda, j) \in \mathbb{N}$ such that $b_j \cdot (\mathbf{1}_{\mathscr{B}})_{|U'_\lambda}$ factors through $\mathscr{O}_{U'_\lambda}^{\oplus n(\lambda,j)}$. Next, by virtue of [65, Ch.IV, Th.8.5.2] we may assume that $\mathscr{B}$ descends to a $\mathscr{O}_{X_i}$-algebra $\mathscr{R}$, finitely presented as an $\mathscr{O}_{X_i}$-module, and such that $b_j \cdot (\mathbf{1}_{\mathscr{R}})_{|U'_{i,\lambda}}$ factors through $\mathscr{O}_{U'_{i,\lambda}}^{\oplus n(\lambda,j)}$ for every $j = 1, \dots, k$ and every $\lambda \in \Lambda$. It follows easily that

$$\mathfrak{m}_1^2 \cdot \mathrm{Tor}^1_{\mathscr{O}_{X_i}(U'_{i,\lambda})}(M, \mathscr{R}(U'_{i,\lambda})) = 0$$

for every $\lambda \in \Lambda$ and every $\mathscr{O}_{X_i}(U'_{i,\lambda})$-module M. Then, again [75, Lemma 2.4.17] and claim 14.3.34 imply that condition (b) holds for $\mathscr{R}$, and by construction we also get condition (a). □

14.4. **Almost pure pairs.** Throughout this section, we keep the notation of (14.3), and *we assume that* $\mathfrak{m}$ *fulfills condition* $(\mathbf{B})$.

Definition 14.4.1. Let X be an S-scheme, $Z \subset X$ a closed subscheme such that $U := X\setminus Z$ is a dense subset of X, and denote by $j : U \to X$ the open immersion.

(i) We say that the pair (X, Z) is *almost pure relative to* $(V, \mathfrak{m})$ (or just *almost pure*, when the underlying basic setup is clear from the context) if the restriction functor

$$\mathscr{O}^a_X\text{-}\mathbf{\acute{E}t}_{\mathrm{afr}} \to \mathscr{O}^a_U\text{-}\mathbf{\acute{E}t}_{\mathrm{afr}} \quad : \quad \mathscr{A} \mapsto \mathscr{A}_{|U} \tag{14.4.2}$$

from the category of étale $\mathscr{O}^a_X$-algebras of almost finite rank, to the category of étale $\mathscr{O}^a_U$-algebras of almost finite rank, is an equivalence.

(ii) We say that the pair (X, Z) is *normal* if Z is a constructible subset of X, and the natural map $\mathscr{O}^a_X \to j_*\mathscr{O}^a_U$ is a monomorphism, whose image is integrally closed in $j_*\mathscr{O}^a_U$.

Remark 14.4.3. Let (X, Z) be a pair as in definition 14.4.1, where Z is a constructible subset of X, and set $U := X\setminus Z$.

(i) Consider the following conditions:

(a) $\mathscr{O}_{X,z}$ is a normal domain, for every $z \in Z$.

(b) the natural map $\mathscr{O}_X \to j_*\mathscr{O}_U$ is a monomorphism, and $\mathscr{O}_X = \mathrm{i.c.}(\mathscr{O}_X, j_*\mathscr{O}_U)$.

(c) The pair (X, Z) is normal.

Then (a)⇒(b), since $(j_*\mathscr{O}_U)_z = \mathscr{O}_{X(z)}(U \cap X(z))$ for every $z \in Z$. Also, (b)⇒(c), by virtue of [75, Lemma 8.2.28].

(ii) Let $\mathscr{A}$ be a quasi-coherent $\mathscr{O}^a_U$-algebra. Then $j_*\mathscr{A}_{!!}$ is a quasi-coherent $\mathscr{O}_X$-algebra ([59, Ch.I, Prop.9.4.2(i)] and lemma 14.3.9), hence the integral closure $\mathscr{A}^\nu$ of the image of $\mathscr{O}^a_X$ in $j_*\mathscr{A}$ is a well defined quasi-coherent $\mathscr{O}^a_X$-algebra (see definition 14.3.11). We call $\mathscr{A}^\nu$ the *normalization* of $\mathscr{A}$ over X.

(iii) Let (X, Z) be any normal pair, and $U \subset X$ any open subset. Directly from the definition, we see that $(U, Z \cap U)$ is still a normal pair.

Lemma 14.4.4. *Let* (X, Z) *be a normal pair as in definition* 14.4.1(ii)*, and set* $U := X\setminus Z$*. Let* $\mathscr{A}$ *be an étale* $\mathscr{O}^a_U$*-algebra whose underlying* $\mathscr{O}^a_U$*-module is almost finitely presented (resp. is of finite rank). The following conditions are equivalent :*

(a) *The normalization* $\mathscr{A}^\nu$ *of* $\mathscr{A}$ *over* X *(see remark* 14.4.3(ii)*) is a weakly unramified* $\mathscr{O}^a_X$*-algebra.*

(b) $\mathscr{A}^\nu$ *is an étale* $\mathscr{O}^a_X$*-algebra, and an almost finitely presented* $\mathscr{O}^a_X$*-module (resp. and an* $\mathscr{O}^a_X$*-module of finite rank).*

Proof. Obviously (b)⇒(a), hence suppose that (a) holds, and let $\mathscr{B} := \mathscr{A}^\nu \otimes_{\mathscr{O}^a_X} \mathscr{A}^\nu$. We may assume that X is affine, and we set

$$A := \mathscr{A}^\nu(X) \qquad B := \mathscr{B}(X).$$

By assumption, the multiplication morphism $\mu : \mathscr{B} \to \mathscr{A}^\nu$ is flat, especially A is a flat B-module. Let also B' be the image of B in $\mathscr{B}(U)$; clearly $\mu(X) : B \to A$ factors through an epimorphism $\mu' : B' \to A$, therefore $A = A \otimes_B B'$, so A is also a flat B'-module. Pick a finite covering $U = U_1 \cup \cdots \cup U_k$ consisting of affine open subsets of U. The induced morphism

$$B' \to \mathscr{B}(U_1) \times \cdots \times \mathscr{B}(U_k)$$

is a monomorphism. On the other hand, notice that $A \otimes_{B'} \mathscr{B}(U_i) = \mathscr{A}(U_i)$ is an almost finitely presented $\mathscr{B}(U_i)$-module. It follows that A is an almost finitely generated projective B'-module

([75, Prop. 2.4.18 and 2.4.19]), and therefore the kernel of μ' is generated by an idempotent $e \in B'_*$ ([75, Rem. 3.1.8]). A simple inspection shows that e is necessarily the diagonal idempotent of the unramified $\mathscr{O}_U^a$-algebra $\mathscr{A}$. Then all the assumptions of proposition 14.3.21 are fulfilled, so $\mathscr{A}^\nu$ is an étale $\mathscr{O}_X^a$-algebra, and an almost finitely presented $\mathscr{O}_X^a$-module.

Lastly, suppose that $\mathscr{A}$ is an $\mathscr{O}_U^a$-module of finite rank, and pick $r \in \mathbb{N}$ such that, for every $i = 1, \dots, k$, the r-th exterior power of the $\mathscr{O}_U^a(U_i)$-module $\mathscr{A}(U) = A \otimes_{\mathscr{O}_X^a(X)} \mathscr{O}_U^a(U_i)$ vanishes. Since the induced morphism $\mathscr{O}_X^a \to \mathscr{O}_U^a(U_1) \times \cdots \times \mathscr{O}_U^a(U_1)$ is a monomorphism, and A is a flat $\mathscr{O}_X^a$-module, we deduce that the r-th exterior power of A vanishes as well. Especially, $\mathscr{A}^\nu$ is an $\mathscr{O}_X^a$-module of finite rank. □

Lemma 14.4.5. *Let (X, Z) be a normal pair, $j : X \setminus Z \to X$ the open immersion, $\mathscr{B}$ any étale almost finitely presented $\mathscr{O}_X^a$-algebra, and $Z' \subset Z$ any constructible closed subset. We have :*

(i) *The natural map $\mathscr{B} \to j_* j^* \mathscr{B}$ factors through an isomorphism $\mathscr{B} \xrightarrow{\sim} (j^*\mathscr{B})^\nu$.*
(ii) *The restriction functor* (14.4.2) *is fully faithful.*
(iii) *The pair (X, Z') is normal.*
(iv) *If the pair (X, Z) is almost pure, the same holds for (X, Z').*

Proof. (i): Set $U := X \setminus Z$ and $\mathscr{A} := \mathscr{B}_{|U}$; the natural map $\mathscr{B} \to j_*\mathscr{A}$ factors through a morphism $\varphi : \mathscr{B} \to \mathscr{A}^\nu$ of $\mathscr{O}_X^a$-algebras (lemma 14.3.9 and remark 14.3.12(ii)). Fix $z \in Z$, and set $R := \mathscr{O}_{X,z}^a$, $R' := (j_*\mathscr{O}_X^a)_z$. It follows that the stalk $\mathscr{B}_z$ is an étale R-algebra. Notice that $(j_*\mathscr{A})_z = R' \otimes_R \mathscr{B}_z$, therefore φ is a monomorphism. Moreover, since $\mathscr{B}$ is also an almost finitely presented $\mathscr{O}_X^a$-module, we have

$$\mathscr{B}_z = \text{i.c.}(\mathscr{B}_z, R' \otimes_R \mathscr{B}_z) \qquad \mathscr{A}_z^\nu = \text{i.c.}(R, R' \otimes_R \mathscr{B}_z)$$

(proposition 14.3.25). On the other hand, $\mathscr{B}_z$ is an integral R-algebra, therefore $\mathscr{B}_z = \mathscr{A}_z^\nu$, and since z is arbitrary, we conclude that φ is an isomorphism, as asserted.

(ii): Let $\mathscr{B}_1$ and $\mathscr{B}_2$ be two étale almost finitely presented $\mathscr{O}_X^a$-algebra, set $\mathscr{A}_i := \mathscr{B}_{i|U}$ for $i = 1, 2$, and let $\psi : \mathscr{A}_1 \to \mathscr{A}_2$ be any morphism of $\mathscr{O}_U^a$-algebras; by (i), ψ extends uniquely to a morphism $\psi^\nu : \mathscr{B}_1 = \mathscr{A}_1^\nu \to \mathscr{A}_2^\nu = \mathscr{B}_2$ of $\mathscr{O}_X^a$-algebras, whence the contention.

(iii): Set $U' := X \setminus Z'$ and let $j' : U' \to X$ be the open immersion; the natural morphism $\mathscr{O}_X^a \to j_*\mathscr{O}_U$ factors through the morphism $\mathscr{O}_X^a \to j'_*\mathscr{O}_{U'}$, so the latter is a monomorphism. In order to show that (X, Z') is normal, it then suffices to check that the image of $\mathscr{O}_X^a$ is integrally closed in $j'_*\mathscr{O}_{U'}^a$, and since $\mathscr{O}_X^a$ is integrally closed in $j_*\mathscr{O}_U^a$, we are reduced to proving that the natural morphism $j'_*\mathscr{O}_{U'}^a \to j_*\mathscr{O}_U^a$ is a monomorphism. However, let $V \subset X$ be any affine open subset, and write $U' \cap V = V_1 \cap \cdots \cap V_n$ for certain affine open subsets $V_1, \dots, V_n$ of X; by assumption, the restriction map $\mathscr{O}_X^a(V_i) \to \mathscr{O}_X^a(U \cap V_i)$ is a monomorphism for every $i = 1, \dots, n$, so the same holds for the restriction map $\mathscr{O}_X^a(U' \cap V) \to \mathscr{O}_X^a(U \cap V)$, whence the assertion.

(iv): Suppose that (X, Z) is almost pure; in view of (ii), in order to check that (X, Z') is almost pure, it suffices to show that every étale $\mathscr{O}_{U'}^a$-algebra $\mathscr{A}$ of finite rank extends to an étale $\mathscr{O}_X^a$-algebra of finite rank. However, the assumption implies that $\mathscr{A}_{|U}$ extends to an $\mathscr{O}_X^a$-algebra $\mathscr{B}$ as sought, and since $(X \setminus Z', Z \setminus Z')$ is normal (remark 14.4.3(iii)), (ii) says that the isomorphism $\mathscr{B}_{|U} \xrightarrow{\sim} \mathscr{A}$ extends to an isomorphism $\mathscr{B}_{|U'} \xrightarrow{\sim} \mathscr{A}$, *i.e.* $\mathscr{B}$ is an extension of $\mathscr{A}$, as required. □

Lemma 14.4.6. *Let $(A_i \mid i \in \mathbb{N})$ be a system of V^a-algebras, and set $A := \prod_{i \in \mathbb{N}} A_i$. Let also P be an A-module, B an A-algebra, and suppose that*

(a) $\lim_{i \to \infty} \mathrm{Ann}_{V^a}(A_i) = V^a$ *for the uniform structure of* [75, Def.2.3.1].
(b) *For every $i \in \mathbb{N}$, the A_i-modules $P_i := P \otimes_A A_i$, $B_i := B \otimes_A A_i$ are almost projective of finite constant rank equal to i, and B_i is an étale A_i-algebra.*

Then P is an almost projective A-module of almost finite rank, and B is an étale A-algebra.

Proof. For every $j \in \mathbb{N}$, the finite product $P_{\leq j} := \prod_{i=1}^{j} P_i$ is an almost projective A-module of finite rank, and clearly the induced morphism $\pi_j : P \to P_{\leq j}$ is an epimorphism. On the other hand, the $(j+1)$-th exterior power of P equals the $(j+1)$-th exterior power of $\mathrm{Ker}\,\pi_j$, and from condition (i) we see that $\lim_{j\to\infty} \mathrm{Ann}_{V^a}(\mathrm{Ker}\,\pi_j) = 0$, whence the assertion for P.

It follows already that B is an almost projective A-module. It remains to show that B is an unramified A-algebra, to which aim, we may apply the criterion of [75, Prop.3.1.4].

Claim 14.4.7. Under the assumptions of the lemma, the natural morphism

$$\varphi : B \otimes_A B \to C := \prod_{i \in \mathbb{N}} B_i \otimes_{A_i} B_i$$

is an isomorphism of A-algebras.

Proof of the claim. For every $j \in \mathbb{N}$, let $\pi_j : C \to \prod_{i \leq j} B_i \otimes_A B_i$ be the natural morphism. Then

$$\lim_{j\to\infty} \mathrm{Ann}_{V^a}\mathrm{Ker}(\pi_j \circ \varphi) = V^a = \lim_{j\to\infty} \mathrm{Ann}_{V^a}\mathrm{Ker}\,\pi_j.$$

The first identity implies that φ is a monomorphism. Next, since $\pi_j \circ \varphi$ is an epimorphism, the natural morphism $\mathrm{Ker}\,\pi_j \to \mathrm{Coker}\,\varphi$ is an epimorphism; then the second identity implies that φ is also an epimorphism. ◊

Now, by [75, Prop.3.1.4], for every $i \in \mathbb{N}$ there exists an idempotent $e_i \in (B_i \otimes_{A_i} B_i)_*$ uniquely characterized by the conditions (i)–(iii) of *loc.cit.* In view of claim 14.4.7, the sequence $(e_i \mid i \in \mathbb{N})$ defines an idempotent in $(B \otimes_A B)_*$, which clearly fulfills the same conditions, whence the contention, again by [75, Prop.3.1.4]. □

Proposition 14.4.8. *Let (X, Z) be a normal pair, and set $U := X \setminus Z$. Then the following conditions are equivalent :*

(a) *The pair (X, Z) is almost pure.*

(b) *The restriction functor*

$$\mathscr{O}_X^a\text{-}\mathbf{\acute{E}t}_{\mathrm{fr}} \to \mathscr{O}_U^a\text{-}\mathbf{\acute{E}t}_{\mathrm{fr}} \quad : \quad \mathscr{A} \mapsto \mathscr{A}_{|U}$$

from the category of étale $\mathscr{O}_X^a$-algebras of finite rank, to the category of étale $\mathscr{O}_U^a$-algebras of finite rank, is an equivalence.

(c) *For every étale $\mathscr{O}_U^a$-algebra $\mathscr{A}$ of finite rank, the normalization $\mathscr{A}^\nu$ of $\mathscr{A}$ over X is an étale $\mathscr{O}_X^a$-algebra of finite rank (see remark* 14.4.3(ii)*).*

(d) *For every étale $\mathscr{O}_U^a$-algebra $\mathscr{A}$ of finite rank, $\mathscr{A}^\nu$ is a weakly unramified $\mathscr{O}_X^a$-algebra.*

(e) *Every étale $\mathscr{O}_U^a$-algebra $\mathscr{A}$ of finite rank extends to an étale almost finite $\mathscr{O}_X^a$-algebra.*

Proof. The equivalence (b)⇔(c) is already clear from lemma 14.4.5(i,ii).

Next, clearly (c)⇒(d). Conversely, suppose that (d) holds, and let $\mathscr{A}$ be any étale $\mathscr{O}_U^a$-algebra of finite rank, so $\mathscr{A}^\nu$ is a weakly unramified $\mathscr{O}_X^a$-algebra. Then $\mathscr{A}^\nu$ is actually an étale $\mathscr{O}_X^a$-algebra, and an almost finitely presented $\mathscr{O}_X^a$-module (lemma 14.4.4). Fix any affine open subset $V \subset X$, and pick a finite covering of $U \cap V$ consisting of affine open subsets $V_1, \dots, V_k$ of U. Let $r \in \mathbb{N}$ be an integer such that the r-th exterior power of the $\mathscr{O}_X^a(V_i)$-modules $\mathscr{A}(V_i)$ vanish. The induced map

$$B := \mathscr{O}_X^a(V) \to B' := \mathscr{O}_X^a(V_1) \times \cdots \times \mathscr{O}_X^a(V_k)$$

is a monomorphism, since (X, Z) is a normal pair. Moreover, the r-th exterior power of the B'-module $\mathscr{A}^\nu(V) \otimes_B B'$ vanishes; since $\mathscr{A}^\nu(V)$ is a flat B-module, we conclude that the r-th exterior power of $\mathscr{A}^\nu(V)$ vanishes as well. Especially, $\mathscr{A}^\nu$ is of finite rank, so (c) holds.

Since (c)⇒(e), we suppose that (e) holds, and deduce that (d) holds as well. Indeed, let $\mathscr{B}$ be an almost finite étale $\mathscr{O}_X^a$-algebra extending $\mathscr{A}$. In the foregoing, we have already remarked that $\mathscr{B} \subset \mathscr{A}^\nu$; especially, the diagonal idempotent of $\mathscr{A}$ lies in the image of the restriction map

$\mathscr{A}^\nu \otimes_{\mathscr{O}_X^a} \mathscr{A}^\nu(X)_* \to \mathscr{A} \otimes_{\mathscr{O}_X^a} \mathscr{A}(U)_*$. Then proposition 14.3.21 implies that $\mathscr{A}^\nu$ is an étale almost finite $\mathscr{O}_X^a$-algebra, as sought.

Obviously, (a)⇒(e); to conclude, it suffices then to check that (c)⇒(a). Thus, let $\mathscr{A}$ be an étale $\mathscr{O}_U^a$-algebra of almost finite rank, $V \subset U$ an affine open subset, and set $A := \mathscr{O}_X^a(V)$, $B := \mathscr{A}(V)$. According to [75, Th.4.3.28], there exists a decomposition of A as an infinite product of a system of V^a-algebras $(A_i \mid i \in \mathbb{N})$ fulfilling condition (a) of lemma 14.4.6. Such a decomposition determines a system of idempotent elements $e_{V,i} \in A_*$, for every $i \in \mathbb{N}$, such that $e_{V,i} \cdot e_{V,j} = 0$ whenever $i \neq j$, and characterized by the identities $e_{V,i}A = A_i$ for every $i \in \mathbb{N}$. Since $\mathscr{O}_{X*}$ is a sheaf ([75, §5.5.4]), condition (ii) ensures that, for any fixed $i \in \mathbb{N}$, and V ranging over the affine open subsets of U, the idempotents $e_{V,i}$ glue to a global section $e_i \in \mathscr{O}_X^a(U)_*$, which will still be an idempotent; moreover, the direct factor $\mathscr{A}_i := e_i\mathscr{A}$ of $\mathscr{A}$ is an étale $\mathscr{O}_U^a$-algebra of finite rank, and the projection $\mathscr{A} \to \mathscr{A}_i$ is a morphism of $\mathscr{O}_U^a$-algebras. Especially, A and B fulfill as well condition (b) of lemma 14.4.6. By (c), we deduce that the normalization $\mathscr{A}_i^\nu$ of $\mathscr{A}_i$ over X is an étale $\mathscr{O}_X^a$-algebra of finite rank, for every $i \in \mathbb{N}$. On the other hand, notice that the natural morphism

$$\mathscr{O}_{X*}^a \to j_*(\mathscr{O}_{U*}^a) = (j_*\mathscr{O}_U^a)_*$$

is a monomorphism, and its image is integrally closed in $j_*\mathscr{O}_{U*}^a$ ([75, Rem.8.2.30(i)]). It follows easily that $e_i \in \mathscr{O}_{X*}^a(X)$; hence

$$e_i\mathscr{A}_i^\nu = \mathscr{A}_i^\nu \qquad \text{for every } i \in \mathbb{N}$$

and the latter is the integral closure of $e_i\mathscr{O}_X^a$ in $j_*(\mathscr{A}_i)$. Consequently, $\mathscr{A}^\nu = \prod_{i\in\mathbb{N}} \mathscr{A}^\nu$ ([75, Rem.8.2.30(ii)]). By lemma 14.4.6, it follows that $\mathscr{A}^\nu$ is an étale $\mathscr{O}_X^a$-algebra of almost finite rank. This shows that the functor (14.4.2) is essentially surjective; combining with lemma 14.4.5(ii), we get (a). □

Corollary 14.4.9. *Let (X, Z) be a normal pair, and suppose that :*

(a) *The scheme $Z(z)$ is finite dimensional, for every $z \in Z$.*
(b) *The pair $(X(z), \{z\})$ is almost pure, for every $z \in Z$.*

Then the pair (X, Z) is almost pure.

Proof. Let $\mathscr{A}$ be any étale $\mathscr{O}_U^a$-algebra of finite rank, and $\mathscr{A}^\nu$ the normalization of $\mathscr{A}$ over X. In light of proposition 14.4.8, it suffices to show that $\mathscr{A}_z^\nu$ is a weakly unramified $\mathscr{O}_{X,z}^a$-algebra, for every $z \in Z$. Suppose, by way of contradiction, that the latter assertion fails; in view of condition (a), we may then find a point $z \in Z$, such that $\mathscr{A}_z^\nu$ is not weakly unramified over $\mathscr{O}_{X,z}^a$, but for every proper generization $w \in Z$ of z, the $\mathscr{O}_{X,w}^a$-algebra $\mathscr{A}_w^\nu$ is weakly unramified.

Now, set $U(z) := U \cap X(z)$, $V(z) := X(z)\setminus\{z\}$, and let $j : V(z) \to U$ be the induced morphism. Notice that $(V(z), Z(z))$ is a normal pair, and our assumption implies that $j^*\mathscr{A}^\nu$ is a weakly unramified $\mathscr{O}_{V(z)}^a$-algebra. By lemma 14.4.4, it follows that $j^*\mathscr{A}^\nu$ is actually an étale $\mathscr{O}_{V(z)}^a$-algebra of finite rank. Since by assumption, $(X(z), \{z\})$ is almost pure, proposition 14.4.8 says that $\mathscr{A}_z^\nu$ is an étale $\mathscr{O}_{X,z}^a$-module of finite rank, a contradiction. □

Proposition 14.4.10. *Let (X, Z) be a normal pair, and $f : X' \to X$ a morphism of S-schemes; set $Z' := f^{-1}Z$, and suppose that :*

(a) *$f(Z') = Z$ and f is pro-smooth at every point of Z' (see definition* 9.8.5*).*
(b) *The pair (X', Z') is almost pure.*

Then the pair (X, Z) is almost pure.

Proof. Set $U := X\setminus Z$, $U' := X'\setminus Z'$, and let $f_{|U} : U' \to U$ be the restriction of f. Notice that the open immersion $j' : U' \to X'$ is quasi-compact ([59, Ch.I, Prop.6.6.4]), and since f is flat at every point of Z' (corollary 9.8.6(i)), U' is dense in X'. Moreover :

Claim 14.4.11. (i) The pair (X', Z') is normal.

(ii) More generally, let $\mathscr{A}$ be any quasi-coherent $\mathscr{O}_U^a$-algebra. Then the natural morphism

$$f^*(\mathscr{A}^\nu) \to (f^*_{|U}\mathscr{A})^\nu$$

is an isomorphism (notation of remark 14.4.3(ii)).

Proof of the claim. Since f is flat at every point of Z', and the unit of adjunction $\mathscr{O}_X^a \to j_*\mathscr{O}_U^a$ is a monomorphism, the induced morphism $\mathscr{O}_{X'}^a \to f^*j_*\mathscr{O}_U^a = j'_*\mathscr{O}_{U'}^a$ is a monomorphism as well (corollary 10.3.8). Also, $(\mathscr{O}_U^a)^\nu = \mathscr{O}_X^a$, since (X, Z) is a normal pair; hence, (ii) implies (i).

(ii): Denote $\mathscr{A}_{!!}^\nu$ the integral closure in $j_*(\mathscr{A}_{!!})$ of the image of of $\mathscr{O}_X$; by corollary 10.3.8, the natural map

$$f^*j_*(\mathscr{A}_{!!}) \to j'_*f^*_{|U}(\mathscr{A}_{!!})$$

is an isomorphism of $\mathscr{O}_{X'}$-algebras. On the other hand, corollary 9.8.6(ii) implies that the integral closure of the image of $\mathscr{O}_{X'}$ in $f^*j_*(\mathscr{A}_{!!})$ equals $f^*(\mathscr{A}_{!!}^\nu)$. The assertion follows. ◊

Suppose now that $\mathscr{A}$ is an étale $\mathscr{O}_U^a$-algebra of finite rank. Since, by assumption, (X', Z') is almost pure, proposition 14.4.8 says that $(f^*_{|U}\mathscr{A})^\nu$ is an étale $\mathscr{O}_{X'}^a$-algebra of finite rank. By claim 14.4.11 and lemma 14.3.26(i) and corollary 9.8.6(ii), we deduce that $\mathscr{A}^\nu$ is a weakly étale $\mathscr{O}_X^a$-algebra. To conclude the proof, it suffices now to invoke proposition 14.4.8. □

Lemma 14.4.12. *Let (A, I) be a tight henselian pair (see* [75, Def.5.1.9]*). Then we have :*

(i) *The base change functor*

$$\mathsf{Cov}(\operatorname{Spec} A) \to \mathsf{Cov}(\operatorname{Spec} A/I)$$

is an equivalence (notation of [75, §8.2.22]*).*

(ii) *More generally, the base change functor A-$\mathbf{\acute{E}t} \to A/I$-$\mathbf{\acute{E}t}$ restricts to an equivalence*

$$A\text{-}\mathbf{\acute{E}t}_{\mathrm{afr}} \xrightarrow{\sim} A/I\text{-}\mathbf{\acute{E}t}_{\mathrm{afr}}$$

on the respective full subcategories of étale algebras of almost finite rank.

Proof. In view of [75, Th.5.5.7(iii)], the assertions follow from lemma 14.3.17(ii). □

14.4.13. Let R be a V-algebra, $I \subset R$ a principal ideal generated by a regular element, $R^\wedge$ the I-adic completion of R. Set

$$X := \operatorname{Spec} R \qquad X^\wedge := \operatorname{Spec} R^\wedge \qquad Z := \operatorname{Spec} R/I \qquad Z^\wedge := \operatorname{Spec} R^\wedge/IR^\wedge.$$

Proposition 14.4.14. *In the situation of* (14.4.13)*, suppose furthermore that :*

(i) *There exist $n \in \mathbb{N}$ and a finitely generated subideal $\mathfrak{m}_0 \subset \mathfrak{m}$ such that $I^n \subset \mathfrak{m}_0 R$.*

(ii) *The pair (R, I) is henselian.*

Then the pair (X, Z) is almost pure if and only if the same holds for the pair $(X^\wedge, Z^\wedge)$.

Proof. Under the current assumptions, the pair (R^a, I^a) is tight henselian ([75, §5.1.12]). Hence the assertion is a straightforward consequence of [75, Prop.5.4.53] and lemma 14.4.12(i) (details left to the reader). □

14.4.15. Let $p \in \mathbb{Z}$ be a prime integer with $pV \subset \mathfrak{m}$, and such that p is a regular element of V, and suppose that V contains an element, denoted $p^{1/p}$, whose p-th power generates the ideal pV. Let A and B be two flat V-algebras, and for every $n \in \mathbb{N}$ set $V_n := V/p^{n+1}V$, $A_n := A/p^{n+1}A$, $B_n := B/p^{n+1}B$. Let also

$$A_0 \xrightarrow{\sim} B_0 \qquad (14.4.16)$$

be a given isomorphism of V_0-algebras, and suppose that the Frobenius endomorphisms of V_0 and A_0 induce isomorphisms

$$V/p^{1/p}V \xrightarrow{\sim} V_0 \qquad A/p^{1/p}A \xrightarrow{\sim} A_0. \qquad (14.4.17)$$

Set $X := \mathrm{Spec}\, A$, $X_0 := \mathrm{Spec}\, A_0$, $Y := \mathrm{Spec}\, B$ and $Y_0 := \mathrm{Spec}\, B_0$.

Corollary 14.4.18. *In the situation of* (14.4.15)*, suppose moreover that the pairs* (A, pA) *and* (B, pB) *are henselian. Then the pair* (X, X_0) *is almost pure if and only if the same holds for the pair* (Y, Y_0).

Proof. In view of proposition 14.4.14, it suffices to show that (14.4.16) lifts to an isomorphism $A^\wedge \stackrel{\sim}{\to} B^\wedge$ between the p-adic completions of A and B. In turns, this reduces to exhibiting a system of isomorphisms $(\varphi_n : A_n \stackrel{\sim}{\to} B_n \mid n \in \mathbb{N})$ such that $\varphi_n \otimes_{V_n} V_{n_1} = \varphi_{n-1}$ for every $n > 0$. To this aim, it suffices to show that

$$\mathbb{L}_{A_n/V_n} = 0 \qquad \text{in } \mathsf{D}(s.A\text{-}\mathbf{Mod}) \text{ for every } n \in \mathbb{N}$$

([75, Prop.3.2.16]). However, in view of [75, Th.2.5.36], for every $n \in \mathbb{N}$, the short exact sequence $0 \to p^{n+1}V/p^{n+2}V \to V_{n+1} \to V_n \to 0$ induces a distinguished triangle in $\mathsf{D}(s.A\text{-}\mathbf{Mod})$

$$\mathbb{L}_{A_0/V_0} \to \mathbb{L}_{A_{n+1}/V_{n+1}} \to \mathbb{L}_{A_n/V_n} \to \sigma\mathbb{L}_{A_0/V_0}.$$

Hence, an easy induction further reduces to checking that $\mathbb{L}_{A_0/V_0} = 0$ in $\mathsf{D}(s.A\text{-}\mathbf{Mod})$. According to [75, Lemma 6.5.13], this will follow, once we have shown that the natural map

$$V_{0,(\Phi)} \overset{\mathbf{L}}{\otimes}_{V_0} A_0 \to A_{0,(\Phi)}$$

is an isomorphism in $\mathsf{D}(V_0\text{-}\mathbf{Mod})$ (notation of *loc.cit.*). Taking into account the isomorphisms (14.4.17), this holds if and only if the natural map

$$V/p^{1/p}V \overset{\mathbf{L}}{\otimes}_{V_0} A_0 \to A/p^{1/p}A$$

is an isomorphism in $\mathsf{D}(V_0\text{-}\mathbf{Mod})$. The latter assertion is clear, since A is a flat V-algebra. □

14.5. **Normalized lengths.** Let $(V, |\cdot|)$ be any valuation ring, with value group Γ_V; according to our general conventions, the composition law of Γ_V is denoted multiplicatively; however, sometimes it is convenient to switch to an additive notation. Hence, we adopt the notation :

$$(\log \Gamma_V, \leq)$$

to denote the ordered group Γ_V with additive composition law and whose ordering is the reverse of the original ordering of Γ_V. The unit of $\log \Gamma_V$ shall be naturally denoted by 0, and we shall extend the ordering of $\log \Gamma_V$ by adding a largest element $+\infty$, as customary. Also, we set : $\log \Gamma_V^+ := \{\gamma \in \log \Gamma_V \mid \gamma \geq 0\}$ and $\log |0| := +\infty$.

14.5.1. In this section, $(K, |\cdot|)$ denotes a valued field of rank one, with value group Γ. We let κ be the residue field of K^+. As usual, we set $S := \mathrm{Spec}\, K^+$, and denote by s (resp. η) the closed (resp. generic) point of S. Let $\mathfrak{m}_K \subset K^+$ be the maximal ideal, and set $\mathfrak{m} := \mathfrak{m}_K$ in case Γ is not discrete, or else $\mathfrak{m} := K^+$, in case $\Gamma \simeq \mathbb{Z}$; in the following, whenever we refer to almost rings or almost modules, we shall assume – unless otherwise stated – that the underlying almost ring theory is the one defined by the standard setup $(K^+, \mathfrak{m})$ (see [75, §6.1.15]).

Let A be any K^{+a}-algebra, and c a cardinal number; following [75, §2.3], we denote by $\mathscr{M}_c(A)$ the set of isomorphism classes of K^{+a}-modules which admit a set of generators of cardinality $\leq c$. The set $\mathscr{M}_c(A)$ carries a natural uniform structure (see [75, Def.2.3.1]), which admits the fundamental system of entourages

$$(E_\gamma \mid \gamma \in \log \Gamma^+ \setminus \{0\})$$

defined as follows. For any $b \in \mathfrak{m} \setminus \{0\}$, we let $E_{|b|}$ be the set of all pairs (M, M') of elements of $\mathscr{M}_c(A)$ such that there exist a third A-module N and A-linear morphisms $N \to M$, $N \to M'$ whose kernel and cokernel are annihilated by b.

14.5.2. The aim of this section is to define and study a well-behaved notion of *normalized length* for torsion modules M over K^+-algebras of a fairly general type. This shall be achieved in several steps. Let us first introduce the categories of algebras with which we will be working.

Definition 14.5.3. Let V be any valuation ring, with maximal ideal $\mathfrak{m}_V$.

(i) We let $V\text{-}\mathbf{m}.\mathbf{Alg}_0$ be the subcategory of V-Alg whose objects are the local and essentially finitely presented V-algebras A whose maximal ideal $\mathfrak{m}_A$ contains $\mathfrak{m}_V A$. The morphisms $\varphi : A \to B$ in $K^+\text{-}\mathbf{m}.\mathbf{Alg}_0$ are the local maps. Notice that every morphism in $V\text{-}\mathbf{m}.\mathbf{Alg}_0$ is an essentially finitely presented ring homomorphism ([63, Ch.IV, Prop.1.4.3(v)]).

Recall as well, that every object of $V\text{-}\mathbf{m}.\mathbf{Alg}_0$ is a coherent ring (see (11.4)). More generally, if $\underline{A} := (A_i \mid i \in I)$ is any filtered system of essentially finitely presented V-algebras with flat transition morphisms, then the colimit of $\underline{A}$ is still a coherent ring (lemma 11.3.7(ii.a)).

(ii) We say that a local V-algebra A is *measurable* if it admits an ind-étale local map of V-algebras $A_0 \to A$, from some object A_0 of $V\text{-}\mathbf{m}.\mathbf{Alg}_0$. The measurable V-algebras form a category $V\text{-}\mathbf{m}.\mathbf{Alg}$, whose morphisms are the local maps of V-algebras. As noted above, every measurable V-algebra is a coherent ring.

Lemma 14.5.4. *In the situation of definition* 14.5.3*, let* $(A, \mathfrak{m}_A)$ *be any measurable* V*-algebra. Then the following holds :*

(i) $A/\mathfrak{m}_V A$ *is a noetherian ring.*

(ii) *If the valuation of* V *has finite rank, every* $\mathfrak{m}_A$*-primary ideal of* A *contains a finitely generated* $\mathfrak{m}_A$*-primary subideal.*

(iii) *For every* A*-module* M *of finite type supported at the closed point of* $\operatorname{Spec} A$*, the* A*-module* $M/\mathfrak{m}_V M$ *has finite length.*

(iv) *For every finitely generated ideal* $I \subset A$*, the* V*-algebra* A/I *is measurable.*

Proof. (i): Let A^{sh} be the strict henselization of A at a geometric point localized at the closed point; then $A^{\mathrm{sh}}/\mathfrak{m}_V A^{\mathrm{sh}}$ is a strict henselization of $A/\mathfrak{m}_V A$ ([66, Ch.IV, Prop.18.8.10]), and it is therefore also a strict henselization of a $V/\mathfrak{m}_V$-algebra of finite type. Hence $A^{\mathrm{sh}}/\mathfrak{m}_V A^{\mathrm{sh}}$ is noetherian, and then the same holds for $A/\mathfrak{m}_V A$ ([66, Ch.IV, Prop.18.8.8(iv)]).

(iii) follows easily from (i) : the details shall be left to the reader.

(ii): If the valuation of V has finite rank, there exists an element $t_0 \in \mathfrak{m}_V$ that generates a $\mathfrak{m}_V$-primary ideal. Let $I \subset A$ be a $\mathfrak{m}_A$-primary ideal; then $t_0^N \in I$, for some integer $N > 0$; moreover, the image $\overline{I}$ of I in $A/\mathfrak{m}_V A$ is finitely generated, by (i). Pick elements $t_1, \dots, t_n \in I$ whose images in $A/\mathfrak{m}_V A$ form a system of generators of $\overline{I}$; it follows that $t_0^N, t_1, \dots, t_n$ form a $\mathfrak{m}_A$-primary ideal contained in I.

(iv): We may find an ind-étale local morphism $A_0 \to A$ from an object A_0 of $V\text{-}\mathbf{m}.\mathbf{Alg}_0$, and a finitely generated ideal $I_0 \subset A_0$ such that $I = I_0 A$. Then A/I_0 is an object of $V\text{-}\mathbf{m}.\mathbf{Alg}_0$ as well, and the induced map $A_0/I_0 \to A/I$ is ind-étale. □

14.5.5. Now, suppose that V is both a valuation ring and a flat, measurable K^+-algebra, and denote $\mathfrak{m}_V$ (resp. $\kappa(V)$, resp. $|\cdot|_V$) the maximal ideal (resp. the residue field, resp. the valuation) of V. We claim that V has rank one, and the ramification index $(\Gamma_V : \Gamma)$ is finite. Indeed, let us write V as the colimit of a filtered system $(V_i \mid i \in I)$ of objects of $K^+\text{-}\mathbf{m}.\mathbf{Alg}_0$ with essentially étale transition maps; it follows that each V_i is a valuation ring of rank one (claim 9.1.36). Also, the transition maps induce isomorphisms on the value groups : indeed, this is clear if K^+ is a discrete valuation ring, since in that case the same holds for the V_i, and the transition maps are unramified by assumption; in the case where Γ is not discrete, the assertion follows from corollary 11.4.42. Therefore, we are reduced to the case where V is an object of $K^+\text{-}\mathbf{m}.\mathbf{Alg}_0$, to which corollary 11.4.42 applies.

Suppose first that M is a finitely generated torsion V-module; in this case the Fitting ideal $F_0(M) \subset V$ is well defined, and it is shown in [75, Lemma 6.3.1 and Rem.6.3.5] that the map

$M \mapsto F_0(M)$ is additive on the set of isomorphism classes of finitely generated V-modules, *i.e.* for every short exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ of such modules, one has :

(14.5.6) $$F_0(M_2) = F_0(M_1) \cdot F_0(M_3).$$

Next, the almost module $F_0(M)^a$ is an element of the group of fractional ideals $\mathrm{Div}(V^a)$ defined in [75, §6.1.16], and there is a natural isomorphism

(14.5.7) $$\mathrm{Div}(V^a) \simeq \log \Gamma_V^\wedge \qquad I \mapsto |I|$$

where $\Gamma_V^\wedge$ is the completion of Γ_V for the uniform structure deduced from the ordering : see [75, Lemma 6.1.19]. Hence we may define :

$$\lambda_V(M) := (\Gamma_V : \Gamma) \cdot |F_0(M)^a| \in \log \Gamma_V^\wedge.$$

In view of (14.5.6), we see that :

(14.5.8) $$\lambda_V(M') \leq \lambda_V(M) \qquad \text{whenever } M' \subset M \text{ are finitely generated.}$$

More generally, if M is any torsion V-module, we let :

(14.5.9) $$\lambda_V(M) := \sup \{\lambda_V(M') \mid M' \subset M,\ M' \text{ finitely generated}\} \in \log \Gamma_V^\wedge \cup \{+\infty\}$$

which, in view of (14.5.8), agrees with the previous definition, in case M is finitely generated.

14.5.10. Suppose now that the V^a-module M^a is uniformly almost finitely generated. Then according to [75, Prop.2.3.23] one has a well defined Fitting ideal $F_0(M^a) \subset V^a$, which agrees with $F_0(M)^a$ in case M is finitely generated.

Lemma 14.5.11. *In the situation of* (14.5.10)*, we have :*

(i) $\lambda_V(M) = (\Gamma_V : \Gamma) \cdot |F_0(M^a)|$.

(ii) *If* $0 \to N_1 \to N_2 \to N_3 \to 0$ *is any short exact sequence of uniformly almost finitely generated* V^a*-modules, then*

$$|F_0(N_2)| = |F_0(N_1)| + |F_0(N_3)|.$$

Proof. (ii): It is a translation of [75, Lemma 6.3.1 and Rem.6.3.5(ii)].

(i): Let k be a uniform bound for M^a, and denote by $\mathscr{I}_k(M)$ the set of all submodules of M generated by at most k elements. Set $e := (\Gamma_V : \Gamma)$; if $N \in \mathscr{I}_k(M)$, then $\lambda_V(N) \leq e \cdot |F_0(M^a)|$, due to (ii). By inspecting the definitions, it then follows that

$$e \cdot |F_0(M^a)| = \sup \{\lambda_V(N) \mid N \in \mathscr{I}_k(M)\} \leq \lambda_V(M).$$

Now, suppose $M' \subset M$ is any submodule generated by, say r elements; for every $\varepsilon \in \mathfrak{m}$ we can find $N \in \mathscr{I}_k(M)$ such that $\varepsilon M \subset N$, hence $\varepsilon M' \subset N$, therefore $\lambda_V(\varepsilon M') \leq \lambda_V(N) \leq e \cdot |F_0(M^a)|$. However, $\lambda_V(M') - \lambda_V(\varepsilon M') = \lambda_V(M'/\varepsilon M') \leq \lambda_V(K^{+\oplus r}/\varepsilon K^{+\oplus r}) = r \cdot \lambda_V(K^+/\varepsilon K^+)$. We deduce easily that $\lambda_V(M') \leq e \cdot |F_0(M^a)|$, whence the claim. □

Proposition 14.5.12. (i) *Let* $0 \to M_1 \to M_2 \to M_3 \to 0$ *be a short exact sequence of torsion* V*-modules. Then :*

$$\lambda_V(M_2) = \lambda_V(M_1) + \lambda_V(M_3).$$

(ii) $\lambda_V(M) = 0$ *if and only if* $M^a = 0$.

(iii) *Let* M *be any torsion* V*-module. Then :*

(a) *If* M *is finitely presented,* $\lambda_V(N) > 0$ *for every non-zero submodule* $N \subset M$.

(b) *If* $(M_i \mid i \in I)$ *is a filtered system of submodules of* M*, then :*

$$\lambda_V(\operatorname*{colim}_{i \in I} M_i) = \lim_{i \in I} \lambda_V(M_i).$$

Proof. (ii): By lemma 14.5.11(i) it is clear that $\lambda_V(M) = 0$ whenever $M^a = 0$. Conversely, suppose that $\lambda_V(M) = 0$, and let $m \in M$ be any element; then necessarily $\lambda_V(Vm) = 0$, and then it follows easily that $\mathfrak{m} \subset \mathrm{Ann}_V(m)$, so $M^a = 0$.

(iii.a): It suffices to show that M does not contain non-zero elements that are annihilated by $\mathfrak{m}$, which follows straightforwardly from [75, Lemma 6.1.14].

(iii.b): Set $M := \mathrm{colim}_{i\in I} M_i$. By inspecting the definitions we see easily that $\lambda_V(M) \geq \sup\{\lambda_V(M_i) \mid i \in I\}$. To show the converse inequality, let $N \subset M$ be any finitely generated submodule; we may find $i \in I$ such that $N \subset M_i$, hence $\lambda_V(N) \leq \lambda_V(M_i)$, and the assertion follows.

(i): Set $e := (\Gamma_V : \Gamma)$. We shall use the following :

Claim 14.5.13. Every submodule of a uniformly almost finitely generated V^a-module is uniformly almost finitely generated.

Proof of the claim. Let $N' \subset N$, with N uniformly almost finitely generated, and let k be a uniform bound for N; for every $\varepsilon \in \mathfrak{m}$ we can find $N'' \subset N$ such that N'' is generated by at most k almost elements and $\varepsilon N \subset N''$. Clearly, it suffices to show that $N' \cap N''$ is uniformly almost finitely generated and admits k as a uniform bound, so we may replace N by N'' and N' by $N' \cap N''$, and assume from start that N is finitely generated. Let us pick an epimorphism $\varphi : (V^a)^{\oplus k} \to N$; it suffices to show that $\varphi^{-1}(N')$ is uniformly almost finitely generated with k as a uniform bound, so we are further reduced to the case where N is free of rank k. Then we can write $N' = L^a$ for some submodule $L \subset V^{\oplus k}$; notice that $(V^{\oplus k}/L)^a$ is almost finitely presented, since it is finitely generated ([75, Prop.6.3.6(i)]), hence N' is almost finitely generated ([75, Lemma 2.3.18(iii)]). Furthermore, L is the colimit of the family $(L_i \mid i \in I)$ of its finitely generated submodules, and each L_i is a free V-module ([34, Ch.VI, §3, n.6, Lemma 1]). Necessarily the rank of L_i is $\leq k$ for every $i \in I$, hence $\Lambda^{k+1}_V L \simeq \mathrm{colim}_{i\in I} \Lambda^{k+1}_V L_i = 0$, and then the claim follows from [75, Prop.6.3.6(ii)]. ◊

Now, let $N \subset M_2$ be any finitely generated submodule, $\overline{N} \subset M_3$ the image of N; by claim 14.5.13, $M_1 \cap N$ is uniformly almost finitely generated, hence lemma 14.5.11(i,ii) shows that :

$$\lambda_V(N) = e \cdot |F_0(N^a)| = e \cdot |F_0(M_1^a \cap N^a)| + e \cdot |F_0(\overline{N}^a)| = \lambda_V(M_1 \cap N) + \lambda_V(\overline{N}).$$

Taking the supremum over the family $(N_i \mid i \in I)$ of all finitely generated submodules of M_2 yields the identity :

$$\lambda_V(M_2) = \lambda_V(M_3) + \sup\{\lambda_V(M_1 \cap N_i) \mid i \in I\}.$$

By definition, $\lambda_V(M_1 \cap N_i) \leq \lambda_V(M_1)$ for every $i \in I$; conversely, every finitely generated submodule of M_1 is of the form N_i for some $i \in I$, whence the contention. □

Remark 14.5.14. Suppose that $\Gamma \simeq \mathbb{Z}$, and let $\gamma_0 \in \log \Gamma^+$ be the positive generator. Then by a direct inspection of the definition one finds the identity :

$$\lambda(M) = (\Gamma_V : \Gamma) \cdot \mathrm{length}_V(M) \cdot \gamma_0$$

for every torsion V-module M. The verification shall be left to the reader.

14.5.15. Let M be a torsion V-module, such that M^a is almost finitely generated; we wish now to explain that $\lambda_V(M)$ can also be computed in terms of a suitable sequence of elementary divisors for M^a. Indeed, suppose first that M is a finitely presented torsion V-module; then we have a decomposition

$$M = (V/a_0V) \oplus \cdots \oplus (V/a_nV) \qquad \text{where } n := \dim_{\kappa(V)} M/\mathfrak{m}_V M - 1 \tag{14.5.16}$$

for certain $a_0, \dots, a_n \in \mathfrak{m}_V$ ([75, Lemma 6.1.14]). Clearly $\gamma_i := \log |a_i|_V > 0$ for every $i = 0, \dots, n$, and after reordering we may assume that $\gamma_0 \geq \cdots \geq \gamma_n$; then we may set

$\gamma_i := 0$ for every $i > n$, and the resulting sequence $(\gamma_i \mid i \in \mathbb{N})$ of *elementary divisors* of M is independent of the chosen decomposition (14.5.16), since we have more precisely :

$$a_i V = \mathrm{Ann}_V(\Lambda_V^{i+1} M) \qquad \text{for every } i \in \mathbb{N}. \tag{14.5.17}$$

Indeed, from a decomposition (14.5.16) with $a_j V \subset a_{j+1} V$ for every $j < n$, we get a V-linear surjection $M \to (V/a_i V)^{\oplus i+1}$, whence a surjection

$$N_i := \Lambda_V^{i+1} M \to \Lambda_V^{i+1} (V/a_i V)^{\oplus i+1} \stackrel{\sim}{\to} V/a_i V$$

which shows that $J_i := \mathrm{Ann}_V N_i \subset a_i V$. For the converse inclusion, notice that N_i is generated by the system of all elements of the form $\omega := x_0 \wedge \cdots \wedge x_i$, where $x_0 \in M_{j(0)}, \dots, x_i \in M_{j(i)}$ for some strictly increasing map $j : \{0, \dots, i\} \to \{0, \dots, n\}$; especially, $j(i) \geq i$, whence $a_i \omega = 0$, and the assertion follows. Moreover, a simple inspection yields the identity

$$\lambda_V(M) = (\Gamma_V : \Gamma) \cdot (\gamma_0 + \cdots + \gamma_n). \tag{14.5.18}$$

We regard the sequence $(\gamma_i \mid i \in \mathbb{N})$ as an element of the $\log \Gamma_V^\wedge$-normed space $\ell^\infty(\Gamma_V^+)$ of *bounded* sequences of elements of $\log \Gamma_V^{\wedge +}$, *i.e.* the set of all sequences $\underline{\delta} := (\delta_i \mid i \in \mathbb{N})$ with

$$\|\underline{\delta}\| := \sup(\delta_i \mid i \in \mathbb{N}) < +\infty.$$

Lemma 14.5.19. *Let $\varphi : N \to N'$ be a map of finitely presented torsion V-modules, and denote by $(\gamma_i \mid i \in \mathbb{N})$ (resp. $(\gamma'_i \mid i \in \mathbb{N})$) the sequence of elementary divisors of N (resp. N'). Then :*

(i) *If φ is injective, we have $\gamma_i \leq \gamma'_i$ for every $i \in \mathbb{N}$.*
(ii) *If φ is surjective, we have $\gamma_i \geq \gamma'_i$ for every $i \in \mathbb{N}$.*

Proof. Denote by K_V the field of fractions of V, and set $P^\dagger := \mathrm{Hom}_V(P, K_V/V)$ for every V-module P. We have a natural bilinear pairing $P \otimes_V P^\dagger \to K_V/V$, whence a V-linear map

$$\omega_P : P \to (P^\dagger)^\dagger.$$

Claim 14.5.20. (i) If φ is injective, $\varphi^\dagger := \mathrm{Hom}_V(\varphi, K_V/V)$ is surjective.

(ii) For every finitely presented V-module P, we have :
  (a) The map ω_P is an isomorphism.
  (b) P and $P^\dagger$ are isomorphic V-modules.

Proof of the claim. (ii) is left to the reader. (i) follows immediately from lemma 11.5.9(i). ◇

By virtue of claim 14.5.20(ii.b), the V-modules N and $N^\dagger$ have the same sequences of elementary divisors. Taking into account claim 14.5.20(i), we deduce that (ii)⇒(i), so it remains only to show that (ii) holds. However, if φ is surjective, the same holds for $\Lambda_V^{i+1}\varphi$, and then the assertion follows immediately from (14.5.17). □

Proposition 14.5.21. *Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of finitely presented torsion V-modules. Then for every $i, j, t \in \mathbb{N}$ we have :*

$$\sum_{k=i+j}^{i+j+t} \gamma_k(M) \leq \sum_{k=i}^{i+t} \gamma_k(M') + \sum_{k=j}^{j+t} \gamma_k(M'').$$

Proof. Let us choose a decomposition for M as in (14.5.16), with $a_0 V \subset a_1 V \subset \cdots \subset a_n V$, and set $\overline{M} := (V/b_0 V) \oplus \cdots \oplus (V/b_{i+j+t} V)$, where $b_i = a_{i+j}$ for $i = 0, \dots, i+j$ and $b_i = a_i$ for $i = i+j+1, \dots, i+j+t$. We have an obvious V-linear surjection $M \to \overline{M}$, and we let as well $\overline{M}'$ be the image of M' in $\overline{M}$, and $\overline{M}'' := \overline{M}/\overline{M}'$. By construction, we have $\sum_{k=i+j}^{i+j+t} \gamma_k(M) = \sum_{k=i+j}^{i+j+t} \gamma_k(\overline{M})$, and taking into account lemma 14.5.19(ii), we are easily reduced to checking the stated inequality for the resulting short exact sequence $0 \to \overline{M}' \to \overline{M} \to \overline{M}'' \to 0$. Next, for $k = 0, \dots, i+j+t$ we consider the V-linear map $\varphi_k : V \to V/a_{i+j}V$ that maps 1 to the class of a_{i+j}/b_k; clearly φ_k factors through a V-linear injection $\overline{\varphi}_k : V/b_k V \to V/a_{i+j}V$, and the direct sum of the latter maps is a V-linear injection $\overline{\varphi} : \overline{M} \to N := (V/a_{i+j}V)^{\oplus i+j+t+1}$. Let $N' \subset N$ be the image of $\overline{M}'$, and set $N'' := N/N'$; we notice that

$$\sum_{k=i+j}^{i+j+t} (\gamma_k(N) - \gamma_k(\overline{M})) = |F_0(N)| - |F_0(\overline{M})| = C := \sum_{k=1}^{t} (\log |a_{i+j}|_V - \log |a_{i+j+k}|_V).$$

Taking into account proposition 14.5.12(i), we deduce that

$$\sum_{k=j}^{j+t} (\gamma_k(N'') - \gamma_k(\overline{M}'')) \leq |F_0(N'')| - |F_0(\overline{M}'')| = C.$$

Summing up, we are further reduced to checking the sought inequality for the short exact sequence $0 \to N' \to N \xrightarrow{\pi} N'' \to 0$. Let us remark, quite generally :

Claim 14.5.22. Let A be a local ring, M a finitely generated A-module, F, F' two free A-modules of the same rank $r \in \mathbb{N}$, and $\pi : F \to M$, $\pi' : F' \to M$ two A-linear surjections. Then there exists an A-linear isomorphism $\omega : F \stackrel{\sim}{\to} F'$ such that $\pi' \circ \omega = \pi$.

Proof of the claim. Let κ be the residue field of A, and set $m := \dim_\kappa(M \otimes_A \kappa)$; by Nakayama's lemma, M admits a minimal system of generators $x_1, \dots, x_m$, and we choose $f_1, \dots, f_m \in F$ with $\pi(f_i) = x_i$ for $i = 1, \dots, m$. Next, notice that $G := \mathrm{Ker}\,(\pi \otimes_A \kappa)$ is a κ-vector space of dimension $r - m$, and pick a system of elements $f_{m+1}, \dots, f_r \in F$ whose image in G is a basis; applying again Nakayama's lemma, it is easily seen that the system $f_1, \dots, f_r$ generates the A-module F, and it is therefore a basis of the latter. Likewise, we may find a basis $f'_1, \dots, f'_r$ of F' such that $\pi'(f_i) = x_i$ for $i = 1, \dots, m$ and $\pi'(f_i) = 0$ for $i = m+1, \dots, r$; then the unique A-linear map $\omega : F \to F'$ such that $\omega(f_i) = f'_i$ for $i = 1, \dots, r$ will do. ◊

Now, say that $N'' = (V/c_0V) \oplus \cdots \oplus (V/c_pV)$ is a minimal decomposition with $c_0V \subset \cdots \subset c_pV$; clearly we must have $p \leq i + j + t$ and $a_{i+j} \in c_0V$. Set $c_{p+1} = \cdots = c_{i+j+t} = 1$, and let $\pi'_k : V/a_{i+j+t} \to V/c_kV$ be the natural projection, for $k = 0, \dots, i + j + t$; the direct sum of the maps π'_k is a V-linear surjection $\pi' : N \to N''$, and by virtue of claim 14.5.22 there exists a V-linear automorphism ω of N such that $\pi' \circ \omega = \pi$. Thus, it suffices to show the stated inequality for the short exact sequence $0 \to Q := \omega(N') \to N \xrightarrow{\pi'} N'' \to 0$. But notice that $Q = \bigoplus_{k=0}^{i+j+t} \ker \pi'_k$, and therefore

$$\gamma_k(Q) = \log |a_{i+j}|_V - \log |c_{i+j+t-k}|_V \qquad \text{for } k = 0, \dots, i + j + t.$$

On the other hand $\gamma_k(N) = \log |a_{i+j}|_V$ and $\gamma_k(N'') = \log |c_k|_V$ for $k = 0, \dots, i + j + t$; summing up, we conclude the the sought inequality holds, and indeed it is an equality for this last short exact sequence. □

14.5.23. We fix now a large cardinal number ω, and write just $\mathscr{M}(V^a)$ instead of $\mathscr{M}_\omega(V^a)$. Also, for any $\gamma \in \log \Gamma_V^+$, set $[-\gamma, \gamma] := \{\delta \in \log \Gamma_V \mid -\gamma \leq \delta \leq \gamma\}$. Suppose that N and N' are two finitely presented V-modules such that $(N^a, N'^a) \in E_\delta$ for some $\delta \in \log \Gamma_V^+ \setminus \{0\}$. Say that $\delta = \log |b|_V$ for some $b \in \mathfrak{m}_V$; by standard arguments, we obtain maps $\varphi : N \to N'$ and $\varphi' : N' \to N$ such that $\varphi' \circ \varphi = b^4 \cdot \mathbf{1}_N$ and $\varphi' \circ \varphi = b^4 \cdot \mathbf{1}_{N'}$. Let now $(\gamma_i \mid i \in \mathbb{N})$ (resp. $(\gamma'_i \mid i \in \mathbb{N})$) be the sequence of elementary divisors for N (resp. for N'). Then the sequence of elementary divisors for b^4N is $(\max(0, \gamma_i - 4\delta) \mid i \in \mathbb{N})$, and likewise for b^4N'. In view of lemma 14.5.19, we deduce easily that

$$\gamma_i - \gamma'_i \in [-4\delta, 4\delta] \qquad \text{for every } i \in \mathbb{N}.$$

Consider a torsion almost finitely generated V^a-module M, and recall that M is almost finitely presented ([75, Prop.6.3.6(i)]); we then may attach to M a net of elements of $\ell^\infty(\log \Gamma_V^+)$, as follows. For every $\delta \in \log \Gamma_V^+ \setminus \{0\}$, pick a finitely presented V-module N_δ such that

$(M, N_\delta^a) \in E_\delta$, and denote by $\underline{\gamma}_\delta$ the sequence of elementary divisors of N_δ. The foregoing easily implies that the system $(\underline{\gamma}_\delta \mid \delta \in \log\Gamma_V^+ \setminus \{0\})$ is a net for the uniform structure of $\ell^\infty(\log\Gamma_V^+)$ induced by the norm $\|\cdot\|$. However, $(\ell^\infty(\log\Gamma_V^+), \|\cdot\|)$ is a complete normed space, hence this net converges to a well defined sequence $\underline{\gamma}_M := (\gamma_i \mid i \in \mathbb{N}) \in \ell^\infty(\log\Gamma_V^+)$. It is easily seen that $\underline{\gamma}_M$ is independent of the chosen net, and defines an invariant which we call the *sequence of elementary divisors* of M. A simple inspection of the construction shows that the sequence $\underline{\gamma}_M$ is monotonically decreasing, and

$$\lim_{i \to +\infty} \gamma_i = 0.$$

14.5.24. Denote again by K_V the field of fractions of V; as in the proof of lemma 14.5.19, we consider the functor

$$V^a\text{-}\mathbf{Mod} \to V^a\text{-}\mathbf{Mod} \quad : \quad M \mapsto M^\dagger := \mathrm{alHom}_{V^a}(M, K_V^a/V^a).$$

For every V^a-module M we have a natural V^a-bilinear pairing

$$M \otimes_{V^a} M^\dagger \to K_V^a/V^a$$

which in turns yields a natural transformation

$$M \to (M^\dagger)^\dagger. \tag{14.5.25}$$

Together with the constructions of (14.5.23), we may now extend lemma 14.5.19 *verbatim* to arbitrary almost finitely generated V^a-modules.

Proposition 14.5.26. *Let $\varphi : M \to M'$ be a morphism of almost finitely generated torsion V^a-modules, and denote by $(\gamma_i \mid i \in \mathbb{N})$ (resp. $(\gamma'_i \mid i \in \mathbb{N})$) the sequence of elementary divisors of M (resp. M'). Then :*

(i) *$M^\dagger$ is an almost finitely generated torsion V^a-module, the morphism* (14.5.25) *is an isomorphism, and the sequences of elementary divisors of M and $M^\dagger$ coincide.*
(ii) *If φ is a monomorphism, we have $\gamma_i \leq \gamma'_i$ for every $i \in \mathbb{N}$.*
(iii) *If φ is an epimorphism, we have $\gamma_i \geq \gamma'_i$ for every $i \in \mathbb{N}$.*

Proof. (i): Since M is almost finitely presented ([75, Prop.6.3.6(i)]), the assertion follows immediately from [75, Lemma 2.3.7(iii)] and claim 14.5.20(ii).

(iii): To begin with, we remark more generally :

Claim 14.5.27. Let P, Q be two torsion almost finitely generated V^a-modules, and $\delta \in \log\Gamma_V^+$ any element such that $(P, Q) \in E_\delta$. Denote by $(\gamma_i^P \mid i \in \mathbb{N})$ (resp. $(\gamma_i^Q \mid i \in \mathbb{N})$) the sequence of elementary divisors of P (resp. of Q). Then

$$\gamma_i^P - \gamma_i^Q \in [-4\delta, 4\delta] \qquad \text{for every } i \in \mathbb{N}.$$

Proof of the claim. For any $\varepsilon \in \log\Gamma_V^+$ pick a torsion finitely presented V-module N such that $(P, N^a) \in E_\varepsilon$, and let $(\gamma_i^N \mid i \in \mathbb{N})$ be the sequence of elementary divisors of N. It follows that $(Q, N^a) \in E_{\delta+\varepsilon}$, and from the construction of (14.5.23) we see that

$$\gamma_i^N - \gamma_i^P \in [-4\varepsilon, 4\varepsilon] \qquad \gamma_i^N - \gamma_i^Q \in [-4(\varepsilon+\delta), 4(\varepsilon+\delta)] \qquad \text{for every } i \in \mathbb{N}$$

so $\gamma_i^P - \gamma_i^Q \in [-8\varepsilon - 4\delta, 8\varepsilon + 4\delta]$ for every $i \in \mathbb{N}$. Since ε is arbitrary, the claim follows. ◊

Fix $\delta \in \log\Gamma_V^+ \setminus \{0\}$, pick finitely presented V-modules N, N' such that

$$(N^a, M), (N'^a, M') \in E_\delta$$

and denote by $(\beta_i \mid i \in \mathbb{N})$ (resp. $(\beta'_i \mid i \in \mathbb{N})$) the sequence of elementary divisors of N (resp. of N'). According to claim 14.5.27 we have :

$$\gamma_i - \beta_i, \gamma'_i - \beta'_i \in [-4\delta, 4\delta] \qquad \text{for every } i \in \mathbb{N}. \tag{14.5.28}$$

On the other hand, by the usual arguments we may find V-linear maps $f : N \to M_*$ and $g : M'_* \to N'$ such that $b^2 \cdot \mathrm{Coker}\, f = b^2 \cdot \mathrm{Coker}\, g = 0$ for every $b \in V$ with $\log |b|_V = \delta$. Set $h := g \circ \psi_* \circ f$; it follows easily that $b^5 \cdot \mathrm{Coker}\, h = 0$. The sequence of elementary divisors of $b^5 N'$ is $(\max(0, \beta'_i - 5\delta) \mid i \in \mathbb{N})$, and since $N'' := \mathrm{Im}\, h$ is a finitely presented V-module, the sequence $(\beta''_i \mid i \in \mathbb{N})$ of its elementary divisors satisfies the inequalities

$$\beta_i \geq \beta''_i \geq \beta'_i - 5\delta \qquad \text{for every } i \in \mathbb{N}$$

by lemma 14.5.19(i,ii). Combining with (14.5.28), we deduce that

$$\gamma_i \geq \gamma'_i - 13\delta \qquad \text{for every } i \in \mathbb{N}$$

and since δ is arbitrary, the assertion follows.

(ii): in light of [75, Lemma 2.3.7(iii)] and claim 14.5.20(i), it is easily seen that $\varphi^\dagger$ is an epimorphism, so the assertion follows formally from (i) and (iii), as in the proof of lemma 14.5.19(i). □

One may ask, to which extent the sequence of elementary divisors of a V^a-module M determines the isomorphism class of M. To address this question, we make the following :

Definition 14.5.29. Let $(V, \mathfrak{m})$ be an arbitrary basic setup, A any V^a-algebra, M an A-module and ω a (very large) cardinal number, such that $\mathscr{M}_\omega(A)$ contains the isomorphism class of M. The topological space underlying the uniform space $\mathscr{M}_\omega(A)$ is not necessarily separated, but it admits a maximal separated quotient space $\mathscr{M}^{\mathrm{sep}}_\omega(A)$. The *$\omega$-type of* M is the image in $\mathscr{M}^{\mathrm{sep}}_\omega(A)$ of the isomorphism class of M. If ω' is any cardinal larger than ω, clearly the ω-type of M maps to the ω'-type of M, under the natural map $\mathscr{M}^{\mathrm{sep}}_\omega(A) \to \mathscr{M}^{\mathrm{sep}}_{\omega'}(A)$. For this reason, we shall usually omit explicit mention of ω, and shall call the *type of* M any one of these invariants.

Lemma 14.5.30. *In the situation of definition* 14.5.29*, suppose that M and M' are two A-modules of the same type. Then we have :*

(i) *M is almost finitely presented (resp. almost finitely generated) if and only if the same holds for M'.*
(ii) *M is a flat (resp. almost projective) A-module if and only if the same holds for M'.*

Proof. (i) follows directly from the definitions.

(ii) follows easily from [75, Lemma 2.3.7(iii,iv)] : details left to the reader. □

We may now answer as follows to the foregoing question :

Proposition 14.5.31. *Let $(V, \mathfrak{m}_V)$ be as in* (14.5.5)*, and M, M' any two almost finitely generated torsion V^a-modules. The following conditions are equivalent :*

(a) *M and M' have the same type.*
(b) *The sequences of elementary divisors of M and M' coincide.*

Proof. (a)⇒(b): Condition (a) means that $(M, M') \in E_\delta$ for every $\delta \in \log \Gamma^+_V \setminus \{0\}$, so the assertion is immediate from the construction of the sequences of elementary divisors.

(b)⇒(a): Let $(\gamma_i \mid i \in \mathbb{N})$ be the common sequence of elementary divisors for M and M'. Fix δ as in the foregoing, pick finitely presented V-modules N, N' such that

$$(M, N), (M', N') \in E_\delta$$

and let $(\beta_i \mid i \in \mathbb{N})$, respectively $(\beta'_i \mid i \in \mathbb{N})$ be the sequences of elementary divisors of N and N'. From claim 14.5.27 we deduce

$$\beta_i - \gamma, \beta'_i - \gamma_i \in [-4\delta, 4\delta] \qquad \text{for every } i \in \mathbb{N}.$$

Thus, $\beta_i - \beta'_i \in [-8\delta, 8\delta]$ for every $i \in \mathbb{N}$. Say that $8\delta = \log |a|_V$ for some $a \in V$.

Claim 14.5.32. There exists a V-linear map $f : N \to N'$ whose kernel and cokernel is annihilated by a.

Proof of the claim. By [75, Lemma 6.1.14] we reduce easily to the case where N and N' are cyclic V-modules, so $N = V/bV$ and $N' = V/b'V$, for $b, b' \in V \setminus \{0\}$ such that $\log |b/b'|_V \in [-2\delta, 2\delta]$. Now, if $b \in b'V$, we let f be the natural projection $V/bV \to V/b'V$. If $b' \in bV$, we notice that $b'b^{-1}\mathbf{1}_N$ factors through a map $N \to N'$ with the sought properties (details left to the reader). ♢

Clearly claim 14.5.32 implies that $(N, N') \in E_{8\delta}$, and therefore $(M, M') \in E_{10\delta}$. Since δ is arbitrary, the assertion follows. □

Corollary 14.5.33. *Let $b \in \mathfrak{m}_V$ be any non-zero element, M an almost finitely generated V^a/bV^a-module, and $(\gamma_i \mid i \in \mathbb{N})$ the sequence of elementary divisors of M. The following conditions are equivalent :*

(a) *M is a flat V^a/bV^a-module.*
(b) *There exists $n \in \mathbb{N}$ such that $\gamma_i = \log |b|_V$ for every $i \leq n$, and $\gamma_i = 0$ for every $i > n$.*

Proof. For every $i \in \mathbb{N}$, pick $a_i \in V$ with $\log |a_i|_V = \gamma_i$. According to proposition 14.5.31, the type of M coincides with that of $M' := \bigoplus_{i\in\mathbb{N}} V^a/a_iV^a$, and by lemma 14.5.30(ii), we may then assume that $M = M'$, in which case clearly (b)⇒(a). For the converse, notice that a standard computation gives

$$\mathrm{Tor}_1^{V/bV}(V/aV, V/aV) \simeq V/(aV + a^{-1}bV) \qquad \text{for every } a \in V \text{ such that } \log |a|_V \leq \log |b|_V$$

from which the assertion follows easily (details left to the reader). □

Proposition 14.5.34. *Let $(V, \mathfrak{m}_V)$ be as in* (14.5.5)*, and M a torsion V-module such that M^a is almost finitely generated. Let $(\gamma_i \mid i \in \mathbb{N})$ be the sequence of elementary divisors of M^a. Then*

$$\lambda_V(M) = (\Gamma_V : \Gamma) \cdot \sum_{i\in\mathbb{N}} \gamma_i.$$

Especially, $\lambda_V(M)$ depends only on the type of M^a.

Proof. Fix a sequence $(a_k \mid k \in \mathbb{N})$ of elements of $\mathfrak{m}_V \setminus \{0\}$ with $\lim_{k\to+\infty} \log |a_k|_V = 0$. For every $k \in \mathbb{N}$, let $M_k \subset M$ be a finitely generated submodule such that $a_kM \subset M_k$ ([75, Prop. 2.3.10(i)]), and denote also by $(\gamma_i^k \mid i \in \mathbb{N})$ the sequence of elementary divisors of M_k^a. After replacing each M_k by $\sum_{j=0}^k M_j$, we may also assume that $M_k \subset M_{k+1}$ for every $k \in \mathbb{N}$.

Claim 14.5.35. For every $i, k \in \mathbb{N}$ we have

$$4\delta_k \geq \gamma_i - \gamma_i^k \geq 0 \qquad \gamma_i^{k+1} \geq \gamma_i^k \qquad \text{where } \delta_k := \log |a_k|.$$

Proof of the claim. The inequalities $\gamma_i \geq \gamma_i^{k+1} \geq \gamma_i^k$ follow from proposition 14.5.26(ii). Next, clearly we have $(M_k^a, M^a) \in E_{\delta_k}$ for every $k \in \mathbb{N}$, so the upper bound for $\gamma_i - \gamma_i^k$ follows from claim 14.5.27. ♢

Set $L^k := \sum_{i\in\mathbb{N}} \gamma_i^k$ and $L := \sum_{i\in\mathbb{N}} \gamma_i$. It follows easily from claim 14.5.35 that

$$\lim_{k\to+\infty} L^k = \sum_{i\in\mathbb{N}} \lim_{k\to+\infty} \gamma_i^k = L.$$

In light of proposition 14.5.12(iii.b), it then suffices to check the proposition with M replaced by M_k, for every $k \in \mathbb{N}$. We may therefore assume from start that M is finitely generated, and let g be the cardinality of a finite system of generators for M. We may then find a surjective V-linear map $f : F := V^{\oplus g} \to M$, and $N := \mathrm{Ker}\, f^a$ is almost finitely generated (claim 14.5.13). Thus, for every $k \in \mathbb{N}$ we may find a finitely generated V-submodule $N_k \subset N$ with $a_kN \subset N_k$, so that $(F^a/N_k^a, M) \in E_{\delta_k}$. After replacing each N_k by $\sum_{j=0}^k N_j$, we may also assume that

$N_k \subset N_{k+1}$ for every $k \in \mathbb{N}$. Denote by $(\beta_i^k \mid i \in \mathbb{N})$ the sequence of elementary divisors of F/N_k, and notice that $\beta_i^k = 0$ for every $i \geq g$. Taking into account (14.5.18), claim 14.5.27 and proposition 14.5.26(iii) we get

$$\lambda_V(F/N_k) = \sum_{i\in\mathbb{N}} \beta_i^k \qquad 4\delta_k \geq \beta_i^k - \gamma_i \geq 0 \qquad \text{for every } i,k \in \mathbb{N}.$$

Therefore $\lim_{k\to+\infty} \lambda_V(F/N_k) = L$, and in view of proposition 14.5.12(i), it remains only to check that $\lim_{k\to+\infty} \lambda_V(N/N_k) = 0$, or equivalently, that $\lim_{k\to+\infty} \lambda_V(N_k) = \lambda_V(N)$ (proposition 14.5.12(i)). However, set $N' := \bigcup_{k\in\mathbb{N}} N_k$; by proposition 14.5.12(iii.b), the latter identity holds if and only $\lambda_V(N') = \lambda_V(N)$, and since $(N/N')^a = 0$, this follows from proposition 14.5.12(i,ii). □

14.5.36. In order to deal with general measurable K^+-algebras, we introduce hereafter some further notation which shall be standing throughout this section.

• To begin with, any ring homomorphism $\varphi : A \to B$ induces functors

$$\text{(14.5.37)} \qquad \varphi_* : B\text{-}\mathbf{Mod} \to A\text{-}\mathbf{Mod} \qquad \text{and} \qquad \varphi^* : A\text{-}\mathbf{Mod} \to B\text{-}\mathbf{Mod}.$$

Namely, φ_* assigns to any B-module M the A-module φ_*M obtained by restriction of scalars, and $\varphi^*(M) := B \otimes_A M$.

• For any local ring $(A, \mathfrak{m}_A)$, we let $\kappa(A) := A/\mathfrak{m}_A$, and we denote by $s(A)$ the closed point of $\operatorname{Spec} A$. If A is also a coherent ring, we denote by $A\text{-}\mathbf{Mod}_{\mathrm{coh},\{s\}}$ the full subcategory of $A\text{-}\mathbf{Mod}$ consisting of all the finitely presented A-modules M such that $\operatorname{Supp} M \subset \{s(A)\}$. Notice that the coherence of A implies that $A\text{-}\mathbf{Mod}_{\mathrm{coh},\{s\}}$ is an abelian category.

• Lastly, let $\mathscr{A}$ be any small abelian category; recall that $K_0(\mathscr{A})$ is the abelian group defined by generators and relations as follows. The generators are the isomorphism classes $[T]$ of objects T of $\mathscr{A}$, and the relations are generated by the elements of the form $[T_1] - [T_2] + [T_3]$, for every short exact sequence $0 \to T_1 \to T_2 \to T_3 \to 0$ of objects of $\mathscr{A}$. One denotes by $K_0^+(\mathscr{A}) \subset K_0(\mathscr{A})$ the submonoid generated by the classes $[T]$ of all objects of $\mathscr{A}$. We shall use the following well known *dévissage* lemma :

Lemma 14.5.38. *Let $\iota : \mathscr{B} \subset \mathscr{A}$ be an additive exact and fully faithful inclusion of abelian categories, and suppose that :*

(a) *If $T \in \mathrm{Ob}(\mathscr{B})$ and T' is a subquotient of $\iota(T)$, then T' is in the essential image of ι.*
(b) *Every object T of $\mathscr{A}$ admits a finite filtration $\mathrm{Fil}^\bullet T$ such that the associated graded object $\mathrm{gr}^\bullet T$ is in the essential image of ι.*

Then ι induces an isomorphism :

$$K_0(\mathscr{B}) \xrightarrow{\sim} K_0(\mathscr{A}).$$

Proof. Left to the reader. □

Proposition 14.5.39. *Let $\varphi : A \to B$ be a morphism of measurable K^+-algebras. We have :*

(i) *If φ induces a finite field extension $\kappa(A) \to \kappa(B)$, then the functor φ_* of* (14.5.37) *restricts to a functor*

$$\varphi_* : B\text{-}\mathbf{Mod}_{\mathrm{coh},\{s\}} \to A\text{-}\mathbf{Mod}_{\mathrm{coh},\{s\}}$$

which induces a group homomorphism of the respective K_0-groups :

$$\varphi_* : K_0(B\text{-}\mathbf{Mod}_{\mathrm{coh},\{s\}}) \to K_0(A\text{-}\mathbf{Mod}_{\mathrm{coh},\{s\}}).$$

(ii) *If φ induces an integral morphism $\kappa(A) \to B/\mathfrak{m}_A B$, then $\mathrm{length}_B(B/\mathfrak{m}_A B)$ is finite, and the functor φ^* of* (14.5.37) *restricts to a functor*

$$\varphi^* : A\text{-}\mathbf{Mod}_{\mathrm{coh},\{s\}} \to B\text{-}\mathbf{Mod}_{\mathrm{coh},\{s\}}$$

and if φ is also a flat morphism, φ^* induces a group homomorphism :

$$\varphi^* : K_0(A\text{-}\mathbf{Mod}_{\mathrm{coh},\{s\}}) \to K_0(B\text{-}\mathbf{Mod}_{\mathrm{coh},\{s\}}).$$

Proof. Write A (resp. B) as the colimit of a filtered system $\underline{A} := (A_i \mid i \in I)$ (resp. $\underline{B} := (B_j \mid j \in J)$) of objects of $K^+\text{-}\mathbf{m}.\mathbf{Alg}_0$, with local and essentially étale transition maps. After replacing J (resp. I) by a cofinal subsets, we may assume that the indexing set admits an initial element $0 \in J$ (resp. $0 \in I$). Furthermore, we may assume that the induced map $A_0 \to B$ factors through a morphism $A_0 \to B_0$ in $K^+\text{-}\mathbf{m}.\mathbf{Alg}_0$.

(i): Let M be any object of $B\text{-}\mathbf{Mod}_{\mathrm{coh},\{s\}}$. We need to show that $\varphi_* M$ is finitely presented. We may find $j \in J$ and a finitely presented B_j-module M_j, with an isomorphism $M \xrightarrow{\sim} M_j \otimes_{B_j} B$ of B-modules. After replacing J by J/j, we may assume that $j = 0$ is the initial index. Since the natural map $B_0 \to B$ is local and ind-étale, it is easily seen that M_0 is an object of $B_0\text{-}\mathbf{Mod}_{\mathrm{coh},\{s\}}$. Especially, there exists a finitely generated $\mathfrak{m}_{B_0}$-primary ideal $I \subset \mathrm{Ann}_{B_0} M_0$. We may then replace the system $\underline{B}$ by $(B_j/IB_j \mid j \in J)$ and assume that each B_j has Krull dimension zero.

Claim 14.5.40. Let $\varphi : A \to B$ be a morphism of measurable K^+-algebras inducing a finite residue field extension $\kappa(A) \to \kappa(B)$, and such that B has dimension zero. Let also M be any finitely presented B-module. Then we may find :

(a) a cocartesian diagram of local maps of K^+-algebras

$$\begin{array}{ccc} A_l & \xrightarrow{\varphi_l} & C_l \\ \downarrow & & \downarrow \\ A & \xrightarrow{\varphi} & B \end{array}$$

whose vertical arrows are ind-étale, and where φ_l is a morphism in $K^+\text{-}\mathbf{m}.\mathbf{Alg}_0$

(b) and an object M_l of $C_l\text{-}\mathbf{Mod}_{\mathrm{coh},\{s\}}$, with an isomorphism $M_l \otimes_{C_l} B \xrightarrow{\sim} M$.

Proof of the claim. Define $\underline{A}$ and $\underline{B}$ as in the foregoing. Notice that – under the current assumptions – B_j is a henselian ring, for every $j \in J$. Moreover, we may find $j \in J$ such that $\kappa(B)$ is generated by the image of $\kappa(A) \otimes_{\kappa(B_0)} \kappa(B_j)$; after replacing again J by J/j, we may then also assume that $\kappa(B) = \kappa(A) \cdot \kappa(B_0)$.

For every $i \in I$, set $B'_i := A_i \otimes_{A_0} B_0$; the natural map $B_0 \to B$ factors through a map $B'_i \to B$, and we let $\mathfrak{p}_i \subset B'_i$ be the preimage of $\mathfrak{m}_B$. Set also $C_i := B'_{i,\mathfrak{p}_i}$, so we deduce a filtered system of local maps $(C_i \to B \mid i \in I)$, whose limit is a local map $\psi : C \to B$ of local ind-étale B_0-algebras, which – by construction – induces an isomorphism $\kappa(C) \xrightarrow{\sim} \kappa(B)$ on residue fields. It follows easily that ψ is itself ind-étale, so say that ψ is the colimit of a filtered system $(\psi_\lambda : C \to D_\lambda \mid \lambda \in \Lambda)$ of étale C-algebras. Notice that C is a henselian local ring; in light of [66, Ch.IV, Th.18.5.11] we may then assume that D_λ is a local ring and ψ_λ is a finite étale map, for every $\lambda \in \Lambda$. Clearly the induced residue field extension $\kappa(C) \to \kappa(D_\lambda)$ is an isomorphism; in view of [66, Ch.IV, Prop.18.5.15] it follows that ψ_λ is an isomorphism, for every $\lambda \in \Lambda$, so the same holds for ψ.

Notice that the sequence of residue degrees $d_i := [\kappa(C_i) : \kappa(A_i)]$ is non-increasing, hence there exists $l \in I$ such that $d_i = d := d_l$ for every index $i \geq l$. Notice as well that, for $i \geq l$, the local algebra C_i is also a localization of $C'_i := A_i \otimes_{A_l} C_l$, and the latter is an essentially finitely presented K^+-algebras of Krull dimension zero, hence its spectrum is finite and discrete (lemma 9.1.29). Moreover, since the image of the map $\mathrm{Spec}\, C_l \to \mathrm{Spec}\, A_l$ is the closed point, it is clear that the same holds for the image of the induced map $\mathrm{Spec}\, C'_i \to \mathrm{Spec}\, A_i$. Since the extension $\kappa(A_l) \to \kappa(A_i)$ is finite and separable, we conclude that

$$\kappa(A_i) \otimes_{\kappa(A_l)} \kappa(C_l) = \prod_{\mathfrak{p}\in\mathrm{Spec}\, C'_i} \kappa(C'_{i,\mathfrak{p}}). \tag{14.5.41}$$

However, clearly the left-hand side of (14.5.41) is a $\kappa(A_i)$-algebra of degree d, whereas one of factors of the right-hand side – namely $\kappa(C_i)$ – is already of degree d over $\kappa(A_i)$. Hence $\mathrm{Spec}\, C'_i$ contains a single element, *i.e.* $C'_i = C_i$ is a local ring, and $C = A \otimes_{A_l} C_l$. Summing up, we have obtained the sought cocartesian diagram, and the claim holds with $M_l := M_0 \otimes_{B_0} C_l$. ◊

Let M_l and φ_l be as in claim 14.5.40; then $\varphi_* M$ is isomorphic to $A \otimes_{A_l} \varphi_{l*} M_l$, so may replace from start φ by φ_l, and assume that φ is a morphism in $K^+\text{-}\mathbf{m}.\mathbf{Alg}_0$, with B of Krull dimension zero. In such situation, one sees easily that φ is integral, hence B/I is a finitely presented A-module, by proposition 9.1.34(i), therefore $\varphi_* M$ is a finitely presented A-module, as required. Lastly, since the functor φ_* is exact, it is clear that it induces a map on K_0-groups as stated.

(ii): Under the current assumptions, the induced map $\kappa(A_0) \to B_0/\mathfrak{m}_{A_0} B_0$ is integral and essentially finitely presented, hence it is finite, so $B_0/\mathfrak{m}_{A_0} B_0$ is a B_0-module of finite length; but this is also the length of the B-module $B/\mathfrak{m}_A B$, whence the first assertion. Next, let M be an object of $A\text{-}\mathbf{Mod}_{\mathrm{coh},\{s\}}$; then $\varphi^* M$ is a finitely presented B-module; moreover, we may find a $\mathfrak{m}_A$-primary ideal $I \subset A$ such that M is a A/I-module, hence $\varphi^* M$ is a B/IB-module. Notice that the induced map $\mathrm{Spec}\, B/\mathfrak{m}_A B \to \mathrm{Spec}\, B/IB$ is bijective, and its target is a local scheme of dimension zero (since $B/\mathfrak{m}_A B$ is integral over a field). It follows easily that IB is a $\mathfrak{m}_B$-primary ideal, so $\varphi^* M$ is an object of $B\text{-}\mathbf{Mod}_{\mathrm{coh},\{s\}}$. The last assertion is then a trivial consequence of the exactness of the functor φ^*, when φ is flat. □

Lemma 14.5.42. *Let A be any measurable K^+-algebra. Then there exists a morphism*

$$V \xrightarrow{\varphi} A/I$$

of measurable K^+-algebras, where :

(a) *$I \subset A$ is a finitely generated $\mathfrak{m}_A$-primary ideal.*
(b) *V is a valuation ring, φ is a finitely presented surjection and the natural map $K^+ \to V$ induces an isomorphism of value groups $\Gamma \xrightarrow{\sim} \Gamma_V$.*

Proof. Suppose first that A is an object of $K^+\text{-}\mathbf{m}.\mathbf{Alg}_0$. In this case, choose an affine finitely presented S-scheme X and a point $x \in X$ such that $A = \mathscr{O}_{X,x}$; next, take a finitely presented closed immersion $h : X \to Y := \mathbb{A}^n_{K^+}$ of S-schemes; set $\overline{Y} := \mathbb{A}^n_\kappa \subset Y$, pick elements $f_1, \dots, f_d \in B := \mathscr{O}_{Y,h(x)}$ whose images in the regular local ring $\mathscr{O}_{\overline{Y},h(x)}$ form a regular system of parameters (*i.e.* a regular sequence that generates the maximal ideal), and let $J \subset B$ be the ideal generated by the f_i, $i = 1, \dots, d$. Let $I \subset A$ be any finitely generated $\mathfrak{m}_A$-primary ideal containing the image of J. We deduce a surjection $\varphi : V := B/J \to A/I$, and by construction $V/\mathfrak{m}_K V$ is a field; moreover, the induced map $K^+ \to V$ is flat by virtue of [65, Ch.IV, Th.11.3.8]. It follows that V is a valuation ring with the sought properties, by proposition 9.1.34(ii). Also, φ is finitely presented, by proposition 9.1.34(i).

Next, let A be a general measurable K^+-algebra, and write A as the colimit of a filtered system $(A_j \mid j \in J)$ of objects of $K^+\text{-}\mathbf{m}.\mathbf{Alg}_0$. We may assume that $0 \in J$ is an initial index, and the foregoing case yields an $\mathfrak{m}_{A_0}$-primary ideal $I_0 \subset A_0$, and a surjective finitely presented morphism $\varphi_0 : V_0 \to A_0/I_0$ in $K^+\text{-}\mathbf{m}.\mathbf{Alg}_0$ from a valuation ring V_0, such that $\Gamma_V = \Gamma$. Notice that A/I_0 is a henselian ring, hence φ_0 extends to a ring homomorphism $\varphi_0^{\mathrm{h}} : V_0^{\mathrm{h}} \to A_0/I_0$ from the henselization V_0^{h} of V_0; more precisely, φ_0 induces an isomorphism of V_0^{h}-algebras :

$$V_0^{\mathrm{h}} \otimes_{V_0} (A_0/I_0) \xrightarrow{\sim} A_0/I_0$$

so $\varphi_0^{\rm h}$ is still finitely presented. On the one hand, $\varphi_0^{\rm h}$ induces an identification

$$\kappa(V_0^{\rm h}) = \kappa(A_0).$$

On the other hand, we have the filtered system of separable field extensions $(\kappa(A_j) \mid j \in J)$, whose colimit is $\kappa(A)$. There follows a corresponding filtered system $(V_j^{\rm h} \mid j \in J)$ of finite étale $V_0^{\rm h}$-algebras, whose colimit we denote V ([66, Ch.IV, Prop.18.5.15]). Then V is a valuation ring, and the map $V_0^{\rm h} \to V$ induces an isomorphism on value groups. Moreover, the induced isomorphisms $\kappa(V_j^{\rm h}) \stackrel{\sim}{\to} \kappa(A_j)$ lift uniquely to morphisms of A_0-algebras $\varphi_j^{\rm h} : V_j^{\rm h} \to A_j/I_0A_j$, for every $j \in J$ ([66, Ch.IV, Cor.18.5.12]). Due to the uniqueness of $\varphi_j^{\rm h}$, we see that the resulting system $(\varphi_j^{\rm h} \mid j \in J)$ is filtered, and its colimit is a morphism $\varphi : V \to A/I_0A$. Moreover, $\varphi_j^{\rm h}$ induces an isomorphism $V_j^{\rm h} \otimes_{V_0^{\rm h}} A_0/IA_0 \stackrel{\sim}{\to} A_j/I_0A_j$, especially $\varphi_j^{\rm h}$ is surjective for every $j \in J$, so the same holds for φ. More precisely, $\varphi_0^{\rm h}$ induces an isomorphism $V \otimes_{V_0^{\rm h}} (A_0/I_0) \stackrel{\sim}{\to} A/I_0A$, hence φ is still finitely presented. □

Theorem 14.5.43. *With the notation of* (14.5.36)*, the following holds :*

(i) *For every measurable K^+-algebra A there is a natural group isomorphism :*

$$\boldsymbol{\lambda}_A : K_0(A\text{-}\mathbf{Mod}_{\rm coh,\{s\}}) \stackrel{\sim}{\to} \log \Gamma$$

which induces an isomorphism $K_0^+(A\text{-}\mathbf{Mod}_{\rm coh,\{s\}}) \stackrel{\sim}{\to} \log \Gamma^+$.

(ii) *The family of isomorphisms $\boldsymbol{\lambda}_A$ (for A ranging over the measurable K^+-algebras) is characterized uniquely by the following two properties.*

(a) *If V is a valuation ring and a flat measurable K^+-algebra, then*

$$\boldsymbol{\lambda}_V([M]) = \lambda_V(M) \qquad \textit{for every object } M \textit{ of } V\text{-}\mathbf{Mod}_{\rm coh,\{s\}}$$

where $\lambda_V(M)$ is defined as in (14.5.2)*.*

(b) *Let $\psi : A \to B$ be a morphism of measurable K^+-algebras inducing a finite residue field extension $\kappa(A) \to \kappa(B)$. Then*

$$\boldsymbol{\lambda}_A(\psi_*[M]) = [\kappa(B) : \kappa(A)] \cdot \boldsymbol{\lambda}_B([M]) \qquad \textit{for every } [M] \in K_0(B\text{-}\mathbf{Mod}_{\rm coh,\{s\}}).$$

(iii) *For every $a \in K^+ \setminus \{0\}$ and any object M of A-$\mathbf{Mod}_{\rm coh,\{s\}}$ we have :*

(a) *$[M] = 0$ in $K_0(A\text{-}\mathbf{Mod}_{\rm coh,\{s\}})$ if and only if $M = 0$.*

(b) *If M is flat over K^+/aK^+, then :*

$$\boldsymbol{\lambda}_A([M]) = |a| \cdot {\rm length}_A(M \otimes_{K^+} \kappa). \qquad (14.5.44)$$

(iv) *Let $\psi : A \to B$ be a flat morphism of measurable K^+-algebras inducing an integral map $\kappa(A) \to B/\mathfrak{m}_AB$. Then :*

$$\boldsymbol{\lambda}_B(\psi^*[M]) = {\rm length}_B(B/\mathfrak{m}_AB) \cdot \boldsymbol{\lambda}_A([M]) \qquad \textit{for every } [M] \in K_0(A\text{-}\mathbf{Mod}_{\rm coh,\{s\}}).$$

Proof. We start out with the following :

Claim 14.5.45. Let $(K, |\cdot|) \to (E, |\cdot|)$ be an extension of valued fields of rank one inducing an isomorphism of value groups, $a \in K^+ \setminus \{0\}$ any element, M an E^+/aE^+ module. Then :

(i) M is a flat E^+/aE^+-module if and only if it is a flat K^+/aK^+-module.

(ii) $M \otimes_{K^+} \kappa = M \otimes_{E^+} \kappa(E)$.

Proof of the claim. According to [126, Th.7.8], in order to show (i) it suffices to prove that

$${\rm Tor}_1^{E^+/aE^+}(E^+/bE^+, M) = {\rm Tor}_1^{K^+/aK^+}(K^+/bK^+, M)$$

for every $b \in K^+$ such that $|b| \geq |a|$. The latter assertion is an easy consequence of the faithful flatness of the extension $K^+ \to E^+$. (ii) follows from the identity : $\mathfrak{m}_E = \mathfrak{m}_K E^+$, which holds since $\Gamma_E = \Gamma$. ◊

Claim 14.5.46. Let $f : K^+ \to V$ be a morphism of measurable K^+-algebras, where V is a valuation ring and f induces an isomorphism on value groups. Then $\boldsymbol{\lambda}_V$ (defined by (ii.a)) is an isomorphism and assertion (iii) holds for $A = V$.

Proof of the claim. According to (ii.a), $\boldsymbol{\lambda}_V([M]) = \lambda_V(M)$ for every finitely presented torsion V-module. However, every such module M admits a decomposition of the form $M \simeq (V/a_1V) \oplus \cdots \oplus (V/a_kV)$, with $a_1, \dots, a_k \in \mathfrak{m}_K$ ([75, Lemma 6.1.14]). Then by claim 14.5.45(i), M is flat over K^+/aK^+ if and only if M is flat over V/aV, if and only if $|a| = |a_i|$ for every $i \leq k$. In this case, an explicit calculation shows that $\lambda_V(M) = |a| \cdot \mathrm{length}(M \otimes_V \kappa(V))$, which is equivalent to (14.5.44), in view of claim 14.5.45(ii). Next, we consider the map :

$$\boldsymbol{\mu} : \log \Gamma^+ \to K_0(V\text{-}\mathbf{Mod}_{\mathrm{coh},\{s\}}) \quad : \quad |a| \mapsto [V/aV] \quad \text{for every } a \in K^+ \setminus \{0\}.$$

We leave to the reader the verification that $\boldsymbol{\mu}$ extends to a group homomorphism well-defined on the whole of Γ, that provides an inverse to $\boldsymbol{\lambda}_V$. Finally, it is clear that $\lambda_V(M) = 0$ if and only if $M = 0$, so also (iii.a) holds. ♢

Claim 14.5.47. Let A be any measurable K^+-algebra. For every object N of $A\text{-}\mathbf{Mod}_{\mathrm{coh},\{s\}}$ there exist a finite filtration $0 = N_0 \subset \cdots \subset N_k = N$ by finitely presented A-submodules, and elements $a_1, \dots, a_k \in \mathfrak{m}$ such that N_i/N_{i-1} is a flat K^+/a_iK^+-module for every $1 \leq i \leq k$.

Proof of the claim. Let us find $I \subset A$ and $\varphi : V \to A/I$ as in lemma 14.5.42. It suffices to show the claim for the finitely presented A-modules $I^nN/I^{n+1}N$ (for every $n \in \mathbb{N}$), hence we may assume that N is an A/I-module. Then φ_*N is a finitely presented V-module, hence of the form $(V/a_1V) \oplus \cdots \oplus (V/a_kV)$ for some $a_i \in \mathfrak{m}$; we may order the summands so that $|a_i| \geq |a_{i+1}|$ for all $i < k$. We argue by induction on $d(N) := \dim_\kappa(N \otimes_V \kappa(V))$. If $d(N) = 0$, then $N = 0$ by Nakayama's lemma. Suppose $d > 0$; we remark that N/a_1N is a flat V/a_1V-module, hence a flat K^+/a_1K^+-module (claim 14.5.45), and $d(a_1N) < d(N)$; the claim follows. ♢

Claim 14.5.48. Let A be any measurable K^+-algebra, $I \subset A$ a finitely generated $\mathfrak{m}_A$-primary ideal, and $\pi_I : A \to A/I$ the natural projection. Then the map :

$$\pi_{I*} : K_0(A/I\text{-}\mathbf{Mod}_{\mathrm{coh}}) \to K_0(A\text{-}\mathbf{Mod}_{\mathrm{coh},\{s\}})$$

is an isomorphism.

Proof of the claim. On the one hand we have :

$$A\text{-}\mathbf{Mod}_{\mathrm{coh},\{s\}} = \bigcup_{n \in \mathbb{N}} A/I^n\text{-}\mathbf{Mod}_{\mathrm{coh}}$$

and on the other hand, in view of lemma 14.5.38, we see that the projections $A/I^{n+1} \to A/I^n$ induce isomorphisms $K_0(A/I^n\text{-}\mathbf{Mod}_{\mathrm{coh}}) \to K_0(A/I^{n+1}\text{-}\mathbf{Mod}_{\mathrm{coh}})$ for every $n > 0$, whence the claim. ♢

Let A, I and $\varphi : V \to A/I$ be as in lemma 14.5.42, and $\pi_I : A \to A/I$ the natural surjection; taking into account claim 14.5.48, we may let :

$$\boldsymbol{\lambda}_A := \boldsymbol{\lambda}_V \circ \varphi_* \circ \pi_{I*}^{-1} \tag{14.5.49}$$

where $\boldsymbol{\lambda}_V$ is given by the rule of (ii.a). In view of claim 14.5.46 we see that $\boldsymbol{\lambda}_A([M]) = 0$ if and only if $M = 0$, so (iii.a) follows already.

Claim 14.5.50. The isomorphism $\boldsymbol{\lambda}_A$ is independent of the choice of I, V and φ.

Proof of the claim. Indeed, suppose that $J \subset A$ is another ideal and $\psi : W \to A/J$ is another surjection from a valuation ring W, fulfilling the foregoing conditions. We consider the commutative diagram

$$\begin{array}{ccccc} & & A/I & \xleftarrow{\ \varphi\ } & V \\ & \nearrow^{\pi_I} & & \searrow^{\overline{\pi}_J} & \downarrow{\scriptstyle \varphi'} \\ A & & \xrightarrow{\ \pi_{I+J}\ } & & A/(I+J) \\ & \searrow_{\pi_J} & & \nearrow_{\overline{\pi}_I} & \uparrow{\scriptstyle \psi'} \\ & & A/J & \xleftarrow[\ \psi\ ]{} & W. \end{array}$$

We compute : $\varphi_* \circ \pi_{I*}^{-1} = \varphi_* \circ \pi_{J*} \circ \overline{\pi}_{J*}^{-1} \circ \overline{\pi}_{I*}^{-1} = \varphi'_* \circ \pi_{I+J,*}^{-1}$, and a similar calculation shows that $\psi_* \circ \pi_{J*}^{-1} = \psi'_* \circ \pi_{I+J,*}^{-1}$. We are thus reduced to showing that $\boldsymbol{\lambda}_V \circ \varphi'_*([N]) = \boldsymbol{\lambda}_W \circ \psi'_*([N])$ for every $A/(I+J)$-module N. In view of claim 14.5.47 we may assume that N is flat over K^+/aK^+, for some $a \in \mathfrak{m}$, in which case the assertion follows from claim 14.5.46. ◇

Claims 14.5.46 and 14.5.50 show already that (i) holds. Next, let $\psi : A \to B$ be as in (ii.b). Choose an ideal $I \subset A$ and a surjection $\varphi : V \to A/I$ as in lemma 14.5.42. Let also $J \subset B$ be a finitely generated $\mathfrak{m}_B$-primary ideal containing $\psi(I)$, and $\overline{\psi} : A/I \to B/J$ the induced map.

By inspecting the definitions we see that (ii.b) amounts to the identity : $\boldsymbol{\lambda}_V((\overline{\psi} \circ \varphi)_*[M]) = [\kappa(B) : \kappa(A)] \cdot \boldsymbol{\lambda}_{B/J}([M])$ for every finitely presented B/J-module M. Furthermore, up to enlarging J, we may assume that there is a finitely presented surjection $\xi : W \to B/J$ of K^+-algebras, where W is a valuation ring with value group Γ, and then we come down to showing:

$$\boldsymbol{\lambda}_V((\overline{\psi} \circ \varphi)_*[M]) = [\kappa(B) : \kappa(A)] \cdot \boldsymbol{\lambda}_W(\xi_*[M]).$$

In view of claim 14.5.47 we may also assume that M is a flat K^+/aK^+-module for some $a \in \mathfrak{m}$, in which case the identity becomes :

$$|a| \cdot \mathrm{length}_V(M \otimes_{K^+} \kappa) = [\kappa(B) : \kappa(A)] \cdot |a| \cdot \mathrm{length}_W(M \otimes_{K^+} \kappa)$$

thanks to claim 14.5.46. However, the latter is an easy consequence of the identities : $\kappa(A) = \kappa(V)$ and $\kappa(B) = \kappa(W)$. Next, we show that (iii.b) holds for a general measurable K^+-algebra A. Indeed, let M be any object of $A\text{-}\mathbf{Mod}_{\mathrm{coh},\{s\}}$; as usual, we may find a local ind-étale map $A_0 \to A$ from some object A_0 of $K^+\text{-}\mathbf{m}.\mathbf{Alg}_0$, and an object M_0 of $A_0\text{-}\mathbf{Mod}_{\mathrm{coh},\{s\}}$ with an isomorphism of A-modules $A \otimes_{A_0} M_0 \xrightarrow{\sim} M$ (cp. the proof of proposition 14.5.39(i)). Since (ii.b) is already proved, we have

$$\boldsymbol{\lambda}_A([M]) = \boldsymbol{\lambda}_{A_0}([M_0]) \quad \text{and} \quad \mathrm{length}_A(M \otimes_{K^+} \kappa) = \mathrm{length}_{A_0}(M \otimes_{K^+} \kappa).$$

Therefore, we may replace A by A_0, and assume that A is an object of $K^+\text{-}\mathbf{m}.\mathbf{Alg}_0$. In this case, by lemma 9.1.32 we may find morphisms $f : K^+ \to V$ and $g : V \to A$ in $K^+\text{-}\mathbf{m}.\mathbf{Alg}_0$ such that V is a valuation ring with value group Γ and g induces a finite extension of residue fields $\kappa(V) \to \kappa(A)$. Let N be an A-module supported at $s(A)$ and flat over K^+/aK^+; we may find a finitely generated $\mathfrak{m}_A$-primary ideal $I \subset A$ such that $I \subset \mathrm{Ann}_A(N)$, and since (ii.b) is already known in general, we reduce to showing that (iii.b) holds for the A/I-module N. However, the induced map $\overline{g} : V \to A/I$ is finite and finitely presented (proposition 9.1.34(i)), so another application of (ii.b) reduces to showing that (14.5.44) holds for $A := V$ and $M := \overline{g}_* N$, in which case the assertion is already known by claim 14.5.46.

(ii.a): Suppose that A is a valuation ring, and let M be any object of $A\text{-}\mathbf{Mod}_{\mathrm{coh},\{s\}}$. The sought assertion is obvious when $\Gamma_A = \Gamma$, since in that case we can choose $A = V$ and $\varphi = \pi_I$ in (14.5.49). However, we know already that the rank of A equals one, and $e := (\Gamma_A : \Gamma)$ is finite (see (14.5.5)); we can then assume that $e > 1$, in which case corollary 11.4.42(ii) implies that $\Gamma \simeq \mathbb{Z}$. Then it suffices to check (ii.a) for $M = \kappa(A)$, which is a (flat) κ-module, so that – by assertion (iii) – one has $\boldsymbol{\lambda}_A([M]) = e \cdot \gamma_0$, where $\gamma_0 \in \log \Gamma_A^+$ is the positive generator. In view of remark 14.5.14, we see that this value agrees with $\lambda_V(M)$, as stated.

(iv): In view of claim 14.5.47, we may assume that M is a flat K^+/aK^+-module, for some $a \in \mathfrak{m}$, and then the same holds for $\psi^* M$, since ψ is flat. In view of (iii.b), it then suffices to show that :

$$\mathrm{length}_B(B \otimes_A M) = \mathrm{length}_A(M) \cdot \mathrm{length}_B(B/\mathfrak{m}_A B)$$

for any A-module M of finite length. In turn, this is easily reduced to the case where $M = \kappa(A)$, for which the identity is obvious. □

14.5.51. Let A be a measurable K^+-algebra. The next step consists in extending the definition of $\boldsymbol{\lambda}_A$ to the category $A\text{-}\mathbf{Mod}_{\{s\}}$ of arbitrary A-modules M supported at $s(A)$. First of all, suppose that M is finitely generated. Let $\mathscr{C}_M$ be the set of isomorphism classes of objects M' of $A\text{-}\mathbf{Mod}_{\mathrm{coh},\{s\}}$ that admit a surjection $M' \to M$. Then we set :

$$\lambda_A^*(M) := \inf \{\boldsymbol{\lambda}_A([M']) \mid M' \in \mathscr{C}_M\} \in \log \Gamma^\wedge.$$

Notice that – by the positivity property of theorem 14.5.43(i) – we have $\lambda_A^*(M) = \boldsymbol{\lambda}_A([M])$ whenever M is finitely presented. Next, for a general object of $A\text{-}\mathbf{Mod}_{\{s\}}$ we let :
(14.5.52)

$$\lambda_A(M) := \sup \{\lambda_A^*(M') \mid M' \subset M \text{ and } M' \text{ is finitely generated}\} \in \log \Gamma^\wedge \cup \{+\infty\}.$$

Lemma 14.5.53. *If M is finitely generated, then $\lambda_A^*(M) = \lambda_A(M)$.*

Proof. Let $M' \subset M$ be a finitely generated submodule; we have to show that $\lambda^*(M') \leq \lambda^*(M)$. To this aim, let $f : N \to M$ and $g : N' \to M'$ be two surjections of A-modules with $N \in \mathscr{C}_M$ and $N' \in \mathscr{C}_{M'}$; by filtering the kernel of f by the system of its finitely generated submodules, we obtain a filtered system $(N_i \mid i \in I)$ of finitely presented quotients of N, with surjective transition maps, such that $\mathrm{colim}_{i\in I} N_i = M$. By [75, Prop.2.3.16(ii)] the induced map $N' \to M$ lifts to a map $h : N' \to N_i$ for some $i \in I$. Since A is coherent, $h(N')$ is a finitely presented A-module with a surjection $h(N') \to M'$, hence $\lambda_A^*(M') \leq \boldsymbol{\lambda}_A([h(N')]) \leq \boldsymbol{\lambda}_A([N_i]) \leq \boldsymbol{\lambda}_A(N)$. Since N is arbitrary, the claim follows. □

Proposition 14.5.54. (i) *If A is a valuation ring,* (14.5.52) *agrees with* (14.5.9).

(ii) *If $(M_i \mid i \in I)$ is a filtered system of objects of $A\text{-}\mathbf{Mod}_{\{s\}}$ with injective (resp. surjective) transition maps, then :*

$$\lambda_A(\mathop{\mathrm{colim}}_{i\in I} M_i) = \lim_{i\in I} \lambda_A(M_i)$$

(resp. provided there exists $i \in I$ such that $\lambda_A(M_i) < +\infty$).

(iii) *If $\psi : A \to B$ is a morphism of measurable K^+-algebras inducing a finite extension $\kappa(A) \to \kappa(B)$ of residue fields, then :*

$$\lambda_B(M) = \frac{\lambda_A(\psi_* M)}{[\kappa(B) : \kappa(A)]} \qquad \textit{for every object } M \textit{ of } B\text{-}\mathbf{Mod}_{\{s\}}.$$

(iv) *Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence in $A\text{-}\mathbf{Mod}_{\{s\}}$. Then :*

$$\lambda_A(M_2) = \lambda_A(M_1) + \lambda_A(M_3).$$

(v) *Let ψ be a flat morphism of measurable K^+-algebras inducing an integral map $\kappa(A) \to B/\mathfrak{m}_A B$. Then :*

$$\lambda_B(\psi^* M) = \mathrm{length}_B(B/\mathfrak{m}_A B) \cdot \lambda_A(M) \qquad \textit{for every object } M \textit{ of } A\text{-}\mathbf{Mod}_{\{s\}}.$$

Proof. (i): Set $e := (\Gamma_V : \Gamma)$. It suffices to check the assertion for a finitely generated A-module M, in which case one has to show the identity :

$$e \cdot |F_0(M)^a| = \lambda^*_A(M).$$

By proposition 14.5.12(i) we have $|F_0(M')^a| \geq |F_0(M)^a|$ for every $M' \in \mathscr{C}_M$, hence $\lambda^*_A(M) \geq e \cdot |F_0(M)^a|$. On the other hand, let us fix a surjection $V^{\oplus n} \to M$, and let us write its kernel in the form $K = \bigcup_{i\in I} K_i$, for a filtered system $(K_i \mid i \in I)$ of a finitely generated V-submodules; it follows that :

$$F_0(M) = \bigcup_{i\in I} F_0(V^{\oplus n}/K_i).$$

In view of proposition 14.5.12(iii.b) we deduce : $e \cdot |F_0(M)^a| = \lim_{i\in I} e \cdot |F_0(V^{\oplus n}/K_i)^a| \geq \lambda^*_A(M)$.

The proof of (ii) in the case where the transition maps are injective, is the same as that of proposition 14.5.12(iii.b).

(iii): Let us write $M = \bigcup_{i\in I} M_i$ for a filtered family $(M_i \mid i \in I)$ of finitely generated B-submodules. By the case already known of (ii) we have : $\lambda_B(M) = \lim_{i\in I} \lambda_B(M_i)$, and likewise for $\lambda_A(\psi_*M)$, hence we may assume from start that M is a finitely generated B-module, in which case the annihilator of M contains a finitely generated $\mathfrak{m}_B$-primary ideal $J' \subset B$. Choose a finitely generated $\mathfrak{m}_A$-primary ideal $J \subset A$ contained in the kernel of the induced map $A \to B/J'$.

Claim 14.5.55. $\lambda_B(M) = \lambda_{B/J'}(M)$ and $\lambda_A(\psi_*M) = \lambda_{A/J}(\overline{\psi}_*M)$.

Proof of the claim. Directly on the definition (and by applying theorem 14.5.43(ii.b) to the surjection $B \to B/J'$) we see that $\lambda_B(M) \leq \lambda_{B/J'}(M)$. On the other hand, any surjection of B-modules $M' \to M$ with M' finitely presented, factors through the natural map $M' \to M'/J'M'$, and by theorem 14.5.43(i,ii.b) we have $\lambda_B(M') - \lambda_{B/J'}(M'/J'M') = \lambda_B(J'M') \geq 0$, whence the first stated identity. The proof of the second identity is analogous. ◊

In view of claim 14.5.55 we are reduced to proving the assertion for the morphism $\overline{\psi}$ and the B/J'-module M, hence we may replace ψ by $\overline{\psi}$, and assume from start that A and B have Krull dimension zero.

Claim 14.5.56. Let A be a measurable K^+-algebra of Krull dimension zero. Then there exists a morphism $\varphi : V \to A$ of measurable K^+-algebras, with V a valuation ring flat over K^+, such that the residue field extension $\kappa(V) \to \kappa(A)$ is finite, and the induced map of value groups $\Gamma \to \Gamma_V$ is an isomorphism.

Proof of the claim. Let $A_0 \to A$ be a local ind-étale map, from an object A_0 of $K^+\text{-}\mathbf{m}.\mathbf{Alg}_0$. By lemma 9.1.32 we may find a K^+-flat valuation ring V_0 in $K^+\text{-}\mathbf{m}.\mathbf{Alg}_0$ and a local map $\varphi_0 : V \to A_0$, inducing a finite residue field extension $\kappa(V_0) \to \kappa(A_0)$ and an isomorphism on value groups $\Gamma \xrightarrow{\sim} \Gamma_V$. Then A_0 has also Krull dimension zero; especially, it is henselian, hence φ_0 factors through a morphism $\varphi_0^{\mathrm{h}} : V_0^{\mathrm{h}} \to A_0$, from the henselization V_0^{h} of V_0. Let $E \subset \kappa(A)$ be the largest separable subextension of $\kappa(V_0) = \kappa(V_0^{\mathrm{h}})$ contained in $\kappa(A)$; there exists a unique (up to unique isomorphism) finite étale morphism $V_0^{\mathrm{h}} \to V'$ with an isomorphism $\kappa(V') \xrightarrow{\sim} E$ of $\kappa(V_0)$-algebras, and φ_0^{h} factors through a morphism $V' \to A_0$. Notice that V' is still a henselian valuation ring and a measurable K^+-algebra, hence we may replace V_0 by V', and assume that V_0 is henselian, and the residue field extension $\overline{\varphi}_0 : \kappa(V_0) \to \kappa(A_0)$ is purely inseparable. Since A_0 is henselian, we may write A as the colimit of a filtered system $(A_i \mid i \in I)$ of finite étale A_0-algebras. Now, on the one hand, $\overline{\varphi}_0$ induces an equivalence from the category of finite étale $\kappa(V_0)$-algebras, to the category of finite étale $\kappa(A_0)$-algebras (lemma 13.1.7(i)). On the other hand, the category of finite étale V_0-algebras is equivalent to the category of finite étale $\kappa(V_0)$-algebras, and likewise for A_0. Therefore, for every $i \in I$ we may find a finite étale morphism

$V_0 \to V_i$, unique up to unique isomorphism, inducing an isomorphism of $\kappa(A_0)$-algebras :

$$\kappa(V_i) \otimes_{\kappa(V_0)} \kappa(A_0) \xrightarrow{\sim} \kappa(A_i) \tag{14.5.57}$$

and the transition maps of residue fields $\kappa(A_i) \to \kappa(A_j)$ induce unique maps $V_i \to V_j$ of V_0-algebras, compatible with the isomorphisms (14.5.57). Hence, the resulting system $(V_i \mid i \in I)$ is filtered, and its colimit is a valuation ring V, which is still a measurable K^+-algebra. Moreover, the field extensions $\kappa(V_i) \to \kappa(A_i)$ deduced from (14.5.57) lift uniquely to maps of V_0-algebras $V_i \to A_i$ ([66, Ch.IV, Cor.18.5.12]); taking colimits, we get finally a map $V \to A$ as sought. $\diamondsuit$

Let φ be as in claim 14.5.56; clearly it suffices to prove the sought identity for the two morphisms $\psi \circ \varphi$ and φ, so we may replace A by V, and assume from start that A is a valuation ring. Let us set $d := [\kappa(B) : \kappa(A)]$; we deduce :

$$\lambda_B(M) = \inf \{d^{-1} \cdot \lambda_A(\psi_* M') \mid M' \in \mathscr{C}_M\} \geq d^{-1} \cdot \lambda_A(\psi_* M)$$

by theorem 14.5.43(ii.b). Furthermore, let us choose a surjection $B^{\oplus k} \to M$, whose kernel we write in the form $K := \bigcup_{i \in I} K_i$ where $(K_i \mid i \in I)$ is a filtered family of finitely generated B-submodules of K. Next, by applying (i), proposition 14.5.12(i,iii.b), and theorem 14.5.43(ii.b) we derive :

$$\begin{aligned}\lambda_B(M) &\leq \inf \{\lambda_B(B^{\oplus k}/K_i) \mid i \in I\} = \inf \{d^{-1} \cdot \lambda_A(\psi_*(B^{\oplus k}/K_i)) \mid i \in I\} \\ &= d^{-1} \cdot (\lambda_A(\psi_* B^{\oplus k}) - \sup \{\lambda_A(\psi_* K_i) \mid i \in I\}) \\ &= d^{-1} \cdot (\lambda_A(\psi_* B^{\oplus k}) - \lambda_A(\psi_* K)) \\ &= d^{-1} \cdot \lambda_A(\psi_* M)\end{aligned}$$

whence the claim.

(iv): Let $(N_i \mid i \in I)$ be the filtered system of finitely generated submodules of M_2. For every $i \in I$ we have short exact sequences : $0 \to M_1 \cap N_i \to N_i \to \overline{N}_i \to 0$, where $\overline{N}_i$ is the image of N_i in M_3. In view of the case of (ii) already known, we may then replace M_2 by N_i, and thus assume from start that M_2 is finitely generated, so that we may find a finitely generated $\mathfrak{m}_A$-primary ideal $J \subset A$ that annihilates M_2. By (iii) we have $\lambda_A(M_2) = \lambda_{A/J}(M_2)$, and likewise for M_1 and M_3, hence we may replace A by A/J. By claim 14.5.56, we may then find a morphism $V \to A$ of measurable K^+-algebras with V a valuation ring, inducing a finite residue field extension $\kappa(V) \to \kappa(A)$; then again (iii) reduces to the case where $A = V$, to which one may apply (i) and proposition 14.5.12(i) to conclude the proof.

Next we consider assertion (ii) for the case where the transition maps are surjective. We may assume that I admits a smallest element i_0; for every $i \in I$ let K_i denote the kernel of the transition map $M_{i_0} \to M_i$. We deduce a short exact sequence :

$$0 \to \bigcup_{i \in I} K_i \to M_{i_0} \to \operatorname*{colim}_{i \in I} M_i \to 0$$

and we may then compute using (iv) and the previous case of (ii) :

$$\lambda_A(\operatorname*{colim}_{i \in I} M_i) = \lambda_A(M_{i_0}) - \lambda_A(\bigcup_{i \in I} K_i) = \lim_{i \in I}(\lambda_A(M_{i_0}) - \lambda_A(K_i)) = \lim_{i \in I} \lambda_A(M_{i_0}/K_i)$$

whence the claim.

(v): Since ψ_* is an exact functor which commutes with colimits, we may use (ii) to reduce to the case where M is finitely presented, for which the assertion is already known, in view of theorem 14.5.43(iv). $\square$

Remark 14.5.58. Suppose that the valuation of K is discrete; then one sees easily that theorem 14.5.43(i,iii) still holds (with simpler proof) when A is replaced by any local noetherian K^+-algebra, and by inspecting the definition, the resulting map λ_A is none else than the standard length function for modules supported on $\{s(A)\}$.

Lemma 14.5.59. *Let A be a measurable K^+-algebra, and M an A-module supported at $s(A)$ with $\lambda_A(M) < \infty$. Then*

$$\{a \in K^+ \mid \log|a| > \lambda_A(M)\} \subset \mathrm{Ann}_{K^+}M.$$

Proof. Using proposition 14.5.54(ii), we easily reduce, first, to the case where M is finitely generated, and second, to the case where M is finitely presented. Pick an ideal $I \subset A$ and a valuation ring V mapping onto A/I, as in lemma 14.5.42; by considering the I-adic filtration of M, the additivity properties of λ_A allow to further reduce to the case where M is an A/I-module. Next, by theorem 14.5.43(ii.b) we may replace A by V, and therefore assume that A is a valuation ring whose value group equals Γ. In this case, λ_A is computed by Fitting ideals, so the assertion follows easily from [75, Prop.6.3.6(iii)]. □

14.5.60. Let us consider a K^+-algebra R_∞ that is the colimit of an inductive system

$$R_0 \to R_1 \to R_2 \to \cdots \qquad (14.5.61)$$

of morphisms of measurable K^+-algebras inducing integral ring homomorphisms $\kappa(R_i) \to R_{i+1}/\mathfrak{m}_{R_i}R_i$ for every $i \in \mathbb{N}$. The final step consists in generalizing the definition of normalized length to the category R_∞-$\mathbf{Mod}_{\{s\}}$ of R_∞-modules supported at the closed point $s(R_\infty)$ of $\mathrm{Spec}\, R_\infty$. To this purpose, we shall axiomatize the general situation in which we can solve this problem. Later we shall see that our axioms are satisfied in many interesting cases.

14.5.62. Hence, let R_∞ be as in (14.5.60). After fixing an order-preserving isomorphism

$$(\mathbb{Q} \otimes_{\mathbb{Z}} \log\Gamma)^\wedge \xrightarrow{\sim} \mathbb{R} \qquad (14.5.63)$$

we may regard the mappings λ_A (for any measurable K^+-algebra A) as real-valued functions on A-modules. To ease notation, for every R_n-module N supported on $\{s(R_n)\}$ we shall write $\lambda_n(N)$ instead of $\lambda_{R_n}(N)$. Notice that, for every such N, and every $m \geq n$, the R_m-module $R_m \otimes_{R_n} N$ is supported at $\{s(R_m)\}$, since by assumption the map $R_n \to R_m/\mathfrak{m}_{R_n}R_m$ is integral.

Definition 14.5.64. In the situation of (14.5.62), we say that R_∞ is an *ind-measurable* K^+-algebra, if there exists a sequence of real *normalizing factors* $(d_n > 0 \mid n \in \mathbb{N})$ such that:

(a) For every $n \in \mathbb{N}$ and every object N of R_n-$\mathbf{Mod}_{\mathrm{coh},\{s\}}$, the sequence :

$$m \mapsto d_m^{-1} \cdot \lambda_m(R_m \otimes_{R_n} N)$$

converges to an element $\lambda_\infty(R_\infty \otimes_{R_n} N) \in \mathbb{R}$.

(b) For every $m \in \mathbb{N}$, every finitely generated $\mathfrak{m}_{R_0}$-primary ideal $I \subset R_0$, and every $\varepsilon > 0$ there exists $\delta(m, \varepsilon, I) > 0$ such that the following holds. For every $n \in \mathbb{N}$ and every surjection $N \to N'$ of finitely presented R_n/IR_n-modules generated by m elements, such that

$$|\lambda_\infty(R_\infty \otimes_{R_n} N) - \lambda_\infty(R_\infty \otimes_{R_n} N')| \leq \delta(m, \varepsilon, I)$$

we have :

$$d_n^{-1} \cdot |\lambda_n(N) - \lambda_n(N')| \leq \varepsilon.$$

14.5.65. Assume now that R_∞ is ind-measurable, and let N be a finitely presented R_n-module supported on $\{s(R_n)\}$. The first observation is that $\lambda_\infty(R_\infty \otimes_{R_n} N)$ only depends on the R_∞-module $R_\infty \otimes_{R_n} N$. Indeed, suppose that $R_\infty \otimes_{R_m} M \simeq R_\infty \otimes_{R_n} N$ for some $m \in \mathbb{N}$ and some finitely presented R_m-module M; then there exists $p \geq m, n$ such that $R_p \otimes_{R_m} M \simeq R_p \otimes_{R_n} N$, and then the assertion is clear.

The second observation – contained in the following lemma 14.5.66 – will show that the conditions of definition 14.5.64 impose some non-trivial restrictions on the inductive system $(R_n \mid n \in \mathbb{N})$.

Lemma 14.5.66. *Let $(R_n; d_n \mid n \in \mathbb{N})$ be an inductive system of measurable K^+-algebras and a sequence of positive reals, fulfilling conditions* (a) *and* (b) *of definition* 14.5.64. *Then:*

(i) *The natural map*
$$M \to R_m \otimes_{R_n} M$$
is injective for every $n, m \in \mathbb{N}$ with $m \geq n$ and every R_n-module M.

(ii) *Especially, the transition maps $R_n \to R_{n+1}$ are injective for every $n \in \mathbb{N}$.*

(iii) *Suppose that $(d'_n \mid n \in \mathbb{N})$ is another sequence of positive reals such that conditions* (a) *and* (b) *hold for the datum $(R_n; d'_n \mid n \in \mathbb{N})$. Then the sequence $(d_n/d'_n \mid n \in \mathbb{N})$ converges to a non-zero real number.*

Proof. (i): We reduce easily to the case where M is finitely presented over R_n. Let $N \subset \mathrm{Ker}\,(M \to R_m \otimes_{R_n} M)$ be a finitely generated R_n-module; since R_n is coherent, N is a finitely presented R_n-module. We suppose first that M is in $R_n\text{-}\mathbf{Mod}_{\mathrm{coh},\{s\}}$.

Claim 14.5.67. The natural map $R_m \otimes_{R_n} M \to R_m \otimes_{R_n} (M/N)$ is an isomorphism.

Proof of the claim. On the one hand, the R_m-module $R_m \otimes_{R_n} M$ represents the functor
$$R_m\text{-}\mathbf{Mod} \to \mathbf{Set} \quad : \quad Q \mapsto \mathrm{Hom}_{R_n}(M, Q).$$
On the other hand, the assumption on N implies that $\mathrm{Hom}_{R_n}(M, Q) = \mathrm{Hom}_{R_n}(M/N, Q)$ for every R_m-module Q, so the claim follows easily. ♢

From claim 14.5.67 we deduce that the natural map $R_\infty \otimes_{R_n} M \to R_\infty \otimes_{R_n} (M/N)$ is an isomorphism, and then condition (b) says that $\lambda_n(M) = \lambda_n(M/N)$, hence $\lambda_n(N) = 0$ and finally $N = 0$, as stated. Next, suppose M is any finitely presented R_n-module, and pick a finitely generated $\mathfrak{m}_{R_n}$-primary ideal $I \subset R_n$.

Claim 14.5.68. There exists $c \in \mathbb{N}$ such that $N \cap I^{k+c}M = I^k(N \cap I^c M)$ for every $k \geq 0$.

Proof of the claim. We may find a local ind-étale map $A \to R_n$ of K^+-algebras, where A is an object of $K^+\text{-}\mathbf{m.Alg}_0$, and such that I, M and N descend respectively to a finitely generated ideal $I_0 \subset A$, a finitely presented A-module M_0, and a finitely generated submodule $N_0 \subset M_0$. Then theorem 11.4.46 ensures the existence of $c \in \mathbb{N}$ such that $N_0 \cap I_0^{k+c} M_0 = I_0^k(N_0 \cap I_0^c M_0)$ for every $k \geq 0$. Since R_n is a faithfully flat A-algebra, the claim follows. ♢

Pick $c \in \mathbb{N}$ as in claim 14.5.68; then $N/(N \cap I^{k+c}M)$ is in the kernel of the natural map $M/I^{k+c}M \to R_m \otimes_{R_n} (M/I^{k+c}M)$, hence $N = N \cap I^{k+c}M$ by the foregoing, so that $N \subset I^k N$ for every $k \geq 0$, and finally $N = 0$ by Nakayama's lemma.

(ii) is a special case of (i). To show (iii), let us denote by $\lambda'_\infty(M)$ the normalized length of any object M of $R_\infty\text{-}\mathbf{Mod}_{\mathrm{coh},\{s\}}$, defined using the sequence $(d'_n \mid n \in \mathbb{N})$. From (b) it is clear that $\lambda_\infty(M), \lambda'_\infty(M) \neq 0$ whenever $M \neq 0$. Then, for any such non-zero M, the quotient $\lambda'_\infty(M)/\lambda_\infty(M)$ is the limit of the sequence $(d_n/d'_n \mid n \in \mathbb{N})$. □

Remark 14.5.69. (i) There is another situation of interest which leads to a well-behaved notion of normalized length. Namely, suppose that $R_\bullet := (R_n \mid n \in \mathbb{N})$ is an inductive system of local homomorphisms of local noetherian rings, such that the fibres of the induced morphisms

$\mathrm{Spec}\, R_{n+1} \to \mathrm{Spec}\, R_n$ have dimension zero, and denote by R_∞ the inductive limit of the system $R_\bullet$. For every $n \in \mathbb{N}$, let also λ_n be the usual length function on the set of isomorphism classes of finitely generated R_n-modules supported on $s(R_n)$. Then for every $m, n \in \mathbb{N}$ with $m \geq n$, and every R_n-module M of finite length, the R_m-module $R_m \otimes_{R_n} M$ has again finite length, so the analogues of conditions (a) and (b) of (14.5.62) can be formulated (cp. remark 14.5.58), and if these conditions hold for $R_\bullet$, we shall say that R_∞ is an *ind-measurable ring*. In such situation, lemma 14.5.66 – as well as the forthcoming lemma 14.5.70 and theorem 14.5.75 – still hold, with simpler proofs : we leave the details to the reader.

(ii) In spite of the uniqueness properties expressed by lemma 14.5.66, we do not know to which extent the normalized length of an ind-measurable K^+-algebra depends on the chosen tower of measurable algebras. Namely, suppose that $(R_n \mid n \in \mathbb{N})$ and $(R'_n \mid n \in \mathbb{N})$ are two such towers, with isomorphic colimit R_∞, and suppose that we have found normalizing factors $(d_n \mid n \in \mathbb{N})$ (resp. $(d'_n \mid n \in \mathbb{N})$) for the first (resp. second) tower, whence a normalized length λ_∞ (resp. λ'_∞) for R_∞-modules. Then we do not know whether the ratio of λ_∞ and λ'_∞ is a constant.

Next, for a given finitely generated R_∞-module M supported on $\{s(R_\infty)\}$, we shall proceed as in (14.5.51) : we denote by $\mathscr{C}_M$ the set of isomorphism classes of finitely presented R_∞-modules supported on $\{s(R_\infty)\}$ that admit a surjection $M' \to M$, and we set

$$\lambda^*_\infty(M) := \inf \{\lambda_\infty(M') \mid M' \in \mathscr{C}_M\} \in \mathbb{R}.$$

Directly on the definitions, one checks that $\lambda^*_\infty(M) = \lambda_\infty(M)$ if M is finitely presented.

Lemma 14.5.70. *Let M be a finitely generated R_∞-module, $\Sigma \subset M$ any finite set of generators, $(N_i \mid i \in I)$ any filtered system of objects of R_∞-$\mathbf{Mod}_{\{s\}}$, with surjective transition maps, such that* $\mathrm{colim}_{i\in I} N_i \simeq M$. *Then :*

(i) $\lambda^*_\infty(M) = \lim\limits_{n\to\infty} d_n^{-1} \cdot \lambda_n(\Sigma R_n)$.

(ii) *If every N_i is finitely generated, we have :*

$$\inf \{\lambda^*_\infty(N_i) \mid i \in I\} = \lambda^*_\infty(M).$$

Proof. To start out, we show assertion (ii) in the special case where all the modules N_i are finitely presented. Indeed, let us pick any surjection $\varphi : M' \to M$ with $M' \in \mathscr{C}_M$. We may find $i \in I$ such that φ lifts to a map $\varphi_i : M' \to N_i$ ([75, Prop.2.3.16(ii)]), and up to replacing I by a cofinal subset, we may assume that φ_i is defined for every $i \in I$. Moreover

$$\mathop{\mathrm{colim}}_{i\in I} \mathrm{Coker}\, \varphi_i = \mathrm{Coker}\, \varphi = 0$$

hence there exists $i \in I$ such that φ_i is surjective. It follows easily that $\lambda_\infty(N_i) \leq \lambda_\infty(M')$, whence (ii) in this case.

(i): Let k be the cardinality of Σ, $\varepsilon > 0$ any real number, $Q \subset \mathrm{Ann}_{R_0}\Sigma$ a finitely generated $\mathfrak{m}_{R_0}$-primary ideal, $S := R_\infty/QR_\infty$ and $\beta : S^{\oplus k} \to M$ a surjection that sends the standard basis onto Σ. By filtering $\mathrm{Ker}\, \beta$ by the system $(K_j \mid j \in J)$ of its finitely generated submodules, we obtain a filtered system $(N_j := S^{\oplus k}/K_j \mid j \in J)$ of finitely presented R_∞-modules supported on $\{s(R_\infty)\}$, with surjective transition maps and colimit isomorphic to M. Let $\psi_j : N_j \to M$ be the natural map; by the foregoing, we may find $j_0 \in J$ such that

$$0 \leq \lambda_\infty(N_{j_0}) - \lambda^*_\infty(M) \leq \min(\delta(k, \varepsilon, Q), \varepsilon) \tag{14.5.71}$$

(notation of definition 14.5.64(b)). We can then find $n \in \mathbb{N}$ and a finitely presented R_n-module M'_n supported on $\{s(R_n)\}$ and generated by at most k elements, such that $N_{j_0} = R_\infty \otimes_{R_n} M'_n$; we set $M'_m := R_m \otimes_{R_n} M_n$ for every $m \geq n$ and let M_m be the image of M'_m in M. Up to replacing n by a larger integer, we may assume that $M_m = \Sigma R_m$ for every $m \geq n$. Choose $m \in \mathbb{N}$ so that :

$$|\lambda_\infty(N_{j_0}) - d_m^{-1} \cdot \lambda_m(M'_m)| \leq \varepsilon. \tag{14.5.72}$$

Let $(M_{m,i} \mid i \in I)$ be a filtered system of finitely presented R_m-modules supported at $\{s(R_m)\}$, with surjective transition maps, such that $\mathrm{colim}_{i\in I} M_{m,i} = M_m$. Arguing as in the foregoing, we show that there exists $i \in I$ such that the natural map $\varphi : M'_m \to M_m$ factors through a surjection $\varphi_i : M'_m \to M_{m,i}$, and up to replacing I by a cofinal subset, we may assume that such a surjection φ_i exists for every $i \in I$. We obtain therefore a compatible system of surjections of R_∞-modules :

$$N_{j_0} \simeq R_\infty \otimes_{R_m} M'_m \to R_\infty \otimes_{R_m} M_{m,i} \to M$$

and combining with (14.5.71) we find :

$$|\lambda_\infty(N_{j_0}) - \lambda_\infty(R_\infty \otimes_{R_m} M_{m,i})| \leq \delta(k, \varepsilon, Q) \qquad \text{for every } i \in I.$$

In such situation, condition (b) of definition 14.5.64 ensures that :

$$d_m^{-1} \cdot |\lambda_m(M'_m) - \lambda_m(M_{m,i})| \leq \varepsilon \qquad \text{for every } i \in I.$$

Therefore : $d_m^{-1} \cdot |\lambda_m(M'_m) - \lambda_m(M_m)| \leq \varepsilon$, by proposition 14.5.54(ii). Combining with (14.5.72) and again (14.5.71) we obtain :

$$|\lambda^*_\infty(M) - d_m^{-1} \cdot \lambda_m(M_m)| \leq 3\varepsilon$$

which implies (i).

(ii): With no loss of generality, we may assume that I admits a smallest element i_0. Let us fix a surjection $F := R_\infty^{\oplus k} \to N_{i_0}$, and for every $i \in I$, let C_i denote the kernel of the induced surjection $F \to N_i$. We consider the filtered system $(D_j \mid j \in J)$ consisting of all finitely generated submodules $D_j \subset F$ such that $D_j \subset C_i$ for some $i \in I$. It is clear that $\mathrm{colim}_{j\in J} F/D_j \simeq M$, hence $\lambda^*_\infty(M) = \inf\{\lambda_\infty(F/D_j) \mid j \in J\}$, by (i). On the other, by construction, for every $j \in J$ we may find $i \in I$ such that $\lambda^*_\infty(N_i) \leq \lambda_\infty(F/D_j)$; since clearly $\lambda^*_\infty(N_i) \geq \lambda^*_\infty(M)$ for every $i \in I$, the assertion follows. □

14.5.73. Let now M be an arbitrary object of the category R_∞-$\mathbf{Mod}_{\{s\}}$. We let :

$$\lambda_\infty(M) := \sup\{\lambda^*_\infty(M') \mid M' \subset M \text{ and } M' \text{ is finitely generated}\} \in \mathbb{R} \cup \{+\infty\}.$$

Lemma 14.5.74. *If M is finitely generated, then $\lambda_\infty(M) = \lambda^*_\infty(M)$.*

Proof. Let $N \subset M$ be any finitely generated submodule. We choose finite sets of generators $\Sigma \subset N$ and $\Sigma' \subset M$ with $\Sigma \subset \Sigma'$. In view of proposition 14.5.54(iv) we have $\lambda_n(\Sigma R_n) \leq \lambda_n(\Sigma' R_n)$ for every $n \in \mathbb{N}$, hence $\lambda^*_\infty(N) \leq \lambda^*_\infty(M)$, by lemma 14.5.70(i). The contention follows easily. □

Theorem 14.5.75. (i) *Let $(M_i \mid i \in I)$ be a filtered system of objects of R_∞-$\mathbf{Mod}_{\{s\}}$, and suppose that either :*

(a) *all the transition maps of the system are injections, or*
(b) *all the transition maps are surjections and $\lambda_\infty(M_i) < +\infty$ for every $i \in I$.*

Then :

$$\lambda_\infty(\mathop{\mathrm{colim}}_{i\in I} M_i) = \lim_{i\in I} \lambda_\infty(M_i).$$

(ii) *Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence in R_∞-$\mathbf{Mod}_{\{s\}}$. Then :*

$$\lambda_\infty(M) = \lambda_\infty(M') + \lambda_\infty(M'').$$

(iii) *Let M be a finitely presented R_∞-module, $N \subset M$ a submodule supported at $s(R_\infty)$. Then $\lambda_\infty(N) = 0$ if and only if $N = 0$.*

Proof. The proof of (i) in case (a) is the same as that of proposition 14.5.54(iii.b).

(ii): We proceed in several steps :

• Suppose first that M and M'' are finitely presented, hence M' is finitely generated ([75, Lemma 2.3.18(ii)]). We may then find an integer $n \in \mathbb{N}$ and finitely presented R_n-modules M_n and M''_n such that $M \simeq R_\infty \otimes_{R_n} M_n$ and $M'' \simeq R_\infty \otimes_{R_n} M''_n$. For every $m \geq n$ we set $M_m := R_m \otimes_{R_n} M_n$ and likewise we define M''_m. Up to replacing n by a larger integer, we may assume that the given map $\varphi : M \to M''$ descends to a surjection $\varphi_n : M_n \to M''_n$, and then φ is the colimit of the induced maps $\varphi_m := \mathbf{1}_{R_m} \otimes_{R_n} \varphi_n$, for every $m \geq n$. Moreover, $M' \simeq \mathrm{colim}_{m \geq n} \mathrm{Ker}\, \varphi_m$, and by the right exactness of the tensor product, the image of $\mathrm{Ker}\, \varphi_m$ generates M' for every $m \geq n$. Furthermore, since the natural maps $M_m \to M_p$ are injective for every $p \geq m \geq n$ (lemma 14.5.66), the same holds for the induced maps $\mathrm{Ker}\, \varphi_m \to \mathrm{Ker}\, \varphi_p$. The latter factors as a composition :

$$\mathrm{Ker}\, \varphi_m \xrightarrow{\alpha} R_p \otimes_{R_m} \mathrm{Ker}\, \varphi_m \xrightarrow{\beta} \mathrm{Ker}\, \varphi_p$$

where α is injective (lemma 14.5.66) and β is surjective. In other words, $R_p \cdot \mathrm{Ker}\, \varphi_m = \mathrm{Ker}\, \varphi_p$ for every $p \geq m \geq n$. In such situation, lemma 14.5.70(i) ensures that :

$$\lambda_\infty(M') = \lim_{m \geq n} d_m^{-1} \cdot \lambda_m(\mathrm{Ker}\, \varphi_m)$$

and likewise :

$$\lambda_\infty(M) = \lim_{m \geq n} d_m^{-1} \cdot \lambda_m(M_m) \qquad \lambda_\infty(M'') = \lim_{m \geq n} d_m^{-1} \cdot \lambda_m(M''_m).$$

To conclude the proof of (ii) in this case, it suffices then to apply proposition 14.5.54(iv).

• Suppose next that M, M' (and hence M'') are finitely generated. We choose a filtered system $(M_i \mid i \in I)$ of finitely presented R_∞-modules, with surjective transition maps, such that $M \simeq \mathrm{colim}_{i \in I} M_i$. After replacing I by a cofinal subset, we may assume that M' is generated by a finitely generated submodule M'_i of M_i, for every $i \in I$, and that $(M'_i \mid i \in I)$ forms a filtered system with surjective transition maps, whose colimit is necessarily M'; set also $M''_i := M_i/M'_i$ for every $i \in I$, so that the colimit of the filtered system $(M''_i \mid i \in I)$ is M''. In view of lemmata 14.5.70(ii) and 14.5.74, we are then reduced to showing the identity :

$$\inf \{\lambda^*_\infty(M_i) \mid i \in I\} = \inf \{\lambda^*_\infty(M'_i) \mid i \in I\} + \inf \{\lambda^*_\infty(M''_i) \mid i \in I\}$$

which follows easily from the previous case.

• Suppose now that M is finitely generated. We let $(M'_i \mid i \in I)$ be the filtered family of finitely generated submodules of M'. Then :

$$M' \simeq \operatorname*{colim}_{i \in I} M'_i \quad \text{and} \quad M'' \simeq \operatorname*{colim}_{i \in I} M/M'_i.$$

Hence :

$$\lambda_\infty(M') = \lim_{i \in I} \lambda_\infty(M'_i) \quad \text{(resp.} \quad \lambda_\infty(M'') = \lim_{i \in I} \lambda_\infty(M/M'_i))$$

by (i.a) (resp. by lemmata 14.5.70(ii) and 14.5.74). However, the foregoing case shows that $\lambda_\infty(M) = \lambda_\infty(M'_i) + \lambda_\infty(M/M'_i)$ for every $i \in I$, so assertion (ii) holds also in this case.

• Finally we deal with the general case. Let $(M_i \mid i \in I)$ be the filtered system of finitely generated submodules of M; we denote by M''_i the image of M_i in M'', and set $M'_i := M' \cap M_i$ for every $i \in I$. By (i.a) we have :

$$\lambda_\infty(M) = \lim_{i \in I} \lambda_\infty(M_i)$$

and likewise for M' and M''. Since we already know that $\lambda_\infty(M_i) = \lambda_\infty(M'_i) + \lambda_\infty(M''_i)$ for every $i \in I$, we are done.

(iii): Let $f \in N$ be any element; in view of (ii) we see that $\lambda_\infty(fR_\infty) = 0$, and it suffices to show that $f = 0$. However, we may find $n \in \mathbb{N}$ and a finitely presented R_n-module M_n such that $M \simeq R_\infty \otimes_{R_n} M_n$; notice that the natural map $M_n \to M$ is injective, by lemma 14.5.66(i). We may also assume that f is in the image of M_n. Let $I \subset \mathrm{Ann}_{R_n}(f)$ be a finitely generated $\mathfrak{m}_{R_n}$-primary ideal; after replacing M by M/IM, we may assume that M_n is supported at $s(R_n)$. In light of (ii), we see that $\lambda_\infty(M) = \lambda_\infty(M/fR_\infty)$, hence $\lambda_n(M_n) = \lambda_n(M_n/fR_n)$, due to condition (b) of definition 14.5.64. Hence $\lambda_n(fR_n) = 0$ by proposition 14.5.54(iv), and finally $f = 0$ by theorem 14.5.43(i,iii.a).

To conclude, we consider assertion (i) in case (b) : set $M := \mathrm{colim}_{i\in I} M_i$; it is clear that $\lambda_\infty(M) \leq \lambda_\infty(M_i) \leq \lambda_\infty(M_j)$ whenever $i \geq j$, hence

$$\lim_{i\in I} \lambda_\infty(M_i) = \inf \{\lambda_\infty(M_i) \mid i \in I\} \geq \lambda_\infty(M).$$

For the converse inequality, fix $\varepsilon > 0$; without loss of generality, we may assume that I admits a smallest element i_0, and we can find a finitely generated submodule $N_{i_0} \subset M_{i_0}$ such that $\lambda_\infty(M_{i_0}) - \lambda_\infty(N_{i_0}) < \varepsilon$. For every $i \in I$, let $N_i \subset M_i$ be the image of N_{i_0}, and let $N \subset M$ be the colimit of the filtered system $(N_i \mid i \in I)$; then M_i/N_i is a quotient of M_{i_0}/N_{i_0}, and the additivity assertion (ii) implies that $\lambda_\infty(M_i) - \lambda_\infty(N_i) < \varepsilon$ for every $i \in I$. According to lemma 14.5.70(ii) (and lemma 14.5.74) we have :

$$\lambda_\infty(M) \geq \lambda_\infty(N) = \inf \{\lambda_\infty(N_i) \mid i \in I\} \geq \inf \{\lambda_\infty(M_i) \mid i \in I\} - \varepsilon$$

whence the claim. □

14.5.76. We wish now to show that the definition of normalized length descends to almost modules (see (14.5)). Namely, we have the following :

Proposition 14.5.77. *Let M, N be two objects of R_∞-$\mathbf{Mod}_{\{s\}}$ such that $M^a \simeq N^a$. Then $\lambda_\infty(M) = \lambda_\infty(N)$.*

Proof. Using additivity (theorem 14.5.75(ii)), we easily reduce to the case where $M^a = 0$, in which case we need to show that $\lambda_\infty(M) = 0$. Using theorem 14.5.75(i) we may further assume that M is finitely generated. Then, in view of lemma 14.5.70(i), we are reduced to showing the following :

Claim 14.5.78. Let A be any measurable K^+-algebra, and M any object of A-$\mathbf{Mod}_{\{s\}}$ such that $M^a = 0$. Then $\lambda_A(M) = 0$.

Proof of the claim. Arguing as in the foregoing we reduce to the case where M is finitely generated. Then, let us pick $I \subset A$ and $\varphi : V \to A/I$ as in lemma 14.5.42. It suffices to show the assertion for the finitely generated module $\bigoplus_{n\in\mathbb{N}} I^n M/I^{n+1}M$, hence we may assume that $I \subset \mathrm{Ann}_A M$, in which case, by proposition 14.5.54(iii) we may replace A by V and assume throughout that A is a valuation ring. Then the claim follows from propositions 14.5.54(i) and 14.5.12(ii). □

14.5.79. Proposition 14.5.77 suggests the following definition. We let R^a_∞-$\mathbf{Mod}_{\{s\}}$ be the full subcategory of R^a_∞-$\mathbf{Mod}$ consisting of all the R^a_∞-modules M such that $M_!$ is supported at $s(R_\infty)$, in which case we say that M *is supported at* $s(R_\infty)$. Then, for every such M we set :

$$\lambda_\infty(M) := \lambda_\infty(M_!).$$

With this definition, it is clear that theorem 14.5.75(i,ii) extends *mutatis mutandis* to almost modules. For future reference we point out :

Lemma 14.5.80. *Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of R^a_∞-modules. We have :*

(i) *If M lies in R^a_∞-$\mathbf{Mod}_{\{s\}}$, then $\lambda_\infty(abM) \leq \lambda_\infty(aM') + \lambda_\infty(bM'')$ for every $a, b \in \mathfrak{m}$.*
(ii) *If M is almost finitely presented, and M' is supported at $s(R_\infty)$, then $\lambda_\infty(M') = 0$ if and only if $M' = 0$.*

Proof. (i): To start out, we deduce a short exact sequence : $0 \to bM \cap M' \to bM \to bM'' \to 0$. Next, let $N := \mathrm{Ker}(a : bM \to bM)$, and denote by N' the image of N in bM''; there follows a short exact sequence : $0 \to a(bM \cap M') \to abM \to bM''/N' \to 0$. The claim follows.

(ii): We reduce easily to the case where M' is a cyclic R^a_∞-module, say $M' = R^a_\infty x$ for some $x \in M_!$. In this case, let $I \subset \mathrm{Ann}_{R_0}(x)$ be a finitely generated $\mathfrak{m}_{R_0}$-primary ideal; we may then replace M by M/IM, and assume that M lies in R^a_∞-$\mathbf{Mod}_{\{s\}}$ as well. We remark :

Claim 14.5.81. M is almost finitely presented if and only if, for every $b \in \mathfrak{m}$ there exist a finitely presented R_∞-module N and a morphism $M \to N^a$ whose kernel and cokernel are annihilated by b. Moreover if M is supported at $s(R_\infty)$, one can choose N to be supported at $s(R_\infty)$.

Proof of the claim. The "if" direction is clear. For the "only if" part, we use [75, Cor.2.3.13], which provides us with a morphism $\varphi : N \to M_!$ with N finitely presented over R_∞, such that $b \cdot \mathrm{Ker}\,\varphi = b \cdot \mathrm{Coker}\,\varphi = 0$. Then $b \cdot \mathbf{1}^a_N$ factors through a morphism $\varphi' : \mathrm{Im}\,\varphi^a \to N^a$, and $b \cdot \mathbf{1}_M$ factors through a morphism $\varphi'' : M \to \mathrm{Im}\,\varphi^a$; the kernel and cokernel of $\varphi' \circ \varphi''$ are annihilated by b^2. Finally, suppose that M is supported on $s(R_\infty)$, and let $I \subset R_\infty$ be any finitely generated ideal such that $V(I) = \{s(R_\infty)\}$. By assumption, for every $f \in I$ and every $m \in M_!$ there exists $n \in \mathbb{N}$ such that $f^n m = 0$; it follows that I^n annihilates $\mathrm{Im}\,\varphi$ for every sufficiently large $n \in \mathbb{N}$, and we may then replace N by $N/I^n N$. ◊

Let M and M' be as in (ii), and choose a morphism $\varphi : M \to N^a$ as in claim 14.5.81; by adjunction we get a map $\psi : M'_! \to M_! \to N$ with $b \cdot \mathfrak{m} \cdot \mathrm{Ker}\,\psi = 0$. It follows that $\lambda_\infty(\mathrm{Im}\,\psi) = 0$, hence $\mathrm{Im}\,\psi = 0$, by theorem 14.5.75(iii). Hence $bM' = 0$; since b is arbitrary, the assertion follows. □

Simple examples show that an almost finitely generated (or even almost finitely presented) R^a_∞-module may fail to have finite normalized length. The useful finiteness condition for almost modules is contained in the following :

Definition 14.5.82. Let M be a R^a_∞-module supported at $s(R_\infty)$. We say that M has *almost finite length* if $\lambda_\infty(bM) < +\infty$ for every $b \in \mathfrak{m}$.

Lemma 14.5.83. (i) *The set of isomorphism classes of R^a_∞-modules of almost finite length forms a closed subset of the uniform space $\mathscr{M}(A)$ (notation of* [75, §2.3]*).*

(ii) *Especially, every almost finitely generated R^a_∞-module supported at $s(R_\infty)$ has almost finite length.*

Proof. Assertion (i) boils down to the following :

Claim 14.5.84. Let $a, b \in \mathfrak{m}$, and $f : N \to M$, $g : N \to M'$ morphisms of R^a_∞-modules, such that the kernel and cokernel of f and g are annihilated by $a \in \mathfrak{m}$, and such that $\lambda_\infty(bM) < +\infty$. Then $\lambda_\infty(a^2bM') < +\infty$.

Proof of the claim. By assumption, $\mathrm{Ker}\,f \subset \mathrm{Ker}\,a \cdot \mathbf{1}_N$, whence an epimorphism $f(N) \to aN$; likewise, we have an epimorphism $g(N) \to aM'$. It follows that

$$\lambda_\infty(a^2bM') \leq \lambda_\infty(ab \cdot g(M)) \leq \lambda_\infty(abN) \leq \lambda_\infty(b \cdot f(N)) < +\infty$$

as stated. ◊

(ii) follows from (i) and the obvious fact that every finitely generated R_∞-module supported at $s(R_\infty)$ has finite normalized length. □

14.5.85. For the case of a measurable K^+-algebra A of dimension zero, we can show a further *Lipschitz type* uniform estimate for the normalized length. Namely, for any integer $k > 0$ define the set $\mathscr{M}_k(A^a)$ with its uniform structure as in (14.5.1); *i.e.* we have the fundamental system of entourages

$$(E_r \mid r \in \mathbb{R}_{>0})$$

where each E_r consists of the pairs (N, N') such that there exist a third A^a-module N'' and A^a-linear maps $N'' \to N$, $N'' \to N'$ whose kernel and cokernels are annihilated by any $b \in K^+$ such that $\log |b| \geq r$.

Lemma 14.5.86. *In the situation of* (14.5.85)*, we have :*

$$|\lambda_{A^a}(N) - \lambda_{A^a}(N')| \leq 4k \cdot \mathrm{length}_A(A/\mathfrak{m}A) \cdot r$$

for every $r \in \mathbb{R}_{>0}$ *and every* $(N, N') \in E_r$.

Proof. We begin with the following

Claim 14.5.87. For any finitely generated A-module N, and every $b \in K^+ \setminus \{0\}$ we have

$$\lambda_A(N/bN) \leq \mathrm{length}_A(N/\mathfrak{m}N) \cdot \log |b|.$$

Proof of the claim. Choose a map $\psi : V \to A$ of measurable K^+-algebras, inducing a finite residue field extension $\kappa(V) \to \kappa(A)$, where V is a K^+-flat valuation ring, and the induced map of value groups $\Gamma \to \Gamma_V$ is an isomorphism (claim 14.5.56). Let $d := \dim_{\kappa(V)} N/\mathfrak{m}_V N$; applying Nakayama's lemma, we get a surjection $(V/bV)^{\oplus d} \to N/bN$ of V-modules, for every $b \in K^+$. Hence

$$\lambda_A(N/bN) = \frac{\lambda_V(\psi_*(N/bN))}{[\kappa(V) : \kappa(A)]} \leq \frac{d \log |b|}{[\kappa(V) : \kappa(A)]} = \mathrm{length}_A(N/\mathfrak{m}N) \cdot \log |b|$$

as stated. ◊

Now, suppose that $(N, N') \in E_{\log |b|}$ for some $b \in K^+$, and pick maps $\varphi : N'' \to N$, $\psi : N'' \to N'$ whose kernel and cokernel are annihilated by b. Especially, if $n_1, \dots, n_k \in N_*$ (resp. $n'_1, \dots, n'_k \in N'_*$) is a system of generators for N (resp. for N'), we may find $n''_1, \dots, n''_{2k} \in N''_*$ such that $\varphi(n''_i) = b^2 n_i$ for $i = 1, \dots, k$ and $\psi(n''_i) = b^2 n'_{i-k}$ for $i = k+1, \dots, 2k$. After replacing N'' by its submodule generated by $n''_1, \dots, n''_{2k}$, we may assume that $N'' \in \mathscr{M}_{2k}(A)$, $\mathrm{Ker}\,\varphi$ is still annihilated by b, but $\mathrm{Coker}\,\varphi$ is only annihilated by b^2. On the one hand, we deduce that

$$\lambda_{A^a}(N'') \geq \lambda_{A^a}(\varphi(N'')) \geq \lambda_{A^a}(b^2 N) = \lambda_{A^a}(N) - \lambda_{A^a}(N/b^2 N)$$

therefore

$$\lambda_{A^a}(N) - \lambda_{A^a}(N'') \leq \lambda_{A^a}(N/b^2 N).$$

On the other hand, the map $N'' \to N''$ given by the rule : $n'' \mapsto bn''$ for every $n'' \in N''_*$, factors through $\varphi(N'')$, therefore $\lambda_{A^a}(N) \geq \lambda_{A^a}(bN'')$ so the same calculation yields the inequality

$$\lambda_{A^a}(N'') - \lambda_{A^a}(N) \leq \lambda_{A^a}(N''/bN'').$$

Taking into account claim 14.5.87 (and proposition 14.5.77) we see that

$$|\lambda_{A^a}(N'') - \lambda_{A^a}(N)| \leq 2k \cdot \mathrm{length}_A(A/\mathfrak{m}A) \cdot \log |b|.$$

Of course, the same holds with N replaced by N', and the lemma follows. □

We conclude this section with some basic examples of the situation contemplated in definition 14.5.64.

Example 14.5.88. Suppose that all the transition maps $R_n \to R_{n+1}$ of the inductive system in (14.5.62) are flat. Then, proposition 14.5.54(v) implies that conditions (a) and (b) hold with:

$$d_n := \mathrm{length}_{R_n}(R_n/\mathfrak{m}_{R_0} R_n) \qquad \text{for every } n \in \mathbb{N}.$$

Example 14.5.89. Suppose that $p := \mathrm{char}\,\kappa > 0$. Let $d \in \mathbb{N}$ be any integer, $f : X \to \mathbb{A}^d_{K^+} := \mathrm{Spec}\,K^+[T_1, \dots, T_d]$ an étale morphism. For every $r \in \mathbb{N}$ we consider the cartesian diagram of schemes

$$\begin{array}{ccc} X_r & \xrightarrow{f_r} & \mathbb{A}^d_{K^+} \\ {\scriptstyle \psi_r}\downarrow & & \downarrow{\scriptstyle \varphi_r} \\ X & \xrightarrow{f} & \mathbb{A}^d_{K^+} \end{array}$$

where φ_r is the morphism corresponding to the K^+-algebra homomorphism

$$\varphi_r^\natural : K^+[T_1, \dots, T_d] \to K^+[T_1, \dots, T_d]$$

defined by the rule: $T_j \mapsto T_j^{p^r}$ for $j = 1, \dots, d$. For every $r, s \in \mathbb{N}$ with $r \geq s$, ψ_r factors through an obvious S-morphism $\psi_{rs} : X_r \to X_s$, and the collection of the schemes X_r and transition morphisms ψ_{sr} gives rise to an inverse system $\underline{X} := (X_r \mid r \in \mathbb{N})$, whose inverse limit is representable by an S-scheme X_∞ ([65, Ch.IV, Prop.8.2.3]). Let $g_\infty : X_\infty \to S$ (resp. $g_r : X_r \to S$ for every $r \in \mathbb{N}$) be the structure morphism, $x \in g_\infty^{-1}(s)$ any point, $x_r \in X_r$ the image of x and $R_r := \mathscr{O}_{X_r, x_r}$ for every $r \in \mathbb{N}$. Clearly the colimit R_∞ of the inductive system $(R_r \mid r \in \mathbb{N})$ is naturally isomorphic to $\mathscr{O}_{X_\infty, x}$. Moreover, notice that the restriction $g_{r+1}^{-1}(s) \to g_r^{-1}(s)$ is a radicial morphism for every $r \in \mathbb{N}$. It follows easily that the transition maps $R_r \to R_{r+1}$ are finite; furthermore, by inspection one sees that $\varphi_r^\natural$ is flat and finitely presented, so R_{r+1} is a free R_r-module of rank p^d, for every $r \in \mathbb{N}$. Hence, the present situation is a special case of example 14.5.88, and therefore conditions (a) and (b) hold if we choose the sequence of integers $(d_i \mid i \in \mathbb{N})$ with

$$d_i := p^{id}/[\kappa(x_i) : \kappa(x_0)] \qquad \text{for every } i \in \mathbb{N}.$$

The foregoing discussion then yields a well-behaved notion of normalized length for arbitrary R_∞-modules supported at $\{s(R_\infty)\}$.

Example 14.5.90. In the situation of example 14.5.89, it is easy to construct R^a_∞-modules $M \neq 0$ such that $\lambda_\infty(M) = 0$. For instance, for $d := 1$, let $x \in X_\infty$ be the point of the special fibre where $T_1 = 0$; then we may take $M := R_\infty/I$, where I is the ideal generated by a non-zero element of $\mathfrak{m}_K$ and by the radical of $T_1 R_\infty$. The verification shall be left to the reader.

14.6. **Finite group actions on almost algebras.** In this section we fix a basic setup $(V, \mathfrak{m})$ such that $\widetilde{\mathfrak{m}} := \mathfrak{m} \otimes_V \mathfrak{m}$ is a flat V-module (see [75, §2.1.1]), and we consider some descent problems for V^a-algebras endowed with a finite group of automorphisms. Hence, the results below overlap with those of [75, §4.5].

14.6.1. Let G be a finite group, A a V^a-algebra, and let $S := \mathrm{Spec}\,V^a$, $X := \mathrm{Spec}\,A$. A *right action of* G *on* X is a group homomorphism :

$$\rho : G \to \mathrm{Aut}_{V^a\text{-}\mathsf{Alg}}(A)$$

from G to the group of automorphisms of A. Let G_S be the affine group S-scheme defined by G; hence every $g \in G$ determines a section $g_S : S \to G_S$ of the structure morphism $G_S \to S$, and the resulting morphism:

$$\coprod_{g \in G} g_S : S \amalg \cdots \amalg S \to G_S$$

is an isomorphism of S-schemes. Then ρ can be also regarded as a right action of G_S on X, as defined in [75, §3.3.6]. Especially, ρ induces morphisms of S-schemes :

$$\partial_i : X \times G := X \times_S G_S \to X \qquad i = 0, 1$$

as in *loc. cit.*, and we may define a *G-action on an A-module M* (covering the given action of G on X) as a morphism of quasi-coherent $\mathscr{O}_{X\times G}$-modules :

$$\beta : \partial_0^* M \to \partial_1^* M$$

fulfilling the conditions of [75, §3.3.7]. One also says that (M,β) is a *G-equivariant A-module*. We denote by $A[G]$-$\mathbf{Mod}$ the category of all G-equivariant A-modules and G-equivariant A-linear morphisms. Notice that $A[G]$-$\mathbf{Mod}$ is an abelian tensor category : indeed, for any two objects (M,β), (M,β') we may set $(M,\beta)\otimes_A (M',\beta') := (M\otimes_A M', \beta\otimes_{\mathscr{O}_{X\times G}} \beta')$.

14.6.2. Likewise, if B is any A-algebra, a G-action on B is a morphism $\beta : \partial_0^* B \to \partial_1^* B$ of quasi-coherent $\mathscr{O}_{X\times G}$-algebras, such that the pair (B,β) is a G-equivariant A-module. We say that (B,β) is a *G-equivariant A-algebra*, and we denote by $A[G]$-Alg the category of such pairs, with G-equivariant morphisms of A-algebras. One verifies easily that the datum (B,β) is the same as a morphism $\psi : A \to B$ of V^a-algebras, together with a G-action $\rho_B : G \to \mathrm{Aut}_{V^a\text{-}\mathsf{Alg}}(B)$ on the affine scheme $\mathrm{Spec}\, B$, such that the diagram :

$$\begin{array}{ccc} A & \xrightarrow{\psi} & B \\ {\scriptstyle \rho(g)}\downarrow & & \downarrow{\scriptstyle \rho_B(g)} \\ A & \xrightarrow{\psi} & B \end{array}$$

commutes for every $g\in G$. We shall also consider the full subcategory $A[G]$-$\mathsf{Alg}_{\mathrm{fl}}$ (resp. $A[G]$-$\mathbf{w.\acute{E}t}$, resp. $A[G]$-$\mathbf{\acute{E}t}$, resp. $A[G]$-$\mathbf{\acute{E}t}_{\mathrm{afp}}$) of all such pairs, where B is a flat (resp. weakly étale, resp. étale, resp. étale and almost finitely presented) A-algebra.

14.6.3. The *trivial G-action* on X is the map ρ with $\rho(g) = \mathbf{1}_A$ for every $g\in G$; this is the same as saying that $\partial_0 = \partial_1$. If G acts trivially on X, a G-equivariant A-module (M,β) is the same as a group homomorphism $\overline{\beta} : G \to \mathrm{Aut}_A(M)$ from G to the group of A-linear automorphisms of M. Namely, for every $g\in G$, one lets :

$$\overline{\beta}(g) := (\mathbf{1}_X \times_S g_S)^*\beta \qquad (14.6.4)$$

and conversely, to a given map $\overline{\beta}$ there corresponds a unique pair (M,β) such that (14.6.4) holds.

Under this correspondence, the *trivial G-action* $\overline{\beta}_0$ (such that $\overline{\beta}(g) = \mathbf{1}_M$ for every $g\in G$) corresponds to the identity morphism $\beta_0 : \partial_0^* M \xrightarrow{\sim} \partial_1^* M$.

More generally, let (M,β) be a G-action on an A-module M, covering the trivial G-action on X; for every A_*-valued character $\chi : G \to A_*^\times$ of G, we let

$$M_\chi := \bigcap_{g\in G} \mathrm{Ker}(\overline{\beta}(g) - \chi(g)\cdot \mathbf{1}_M).$$

The restriction of $\overline{\beta}$ defines a G-action on M_χ, such that the monomorphism $M_\chi \subset M$ is G-equivariant. In the special case where χ is the trivial character, we have $M_\chi = (M,\beta)^G$, the largest G-equivariant A-submodule of (M,β) on which β restricts to the trivial G-action (*i.e.* the submodule fixed by G). When the notation is not ambiguous, we shall often just write M^G instead of $(M,\beta)^G$.

14.6.5. If $H\subset G$ is a subgroup, any G-action ρ on X induces by restriction an H-action $\rho_{|H}$; then the morphisms $\partial_i : X\times H \to X$ are just the restrictions of the corresponding morphisms for G (under the natural closed immersion $X\times H \to X\times G$). Similarly, a G-action β on an A-module M induces by restriction an H-action $\beta_{|H}$ on the same module.

Let $Y := \operatorname{Spec} B$ be any affine S-scheme. The G-action ρ induces a G-action $Y \times_S \rho$ on $Y \times_S X$; namely, $Y \times_S \rho(g) := \mathbf{1}_Y \times_S \rho(g)$ for every $g \in G$. In terms of the group scheme G_S, this is the action given by the morphisms :

$$\partial_{Y,i} := \mathbf{1}_Y \times_S \partial_i : (Y \times_S X) \times G \to (Y \times_S X) \qquad i = 0, 1.$$

Let $\pi_X : Y \times_S X \to X$ be the natural morphism; then every G-equivariant A-module (M, β) induces a G-equivariant $B \otimes_{V^a} A$-module $\pi_X^*(M, \beta) := (\pi_X^* M, \pi_X^* \beta)$, whose action covers the G-action $Y \times_S \rho$ on $Y \times_S X$.

14.6.6. Let us set :

$$X/G := \mathrm{Coequal}(X \times G \overset{\partial_1}{\underset{\partial_1}{\rightrightarrows}} X) \quad \text{and} \quad X^{\langle g \rangle} := \mathrm{Equal}(X \overset{\rho(g)}{\underset{\mathbf{1}_X}{\rightrightarrows}} X) \quad \text{for every } g \in G.$$

In other words, $X/G = \operatorname{Spec} A^G$, the subalgebra fixed by G, and $X^{\langle g \rangle}$ is the closed subscheme fixed by the subgroup $\langle g \rangle \subset G$ generated by g. Thus, $X^{\langle g \rangle}$ is the spectrum of a quotient A/I_g of A, where $I_g \subset A$ is the ideal generated by the almost elements of the form $a - \rho(g)(a)$, for every $a \in A_*$.

Let $\pi : X \to X/G$ be the natural morphism, N any quasi-coherent $\mathscr{O}_{X/G}$-module; the pull-back $\pi^* N$ is the quasi-coherent $\mathscr{O}_X$-module $A \otimes_{A^G} N$. By construction, there is a natural isomorphism $\beta_N : \partial_0^*(\pi^* N) \xrightarrow{\sim} \partial_1^*(\pi^* N)$ (deduced from the natural isomorphisms of functors $\partial_i^* \circ \pi^* \simeq (\pi \circ \partial_i)^*$, for $i = 0, 1$), and one verifies easily that β_N is a G-action on $\pi^* N$.

Moreover, let $i_g : X^{\langle g \rangle} \to X$ be the natural closed immersion; if M is any A-module, then $i_g^* M$ is the A/I_g-module $M/I_g M$. Especially, take $M := \pi^* N$; we notice that the restriction $\rho_{|\langle g \rangle}$ of the given action ρ, induces the trivial $\langle g \rangle$-action on $X^{\langle g \rangle}$, and directly from the construction, we see that the natural action $\beta_{N|\langle g \rangle}$ restricts to the trivial $\langle g \rangle$-action on $i_g^* M$.

We are thus led to the :

Definition 14.6.7. Let G be a finite group, A a V^a-algebra, ρ a G-action on $X := \operatorname{Spec} A$.

(i) The category $A[G]\text{-}\mathbf{Mod}_{\mathrm{hor}}$ of *A-modules with horizontal G-action* is the full subcategory of $A[G]\text{-}\mathbf{Mod}$ consisting of all pairs (M, β) subject to the following condition. For every $g \in G$, the restriction $\beta_{|\langle g \rangle}$ induces the trivial action on $i_g^* M$.

(ii) We denote by $A[G]\text{-}\mathbf{Mod}_{\mathrm{hor.fl}}$ the full subcategory of $A[G]\text{-}\mathbf{Mod}_{\mathrm{hor}}$ consisting of the pairs (M, β) as above, such that M is a flat A-module.

(iii) Likewise, we denote by $A[G]\text{-}\mathsf{Alg}_{\mathrm{hor}}$ (resp. $A[G]\text{-}\mathsf{Alg}_{\mathrm{hor.fl}}$, resp. $A[G]\text{-}\mathbf{w.\acute{E}t}_{\mathrm{hor}}$, resp. $A[G]\text{-}\mathbf{\acute{E}t}_{\mathrm{hor}}$, resp. $A[G]\text{-}\mathbf{\acute{E}t}_{\mathrm{hor.afp}}$) the full subcategory of $A[G]\text{-}\mathsf{Alg}$ (resp. $A[G]\text{-}\mathsf{Alg}_{\mathrm{fl}}$, resp. $A[G]\text{-}\mathbf{w.\acute{E}t}$, resp. $A[G]\text{-}\mathbf{\acute{E}t}$, resp. $A[G]\text{-}\mathbf{\acute{E}t}_{\mathrm{afp}}$) consisting of all pairs (B, β) which are horizontal, when regarded as G-equivariant A-modules.

Notice that the tensor product of horizontal modules (resp. algebras) is again horizontal. By the foregoing, the rule $N \mapsto (\pi^* N, \beta_N)$ defines a functor :

$$A^G\text{-}\mathbf{Mod} \to A[G]\text{-}\mathbf{Mod}_{\mathrm{hor}}. \tag{14.6.8}$$

On the other hand, if (M, β) is any G-equivariant A-module, the pair $\pi_*(M, \beta) := (\pi_* M, \pi_* \beta)$ may be regarded as a G-action on $\pi_* M$, covering the trivial G-action on $\operatorname{Spec} A^G$, *i.e.* a group homomorphism $G \to \operatorname{Aut}_{A^G}(M)$ (see (14.6.3)). One verifies easily that the functor $N \mapsto (\pi^* N, \beta_N)$ is left adjoint to the functor $A[G]\text{-}\mathbf{Mod} \to A^G\text{-}\mathbf{Mod} : (M, \beta) \mapsto \pi_*(M, \beta)^G$ (details left to the reader). Hence, also (14.6.8) admits a right adjoint, given by the same rule. Similar assertions hold for the analogous functors :

$$A^G\text{-}\mathsf{Alg} \to A[G]\text{-}\mathsf{Alg}_{\mathrm{hor}} \tag{14.6.9}$$

and the variants considered in definition 14.6.7(iii).

Lemma 14.6.10. *In the situation of definition* 14.6.7*, suppose furthermore that the order* $o(G)$ *of* G *is invertible in* A_**, and let* (M, β) *be any* G*-equivariant* A*-module. Then :*

(i) *For every* A_*^G*-valued character* $\chi : G \to (A_*^G)^\times$*, the natural* A^G*-linear monomorphism* $\pi_* M_\chi \to \pi_* M$ *admits a* G*-equivariant* A^G*-linear right inverse* $\pi_* M \to \pi_* M_\chi$*.*

(ii) *For every* A^G*-module* N*, the unit of adjunction :*

$$\varepsilon_N : N \to (A \otimes_{A^G} N)^G$$

is an isomorphism.

(iii) *Let* $Y := \mathrm{Spec}\, B$ *be any affine* S*-scheme; denote by* $\pi_X : Y \times_S X \to X$ *and* $\pi_{X/G} : Y \times_S (X/G) \to X/G$ *the natural projections. Then the natural morphism :*

$$\pi_{X/G}^*(M, \beta)^G \to (\pi_X^* M, \pi_X^* \beta)^G$$

is an isomorphism of $B \otimes_{V^a} A^G$*-modules.*

Proof. This is standard : for every χ as in (i), the group algebra $A_*[G]$ admits the central idempotent :

$$e_\chi := \frac{1}{o(G)} \cdot \sum_{g \in G} \chi(g) \cdot g \qquad (14.6.11)$$

and $M_\chi = e_\chi M$ for any G-equivariant A-module M. Especially, we may take $M := N \otimes_{A^G} A$, and e_0 the central idempotent associated with the trivial character, in which case $e_0 A = A^G$, and $M = N \oplus (N \otimes_{A^G} (1 - e_0)A)$, so all the claims follow easily. ☐

Definition 14.6.12. Let Γ be any finite abelian group with neutral element $0 \in \Gamma$.

(i) A Γ-*graded* V^a-*algebra* is a pair $\underline{A} := (A, \mathrm{gr}_\bullet A)$ consisting of a V^a-algebra A and a decomposition $A = \bigoplus_{\chi \in \Gamma} \mathrm{gr}_\chi A$ as a direct sum of V^a-modules, such that :

$$1 \in \mathrm{gr}_0 A_* \quad \text{and} \quad \mathrm{gr}_\chi A \cdot \mathrm{gr}_{\chi'} A \subset \mathrm{gr}_{\chi + \chi'} A \qquad \text{for every } \chi, \chi' \in \Gamma$$

(where as usual $\mathrm{gr}_\chi A \cdot \mathrm{gr}_{\chi'} A$ denotes the image of the restriction $\mathrm{gr}_\chi A \otimes_{V^a} \mathrm{gr}_{\chi'} A \to A$ of the multiplication morphism μ_A). Especially, $\mathrm{gr}_0 A$ is a V^a-subalgebra of A, and every submodule $\mathrm{gr}_\chi A$ is a $\mathrm{gr}_0 A$-module.

(ii) A Γ-*graded* $\underline{A}$-*module* is a pair $\underline{N} := (N, \mathrm{gr}_\bullet N)$ consisting of an A-module N and a decomposition $N = \bigoplus_{\chi \in \Gamma} \mathrm{gr}_\chi N$ as a direct sum of V^a-modules, such that :

$$\mathrm{gr}_\chi A \cdot \mathrm{gr}_{\chi'} N \subset \mathrm{gr}_{\chi + \chi'} N \qquad \text{for every } \chi, \chi' \in \Gamma.$$

Of course, a morphism of Γ-graded $\underline{A}$-modules $\underline{N} \to \underline{N}' := (N', \mathrm{gr}_\bullet N')$ is an A-linear morphism $N \to N'$ that respects the gradings.

(iii) For every subgroup $\Delta \subset \Gamma$, let $J_\Delta \subset A$ be the graded ideal generated by $\bigoplus_{\chi \notin \Delta} \mathrm{gr}_\chi A$. We say that $\underline{N}$ is *horizontal* if $\mathrm{gr}_\chi(N/J_\Delta N) := \mathrm{gr}_\chi N / (\mathrm{gr}_\chi N \cap J_\Delta N) = 0$ for every subgroup $\Delta \subset \Gamma$ and every $\chi \notin \Delta$.

(iv) If $\Delta \subset \Gamma$ is any subgroup, $p : \Gamma \to \Gamma/\Delta$ the natural projection, and $\rho \in \Gamma/\Delta$ any element, we let :

$$\mathrm{gr}_\rho^{\Gamma/\Delta} N := \bigoplus_{\chi \in p^{-1}(\rho)} \mathrm{gr}_\chi N$$

and set $\underline{N}_{\Gamma/\Delta} := (N, \mathrm{gr}_\bullet^{\Gamma/\Delta} N)$. Then $\underline{A}_{\Gamma/\Delta}$ is a Γ/Δ-graded V^a-algebra, and $\underline{N}_{\Gamma/\Delta}$ is a Γ/Δ-graded $\underline{A}_\Delta$-module. Moreover, let also $\underline{N}_{|\Delta}$ be the pair consisting of $N_{|\Delta} := \mathrm{gr}_0^{\Gamma/\Delta} N$ together with its decomposition $N_{|\Delta} = \bigoplus_{\chi \in \Delta} \mathrm{gr}_\chi N$; then $\underline{A}_{|\Delta}$ is a Δ-graded V^a-algebra, and $\underline{N}_{|\Delta}$ is a Δ-graded $\underline{A}_{|\Delta}$-module.

Proposition 14.6.13. *In the situation of definition* 14.6.12*, let* $\underline{N}$ *be any* Γ*-graded* $\underline{A}$*-module. Then the following conditions are equivalent :*

(a) $\underline{N}$ *is horizontal.*
(b) *The natural morphism* $A \otimes_{\mathrm{gr}_0 A} \mathrm{gr}_0 N \to N$ *is an epimorphism.*

Proof. (b)⇒(a): The assertion is obvious for the Γ-graded $\underline{A}$-module consisting of $A \otimes_{\mathrm{gr}_0 A} \mathrm{gr}_0 N$ and its natural grading deduced from $\mathrm{gr}_\bullet A$; however any (graded) quotient of a horizontal module is horizontal, hence the assertion follows also for $\underline{N}$.

(a)⇒(b): We argue by induction on $o(\Gamma)$. The first case is covered by the following :

Claim 14.6.14. The proposition holds if $o(\Gamma)$ is a prime number.

Proof of the claim. Indeed, suppose that $\underline{N}$ is horizontal. We may replace N by $N/(A \cdot \mathrm{gr}_0 N)$, which is a horizontal Γ-graded $\underline{A}$-module, when endowed with the grading induced from $\mathrm{gr}_\bullet N$. Then $\mathrm{gr}_0 N = 0$, and we have to show that $N = 0$. By assumption, $N \subset J_{\{0\}} N$, *i.e.* :

$$\mathrm{gr}_\chi N \subset \sum_{\sigma \neq 0, \chi} \mathrm{gr}_\sigma A \cdot \mathrm{gr}_{\chi - \sigma} N \qquad \text{for every } \chi \neq 0.$$

For every $n > 0$, the symmetric group S_n acts on the set $(\Gamma \setminus \{0\})^n$ by permutations; we let $Q_n := (\Gamma \setminus \{0\})^n / S_n$, the set of equivalence classes under this action. For every $\underline{\sigma} := (\sigma_1, \dots, \sigma_n) \in Q_n$, let $|\underline{\sigma}| := \sum_{i=1}^n \sigma_i$. By an easy induction, it follows that :

$$\mathrm{gr}_\chi N \subset \sum_{\underline{\sigma} \in Q_n} \mathrm{gr}_{\sigma_1} A \cdots \mathrm{gr}_{\sigma_n} A \cdot \mathrm{gr}_{\chi - |\underline{\sigma}|} N.$$

Since every element of $\Gamma \setminus \{0\}$ generates Γ, it is also clear that there exists $n \in \mathbb{N}$ large enough such that every sequence $\underline{\sigma}$ in Q_n admits a subsequence, say $\underline{\tau} := (\tau_1, \dots, \tau_m)$ for some $m \leq n$, with $|\underline{\tau}| = \chi$ (details left to the reader). Up to a permutation, we may assume that $\underline{\tau}$ is the final segment of $\underline{\sigma}$; then we have :

$$\mathrm{gr}_{\sigma_1} A \cdots \mathrm{gr}_{\sigma_n} A \cdot \mathrm{gr}_{\chi - |\underline{\sigma}|} N \subset \mathrm{gr}_{\sigma_1} A \cdots \mathrm{gr}_{\sigma_{n-m}} A \cdot \mathrm{gr}_0 N = 0$$

whence the claim. ◊

Next, suppose that the assertion is already known for every subgroup $\Gamma' \subset \Gamma$, every graded Γ'-algebra $\underline{B}$, and every Γ'-graded $\underline{B}$-module $\underline{P}$. We choose a subgroup $\Gamma' \subset \Gamma$ such that $(\Gamma : \Gamma')$ is a prime number. We shall use the following :

Claim 14.6.15. Let G be a group, $0 \in G$ the neutral element, $H \subset G$ a subgroup. For every subgroup $L \subset G$ with $H \cap L = \{0\}$, choose an element $g_L \in G \setminus L$; denote by S the subgroup generated by all these elements g_L. Then $S \cap H \neq \{0\}$.

Proof of the claim. Indeed, if $S \cap H = \{0\}$, we would have $g_S \in S$, a contradiction. ◊

Claim 14.6.16. Suppose that $\underline{N}$ is horizontal. Then $\underline{N}_{|\Gamma'}$ is a horizontal Γ'-graded $\underline{A}_{|\Gamma'}$-module.

Proof of the claim. For any given subgroup $\Delta \subset \Gamma'$, let $J'_\Delta \subset A_{|\Gamma'}$ be the ideal generated by $\bigoplus_{\chi \in \Gamma' \setminus \Delta} \mathrm{gr}_\chi A$; have to show that $\mathrm{gr}_\chi(N/J'_\Delta N) = 0$ for every $\chi \in \Gamma' \setminus \Delta$. To this aim, we may replace Γ, Γ', $\underline{A}$, $\underline{A}_{|\Gamma'}$, $\underline{N}$ and $\underline{N}_{|\Gamma'}$, by respectively Γ/Δ, Γ'/Δ, $\underline{A}_{\Gamma/\Delta}$, $(\underline{A}_{|\Gamma'})_{\Gamma'/\Delta}$, $\underline{N}_{\Gamma/\Delta}$ and $(\underline{N}_{|\Gamma'})_{\Gamma'/\Delta}$, which allows to assume that $\Delta = \{0\}$. In this case, we have to show that :

$$N_{|\Gamma'} = \mathrm{gr}_0 N + J'_{\{0\}} \cdot N_{|\Gamma'}. \tag{14.6.17}$$

Notice that $A_{|\Gamma'} \cdot \mathrm{gr}_0 N = \mathrm{gr}_0 N + J'_{\{0\}} \cdot \mathrm{gr}_0 N$; the quotient $P := N/(A \cdot \mathrm{gr}_0 N)$ carries a unique Γ-grading $\mathrm{gr}_\bullet P$ such that the projection $\underline{N} \to \underline{P} := (P, \mathrm{gr}_\bullet P)$ is a morphism of Γ-graded $\underline{A}$-modules (namely, $\mathrm{gr}_\chi P := N_\chi / \mathrm{gr}_\chi A \cdot \mathrm{gr}_0 N$ for every $\chi \in \Gamma$). Moreover, P is horizontal, $\mathrm{gr}_0 P = 0$, and (14.6.17) is equivalent to : $P_{|\Gamma'} = J'_{\{0\}} \cdot P_{|\Gamma'}$. Therefore we may replace $\underline{N}$ by $\underline{P}$, and assume from start that $\mathrm{gr}_0 N = 0$. Similarly, notice that $J'_{\{0\}} \cdot N$ is a Γ-graded $\underline{A}$-submodule of $\underline{N}$, and the pair $\underline{Q}$ consisting of $Q := N/(J'_{\{0\}} \cdot N)$ and its quotient grading, is horizontal.

Moreover, $\mathrm{gr}_\chi Q = \mathrm{gr}_\chi(N_{|\Gamma'}/J'_{\{0\}}N_{|\Gamma'})$, for every $\chi \in \Gamma'$. Hence, we may replace N by Q, which allows to assume as well that

$$J'_{\{0\}}N_{|\Gamma'} = 0 \qquad (14.6.18)$$

in which case, we are reduced to showing that $N = 0$. Now, by assumption, for every subgroup $H \subset \Gamma$ we have :

$$N = N_{|H} + J_H N.$$

Let $\mathscr{C}$ be the set of all subgroups $H \subset \Gamma$ with $H \cap \Gamma' = \{0\}$; we deduce that :

$$\mathrm{gr}_\chi N \subset \sum_{\sigma\notin H} \mathrm{gr}_\sigma A \cdot \mathrm{gr}_{\chi-\sigma} N \qquad \text{for every } \chi \in \Gamma' \setminus \{0\} \text{ and every } H \in \mathscr{C}. \qquad (14.6.19)$$

Moreover, in view of (14.6.18) we may omit from the sum in (14.6.19) all the elements σ that lie in Γ'; for the remaining elements we have $\chi - \sigma \notin \Gamma'$, hence we may apply claim 14.6.14 to the horizontal Γ/Γ'-graded $\underline{A}_{\Gamma/\Gamma'}$-module $\underline{N}_{\Gamma/\Gamma'}$, to deduce that :

$$\mathrm{gr}_{\chi-\sigma} N \subset \sum_{\delta\in\Gamma'\setminus\{0\}} \mathrm{gr}_{\chi-\sigma-\delta} A \cdot \mathrm{gr}_\delta N.$$

Therefore :

$$\mathrm{gr}_\chi N \subset \sum_{\sigma\notin H\cup\Gamma'} \sum_{\delta\in\Gamma'\setminus\{0\}} \mathrm{gr}_\sigma A \cdot \mathrm{gr}_{\chi-\sigma-\delta} A \cdot \mathrm{gr}_\delta N \qquad \text{for every } \chi \in \Gamma' \setminus \{0\} \text{ and every } H \in \mathscr{C}.$$

However – again due to (14.6.18) – we may omit from this sum all the terms corresponding to the pairs (σ, δ) with $\chi \neq \delta$, hence we conclude that :

$$\mathrm{gr}_\chi N \subset \sum_{\sigma\notin H\cup\Gamma'} \mathrm{gr}_\sigma A \cdot \mathrm{gr}_{-\sigma} A \cdot \mathrm{gr}_\chi N \qquad \text{for every } \chi \in \Gamma' \setminus \{0\} \text{ and every } H \in \mathscr{C}.$$

Denote by Σ the set of all mappings $\underline{\sigma} : \mathscr{C} \to \Gamma$ such that $\underline{\sigma}(H) \notin H \cup \Gamma'$ for every $H \in \mathscr{C}$. Moreover, for every $\underline{\sigma} \in \Sigma$, set :

$$B_{\underline{\sigma}} := \prod_{H\in\mathscr{C}} \mathrm{gr}_{\underline{\sigma}(H)} A \cdot \mathrm{gr}_{-\underline{\sigma}(H)} A$$

(this is an ideal of $\mathrm{gr}_0 A$) and let $S_{\underline{\sigma}} \subset \Gamma$ be the subgroup generated by the image of $\underline{\sigma}$. By an easy induction we deduce that :

$$\mathrm{gr}_\chi N \subset \sum_{\underline{\sigma}\in\Sigma} B^n_{\underline{\sigma}} \cdot \mathrm{gr}_\chi N \qquad \text{for every } n > 0.$$

By claim 14.6.15, for every $\underline{\sigma} \in \Sigma$ we may find $\gamma(\underline{\sigma}) \in S_{\underline{\sigma}} \cap \Gamma' \setminus \{0\}$. On the other hand, since Γ is finite and abelian, it is easy to verify that there exists $n \in \mathbb{N}$ large enough such that

$$B^n_{\underline{\sigma}} \subset \mathrm{gr}_{\gamma(\underline{\sigma})} A \cdot \mathrm{gr}_{-\gamma(\underline{\sigma})} A \qquad \text{for every } \underline{\sigma} \in \Sigma$$

(details left to the reader). But (14.6.18) implies that $\mathrm{gr}_{\gamma(\underline{\sigma})} A \cdot \mathrm{gr}_\chi N = 0$ whenever $\chi \in \Gamma' \setminus \{0\}$, so the claim follows. ◊

To conclude, we apply first claim 14.6.14 to the horizontal Γ/Γ'-graded $\underline{A}_{\Gamma/\Gamma'}$-module $\underline{N}_{\Gamma/\Gamma'}$, to see that $N_{|\Gamma/\Gamma'}$ generates the A-module N, and then claim 14.6.16 – together with our inductive assumption – to deduce that $\mathrm{gr}_0 N$ generates that $A_{|\Gamma/\Gamma'}$-module $N_{|\Gamma/\Gamma'}$. The proposition follows. □

14.6.20. Recall that a morphism $M \to N$ of A-modules is said to be *pure* if the natural morphism $Q \otimes_A M \to Q \otimes_A N$ is a monomorphism for every A-module Q. A morphism $A \to B$ of V^a-algebras is called *pure* if it is pure when regarded as a morphism of A-modules.

Lemma 14.6.21. *Let $f : A \to B$ be a pure morphism of V^a-algebras, M an A-module. Then :*

(i) *If $B \otimes_A M$ is a flat B-module, then M is a flat A-module (*i.e. *f descends flatness).*

(ii) *If $B \otimes_A M$ is almost finitely generated (resp. almost finitely presented) as a B-module, then M is almost finitely generated (resp. almost finitely presented) as an A-module.*

Proof. (i): To begin with, we remark :

Claim 14.6.22. Let R be any V-algebra such that $A = R^a$.

(i) A morphism $\varphi : M_1 \to M_2$ of A-modules is pure if and only if the same holds for the induced morphism $\varphi_! : M_{1!} \to M_{2!}$ of R-modules.

(ii) A morphism $g : A \to B'$ of V^a-algebras is pure if and only if the same holds for the induced morphism $g_{!!} : A_{!!} \to B'_{!!}$.

Proof of the claim. (i): Suppose that φ is pure, and let Q be any R-module. Then $Q \otimes_R M_{i!} \simeq (Q^a \otimes_A M_i)_!$ for $i = 1, 2$. Since the functor $M \mapsto M_!$ is exact ([75, Cor.2.2.24(i)]), we deduce that $\varphi_!$ is pure. The converse is easy, and shall be left to the reader.

(ii): From (i) we already see that g is pure whenever $g_{!!}$ is. Next, suppose that g is pure; we may assume that $R = A_{!!}$, and then (i) says that $g_!$ is a pure morphism of $A_{!!}$-modules. However, the natural diagram of $A_{!!}$-modules :

$$\begin{array}{ccc} A_! & \longrightarrow & B'_! \\ \downarrow & & \downarrow \\ A_{!!} & \longrightarrow & B'_{!!} \end{array}$$

is cofibred; since tensor products are right exact functors, the claim follows. ◊

The assertion now follows from claim 14.6.22(ii) and [86, Partie II, Lemme 1.2.1].

(ii): Suppose first that $B \otimes_A M$ is an almost finitely generated B-module. The assumption on f implies that $\mathrm{Ann}_A(B \otimes_A M) \subset \mathrm{Ann}_A M$; then [75, Rem.3.2.26(i)] shows that M is an almost finitely generated A-module.

Finally, we suppose that $B \otimes_A M$ is an almost finitely presented B-module, and we wish to show that M is an almost finitely presented A-module. To this aim, let $\varphi : N \to N'$ be a morphism of A-modules. The assumption on f implies that the natural morphism $\mathrm{Ker}\,\varphi \to \mathrm{Ker}(\mathbf{1}_B \otimes_A \varphi)$ is a monomorphism; especially :

$$\mathrm{Ann}_A(\mathrm{Ker}(\mathbf{1}_B \otimes_A \varphi)) \subset \mathrm{Ann}_A(\mathrm{Ker}\,\varphi).$$

Then one may easily adapt the proof of [75, Lemma 3.2.25(iii)], to derive the assertion. □

Theorem 14.6.23. *In the situation of definition* 14.6.7*, suppose furthermore that G is abelian and the order $o(G)$ of G is invertible in A_*. Then the following holds :*

(i) *Let M be any G-equivariant A-module. The G-action on M is horizontal if and only if the counit of adjunction :*

$$\eta_M : A \otimes_{A^G} M^G \to M$$

is an epimorphism (of A-modules).

(ii) *The functor* (14.6.8) *restricts to an equivalence on the full subcategory of flat A^G-modules :*

$$A^G\text{-}\mathbf{Mod}_{\mathrm{fl}} \stackrel{\sim}{\to} A[G]\text{-}\mathbf{Mod}_{\mathrm{hor.fl}}. \tag{14.6.24}$$

Proof. (i): For $m := o(G)$, let $\boldsymbol{\mu}_m \subset \overline{\mathbb{Q}}^\times$ be the group of m-th roots of 1, and set $B := V^a[\boldsymbol{\mu}_m] := (V[T]/(T^m - 1))^a$. Since B is a faithfully flat V^a-algebra, lemma 14.6.10(iii) allows to replace A by $B \otimes_{V^a} A$ and M by $B \otimes_{V^a} M$, and therefore we may assume from start that $\boldsymbol{\mu}_m \subset (A_*^G)^\times$. Set $\Gamma := \mathrm{Hom}_\mathbb{Z}(G, \boldsymbol{\mu}_m)$. For every $\chi \in \Gamma$, let $e_\chi \in A_*[G]$ be the central idempotent defined as in (14.6.11). A standard calculation shows that :

$$\sum_{\chi \in \Gamma} e_\chi = 1.$$

Hence, every G-equivariant A^G-module (N, β) admits the G-equivariant decomposition :

$$N \simeq \bigoplus_{\chi \in \Gamma} N_\chi. \tag{14.6.25}$$

Especially, $A = \bigoplus_{\chi \in \Gamma} A_\chi$, and clearly the datum $\underline{A}$ consisting of A and its decomposition, is a Γ-graded V^a-algebra. Furthermore, the datum $\underline{N}$ consisting of N and its decomposition (14.6.25) is a Γ-graded $\underline{A}$-module.

Claim 14.6.26. The functor $(N, \beta) \mapsto \underline{N}$ is an equivalence from $A[G]$-$\mathbf{Mod}$ to the category of Γ-graded $\underline{A}$-modules.

Proof of the claim. Indeed, if Q is a Γ-graded $\underline{A}$-module, we may define a G-action on Q by requiring that $Q_\chi = \mathrm{gr}_\chi Q$, the χ-graded direct summand of Q. This gives a quasi-inverse for the functor $(N, \beta) \mapsto \underline{N}$. (Details left to the reader.) ◊

Claim 14.6.27. Let (N, β) be any G-equivariant A-module, and $\underline{N}$ its associated Γ-graded $\underline{A}$-module. The following conditions are equivalent :

(a) (N, β) is horizontal.
(b) $\underline{N}$ is horizontal.

Proof of the claim. For every $g \in G$, let $\Delta(g) \subset \Gamma$ be the subgroup consisting of all $\chi \in \Gamma$ such that $\chi(g) = 1$; a direct inspection of the definitions shows that $J_{\Delta(g)}$ is the ideal I_g, as defined in (14.6.6), and (N, β) is horizontal if and only if $N_\chi/(N_\chi \cap I_g N) = 0$ for every $g \in G$ and every $\chi \notin \Delta(g)$. This already shows that (b)⇒(a); it also shows that condition (b) holds for the subgroups $\Delta(g)$, when (N, β) is horizontal. However, every subgroup of Γ can be written in the form $\Delta = \Delta(g_1) \cap \cdots \cap \Delta(g_n)$, for appropriate $g_1, \ldots, g_n \in G$, and then $J_{\Delta(g_1)} + \cdots + J_{\Delta(g_n)} \subset J_\Delta$, hence (b) follows for all subgroups. ◊

Assertion (i) now follows from claim 14.6.27 and proposition 14.6.13.

(ii): Let (M, β) be any object of $A[G]$-$\mathbf{Mod}_{\mathrm{hor.fl}}$, and denote by L the kernel of the counit η_M. Since M is a flat A-module and η_M is an epimorphism by (i), it follows that $i_g^* L$ is the kernel of $i_g^* \eta_M$, for every $g \in G$ (notation of (14.6.6)). Since the category $A[G]$-$\mathbf{Mod}_{\mathrm{hor}}$ is abelian, we deduce that the natural G-action on L is horizontal, and then (i) says that L is generated by L^G. But lemma 14.6.10(ii) easily implies that $L^G = 0$, so η_M is an isomorphism. Next, letting $M := A$ in lemma 14.6.10(i), we deduce easily that the natural morphism $A^G \to A$ is pure, hence M^G is a flat A^G-module, by lemma 14.6.21(i). Now the assertion follows from lemma 14.6.10(ii) and [28, Prop.3.4.3]. □

Corollary 14.6.28. *In the situation of theorem* 14.6.23*, the following holds :*

(i) *The functor* (14.6.24) *restricts to an equivalence from the subcategory of flat, almost finitely generated (resp. almost finitely presented) A^G-modules, onto the subcategory of G-equivariant, flat, horizontal and almost finitely generated (resp. almost finitely presented) $A[G]$-modules.*

(ii) *The functor* (14.6.9) *restricts to an equivalence :*

$$A^G\text{-}\mathsf{Alg}_{\text{fl}} \to A[G]\text{-}\mathsf{Alg}_{\text{hor.fl}}$$

and likewise for the subcategories of weakly étale (resp. étale, resp. étale and almost finitely presented) algebras.

Proof. (i) follows from theorem 14.6.23(ii), lemma 14.6.21(ii), and the fact that the morphism $A^G \to A$ is pure.

(ii): The assertion concerning A^G-Alg_{fl} is an immediate consequence of theorem 14.6.23. Next, let (B, β) be an object of $A[G]$-$\mathbf{w.\acute{E}t}$; by the foregoing, B descends to a flat A^G-algebra B^G with a G-equivariant isomorphism : $B \simeq A \otimes_{A^G} B^G$. However, on the one hand, B is – by assumption – a flat $B \otimes_A B$-algebra, and on the other hand, $B \otimes_A B$ underlies a flat, horizontal $A[G]$-algebra with $(B \otimes_A B)^G \simeq B^G \otimes_{A^G} B^G$; theorem 14.6.23 then says that B^G is a flat $B^G \otimes_{A^G} B^G$-algebra, whence the assertion for A^G-$\mathbf{w.\acute{E}t}$. Next, since an almost finitely generated module is almost projective if and only if it is flat and almost finitely presented ([75, Prop.2.4.18]), the assertion for A^G-$\mathbf{\acute{E}t}$ follows from the same assertion for A^G-$\mathbf{w.\acute{E}t}$ and lemma 14.6.21(ii). Likewise, the assertion for étale almost finitely presented A^G-algebras follows from the assertion for A^G-$\mathbf{\acute{E}t}$ and lemma 14.6.21(ii). □

Remark 14.6.29. In case the G-action on X is *free*, *i.e.* when $(\partial_0, \partial_1) : X \times G \to X \times_S X$ is a monomorphism, corollary 14.6.28 also follows from [75, Prop.4.5.25].

14.7. **Almost Witt vectors.** In this section we show that the construction of the ring of Witt vectors descends to almost rings, and we study some properties of the resulting functor of *almost Witt vectors*. We begin with some general observation concerning liftings of basic setups (in the sense of [75, §2.1.1]) along ring homomorphisms. These preliminaries shall then be applied to find natural basic setups on truncated rings of Witt vectors of arbitrary rings R, by lifting given setups on R along the 0-th ghost map.

Lemma 14.7.1. *Let A be a ring, $I \subset A$ a nilpotent ideal, $\pi : A \to A_0 := A/I$ the projection, and $\mathfrak{m}_0 \subset A_0$ an ideal such that $\mathfrak{m}_0 = \mathfrak{m}_0^2$. We have :*

(i) *There exists a unique ideal $\mathfrak{m} \subset A$ such that $\mathfrak{m}^2 = \mathfrak{m}$ and $\pi(\mathfrak{m}) = \mathfrak{m}_0$.*
(ii) *$\mathfrak{m}$ is the smallest of the ideals $J \subset A$ such that $\pi(J) = \mathfrak{m}_0$.*
(iii) *$\mathfrak{m}$ fulfills condition* $(\mathbf{B})$ *of* [75, §2.1.6] *if and only if the same holds for $\mathfrak{m}_0$.*

Proof. (i): Say that $I^N = 0$ for some $N \in \mathbb{N}$, and set $J := \pi^{-1}\mathfrak{m}_0$. We remark :

Claim 14.7.2. $J^{2N+1} = J^{2N+2}$.

Proof of the claim. Indeed, let $a_1, \dots, a_{2N+1} \in J$; we may find an integer $r \geq 0$ and elements $b_{1i}, c_{1i}, \dots, b_{ri}, c_{ri} \in \mathfrak{m}_0$ for every $i = 1, \dots, 2N+1$, such that $\pi(a_i) = \sum_{j=1}^r b_{ji}c_{ji}$. Lift each b_{ji} and each c_{ji} to elements $b'_{ji}, c'_{ji} \in J$; it follows that $x_i := a_i - \sum_{j=1} b'_{ji}c'_{ji} \in I$ for every $i = 1, \dots, 2N+1$. Thus :

$$\prod_{i=1}^{2N+1} a_i = \prod_{i=1}^{2N+1} \Big(x_i + \sum_{j=1}^{r} b'_{ji}c'_{ji} \Big) \in J^{2(N+1)} + I^N = J^{2N+2}$$

whence the claim. ◊

Set $\mathfrak{m} := J^{2N+1}$; claim 14.7.2 implies that $\mathfrak{m} = \mathfrak{m}^2$, and clearly $\pi(\mathfrak{m}) = \mathfrak{m}_0^{2N+1} = \mathfrak{m}$. Next, let $\mathfrak{m}' \subset A$ be any ideal such that $\mathfrak{m}'^2 = \mathfrak{m}'$ and $\pi(\mathfrak{m}') = \mathfrak{m}_0$; it follows that

$$\mathfrak{m}' = \mathfrak{m}'^{2N+1} \subset J^{2N+1} = (\mathfrak{m}' + I)^{2N+1} \subset \mathfrak{m}'^{N+1} = \mathfrak{m}'$$

whence $\mathfrak{m}' = \mathfrak{m}$.

(ii): Let $J \subset A$ be any ideal such that $\pi(J) = \mathfrak{m}_0$; it follows that

$$\mathfrak{m} = \mathfrak{m}^N \subset (J+I)^N \subset J + I^N = J.$$

(iii): It is clear that if $\mathfrak{m}$ fulfills condition $(\mathbf{B})$, then the same holds for $\mathfrak{m}_0$. Conversely, if condition $(\mathbf{B})$ holds for $\mathfrak{m}_0$, then for every integer $k > 1$ the system of elements $(x^k \mid x \in \mathfrak{m})$ generates an ideal $J \subset A$ such that $\pi(J) = \mathfrak{m}_0$ and $J \subset \mathfrak{m}$. By (ii), we must then have $J = \mathfrak{m}$, which shows that condition $(\mathbf{B})$ holds for $\mathfrak{m}$. □

Let us also point out the following result, which shall not be needed in the sequel :

Lemma 14.7.3. *Let* $(A, \mathfrak{m})$ *be a basic setup and* $f : B \to A$ *a ring homomorphism such that :*

(a) $p^k A = 0$ *for some integer* $k \in \mathbb{N}$.
(b) $\mathfrak{m}$ *fulfills condition* $(\mathbf{B})$ *of* [75, §2.1.6].
(c) $f \otimes_{\mathbb{Z}} \mathbb{F}_p$ *is invertible up to* Φ^n *for some* $n \in \mathbb{N}$, *in the sense of* [75, Def.3.5.8].

Then there exists a unique basic setup $(B, \mathfrak{n})$ *with* $\mathfrak{n}A = \mathfrak{m}$ *and with* $\mathfrak{n}$ *fulfilling condition* $(\mathbf{B})$.

Proof. Using lemma 14.7.1, we are easily reduced to the case where A and B are $\mathbb{F}_p$-algebras, and f is invertible up to Φ^n. The latter means that there exists a morphism $g : A \to B$ such that $g \circ f = \Phi^n_B$ and $f \circ g = \Phi^n_A$.

Suppose first that $A = B$ and $f = \Phi^n_A$; in this case, we claim that $\mathfrak{n} := \mathfrak{m}$ will do. Indeed, obviously $\mathfrak{n}^2 = \mathfrak{n}$, and assumption (b) says that $\Phi^n_A(\mathfrak{m})$ generates the ideal $\mathfrak{m}$; moreover, if $\mathfrak{n}' \subset A$ is another ideal fulfilling these conditions, we see that $x^p \in \mathfrak{m}$ for every $x \in \mathfrak{n}'$, whence $\mathfrak{n}' \subset \mathfrak{m}$, since $\mathfrak{n}'$ fulfills condition $(\mathbf{B})$, and conversely, $\mathfrak{m} = \Phi^n_A(\mathfrak{n}') \cdot A \subset \mathfrak{n}'$.

In the general case, we claim that $\mathfrak{n} := g(\mathfrak{m}) \cdot B$ will do. Indeed, clearly $\mathfrak{n}^2 = \mathfrak{n}$ and $f(\mathfrak{n}) \cdot A = \Phi^n_A(\mathfrak{m}) \cdot A = \mathfrak{m}$, due to assumption (b); it is also easily seen that $\mathfrak{n}$ fulfills condition $(\mathbf{B})$. Moreover, if $\mathfrak{n}' \subset B$ is another ideal fulfilling these conditions, it follows that

$$\Phi^n_B(\mathfrak{n}') \cdot B = g(f(\mathfrak{n}') \cdot A) \cdot B = g(\mathfrak{m}) \cdot B = \mathfrak{n}$$

whence $\mathfrak{n} = \mathfrak{n}'$, by the foregoing case. □

14.7.4. Consider now a cartesian diagram of rings

$$\begin{array}{ccc} A & \xrightarrow{p_1} & A_1 \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ A_2 & \xrightarrow{\pi_2} & A_3 \end{array}$$

such that π_2 (and hence p_1) is surjective.

Proposition 14.7.5. (i) *In the situation of* (14.7.4)*, let* $\mathfrak{m}_1 \subset A_1$, $\mathfrak{m}_2 \subset A_2$ *be two ideals with*

$$\mathfrak{m}_1^2 = \mathfrak{m}_1 \qquad \mathfrak{m}_2^2 = \mathfrak{m}_2 \qquad \mathfrak{m}_1 A_3 = \mathfrak{m}_2 A_3.$$

Then there exists a unique ideal $\mathfrak{m} \subset A$ *such that*

$$\mathfrak{m}^2 = \mathfrak{m} \qquad \mathfrak{m}A_1 = \mathfrak{m}_1 \qquad \mathfrak{m}A_2 = \mathfrak{m}_2. \tag{14.7.6}$$

(ii) *Moreover,* $\mathfrak{m}$ *is the smallest of the ideals* $J \subset A$ *such that* $JA_1 = \mathfrak{m}_1$ *and* $JA_2 = \mathfrak{m}_2$.
(iii) *Furthermore,* $\mathfrak{m}$ *fulfills condition* $(\mathbf{B})$ *if and only if the same holds for both* $\mathfrak{m}_1$ *and* $\mathfrak{m}_2$.

Proof. (i): We shall regard A as a subset of $A_1 \times A_2$, in the natural fashion. Now, set $I := \mathfrak{m}_1 \times_{A_3} \mathfrak{m}_2$; then I is an ideal of A, and we remark :

Claim 14.7.7. $IA_1 = \mathfrak{m}_1$ and $IA_2 = \mathfrak{m}_2$.

Proof of the claim. Since π_2 is surjective, it is easily seen that p_1 restricts to a surjection $I \to \mathfrak{m}_1$. Next, let $a \in \mathfrak{m}_2$ be any element; we may find an integer $r \in \mathbb{N}$ and elements $x_1, \dots, x_r \in \mathfrak{m}_1$ and $y_1, \dots, y_r \in A_3$ such that $\pi_2(a) = \sum_{i=1}^r y_i \cdot \pi_1(x_i)$. For every $i = 1, \dots, r$, pick $b_i \in \mathfrak{m}_2$ and $c_i \in A_2$ such that $\pi_2(b_i) = \pi_1(x_i)$ and $\pi_2(c_i) = y_i$; then $(x_i, b_i) \in I$ for $i = 1, \dots, r$, and $a - \sum_{i=1}^r p_2(x_i, b_i) \cdot c_i \in \operatorname{Ker} \pi_2$. However, $0 \times (\mathfrak{m}_2 \cap \operatorname{Ker} \pi_2) \subset I$, whence $IA_2 = \mathfrak{m}_2$. ◊

Claim 14.7.8. Let $J \subset A$ be any ideal such that $JA_1 = \mathfrak{m}_1$ and $JA_2 = \mathfrak{m}_2$. Then

$$(A_1 \times_{A_3} \mathfrak{m}_2) \cdot (\mathfrak{m}_1 \times_{A_3} A_2) \subset J.$$

Proof of the claim. Indeed, let $(a_1, a_2), (b_1, b_2) \in A$ be two elements such that $a_2 \in \mathfrak{m}_2$ and $b_1 \in \mathfrak{m}_1$. Since p_1 is surjective, we may find $x \in A_2$ such that $(b_1, x) \in J$, so $\pi_2(x) = \pi_1(b_1)$, and we have

$$(a_1, a_2) \cdot (b_1, b_2) - (a_1, a_2) \cdot (b_1, x) = (a_1, a_2) \cdot (0, b_2 - x) = (0, a_2 \cdot y)$$

where $y := b_2 - x \in \operatorname{Ker} \pi_2$. We are thus reduced to checking that $(0, ab) \in J$ for every $a \in \mathfrak{m}_2$ and $b \in \operatorname{Ker} \pi_2$. However, say that $a = \sum_{i=1}^r p_2(c_i) d_i$ for elements $c_1, \dots, c_r \in J$ and $d_1, \dots, d_r \in A_2$; it follows that $(0, d_i b) \in A$ for every $i = 1, \dots, r$, and $\sum_{i=1}^r c_i \cdot (0, d_i b) = (0, ab)$, whence the contention. ◊

From claim 14.7.7 we deduce that $I^3 A_1 = \mathfrak{m}_1^3 = \mathfrak{m}_1$, and likewise, $I^3 A_2 = \mathfrak{m}_2$; from claim 14.7.8, it then follows that $(\mathfrak{m}_1 \times_{A_3} A_2) \cdot (A_1 \times_{A_3} \mathfrak{m}_2) \subset I^3$. Especially, $I^2 \subset I^3$, so that $I^3 = I^2$, and therefore the ideal $\mathfrak{m} := I^2$ fulfills conditions (14.7.6).

(ii): For every ideal J as in claim 14.7.8 we also get : $\mathfrak{m} \subset J \subset \mathfrak{m}_1 \times_{A_3} \mathfrak{m}_2$; if we have as well $J^2 = J$, it follows that $J \subset \mathfrak{m}$, *i.e.* $J = \mathfrak{m}$, as required.

(iii): Due to [75, Claim 2.19], it suffices to show, for every prime p, that the following conditions are equivalent :

(a) The A-module $\mathfrak{m}/p\mathfrak{m}$ is generated by the p-th powers of its elements
(b) For $i = 1, 2$, the A_i-module $\mathfrak{m}_i/p\mathfrak{m}_i$ is generated by the p-th powers of its elements.

However, it is easily seen that (a)⇒(b). For the converse, consider the ideal $J \subset A$ generated by the system $\{x^p \mid x \in \mathfrak{m}\} \cup \{px \mid x \in \mathfrak{m}\}$; since p_1 is surjective, (b) implies that $JA_1 = \mathfrak{m}_1$. Likewise, $JA_2 = \mathfrak{m}_2$: indeed, since $\mathfrak{m}_2/p\mathfrak{m}_2$ is generated by the p-th powers of its elements, it suffices to check that for every $a \in \mathfrak{m}_2$ we have $a^p \in JA_2$. But say that $a = \sum_{j=1}^n x_i a_i$ for some $x_1, \dots, x_n \in \mathfrak{m}$ and $a_1, \dots, a_n \in A_2$; then $a^p = \sum_{i=1} x_i^p a_i^p + py$ for some $y \in \mathfrak{m}_2$, whence the contention. In view of (ii), we conclude that $J = \mathfrak{m}$, whence (a). □

14.7.9. Henceforth, we fix a prime number p, and the notation $W(A)$ and $W_{n+1}(A)$ will refer to the ring of *p-typical* Witt vector of section (9.3), for every ring A and every integer $n \in \mathbb{N}$. Notice that the functors W and W_{n+1} introduced in section 9.3 are defined on the category of topological rings; however, in this section we shall be interested only in the underlying rings, and the topologies will play no role; if one wishes, one may assume that all the rings in this section carry the discrete topology. We consider first the case where p is nilpotent on A :

Corollary 14.7.10. *Let A be any ring such that $p^k A = 0$ for some $k \in \mathbb{N}$, and $\mathfrak{m} \subset A$ an ideal with $\mathfrak{m}^2 = \mathfrak{m}$. For every $n \in \mathbb{N}$ we have :*

(i) *There exists a unique ideal $\mathfrak{n}_{n+1} \subset W_{n+1}A$ such that*

$$\mathfrak{n}_{n+1}^2 = \mathfrak{n}_{n+1} \qquad \textit{and} \qquad \overline{\boldsymbol{\omega}}_0(\mathfrak{n}_{n+1}) = \mathfrak{m}.$$

(ii) *The image of $\mathfrak{n}_{n+2}$ under the projection $W_{n+2}A \to W_{n+1}A$ agrees with $\mathfrak{n}_{n+1}$.*

Proof. (See remark 9.3.28(i) for the definition of the ring homomorphism $\overline{\boldsymbol{\omega}}_0$.)

(i) follows immediately from corollary 9.3.32 and lemma 14.7.1.

(ii) follows immediately from (i). □

For a general ring A, we may state :

Proposition 14.7.11. *Let A be a ring, $n \in \mathbb{N}$ an integer, $\mathfrak{m} \subset A$ an ideal with $\mathfrak{m} = \mathfrak{m}^2$ and fulfilling condition $(\mathbf{B})$ of* [75, §2.1.6]. *Set $\mathfrak{n}_{n+1} = \{\underline{a} \in W_{n+1}A \mid a_0, \dots, a_n \in \mathfrak{m}\}$. We have :*

(i) *$\mathfrak{n}_{n+1}$ is an ideal of $W_{n+1}A$ with $\mathfrak{n}_{n+1}^2 = \mathfrak{n}_{n+1}$ and fulfilling condition $(\mathbf{B})$.*
(ii) *Moreover, $\mathfrak{n}_{n+1}$ is the unique ideal with $\mathfrak{n}_{n+1}^2 = \mathfrak{n}_{n+1}$ and such that*

$$\overline{\boldsymbol{\omega}}_i(\mathfrak{n}_{n+1}) \cdot A = \mathfrak{m} \qquad \text{for } i = 0, \dots, n. \tag{14.7.12}$$

Proof. We consider the ring homomorphism

$$\pi_n : W_{n+1}A \to A^{n+1}$$

of proposition 9.3.30(i). By proposition 9.3.30(i), the kernel of the surjection $W_{n+1}A \to R := \operatorname{Im} \pi_n$ is nilpotent, hence every basic setup $(R, \mathfrak{m}_R)$ lifts uniquely to a basic setup $(W_{n+1}A, \widetilde{\mathfrak{m}})$, and $\widetilde{\mathfrak{m}}$ satisfies condition $(\mathbf{B})$ if and only if the same holds for $\mathfrak{m}_R$ (lemma 14.7.1(i,iii)). Moreover, by proposition 9.3.30(ii), the subring R of A^{n+1} contains the ideal $p^{n+1}A^{n+1}$, and lies in the larger subring $S \subset A^{n+1}$ consisting of all sequences $(a_0, \dots, a_n)$ such that $a_i \equiv a_0^{p^i}$ $(\mathrm{mod}\ pA)$ for $i = 0, \dots, n$; there follows a cartesian diagram of rings

$$\begin{array}{ccc} R & \longrightarrow & R_0 := R/p^{n+1}A^{n+1} \\ \downarrow & & \downarrow \\ S & \longrightarrow & S_0 := S/p^{n+1}A^{n+1} \end{array} \tag{14.7.13}$$

whose horizontal arrows are surjections. By proposition 14.7.5(i,iii), the datum of $(R, \mathfrak{m}_R)$ is then equivalent to that of a pair of basic setups $(R_0, \mathfrak{m}_{R_0})$ and $(S, \mathfrak{m}_S)$ with $\mathfrak{m}_{R_0}S_0 = \mathfrak{m}_S S_0$. Also, $\mathfrak{m}_R$ fulfills condition $(\mathbf{B})$ if and only if the same holds for both $\mathfrak{m}_{R_0}$ and $\mathfrak{m}_S$. Next, set $A_0 := A/pA$, and notice that we have a cartesian diagram of rings

$$\begin{array}{ccc} S & \longrightarrow & A^{n+1} \\ {\scriptstyle\psi}\downarrow & & \downarrow \\ A_0 & \xrightarrow{\ \varphi\ } & A_0^{n+1} \end{array} \tag{14.7.14}$$

where φ is given by the rule : $x \mapsto (x, x^p, \dots, x^{p^n})$ for every $x \in A_0$; the top horizontal arrow is the inclusion map, and the right vertical arrow is the projection. Moreover, ψ is a surjection. Hence – again by virtue of proposition 14.7.5(i) – the datum of $(S, \mathfrak{m}_S)$ is equivalent to that of a system of basic setups $(A, \mathfrak{m}_i)$ such that

$$\mathfrak{m}_i A_0 = \Phi^i_{A_0}(\mathfrak{m}_0 A_0) \cdot A_0 \qquad \text{for } i = 1, \dots, n$$

where $\Phi_{A_0} : A_0 \to A_0$ is the Frobenius endomorphism. Especially, if $(A, \mathfrak{m})$ is a basic setup on A such that $\mathfrak{m}$ fulfills condition $(\mathbf{B})$, then we can take $\mathfrak{m}_i := \mathfrak{m}$ for every $i = 0, \dots, n$, and in this way we obtain a natural basic setup $(S, \mathfrak{m}_S)$ on S with $\mathfrak{m}_S A^{n+1} = \mathfrak{m}^{n+1}$, and such that condition $(\mathbf{B})$ still holds for $\mathfrak{m}_S$. Furthermore, the restriction of ψ to R factors through a surjection $R_0 \to A_0$ whose kernel is also nilpotent, so that the basic setup $(A_0, \mathfrak{m}A_0)$ lifts uniquely to a basic setup $(R_0, \mathfrak{m}_{R_0})$ with $\mathfrak{m}_{R_0}$ fulfilling condition $(\mathbf{B})$ (lemma 14.7.1(i)). Lastly, both ideals $\mathfrak{m}_{R_0}S_0$ and $\mathfrak{m}_S S_0$ lift $\mathfrak{m}A_0$ along the map $S_0 \to A_0$ induced by ψ, whose kernel is also nipotent; by invoking again 14.7.1(i), we deduce that $\mathfrak{m}_{R_0}S_0 = \mathfrak{m}_S S_0$. Summing up, we have finally associated with the basic setup $(A, \mathfrak{m})$ a natural basic setup $(W_{n+1}A, \widetilde{\mathfrak{m}})$ which is the unique one with property (14.7.12), and with $\widetilde{\mathfrak{m}}$ fulfilling condition $(\mathbf{B})$. To conclude the proof of both (i) and (ii), it then suffices to check that $\widetilde{\mathfrak{m}} = \mathfrak{n}_{n+1}$. However, it is clear that $\mathfrak{n}_{n+1}$ is an ideal of $W_{n+1}A$ enjoying property (14.7.12), since $\overline{\boldsymbol{\omega}}_i(\mathfrak{n}_{n+1})$ is contained in $\mathfrak{m}$ and contains the system $(x^{p^i} \mid x \in \mathfrak{m})$, which generates $\mathfrak{m}$, under condition $(\mathbf{B})$. It remains to check that $\mathfrak{n}_{n+1} = \mathfrak{n}_{n+1}^2$. Now, let us define a descending filtration by subideals of $\mathfrak{n}_{n+1}$, by setting $\mathrm{Fil}_i\mathfrak{n}_{n+1} := \mathfrak{n}_{n+1} \cap \overline{V}_iA$, where $\overline{V}_iA := V_iA/V_{n+1}A$ for every $i = 0, \dots, n$. Let also $\mathrm{gr}_\bullet\mathfrak{n}_{n+1}$ be the associated graded W_{n+1}-module. We are then further reduced to showing that $\mathfrak{n}_{n+1} \cdot \mathrm{gr}_i\mathfrak{n}_{n+1} = \mathrm{gr}_i\mathfrak{n}_{n+1}$ for $i = 0, \dots, n$. However, we have a natural identification of $W_{n+1}A$-modules : $\mathrm{gr}_i\mathfrak{n}_{n+1} \xrightarrow{\sim} \mathfrak{m}$, for the $W_{n+1}A$-module structure on $\mathfrak{m}$ induced by restriction of scalars along the ghost map $\overline{\boldsymbol{\omega}}_i : W_{n+1}A \to A$ (claim 9.3.31). Since $\mathfrak{m}^2 = \mathfrak{m}$, we then come down to the assertion that $\overline{\boldsymbol{\omega}}_i(\mathfrak{n}_{n+1}) \cdot A = \mathfrak{m}$, which was already remarked. □

14.7.15. Let $(A, \mathfrak{m})$ be a basic setup such that either $p^kA = 0$ for some integer $k \in \mathbb{N}$, or else such that $\mathfrak{m}$ fulfills condition $(\mathbf{B})$. For every $n \in \mathbb{N}$ let $\mathfrak{n}_{n+1} \subset W_{n+1}A$ be the ideal provided by corollary 14.7.10(i), or respectively by proposition 14.7.11; especially, $(W_{n+1}A, \mathfrak{n}_{n+1})$ is also a basic setup. Let also $f : B \to C$ be a morphism of A-algebras such that $f^a : B^a \to C^a$ is an isomorphism of $(A, \mathfrak{m})^a$-algebras. Then we claim that f induces an isomorphism of $(W_{n+1}A, \mathfrak{n}_{n+1})^a$-algebras

$$(W_{n+1}f)^a : (W_{n+1}B)^a \xrightarrow{\sim} (W_{n+1}C)^a \qquad \text{for every } n \in \mathbb{N}.$$

Indeed, arguing by induction on $n \in \mathbb{N}$, we are easily reduced to checking that f induces an isomorphism of $(W_{n+1}A)^a$-modules $(V_nB/V_{n+1}B)^a \xrightarrow{\sim} (V_nC/V_{n+1}C)^a$ for every $n \in \mathbb{N}$. However, in light of claim 9.3.31 we have a commutative diagram of $W_{n+1}A$-modules

$$\begin{array}{ccc} B & \longrightarrow & V_nB/V_{n+1}B \\ {\scriptstyle f}\downarrow & & \downarrow \\ C & \longrightarrow & V_nC/V_{n+1}C \end{array}$$

whose horizontal arrows are isomorphisms of $W_{n+1}A$-modules, and where the $W_{n+1}A$-module structures on B and C are induced by the n-th ghost maps. By assumption, $\mathfrak{m}$ annihilates $\mathrm{Ker}\, f$ and $\mathrm{Coker}\, f$; since $\overline{\boldsymbol{\omega}}_n(\mathfrak{n}_{n+1}) \subset \mathfrak{m}$, it follows that $\mathfrak{n}_{n+1}$ annihilates these $W_{n+1}A$-modules as well, whence the contention. Thus, we obtain a well defined functor

$$W_{n+1} : (A, \mathfrak{m})^a\text{-}\mathsf{Alg} \to (W_{n+1}A, \mathfrak{n}_{n+1})^a\text{-}\mathsf{Alg} \qquad B^a \mapsto (W_{n+1}B)^a \qquad \text{for every } n \in \mathbb{N}.$$

Moreover, for every $k \leq n$, the ghost component $\overline{\boldsymbol{\omega}}_k : W_nB \to B$ induces pull-back functors

$$\overline{\boldsymbol{\omega}}_k^* : B^a\text{-}\mathsf{Alg} \to W_{n+1}B^a\text{-}\mathsf{Alg} \qquad \overline{\boldsymbol{\omega}}_k^* : B^a\text{-}\mathbf{Mod} \to W_{n+1}B^a\text{-}\mathbf{Mod}.$$

Namely, for every B-algebra C we let $\overline{\boldsymbol{\omega}}_k^*C$ be the $W_{n+1}B$-algebra whose underlying ring is C, and whose structure morphism is the composition of $\overline{\boldsymbol{\omega}}_k$ and the structure morphism $B \to C$ of C. Since $\overline{\boldsymbol{\omega}}_k(\mathfrak{n}_{n+1}) \subset \mathfrak{m}$, it is clear that this functor on B-algebras descends to a well defined functor $\overline{\boldsymbol{\omega}}_k^*$ on B^a-algebras, as stated. Likewise one argues to define the functor $\overline{\boldsymbol{\omega}}_k^*$ on B^a-modules. Furthermore, we have a well defined ideal

$$\overline{V}_kB^a := (V_kB/V_{n+1}B)^a \subset W_{n+1}B^a \qquad \text{for every } k = 0, \dots, n$$

and claim 9.3.31 yields a natural isomorphism of $W_{n+1}B^a$-modules :

$$\overline{V}_kB^a/\overline{V}_{k+1}B^a \xrightarrow{\sim} \overline{\boldsymbol{\omega}}_k^*B^a \qquad \text{for every } k = 0, \dots, n. \tag{14.7.16}$$

14.7.17. In the situation of (14.7.15), suppose that $\mathfrak{m}$ fulfills condition $(\mathbf{B})$, and let B be any A-algebra; denote by

$$B^a_{!!} \xrightarrow{\varepsilon_B} B \xrightarrow{\eta_B} B^a_* \qquad \text{and} \qquad (W_{n+1}B)^a_{!!} \xrightarrow{\varepsilon_{W_{n+1}B}} W_{n+1}B \xrightarrow{\eta_{W_{n+1}B}} (W_{n+1}B)^a_*$$

the units and counits of adjunction. We have :

Proposition 14.7.18. *In the situation of* (14.7.17)*, the following holds :*

(i) *There exist isomorphisms of $W_{n+1}A$-algebras*

$$\omega : W_{n+1}(B^a_*) \xrightarrow{\sim} (W_{n+1}B)^a_* \qquad \textit{and} \qquad \tau : W_{n+1}(B^a_{!!}) \xrightarrow{\sim} (W_{n+1}B)^a_{!!}$$

such that $\omega \circ W_{n+1}(\eta_B) = \eta_{W_{n+1}(B)}$ *and* $\varepsilon_{W_{n+1}(B)} \circ \tau = W_{n+1}(\varepsilon_B)$.

(ii) *Moreover, we have a natural isomorphism of non-unital rings (see remark* 9.3.16*)*

$$\widetilde{\mathfrak{n}}_{n+1} := \mathfrak{n}_{n+1} \otimes_{W_{n+1}A} \mathfrak{n}_{n+1} \xrightarrow{\sim} W_{n+1}(\widetilde{\mathfrak{m}}).$$

Proof. (i): We consider the commutative diagrams

$$\begin{array}{ccc} W_{n+1}B & \xrightarrow{W_{n+1}(\eta_B)} & W_{n+1}(B^a_*) \\ {\scriptstyle \eta_{W_{n+1}B}}\downarrow & & \downarrow{\scriptstyle \eta_{W_{n+1}(B^a_*)}} \\ (W_{n+1}B)^a_* & \xrightarrow{(W_{n+1}\eta_B)^a_*} & W_{n+1}(B^a_*)^a_* \end{array} \qquad \begin{array}{ccc} W_{n+1}(B^a_{!!})^a_{!!} & \xrightarrow{(W_{n+1}\varepsilon_B)^a_{!!}} & W_{n+1}(B)^a_{!!} \\ {\scriptstyle \varepsilon_{W_{n+1}(B^a_{!!})}}\downarrow & & \downarrow{\scriptstyle \varepsilon_{W_{n+1}(B)}} \\ W_{n+1}(B^a_{!!}) & \xrightarrow{W_{n+1}(\varepsilon_B)} & W_{n+1}(B) \end{array}$$

and notice that $(W_{n+1}\eta_B)^a_*$ and $(W_{n+1}\varepsilon_B)^a_{!!}$ are isomorphisms, by the discussion of (14.7.15). Hence, it suffices to show that $\eta_{W_{n+1}(B^a_*)}$ and $\varepsilon_{W_{n+1}(B^a_{!!})}$ are isomorphisms, which follows from :

Claim 14.7.19. If η_B (resp. ε_B) is an isomorphism, the same holds for $\eta_{W_{n+1}B}$ (resp. $\varepsilon_{W_{n+1}B}$).

Proof of the claim. Let $\mathrm{gr}_\bullet(W_{n+1}B)$ be the graded ring associated with the filtration $(\overline{V}_k B \mid k = 0, \dots, n)$ of $W_{n+1}B$; consider also the filtration $((\overline{V}_k B^a)_* \mid k = 0, \dots, n)$ of $(W_{n+1}B)^a_*$, and let $\mathrm{gr}_\bullet(W_{n+1}B)^a_*$ be the associated graded ring. We get a commutative diagram

$$\begin{array}{ccc} & \mathrm{gr}_i(W_{n+1}B) & \\ {\scriptstyle \mathrm{gr}_i(\eta_{W_{n+1}B})}\swarrow & & \searrow{\scriptstyle \eta_{\mathrm{gr}_i(W_{n+1}B)}} \\ \mathrm{gr}_i(W_{n+1}B)^a_* & \longrightarrow & (\mathrm{gr}_i W_{n+1}B)^a_* \end{array}$$

whose bottom horizontal arrow is injective, since the functor $(-)^a_*$ is left exact. Here $\eta_{\mathrm{gr}_i(W_{n+1}B)}$ is again the unit of adjunction, which, by virtue of (14.7.16), is naturally identified with the unit of adjunction

$$\eta_{\overline{\boldsymbol{\omega}}^*_i B} : \overline{\boldsymbol{\omega}}^*_i B \to (\overline{\boldsymbol{\omega}}^*_i B)^a_* \qquad \text{for every } i = 0, \dots, n.$$

Let now $\mathscr{B}$ be the category of basic setups, and $\mathscr{B}$-$\mathsf{Alg} \to \mathscr{B}$ (resp. $\mathscr{B}^a$-$\mathsf{Alg} \to \mathscr{B}$) the fibred and cofibred category of $\mathscr{B}$-algebras (resp. of almost $\mathscr{B}$-algebras), as in [75, §3.5]. The localization $\mathscr{B}$-$\mathsf{Alg} \to \mathscr{B}^a$-$\mathsf{Alg}$ admits a right adjoint

$$(-)_* : \mathscr{B}^a\text{-}\mathsf{Alg} \to \mathscr{B}\text{-}\mathsf{Alg} \qquad ((V, \mathfrak{m}), R) \mapsto ((V, \mathfrak{m}), R_*)$$

which is a $\mathscr{B}$-cartesian functor ([75, §3.5.4]). By (14.7.12), we have the morphism $\overline{\boldsymbol{\omega}}_i : (W_{n+1}A, \mathfrak{n}_{n+1}) \to (A, \mathfrak{m})$ in $\mathscr{B}$, whence a commutative diagram in $\mathscr{B}$-Alg :

$$\begin{array}{ccc} ((W_{n+1}A, \mathfrak{n}_{n+1}), \overline{\boldsymbol{\omega}}^*_i B) & \longrightarrow & ((A, \mathfrak{m}), B) \\ {\scriptstyle \eta_{\overline{\boldsymbol{\omega}}^*_i B}}\downarrow & & \downarrow{\scriptstyle \eta_B} \\ ((W_{n+1}A, \mathfrak{n}_{n+1}), (\overline{\boldsymbol{\omega}}^*_i B)^a_*) & \longrightarrow & ((A, \mathfrak{m}), B^a_*) \end{array}$$

whose top horizontal arrow is given by the identity map of $\overline{\boldsymbol{\omega}}^*_i B$, and is therefore a cartesian morphism of $\mathscr{B}$-Alg. Then also the bottom vertical arrow is a cartesian morphism, since $(-)_*$ is a cartesian functor. We conclude that $\eta_{\mathrm{gr}_i(W_{n+1}B)}$ is an isomorphism, under our assumptions. Then the same holds for $\mathrm{gr}_i(\eta_{W_{n+1}B})$, for every $i = 0, \dots, n$, whence the assertion for $\eta_{W_{n+1}B}$.

To show the assertion concerning $\varepsilon_{W_{n+1}B}$, we argue by induction on $n \in \mathbb{N}$; the case $n = 0$ is trivial. Let then $n > 0$, and suppose that the assertion is already known for $\varepsilon_{W_n B}$; recall that for every $i \in \mathbb{N}$, the ring $(W_{i+1}B)^a_{!!}$ is the cokernel of a map of $W_{i+1}A$-modules

$$\delta_i : \widetilde{\mathfrak{n}}_{i+1} \to W_{i+1}A \oplus (\widetilde{\mathfrak{n}}_{i+1} \otimes_{W_{i+1}A} W_{i+1}B) \qquad x_\bullet \otimes y_\bullet \mapsto (x_\bullet \cdot y_\bullet, x_\bullet \otimes y_\bullet \otimes 1).$$

We consider the commutative ladder with right exact rows :

$$\begin{array}{ccccccc} \widetilde{\mathfrak{n}}_{n+1} & \xrightarrow{\delta_n} & W_{n+1}A \oplus (\widetilde{\mathfrak{n}}_{n+1} \otimes_{W_{n+1}A} W_{n+1}B) & \longrightarrow & (W_{n+1}B)^a_{!!} & \longrightarrow & 0 \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ \widetilde{\mathfrak{n}}_n & \xrightarrow{\delta_{n-1}} & W_nA \oplus (\widetilde{\mathfrak{n}}_n \otimes_{W_nA} W_nB) & \longrightarrow & (W_nB)^a_{!!} & \longrightarrow & 0 \end{array}$$

whose vertical arrows are induced by the projections $\pi^A_n : W_{n+1}A \to W_nA$ and $\pi^B_n : W_{n+1}B \to W_nB$ (notice that the almost structure used to compute $(W_{n+1}B)^a_{!!}$ is the one relative to the basic setup $(W_{n+1}A, \mathfrak{n}_{n+1})$, whereas for $(W_nB)^a_{!!}$ we use the almost structure relative to the basic setup $(W_nA, \mathfrak{n}_n)$). Since $\pi^A_n(\mathfrak{n}_{n+1}) = \mathfrak{n}_n$, by [75, Rem.2.1.4(ii)] we have a natural isomorphism

$$\widetilde{\mathfrak{n}}_{n+1} \otimes_{W_{n+1}A} W_nA \xrightarrow{\sim} \widetilde{\mathfrak{n}}_n \qquad x_\bullet \otimes y_\bullet \otimes a_\bullet \mapsto a_\bullet \cdot \pi^A_n(x_\bullet) \otimes \pi^A_n(y_\bullet)$$

which identifies α with $\widetilde{\mathfrak{n}}_{n+1} \otimes_{W_{n+1}A} \pi^A_n$, and β with $\pi^A_n \oplus (\widetilde{\mathfrak{n}}_{n+1} \otimes_{W_{n+1}A} \pi^B_n)$. Taking claim 9.3.31 into account, we deduce natural $W_{n+1}A$-linear surjections :

$$(14.7.20) \qquad \widetilde{\mathfrak{n}}_{n+1} \otimes_{W_{n+1}A} \overline{\boldsymbol{\omega}}^*_n(A) \to \operatorname{Ker}\alpha \qquad \overline{\boldsymbol{\omega}}^*_n(A) \oplus (\widetilde{\mathfrak{n}}_{n+1} \otimes_{W_{n+1}A} \overline{\boldsymbol{\omega}}^*_n(B)) \to \operatorname{Ker}\beta.$$

Then, in light of (14.7.12) and [75, Rem.2.1.4(ii)], we have as well the natural isomorphism

$$\widetilde{\mathfrak{n}}_{n+1} \otimes_{W_{n+1}A} \overline{\boldsymbol{\omega}}^*_n(A) \xrightarrow{\sim} \widetilde{\mathfrak{m}} \qquad x_\bullet \otimes y_\bullet \otimes a \mapsto a \cdot \overline{\boldsymbol{\omega}}_n(x_\bullet) \otimes \overline{\boldsymbol{\omega}}_n(y_\bullet)$$

which identifies (14.7.20) with $W_{n+1}A$-linear surjections :

$$(14.7.21) \qquad \overline{\boldsymbol{\omega}}^*_n(\widetilde{\mathfrak{m}}) \to \operatorname{Ker}\alpha \qquad \overline{\boldsymbol{\omega}}^*_n(A \oplus (\widetilde{\mathfrak{m}} \otimes_A B)) \to \operatorname{Ker}\beta.$$

By a direct inspection, we then get a commutative diagram :

$$\begin{array}{ccc} \overline{\boldsymbol{\omega}}^*_n(\widetilde{\mathfrak{m}}) & \xrightarrow{\overline{\boldsymbol{\omega}}^*_n(\delta_0)} & \overline{\boldsymbol{\omega}}^*_n(A \oplus (\widetilde{\mathfrak{m}} \otimes_A B)) \\ \downarrow & & \downarrow \\ \operatorname{Ker}\alpha & \xrightarrow{\delta'_n} & \operatorname{Ker}\beta \end{array}$$

whose vertical arrows are the surjections (14.7.21), and δ'_n is the restriction of δ_n. Since α and β are surjective, the induced map $\operatorname{Coker}\delta'_n \to \operatorname{Ker}\gamma$ is bijective, by the snake lemma, and γ is surjective, so finally we get a right exact sequence of $W_{n+1}A$-modules :

$$\overline{\boldsymbol{\omega}}^*_n(B^a_{!!}) \xrightarrow{\rho} (W_{n+1}B)^a_{!!} \xrightarrow{\gamma} (W_nB)^a_{!!} \to 0.$$

We consider then the diagram of $W_{n+1}A$-modules :

$$\begin{array}{ccccccccc} & & \overline{\boldsymbol{\omega}}^*_n(B^a_{!!}) & \xrightarrow{\rho} & (W_{n+1}B)^a_{!!} & \xrightarrow{\gamma} & (W_nB)^a_{!!} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\overline{\boldsymbol{\omega}}^*_n\varepsilon_B} & & \downarrow{\scriptstyle\varepsilon_{W_{n+1}B}} & & \downarrow{\scriptstyle\varepsilon_{W_nB}} & & \\ 0 & \longrightarrow & \overline{\boldsymbol{\omega}}^*_nB & \xrightarrow{\rho'} & W_{n+1}B & \xrightarrow{\gamma'} & W_nB & \longrightarrow & 0 \end{array}$$

where γ' is the projection, and ρ' is the natural identification of $\overline{\boldsymbol{\omega}}^*_nB$ with $\operatorname{Ker}\gamma'$ given by claim 9.3.31. Thus, the bottom horizontal sequence is exact, and by inductive assumption the first and third vertical arrows are bijective; therefore, in order to conclude the proof, it will suffice to check the diagram commutes. The commutativity of the right square subdiagram is clear. To see the commutativity of the left square subdiagram, let $w \in \overline{\boldsymbol{\omega}}^*_n(B^a_{!!})$ be any element; hence w is the class $[a, z]$ of a pair $(a, z) \in A \oplus (\widetilde{\mathfrak{m}} \otimes_A B)$, and we may assume that $z = x \otimes y \otimes b$ for some $x, y \in \mathfrak{m}$ and $b \in B$. Then, by inspecting the constructions, we see that $w' := \rho(w)$ is the class $[(0, \dots, 0, a), (0, \dots, 0, xyb)]$ of the corresponding pair in $W_{n+1}A \oplus (\widetilde{\mathfrak{n}}_{n+1} \otimes_{W_{n+1}A} W_{n+1}B)$. On the other hand, $\overline{\boldsymbol{\omega}}^*_n\varepsilon_B(w) = a + xyb$; then, again by claim 9.3.31, we deduce that $\varepsilon_{W_{n+1}B}(w') = (0, \dots, 0, a + xyb) = \rho'(a + xyb)$, as required. ◊

(ii): For every $n \in \mathbb{N}$, we have a natural map of non-unital $W_{n+1}A$-algebras :

$$\mu_n : W_{n+1}(\mathfrak{m}) \otimes_{W_{n+1}A} W_{n+1}(\mathfrak{m}) \to W_{n+1}(\widetilde{\mathfrak{m}}) \qquad a_\bullet \otimes b_\bullet \mapsto (P_i(a_\bullet \otimes 1, 1 \otimes b_\bullet) \mid i = 0, \dots, n)$$

for the non-unital ring structure of $W_{n+1}(\mathfrak{m}) = \mathfrak{n}_{n+1}$ and $W_{n+1}(\widetilde{\mathfrak{m}})$ induced by those of $\mathfrak{m}$ and respectively $\widetilde{\mathfrak{m}}$ (remark 9.3.16). We show by induction on $n \in \mathbb{N}$, that μ_n is an isomorphism; the assertion is trivial for $n = 0$, since $\mu_0 = \mathbf{1}_{\widetilde{\mathfrak{m}}}$. Thus, let $n > 0$, and suppose that the assertion is already known for $n - 1$; we consider the commutative diagram of $W_{n+1}A$-modules :

$$\begin{array}{ccc} \mathfrak{n}_{n+1} \otimes_{W_{n+1}A} \mathfrak{n}_{n+1} & \xrightarrow{\mu_n} & W_{n+1}(\widetilde{\mathfrak{m}}) \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ \mathfrak{n}_n \otimes_{W_n A} \mathfrak{n}_n & \xrightarrow{\mu_{n-1}} & W_n(\widetilde{\mathfrak{m}}) \end{array}$$

whose left vertical arrows is induced by the projection $W_{n+1}(\mathfrak{m}) \to W_n(\mathfrak{m})$, and whose right vertical arrow is likewise the natural projection. Claim 9.3.31 yields a natural isomorphism of $W_{n+1}A$-modules :

$$(14.7.22) \qquad \overline{\boldsymbol{\omega}}_n^*(\widetilde{\mathfrak{m}}) \xrightarrow{\sim} \operatorname{Ker} \beta.$$

Claim 14.7.23. We have a commutative diagram of $W_{n+1}A$-modules :

$$\begin{array}{ccc} \overline{\boldsymbol{\omega}}_n^*(\widetilde{\mathfrak{m}}) & \xrightarrow{\overline{\boldsymbol{\omega}}_n^*(\mu_0)} & \overline{\boldsymbol{\omega}}_n^*(\widetilde{\mathfrak{m}}) \\ \downarrow & & \downarrow \\ \operatorname{Ker} \alpha & \longrightarrow & \operatorname{Ker} \beta \end{array}$$

whose bottom horizontal arrow is the restriction of μ_n, and whose right (resp. left) vertical arrow is (14.7.22) (resp. is (14.7.21)).

Proof of the claim. By construction, the left vertical arrow is characterized as the map such that $a \cdot \overline{\boldsymbol{\omega}}_n(x_\bullet) \otimes \overline{\boldsymbol{\omega}}_n(y_\bullet) \mapsto (0, \dots, 0, a) \cdot (x_\bullet \otimes y_\bullet)$ for every $a \in A$ and every $x_\bullet, y_\bullet \in \mathfrak{n}_{n+1}$. On the other hand, the right vertical arrow is the map such that $x \otimes y \mapsto (0, \dots, 0, x \otimes y)$ for every $x, y \in \mathfrak{m}$. Hence, we need to check the identity :

$$\mu_n((0, \dots, 0, a) \cdot (x_\bullet \otimes y_\bullet)) = (0, \dots, 0, a \cdot \overline{\boldsymbol{\omega}}_n(x_\bullet) \otimes \overline{\boldsymbol{\omega}}_n(y_\bullet))$$

for every $a \in A$ and every $x_\bullet, y_\bullet \in \mathfrak{n}_{n+1}$. To this aim, endow $R := A \oplus \mathfrak{m}$ with its natural A-algebra structure as in remark 9.3.16(iii), and recall that μ_n is the restriction of the map of unital $W_{n+1}A$-algebras

$$W_{n+1}R \otimes_{W_{n+1}A} W_{n+1}R \to W_{n+1}(R \otimes_A R) \qquad x_\bullet \otimes y_\bullet \mapsto (x_\bullet \otimes 1) \cdot (1 \otimes y_\bullet).$$

Thus, we are reduced to checking that

$$((V_R^n(\tau_R(a)) \cdot x_\bullet) \otimes 1) \cdot (1 \otimes y_\bullet) = V_{R\otimes_A R}^n(\tau_{R\otimes_A R}(a \cdot \overline{\boldsymbol{\omega}}_n(x_\bullet) \otimes \overline{\boldsymbol{\omega}}_n(y_\bullet)))$$

for every $a \in A$ and every $x_\bullet, y_\bullet \in W_{n+1}R$, where V_R and τ_R denote respectively the Verschiebung and the Teichmüller maps, and likewise for $V_{R\otimes_A R}$ and $\tau_{R\otimes_A R}$ (see (9.3.18)). Let also F_R be the Frobenius map; using proposition 9.3.22(ii) we compute :

$$\begin{aligned} ((V_R^n(\tau_R(a)) \cdot x_\bullet) \otimes 1) \cdot (1 \otimes y_\bullet) &= (V_R^n(\tau_R(a) \cdot F_R^n(x_\bullet)) \otimes 1) \cdot (1 \otimes y_\bullet) \\ &= V_R^n((\tau_R(a) \cdot F_R^n(x_\bullet)) \otimes 1) \cdot (1 \otimes y_\bullet) \\ &= V_{R\otimes_A R}^n(((\tau_R(a) \cdot F_R^n(x_\bullet)) \otimes 1) \cdot (1 \otimes F_R^n(y_\bullet))) \\ &= V_{R\otimes_A R}^n(\tau_{R\otimes_A R}(a) \cdot F_{R\otimes_A R}^n(x_\bullet \otimes 1) \cdot F_{R\otimes_A R}^n(1 \otimes y_\bullet)). \end{aligned}$$

Hence, we are reduced to checking that :

$$\overline{\boldsymbol{\omega}}_n(x_\bullet) \otimes \overline{\boldsymbol{\omega}}_n(y_\bullet) = \overline{\boldsymbol{\omega}}_0(F_{R\otimes_A R}^n(x_\bullet \otimes 1) \cdot F_{R\otimes_A R}^n(1 \otimes y_\bullet)) \qquad \text{in } W_{n+1}(R \otimes_A R).$$

The latter follows easily from (9.3.19) : details left to the reader. ◊

Claim 14.7.23 implies that also (14.7.21) is an isomorphism, and the same for μ_{n-1}, by inductive assumption; so the same follows for μ_n and the proof is concluded. □

14.7.24. Let now A_0 be any $\mathbb{F}_p$-algebra, $(A_0, \mathfrak{m}_0)$ any basic setup, and R any A_0-algebra. Recall that the Frobenius endomorphism $\Phi_R : R \to R$ induces an endofunctor

$$\Phi_R^{k*} : R\text{-}\mathsf{Alg} \to R\text{-}\mathsf{Alg} \qquad \text{for every } k \in \mathbb{N}$$

that assigns to every R-algebra S the R-algebra $\Phi_R^{k*}S$ whose underlying ring is the same as S, and whose structure morphism is the composition of Φ_R^k with the structure morphism $R \to S$ of S. Moreover, the Frobenius endomorphism Φ_S^k induces a natural transformation

$$\Phi_{S/R}^k : S \otimes_R \Phi_R^{k*}R \to \Phi_R^{k*}S \qquad x \otimes y \mapsto x^{p^k}y \qquad \text{for every } R\text{-algebra } S$$

Let R^a be the $(A_0, \mathfrak{m})^a$-algebra represented by R; then Φ_R^{k*} descends to an endofunctor

$$\Phi_{R^a}^{k*} : R^a\text{-}\mathsf{Alg} \to R^a\text{-}\mathsf{Alg}$$

that assigns to every R^a-algebra S^a the R^a-algebra $\Phi_{R^a}^{k*}(S^a) := (\Phi_R^{k*}S)^a$, where S is any R-algebra representing S^a : see [75, §3.5.7]. Likewise, the Frobenius endomorphism Φ_S^k induces a natural transformation

$$\Phi_{S^a/R^a}^k := (\Phi_{S/R}^k)^a : (S^a \otimes_{R^a} \Phi_{R^a}^{k*}R^a) \to \Phi_{R^a}^{k*}S^a \qquad \text{for every } R^a\text{-algebra } S^a.$$

Moreover, notice that Φ_{S^a/R^a}^{k+1} also equals the composition of $\Phi_{S^a/R^a} \otimes_{\Phi_{R^a}^* R^a} \Phi_{R^a}^{k+1*}R^a$:

$$S^a \otimes_{R^a} \Phi_{R^a}^{k+1*}R^a \xrightarrow{\sim} S^a \otimes_{R^a} \Phi_{R^a}^{*}R^a \otimes_{\Phi_{R^a}^* R^a} \Phi_{R^a}^{k+1*}R^a \to \Phi_{R^a}^{*}S^a \otimes_{\Phi_{R^a}^* R^a} \Phi_{R^a}^{k+1*}R^a$$

together with $\Phi_{R^a}^*(\Phi_{S^a/R^a}^k) : \Phi_{R^a}^*S^a \otimes_{\Phi_{R^a}^* R^a} \Phi_{R^a}^{k+1*}R^a = \Phi_{R^a}^*(S^a \otimes_{R^a} \Phi_{R^a}^{k*}R^a) \to \Phi_{R^a}^{k+1*}S^a$. Especially, if Φ_{S^a/R^a} is an isomorphism, the same holds for Φ_{S^a/R^a}^k, for every $k \in \mathbb{N}$.

Remark 14.7.25. In the situation of (14.7.24), suppose that $\mathfrak{m}$ fulfills condition $(\mathbf{B})$ of [75, §2.1.6], and let $f : R \to S$ be any weakly étale morphism of A_0^a-algebras. Then $\Phi_{S/R}$ is an isomorphism. Indeed, the assertion is [75, Th.3.5.13(ii)] in case $\mathfrak{m} \otimes_{A_0} \mathfrak{m}$ is a flat A_0-module, but by direct inspection one sees easily that the proof of *loc.cit.* is valid more generally, whenever condition $(\mathbf{B})$ holds.

Theorem 14.7.26. *In the situation of* (14.7.15)*, let $f : B \to C$ be a flat morphism of A^a-algebras; set $B_0 := B/pB$, $C_0 := C/pC$, and suppose that Φ_{C_0/B_0} is an isomorphism. Then:*

(i) *f induces a flat morphism of $W_{n+1}A^a$-algebras*

$$W_{n+1}f : W_{n+1}B \to W_{n+1}C \qquad \textit{for every } n \in \mathbb{N}.$$

(ii) *For every $n \in \mathbb{N}$ and every $i = 0, \dots, n$, the following diagram is cocartesian :*

$$\begin{array}{ccc} W_{n+1}B & \xrightarrow{W_{n+1}f} & W_{n+1}C \\ {\scriptstyle\overline{\boldsymbol{\omega}}_i}\downarrow & & \downarrow{\scriptstyle\overline{\boldsymbol{\omega}}_i} \\ \overline{\boldsymbol{\omega}}_i^*B & \xrightarrow{\overline{\boldsymbol{\omega}}_i^*f} & \overline{\boldsymbol{\omega}}_i^*C. \end{array}$$

Proof. We prove first both assertions in the case where $p^kA = 0$ for some integer $k \in \mathbb{N}$. For every $j \in \mathbb{N}$, and every A^a-algebra R set

$$W_{n,j}R := W_{n+1}(R)/p^j\overline{V}_n(R) \qquad \text{with } \overline{V}_n(R) := V_nR/V_{n+1}R \subset W_{n+1}R.$$

Hence, $W_{n,0}R = W_nR$, and in light of corollary 9.3.32, we see that $W_{n,j}R = W_{n+1}R$ for every sufficiently large $j \in \mathbb{N}$. We shall show, by induction on j and n, the following assertions :

$(\mathrm{a})_{n,j}$ The induced map $W_{n,j}f : W_{n,j}B \to W_{n,j}C$ is flat for every $j, n \in \mathbb{N}$.

$(\mathrm{b})_{n,j}$ The morphism f induces a cocartesian diagram of $W_{n+1}B$-algebras :

$$\begin{array}{ccc} W_{n,j}B & \longrightarrow & \overline{\boldsymbol{\omega}}_0^*B \\ {\scriptstyle W_{n,j}f}\downarrow & & \downarrow{\scriptstyle \overline{\boldsymbol{\omega}}_0^*f} \\ W_{n,j}C & \longrightarrow & \overline{\boldsymbol{\omega}}_0^*C. \end{array}$$

Both assertions are trivial for $n = 0$ and every $j \in \mathbb{N}$. Let then $n, j \in \mathbb{N}$ be any integers, and suppose that $(\mathrm{a})_{n,j}$ and $(\mathrm{b})_{n,j}$ hold. According to (14.7.16), the kernel $K_{n,j}$ of the projection $W_{n,j+1}B \to W_{n,j}B$ is isomorphic, as a $W_{n+1}B$-module, to $\overline{\boldsymbol{\omega}}_n^*(p^jB/p^{j+1}B)$. Likewise for the kernel $K'_{n,j}$ of the projection $W_{n,j+1}C \to W_{n,j}C$, and the induced morphism $K_{n,j} \to K'_{n,j}$ corresponds to the restriction of $\overline{\boldsymbol{\omega}}_n^*(f \otimes_{\mathbb{Z}} \mathbb{Z}/p^{j+1}\mathbb{Z})$, under these identifications. By the (almost version of the) local flatness criterion (see [126, Th.22.3]), assertion $(\mathrm{a})_{n,j+1}$ will follow from :

Claim 14.7.27. The induced morphism $\overline{\boldsymbol{\omega}}_n^*(p^jB/p^{j+1}B) \otimes_{W_{n,j}B} W_{n,j}C \to \overline{\boldsymbol{\omega}}_n^*(p^jC/p^{j+1}C)$ is an isomorphism.

Proof of the claim. Clearly $\overline{\boldsymbol{\omega}}_n^*(p^jB/p^{j+1}B)$ is a $\overline{\boldsymbol{\omega}}_n^*B_0$-module and $\overline{\boldsymbol{\omega}}_n^*(p^jC/p^{j+1}C)$ is a C_0-module, and notice that $\overline{\boldsymbol{\omega}}_n^*(B_0) = \overline{\boldsymbol{\omega}}_0^* \circ \Phi_{B_0}^{n*}(B_0)$, and likewise for $\overline{\boldsymbol{\omega}}_n^*(C_0)$. Then the stated morphism agrees with the composition

$$\begin{aligned} \overline{\boldsymbol{\omega}}_n^*(p^jB/p^{j+1}B) \otimes_{W_{n,j}B} W_{n,j}C &\xrightarrow{\sim} \overline{\boldsymbol{\omega}}_0^*\Phi_{B_0}^{n*}(p^jB/p^{j+1}B) \otimes_{\overline{\boldsymbol{\omega}}_0^*B_0} \overline{\boldsymbol{\omega}}_0^*B_0 \otimes_{W_{n,j}B} W_{n,j}C \\ &\xrightarrow{\sim} \overline{\boldsymbol{\omega}}_0^*\Phi_{B_0}^{n*}(p^jB/p^{j+1}B) \otimes_{\overline{\boldsymbol{\omega}}_0^*B_0} \overline{\boldsymbol{\omega}}_0^*C_0 \\ &= \overline{\boldsymbol{\omega}}_0^*(\Phi_{B_0}^{n*}(p^jB/p^{j+1}B) \otimes_{B_0} C_0) \\ &\xrightarrow{\sim} \overline{\boldsymbol{\omega}}_0^*(\Phi_{B_0}^{n*}(p^jB/p^{j+1}B) \otimes_{\Phi_{B_0}^{n*}B_0} \Phi_{B_0}^{n*}B_0 \otimes_{B_0} C_0) \\ &\xrightarrow{\sim} \overline{\boldsymbol{\omega}}_0^*(\Phi_{B_0}^{n*}(p^jB/p^{j+1}B) \otimes_{\Phi_{B_0}^{n*}B_0} \Phi_{B_0}^{n*}C_0) \\ &= \overline{\boldsymbol{\omega}}_0^*\Phi_{B_0}^{n*}((p^jB/p^{j+1}B) \otimes_{B_0} C_0) \\ &\xrightarrow{\sim} \overline{\boldsymbol{\omega}}_0^*\Phi_{B_0}^{n*}(p^jC/p^{j+1}C) \\ &= \overline{\boldsymbol{\omega}}_n^*(p^jC/p^{j+1}C) \end{aligned}$$

where the second isomorphism follows from $(\mathrm{b})_{n,j}$, the fourth follows from our assumption about Φ_{C_0/B_0}, and the fifth follows from the flatness of f. ◊

Claim 14.7.27 also implies that the diagram

$$\begin{array}{ccc} W_{n,j+1}B & \longrightarrow & W_{n,j}B \\ {\scriptstyle W_{n,j+1}f}\downarrow & & \downarrow{\scriptstyle W_{n,j}f} \\ W_{n,j+1}C & \longrightarrow & W_{n,j}C \end{array}$$

is cocartesian; combining with $(\mathrm{b})_{n,j}$, we see that $(\mathrm{b})_{n,j+1}$ holds. Thus, we conclude that $(\mathrm{a})_{n,j}$ and $(\mathrm{b})_{n,j}$ hold for every $j \in \mathbb{N}$; for large values of j, we deduce that $(\mathrm{a})_{n+1,0}$ and $(\mathrm{b})_{n+1,0}$ both hold. By induction on $n \in \mathbb{N}$, the assertion follows. This concludes the proof of (i) in case $p^kA = 0$. To prove (ii) under the same assumption, we show first that the natural morphism

$$\overline{\boldsymbol{\omega}}_i^*B_0 \otimes_{W_{n+1}B} W_{n+1}C \to \overline{\boldsymbol{\omega}}_i^*C_0 \tag{14.7.28}$$

is an isomorphism. Indeed, notice that $\overline{\boldsymbol{\omega}}_i^* B_0 = \overline{\boldsymbol{\omega}}_0^*(\Phi_{B_0}^{i*} B_0)$, and likewise for $\overline{\boldsymbol{\omega}}_i^* C_0$. Then the foregoing morphism is the composition :

$$\begin{aligned}\overline{\boldsymbol{\omega}}_i^* B_0 \otimes_{W_{n+1}B} W_{n+1}C &= \overline{\boldsymbol{\omega}}_0^*(\Phi_{B_0}^{i*} B_0) \otimes_{W_{n+1}B} W_{n+1}C \\ &\stackrel{\sim}{\to} \overline{\boldsymbol{\omega}}_0^*(\Phi_{B_0}^{i*} B_0) \otimes_{\overline{\boldsymbol{\omega}}_0^* B_0} \overline{\boldsymbol{\omega}}_0^* B_0 \otimes_{W_{n+1}B} W_{n+1}C \\ &\stackrel{\sim}{\to} \overline{\boldsymbol{\omega}}_0^*(\Phi_{B_0}^{i*} B_0) \otimes_{\overline{\boldsymbol{\omega}}_0^* B_0} \overline{\boldsymbol{\omega}}_0^* C_0 \\ &= \overline{\boldsymbol{\omega}}_0^*(\Phi_{B_0}^{i*} B_0 \otimes_{B_0} C_0) \\ &\stackrel{\sim}{\to} \overline{\boldsymbol{\omega}}_0^*(\Phi_{B_0}^{i*} C_0) \\ &= \overline{\boldsymbol{\omega}}_i^* C_0\end{aligned}$$

where the second isomorphism follows from the foregoing condition $(\mathrm{b})_{n+1,0}$, and the third isomorphism follows from our assumption on Φ_{C_0/B_0}. Now, since $p^k B = 0$, the assertion follows from the isomorphism (14.7.28) together with the following :

Claim 14.7.29. Let $g : R \to S$ be a flat morphism of A^a-algebras, $I \subset R$ a nilpotent ideal, and suppose that $g \otimes_R R/I$ is an isomorphism. Then g is an isomorphism.

Proof of the claim. By a simple induction, we are reduced to the case where $I^2 = 0$. Then we have the commutative ladder with exact rows :

$$\begin{array}{ccccccccc} 0 & \to & I & \to & R & \to & R/I & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & I \otimes_R S & \to & S & \to & S/IS & \to & 0 \end{array}$$

and a natural identification $I \otimes_R S \stackrel{\sim}{\to} I \otimes_{R/I} S/IS \stackrel{\sim}{\to} I$. Thus, the first and third vertical arrows are both isomorphisms, and the same then follows for the middle one. ◊

We consider next the case where A is a general ring, and $\mathfrak{m}$ fulfills condition $(\mathbf{B})$. We shall use the following criterion :

Claim 14.7.30. Let $(V, \mathfrak{m}_V)$ be any basic setup, R a $(V, \mathfrak{m}_V)^a$-algebra and M an R-module. Set

$$M_{\mathrm{tor}} := \bigcup_{n \in \mathbb{N}} \mathrm{Ann}_M(p^n) \qquad R_{\mathrm{tor}} := \bigcup_{n \in \mathbb{N}} \mathrm{Ann}_R(p^n).$$

Then M is a flat R-module if and only if the following three conditions hold :

(a) $M/p^i M$ is a flat $R/p^i R$-module for every $i \in \mathbb{N}$.
(b) $M \otimes_R R[p^{-1}]$ is a flat $R[p^{-1}]$-module.
(c) The natural morphism $R_{\mathrm{tor}} \otimes_R M \to M_{\mathrm{tor}}$ is an isomorphism.

Proof of the claim. Clearly if M is flat, conditions (a)–(c) hold. Thus, suppose that (a)–(c) hold; according to [75, Lemma 5.2.1] it suffices to prove that

(14.7.31) $$\mathrm{Tor}_i^R(M, R/pR) = 0 \qquad \text{for } i = 1, 2.$$

However, a standard calculation shows that

$$\mathrm{Tor}_1^R(M, R/pR) = \frac{\mathrm{Ann}_M(p)}{\mathrm{Ann}_R(p)M} \qquad \text{and} \qquad \mathrm{Tor}_2^R(M, R/pR) = \mathrm{Ker}\,(\mathrm{Ann}_R(p) \otimes_R M \to M).$$

Let us then consider the commutative ladder :

$$\begin{array}{ccccccc} 0 & \to & \mathrm{Ann}_R(p) \otimes_R M & \to & R_{\mathrm{tor}} \otimes_R M & \xrightarrow{p} & R_{\mathrm{tor}} \otimes_R M \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \to & \mathrm{Ann}_M(p) & \to & M_{\mathrm{tor}} & \xrightarrow{p} & M_{\mathrm{tor}} \end{array}$$

whose bottom horizontal row is exact, and whose central and right vertical arrows are isomorphisms; in order to verify (14.7.31), it suffices therefore to check that the top horizontal row is also exact; the latter is the inductive limit of the system of morphisms :

$$\mathrm{Ann}_R(p) \otimes_R M \to \mathrm{Ann}_R(p^n) \otimes_R M \to \mathrm{Ann}_R(p^n) \otimes_R M \qquad \text{for every } n \in \mathbb{N}.$$

But clearly $\mathrm{Ann}_R(p^i) \otimes_R M = \mathrm{Ann}_R(p^i) \otimes_{R/p^nR} M/p^nM$ for every $i, n \in \mathbb{N}$ with $i \leq n$; in view of (a), the assertion follows. ◊

Now, example 9.3.38(ii), implies easily that $W_{n+1}(C)[p^{-1}]$ is a flat $W_{n+1}(B)[p^{-1}]$-algebra. Next, for every $k \in \mathbb{N}$ let $\pi_k : C \to C/p^kC$ be the projection; from example 9.3.37(iii) we get

$$\mathrm{Ker}\, W_{n+1}(\pi_{n+k}) \subset p^k W_{n+1}(C) \qquad \text{for every } k \in \mathbb{N}$$

so that π_{n+k} induces an isomorphism of $W_{n+1}A^a$-algebras

$$W_{n+1}(C) \otimes_{\mathbb{Z}} \mathbb{Z}/p^k\mathbb{Z} \xrightarrow{\sim} W_{n+1}(C/p^{n+k}C) \otimes_{\mathbb{Z}} \mathbb{Z}/p^k\mathbb{Z} \qquad \text{for every } k \in \mathbb{N}. \tag{14.7.32}$$

But, by the foregoing case, we know already that $W_{n+1}(C/p^{n+k}C)$ is a flat $W_{n+1}(B/p^{n+k}B)$-algebra for every $k \in \mathbb{N}$, so we conclude that $W_{n+1}(C) \otimes_{\mathbb{Z}} \mathbb{Z}/p^k\mathbb{Z}$ is a flat $W_{n+1}(B) \otimes_{\mathbb{Z}} \mathbb{Z}/p^k\mathbb{Z}$-algebra for every $k \in \mathbb{N}$. In light of claim 14.7.30, we are therefore reduced to checking that the natural morphism

$$W_{n+1}(B)_{\mathrm{tor}} \otimes_{W_{n+1}B} W_{n+1}C \to W_{n+1}(C)_{\mathrm{tor}}$$

is an isomorphism. More precisely, we shall show, by descending induction on i, that :

- the induced morphism $(\overline{V}_iB)_{\mathrm{tor}} \otimes_{W_{n+1}B} W_{n+1}C \to (\overline{V}_iC)_{\mathrm{tor}}$ is an isomorphism of $W_{n+1}B$-modules for every $i = 0, \dots, n+1$
- the induced morphism $(\overline{V}_{i+1}B)_{\mathrm{tor}} \otimes_{W_{n+1}B} W_{n+1}C \to (\overline{V}_iB)_{\mathrm{tor}} \otimes_{W_{n+1}B} W_{n+1}C$ is a monomorphism of $W_{n+1}B$-modules for $i = 0, \dots, n$.

Indeed, both assertions are trivial for $i = n+1$. Suppose that both assertions are already known for some strictly positive integer $i \leq n+1$; from the commutative diagram

$$\begin{array}{ccc} (\overline{V}_iB)_{\mathrm{tor}} \otimes_{W_{n+1}B} W_{n+1}C & \longrightarrow & (\overline{V}_iC)_{\mathrm{tor}} \\ \downarrow & & \downarrow \\ (\overline{V}_{i-1}B)_{\mathrm{tor}} \otimes_{W_{n+1}B} W_{n+1}C & \longrightarrow & (\overline{V}_{i-1}C)_{\mathrm{tor}} \end{array}$$

we then deduce that the left vertical arrow is a monomorphism, *i.e.* the second assertion holds for $i-1$. Moreover, in order to prove the first assertion for $i-1$, it suffices to check that the induced morphism

$$(\overline{V}_{i-1}B)_{\mathrm{tor}}/(\overline{V}_iB)_{\mathrm{tor}} \otimes_{W_{n+1}B} W_{n+1}C \to (\overline{V}_{i-1}C)_{\mathrm{tor}}/(\overline{V}_{i-1}C)_{\mathrm{tor}}$$

is an isomorphism of $W_{n+1}B$-modules. On the other hand, from example 9.3.38(i) we see that $W_{n+1}(B)_{\mathrm{tor}}$ is the kernel of the natural morphism $W_{n+1}B \to W_{n+1}(B[p^{-1}])$, and likewise for $W_{n+1}(C)_{\mathrm{tor}}$; taking into account (14.7.16), we deduce a natural isomorphism :

$$(\overline{V}_{i-1}B)_{\mathrm{tor}}/(\overline{V}_iB)_{\mathrm{tor}} \xrightarrow{\sim} (\overline{\boldsymbol{\omega}}^*_{i-1}B)_{\mathrm{tor}}$$

of $W_{n+1}B$-modules, and likewise for $(\overline{V}_{i-1}C)_{\mathrm{tor}}/(\overline{V}_{i-1}C)_{\mathrm{tor}}$. We are then further reduced to checking that the induced morphism

$$\overline{\boldsymbol{\omega}}^*_{i-1}(\mathrm{Ann}_B(p^k)) \otimes_{W_{n+1}(B)} W_{n+1}C \to \overline{\boldsymbol{\omega}}^*_{i-1}\mathrm{Ann}_C(p^k)$$

is an isomorphism for every $k \in \mathbb{N}$. However, the latter is the composition of the isomorphisms

$$\begin{aligned}\overline{\boldsymbol{\omega}}^*_{i-1}(\mathrm{Ann}_B(p^k)) \otimes_{W_{n+1}(B)} W_{n+1}C &\xrightarrow{\sim} \overline{\boldsymbol{\omega}}^*_{i-1}(\mathrm{Ann}_B(p^k)) \otimes_{W_{n+1}(B/p^{n+k}B)} W_{n+1}(C/p^{n+k}C) \\ &\xrightarrow{\sim} \overline{\boldsymbol{\omega}}^*_{i-1}(\mathrm{Ann}_B(p^k)) \otimes_{\overline{\boldsymbol{\omega}}^*_{i-1}(B/p^{n+k}B)} \overline{\boldsymbol{\omega}}^*_{i-1}(C/p^{n+k}C) \\ &= \overline{\boldsymbol{\omega}}^*_{i-1}(\mathrm{Ann}_B(p^k) \otimes_B C) \\ &\xrightarrow{\sim} \overline{\boldsymbol{\omega}}^*_{i-1}\mathrm{Ann}_C(p^k)\end{aligned}$$

where the first isomorphism is due to (14.7.32), the second one follows from the part of assertion (ii) that we have already proved, and the third one follows from the flatness of f.

To conclude the proof of (ii) for a general ring A, we shall need the following :

Claim 14.7.33. Let $(V, \mathfrak{m}_V)$ be a basic setup, R a $(V, \mathfrak{m})^a$-algebra and $\varphi : M \to N$ a morphism of flat R-modules. Suppose that :

(a) $\varphi \otimes_{\mathbb{Z}} \mathbb{F}_p$ is an isomorphism of R/pR-modules.
(b) $\varphi \otimes_{\mathbb{Z}} \mathbb{Z}[p^{-1}]$ is an isomorphism of $R[p^{-1}]$-modules.

Then φ is an isomorphism.

Proof of the claim. Denote by $C_\bullet$ the complex of R-modules $[M \xrightarrow{\varphi} N]$, say with M placed in degree 0. Consider as well the acyclic complex

$$D_\bullet \quad : \quad 0 \to \mathrm{Ann}_R(p) \to R \xrightarrow{p\mathbf{1}_R} R \to R/pR \to 0.$$

Since $C_\bullet$ is flat in every degree, the complex $D_\bullet \otimes_R C_\bullet$ is still exact; the latter is also the total complex of the double complex

$$0 \to E_\bullet := \mathrm{Ann}_R(p) \otimes_{R/pR} (C_\bullet \otimes_{\mathbb{Z}} \mathbb{F}_p) \to C_\bullet \xrightarrow{p\mathbf{1}_{C_\bullet}} C_\bullet \to C_\bullet \otimes_{\mathbb{Z}} \mathbb{F}_p \to 0.$$

On the other hand, assumption (a) says that $C_\bullet \otimes_{\mathbb{Z}} \mathbb{F}_p$ is an acyclic complex of flat R/pR-modules, hence also $E_\bullet$ is acyclic. Summing up, we find that the morphism of complexes $p\mathbf{1}_{C_\bullet} : C_\bullet \to C_\bullet$ is a quasi-isomorphism, and hence induces isomorphisms $p \cdot \mathbf{1}_{H_i(C_\bullet)} : H_i(C_\bullet) \xrightarrow{\sim} H_i(C_\bullet)$ for $i = 0, 1$. In other words, $H_i(C_\bullet) = H_i(C_\bullet) \otimes_{\mathbb{Z}} \mathbb{Z}[p^{-1}]$ for $i = 0, 1$; combining with assumption (b), we conclude that $\mathrm{Ker}\,\varphi = \mathrm{Coker}\,\varphi = 0$, whence the claim. ◊

Now, we need to show that the morphism

$$\varphi_f : \overline{\boldsymbol{\omega}}^*_i B \otimes_{W_{n+1}B} W_{n+1}C \to \overline{\boldsymbol{\omega}}^*_i C$$

resulting from the diagram of (ii) is an isomorphism of $W_{n+1}B$-modules; however, set as well $g := f \otimes_{\mathbb{Z}} \mathbb{Z}/p^{n+1}\mathbb{Z} : B/p^{n+1}B \to C/p^{n+1}C$; according to the part of (ii) that has already been proved, the corresponding morphism

$$\varphi_g : \overline{\boldsymbol{\omega}}^*_i(B/p^{n+1}B) \otimes_{W_{n+1}(B/p^{n+1}B)} W_{n+1}(C/p^{n+1}C) \to \overline{\boldsymbol{\omega}}^*_i(C/p^{n+1}C)$$

is an isomorphism; on the other hand, in view of (14.7.32) the morphism $\varphi_f \otimes_{\mathbb{Z}} \mathbb{F}_p$ is naturally identified with $\varphi_g \otimes_{\mathbb{Z}} \mathbb{F}_p$, so $\varphi_f \otimes_{\mathbb{Z}} \mathbb{F}_p$ is an isomorphism as well. Next, recall that the ghost map induces an isomorphism of $W_{n+1}A^a$-algebras

$$(W_{n+1}B)[p^{-1}] \xrightarrow{\sim} B[p^{-1}]^{n+1} := \prod_{i=0}^{n} \overline{\boldsymbol{\omega}}^*_i B[p^{-1}]$$

and likewise for $(W_{n+1}C)[p^{-1}]$ (example 9.3.38(ii)); under these isomorphisms, the morphism $\overline{\boldsymbol{\omega}}_i \otimes_{\mathbb{Z}} \mathbb{Z}[p^{-1}]$ on $(W_{n+1}B)[p^{-1}]$ is identified with the projection $\pi_i : B[p^{-1}]^{n+1} \to B[p^{-1}]$ on the $(i+1)$-th factor, and likewise for the corresponding morphism on $(W_{n+1}C)[p^{-1}]$. Thus, $\varphi_f \otimes_{\mathbb{Z}} \mathbb{Z}[p^{-1}]$ is identified with the natural isomorphism of $B[p^{-1}]^{n+1}$-modules

$$\pi^*_i B[p^{-1}] \otimes_{B[p^{-1}]^{n+1}} C[p^{-1}]^{n+1} \xrightarrow{\sim} \pi^*_i C[p^{-1}].$$

From claim 14.7.33 and (i) we deduce that φ_f is an isomorphism, as required. □

Corollary 14.7.34. *In the situation of theorem* 14.7.26*, for every $n \in \mathbb{N}$ we have :*

(i) *Every B-algebra D induces a cocartesian diagram of $W_{n+1}A^a$-algebras :*

$$\begin{array}{ccc} W_{n+1}B & \longrightarrow & W_{n+1}C \\ \downarrow & & \downarrow \\ W_{n+1}D & \longrightarrow & W_{n+1}(C \otimes_B D). \end{array}$$

(ii) *The $W_{n+1}B$-module $W_{n+1}C$ is almost finitely generated (resp. almost finitely presented, resp. almost projective) if and only if the same holds for the B-module C.*

(iii) *Suppose that condition* $(\mathbf{B})$ *of* [75, §2.1.6] *holds for* $\mathfrak{m}$*. Then $W_{n+1}(f)$ is étale (resp. weakly étale) if and only if the same holds for the morphism f.*

(iv) *The projection $W_{n+2}C \to W_{n+1}C$ induces an isomorphism of $W_{n+1}B$-algebras :*

$$W_{n+2}C \otimes_{W_{n+2}B} W_{n+1}B \xrightarrow{\sim} W_{n+1}C.$$

Proof. (i): Notice that the natural morphism $h : W_{n+1}C \otimes_{W_{n+1}B} W_{n+1}D \to W_{n+1}(C \otimes_B D)$ restricts to morphisms of $W_{n+1}B$-modules

$$W_{n+1}C \otimes_{W_{n+1}B} \overline{V}_k D \to \overline{V}_k(C \otimes_B D) \qquad \text{for every } k = 0, \dots, n.$$

Taking into account (14.7.16), we are then reduced to checking that h induces isomorphisms

$$\mathrm{gr}_k h : W_{n+1}C \otimes_{W_{n+1}B} \overline{\boldsymbol{\omega}}_k^* D \to \overline{\boldsymbol{\omega}}_k^*(C \otimes_B D) \qquad \text{for every } k = 0, \dots, n.$$

However, a simple inspection shows that $\mathrm{gr}_k h$ agrees with the composition

$$W_{n+1}C \otimes_{W_{n+1}B} \overline{\boldsymbol{\omega}}_k^* D \xrightarrow{\sim} W_{n+1}C \otimes_{W_{n+1}B} \overline{\boldsymbol{\omega}}_k^* B \otimes_{\overline{\boldsymbol{\omega}}_k^* B} \overline{\boldsymbol{\omega}}_k^* D \xrightarrow{\sim} \overline{\boldsymbol{\omega}}_k^* C \otimes_{\overline{\boldsymbol{\omega}}_k^* B} \overline{\boldsymbol{\omega}}_k^* D = \overline{\boldsymbol{\omega}}_k^*(C \otimes_B D)$$

where the second isomorphism follows from theorem 14.7.26(ii).

(iv): The proof of (i) also shows that the natural morphism

$$W_{n+2}C \otimes_{W_{n+2}B} \overline{V}_{n+1} B \to \overline{V}_{n+1} C$$

is an isomorphism. The assertion is an immediate consequence.

(ii): Let $\psi : W_{n+1}B \to B^{n+1} := \prod_{i=0}^n \overline{\boldsymbol{\omega}}_i^* B$ be the morphism of $W_{n+1}A^a$-algebras whose composition with the projection $B^{n+1} \to \overline{\boldsymbol{\omega}}_i^* B$ is the morphism $\overline{\boldsymbol{\omega}}_i$, for $i = 0, \dots, n$; denote by $R \subset B^{n+1}$ the image of ψ. Let also $\mathbf{P}$ be one of the properties : "almost finitely generated", "almost finitely presented", or "almost projective", and suppose that f enjoys $\mathbf{P}$; in view of theorem 14.7.26(i), proposition 9.3.30(i) and [75, Lemma 3.2.25], it suffices to prove that the R-module $W_{n+1}C \otimes_{W_{n+1}B} R$ enjoys property $\mathbf{P}$. Next, by proposition 9.3.30(ii) we know that

$$p^{n+1}B^{n+1} \subset R \qquad \text{and we let} \qquad R_0 := R/p^{n+1}B^{n+1} \qquad B_0^{n+1} := B^{n+1}/pB^{n+1}.$$

With this notation, it is easily seen that there exists a well defined morphism of $W_{n+1}B$-algebras

$$\varphi : B_0 \to B_0^{n+1} \qquad \text{such that} \qquad \pi_i \circ \varphi = \Phi_{B_0}^i : B_0 \to \overline{\boldsymbol{\omega}}_i^* B_0 = \Phi_{B_0}^{i*} B_0 \qquad \text{for } i = 0, \dots, n$$

where $\pi_i : B_0^{n+1} \to \overline{\boldsymbol{\omega}}_i^* B_0$ is the projection on the $(i+1)$-th factor. Arguing as in the proof of proposition 14.7.11, we get cartesian diagrams

$$\begin{array}{ccc} S & \longrightarrow & B^{n+1} \\ \downarrow & & \downarrow \\ B_0 & \xrightarrow{\varphi} & B_0^{n+1} \end{array} \qquad \begin{array}{ccc} R & \longrightarrow & R_0 \\ \downarrow & & \downarrow \\ S & \longrightarrow & S_0 := S/p^{n+1}B^{n+1}. \end{array}$$

In case $\mathbf{P}$ is "almost finitely generated" or "almost finitely presented" we may then apply [75, Rem.3.2.26(i) and Lemma 3.4.18(i)] to the second of these cartesian diagrams, and thereby reduce to checking property $\mathbf{P}$ for the $(S \times R_0)$-module $W_{n+1}C \otimes_{W_{n+1}B} (S \times R_0)$. In case $\mathbf{P}$ is the property "almost projective", we may argue as in the proof of [75, Prop.3.4.21] to achieve the same reduction. But notice as well that the restriction $R_0 \to \overline{\boldsymbol{\omega}}_0^* B_0$ of π_0 is an epimorphism (on the underlying modules), and has nilpotent kernel (see the proof of proposition 14.7.11), so the R_0-module $W_{n+1}C \otimes_{W_{n+1}B} R_0$ enjoys $\mathbf{P}$ if and only the same holds for the $\overline{\boldsymbol{\omega}}_0^* B_0$-module $W_{n+1}C \otimes_{W_{n+1}B} \overline{\boldsymbol{\omega}}_0^* B_0$. Next, by invoking again [75, Rem.3.2.26(i) and Lemma 3.4.18(i)] or the proof of [75, Prop.3.4.21] to the first of the two above diagrams we deduce that the S-module $W_{n+1}C \otimes_{W_{n+1}B} S$ enjoys $\mathbf{P}$ if and only if the same holds for the $(\overline{\boldsymbol{\omega}}_0^* B_0 \times B^{n+1})$-module $W_{n+1}C \otimes_{W_{n+1}B} (\overline{\boldsymbol{\omega}}_0^* B_0 \times B^{n+1})$. Since $\overline{\boldsymbol{\omega}}_0^* B_0$ is a quotient of B^{n+1}, we are finally reduced to checking that the B^{n+1}-module $W_{n+1}C \otimes_{W_{n+1}B} B^{n+1}$ enjoys $\mathbf{P}$; but according to theorem 14.7.26(ii), the latter is isomorphic to the B^{n+1}-module $\prod_{i=0}^n \overline{\boldsymbol{\omega}}_i^* C$, so the assertion is clear.

(iii): Suppose first that f is weakly étale, so that the multiplication morphism $\mu_{C/B} : C \otimes_B C \to C$ is flat; since it is also an epimorphism on the underlying B-modules, it follows easily that $\mu_{C/B}$ is weakly étale. Especially, $\Phi_{C/C\otimes_B C}$ is an isomorphism, by remark 14.7.25 and our assumption on $\mathfrak{m}$. Then theorem 14.7.26 says that $W_{n+1}(\mu_{C/C\otimes_B C})$ is flat. Lastly, according to (i), the multiplication morphism $\mu_{W_{n+1}C/W_{n+1}B}$ of $W_{n+1}C$ factors through $W_{n+1}(\mu_{C/C\otimes_B C})$ and an isomorphism

$$W_{n+1}C \otimes_{W_{n+1}B} W_{n+1}C \xrightarrow{\sim} W_{n+1}(C \otimes_B C). \tag{14.7.35}$$

Hence $\mu_{W_{n+1}C/W_{n+1}B}$ is flat, and combining with theorem 14.7.26, we conclude that $W_{n+1}f$ is weakly étale. In case f is étale, we deduce from (ii), the isomorphism (14.7.35), and [75, Prop.2.4.18] that $W_{n+1}f$ is étale as well. Conversely, (iv) implies that if $W_{n+1}f$ is weakly étale (resp. étale), then the same holds for f. □

Remark 14.7.36. (i) It is shown in [104, 1.5.8] that if $f : B \to C$ is a homomorphism of $\mathbb{F}_p$-algebras and $n \in \mathbb{N}$ any integer, then $W_{n+1}f$ is étale (in the usual sense of [66], which includes the condition that f is finitely presented), if and only if the same holds for f. This result can be deduced from corollary 14.7.34(iii) and extended to arbitrary rings, as follows. Resume the notation of the proof of the corollary, and suppose that $\mathfrak{m} = A$, so we are dealing with the "classical limit" for which almost rings are just usual rings; then, $W_{n+1}f$ is a weakly étale ring homomorphism, and by [75, §3.4.44] we know already that $W_{n+1}f$ is finitely presented if and only if the same holds for $W_{n+1}f \otimes_{W_{n+1}B} R$. We claim next that $W_{n+1}f \otimes_{W_{n+1}B} R$ is finitely presented if and only if the same holds for $W_{n+1}f \otimes_{W_{n+1}B} (R_0 \times S)$. For the proof, arguing as in [75, §3.4.44] we reduce to checking that the natural morphism $R \to R_0 \times S$ is of universal effective descent for the fibred category of weakly étale morphisms of rings. The latter assertion is already known by corollary 14.1.92. By the same token, $W_{n+1}f \otimes_{W_{n+1}B} S$ is finitely presented if and only if the same holds for $W_{n+1}f \otimes_{W_{n+1}B} (B^{n+1} \times B_0)$. Moreover, the surjective ring homomorphism $R_0 \to B_0$ has nilpotent kernel, so again by [75, §3.4.44] we know that $W_{n+1}f \otimes_{W_{n+1}B} R_0$ is finitely presented if and only if the same holds for $W_{n+1}f \otimes_{W_{n+1}B} B_0$. Since B_0 is a quotient of B^{n+1}, we are finally reduced to checking that $W_{n+1}f \otimes_{W_{n+1}B} B^{n+1}$ is finitely presented; but it was observed in the proof of corollary 14.7.34(ii) that the latter is isomorphic to the B^{n+1}-module $\prod_{i=0}^n \overline{\boldsymbol{\omega}}_i^* C$, so the assertion is clear.

(ii) A version of corollary 14.7.34(iii) for truncated big Witt vectors of usual rings is found in [109, Th.2.4]. Another proof and generalization is given by [31, Th.9.2].

14.8. **Complements : locally measurable algebras.** This section studies the global counterpart of the class of measurable algebras introduced in section 14.5. To begin with – and until (14.8.42) – we consider an arbitrary valued field $(K, |\cdot|)$, and we resume the notation of (11.4) and (14.5). Our first result is the following generalization of proposition 9.1.27 :

Proposition 14.8.1. *Let A be a measurable K^+-algebra, M a K^+-flat and finitely generated A-module. Then M is a finitely presented A-module.*

Proof. Let Σ be a finite system of generators for M; also let us write A as the colimit of a filtered system $(A_i \mid i \in I)$ of finitely presented K^+-algebras, with étale transition maps. For every $i \in I$, let M_i be the A_i-submodule of M generated by Σ; notice that M_i is still K^+-flat, hence it is a finitely presented A_i-module (proposition 9.1.27). We may then write M as the colimit of the filtered system $(M_i \otimes_{A_i} A \mid i \in I)$ of finitely presented A-modules, with surjective transition maps. Moreover, since $\overline{A} := A/\mathfrak{m}_K A$ is noetherian (lemma 14.5.4(i)), there exists $i \in I$ such that $M_j \otimes_{A_j} \overline{A} = M/\mathfrak{m}_K M$ for every $j \geq i$. Since the ring homomorphism $\overline{A}_j := A_j/\mathfrak{m}_K A_j \to \overline{A}$ is faithfully flat, we deduce that

$$M_i \otimes_{A_i} \overline{A}_j = M_j \otimes_{A_j} \overline{A}_j \qquad \text{for every } j \geq i. \tag{14.8.2}$$

Consider the short exact sequence

$$C \quad : \quad 0 \to N \to M_i \otimes_{A_i} A_j \to M_j \to 0.$$

Since M_i is K^+-flat, the same holds for N, and the latter is a finitely generated A_j-module, since both M_j and $M_i \otimes_{A_i} A_j$ are finitely presented; therefore, the sequence $C \otimes_{K^+} \kappa$ is still exact. Taking into account (14.8.2), we see that $N/\mathfrak{m}_K N = 0$. By Nakayama's lemma, it follows that $N = 0$, and finally, $M = M_i \otimes_{A_i} A$ is finitely presented, as stated. □

Definition 14.8.3. Let A be a K^+-algebra. Set $X := \operatorname{Spec} A$, $S := \operatorname{Spec} K^+$, and denote by $f : X \to S$ the structure morphism. We say that A is *locally measurable*, if the following holds. For every $x \in X$ and every point ξ of X localized at x, the strict henselization of A at ξ is a measurable $\mathscr{O}_{S,f(x)}$-algebra.

Remark 14.8.4. Let A be a local K^+-algebra, A^{sh} the strict henselization of A at a geometric point localized at the closed point, and M an A-module.

(i) Clearly, A is locally measurable if and only if A^{sh} is measurable.

(ii) However, if A^{sh} is measurable, it does not necessarily follow that A is measurable.

(iii) On the other hand, if A is a normal local domain, one can show that A is measurable if and only if the same holds for A^{sh}.

(iv) Suppose that A is local and locally measurable. Since the natural map $A \to A^{\mathrm{sh}}$ is faithfully flat, and since every measurable K^+-algebra is a coherent ring, it is easily seen that A is coherent. If furthermore, the structure map $K^+ \to A$ is local, lemma 14.5.4(i) implies that $A/\mathfrak{m}_K A$ is a noetherian ring.

Remark 14.8.5. Let A be a locally measurable K^+-algebra. Then, for every finitely generated ideal $I \subset A$, the K^+-algebra A/I is also locally measurable. Indeed, for every geometric point ξ of $\operatorname{Spec} A$, let A^{sh}_ξ (resp. $(A/I)^{\mathrm{sh}}_\xi$) be the strict henselization of A (resp. of A/I) at ξ. Then the natural map $A^{\mathrm{sh}}_\xi / I A^{\mathrm{sh}}_\xi \to (A/I)^{\mathrm{sh}}_\xi$ is an isomorphism for every such ξ ([66, Ch.IV, Prop.18.8.10]), so the assertion follows from lemma 14.5.4(iv).

Definition 14.8.6. Let A be a K^+-algebra, M an A-module, and $\gamma \in \log \Gamma^+$ any element.

(i) A *K^+-flattening sequence* for the A-module M is a finite sequence $\underline{b} := (b_0, \dots, b_n)$ of elements of K^+ such that
   (a) $\log |b_{i+1}| > \log |b_i|$ for every $i = 0, \dots, n-1$, $b_0 = 1$ and $b_n = 0$.
   (b) $b_i M / b_{i+1} M$ is a $K^+/b_i^{-1} b_{i+1} K^+$-flat module, for every $i = 0, \dots, n-1$.

   We say that a K^+-flattening sequence $\underline{b}$ for M is *minimal*, if no proper subsequence of $\underline{b}$ is still flattening for M.

(ii) Say that $\gamma = \log |b|$ for some $b \in K^+$, and let $b_M : M \to bM$ be the map given by the rule : $m \mapsto bm$, for every $m \in M$. We say that γ *breaks* M if the κ-linear map

$$b_M \otimes_{K^+} \kappa : M \otimes_{K^+} \kappa \to bM \otimes_{K^+} \kappa$$

is not an isomorphism.

Remark 14.8.7. Let A be a K^+-algebra, M an A-module.

(i) Suppose that $\gamma \in \log \Gamma^+$ breaks M, and say that $\gamma = \log |c|$ for some $c \in K^+$. Clearly, for every $b \in cK^+$, the map b_M factors through c_M. We deduce that every $\gamma' \in \log \Gamma^+$ with $\gamma' \geq \gamma$ also breaks M.

(ii) For given $b \in K^+$, suppose that M is a flat K^+/bK^+-module. Then we claim that no $\gamma < \log |b|$ breaks M. Indeed, for such γ pick $c \in K^+$ with $\log |c| = \gamma$, and set $W := K^+/bK^+$; notice that the map $c_W : W \to cW$ induces an isomorphism $c_W \otimes_W \mathbf{1}_\kappa : \kappa \xrightarrow{\sim} cW \otimes_W \kappa$ (notation of definition 14.8.6(ii)), whence an isomorphism

$$\mathbf{1}_M \otimes_W c_W \otimes_W \mathbf{1}_\kappa : M \otimes_W \kappa \xrightarrow{\sim} M \otimes_W (cW \otimes_W \kappa).$$

Since M is a flat W-module, the natural map $M \otimes_W cW \to cM$ is an isomorphism, and the resulting isomorphism $M \otimes_W \kappa \xrightarrow{\sim} cM \otimes_W \kappa$ is naturally identified with c_M, whence the contention.

(iii) Let $\underline{b} := (b_0, \dots, b_n)$ be a sequence of elements of K^+ fulfilling condition (a) of definition 14.8.6(i), and suppose that $\underline{b}$ admits a subsequence that is K^+-flattening for M. Then $\underline{b}$ is K^+-flattening for M as well. Indeed, an easy induction reduces the contention to the following. Let $b, c \in K^+$ be any two elements such that $\log |c| < \log |b|$, and suppose that M is K^+/bK^+-flat; then M/cM is K^+/cK^+-flat, and cM is $K^+/c^{-1}bK^+$-flat. Of this two assertion, the first is trivial; to show the second, recall that M can be written as the colimit of a filtered system of free K^+/bK^+-modules ([120, Ch.I, Th.1.2]). Then we may assume that M is free, in which case the assertion is easily verified.

Lemma 14.8.8. *Let A be a K^+-algebra, M a coherent A-module that admits a K^+-flattening sequence, and suppose that the Jacobson radical of A contains $\mathfrak{m}_K A$. Then we have :*

(i) *M admits a minimal K^+-flattening sequence, unique up to units of K^+.*

(ii) *Let $(b_0, \dots, b_n)$ be a minimal K^+-flattening sequence for M. For every $\gamma \in \log \Gamma^+$ and every $i = 0, \dots, n-1$, the following conditions are equivalent :*

(a) *γ breaks $b_i M$.*

(b) *$\gamma \geq \log |b_i^{-1} b_{i+1}|$. (As usual, we set $\log |0| := +\infty$: see (14.5).)*

Proof. It is clear that M admits a minimal K^+-flattening sequence $(1, b_1, \dots, b_{n-1}, 0)$, and (i) asserts that every other such minimal sequence is of the type $(1, u_1 b_1, \dots, u_{n-1} b_{n-1}, 0)$ for some elements $u_1, \dots, u_{n-1} \in (K^+)^\times$. However, notice that the condition of (ii) characterize uniquely such a sequence, so we only have to show that (ii) holds. We may also assume that $n > 1$; indeed, when $n = 1$, the module M is K^+-flat, and the assertion is immediate. Moreover, since M is coherent, the same holds for bM, for every $b \in K^+$. Thus, an easy induction reduces to showing the equivalence of conditions (ii.a) and (ii.b) for $i = 0$.

However, notice that if γ breaks M, then it obviously also breaks $M/b_1 M$; in light of remark 14.8.7(ii), it follows already that (a)$\Rightarrow$(b). Therefore, taking into account remark 14.8.7(i), it remains only to show that $\log |b_1|$ breaks M.

If $n = 2$, then $b_1 M$ is a flat K^+-module; let $N := \operatorname{Ker} b_{1,M}$. We have already observed that $b_1 M$ is a finitely presented A-module, hence N is a finitely generated A-module, and $N \otimes_{K^+} \kappa$ is the kernel of $b_{1,M} \otimes_{K^+} \kappa$. Suppose that $\log |b_1|$ does not break M; then $N \otimes_{K^+} \kappa$ vanishes, and then $N = 0$, by Nakayama's lemma. It follows that M is K^+-flat, which contradicts the minimality of the sequence $(1, b_1, 0)$.

Next, we consider the case where $n > 2$ and set $M' := M/b_2 M$, $b := b_1^{-1} b_2$; since $b_2 M \subset b_1 \mathfrak{m} M$, it suffices to show that $\log |b_1|$ breaks M', hence we may assume that $b_2 M = 0$, $b_1 M$ is a flat K^+/bK^+-module, and the flattening sequence is $(1, b_1, b_2, 0)$. We remark :

Claim 14.8.9. If $\log |b_1|$ does not break M, the map $b_{1,M/bM} : M/bM \to b_1 M$ is an isomorphism of K^+/bK^+-modules.

Proof of the claim. Indeed, let $N := \mathrm{Ker}\, b_{1,M/bM}$; since b_1M is K^+/bK^+-flat, $N \otimes_{K^+} \kappa$ is the kernel of $b_{1,M/bM} \otimes_{K^+} \kappa$, and the latter vanishes if $\log |b_1|$ does not break M; on the other hand, N is a finitely generated A-module, hence $N = 0$, by Nakayama's lemma. $\diamondsuit$

We shall use the following variant of the local flatness criterion :

Claim 14.8.10. Let R be a ring, $I_1, I_2 \subset A$ two ideals, and M an R-module such that :

(a) $I_1 I_2 = 0$
(b) M/I_iM is R/I_i-flat for $i = 1, 2$
(c) the natural map $I_1 \otimes_R M/I_2M \to I_1M$ is an isomorphism.

Then M is a flat R-module.

Proof of the claim. Set $I_3 := I_1 \cap I_2$; to begin with, [75, Lemma 3.4.18] and (b) imply that M/I_3M is a flat R/I_3-module. We have an obviously commutative diagram

$$\begin{array}{ccccc} I_3 \otimes_R M/I_3M & & \xrightarrow{\quad\alpha\quad} & & I_3M \\ \downarrow{\scriptstyle\beta} & & & & \downarrow{\scriptstyle\gamma} \\ I_3 \otimes_{R/I_2} M/I_2M & \xrightarrow{\delta} & I_1 \otimes_{R/I_2} M/I_2M & \xrightarrow{\tau} & I_1M \end{array}$$

of R-linear maps. From (a) we see that $I_3^2 = 0$, and it follows easily that β is an isomorphism; γ is clearly injective, and the same holds for δ, since M/I_2M is a flat R/I_2-module; by the same token, τ is an isomorphism. We conclude that α is an isomorphism, and then the local flatness criterion ([126, Th.22.3]) yields the contention. $\diamondsuit$

We shall apply claim 14.8.10 with $R := K^+/b_2K^+$, $I_1 := b_1R$, $I_2 := bR$. Indeed, condition (a) obviously holds with these choices; by assumption, M/I_1M is a R/I_1-flat module, and if $\log |b_1|$ does not break M, claim 14.8.9 implies that M/I_2M is a flat R/I_2-module. Lastly, condition (c) is equivalent to claim 14.8.9. Summing up, we have shown that if $\log |b_1|$ does not break M, then M is a flat K^+/b_2-module, contradicting again the minimality of our flattening sequence. $\square$

Proposition 14.8.11. *Let A be a K^+-algebra, M a finitely presented A-module, and suppose that either one of the following two conditions holds :*

(a) *A is an essentially finitely presented K^+-algebra.*
(b) *A is a local and locally measurable K^+-algebra.*

Then M admits a K^+-flattening sequence.

Proof. Suppose first that (b) holds, and let A^{sh} be the strict henselization of A at a geometric point localized at the closed point. Since A^{sh} is a faithfully flat A-algebra, it suffices to show that $M \otimes_A A^{\mathrm{sh}}$ admits a K^+-flattening sequence. Hence, we may assume that A is measurable, in which case we may find a finitely presented K^+-algebra A_0 with an ind-étale map $A_0 \to A$ and a finitely presented A_0-module M_0 with an isomorphism $M_0 \otimes_{A_0} A \xrightarrow{\sim} M$ of A-modules. We may then replace A by A_0, M by M_0, and therefore assume that A is finitely presented over K^+, especially, we are reduced to showing the assertion in the case where (a) holds. To this aim, let us remark :

Claim 14.8.12. Let $B := \bigoplus_{n \in \mathbb{N}} B_n$ be a $\mathbb{N}$-graded finitely presented K^+-algebra with $B_0 = K^+$, and $N := \bigoplus_{n \in \mathbb{N}} N_n$ a $\mathbb{N}$-graded finitely presented B-module. We have :

(i) The K^+-module N_n is finitely presented, for every $n \in \mathbb{N}$.
(ii) For every $n \in \mathbb{N}$, let $(\gamma_{n,i} \mid i \in \mathbb{N})$ be the sequence of elementary divisors of N_n (see (14.5.15)). Then $\Gamma(N) := \{\gamma_{n,i} \mid n, i \in \mathbb{N}\}$ is a finite set.
(iii) N admits a K^+-flattening sequence.

Proof of the claim. (i) is just a special case of proposition 7.6.11(iii).

(ii): Let $\mathscr{Q}$ be the set of all finitely presented graded quotients Q of the B-modules N, for which $\Gamma(Q)$ is infinite. We have to show that $\mathscr{Q} = \varnothing$. However, for every $Q \in \mathscr{Q}$, the $B \otimes_{K^+} \kappa$-module $\overline{Q} := Q \otimes_{K^+} \kappa$ is a quotient of $\overline{N} := N \otimes_{K^+} \kappa$; since $B \otimes_{K^+} \kappa$ is a noetherian ring, the set $\overline{\mathscr{Q}} := \{\overline{Q} \mid Q \in \mathscr{Q}\}$ admits minimal elements, if it is not empty. In the latter case, we may then replace N by any $Q_0 \in \mathscr{Q}$ such that $\overline{Q}_0$ is a minimal element of $\overline{\mathscr{Q}}$, and assume that, for every graded quotient Q of N, either $\overline{Q} = \overline{N}$, or else $\Gamma(Q)$ is a finite set. Now, for every $n \in \mathbb{N}$, let γ_n be the minimal non-zero elementary divisor of N_n, and pick $a_n \in \mathfrak{m}_K$ with $\log|a_n| = \gamma_n$ (if $N_n = 0$, set $a_n = 0$). Let $I \subset K^+$ be the ideal generated by $\{a_n \mid n \in \mathbb{N}\}$; it is easily seen that N_n/IN_n is a free K^+/I-module for every $n \in \mathbb{N}$, hence N/IN is a K^+/I-flat finitely presented B/IB-module. We may then find $a \in I$ such that N/aN is already a K^+/aK^+-flat B/aB-module ([65, Ch.IV, Cor.11.2.6.1]), so N_n/aN_n is a free K^+/aK^+-module for every $n \in \mathbb{N}$, and we easily deduce that $\log|a| \leq \gamma_n$ for every $n \in \mathbb{N}$, *i.e.* $I = aK^+$. Notice that aN is a finitely presented B-module (corollary 9.1.28); it follows that

$$\Gamma(N) = \{\gamma + \log|a| \mid \gamma \in \Gamma(aN)\} \cup \{0\}.$$

Especially, $\Gamma(aN)$ is an infinite set. On the other hand, there exists some $n \in \mathbb{N}$ such that $\dim_\kappa(aN_n) \otimes_{K^+} \kappa < \dim_\kappa N_n \otimes_{K^+} \kappa$, so $(aN) \otimes_{K^+} \kappa$ is a proper quotient of $\overline{N}$, a contradiction.

(iii): Let $|b_0|, \dots, |b_{n-1}|$ be the finitely many elements of $\Gamma(N)$ (for suitable $b_1, \dots, b_n \in K^+$), and set $b_n := 0$; after permutation, we may assume that $b_{i+1} \in b_i\mathfrak{m}_K$ for every $i = 0, \dots, n-1$, and $b_0 = 1$. Then we claim that $(b_0, \dots, b_n)$ is a K^+-flattening sequence for N, *i.e.* $b_iN_k/b_{i+1}N_k$ is $K^+/b_i^{-1}b_{i+1}K^+$-flat for every $i = 0, \dots, n$ and every $k \in \mathbb{N}$. However, say that $(\log|c_j| \mid j \in \mathbb{N})$ is the sequence of elementary divisors of N_k; we are reduced to checking that $(c_jK^+ + b_iK^+)/(c_jK^+ + b_{i+1}K^+)$ is a flat $K^+/b_i^{-1}b_{i+1}K^+$-module for every $j \in \mathbb{N}$. However, by construction we have either $c_jK^+ \subset b_{i+1}K^+$, or $b_iK^+ \subset c_jK^+$; in either case the assertion is clear. ◊

Let now A be an arbitrary essentially finitely presented K^+-algebra, and M an arbitrary finitely presented A-module. We easily reduce to the case where $A = K^+[T_1, , \dots, T_r]$ is a free polynomial K^+-algebra. In this case, we define a filtration $\mathrm{Fil}_\bullet A$ on A, by declaring that $\mathrm{Fil}_k A$ is the K^+-submodule of all polynomials of total degree $\leq k$, for every $k \in \mathbb{N}$; then $R := \mathsf{R}(A, \mathrm{Fil}_\bullet A)_\bullet$ is a free polynomial K^+-algebra as well (see example 7.9.5). Let

$$L_1 \xrightarrow{\varphi} L_0 \to M$$

be a presentation of M as quotient of free A-modules of finite rank. Let

$$\mathbf{e} := (e_1, \dots, e_n) \qquad (\text{resp. } \mathbf{f} := (f_1, \dots, f_m))$$

be a basis of L_0 (resp. of L_1); we endow L_0 with the good $(A, \mathrm{Fil}_\bullet A)$-filtration $\mathrm{Fil}_\bullet L_0$ associated with the pair $(\mathbf{e}, (1, \dots, 1))$ as in (7.9.6) (this means that $e_i \in \mathrm{Fil}_1 L_0$ for every $i = 1, \dots, n$). Also, for every $i = 1, \dots, m$, pick $j_i \in \mathbb{N}$ such that $\varphi(f_i) \in \mathrm{Fil}_{j_i} L_0$, and endow L_1 with the good $(A, \mathrm{Fil}_\bullet A)$-filtration $\mathrm{Fil}_\bullet L_1$ associated with the pair $(\mathbf{f}, (j_1, \dots, j_m))$. Set $L'_i := \mathsf{R}(L_i, \mathrm{Fil}_\bullet L_i)$, and notice that L'_i is a free R-module of finite rank, for $i = 0, 1$. With these choices, φ is a map of filtered A-modules, and there follows an R-linear map of $\mathbb{N}$-graded R-modules $\mathsf{R}(\varphi)_\bullet : L'_1 \to L'_0$, whose cokernel is a $\mathbb{N}$-graded finitely presented R-module $N_\bullet$. By inspecting the construction, it is easily seen that the inclusion $\mathrm{Fil}_k L_0 \subset \mathrm{Fil}_{k+1} L_0$ induces a K^+-linear map $N_k \to N_{k+1}$, for every $k \in \mathbb{N}$, as well as an isomorphism of K^+-modules

$$\operatorname*{colim}_{k \in \mathbb{N}} N_k \xrightarrow{\sim} M.$$

On the other hand, claim 14.8.12 ensures that $N_\bullet$ admits a K^+-flattening sequence. We easily deduce that the same sequence is also flattening for M. □

Corollary 14.8.13. *Let A be a local and locally measurable K^+-algebra, $I \subset K^+$ any ideal, M a finitely generated A/IA-module, and suppose that the structure map $K^+ \to A$ is local. Then the following conditions are equivalent :*

(a) *M is a flat K^+/I-module.*
(b) *For every $c \in K^+$ such that $I \subset c \cdot \mathfrak{m}_K$, the value $\log |c|$ does not break M.*
(c) *M is a K^+/I-flat finitely presented A/IA-module.*

Proof. (a)⇒(b): in light of remark 14.8.7(ii), it suffices to remark that $M \otimes_{K^+} /bK^+$ is a flat K^+/bK^+-module, for every $b \in K^+$ such that $I \subset K^+b$.

(b)⇒(c): Let us write M as the colimit of a filtered system $(M_\lambda \mid \lambda \in \Lambda)$ of finitely presented A-modules, with surjective transition maps. Then each M_λ is a coherent A-module, and $A/\mathfrak{m}_K A$ is noetherian (remark 14.8.4(iv)), so we may also assume that the induced maps $\varphi_\lambda : M_\lambda/\mathfrak{m}_K M_\lambda \to M/\mathfrak{m}_K M$ are isomorphisms for every $\lambda \in \Lambda$. In view of the commutative diagram

$$\begin{array}{ccc} M_\lambda \otimes_{K^+} \kappa & \xrightarrow{c_{M_\lambda}} & cM_\lambda \otimes_{K^+} \kappa \\ {\scriptstyle \varphi_\lambda}\downarrow & & \downarrow \\ M \otimes_{K^+} \kappa & \xrightarrow{c_M} & cM \otimes_{K^+} \kappa \end{array}$$

it easily follows that $\log |c|$ does not break M_λ, for any $\lambda \in \Lambda$ and any $c \in K^+$ such that $I \subset c{\cdot}\mathfrak{m}_K$. Thus, if $(b_0, \dots, b_n)$ is the minimal flattening sequence for M_λ, we see that $b_1 \in I$ (lemma 14.8.8(ii)), and therefore M_λ/IM_λ is a flat K^+/I-module for every $\lambda \in \Lambda$. Now, for $\lambda, \mu \in \Lambda$ with $\mu \geq \lambda$, let $\psi_{\lambda\mu} : M_\lambda \to M_\mu$ be the transition map, and set $N_{\lambda\mu} := \mathrm{Ker}(\psi_{\lambda\mu} \otimes_{K^+} K^+/I)$; it follows that $N_{\lambda\mu} \otimes_{K^+} \kappa$ is the kernel of $\psi_{\lambda\mu} \otimes_{K^+} \kappa$. But the latter map is an isomorphism, since the same holds for φ_λ and φ_μ. By Nakayama's lemma, we deduce that $N_{\lambda\mu} = 0$, therefore $M = M_\lambda/IM_\lambda$ for any $\lambda \in \Lambda$, whence (c).

Lastly, (c)⇒(a) is obvious. □

Proposition 14.8.14. *Let A be a locally measurable K^+-algebra. Suppose that*

(a) *$A/\mathfrak{m}_K A$ is a noetherian ring.*
(b) *The Jacobson radical of A contains $\mathfrak{m}_K A$.*

Then we have :

(i) *$A \otimes_{K^+} K$ is a noetherian ring.*
(ii) *Every finitely generated K^+-flat A-module is finitely presented.*
(iii) *If the valuation of K is discrete, A is noetherian.*

Proof. For any K^+-algebra B, and any ideal $I \subset B_K := B \otimes_{K^+} K$, let us set $I^{\mathrm{sat}} := \mathrm{Ker}(B \to B_K/I)$, and notice that $I^{\mathrm{sat}} \otimes_{K^=} K = I$. To begin with, we remark :

Claim 14.8.15. Let B be a measurable K^+-algebra, $I \subset B_K$ an ideal. Then I^{sat} is a finitely generated ideal of B.

Proof of the claim. Clearly B/I^{sat} is a K^+-flat finitely generated B-module, hence it is finitely presented, by proposition 14.8.1; now the claim follows from [75, Lemma 2.3.18(ii)]. ◊

(i): Suppose $(I_k \mid k \in \mathbb{N})$ is an increasing sequence of ideals of $A \otimes_{K^+} K$; assumption (a) implies that there exists $n \in \mathbb{N}$ such that the images of I_n^{sat} and I_m^{sat} agree in $A/\mathfrak{m}_K A$, for every $m \geq n$. This means that

$$(I_m^{\mathrm{sat}}/I_n^{\mathrm{sat}}) \otimes_{K^+} \kappa = 0 \qquad \text{for every } m \geq n. \tag{14.8.16}$$

Next, for any geometric point ξ of $\mathrm{Spec}\, A$, let A_ξ^{sh} be the strict henselization of A at ξ; since the natural map $A \to A_\xi^{\mathrm{sh}}$ is flat, $I_k \otimes_A A_\xi^{\mathrm{sh}}$ is an ideal of $A_\xi^{\mathrm{sh}} \otimes_{K^+} K$, and clearly $I_k^{\mathrm{sat}} \otimes_A A_\xi^{\mathrm{sh}} = (I_k \otimes_A A_\xi^{\mathrm{sh}})^{\mathrm{sat}}$, for every $k \in \mathbb{N}$. Especially $I_k^{\mathrm{sat}} \otimes_A A_\xi^{\mathrm{sh}}$ is a finitely generated ideal of A_ξ^{sh}, by claim 14.8.15. In view of (14.8.16), and assumption (b), Nakayama's lemma then says that $(I_m^{\rm sat}/I_n^{\rm sat}) \otimes_A A_\xi^{\rm sh} = 0$ for every $m \geq n$. Since ξ is arbitrary, we conclude that $I_m^{\rm sat}/I_n^{\rm sat} = 0$ for every $m \geq n$, therefore the sequence $(I_k \mid k \in \mathbb{N})$ is stationary.

(ii): Let M be a K^+-flat and finitely generated A-module, pick an A-linear surjection $\varphi : A^{\oplus n} \to M$, and set $N := \operatorname{Ker} \varphi$. Since M is K^+-flat, $N/\mathfrak{m}_K N$ is the kernel of $\varphi \otimes_{K^+} \kappa$, and assumption (a) implies that $N/\mathfrak{m}_K N$ is a finitely generated $A/\mathfrak{m}_K A$-module. Hence we may find a finitely generated A-submodule $N' \subset N$ such that $N = N' + \mathfrak{m}_K N$. For any geometric point ξ of $\operatorname{Spec} A$, define $A_\xi^{\rm sh}$ as in the foregoing; by proposition 14.8.1 (and by [75, Lemma 2.3.18(ii)]), $N \otimes_A A_\xi^{\rm sh}$ is a finitely generated $A_\xi^{\rm sh}$-module; then (b) and Nakayama's lemma imply that $N' \otimes_A A_\xi^{\rm sh} = N \otimes_A A_\xi^{\rm sh}$. Since ξ is arbitrary, it follows that $N = N'$; especially, M is finitely presented, as stated.

(iii): It suffices to show that every prime ideal of A is finitely generated ([126, Th.3.4]). However, let $\mathfrak{p} \subset A$ be such a prime ideal, and fix a generator t of $\mathfrak{m}_K$. Suppose first that $t \in \mathfrak{p}$; in that case (a) implies that $\mathfrak{p}/tA$ is a finitely generated ideal of A/tA, so then clearly $\mathfrak{p}$ is finitely generated as well. Next, in case $t \notin \mathfrak{p}$, the quotient $A/\mathfrak{p}$ is a K^+-flat finitely generated A-module, hence it is finitely presented (proposition 14.8.1), so again $\mathfrak{p}$ is finitely generated ([75, Lemma 2.3.18(ii)]). □

Theorem 14.8.17. *Let A be a locally measurable K^+-algebra, M a finitely presented A-module, and suppose that :*

(a) *The valuation of K has finite rank.*
(b) *For every $t \in \operatorname{Spec} K^+$, the ring $A \otimes_{K^+} \kappa(t)$ is noetherian.*

Then we have :

(i) *M admits a K^+-flattening sequence.*
(ii) *$\operatorname{Ass} M$ is a finite set.*

Proof. (i): Set $X := \operatorname{Spec} A$, $S := \operatorname{Spec} K^+$, let $f : X \to S$ be the structure morphism, and denote by $\mathscr{M}$ the quasi-coherent $\mathscr{O}_X$-module associated with M. Also, for every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ and every $c \in K^+$, let $c_\mathscr{F} : \mathscr{F} \to c\mathscr{F}$ be the unique $\mathscr{O}_X$-linear morphism such that $c_\mathscr{F}(U) = c_{\mathscr{F}(U)}$ for every affine open subset $U \subset X$ (notation of definition 14.8.6(ii)). For every $x \in X$, set $K^+_{f(x)} := \mathscr{O}_{S,f(x)}$; then $K^+_{f(x)}$ is a valuation ring whose valuation we denote $|\cdot|_{f(x)}$, and $\mathscr{O}_{X,x}$ is a locally K^+_x-measurable algebra. By proposition 14.8.11, the $\mathscr{O}_{X,x}$-module $\mathscr{M}_x$ admits a $K^+_{f(x)}$-flattening sequence $(b_{x,0}, \dots, b_{x,n})$; we wish to show that there exists an open neighborhood $U(x)$ of x in X, such that $(b_{x,i}\mathscr{M}/b_{x,i+1}\mathscr{M})_y$ is a $K^+/b_{x,i}^{-1}b_{x,i+1}K^+$-flat module, for every $y \in U(x)$ and every $i = 0, \dots, n-1$. To this aim, we remark :

Claim 14.8.18. Let $\mathscr{N}$ be a quasi-coherent $\mathscr{O}_X$-module of finite type, $x \in X$ a point, and $b \in K^+$ such that $\mathscr{N}_x$ is a flat K^+/bK^+-module. Then there exists an open neighborhood $U \subset X$ of x such that the map

$$c_{\mathscr{N},y} \otimes_{K^+} \kappa(f(y)) : \mathscr{N}_y \otimes_{K^+} \kappa(f(y)) \to c\mathscr{N}_y \otimes_{K^+} \kappa(f(y))$$

is an isomorphism for every $y \in U$ and every $c \in K^+$ with $\log |c|_{f(y)} < \log |b|_{f(y)}$.

Proof of the claim. Let $z \in X(x)$ be any point, and set $t := f(z)$; under our assumptions, $\mathscr{N}_z$ is a flat K^+/bK^+-module. Therefore, the sequence of $\mathscr{O}_{X,z}$-modules

$$0 \to c^{-1}b\mathscr{N}_z \xrightarrow{j} \mathscr{N}_z \xrightarrow{c_{\mathscr{N},z}} c\mathscr{N}_z \to 0$$

is exact, for every $c \in K^+$ with $\log |c| < \log |b|$, and *a fortiori*, whenever $\log |c|_t < \log |b|_t$. If the latter inequality holds, the map $j \otimes_{K^+} \kappa(t)$ vanishes, hence $c_{\mathscr{N},z} \otimes_{K^+} \kappa(t)$ is injective.

Let now $y \in X$ be any point; set $u := f(y)$, pick $c \in K^+$ with $\log |c|_u < \log |b|_u$, set $\mathscr{K}(u) := \operatorname{Ker}(c_\mathscr{N} \otimes_{K^+} \kappa(u))$, and suppose that $\mathscr{K}(u)_y \neq 0$. The foregoing shows that in this case $y \notin X(x)$. On the other hand, by assumption $f^{-1}(u)$ is a noetherian scheme, and $\mathscr{N}(u) := \mathscr{N} \otimes_{K^+} \kappa(u)$ is a coherent $\mathscr{O}_{f^{-1}(u)}$-module, therefore $\mathrm{Ass}\,\mathscr{N}(u)$ is a finite set ([126, Th.6.5(i)]). In view of proposition 10.5.6(ii), we conclude that $\mathrm{Ass}\,\mathscr{K}(u)$ is contained in the finite set $\mathrm{Ass}\,\mathscr{N}(u)\backslash X(x)$. Let Z be the topological closure in X of the (finite) set $\bigcup_{u\in S} \mathrm{Ass}\,\mathscr{N}(u)\backslash X(x)$; taking into account lemma 10.5.5(ii,iii), we see that $U := X \setminus Z$ will do. $\Diamond$

Fix $x \in X$ and $i \leq n$; taking $\mathscr{N} := b_{x,i}\mathscr{M}/b_{x,i+1}\mathscr{M}$ and $b := b_{x,i}^{-1}b_{x,i+1}$ in claim 14.8.18, and invoking the criterion of corollary 14.8.13, we obtain an open neighborhood U_i of x in X such that $\mathscr{N}_y$ is K^+/bK^+-flat for every $y \in U_i$. Clearly the subset $U(x) := U_1 \cap \cdots \cap U_n$ fulfills the sought condition. Next, pick finitely many points $x_1, \dots, x_k \in X$ and corresponding K^+-flattening sequences $\underline{b}_i$ for $\mathscr{M}_{x_i}$, for each $i = 1, \dots, k$, such that $U(x_1) \cup \cdots \cup U(x_k) = X$; after reordering, the sequence $(\underline{b}_1, \dots, \underline{b}_k)$ becomes K^+-flattening for M (see remark 14.8.7(iii)).

(ii): Let $(b_0, \dots, b_n)$ be a K^+-flattening sequence for M; in view of proposition 10.5.6(ii), it suffices to prove that $\mathrm{Ass}\, b_iM/b_{i+1}M$ is a finite set, for every $i = 0, \dots, n-1$. Taking into account remark 14.8.4(iv), we are then reduced to showing

Claim 14.8.19. Let $\mathscr{N}$ be a quasi-coherent $\mathscr{O}_X$-module, and $b \in K^+$ any element such that $\mathscr{N}_x$ is a K^+/bK^+-flat and finitely presented $\mathscr{O}_{X,x}$-module for every $x \in X$. Then $\mathrm{Ass}\,\mathscr{N}$ is finite.

Proof of the claim. For given $x \in X$ and any geometric point ξ of X localized at x, let A^{sh} be the strict henselization of A at ξ, and B a local and essentially finitely presented K^+-algebra with a local and ind-étale map $B \to A^{\mathrm{sh}}$. We may assume that $\mathscr{N}_x \otimes_A A^{\mathrm{sh}}$ descends to a finitely presented K^+/bK^+-flat B-module N_B ([65, Ch.IV, Cor.11.2.6.1(ii)]). Denote by $\overline{x}$ (resp. by x_B) the closed point of $X(\xi) = \mathrm{Spec}\, A^{\mathrm{sh}}$ (resp. of $\mathrm{Spec}\, B$); in light of corollary 10.4.37 we have

$$x \in \mathrm{Ass}\,\mathscr{N} \Leftrightarrow x \in \mathrm{Ass}_{\mathscr{O}_{X,x}}\mathscr{N}_x \Leftrightarrow \overline{x} \in \mathrm{Ass}_{A^{\mathrm{sh}}}\mathscr{N}_x \otimes_A A^{\mathrm{sh}} \Leftrightarrow x_B \in \mathrm{Ass}_B N_B. \tag{14.8.20}$$

On the other hand, it is easily seen that $\mathrm{Ass}_{K^+} K^+/bK^+$ consists of only one point, namely the maximal point t of $\mathrm{Spec}\, K^+/bK^+$. From corollary 10.5.9, we deduce :

$$x_B \in \mathrm{Ass}_B N_B \Leftrightarrow x_B \in \mathrm{Ass}_B N_B \otimes_{K^+} \kappa(t)$$

and by applying again repeatedly corollary 10.4.37 as in (14.8.20), we see that

$$x_B \in \mathrm{Ass}_B N_B \otimes_{K^+} \kappa(t) \Leftrightarrow x \in \mathrm{Ass}\,\mathscr{N} \otimes_{K^+} \kappa(t).$$

Summing up, we conclude that $\mathrm{Ass}\mathscr{N} = \mathrm{Ass}\mathscr{N} \otimes_{K^+} \kappa(t)$, and the latter is a finite set, by [126, Th.6.5(i)]. $\square$

Corollary 14.8.21. *Let $(K, |\cdot|)$ be a valued field, and A a locally measurable K^+-algebra fulfilling conditions* (a) *and* (b) *of theorem* 14.8.17. *Let also M be a finitely generated A-module, and $\mathfrak{p} \subset A$ any prime ideal such that $M_{\mathfrak{p}}$ is a finitely presented $A_{\mathfrak{p}}$-module. We have:*

(i) *There exists $f \in A\backslash\mathfrak{p}$ such that M_f is a finitely presented A_f-module.*
(ii) *A is coherent.*

Proof. (i): We may find a finitely presented A-module M' with an A-linear surjection $\varphi : M' \to M$ such that $\varphi_{\mathfrak{p}}$ is an isomorphism. Set $M'' := \mathrm{Ker}\,\varphi$; it follows that $\mathrm{Ass}\, M'' \subset \mathrm{Ass}\, M' \setminus \mathrm{Spec}\, A_{\mathfrak{p}}$. However, $\mathrm{Ass}\, M'$ is a finite set (theorem 14.8.17(ii)); therefore the support of $\mathrm{Ker}\,\varphi$ is contained in a closed subset of $X := \mathrm{Spec}\, A$ that does not contain $\mathfrak{p}$, and the assertion follows.

(ii): Let M be a finitely presented A-module, $M' \subset M$ a finitely generated submodule, and let $\mathscr{M}'$ be the quasi-coherent $\mathscr{O}_X$-module associated with M'. By remark 14.8.4(iv), we know that $\mathscr{M}'_x$ is a finitely presented $\mathscr{O}_{X,x}$-module for every $x \in X$; from (i), it follows that $\mathscr{M}'$ is a finitely presented $\mathscr{O}_X$-module, and the assertion follows easily. $\square$

In view of theorem 14.8.17 and corollary 14.8.21, it is interesting to have criteria ensuring that the fibres over $\mathrm{Spec}\, K$ of the spectrum of a locally measurable K^+-algebra are noetherian. We present two results in this direction. To state them, let us make first the following :

Definition 14.8.22. Let $f : X \to Y$ be a morphism of schemes.

(i) We say that f is *absolutely flat* if the following holds. For every geometric point ξ of X, the induced morphism $f_\xi : X(\xi) \to Y(f(\xi))$ is an isomorphism.
(ii) We say that f *has finite fibres* if the set $f^{-1}(y)$ is finite, for every $y \in Y$.
(iii) We say that Y *admits a geometrically unibranch stratification* if every irreducible closed subset Z of Y contains a subset $U \neq \varnothing$, open in Z, and geometrically unibranch.
(iv) We say that a ring homomorphism $\varphi : A \to B$ is *absolutely flat* if $\mathrm{Spec}\,\varphi$ is absolutely flat.

Remark 14.8.23. In practice, it is often the case that a noetherian scheme admits a geometrically unibranch stratification; for instance, this holds (essentially by definition) for the spectrum of a quasi-excellent noetherian ring (see definition 9.7.3(ii)).

Lemma 14.8.24. *Let $f : X \to Y$ be an absolutely flat morphism of schemes. Then $\mathbb{L}_{X/Y} \simeq 0$ in $\mathsf{D}(\mathscr{O}_X\text{-}\mathbf{Mod})$.*

Proof. We easily reduce to the case where both X and Y are local schemes. Let ξ be a geometric point of X localized at the closed point; then $\mathbb{L}_{X(\xi)/X} \simeq 0$ in $\mathsf{D}(\mathscr{O}_{X(\xi)}\text{-}\mathbf{Mod})$ ([75, Th.2.5.37]). Hence, by transitivity ([75, Th.2.5.33]), we are reduced to showing that $\mathbb{L}_{X(\xi)/Y} \simeq 0$ in $\mathsf{D}(\mathscr{O}_{X(\xi)}\text{-}\mathbf{Mod})$. The latter assertion assertion holds, again by virtue of [75, Th.2.5.37], since f_ξ is an isomorphism. $\square$

Proposition 14.8.25. *Let $\varphi : A \to B$ be a local and absolutely flat morphism of local rings. Assume that either one of the following two conditions holds :*

(a) *A is a normal local domain.*
(b) *B is a henselian local ring.*

Then φ is an ind-étale ring homomorphism.

Proof. Pick a geometric point ξ of $\mathrm{Spec}\, B$ localized at the closed point, and let ξ' denote the image of ξ in $\mathrm{Spec}\, A$. Let also A^{sh} (resp. B^{sh}) be the strict henselization of A at ξ' (resp. of B at ξ). Notice first that A is a normal local domain if and only if the same holds for B ([66, Ch.IV, Prop.18.8.12(i)]). In case (a) holds, let F_A (resp. F_B) be the field of fractions of A (resp. B). We remark :

Claim 14.8.26. Let R be any normal local domain, and R^{sh} the strict henselization of R at a geometric point localized at the closed point of $\mathrm{Spec}\, R$. We have :

(i) $R = R^{\mathrm{sh}} \cap \mathrm{Frac}\, R$ (where the intersection takes place in $\mathrm{Frac}\, R^{\mathrm{sh}}$).
(ii) For every field extension F of $\mathrm{Frac}\, R$ contained in $\mathrm{Frac}\, R^{\mathrm{sh}}$, the R-algebra $F \cap R^{\mathrm{sh}}$ is ind-étale.

Proof of the claim. (i): More generally, let $C \to D$ be any faithfully flat ring homomorphism, and denote by $\mathrm{Frac}\, C$ (resp. $\mathrm{Frac}\, D$) the total ring of fractions of C (resp. of D); then it is easily seen that $C = D \cap \mathrm{Frac}\, C$, where the intersection takes place in $\mathrm{Frac}\, D$: the proof shall be left as an exercise for the reader.

(ii): We easily reduce to the case where F is a finite extension of $F_R := \mathrm{Frac}\, R$. Then, let E be a finite Galois extension of F_R, contained in a fixed separable closure F^{sep} of $F_R^{\mathrm{sh}} := \mathrm{Frac}\, R^{\mathrm{sh}}$, and containing F; also, denote by R^ν (resp. R^ν_E) the integral closure of R in F^{sep} (resp. in E). Recall that there exists a unique maximal ideal $\mathfrak{p}^{\mathrm{sep}} \subset R^\nu$ lying over the maximal ideal of R, such that $R^{\mathrm{sh}} = (R^\nu)^I_{\mathfrak{p}^{\mathrm{sep}}}$, where $I \subset \mathrm{Gal}(F^{\mathrm{sep}}/F_R)$ is the inertia subgroup associated with $\mathfrak{p}^{\mathrm{sep}}$ ([141, Ch.X, §2, Th.2]); therefore, $F^{\mathrm{sh}}_R = (F^{\mathrm{sep}})^I$. On the other hand, $F = E^H$ for a subgroup $H \subset \mathrm{Gal}(E/F_R)$, and recall that the natural surjection $\mathrm{Gal}(F^{\mathrm{sep}}/F_R) \to \mathrm{Gal}(E/F_R)$ maps I onto the inertia subgroup I_E associated with the maximal ideal $\mathfrak{p}_E := \mathfrak{p}^{\mathrm{sep}} \cap E \subset R^\nu_E$. It follows that H contains I_E. Set $\mathfrak{p}_F := \mathfrak{p}_E \cap F$; then $R_F := (R^\nu_E \cap F)_{\mathfrak{p}_F}$ is a faithfully flat and essentially étale R-algebra ([141, Ch.X, §1, Th.1(1)]). Especially, R^{sh} is also a strict henselization of R_F, and notice that $\mathrm{Frac}\, R_F = F$, so the assertion follows from (i). ◇

By assumption, φ extends to an isomorphism of local domains $A^{\mathrm{sh}} \xrightarrow{\sim} B^{\mathrm{sh}}$; in view of claim 14.8.26(i), we see that $B = F_B \cap A^{\mathrm{sh}}$, and then the proposition follows from claim 14.8.26(ii).

Next, suppose assumption (b) holds. Then, φ extends uniquely to a local ring homomorphism $\varphi^{\mathrm{h}} : A^{\mathrm{h}} \to B$ from the henselization of A ([66, Ch.IV, Th.18.6.6(ii)]); clearly φ^{h} is still absolutely flat, hence we may replace A by A^{h} and assume that both A and B are henselian. Since φ is absolutely flat, the residue field extension $\kappa(A) \to \kappa(B)$ is separable and algebraic; then for every finite extension E of $\kappa(A)$ contained in $\kappa(B)$ there exists a finite étale local A-algebra A_E, unique up to isomorphism, whose residue field is E ([66, Ch.IV, Prop.18.5.15]). The natural map $\kappa(B) \to E \otimes_{\kappa(A)} \kappa(B)$ admits a well defined section, given by the multiplication in $\kappa(B)$, and the latter extends to a section s_E of the induced finite étale ring homomorphism $B \to A_E \otimes_A B$ ([66, Ch.IV, Th.18.5.11]). The composition of s_E with the natural map $A_E \to A_E \otimes_A B$ is a ring homomorphism $A_E \to B$ that extends φ. The construction is clearly compatible with inclusion of subextensions $E \subset E'$, hence let A' be the colimit of the system $(A_E \mid E \subset \kappa(B))$; summing up, φ extends to a local and absolutely flat map $A' \to B$. We may therefore replace A by A' and assume from start that $\kappa(A) = \kappa(B)$. In this case, set $B' := A^{\mathrm{sh}} \otimes_A B$, and notice that $\kappa(A^{\mathrm{sh}}) \otimes_{\kappa(A)} \kappa(B) = \kappa(A^{\mathrm{sh}})$, so that B' is a local ring, ind-étale, faithfully flat over B, and with separably closed residue field. Since B is already henselian, it follows that B' is a strict henselization of B, and therefore the induced map $A^{\mathrm{sh}} \to B'$ is an isomorphism; by faithfully flat descent, we deduce that φ is already an isomorphism, and the proof of the proposition is concluded. □

Lemma 14.8.27. *Let A be a locally measurable K^+-algebra, and set $X := \mathrm{Spec}\, A$. Let also $f : X \to Y$ be a morphism of K^+-schemes with $\Omega_{X/Y} = 0$, and Y a finitely presented K^+-scheme. Then f_ξ is a finitely presented closed immersion, for every geometric point ξ of X.*

Proof. Fix such a geometric point ξ; by definition, we may find a local K^+-scheme Z essentially finitely presented, and a pro-étale morphism $g : X(\xi) \to Z$ of K^+-schemes. We may also assume that the morphism $X(\xi) \to Y$ deduced from f factors through g, in which case the resulting morphism $h : Z \to Y$ is essentially finitely presented. Under the current assumptions, $\Omega_{X(\xi)/Y} = 0$, and then lemma 14.8.24 easily implies that $\Omega_{Z/Y}$ vanishes as well, so h is essentially unramified, and therefore it factors as the composition of a finitely presented closed immersion $Z \to Z'$ followed by an essentially étale morphism $Z' \to Y$ ([66, Ch.IV, Cor.18.4.7]). The lemma is an immediate consequence. □

Proposition 14.8.28. *Let A be a local, henselian, and locally measurable K^+-algebra, whose structure map $K^+ \to A$ is local. Then A is a measurable K^+-algebra.*

Proof. Pick a local and essentally finitely presented K^+-algebra B and a local and ind-étale map $\varphi : B \to A$ of K^+-algebras. Let ξ be a geometric point of $\mathrm{Spec}\, A$ localized at the closed point, and denote by ξ' the image of ξ in $\mathrm{Spec}\, B$; also, let A^{sh} (resp. B^{sh}) be the strict henselization of A at ξ (resp. of B at ξ'). Then $\Omega_{A/B} = 0$, hence the induced map $\varphi^{\mathrm{sh}} : B^{\mathrm{sh}} \to A^{\mathrm{sh}}$ is a finitely presented surjection (lemma 14.8.27), especially $I := \mathrm{Ker}\, \varphi^{\mathrm{sh}}$ is a finitely generated ideal, and it follows as well that the residue field extension $\kappa(B) \to \kappa(A)$ is separable and algebraic. Let B^{h} be the henselization of B, and for every field extension E of $\kappa(B)$ contained in $\kappa(A)^{\mathrm{sep}} = \kappa(B)^{\mathrm{sep}}$, let B^{h}_E denote the local ind-étale B^{h}-algebra –

determined up to isomorphism – whose residue field is E ([66, Ch.IV, Prop.18.5.15]). We may find a finite Galois extension E of $\kappa(B)$ such that I descends to a finitely generated ideal $I_E \subset B^{\mathrm{h}}_E$. For any automorphism $\sigma \in G := \mathrm{Gal}(\kappa(B)^{\mathrm{sep}}/\kappa(B))$, let $\overline{\sigma} \in G_E := \mathrm{Gal}(E/\kappa(B))$ be the image of σ. Recall that G (resp. G_E) is the group of automorphisms of the B^{h}-algebra B^{sh} (resp. B^{h}_E). With this notation, we have the identity

$$\overline{\sigma}(I_E)B^{\mathrm{sh}} = \sigma(I) = I \qquad \text{for every } \sigma \in G$$

and since B^{sh} is a faithfully flat B^{h}-algebra, it follows that I_E is invariant under the action of G_E; by Galois descent, we conclude that I descends to a finitely generated ideal $I_0 \subset B^{\mathrm{h}}$. Set $C := B^{\mathrm{h}}/I_0$, and let C^{sh} denote the strict henselization of C at (the unique lifting of) the geometric point ξ'. Since A is henselian, φ extends uniquely to a local homomorphism $\varphi^{\mathrm{h}} : B^{\mathrm{h}} \to A$ ([66, Ch.IV, Th.18.6.6(ii)]), and it is easily seen that φ^{h} factors through C. By construction, the resulting map $C \to A$ is absolutely flat, so the assertion follows from proposition 14.8.25. □

Lemma 14.8.29. *Let X be a noetherian scheme that admits a geometrically unibranch stratification, and $\mathscr{F}$ a coherent $\mathscr{O}_{X_{\mathrm{\acute{e}t}}}$-module. Then there exists a partition*

$$X = X_1 \cup \cdots \cup X_k$$

of X into finitely many disjoint irreducible locally closed subsets, such that the following holds. For every $i = 1, \dots, k$, every geometric point $\overline{x}$ of X_i, and every generization $\overline{u}$ of $\overline{x}$ in X_i, every strict specialization map

$$s_{\overline{x},\overline{u}} : \mathscr{F}_{\overline{x}} \to \mathscr{F}_{\overline{u}}$$

is injective (see (4.9.23)).

Proof. Arguing by noetherian induction, it suffices to show that every reduced and irreducible closed subscheme W of X contains a subset $U \neq \varnothing$ that is open in W, and such that every strict specialization map $s_{\overline{x},\overline{u}}$ with $\overline{x}$, $\overline{u}$ localized in U, is injective. However, W contains an open subset $U \neq \varnothing$ such that

(a) U is geometrically unibranch
(b) U is affine and irreducible
(c) $\mathscr{F}$ is normally flat along W at every point of U (see [64, Ch.IV,§6.10.1]).

Indeed, (a) holds by assumption; (b) can be easily arranged by shrinking U, since X is noetherian. Lastly, (c) follows from [64, Ch.IV, Prop.6.10.2]. We claim that such U will do. Indeed, let $i : W \to W$ be the closed immersion; set

$$\mathscr{I} := \mathrm{Ker}(\mathscr{O}_X \to i_*\mathscr{O}_W) \qquad \mathscr{R}^\bullet := \bigoplus_{n\in\mathbb{N}} \mathscr{I}^n\mathscr{F}/\mathscr{I}^{n+1}\mathscr{F}$$

and recall that condition (c) means that $\mathscr{R}^\bullet_u$ is a flat $\mathscr{O}_{W,u}$-module, for every $u \in U$. Now, for a given $s_{\overline{x},\overline{u}}$ as in the foregoing, denote by x (resp. u) the support of $\overline{x}$ (resp. of $\overline{u}$) and define descending filtrations by the rule :

$$\mathrm{Fil}^k\mathscr{F}_{\overline{x}} := \mathscr{I}^k_x \cdot \mathscr{F}_{\overline{x}} \qquad \mathrm{Fil}^k\mathscr{F}_{\overline{u}} := \mathscr{I}^k_u \cdot \mathscr{F}_{\overline{u}} \qquad \text{for every } k \in \mathbb{N}.$$

By [126, Th.8.9], both these filtrations are separated, and obviously $s_{\overline{x},\overline{u}}$ is a map of filtered modules; thus, it suffices to show that the induced maps $\mathrm{gr}^k\mathscr{F}_{\overline{x}} \to \mathrm{gr}^k\mathscr{F}_{\overline{u}}$ of associated graded modules are injective, for every $k \in \mathbb{N}$. However, notice the natural identifications :

$$\mathrm{gr}^\bullet\mathscr{F}_{\overline{x}} = \mathscr{R}^\bullet_x \otimes_{\mathscr{O}_{W,x}} \mathscr{O}_{W_{\mathrm{\acute{e}t}},\overline{x}} \qquad \mathrm{gr}^\bullet\mathscr{F}_{\overline{u}} = \mathscr{R}^\bullet_x \otimes_{\mathscr{O}_{W,x}} \mathscr{O}_{W_{\mathrm{\acute{e}t}},\overline{u}}.$$

Then, the normal flatness condition reduces to checking that the induced strict specialization map $\mathscr{O}_{W_{\mathrm{\acute{e}t}},\overline{x}} \to \mathscr{O}_{W_{\mathrm{\acute{e}t}},\overline{u}}$ is injective; the latter assertion holds by the following :

Claim 14.8.30. Let W be a reduced, irreducible scheme, $\overline{w}$ a geometric point of W, and suppose that W is unibranch at the support w of $\overline{w}$. Then, for every generization $\overline{u}$ of $\overline{w}$ in W, every strict specialization map $\mathscr{O}_{W_{\text{ét}},\overline{w}} \to \mathscr{O}_{W_{\text{ét}},\overline{u}}$ is injective.

Proof of the claim. Let W^ν be the normalization of W, and $\overline{w}^\nu$ a geometric point of W^ν whose image in X is isomorphic to $\overline{w}$, and denote by w^ν the support of $\overline{w}^\nu$. The assumption on w means that the induced morphism $W^\nu(w^\nu) \to W(w)$ is integral, and the residue field extension $\kappa(w) \to \kappa(w^\nu)$ is radicial, hence the natural morphism of $W^\nu(\overline{w}^\nu)$-schemes

$$W(\overline{w}) \times_{W(w)} W^\nu(w^\nu) \to W^\nu(\overline{w}^\nu)$$

is an isomorphism ([66, Ch.IV, Prop.18.8.10]). Since $W^\nu(\overline{w}^\nu)$ is a normal local scheme ([66, Prop.18.8.12(i)]), it follows easily that $W(\overline{w})$ is reduced and irreducible. However, any specialization map is the composition of a localization map, followed by a local ind-étale map of local rings, whence the claim. □

14.8.31. Now, consider a ring homomorphism $A \to B$, with A noetherian. Set $Y := \operatorname{Spec} A$, $X := \operatorname{Spec} B$, and denote by $f : X \to Y$ the associated morphism of affine schemes.

Lemma 14.8.32. *In the situation of* (14.8.31)*, suppose moreover that f is absolutely flat. Then the following conditions are equivalent :*

(a) *B is noetherian.*
(b) *f has finite fibres.*

Proof. Let ξ be any geometric point of X, and x (resp. y) the support of ξ (resp. of $f(\xi)$).

(a)⇒(b): $B(y) := B \otimes_A \kappa(y)$ is a noetherian ind-étale $\kappa(y)$-algebra (proposition 14.8.25), so of dimension ≤ 0, and then $\operatorname{Spec} B(y)$ is a finite set (proposition 8.1.4). Since x is arbitrary, (b) follows.

(b)⇒(a): Let $I \subset B$ be any ideal, and denote by $\mathscr{I}$ the associated quasi-coherent $\mathscr{O}_X$-module. We need to show that $\mathscr{I}$ is an $\mathscr{O}_X$-module of finite type. Since f_ξ is an isomorphism, we may find a commutative diagram of affine schemes

$$\begin{array}{ccc} X' & \xrightarrow{\ g\ } & Y' \\ {\scriptstyle h}\big\downarrow & & \big\downarrow \\ X & \xrightarrow{\ f\ } & Y \end{array}$$

whose left (resp. right) vertical arrow is an étale neighborhood of ξ (resp. of $f(\xi)$), and a quasi-coherent ideal $\mathscr{J} \subset \mathscr{O}_{Y'}$ such that $g^*\mathscr{J} \subset h^*\mathscr{I}$ and such that $(g^*\mathscr{J})_{x'} = (h^*\mathscr{I})_{x'}$ for some $x' \in h^{-1}(x)$ (notice that g is a flat morphism). Set $\mathscr{M} := \mathscr{O}_{Y'}/\mathscr{J}$, let $z' \in X'$ an arbitrary point, and set $y' := g(z')$; in light of corollary 10.4.37, we have

$$z' \in \operatorname{Ass} g^*\mathscr{M} \Leftrightarrow z' \in \operatorname{Ass} g^*\mathscr{M}_{|X'(x')} \Leftrightarrow y' \in \operatorname{Ass} \mathscr{M}_{|Y'(y')} \Leftrightarrow y' \in \operatorname{Ass} \mathscr{M}.$$

In other words, $\operatorname{Ass} g^*\mathscr{M} = g^{-1}\operatorname{Ass} \mathscr{M}$. However, $\operatorname{Ass} \mathscr{M}$ is finite ([126, Th.6.5(i)]), hence the same holds for $\operatorname{Ass} g^*\mathscr{M}$, in view of (b). Now, notice that $\operatorname{Ass} h^*\mathscr{I}/g^*\mathscr{J} \subset \operatorname{Ass} g^*\mathscr{M} \setminus X'(x')$ (proposition 10.5.6(ii)). It follows that there exists an open neighborhood U of x' in X' such that $(h^*\mathscr{I}/g^*\mathscr{J})_{|U} = 0$ (lemma 10.5.5(iii)), *i.e.* $(g^*\mathscr{J})_{|U} = h^*\mathscr{I}_{|U}$; especially, $h^*\mathscr{I}_{|U}$ is an $\mathscr{O}_U$-module of finite type. Since h is an open map, we deduce that $h(U)$ is an open neighborhood of x in X, and $\mathscr{I}_{|h(U)}$ is an $\mathscr{O}_{h(U)}$-module of finite type. Since x is arbitrary, the lemma follows. □

14.8.33. For our second criterion, keep the situation of (14.8.31), and suppose additionally that, for every geometric point ξ of X, the induced morphism $f_\xi : X(\xi) \to Y(\xi)$ is a closed immersion. Under this weaker assumption, it is not necessarily true that conditions (a) and (b) of lemma 14.8.32 are equivalent. For instance, we have :

Example 14.8.34. Take $A := k[X]$, the free polynomial algebra over a given infinite field k; also, let $(a_i \mid i \in \mathbb{N})$ be a sequence of distinct elements of k. We construct an A-algebra B, as the colimit of the inductive system $(B_i \mid i \in \mathbb{N})$ of A-algebras, such that

- $B_i := k[X] \times k^{i+1}$ (the product of $k[x]$ and $i+1$ copies of k, in the category of rings)
- the structure map $A \to B_i$ is the unique map of k-algebras given by the rule : $X \mapsto (X, a_0, \dots, a_i)$
- the transition maps $B_i \to B_{i+1}$ are given by the rule : $X \mapsto (X, 0, \dots, 0, a_{i+1})$ and $(0, e_i) \mapsto (0, e_i, 0)$ for $i = 0, \dots, i$. (Here $e_0, \dots, e_i$ is the standard basis of the k-vector space k^{i+1}.)

Then one can check that $X := \operatorname{Spec} B = \operatorname{Spec} k[X] \cup \mathbb{N}$, and $\mathbb{N}$ is an open subset of X with the discrete topology. Moreover, the induced map $\operatorname{Spec} B \to Y := \operatorname{Spec} A$ restricts to the continuous map $\mathbb{N} \to Y$ given by the rule $i \mapsto \mathfrak{p}_i$, for every $i \in \mathbb{N}$, where $\mathfrak{p}_i$ is the prime ideal generated by $X - a_i$. It follows easily that the condition of (14.8.33) is fulfilled; nevertheless, clearly X has infinitely many maximal points, hence its underlying topological space is not noetherian, and *a fortiori*, B cannot be noetherian.

However, we have the following positive result :

Proposition 14.8.35. *In the situation of* (14.8.33)*, suppose additionally that Y admits a geometrically unibranch stratification. Then the following conditions are equivalent :*

(a) *B is noetherian.*
(b) *The topological space underlying X is noetherian.*
(c) *For every geometric point ξ of X there exist a neighborhood X' of ξ in $X_{\text{ét}}$, an unramified Y-scheme Y', and an absolutely flat morphism $X' \to Y'$ of Y-schemes, with finite fibres.*

Proof. Obviously (a)⇒(b).

(c)⇒(a): Indeed, under assumption (c), we may find finitely many geometric points $\xi_1, \dots, \xi_k$ of X, and for every $i = 1, \dots, k$, a neighborhood X'_i of ξ_i in $X_{\text{ét}}$, and a Y-morphism $X'_i \to Y'_i$ with the stated properties, such that moreover, the family $(X'_i \mid i = 1, \dots, k)$ is an étale covering of X. Furthermore, we may assume that X'_i and Y'_i are affine for every $i \leq k$. In this case, lemma 14.8.32 shows that every X'_i is noetherian, and then the same holds for X.

(b)⇒(c): Fix a geometric point ξ of X, and let B^{sh}_ξ (resp. A^{sh}_ξ) denote the strict henselization of B at ξ (resp. of A at $f(\xi)$); by assumption, we may find a (finitely generated) ideal $I \subset A^{\text{sh}}_\xi$ such that f_ξ induces an isomorphism $A^{\text{sh}}_\xi/I \xrightarrow{\sim} B^{\text{sh}}_\xi$. Then we may find an affine étale neighborhood Y' of $f(\xi)$, say $Y' := \operatorname{Spec} A'$ for some étale A-algebra A', and an ideal $I' \subset A'$ such that $I'A^{\text{sh}}_\xi = I$. Next, we may find an affine étale neighborhood $X' := \operatorname{Spec} B'$ of ξ such that f_ξ extends to a morphism $g : X' \to Y'$, and we may further suppose that the corresponding ring homomorphism $A' \to B'$ factors through A'/I', so g factors through a morphism

$$h : X' \to Z := \operatorname{Spec} A'/I'$$

and the closed immersion $Z \to Y'$. By construction, ξ lifts to a geometric point ξ' of X', and $h_{\xi'} : X'(\xi') \to Z(h(\xi'))$ is an isomorphism ([66, Ch.IV, Prop.18.8.10]). To conclude the proof, it then suffices to exhibit an open subset $U \subset X'$ containing the support of ξ', and such that the restriction $U \to Z$ of h is absolutely flat with finite fibres.

Claim 14.8.36. Let $\varphi : W \to W'$ be a quasi-finite, separated, dominant and finitely presented morphism of reduced, irreducible schemes, and suppose that W' contains a non-empty geometrically unibranch open subset. Then the same holds for W.

Proof of the claim. After replacing W' by some open subset $U' \subset W'$, and W by $\varphi^{-1}U'$, may assume that W' is affine and unibranch; then we may also suppose that φ is finite ([65, Ch.IV, Th.8.12.6]), in which case W is affine as well, and φ is surjective.

Let $\eta \in W$ and $\eta' \in W'$ be the respective generic points, and denote by $E \subset \kappa(\eta)$ the maximal subfield that is separable over $\kappa(\eta')$. We may then find a reduced and irreducible scheme W'', with generic point η'', such that φ factors as the composition of finite surjective morphisms $\varphi' : W \to W''$, $\varphi'' : W'' \to W'$, and such that $\kappa(\eta'') = E$. By virtue of [65, Ch.IV, Th.8.10.5] and [66, Ch.IV, Prop.17.7.8(ii)], we may then replace W' by a non-empty open subset, and assume that φ' is radicial, and φ'' is étale. Then W'' is geometrically unibranch ([64, Ch.IV, Prop.6.15.10]); hence, we may replace W' by W'', and reduce to the case where φ is radicial. Let p be the characteristic of $\kappa(\eta')$; if $p = 0$, φ is birational, in which case the assertion follows from [64, Prop.6.15.5(ii)]. In case $p > 0$, write $W = \mathrm{Spec}\, C$, $W' = \mathrm{Spec}\, C'$; the induced ring homomorphism $C' \to C$ is finite and injective, and we have $C^{p^n} \subset C'$ for $n \in \mathbb{N}$ large enough. Denote by C^ν (resp. C'^ν) the normalization of the domain C (resp. of C'); it follows easily that $(C^\nu)^{p^n} \subset C'^\nu$, so the morphism $\mathrm{Spec}\, C^\nu \to \mathrm{Spec}\, C'^\nu$ is radicial. On the other hand, since W' is geometrically unibranch, the normalization morphism $\mathrm{Spec}\, C'^\nu \to W'$ is radicial ([63, Ch.0, Lemme 23.2.2]) and therefore the normalization map $\mathrm{Spec}\, C^\nu \to W$ is radicial as well ([64, Ch.IV, Lemme 6.15.3.1(i)]). Then W is geometrically unibranch, again by [63, Ch.0, Lemme 23.2.2]. ◊

From claim 14.8.36 and our assumption on Y, it follows easily that Z admits a geometrically unibranch stratification. The morphism h induces as usual a morphism of étale topoi

(14.8.37) $$(X')^{\sim}_{\text{ét}} \underset{h^*}{\overset{h_*}{\rightleftarrows}} Z^{\sim}_{\text{ét}}$$

as well as a morphism $h^\natural : h^* \mathscr{O}_{Z_{\text{ét}}} \to \mathscr{O}_{X'_{\text{ét}}}$ of $(X')^{\sim}_{\text{ét}}$-rings. By construction, for every geometric point τ of X', the induced map on stalks $h^\natural_\tau : \mathscr{O}_{Z_{\text{ét}}, h(\tau)} \to \mathscr{O}_{X'_{\text{ét}}, \tau}$ is surjective, and $h^\natural_{\xi'}$ is a bijection. It follows easily that $h^\natural_\tau$ is also bijective for every generization τ of ξ'. Now, choose a partition $Z = Z_1 \cup \cdots \cup Z_k$ as in lemma 14.8.29 (with $\mathscr{F} := \mathscr{O}_{Z_{\text{ét}}}$), and for given $i \leq k$, suppose that τ and η are two geometric points of $h^{-1} Z_i$, with η a generization of τ. The choice of a strict specialization morphism $X'(\eta) \to X'(\tau)$ yields a commutative diagram

(14.8.38) $$\begin{array}{ccc} \mathscr{O}_{Z_{\text{ét}}, h(\tau)} & \xrightarrow{h^\natural_\tau} & \mathscr{O}_{X'_{\text{ét}}, \tau} \\ \downarrow & & \downarrow \\ \mathscr{O}_{Z_{\text{ét}}, h(\eta)} & \xrightarrow{h^\natural_\eta} & \mathscr{O}_{X'_{\text{ét}}, \eta} \end{array}$$

whose vertical arrows are strict specialization maps (see remark 4.9.26(i)); in light of lemma 14.8.29, we deduce that $h^\natural_\tau$ is injective, whenever the same holds for $h^\natural_\eta$. Now, let $x' \in X'$ be the support of ξ', and Σ_i the set of maximal points of $h^{-1} Z_i$. Condition (b) and proposition 8.1.65 imply that $h^{-1} Z_i$ is a noetherian topological space, hence Σ_i is a finite set; it follows that the topological closure W of $\bigcup_{i=1}^k \Sigma_i \setminus X'(x')$ in X' is a closed subset that does not contain x'; by construction, $h^\natural$ restricts to a monomorphism on $U := X' \setminus W$, *i.e.* the restriction $h_U : U \to Z$ of h is absolutely flat, as required. It also follows that the fibres of h_U are noetherian topological spaces of dimension zero (cp. the proof of lemma 14.8.32), hence they are finite, and the proof is complete. □

Lemma 14.8.39. *Suppose that the valuation of K has finite rank, and let X be a finitely presented K^+-scheme, $\mathscr{F}$ a coherent $\mathscr{O}_{X_{\text{ét}}}$-module. Then there exists a partition*

$$X = X_1 \cup \cdots \cup X_k$$

of X into finitely many disjoint irreducible locally closed subsets, such that the following holds. For every $i = 1, \ldots, k$, every geometric point $\overline{x}$ of X_i, and every generization $\overline{u}$ of $\overline{x}$ in X_i, *every strict specialization map*

$$s_{\overline{x},\overline{u}} : \mathscr{F}_{\overline{x}} \to \mathscr{F}_{\overline{u}}$$

is injective (see (4.9.23)*).*

Proof. We easily reduce to the case where X is affine, say $X = \operatorname{Spec} A$, and then $\mathscr{F}$ is the coherent $\mathscr{O}_{X_{\text{ét}}}$-module arising from a finitely presented A-module M. By theorem 14.8.17(i), M admits a K^+-flattening sequence $(b_0, \dots, b_n)$. Then we are further reduced to showing the assertion for the subquotients $b_i M/b_{i+1}M$ (that are finitely presented, by corollary 14.8.21(ii)). So, we may assume from start that $f : X \to S_0 := \operatorname{Spec} K^+/bK^+$ is a finitely presented morphism for some $b \in K^+$, and $\mathscr{F}$ is f-flat. For every $t \in S_0$, let

$$i_t : X_t := f^{-1}(t) \to X$$

be the locally closed immersion; since X_t is an excellent noetherian scheme, we may apply lemma 14.8.29 and remark 14.8.23 to produce a partition $X_t = X_{t,1} \cup \dots \cup X_{t,k}$ by finitely many disjoint irreducible locally closed subsets such that, for every $i = 1, \dots, k$, every geometric point $\overline{x}$ of $X_{t,i}$ and every generization $\overline{u}$ of $\overline{x}$ in $X_{t,i}$, every specialization map

$$(i_t^* \mathscr{F})_{\overline{x}} \to (i_t^* \mathscr{F})_{\overline{u}}$$

is injective. Since $|S_0|$ is a finite set, the lemma will then follow from :

Claim 14.8.40. Let $g : Y \to T$ be any finitely presented morphism of schemes, $\mathscr{G}$ a finitely presented, quasi-coherent g-flat $\mathscr{O}_Y$-module, and $t \in T$. Let also $\overline{y}, \overline{u}$ be two geometric points of $g^{-1}(t)$, such that $\overline{u}$ is a generization of $\overline{y}$. Let $s_{\overline{y},\overline{u}} : \mathscr{G}_{\overline{y}} \to \mathscr{G}_{\overline{u}}$ be a strict specialization map, and suppose that $s_{\overline{y},\overline{u}} \otimes_{\mathscr{O}_{T,t}} \kappa(t)$ is injective. Then the same holds for $s_{\overline{y},\overline{u}}$.

Proof of the claim. Set $Y' := Y(\overline{y})$, denote by $j : Y' \to Y$ the natural morphism, and set $\mathscr{G}' := j^* \mathscr{G}$. The map $s_{\overline{y},\overline{u}}$ is deduced from a morphism $Y(\overline{u}) \to Y'$ of Y-schemes; the latter factors through a faithfully flat morphism $Y(\overline{u}) \to Y'(u)$, where $u \in Y'$ is the image of the closed point of $Y(\overline{u})$. Hence, $s_{\overline{y},\overline{u}}$ is the composition of the specialization map $s' : \mathscr{G}'_{\overline{y}} \to \mathscr{G}'_u$, and the injective map $\mathscr{G}'_u \to \mathscr{G}'_{\overline{u}}$. Our assumption implies that $s' \otimes_{\mathscr{O}_{T,t}} \kappa(t)$ is injective, and it suffices to show that the same holds for s'. However, Y' is the limit of a cofiltered system $(j_\lambda : Y_\lambda \to Y \mid \lambda \in \Lambda)$ of local, essentially étale Y-schemes. Write $\mathscr{G}_\lambda := j_\lambda^* \mathscr{G}$ for every $\lambda \in \Lambda$, and notice that the transition morphisms $Y_\lambda \to Y_\mu$ are faithfully flat, for every $\lambda \geq \mu$; it follows that $\mathscr{G}'_{\overline{y}}$ is the filtered union of the system of modules $(G_\lambda := \Gamma(Y_\lambda, \mathscr{G}_\lambda) \mid \lambda \in \Lambda)$ (proposition 10.1.10(i)); likewise, $\mathscr{G}'_{\overline{y}} \otimes_{\mathscr{O}_{T,t}} \kappa(t)$ is the filtered union of the submodules $(G_\lambda \otimes_{\mathscr{O}_{T,t}} \kappa(t) \mid \lambda \in \Lambda)$. We are then reduced to checking that all the restrictions $s_\lambda : G_\lambda \to \mathscr{G}'_u$ of s' are injective, and we know already that $s_\lambda \otimes_{\mathscr{O}_{T,t}} \kappa(t)$ is injective for every $\lambda \in \Lambda$. For every such λ, let $u_\lambda \in Y_\lambda$ be the image of u; then s_λ factors through the injective map $\mathscr{G}_{\lambda,u_\lambda} \to \mathscr{G}'_u$ and the specialization map $s'_\lambda : G_\lambda \to \mathscr{G}_{\lambda,u_\lambda}$. Consequently, it suffices to show that s'_λ is injective, and we know already that the same holds for $s'_\lambda \otimes_{\mathscr{O}_{T,t}} \kappa(t)$. However, $\mathscr{G}_{\lambda,u_\lambda}$ is a localization $Q_\lambda^{-1} G_\lambda$, for a multiplicative set $Q_\lambda \subset \Gamma(Y_\lambda, \mathscr{O}_{Y_\lambda})$, and s'_λ is the localization map. The claim therefore boils down to the assertion that, for every $\lambda \in \Lambda$ and every $q \in Q_\lambda$, the endomorphism $q \cdot \mathbf{1}_{G_\lambda}$ is injective on G_λ, and our assumption already ensures that $(q \cdot \mathbf{1}_{G_\lambda}) \otimes_{\mathscr{O}_{T,t}} \kappa(t)$ is injective on $G_\lambda \otimes_{\mathscr{O}_{T,t}} \kappa(t)$. However, let $g_\lambda : Y_\lambda \to T$ be the morphism induced by g; by construction, $\mathscr{G}_\lambda$ is a g_λ-flat $\mathscr{O}_{Y_\lambda}$-module, hence the contention follows from [65, Prop.11.3.7]. □

Proposition 14.8.41. *Let* $(K, |\cdot|)$ *be a valued field, and* A *a locally measurable* K^+*-algebra fulfilling conditions* (a) *and* (b) *of theorem* 14.8.17. *The following conditions are equivalent:*

- (c) *For every geometric point* ξ *of* $X := \operatorname{Spec} A$ *there exist a neighborhood* U *of* ξ *in* $X_{\text{ét}}$*, and a finitely presented* K^+*-scheme* Z *with an absolutely flat morphism of* K^+*-schemes* $X' \to Z$.
- (d) Ω_{A/K^+} *is an* A*-module of finite type.*

Proof. (c)⇒(d) follows easily from lemma 14.8.24.

(d)⇒(c): Let $a_1, \dots, a_k \in A$ be a finite system of elements such that $da_1, \dots, da_k$ generate the A-module Ω_{A/K^+}. We define a map of K^+-algebras $A_0 := K^+[T_1, \dots, T_k] \to A$ by the rule: $T_i \mapsto a_i$ for $i = 1, \dots, k$. Clearly $\Omega_{A/A_0} = 0$. Let $Z_0 := \mathrm{Spec}\, A_0$, and denote by $f : X \to Z_0$ the induced morphism of schemes. Let A^{sh}_ξ (resp. $A^{\mathrm{sh}}_{0,\xi}$) be the strict henselization of A at ξ (resp. of A_0 at $f(\xi)$). According to lemma 14.8.27, the induced map $A^{\mathrm{sh}}_{0,\xi} \to A^{\mathrm{sh}}_\xi$ is surjective, and its kernel is a finitely generated ideal $I \subset A^{\mathrm{sh}}_{0,\xi}$. In this situation, we may argue as in the proof of proposition 14.8.35, to produce an affine étale neighborhood X' of ξ, a finitely presented affine unramified Z_0-scheme Z, and a morphism of Z_0-schemes $h : X' \to Z$ such that $h_\tau : X'(\tau) \to Z(h(\tau))$ is a closed immersion for every geometric point τ of X', and $h_{\xi'}$ is an isomorphism for some lifting ξ' of ξ.

Then we consider the associated morphism of étale topoi as in (14.8.37) and the morphism $h^\natural : h^*\mathscr{O}_{Z_{\mathrm{\acute{e}t}}} \to \mathscr{O}_{X'_{\mathrm{\acute{e}t}}}$ of $(X')^\sim_{\mathrm{\acute{e}t}}$-rings. Again, the induced map on stalks $h^\natural_\tau$ is surjective for every geometric point τ of X', and is bijective if τ is a generization of ξ'. We pick a finite partition $Z = Z_1 \cup \dots \cup Z_k$ as in lemma 14.8.39 (for $\mathscr{F} := \mathscr{O}_Z$). For any $i \leq k$, let τ, η be two geometric points of $h^{-1}Z_i$, such that η is a generization of τ; by considering the commutative diagram (14.8.38), we see again that $h^\natural_\tau$ is injective whenever the same holds for $h^\natural_\eta$.

Now, condition (b) of theorem 14.8.17 easily implies that $h^{-1}Z_i$ is a noetherian topological space, hence its set Σ_i of maximal points is finite. Again we let $x' \in X'$ be the support of ξ', and W the topological closure of $\bigcup_{i=1}^k \Sigma_i \setminus X'(x')$ in X', and it is easily seen that the restriction $U \to Z$ of h is absolutely flat, so (c) holds. □

14.8.42. Henceforth we restrict to the case where the value group Γ of K is not discrete and of rank one. As usual, we consider the almost structure attached to the standard setup attached to $(K, |\cdot|)$.

Definition 14.8.43. In the situation of (14.8.42), let A be a K^{+a}-algebra and M an A-module.

(i) We say M is an *almost noetherian* A-module, if every A-submodule of M is almost finitely generated.

(ii) We say that A is an *almost noetherian* K^{+a}-algebra, if A is an almost noetherian A-module.

Remark 14.8.44. (i) In the situation of (14.8.42), suppose that A is an almost noetherian K^{+a}-algebra. Then the same argument as in the "classical limit" case shows that every almost finitely generated A-module M is almost noetherian. The details shall be left to the reader.

(ii) The notion of almost noetherian module goes back to the appendix of Kiehl's article [117]; in our language, he proved in Satz 5.1 that for a rank 1 complete valuation ring V with dense value group, every finitely generated module over a ring of restricted formal series $V\langle X_1, ..., X_n\rangle$ (or more generally, over V-algebras of topologically finite type) is almost noetherian.

Theorem 14.8.45. *Let A be a locally measurable K^+-algebra. Suppose that both $A \otimes_{K^+} \kappa$ and $A \otimes_{K^+} K$ are noetherian rings. Then A^a is an almost noetherian K^{+a}-algebra.*

Proof. If the valuation of K is discrete, the assertion is proposition 14.8.14(iii). Hence, we may assume that the valuation of K is not discrete. Moreover, let $\mathscr{Q}$ be the set of all locally measurable K^+-algebras B that are quotients of A, and such that B^a is not almost noetherian. We have to show that $\mathscr{Q} = \varnothing$. However, for every $B \in \mathscr{Q}$ the κ-algebra $\overline{B} := B \otimes_{K^+} \kappa$ is a quotient of the noetherian κ-algebra $\overline{A} := A \otimes_{K^+} \kappa$; it follows that the set $\overline{\mathscr{Q}} := \{\overline{B} \mid B \in \mathscr{Q}\}$ admits minimal elements, if it is not empty. In the latter case, we may then replace A by any quotient $B \in \mathscr{Q}$ such that $\overline{B}$ is minimal in $\overline{\mathscr{Q}}$, and therefore assume that for every locally measurable quotient B of A, either $\overline{B} = \overline{A}$, or else B^a is almost noetherian.

Let $I \subset A$ be any ideal; we have to show that I^a is almost finitely generated. By assumption, the image $\overline{I}$ of I in $\overline{A}$ is finitely generated, and the same holds for the ideal $I_K := I \otimes_{K^+} K$ of $A_K := A \otimes_{K^+} K$. Thus, we may find a finitely generated subideal $I_0 \subset I$ whose image in $\overline{A}$ agrees with $\overline{I}$, and such that $I_0 \otimes_{K^+} K = I_K$. After replacing A by A/I_0 and I by I/I_0, we are then reduced to the case where both $\overline{I}$ and I_K vanish. Let $J \subset A$ denote the kernel of the localization map $A \to A_K$, set $S := 1 + \mathfrak{m}_K A$, and let $B := S^{-1}A \times A_K$. Clearly B is a faithfully flat A-algebra, and A/J is a K^+-flat A-module; therefore $(A/J) \otimes_A B$ is a K^+-flat B-module of finite type, and since A_K is noetherian, proposition 14.8.14(ii) implies that $(A/J) \otimes_A B$ is finitely presented. Then A/J is finitely presented as well, and therefore J is a finitely generated ideal. We conclude that there exists $c \in \mathfrak{m}_K$ such that $cJ = 0$. Then notice that $J \cap cA = 0$: indeed, if $a \in J \cap cA$, we have $a = cx$ for some $x \in A$, and $ca = 0$, therefore $c^2 x = 0$, so $x \in J$, and consequently $a = cx = 0$. Now, fix $b \in \mathfrak{m}_K$; since the valuation of K has rank one, and since $I \subset J$, it follows that there exists $n \in \mathbb{N}$ large enough, so that

$$I \cap b^n A = 0.$$

Let $i_0 := \max\{i \in \mathbb{N} \mid I \subset b^i A\}$, and set

$$N := b^{i_0}A/(I + b^{i_0+1}A) \qquad N' := b^{i_0}A/b^{i_0+1}A.$$

Let $\varphi : N' \to N$ be the natural surjection, and set $c^*_N := c_N \circ \varphi : N' \to cN$ for every $c \in K^+$ (notation of definition 14.8.6(ii)).

Claim 14.8.46. There exists $c \in K^+$ with $\log |c| < \log |b|$ such that $c^*_N \otimes_{K^+} \kappa$ is not an isomorphism.

Proof of the claim. Suppose that the claim fails; then it is easily seen that no $\gamma \in \log \Gamma^+$ with $\gamma < \log |b|$ breaks N. For every geometric point ξ of $\operatorname{Spec} A$, let A^{sh}_ξ denote the strict henselization of A at ξ; since A^{sh}_ξ is a flat A-algebra, we have a natural identification

$$cN \otimes_A A^{\mathrm{sh}}_\xi = c(N \otimes_A A^{\mathrm{sh}})$$

of A^{sh}-modules; therefore, no $\gamma < \log |b|$ breaks $N \otimes_A A^{\mathrm{sh}}_\xi$. By corollary 14.8.13, we deduce that $N \otimes_A A^{\mathrm{sh}}_\xi$ is a K^+/bK^+-flat and finitely presented A^{sh}_ξ-module, for every geometric point ξ. Hence, $C_\xi := \operatorname{Ker} \varphi \otimes_A A^{\mathrm{sh}}_\xi$ is a finitely generated A^{sh}_ξ-module, and $C_\xi \otimes_{K^+} \kappa = 0$, for every geometric point ξ. By Nakayama's lemma, it follows that $C_\xi = 0$ for every such ξ, so finally $\operatorname{Ker} \varphi = 0$, which means that $I \subset b^{i_0+1}A$, contradicting the choice of i_0. ◊

Let c be as in claim 14.8.46, and set $d := cb^{i_0}$; notice the natural isomorphism of $\overline{A}$-modules

$$cN \otimes_{K^+} \kappa \xrightarrow{\sim} \frac{dA}{I \cap dA} \otimes_{K^+} \kappa.$$

Let $\varphi : A \to dA/(I \cap dA)$ be the composition of $d_A : A \to dA$ and the projection $dA \to dA/(I \cap dA)$ (notation of definition 14.8.6(ii)); by construction, $\varphi \otimes_{K^+} \kappa$ is not an isomorphism, hence there exists $x \in I \cap dA$ such that the composition $\varphi_x : A \to dA/xA$ of d_A and the projection $dA \to dA/xA$ induces a map $\varphi_x \otimes_{K^+} \kappa$ with non-trivial kernel. In other words, dA/xA is a cyclic module over a locally measurable quotient B of A such that the projection $\overline{A} \to \overline{B}$ is not an isomorphism, so B^a is almost noetherian. Set $I_0 := (I \cap dA)/xA$; then I_0 is a submodule of dA/xA, and consequently I^a_0 is an almost finitely generated B^a-module (remark 14.8.44(i)). Then clearly $(I \cap dA)^a$ is an almost finitely generated ideal of A^a. But by construction, $cI \subset I \cap dA$, and b annihilates $(I \cap dA)/cI$. Since b is arbitrary, this easily implies that I^a is almost finitely generated, as required. □

Corollary 14.8.47. *In the situation of theorem* 14.8.45*, the following holds :*

(i) *Every almost finitely generated A^a-module is almost finitely presented.*
(ii) *Every flat almost finitely generated A^a-module is almost projective of finite rank.*

Proof. Assertion (i) is an easy consequence of theorem 14.8.45 and remark 14.8.44(i) : the details shall be left to the reader.

(ii): Let M be a flat and almost finitely generated A^a-module; by (i) and [75, Prop.2.4.18(ii)], M is almost projective. It remains to show that there exists $n \in \mathbb{N}$ such that $\Lambda^n_{A^a} M = 0$, or equivalently, that $(\Lambda^n_A M_!)^a = 0$. However, $M_!$ is a flat A-module, so $\mathrm{Ass}\, M_! \subset \mathrm{Ass}\, A$; in view of theorem 14.8.17(ii), we may argue by induction on the cardinality c of $\mathrm{Ass}\, A$, and it suffices to check that, for every $\mathfrak{p} \in \mathrm{Ass}\, A$ there exists $n \in \mathbb{N}$ such that $\Lambda^n_{A_\mathfrak{p}}(M_!)_\mathfrak{p} = 0$. If $c = 0$, we have $A = 0$, and there is nothing to prove. Suppose that $c > 0$ and the assertion is known for every locally measurable K^+-algebra B such that $B \otimes_{K^+} K$ and $B \otimes_{K^+} \kappa$ are noetherian, and such that $\mathrm{Ass}\, B$ has cardinality $< c$. Especially, for a fixed $\mathfrak{p} \in \mathrm{Ass}\, A$, we can cover $\mathrm{Spec}\, A_\mathfrak{p} \setminus \{\mathfrak{p}\}$ by finitely many affine open subsets $\mathrm{Spec}\, B_1, \dots, \mathrm{Spec}\, B_k$, and then the inductive assumption yields $n \in \mathbb{N}$ such that $\Lambda^n_{B_i}(M_! \otimes_A B_i) = \Lambda^n_{B_i}(M \otimes_{A^a} B_i^a)_! = 0$ for every $i = 1, \dots, k$. In other words, $N := \Lambda^n_{A_\mathfrak{p}}(M_\mathfrak{p})$ is a flat $A^a_\mathfrak{p}$-module with $\mathrm{Supp}\, N_! \subset \{\mathfrak{p}\}$.

Let $A^{\mathrm{sh}}_\mathfrak{p}$ denote the strict henselization of A at some geometric point localized at $\mathfrak{p}$; on the one hand, the $A_\mathfrak{p}$-algebra $A^{\mathrm{sh}}_\mathfrak{p}$ is faithfully flat, and $M_! \otimes_A A^{\mathrm{sh}}_\mathfrak{p} = (M \otimes_{A^a} (A^{\mathrm{sh}}_\mathfrak{p})^a)_!$. On the other hand, exterior powers commute with arbitrary base change; thus, we are reduced to showing :

Claim 14.8.48. Let B be a measurable K^+-algebra, and N a flat almost finitely generated B^a-module whose support is contained in $\{s(B)\}$ (notation of (14.5.36)). Then $\Lambda^n_B N_! = 0$ for every sufficiently large $n \in \mathbb{N}$.

Proof of the claim. By assumption, we may find an essentially finitely presented K^+-algebra B_0 and an ind-étale and faithfully flat map $B_0 \to B$ of K^+-algebras. Set $X_0 := \mathrm{Spec}\, B_0$ and $X := \mathrm{Spec}\, B$; since $\{s(B_0)\}$ is a constructible subset of X_0, the natural map

$$B \otimes_{B_0} \Gamma_{\{s(B_0)\}} \mathscr{O}_{X_0} \to \Gamma_{\{s(B)\}} \mathscr{O}_X$$

is an isomorphism (lemma 10.4.17(iii)). On the other hand, there exists a finitely generated $s(B_0)$-primary ideal $J \subset B_0$ such that the natural map $\Gamma_{\{s(B_0)\}} \mathscr{O}_{X_0} \to B_0/J$ is injective (lemma 10.5.15 and theorem 11.4.37(i)). It follows that the natural map $\Gamma_{\{s(B)\}} \mathscr{O}_X \to B/JB$ is injective as well. Now, $N_!$ can be written as the colimit of a filtered system $(L_\lambda \mid \lambda \in \Lambda)$ of free B-modules of finite rank; for each $\lambda \in \Lambda$, let $L^\sim_\lambda$ be the quasi-coherent $\mathscr{O}_X$-module arising from L_λ, and define likewise $N^\sim_!$; taking into account lemma 10.4.4(iii.b) we deduce that the natural map

$$N_! = \Gamma_{\{s(B)\}} N^\sim_! = \operatorname*{colim}_{\lambda \in \Lambda} \Gamma_{\{s(B)\}} L^\sim_\lambda \to \operatorname*{colim}_{\lambda \in \Lambda} L_\lambda / J L_\lambda = N_!/JN_!$$

is injective. In other words, N is a B^a/JB^a-module. We may then replace B by B/JB, and assume from start that B has Krull dimension zero. In this situation, we may find a nilpotent ideal $I \subset B$, a valuation ring V that is a measurable K^+-algebra, and a finitely presented surjection $V \to B/I$ (lemma 14.5.42). It suffices to find $n \in \mathbb{N}$ such that $(\Lambda^n_B N_!) \otimes_B B/I = 0$; hence, we may replace N by N/IN and B by B/I, and assume as well that $B = V/bV$ for some $b \in V$. In this case, the assertion follows easily from corollary 14.5.33. ☐

15. Continuous valuations and adic spaces

In this chapter we present Huber's theory of adic spaces.

15.1. **Formal schemes.** In this section we define a category of topologically ringed spaces that generalize the usual formal schemes from [59]. This generalization was crucial for the previous releases of this work, that followed closely Faltings's original method for proving the almost purity theorem. In the current release, where we have switched to the new approach invented by Scholze, and based on his theory of perfectoid spaces, formal schemes are still useful, but they play a rather different role : we will employ them to establish some foundational properties of adic spaces (and later, of perfectoid spaces); for such purposes, the standard adic formal schemes of [59] would already suffice, but the generalization that we worked out in the previous releases might be interesting in its own right, and for other applications. Besides, our former treatment included some complements on the cohomology of formal schemes that do not appear explicitly in [61], which will come in handy in section 15.5.

15.1.1. In this section we shall deal with topological rings whose topology is linear, so that $0 \in A$ admits a fundamental system of open neighborhoods $I_\bullet := (I_\lambda \mid \lambda \in \Lambda)$ consisting of ideals of A (see definition 8.3.1(v)). Then the separated completion $A^\wedge$ of A is also a topological ring of this type : namely, its topology is linear, defined by the system $I^\wedge_\bullet := (I^\wedge_\lambda \mid \lambda \in \Lambda)$, where $I^\wedge_\lambda$ denotes the topological closure of I_λ in $A^\wedge$, for every $\lambda \in \Lambda$.

15.1.2. Keep the notation of (15.1.1). We shall consider topological A-modules $(M, \mathscr{T}_M)$ whose topology is A-linear (see definition 8.3.1(iv)); additionally, we shall assume that $\mathscr{T}_M$ *is coarser than the* $I_\bullet$*-adic topology*, *i.e.* for every open submodule $N \subset M$, there exists $\lambda \in \Lambda$ such that $I_\lambda M \subset N$ (see remark 8.3.2(ii)). (Notice that [59, Ch.0, §7.7.1] is slightly ambiguous: it is not clear whether all the topological modules considered there are supposed to satisfy the foregoing additional condition.) Then, the topology of the separated completion $(M^\wedge, \mathscr{T}^\wedge_M)$ is $A^\wedge$-linear and coarser than the $I^\wedge_\bullet$-adic topology. Let N be any other topological A-module of this type; notice that the topology $\mathscr{T}^\otimes_{M,N}$ on $M \otimes_A N$ (see (8.3.7)) is also coarser than the $I_\bullet$-adic topology, so the topology of $M \widehat{\otimes}_A N$ is coarser than the $I^\wedge_\bullet$-adic topology. We also denote

$$\mathrm{top.Hom}_A(M, N)$$

the A-module of all continuous A-linear maps $M \to N$. Notice the natural identification :

$$\mathrm{top.Hom}_A(M, N^\wedge) \simeq \lim_{N' \subset N} \operatorname*{colim}_{M' \subset M} \mathrm{Hom}_A(M/M', N/N') \tag{15.1.3}$$

where N' (resp. M') ranges over the family of open submodules of N (resp. of M).

Definition 15.1.4. Let $X := (\mathscr{X}, J)$ be any site, and $\mathscr{C}$ any other category.

(i) A *sheaf of topological spaces on* X, any sheaf on X with values in the category **Top** of topological spaces (see definition 5.5.1(ii)). Likewise, a *sheaf of topological groups* (resp. *of topological rings*) on X is a sheaf $\mathscr{A}$ on X with values in the category of topological groups (resp. topological rings).

(ii) Let $\mathscr{A}$ be a sheaf of topological rings on X. A *sheaf of topological* $\mathscr{A}$*-modules* – or briefly, a *topological* $\mathscr{A}$*-module* – is the datum of an $\mathscr{A}$-module $\mathscr{F}$, and for every $U \in \mathrm{Ob}(\mathscr{X})$, a topology $\mathscr{T}_U$ on $\mathscr{F}(U)$, such that $\underline{\mathscr{F}}(U) := (\mathscr{F}(U), \mathscr{T}_U)$ is a topological $\mathscr{A}(U)$-module, and the rule $U \mapsto \underline{\mathscr{F}}(U)$ defines a sheaf of topological groups on X.

(iii) An $\mathscr{A}$-linear map $\varphi : \mathscr{F} \to \mathscr{G}$ between topological $\mathscr{A}$-modules is said to be *continuous* if it is a morphism of sheaves of topological groups, *i.e.* if $\varphi_U : \mathscr{F}(U) \to \mathscr{G}(U)$ is a continuous map for every $U \in \mathrm{Ob}(\mathscr{X})$. We denote by :

$$\mathrm{top.Hom}_{\mathscr{A}}(\mathscr{F}, \mathscr{G})$$

the A-module of all continuous $\mathscr{A}$-linear morphisms $\mathscr{F} \to \mathscr{G}$.

Remark 15.1.5. Keep the notation of definition 15.1.4.

(i) Since the forgetful functor **Top** $\to$ **Set** commutes with all limits, remark 5.5.2(i,ii) implies that the presheaf (of sets) underlying any sheaf of topological spaces on X is a sheaf (of sets). Likewise, the presheaf underlying any sheaf of topological groups or topological rings is also a sheaf of sets.

(ii) For any sheaf $\mathscr{A}$ of topological rings on X, and any topological $\mathscr{A}$-modules $\mathscr{F}$ and $\mathscr{G}$, consider the presheaf on X defined by the rule :

$$U \mapsto \mathrm{top.Hom}_{\mathscr{A}_{|U}}(\mathscr{F}_{|U}, \mathscr{G}_{|U}). \tag{15.1.6}$$

From the discussion of remark 5.5.2(v) we deduce that (15.1.6) is a sheaf on X, which we denote by :

$$top.\mathscr{H}om_{\mathscr{A}}(\mathscr{F},\mathscr{G}).$$

15.1.7. In the situation of (15.1.1), set $X_\lambda := \operatorname{Spec} A/I_\lambda$ for every $\lambda \in \Lambda$; we define the *formal spectrum* of A as the colimit of topological spaces :

$$\operatorname{Spf} A := \operatorname*{colim}_{\lambda\in\Lambda} X_\lambda.$$

Hence, the set underlying $\operatorname{Spf} A$ is the filtered union of the X_λ, and a subset $U \subset \operatorname{Spf} A$ is open (resp. closed) in $\operatorname{Spf} A$ if and only if $U \cap X_\lambda$ is open (resp. closed) in X_λ for every $\lambda \in \Lambda$.

Let $I \subset A$ be any open ideal of A; then I contains an ideal I_λ, and therefore $\operatorname{Spec} A/I$ is a closed subset of $\operatorname{Spec} A/I_\lambda$, hence also a closed subset of $\operatorname{Spf} A$.

15.1.8. We endow $\operatorname{Spf} A$ with a sheaf of topological rings as follows. For every $\lambda \in \Lambda$, the structure sheaf $\mathscr{O}_{X_\lambda}$ carries a natural *pseudo-discrete topology* that makes it a sheaf of topological rings (see [59, Ch.0, §3.8]). Let $j_\lambda : X_\lambda \to \operatorname{Spf} A$ be the closed immersion; we set

$$\mathscr{O}_{\operatorname{Spf} A} := \lim_{\lambda\in\Lambda} j_{\lambda *}\mathscr{O}_{X_\lambda}$$

where the limit is taken in the category of sheaves of topological rings ([59, Ch.0, §3.2.6]). By remark 5.5.2(iii), it follows that :

$$\mathscr{O}_{\operatorname{Spf} A}(U) = \lim_{\lambda\in\Lambda} \mathscr{O}_{X_\lambda}(U \cap X_\lambda) \tag{15.1.9}$$

for every open subset $U \subset \operatorname{Spf} A$. In this equality, the right-hand side is endowed with the topology of the (projective) limit, and the identification with the left-hand side is a homeomorphism. Set $X := \operatorname{Spf} A$; directly from the definitions we get natural maps of locally ringed spaces :

$$(X_\lambda, \mathscr{O}_{X_\lambda}) \xrightarrow{j_\lambda} (X, \mathscr{O}_X) \xrightarrow{i_X} (\operatorname{Spec} A^\wedge, \mathscr{O}_{\operatorname{Spec} A^\wedge}) \qquad \text{for every } \lambda \in \Lambda \tag{15.1.10}$$

where $A^\wedge := \mathscr{O}_X(X)$ is the (separated) completion of A, and i_X is given by the universal property [58, Ch.I, Prop.1.6.3] of the spectrum of a ring. Clearly the composition $i_X \circ j_\lambda$ is the map of affine schemes induced by the surjection $A^\wedge \to A/I_\lambda$. Therefore i_X identifies the set underlying X with the subset $\bigcup_{\lambda\in\Lambda} X_\lambda \subset \operatorname{Spec} A^\wedge$. However, the topology of X is usually strictly finer than the subspace topology on the image of i_X. For any $f \in A^\wedge$ we let:

$$\mathfrak{D}(f) := i_X^{-1} D(f)$$

where as usual, $D(f) \subset \operatorname{Spec} A^\wedge$ is the open subset consisting of all prime ideals that do not contain f.

15.1.11. Let $f : A \to B$ be a continuous map of topological rings whose topologies are linear. For every open ideal $J \subset B$, we have an induced map $A/f^{-1}J \to B/J$, and after taking colimits, a natural continuous map of topologically ringed spaces :

$$\operatorname{Spf} f : (\operatorname{Spf} B, \mathscr{O}_{\operatorname{Spf} B}) \to (\operatorname{Spf} A, \mathscr{O}_{\operatorname{Spf} A}).$$

Lemma 15.1.12. *In the situation of* (15.1.11)*, we have :*

(i) $\operatorname{Spf} A$ *is a locally and topologically ringed space.*

(ii) $\varphi := \operatorname{Spf} f : \operatorname{Spf} B \to \operatorname{Spf} A$ *is a morphism of locally and topologically ringed spaces; in particular, the induced map on stalks* $\mathscr{O}_{\operatorname{Spf} A,\varphi(x)} \to \mathscr{O}_{\operatorname{Spf} B,x}$ *is a local ring homomorphism, for every* $x \in \operatorname{Spf} B$.

(iii) *Let* $I \subset A$ *be any open ideal,* $x \in \operatorname{Spec} A/I \subset \operatorname{Spf} A$ *any point. The induced map :*

$$\mathscr{O}_{\operatorname{Spf} A,x} \to \mathscr{O}_{\operatorname{Spec} A/I,x} \tag{15.1.13}$$

is a surjection.

Proof. (i): We need to show that the stalk $\mathscr{O}_{X,x}$ is a local ring, for every $x \in X := \mathrm{Spf}\, A$. Let $(I_\lambda \mid \lambda \in \Lambda)$ be a cofiltered fundamental system of open ideals. Then $x \in X_\mu := \mathrm{Spec}\, A/I_\mu$ for some $\mu \in \Lambda$; let $\kappa(x)$ be the residue field of the stalk $\mathscr{O}_{X_\mu,x}$; there follows a ring homomorphism

$$\mathscr{O}_{X,x} \to (j_{\mu*}\mathscr{O}_{X_\mu})_x = \mathscr{O}_{X_\mu,x} \to \kappa(x). \tag{15.1.14}$$

Moreover, for every $\lambda \geq \mu$, the closed immersion $X_\mu \to X_\lambda$ induces an identification of $\kappa(x)$ with the residue field of $\mathscr{O}_{X_\lambda,x}$, so (15.1.14) is independent of μ.

Suppose now that $g \in \mathscr{O}_{X,x}$ is mapped to a non-zero element in $\kappa(x)$; according to (15.1.9) we may find an open subset $U \subset X$, such that g is represented as a compatible system $(g_\lambda \mid \lambda \geq \mu)$ of sections $g_\lambda \in \mathscr{O}_{X_\lambda}(U \cap X_\lambda)$. For every $\lambda \geq \mu$, let $V_\lambda \subset X_\lambda$ denote the open subset of all $y \in X_\lambda$ such that $g_\lambda(y) \neq 0$ in $\kappa(y)$. Then $V_\eta \cap X_\lambda = V_\lambda$ whenever $\eta \geq \lambda \geq \mu$. It follows that $V := \bigcup_{\lambda \geq \mu} V_\lambda$ is an open subset of X, and clearly g is invertible at every point of V, hence g is invertible in $\mathscr{O}_{X,x}$, which implies the contention.

(ii): The assertion is easily reduced to the corresponding statement for the induced morphisms of schemes : $\mathrm{Spec}\, B/J \to \mathrm{Spec}\, A/f^{-1}J$, for any open ideal $J \subset B$. The details shall be left to the reader.

(iii): Indeed, using (15.1.10) with $X := \mathrm{Spf}\, A$ and $I_\lambda := I$, we see that the natural surjection $\mathscr{O}_{\mathrm{Spec}\, A^\wedge,x} \to \mathscr{O}_{\mathrm{Spec}\, A/I,x}$ factors through (15.1.13). □

15.1.15. Let A and M be as in (15.1.2), and fix a (cofiltered) fundamental system $(M_\lambda \mid \lambda \in \Lambda)$ of open submodules $M_\lambda \subset M$. For every $\lambda \in \Lambda$ we may find an open ideal $I_\lambda \subset A$ such that M/M_λ is an A/I_λ-module. Let $j_\lambda : X_\lambda := \mathrm{Spec}\, A/I_\lambda \to X := \mathrm{Spf}\, A$ be the natural closed immersion (of ringed spaces). We define the topological $\mathscr{O}_X$-module :

$$M^\sim := \lim_{\lambda \in \Lambda} j_{\lambda*}(M/M_\lambda)^\sim$$

where, as usual, $(M/M_\lambda)^\sim$ denotes the quasi-coherent pseudo-discrete $\mathscr{O}_{X_\lambda}$-module associated with M/M_λ, and the limit is formed in the category of sheaves of topological $\mathscr{O}_X$-modules ([59, Ch.0, §3.2.6]). Thus, for every open subset $U \subset X$ one has the identity of topological modules

$$M^\sim(U) = \lim_{\lambda \in \Lambda} (M/M_\lambda)^\sim(U \cap X_\lambda) \tag{15.1.16}$$

that generalizes (15.1.9), and follows likewise from remark 5.5.2(iii).

To proceed beyond these simple generalities, we need to add further assumptions. The following definition covers all the situations that we shall find in the sequel.

Definition 15.1.17. Let A be a topological ring whose topology is linear.

(i) We say that A is *ω-admissible* if A is separated and complete, and $0 \in A$ admits a countable fundamental system of open neighborhoods.
(ii) We say that an open subset $U \subset \mathrm{Spf}\, A$ is *affine* if there exist an ω-admissible topological ring B and an isomorphism $(\mathrm{Spf}\, B, \mathscr{O}_{\mathrm{Spf}\, B}) \xrightarrow{\sim} (U, \mathscr{O}_{\mathrm{Spf}\, A|U})$ of topologically ringed spaces.
(iii) We say that an open subset $U \subset \mathrm{Spf}\, A$ is *truly affine* if $U \cap \mathrm{Spec}\, A/I$ is an affine open subset of $\mathrm{Spec}\, A/I$ for every open ideal $I \subset A$.
(iv) An *affine ω-formal scheme* is a topologically and locally ringed space $(X, \mathscr{O}_X)$ that is isomorphic to the formal spectrum of an ω-admissible topological ring.
(v) An *ω-formal scheme* is a topologically and locally ringed space $(X, \mathscr{O}_X)$ that admits an open covering $X = \bigcup_{i \in I} U_i$ such that, for every $i \in I$, the restriction $(U_i, \mathscr{O}_{X|U_i})$ is an affine ω-formal scheme.
(vi) A *morphism of ω-formal schemes* $f : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is a map of topologically and locally ringed spaces, *i.e.* a morphism of locally ringed spaces such that the corresponding map $\mathscr{O}_Y \to f_*\mathscr{O}_X$ induces continuous ring homomorphisms $\mathscr{O}_Y(U) \to \mathscr{O}_X(f^{-1}U)$, for every open subset $U \subset Y$.

Remark 15.1.18. (i) Let X be any ω-formal scheme, and $U \subset X$ any open subset. Using the condition of definition 5.5.1(ii), it is easily seen that :

(a) $\mathscr{O}_X(U)$ is a complete and separated topological ring.
(b) If additionally, $U = \bigcup_{i\in I} U_i$ for a countable family $(U_i \mid i \in I)$ of open subsets, such that $(U_i, \mathscr{O}_{X|U_i})$ is an affine ω-formal scheme for every $i \in I$ (briefly, each U_i is an *affine open subset* of X), then $\mathscr{O}_X(U)$ is ω-admissible.

(ii) In [58] one finds a notion of *admissible topological ring*, and to any such ring A it is assigned an affine formal scheme $\mathrm{Spf}\, A$. These notions relate to ours as follows. Let us say that a topological ring is c-admissible if it is admissible in the sense of [58, Ch.0, Déf.7.1.2] and $0 \in A$ admits a countable fundamental system of open neighborhoods. Then, an admissible topological ring A is ω-admissible if and only if it is c-admissible. Likewise, if A is admissible, then the formal scheme $\mathrm{Spf}\, A$ defined in [58, Ch.I, Déf.10.1.2] is an ω-formal scheme if and only if A is c-admissible. Indeed, if A is c-admissible, obviously $\mathrm{Spf}\, A$ coincides with the affine ω-formal scheme attached to A as in (15.1.7) and (15.1.8). For the converse, notice that $\mathrm{Spf}\, A$ (in the sense of [58]) is quasi-compact, so if it is an ω-formal scheme we may cover it by finitely many of its affine open subsets $U_1, \dots, U_n$; say that $U_i = \mathrm{Spf}\, B_i$ for $i = 1, \dots, n$, where $B_1, \dots, B_n$ are ω-admissible. There follows a natural continuous map $\rho : A \to B := \prod_{i=1}^n B_i$, and the topology of A agrees with the topology induced by B via ρ (remark 5.5.2(i)); however, clearly $0 \in B$ admits a countable fundamental system of open neighborhoods, so the same holds for A.

(iii) In the same vein, if A is ω-admissible, then $\mathrm{Spf}\, A$ as defined in (15.1.7) and (15.1.8) is a quasi-compact formal scheme in the sense of [58] if and only if A is c-admissible. Indeed, the condition is obviously sufficient; conversely, if $\mathrm{Spf}\, A$ is a quasi-compact formal scheme in the sense of [58], we may cover it by finitely many open subsets $U_1, \dots, U_n$ such that, for every $i = 1, \dots, n$ we have $U_i = \mathrm{Spf}\, B_i$ for some topological ring that is admissible in the sense of [58], and the topology of A is induced by the natural inclusion map into $B := \prod_{i=1}^n B_i$. For each such B_i, pick an open and nilpotent ideal $I_i \subset B_i$; then $I := \prod_{i=1}^n I_i$ is an open and nilpotent ideal of B, and $I \cap A$ is open and nilpotent in A, so the latter is c-admissible.

Proposition 15.1.19. *Suppose that A is an ω-admissible topological ring. Then :*

(i) *The truly affine open subsets form a basis for the topology of $X := \mathrm{Spf}\, A$.*
(ii) *Let $I \subset A$ be an open ideal, $U \subset X$ a truly affine open subset. Then :*
   (a) *The natural map $\rho_U : A \to A_U := \mathscr{O}_X(U)$ induces an isomorphism :*
$$(\mathrm{Spf}\, A_U, \mathscr{O}_{\mathrm{Spf}\, A_U}) \xrightarrow{\sim} (U, \mathscr{O}_{X|U}).$$
   (b) *The topological closure I_U of IA_U in A_U is an open ideal, and the natural map : $\mathrm{Spec}\, A_U/I_U \to \mathrm{Spec}\, A/I$ is an open immersion.*
   (c) *U is affine.*
   (d) *Every open covering of U admits a countable subcovering.*

Proof. (i): By assumption, we may find a countable fundamental system $(I_n \mid n \in \mathbb{N})$ of open ideals of A, and clearly we may assume that this system is ordered under inclusion (so that $I_n \subset I_m$ whenever $n \geq m$). Let $x \in \mathrm{Spf}\, A$ be any point, and $U \subset \mathrm{Spf}\, A$ an open neighborhood of x. Then $x \in X_n := \mathrm{Spec}\, A/I_n$ for sufficiently large $n \in \mathbb{N}$. We are going to exhibit, by induction on $m \in \mathbb{N}$, a sequence $(f_m \mid m \geq n)$ of elements of A, such that :

$$x \in \mathfrak{D}(f_p) \cap X_m = \mathfrak{D}(f_m) \cap X_m \subset U \qquad \text{whenever } p \geq m \geq n \tag{15.1.20}$$

To start out, we may find $f_n \in A/I_n$ such that $x \in \mathfrak{D}(f_n) \cap X_n \subset U \cap X_n$. Next, let $m \geq n$ and suppose that f_m has already been found; we may write

$$U \cap X_{m+1} = X_{m+1} \setminus V(J) \qquad \text{for some ideal } J \subset A/I_{m+1}.$$

Let $\overline{J} \subset A/I_m$ and $\overline{f}_m \in A/I_m$ be the images of J and f_m; then $V(\overline{J}) \subset V(\overline{f}_m)$, hence there exists $k \in \mathbb{N}$ such that $\overline{f}_m^k \in \overline{J}$. Pick $\overline{f}_{m+1} \in J$ such that the image of $\overline{f}_{m+1}$ in A/I_m agrees with $\overline{f}_m^k$, and let $f_{m+1} \in A$ be any lifting of $\overline{f}_{m+1}$; with this choice, one verifies easily that (15.1.20) holds for $p := m+1$. Finally, the subset :

$$U' := \bigcup_{m \geq n} \mathfrak{D}(f_m) \cap X_m$$

is an admissible affine open neighborhood of x contained in U.

(ii.a): For every $n, m \in \mathbb{N}$ with $n \geq m$, we have a closed immersion of affine schemes: $U \cap X_m \subset U \cap X_n$; whence induced surjections :

$$A_{U,n} := \mathscr{O}_{X_n}(U \cap X_n) \to A_{U,m} := \mathscr{O}_{X_m}(U \cap X_m).$$

By [59, Ch.0, §3.8.1], $A_{U,n}$ is a discrete topological ring, for every $n \in \mathbb{N}$, hence $A_U \simeq \lim_{n \in \mathbb{N}} A_{U,n}$ carries the linear topology that admits the fundamental system of open ideals $(\mathrm{Ker}\,(A_U \to A_{U,n}) \mid n \in \mathbb{N})$, especially A_U is ω-admissible. It follows that the topological space underlying $\mathrm{Spf}\,A_U$ is $\mathrm{colim}_{n \in \mathbb{N}}(U \cap X_n)$, which is naturally identified with U, under $\mathrm{Spf}\,\rho_U$. Likewise, let $i : U \to X$ be the open immersion; one verifies easily (*e.g.* using (15.1.9)) that the natural map :

$$i^* \lim_{n \in \mathbb{N}} j_{n*}\mathscr{O}_{X_n} \to \lim_{n \in \mathbb{N}} i^* j_{n*}\mathscr{O}_{X_n}$$

is an isomorphism of topological sheaves, which implies the assertion.

(ii.b): We may assume that $I_0 = I$. Since $U \cap X_n$ is affine for every $n \in \mathbb{N}$, we deduce short exact sequences :

$$\mathscr{E}_n := (0 \to I \cdot \mathscr{O}_{X_n}(X_n \cap U) \to \mathscr{O}_{X_n}(U \cap X_n) \to \mathscr{O}_{X_0}(U \cap X_0) \to 0)$$

and $\lim_{n \in \mathbb{N}} \mathscr{E}_n$ is the exact sequence :

$$0 \to I_U \to A_U \to \mathscr{O}_{X_0}(U \cap X_0) \to 0.$$

Since $U \cap X_0$ is an open subset of X_0, both assertions follow easily.

(ii.c): It has already been remarked that A_U is ω-admissible. More precisely, for every $n \in \mathbb{N}$ let $(I_n A_U)^c$ be the topological closure of $I_n A_U$ in A_U; then the proof of (ii.b) shows that the family of ideals $((I_n A_U)^c \mid n \in \mathbb{N})$ is a fundamental system of open neighborhoods of $0 \in A_U$. Hence the assertion follows from (ii.a).

(ii.d): By definition, U is a countable union of quasi-compact subsets, so the assertion is immediate. □

Corollary 15.1.21. *Let X be an ω-formal scheme, $U \subset X$ any open subset. Then $(U, \mathscr{O}_{X|U})$ is an ω-formal scheme.* □

Proposition 15.1.22. *Let X be an ω-formal scheme, A an ω-admissible topological ring. Then the rule :*

$$(f : X \to \mathrm{Spf}\,A) \mapsto (f^\natural : A \to \Gamma(X, \mathscr{O}_X)) \qquad (15.1.23)$$

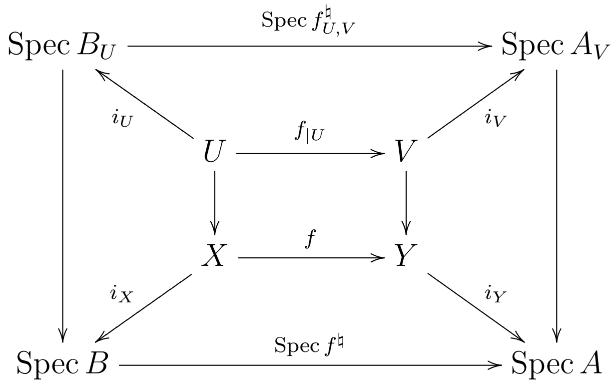
establishes a natural bijection between the set of morphisms of ω-formal schemes $X \to \mathrm{Spf}\,A$ and the set of continuous ring homomorphisms $A \to \Gamma(X, \mathscr{O}_X)$.

Proof. We reduce easily to the case where $X = \mathrm{Spf}\,B$ for some ω-admissible topological ring B. Let $f : X \to Y := \mathrm{Spf}\,A$ be a morphism of ω-formal schemes; we have to show that $f = \mathrm{Spf}\,f^\natural$, where $f^\natural : A \to B$ is the map on global sections induced by the morphism of sheaves $\mathscr{O}_Y \to f^*\mathscr{O}_X$ that defines f. Let $U \subset X$ and $V \subset Y$ be two truly affine open subsets, such that $f(U) \subset V$, and let likewise :

$$f^\natural_{U,V} : A_V := \mathscr{O}_Y(V) \to B_U := \mathscr{O}_X(U)$$

be the map induced by $f_{|U}$. Using the universal property of [58, Ch.I, Prop.1.6.3], we obtain a commutative diagram of morphisms of locally ringed spaces:

(15.1.24)



where i_X, i_Y, i_U and i_V are the morphisms of (15.1.10). Since i_X and i_Y are injective on the underlying sets, it follows already that f and $\mathrm{Spf}\, f^\natural$ induce the same continuous map of topological spaces. Let now $J \subset B$ be any open ideal, $I \subset f^{\natural -1}(J)$ an open ideal of A, and let J_U (resp. I_V) be the topological closure of JB_U in B_U (resp. of IA_V in A_V). Since the map $f^\natural_{U,V}$ is continuous, we derive a commutative diagram of schemes :

$$\begin{array}{ccc} \mathrm{Spec}\, B_U/J_U & \xrightarrow{\varphi} & \mathrm{Spec}\, A_V/I_V \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ \mathrm{Spec}\, B/J & \xrightarrow{\psi} & \mathrm{Spec}\, A/I \end{array}$$

where α and β are open immersions, by proposition 15.1.19(ii.b), and φ (resp. ψ) is induced by $f^\natural_{U,V}$ (resp. by $f^\natural$). Let $\mathscr{I}$ be a fundamental system of open neighborhoods of $0 \in A$ consisting of ideals, and for every $I \in \mathscr{I}$ denote by $(IA_V)^c$ the topological closure of IA_V in A_V; from proposition 15.1.19(ii.b) (and its proof) it follows as well that $A_V \simeq \lim_{I\in\mathscr{I}} A_V/(IA_V)^c$. Summing up, this shows that $f^\natural_{U,V}$ is determined by $f^\natural$, whence the contention. □

Corollary 15.1.25. *Let A be an ω-admissible topological ring, $U \subset \mathrm{Spf}\, A$ an affine open subset, $V \subset U$ a truly affine open subset of X. Then V is a truly affine open subset of U.*

Proof. Say that $U = \mathrm{Spf}\, B$, for some ω-admissible topological ring B; by proposition 15.1.22, the immersion $j : U \to X$ is of the form $j = \mathrm{Spf}\,\varphi$ for a unique map $\varphi : A \to B$. Let now $J \subset B$ be any open ideal, and set $I := \varphi^{-1}J$; there follows a commutative diagram of locally ringed spaces :

$$\begin{array}{ccc} U_0 := \mathrm{Spec}\, B/J & \longrightarrow & X_0 := \mathrm{Spec}\, A/I \\ \downarrow & & \downarrow \\ U & \xrightarrow{\ j\ } & X. \end{array}$$

By assumption, $V \cap X_0 = \mathrm{Spec}\, C$ for some A/I-algebra C; it follows that :

$$V \cap U_0 = \mathrm{Spec}\, B/J \otimes_{A/I} C$$

and since J is arbitrary, the claim follows. □

Remark 15.1.26. (i) Let A be an ω-admissible topological ring, and B, C two topological A-algebras whose topologies are linear (and the structure maps $A \to B$, $A \to C$ are continuous). Recall that $B \widehat{\otimes}_A C$ represents the product of B and C in the category of ω-admissible A-algebras (see (8.3.7)). By standard arguments, we deduce from proposition 15.1.22 that $\mathrm{Spf}\, B \widehat{\otimes}_A C$ represents the product of $\mathrm{Spf}\, B$ and $\mathrm{Spf}\, C$ in the category of ω-formal $\mathrm{Spf}\, A$-schemes; especially, the category of ω-formal schemes admits arbitrary fibre products.

(ii) We shall say that a topological A-module M is *ω-admissible* if M is complete and separated, and admits a countable fundamental system of open neighborhoods of $0 \in M$. Notice that the completion functor $N \mapsto N^\wedge$ on topological A-modules is not always "right exact", in the following sense. Suppose $N \to N'$ is a quotient map; then the induced map $N^\wedge \to N'^\wedge$ is not necessarily onto. However, this is the case if $N^\wedge$ is ω-admissible (see proposition 8.2.13(v)). For such A-modules, we have moreover the following :

Lemma 15.1.27. *Let A be a topological ring, M, M', N three ω-admissible topological A-modules, and $f : M \to M'$ a quotient map. Then :*

(i) *The homomorphism $f \widehat{\otimes}_A \mathbf{1}_N : M \widehat{\otimes}_A N \to M' \widehat{\otimes}_A N$ is a quotient map.*

(ii) *Let us endow $\mathrm{Ker}\, f$ with the subspace topology induced from M, and suppose additionally that, for every open ideal $I \subset A$:*

(a) *The topological closure $(IN)^c$ of IN in N is an open submodule of N.*

(b) *$N/(IN)^c$ is a flat A/I-module.*

Then the complex :

$$0 \to (\mathrm{Ker}\, f) \widehat{\otimes}_A N \to M \widehat{\otimes}_A N \to M' \widehat{\otimes}_A N \to 0$$

is an admissible short exact sequence of topological $A^\wedge$-modules.

Proof. By assumption, we may find an inverse system of discrete A-modules, with surjective transition maps $(M_n \mid n \in \mathbb{N})$ (resp. $(N_n \mid n \in \mathbb{N})$), and an isomorphism of topological A-modules : $M \simeq \lim_{n\in\mathbb{N}} M_n$ (resp. $N \simeq \lim_{n\in\mathbb{N}} N_n$); let us define $M'_n := M' \amalg_M M_n$ for every $n \in \mathbb{N}$ (the cofibred sum of M' and M_n over M). Since f is a quotient map, M'_n is a discrete A-module for every $n \in \mathbb{N}$, and the natural map : $M' \to \lim_{n\in\mathbb{N}} M'_n$ is a topological isomorphism. We deduce an inverse system of short exact sequences of discrete A-modules :

$$0 \to K_n \to M_n \otimes_A N_n \to M'_n \otimes_A N_n \to 0$$

where K_n is naturally a quotient of $\mathrm{Ker}(M_n \to M'_n) \otimes_A N_n$, for every $n \in \mathbb{N}$; especially, the transition maps $K_j \to K_i$ are surjective whenever $j \geq i$. Then assertion (i) follows from lemma 8.6.2.

(ii): We may find open ideals $I_n \subset A$ such that $I_n N_n = I_n M_n = 0$ for every $n \in \mathbb{N}$, hence $(I_n N)^c \subset \mathrm{Ker}(N \to N_n)$, and from (a) we deduce that $((I_n N)^c \mid n \in \mathbb{N})$ is a fundamental system of open neighborhoods of $0 \in N$. Due to (b), we obtain short exact sequences :

$$0 \to \mathrm{Ker}(M_n \to M'_n) \otimes_A N/(I_n N)^c \to M_n \otimes_A N/(I_n N)^c \to M'_n \otimes_A N/(I_n N)^c \to 0$$

for every $n \in \mathbb{N}$. Then it suffices to invoke again lemma 8.6.2. □

Lemma 15.1.28. *Let A be an ω-admissible topological ring, M a topological A-module, and $U \subset X := \mathrm{Spf}\, A$ any affine open subset. Then the following holds :*

(i) *There is a natural isomorphism of topological $\mathscr{O}_X(U)$-modules :*

$$M^\sim(U) \simeq M \widehat{\otimes}_A \mathscr{O}_X(U).$$

(ii) *If L is any other topological A-module, the natural map :*

$$\mathrm{top.Hom}_A(L, M^\wedge) \to \mathrm{top.Hom}_{\mathscr{O}_X}(L^\sim, M^\sim) \quad \varphi \mapsto \varphi^\sim$$

is an isomorphism.

(iii) *The functor $M \mapsto M^\sim$ on topological A-modules, is left adjoint to the global sections functor $\mathscr{F} \mapsto \mathscr{F}(X)$, defined on the category of complete and separated topological $\mathscr{O}_X$-modules and continuous maps.*

Proof. (i): First we remark that the assertion holds whenever U is a truly affine open subset of X; the easy verification shall be left to the reader. For a general U, set $M_U := M \widehat{\otimes}_A \mathscr{O}_X(U)$, and denote by $M_U^\sim$ the associated $\mathscr{O}_U$-module. Let $V \subset U$ be any open subset which is truly affine in X; by corollary 15.1.25, V is truly affine in U as well. We deduce natural isomorphisms:

$$M^\sim(V) \simeq M_U \widehat{\otimes}_{\mathscr{O}_X(U)} \mathscr{O}_X(V) \simeq M_U^\sim(V)$$

which – in view of proposition 15.1.19(i) – amount to a natural isomorphism of topological $\mathscr{O}_U$-modules : $(M^\sim)_{|U} \xrightarrow{\sim} M_U^\sim$. Assertion (i) follows easily.

(iii): Given a continuous map $M^\sim \to \mathscr{F}$, we get a map of global sections $M = M^\sim(X) \to \mathscr{F}(X)$. Conversely, suppose $f : M \to \mathscr{F}(X)$ is a given continuous map; let $U \subset X$ be any affine open subset, and $f_U : M \to \mathscr{F}(U)$ the composition of f and the restriction map $\mathscr{F}(X) \to \mathscr{F}(U)$. Then f_U extends first – by linearity – to a map $M \otimes_A \mathscr{O}_X(U) \to \mathscr{F}(U)$, and second – by continuity – to a map $M \widehat{\otimes}_A \mathscr{O}_X(U) \to \mathscr{F}(U)$; the latter, in view of (i), is the same as a map $f_U^\sim : M^\sim(U) \to \mathscr{F}(U)$. Clearly the rule $U \mapsto f_U^\sim$ thus defined is functorial for inclusion of open subsets $U \subset U'$, whence (iii).

(ii) is a straightforward consequence of (iii). □

Proposition 15.1.29. *Let A be an ω-admissible topological ring, M an ω-admissible topological A-module, $N \subset M$ a submodule, and $U \subset X := \mathrm{Spf}\, A$ an affine open subset. Then:*

(i) *If we endow M/N with the quotient topology, the sequence of $\mathscr{O}_X$-modules :*

$$0 \to N^\sim \to M^\sim \to (M/N)^\sim \to 0$$

is short exact.

(ii) *The induced sequence*

$$0 \to N^\sim(U) \to M^\sim(U) \to (M/N)^\sim(U) \to 0$$

is short exact and admissible in the sense of (8.6).

Proof. (i): We recall the following :

Claim 15.1.30. Let $j : Z' := \mathrm{Spec}\, R' \to Z := \mathrm{Spec}\, R$ be an open immersion of affine schemes. Then R' is a flat R-algebra.

Proof of the claim. The assertion can be checked on the localizations at the prime ideals of R'; however, the induced maps $\mathscr{O}_{Z,j(z)} \to \mathscr{O}_{Z',z}$ are isomorphisms for every $z \in Z'$, so the claim is clear. ◊

Let $V \subset X$ be any truly affine open subset. It follows easily from proposition 15.1.19(ii.b) and claim 15.1.30 that $\mathscr{O}_X(V)$ fulfills conditions (a) and (b) of lemma 15.1.27(ii). In light of lemma 15.1.28(i), we deduce that the sequence :

$$0 \to N^\sim(V) \to M^\sim(V) \to (M/N)^\sim(V) \to 0$$

is admissible short exact, whence the contention.

(ii): Clearly the sequence is left exact; using lemmata 15.1.27(i) and 15.1.28(i), we see that it is also right exact, and moreover $M^\sim(U) \to (M/N)^\sim(U)$ is a quotient map. It remains only to show that the topology on $N^\sim(U)$ is induced from $M^\sim(U)$, and to this aim we may assume – thanks to the condition of definition 5.5.1(ii) – that U is a truly affine open subset of X, in which case the assertion has already been observed in the proof of (i). □

Definition 15.1.31. Let X be an ω-formal scheme, and $\mathscr{F}$ a topological $\mathscr{O}_X$-module $\mathscr{F}$.

(i) We say that $\mathscr{F}$ is *quasi-coherent* if there exist a covering $\mathfrak{U} := (U_i \mid i \in I)$ of X consisting of affine open subsets, and for every $i \in I$, an ω-admissible topological $\mathscr{O}_X(U_i)$-module M_i with an isomorphism $\mathscr{F}_{|U_i} \xrightarrow{\sim} M_i^\sim$ of topological $\mathscr{O}_{U_i}$-modules (in the sense of definition 15.1.4(ii,iii)).

(ii) We denote by $\mathscr{O}_X\text{-}\mathbf{Mod}_{\mathrm{qcoh}}$ the category of quasi-coherent $\mathscr{O}_X$-modules and continuous $\mathscr{O}_X$-linear morphisms.

(iii) We say that a continuous morphism $f : \mathscr{F} \to \mathscr{G}$ of quasi-coherent $\mathscr{O}_X$-modules is a *quotient map* if the induced map $f(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ is a quotient map of topological $\mathscr{O}_X(U)$-modules, for every affine open subset $U \subset X$.

Remark 15.1.32. (i) The category $\mathscr{O}_X$-$\mathbf{Mod}_{\mathrm{qcoh}}$ is usually not abelian (see (8.6)); more than that, the kernel (in the category of abelian sheaves) of a continuous map $f : \mathscr{F} \to \mathscr{G}$ of quasi-coherent $\mathscr{O}_X$-modules may fail to be quasi-coherent. However, using proposition 15.1.29 one may show that $\mathrm{Ker}\, f$ is quasi-coherent whenever f is a quotient map, and in this case $\mathrm{Ker}\, f$ is also the kernel of f in the category $\mathscr{O}_X$-$\mathbf{Mod}_{\mathrm{qcoh}}$.

(ii) Furthermore, any (continuous) morphism f in $\mathscr{O}_X$-$\mathbf{Mod}_{\mathrm{qcoh}}$ admits a cokernel. This can be exhibited as follows. To start out, let us define presheaves $\mathscr{I}$ and $\mathscr{L}$ by declaring that $\mathscr{I}(U) \subset \mathscr{G}(U)$ is the topological closure of the $\mathscr{O}_X(U)$-submodule $\mathrm{Im}(f(U) : \mathscr{F}(U) \to \mathscr{G}(U))$, and $\mathscr{L}(U) := \mathscr{G}(U)/\mathscr{I}(U)$, which we endow with its natural quotient topology, for every open subset $U \subset X$. Now, suppose that $V \subset U$ is an inclusion of sufficiently small affine open subsets of X (so that $\mathscr{F}_{|U}$ and $\mathscr{G}_{|U}$ are of the form $M^\sim$ for some topological $\mathscr{O}_X(U)$-module M); since $\mathscr{F}(U)\widehat{\otimes}_{\mathscr{O}_X(U)}\mathscr{O}_X(V) = \mathscr{F}(V)$, we see that the image of $\mathscr{I}(U)$ in $\mathscr{I}(V)$ generates a dense $\mathscr{O}_X(V)$-submodule. On the other hand, by construction the exact sequence $\mathscr{E} := (0 \to \mathscr{I}(U) \to \mathscr{G}(U) \to \mathscr{L}(U) \to 0)$ is admissible, hence the same holds for the sequence $\mathscr{E}\widehat{\otimes}_{\mathscr{O}_X(U)}\mathscr{O}_X(V)$ (lemma 15.1.28(i) and proposition 15.1.29(ii)). It follows that $\mathscr{I}(V) = \mathscr{I}(U)\widehat{\otimes}_{\mathscr{O}_X(U)}\mathscr{O}_X(V)$ and $\mathscr{L}(V) = \mathscr{L}(U)\widehat{\otimes}_{\mathscr{O}_X(U)}\mathscr{O}_X(V)$. Thus $\mathscr{I}$ and $\mathscr{L}$ are sheaves of topological $\mathscr{O}_X$-modules on the site C of all sufficiently small affine open subsets of X, and their sheafifications $\mathscr{I}'$ and $\mathscr{L}'$ are the topological $\mathscr{O}_X$-modules obtained as in [59, Ch.0, §3.2.1], by extension of $\mathscr{I}$ and $\mathscr{L}$ from the site C to the whole topology of X. It follow that $\mathscr{I}'$ and $\mathscr{L}'$ are quasi-coherent $\mathscr{O}_X$-modules; then it is easy to check that $\mathscr{L}'$ is the cokernel of f in the category $\mathscr{O}_X$-$\mathbf{Mod}_{\mathrm{qcoh}}$ (briefly : the *topological cokernel of* f), and shall be denoted

$$\mathrm{top.Coker}\, f.$$

The sheaf $\mathscr{I}'$ shall be called *the topological closure of the image of* f, and denoted

$$\overline{\mathrm{Im}}(f).$$

A morphism f with $\mathrm{top.Coker}\, f = 0$ shall be called a *topological epimorphism.*

(iii) Notice also that the natural map $\mathrm{Coker}\, f \to \mathrm{top.Coker}\, f$ is an epimorphism of abelian sheaves, hence a continuous morphism of quasi-coherent $\mathscr{O}_X$-modules which is an epimorphism of $\mathscr{O}_X$-modules, is also a topological epimorphism.

Proposition 15.1.33. *Let X be an affine ω-formal scheme, $\mathscr{F}$ a quasi-coherent $\mathscr{O}_X$-module. Then $\mathscr{F}(X)$ is an ω-admissible $\mathscr{O}_X(X)$-module, and the natural map*

$$\mathscr{F}(X)^\sim \to \mathscr{F}$$

is an isomorphism of topological $\mathscr{O}_X$-modules.

Proof. By assumption we may find an affine open covering $\mathfrak{U} := (U_i \mid i \in I)$ of X such that, for every $i \in I$, $\mathscr{F}_{|U_i} \simeq M_i^\sim$ for some $\mathscr{O}_X(U_i)$-module M_i. In view of lemma 15.1.28(i) we may assume – up to replacing $\mathfrak{U}$ by a refinement – that U_i is a truly affine subset of X for every $i \in I$. Furthermore, we may write $X = \bigcup_{n\in\mathbb{N}} X_n$ for an increasing countable family of quasi-compact subsets; for each $n \in \mathbb{N}$ we may then find a finite subset $I(n) \subset I$ such that $X_n \subset \bigcup_{i\in I(n)} U_i$, and therefore we may replace I by $\bigcup_{n\in\mathbb{N}} I_n$, which allows to assume that I is countable.

Next, for every $i \in I$ we may find a countable fundamental system of open submodules $(M_{i,n} \mid n \in \mathbb{N})$ of M_i, and for every $n \in \mathbb{N}$ an open ideal $J_{i,n} \subset A_i := \mathscr{O}_X(U_i)$ such that $N_{i,n} := M_i/M_{i,n}$ is an $A_i/J_{i,n}$-module. Let $j_{i,n} : U_{i,n} := \mathrm{Spec}\, A_i/J_{i,n} \to U_i$ be the natural closed immersion; we may write :

$$\mathscr{F}_{|U_i} \simeq \lim_{n\in\mathbb{N}} j_{i,n*}N_{i,n}^\sim. \qquad (15.1.34)$$

Let also $\iota_{i,n} : U_{i,n} \to X$ be the locally closed immersion obtained as the composition of $j_{i,n}$ and the open immersion $j_i : U_i \to X$; we deduce natural maps of $\mathscr{O}_X$-modules :

$$\varphi_{i,n} : \mathscr{F} \to j_{i*}\mathscr{F}_{|U_i} \to \mathscr{G}_{i,n} := \iota_{i,n*}N^{\sim}_{i,n}.$$

Claim 15.1.35. (i) There exists $m \in \mathbb{N}$ such that $\mathscr{G}_{i,n}$ is the extension by zero of a quasi-coherent $\mathscr{O}_{X_m}$-module.

(ii) $\varphi_{i,n}$ is continuous for the pseudo-discrete topology on $\mathscr{G}_{i,n}$.

Proof of the claim. (i): It is easy to see that $U_{i,n} \subset X_m$ for $m \in \mathbb{N}$ large enough; then the assertion follows from ([59, Ch.I, Cor.9.2.2]).

(ii): We need to check that the map $\varphi_{i,n,V} : \mathscr{F}(V) \to \mathscr{G}_{i,n}(V)$ is continuous for every open subset $V \subset X$. However, the condition of definition 5.5.1(ii) implies that the assertion is local on X, hence we may assume that V is a truly affine open subset of X, in which case $\mathscr{G}_{i,n}(V)$ is a discrete space. We may factor $\varphi_{i,n,V}$ as a composition :

$$\mathscr{F}(V) \xrightarrow{\alpha} \mathscr{F}(V \cap U_i) \xrightarrow{\beta} N^{\sim}_{i,n}(U_{i,n} \cap V)$$

where the restriction map α is continuous, and β is continuous for the pseudo-discrete topology on $N^{\sim}_{i,n}(U_{i,n} \cap V)$. However, $U_i \cap V$ is a truly affine open subset of U_i; therefore $U_{i,n} \cap V$ is quasi-compact, and the pseudo-discrete topology on $N^{\sim}_{i,n}$ induces the discrete topology on $N^{\sim}_{i,n}(U_{i,n} \cap V)$. The claim follows. ◊

For every finite subset $S \subset I \times \mathbb{N}$, let $\varphi_S : \mathscr{F} \to \mathscr{G}_S := \prod_{(i,n)\in S} \mathscr{G}_{i,n}$ be the product of the maps $\varphi_{i,n}$; according to claim 15.1.35(ii), φ_S is continuous for the pseudo-discrete topology on $\mathscr{G}_S$. Hence, for every $i \in I$, the restriction $\varphi_{S|U_i} : \mathscr{F}_{|U_i} \to \mathscr{G}_{S|U_i}$ is of the form $f^{\sim}_{i,S}$, for some continuous map $f_{i,S} : M_i \to \mathscr{G}_S(U_i)$ (lemma 15.1.28(ii)). It follows easily that $(\mathrm{Im}\,\varphi_S)_{|U_i}$ is the quasi-coherent $\mathscr{O}_{U_i}$-module $(\mathrm{Im}\, f_{i,S})^{\sim}$, especially $\mathrm{Im}\,\varphi_S$ is quasi-coherent. Notice that $\mathscr{G}_S$ is already a quasi-coherent $\mathscr{O}_{X_m}$-module for $m \in \mathbb{N}$ large enough, hence the same holds for $\mathrm{Im}\,\varphi_S$; we may therefore find an $\mathscr{O}_{X_m}(X_m)$-module G_S such that $G^{\sim}_S \simeq \mathrm{Im}\,\varphi_S$. Furthermore, for every other finite subset $S' \subset I \times \mathbb{N}$ containing S, we deduce a natural $\mathscr{O}_X(X)$-linear surjection $G_{S'} \to G_S$, compatible with compositions of inclusions $S \subset S' \subset S''$. Let $(S_n \mid n \in \mathbb{N})$ be a countable increasing family of finite subsets, whose union is $I \times \mathbb{N}$; we endow $G := \lim_{n\in\mathbb{N}} G_{S_n}$ with the pro-discrete topology. It is then easy to check that G is an ω-admissible $\mathscr{O}_X(X)$-module; furthermore, by construction we get a unique continuous map $\varphi : \mathscr{F} \to G^{\sim}$ whose composition with the projection onto $G^{\sim}_{S_n}$ agrees with φ_{S_n} for every $n \in \mathbb{N}$. In light of (15.1.34) we see easily that $\varphi_{|U_i}$ is an isomorphism of topological $\mathscr{O}_{U_i}$-modules for every $i \in I$, hence the same holds for φ. Then φ necessarily induces an isomorphism $\mathscr{F}(X) \xrightarrow{\sim} G$. □

Corollary 15.1.36. *Let X be an ω-formal scheme, $(\mathscr{F}_n \mid n \in \mathbb{N})$ an inverse system of quasi-coherent $\mathscr{O}_X$-modules, whose transition maps are topological epimorphisms. Then $\lim_{n\in\mathbb{N}} \mathscr{F}_n$ (with its inverse limit topology) is a quasi-coherent $\mathscr{O}_X$-module.*

Proof. We may assume that X is affine; then, for every $n \in \mathbb{N}$ we have $\mathscr{F}_n = M^{\sim}_n$ for some complete and separated $\mathscr{O}_X(X)$-module M_n (by proposition 15.1.33), and the transition maps $\mathscr{F}_{n+1} \to \mathscr{F}_n$ come from corresponding continuous linear maps $f_n : M_{n+1} \to M_n$. We choose inductively, for every $n \in \mathbb{N}$, a descending fundamental system of open submodules $(N_{n,k} \mid k \in \mathbb{N})$ of M_n, such that $f_n(N_{n+1,k}) \subset N_{n,k}$ for every $n, k \in \mathbb{N}$. Set $M_{n,k} := M_n/N_{n,k}$ for every $n, k \in \mathbb{N}$. By assumption, $\mathrm{top.Coker}\, f^{\sim}_n = 0$; hence the induced maps $M^{\sim}_{n+1,k} \to M^{\sim}_{n,k}$ are topological epimorphisms; since $M^{\sim}_{n,k}$ is pseudo-discrete (on the closure of its support), it follows that the latter maps are even epimorphisms (of abelian sheaves) so the corresponding maps $M_{n+1,k} \to M_{n,k}$ are onto for every $n, k \in \mathbb{N}$. Thus :

$$\lim_{n\in\mathbb{N}} \mathscr{F}_n \simeq \lim_{n\in\mathbb{N}} \lim_{k\in\mathbb{N}} M^{\sim}_{n,k} \simeq \lim_{n\in\mathbb{N}} M^{\sim}_{n,n} \simeq (\lim_{n\in\mathbb{N}} M_{n,n})^{\sim}$$

and it is clear that the $\mathscr{O}_X(X)$-module $\lim_{n\in\mathbb{N}} M_{n,n}$ is ω-admissible. $\square$

Theorem 15.1.37. *Let X be an ω-formal scheme, $\mathscr{F}$ a quasi-coherent $\mathscr{O}_X$-module, $\mathfrak{U} := (U_i \mid i \in I)$ an open covering of X, such that the intersection $U_{t_0} \cap \cdots \cap U_{t_n}$ is affine for every $n \in \mathbb{N}$ and every $(t_0, \dots, t_n) \in I^{n+1}$. Then :*

(i) *There is a natural isomorphism (notation of* (10.2.18)*)*

$$H^\bullet_{\mathrm{alt}}(\mathfrak{U}, \mathscr{F}) \xrightarrow{\sim} H^\bullet(X, \mathscr{F}).$$

(ii) *If moreover X is affine, we have $H^i(X, \mathscr{F}) = 0$ for every $i > 0$.*

Proof. (The statements refer to the cohomology of the abelian sheaf underlying $\mathscr{F}$, in other words, we forget the topology of the modules $\mathscr{F}(U)$.)

(ii): Say that $X = \operatorname{Spf} A$ for an ω-admissible topological ring A. Let $U \subset X$ be any truly affine open subset, and $\mathfrak{U} := (U_i \mid i \in I)$ a covering of U consisting of truly affine open subsets. By proposition 15.1.33 we have $\mathscr{F}_{|U} \simeq M^\sim$, where $M := \mathscr{F}(U)$.

Claim 15.1.38. The augmented alternating Čech complex $C^\bullet_{\mathrm{alt}}(\mathfrak{U}, M^\sim)$ is acyclic.

Proof of the claim. Let $(M_n \mid n \in \mathbb{N})$ be a fundamental system of neighborhoods of $0 \in M$, consisting of open submodules, and for every $n \in \mathbb{N}$, choose an open ideal $I_n \subset A$ such that $I_n M \subset M_n$. Set $X_n := \operatorname{Spec} A/I_n$ and $\mathfrak{U}_n := (U_i \cap X_n \mid i \in I)$ for every $n \in \mathbb{N}$. For every $n \in \mathbb{N}$ we may consider the augmented alternating Čech complex $C^\bullet_{\mathrm{alt}}(\mathfrak{U}_n, (M/M_n)^\sim)$, and in view of (15.1.16) we obtain a natural isomorphism of complexes :

$$C^\bullet_{\mathrm{alt}}(\mathfrak{U}, M^\sim) \xrightarrow{\sim} \lim_{n\in\mathbb{N}} C^\bullet_{\mathrm{alt}}(\mathfrak{U}_n, (M/M_n)^\sim).$$

We may view the double complex $C^\bullet_{\mathrm{alt}}(\mathfrak{U}_\bullet, (M/M_\bullet)^\sim)$ also as a complex of inverse systems of modules, whose term in degree $i \in \mathbb{N}$ is $C^i_{\mathrm{alt}}(\mathfrak{U}_\bullet, (M/M_\bullet)^\sim)$. Notice that, for every $i \in \mathbb{N}$, all the transition maps of this latter inverse system are surjective. Hence ([163, Lemma 3.5.3]) :

$$\lim_{n\in\mathbb{N}}{}^q\, C^i_{\mathrm{alt}}(\mathfrak{U}_n, (M/M_n)^\sim) = 0 \qquad \text{for every } q > 0.$$

In other words, these inverse systems are acyclic for the inverse limit functor. It follows (*e.g.* by means of [163, Th.10.5.9]) that :

$$R\lim_{n\in\mathbb{N}} C^\bullet_{\mathrm{alt}}(\mathfrak{U}_\bullet, (M/M_\bullet)^\sim) \simeq C^\bullet_{\mathrm{alt}}(\mathfrak{U}, M^\sim).$$

On the other hand, the complexes $C^\bullet_{\mathrm{alt}}(\mathfrak{U}_n, (M/M_n)^\sim)$ are acyclic for every $n \in \mathbb{N}$ ([61, Ch.III, Th.1.3.1 and Prop.1.4.1]), hence $C^\bullet_{\mathrm{alt}}(\mathfrak{U}_\bullet, (M/M_\bullet)^\sim)$ is acyclic, when viewed as a cochain complex of inverse systems of modules. The claim follows. $\diamondsuit$

Assertion (ii) follows from claim 15.1.38, proposition 15.1.19(i) and theorem 10.2.24(i).
Assertion (i) follows from (ii) and corollary 10.2.21(ii). $\square$

Corollary 15.1.39. *In the situation of* (15.1.15)*, suppose that A and M are ω-admissible. Then:*

$$\lim_{\lambda\in\Lambda}{}^q\, j_{\lambda *}(M/M_\lambda)^\sim = 0 \qquad \textit{for every } q > 0.$$

Proof. For every truly affine open subset $U \subset X$, we have a topos U^Λ defined as in [75, §7.3.4], and the cofiltered system $\mathscr{M} := (j_{\lambda *}(M/M_\lambda)^\sim \mid \lambda \in \Lambda)$ defines an abelian sheaf on U^Λ. According to *loc.cit.* there are two spectral sequences :

$$E_2^{pq} := R^p\Gamma(U, \lim_{\lambda\in\Lambda}{}^q\, j_{\lambda *}(M/M_\lambda)^\sim) \Rightarrow H^{p+q}(U^\Lambda, \mathscr{M})$$

$$F_2^{pq} := \lim_{\lambda\in\Lambda}{}^q\, R^p\Gamma(U, j_{\lambda *}(M/M_\lambda)^\sim) \Rightarrow H^{p+q}(U^\Lambda, \mathscr{M})$$

and we notice that $F_2^{pq} = 0$ whenever $p > 0$ (since $U \cap X_\lambda$ is affine for every $\lambda \in \Lambda$) and whenever $q > 0$, since the cofiltered system $(\Gamma(U, j_{\lambda *}(M/M_\lambda)^\sim) \mid \lambda \in \Lambda)$ has surjective transition maps. One can then argue as in the proof of [75, Lemma 7.3.5] : the sheaf $L^q := \lim^q_{\lambda \in \Lambda} j_{\lambda *}(M/M_\lambda)^\sim$ is the sheafification of the presheaf : $U \mapsto H^q(U^\Lambda, \mathscr{M})$ and the latter vanishes by the foregoing. We supply an alternative argument. Since the truly affine open subsets form a basis of X, it suffices to show that $E_2^{0q} = 0$ whenever $q > 0$. We proceed by induction on q. For $q = 1$, we look at the differential $d_2^{01} : E_2^{0,1} \to E_2^{2,0}$; by theorem 15.1.37 we have $E_2^{2,0} = 0$, hence $E_2^{0,1} = E_\infty^{0,1}$, and the latter vanishes by the foregoing. Next, suppose that $q > 1$, and that we have shown the vanishing of L^j for $1 \leq j < q$. It follows that $E_2^{pj} = 0$ whenever $1 \leq j < q$, hence $E_r^{pj} = 0$ for every $r \geq 2$ and the same values of j. Consequently :

$$0 = E_\infty^{0q} \simeq \mathrm{Ker}(d_{q+1}^{0q} : E_2^{0q} = E_{q+1}^{0q} \to E_{q+1}^{q+1,0}).$$

However, theorem 15.1.37 implies that $E_r^{p0} = 0$ whenever $p > 0$ and $r \geq 2$, therefore $E_2^{0q} = E_\infty^{0q}$, *i.e.* $E_2^{0q} = 0$, which completes the inductive step. □

15.1.40. Let X be an ω-formal scheme, $\mathscr{F}$ a quasi-coherent $\mathscr{O}_X$-module. For every affine open subset $U \subset X$, we let $\mathrm{Cl}_\mathscr{F}(U)$ be the set consisting of all closed $\mathscr{O}_X(U)$-submodules of $\mathscr{F}(U)$. It follows from proposition 15.1.29(ii) that the rule

$$U \mapsto \mathrm{Cl}_\mathscr{F}(U)$$

defines a presheaf on the site of all affine open subsets of X. Namely, for an inclusion $U' \subset U$ of affine open subsets, the restriction map $\mathrm{Cl}_\mathscr{F}(U) \to \mathrm{Cl}_\mathscr{F}(U')$ assigns to $N \subset \mathscr{F}(U)$ the submodule $N^\sim(U') \subset \mathscr{F}(U')$ (here $N^\sim$ is a quasi-coherent $\mathscr{O}_U$-module).

Proposition 15.1.41. *With the notation of* (15.1.40)*, the presheaf* $\mathrm{Cl}_\mathscr{F}$ *is a sheaf on the site of affine open subsets of* X.

Proof. Let $U \subset X$ be an affine open subset, and $U = \bigcup_{i \in I} U_i$ a covering of U by affine open subsets $U_i \subset X$. For every $i, j \in I$ we let $U_{ij} := U_i \cap U_j$. Suppose there is given, for every $i \in I$, a closed $\mathscr{O}_X(U_i)$-submodule $N_i \subset \mathscr{F}(U_i)$, with the property that :

$$N_{ij} := N_i^\sim(U_{ij}) = N_j^\sim(U_{ij}) \qquad \text{for every } i, j \in I$$

(an equality of topological $\mathscr{O}_X(U_{ij})$-submodules of $\mathscr{F}(U_{ij})$). Then $(N_i^\sim)_{|U_{ij}} = (N_j^\sim)_{|U_{ij}}$, by proposition 15.1.33, hence there exist a quasi-coherent $\mathscr{O}_U$-module $\mathscr{N}$, and isomorphisms $\mathscr{N}_{|U_i} \xrightarrow{\sim} N_i^\sim$, for every $i \in I$, such that the induced $\mathscr{O}_{U_i}$-linear maps $N_i^\sim \to \mathscr{F}_{|U_i}$ assemble into a continuous $\mathscr{O}_U$-linear morphism $\varphi : \mathscr{N} \to \mathscr{F}_{|U}$. By construction, we have a commutative diagram of continuous maps with exact rows :

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathscr{N}(U) & \xrightarrow{\rho_\mathscr{N}} & \prod_{i \in I} N_i & \longrightarrow & \prod_{i,j \in I} N_{ij} \\
& \downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow & \mathscr{F}(U) & \xrightarrow{\rho_\mathscr{F}} & \prod_{i \in I} \mathscr{F}(U_i) & \longrightarrow & \prod_{i,j \in I} \mathscr{F}(U_{ij})
\end{array}$$

where the central vertical arrow is a closed immersion. However, the condition of definition 5.5.1(ii) implies that both $\rho_\mathscr{N}$ and $\rho_\mathscr{F}$ are admissible monomorphisms (in the sense of (8.6)), and moreover the image of $\rho_\mathscr{N}$ is a closed submodule, since each N_{ij} is a separated module. Hence also the left-most vertical arrow is a closed immersion, and the assertion follows. □

Definition 15.1.42. Let $f : X \to Y$ be a morphism of ω-formal schemes. We say that f is *affine* (resp. a *closed immersion*) if there exists a covering $Y = \bigcup_{i \in I} U_i$ by affine open subsets, such that for every $i \in I$, the open subset $f^{-1}U_i \subset X$ is an affine ω-formal scheme (resp. is isomorphic, as a U_i-scheme, to an ω-formal scheme of the form $\mathrm{Spf}\, A_i/J_i$, where $A_i := \mathscr{O}_Y(U_i)$ and $J_i \subset A_i$ is a closed ideal).

Corollary 15.1.43. *Let $f : X \to Y$ be a morphism of ω-formal schemes. The following conditions are equivalent :*

(i) *f is an affine morphism (resp. a closed immersion).*
(ii) *For every affine open subset $U \subset Y$, the preimage $f^{-1}U$ is an affine ω-formal scheme (resp. is isomorphic, as a U-scheme, to the ω-formal scheme $\mathrm{Spf}\, A/J$, where $A := \mathscr{O}_Y(U)$, and $J := \mathrm{Ker}(A \to \mathscr{O}_X(f^{-1}U)) \subset A$ is a closed ideal).*

Proof. Of course, it suffices to show that (i)⇒(ii). Hence, suppose that f is an affine morphism, and choose an affine open covering $Y = \bigcup_{i\in I} U_i$ such that $f^{-1}U_i$ is an affine ω-formal scheme for every $i \in I$. We may find an affine open covering $U = \bigcup_{j\in J} V_j$ such that, for every $j \in J$ there exists $i \in I$ with $V_j \subset I_i$; moreover, we may assume that J is countable, by proposition 15.1.19(ii.d). The assumption implies that $f^{-1}V_j$ is affine for every $j \in J$, and then remark 15.1.18(i.b) says that $A := \mathscr{O}_X(f^{-1}U)$ is an ω-admissible topological ring, so there exists a unique morphism $g : f^{-1}U \to \mathrm{Spf}\, A$ such that $g^\natural : A \to \mathscr{O}_X(f^{-1}U)$ is the identity (proposition 15.1.22), and $f_{|f^{-1}U}$ factors as the composition of g and the morphism $h : \mathrm{Spf}\, A \to U$ induced by the natural map $B := \mathscr{O}_Y(U) \to A$. It remains to check that g is an isomorphism. To this aim, it suffices to verify that the restriction $g_{|V_j} : f^{-1}V_j \to h^{-1}V_j$ is an isomorphism for every $j \in J$. However, it is clear that $f_*\mathscr{O}_X$ is a quasi-coherent $\mathscr{O}_Y$-module, whence a natural isomorphism of A-algebras :

$$\mathscr{O}_X(f^{-1}V_j) = f_*\mathscr{O}_X(V_j) \xrightarrow{\sim} A_j := A\widehat{\otimes}_B \mathscr{O}_Y(V_j)$$

(proposition 15.1.33) as well as an isomorphism $f^{-1}V_j \xrightarrow{\sim} \mathrm{Spf}\, A_j$. On the other hand, $h^{-1}V_j = \mathrm{Spf}\, A_j$, and by construction $g_{|V_j}$ is the unique morphism such that $(g_{|V_j})^\natural$ is the identity map of A_j. The assertion follows.

Next, suppose that f is a closed immersion, and choose an affine open covering $Y = \bigcup_{i\in I} U_i$, and closed ideals $J_i \subset A_i := \mathscr{O}_Y(U_i)$ such that we have isomorphisms $f^{-1}U_i \xrightarrow{\sim} \mathrm{Spf}\, A_i/J_i$ for every $i \in I$. Set $U_{ij} := U_i \cap U_j$ for every $i, j \in I$. Clearly, $J_i^\sim(U_{ij}) = J_j^\sim(U_{ij})$ for every $i, j \in I$, hence there exists a unique closed ideal $J \subset A := \mathscr{O}_Y(U)$ such that :

$$J^\sim(U_i) = J_i \qquad \text{for every } i \in I \tag{15.1.44}$$

(proposition 15.1.41). Especially, we have $J\mathscr{O}_Y(U_i) \subset J_i$, whence a unique morphism of U-schemes : $g_i : f^{-1}U_i \to Z := \mathrm{Spf}\, A/J$, for every $i \in I$. The uniqueness of g_i implies in particular that $g_{i|U_{ij}} = g_{j|U_{ij}}$ for every $i, j \in I$, whence a unique morphism $g : f^{-1}U \to Z$ of U-schemes. It remains to verify that g is an isomorphism, and to this aim it suffices to check that the restriction $g^{-1}(U_i \cap Z) \to U_i \cap Z$ is an isomorphism for every $i \in I$. The latter assertion is clear, in view of (15.1.44). □

Corollary 15.1.45. *Let A be an ω-admissible topological ring, $U \subset X := \mathrm{Spf}\, A$ an affine open subset. We have :*

(i) *The following conditions are equivalent :*
  (a) *U is truly affine.*
  (b) *$U \cap \mathrm{Spec}\, A/I$ is quasi-compact, for every open ideal $I \subset A$.*
(ii) *Especially, every quasi-compact affine open subset of X is truly affine.*

Proof. (i): Obviously (i.a)⇒(i.b). Conversely, let $j : Y := \mathrm{Spf}\, A/I \to X$ be the closed immersion, and set $A_U := \mathscr{O}_X(U)$. On the one hand, we have $j_*\mathscr{O}_Y = (A/I)^\sim$; on the other hand, proposition 15.1.29(ii) gives an admissible short exact sequence of topological A_U-modules $0 \to I^\sim(U) \to A_U \to B := (A/I)^\sim(U) \to 0$, and notice that the topology of B is discrete, since U is quasi-compact. Thus, $(Y \cap U, (\mathscr{O}_Y)_{|U}) = \mathrm{Spf}\, B = \mathrm{Spec}\, B$, by corollary 15.1.43(ii).

(ii) is an immediate consequence of (i). □

Remark 15.1.46. (i) Let A be an ω-admissible topological ring. Then $\mathrm{Spf}\, A$ may well contain affine subsets that are not truly affine. As an example, consider the one point compactification $X := \mathbb{N} \cup \{\infty\}$ of the discrete topological space $\mathbb{N}$ (this is the space which induces the discrete topology on its subset $\mathbb{N}$, and such that the open neighborhoods of ∞ are the complements of the finite subsets of $\mathbb{N}$). We choose any field F, which we endow with the discrete topology, and let A be the ring of all continuous functions $X \to F$. We endow A with the discrete topology, in which case $\mathrm{Spf}\, A = \mathrm{Spec}\, A$, and one can exhibit a natural homeomorphism $\mathrm{Spec}\, A \stackrel{\sim}{\to} X$ (exercise for the reader). On the other hand, we have an isomorphism of topological rings :

$$\mathscr{O}_{\mathrm{Spf}\, A}(\mathbb{N}) = \lim_{b \in \mathbb{N}} \mathscr{O}_{\mathrm{Spec}\, A}(\{0, \dots, b\}) \simeq k^{\mathbb{N}}$$

where $k^{\mathbb{N}}$ is endowed with the product topology. A verification that we leave to the reader, shows that the natural injective ring homomorphism $A \to k^{\mathbb{N}}$ induces an isomorphism of ringed spaces

$$\mathrm{Spf}\, k^{\mathbb{N}} \stackrel{\sim}{\to} (\mathbb{N}, \mathscr{O}_{\mathrm{Spf}\, A|\mathbb{N}})$$

hence $\mathbb{N} \subset \mathrm{Spf}\, A$ is an affine subset. However, $\mathbb{N}$ is not an affine subset of $\mathrm{Spec}\, A$, hence $\mathbb{N}$ is not a truly affine open subset of $\mathrm{Spf}\, A$.

(ii) On the other hand, suppose that A is c-admissible, in the sense of remark 15.1.18(ii), and let $U \subset \mathrm{Spf}\, A$ be any open subset. Combining corollary 15.1.45(ii) and remark 15.1.18, we see that U is an affine formal scheme in the sense of [58] if and only if it is an affine and quasi-compact ω-formal scheme, if and only if it is truly affine.

We conclude this section by reviewing briefly a standard method for getting formal schemes out of usual schemes.

15.1.47. Let X be a scheme, $\mathscr{I} \subset \mathscr{O}_X$ a quasi-coherent sheaf of ideals of finite type, and $\mathscr{F}$ a quasi-coherent $\mathscr{O}_X$-module. Let also $i : X_0 := \mathrm{Spec}\, \mathscr{O}_X/\mathscr{I} \to X$ be the closed immersion. For every $n \in \mathbb{N}$, set $\mathscr{F}_n := \mathscr{F}/\mathscr{I}^{n+1}\mathscr{F}$; we endow the $i^{-1}\mathscr{O}_X$-module $i^{-1}(\mathscr{F}_n)$ with its pseudo-discrete topology. Following [58, Ch.I, Déf.10.8.2], we define the *completion of $\mathscr{F}$ along X_0*

$$\mathscr{F}^\wedge := \lim_{n \in \mathbb{N}} i^{-1}\mathscr{F}_n$$

where the limit is taken in the category of sheaves of topological abelian groups. Especially, we may consider the completion $\mathscr{O}_{X^\wedge} := \mathscr{O}_X^\wedge$ which is naturally a sheaf of topological rings on X_0, and clearly $\mathscr{F}^\wedge$ is naturally an $\mathscr{O}_{X^\wedge}$-module. For every open subset $U \subset X$, set $U_0 := U \cap X_0$; notice that, if $U \subset U'$ are two such open subsets with $U_0 = U'_0$, then the restriction map $\mathscr{F}_n(U') \to \mathscr{F}_n(U)$ is an isomorphism for every $n \in \mathbb{N}$, since the support of $\mathscr{F}_n$ lies in X_0. There follows a natural identification

$$i^{-1}\mathscr{F}_n(U_0) \stackrel{\sim}{\to} \mathscr{F}_n(U) \qquad \text{for every } n \in \mathbb{N} \text{ and every open subset } U \subset X$$

whence, by virtue of remark 5.5.2(iii) a natural isomorphism of topological groups

$$\mathscr{F}^\wedge(U_0) \stackrel{\sim}{\to} \lim_{n \in \mathbb{N}} \mathscr{F}_n(U) \qquad \text{for every open subset } U \subset X.$$

If moreover U is quasi-compact, then $\mathscr{F}_n(U)$ is a discrete topological group for every $n \in \mathbb{N}$. If U is affine, and $A_U := \mathscr{O}_X(U)$, $I_U := \mathscr{I}(U)$, we have $\mathscr{F}_n(U) = A_U/I_U^n \otimes_{A_U} \mathscr{F}(U)$ for every $n \in \mathbb{N}$, so that

$$\mathscr{F}^\wedge(U_0) = \mathscr{F}(U)^\wedge$$

where $\mathscr{F}(U)^\wedge$ denotes the I_U-adic completion of $\mathscr{F}(U)$. Especially, the ringed space

$$(U_0, (\mathscr{O}_{X^\wedge})_{|U_0})$$

is an affine ω-formal scheme (see definition 15.1.17(iv)), isomorphic to $\mathrm{Spf}\, \mathscr{O}_X(U)^\wedge$. Thus

$$X^\wedge := (X_0, \mathscr{O}_{X^\wedge})$$

is an ω-formal scheme called the *completion of* X *along* X_0, and $\mathscr{F}^\wedge$ is a quasi-coherent $\mathscr{O}_{X^\wedge}$-module (see definition 15.1.31(i)). Furthermore, the system of projections $(i^{-1}\mathscr{F} \to \mathscr{F}_n)$ yields a morphism of $i^{-1}\mathscr{O}_X$-modules

$$i^{-1}\mathscr{F} \to \mathscr{F}^\wedge.$$

Especially, the map $\vartheta : i^{-1}\mathscr{O}_X \to \mathscr{O}_{X^\wedge}$ is a morphism of sheaves of rings, and the pair

$$\pi_X := (i, \vartheta) : X^\wedge \to X$$

is a natural morphism of ringed spaces. By inspecting the proof of lemma 15.1.12(i), we see more precisely that π_X is a morphism of locally ringed spaces.

Remark 15.1.48. In the situation of (15.1.47), let $U \subset X^\wedge$ be any quasi-compact open subset. Then the topology of $\mathscr{F}(U)$ is the linear topology defined by the system of submodules

$$(\Gamma(U, (\mathscr{I}^n\mathscr{F})^\wedge) \mid n \in \mathbb{N}).$$

Indeed, the assertion is clear if U is affine, since in this case the ideal I_U is finitely generated (see remark 8.3.3(ii,iv)). For the general case, let $(U_\lambda \mid \lambda \in \Lambda)$ be a finite affine covering of U, and endow $M := \prod_{\lambda\in\Lambda} \mathscr{F}(U_\lambda)$ with the product topology; according to remark 5.5.2(i), the topology of $\mathscr{F}(U)$ agrees with the topology induced by M via the natural injection $\mathscr{F}(U) \to M$. However, the topology of M is I_U-adic, and we have $\mathscr{F}(U) \cap I_U^n M = \Gamma(U, (\mathscr{I}^n\mathscr{F})^\wedge)$ for every $n \in \mathbb{N}$, whence the contention.

15.1.49. Keep the notation of (15.1.47), and let Y be another scheme, $\mathscr{J} \subset \mathscr{O}_Y$ another quasi-coherent sheaf of ideals, and $f : X \to Y$ a morphism of schemes such that the associated map of sheaves $\mathscr{O}_Y \to f_*\mathscr{O}_X$ restricts to a morphism $\mathscr{J} \to f_*\mathscr{I}$. We may then consider the completion $Y^\wedge$ of Y along $Y_0 := \mathrm{Spec}\mathscr{O}_Y/\mathscr{J}$. Let also $U \subset X$ and $V \subset Y$ be two affine open subsets with $f(U) \subset V$, set $I_U := \mathscr{I}(U) \subset A_U := \mathscr{O}_X(U)$, $J_V := \mathscr{J}(V) \subset B_V := \mathscr{O}_Y(V)$, and endow A_U (resp. B_V) with its I_U-adic (resp. J_V-adic) topology; it follows that the resulting map $B_V \to A_U$ is continuous, and therefore we get an induced morphism

$$f^\wedge_{U,V} : (U_0, (\mathscr{O}_{X^\wedge})_{|U_0}) \to (V_0, (\mathscr{O}_{Y^\wedge})_{|V_0}).$$

If moreover, $U' \subset U$ and $V' \subset V$ is another pair of affine open subsets such that $f(U') \subset V'$, it is clear that $f^\wedge_{U',V'}$ agrees with the restriction of $f^\wedge_{U,V}$. We deduce a well defined morphism of formal schemes

$$f^\wedge : X^\wedge \to Y^\wedge$$

that makes commute the diagram

$$\begin{array}{ccc} X^\wedge & \xrightarrow{f^\wedge} & Y^\wedge \\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle \pi_Y} \\ X & \xrightarrow{f} & Y. \end{array}$$

Especially, if f is an open immersion and $\mathscr{I} = f^{-1}\mathscr{J}$, then clearly $f^\wedge$ is an open immersion.

15.2. **Analytically noetherian rings.**

Definition 15.2.1. Let A be an adic topological ring with a finitely generated ideal I of adic definition.

(i) We say that A is *analytically noetherian* if the following two conditions hold :

  (a) The analytic locus of $\mathrm{Spec}\, A$ is a noetherian scheme (see definition 8.3.28).

  (b) For every finitely generated A-module M, the increasing sequence of submodules

$$(\mathrm{Ann}_M(I^n) \mid n \in \mathbb{N})$$

  is stationary.

(ii) We say that A is *universally analytically noetherian* if, for every $n \in \mathbb{N}$, the polynomial A-algebra $A[X_1, \dots, X_n]$, endowed with its I-adic topology, is analytically noetherian.

(iii) We say that A *satisfies the topological Artin-Rees condition*, if the following holds. For every A-module M of finite type, and every submodule $N \subset M$, the I-adic topology on N agrees with the topology induced by the I-adic topology of M.

(iv) We say that an A-module M is *analytically of finite type*, if there exists a submodule $N \subset M$ of finite type and an integer $n \in \mathbb{N}$ such that $I^n M \subset N$.

Remark 15.2.2. (i) It is easily seen that condition (b) of definition 15.2.1(i) does not depend on the chosen finitely generated ideal I of adic definition of A. Likewise, the class of A-modules of analytically finite type is determined solely by the topology of A.

(ii) Obviously, every noetherian ring is analytically noetherian when endowed with the adic topology defined by any of its ideals.

(iii) Let A be an analytically noetherian ring, and M an A-module of analytically finite type; then the sequence $(\mathrm{Ann}_M(I^k) \mid k \in \mathbb{N})$ is stationary. Indeed, pick a finitely generated submodule $N \subset M$ such that $I^n M \subset N$ for some $n \in \mathbb{N}$; by assumption, there exists $t \in \mathbb{N}$ such that $\mathrm{Ann}_N(I^t) = \mathrm{Ann}_N(I^s)$ for every $s \geq t$. It follows easily that $\mathrm{Ann}_M(I^k) = \mathrm{Ann}_M(I^{t+n})$ for every $k \geq t+n$.

Lemma 15.2.3. *Let A be an adic topological ring that admits a finitely generated ideal I of adic definition. Then we have :*

(i) *The following conditions are equivalent :*
  (a) *A satisfies the topological Artin-Rees condition.*
  (b) *For every A-module M of finite type, every $n \in \mathbb{N}$, and every submodule $N \subset M$ such that $I^n N = 0$, there exists $m \in \mathbb{N}$ such that $N \cap I^m M = 0$.*

(ii) *Suppose that the equivalent conditions of* (i) *hold for A, let M be any A-module of analytically finite type, and $N \subset M$ any submodule. Then the I-adic topology on N agrees with the topology induced by the I-adic topology of M.*

Proof. (i): Clearly (a)⇒(b). Conversely, let M be an A-module of finite type, $N \subset M$ a submodule, and $n \in \mathbb{N}$ any integer. Set $M' := M/I^n N$ and $N' := N/I^n N$; clearly $I^n N' = 0$, so (b) implies that $N' \cap I^m M' = 0$ for some $m \in \mathbb{N}$, and the latter means that $N \cap I^m M \subset I^n N$, whence (a).

(ii): By assumption, there exist $n \in \mathbb{N}$ and a submodule $M' \subset M$ of finite type such that $I^n M \subset M'$; set $N' := M' \cap N$. Then, for every $t \in \mathbb{N}$ there exists $s \in \mathbb{N}$ such that $I^s M' \cap N = I^s M' \cap N' \subset I^t N' \subset I^t N$. We conclude that $I^{s+n} M \cap N \subset I^t N$, whence the assertion. □

Lemma 15.2.4. *Let A be any analytically noetherian (resp. universally analytically noetherian) topological ring, and $I \subset A$ any ideal of adic definition. We have :*

(i) *For every multiplicative subset $S \subset A$, the localization $S^{-1}A$ is analytically noetherian (resp. universally analytically noetherian) for its $S^{-1}I$-adic topology.*

(ii) *Let $f : A \to B$ be a ring homomorphism, and $\mathscr{T}_B$ the IB-adic topology on B. If f is finite (resp. of finite type) Then $(B, \mathscr{T}_B)$ is analytically noetherian (resp. universally analytically noetherian).*

Proof. (i): Suppose that A is analytically noetherian. Clearly the analytic locus of $\mathrm{Spec}\, S^{-1}A$ is noetherian. Next, let M be any $S^{-1}A$-module of finite type; pick a finite system of generators $x_\bullet := (x_1, \dots, x_k)$ for M, and let N be the A-submodule of M generated by $x_\bullet$. Then $S^{-1}N = M$, and $\mathrm{Ann}_M(S^{-1}I^n) = S^{-1}\mathrm{Ann}_N(I^n)$ for every $n \in \mathbb{N}$, whence the assertion. The assertion for the case where A is universally analytically noetherian case follows immediately,

(ii): Again, suppose first that A is analytically noetherian, and f is finite. The analytic locus of $\mathrm{Spec}\, B$ is finite over the analytic locus of $\mathrm{Spec}\, A$, hence it is noetherian. Next, if M is a B-module of finite type, then it is also an A-module of finite type, and $\mathrm{Ann}_M(I^n B) = \mathrm{Ann}_M(I^n)$

for every $n \in \mathbb{N}$, whence the assertion. Lastly, suppose that A is universally analytically noetherian and f is of finite type. For every $n \in \mathbb{N}$, the A-algebra $B[T_1, \dots, T_n]$ is a quotient of a free polynomial A-algebra of finite type, so it is analytically noetherian, by the foregoing case, whence the assertion. □

Proposition 15.2.5. *Let A be a ring, $I, J \subset A$ two finitely generated ideals, and denote by $\mathscr{T}_I$ (resp. $\mathscr{T}_J$, resp. $\mathscr{T}_{I+J}$) the I-adic (resp. J-adic, resp. $(I+J)$-adic) topology on A. We have :*

(i) *If $(A, \mathscr{T}_I)$ is analytically noetherian, and $\mathscr{T}_J$ is finer than $\mathscr{T}_I$, then $(A, \mathscr{T}_J)$ is analytically noetherian.*
(ii) *If $(A, \mathscr{T}_I)$ and $(A, \mathscr{T}_J)$ are analytically noetherian, then the same holds for $(A, \mathscr{T}_{I+J})$.*
(iii) *If $(A, \mathscr{T}_I)$ and $(A, \mathscr{T}_J)$ satisfy the topological Artin-Rees condition, the same holds for $(A, \mathscr{T}_{I+J})$.*

Proof. (i): After replacing J by J^n for a suitable $n \in \mathbb{N}$, we may assume that $J \subset I$, in which case it is clear that the analytic locus $U_I \subset X := \operatorname{Spec} A$ of $(A, \mathscr{T}_I)$ contains the analytic locus U_J of $(A, \mathscr{T}_J)$. Since by assumption U_I is noetherian, the same holds for the scheme U_J. Next, let M be any A-module of finite type, and for every $n \in \mathbb{N}$, denote by $\mathscr{M}_n$ the quasi-coherent $\mathscr{O}_X$-module associated with $M_n := \operatorname{Ann}_M(J^n)$. Since U_I is noetherian, there exists $t \in \mathbb{N}$ such that $\mathscr{M}_{n|U_I} = \mathscr{M}_{t|U_I}$ for every $n \geq t$. Set $M' := M/M_t$, and $M'_n := \operatorname{Ann}_{M'}(J^n)$ for every $n \in \mathbb{N}$. It is easily seen that

$$M'_n = M_{n+t}/M_t \qquad \text{for every } n \in \mathbb{N}. \tag{15.2.6}$$

On the other hand, since $(\mathscr{M}_{n+t}/\mathscr{M}_t)_{|U_I} = 0$ for every $n \in \mathbb{N}$, we get

$$\bigcup_{n\in\mathbb{N}} M'_n = N := \bigcup_{n\in\mathbb{N}} \operatorname{Ann}_{M'}(I^n).$$

By assumption, there exists $s \in \mathbb{N}$ such that $N = \operatorname{Ann}_{M'}(I^s)$, and therefore $N \subset M'_s \subset N$. Combining with (15.2.6) we conclude that $M_n = M_{s+t}$ for every $n \geq s+t$, whence the assertion.

(ii): Let U_I and U_J be as in the foregoing, and define U_{I+J} likewise as the analytic locus of $(A, \mathscr{T}_{I+J})$. Then $U_{I+J} = U_I \cup U_J$; since by assumption U_I and U_J are noetherian, the same then holds for U_{I+J}. Next, let M be as in the foregoing; by assumption, there exists $t \in \mathbb{N}$ such that $\operatorname{Ann}_M(I^n) = \operatorname{Ann}_M(I^t)$ and $\operatorname{Ann}_M(J^n) = \operatorname{Ann}_M(J^t)$ for every $n \geq t$. It then follows easily that $\operatorname{Ann}_M((I+J)^n) = \operatorname{Ann}_M((I+J)^{2t-1})$ for every $n \geq 2t-1$, whence the assertion.

(iii): Let M be any A-module of finite type, $N \subset M$ a submodule such that $(I+J)^n N = 0$ for some $n \in \mathbb{N}$; in light of lemma 15.2.3(i), it suffices to show that there exists $m \in \mathbb{N}$ such that $N \cap (I+J)^m M = 0$. However, again by lemma 15.2.3(i), our assumption on $(A, \mathscr{T}_I)$ implies that $N \cap I^t M = 0$ for some $t \in \mathbb{N}$; set $M' := M/I^t M$ and let $N' \subset M'$ be the image of N. Notice that $J^n N' = 0$; invoking again lemma 15.2.3(i), our assumption on $(A, \mathscr{T}_J)$ implies that $N' \cap J^s M' = 0$ for some $s \in \mathbb{N}$. We conclude that $N \cap (I^t + J^s) M = 0$, and then $m := t+s-1$ will do. □

As a corollary, we get the following weak form of the Artin-Rees lemma :

Corollary 15.2.7. *Let $(A, \mathscr{T})$ be an analytically noetherian topological ring, $\mathbf{f} := (f_1, \dots, f_n)$ a finite sequence of elements of A that generates an ideal of adic definition. We have :*

(i) *A satisfies the topological Artin-Rees condition.*
(ii) *A satisfies condition $(\mathrm{d})_{\mathbf{f}}$ of (7.8.21).*

Proof. For every $i = 1, \dots, n$, let $\mathscr{T}_i$ be the $f_i A$-adic topology on A; by proposition 15.2.5(i), $(A, \mathscr{T}_i)$ is analytically noetherian for every $i = 1, \dots, n$. Taking into account lemma 15.2.4(ii), we see that A fulfills the assumptions of lemma 7.8.44, whence (ii).

Next, if the topological Artin-Rees condition holds for each $(A, \mathscr{T}_i)$, proposition 15.2.5(iii) and a simple induction show that the same holds for $(A, \mathscr{T})$. Thus, we may assume from start that $n = 1$ and A admits a principal ideal $I = fA$ of adic definition.

Now, let M be any A-module of finite type, $N \subset M$ a submodule, and set $M' := M/N$. Pick $t \in \mathbb{N}$ such that $\mathrm{Ann}_{M'}(f^n) = \mathrm{Ann}_{M'}(f^t)$ for every $n \geq t$; it follows easily that

$$N \cap f^{n+t}M = f^n(N \cap f^t M) \qquad \text{for every } n \in \mathbb{N}$$

whence the contention. □

Corollary 15.2.8. *Let A be an analytically noetherian ring, $I \subset A$ a finitely generated ideal of adic definition, and*

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

a short exact sequence of A-modules, which we endow with their I-adic topologies. We have :

(i) *M_2 is analytically of finite type if and only if the same holds for both M_1 and M_3.*

(ii) *Suppose that M_2 is analytically of finite type. Then the induced sequence of separated completions*

$$0 \to M_1^\wedge \to M_2^\wedge \to M_3^\wedge \to 0 \tag{15.2.9}$$

is short exact.

Proof. (i): Suppose that M_2 is analytically of finite type. Then it is easily seen that the same holds for M_3. Next, set $X := \mathrm{Spec}\, A$, so that $U := X \setminus \mathrm{Spec}\, A/I$ is the analytic locus of X. Let also $\mathscr{M}_1$ and $\mathscr{M}_2$ be the quasi-coherent $\mathscr{O}_X$-modules associated with M_1 and respectively M_2. Then $\mathscr{M}_{2|U}$ is a quasi-coherent $\mathscr{O}_U$-module of finite type, and since U is noetherian, the same then holds for its submodule $\mathscr{M}_{1|U}$. Hence, we may find a quasi-coherent $\mathscr{O}_X$-submodule $\mathscr{P} \subset \mathscr{M}_1$ of finite type such that $\mathscr{P}_{|U} = \mathscr{M}_{1|U}$, and we denote by $P \subset M_1$ the submodule corresponding to $\mathscr{P}$. Set $M_1' := M_1/P$ and $M_2' := M_2/P$; by construction, the support of M_1' lies in $\mathrm{Spec}\, A/I$, and since A is analytically noetherian and M_2' is analytically of finite type, it follows that $I^n M_1' = 0$ for some $n \in \mathbb{N}$ (remark 15.2.2(iii)). Thus, $I^n M_1 \subset P$, which shows that M_1 is analytically of finite type.

Lastly, suppose that M_1 and M_3 are analytically of finite type, and pick $n \in \mathbb{N}$ and finitely generated submodules $N_i \subset M_i$ such that $I^n M_i \subset N_i$ for $i = 1, 3$. Then we may find a submodule $P \subset M_2$ of finite type whose image in M_3 equals N_3, and it is easily seen that $I^{2n} M_2 \subset P + N_1$.

(ii): This follows from corollary 15.2.7(i), lemma 15.2.3(ii) and proposition 8.2.13(v). □

Proposition 15.2.10. *Let A and I be as in corollary* 15.2.8*, and M any A-module of analytically finite type. Endow M with its I-adic topology, and denote by $A^\wedge$ and $M^\wedge$ the separated completions of A and M. We have :*

(i) *The natural map $\varphi_M : A^\wedge \otimes_A M \to M^\wedge$ is an isomorphism.*

(ii) *The completion map $A \to A^\wedge$ is flat.*

Proof. (i): We consider first the case where M is of finite type. Pick any surjective A-linear map $\psi : L \to M$, from a finitely presented A-module L, and endow $K := \mathrm{Ker}\, \psi$ with its I-adic topology. We consider the induced commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^\wedge \otimes_A K & \longrightarrow & A^\wedge \otimes_A L & \longrightarrow & A^\wedge \otimes_A M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi_K} & & \downarrow{\scriptstyle \varphi_L} & & \downarrow{\scriptstyle \varphi_M} & & \\ 0 & \longrightarrow & K^\wedge & \longrightarrow & L^\wedge & \longrightarrow & M^\wedge & \longrightarrow & 0 \end{array}$$

whose bottom row is short exact, by corollary 15.2.8. If L is free of finite rank, φ_L is an isomorphism; then φ_M is surjective and φ_K is injective. Since we can always find such a surjection φ with L free of finite rank, we conclude already that φ_M is surjective for every finitely generated A-module M.

Next, suppose that M is finitely presented and L is still free of finite rank; in this case, K is an A-module of finite type ([75, Lemma 2.3.18(iii)]), so φ_K is also surjective, by the foregoing, hence φ_K is an isomorphism and therefore the same holds for φ_M.

For a general M of finite type, in view of corollary 15.2.8(i) we may find a submodule $K' \subset K$ of finite type such that $I^n K \subset K'$. Set $L' := L/K'$, and let $\psi' : L' \to M$ be the surjective map induced by ψ; then L' is still finitely presented, so we may replace L by L' and ψ by ψ', and assume from start that the I-adic topology is discrete on K, in which case the completion map $K \to K^\wedge$ is an isomorphism, consequently φ_K is again surjective, and we conclude as in the foregoing that φ_M is an isomorphism, as required.

Lastly, suppose that M is of analytically finite type, and pick a submodule $N \subset M$ of finite type such that $I^n(M/N) = 0$ for some $n \in \mathbb{N}$. We consider the analogous commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^\wedge \otimes_A N & \longrightarrow & A^\wedge \otimes_A M & \longrightarrow & A^\wedge \otimes_A (M/N) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi_N} & & \downarrow{\scriptstyle \varphi_M} & & \downarrow{\scriptstyle \varphi_{M/N}} & & \\ 0 & \longrightarrow & N^\wedge & \longrightarrow & M^\wedge & \longrightarrow & (M/N)^\wedge & \longrightarrow & 0 \end{array}$$

and it is easily seen that both $\varphi_{M/N}$ and the completion map $M/N \to (M/N)^\wedge$ are isomorphisms. The same holds for φ_N, by the foregoing, so finally φ_M is an isomorphism.

(ii): Let $N \to M$ be any injective homomorphism of A-modules; we need to check that the induced map $A^\wedge \otimes_A N \to A^\wedge \otimes_A M$ is still injective. Since the tensor product commutes with all colimits, we are easily reduced to the case where M and N are A-modules of finite type. In this case, endow M and N with their I-adic topologies; by corollary 15.2.8, the induced map on I-adic completions $N^\wedge \to M^\wedge$ is injective. Then it suffices to apply (i) to conclude. □

15.2.11. Let $(A, \mathscr{T})$ be a complete and separated adic topological ring that admits a principal ideal $I = Aa$ of adic definition. Let also M be an A-module whose I-adic topology $\mathscr{T}_M$ is complete and separated. For a submodule $N \subset M$ let N^c be the topological closure of N in M, and for every $k \in \mathbb{N}$ denote by $N_k \subset M$ the submodule of all $x \in M$ such that $a^k x \in N$. Set

$$N^s := \bigcup_{k \in \mathbb{N}} N_k.$$

Lemma 15.2.12. *In the situation of* (15.2.11)*, let $N' \subset N \subset M$ be two submodules, such that N' is dense in N (for the topology $\mathscr{T}_M$) and $N = N^s$. The following holds :*

(i) *The I-adic topology on N agrees with the topology induced by $\mathscr{T}_M$.*
(ii) $N^c = (N^c)^s$.
(iii) *If $N = N^c$ as well, and $N[a^{-1}]$ is an $A[a^{-1}]$-module of finite type, then N is analytically of finite type, and $N' = N$.*

Proof. (i): Since $N = N^s$ we have $a^m N = a^m M \cap N$ for every $m \in \mathbb{N}$, whence the assertion.

(ii): If $x \in M$ and $ax \in N^c$, then for every $n \in \mathbb{N}$ there exists $z_n \in M$ such that $ax - a^n z_n \in N$, and therefore $x - a^{n-1} z_n \in N$ for every $n \geq 1$, whence $x \in N^c$.

(iii): By assumption, there exists a finitely generated submodule $L \subset N$ with

$$N = L^s = \bigcup_{k \in \mathbb{N}} (L_k)^c$$

(indeed, the first identity holds if we choose L such that $N[a^{-1}] = L[a^{-1}]$, and the second follows, since $N = N^c$). Notice next that we may regard M as a complete metric space, with the metric defined by the rule

$$d(x,y) := \begin{cases} 0 & \text{if } x = y \\ 2^{-b} & \text{if } x \neq y \text{, where } b := \max(n \in \mathbb{N} \mid x - y \in a^n M). \end{cases}$$

By Baire's category theorem ([41, Ch.IX, §5, n.3, Th.1]), it follows that $(L_k)^c$ is open in N for some $k \in \mathbb{N}$ (for the topology of N induced by $\mathscr{T}_M$), and therefore it contains $a^m N$ for some $m \in \mathbb{N}$; thus $a^{k+m}N \subset a^k(L_k)^c \subset L^c$. Combining with (i) we deduce that the I-adic topology of L^c agrees with the topology induced by $\mathscr{T}_M$; then by lemma 8.3.22(i) we conclude that $L = L^c$, so $a^{k+m}N \subset L$, which shows that N is analytically of finite type.

Lastly, since N' is dense in N, from (i) we see that $N = N' + a^{m+k+1}N \subset N' + aL$. It follows that $L = (N' \cap L) + aL$, and by Nakayama's lemma we get $L = N' \cap L$, *i.e.* $L \subset N'$, so N' is open in N and thus $N = N'$. □

Theorem 15.2.13. *Let $(A, \mathscr{T})$ be a complete and separated adic topological ring that admits a finitely generated ideal I of adic definition, and suppose that the analytic locus of* Spec A *is noetherian. Then A is analytically noetherian.*

Proof. (This is [71, Th.5.1.2].) Let $a_1, \dots, a_k$ be a finite system of elements of A that generates I, and for every $i = 1, \dots, k$ denote by $\mathscr{T}_i$ that Aa_i-adic topology on A; taking into account proposition 15.2.5(ii), it suffices to show that $(A, \mathscr{T}_i)$ is analytically noetherian for $i = 1, \dots, k$. However, since the analytic locus of $(A, \mathscr{T})$ is noetherian, the same holds for the analytic locus of $(A, \mathscr{T}_i)$, and the topology $\mathscr{T}_i$ is also complete and separated for every $i = 1, \dots, k$, by virtue of lemma 8.3.12. We are then reduced to the case where $I = Aa$ is a principal ideal.

Now, let M be an A-module of finite type; we need to show that the sequence of submodules $(\mathrm{Ann}_M(a^n) \mid n \in \mathbb{N})$ is stationary. Let us write $M = L/Q$ for a free A-module L of finite rank, and a submodule $Q \subset L$, and set $N := Q^s$ (notation of (15.2.11)). Notice that

$$N/Q = \bigcup_{n \in \mathbb{N}} \mathrm{Ann}_M(a^n)$$

so it suffices to show that $a^k N \subset Q$ for some $k \in \mathbb{N}$. Now, obviously $N^s = N$, so $N^c = (N^c)^s$ as well, by lemma 15.2.12(ii). Moreover, since $A[a^{-1}]$ is noetherian, $N^c[a^{-1}]$ is an $A[a^{-1}]$-module of finite type, and consequently N^c is analytically of finite type, and $N = N^c$, by lemma 15.2.12(iii). It follows that N/Q is also analytically of finite type; so, pick a submodule $N' \subset N/Q$ of finite type and an integer $k \in \mathbb{N}$ such that $a^k(N/Q) \subset N'$; since N' is finitely generated, there exists as well $n \in \mathbb{N}$ such that $a^n N' = 0$, so finally $a^{k+n}(N/Q) = 0$, as required. □

Next, we globalize definition 15.2.1 to schemes as follows.

Definition 15.2.14. Let A be an adic topological ring that has a finitely generated ideal I of adic definition, X an A-scheme, and $\mathscr{F}$ a quasi-coherent $\mathscr{O}_X$-module.

(i) We say that X is *locally analytically noetherian* if every point of X admits an affine open neighborhood U such that the following holds. Endow $A_U := \mathscr{O}_X(U)$ with the I-adic topology $\mathscr{T}_U$; then $(A_U, \mathscr{T}_U)$ is analytically noetherian.

(ii) We say that X is *analytically noetherian* if it is quasi-compact, quasi-separated and locally analytically noetherian.

(iii) We say that $\mathscr{F}$ is *analytically of finite type* if every point of X admits an affine open neighborhood U such that $\mathscr{F}(U)$ is an $(A_U, \mathscr{T}_U)$-module of analytically finite type.

Remark 15.2.15. If A is a universally analytically noetherian ring, then every A-scheme of finite type is analytically noetherian, by virtue of lemma 15.2.4.

Lemma 15.2.16. *Let A and I be as in definition* 15.2.14. *Let also X be a quasi-compact and quasi-separated A-scheme, and $\mathscr{F}$ a quasi-coherent $\mathscr{O}_X$-module. We have :*

(i) *If $\mathscr{F}$ is analytically of finite type, there exist an integer $n \in \mathbb{N}$ and a quasi-coherent $\mathscr{O}_X$-submodule of finite type $\mathscr{G} \subset \mathscr{F}$ such that $I^n\mathscr{F} \subset \mathscr{G}$.*

(ii) *If $\mathscr{F}$ is analytically of finite type and X is analytically noetherian, there exists $p \in \mathbb{N}$ such that $\mathrm{Ann}_{\mathscr{F}}(I^q) = \mathrm{Ann}_{\mathscr{F}}(I^p)$ for every integer $q \geq p$.*

(iii) *If $\mathscr{F}$ is of finite type and X is analytically noetherian, there exist a finitely presented $\mathscr{O}_X$-module $\mathscr{G}$, an epimorphism $\varphi : \mathscr{G} \to \mathscr{F}$ of $\mathscr{O}_X$-modules, and an integer $n \in \mathbb{N}$ such that $I^n\mathrm{Ker}\,\varphi = 0$.*

Proof. (i): By assumption, we may find a finite affine open covering $(U_\lambda \mid \lambda \in \Lambda)$ of X, and for every $\lambda \in \Lambda$ a quasi-coherent $\mathscr{O}_{U_\lambda}$-submodule $\mathscr{H}_\lambda$ of finite type of $\mathscr{F}_\lambda := \mathscr{F}_{|U_\lambda}$ and an integer $n_\lambda \in \mathbb{N}$ such that $I^{n_\lambda}\mathscr{F}_\lambda \subset \mathscr{H}_\lambda$. We may also find, for every $\lambda \in \Lambda$, a finitely presented quasi-coherent $\mathscr{O}_{U_\lambda}$-module $\mathscr{H}'_\lambda$ with an $\mathscr{O}_{U_\lambda}$-linear morphism $\varphi_\lambda : \mathscr{H}'_\lambda \to \mathscr{F}_\lambda$ whose image is $\mathscr{H}_\lambda$. According to lemma 10.3.27(i) there exist a finitely presented quasi-coherent $\mathscr{O}_X$-module $\mathscr{G}_\lambda$ with $\mathscr{G}_{\lambda|U_\lambda} = \mathscr{H}'_\lambda$ and a morphism $\psi_\lambda : \mathscr{G}_\lambda \to \mathscr{F}_\lambda$ of $\mathscr{O}_X$-modules such that $\psi_{\lambda|U_\lambda} = \varphi_\lambda$. Then we may take $\mathscr{G} := \sum_{\lambda\in\Lambda} \mathrm{Im}\,\psi_\lambda$.

(ii): Suppose that for every $\lambda \in \Lambda$ there exists $p_\lambda \in \mathbb{N}$ such that $\mathrm{Ann}_{\mathscr{F}_\lambda}(I^q) = \mathrm{Ann}_{\mathscr{F}_\lambda}(I^{p_\lambda})$ for every $q \geq p_\lambda$. Then $p := \max(p_\lambda \mid \lambda \in \Lambda)$ fulfills the stated condition. Thus, we may suppose that $X = \mathrm{Spec}\,A$ for an analytically noetherian ring A, in which case the assertion follows from remark 15.2.2(iii).

(iii): By assumption, we may find a finite affine covering $(U_\lambda \mid \lambda \in \Lambda)$ of X such that $A_\lambda := \mathscr{O}_X(U_\lambda)$ is analytically noetherian relative to its IA_λ-adic topology, for every $\lambda \in \Lambda$. Pick a finitely presented quasi-coherent $\mathscr{O}_{U_\lambda}$-module $\mathscr{H}_\lambda$ and an epimorphism $\psi_\lambda : \mathscr{H}_\lambda \to \mathscr{F}_{|U_\lambda}$ for every $\lambda \in \Lambda$. By lemma 10.3.27(i) there exist a finitely presented quasi-coherent $\mathscr{O}_X$-module $\mathscr{H}'_\lambda$ with $\mathscr{H}'_{\lambda|U_\lambda} = \mathscr{H}_\lambda$ and a morphism of $\mathscr{O}_X$-modules $\psi'_\lambda : \mathscr{H}'_\lambda \to \mathscr{F}$ such that $\psi'_{\lambda|U_\lambda} = \psi_\lambda$, for every $\lambda \in \Lambda$. Set $\mathscr{H} := \bigoplus_{\lambda\in\Lambda} \mathscr{H}'_\lambda$; the sum of the morphisms ψ'_λ is an epimorphism $\psi : \mathscr{H} \to \mathscr{F}$, and $\mathscr{H}$ is clearly finitely presented. For every $\lambda \in \Lambda$, let also $V_\lambda \subset U_\lambda$ be the analytic locus; then $V := \bigcup_{\lambda\in\Lambda} V_\lambda$ is locally noetherian, quasi-compact and quasi-separated, and therefore $\mathscr{K} := \mathrm{Ker}\,\psi$ restricts to a finitely presented quasi-coherent $\mathscr{O}_V$-module $\mathscr{K}_{|V}$. Then, invoking again lemma 10.3.27(i) we find a quasi-coherent finitely presented $\mathscr{O}_X$-module $\mathscr{K}'$ and an $\mathscr{O}_X$-linear morphism $\mathscr{K}' \to \mathscr{K}$ that restricts to an isomorphism on V. Let $\mathscr{K}'' \subset \mathscr{K}$ be the image of $\mathscr{K}'$, and set $\mathscr{G} := \mathscr{H}/\mathscr{K}''$. Clearly ψ factors through an epimorphism $\varphi : \mathscr{G} \to \mathscr{F}$, and the support of $\mathrm{Ker}\,\varphi$ lies in $X \setminus V$. Lastly, since A_λ is analytically noetherian, we may find an integer $n_\lambda \in \mathbb{N}$ such that $I^{n_\lambda} \cdot (\mathrm{Ker}\,\varphi)_{|V_\lambda} = 0$, for every $\lambda \in \Lambda$. The assertion then holds with $n := \max(n_\lambda \mid \lambda \in \Lambda)$. ☐

Proposition 15.2.17. *Let A be an adic topological ring with a finitely generated ideal I of adic definition, B an A-algebra, that we endow with its IB-adic topology $\mathscr{T}_B$. Let also F be any B-module, and denote by $\mathscr{F}$ the quasi-coherent $\mathscr{O}_{\mathrm{Spec}\,B}$-module arising from F. We have :*

(i) *The following conditions are equivalent :*
   (a) *$(B, \mathscr{T}_B)$ is analytically noetherian.*
   (b) *$\mathrm{Spec}\,B$ is an analytically noetherian A-scheme.*

(ii) *The following conditions are equivalent :*
   (a) *F is a B-module of analytically finite type.*
   (b) *$\mathscr{F}$ is an $\mathscr{O}_{\mathrm{Spec}\,B}$-module of analytically finite type.*

Proof. (i): Clearly (a)⇒(b). Hence, suppose that $X := \mathrm{Spec}\,B$ is analytically noetherian, and let $(U_i \mid i = 1, \dots, k)$ be a finite affine covering of X such that $B_i := \mathscr{O}_X(U_i)$ is analytically noetherian for its IB_i-adic topology for every $i = 1, \dots, k$. By virtue of lemma 15.2.4, we may assume that for every $i = 1, \dots, k$ there exists $b_i \in B$ such that $B_i = B[b_i^{-1}]$. Clearly the analytic locus U of X is the union of the analytic loci of $\mathrm{Spec}\,B_1, \dots, \mathrm{Spec}\,B_k$, so U is noetherian. Next, let M be any B-module of finite type, and for every $i = 1, \dots, k$, set $M_i :=$

$B_i \otimes_B M$; choose $n \in \mathbb{N}$ such that $\mathrm{Ann}_{M_i}(I^r B_i) = \mathrm{Ann}_{M_i}(I^n B_i)$ for every $i = 1, \dots, n$ and every $r \geq n$. It follows that $\mathrm{Ann}_M(I^r) = \mathrm{Ann}_M(I^n)$ for every $r \geq n$, whence the contention.

(ii) follows easily from lemma 15.2.16(i). □

15.2.18. Let A be a universally analytically noetherian ring, $I \subset A$ a finitely generated ideal of adic definition, X a quasi-separated A-scheme locally of finite type, and $\mathscr{F}$ a quasi-coherent $\mathscr{O}_X$-module. According to (15.1.47) we may consider the completions $X^\wedge$ and $\mathscr{F}^\wedge$ of X and $\mathscr{F}$ along the closed subscheme $X_0 := \mathrm{Spec}\,\mathscr{O}_X/I\mathscr{O}_X = \mathrm{Spec}\,A/I \times_{\mathrm{Spec}\,A} X$, which are respectively a formal scheme endowed with a natural morphism of locally ringed spaces

$$\pi : X^\wedge \to X$$

and a quasi-coherent $\mathscr{O}_{X^\wedge}$-module with a natural map of $\mathscr{O}_{X^\wedge}$-modules

$$\pi^*\mathscr{F} \to \mathscr{F}^\wedge. \tag{15.2.19}$$

Proposition 15.2.20. *With the notation of* (15.2.18), *the following holds :*

(i) *If $\mathscr{F}$ is analytically of finite type, the map* (15.2.19) *is an isomorphism.*

(ii) *We have $H^i(U, \pi^*\mathscr{F}) = 0$ for every affine quasi-compact open subset $U \subset X^\wedge$ and every $i > 0$.*

(iii) *If X is affine, the natural map*

$$\mathscr{O}_{X^\wedge}(X^\wedge) \otimes_{\mathscr{O}_X(X)} H^0(X, \mathscr{F}) \to H^0(X^\wedge, \pi^*\mathscr{F})$$

is an isomorphism.

(iv) *Let $\mathfrak{U} := (U_i \mid i \in I)$ be an open covering of $X^\wedge$, and suppose that $U_i \cap U_j$ is affine for every $i, j \in I$. Then there is a natural isomorphism (notation of* (10.2.18)*) :*

$$H^\bullet_{\mathrm{alt}}(\mathfrak{U}, \pi^*\mathscr{F}) \xrightarrow{\sim} H^\bullet(X^\wedge, \pi^*\mathscr{F}).$$

Proof. (i): The assertion is local on X, hence we may assume that X is local, say $X = \mathrm{Spec}\,B$, and $\mathscr{F}$ is the quasi-coherent $\mathscr{O}_X$-module attached to the module $F := \mathscr{F}(X)$. Let T be any final object in the category of topological spaces (*i.e.* T is the unique topology over a set that has only one point), and endow T with a structure sheaf $\mathscr{O}_T$ by declaring that $\mathscr{O}_T(T) := B$ and $\mathscr{O}_T(\varnothing) := 0$. The module F defines an $\mathscr{O}_T$-module F_T by the rule : $F_T(T) := F$ and $F_T(\varnothing) := 0$. The system of restriction maps $B \to \mathscr{O}_X(U)$ for U ranging over the open subsets of X, may be viewed as a morphism of ringed spaces

$$\varepsilon : X \to T$$

and with this notation we then have $\mathscr{F} = \varepsilon^* F_T$. Set as well $\varepsilon^\wedge := \varepsilon \circ \pi$; there follows a natural isomorphism of $\mathscr{O}_{X^\wedge}$-modules

$$\pi^*\mathscr{F} \xrightarrow{\sim} \varepsilon^{\wedge *} F_T.$$

In other words, $\pi^*\mathscr{F}$ is the sheaf associated to the presheaf given by the rule :

$$U_0 \mapsto \mathscr{O}_{X^\wedge}(U_0) \otimes_B F \qquad \text{for every open subset } U_0 \subset X^\wedge = X_0$$

and the discussion of (15.1.47) shows that (15.2.19) is the morphism associated to the morphism of presheaves given, on every affine open subset $U_0 \subset X_0$, by the natural map

$$\mathscr{O}_X(U)^\wedge \otimes_B F \to \mathscr{F}(U)^\wedge \tag{15.2.21}$$

where $U \subset X$ is any affine open subset such that $U_0 = U \cap X_0$, and where $\mathscr{O}_X(U)^\wedge$ and $\mathscr{F}(U)^\wedge$ denote the I-adic completions of $\mathscr{O}_X(U)$ and respectively $\mathscr{F}(U)$. However, since $\mathscr{F}$ is quasi-coherent, we have $\mathscr{F}(U) = \mathscr{O}_X(U) \otimes_B F$; taking into account remark 15.2.15 and propositions 15.2.10 and 15.2.17(ii) we deduce that (15.2.21) is an isomorphism for every such U_0. Since the affine open subsets are a basis of the topology of $X^\wedge$ that is closed under finite intersections, the assertion follows.

(ii): Since U is quasi-compact, $\pi(U)$ is contained in a quasi-compact open subset $V \subset X$; we may then replace X by V, and assume from start that X is quasi-compact and quasi-separated. In this case, $\mathscr{F}$ is the colimit of the filtered family $(\mathscr{F}_\lambda \mid \lambda \in \Lambda)$ of its quasi-coherent $\mathscr{O}_X$-submodules of finite type (proposition 10.3.31), hence $\pi^*\mathscr{F}$ is the colimit of the system $(\pi^*\mathscr{F}_\lambda \mid \lambda \in \Lambda)$. Since U is a spectral topological space, proposition 10.1.10 then reduces to showing that $H^i(U, \pi^*\mathscr{F}_\lambda) = 0$ for every $\lambda \in \Lambda$. Hence, we may assume from start that $\mathscr{F}$ is of finite type, in which case $\pi^*\mathscr{F}$ is a quasi-coherent $\mathscr{O}_{X^\wedge}$-module, by (i), and then the assertion follows from theorem 15.1.37(ii).

(iii): Arguing as in the proof of (ii), we reduce to the case where $\mathscr{F}$ is analytically of finite type; then $\mathscr{F}(X)$ is an $\mathscr{O}_X(X)$-module of analytically finite type (proposition 15.2.17(ii)), and the discussion of (15.1.47) says that $\mathscr{O}_{X^\wedge}(X^\wedge)$ (resp. $\mathscr{F}^\wedge(X^\wedge)$) is the I-adic completion of $\mathscr{O}_X(X)$ (resp. of $\mathscr{F}(X)$). Then the assertion follows from (i), lemma 15.1.28(i) and proposition 15.2.10(i).

(iv) follows from (ii), together corollary 10.2.21(ii) and [58, Ch.I, Prop.10.7.2]. □

Theorem 15.2.22. *Let A be a universally analytically noetherian ring, X a proper and finitely presented A-scheme, and $\mathscr{F}$ a quasi-coherent $\mathscr{O}_X$-module of analytically finite type. Then the A-module $H^i(X, \mathscr{F})$ is analytically of finite type for every $i \in \mathbb{N}$.*

Proof. Let $I \subset A$ be any ideal of adic definition. The first observation is the following :

Claim 15.2.23. Let $i \in \mathbb{N}$ be any integer, and suppose that $H^i(X, \mathscr{F}')$ is analytically of finite type for every finitely presented $\mathscr{O}_X$-module $\mathscr{F}'$. Then $H^i(X, \mathscr{F})$ is analytically of finite type for every $\mathscr{O}_X$-module $\mathscr{F}$ of analytically finite type.

Proof of the claim. Let $\mathscr{G} \subset \mathscr{F}$ be an $\mathscr{O}_X$-submodule as in lemma 15.2.16(i), so that $I^n\mathscr{F} \subset \mathscr{G}$ for some $n \in \mathbb{N}$, and denote by $j : \mathscr{G} \to \mathscr{F}$ the inclusion morphism. Then, for every $a \in I^n$ the endomorphism $a \cdot \mathbf{1}_\mathscr{F}$ factors through an $\mathscr{O}_X$-linear morphism $\varphi_a : \mathscr{F} \to \mathscr{G}$. Set $H^i := H^i(X, \mathscr{F})$; we deduce that $a \cdot \mathbf{1}_{H^i} = H^i(X, j) \circ H^i(X, \varphi_a)$. Especially, $I^n H^i$ is contained in the image of $H^i(X, j)$. Hence, if $H^i(X, \mathscr{G})$ is analytically of finite type, the same follows for H^i. We may then replace $\mathscr{F}$ by $\mathscr{G}$, and assume from start that $\mathscr{F}$ is quasi-coherent of finite type. Next, by lemma 15.2.16(iii) and remark 15.2.15 we may find a short exact sequence of quasi-coherent $\mathscr{O}_X$-modules $0 \to \mathscr{K} \to \mathscr{F}' \to \mathscr{F} \to 0$ such that $\mathscr{F}'$ is finitely presented and $I^n\mathscr{K} = 0$ for some $n \in \mathbb{N}$. It follows easily that I^n annihilates the cokernel of the induced map $H^i(X, \mathscr{F}') \to H^i$, whence the claim. ◊

Thus, suppose henceforth that $\mathscr{F}$ is finitely presented, and set $S := \operatorname{Spec} A$; according to [65, Ch.IV, Prop.8.9.1(i,ii)] we may assume that there exist a noetherian subring $A_0 \subset A$, a finitely presented morphism of schemes $X_0 \to S_0 := \operatorname{Spec} A_0$ and a coherent $\mathscr{O}_{X_0}$-module $\mathscr{F}_0$ such that $X = S \times_{S_0} X_0$ and $\mathscr{F} = \pi^*\mathscr{F}_0$, where $\pi : X \to X_0$ is the projection. By [65, Ch.IV, Th.8.10.5], we may moreover assume that X_0 is a proper A_0-scheme.

Claim 15.2.24. The $\mathscr{O}_X$-module $\mathscr{T}or_q^{S_0}(\mathscr{O}_S, \mathscr{F}_0)$ is analytically of finite type for every $q \in \mathbb{N}$.

Proof of the claim. The assertion is local on X_0, so we may assume that X_0 is and affine and finitely presented S_0-scheme; then we may find an integer $n \in \mathbb{N}$ and a closed immersion $i : X_0 \to \mathbb{A}^n_{S_0}$. Set $i_S := S \times_{S_0} i : X \to \mathbb{A}^n_S$; according to [62, Ch.III, Prop.6.5.11] there is a natural isomorphism of $\mathscr{O}_X$-modules :

$$i_{S*}\mathscr{T}or_q^{S_0}(\mathscr{O}_S, \mathscr{F}_0) \xrightarrow{\sim} \mathscr{T}or_q^{S_0}(\mathscr{O}_S, i_*\mathscr{F}_0)$$

so we may replace X_0 by $\mathbb{A}^n_{S_0}$ and $\mathscr{F}_0$ by $i_*\mathscr{F}_0$, and assume from start that X_0 is a flat and affine S_0-scheme. In this situation, there exists as well a natural isomorphism of $\mathscr{O}_X$-modules :

$$\mathscr{T}or_q^{S_0}(\mathscr{O}_S, \mathscr{F}_0) \xrightarrow{\sim} \mathscr{T}_q := \mathscr{T}or_q^{X_0}(\mathscr{O}_X, \mathscr{F}_0) \qquad \text{for every } n \in \mathbb{N}.$$

Indeed, for every V_0, V and U as in the foregoing condition (a) we have an isomorphism

$$\gamma_{V,U} : \mathrm{Tor}_q^{\mathscr{O}_{S_0}(V_0)}(\mathscr{O}_S(V), \mathscr{F}_0(U)) \xrightarrow{\sim} \mathrm{Tor}_q^{\mathscr{O}_{X_0}(U)}(\mathscr{O}_X(V \times_{S_0} U), \mathscr{F}_0(U))$$

of $\mathscr{O}_X(V \times_{S_0} U)$-modules, such that for every inclusion of open subsets $V'_0 \subset V_0$, $V' \subset V \cap (S \times_{S_0} V'_0)$ and $U' \subset U \cap (S \times_{S_0} V'_0)$ the resulting diagram commutes :

$$\begin{array}{ccc} \mathrm{Tor}_q^{\mathscr{O}_{S_0}(V_0)}(\mathscr{O}_S(V), \mathscr{F}_0(U)) & \xrightarrow{\gamma_{V,U}} & \mathrm{Tor}_q^{\mathscr{O}_{X_0}(U)}(\mathscr{O}_X(V \times_{S_0} U), \mathscr{F}_0(U)) \\ \downarrow & & \downarrow \\ \mathrm{Tor}_q^{\mathscr{O}_{S_0}(V'_0)}(\mathscr{O}_S(V'), \mathscr{F}_0(U')) & \xrightarrow{\gamma_{V',U'}} & \mathrm{Tor}_q^{\mathscr{O}_{X_0}(U')}(\mathscr{O}_X(V' \times_{S_0} U'), \mathscr{F}_0(U')) \end{array}$$

([163, Prop.3.2.9]) whence the assertion. Thus, it suffices to check that $\mathscr{T}_q$ is analytically of finite type; however, since X_0 is affine, $\mathscr{F}_0$ admits a resolution $\mathscr{P}_\bullet \xrightarrow{\sim} \mathscr{F}[0]$ consisting of free $\mathscr{O}_{X_0}$-modules of finite type, and $\mathscr{T}_q$ is isomorphic to $H_q(\mathscr{O}_X \otimes_{\mathscr{O}_{X_0}} \mathscr{P}_\bullet)$. The claim then follows easily from corollary 15.2.8(i). ◊

Let $\mathfrak{U} := (U_i \mid i = 1, \dots, n)$ be a finite affine covering of X_0; we have natural isomorphisms in $\mathsf{D}(A_0\text{-}\mathbf{Mod})$ and respectively $\mathsf{D}(A\text{-}\mathbf{Mod})$

$$R\Gamma(X_0, \mathscr{F}_0) \xrightarrow{\sim} \overline{C}^\bullet_{\mathrm{alt}}(\mathfrak{U}, \mathscr{F}_0) \qquad R\Gamma(X, \mathscr{F}) \xrightarrow{\sim} A \otimes_{A_0} \overline{C}^\bullet_{\mathrm{alt}}(\mathfrak{U}, \mathscr{F}_0)$$

where $\overline{C}^\bullet_{\mathrm{alt}}(\mathfrak{U}, -)$ denotes the truncated alternating Čech complex associated with the covering $\mathfrak{U}$ (theorem 10.2.28(ii)). Let us choose a Cartan-Eilenberg projective resolution $P^{\bullet\bullet} \xrightarrow{\sim} \overline{C}^\bullet_{\mathrm{alt}}(\mathfrak{U}, \mathscr{F}_0)[0]$ such that $P^{pq} = 0$ whenever $q > 0$ (see [163, Lemma 5.7.2]); the double complex $A \otimes_{A_0} P^{\bullet\bullet}$ gives rise to a spectral sequence

$$E_2^{p,-q} := \mathrm{Tor}_q^{A_0}(A, H^p\overline{C}^\bullet_{\mathrm{alt}}(\mathfrak{U}, \mathscr{F}_0)) \Rightarrow M_{p-q} := H_{p-q}(A \overset{\mathbf{L}}{\otimes}_{A_0} \overline{C}^\bullet_{\mathrm{alt}}(\mathfrak{U}, \mathscr{F}_0))$$

as well as a spectral sequence

$$F_1^{p,-q} := \mathrm{Tor}_q^{A_0}(A, \overline{C}^p_{\mathrm{alt}}(\mathfrak{U}, \mathscr{F}_0)) \Rightarrow M_{p-q}$$

whose differentials $d_1^{pq} : F_1^{pq} \to F_1^{p+1,q}$ are induced by those of $\overline{C}^\bullet_{\mathrm{alt}}(\mathfrak{U}, \mathscr{F}_0)$, whence natural isomorphisms for every $p, q \in \mathbb{N}$:

$$(15.2.25) \qquad E_2^{p,-q} \xrightarrow{\sim} \mathrm{Tor}_q^{A_0}(A, H^p(X_0, \mathscr{F}_0)) \qquad F_2^{p,-q} \xrightarrow{\sim} H^p(X, \mathscr{T}\!or_q^{S_0}(\mathscr{O}_S, \mathscr{F}_0))$$

and especially, we have natural isomorphisms

$$(15.2.26) \qquad F_2^{i,0} \xrightarrow{\sim} H^i \qquad \text{for every } i \in \mathbb{N}.$$

Claim 15.2.27. M_{p-q} is analytically of finite type, for every $p, q \in \mathbb{N}$.

Proof of the claim. The A_0-module $H^p(X_0, \mathscr{F}_0)$ is finitely generated for every $p \in \mathbb{N}$, because X_0 is a proper A_0-scheme ([61, Ch.III, Th.3.2.1]); since A_0 is noetherian, we may then find a resolution $P_\bullet$ of $H^p(X_0, \mathscr{F}_0)$ consisting of free A_0-modules of finite type, and there are natural isomorphisms $E_2^{p,-q} \xrightarrow{\sim} H_q(A \otimes_{A_0} P_\bullet)$; taking into account corollary 15.2.8(i), we conclude that $E_2^{p,-q}$ is an A-module of analytically finite type, for every $p, q \in \mathbb{N}$. Then, again by corollary 15.2.8(i), a simple induction on $r \in \mathbb{N}$ shows that $E_r^{p,-q}$ is analytically of finite type, for every $p, q, r \in \mathbb{N}$. We deduce that the A-module M_{p-q} admits a finite filtration whose subquotients are analytically of finite type, and to conclude it suffices to invoke again corollary 15.2.8(i). ◊

Now the theorem is a special case of the following more general

Claim 15.2.28. $F_{r+2}^{p,-q}$ is an A-module of analytically finite type, for every $p, q, r \in \mathbb{N}$.

Proof of the claim. We argue by descending induction on p, and notice that $F_1^{p,-q} = 0$ for every $p \geq n-1$, so the assertion is clear for $p \geq n-1$. Next, let $t \leq n-1$, and suppose that the claim has already been shown for every $p, q, r \in \mathbb{N}$ with $r \geq 2$ and $p \geq t$. First we prove, by descending induction on $r \geq 2$, that $F_r^{t-1,0}$ is analytically of finite type. Indeed, there exists $s \in \mathbb{N}$ large enough, such that $F_r^{t-1,0}$ is a subquotient of M_{t-1} for every $r \geq s$, whence the assertion for every $r \geq s$, by virtue of claim 15.2.27 and corollary 15.2.8(i). Thus, suppose that $2 < r \leq s$, and we know already that $F_r^{t-1,0}$ is analytically of finite type; we have an exact sequence

$$0 \to F_r^{t-1,0} \to F_{r-1}^{t-1,0} \to F_{r-1}^{t-2+r,2-r}$$

and by inductive assumption we also know that $F_{r-1}^{t-2+r,2-r}$ is analytically of finite type. From corollary 15.2.8(i), it follows that the same holds for $F_{r-1}^{t-1,0}$, as required. Especially, in light of (15.2.26), the foregoing implies that $H^{t-1}(X, \mathscr{F})$ is analytically of finite type, for every finitely presented $\mathscr{O}_X$-module $\mathscr{F}$, hence the same holds more generally whenever $\mathscr{F}$ is analytically of finite type, by virtue of claim 15.2.23. Taking into account claim 15.2.24 and (15.2.25), we deduce that $F_2^{t-1,-q}$ is analytically of finite type for every $q \in \mathbb{N}$, and finally, the same follows for $F_r^{t-1,-q}$, whenever $q, r \in \mathbb{N}$ and $r \geq 2$, after invoking once more corollary 15.2.8(i). □

15.2.29. In the situation of theorem 15.2.22, let $I \subset A$ be any finitely generated ideal of adic definition; we notice that $I^n\mathscr{F}$ is a quasi-coherent $\mathscr{O}_X$-module of analytically finite type for every $\in \mathbb{N}$, and we define a descending filtration $\mathrm{Fil}_I^\bullet H^i$ on $H^i := H^i(X, \mathscr{F})$ by setting

$$\mathrm{Fil}_I^n H^i := \mathrm{Im}(\psi_{i,n} : H^i(X, I^n\mathscr{F}) \to H^i) \qquad \text{for every } i, n \in \mathbb{N}.$$

Corollary 15.2.30. *With the notation of* (15.2.29)*, the following holds for every $i \in \mathbb{N}$:*

(i) *The linear topology on H^i defined by the descending filtration $\mathrm{Fil}_I^\bullet H^i$ agrees with the I-adic topology.*
(ii) *The system* $(\mathrm{Ker}\,\psi_{i,n} \mid n \in \mathbb{N})$ *is essentially zero.*

Proof. Let us show first that (i)⇒(ii). To ease notation, set $K_n^i := \mathrm{Ker}\,\psi_{i,n}$ for every $i, n \in \mathbb{N}$. For every $a \in I^n$, the endomorphism $a \cdot \mathbf{1}_\mathscr{F}$ is the composition of the inclusion map $j_n : I^n\mathscr{F} \to \mathscr{F}$ and a morphism $\varphi_{a,n} : \mathscr{F} \to I^n\mathscr{F}$ of $\mathscr{O}_X$-modules, and we have as well

$$\varphi_{a,n} \circ j_n = a \cdot \mathbf{1}_{I^n\mathscr{F}}$$

whence $H^i(X, \varphi_{a,n}) \circ \psi_{i,n} = a \cdot \mathbf{1}_{H^i(X, I^n\mathscr{F})}$. We deduce immediately that

$$I^n K_n^i = 0 \qquad \text{for every } p, n \in \mathbb{N}. \tag{15.2.31}$$

Next, notice that

$$\mathrm{Im}(K_p^i \to K_n^i) = K_n^i \cap \mathrm{Im}(H^i(X, I^p\mathscr{F}) \to H^i(X, I^n\mathscr{F})) \qquad \text{for all integers } p \geq n \geq 0.$$

Suppose now that (i) holds for every $\mathscr{O}_X$-module $\mathscr{F}$ of analytically finite type; especially, it applies to $I^n\mathscr{F}$, and therefore for every $k \in \mathbb{N}$ we may find an integer $p \geq n$ such that $\mathrm{Im}(H^i(X, I^p\mathscr{F}) \to H^i(X, I^n\mathscr{F})) \subset I^k H^i(X, I^n\mathscr{F})$, whence

$$\mathrm{Im}(K_p^i \to K_n^i) \subset K_n^i \cap I^k H^i(X, I^n\mathscr{F}).$$

By theorem 15.2.22, we know that $H^i(X, I^n\mathscr{F})$ is analytically of finite type; by corollary 15.2.7 and lemma 15.2.3(ii), it follows that there exists $k \in \mathbb{N}$ large enough, such that $K_n^i \cap I^k H^i(X, I^n\mathscr{F}) \subset I^n K_n^i$, and then the assertion follows from (15.2.31).

(i): Let $a_1, \dots, a_k$ be a finite system of generators for I; we argue by induction on k. If $k = 1$, let $p \in \mathbb{N}$ such that $\mathrm{Ann}_\mathscr{F}(a_1^q) = \mathrm{Ann}_\mathscr{F}(a_1^p)$ for every $q \geq p$ (lemma 15.2.16(ii)); there follows a commutative diagram of $\mathscr{O}_X$-modules

$$\begin{array}{ccc} a_1^p \mathscr{F} & \longrightarrow & a_1^q \mathscr{F} \\ \downarrow & & \downarrow \\ \mathscr{F} & \xrightarrow{a_1^{q-p} \cdot \mathbf{1}_{\mathscr{F}}} & \mathscr{F} \end{array}$$

whose top horizontal arrow is an isomorphism, and whose vertical arrows are the inclusion maps. We deduce easily that $\mathrm{Fil}^q_I H^i = a_1^{q-p} \cdot \mathrm{Fil}^p_I H^i$ for every $q \geq p$, whence

$$a_1^q H^i \subset \mathrm{Fil}^q_I H^i \subset a_1^{q-p} H^i$$

as required. Next, let $k > 1$, and set $J := Aa_1 + \cdots + Aa_{k-1}$ and $L := Aa_k$. Denote by $\mathscr{T}_I$, $\mathscr{T}_J$ and $\mathscr{T}_L$ respectively the I-adic, J-adic and L-adic topologies on A.

Fix $p \in \mathbb{N}$; it is easily seen that $I^p H^i \subset \mathrm{Fil}^p H^i$, so it remains only to check that there exists $n \in \mathbb{N}$ such that $\mathrm{Fil}^n H^i \subset I^p H^i$. To this aim, notice first that X is also an analytically noetherian $(A, \mathscr{T}_L)$-scheme (proposition 15.2.5(i)), and $\mathscr{F}$ is also analytically of finite type relative to the topology $\mathscr{T}_L$; since L is principal, it follows that we may find $t \in \mathbb{N}$ such that

$$\mathrm{Fil}^t_L H^i \subset L^p H^i. \tag{15.2.32}$$

Notice also that

$$J^{r+t} \subset I^{r+t} \subset J^r + L^t \qquad \text{for every } r, t \in \mathbb{N}$$

which implies that the I-adic topology on any A/L^t-module agrees with the J-adic topology. By the same token, an A/L^t-module is an $(A, \mathscr{T}_I)$-module of analytically finite type if and only if it is an $(A, \mathscr{T}_J)$-module of analytically finite type. Taking into account proposition 15.2.5(i) we deduce that X is also an analytically noetherian $(A, \mathscr{T}_J)$-scheme and $\mathscr{F}/L^t\mathscr{F}$ is analytically of finite type relative to the topology $\mathscr{T}_J$ as well, for every $t \in \mathbb{N}$. Then, theorem 15.2.22 says that $H^i_t := H^i(X, \mathscr{F}/L^t\mathscr{F})$ is an A-module analytically of finite type relative to the topology $\mathscr{T}_J$, hence also relative to the topology $\mathscr{T}_I$, since it is an A/L^tA-module. Moreover, by inductive assumption the linear topology on H^i_t defined by the filtration $\mathrm{Fil}^\bullet_J H^i_t$ agrees with the J-adic topology, hence the linear topology on H^i_t defined by the filtration $\mathrm{Fil}^\bullet_I H^i_t$ agrees with the I-adic topology, by the foregoing observation.

Now, the image of $\mathrm{Fil}^n_I H^i$ in H^i_t lies in the intersection of $\mathrm{Fil}^n_I H^i_t$ with the submodule $H^i/\mathrm{Fil}^t_L H^i \subset H^i_t$, and the I-adic topology on the latter agrees with the one induced by the I-adic topology of H^i_t (corollary 15.2.7 and lemma 15.2.3(ii)), so

$$\mathrm{Im}(\mathrm{Fil}^n_I H^i \to H^i/\mathrm{Fil}^t_L H^i) \subset I^p(H^i/\mathrm{Fil}^t_L H^i) \qquad \text{for some } n \in \mathbb{N}$$

i.e. $\mathrm{Fil}^n_I H^i \subset I^p H^i + \mathrm{Fil}^t_L H^i$, and combining with (15.2.32), the assertion follows. □

15.2.33. Keep the notation of (15.2.29), and define $\pi : X^\wedge \to X$ and $\mathscr{F}^\wedge$ as in (15.2.18). Moreover, for every $i, n \in \mathbb{N}$ endow $H^i(X, \mathscr{F})$ with its I-adic topology and $H^i(X, \mathscr{F}/I^n\mathscr{F})$ with the discrete topology, and denote by $A^\wedge$ and $H^i(X, \mathscr{F})^\wedge$ the separated completions of A and respectively $H^i(X, \mathscr{F})$.

Corollary 15.2.34. *In the situation of* (15.2.33)*, the following holds :*

(i) *There exist natural $A^\wedge$-linear isomorphisms*

$$H^i(X^\wedge, \mathscr{F}^\wedge) \xrightarrow{\sim} \lim_{n \in \mathbb{N}} H^i(X, \mathscr{F}/I^n\mathscr{F}) \xleftarrow{\sim} H^i(X, \mathscr{F})^\wedge \qquad \textit{for every } i \in \mathbb{N}.$$

(ii) *For $i = 0$, the maps of* (i) *are even isomorphisms of topological $A^\wedge$-modules.*

(iii) *For every quasi-coherent $\mathscr{O}_X$-module $\mathscr{G}$, there exists a natural isomorphism*

$$H^i(X^\wedge, \pi^*\mathscr{G}) \xrightarrow{\sim} A^\wedge \otimes_A H^i(X, \mathscr{G}) \qquad \textit{for every } i \in \mathbb{N}.$$

Proof. (i): Fix a finite affine covering $U_\bullet := (U_\lambda \mid \lambda \in \Lambda)$ of X, and for every $\lambda \in \Lambda$ let $U_\lambda^\wedge$ be the completion of U_λ along its closed subscheme $\mathrm{Spec}\, A/I \times_{\mathrm{Spec}\, A} U$, . In light of lemma 15.1.28(i) we have a natural identification of complexes of topological $A^\wedge$-modules

$$\overline{C}^\bullet_{\mathrm{alt}}(U^\wedge_\bullet, \mathscr{F}^\wedge) \xrightarrow{\sim} \lim_{n\in\mathbb{N}} \overline{C}^\bullet_{\mathrm{alt}}(U_\bullet, \mathscr{F}/I^n\mathscr{F})$$

(where $\overline{C}^q_{\mathrm{alt}}(U_\bullet, \mathscr{F}/I^n\mathscr{F})$ is endowed with the discrete topology, for every $n, q \in \mathbb{N}$). On the other hand, by theorems 15.1.37(i) and 10.2.28(ii), we have natural isomorphisms

$$H^i(X, \mathscr{F}/I^n\mathscr{F}) \xrightarrow{\sim} H^i\overline{C}^\bullet_{\mathrm{alt}}(U_\bullet, \mathscr{F}/I^n\mathscr{F}) \qquad H^i(X^\wedge, \mathscr{F}^\wedge) \xrightarrow{\sim} H^i\overline{C}^\bullet_{\mathrm{alt}}(U^\wedge_\bullet, \mathscr{F}^\wedge)$$

for every $i, n \in \mathbb{N}$. Taking into account [163, Th.3.5.8] there follows a short exact sequence

$$(15.2.35) \qquad 0 \to \lim_{n\in\mathbb{N}}{}^1 H^{i-1}(X, \mathscr{F}/I^n\mathscr{F}) \to H^i(X^\wedge, \mathscr{F}^\wedge) \to \lim_{n\in\mathbb{N}} H^i(X, \mathscr{F}/I^n\mathscr{F}) \to 0$$

of $A^\wedge$-modules, for every $i \in \mathbb{N}$. Set also

$$F^i_k := \mathrm{Im}(H^i(X, \mathscr{F}) \to H^i(X, \mathscr{F}/I^k\mathscr{F})) \qquad \text{for every } i, k \in \mathbb{N}.$$

We notice :

Claim 15.2.36. $\lim_{n\in\mathbb{N}}{}^1 H^{i-1}(X, \mathscr{F}/I^n\mathscr{F}) = 0$ for every $i \in \mathbb{N}$.

Proof of the claim. By virtue of [163, Prop.3.5.7], it suffices to check that the descending system of submodules

$$(M^{i-1}_{k,k+n} := \mathrm{Im}(H^{i-1}(X, \mathscr{F}/I^{k+n}\mathscr{F}) \to H^{i-1}(X, \mathscr{F}/I^k\mathscr{F})) \mid n \in \mathbb{N})$$

is stationary, for every $i, k \in \mathbb{N}$. However, the short exact sequence of $\mathscr{O}_X$-modules

$$0 \to I^n\mathscr{F} \to \mathscr{F} \to \mathscr{F}/I^n\mathscr{F} \to 0$$

induces a commutative diagram of A-modules with exact rows

$$\begin{array}{ccccccc} H^{i-1}(X, \mathscr{F}) & \longrightarrow & H^{i-1}(X, \mathscr{F}/I^{n+k}\mathscr{F}) & \longrightarrow & H^i(X, I^{n+k}\mathscr{F}) & \longrightarrow & H^i(X, \mathscr{F}) \\ \| & & \downarrow & & \downarrow & & \| \\ H^{i-1}(X, \mathscr{F}) & \longrightarrow & H^{i-1}(X, \mathscr{F}/I^k\mathscr{F}) & \longrightarrow & H^i(X, I^k\mathscr{F}) & \longrightarrow & H^i(X, \mathscr{F}) \end{array}$$

which shows that

$$F^{i-1}_k \subset M^{i-1}_{k,k+n} \qquad \text{for every } i, k, n \in \mathbb{N}.$$

On the other hand, corollary 15.2.30(ii) implies that

$$M^{i-1}_{k,k+n} \subset \mathrm{Ker}(H^{i-1}(X, \mathscr{F}/I^k\mathscr{F}) \to H^i(X, I^k\mathscr{F})) = F^{i-1}_k$$

for every $k \in \mathbb{N}$ and every sufficiently large $n \in \mathbb{N}$. The claim follows. ◊

From (15.2.35) and claim 15.2.36, we already get the first stated isomorphism. Next, the proof of claim 15.2.36 shows that the system of inclusion maps $(F^i_n \to H^i(X, \mathscr{F}/I^n\mathscr{F}) \mid n \in \mathbb{N})$ induces an isomorphism of $A^\wedge$-modules

$$\lim_{n\in\mathbb{N}} F^i_n \xrightarrow{\sim} \lim_{n\in\mathbb{N}} H^i(X, \mathscr{F}/I^n\mathscr{F}) \qquad \text{for every } i \in \mathbb{N}.$$

But the system $(F^i_n \mid n \in \mathbb{N})$ is also isomorphic to the system $(H^i/\mathrm{Fil}^n_I H^i \mid n \in \mathbb{N})$ (notation of (15.2.29)), and lastly, from corollary 15.2.30(i) we obtain a natural isomorphism

$$\lim_{n\in\mathbb{N}} H^i/\mathrm{Fil}^n_I H^i \xrightarrow{\sim} H^i(X, \mathscr{F})^\wedge$$

whence the second stated isomorphism.

(ii): The topology of $H^0(X,\mathscr{F})^\wedge$ is I-adic, by remark 8.3.3(ii,iv). On the other hand, the topology of $H^0(X^\wedge,\mathscr{F}^\wedge)$ is the linear topology given by the system of submodules

$$(H^0(X^\wedge,(I^n\mathscr{F})^\wedge) \mid n \in \mathbb{N})$$

by remark 15.1.48. Denote $H^0(X, I^n\mathscr{F})^\wedge$ the I-adic completion of $H^0(X, I^n\mathscr{F})$, for every $n \in \mathbb{N}$; by (i) we get a commutative diagram of $A^\wedge$-modules

$$\begin{array}{ccc} H^0(X^\wedge,(I^n\mathscr{F})^\wedge) & \longrightarrow & H^0(X, I^n\mathscr{F})^\wedge \\ \downarrow & & \downarrow \\ H^0(X^\wedge,\mathscr{F}^\wedge) & \longrightarrow & H^0(X,\mathscr{F})^\wedge \end{array} \tag{15.2.37}$$

whose horizontal arrows are isomorphisms, and whose left (resp. right) vertical arrow is induced by the inclusion $(I^n\mathscr{F})^\wedge \subset \mathscr{F}^\wedge$ (resp. is the I-adic completion of the natural map $H^0(X, I^n\mathscr{F}) \to H^0(X,\mathscr{F})$). By corollary 15.2.30(i), the topology on $H^0(X,\mathscr{F})^\wedge$ given by the filtration $(H^0(X, I^n\mathscr{F})^\wedge \mid n \in \mathbb{N})$ agrees with the topology given by the filtration $((I^n H^0(X,\mathscr{F}))^\wedge \mid n \in \mathbb{N})$, and the latter is none else than the I-adic filtration, again by remark 8.3.3(ii,iv). We conclude already that the bottom horizontal arrow of (15.2.37) is an isomorphism of topological $A^\wedge$-modules. The same holds for the natural map $H^0(X^\wedge,\mathscr{F}^\wedge) \to \lim_{n\in\mathbb{N}} H^0(X,\mathscr{F}/I^n\mathscr{F})$, by the discussion of (15.1.47).

(iii): Since X is quasi-compact and quasi-separated, $\mathscr{G}$ is the colimit of the filtered family $(\mathscr{G}_\lambda \mid \lambda \in \Lambda)$ of its quasi-coherent $\mathscr{O}_X$-submodules of finite type (proposition 10.3.31), hence $\pi^*\mathscr{G}$ is the colimit of the system $(\pi^*\mathscr{G}_\lambda \mid \lambda \in \Lambda)$. Since $X^\wedge$ is a spectral topological space, proposition 10.1.10 then reduces to showing the assertion for each $\mathscr{G}_\lambda$, so we may assume from start that $\mathscr{G}$ is analytically of finite type, in which case it suffices to apply (i), propositions 15.2.10(i) and 15.2.20(i), and theorem 15.2.22. □

Remark 15.2.38. (i) The assertions of theorem 15.2.22 and of its corollaries 15.2.30 and 15.2.34 hold also in case X is projective but not necessarily finitely presented : indeed, if $i : X \to \mathbb{P}^n_A$ is a closed immersion in a projective space over $\operatorname{Spec} A$, then $H^\bullet(X,\mathscr{F}) = H^\bullet(\mathbb{P}^n_A, i_*\mathscr{F})$, and $i_*\mathscr{F}$ is clearly an $\mathscr{O}_{\mathbb{P}^n_A}$-module of analytically finite type, so we are reduced to the case where $X = \mathbb{P}^n_A$, which is covered by theorem 15.2.22.

(ii) More generally, theorem 15.2.22 and its two corollaries hold for any proper A-scheme X (and any $\mathscr{O}_X$-module $\mathscr{F}$ of analytically finite type) : indeed, according to [49, Th.4.1 and Th.4.3] there exists a locally closed immersion $i : X \to X'$ of A-scheme, with X' proper and finitely presented over $\operatorname{Spec} A$. Since X is proper, i is even a closed immersion ([60, Ch.II, Cor.5.4.3]), so again it suffices to apply the theorem to the $\mathscr{O}_{X'}$-module $i_*\mathscr{F}$.

(iii) Theorem 15.2.22 and corollary 15.2.34 have also been announced, respectively as theorem C.3.1 and theorem C.3.3 of Chapter 1, in K.Fujiwara and F.Kato's treatise [72]; however, their proofs have been postponed to a forthcoming article, which has not appeared yet.

Corollary 15.2.39. *Let A be a universally analytically noetherian ring, $\mathbf{f} := (f_1, \dots, f_r)$ a sequence of elements of A that generates an ideal of adic definition. Then A satisfies condition* $(\mathrm{a})^{\mathrm{un}}_{\mathbf{f}}$ *of* (7.8.21).

Proof. Let $X := \operatorname{Spec} A$, denote by $I \subset A$ the ideal generated by $\mathbf{f}$, and by $\mathscr{I} \subset \mathscr{O}_X$ the quasi-coherent ideal arising from I. Let also $\pi : Y \to X$ be the blowing up morphism of $\mathscr{I}$. It follows easily from remark 10.6.39(iii) that the support of the A-module $H^p := H^p(Y,\mathscr{O}_Y)$ is contained in $\operatorname{Spec} A/I$, for every $p > 0$, and the same holds for the kernel and cokernel of the natural map $\varphi : A \to H^0(Y,\mathscr{O}_Y)$. Combining with theorem 15.2.22 and remark 15.2.2(iii), we deduce that there exists $n \in \mathbb{N}$ such that $I^n \cdot H^p = 0$ for every $p > 0$, and $I^n \cdot \operatorname{Ker}\varphi = I^n \cdot \operatorname{Coker}\varphi = 0$. Then it suffices to invoke theorem 10.6.50. □

15.3. **Continuous valuations.** As already mentioned in (8.1.54), the study of the valuation spectrum of a topological ring A will lead us to consider certain subsets of $\mathrm{Spv}\, A$ that are not pro-constructible, but nevertheless are spectral spaces, with the topology induced via the inclusion map into $\mathrm{Spv}\, A$.

15.3.1. As a first step, let A be any ring, $I \subset A$ a finitely generated ideal; we attach to I a set of specializations S_I of $\mathrm{Spv}\, A$ (see (8.1.54)), by declaring that $(v, w) \in S_I$ if and only if :

- either, $w(I) \neq \{0\}$ and w is a primary specialization of v
- or else, $v = w$.

The S_I-admissible specializations in $\mathrm{Spv}\, A$ will be simply called *I-admissible*, and the S_I-closed subsets of $\mathrm{Spv}\, A$ shall be called *I-closed.* It is clear that the set of I-admissible specializations of $\mathrm{Spv}\, A$ is transitive, and it satisfies condition (S3) of corollary 8.1.57, by lemma 9.2.15(i). We shall show hereafter that it fulfills also conditions (S1) and (S2) of proposition 8.1.55. This shall be achieved in several steps; to start out, for any abelian ordered group Δ, let us say that an element $\delta \in \Delta_\circ$ is *final in* Δ if the following holds. For every $\gamma \in \Delta$ there exists $n \in \mathbb{N}$ such that $\delta^n < \gamma$. We remark :

Lemma 15.3.2. *In the situation of* (15.3.1)*, let v be a valuation of A with*

$$v(I) \neq \{0\} \qquad \textit{and} \qquad v(I) \cap c\Gamma_v = \varnothing.$$

Then we have :

(i) *There exists a smallest convex subgroup Δ of Γ_v with $v(I) \cap \Delta \neq \varnothing$.*
(ii) $c\Gamma_v \subset \Delta$.
(iii) *For every $i \in I$, either $v(i) < \delta$ for every $\delta \in \Delta$, or else $v(i)$ is final in Δ.*

Proof. (i): Let $a_1, \dots, a_n$ be a finite system of generators of I; the stated assumptions on v imply that $v(a_i) < 1$ for every $i = 1, \dots, n$, and $v(a_k) > 0$ for at least one index $k \leq n$. Let Δ be the convex hull of the subgroup Δ_0 of Γ_v generated by $\max\{v(a_i) \mid i = 1, \dots, n\}$. It remains only to check that $\Delta \subset \Delta'$ for every convex subgroup $\Delta' \subset \Gamma_v$ with $\Delta' \cap v(I) \neq \varnothing$. To this aim, notice first that $c\Gamma_v \subset \Delta'$ for every such Δ' : indeed, otherwise there exists $\gamma \in c\Gamma_v$ such that $\gamma < \delta$ for every $\delta \in \Delta'$. Then, pick $i \in I$ with $v(i) \in \Delta'$; it follows that $1 > v(i) > \gamma$, so $v(i) \in c\Gamma_v$, contradicting our assumptions. Now, let $i \in I$ be any element with $v(i) \in \Delta'$, and write $i = a_1 b_1 + \cdots + a_n b_n$ for some $b_1, \dots, b_n \in A$; since $c\Gamma_v \subset \Delta'$ and $v(i) \leq \max\{v(a_i) \cdot v(b_i) \mid i = 1, \dots, n\}$, it follows that $v(a_k) \in T$ for some $k \leq n$ (details left to the reader). Then clearly $\Delta_0 \subset \Delta'$, and the contention follows. Moreover, the argument also shows (ii).

(iii): Let $a_0 \in I$ be any element; the sequence $a_0, a_1, \dots, a_n$ is also a system of generators for I, and the foregoing shows that Δ is the convex hull of the subgroup generated by $\max\{v(a_i) \mid i = 0, \dots, n\}$. The assertion follows immediately. □

Definition 15.3.3. Keep the situation of (15.3.1). To every $v \in \mathrm{Spv}\, A$ we attach a convex subgroup

$$c\Gamma_v(I) \subset \Gamma_v$$

as follows :

- If $v(I) = \{0\}$, we set $c\Gamma_v(I) := \Gamma_v$.
- If $v(I) \cap c\Gamma_v \neq \varnothing$, we set $c\Gamma_v(I) := c\Gamma_v$.
- Otherwise, v fulfills the conditions of lemma 15.3.2, and we let $c\Gamma_v(I)$ be the smallest of the convex subgroups Δ such that $v(I) \cap \Delta \neq \varnothing$.

Lemma 15.3.4. *Keep the notation of definition* 15.3.3*, and let $a_1, \dots, a_n$ be a system of generators for the ideal I. We have :*

(i) *The following conditions are equivalent :*

(a) $\Gamma_v = c\Gamma_v(I)$.
(b) $\Gamma_v = c\Gamma_v$*, or else* $v(a)$ *is final in* Γ_v *for every* $a \in I$.
(c) $\Gamma_v = c\Gamma_v$*, or else* $v(a_i)$ *is final in* Γ_v*, for every* $i = 1, \dots, n$.

(ii) *An element* $w \in \mathrm{Spv}\, A$ *is an* I*-admissible specialization of* v *if and only if* $w = v^\Delta$ *for a convex subgroup* Δ *of* Γ_v *with* $c\Gamma_v(I) \subset \Delta$.

(iii) v *has no proper* I*-admissible specializations if and only if* $\Gamma_v = c\Gamma_v(I)$.

Proof. (i): The equivalence of (a) and (b) follows directly from the definitions and lemma 15.3.2(iii). The equivalence of (b) and (c) follows by remarking that the set

$$\{a \in A \mid v(a) = 0 \text{ or else } v(a) \text{ is final in } \Gamma_v\}$$

is an ideal in A, provided $\Gamma_v \neq c\Gamma_v$ (details left to the reader).

(ii) and (iii) are immediate from the definitions. □

Lemma 15.3.5. *Let* A *be a ring,* $I \subset A$ *a finitely generated ideal. We have :*

(i) *Let also* $f_\bullet := (f_0, f_1, \dots, f_n)$ *be a finite sequence of elements of* A*, and* $J \subset A$ *the ideal generated by* $f_\bullet$*. Suppose that the radical of* J *contains* I*. Then the rational subset* $R_A\big(\frac{f_1}{f_0}, \dots, \frac{f_n}{f_0}\big)$ *is* I*-closed.*

(ii) *Let* v *be any valuation of* A *such that* $\Gamma_v = c\Gamma_v(I)$*. Then for every open neighborhood* U *of* v *in* $\mathrm{Spv}\, A$ *there exists a finite sequence* $f_\bullet := (f_0, f_1, \dots, f_n)$ *of elements of* A *such that :*
(a) $v \in R_A\big(\frac{f_1}{f_0}, \dots, \frac{f_n}{f_0}\big) \subset U$.
(b) I *is contained in the radical of the ideal generated by* $f_\bullet$.

Proof. (i): Let $w \in \mathrm{Spv}\, A$ be an I-admissible specialization of some $v \in R := R_A\big(\frac{f_1}{f_0}, \dots, \frac{f_n}{f_0}\big)$. If $v(I) = \{0\}$, then $v = w$, and there is nothing to prove. Thus, we may assume $v(I) \neq \{0\}$, in which case $w(I) \neq \{0\}$ as well. Since w is a primary specialization of v, we have $w(f_i) \leq w(f_0)$ for every $i = 1, \dots, n$. Suppose, by way of contradiction, that $w \notin R$; then we must have $w(f_0) = 0$, and consequently $w(f_i) = 0$ for every $i \leq n$ as well. Thus, $J \subset \mathrm{Ker}\, w$, and since the support of w is a prime ideal, it contains also the radical of J; hence, $I \subset \mathrm{Ker}\, w$, a contradiction.

(ii): We may assume that $U = R_A\big(\frac{g_1}{g_0}, \dots, \frac{g_k}{g_0}\big)$ for some $g_0, g_1, \dots, g_k \in A$. Pick also a finite system $a_1, \dots, a_n$ of generators of I. We distinguish two cases :

• Suppose first that $\Gamma_v = c\Gamma_v$. In this case, since $v(g_0) \neq 0$, there exists $d \in A$ such that $v(g_0) \cdot v(d) = v(g_0 d) > 1$. Then $v \in R_A\big(\frac{1}{g_0 d}\big) \cap U$, and we may take $f_\bullet := (g_0 d, \dots, g_k d, 1)$.

• Lastly, suppose that $\Gamma_v \neq c\Gamma_v$. In this case, since $v(g_0) \neq 0$, lemma 15.3.4(i) implies that there exists $r \in \mathbb{N}$ with $v(g_0) \geq v(a_i^r)$ for every $i = 1, \dots, n$. Then $v \in R_A\big(\frac{a_1^r}{g_0}, \dots, \frac{a_n^r}{g_0}\big) \cap U$, and the sequence $f_\bullet := (g_0, \dots, g_k, a_1^r, \dots, a_n^r)$ will do. □

Definition 15.3.6. Let A be any ring and $I \subset A$ any finitely generated ideal.

(i) We set

$$\mathrm{Spv}(A, I) := \{v \in \mathrm{Spv}\, A \mid \Gamma_v = c\Gamma_v(I)\}$$

and we endow $\mathrm{Spv}(A, I)$ with the topology induced by $\mathrm{Spv}\, A$ via the inclusion map.

(ii) The *rational subsets* of $\mathrm{Spv}(A, I)$ are the subsets of the form

$$\mathrm{Spv}(A, I) \cap R_A\Big(\frac{f_1}{f_0}, \dots, \frac{f_n}{f_0}\Big)$$

where $f_\bullet := (f_0, f_1, \dots, f_n)$ is any sequence of elements of A such that I is contained in the radical of the ideal generated by $f_\bullet$.

Theorem 15.3.7. *With the notation of definition* 15.3.6*, we have :*

(i) *The topological space* $\mathrm{Spv}(A,I)$ *is spectral, and the retraction*

$$r_I : \mathrm{Spv}\, A \to \mathrm{Spv}(A,I) \qquad v \mapsto v^{c\Gamma_v(I)}$$

is spectral.

(ii) *The rational subsets of* $\mathrm{Spv}(A,I)$ *are constructible in* $\mathrm{Spv}(A,I)$*, and form a basis of the topology of* $\mathrm{Spv}(A,I)$ *that is closed under finite intersections.*

(iii) *A subset* $T \subset \mathrm{Spv}(A,I)$ *is constructible in* $\mathrm{Spv}(A,I)$ *if and only if* $r_I^{-1}T$ *is constructible in* $\mathrm{Spv}\, A$.

Proof. (i): From lemma 15.3.5 we see that the set of I-admissible specializations of $\mathrm{Spv}\, A$ fulfills conditions (S1) and (S2) of proposition 8.1.55, whence the assertion.

(ii): Lemma 15.3.5 already shows that the rational subsets form a basis $\mathscr{B}$ of the topology of $\mathrm{Spv}(A,I)$. If $R_A\big(\frac{f_1}{f_0},\dots,\frac{f_n}{f_0}\big)$ and $R_A\big(\frac{g_1}{g_0},\dots,\frac{g_m}{g_0}\big)$ are any two rational open subset of $\mathrm{Spv}\, A$ such that the radicals of the ideals J and J' generated respectively by $f_\bullet$ and $g_\bullet$ contain I, then the same holds for the radical of the ideal JJ'; taking into account remark 9.2.4(i), it follows that $\mathscr{B}$ is closed under finite intersections.

Lastly, let $f_\bullet$ and J be as in the foregoing, with I contained in the radical of J, and set $R := R_A\big(\frac{f_1}{f_0},\dots,\frac{f_n}{f_0}\big)$. By lemma 15.3.5, the set R is I-closed, so $r_I(R) = R\cap\mathrm{Spv}(A,I)$; since R is quasi-compact in $\mathrm{Spv}\, A$, the subset $r_I(R)$ is quasi-compact in $\mathrm{Spv}(A,I)$, so $R\cap\mathrm{Spv}(A,I)$ is a constructible open subset of $\mathrm{Spv}(A,I)$.

(iii) follows from proposition 8.1.55(iii). □

Example 15.3.8. Let A be any ring.

(i) We may take $I = 0$, in which case $c\Gamma_v(I) = \Gamma_v$ for every $v \in \mathrm{Spv}\, A$, so that $\mathrm{Spv}(A,0) = \mathrm{Spv}\, A$. Also, the rational subsets of $\mathrm{Spv}(A,0)$ are the same as the rational subsets of $\mathrm{Spv}\, A$.

(ii) We may also take $I = A$, in which case $c\Gamma_v(I) = c\Gamma_v$ for every $v \in \mathrm{Spv}\, A$, so that $\mathrm{Spv}(A,A) = \{v \in \mathrm{Spv}\, A \mid \Gamma_v = c\Gamma_v\}$. The rational subsets of $\mathrm{Spv}(A,A)$ can be described as the sets of the form

$$\mathrm{Spv}(A,A) \cap \{v \in \mathrm{Spv}\, A \mid v(f_1) \le v(f_0),\dots,v(f_n) \le v(f_0)\}$$

where $f_0, f_1, \dots, f_n$ is any sequence of elements with $\sum_{i=1}^n f_iA = A$ (details left to the reader).

Lemma 15.3.9. *Let* A *be any ring, and* $I, J \subset A$ *two finitely generated ideals. We have :*

(i) *If* I *is contained in the radical of* J*, then* $\mathrm{Spv}(A,J) \subset \mathrm{Spv}(A,I)$*. The inclusion map* $\mathrm{Spv}(A,J) \to \mathrm{Spv}(A,I)$ *is not spectral in general. On the other hand, there is a spectral retraction*

$$r_{I,J} : \mathrm{Spv}(A,I) \to \mathrm{Spv}(A,J) \qquad \textit{such that} \qquad r_{I,J}\circ r_I = r_J.$$

(ii) *Especially, the spectral map* $r_{I,I+J}$ *is well defined, and it coincides with the restriction of* r_J *to* $\mathrm{Spv}(A,I)$.

(iii) $\mathrm{Spv}(A, I\cdot J) = \mathrm{Spv}(A,I) \cup \mathrm{Spv}(A,J)$.

(iv) $\mathrm{Spv}(A, I+J) = \mathrm{Spv}(A,I) \cap \mathrm{Spv}(A,J)$.

Proof. (i): Indeed, the assumption easily implies that $c\Gamma_v(J) \subset c\Gamma_v(I)$ for every $v \in \mathrm{Spv}\, A$, from which the assertion follows immediately.

(ii): Let $v \in \mathrm{Spv}\, A$ be any element, and set $w := r_I(v)$, $u := r_J(w)$; then u does not have proper J-admissible specializations, nor any proper I-admissible specialization, because any I-admissible specialization of u would be also an I-admissible specialization of w. It follows easily that u is a primary specialization of v that does not have any proper $(I+J)$-specialization, whence the assertion.

Claim 15.3.10. For every $v \in \mathrm{Spv}\, A$ we have :

$$c\Gamma_v(I+J) = c\Gamma_v(I) \cap c\Gamma_v(J) \qquad \text{and} \qquad c\Gamma_v(I\cdot J) = c\Gamma_v(I) \cup c\Gamma_v(J).$$

Proof of the claim. We consider first the assertion for $c\Gamma_v(I \cdot J)$: it is easily seen that $v(IJ) = \{0\}$ if and only if either $v(I) = \{0\}$ or $v(J) = \{0\}$, so the assertion holds in case $v(IJ) = \{0\}$. Similarly, $v(IJ) \cap c\Gamma_v \neq \varnothing$ if and only if both $v(I)$ and $v(J)$ intersect $c\Gamma_v$, so the assertion holds also in case $v(IJ)$ intersects $c\Gamma_v$. Lastly, suppose that $v(IJ) \neq \{0\}$ and $v(IJ) \cap c\Gamma_v = \varnothing$, and pick a finite system of generators $a_1, \dots, a_n$ (resp. $b_1, \dots, b_m$) for I (resp. for J), with

$$v(a_1) = \max(v(a_i) \mid i = 1, \dots, n) \qquad \text{and} \qquad v(b_1) = \max(v(b_i) \mid i = 1, \dots, m).$$

Then the system $(a_i b_j \mid i \leq n,\ j \leq m)$ generates IJ, and $v(ab) = \max(v(a_i b_j) \mid i \leq n,\ j \leq m)$. Hence $v(a_1)$ (resp. $v(b_1)$, resp. $v(a_1 b_1)$) is final in $c\Gamma_v(I)$ (resp. in $c\Gamma_v(J)$, resp. in $c\Gamma_v(IJ)$) by lemma 15.3.2(iii). Without loss of generality we may assume that $c\Gamma_v(J) \subset c\Gamma_w(I)$, in which case there exists $n \in \mathbb{N}$ such that $v(a_1^n) \leq v(b_1)$, hence $v(a_1 b_1)^n \geq v(a_1)^{2N}$, which shows that $c\Gamma_v(IJ) \subset c\Gamma_v(I)$, and the converse inclusion is clear.

We argue similarly for the calculation of $c\Gamma_v(I + J)$: first, it is clear that $v(I + J) = \{0\}$ if and only if $v(I) = v(J) = \{0\}$. Next, we have $v(I + J) \cap c\Gamma_v \neq \varnothing$ if and only if $(v(I) \cup v(J)) \cap c\Gamma_v \neq \varnothing$ (details left to the reader). Lastly, if $v(IJ) \neq \{0\}$ and $v(IJ) \cap c\Gamma_v = \varnothing$, pick $a_1, \dots, a_n$ and $b_1, \dots, b_m$ as in the foregoing; without loss of generality we may assume that $v(a_1) \geq v(b_1)$. Since the system $a_1, \dots, a_n, b_1, \dots, b_m$ generates $I + J$, we deduce that $v(a_1)$ is final in $c\Gamma_v(I + J)$ (lemma 15.3.2(iii)), whence the contention. ◊

Assertion (iii) and (iv) follow immediately from claim 15.3.10 : details left to the reader. □

Example 15.3.11. Let A be any ring, $I \subset A$ a finitely generated ideal.

(i) Every element of the subset $L := \{v \in \mathrm{Spv}\, A \mid v(I) = \{0\}\}$ has no proper I-admissible specializations, so L lies in $\mathrm{Spv}(A, I)$. Clearly, L is constructible in $\mathrm{Spv}\, A$, and $r_I^{-1}(L) = L$, so L is constructible in $\mathrm{Spv}(A, I)$, by theorem 15.3.7(iii).

(ii) Let $(\mathrm{Spv}\, A)_0$ be the set of trivial valuations of A; from remark 9.2.4(iii) we know already that $(\mathrm{Spv}\, A)_0$ is a spectral space, with the topology induced by the inclusion into $\mathrm{Spv}\, A$. Moreover, if $R := R_A\big(\frac{f_1}{f_0}, \dots, \frac{f_n}{f_0}\big)$ is any rational subset of $\mathrm{Spv}(A, I)$, then $R \cap (\mathrm{Spv}\, A)_0 = \{v \in (\mathrm{Spv}\, A)_0 \mid v(f_0) = 1\}$, which is a constructible (open) subset of $(\mathrm{Spv}\, A)_0$. Hence, the inclusion map $(\mathrm{Spv}\, A)_0 \to \mathrm{Spv}(A, I)$ is spectral.

(iii) Pick a finite system $a_1, \dots, a_n$ of generators for I, and set

$$E_0 := \{v \in (\mathrm{Spv}\, A)_0 \mid v(I) = \{0\}\}$$
$$E_i := \{v \in \mathrm{Spv}\, A \mid v(a_i) = 1 \text{ and } v(a) \leq 1 \text{ for every } a \in A\} \quad \text{for every } i = 1, \dots, n.$$

Then $E_0, \dots, E_n$ are pro-constructible subsets of $\mathrm{Spv}\, A$ that are closed under I-admissible specializations and generizations. We have $(\mathrm{Spv}\, A)_0 = \mathrm{Spv}(A, I) \cap \bigcup_{i=0}^n E_i$, and therefore $r_I^{-1}(\mathrm{Spv}\, A)_0 = \bigcup_{i=0}^n E_i$, by corollary 8.1.57.

We are now ready to introduce the *spectrum of continuous valuations* of a topological ring.

Definition 15.3.12. Let $(A, \mathscr{T}_A)$ be any topological ring, and v a valuation of A.

(i) We endow $\Gamma_{v\circ}$ with a topology $\mathscr{T}_{\Gamma_v}$, by ruling that a subset $U \subset \Gamma_{v\circ}$ is open if either $0 \notin U$, or else there exists $\delta \in \Gamma_v$ such that $\{\gamma \in \Gamma_{v\circ} \mid \gamma < \delta\} \subset U$. Then, we say that v is *continuous* if it is a continuous map $(A, \mathscr{T}_A) \to (\Gamma_{v\circ}, \mathscr{T}_{\Gamma_v})$.

(ii) We set

$$\mathrm{Cont}(A) := \{v \in \mathrm{Spv}\, A \mid v \text{ is continuous}\} \qquad \mathrm{Cont}^+(A) := \mathrm{Cont}(A) \cap \mathrm{Spv}^+(A)$$

and we endow $\mathrm{Cont}(A)$ and $\mathrm{Cont}^+(A)$ with the topology induced by $\mathrm{Spv}\, A$.

Remark 15.3.13. Let A be any topological ring, any v any valuation of A.

(i) Endow $\kappa(v)$ with its valuation topology $\mathscr{T}_v$ (see definition 9.1.14(i)). A simple inspection of the definitions shows that the residual valuation $\overline{v} : (\kappa(v), \mathscr{T}_v) \to \Gamma_{v\circ}$ of v is continuous. Moreover, v is continuous if and only if the same holds for the natural map $\pi_v : A \to (\kappa(v), \mathscr{T}_v)$.

Indeed, clearly if π_v is continuous, the same holds for $v = \overline{v} \circ \pi_v$. Conversely, suppose that v is continuous; it suffices to check that π_v is continuous at the point $0 \in A$. However, for every $\delta \in \Gamma_v$ let $U_\delta := \{\gamma \in \Gamma_{v\circ} \mid \gamma < \delta\}$; then the family $(U_\delta \mid \delta \in \Gamma_v)$ is a fundamental system of open neighborhoods of $0 \in \Gamma_{v\circ}$, and $(\overline{v}^{-1} U_\delta \mid \delta \in \Gamma_v)$ is a fundamental system of open neighborhoods of $0 \in \kappa(v)$, whence the contention.

(ii) It follows easily from (i) that the valuation v is continuous if and only if it is continuous at the point $0 \in A$.

(iii) Let $\varphi : \Gamma \to \Gamma'$ be a morphism of ordered groups, and endow $\Gamma_\circ$ and $\Gamma'_\circ$ with the topologies $\mathscr{T}_\Gamma$ and $\mathscr{T}_{\Gamma'}$ described in definition 15.3.12(i). In general, the induced map $\varphi_\circ : \Gamma_\circ \to \Gamma'_\circ$ is *not* necessarily continuous for these topologies; for instance :

- If Γ is a subgroup of Γ', and φ is the inclusion map, then $\varphi_\circ$ is continuous if and only if the convex hull of Γ equals Γ' (see remark 9.1.2(iii)), and if the latter condition holds, then the topology $\mathscr{T}_\Gamma$ agrees with the topology induced by $\mathscr{T}_{\Gamma'}$.
- If $\Gamma' = \Gamma/\Delta$ for a proper convex subgroup $\Delta \subset \Gamma$, and φ is the projection, then $\varphi_\circ$ is continuous.
- If $\Gamma \neq \{1\}$ and $\Gamma' = \{1\}$, then $\varphi_\circ$ is not continuous.

Especially, a secondary generization w of v is continuous if either $v = w$ or else $\Gamma_w \neq \{1\}$, but w may fail to be continuous in case $\Gamma_v \neq \{1\}$ and $\Gamma_w = \{1\}$. On the other hand, if $\varphi_\circ \circ v$ is a continuous map, then it is easily seen that v is continuous as well.

(iv) Let $\Delta \subset \Gamma_v$ be any convex subgroup containing $c\Gamma_v$, denote by $\Gamma^+_{v\circ} \cdot \Delta$ the smallest submonoid of $\Gamma_{v\circ}$ containing $\Gamma^+_{v\circ}$ and Δ, and endow $\Gamma^+_{v\circ} \cdot \Delta$ with the topology $\mathscr{T}_{\Gamma^+_{v\circ}\cdot\Delta}$ induced by $\mathscr{T}_{\Gamma_v}$ via the inclusion map. Let also $\mathscr{T}_\Delta$ be the topology on $\Delta_\circ$ described in definition 15.3.12(i). Then the inclusion map $\Delta_\circ \to \Gamma^+_{v\circ} \cdot \Delta$ admits a unique continuous retraction

$$\rho : (\Gamma^+_{v\circ} \cdot \Delta, \mathscr{T}_{\Gamma^+_{v\circ}\cdot\Delta}) \to (\Delta_\circ, \mathscr{T}_\Delta) \qquad \text{such that } \rho(\gamma) = 0 \text{ whenever } \gamma \notin \Delta_\circ.$$

On the other hand, since $c\Gamma_v \subset \Delta$, the map v factors through a (unique) mapping $v' : A \to \Gamma^+_{v\circ} \cdot \Delta$, and if v is continuous, the same holds for v'. Clearly $\rho \circ v' = v^\Delta$, so the latter is continuous as well. Combining with (iii) and proposition 9.2.24, we conclude that every specialization of a continuous valuation is continuous. However, in general $\mathrm{Cont}(A)$ is not a closed subset (not even a pro-constructible subset) of $\mathrm{Spv}\, A$.

(v) Since $\mathrm{Spv}\, A$ is a T_0 topological space, the same holds for $\mathrm{Cont}(A)$. Especially, the specializations in $\mathrm{Cont}(A)$ define a partial ordering on $\mathrm{Cont}(A)$: see remark 8.1.45(ii).

(vi) It follows also from (ii) that if $\mathrm{Ker}\, v$ is an open prime ideal, then v is continuous. We shall say that v is *non-analytic* if $\mathrm{Ker}\, v$ is an open prime ideal. If v is continuous and its support is not an open prime ideal, we shall say that v is *analytic*. We denote by

$$\mathrm{Cont}(A)_{\mathrm{a}} \qquad \text{and} \qquad \mathrm{Cont}(A)_{\mathrm{na}}$$

the sets of all analytic and respectively non-analytic valuations of A.

(vii) Let $f : A \to B$ be any continuous ring homomorphism of topological rings. Then $\mathrm{Spv}(f)$ restricts to continuous maps

$$\mathrm{Cont}(f) : \mathrm{Cont}(B) \to \mathrm{Cont}(A) \qquad \mathrm{Cont}^+(f) : \mathrm{Cont}^+(B) \to \mathrm{Cont}^+(A).$$

Indeed, let w be a continuous valuation of B; obviously $v := w \circ f : A \to \Gamma_{w\circ}$ is a continuous map. The value group Γ_v of v is a subgroup of Γ_w, and v factors uniquely through a valuation $v' : A \to \Gamma_v$ that is obviously equivalent to v; by (iii), the continuity of w implies the continuity of v', whence the claim. It is then also clear that $\mathrm{Cont}(f)$ in turns restricts to a mapping

$$\mathrm{Cont}(f)_{\mathrm{na}} : \mathrm{Cont}(B)_{\mathrm{na}} \to \mathrm{Cont}(A)_{\mathrm{na}}.$$

Lemma 15.3.14. *Let A be any f-adic topological ring, and $v \in \mathrm{Cont}(A)$. We have :*

(i) $\mathrm{Cont}(A) = \{w \in \mathrm{Spv}\, A \mid w(a)$ *is final in* Γ_w*, for every* $a \in A^{\circ\circ}\}$.

(ii) *Suppose that v is analytic, and endow $\kappa(v)$ with its valuation topology $\mathscr{T}_v$. Then v is a Tate valuation (see definition* 9.1.14(iv)*), and the natural map*

$$\pi_v : A \to (\kappa(v), \mathscr{T}_v)$$

is an f-adic continuous ring homomorphism.

(iii) *Let $w \in \mathrm{Cont}(A)$ be any proper primary specialization of v. Then w is non-analytic.*

(iv) *Let $w \in \mathrm{Cont}(A)_{\mathrm{a}}$ be a specialization of v. Then $v \in \mathrm{Cont}(A)_{\mathrm{a}}$ and w is a secondary specialization of v.*

(v) *If v is analytic, the totally ordered set of continuous generizations of v admits a maximal element, which is a rank one analytic valuation.*

Proof. Let $A_0 \subset A$ be a subring of definition, $I \subset A_0$ a finitely generated ideal of adic definition, and $a_1, \dots, a_n$ a finite system of generators of I; for every $w \in \mathrm{Spv}\, A$ we set $\gamma_w := \max(w(a_1), \dots, w(a_n))$

(i): The condition is clearly necessary. Conversely, suppose that $w \in \mathrm{Spv}\, A$ fulfills this condition; by remark 15.3.13(ii), it suffices to check that, for every $\delta \in \Gamma_w$ there exists an open neighborhood U_δ of 0 in A such that $w(a) < \delta$ for every $a \in U_\delta$. However, we have $w(a) < 1$ for every $a \in I$, since $I \subset A^{\circ\circ}$, and we may find $k \in \mathbb{N}$ such that $\delta > \gamma_w^k$. Hence $U_\delta := I^{k+1}$ will do.

(ii): Denote by $\overline{v}$ the residual valuation of v; if v is analytic, the value γ_v lies in Γ_v, and if $x \in \kappa(v)^+$ is any element with $\overline{v}(x) = \gamma$, it follows easily that the valuation topology on $\kappa(v)^+$ agrees with the x-adic topology, so v is a Tate valuation. Next, after replacing A_0 by the smallest $\mathbb{Z}$-subalgebra of A_0 containing I, we may assume that $\pi_v(A_0) \subset \kappa(v)^+$, and notice that $\pi_v(I^n)$ generates $x^n \cdot \kappa(v)^+$ for every $n \in \mathbb{N}$, so that π_v is continuous and f-adic (details left to the reader).

(iii): Say that $w = v^\Delta$ for some convex subgroup Δ strictly contained in Γ_v. Then the value γ_v cannot lie in Δ, so $w(I) = 0$, whence the assertion.

(iv): Proposition 9.2.24 says that w is a primary specialization of a secondary specialization u of v, and u is a continuous valuation of A, by remark 15.3.13(iv). In light of (iii), it follows that $u = w$, whence the contention.

(v): First, from (iv) we see that every continuous generization of v must be a secondary generization and must be analytic; hence, let $\Delta \subset \Gamma_v$ be the largest convex subgroup that does not contain γ_v, and set $w := v_\Delta$. It is easily seen that the image $\overline{\gamma}_v$ of γ_v in $\Gamma_w = \Gamma_v/\Delta$ is still final, and by definition $\gamma_w = \overline{\gamma}_v$, so w is continuous, by (i). It remains only to check that w does not admit any proper (secondary) generization; however, by construction, every non-trivial convex subgroup $\Delta' \subset \Gamma_w$ contains γ_w, and on the other hand, γ_w cannot lie in any proper convex subgroup of Γ_w. Thus, the only convex subgroups of Γ_w are $\{1\}$ and Γ_w, whence the assertion. □

Theorem 15.3.15. *Let A be any f-adic ring, and $J \subset A$ any finitely generated ideal such that* $\mathrm{Spec}\, A/J$ *is the non-analytic locus of* $\mathrm{Spec}\, A$. *Then we have :*

(i) $\mathrm{Cont}(A) = \{v \in \mathrm{Spv}(A, J) \mid v(a) < 1$ *for every* $a \in A^{\circ\circ}\}$.

(ii) $\mathrm{Cont}(A)$ *is a closed (in particular, pro-constructible) subset of* $\mathrm{Spv}(A, J)$. *Especially,* $\mathrm{Cont}(A)$ *is a spectral topological space.*

Proof. (i): Let $A_0 \subset A$ be a subring of definition, and $I \subset A_0$ an ideal of adic definition; by lemma 8.3.29(i) we have $\mathrm{Spec}\, A/IA = \mathrm{Spec}\, A/J$, so that the ideals J and IA have the same radical; from lemma 15.3.9(i) we deduce that $\mathrm{Spv}(A, J) = \mathrm{Spv}(A, IA)$, so we may assume from start that $J = IA$. Since I is an open subset of A, for every $a \in A^{\circ\circ}$ we may find $n \in \mathbb{N}$ such that $a^n \in I$; combining with lemma 15.3.14(i) we easily deduce that

$$\mathrm{Cont}(A) = \{v \in \mathrm{Spv}\, A \mid v(a) \text{ is final in } \Gamma_v, \text{for every } a \in I\}.$$

In light of lemma 15.3.4(i) we then see that

$$\mathrm{Cont}(A) \subset \{v \in \mathrm{Spv}(A, J) \mid v(a) < 1 \text{ for every } a \in I\}$$

and to conclude, it suffices to prove the converse inclusion. Hence, let $v \in \mathrm{Spv}(A, J)$ be any valuation such that $v(a) < 1$ for every $a \in I$. If $\Gamma_v \neq c\Gamma_v$, lemma 15.3.4(i) implies that either $v(a) = 0$ or $v(a)$ is final in Γ_v for every $a \in J$, and then $v \in \mathrm{Cont}(A)$, by the foregoing.

Lastly, suppose that $\Gamma_v = c\Gamma_v$, and let $a \in I$ be any element; in this case, it suffices to show that, for every $b \in A$ with $v(b) \neq 0$, there exists $n \in \mathbb{N}$ such that $v(a)^n < v(b)^{-1}$. However, we may find $n \in \mathbb{N}$ such that $a^n b \in I$, so $v(a^n b) < 1$ by assumption, whence the contention.

(ii): From (i) we see that $\mathrm{Cont}(A) = \bigcap_{a \in A^{\circ\circ}} \big(\mathrm{Spv}(A, J) \backslash R_A\big(\frac{1}{a}\big)\big)$. Combining with corollary 8.1.42, we obtain the assertion. □

15.3.16. Let A be and J be as in theorem 15.3.15. We say that a subset of $\mathrm{Cont}(A)$ is *rational*, if it can be written as an intersection $\mathrm{Cont}(A) \cap R$, where R is a rational subset of $\mathrm{Spv}(A, J)$ (see definition 15.3.6(ii)). Hence, a rational subset is of the form $\mathrm{Cont}(A) \cap R_A\big(\frac{f_1}{f_0}, \dots, \frac{f_n}{f_0}\big)$, where $(f_0, f_1, \dots, f_n)$ is any sequence of elements of A that generates an open ideal. Especially, the class of rational subsets of $\mathrm{Cont}(A)$ does not depend on the choice of the ideal J, and provides a basis of quasi-compact open subsets of $\mathrm{Cont}(A)$ that is closed under finite intersections (theorem 15.3.7(ii)).

Corollary 15.3.17. *Let $f : A \to B$ be an f-adic morphism of f-adic topological rings. We have:*

(i) $\mathrm{Cont}(f)$ *is a spectral map. More precisely, if R is a rational subset of* $\mathrm{Cont}(A)$*, then* $\mathrm{Cont}(f)^{-1}(R)$ *is a rational subset of* $\mathrm{Cont}(B)$.

(ii) $\mathrm{Cont}(f)$ *restricts to maps* $\mathrm{Cont}(B)_{\mathrm{a}} \to \mathrm{Cont}(A)_{\mathrm{a}}$ *and* $\mathrm{Cont}(B)_{\mathrm{na}} \to \mathrm{Cont}(A)_{\mathrm{na}}$.

Proof. (i): Indeed, if $R = \mathrm{Cont}(A) \cap R_A\big(\frac{a_1}{a_0}, \dots, \frac{a_n}{a_0}\big)$ for a sequence $a_0, a_1, \dots, a_n$ that generates an open ideal of A, then $\mathrm{Cont}(f)^{-1}(R) = \mathrm{Cont}(B) \cap R_A\big(\frac{f(a_1)}{f(a_0)}, \dots, \frac{f(a_n)}{f(a_0)}\big)$, and since f is adic, it is easily seen that the sequence $f(a_0), f(a_1), \dots, f(a_n)$ generates an open ideal of B.

(ii) is clear. □

Proposition 15.3.18. *Let A be an f-adic ring, $B \subset A$ an open subring, and denote by $i : B \to A$ the inclusion map. Then we have :*

(i) $\mathrm{Cont}(A) = \mathrm{Spv}(i)^{-1}\mathrm{Cont}(B)$.

(ii) $\mathrm{Cont}(i)$ *restricts to a homeomorphism* $\mathrm{Cont}(A)_{\mathrm{a}} \xrightarrow{\sim} \mathrm{Cont}(B)_{\mathrm{a}}$.

Proof. (i): Let v be a valuation of A such that $w := \mathrm{Spv}(i)(v) = v_{|B}$ is a continuous valuation of B; we have to show that v is a continuous valuation of A, and by remark 15.3.13(vi), we may assume that $\mathrm{Ker}\, v$ is not open. In this case, lemma 8.3.29(iii) implies that $\kappa(v) = \kappa(w)$, and therefore v and w have the same value group; the assertion follows immediately.

(ii): Define X_A and $X_A^{\circ\circ}$ as in definition 8.3.28; we have

$$\mathrm{Cont}(A)_{\mathrm{a}} = \mathrm{Spv}(X_A \setminus X_A^{\circ\circ}) \cap \mathrm{Cont}(A)$$

and likewise for $\mathrm{Cont}(B)_{\mathrm{a}}$, so the assertion follows from (i) and lemma 8.3.29(iii). □

Proposition 15.3.19. *Let A be a topological ring, $A^\wedge$ the separated completion of A. We have:*

(i) *The completion map $i : A \to A^\wedge$ induces a bijective and continuous map*

$$\mathrm{Cont}(i) : \mathrm{Cont}(A^\wedge) \to \mathrm{Cont}(A).$$

(ii) *Moreover, with the ordering given by specializations on* $\mathrm{Cont}(A)$ *and* $\mathrm{Cont}(A^\wedge)$*, the map* $\mathrm{Cont}(i)$ *is also an isomorphism of partially ordered sets (see remark* 15.3.13(v)*)*

$$(\mathrm{Cont}(A^\wedge), \leq) \xrightarrow{\sim} (\mathrm{Cont}(A), \leq).$$

(iii) *Furthermore,* $\mathrm{Cont}(i)$ *restricts to bijections*

$$\mathrm{Cont}(A^\wedge)_{\mathrm{a}} \xrightarrow{\sim} \mathrm{Cont}(A)_{\mathrm{a}} \qquad \textit{and} \qquad \mathrm{Cont}(A^\wedge)_{\mathrm{na}} \xrightarrow{\sim} \mathrm{Cont}(A)_{\mathrm{na}}.$$

(iv) *If A is f-adic,* $\mathrm{Cont}(i)$ *is a homeomorphism.*

Proof. (i): For the injectivity of $\mathrm{Cont}(i)$ we show the following more precise :

Claim 15.3.20. Let $w \in \mathrm{Cont}(A^\wedge)$ be any element, and set $v := \mathrm{Cont}(i)(w)$. Let Γ_w and Γ_v be the value groups of w and respectively v. Then we have:

(i) $\Gamma_w = \Gamma_v$ and $c\Gamma_w = c\Gamma_v$.
(ii) The induced map of residue fields $\kappa(v) \to \kappa(w)$ has dense image, for the valuation topology of $\kappa(w)$ (notation of remark 9.1.4(v)).

Proof of the claim. (i): Let $a \in A^\wedge$ be any element such that $\gamma := w(a) \in \Gamma_w$; by assumption, there exists an open neighborhood U_γ of 0 in $A^\wedge$ such that $w(x) < \gamma$ for every $x \in U_\gamma$, hence $w(a + U_\gamma) = \{\gamma\}$, and since $a + U_\gamma$ intersects the image of A in $A^\wedge$, we deduce that $\gamma \in \Gamma_v$, whence the claim.

(ii): According to remark 15.3.13(i), the natural map $\pi_v : A \to \kappa(v)$ is continuous for the valuation topology $\mathscr{T}_v$ of $\kappa(v)$, hence it extends uniquely to a continuous ring homomorphism $\pi_v^\wedge : A^\wedge \to \kappa(v)^\wedge$, where $(\kappa(v)^\wedge, \mathscr{T}_v^\wedge)$ denotes the completion of $(\kappa(v), \mathscr{T}_v)$. However, v extends uniquely to a valuation $v^\wedge : \kappa(v)^\wedge \to \Gamma_{v\circ}$ and $\mathscr{T}_v^\wedge$ agrees with the corresponding valuation topology of $\kappa(v)^\wedge$ (proposition 9.1.16(iv,v)). It follows that $v^\wedge \circ \pi_v^\wedge : A^\wedge \to \Gamma_{v\circ}$ is the unique continuous valuation whose composition with i agrees with v. Consequently, there exists as well a unique inclusion of fields $j : \kappa(w) \to \kappa(v)^\wedge$ whose composition with the projection $\pi_w : A^\wedge \to \kappa(w)$ agrees with $\pi_v^\wedge$, and $v^\wedge \circ j$ is the residual valuation $\mathscr{T}_w$ of $\kappa(w)$. Especially, $\mathscr{T}_w$ agrees with the topology induced by $\mathscr{T}_v^\wedge$ via j; since the image of $\kappa(v)$ is dense in $\kappa(v)^\wedge$ relative to the topology $\mathscr{T}_v^\wedge$ (theorem 8.2.8(ii)), the assertion follows. ◊

Now, let $w, w' \in \mathrm{Cont}(A^\wedge)$ be two valuations such that $w \circ i$ is equivalent to $w' \circ i$, and denote by Γ_w (resp. $\Gamma_{w'}$) the value group of w (resp. of w'). From claim 15.3.20 we may assume that $\Gamma_w = \Gamma_{w'}$ and $w \circ i = w' \circ i$. Since the image of A is dense in $A^\wedge$ and the topology of Γ_w is separated, we conclude that $w = w'$, as required.

Next, we show that $\mathrm{Cont}(i)$ is surjective. Indeed, let v be any continuous valuation of A with value group Γ, so that, for every $\gamma \in \Gamma$ there exists an open neighborhood U_γ of 0 in A such that $v(x) < \gamma$ for every $x \in U_\gamma$; we need to show that there exists a valuation $v^\wedge : A^\wedge \to \Gamma_\circ$ such that $v = v^\wedge \circ i$. We define $v^\wedge$ as follows. Let $\mathscr{C} := (a_i \mid i \in I)$ be any Cauchy net in A (indexed by some filtered ordered set I); for every $\gamma \in \Gamma$, we may then find $i(\gamma) \in I$ such that $a_j - a_k \in U_\gamma$ for every $j, k \in I$ with $j, k \geq i(\gamma)$. We let $\mathscr{C}_\gamma$ be the Cauchy net $(a_j \mid j \in I,\ j \geq i(\gamma))$ for every such γ. Now, suppose that there exists $\gamma \in \Gamma$ and $a \in \mathscr{C}_\gamma$ with $\delta := v(a) \geq \gamma$; then $v(b) = \delta$ for every $b \in \mathscr{C}_\gamma$, and in this case we set $v^\wedge(\mathscr{C}) := \delta$. Otherwise, we set $v^\wedge(\mathscr{C}) := 0$. It is easily seen that this definition depends only on the equivalence class of $\mathscr{C}$ in $A^\wedge$, so it yields a well defined map $v^\wedge : A^\wedge \to \Gamma_\circ$, and it is clear that $v^\wedge \circ i = v$. We leave to the reader the verification that $v^\wedge$ is indeed a continuous valuation on $A^\wedge$.

(ii): Let $u \in \mathrm{Cont}(A^\wedge)$ be any element, and w' a specialization of $u' := \mathrm{Cont}(i)(u)$ in $\mathrm{Cont}(A)$. By proposition 9.2.24 there exists $v' \in \mathrm{Spv}\,A$ that is both a secondary specialization of u' and a primary generization of w' in $\mathrm{Spv}\,A$. By lemma 9.2.25(iii) we may find a secondary specialization v of u in $\mathrm{Spv}\,A^\wedge$ with $\mathrm{Spv}(i)(v) = v'$; then $v \in \mathrm{Cont}(A^\wedge)$ and $v' \in \mathrm{Cont}(A)$ (remark 15.3.13(iv)). By claim 15.3.20, we know that v and v' have the same value groups and the same characteristic subgroups; hence, we may find a primary specialization w of v in $\mathrm{Spv}\,A$ with $w' = \mathrm{Spv}(i)(w)$, and $w \in \mathrm{Cont}(A)$ (remark 15.3.13(iv)), as required.

(iii): We have already observed that $\mathrm{Cont}(i)$ restricts to a map $\mathrm{Cont}(A^\wedge)_{\mathrm{na}} \to \mathrm{Cont}(A)_{\mathrm{na}}$ (remark 15.3.13(vii)); in light of (i), it then suffices to show that $\mathrm{Cont}(i)$ restricts as well to a map $\mathrm{Cont}(A^\wedge)_\mathrm{a} \to \mathrm{Cont}(A)_\mathrm{a}$. Thus, let $v : A^\wedge \to \Gamma_\circ$ be any continuous analytic valuation, and suppose that $\mathfrak{p} := \mathrm{Ker}\, v \circ i$ is open in A; by continuity, the topological closure $i(\mathfrak{p})^c$ of $i(\mathfrak{p})$ in $A^\wedge$ lies in $\mathrm{Ker}\, v$. However, $i(\mathfrak{p})^c$ contains an open subset of $A^\wedge$ (claim 8.2.11(ii)), and therefore it is open, a contradiction.

(iv): From (i), we know already that $\mathrm{Cont}(i)$ is continuous and bijective. It then suffices to show that $\mathrm{Cont}(i)$ is a closed map, if A is f-adic. However, in this case $\mathrm{Cont}(A)$ and $\mathrm{Cont}(A^\wedge)$ are spectral topological spaces (theorem 15.3.15(ii)) and $\mathrm{Cont}(i)$ is a spectral map (corollary 15.3.17(i) and proposition 8.3.33(ii)), so it suffices to check that $\mathrm{Cont}(i)$ is specializing (proposition 8.1.47(i) and corollary 8.1.50(i)). The latter follows immediately from (ii). □

Corollary 15.3.21. *Let $f : A \to B$ be a continuous ring homomorphism of f-adic rings. Suppose that B is topologically local, and denote by $B^\wedge$ the separated completion of B. Then the following conditions are equivalent :*

(a) *f is an f-adic map.*
(b) *The composition $A \to B^\wedge$ of f with the completion map $B \to B^\wedge$ is f-adic.*
(c) *$\mathrm{Cont}(f)$ restricts to a map $\mathrm{Cont}(B)_\mathrm{a} \to \mathrm{Cont}(A)_\mathrm{a}$.*

Proof. (a)⇒(b) since the completion map $B \to B^\wedge$ is f-adic (proposition 8.3.33(i,ii)).

(b)⇒(c) by corollary 15.3.17(ii) and proposition 15.3.19(i).

(c)⇒(a): Let B_0 be a subring of definition of B, and A_0 a subring of definition of A such that $A_0 \subset f^{-1}B_0$ (corollary 8.3.19(ii)), and let $f_0 : A_0 \to B_0$ be the restriction of f. From (c) and proposition 15.3.18(ii) we deduce that $\mathrm{Cont}(f_0)$ restricts to a map $\mathrm{Cont}(B_0)_\mathrm{a} \to \mathrm{Cont}(A_0)_\mathrm{a}$, and it suffices to check that f_0 is f-adic. Moreover, B_0 is still topologically local (proposition 8.4.2(i)). Thus, we may replace f by f_0, and assume from start that $B = B^\circ$, so that $B^{\circ\circ}$ lies in the Jacobson radical of B. Now, define X_A and $X_A^{\circ\circ}$ as in definition 8.3.28, and likewise for the ring B; also, set $\varphi := \mathrm{Spec}\, f$. By lemma 8.3.29(iv), it suffices to show that $\varphi^{-1}X_A^{\circ\circ} = X_B^{\circ\circ}$. However, suppose that the latter fails; then there exists a prime ideal $\mathfrak{p} \in X_B \setminus X_B^{\circ\circ}$ with $\varphi(\mathfrak{p}) \in X_A^{\circ\circ}$, and we pick any maximal ideal $\mathfrak{m}$ of B containing $\mathfrak{p}$. Set $C := (B/\mathfrak{p})_\mathfrak{m}$, and choose a valuation ring V of $\mathrm{Frac}\, C$ that dominates C (corollary 9.1.25). The valuation ring V corresponds to a point $v \in \mathrm{Spv}\, B$; next, pick any finitely generated ideal of definition I of B, and set $w := v^{\mathrm{c}\Gamma_v(I)}$. Since $B^{\circ\circ} \subset \mathfrak{m}$, theorem 15.3.15(i) implies that w is a continuous valuation of B. Since $\mathfrak{p} = \mathrm{Ker}\, v$ and I is not contained in $\mathfrak{p}$, it follows that I is not contained in the support of w, therefore $w \in \mathrm{Cont}(B)_\mathrm{a}$. However, $w' := \mathrm{Cont}(f)(w)$ is a primary specialization of $\mathrm{Spv}\, f(v)$, and the support of the latter is an open ideal of A, by construction, so $w' \in \mathrm{Cont}(A)_\mathrm{na}$, which contradicts our assumptions. □

Proposition 15.3.22. *Let A be any f-adic topological ring, and denote by A^sep the maximal separated quotient of A. The following holds :*

(i) *$\mathrm{Spv}\, A = \varnothing$ if and only if $A = 0$.*
(ii) *$\mathrm{Cont}(A) = \varnothing$ if and only if $A^\mathrm{sep} = 0$.*
(iii) *$\mathrm{Cont}(A)_\mathrm{a} = \varnothing$ if and only if the topology of A^sep is discrete.*

Proof. (i): Indeed, if $A \neq 0$, it has a prime ideal $\mathfrak{p}$, and the trivial valuation on $A/\mathfrak{p}$ yields an element of $\mathrm{Spv}\, A$.

(iii): Let $A^\wedge$ be the separated completion of A; if $\mathrm{Cont}(A)_\mathrm{a} = \varnothing$, we have $\mathrm{Cont}(A^\wedge)_\mathrm{a} = \varnothing$ as well, by proposition 15.3.19(iii), and it suffices to show that the topology of $A^\wedge$ is discrete. Let $B \subset A^\wedge$ be a subring of definition, and $I \subset B$ a finitely generated ideal of adic definition (proposition 8.3.33(i)); then

$$\mathrm{Cont}(B)_\mathrm{a} = \varnothing \qquad (15.3.23)$$

as well (proposition 15.3.18(ii)), and we remark :

Claim 15.3.24. I is a nilpotent ideal.

Proof of the claim. Let $\mathfrak{p} \subset B$ be any prime ideal, and suppose by way of contradiction, that I is not contained in $\mathfrak{p}$. Choose a maximal ideal $\mathfrak{m}$ of $C := B/\mathfrak{p}$, and pick a valuation ring V of $\mathrm{Frac}\, C$ that dominates $C_\mathfrak{m}$ (corollary 9.1.25). Let $\mathfrak{n} \subset B$ be the maximal ideal such that $\mathfrak{n}/\mathfrak{p} = \mathfrak{m}$; then V corresponds to a valuation v of B with support equal to $\mathfrak{p}$, and such that $v(a) < 1$ for every $a \in \mathfrak{n}$. Since B is I-adically complete, we have $I \subset \mathfrak{n}$ (remark 8.3.10(v)), so it follows easily that $v(a) < 1$ for every $a \in B^{\circ\circ}$. Let Γ_v be the value group of v, and set $w := v^{c\Gamma_v(I)}$ (notation of definition 15.3.3). Then $w \in \mathrm{Spv}(B, I)$ and $w(a) < 1$ for every $a \in B^{\circ\circ}$, so w is continuous (theorem 15.3.15(i)). Lastly, since $v(I) \neq \{0\}$ by construction, we have $w(I) \neq \{0\}$ as well, since w is an I-admissible specialization of v. Therefore $w \in \mathrm{Cont}(B)_\mathrm{a}$, which contradicts (15.3.23). ◊

Claim 15.3.24 means that the topology of B is discrete, hence the same holds for the topology of $A^\wedge$, as required.

(ii): From (iii) we know that the topological closure $\{0\}^c$ of the ideal $\{0\}$ of A is open in A; hence, every prime ideal containing $\{0\}^c$ is open as well, and therefore every valuation with support equal to such a prime ideal would be continuous (remark 15.3.13(vi)); since $\mathrm{Cont}(A) = \varnothing$, we conclude that no prime ideal of A contains $\{0\}^c$, *i.e.* $\{0\}^c = A$, whence the assertion. □

Lemma 15.3.25. *Let A be a topologically local f-adic ring, $J \subset A$ an ideal. The following conditions are equivalent :*

(a) *J is an open ideal.*
(b) *For every $v \in \mathrm{Cont}(A)_\mathrm{a}$ we have $J \not\subset \mathrm{Ker}\, v$.*
(c) *For every $v \in \mathrm{Cont}(A)_\mathrm{a}$ of rank one, we have $J \not\subset \mathrm{Ker}\, v$.*

Proof. Clearly (a)⇒(b)⇒(c).

Next, suppose that J is not open; to conclude, it suffices to exhibit a rank one continuous analytic valuation on A whose support contains J. Now, let $B \subset A$ be a subring of definition, and set $J_B := B \cap J$; since J is not open in A, the ideal J_B cannot be open in B. Moreover, B is also topologically local, by proposition 8.4.2(i).

Claim 15.3.26. It suffices to exhibit a rank one continuous analytic valuation on B whose support contains J_B.

Proof of the claim. Indeed, let $v : B \to \Gamma_\circ$ be such a valuation, and denote by $\mathfrak{p} \subset B$ the support of v; with the notation of definition 8.3.28, we have $\mathfrak{p} \in X_B \setminus X_B^{\circ\circ}$, and by lemma 8.3.29(iii) the inclusion map $i : B \to A$ induces an isomorphism of schemes $\varphi : X_A \setminus X_A^{\circ\circ} \xrightarrow{\sim} X_B \setminus X_B^{\circ\circ}$. Set $\mathfrak{q} := \varphi^{-1}(\mathfrak{p})$; it follows that i extends to a ring isomorphism $i_\mathfrak{p} : B_\mathfrak{p} \xrightarrow{\sim} A_\mathfrak{q}$. Let also $j : A \to A_\mathfrak{q}$ be the localization map; v extends uniquely to a valuation $\overline{v} : B_\mathfrak{p} \to \Gamma_\circ$, and composing with $i_\mathfrak{p}^{-1} \circ j$ we deduce a valuation $w : A \to \Gamma_\circ$. Moreover, $\mathrm{Cont}(i) : \mathrm{Cont}(A) \to \mathrm{Cont}(B)$ is a homeomorphism (proposition 15.3.18(ii)), and a direct inspection of the definitions yields

$$\mathrm{Cont}(i)^{-1}(v) = w.$$

Especially, w is continuous, analytic and of rank one. Lastly, the support of $\overline{v}$ contains $J_{B,\mathfrak{p}} = i_\mathfrak{p}^{-1} J_\mathfrak{q}$, so the support of w contains J. ◊

Due to claim 15.3.26, we may replace A by B and J by J_B, and assume from start that $A = A^\circ$; especially, $A^{\circ\circ}$ lies in the Jacobson radical of A. By lemma 8.3.29(v), there exists $x \in \mathrm{Spec}\, A/J \setminus X_A^{\circ\circ}$; according to remark 8.1.45(ii) we may find a minimal specialization y of x in $\mathrm{Spec}\, A/J \setminus X_A^{\circ\circ}$, and if $\{y\}^c$ denotes the topological closure of $\{y\}$ in $\mathrm{Spec}\, A/J$, the closed subset $Z := \{y\}^c \cap X_A^{\circ\circ}$ of $\mathrm{Spec}\, A/J$ is non-empty, since $X_A^{\circ\circ}$ contains all the maximal ideals of A. We let $\mathfrak{p} \subset A$ be any maximal point of Z, and we notice that the image C of $A_\mathfrak{p}$ in $\kappa(y)$ is a one-dimensional local domain.

Claim 15.3.27. For any one-dimensional local domain $(R, \mathfrak{m}_R)$ there exists a one-dimensional valuation ring of $\mathrm{Frac}\, R$ that dominates R.

Proof of the claim. By corollary 9.1.25 we may find a valuation ring V of $\mathrm{Frac}\, R$ that dominates R. Let $\mathscr{F}$ be the set of all prime ideals $\mathfrak{q}$ of V such that $\mathfrak{q} \cap R = \mathfrak{m}_R$; then $\mathscr{F}$ is non-empty, and it admits a minimal element $\mathfrak{r} := \bigcap_{\mathfrak{q} \in \mathscr{F}} \mathfrak{q}$. It then suffices to check that the localization $V_\mathfrak{r}$ is one-dimensional. However, let $\mathfrak{p} \subset V_\mathfrak{r}$ be any non-zero prime ideal, and pick any $a \in \mathfrak{p} \setminus \{0\}$; we may write $a := b/b'$ for some $b, b' \in R$, and then $b \in \mathfrak{p}$ as well, so $\mathfrak{p} \cap R = \mathfrak{m}_R$, in which case $\mathfrak{p}$ must be the maximal ideal of $V_\mathfrak{r}$, and the assertion follows. ◊

By claim 15.3.27, we may find a one-dimensional valuation ring V of $\kappa(y)$ that dominates C, and denote by $v \in \mathrm{Spv}\, A$ the corresponding valuation. Let also $I \subset A$ be any finitely generated ideal of adic definition; by construction, the support of v contains J and is not an open ideal, and v does not admit any proper I-admissible specializations (see (15.3.1)), so $v \in \mathrm{Spv}(A, I)$. Moreover $A^{\circ\circ} \subset \mathfrak{p}$, so that $v(b) < 1$ for every $b \in A^{\circ\circ}$, and consequently $v \in \mathrm{Cont}(A)_\mathrm{a}$. □

15.4. **Affinoid rings and affinoid schemes.** In this section we present an extension and refinement of Huber's theory of affinoid rings.

Definition 15.4.1. (i) Let A be any f-adic ring. An open subring $B \subset A$ is called a *ring of integral elements of* A, if $B \subset A^\circ$ and B is integrally closed in A.

(ii) An *affinoid ring* is a datum (A, A^+) consisting of an f-adic ring A and a ring A^+ of integral elements of A. A *morphism of affinoid rings* $f : (A, A^+) \to (B, B^+)$ is a continuous ring homomorphism $f : A \to B$ such that $f(A^+) \subset B^+$. Then we say that f is *f-adic*, if the same holds for the underlying map $A \to B$.

(iii) A *quasi-affinoid ring* is a datum (A, A^+, U) consisting of an affinoid ring (A, A^+) and a constructible open subset $U \subset X_A$ that contains the analytic locus (see definition 8.3.28). A *morphism of quasi-affinoid rings* $f : (A, A^+, U) \to (B, B^+, V)$ is a morphism $f : (A, A^+) \to (B, B^+)$ of affinoid rings such that $\mathrm{Spec}\, f$ restricts to a morphism of schemes $V \to U$. We say that such f is *f-adic*, if the same holds for the underlying map $A \to B$.

(iv) A *quasi-affinoid scheme* is a datum $\underline{X} := (X, \mathscr{T}_X, A^+_X)$ consisting of a (quasi-compact) quasi-affine scheme X (see [60, Ch.II, Déf.5.1.1]), an f-adic topology $\mathscr{T}_X$ on $A_X := \mathscr{O}_X(X)$, and a subring $A^+_X \subset A_X$ such that the following holds :

- (A_X, A^+_X) is an affinoid ring.
- For every $s \in A^{\circ\circ}_X$, the open subscheme $X_s := X \times_{\mathrm{Spec}\, A_X} \mathrm{Spec}\, A_X[s^{-1}]$ is affine.

A *morphism of quasi-affinoid schemes* $\varphi : (X, \mathscr{T}_X, A^+_X) \to (Y, \mathscr{T}_Y, A^+_Y)$ is a morphism of schemes $\varphi : X \to Y$ whose associated morphism of sheaves of rings $\varphi^\flat : \mathscr{O}_Y \to \varphi_*\mathscr{O}_X$ induces a morphism of affinoid rings $\varphi^\flat_Y : (\mathscr{O}_Y(Y), A^+_Y) \to (\mathscr{O}_X(X), A^+_X)$. We say that φ is *f-adic* if $\varphi^\flat_Y$ is f-adic. We say that $\underline{X}$ is an *affinoid scheme* if X is an affine scheme.

(v) Let $\underline{A} := (A, A^+, U)$ (resp. $\underline{X} := (X, \mathscr{T}_X, A^+_X)$) be any quasi-affinoid ring (resp. quasi-affinoid scheme). We say that $\underline{A}$ (resp. $\underline{X}$) is *topologically local* (resp. *topologically henselian*, resp. *complete*, resp. *separated*) if the underlying f-adic ring A (resp. $\mathscr{O}_X(X)$) enjoys the corresponding property.

Remark 15.4.2. (i) Clearly the affinoid rings, the quasi-affinoid rings, the affinoid schemes and the quasi-affinoid schemes with their morphisms as in definition 15.4.1 form categories denoted respectively :

$$\mathsf{Afd.Ring} \qquad \mathsf{q.Afd.Ring} \qquad \mathsf{Afd.Sch} \qquad \mathsf{q.Afd.Sch}$$

(Afd.Sch is a full subcategory of q.Afd.Sch). Also, we have a natural fully faithful functor

$$\mathsf{Afd.Ring} \to \mathsf{q.Afd.Ring} \qquad (A, A^+) \mapsto (A, A^+, \mathrm{Spec}\, A)$$

so we may regard Afd.Ring as a full subcategory of q.Afd.Ring.

(ii) By virtue of [60, Ch.II, Prop.5.1.2], we have a natural fully faithful functor

$$\Gamma : \mathsf{q.Afd.Sch}^o \to \mathsf{q.Afd.Ring} \qquad (X, \mathscr{T}_X, A^+_X) \mapsto (A_X, A^+_X, X)$$

(where $A_X := \mathscr{O}_X(X)$ and X is naturally identified with the image of the open immersion $X \to \mathrm{Spec}\, A_X$). Indeed, let $s \in A_X^{\circ\circ}$ be any element, and define the open subscheme $X_s \subset X$ as in definition 15.4.1(iv); by assumption, X_s is affine, and on the other hand we have a natural isomorphism of A_X-algebras

$$\Gamma(X_s, \mathscr{O}_X) \xrightarrow{\sim} A_X[s^{-1}]$$

by [59, Ch.I, Cor.9.2.2]. Therefore the image of the open immersion $\mathrm{Spec}\, A_X[s^{-1}] \to \mathrm{Spec}\, A_X$ equals X_s; especially, $\mathrm{Spec}\, A_X[s^{-1}] \subset X$ for every $s \in A_X^{\circ\circ}$, so (A_X, A_X^+, X) is indeed a quasi-affinoid ring. Especially, the analytic locus of $\mathrm{Spec}\, A_X$ lies in X, and shall also be called the *analytic locus of* $\underline{X}$.

(iii) Let (A, A^+) be any affinoid ring, $B \subset A$ an open subring, and endow B with the topology induced by the inclusion map $B \to A$. Then corollary 8.3.20(i) implies that the pair $(B, A^+ \cap B)$ is an affinoid ring.

(iv) Let A be any f-adic ring, and endow $R := \mathbb{Z} \oplus A^{\circ\circ}$ with the multiplication map given by the rule : $(n, a) \cdot (m, b) := (nm, ma + nb + ab)$ for every $n, m \in \mathbb{Z}$ and $a, b \in A^{\circ\circ}$. Then R is a ring, and we have a unique ring homomorphism $f : R \to A$ that restricts to the inclusion map on $A^{\circ\circ}$. The integral closure B of the image of f is the smallest ring of integral elements of A. On the other hand, A° is the largest subring of integral elements of A.

Proposition 15.4.3. (i) *The functor* Γ *of remark* 15.4.2(ii) *admits a left adjoint :*

$$\mathsf{Spec} : \mathsf{q.Afd.Ring} \to \mathsf{q.Afd.Sch}^o \qquad (A, A^+, U) \mapsto (U, \mathscr{T}_U, A_U^+)$$

(ii) *If* $f : \underline{A} \to \underline{B}$ *is an f-adic morphism of quasi-affinoid rings, then* $\mathsf{Spec}(f) : \mathsf{Spec}(\underline{B}) \to \mathsf{Spec}(\underline{A})$ *is an f-adic morphism of quasi-affinoid rings.*

(iii) *The functor* Spec *is not fully faithful, but it restricts to an equivalence of categories*

$$\mathsf{Spec} : \mathsf{Afd.Ring} \to \mathsf{Afd.Sch}^o.$$

Proof. (i): For a quasi-affinoid ring $\underline{A} := (A, A^+, U)$, we let $\mathscr{T}_U$ be the f-adic topology on $A_U := \mathscr{O}_U(U)$ provided by proposition 8.3.30(i), and let A_U^+ be the integral closure of A^+ in A_U. Indeed, notice that the restriction map $A \to A_U$ sends A° into A_U° (lemma 8.3.24(iii.a)); since $A^+ \subset A^\circ$, remark 8.3.10(iii) easily implies that $A_U^+ \subset A_U^\circ$. Next, let $\overline{A} := \mathrm{Im}(A \to A_U)$ and endow $\overline{A}$ with the topology induced by A_U via the open inclusion map $\overline{A} \to A_U$; since $A^{\circ\circ}$ is an open subgroup of A, for every $s \in A_U^{\circ\circ}$ we may find $n \in \mathbb{N}$ and $t \in A^{\circ\circ}$ such that s^n equals the image of t in $\overline{A}$. Notice that t annihilates the kernel of the projection $\pi : A \to \overline{A}$, so $\mathrm{Spec}\,\pi$ restricts to an isomorphism of schemes $\mathrm{Spec}\,\overline{A}[s^{-n}] \xrightarrow{\sim} \mathrm{Spec}\, A[t^{-1}] \subset U$. It follows that

$$U \times_{\mathrm{Spec}\, A_U} \mathrm{Spec}\, A_U[s^{-1}] = U \times_{\mathrm{Spec}\, A} \mathrm{Spec}\, A[t^{-1}] = \mathrm{Spec}\, A[t^{-1}]$$

so the construction of $\mathsf{Spec}(\underline{A})$ is achieved. Next, let $f : \underline{A} \to \underline{B} := (B, B^+, V)$ be any morphism of quasi-affinoid rings; set $(V, \mathscr{T}_V, B_V^+) := \mathsf{Spec}(\underline{B})$ and denote by $\overline{B}$ the image of B in $B_V := \mathscr{O}_V(V)$. We deduce easily from proposition 8.3.30(i) that (a) the subrings $\overline{A}$ and $\overline{B}$ are open in A_U and B_V, so the induced map $\varphi : A_U \to B_V$ is continuous if and only the same holds for its restriction $\overline{f} : \overline{A} \to \overline{B}$, and (b) the topologies of $\overline{A}$ and $\overline{B}$ are induced from the surjections $A \to \overline{A}$ and $B \to \overline{B}$, so the continuity of $\overline{f}$ follows from that of f. Thus, φ is continuous, and clearly $\varphi(A_U^+) \subset B_V^+$, so $\varphi : \mathsf{Spec}(\underline{B}) \to \mathsf{Spec}(\underline{A})$ is a well defined morphism of quasi-affinoid schemes, and we let $\mathsf{Spec}(f) := \varphi$.

Now, a simple inspection shows that $\mathsf{Spec} \circ \Gamma$ is the identity automorphism of the category $\mathsf{q.Afd.Sch}^o$; on the other hand, for every quasi-affinoid ring $\underline{A} := (A, A^+, U)$, the restriction map $A \to A_U$ determines a natural morphism $\eta_{\underline{A}} : \underline{A} \to \Gamma \circ \mathsf{Spec}(\underline{A})$. Lastly, it is easily seen that the pair of natural transformations $(\varepsilon := \mathbf{1}_{\mathsf{Spec}\circ\Gamma}, \eta)$ fulfills the triangular identities of (1.1.13), so the assertion follows from proposition 1.1.15(i).

(ii): Indeed, set $A_U := \mathscr{O}_U(U)$ and $B_V := \mathscr{O}_V(V)$; by assumption there exist open and bounded subrings $A_0 \subset A$ and $B_0 \subset B$ such that $f(A_0) \subset B_0$, and f restricts to an adic ring homomorphism $A_0 \to B_0$. Then, let $\overline{A}_0$ be the image of A_0 in A_U, and define likewise $\overline{B}_0 \subset B_V$; we know that $\overline{A}_0$ and $\overline{B}_0$ are open and bounded subrings of A_U and respectively B_V, for the topologies induced by the latter topological rings, and the image of every ideal of adic definition of A_0 (resp. of B_0) is an ideal of adic definition of $\overline{A}_0$ (resp. of $\overline{B}_0$) (proposition 8.3.30(i)). It follows easily that the continuous map $A_U \to B_V$ restricts to an adic ring homomorphism $\overline{A}_0 \to \overline{B}_0$, whence the contention.

(iii) follows easily from the definitions. □

Example 15.4.4. Let $\underline{X} := (X, \mathscr{T}_X, A^+_X)$ be any quasi-affinoid scheme, set $A_X := \mathscr{O}_X(X)$, and let $U \subset X$ be any quasi-compact open subset containing the analytic locus of $\mathrm{Spec}\, A_X$. Let us denote by F_U the presheaf on the category $\mathsf{q.Afd.Sch}$ such that

$$F_U(\underline{Z}) := \{\varphi : \underline{Z} \to \underline{X} \mid \varphi(Z) \subset U\} \qquad \text{for every quasi-affinoid scheme } \underline{Z} := (Z, \mathscr{T}_Z, A^+_Z)$$

so that F_U is a sub-presheaf of the presheaf $h_{\underline{X}}$ represented by $\underline{X}$. Let $\underline{A}$ be the quasi-affinoid ring (A_X, A^+_X, U); we claim that F_U is representable by the quasi-affinoid scheme

$$U \times_X \underline{X} := \mathsf{Spec}\, \underline{A}.$$

Indeed, the identity map of A_X yields a morphism of quasi-affinoid rings $i : \Gamma(\underline{X}) \to \underline{A}$; now, for every quasi-affinoid scheme $\underline{Z} := (Z, \mathscr{T}_Z, A^+_Z)$, the datum of a morphism $\psi : \underline{Z} \to U \times_X \underline{X}$ of quasi-affinoid schemes is equivalent, by adjunction, to that of a morphism $g : \underline{A} \to \Gamma(\underline{Z})$ of quasi-affinoid rings, and the composition $f := g \circ i : \Gamma(\underline{X}) \to \Gamma(\underline{Z})$ is a morphism such that $\mathrm{Spec}(f)(Z) \subset U$, whence the morphism $\varphi := \mathsf{Spec}(f) : \underline{Z} \to \underline{X}$ with $\varphi(Z) \subset U$. Conversely, any morphism $\varphi : \underline{Z} \to \underline{X}$ with $\varphi(Z) \subset U$ yields a morphism $\Gamma(\varphi) : \Gamma(\underline{X}) \to \Gamma(\underline{Z})$ that factors uniquely through $\underline{A}$, whence the contention.

Example 15.4.5. (i) Consider two f-adic morphisms of affinoid rings

$$\underline{B}' := (B', B'^+) \xleftarrow{g} \underline{B} := (B, B^+) \xrightarrow{f} \underline{A} := (A, A^+)$$

and endow $A' := B' \otimes_B A$ with the f-adic topology $\mathscr{T}_{A'}$ characterized by proposition 8.3.34(i), so that the natural maps $f' : B' \to A'$ and $g_A : A \to A'$ are both f-adic. Let also $R \subset A'$ be the image of $B'^+ \otimes_{B^+} A^+$, and denote by A'^+ the integral closure of R in A'. Then the datum

$$\underline{B}' \otimes_{\underline{B}} \underline{A} := (A', A'^+)$$

is an affinoid ring. Indeed, a simple inspection of the definition of $\mathscr{T}_{A'}$ shows that R is open in A', and we have $g_A(A^+), f(B^+) \subset A'^\circ$ by lemma 8.3.24(iii), so also $R \subset A'^\circ$, and then $A'^+ \subset A'^\circ$ as well, by remark 8.3.10(iv).

(ii) In the situation of (i), let also $U_B \subset X_B := \mathrm{Spec}\, B$, $U_A \subset X_A := \mathrm{Spec}\, A$ and $U_{B'} \subset X_{B'} := \mathrm{Spec}\, B'$ be three open subsets, such that f and g are morphisms of quasi-affinoid rings

$$(\underline{B}', U_{B'}) \xleftarrow{g} (\underline{B}, U_B) \xrightarrow{f} (\underline{A}, U_A)$$

and set $U_{A'} := U_{B'} \times_{U_B} U_A$. Then the datum

$$(\underline{B}', U_{B'}) \otimes_{(\underline{B}, U_B)} (\underline{A}, U_A) := (\underline{B}' \otimes_{\underline{B}} \underline{A}, U_{A'})$$

is a quasi-affinoid ring as well. Indeed, let $B_0 \subset B$ be a ring of definition, and $I_0 \subset B_0$ an ideal of adic definition; we need to show that $\mathrm{Spec}\, A' \setminus \mathrm{Spec}\, A'/I_0A' \subset U_{A'}$. Set $Z_{B'} := X_{B'} \setminus U_{B'}$ and $Z_A := X_A \setminus U_A$; the assertion is equivalent to $(Z_{B'} \times_{X_B} X_A) \cup (X_{B'} \times_{X_B} Z_A) \subset \mathrm{Spec}\, A'/I_0A'$. The latter is clear, since by assumption $Z_{B'} \subset \mathrm{Spec}\, B'/I_0B'$ and $Z_A \subset \mathrm{Spec}\, A/I_0A$.

Lemma 15.4.6. *With the notation of example* 15.4.5*, the resulting commutative diagrams*

$$\begin{array}{ccc} \underline{B} & \xrightarrow{\ f\ } & \underline{A} \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle g_A} \\ \underline{B}' & \xrightarrow{\ f'\ } & \underline{B}'\otimes_{\underline{B}}\underline{A} \end{array} \qquad\qquad \begin{array}{ccc} (\underline{B},U_B) & \xrightarrow{\qquad f\qquad} & (\underline{A},U_A) \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle g_A} \\ (\underline{B}',U_{B'}) & \xrightarrow{\ f'\ } & (\underline{B}',U_{B'})\otimes_{(\underline{B},U_B)}(\underline{A},U_A) \end{array}$$

are cocartesian in the categories $\mathsf{Afd.Ring}$ *and respectively* $\mathsf{q.Afd.Ring}$.

Proof. Indeed, consider any two morphisms $h:(A,A^+)\to(C,C^+)$, $k:(B,B^+)\to(C,C^+)$ of affinoid rings such that $h\circ f=k\circ g$; by proposition 8.3.34(i) there exists a unique continuous ring homomorphism $l:A'\to C$ such that $l\circ g_A=h$ and $l\circ f'=k$, and it is easily seen that $l(A'^+)\subset C^+$, *i.e.* $l:(A',A'^+)\to(C,C^+)$ is a morphism of affinoid rings, whence the assertion for the left diagram. The assertion for the right diagram is an immediate consequence. $\square$

Example 15.4.7. Consider a quasi-affinoid ring $\underline{A}:=(A,A^+)$ and any two f-adic morphisms $f:B\to A$, $g:B\to B'$ of f-adic rings.

(i) As a special case of example 15.4.5(i), take for B^+ and B'^+ the smallest subrings of integral elements of B and respectively B' (see remark 15.4.2(iv)) and set $\underline{B}:=(B,B^+)$ and $\underline{B}':=(B',B'^+)$; we may regard f and g as morphisms $\underline{B}\to\underline{A}$ and $\underline{B}\to\underline{B}'$ respectively. The resulting tensor product $\underline{B}'\otimes_{\underline{B}}\underline{A}$ shall be simply denoted

$$B'\otimes_B\underline{A}.$$

By lemma 15.4.6, the resulting morphisms $\underline{A}\xrightarrow{g_A}B'\otimes_B\underline{A}\xleftarrow{f'}\underline{B}'$ enjoy the following universal property. Let (h,k) be any pair consisting of a morphism $h:\underline{A}\to(C,C^+)$ of affinoid rings, and a continuous ring homomorphism $k:B'\to C$ such that $h\circ f=k\circ g$; then there exists a unique morphism $l:B'\otimes_B\underline{A}\to(C,C^+)$ of affinoid rings such that $l\circ g_A=h$ and $l\circ f'=k$.

(ii) Likewise, let $(\underline{A},U)$ be a quasi-affinoid ring. As a special case of example 15.4.5(ii), we may define $\underline{B}$ and $\underline{B}'$ as in (i), and take $U_B:=\operatorname{Spec}B$, $U_{B'}:=\operatorname{Spec}B'$. There follow morphisms of quasi-affinoid rings $f:(\underline{B},U_B)\to(\underline{A},U)$ and $f:(\underline{B},U_B)\to(\underline{B}',U_{B'})$. The resulting tensor product $(B'\otimes_B\underline{A},U_{B'})$ will be denoted simply

$$B'\otimes_B(\underline{A},U)$$

and notice that $U_{B'}=\operatorname{Spec}A'\times_{\operatorname{Spec}A}U$. It follows easily from lemma 8.3.29(iv) that this quasi-affinoid ring enjoys the corresponding universal property as in (i). Namely, let (h,k) be any pair consisting of a morphism $h:(\underline{A},U)\to(C,C^+,V)$ of quasi-affinoid rings, and $k:B'\to C$ a continuous ring homomorphism such that $h\circ f=k\circ g$; then there exists a unique morphism $l:B'\otimes_B(\underline{A},U)\to(C,C^+,V)$ of quasi-affinoid rings such that $l\circ g_A=h$ and $l\circ f'=k$.

Example 15.4.8. Let $\underline{Y}\to\underline{X}$ and $\underline{Y}'\to\underline{X}$ be two f-adic morphisms of quasi-affinoid schemes. Then the fibre product $\underline{Y}\times_{\underline{X}}\underline{Y}'$ is representable in the category $\mathsf{q.Afd.Sch}$. Indeed, the induced morphisms of quasi-affinoid rings $\Gamma(\underline{X})\to\Gamma(\underline{Y})$ and $\Gamma(\underline{X})\to\Gamma(\underline{Y}')$ are f-adic, hence we may form the tensor product $\underline{A}:=\Gamma(\underline{Y})\otimes_{\Gamma(\underline{X})}\Gamma(\underline{Y}')$ as in example 15.4.5(ii), and in view of lemma 15.4.6, the quasi-affinoid scheme $\mathsf{Spec}\,\underline{A}$ represents the sought fibre product (details left to the reader).

Example 15.4.9. (i) Let (A,A^+) be an affinoid ring and B a finite A-algebra. According to remark 8.3.43(iii), the canonical topology $\mathscr{T}_B^A$ on B is f-adic, and the structure map $\varphi:A\to B$ is f-adic and restricts to a finite map $A_0\to B_0$ of suitable subrings of definition. Let B^+ be the integral closure of $\varphi(A^+)$ in B. We claim that B^+ is a ring of integral elements of B. Indeed, $B^+\subset B^\circ$ by lemma 8.3.24(iii.a) and remark 8.3.10(iv). To check that B^+ is open in B, pick a finite system $b_1,\dots,b_n$ of generators of the A_0-module B_0; for every $i=1,\dots,n$ there exist $m_i\in\mathbb{N}$ and a polynomial $P_i=X^{m_i}+\sum_{j=0}^{m_i-1}a_{ij}X^j\in A_0[X]$ with $P_i(b_i)=0$. Let $I_0\subset A_0$

be an ideal of adic definition; since A^+ is open, there exists $k \in \mathbb{N}$ such that $I_0^k a_{ij} \in A^+$ for $i = 1, \dots, n$ and $j = 0, \dots, m_i - 1$. Thus, for every $c \in I_0^k$ and $i = 1, \dots, n$ we have

$$0 = c^{m_i} P_i(b_i) = Q_i(cb_i) \qquad \text{where } Q_i := X^{m_i} + \sum_{j=0}^{m_i-1} c^{m_i - j} a_{ij} X^j \in A^+[X].$$

This shows that $I_0^k B_0 \subset B^+$, whence the contention. We call (B, B^+) *the affinoid ring associated with the finite A-algebra B*, and we denote it by

$$B \otimes_A (A, A^+).$$

(ii) Thus, we get an f-adic morphism $\varphi : (A, A^+) \to B \otimes_A (A, A^+)$ of affinoid rings. We claim that the morphism of affinoid rings φ enjoys the following universal property. For every morphism $f : (A, A^+) \to (C, C^+)$ of affinoid rings, every ring homomorphism $g : B \to C$ such that $g \circ \varphi = f$ is a morphism $g : (B, B^+) \to (C, C^+)$ of affinoid rings. Indeed, a simple inspection shows that $g(B^+) \subset C^+$, and g is continuous by virtue of proposition 8.3.41(ii).

(iii) As usual, the universal property determines the pair $(B \otimes_A (A, A^+), \varphi)$ up to unique isomorphism of affinoid rings. Especially, if $\psi : B \to B'$ is any finite ring homomorphism, then $\psi \circ \varphi : A \to B'$ is also finite, and it follows easily that

$$B' \otimes_B (B \otimes_A (A, A^+)) = B' \otimes_A (A, A^+).$$

(iv) Likewise, if $\underline{A} := (A, A^+, U)$ is any quasi-affinoid ring, and $\varphi : A \to B$ is as in (i), let us set $U_B := \mathrm{Spec}\, B \times_{\mathrm{Spec}\, A} U$. According to example 15.4.7(ii) we get a quasi-affinoid ring

$$B \otimes_A \underline{A} := (B, B^+, U_B)$$

(for the topology $\mathscr{T}_B^A$ on B) and φ is an f-adic morphism of quasi-affinoid rings $\underline{A} \to B \otimes_A \underline{A}$ enjoying a corresponding universal property that the reader may spell out.

(v) In the same vein, let $\underline{X} := (X, \mathscr{T}_X, A_X^+)$ be a quasi-affinoid scheme and $f : X' \to X$ a finite morphism of schemes; set $A_X := \mathscr{O}_X(X)$, $Y := \mathrm{Spec}\, A_X$ and let $i : X \to Y$ be the open immersion and $g := i \circ f : X' \to Y$. We have natural morphisms of quasi-coherent $\mathscr{O}_Y$-algebras $i^\flat : \mathscr{O}_Y \to i_*\mathscr{O}_X$ and $g^\flat : i_*\mathscr{O}_X \to g_*\mathscr{O}_{X'}$, and we let $\mathscr{A}$ be the integral closure of the image of $i_*\mathscr{O}_X$ in $g_*\mathscr{O}_{X'}$. Notice that $i^\flat$ is an isomorphism, and $\mathscr{A}$ is a quasi-coherent $\mathscr{O}_Y$-algebra; hence $\mathscr{A}$ is the filtered union of its finite quasi-coherent $\mathscr{O}_Y$-subalgebras. Clearly $\mathscr{A}_{|X} = f_*\mathscr{O}_{X'}$ is a finite $\mathscr{O}_X$-algebra; since X is quasi-compact, it follows that there exists a finite quasi-coherent $\mathscr{O}_Y$-subalgebra $\mathscr{B} \subset \mathscr{A}$ such that $\mathscr{B}_{|X} = f_*\mathscr{O}_{X'}$, and $B := \mathscr{B}(Y)$ is a finite A_X-algebra. We claim that the quasi-affinoid scheme

$$X' \times_X \underline{X} := \mathsf{Spec}\,(B \otimes_{A_X} \Gamma(\underline{X})).$$

is independent, up to unique isomorphism, of the choice of $\mathscr{B}$ and enjoys the following universal property. For every morphism of quasi-affinoid schemes $h : \underline{Z} := (Z, \mathscr{T}_Z, A_Z^+) \to \underline{X}$, every morphism $k : Z \to X'$ of schemes such that $f \circ k = h$ is a morphism of quasi-affinoid schemes $\underline{Z} \to X' \times_X \underline{X}$. Indeed notice that, by construction, the commutative diagram of schemes

$$\begin{array}{ccc} X' & \xrightarrow{\ j\ } & \mathrm{Spec}\, B \\ {\scriptstyle f}\big\downarrow & & \big\downarrow \\ X & \xrightarrow{\ i\ } & Y \end{array}$$

is cartesian, hence X' is the scheme underlying $X' \times_X \underline{X}$. The morphisms k and j induce homomorphisms of A_X-algebras $\mathscr{O}_{X'} \to \mathscr{O}_Z(Z)$ and $B \to \mathscr{O}_{X'}$; their composition is a homomorphism of A_X-algebras $u : B \to \mathscr{O}_Z(Z)$, and since $\mathscr{O}_Z(Z)$ is a topological $(A_X, \mathscr{T}_X)$-algebra, (iv) implies that u is a morphism of quasi-affinoid rings $B \otimes_{A_X} \Gamma(\underline{X}) \to \Gamma(\underline{Z})$. By adjunction, u induces a morphism of quasi-affinoid schemes $\underline{Z} \to X' \times_X \underline{X}$, and a simple inspection shows that this morphism is precisely given by k. Then the universal property implies as usual the stated independence of the choice of $\mathscr{B}$. By the same token, we also deduce that if $X'' \to X'$ is another finite morphism of schemes, we have a natural identification of quasi-affinoid schemes

$$X'' \times_{X'} (X' \times_X \underline{X}) \xrightarrow{\sim} X'' \times_X \underline{X}$$

(details left to the reader). Lastly, notice that the induced projection $X' \times_X \underline{X} \to \underline{X}$ is f-adic, by proposition 15.4.3(ii).

(vi) In the situation of (i) (resp. of (iv), resp. of (v)), suppose that (A, A^+) (resp. $\underline{A}$, resp. $\underline{X}$) is topologically henselian; combining with proposition 8.4.2(iv) we deduce that the same holds for $B \otimes_A (A, A^+)$ (resp. for $B \otimes_A \underline{A}$, resp. for $X' \times_X \underline{X}$).

Remark 15.4.10. (i) Denote by $\mathsf{Afd.Ring}_{\mathrm{comp}}$ (resp. $\mathsf{q.Afd.Ring}_{\mathrm{comp}}$) the full subcategory of $\mathsf{Afd.Ring}$ (resp. of $\mathsf{q.Afd.Ring}$) whose objects are the complete and separated affinoid (resp. quasi-affinoid) rings. Then the inclusion functor

$$\mathsf{Afd.Ring}_{\mathrm{comp}} \to \mathsf{Afd.Ring} \qquad (\text{resp. } \mathsf{q.Afd.Ring}_{\mathrm{comp}} \to \mathsf{q.Afd.Ring})$$

admits a left adjoint, that assigns to every affinoid ring (A, A^+) (resp. quasi-affinoid ring (A, A^+, U)) its *completion*, which is the datum

$$(A, A^+)^\wedge := (A^\wedge, (A^+)^\wedge) \qquad (\text{resp. } (A, A^+, U)^\wedge := (A^\wedge, (A^+)^\wedge, U^\wedge))$$

consisting of the separated completions of A and A^+ (resp. and of the open subset $U^\wedge := U \times_{\mathrm{Spec}\, A} \mathrm{Spec}\, A^\wedge$). Indeed, taking into account proposition 8.3.33(i,iii) and lemma 8.3.6 we see that $(A, A^+)^\wedge = A^\wedge \otimes_A (A, A^+)$ and $(A, A^+, U)^\wedge = A^\wedge \otimes_A (A, A^+, U)$, so the sought adjunctions follow from example 15.4.7(i,ii).

(ii) Let $\mathsf{Afd.Ring}_{\mathrm{hens}}$ (resp. $\mathsf{q.Afd.Ring}_{\mathrm{hens}}$) be the full subcategory of $\mathsf{Afd.Ring}$ (resp. of $\mathsf{q.Afd.Ring}$) whose objects are the topologically henselian affinoid (resp. quasi-affinoid) rings. Then the inclusion functor

$$\mathsf{Afd.Ring}_{\mathrm{hens}} \to \mathsf{Afd.Ring} \qquad (\text{resp. } \mathsf{q.Afd.Ring}_{\mathrm{hens}} \to \mathsf{q.Afd.Ring})$$

admits a left adjoint, that assigns to every affinoid ring (A, A^+) (resp. quasi-affinoid ring (A, A^+, U)) its *topological henselization*, which is the datum

$$(A, A^+)^{\mathrm{h}} := (A^{\mathrm{h}}, A^{+\mathrm{h}}) \qquad (\text{resp. } (A^{\mathrm{h}}, A^{+\mathrm{h}}, U^{\mathrm{h}}))$$

consisting of the topological henselizations of A and A^+ (resp. and of the open subset $U^{\mathrm{h}} := U \times_{\mathrm{Spec}\, A} \mathrm{Spec}\, A^{\mathrm{h}}$: see definition 8.4.13). Indeed, by inspecting the construction of (8.4.9) it is easily seen that $A^{+\mathrm{h}}$ identifies naturally with an open subring of A^{h} contained in $(A^{\mathrm{h}})^\circ$. Moreover, $A^{+\mathrm{h}}$ is integrally closed in A^{h}, by corollary 9.8.6(ii). Hence $(A, A^+)^{\mathrm{h}}$ is a topologically henselian affinoid ring, and theorem 8.4.10(ii) easily implies that every morphism $(A, A^+) \to (B, B^+)$ of affinoid rings with (B, B^+) topologically henselian, factors uniquely through a morphism $(A, A^+)^{\mathrm{h}} \to (B, B^+)$. By the same token, we deduce that

$$(A, A^+)^{\mathrm{h}} = A^{\mathrm{h}} \otimes_A (A, A^+)$$

from which it follows that $(A, A^+, U)^{\mathrm{h}} = A^{\mathrm{h}} \otimes_A (A, A^+, U)$, whence also the second sought adjunction.

(iii) In the same vein, if (A, A^+) (resp. (A, A^+, U)) is any affinoid (resp. quasi-affinoid) ring, we may define its *topological localization* by considering the datum $(A, A^+)_{\mathrm{loc}}$ (resp. $(A, A^+, U)_{\mathrm{loc}}$) consisting of the topological localizations A_{loc} and A^+_{loc} of A and respectively A^+, defined as in (8.4.8) (resp. and of the open subset $U \times_{\mathrm{Spec}\, A} \mathrm{Spec}\, A_{\mathrm{loc}}$). Then it is easily seen that the rule $(A, A^+) \mapsto (A, A^+)_{\mathrm{loc}}$ (resp. $(A, A^+, U) \mapsto (A, A^+, U)_{\mathrm{loc}}$) yields a left adjoint to the inclusion functor

$$\mathsf{Afd.Ring}_{\mathrm{loc}} \to \mathsf{Afd.Ring} \qquad (\text{resp. } \mathsf{q.Afd.Ring}_{\mathrm{loc}} \to \mathsf{q.Afd.Ring})$$

from the full subcategory of Afd.Ring (resp. of q.Afd.Ring) whose objects are the topologically local affinoid (resp. quasi-affinoid) rings. Notice also that $(A, A^+)_{\mathrm{loc}} = A_{\mathrm{loc}} \otimes_A (A, A^+)$, and likewise for $(A, A^+, U)_{\mathrm{loc}}$.

(iv) Let $\underline{X}$ be any quasi-affinoid scheme; by definition, $\underline{X}$ is topologically local (resp. topologically henselian) if and only if the same holds for the quasi-affinoid ring $\Gamma(\underline{X})$. On the other hand, let $\underline{A} := (A, A^+, U)$ be any quasi-affinoid ring; then $\underline{A}$ is topologically local (resp. topologically henselian) if and only if the same holds for the quasi-affinoid scheme $\mathsf{Spec}\, \underline{A}$ (notation of proposition 15.4.3). Indeed, set $A_U := \mathscr{O}_U(U)$, and recall that the image B of A in A_U is open in the f-adic topology $\mathscr{T}_U$ of A_U (proposition 8.3.30(i)). In light of proposition 8.4.2(i,iii), we are then reduced to showing that A is topologically local (resp. topologically henselian) if and only if the same holds for B, where the latter is endowed with the f-adic topology induced by the inclusion map into A_U, which is the same as the topology induced by A via the surjection $\pi : A \to B$. Let U_A and U_B be the analytic loci of $\mathrm{Spec}\, A$ and respectively $\mathrm{Spec}\, B$; since U contains U_A, it is easily seen that $\mathrm{Spec}\, \pi$ induces a homeomorphism $U_B \xrightarrow{\sim} U_A$ (details left to the reader). On the other hand, notice that $B^{\circ\circ}$ is the radical of the ideal $A^{\circ\circ}B$; then the assertion follows from claim 8.4.4 and [75, Rem.5.1.10(i)].

(v) For every quasi-affinoid scheme $\underline{X}$, set

$$\underline{X}_{\mathrm{loc}} := \mathsf{Spec}(\Gamma(\underline{X})_{\mathrm{loc}}) \qquad \underline{X}^{\mathrm{h}} := \mathsf{Spec}(\Gamma(\underline{X})^{\mathrm{h}}).$$

It follows easily from (iii),(iv), remark 15.4.2(ii) and proposition 15.4.3(i) that $\underline{X}_{\mathrm{loc}}$ is a topologically local quasi-affinoid scheme endowed with a natural morphism $j_{\underline{X}} : \underline{X}_{\mathrm{loc}} \to \underline{X}$ with the following property. Every morphism of quasi-affinoid schemes $\underline{Y} \to \underline{X}$ with $\underline{Y}$ topologically local, factors uniquely through $j_{\underline{X}}$. In other words, the rule $\underline{X} \mapsto \underline{X}_{\mathrm{loc}}$ yields a right adjoint

$$\mathsf{q.Afd.Sch} \to \mathsf{q.Afd.Sch}_{\mathrm{loc}}$$

to the inclusion functor of the full subcategory $\mathsf{q.Afd.Sch}_{\mathrm{loc}}$ of topologically local quasi-affinoid schemes. Likewise, $\underline{X}^{\mathrm{h}}$ is a topologically henselian quasi-affinoid scheme endowed with a universal morphism $h_{\underline{X}} : \underline{X}^{\mathrm{h}} \to \underline{X}$, so the rule $\underline{X} \mapsto \underline{X}^{\mathrm{h}}$ yields a right adjoint

$$\mathsf{q.Afd.Sch} \to \mathsf{q.Afd.Sch}_{\mathrm{hens}}$$

to the inclusion functor of the full subcategory $\mathsf{q.Afd.Sch}_{\mathrm{hens}}$ of topologically henselian quasi-affinoid schemes.

(vi) From (iv), we deduce that the adjunction (Spec, Γ) of proposition 15.4.3(i) restricts to two adjoint pairs of functors

$$\mathsf{q.Afd.Ring}_{\mathrm{loc}} \underset{\Gamma}{\overset{\mathsf{Spec}}{\rightleftarrows}} \mathsf{q.Afd.Sch}^o_{\mathrm{loc}} \qquad \mathsf{q.Afd.Ring}_{\mathrm{hens}} \underset{\Gamma}{\overset{\mathsf{Spec}}{\rightleftarrows}} \mathsf{q.Afd.Sch}^o_{\mathrm{hens}}.$$

Taking into account (v) and remark 1.1.17(i), we then obtain natural isomorphisms

$$\mathsf{Spec}(\underline{A}_{\mathrm{loc}}) \xrightarrow{\sim} (\mathsf{Spec}\, \underline{A})_{\mathrm{loc}} \qquad \mathsf{Spec}(\underline{A}_{\mathrm{hens}}) \xrightarrow{\sim} (\mathsf{Spec}\, \underline{A})_{\mathrm{hens}}$$

of quasi-affinoid schemes, for every quasi-affinoid ring $\underline{A}$. On the other hand, we have as well natural identifications

$$\Gamma(\underline{X}_{\mathrm{loc}}) \xrightarrow{\sim} \Gamma(\underline{X})_{\mathrm{loc}} \qquad \Gamma(\underline{X}^{\mathrm{h}}) \xrightarrow{\sim} \Gamma(\underline{X})^{\mathrm{h}} \tag{15.4.11}$$

for every quasi-affinoid scheme $\underline{X}$. Indeed, if $\Gamma(\underline{X}) = (A, A^+, U)$, we have $\Gamma(\underline{X})_{\mathrm{loc}} = (A_{\mathrm{loc}}, A^+_{\mathrm{loc}}, U_{\mathrm{loc}})$, where $U_{\mathrm{loc}} = U \times_{\mathrm{Spec}\, A} \mathrm{Spec}\, A_{\mathrm{loc}}$. Therefore $\Gamma(\underline{X}_{\mathrm{loc}}) = (B, B^+, U_{\mathrm{loc}})$, where $B := \mathscr{O}_{U_{\mathrm{loc}}}(U_{\mathrm{loc}})$, and with B^+ equal to the integral closure of A^+_{loc} in B. Since the localization map is flat, and since $A = \mathscr{O}_U(U)$, the assertion for $\Gamma(\underline{X}_{\mathrm{loc}})$ then follows from corollary 10.3.8. The same argument applies to $\Gamma(\underline{X}^{\mathrm{h}})$.

(vii) On the other hand, if $\underline{A} := (A, A^+, U)$ is a complete and separated quasi-affinoid ring, the quasi-affinoid scheme $\mathsf{Spec}\, \underline{A}$ is not necessarily complete and separated (essentially, this is because the completion map $A \to A^\wedge$ is not necessarily flat, in this generality). Thus, in order to produce a completion functor for quasi-affinoid schemes, we have to proceed more carefully, as explained in the following proposition.

Proposition 15.4.12. *Let* $\mathsf{q.Afd.Sch}_{\mathrm{comp}}$ *be the full subcategory of* $\mathsf{q.Afd.Sch}$ *whose objects are the complete and separated quasi-affinoid schemes* $(X, \mathscr{T}_X, A^+_X)$*. Then the inclusion functor*

$$\mathsf{q.Afd.Sch}_{\mathrm{comp}} \to \mathsf{q.Afd.Sch}$$

admits a right adjoint, called the completion functor *:*

$$\mathsf{q.Afd.Sch} \to \mathsf{q.Afd.Sch}_{\mathrm{comp}} \qquad (X, \mathscr{T}_X, A^+_X) \mapsto (X, \mathscr{T}_X, A^+_X)^\wedge.$$

Proof. For any given quasi-affinoid scheme $(X, \mathscr{T}_X, A^+_X)$, set $A_X := \mathscr{O}_X(X)$, and denote by $(A^\wedge_X, \mathscr{T}^\wedge_X)$ the completion of $(A_X, \mathscr{T}_X)$. For every ideal $I \subset A^\wedge_X$, set as well

$$V(I) := \mathrm{Spec}\, A^\wedge_X/I \qquad \text{and} \qquad X^\wedge_I := X \times_{\mathrm{Spec}\, A_X} V(I).$$

We let $\mathscr{F}$ be the family of all topologically closed ideals $I \subset A^\wedge_X$ such that the induced map $\rho_I : A^\wedge_X/I \to \mathscr{O}_{V(I)}(X^\wedge_I)$ is injective. Set $J := \bigcap_{I \in \mathscr{F}} I$, so that the natural map $i : A^\wedge_X/J \to \prod_{I \in \mathscr{F}} A^\wedge_X/I$ is injective, and notice that $(\prod_{I \in \mathscr{F}} \rho_I) \circ i$ factors naturally through ρ_J, therefore the latter is injective, and thus $J \in \mathscr{F}$. We endow $A' := A^\wedge_X/J$ with the quotient topology induced by the projection $\pi : A^\wedge_X \to A'$; it is clear that A' is a complete and separated f-adic ring, and π is f-adic (lemma 8.3.24(iv)), so we may define

$$(X, \mathscr{T}_X, A^+_X)^\wedge := \mathsf{Spec}\,(A' \otimes_{A^\wedge_X} (A_X, A^+_X, X)^\wedge).$$

Since the restriction map $A' \to A'_X := \mathscr{O}_{V(J)}(X^\wedge_J)$ is injective and open, the topology of A'_X is complete and separated, so the datum $(X, \mathscr{T}_X, A^+_X)^\wedge$ is indeed an object of $\mathsf{q.Afd.Sch}_{\mathrm{comp}}$.

On the other hand, let $(Y, \mathscr{T}_Y, A^+_Y)$ be any complete and separated quasi-affinoid scheme, and $f : (Y, \mathscr{T}_Y, A^+_Y) \to (X, \mathscr{T}_X, A^+_X)$ any morphism of quasi-affinoid schemes; by adjunction, $\Gamma(f)$ factors uniquely through a morphism of complete quasi-affinoid rings

$$\Gamma(f)^\wedge : (A_X, A^+_X, X)^\wedge \to \Gamma(Y, \mathscr{T}_Y, A^+_Y)$$

and we let K be the kernel of the underlying continuous ring homomorphism $h : A^\wedge_X \to A_Y$. Clearly K is a closed ideal of A_X, and $\mathrm{Spec}\, h$ restricts to a morphism of schemes $g : Y \to V(K)$; then, the pair (f, g) determines a unique morphism of X-schemes $Y \to X^\wedge_K$. Therefore, the resulting map $\overline{h} : A^\wedge_X/K \to A_Y$ is injective by construction, and it factors through ρ_K; so ρ_K is injective as well, and thus $K \in \mathscr{F}$. Hence, h factors uniquely through a continuous ring homomorphism $A' \to A_Y$, so in turns $\Gamma(f)^\wedge$ factors uniquely through a morphism of quasi-affinoid rings

$$\varphi : A' \otimes_{A_X} (A_X, A^+_X, X)^\wedge \to \Gamma(Y, \mathscr{T}_Y, A^+_Y).$$

and $\mathsf{Spec}(\varphi) : (Y, \mathscr{T}_Y, A^+_Y) \to (X, \mathscr{T}_X, A^+_X)^\wedge$ is the unique morphism of complete quasi-affinoid schemes that lifts f, whence the proposition. □

Remark 15.4.13. (i) Let $\underline{X} := (X, \mathscr{T}_X, A^+_X)$ be any quasi-affinoid scheme, set $(A_X, A^+_X, X) := \Gamma(\underline{X})$, and let $A^\wedge_X$ be the separated completion of A_X. Let also $I \subset A_X$ be any finitely generated ideal such that $\mathrm{Spec}\, A_X/I = \mathrm{Spec}\, A_X \setminus X$. By inspecting the proof of proposition 15.4.12 we obtain a natural morphism of quasi-affinoid schemes

$$\underline{X}^\wedge \to \mathsf{Spec}(\Gamma(\underline{X})^\wedge)$$

which is an isomorphism if and only if $\mathrm{depth}_I A^\wedge_X > 0$ (notation of (10.4.29)).

(ii) The morphism of (i) induces by adjunction a morphism of quasi-affinoid rings

$$\Gamma(\underline{X})^\wedge \to \Gamma(\underline{X}^\wedge)$$

which is an isomorphism if and only if $\mathrm{depth}_I A^\wedge_X > 1$.

Definition 15.4.14. Let $\underline{A} := (A, A^+, U)$ be any quasi-affinoid ring, and $\underline{X} := (X, \mathscr{T}_X, A^+_X)$ any quasi-affinoid scheme.

(i) The *adic spectrum* of the affinoid ring (A, A^+) is the set

$$\mathrm{Spa}(A, A^+) := \{v \in \mathrm{Cont}(A) \mid v(a) \leq 1 \text{ for every } a \in A^+\}$$

(notation of definition 15.3.12). Hence, $\mathrm{Spa}(A, A^+)$ is a pro-constructible subset of $\mathrm{Cont}(A)$, and we endow $\mathrm{Spa}(A, A^+)$ with the topology induced by $\mathrm{Cont}(A)$. We also let

$$\mathrm{Spa}(A, A^+)_{\mathrm{a}} := \mathrm{Spa}(A, A^+) \cap \mathrm{Cont}(A)_{\mathrm{a}} \qquad \mathrm{Spa}(A, A^+)_{\mathrm{na}} := \mathrm{Spa}(A, A^+) \cap \mathrm{Cont}(A)_{\mathrm{na}}$$

(notation of remark 15.3.13(vi)).

(ii) The *adic spectrum* of $\underline{A}$ is the set

$$\mathrm{Spa}\,\underline{A} := \mathrm{Spa}(A, A^+) \cap \mathrm{Spv}\,U.$$

We endow this subset with the topology induced by $\mathrm{Spa}(A, A^+)$.

(iii) The *adic spectrum* of $\underline{X}$ is the topological space

$$\mathrm{Spa}\,\underline{X} := \mathrm{Spa}\,\Gamma(\underline{X})$$

and we set as well

$$\mathrm{Spa}\,(\underline{X})_{\mathrm{a}} := \mathrm{Spa}\,\Gamma(\underline{X})_{\mathrm{a}} \qquad \text{and} \qquad \mathrm{Spa}\,(\underline{X})_{\mathrm{na}} := \mathrm{Spa}\,\Gamma(\underline{X})_{\mathrm{na}}.$$

(iv) A *rational subset* of $\mathrm{Spa}(A, A^+, U)$ is a subset of the form $R \cap \mathrm{Spa}(A, A^+, U)$, where R is a rational subset of $\mathrm{Cont}(A)$ (see (15.3.16)). Likewise, a *rational subset* of $\mathrm{Spa}\,\underline{X}$ is a subset of the form $R \cap \mathrm{Spa}\,\underline{X}$, where R is a rational subset of $\mathrm{Cont}(\mathscr{O}_X(X), \mathscr{T}_X)$.

Remark 15.4.15. (i) Let $f : (A, A^+) \to (B, B^+)$ be any morphism of affinoid rings. A simple inspection of definition 15.4.14(i) shows that the map $\mathrm{Cont}(f)$ restricts to a continuous map

$$\mathrm{Spa}\,f : \mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+).$$

(ii) More generally, a morphism $g : (A, A^+, U) \to (B, B^+, V)$ of quasi-affinoid rings induces a continuous map

$$\mathrm{Spa}\,g : \mathrm{Spa}(B, B^+, V) \to \mathrm{Spa}(A, A^+, U)$$

that agrees with the restriction of the map $\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ attached to g as in (i).

(iii) Likewise, any morphism $\varphi : (X, \mathscr{T}_X, A^+_X) \to (Y, \mathscr{T}_Y, A^+_Y)$ of quasi-affinoid schemes induces a continuous map

$$\mathrm{Spa}\,\varphi := \mathrm{Spa}\,\Gamma(\varphi) : \mathrm{Spa}(X, \mathscr{T}_X, A^+_X) \to \mathrm{Spa}(Y, \mathscr{T}_Y, A^+_Y).$$

(iv) In light of theorem 15.3.15(i), it is easily seen that $v(x) \leq 1$ for every $x \in A^\circ$ and every $v \in \mathrm{Cont}(A)$ of rank ≤ 1. Especially, the subset of $\mathrm{Spa}(A, A^+, U)$ consisting of rank one valuations is independent of the ring of integral elements A^+.

Example 15.4.16. For any f-adic ring A, let $A' \subset A$ be the smallest subring of integral elements (see remark 15.4.2(iv)). It is easily seen that $\mathrm{Spa}(A, A') = \mathrm{Cont}(A)$. Also, every continuous map $f : A \to B$ of f-adic rings yields a morphism of affinoid rings $f : (A, A') \to (B, B')$, and clearly $\mathrm{Spa}\,\varphi = \mathrm{Cont}(f)$.

(ii) On the other hand, every f-adic map $f : A \to B$ of f-adic rings induces a continuous map $\mathrm{Spa}\,f : \mathrm{Spa}(B, B^\circ) \to \mathrm{Spa}(A, A^\circ)$ (lemma 8.3.24(iii.a)).

Lemma 15.4.17. *Let $\underline{A} := (A, A^+, U)$ be any quasi-affinoid ring.*

(i) *The natural morphisms $\underline{A} \to \underline{A}_{\mathrm{loc}} \to \underline{A}^{\mathrm{h}} \to \underline{A}^\wedge$ (notation of remark 15.4.10(ii,iii)) induce homeomorphisms*

$$\mathrm{Spa}\,\underline{A}^\wedge \xrightarrow{\sim} \mathrm{Spa}\,\underline{A}^{\mathrm{h}} \xrightarrow{\sim} \mathrm{Spa}\,\underline{A}_{\mathrm{loc}} \xrightarrow{\sim} \mathrm{Spa}\,\underline{A}.$$

(ii) *The unit of the adjunction* (Spec, Γ) *of proposition* 15.4.3(i) *induces a homeomorphism*

$$\mathrm{Spa}(\mathsf{Spec}\,\underline{A}) \xrightarrow{\sim} \mathrm{Spa}\,\underline{A}.$$

(iii) *Let* $\underline{X} := (X, \mathscr{T}_X, A^+_X)$ *be any quasi-affinoid scheme, and* $\underline{X}^\wedge$ *the completion of* $\underline{X}$ *(see proposition* 15.4.12*). The completion map* $\underline{X} \to \underline{X}^\wedge$ *induces a homeomorphism*

$$\mathrm{Spa}\,\underline{X}^\wedge \xrightarrow{\sim} \mathrm{Spa}\,\underline{X}.$$

(iv) *Let* $\varphi : \mathrm{Spa}\,\underline{A}^\wedge \xrightarrow{\sim} \mathrm{Spa}\,\underline{A}$ *be the homeomorphism of* (i)*, and* $R \subset \mathrm{Spa}\,\underline{A}^\wedge$ *any rational subset. Then* $\varphi(R)$ *is a rational subset of* $\mathrm{Spa}\,\underline{A}$*.*

Proof. (i): Let A_{loc} (resp. A^{h}) be the topological localization (resp. henselization) of A; from corollary 8.4.15(i) and proposition 15.3.19(iv), we deduce that the natural maps $A \to A_{\mathrm{loc}} \to A^{\mathrm{h}} \to A^\wedge$ induce homeomorphisms

$$\mathrm{Cont}(A^\wedge) \xrightarrow{\sim} \mathrm{Cont}(A^{\mathrm{h}}) \xrightarrow{\sim} \mathrm{Cont}(A_{\mathrm{loc}}) \xrightarrow{\sim} \mathrm{Cont}(A).$$

Moreover, it was already remarked that the localization map induces a natural identification $\underline{A}_{\mathrm{loc}} \xrightarrow{\sim} A_{\mathrm{loc}} \otimes_A (A, A^+)$, and likewise for the topological henselization and the completion of $\underline{A}$; the assertion follows easily (details left to the reader).

(ii): Set $(U, \mathscr{T}_U, A^+_U) := \mathsf{Spec}\,\underline{A}$ and $A_U := \mathscr{O}_U(U)$; the assertion boils down to the following:

Claim 15.4.18. The natural morphism of quasi-affinoid rings $\rho : (A, A^+, U) \to (A_U, A^+_U, U)$ induces a homeomorphism $\mathrm{Spa}\,\rho : \mathrm{Spa}(A_U, A^+_U, U) \xrightarrow{\sim} \mathrm{Spa}(A, A^+, U)$.

Proof of the claim. Let us first check that ρ induces a homeomorphism

$$\mathrm{Cont}(A_U) \cap \mathrm{Spv}\,U \xrightarrow{\sim} \mathrm{Cont}(A) \cap \mathrm{Spv}\,U. \tag{15.4.19}$$

Indeed, since the topology of both of these spaces are induced from the inclusion into $\mathrm{Spv}\,U$, it suffices to check that (15.4.19) is surjective. Hence, let $v : A \to \Gamma_{v\circ}$ be any continuous valuation with support given by a prime ideal $\mathfrak{p} \subset A$, and suppose that $\mathfrak{p} \in U$; then there exists a unique prime ideal $\mathfrak{p}' \subset A_U$ such that $\rho^{-1}\mathfrak{p}' = \mathfrak{p}$. Set $B := A/\mathfrak{p}$ and $C := A_U/\mathfrak{p}'$; the map v factors through a continuous valuation $\overline{v} : B \to \Gamma_{v\circ}$ (for the quotient topology on B induced by the projection $A \to B$) and C is naturally identified with a subring of $\mathrm{Frac}\,B$, so $\overline{v}$ extends uniquely to a valuation $w : C \to \Gamma_{v\circ}$. It remains only to show that w is continuous, for the quotient topology on C induced by the projection $A_U \to C$. By remark 15.3.13(ii) it suffices to prove the continuity of w at the point $0 \in C$; however, ρ is an open map (proposition 8.3.30(i)), so the same holds for the induced injective map $B \to C$, and the assertion follows.

Since A^+_U is the integral closure of the image of A^+ in A_U, the claim is an immediate consequence of the foregoing : details left to the reader. ◊

(iii): Let $A_X := \mathscr{O}_X(X)$, and set $(A^\wedge_X, A^{+\wedge}_X, X^\wedge) := (A_X, A^+_X, X)^\wedge$. Define the family $\mathscr{F}$ of ideals of $A^\wedge_X$ as in the proof of proposition 15.4.12, and denote by J the minimal element of $\mathscr{F}$; also, endow $A' := A^\wedge_X/J$ with the quotient topology induced via the projection $A^\wedge_X \to A'$. Then $\underline{X}^\wedge = \mathsf{Spec}(A' \otimes_{A^\wedge_X} (A^\wedge_X, A^{+\wedge}_X, X^\wedge))$. Taking into account (i) and (ii), we are then reduced to checking that the natural projection $\pi : (A^\wedge_X, A^{+\wedge}_X, X^\wedge) \to A' \otimes_{A^\wedge_X} (A^\wedge_X, A^{+\wedge}_X, X^\wedge)$ induces a homeomorphism

$$\mathrm{Spa}\,\pi : \mathrm{Spa}(A' \otimes_{A^\wedge_X} (A^\wedge_X, A^{+\wedge}_X, X^\wedge)) \xrightarrow{\sim} \mathrm{Spa}(A^\wedge_X, A^{+\wedge}_X, X^\wedge).$$

However, $\mathrm{Spa}\,\pi$ is a closed immersion, so it suffices to show that it is a bijection. To this aim, let $v : A^\wedge_X \to \Gamma_{v\circ}$ be any continuous valuation whose support $\mathfrak{p}$ lies in $X^\wedge$; we come down to checking that $\mathfrak{p} \in \mathscr{F}$. However, $\mathfrak{p}$ is obviously a topologically closed ideal of $A^\wedge_X$. Next, set $B := A^\wedge_X/\mathfrak{p}$ and $X^\wedge_{\mathfrak{p}} := X \times_{\mathrm{Spec}\,A_X} \mathrm{Spec}\,B$; since B is a domain and $X^\wedge_{\mathfrak{p}} \neq \varnothing$, the restriction map $B \to \mathscr{O}_{\mathrm{Spec}\,B}(X^\wedge_{\mathfrak{p}})$ is injective, whence the contention.

(iv): Say that $R = R_{A^\wedge}\big(\frac{a_1}{a_0},\dots,\frac{a_n}{a_0}\big) \cap \mathrm{Spa}\,\underline{A}^\wedge$ for a sequence of elements $a_0,\dots,a_n \in A^\wedge$ that generates an open ideal J. Let also B be a ring of definition of A, and $I \subset B$ an ideal of adic definition. To begin with, we remark :

Claim 15.4.20. (i) Let T be a quasi-compact subset of $\mathrm{Spa}\,\underline{A}$, and t an element of $A^{\circ\circ}$ such that $v(t) \neq 0$ for every $v \in T$. Then there exists $k \in \mathbb{N}$ such that $v(x) < v(t)$ for every $v \in T$ and every $x \in I^k$.

(ii) There exists $q \in \mathbb{N}$ such that every sequence $b_0,\dots,b_n$ with $b_i \in a_i + I^q B^\wedge$ for $i = 0,\dots,n$, generates an open ideal of $A^\wedge$, and $R' := R_{A^\wedge}\big(\frac{b_1}{b_0},\dots,\frac{b_n}{b_0}\big) \cap \mathrm{Spa}\,\underline{A}^\wedge = R$.

Proof of the claim. (i): Let $x_1,\dots,x_m$ be a finite system of generators of I. For every $r \in \mathbb{N}$ set

$$T_r := \{v \in \mathrm{Spa}\,\underline{A} \mid v(x_i^r) \leq v(t) \text{ for every } i = 1,\dots,m\}.$$

Then T_r is open in $\mathrm{Spa}\,\underline{A}$ for every $r \in \mathbb{N}$, and $T \subset \bigcup_{r\in\mathbb{N}} T_r$. Thus, $T \subset T_r$ for some $r \in \mathbb{N}$, and the assertion holds with $k := r + 1$.

(ii): Pick elements $c_1,\dots,c_m \in J \cap B^\wedge$ that generate an open ideal J_0 of $B^\wedge$, and $r \in \mathbb{N}$ such that $I^r \subset J_0$. Then, every sequence $d_1,\dots,d_m$ of elements of $B^\wedge$ with $d_i \in c_i + I^r B^\wedge$ for $i = 1,\dots,m$, also generates J_0 ([126, Th.8.4]). Next, for every $i = 1,\dots,m$ pick $x_{i0},\dots,x_{in} \in A^\wedge$ such that $c_i = \sum_{j=0}^n a_j x_{ij}$, and let $p \in \mathbb{N}$ be large enough so that $I^p x_{ij} \subset I^r B^\wedge$ for every $i = 1,\dots,m$ and every $j = 0,\dots,n$; it follows already that every sequence $b_0,\dots,b_n$ of elements of $A^\wedge$ with $b_i \in a_i + I^p B^\wedge$ for $i = 0,\dots,n$ generates an open ideal of $A^\wedge$.

Next, we apply part (i) of the claim to the rational subsets $R_i := R_{A^\wedge}\big(\frac{a_1}{a_i},\dots,\frac{a_n}{a_i}\big) \cap \mathrm{Spa}\,\underline{A}^\wedge$, to find an integer $q \geq p$ such that $v(x) < v(a_i)$ for every $x \in I^q B^\wedge$, every $i = 0,\dots,n$ and every $v \in R_i$. We claim that this q will do. Indeed, let $v \in \mathrm{Spa}\,\underline{A}^\wedge$, and say that $v \in R = R_0$; since $b_i - a_i \in I^q B^\wedge$, we have $v(b_i - a_i) < v(a_0)$ for $i = 0,\dots,n$. This implies that

$$v(b_i) = v(a_i + (b_i - a_i)) \leq \max(v(a_i), v(b_i - a_i)) \leq v(a_0) = v(a_0 + (b_0 - a_0)) = v(b_0)$$

for every $i = 0,\dots,n$, which shows that $R \subset R'$. Next, suppose $v \notin R$, and consider first the case where $v(a_i) = 0$ for every $i = 0,\dots,n$. Then v is non-analytic, so that $v(b_0 - a_0) = 0$, whence $v(b_0) = 0$, and finally $v \notin R'$. Lastly, suppose that $v(a_i) \neq 0$ for some $i \leq n$, and choose $j \leq n$ so that $v \in R_j$. We must have $v(a_0) < v(a_j)$, since $v \notin R$; also, by construction $v(b_i - a_i) < v(a_j)$ for every $i = 0,\dots,n$. Then

$$v(b_0) = v(a_0 + (b_0 - a_0)) \leq \max(v(a_0), v(b_0 - a_0)) < v(a_j) = v(a_j + (b_j - a_j)) = v(b_j)$$

which yields again $v \notin R'$, and the claim follows. ◊

Now, by claim 15.4.20(ii) there exist $b_0,\dots,b_n$ in A such that $R = R_{A^\wedge}\big(\frac{b_1}{b_0},\dots,\frac{b_n}{b_0}\big) \cap \mathrm{Spa}\,\underline{A}^\wedge$. Also, by claim 15.4.20(i) there exists $k \in \mathbb{N}$ such that $v(x) < v(b_0)$ for every $v \in R$ and every $x \in I^k$. Pick a finite system $b_{n+1},\dots,b_p$ of generators of I^k; it follows that $\varphi(R) = R_A\big(\frac{b_1}{b_0},\dots,\frac{b_p}{b_0}\big)$ whence the contention. □

Remark 15.4.21. By combining lemma 15.4.17(iv) and proposition 15.3.19(i) one may obtain another, more constructive, proof of lemma 15.4.17(i) (and of proposition 15.3.19(iv)). This is the route followed in [99].

Proposition 15.4.22. *Let (A, A^+, U) be any quasi-affinoid ring. The following holds :*

(i) $\mathrm{Spa}(A, A^+, U)$ *is a constructible open subset of* $\mathrm{Spa}(A, A^+)$*, and both these topological spaces are spectral.*

(ii) *The rational subsets are a basis of quasi-compact open subsets of* $\mathrm{Spa}(A, A^+)$ *that is closed under finite intersections.*

(iii) *For any morphism $f : (A, A^+, U) \to (B, B^+, V)$ of quasi-affinoid rings, we have :*

(a) *If f is f-adic,* $\mathrm{Spa}\,f$ *is a spectral map. More precisely, if R is a rational subset of* $\mathrm{Spa}(A, A^+, U)$*, then* $(\mathrm{Spa}\,f)^{-1}(R)$ *is a rational subset of* $\mathrm{Spa}(B, B^+, V)$.

(b) *Suppose that B is topologically local. Then f is f-adic if and only if* $\mathrm{Spa}\, f$ *restricts to a map* $\mathrm{Spa}(B, B^+)_{\mathrm{a}} \to \mathrm{Spa}(A, A^+)_{\mathrm{a}}$.

(iv) *For any quasi-affinoid scheme* $(X, \mathscr{T}_X, A^+_X)$*, the topological space* $\mathrm{Spa}(X, \mathscr{T}_X, A^+_X)$ *is spectral.*

(v) *For any f-adic morphism* $\varphi : (X, \mathscr{T}_X, A^+_X) \to (Y, \mathscr{T}_Y, A^+_Y)$ *of quasi-affinoid schemes, the induced map* $\mathrm{Spa}\,\varphi$ *is spectral.*

Proof. Assertion (i) for $\mathrm{Spa}(A, A^+)$ follows from theorem 15.3.15(ii) and corollary 8.1.42. According to (15.3.16), the rational subsets of $\mathrm{Cont}(A)$ form a basis of quasi-compact open subsets for the latter space; invoking again corollary 8.1.42, we obtain (ii) as well.

Next, let $I \subset A$ be any finitely generated ideal such that $\mathrm{Spec}\, A/I = \mathrm{Spec}\, A \setminus U$, and pick a finite system of generators $a_1, \dots, a_n$ for I. Then

$$\mathrm{Spa}(A, A^+, U) = \mathrm{Spa}(A, A^+) \cap \bigcup_{i=1}^{n} R_A\Big(\frac{a_1}{a_i}, \dots, \frac{a_n}{a_i}\Big).$$

Since I is open in A, this identity presents $\mathrm{Spa}(A, A^+, U)$ as a finite union of rational subsets of $\mathrm{Spa}(A, A^+)$, and the latter are open and quasi-compact, by (ii), so the proof of (i) is complete.

(iii.a) follows immediately from corollary 15.3.17(i). The proofs of (iv) and (v) are similar, and the details shall be left to the reader.

(iii.b): Set $\varphi := \mathrm{Spec}\, f$; in light of lemma 8.3.29(iv) and corollary 15.3.21, we need only check that $\varphi^{-1} X^{\circ\circ}_A = X^{\circ\circ}_B$, if $\mathrm{Spa}\, f$ restricts to a map $\mathrm{Spa}(B, B^+)_{\mathrm{a}} \to \mathrm{Spa}(A, A^+)_{\mathrm{a}}$.

Thus, suppose that there exists $\mathfrak{p} \in X_B \setminus X^{\circ\circ}_B$ with $\varphi(\mathfrak{p}) \in X^{\circ\circ}_A$. Then $\mathfrak{p}$ is a non-open ideal of B, so there exists a rank one continuous analytic valuation v of B whose support contains $\mathfrak{p}$ (lemma 15.3.25); clearly the support of $\mathrm{Cont}(f)(v)$ contains $\varphi(\mathfrak{p})$, and it is therefore open, *i.e.* $\mathrm{Cont}(f)(v) \in \mathrm{Cont}(A)_{\mathrm{na}}$. To conclude it suffices to remark :

Claim 15.4.23. Let R be any f-adic ring, $v \in \mathrm{Cont}(R)_{\mathrm{a}}$ any element such that Γ_v has rank one. Then $v \in \mathrm{Spa}(R, R^\circ)$.

Proof of the claim. Since v is analytic, there exists $a \in R$ such that $0 < v(a) < 1$. Let $b \in R^\circ$ be any element, and suppose that $v(b) > 1$; then there exists $n \in \mathbb{N}$ large enough, such that $v(b^n a) > 1$, which is absurd, since $b^n a \in R^{\circ\circ}$ (remark 8.3.10(iv)). ☐

Proposition 15.4.24. *Let (A, A^+, U) be any quasi-affinoid ring, $I \subset A^+$ an ideal, $g_\bullet := (g_i \mid i \in \Sigma)$ and $h_\bullet := (h_j \mid j \in \Sigma')$ two systems of elements of A such that :*

(a) *The system $(g_\bullet, h_\bullet)$ generates an open ideal of A.*

(b) $\mathrm{Spec}\, A/IA$ *is the set of open prime ideals of A.*

(c) *U is dense in* $\mathrm{Spec}\, A$.

Then for every $f \in A$ the following conditions are equivalent :

(d) *For every $v \in \mathrm{Spa}(A, A^+, U)$ with $v(f) \neq 0$ there exists either $i \in \Sigma$ such that $v(f) \leq v(g_i)$ or else $j \in \Sigma'$ such that $v(f) < v(h_j)$.*

(e) *There exists a polynomial $P \in \mathscr{P}_I$ such that $P(f, g_\bullet, h_\bullet) = 0$ (notation of* (9.2.39)*).*

Proof. From theorem 15.3.15(i) and lemmata 15.4.17(i), 9.2.40 we already see that (e)⇒(d).

(d)⇒(e): Let (A', A'^+, U') be the topological localization of (A, A^+, U), and set $I' := IA'^+$. Notice that U' is dense in $\mathrm{Spec}\, A'$ (proposition 8.1.47(iii)), and $\mathrm{Spec}\, A'/I'A'$ is the set of open prime ideals of A', since the localization map $A \to A'$ is f-adic (lemma 8.3.29(iv)). We may then consider also the set $\mathscr{P}_{I'} \subset A'^+[Z, X_\bullet, Y_\bullet]$ defined as in (9.2.39), and we remark

Claim 15.4.25. Condition (e) is equivalent to :

(f) There exists a polynomial $Q \in \mathscr{P}_{I'}$ such that $Q(f, g_\bullet, h_\bullet) = 0$ in A'^+.

Proof of the claim. Obviously (e)⇒(f). Conversely, let $A_0 \subset A^+$ be a subring of definition, and J any ideal of adic definition of A_0, so that $A'^+ = (1+J)^{-1}A^+$ and $I' = (1+J)^{-1}I$. If (f) holds, we deduce that there exists a polynomial $G \in A^+[Z, X_\bullet, Y_\bullet]$ of the form

$$G(Z, X_\bullet, Y_\bullet) = (1-a) \cdot Z^n + \sum_{k=1}^{n} Z^{n-k} P_k(X_\bullet, Y_\bullet)$$

for some $a \in J$ and some $P_1, \dots, P_n \in A^+[X_\bullet, Y_\bullet]$ fulfilling the conditions of (9.2.39), and such that $G(f, g_\bullet, h_\bullet) = 0$ in A. Since A is f-adic, condition (a) implies that the system $(g_i^n, h_j^n \mid i \in \Sigma,\ j \in \Sigma')$ generates an open ideal of A (details left to the reader); then we may find $N \in \mathbb{N}$ large enough, finite subsets $\Sigma_0 \subset \Sigma'$, $\Sigma'_0 \subset \Sigma'$, and systems $(b_i \mid i \in \Sigma_0)$, $(c_j \mid j \in \Sigma'_0)$ of elements of A such that

$$a^N f^n = \sum_{i \in \Sigma_0} b_i g_i^n + \sum_{j \in \Sigma'_0} c_j h_j^n.$$

In light of (b), we may next find $M \in \mathbb{N}$ sufficiently large, so that $a^M b_i \in A^+$ and $a^M c_j \in I$ for every $i \in \Sigma_0$ and $j \in \Sigma'_0$. Summing up, we see that $a^{M+N} f^n = H(g_\bullet, h_\bullet)$ in A for some homogeneous polynomial $H(X_\bullet, Y_\bullet) \in A^+[X_\bullet, Y_\bullet]$ of degree n and with $H(0, Y_\bullet) \in I[Y_\bullet]$. Then set $u := 1 + a + \cdots + a^{M+N-1}$, so that $u \cdot (1-a) = 1 - a^{M+N}$; it follows that the polynomial

$$P := u \cdot G + a^{M+N} Z^n - H$$

fulfills condition (e). ◊

Now, if (e) fails for some $f \in A$, claim 15.4.25 and lemma 9.2.40 show that there exists $w \in \mathrm{Spv}\, A'$ whose support is a minimal prime ideal of A', and with $w(f) \neq 0$ and

$$w(a) \leq 1 \qquad w(b) < 1 \qquad \text{for every } a \in A'^+ \text{ and } b \in A'^{\circ\circ}$$
$$w(g_i) < w(f) \qquad w(h_j) \leq w(f) \qquad \text{for every } i \in \Sigma \text{ and } j \in \Sigma'.$$

Since U' is dense in $\mathrm{Spec}\, A'$, we have $w \in \mathrm{Spv}\, U'$ (proposition 8.1.47(iii)); if w is continuous, it follows that $w \in \mathrm{Spa}(A', A'^+, U') = \mathrm{Spa}(A, A^+, U)$ (lemma 15.4.17(i)), and (d) fails for w. In case w is not continuous, set $L := IA$ and $v := w^{c\Gamma_v(L)}$ (notation of definition 15.3.3).

Claim 15.4.26. (i) $v(L) \neq \{0\}$.
(ii) $v(f) \neq 0$.

Proof of the claim. (i): Since v is an L-admissible specialization of w, it suffices to show that $w(L) \neq \{0\}$. But if the latter fails, then the support of w is an open prime ideal, so w is continuous (remark 15.3.13(vi)), contradicting our assumption.

(ii): Suppose that $v(f) = 0$; then $v(g_i) = v(h_j) = 0$ as well, for every $i \in \Sigma$ and $j \in \Sigma'$. Hence the support of v is an open ideal of A, and especially, it contains L, contradicting (i). ◊

Claim 15.4.26(ii), theorem 15.3.15(i) and lemma 15.4.17(i) imply that v is an analytic point of $\mathrm{Spa}(A, A^+)$ and condition (d) fails for v; lastly, notice that $\mathrm{Cont}(A)_{\mathrm{a}} \subset \mathrm{Spv}\, U$, so $v \in \mathrm{Spa}(A, A^+, U)$ whence the proposition. □

Corollary 15.4.27. *Let* $\underline{A} := (A, A^+, U)$ *be any quasi-affinoid ring such that* U *is dense in* $\mathrm{Spec}\, A$, *and* $f \in A$ *any element. We have :*

(i) $f \in A^+$ *if and only if* $v(f) \leq 1$ *for every* $v \in \mathrm{Spa}\, \underline{A}$.
(ii) $f \in A^{\circ\circ}$ *if and only if* $v(f) < 1$ *for every* $v \in \mathrm{Spa}\, \underline{A}$.
(iii) *Let also* $J \subset A^+$ *be any finitely generated ideal whose radical is* $A^{\circ\circ}$. *Then the following conditions are equivalent :*
  (a) $|f|_J^* = 0$ *(notation of example* 9.1.9*).*
  (b) $v(f) = 0$ *for every* $v \in \mathrm{Spa}\, \underline{A}$.

Proof. (i): We may assume that $v(f) \leq 1$ for every $v \in \mathrm{Spa}\,\underline{A}$, and we show that $f \in A^+$. To this aim, we apply proposition 15.4.24 with $g_\bullet = \{1\}$, $h_\bullet$ the empty subset and $I := A^{\circ\circ}$; we deduce that there exists a monic polynomial $P(Z) \in A^+[Z]$ with $P(f) = 0$, *i.e.* f is integral over A^+, whence the contention, since A^+ is integrally closed in A.

(ii): Again, we may assume that $v(f) < 1$ for every $v \in \mathrm{Spa}\,\underline{A}$, and we check that $f \in A^{\circ\circ}$. To this aim, we apply proposition 15.4.24 with $h_\bullet = \{1\}$, $g_\bullet$ the empty subset and $I := A^{\circ\circ}$; we deduce that there exist $n \in \mathbb{N}$ and $a_1, \dots, a_n \in A^{\circ\circ}$ with $f^n + f^{n-1}a_1 + \cdots + a_n = 0$. By (i), we know already that $f \in A^+$, so that $f^{n-1}a_1 + \cdots + a_n \in A^{\circ\circ}$, and combining with remark 8.3.10(iv), the claim follows.

(iii.a)⇒(iii.b): Let $a_1, \dots, a_r$ be any finite system of generators of J, and set

$$\gamma_v := \max(v(a_1), \dots, v(a_r)) \in \Gamma_{v\circ} \qquad \text{for every } v \in \mathrm{Spa}(A, A^+, U).$$

The assumption implies that for every $n \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that $f^k \in J^{kn}$, hence $v(f^k) \leq \gamma_v^{kn}$, and therefore $v(f) \leq \gamma_v^n$ for every $n \in \mathbb{N}$ and every $v \in \mathrm{Spa}\,\underline{A}$ (details left to the reader). However γ_v is final in Γ_v for every such v (lemma 15.3.14(i)), whence the assertion.

(iii.b)⇒(iii.a): For every $n \in \mathbb{N}$, we apply the criterion of proposition 15.4.24 with $g_\bullet$ equal to the empty set, $h_\bullet^{(n)} := \{1\}$, and $I := J^n$. We deduce that for every $n \in \mathbb{N}$ there exist $k \in \mathbb{N}$ and elements $a_i^{(n)} \in J^{ni}$ for $i = 1, \dots, k$, such that $f^{k+1} + a_1^{(n)} f^k + \cdots + a_k^{(n)} = 0$. Therefore :

$$|f^{k+1}|_J^* \leq \max(|a_i^{(n)}|_J^* \cdot |f^{k+1-i}|_J^* \mid i = 1, \dots, k)$$

(lemma 9.1.8(ii)) whence $(|f|_J^*)^i \leq |a_i^{(n)}|_J^* \leq \rho^{ni}$ for some $i \leq k$, (lemma 9.1.8(iii)), and consequently $|f|_J^* \leq \rho^n$ for every $n \in \mathbb{N}$, whence (iii.a). □

Corollary 15.4.28. *Let $\underline{X} := (X, \mathscr{T}_X, A_X^+)$ be any quasi-affinoid scheme, and $(X^\wedge, \mathscr{T}_X^\wedge, A_{X^\wedge}^+)$ the completion of $\underline{X}$. Denote by $A_X^\wedge$ (resp. $A_X^{\wedge+}$) the separated completion of $A_X := \mathscr{O}_X(X)$ (resp. of A_X^+). Then we have :*

(i) *The kernel $\mathscr{J}$ of the natural map $\pi : A_X^\wedge \to \mathscr{O}_{X^\wedge}(X^\wedge)$ is contained in $(A_X^\wedge)^{\circ\circ}$.*
(ii) $\pi^{-1}A_{X^\wedge}^+ = A_X^{\wedge+}$.

Proof. (i): To begin with, we remark, quite generally :

Claim 15.4.29. Let A be any ring, $I \subset A$ a finitely generated ideal, and set

$$X_A := \mathrm{Spec}\,A \qquad Z := \mathrm{Spec}\,A/I \qquad U := X_A \setminus Z.$$

Then the following conditions are equivalent :

(a) U is dense in X_A.
(b) The support of the A-module I^n/I^{n+1} is Z, for every $n \in \mathbb{N}$.

Proof of the claim. Since U is constructible in X_A, condition (a) is equivalent to :

(c) $I_\mathfrak{q} = A_\mathfrak{q}$ for every minimal prime ideal $\mathfrak{q}$ of A

(proposition 8.1.47(iii)). Now, if $A = 0$ there is nothing to prove, so assume that $A \neq 0$. Suppose that (c) holds. Clearly the support of I^n/I^{n+1} lies in Z, so we consider any $\mathfrak{p} \in Z$, and we have to show that $I_\mathfrak{p}^n/I_\mathfrak{p}^{n+1} \neq 0$ for every $n \in \mathbb{N}$. However, if the latter fails for some $n \in \mathbb{N}$, Nakayama's lemma implies that $I_\mathfrak{p}^n = 0$; then $I_\mathfrak{q}^n = 0$ for every minimal prime ideal $\mathfrak{q}$ of A contained in $\mathfrak{p}$, contradicting (c).

Conversely, if (c) fails for some minimal prime ideal $\mathfrak{q}$, it follows that $I_\mathfrak{q}$ is a nilpotent ideal of $A_\mathfrak{q}$. Say that $I_\mathfrak{q}^n \neq 0$ and $I_\mathfrak{q}^{n+1} = 0$ for some $n \in \mathbb{N}$; then $\mathfrak{q}$ lies in the support of I^n/I^{n+1}, hence $\mathfrak{q} \in Z$, and therefore U cannot be dense in X_A. ◊

Set $Y := \mathrm{Spec}\,A_X$, $Y := \mathrm{Spec}\,A_X^\wedge$, $X' := X \times_Y Y'$, and let $I \subset A_X$ be any ideal such that $A_X^{\circ\circ} \subset I$ and $Y \setminus X = \mathrm{Spec}\,A_X/I$. Then $Y' \setminus X' = \mathrm{Spec}\,A_X^\wedge/I'$, where $I' := IA_X^\wedge$, and since I is open in A_X, the natural maps $I^n/I^{n+1} \to I'^n/I'^{n+1}$ are bijective for every $n \in \mathbb{N}$. Moreover, by assumption X is dense in Y; taking into account claim 15.4.29, we deduce that X' is dense in Y'. Pick any finitely generated ideal J of $A^\wedge_X$ whose radical is $A^{\wedge\circ\circ}_X$; in view of corollary 15.4.27(ii,iii), we see that

$$\bigcap_{v\in\mathrm{Spa}\,(A^\wedge_X,A^{\wedge+}_X,X')} \mathrm{Ker}\, v = \{a\in A^\wedge_X \mid |a|^*_J = 0\} \subset (A^\wedge_X)^{\circ\circ}.$$

But we have already noticed that $\mathscr{J}$ is contained in the support of every $v\in\mathrm{Spa}(A^\wedge_X, A^{\wedge+}_X, X')$ (see the proof of lemma 15.4.17(iii)), whence (i).

(ii) follows immediately from (i) and corollary 15.4.27(i) and lemma 15.4.17(iii) (details left to the reader; more directly, one can also argue as in the proof of theorem 16.5.13(ii)). □

Proposition 15.4.30. *Let $(X,\mathscr{T}_X,A^+_X)$ be any topologically local quasi-affinoid scheme, and $f\in\mathscr{O}_X(X)$ any element. The following conditions are equivalent :*

(a) $f\in\mathscr{O}_X(X)^\times$.
(b) $v(f)\neq 0$ *for every* $v\in\mathrm{Spa}(X,\mathscr{T}_X,A^+_X)$.

Proof. Clearly, it suffices to check that (b)⇒(a). Hence, suppose that (b) holds, and set

$$A_X := \mathscr{O}_X(X) \qquad Y:=\mathrm{Spec}\,A_X \qquad Y^{\circ\circ} := \mathrm{Spec}\,A_X/A^{\circ\circ}_X\cdot A_X.$$

Let $x\in X$ be any point; we need to show that the image $f(x)$ of f does not vanish in $\kappa(x)$. If $x\in Y^{\circ\circ}$, let v be the trivial valuation with support equal to x; clearly $v\in\mathrm{Spa}(X,\mathscr{T}_X,A^+_X)$, so (b) says that $v(f)\neq 0$, whence the contention, in this case.

Next, if $x\in X\setminus Y^{\circ\circ}$, we proceed as in the proof of proposition 15.4.22(iii.b) : we pick first a minimal specialization y of x in $X\setminus Y^{\circ\circ}$, then a maximal point $\mathfrak{p}$ of the non-empty closed subset $\{y\}^c\cap Y^{\circ\circ}$ of Y (where $\{y\}^c$ denotes the topological closure of $\{y\}$ in Y). The image C of $A_{X,\mathfrak{p}}$ in $\kappa(y)$ is a one-dimensional local domain, so we may find a valuation ring V of $\kappa(y)$ with value group of rank one, that dominates C (claim 15.3.27). Let $v\in\mathrm{Spv}\,A_X$ be the valuation corresponding to V; arguing as in *loc. cit.* we see that $v\in\mathrm{Spa}(X,\mathscr{T}_X,A^+_X)$, hence $v(f)\neq 0$, so $f(y)\neq 0$ and finally $f(x)\neq 0$ as well. □

15.5. **Adic spaces.** In this section we shall endow the adic spectra introduced in section 15.3 with certain natural presheaves of topological rings, and we shall show that, under suitable conditions, these presheaves are sheaves of rings or of topological rings. The first step is the following universal construction :

15.5.1. Let $\underline{A} := (A, A^+, U)$ be any quasi-affinoid ring, Λ a (small) set, $S:=(s_\lambda\mid\lambda\in\Lambda)$ a system of elements of A, and $T:=(T_\lambda\mid\lambda\in\Lambda)$ a family of subsets of A such that T_λ generates an open ideal of A, for every $\lambda\in\Lambda$. We consider the f-adic ring $A[X_\lambda\mid\lambda\in\Lambda]_T$ provided by proposition 8.3.37(ii), and its ideal J generated by the system $(1-s_\lambda X_\lambda\mid\lambda\in\Lambda)$. We set

$$A\Big(\frac{T}{S}\Big) := A[X_\bullet]_T/J$$

and we endow this ring with the quotient topology induced by $A[X_\bullet]_T$ via the natural projection. In other words, $A(\frac{T}{S})$ is an f-adic topological ring whose underlying A-algebra is naturally identified with the localization $A[s^{-1}_\lambda\mid\lambda\in\Lambda]$. From proposition 8.3.37(iii,iv) we see that – under this identification – the subset $\{t/s_\lambda\mid\lambda\in\Lambda,\ t\in T_\lambda\}$ is power bounded in $A(\frac{T}{S})$. Moreover, the localization map $h:A\to A(\frac{T}{S})$ is f-adic, and enjoys the following universal property. For every f-adic ring B and every continuous map $f:A\to B$ such that :

- $f(s_\lambda)\in B^\times$ for every $\lambda\in\Lambda$
- the subset $\{f(t)/f(s_\lambda)\mid\lambda\in\Lambda,\ t\in T_\lambda\}$ is power bounded in B

there exists a unique continuous map $g : A(\frac{T}{S}) \to B$ such that $g \circ h = f$. Furthermore, if $A_0 \subset A$ is a subring of definition, $A_0[\frac{t}{s_\lambda} \mid \lambda \in \Lambda,\ t \in T_\lambda]$ is a subring of definition of $A(\frac{T}{S})$.

Next, notice that $C := A^+[\frac{t}{s_\lambda} \mid \lambda \in \Lambda,\ t \in T_\lambda]$ is an open subring of $A(\frac{T}{S})^\circ$, and denote by C' the integral closure of C in $A(\frac{T}{S})$. Moreover, set $X := \mathrm{Spec}\, A[s_\lambda^{-1} | \lambda \in \Lambda]$, and notice that $U' := U \cap X$ contains the analytic locus of X, since h is f-adic (lemma 8.3.29(iv)). We obtain therefore a quasi-affinoid ring

$$\underline{A}\Big(\frac{T}{S}\Big) := \Big(A\Big(\frac{T}{S}\Big), C', U'\Big)$$

and the localization map yields a natural f-adic morphism of quasi-affinoid rings

$$\underline{A} \to \underline{A}\Big(\frac{T}{S}\Big). \tag{15.5.2}$$

In case $\Lambda = \{\lambda\}$ has only one element, $T_\lambda = \{f_0, \dots, f_n\}$ is a finite subset of A which generates an open ideal, and $S = \{s\}$ for some $s \in A$, we also denote this quasi-affinoid ring by :

$$\underline{A}\Big(\frac{f_0, \dots, f_n}{s}\Big).$$

15.5.3. We consider now the Yoneda embedding (see (1.2.4))

$$h : \mathsf{q.Afd.Sch}_{\mathrm{loc}} \to \mathsf{q.Afd.Sch}^\wedge_{\mathrm{loc}} \qquad \underline{X} \mapsto h_{\underline{X}}$$

(notation of remark 15.4.10(v)). Let $\underline{X}$ be any topologically local quasi-affinoid scheme, and $U \subset \mathrm{Spa}\, \underline{X}$ any open subset. We attach to U the sub-presheaf $h_U \subset h_{\underline{X}}$ given by the rule :

$$h_U(\underline{Y}) := \{\varphi \in h_{\underline{X}}(\underline{Y}) \mid \mathrm{Im}\, \mathrm{Spa}(\varphi) \subset U\} \qquad \text{for every } \underline{Y} \in \mathrm{Ob}(\mathsf{q.Afd.Sch}_{\mathrm{loc}}).$$

Definition 15.5.4. (i) In the situation of (15.5.3), we say that U is a *quasi-affinoid open subset* of $\mathrm{Spa}\, \underline{X}$, if the presheaf h_U is representable by some $\underline{Y} \in \mathrm{Ob}(\mathsf{q.Afd.Sch}_{\mathrm{loc}})$.

(ii) We say that U is an *affinoid open subset* of $\mathrm{Spa}\, \underline{X}$, if $\underline{Y}$ as in (i) is an affinoid scheme.

Theorem 15.5.5. *Every rational subset of* $\mathrm{Spa}\, \underline{X}$ *is a quasi-affinoid open subset.*

Proof. Let R be such a rational subset, and say that

$$\Gamma(\underline{X}) = (A_X, A_X^+, X) \qquad \text{and} \qquad R = R_{A_X}\Big(\frac{f_1}{f_0}, \dots, \frac{f_n}{f_0}\Big) \cap \mathrm{Spa}\, \underline{X}$$

for a sequence $(f_0, f_1, \dots, f_n)$ of elements of A_X that generates an open ideal. We consider the quasi-affinoid rings

$$\underline{A}_R := \Gamma(\underline{X})\Big(\frac{f_0, \dots, f_n}{f_0}\Big)$$

defined as in (15.5.1). Also, we let

$$i_R : \Gamma(\underline{X}) \to \underline{A}_{R,\mathrm{loc}}$$

be the composition of the natural morphism $\Gamma(\underline{X}) \to \underline{A}_R$ of (15.5.2) with the topological localization $\underline{A}_R \to \underline{A}_{R,\mathrm{loc}}$. Now, let $\underline{Y}$ be any topologically local quasi-affinoid scheme, and $\varphi \in h_R(\underline{Y})$ any element. Set $(A_Y, A_Y^+, Y) := \Gamma(\underline{Y})$, and let $\varphi^\flat : A_X \to A_Y$ be the continuous map associated with φ.

Claim 15.5.6. $\varphi^\flat(f_0) \in A_Y^\times$ and $\varphi^\flat(f_i)/\varphi^\flat(f_0) \in A_Y^+$ for every $i = 1, \dots, n$.

Proof of the claim. Indeed, for every $v \in \mathrm{Spa}\, \underline{Y}$ we have by assumption $v \circ \varphi^\flat \in R$, so that $v(\varphi^\flat(f_0)) \neq 0$ and $v(\varphi^\flat(f_i)) \leq v(\varphi^\flat(f_0))$ for $i = 1, \dots, n$. Then proposition 15.4.30 implies that $\varphi^\flat(f_0) \in A_Y^\times$, so we may write $v(\varphi^\flat(f_i)/\varphi^\flat(f_0)) \leq 1$ for every $i = 1, \dots, n$ and every $v \in \mathrm{Spa}\, \underline{Y}$. To conclude, it then suffices to invoke corollary 15.4.27(i). ◊

From claim 15.5.6, remark 8.3.9(ii) and the discussion of (15.5.1) we see that $\varphi^\flat$ factors uniquely through a continuous ring homomorphism $g : A_R \to A_Y$, and $g(f_i/f_0) \in A^+_Y$ for every $i = 1, \dots, n$. Thus, g defines a morphism of quasi-affinoid rings

$$g : \underline{A}_R \to \Gamma(\underline{Y})$$

and summing up, we easily conclude that φ factors uniquely through $\mathsf{Spec}(i_R)$ and the morphism of topologically local quasi-affinoid schemes

$$\psi := \mathsf{Spec}(g_{\mathrm{loc}}) : \underline{Y} \to \underline{R} := \mathsf{Spec}\,(\underline{A}_{R,\mathrm{loc}}).$$

Lastly, taking into account lemma 15.4.17(i) it is easily seen that $\mathrm{Spa}(i_R)$ is an injective map inducing a homeomorphism $\mathrm{Spa}\,\underline{R} \stackrel{\sim}{\to} R$, and the theorem follows. □

15.5.7. In the situation of (15.5.3), let $U \subset \mathrm{Spa}\,\underline{X}$ be a quasi-affinoid open subset, and $\underline{Y}$ any topologically local quasi-affinoid scheme that represents h_U. Then the morphism $\mathbf{1}_{\underline{Y}}$ corresponds to a well defined section $\varphi_{\underline{Y}/\underline{X}} \in h_U(\underline{Y})$, *i.e.* to a morphism $\varphi_{\underline{Y}/\underline{X}} : \underline{Y} \to \underline{X}$ of quasi-affinoid schemes with $\mathrm{Im}(\mathrm{Spa}\,\varphi_{\underline{Y}/\underline{X}}) \subset U$.

Corollary 15.5.8. *With the notation of* (15.5.7)*, the following holds :*

(i) *The map* $\mathrm{Spa}\,\varphi_{\underline{Y}/\underline{X}} : \mathrm{Spa}\,\underline{Y} \to \mathrm{Spa}\,\underline{X}$ *induces a homeomorphism* $\mathrm{Spa}\,\underline{Y} \stackrel{\sim}{\to} U$.

(ii) *Especially, every quasi-affinoid open subset is quasi-compact.*

(iii) *More precisely,* $\mathrm{Spa}\,\varphi_{\underline{Y}/\underline{X}}$ *induces homeomorphisms*

$$(\mathrm{Spa}\,\underline{Y})_{\mathrm{a}} \stackrel{\sim}{\to} U \cap \mathrm{Spa}(\underline{X})_{\mathrm{a}} \qquad (\mathrm{Spa}\,\underline{Y})_{\mathrm{na}} \stackrel{\sim}{\to} U \cap \mathrm{Spa}(\underline{X})_{\mathrm{na}}.$$

(iv) *The morphism* $\varphi_{\underline{Y}/\underline{X}}$ *is f-adic.*

Proof. (i): Let $(R_i \mid i \in I)$ be a system of rational subsets of $\mathrm{Spa}\,\underline{X}$ such that $U = \bigcup_{i \in I} R_i$, and for every $i, j \in I$ set $R_{ij} := R_i \cap R_j$ and choose topologically local quasi-affinoid schemes $\underline{Z}_i$ and $\underline{Z}_{ij}$ that represent h_{R_i} and $h_{R_{ij}}$. The inclusions $R_i \subset U$ and $R_{ij} \subset U$ are then represented by morphisms $\psi_i : \underline{Z}_i \to \underline{Y}$ and respectively $\psi_{ij} : \underline{Z}_{ij} \to \underline{Y}$ of quasi-affinoid schemes such that $\varphi_{\underline{Z}_i/\underline{X}} = \varphi_{\underline{Y}/\underline{X}} \circ \psi_i$ and $\varphi_{\underline{Z}_{ij}/\underline{X}} = \varphi_{\underline{Y}/\underline{X}} \circ \psi_{ij}$, for every $i, j \in I$. For every $i \in I$ set $U_i := \mathrm{Im}\,\mathrm{Spa}\,\psi_i \subset \mathrm{Spa}\,\underline{Y}$; it was observed in the proof of theorem 15.5.5 that $\mathrm{Spa}\,\varphi_{\underline{Z}_i/\underline{X}}$ induces a homeomorphism $\mathrm{Spa}\,\underline{Z}_i \stackrel{\sim}{\to} R_i$; it follows easily that $\mathrm{Spa}\,\psi_i$ and $\mathrm{Spa}\,\varphi_{\underline{Y}/\underline{X}}$ induce homeomorphisms $\mathrm{Spa}\,\underline{Z}_i \stackrel{\sim}{\to} U_i$ and respectively $U_i \stackrel{\sim}{\to} R_i$. It is also easily seen that $U_i \cap U_j = \mathrm{Im}\,\mathrm{Spa}\,\psi_{ij}$, so the map $\mathrm{Spa}\,\varphi_{\underline{Y}/\underline{X}}$ restricts to a homeomorphism $U' := \bigcup_{i \in I} U_i \stackrel{\sim}{\to} U$. To conclude, it suffices to check that $U' = \mathrm{Spa}\,\underline{Y}$. However, suppose that $v \in \mathrm{Spa}\,\underline{Y} \setminus U'$; we may find a rational subset R' of $\mathrm{Spa}\,\underline{Y}$ that contains v, and we may assume that $\mathrm{Spa}\,\varphi_{\underline{Y}/\underline{X}}(R') \subset R_i$ for some $i \in I$. Pick a topologically local quasi-affinoid scheme $\underline{Y}'$ that represents the subsheaf $h_{R'}$ of $h_{\underline{Y}}$, and let $\varphi' : \underline{Y}' \to \underline{Y}$ be the morphism of quasi-affinoid schemes that represents the inclusion $R' \subset \mathrm{Spa}\,\underline{Y}$. Then $\varphi_{\underline{Y}/\underline{X}} \circ \varphi' : \underline{Y}' \to \underline{X}$ lies in $h_{R_i}(\underline{Y}')$, so it corresponds to a unique morphism $\beta : \underline{Y}' \to \underline{Z}_i$ such that $\varphi_{\underline{Y}/\underline{X}} \circ \varphi' = \varphi_{\underline{Z}_i/\underline{X}} \circ \beta = \varphi_{\underline{Y}/\underline{X}} \circ \psi_i \circ \beta$. Since $\varphi_{\underline{Y}/\underline{X}}$ induces an injective morphism $h_{\underline{Y}} \to h_{\underline{X}}$ (whose image is h_U), we deduce that $\varphi' = \psi_i \circ \beta$. Especially, U_i contains the image of $\mathrm{Spa}\,\varphi'$; but we know that the latter coincides with R', a contradiction.

(ii) is an immediate consequence of (i).

(iii): By virtue of remark 15.3.13(vii), it suffices to show that $\mathrm{Spa}\,\varphi_{\underline{Y}/\underline{X}}$ maps $(\mathrm{Spa}\,\underline{Y})_{\mathrm{a}}$ into $(\mathrm{Spa}\,X)_{\mathrm{a}}$. Thus, let $v \in (\mathrm{Spa}\,\underline{Y})$, pick $i \in I$ such that $w := \mathrm{Spa}\,\varphi_{\underline{Y}/\underline{X}}(v) \in R_i$, and let $u := (\mathrm{Spa}\,\varphi_{\underline{Z}_i/\underline{X}})^{-1}(w)$. Suppose that $w \in (\mathrm{Spa}\,X)_{\mathrm{na}}$; by inspecting the proof of theorem 15.5.5, it is easily seen that $\varphi_{\underline{Z}_i/\underline{X}}$ is f-adic, so $u \in \mathrm{Spa}(\underline{Z}_i)_{\mathrm{na}}$, by corollary 15.3.17(ii). By the foregoing, we also know that $\mathrm{Spa}\,\psi_i(u) = v$, and then $v \in (\mathrm{Spa}\,\underline{Y})_{\mathrm{na}}$, again by remark 15.3.13(vii), whence the assertion.

(iv) follows immediately from (iii) and proposition 15.4.22(iii.b). □

Remark 15.5.9. (i) In the situation of (15.5.3), we can also consider the Yoneda imbeddings for the categories of topologically henselian and of complete, separated quasi-affinoid schemes

$$h' : \mathsf{q.Afd.Sch}_{\mathrm{hens}} \to \mathsf{q.Afd.Sch}^{\wedge}_{\mathrm{hens}} \qquad h'' : \mathsf{q.Afd.Sch}_{\mathrm{comp}} \to \mathsf{q.Afd.Sch}^{\wedge}_{\mathrm{comp}}$$

and if $\underline{X}$ is topologically henselian (resp. complete and separated) we set

$$h'_U(\underline{Y}') := h_U(\underline{Y}') \subset h'_{\underline{X}}(\underline{Y}') \qquad (\text{resp. } h''_U(\underline{Y}'') := h_U(\underline{Y}'') \subset h''_{\underline{X}}(\underline{Y}'')\ )$$

for every topologically henselian quasi-affinoid scheme $\underline{Y}'$ and every complete and separated quasi-affinoid scheme $\underline{Y}''$. Suppose that U is quasi-affinoid, and let $\varphi_{\underline{Y}/\underline{X}} : \underline{Y} \to \underline{X}$ be a morphism of topologically local quasi-affinoid schemes representing the sub-presheaf $h_U \subset h_{\underline{X}}$, as in (15.5.7); from remark 15.4.10(v) and lemma 15.4.17(i) we see that if $\underline{X}$ is topologically henselian, the topological henselization $\varphi^{\mathrm{h}}_{\underline{Y}/\underline{X}} : \underline{Y}^{\mathrm{h}} \to \underline{X}$ of $\varphi_{\underline{Y}/\underline{X}}$ represents h'_U. Likewise, taking into account proposition 15.4.12 and lemma 15.4.17(iii), we conclude that if $\underline{X}$ is complete and separated, the completion $\varphi^{\wedge}_{\underline{Y}/\underline{X}} : \underline{Y}^{\wedge} \to \underline{X}$ of $\varphi_{\underline{Y}/\underline{X}}$ represents h''_U.

(ii) Moreover, we may consider the Yoneda imbedding for the opposite of the category of topologically local affinoid rings :

$$h''' : \mathsf{Afd.Ring}^o_{\mathrm{loc}} \to \mathsf{Afd.Ring}^{o\wedge}_{\mathrm{loc}}$$

and if $\underline{A} := (A, A^+)$ is any topologically local affinoid ring, and $U \subset \mathrm{Spa}\,\underline{A}$ any open subset, we may form again the sub-presheaf $h'''_U :\subset h'''_{\underline{A}}$ by the rule : $h'''_U(\underline{B}) := h_U(\mathsf{Spec}\,\underline{B})$ for every topologically local affinoid ring $\underline{B}$. We may regard U as an open subset of $\mathrm{Spa}(\mathsf{Spec}\,\underline{A})$, and we say that U is a quasi-affinoid open subset of $\mathrm{Spa}\,\underline{A}$ if it is such, as an open subset of $\mathrm{Spa}\,\mathsf{Spec}\,\underline{A}$. Since Spec is fully faithful on the full subcategory $\mathsf{Afd.Ring}$, it follows that if U is quasi-affinoid, h'''_U is representable : namely, a representing affinoid ring is $\Gamma(\underline{Y})$, where $\underline{Y}$ is any topologically local quasi-affinoid scheme that represents the presheaf h_U. Lastly, we may in the same way consider variants for topologically henselian (resp. complete and separated) affinoid rings : the reader may spell out the details. (However, the method does not produce representing objects on the whole category of quasi-affinoid rings, essentially because there is – in this generality – no counterpart of proposition 15.4.30 for quasi-affinoid rings).

(iii) It follows from lemma 15.4.6 and corollary 15.5.8(iv) that the intersection of two quasi-affinoid open subsets $U, U' \subset \mathrm{Spa}\,\underline{X}$ is quasi-affinoid. Indeed, if $\underline{Y}$ and $\underline{Y}'$ are topologically local quasi-affinoid schemes that represent h_U and respectively $h_{U'}$, it is easily seen that

$$(\underline{Y} \times_{\underline{X}} \underline{Y}')_{\mathrm{loc}}$$

represents $h_U \cap h_{U'} = h_{U \cap U'}$ (see example 15.4.8).

(iv) In the same vein, if $f : \underline{X}' \to \underline{X}$ is any f-adic morphism of topologically local quasi-affinoid schemes, and $U \subset \mathrm{Spa}\,\underline{X}$ is any quasi-affinoid open subset, then $U' := (\mathrm{Spa}\,f)^{-1}U$ is a quasi-affinoid open subset of $\mathrm{Spa}\,\underline{X}'$. Indeed, if $\underline{Y}$ represents h_U, then $h_{U'}$ is represented by $(\underline{Y} \times_{\underline{X}} \underline{X}')_{\mathrm{loc}}$, where the fibre product is given as in example 15.4.8.

(v) Lastly, let U be a quasi-affinoid open subset of $\mathrm{Spa}\,\underline{X}$ such that the inclusion $h_U \subset h_{\underline{X}}$ is represented by the morphism of topologically local quasi-affinoid schemes $\varphi_{\underline{Y}/\underline{X}} : \underline{Y} \to \underline{X}$, and V a quasi-affinoid open subset of $\mathrm{Spa}\,\underline{Y}$ such that the inclusion $h_V \subset h_{\underline{Y}}$ is represented by the morphism of topologically local affinoid schemes $\varphi_{\underline{Z}/\underline{Y}} : \underline{Z} \to \underline{Y}$, and set $U' := \mathrm{Spa}\,\varphi_{\underline{Y}/\underline{X}}(V) \subset U$. Then $\varphi_{\underline{Y}/\underline{X}}$ induces an isomorphism $h_{\underline{Y}} \xrightarrow{\sim} h_U$ that identifies h_V with the sub-presheaf $h_{U'}$ of h_U. Especially, U' is a quasi-affinoid open subset of $\mathrm{Spa}\,X$, and the inclusion $h_{U'} \subset h_{\underline{X}}$ is represented by $\varphi_{\underline{Y}/\underline{X}} \circ \varphi_{\underline{Z}/\underline{Y}}$.

Lemma 15.5.10. *In the situation of remark* 15.5.9(v)*, suppose furthermore that U is a rational subset of* $\mathrm{Spa}\,\underline{X}$ *and V is a rational subset of* $\mathrm{Spa}\,\underline{Y}$*. Then V is a rational subset of* $\mathrm{Spa}\,\underline{X}$.

Proof. Say that $\Gamma(\underline{X}) = (A, A^+, X)$, and $U = R_A(\frac{f_1}{f_0}, \dots, \frac{f_n}{f_0}) \cap \mathrm{Spa}\,\underline{X}$, with $(f_0, \dots, f_n)$ a sequence of elements of A that generates an open ideal. Set $(A_U, A_U^+) := \underline{A}(\frac{f_1}{f_0}, \dots, \frac{f_n}{f_0})$, where $\underline{A} := (A, A^+)$, so that $A_U = A[f_0^{-1}]$. By inspecting the proof of theorem 15.5.5, we get an isomorphism of quasi-affinoid schemes

$$\underline{Y} \xrightarrow{\sim} \mathsf{Spec}\,(A_U, A_U^+, X \cap \mathrm{Spec}\,A_U)_{\mathrm{loc}}$$

Since X is quasi-compact, and $\mathrm{Spec}\,A_U$ is an open affine subset of $\mathrm{Spec}\,A$, we have $\mathscr{O}_X(X \cap \mathrm{Spec}\,A_U) = \mathscr{O}_X(X)[f_0^{-1}] = A_U$ ([59, Ch.I, Prop.9.2.1]). Taking into account (15.4.11), we deduce that $\Gamma(\underline{Y}) = (B, B^+, X \cap \mathrm{Spec}\,A_U)$, where $B := A_{U,\mathrm{loc}}$ and $B^+ := A^+_{U,\mathrm{loc}}$. Consequently, we have $V = R_B(\frac{g_1}{g_0}, \dots, \frac{g_m}{g_0}) \cap \mathrm{Spa}\,\underline{Y}$ for a sequence $g_\bullet := (g_0, \dots, g_m)$ of elements of B that generates an open ideal J. Since B is a localization of A, we may also find $b \in B^\times$, such that $bg_i \in A' := \mathrm{Im}(A \to B)$; after replacing $g_\bullet$ by the sequence $(bg_0, \dots, bg_n)$, we may then assume that $g_0, \dots, g_m \in A'$, and for every $i = 0, \dots, m$ we pick $h_i \in A$ whose image in B agrees with g_i. Moreover, let $A_0 \subset A$ be a ring of definition, and $I \subset A_0$ a finitely generated ideal of adic definition; according to claim 15.4.20(i), there exists $k \in \mathbb{N}$ such that $v(a) < v(h_0)$ for every $a \in I^k$ and every $v \in V$. Pick a finite system $x_1, \dots, x_r$ of generators for I^k and set $h_{i+m} := x_i$ for every $i = 1, \dots, r$; by construction, we easily see that

$$V = U \cap R \qquad \text{where } R := R_A\Big(\frac{h_1}{h_0}, \dots, \frac{h_{m+r}}{h_0}\Big)$$

and the system $(h_i \mid i = 1, \dots, m + r)$ generates an open ideal of A, so R is a rational subset of $\mathrm{Spa}\,\underline{X}$, and finally, the same holds for V. $\square$

15.5.11. Let $\underline{X} := (X, \mathscr{T}_X, A_X^+)$ be any topologically local quasi-affinoid scheme; in view of remark 15.5.9(iii) we may associate with $\underline{X}$ a site

$$(\mathscr{Q}(\underline{X}), J_{\mathscr{Q}})$$

where $\mathscr{Q}(\underline{X})$ is the category of all quasi-affinoid open subsets of $\mathrm{Spa}\,\underline{X}$; for every $U, U' \in \mathrm{Ob}(\mathscr{Q}(\underline{X}))$, the set $\mathrm{Hom}_{\mathscr{Q}(\underline{X})}(U, U')$ contains exactly one morphism if $U \subset U'$, and is empty otherwise. For every such U, the sieves covering U for the topology $J_{\mathscr{Q}}$ are precisely those generated by the families $(U_\lambda \mid \lambda \in \Lambda)$ of quasi-affinoid open subsets of $\mathrm{Spa}\,\underline{X}$ such that $\bigcup_{\lambda\in\Lambda} U_\lambda = U$. We consider nine presheaves of topological $\mathscr{O}_X(X)$-algebras on $\mathscr{Q}(\underline{X})$, related by natural morphisms of presheaves :

$$\begin{array}{ccccc}
\mathscr{O}^{\mathrm{loc}+}_{\mathrm{Spa}\,\underline{X}} & \hookrightarrow & \mathscr{O}^{\mathrm{loc}\,\circ}_{\mathrm{Spa}\,\underline{X}} & \hookrightarrow & \mathscr{O}^{\mathrm{loc}}_{\mathrm{Spa}\,\underline{X}} \\
\downarrow & & \downarrow & & \downarrow \\
\mathscr{O}^{\mathrm{h}+}_{\mathrm{Spa}\,\underline{X}} & \hookrightarrow & \mathscr{O}^{\mathrm{h}\,\circ}_{\mathrm{Spa}\,\underline{X}} & \hookrightarrow & \mathscr{O}^{\mathrm{h}}_{\mathrm{Spa}\,\underline{X}} \\
\downarrow & & \downarrow & & \downarrow \\
\mathscr{O}^{\wedge+}_{\mathrm{Spa}\,\underline{X}} & \hookrightarrow & \mathscr{O}^{\wedge\circ}_{\mathrm{Spa}\,\underline{X}} & \hookrightarrow & \mathscr{O}^{\wedge}_{\mathrm{Spa}\,\underline{X}}
\end{array}$$

and constructed as follows. By theorem 15.5.5, for every $U \in \mathrm{Ob}(\mathscr{Q}(\underline{X}))$ the sub-presheaf h_U of $h_{\underline{X}}$ is representable (notation of (15.5.3)); we choose a topologically local quasi-affinoid scheme $\underline{X}_U := (X_U, \mathscr{T}_U, A_U^+)$ that represents this presheaf, and we set

$$\mathscr{O}^{\mathrm{loc}}_{\mathrm{Spa}\,\underline{X}}(U) := \mathscr{O}_{X_U}(X_U) \qquad \mathscr{O}^{\mathrm{loc}\,\circ}_{\mathrm{Spa}\,\underline{X}}(U) := \mathscr{O}_{X_U}(X_U)^\circ \qquad \mathscr{O}^{\mathrm{loc}+}_{\mathrm{Spa}\,\underline{X}}(U) := A_U^+.$$

Any inclusion $U \subset U'$ of quasi-affinoid open subsets induces a morphism $h_U \to h_{U'}$ of presheaves, which is represented by a unique morphism $\underline{X}_U \to \underline{X}_{U'}$ of topologically local quasi-affinoid schemes (see remark 1.2.8(i)), whence well defined continuous homomorphisms

$$\mathscr{O}^{\mathrm{loc}+}_{\mathrm{Spa}\,\underline{X}}(U') \to \mathscr{O}^{\mathrm{loc}+}_{\mathrm{Spa}\,\underline{X}}(U) \qquad \mathscr{O}^{\mathrm{loc}}_{\mathrm{Spa}\,\underline{X}}(U') \to \mathscr{O}^{\mathrm{loc}}_{\mathrm{Spa}\,\underline{X}}(U)$$

of topological $\mathscr{O}_X(X)$-algebras, and clearly the rules $U \mapsto \mathscr{O}^{\mathrm{loc}+}_{\mathrm{Spa}\,\underline{X}}(U)$ and $U \mapsto \mathscr{O}^{\mathrm{loc}}_{\mathrm{Spa}\,\underline{X}}(U)$ yield two functors from $\mathscr{Q}(\underline{X})^o$ to the category of topologically local f-adic $\mathscr{O}_X(X)$-algebras. Likewise, in view of corollary 15.5.8(iv) and lemma 8.3.24(iii.a) we also get a continuous map $\mathscr{O}^{\mathrm{loc}\,\circ}_{\mathrm{Spa}\,\underline{X}}(U') \to \mathscr{O}^{\mathrm{loc}\,\circ}_{\mathrm{Spa}\,\underline{X}}(U)$, whence the presheaf $\mathscr{O}^{\mathrm{loc}\,\circ}_{\mathrm{Spa}\,\underline{X}}$.

Similarly, let $\underline{X}^{\mathrm{h}}_U := (X^{\mathrm{h}}_U, \mathscr{T}^{\mathrm{h}}_U, A^{\mathrm{h}+}_U)$ (resp. $\underline{X}^{\wedge}_U := (X^{\wedge}_U, \mathscr{T}^{\wedge}_U, A^{\wedge+}_U)$) be a topologically henselian (resp. complete and separated) quasi-affinoid scheme representing the sub-presheaf h'_U of $h'_{\underline{X}^{\mathrm{h}}}$ (resp. h''_U of $h''_{\underline{X}^{\wedge}}$) as in remark 15.5.9(i); we define

$$\mathscr{O}^{\mathrm{h}}_{\mathrm{Spa}\,\underline{X}}(U) := \mathscr{O}_{X^{\mathrm{h}}_U}(X^{\mathrm{h}}_U) \qquad \mathscr{O}^{\mathrm{h}\,\circ}_{\mathrm{Spa}\,\underline{X}}(U) := \mathscr{O}_{X^{\mathrm{h}}_U}(X^{\mathrm{h}}_U)^{\circ} \qquad \mathscr{O}^{\mathrm{h}+}_{\mathrm{Spa}\,\underline{X}}(U) := A^{\mathrm{h}+}_U$$
$$\mathscr{O}^{\wedge}_{\mathrm{Spa}\,\underline{X}}(U) := \mathscr{O}_{X^{\wedge}_U}(X^{\wedge}_U) \qquad \mathscr{O}^{\wedge\circ}_{\mathrm{Spa}\,\underline{X}}(U) := \mathscr{O}_{X^{\wedge}_U}(X^{\wedge}_U)^{\circ} \qquad \mathscr{O}^{\wedge+}_{\mathrm{Spa}\,\underline{X}}(U) := A^{\wedge+}_U$$

and arguing as in the foregoing, it is easily seen that these rules yield well defined presheaves of topologically henselian (resp. complete and separated) f-adic $\mathscr{O}_X(X)$-algebras. All these presheaves depend on the choices of representing quasi-affinoid schemes, but for any two such sets of choices there exists a unique isomorphism of presheaves of topological $\mathscr{O}_X(X)$-algebras between the corresponding presheaves. We notice as well that the rule

$$U \mapsto \mathscr{O}^{\mathrm{loc}}_{\mathrm{Spa}\underline{X}}(U)^{\circ\circ}$$

defines a sub-presheaf $\mathscr{O}^{\mathrm{loc}\,\circ\circ}_{\mathrm{Spa}\,\underline{X}} \subset \mathscr{O}^{\mathrm{loc}}_{\mathrm{Spa}\,\underline{X}}$ that is a presheaf of ideals in both $\mathscr{O}^{\mathrm{loc}+}_{\mathrm{Spa}\,\underline{X}}$ and $\mathscr{O}^{\mathrm{loc}\,\circ}_{\mathrm{Spa}\,\underline{X}}$. Likewise we get presheaves of topologically nilpotent sections $\mathscr{O}^{\mathrm{h}\,\circ\circ}_{\mathrm{Spa}\,\underline{X}} \subset \mathscr{O}^{\mathrm{h}}_{\mathrm{Spa}\,\underline{X}}$ (resp. $\mathscr{O}^{\wedge\,\circ\circ}_{\mathrm{Spa}\,\underline{X}} \subset \mathscr{O}^{\wedge}_{\mathrm{Spa}\,\underline{X}}$) that are presheaves of ideals in $\mathscr{O}^{\mathrm{h}\,\circ}_{\mathrm{Spa}\,\underline{X}}$ and $\mathscr{O}^{\mathrm{h}+}_{\mathrm{Spa}\,\underline{X}}$ (resp. in $\mathscr{O}^{\wedge\circ}_{\mathrm{Spa}\,\underline{X}}$ and $\mathscr{O}^{\wedge+}_{\mathrm{Spa}\,\underline{X}}$).

15.5.12. With the notation of (15.5.11), let $x \in \mathrm{Spa}\,\underline{X}$ be any point; recall that x is the equivalence class of a continuous valuation $v_x : A := \mathscr{O}_X(X) \to \Gamma_x$; if $R \subset \mathrm{Spa}\,\underline{X}$ is any rational subset containing x, the valuation v_x extends uniquely to a continuous valuation $v^{\wedge}_{x,R}$ on $\mathscr{O}^{\wedge}_{\mathrm{Spa}\,\underline{X}}(R)$ with value group Γ_x (claim 15.3.20(i)). Then, by restricting along the natural maps $\mathscr{O}^{\mathrm{loc}}_{\mathrm{Spa}\,\underline{X}}(R) \to \mathscr{O}^{\mathrm{h}}_{\mathrm{Spa}\,\underline{X}}(R) \to \mathscr{O}^{\wedge}_{\mathrm{Spa}\,\underline{X}}(R)$ we get corresponding valuations $v^{\mathrm{h}}_{x,R}$, $v^{\mathrm{loc}}_{x,R}$, and – after taking colimits – unique valuations on the stalks of these presheaves

(15.5.13)
$$\begin{array}{ccccc} \mathscr{O}^{\mathrm{loc}}_{\mathrm{Spa}\,\underline{X},x} & \longrightarrow & \mathscr{O}^{\mathrm{h}}_{\mathrm{Spa}\,\underline{X},x} & \longrightarrow & \mathscr{O}^{\wedge}_{\mathrm{Spa}\,\underline{X},x} \\ & {\scriptstyle |\cdot|^{\mathrm{loc}}_x} \searrow & \downarrow{\scriptstyle |\cdot|^{\mathrm{h}}_x} & \swarrow {\scriptstyle |\cdot|^{\wedge}_x} & \\ & & \Gamma_x. & & \end{array}$$

We denote by $\kappa(x^{\mathrm{loc}})$, $\kappa(x^{\mathrm{h}})$ and $\kappa(x^{\wedge})$ the residue fields of $|\cdot|^{\mathrm{loc}}_x$, $|\cdot|^{\mathrm{h}}_x$ and respectively $|\cdot|^{\wedge}_x$, and recall that the valuations of (15.5.13) induce natural residual valuations on these fields : see remark 9.1.4(v). There follow natural inclusion of valued fields

$$\kappa(x^{\mathrm{loc}}) \to \kappa(x^{\mathrm{h}}) \to \kappa(x^{\wedge})$$

and taking into account claim 15.3.20(ii) we see that the images of $\kappa(x^{\mathrm{loc}})$ and of $\kappa(x^{\mathrm{h}})$ in $\kappa(x^{\wedge})$ are both dense for the valuation topology of the latter field. Especially, these inclusions induce natural identifications of the completions of these fields for their respective valuation topologies (theorem 8.2.8(iii)), and we denote this common completion by

$$(\kappa(x)^{\wedge}, |\cdot|^{\wedge}_x)$$

which is a valued field with value group Γ_x, by proposition 9.1.16(iii,v).

Lemma 15.5.14. *In the situation of* (15.5.12)*, we have :*

(i) *The stalks* $\mathscr{O}^{\wedge}_{\mathrm{Spa}\,\underline{X},x}$, $\mathscr{O}^{\mathrm{h}}_{\mathrm{Spa}\,\underline{X},x}$ *and* $\mathscr{O}^{\mathrm{loc}}_{\mathrm{Spa}\,\underline{X},x}$ *are local rings, and* $\kappa(x^{\mathrm{loc}})$, $\kappa(x^{\mathrm{h}})$, $\kappa(x^{\wedge})$ *are their respective residue fields.*

(ii) *The stalks* $\mathscr{O}^{\wedge+}_{\mathrm{Spa}\,\underline{X},x}$, $\mathscr{O}^{\mathrm{h}+}_{\mathrm{Spa}\,\underline{X},x}$ *and* $\mathscr{O}^{\mathrm{loc}+}_{\mathrm{Spa}\,\underline{X},x}$ *are local rings, and the induced diagrams*

$$\begin{array}{ccc} \mathscr{O}^{\mathrm{loc}+}_{\mathrm{Spa}\,\underline{X},x} & \longrightarrow & \mathscr{O}^{\mathrm{loc}}_{\mathrm{Spa}\,\underline{X},x} \\ \downarrow & & \downarrow{\scriptstyle \pi^{\mathrm{loc}}_x} \\ \kappa(x^{\mathrm{loc}})^+ & \longrightarrow & \kappa(x^{\mathrm{loc}}) \end{array} \qquad \begin{array}{ccc} \mathscr{O}^{\mathrm{h}+}_{\mathrm{Spa}\,\underline{X},x} & \longrightarrow & \mathscr{O}^{\mathrm{h}}_{\mathrm{Spa}\,\underline{X},x} \\ \downarrow & & \downarrow{\scriptstyle \pi^{\mathrm{h}}_x} \\ \kappa(x^{\mathrm{h}})^+ & \longrightarrow & \kappa(x^{\mathrm{h}}) \end{array} \qquad \begin{array}{ccc} \mathscr{O}^{\wedge+}_{\mathrm{Spa}\,\underline{X},x} & \longrightarrow & \mathscr{O}^{\wedge}_{\mathrm{Spa}\,\underline{X},x} \\ \downarrow & & \downarrow{\scriptstyle \pi^{\wedge}_x} \\ \kappa(x^{\wedge})^+ & \longrightarrow & \kappa(x^{\wedge}) \end{array}$$

are cartesian (here $\kappa(x^{\mathrm{loc}})^+$ *is the valuation ring of the residual valuation on* $\kappa(x^{\mathrm{loc}})$, *and likewise for* $\kappa(x^{\mathrm{h}})^+$ *and* $\kappa(x^{\wedge})^+$*).*

(iii) *The stalk* $\mathscr{O}^{\mathrm{loc}\,\circ\circ}_{\mathrm{Spa}\,\underline{X},x}$ *(resp.* $\mathscr{O}^{\mathrm{h}\,\circ\circ}_{\mathrm{Spa}\,\underline{X},x}$, *resp.* $\mathscr{O}^{\wedge\,\circ\circ}_{\mathrm{Spa}\,\underline{X},x}$*) is the radical of the ideal* $A^{\circ\circ}\cdot\mathscr{O}^{\mathrm{loc}+}_{\mathrm{Spa}\,\underline{X},x}$ *of* $\mathscr{O}^{\mathrm{loc}+}_{\mathrm{Spa}\,\underline{X},x}$ *(resp.* $A^{\circ\circ}\cdot\mathscr{O}^{\mathrm{h}+}_{\mathrm{Spa}\,\underline{X},x}$ *of* $\mathscr{O}^{\mathrm{h}+}_{\mathrm{Spa}\,\underline{X},x}$, *resp.* $A^{\circ\circ}\cdot\mathscr{O}^{\wedge+}_{\mathrm{Spa}\,\underline{X},x}$ *of* $\mathscr{O}^{\wedge+}_{\mathrm{Spa}\,\underline{X},x}$*).*

(iv) *The pairs* $(\mathscr{O}^{\mathrm{h}+}_{\mathrm{Spa}\,\underline{X},x},\mathscr{O}^{\mathrm{h}\,\circ\circ}_{\mathrm{Spa}\,\underline{X},x})$ *and* $(\mathscr{O}^{\wedge+}_{\mathrm{Spa}\,\underline{X},x},\mathscr{O}^{\wedge\,\circ\circ}_{\mathrm{Spa}\,\underline{X},x})$ *are henselian.*

(v) *Suppose that* x *is analytic, and let*

$$\mathfrak{p}^{\mathrm{loc}}\subset\kappa(x^{\mathrm{loc}})^+ \qquad \mathfrak{p}^{\mathrm{h}}\subset\kappa(x^{\mathrm{h}})^+ \qquad \mathfrak{p}^{\wedge}\subset\kappa(x^{\wedge})^+$$

be the (unique) prime ideals of height one. Then :

(a) $\mathscr{O}^{\mathrm{loc}\,\circ\circ}_{\mathrm{Spa}\,\underline{X},x}=\pi^{\mathrm{loc}\,-1}_x(\mathfrak{p}^{\mathrm{loc}})$ $\qquad \mathscr{O}^{\mathrm{h}\,\circ\circ}_{\mathrm{Spa}\,\underline{X},x}=\pi^{\mathrm{h}\,-1}_x(\mathfrak{p}^{\mathrm{h}})$ $\qquad \mathscr{O}^{\wedge\circ\circ}_{\mathrm{Spa}\,\underline{X},x}=\pi^{\wedge\,-1}_x(\mathfrak{p}^{\wedge})$.

(b) *The pairs* $(\kappa(x^{\mathrm{h}})^+,\mathfrak{p}^{\mathrm{h}})$ *and* $(\kappa(x^{\wedge})^+,\mathfrak{p}^{\wedge})$ *are henselian.*

(c) *The local rings* $\mathscr{O}^{\mathrm{h}}_{\mathrm{Spa}\,\underline{X},x}$ *and* $\mathscr{O}^{\wedge}_{\mathrm{Spa}\,\underline{X},x}$ *are henselian.*

(d) *The valuation rings* $\kappa(x^{\mathrm{h}})^+_{\mathfrak{p}^{\mathrm{h}}}$ *and* $\kappa(x^{\wedge})^+_{\mathfrak{p}^{\wedge}}$ *are henselian.*

(vi) *Suppose that* x *is non-analytic. Then :*

(a) $\mathscr{O}^{\mathrm{loc}\,\circ\circ}_{\mathrm{Spa}\,\underline{X},x}$ *(resp.* $\mathscr{O}^{\mathrm{h}\,\circ\circ}_{\mathrm{Spa}\,\underline{X},x}$, *resp.* $\mathscr{O}^{\wedge\,\circ\circ}_{\mathrm{Spa}\,\underline{X},x}$*) is also an ideal of* $\mathscr{O}^{\mathrm{loc}}_{\mathrm{Spa}\,\underline{X},x}$ *(resp. of* $\mathscr{O}^{\mathrm{h}}_{\mathrm{Spa}\,\underline{X},x}$, *resp. of* $\mathscr{O}^{\wedge}_{\mathrm{Spa}\,\underline{X},x}$*), and it is the radical of the ideal generated by* $A^{\circ\circ}$.

(b) *The pairs* $(\mathscr{O}^{\mathrm{h}}_{\mathrm{Spa}\,\underline{X},x},\mathscr{O}^{\mathrm{h}\,\circ\circ}_{\mathrm{Spa}\,\underline{X},x})$ *and* $(\mathscr{O}^{\wedge}_{\mathrm{Spa}\,\underline{X},x},\mathscr{O}^{\wedge\circ\circ}_{\mathrm{Spa}\,\underline{X},x})$ *are henselian.*

Proof. Let $f\in\mathscr{O}^{\wedge}_{\mathrm{Spa}\,\underline{X},x}$, and pick any rational subset R in $\mathrm{Spa}\underline{X}$ with $x\in R$, and such that f extends to a section $f_R\in B:=\mathscr{O}^{\wedge}_{\mathrm{Spa}\,\underline{X}}(R)$. Let $\underline{Y}^{\wedge}:=(Y^{\wedge},\mathscr{T}^{\wedge}_Y,A^{\wedge+}_Y)$ be a complete and separated quasi-affinoid scheme representing the subsheaf h''_R of $h''_{\underline{X}^{\wedge}}$, so that $B=\mathscr{O}_{Y^{\wedge}}(Y^{\wedge})$.

(i): Suppose now that $|f|^{\wedge}_x\neq 0$; then clearly $v^{\wedge}_{x,R}(f_R)\neq 0$. We set $U:=R_B\big(\frac{f_R}{f_R}\big)\cap\mathrm{Spa}\,\underline{Y}^{\wedge}$. In view of lemma 15.4.17(iii) and corollary 15.5.8(i), we may then identify U with an open subset of $\mathrm{Spa}\,X$ containing x, and we pick a rational subset $U'\subset U$ of $\mathrm{Spa}\,\underline{X}$ with $x\in U'$; by construction the image of f_R under the restriction map $\rho_{U'}:\mathscr{O}^{\wedge}_{\mathrm{Spa}\,\underline{X}}(R)\to\mathscr{O}^{\wedge}_{\mathrm{Spa}\,\underline{X}}(U')$ is an invertible element. This shows that $\mathrm{Ker}\,|\cdot|^{\wedge}_x$ is the unique maximal ideal of $\mathscr{O}^{\wedge}_{\mathrm{Spa}\,\underline{X},x}$, and it also easily implies that the projection $\pi^{\wedge}_x:\mathscr{O}^{\wedge}_{\mathrm{Spa}\,\underline{X},x}\to\kappa(x^{\wedge})$ is surjective (details left to the reader). The same argument applies to $\mathscr{O}^{\mathrm{h}}_{\mathrm{Spa}\,\underline{X},x}$ and $\mathscr{O}^{\mathrm{loc}}_{\mathrm{Spa}\,\underline{X},x}$, whence the contention.

(ii): Suppose next that $|f|^{\wedge}_x\leq 1$. Then we let $U:=R_B\big(\frac{f_R}{1}\big)\cap\mathrm{Spa}\,\underline{Y}^{\wedge}$, which again we identify naturally with an open subset of $\mathrm{Spa}\,\underline{X}$ containing x, and we pick a rational subset $U'\subset U$ of $\mathrm{Spa}\,\underline{X}$ with $x\in U'$; then $\rho_{U'}(f_R)\in\mathscr{O}^{\wedge+}_{\mathrm{Spa}\,\underline{X}}(U')$. This shows that $f\in\mathscr{O}^{\wedge+}_{\mathrm{Spa}\,\underline{X},x}$, whence the assertion concerning the third square diagram. The same argument applies to the other two diagrams. It follows easily that the preimage in $\mathscr{O}^{\wedge+}_{\mathrm{Spa}\,\underline{X},x}$ of the maximal ideal of $\kappa(x^{\wedge})^+$ is the unique maximal ideal of $\mathscr{O}^{\wedge+}_{\mathrm{Spa}\,\underline{X},x}$ (details left to the reader); especially, the latter is a local ring. Again, the same applies to the other two rings.

(iii): It follows easily from corollary 15.5.8(iv) that the natural map $A\to\mathscr{O}^{\mathrm{loc}}_{\mathrm{Spa}\,\underline{X}}(U)$ is f-adic, for every rational subset $U\subset\mathrm{Spa}\,\underline{X}$. Hence, $A^{\circ\circ}\cdot\mathscr{O}^{\mathrm{loc}+}_{\mathrm{Spa}\underline{X}}(U)$ is an open ideal of $\mathscr{O}^{\mathrm{loc}+}_{\mathrm{Spa}\underline{X}}(U)$ consisting of topologically nilpotent section, and therefore its radical equals $\mathscr{O}^{\mathrm{loc}\,\circ\circ}_{\mathrm{Spa}\underline{X}}(U)$. It suffices then to observe that the radical of $A^{\circ\circ}\cdot\mathscr{O}^{\mathrm{loc}+}_{\mathrm{Spa}\underline{X},x}$ is the colimit of the filtered system of the radicals of the ideals $(A^{\circ\circ}\cdot\mathscr{O}^{\mathrm{loc}+}_{\mathrm{Spa}\underline{X}}(U)\mid x\in U)$. The same applies to the other ideals in (iii).

(iv): By proposition 8.4.2(iii), both $(\mathscr{O}^{\mathrm{h}+}_{\mathrm{Spa}\,\underline{X}}(U),\mathscr{O}^{\mathrm{h}\,\circ\circ}_{\mathrm{Spa}\,\underline{X}}(U))$ and $(\mathscr{O}^{\wedge+}_{\mathrm{Spa}\,\underline{X}}(U),\mathscr{O}^{\wedge\circ\circ}_{\mathrm{Spa}\,\underline{X}}(U))$ are henselian pairs for every rational subset $U\subset\mathrm{Spa}\,\underline{X}$ containing x. The assertion follows, after taking colimits over the filtered family of such rational subsets.

(v.a): First, recall that the existence of a rank one prime ideal $\mathfrak{p}^{\mathrm{loc}}\subset\kappa(x^{\mathrm{loc}})^+$ is ensured by lemma 15.3.14(v); it follows easily that $\mathfrak{p}^{\mathrm{loc}}=\kappa(x^{\mathrm{loc}})^{\circ\circ}$, and taking into account remark 15.3.13(i) we deduce already that $\pi^{\mathrm{loc}}_x(\mathscr{O}^{\mathrm{loc}\,\circ\circ}_{\mathrm{Spa}\,\underline{X},x})\subset\mathfrak{p}^{\mathrm{loc}}$. Conversely, let $f\in\mathscr{O}^{\mathrm{loc}}_{\mathrm{Spa}\,\underline{X},x}\setminus\mathscr{O}^{\mathrm{loc}\,\circ\circ}_{\mathrm{Spa}\,\underline{X},x}$ be any element; denote by $\mathscr{U}$ the set of all rational subsets of $\mathrm{Spa}\,\underline{X}$ containing x, and for every $U\in\mathscr{U}$ set $U':=U\cap R_A\big(\frac{1}{f}\big)$. The assumption on f implies that U' is a non-empty constructible subset of $\mathrm{Spa}\,\underline{X}$ for every $U\in\mathscr{U}$ (corollary 15.4.27(ii)), and therefore $T:=\bigcap_{U\in\mathscr{U}}U'\neq\varnothing$ (proposition 8.1.23(ii.a)). However, T lies in the set $\mathrm{Spa}\,\underline{X}(x)$ of all generization of x in $\mathrm{Spa}\,\underline{X}$ (remark 8.1.45(i)), and must then contain the unique maximal generization of x; the latter means precisely that $\pi^{\mathrm{loc}}_x(f)\notin\mathfrak{p}^{\mathrm{loc}}$, as required.

Next, since $|\cdot|^{\mathrm{loc}}_x$ and $|\cdot|^{\wedge}_x$ have the same value group, the existence of a (unique) height one prime ideal $\mathfrak{p}^{\wedge}\subset\kappa(v^{\wedge})^+$ follows from the existence of $\mathfrak{p}^{\mathrm{loc}}$ and remark 9.1.13(vii); then, since $v^{\wedge}_{x,R}$ is continuous, we deduce likewise that $\pi^{\wedge}_x(\mathscr{O}^{\wedge\circ\circ}_{\mathrm{Spa}\,\underline{X},x})\subset\mathfrak{p}^{\wedge}$ for every rational subset $R\subset\mathrm{Spa}\,\underline{X}$. If now $f\in\mathscr{O}^{\wedge}_{\mathrm{Spa}\,\underline{X},x}\setminus\mathscr{O}^{\wedge\circ\circ}_{\mathrm{Spa}\,\underline{X},x}$, pick R as in the foregoing and $\underline{Y}$ representing h''_R, so that $x\in R$, and f extends to a section $f_R\in B:=\mathscr{O}^{\wedge}_{\mathrm{Spa}\,\underline{Y}}(Y)$; let $\mathscr{U}$ be the set of all rational subsets of $\mathrm{Spa}\,\underline{Y}$ containing x, and for every $U\in\mathscr{U}$ set $U':=U\cap R_B\big(\frac{1}{f_R}\big)$. Due to corollary 15.5.8(i) and lemma 15.4.17(iii), the image of $T:=\bigcap_{U\in\mathscr{U}}U'$ in $\mathrm{Spa}\,\underline{X}$ lies in $\mathrm{Spa}\,\underline{X}(x)$, and is not empty, and arguing as in the foregoing we deduce again that $\pi^{\wedge}_x(f)\notin\mathfrak{p}^{\wedge}$. The same argument applies to $\mathfrak{p}^{\mathrm{h}}$.

(v.b): Let us remark more generally :

Claim 15.5.15. Let A be any ring, $J\subset I\subset A$ two ideals. Then the pair (A,I) is henselian if and only if the same holds for both $(A/J,I/J)$ and (A,J).

Proof of the claim. From [75, Rem.5.1.10(iv,v)] we see that if (A,I) is henselian, the same holds for the pairs (A,J) and $(A/J,I/J)$. Conversely, let $A\to B$ be a finite ring homomorphism; if (A,J) is henselian, the projection $B\to B/JB$ restricts to a bijection $\mathrm{Idemp}\,(B)\stackrel{\sim}{\to}\mathrm{Idemp}\,(B/JB)$ on the respective subsets of idempotent elements. Then, if also $(A/J,I/J)$ is henselian, the induced map $\mathrm{Idemp}\,(B/JB)\to\mathrm{Idemp}\,(B/IB)$ is bijective as well. Summing up, we conclude that (A,I) is henselian. ◊

In view of (iv) and claim 15.5.15, we are easily reduced to showing that the projections $\mathscr{O}^{\mathrm{h}+}_{\mathrm{Spa}\,\underline{X},x}\to\kappa(x^{\mathrm{h}})^+$ and $\mathscr{O}^{\wedge+}_{\mathrm{Spa}\,\underline{X},x}\to\kappa(x^{\wedge})^+$ are surjective. The latter follows from (ii) and from the surjectivity of π^{h}_x and $\pi^{\wedge}_x$, which has already been remarked in the proof of (i).

(v.c): Let $\mathfrak{m}^{\mathrm{h}}:=\mathrm{Ker}\,\pi^{\mathrm{h}}_x$; the proof of (i) shows that $\mathfrak{m}^{\mathrm{h}}$ is the maximal ideal of $\mathscr{O}^{\mathrm{h}}_{\mathrm{Spa}\,\underline{X},x}$, and (ii) implies that $\mathfrak{m}^{\mathrm{h}}$ is also an ideal in $\mathscr{O}^{\mathrm{h}+}_{\mathrm{Spa}\,\underline{X},x}$; moreover, in light of (iv) and [75, Rem.5.1.10(iv)] the pair $(\mathscr{O}^{\mathrm{h}+}_{\mathrm{Spa}\,\underline{X},x},\mathfrak{m}^{\mathrm{h}})$ is henselian. Then the assertion follows from [75, Rem.5.1.10(ii)]. The same argument applies to $\mathscr{O}^{\wedge}_{\mathrm{Spa}\,\underline{X},x}$.

(v.d): It is easily seen that the localization map $\kappa(x^{\mathrm{h}})^+\to\kappa(x^{\mathrm{h}})^+_{\mathfrak{p}^{\mathrm{h}}}$ restricts to a bijection from $\mathfrak{p}^{\mathrm{h}}$ to the maximal ideal of $\kappa(x^{\mathrm{h}})^+_{\mathfrak{p}^{\mathrm{h}}}$; then the assertion follows from [75, Rem.5.1.10(ii)]. The same argument applies to $\kappa(x^{\wedge})^+_{\mathfrak{p}^{\wedge}}$.

(vi.a): By (iii) we know already that $\mathscr{O}^{\wedge\circ\circ}_{\mathrm{Spa}\,\underline{X},x}$ lies in the radical of $A^{\circ\circ}\cdot\mathscr{O}^{\wedge}_{\mathrm{Spa}\,\underline{X},x}$; hence, suppose that $f^n\in A^{\circ\circ}\cdot\mathscr{O}^{\wedge}_{\mathrm{Spa}\,\underline{X},x}$ for some $n\in\mathbb{N}$, and say that $f^n=\sum_{i=1}^k a_ig_i$ for some $a_1,\dots,a_k\in A^{\circ\circ}$ and $g_1,\dots,g_k\in\mathscr{O}^{\wedge}_{\mathrm{Spa}\,\underline{X},x}$. Notice that since x is non-analytic, we have $\pi^{\wedge}_x(a_i)=0$ for $i=1,\dots,k$, whence $\pi^{\wedge}_x(f^n)=0$, and therefore $\pi^{\wedge}_x(f^ng_i)=0$ as well for

$i = 1, \dots, k$; then (ii) implies that $f^n g_i \in \mathscr{O}^{\wedge+}_{\mathrm{Spa}\,\underline{X},x}$, so $f^{2n} = \sum_{i=1}^k a_i \cdot (f^n g_i) \in A^{\circ\circ} \cdot \mathscr{O}^{\wedge+}_{\mathrm{Spa}\,\underline{X},x}$. The same argument applies to $\mathscr{O}^{\mathrm{loc}\,\circ\circ}_{\mathrm{Spa}\,\underline{X},x}$ and $\mathscr{O}^{\mathrm{h},\circ\circ}_{\mathrm{Spa}\,\underline{X},x}$.

(vi.b) follows directly from (vi.a), (iv) and [75, Rem.5.1.10(ii)]. □

15.5.16. Let $\underline{A} := (A, A^+, U)$ be any quasi-affinoid ring, and $J \subset A$ a finitely generated ideal such that $U = \mathrm{Spec}\,A \setminus \mathrm{Spec}\,A/J$. The *Zariski-Riemann spectrum* of $\underline{A}$ is the subset of $\mathrm{Spv}\,A$

$$\mathrm{ZR}(\underline{A}) := \mathrm{Spa}(\underline{A}) \cap \mathrm{Spv}(A, J)$$

(notation of definition 15.3.6(i)) that we endow with the topology induced by the inclusion into $\mathrm{Spv}\,A$. A *rational subset* of $\mathrm{ZR}(\underline{A})$ is a subset of the form $R \cap \mathrm{ZR}(\underline{A})$, where R is a rational subset of $\mathrm{Spv}(A, J)$ (see definition 15.3.6(ii)). Notice that neither $\mathrm{ZR}(\underline{A})$ nor the class of its rational subsets depend on the choice of J (lemma 15.3.9(i)).

Proposition 15.5.17. *With the notation of* (15.5.16)*, the following holds :*

(i) $\mathrm{ZR}(\underline{A}) = \{v \in \mathrm{Spa}(\underline{A}) \mid v$ *has no proper primary specializations in* $\mathrm{Spa}(\underline{A})\}$.

(ii) $\mathrm{ZR}(\underline{A})$ *is a pro-constructible subset of* $\mathrm{Spv}(A, J)$. *In particular, it is a spectral topological space.*

(iii) *The rational subsets of* $\mathrm{ZR}(\underline{A})$ *are constructible in* $\mathrm{ZR}(\underline{A})$*, and form a basis of the topology of* $\mathrm{ZR}(\underline{A})$ *that is closed under finite intersections.*

(iv) *The unit of adjunction* $\underline{A} \to \Gamma \circ \mathsf{Spec}\,\underline{A}$ *induces a homeomorphism*

$$\mathrm{ZR}(\Gamma \circ \mathsf{Spec}\,\underline{A}) \xrightarrow{\sim} \mathrm{ZR}(\underline{A}).$$

Especially, $\mathrm{ZR}(\underline{A})$ *depends only on the quasi-affinoid scheme* $\mathsf{Spec}\,\underline{A}$.

(v) *The homeomorphism* $\mathrm{Spa}(\underline{A}_{\mathrm{loc}}) \xrightarrow{\sim} \mathrm{Spa}(\underline{A})$ *of lemma* 15.4.17(i) *restricts to a homeomorphism :*

$$\mathrm{ZR}(\underline{A}_{\mathrm{loc}}) \xrightarrow{\sim} \mathrm{ZR}(\underline{A}).$$

Proof. (i): Recall that $\mathrm{Cont}(A)$ is closed under specializations in $\mathrm{Spv}\,A$ (remark 15.3.13(iv)); it follows easily that $\mathrm{Spa}(A, A^+)$ is closed under primary specializations. Recall as well that a proper specialization (v, w) in $\mathrm{Spv}\,A$ is J-admissible if and only if w is a primary specialization of v and the supports of v and w lie in U (see (15.3.1)); hence, if $v \in \mathrm{Spa}\,\underline{A}$, the proper specialization (v, w) is J-admissible if and only if w is a primary specialisation of v that lies in $\mathrm{Spa}\,\underline{A}$. Then the assertion follows from lemma 15.3.4(iii).

(ii): Let $A_0 \subset A$ be a subring of definition, and $I \subset A_0$ a finitely generated ideal of adic definition; recall that $\mathrm{Spa}(\underline{A})$ is a pro-constructible subset of $\mathrm{Cont}(A)$ (proposition 15.4.22(i)) and the inclusion map $\mathrm{Cont}(A) \to \mathrm{Spv}(A, IA)$ is a closed immersion (theorem 15.3.15(ii)) so $\mathrm{Spv}(\underline{A})$ is a pro-constructible subset of $\mathrm{Spv}(A, IA)$ (corollary 8.1.50(i)). Moreover, the radical of J contains IA, hence the inclusion map $\mathrm{Spv}(A, J) \to \mathrm{Spv}(A, IA)$ admits a spectral retraction (lemma 15.3.9(i))

$$r : \mathrm{Spv}(A, IA) \to \mathrm{Spv}(A, J). \qquad (15.5.18)$$

Namely, for every $v \in \mathrm{Spv}(A, IA)$, the valuation $r(v)$ is the unique J-admissible specialization of v that does not admit further proper J-admissible specializations. We claim that

$$\mathrm{ZR}(\underline{A}) = r(\mathrm{Spa}\,\underline{A}).$$

Indeed, clearly $\mathrm{ZR}(\underline{A}) = r(\mathrm{ZR}(\underline{A})) \subset r(\mathrm{Spa}\,\underline{A})$. The converse inclusion follows from the proof of (i). Then, the assertion follows from corollary 8.1.50(i).

(iii) follows immediately from (ii) and theorem 15.3.7(ii).

(iv) follows directly from (i) and lemmata 15.4.17(ii) and 9.2.25(i).

(v): Clearly the homeomorphism $\mathrm{Spa}\,\underline{A}_{\mathrm{loc}} \xrightarrow{\sim} \mathrm{Spa}\,\underline{A}$ preserves primary specializations, so the assertion follows from (i). □

Remark 15.5.19. In [157], Temkin associates with every separated morphism $f : Y \to X$ of quasi-compact and quasi-separated schemes, a spectral space that he calls the Riemann-Zariski space of f, and denotes $\mathrm{RZ}_Y(X)$, together with a continuous map $\mathrm{RZ}_Y(X) \to X$. His construction generalizes earlier work by Zariski, and is related as follows to our $\mathrm{ZR}(\underline{A})$. Let $U \to \operatorname{Spec} A^+$ be the composition of the open immersion $U \to \operatorname{Spec} A$ with the natural morphism $\operatorname{Spec} A \to \operatorname{Spec} A^+$; then we may consider the induced map $\varphi : \mathrm{RZ}_U(\operatorname{Spec} A^+) \to \operatorname{Spec} A^+$, and there is a natural homeomorphism $\varphi^{-1}(\operatorname{Spec} A^+/A^{\circ\circ}) \xrightarrow{\sim} \mathrm{ZR}(\underline{A})$.

15.5.20. Keep the notation of (15.5.16), and for some integer $n \geq -1$, let $R_\bullet := (R_i \mid i = 0, \dots, n)$ be a finite system of open subsets of $\mathrm{ZR}(\underline{A})$ such that $\mathrm{ZR}(\underline{A}) = \bigcup_{i=0}^n R_i$ (the case $n = -1$ occurs for the empty covering of the empty Zariski-Riemann spectrum); let also $A' \subset A$ be a subring, and J' an ideal of A'. Then we say that $R_\bullet$ is an (A', J')-*standard covering* of $\mathrm{ZR}(\underline{A})$ if there exists a finite system $f_\bullet := (f_0, \dots, f_n)$ of elements of A' such that :

- $R_i = R_A\big(\frac{f_0}{f_i}, \dots, \frac{f_n}{f_i}\big) \cap \mathrm{ZR}(\underline{A})$ for $i = 0, \dots, n$
- J' is contained in the radical of the ideal of A generated by the system $f_\bullet$.

Especially, each R_i is a rational subset of $\mathrm{ZR}(\underline{A})$. By construction, the retraction r of (15.5.18) restricts to a surjective continuous map

$$s : \operatorname{Spa} \underline{A} \to \mathrm{ZR}(\underline{A})$$

and if $f_\bullet$ fulfills the foregoing conditions, lemma 15.3.5(i) says that

$$s^{-1}R_i = R_A\Big(\frac{f_0}{f_i}, \dots, \frac{f_n}{f_i}\Big) \cap \operatorname{Spa} \underline{A} \qquad \text{for every } i = 0, \dots, n.$$

Especially, the system $(R_A\big(\frac{f_0}{f_i}, \dots, \frac{f_n}{f_i}\big) \cap \operatorname{Spa} \underline{A} \mid i = 0, \dots, n)$ is a covering of $\operatorname{Spa} \underline{A}$. A covering of $\operatorname{Spa} \underline{A}$ of this type shall also be called a (A', J')-*standard covering*. In case $A' = A$ and $J' = J$, we shall write simply *standard covering* for this class of coverings of $\mathrm{ZR}(\underline{A})$ and of $\operatorname{Spa} \underline{A}$. The following lemma refines [100, Lemma 2.6].

Lemma 15.5.21. *Let $j : A' \to A$ be the inclusion of a subring, and $J' \subset A'$ a finitely generated ideal with $J'A \subset J$, and such that* $\operatorname{Spec}(j)$ *restricts to an isomorphism of schemes*

$$U \xrightarrow{\sim} \operatorname{Spec}(A') \setminus \operatorname{Spec}(A'/J'). \tag{15.5.22}$$

Then every open covering of $\mathrm{ZR}(\underline{A})$ *or* $\operatorname{Spa} \underline{A}$ *is refined by an (A', J')-standard covering.*

Proof. We consider first the case where $A' = A$ and $J' = J$. Let $U_\bullet := (U_\lambda \mid \lambda \in \Lambda)$ be an open covering of $\mathrm{ZR}(\underline{A})$; we need to find a standard covering $(R_i \mid i = 0, \dots, n)$ such that, for every $i = 0, \dots, n$ there exists $\lambda \in \Lambda$ with $R_i \subset U_\lambda$. To this aim, we suppose first that A is topologically local; in light of proposition 15.5.17(i,ii) we may assume that Λ is a finite set, and each U_λ is rational, say

$$U_\lambda = R_A\Big(\frac{g_{\lambda,1}}{g_{\lambda,0}}, \dots, \frac{g_{\lambda,n_\lambda}}{g_{\lambda,0}}\Big) \cap \mathrm{ZR}(\underline{A})$$

for a finite system $g_{\lambda,\bullet} := (g_{\lambda,j} \mid j = 0, \dots, n_\lambda)$ of elements of A that generates an ideal whose radical contains J. Define S as the set of all the sequences of integers $j_\bullet := (j_\lambda \mid \lambda \in \Lambda)$ such that $0 \leq j_\lambda \leq n_\lambda$ for every $\lambda \in \Lambda$. Let also $T \subset S$ be the subset of all sequences $j_\bullet$ such that $j_\lambda = 0$ for at least one $\lambda \in \Lambda$. Set $g_{j_\bullet} := \prod_{\lambda \in \Lambda} g_{\lambda, j_\lambda}$ for every $j_\bullet \in S$. Notice that :

$$R_{k_\bullet} := R_A\Big(\frac{g_{j_\bullet}}{g_{k_\bullet}} \mid j_\bullet \in S\Big) = \bigcap_{\lambda \in \Lambda} R_A\Big(\frac{g_{\lambda,1}}{g_{\lambda,k_\lambda}}, \dots, \frac{g_{\lambda,n_\lambda}}{g_{\lambda,k_\lambda}}\Big) \qquad \text{for every } k_\bullet \in S.$$

Moreover, each $g_{\lambda,\bullet}$ defines a standard open covering of $\mathrm{ZR}(\underline{A})$. It follows easily that

$$\mathrm{ZR}(\underline{A}) = \bigcup_{k_\bullet \in T} (R_{k_\bullet} \cap \mathrm{ZR}(\underline{A})) \tag{15.5.23}$$

and furthermore, for every $k_\bullet \in T$, the subset $R_{k_\bullet} \cap \mathrm{ZR}(\underline{A})$ is contained in some U_λ.

Claim 15.5.24. $R_{k_\bullet} \cap \mathrm{ZR}(\underline{A}) = R_A\big(\frac{g_{j_\bullet}}{g_{k_\bullet}} \mid j_\bullet \in T\big) \cap \mathrm{ZR}(\underline{A})$ for every $k_\bullet \in T$.

Proof of the claim. Indeed, the inclusion $R_{k_\bullet} \subset R'_{k_\bullet} := R_A\big(\frac{g_{j_\bullet}}{g_{k_\bullet}} \mid j_\bullet \in T\big)$ is obvious. Conversely, let $v \in R'_{k_\bullet} \cap \mathrm{ZR}(A)$; we see from (15.5.23) that for every $j_\bullet \in S$ there exists $i_\bullet \in T$ such that $v(g_{j_\bullet}) \leq v(i_\bullet) \neq 0$, whence $v(g_{j_\bullet}) \leq v(k_\bullet) \neq 0$, so $v \in R_{k_\bullet}$. ◊

In light of (15.5.23) and claim 15.5.24, to conclude it now suffices to show :

Claim 15.5.25. J is contained in the radical of the ideal J' of A generated by $(g_{k_\bullet} \mid k_\bullet \in T)$.

Proof of the claim. First, we check that J' is an open ideal, using the criterion of lemma 15.3.25. Indeed, by claim 15.4.23, every continuous analytic rank one valuation of A lies in $\mathrm{Spa}\,\underline{A}$, and from (15.5.23) and claim 15.5.24 we see that for every such valuation v there exists $k_\bullet \in T$ such that $v(g_{k_\bullet}) \neq 0$, whence the assertion. By lemma 8.3.29(v), it follows that $\mathrm{Spec}A/J' \subset X^{\circ\circ}_A := \mathrm{Spec}\,A/A^{\circ\circ}A$. Hence, let $\mathfrak{p} \in X^{\circ\circ}_A \setminus \mathrm{Spec}A/J'$; it suffices to show that $J' \not\subset \mathfrak{p}$. To this aim, denote by $v_\mathfrak{p}$ the trivial valuation of A supported at $\mathfrak{p}$ (see remark 9.1.4(vii)); since $\mathfrak{p}$ is an open ideal (lemma 8.3.29(i)), the valuation $v_\mathfrak{p}$ is continuous (remark 15.3.13(vi)), and then obviously $v \in \mathrm{Spa}\,\underline{A}$. As in the foregoing, we deduce that $v_\mathfrak{p}(g_{k_\bullet}) \neq 0$ for some $k_\bullet \in T$, *i.e.* $g_{k_\bullet} \notin \mathfrak{p}$, whence the claim. ◊

• Next, let A be any f-adic ring; let us choose a subring of definition A_0 of A such that $A_0 \subset A^+$, and let $I_0 \subset A_0$ be a finitely generated ideal of definition. We apply the foregoing to the topological localization $(A, A^+, U)_{\mathrm{loc}}$: notice that proposition 15.5.17(v) naturally identifies $U_\bullet$ with an open covering of $\mathrm{ZR}(\underline{A}_{\mathrm{loc}})$. By the foregoing, there exists therefore a finite system $f_0, \dots, f_n$ of elements of $A_{\mathrm{loc}} = (1 + I_0)^{-1}A$ such that JA_{loc} lies in the radical of $\sum_{i=0}^n A_{\mathrm{loc}} f_i$, and the standard covering $(R_{A_{\mathrm{loc}}}(\frac{f_0}{f_i}, \cdots, \frac{f_n}{f_i}) \cap \mathrm{ZR}(\underline{A}_{\mathrm{loc}}))$ refines $U_\bullet$. After clearing some denominators, we may assume that $f_0, \dots, f_n \in A$, and it follows easily that there exists $s \in 1 + I_0$ such that sJ lies in the radical of $\sum_{i=0}^n Af_i$. Since J is an open ideal, we have $I_0^k \subset J$ for some $k \in \mathbb{N}$; then, after replacing k by a larger integer, we also get $sI_0^k \subset \sum_{i=0}^n Af_i$. Next, after replacing k by a still larger integer, we can assume that $sI_0^k \subset \sum_{i=0}^n A_0 f_i$. Let also $g_1, \dots, g_m$ be a finite system of generators of I_0; since $A_0 \subset A^+$, it follows that

$$\max(v(g_1)^k, \dots, v(g_m)^k) \leq \max(v(f_0), \dots, v(f_n)) \qquad \text{for every } v \in \mathrm{Spa}(A, A^+)$$

and since J is in the radical of $\sum_{i=0}^n Af_i$, we deduce :

$$\max(v(g_1)^{k+1}, \dots, v(g_m)^{k+1}) < \max(v(f_0), \dots, v(f_n)) \qquad \text{for every } v \in \mathrm{Spa}(A, A^+, U).$$

Since $s \in 1 + I_0$, it is easily seen that the radical of $\sum_{i=0} Af_i + \sum_{j=1}^m Ag_j$ contains J. Then, consider the standard covering $R'_\bullet$ of $\mathrm{ZR}(\underline{A})$ associated with the sequence $(f_\bullet, g_\bullet^{k+1})$: every element of $R'_\bullet$ of the form $R_A(\frac{f_\bullet}{f_i}, \frac{g_\bullet^{k+1}}{f_i})$ (for any $i = 0, \dots, n$) equals $R_A(\frac{f_\bullet}{f_i})$, and on the other hand $R_A(\frac{f_\bullet}{g_j^{k+1}}, \frac{g_\bullet^{k+1}}{g_j^{k+1}}) = \varnothing$ for every $j = 1, \dots, m$. Hence, $R'_\bullet$ refines $U_\bullet$.

• Lastly, consider any open covering $(U'_\lambda \mid \lambda \in \Lambda)$ of $\mathrm{Spa}\,\underline{A}$, and set $U_\lambda := U'_\lambda \cap \mathrm{ZR}(\underline{A})$ for every $\lambda \in \Lambda$; clearly $(U_\lambda \mid \lambda \in \Lambda)$ is an open covering of $\mathrm{ZR}(\underline{A})$, which can be refined by a standard covering $(R_0, \dots, R_n)$; but then the system $(s^{-1}R_0, \dots, s^{-1}R_n)$ is a standard covering of $\mathrm{Spa}\,\underline{A}$, and notice that if $R_i \subset U_\lambda$, we get

$$s^{-1}R_i \subset s^{-1}U_\lambda \subset U'_\lambda$$

where the last inclusion holds, since U'_λ is open in $\mathrm{Spa}\,\underline{A}$ and $s^{-1}U_\lambda$ is the set of all primary generizations of the elements of U_λ in $\mathrm{Spa}\,\underline{A}$. Thus, $(s^{-1}R_0, \dots, s^{-1}R_n)$ refines the covering $(U'_\lambda \mid \lambda \in \Lambda)$, as required.

• We consider next the case of a general pair (A', J'). By the foregoing, the given covering $U_\bullet$ of $\mathrm{ZR}(\underline{A})$ or $\mathrm{Spa}\,\underline{A}$ is refined by the standard covering associated with a sequence $g_0, \dots, g_n$

of elements of A. Set $I := \sum_{i=0}^n Ag_i$, and let $h_1, \dots, h_m$ be a finite system of generators of J'; under the stated assumptions, j induces an isomorphism $A'[h_k^{-1}] \xrightarrow{\sim} A[h_k^{-1}]$ for every $i = 1, \dots, m$, so there exists $N \in \mathbb{N}$ such that $f_{ki} := h_k^N g_i \in A'$ for every $k = 1, \dots, m$ and every $i = 0, \dots, n$. Let $I' \subset A'$ be the ideal generated by $f_{\bullet\bullet} := (f_{ki} \mid 1 \le k \le m, 0 \le i \le n)$; notice that $\mathrm{Spec}(A/I'A) = \mathrm{Spec}(A/I) \cup \mathrm{Spec}(A/J'A)$, and by assumption $\mathrm{Spec}(A/I) \subset \mathrm{Spec}(A/J) \subset \mathrm{Spec}(A/J'A)$. So, $\mathrm{Spec}(A/I'A) \subset \mathrm{Spec}(A/J'A)$, and notice that

$$\mathrm{Spec}(A/I'A) = (\mathrm{Spec}\, j)^{-1}(\mathrm{Spec}\, A'/I') \quad \text{and} \quad \mathrm{Spec}(A/J'A) = (\mathrm{Spec}\, j)^{-1}(\mathrm{Spec}\, A'/J').$$

In view of the isomorphism (15.5.22), it follows that $\mathrm{Spec}(A'/I') \subset \mathrm{Spec}(A'/J')$, *i.e.* the radical of I' contains J'. Moreover, by the same token we have :

$$\mathrm{Spa}\, \underline{A} = \bigcup_{j=1}^m \mathrm{Spa}(A, A^+) \cap R_A\Big(\frac{h_1^N}{h_j^N}, \dots, \frac{h_m^N}{h_j^N}\Big).$$

Taking into account remark 9.2.4(i), we conclude easily that the (A', J')-standard covering associated with the sequence $f_{\bullet\bullet}$ is the sought refinement of $U_\bullet$. □

Remark 15.5.26. The following constructions shall be exploited in the proofs of both theorem 15.5.27 and theorem 15.5.35.

(i) Let $\underline{X}$ be a topologically local quasi-affinoid scheme, set $(A, A^+, X) := \Gamma(\underline{X})$, and choose a subring of definition $A_0 \subset A$ and a finitely generated ideal $I_0 \subset A_0$ of adic definition. Consider a sequence $f_\bullet := (f_0, \dots, f_n)$ of elements of A that generates an ideal $J \subset A$ such that $X \cap \mathrm{Spec}\, A/J = \varnothing$, and let $R_\bullet := (R_0, \dots, R_n)$ be the standard covering of $\mathrm{Spa}\, \underline{X}$ associated with $f_\bullet$. For every subset $\Lambda \subset \{0, \dots, n\}$, let $A_\Lambda := A[f_i^{-1} \mid i \in \Lambda]$ and $X_\Lambda := X \cap \mathrm{Spec}\, A_\Lambda$; by inspecting the definitions we see that

$$R_\Lambda := \bigcap_{i \in \Lambda} R_i = \mathrm{Spa}\, \underline{A}_\Lambda \qquad \text{with} \qquad \underline{A}_\Lambda := (A_\Lambda, A_\Lambda^+, X_\Lambda)$$

where A_Λ^+ is the integral closure of $A^+[f_k/f_i \mid (k,i) \in \{0, \dots, n\} \times \Lambda]$ in A_Λ; moreover, $A_{0,\Lambda} := A_0[f_k/f_i \mid (k,i) \in \{0, \dots, n\} \times \Lambda]$ is a subring of definition of A_Λ, and the natural ring homomorphism $A_0 \to A_{0,\Lambda}$ is adic (here we let $R_\varnothing := \mathrm{Spa}\, \underline{X}$). Furthermore, since X is quasi-compact, we have $\mathscr{O}_X(X_\Lambda) = A_\Lambda$ ([59, Ch.I, Prop.9.2.1]), and therefore

$$\underline{A}_\Lambda = \Gamma \circ \mathsf{Spec}\, (\underline{A}_\Lambda) \qquad \text{for every } \Lambda \subset \{0, \dots, n\}.$$

(ii) Denote by $(A_{0,\Lambda}^{\mathrm{h}}, I_{0,\Lambda}^{\mathrm{h}})$ the henselization of the pair $(A_{0,\Lambda}, I_0 A_{0,\Lambda})$, and endow $A_{0,\Lambda}^{\mathrm{h}}$ with the $I_{0,\Lambda}^{\mathrm{h}}$-adic topology; we deduce a unique isomorphism of topological A-algebras :

$$\omega_\Lambda^{\mathrm{h}} : A_\Lambda^{\mathrm{h}} := A_\Lambda \otimes_{A_{0,\Lambda}} A_{0,\Lambda}^{\mathrm{h}} \xrightarrow{\sim} \mathscr{O}_{\mathrm{Spa}\, \underline{X}}^{\mathrm{h}}(R_\Lambda) \qquad \text{for every } \Lambda \subset \{0, \dots, n\}$$

where the topology on A_Λ^{h} is the unique one such that the natural map $A_{0,\Lambda}^{\mathrm{h}} \to A_\Lambda^{\mathrm{h}}$ is open (proposition 8.3.34(ii)). Likewise, let $A_\Lambda^\wedge$ be the separated completion of A_Λ; we deduce a natural continuous homomorphism of A-algebras

$$\omega_\Lambda^\wedge : A_\Lambda^\wedge \to \mathscr{O}_{\mathrm{Spa}\, \underline{X}}^\wedge(R_\Lambda) \qquad \text{for every } \Lambda \subset \{0, \dots, n\}.$$

Notice also that for every $\Lambda' \subset \Lambda$ there exist unique continuous homomorphisms of A-algebras and respectively $A_{0,\Lambda}$-algebras

$$\mu_{\Lambda',\Lambda} : A_{\Lambda'} \to A_\Lambda \qquad \mu_{0,\Lambda',\Lambda}^{\mathrm{h}} : A_{0,\Lambda'}^{\mathrm{h}} \to A_{0,\Lambda}^{\mathrm{h}}$$

and set $\mu_{\Lambda',\Lambda}^{\mathrm{h}} := \mu_{\Lambda',\Lambda} \otimes_A \mu_{0,\Lambda',\Lambda}^{\mathrm{h}}$. Let also $\mu_{\Lambda',\Lambda}^\wedge : A_{\Lambda'}^\wedge \to A_\Lambda^\wedge$ be the completion of $\mu_{\Lambda',\Lambda}$, and

$$\rho_{\Lambda',\Lambda}^{\mathrm{h}} : \mathscr{O}_{\mathrm{Spa}\, \underline{X}}^{\mathrm{h}}(R_{\Lambda'}) \to \mathscr{O}_{\mathrm{Spa}\, \underline{X}}^{\mathrm{h}}(R_\Lambda) \qquad \rho_{\Lambda',\Lambda}^\wedge : \mathscr{O}_{\mathrm{Spa}\, \underline{X}}^\wedge(R_{\Lambda'}) \to \mathscr{O}_{\mathrm{Spa}\, \underline{X}}^\wedge(R_\Lambda)$$

the restriction homomorphisms associated with the inclusion $R_\Lambda \subset R_{\Lambda'}$.

(iii) We claim that the diagrams of A-algebras:

$$\begin{array}{ccc} A^{\mathrm{h}}_{\Lambda'} & \xrightarrow{\omega^{\mathrm{h}}_{\Lambda'}} & \mathscr{O}^{\mathrm{h}}_{\mathrm{Spa}\,\underline{X}}(R_{\Lambda'}) \\ {\scriptstyle \mu^{\mathrm{h}}_{\Lambda',\Lambda}}\downarrow & & \downarrow{\scriptstyle \rho^{\mathrm{h}}_{\Lambda',\Lambda}} \\ A^{\mathrm{h}}_{\Lambda} & \xrightarrow{\omega^{\mathrm{h}}_{\Lambda}} & \mathscr{O}^{\mathrm{h}}_{\mathrm{Spa}\,\underline{X}}(R_{\Lambda}) \end{array} \qquad \begin{array}{ccc} A^{\wedge}_{\Lambda'} & \xrightarrow{\omega^{\wedge}_{\Lambda'}} & \mathscr{O}^{\wedge}_{\mathrm{Spa}\,\underline{X}}(R_{\Lambda'}) \\ {\scriptstyle \mu^{\wedge}_{\Lambda',\Lambda}}\downarrow & & \downarrow{\scriptstyle \rho^{\wedge}_{\Lambda',\Lambda}} \\ A^{\wedge}_{\Lambda} & \xrightarrow{\omega^{\wedge}_{\Lambda}} & \mathscr{O}^{\wedge}_{\mathrm{Spa}\,\underline{X}}(R_{\Lambda}) \end{array}$$

commute for every $\Lambda' \subset \Lambda \subset \{0, \dots, n\}$. Indeed, a simple inspection shows that $\mu^{\mathrm{h}}_{\Lambda',\Lambda}$ is continuous. On the other hand, all the arrows in the diagram are A-algebra homomorphisms, therefore also $A_{\Lambda'}$-algebra homomorphisms, and by the universal property of the topological henselization there is a unique continuous $A_{\Lambda'}$-algebra homomorphism $A^{\mathrm{h}}_{\Lambda'} \to \mathscr{O}^{\mathrm{h}}_{\mathrm{Spa}\,\underline{X}}(R_{\Lambda})$, whence the contention for the left diagram. For the right diagram one argues similarly, using the universal property of the separated completions.

(iv) Endow $A[Y]$ and $B := A_0[T_0, \dots, T_n]$ with the standard $\mathbb{N}$-gradings such that $\mathrm{gr}_k A[Y] = AY^k$ and $\mathrm{gr}_k B$ is the free A-module generated by the monomials $T_0^{\nu_0} \cdots T_n^{\nu_n}$ of total degree $\nu_0 + \cdots + \nu_n = k$, for every $k \in \mathbb{N}$. Set as well

$$S := \mathrm{Spec}\, A_0 \qquad \mathbb{P}^n_S := \mathrm{Proj}\, B$$

and recall that $\mathbb{P}^n_S$ admits the following finite affine open covering (see (10.6.1)). For every $i = 0, \dots, n$ we have the subring $B_i := A_0[T_0/T_i, \dots, T_n/T_i]$ of $B[1/T_i]$, and $\Omega_i := \mathrm{Spec}\, B_i$ is naturally identified with the open subscheme of $\mathbb{P}^n_S$ consisting of all homogeneous prime ideals of B that do not contain T_i; then

$$\mathbb{P}^n_S = \Omega_0 \cup \cdots \cup \Omega_n.$$

Next, according to (10.6.5), the homomorphism of $\mathbb{N}$-graded A_0-algebras

$$h : B \to A[Y] \qquad T_j \mapsto f_j Y \qquad \text{for every } j = 0, \dots, n$$

induces a morphism of A_0-schemes

$$\varphi : \mathrm{Spec}\, A \setminus \mathrm{Spec}\, A/J \to \mathbb{P}^n_S.$$

Explicitly, $\varphi^{-1}\Omega_i = \mathrm{Spec}\, A_{\{i\}}$ for every $i = 0, \dots, n$, and the restriction $\varphi^{-1}\Omega_i \to \Omega_i$ of φ corresponds to the homomorphism of A_0-algebras

$$h_i : B_i \to A_{\{i\}} \qquad T_j/T_i \mapsto f_j/f_i \qquad \text{for every } j = 0, \dots, n.$$

(v) Now, quite generally, let $\beta : \mathsf{X} \to \mathsf{Y}$ be any quasi-compact and quasi-separated morphism of schemes. Then $\beta_*\mathscr{O}_{\mathsf{X}}$ is a quasi-coherent $\mathscr{O}_{\mathsf{Y}}$-module ([59, Ch.I, Prop.9.2.1]), hence $\mathscr{I} := \mathrm{Ker}\, \beta^\sharp : \mathscr{O}_{\mathsf{Y}} \to \beta_*\mathscr{O}_{\mathsf{X}}$ is a quasi-coherent sheaf of ideals of $\mathscr{O}_{\mathsf{Y}}$, and therefore $\mathrm{Spec}\, \mathscr{O}_{\mathsf{Y}}/\mathscr{I}$ is a well defined closed subscheme of Y, called the *schematic image of* ψ. From the definition, it is clear that the construction of the schematic image is local on Y : namely, if $\mathsf{Y}' \subset \mathsf{Y}$ is any open subscheme, and $\mathsf{X}' := \beta^{-1}\mathsf{Y}'$, then the schematic image of $\beta_{|\mathsf{X}'} : \mathsf{X}' \to \mathsf{Y}'$ is the intersection of Y' with the schematic image of β.

Thus, if V is the schematic image of our φ, we get a commutative diagram of schemes

$$\begin{array}{ccc} X & \xrightarrow{\quad\psi\quad} & V \\ \downarrow & \nearrow & \downarrow{\scriptstyle \iota} \\ \mathrm{Spec}\, A \setminus \mathrm{Spec}\, A/J & \xrightarrow{\quad\varphi\quad} & \mathbb{P}^n_S \end{array}$$

whose left vertical arrow is an open immersion, and with ι a closed immersion. Lastly, set $V_\varnothing := V$ and for every subset $\Lambda \subset \{0, \dots, n\}$ let $V_\Lambda := \iota^{-1}\left(\bigcap_{i\in\Lambda} \Omega_i\right)$; then V_Λ is naturally identified with $\mathrm{Spec}\, A_{0,\Lambda}$, for every non-empty Λ.

(vi) Furthermore, let U be the analytic locus of S; recall that the morphism $\mathrm{Spec}\, A \to S$ (associated with the inclusion map $A_0 \to A$) identifies U with the analytic locus of $\mathrm{Spec}\, A$ (lemma 8.3.29(iii)), and U is an open subset of X, under this identification. Then $\mathbb{P}^n_U := \mathbb{P}^n_S \times_S U$ is an open subset of $\mathbb{P}^n_S$, and we claim that ψ restricts to an isomorphism of schemes

$$\psi_{|U} : U \xrightarrow{\sim} V \cap \mathbb{P}^n_U.$$

Indeed, $\psi_{|U}$ is a closed immersion, by [59, Ch.I, Cor.5.4.6]. On the other hand, $\psi_{|U}$ is also the schematic closure of the restriction $U \to \mathbb{P}^n_U$ of φ, whence the claim.

Theorem 15.5.27. *For any topologically local quasi-affinoid scheme $\underline{X} := (X, \mathscr{T}_X, A^+)$, the presheaves $\mathscr{O}^{\mathrm{h}}_{\mathrm{Spa}\, \underline{X}}$ and $\mathscr{O}^{\mathrm{h}+}_{\mathrm{Spa}\, \underline{X}}$ are sheaves of rings on $(\mathscr{Q}(\underline{X}), J_{\mathscr{Q}})$.*

Proof. Notice first that – by virtue of corollary 15.4.27(i) – if $\mathscr{O}^{\mathrm{h}}_{\mathrm{Spa}\, \underline{X}}$ is a sheaf, the same holds for $\mathscr{O}^{\mathrm{h}+}_{\mathrm{Spa}\, \underline{X}}$, so it suffices to check the assertion for $\mathscr{O}^{\mathrm{h}}_{\mathrm{Spa}\, \underline{X}}$. To this aim, for every $U \in \mathrm{Ob}(\mathscr{Q}(\underline{X}))$ and every $\mathscr{S} \in J_{\mathscr{Q}}(U)$, let $h_{\mathscr{S}}$ be the corresponding sub-presheaf of the presheaf h_U on $\mathscr{Q}(\underline{X})$ represented by U. According to claim 4.1.16(ii,iii), it suffices to show that the natural map

$$r_U : \mathscr{O}^{\mathrm{h}}_{\mathrm{Spa}\, \underline{X}}(U) \to \operatorname*{colim}_{\mathscr{S} \in J_{\mathscr{Q}}(U)} \mathrm{Hom}_{\mathscr{Q}(\underline{X})^\wedge}(h_{\mathscr{S}}, \mathscr{O}^{\mathrm{h}}_{\mathrm{Spa}\, \underline{X}})$$

is a bijection for every such U. However, let $\varphi_{\underline{T}/\underline{X}} : \underline{T} \to \underline{X}$ be any morphism of topologically local quasi-affinoid schemes representing the inclusion $h_U \subset h_{\underline{X}}$ (see (15.5.7)); by remark 15.5.9(v) it follows that $\varphi_{\underline{T}/\underline{X}}$ induces an equivalence of categories

$$\mathscr{Q}(\underline{T}) \xrightarrow{\sim} \mathscr{Q}(\underline{X})/U \qquad V \mapsto \mathrm{Spa}\, \varphi_{\underline{T}/\underline{X}}(V) \subset U$$

so we may replace $\underline{X}$ by $\underline{T}$, and reduce to checking that $r_{\mathrm{Spa}\, \underline{X}}$ is an isomorphism. Next, lemma 15.5.21 implies that the sieves of $\mathscr{Q}(\underline{X})$ generated by the standard coverings of $\mathrm{Spa}\, \underline{X}$ form a coinitial subset of $J_{\mathscr{Q}}(\mathrm{Spa}\, \underline{X})$. Hence, let $f_\bullet$ and $R_\bullet$ be as in remark 15.5.26(i); taking into account the discussion of (4.1.6), we are then reduced to checking that the natural map

$$\mathscr{O}^{\mathrm{h}}_{\mathrm{Spa}\, \underline{X}}(\mathrm{Spa}\, \underline{X}) \to \mathrm{Equal}\Big(\prod_{i=0}^{n} \mathscr{O}^{\mathrm{h}}_{\mathrm{Spa}\, \underline{X}}(R_i) \rightrightarrows \prod_{i,j=0}^{n} \mathscr{O}^{\mathrm{h}}_{\mathrm{Spa}\, \underline{X}}(R_{\{i,j\}})\Big)$$

is a bijection. Now, say that $(A, A^+, X) = \Gamma(\underline{X})$. Recall that the henselization morphism

$$\lambda : \underline{X}^{\mathrm{h}} := (X^{\mathrm{h}}, \mathscr{T}^{\mathrm{h}}_X, A^{\mathrm{h}+}) \to \underline{X}$$

is f-adic and induces a homeomorphism $\mathrm{Spa}\, \lambda : \mathrm{Spa}\, \underline{X}^{\mathrm{h}} \xrightarrow{\sim} \mathrm{Spa}\, \underline{X}$, as well as an isomorphism of schemes $X^{\mathrm{h}} \xrightarrow{\sim} X \times_A A^{\mathrm{h}}$; it follows easily that the system $((\mathrm{Spa}\, \lambda)^{-1} R_i \mid i = 0, \dots, n)$ is a rational covering of $\mathrm{Spa}\, \underline{X}^{\mathrm{h}}$, and moreover there is a unique isomorphism of presheaves of topological A-algebras

$$\mathscr{O}^{\mathrm{h}}_{\mathrm{Spa}\, \underline{X}^{\mathrm{h}}} \xrightarrow{\sim} (\mathrm{Spa}\, \lambda)^{-1} \mathscr{O}^{\mathrm{h}}_{\mathrm{Spa}\, \underline{X}}.$$

We may then replace $\underline{X}$ by $\underline{X}^{\mathrm{h}}$, and assume from start that $\underline{X}$ is topologically henselian.

In this situation, let $A_0 \subset A$ be a subring of definition, and $I_0 \subset A_0$ an ideal of adic definition; consider the system of quasi-affinoid rings $(\underline{A}_\Lambda \mid \Lambda \subset \{0, \dots, n\})$, their rings of definition $A_{0,\Lambda} \subset A_\Lambda$, and the systems of their topological henselizations $(A^{\mathrm{h}}_\Lambda \mid \Lambda \subset \{0, \dots, n\})$ and $(A^{\mathrm{h}}_{0,\Lambda} \mid \Lambda \subset \{0, \dots, n\})$ constructed in remark 15.5.26(i). Define also S, V, $\psi : X \to V$ and V_Λ for every subset $\Lambda \subset \{0, \dots, n\}$ as in remark 15.5.26(iv,v); set $S' := \mathrm{Spec}\, A_0/I_0$, $V'_\Lambda := V_\Lambda \times_S S'$ for every such Λ, and consider the cartesian diagram of schemes

$$\begin{array}{ccc} V' & \xrightarrow{\tau_V} & V \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ S' & \xrightarrow{\tau_S} & S \end{array}$$

where π is the restriction to V of the projection $\mathbb{P}^n_S \to S$ and $\tau_S : S' \to S$ is the closed immersion. For every scheme X, let $\mathsf{X}_{\mathrm{\acute{e}t}}$ be the étale site of X, and $\mathscr{O}_{\mathsf{X}_{\mathrm{\acute{e}t}}}$ the structure sheaf of rings on $\mathsf{X}_{\mathrm{\acute{e}t}}$. We define the sheaf on $V'_{\mathrm{\acute{e}t}}$

$$\mathscr{F} := \tau_V^* \psi_* \mathscr{O}_{X_{\mathrm{\acute{e}t}}}.$$

Claim 15.5.28. (i) For every $\Lambda \subset \{0, \dots, n\}$ there exists a commutative diagram of rings

$$\mathscr{D}_\Lambda \quad : \quad \begin{array}{ccc} A^{\mathrm{h}}_{0,\Lambda} & \xrightarrow{\omega_{0,\Lambda}} & \Gamma(V'_\Lambda, \tau_V^* \mathscr{O}_{V,\mathrm{\acute{e}t}}) \\ \downarrow & & \downarrow \\ A^{\mathrm{h}}_\Lambda & \xrightarrow{\omega_\Lambda} & \Gamma(V'_\Lambda, \mathscr{F}) \end{array}$$

whose left vertical arrow is the inclusion map, and whose right vertical arrow is deduced from $\psi^\sharp : \mathscr{O}_{V,\mathrm{\acute{e}t}} \to \psi_* \mathscr{O}_{X,\mathrm{\acute{e}t}}$. Moreover, $\omega_{0,\Lambda}$ is an isomorphism of A_0-algebras for every $\Lambda \neq \varnothing$, and ω_Λ is an isomorphism of A-algebras for every Λ.

(ii) For every $\Lambda' \subset \Lambda$, let $\rho_{\Lambda',\Lambda} : \Gamma(V'_{\Lambda'}, \mathscr{F}) \to \Gamma(V'_\Lambda, \mathscr{F})$ be the restriction homomorphism associated with the inclusion $V'_\Lambda \subset V'_{\Lambda'}$. Then we have a commutative diagram

$$\begin{array}{ccc} A^{\mathrm{h}}_{\Lambda'} & \xrightarrow{\omega_{\Lambda'}} & \Gamma(V'_{\Lambda'}, \mathscr{F}) \\ {\scriptstyle \mu^{\mathrm{h}}_{\Lambda',\Lambda}} \downarrow & & \downarrow {\scriptstyle \rho_{\Lambda',\Lambda}} \\ A^{\mathrm{h}}_\Lambda & \xrightarrow{\omega_\Lambda} & \Gamma(V'_\Lambda, \mathscr{F}). \end{array}$$

Proof of the claim. (i): We consider first the case where $\Lambda = \varnothing$, and notice that $V_\varnothing = V$, $A^{\mathrm{h}}_{0,\varnothing} = A_0$ and $A^{\mathrm{h}}_\varnothing = A$. Under the current assumptions, the pair (A_0, I_0) is henselian; by [10, Exp.XII, Th.5.1(i) and Prop.6.5(i)] we then get for every sheaf $\mathscr{G}$ on $V_{\mathrm{\acute{e}t}}$ natural isomorphisms

$$\Gamma(V, \mathscr{G}) \xrightarrow{\sim} \Gamma(S, \pi_* \mathscr{G}) \xrightarrow{\sim} \Gamma(S', \tau_S^* \pi_* \mathscr{G}) \xrightarrow{\sim} \Gamma(S', \pi'_* \tau_V^* \mathscr{G}) \xrightarrow{\sim} \Gamma(V', \tau_V^* \mathscr{G}).$$

Taking $\mathscr{G} := \psi_* \mathscr{O}_{X_{\mathrm{\acute{e}t}}}$, we then obtain the sought isomorphism $\omega_\varnothing$

$$A = \Gamma(X, \mathscr{O}_{X_{\mathrm{\acute{e}t}}}) \xrightarrow{\sim} \Gamma(V, \psi_* \mathscr{O}_{X_{\mathrm{\acute{e}t}}}) \xrightarrow{\sim} \Gamma(V', \mathscr{F}).$$

Likewise, the map $\omega_{0,\varnothing}$ is defined as the composition

$$A_0 = \Gamma(S, \mathscr{O}_S) \to \Gamma(S, \pi_* \mathscr{O}_V) \xrightarrow{\sim} \Gamma(V', \tau_V^* \mathscr{O}_V)$$

from which the commutativity of $\mathscr{D}_\varnothing$ is straightforward. Next, let $\Lambda \neq \varnothing$, so that V_Λ is an affine scheme; set $V^{\mathrm{h}}_\Lambda := \operatorname{Spec} A^{\mathrm{h}}_{0,\Lambda}$ and $X_\Lambda := X \cap \operatorname{Spec} A_\Lambda$. We let $\psi_\Lambda : X_\Lambda \to V_\Lambda$ be the restriction of ψ, and we consider the cartesian diagram of schemes

$$\begin{array}{ccccc} & & X^{\mathrm{h}}_\Lambda & \xrightarrow{\tau''_{X_\Lambda}} & X_\Lambda \\ & & {\scriptstyle \psi^{\mathrm{h}}_\Lambda} \downarrow & & \downarrow {\scriptstyle \psi_\Lambda} \\ V'_\Lambda & \xrightarrow{\tau'_{V_\Lambda}} & V^{\mathrm{h}}_\Lambda & \xrightarrow{\tau''_{V_\Lambda}} & V_\Lambda \end{array}$$

where τ'_{V_Λ} and τ''_{V_Λ} are induced by the natural projection $A^{\mathrm{h}}_{0,\Lambda} \to A^{\mathrm{h}}_{0,\Lambda}/I^{\mathrm{h}}_{0,\Lambda} \xrightarrow{\sim} A_{0,\Lambda}/I_0 A_{0,\Lambda}$, and respectively the henselization map $A_{0,\Lambda} \to A^{\mathrm{h}}_{0,\Lambda}$. Then ω_Λ is defined as the composition of isomorphisms of A-algebras

$$\begin{aligned} A^{\mathrm{h}}_\Lambda = \Gamma(X^{\mathrm{h}}_\Lambda, \mathscr{O}_{X^{\mathrm{h}}_{\Lambda,\mathrm{\acute{e}t}}}) \xrightarrow{\sim} \Gamma(V^{\mathrm{h}}_\Lambda, \psi^{\mathrm{h}}_{\Lambda*} \circ \tau''^*_{X_\Lambda} \mathscr{O}_{X_{\Lambda,\mathrm{\acute{e}t}}}) &\xrightarrow{\sim} \Gamma(V^{\mathrm{h}}_\Lambda, \tau''^*_{V_\Lambda} \circ \psi_{\Lambda*} \mathscr{O}_{X_{\Lambda,\mathrm{\acute{e}t}}}) \\ &\xrightarrow{\sim} \Gamma(V'_\Lambda, \tau'^*_{V_\Lambda} \circ \tau''^*_{V_\Lambda} \circ \psi_{\Lambda*} \mathscr{O}_{X_{\Lambda,\mathrm{\acute{e}t}}}) \\ &\xrightarrow{\sim} \Gamma(V'_\Lambda, \mathscr{F}) \end{aligned}$$

where the bijectivity of the third map is due to [10, Exp.XII, Prop.6.5(i)], and that of the second one is due to the fact that τ''_{V_Λ} is the limit of a cofiltered system of affine étale V_Λ-schemes. Likewise, $\omega_{0,\Lambda}$ is obtained as the composition

$$A^{\mathrm{h}}_{0,\Lambda} = \Gamma(V^{\mathrm{h}}_\Lambda, \mathscr{O}_{V^{\mathrm{h}}_\Lambda}) \xrightarrow{\sim} \Gamma(V'_\Lambda, \tau'^*_{V_\Lambda}\mathscr{O}_{V^{\mathrm{h}}_\Lambda}) \xrightarrow{\sim} \Gamma(V'_\Lambda, \tau^*_V\mathscr{O}_V).$$

Then the commutativity of $\mathscr{D}_\Lambda$ follows by a direct inspection.

(ii): We consider the diagram of A_0-algebras

$$\begin{array}{ccccccc} A^{\mathrm{h}}_{0,\Lambda'} & \xrightarrow{\omega_{0,\Lambda'}} & \Gamma(V'_{\Lambda'}, \tau^*_V\mathscr{O}_{V,\text{ét}}) & \longrightarrow & \Gamma(V'_{\Lambda'}, \mathscr{F}) & \longleftarrow & \Gamma(V_{\Lambda'}, \psi_*\mathscr{O}_{X,\text{ét}}) = A_{\Lambda'} \\ \downarrow{\scriptstyle \mu^{\mathrm{h}}_{0,\Lambda',\Lambda}} & & \downarrow & & \downarrow{\scriptstyle \rho_{\Lambda',\Lambda}} & & \downarrow{\scriptstyle \mu_{\Lambda',\Lambda}} \\ A^{\mathrm{h}}_{0,\Lambda} & \xrightarrow{\omega_{0,\Lambda'}} & \Gamma(V'_\Lambda, \tau^*_V\mathscr{O}_{V,\text{ét}}) & \longrightarrow & \Gamma(V'_\Lambda, \mathscr{F}) & \longleftarrow & \Gamma(V_\Lambda, \psi_*\mathscr{O}_{X,\text{ét}}) = A_\Lambda \end{array}$$

whose central square subdiagram is induced by the natural morphism $\mathscr{O}_{V,\text{ét}} \to \psi_*\mathscr{O}_{X,\text{ét}}$, and the right square subdiagram is induced by the unit of adjunction $\psi_*\mathscr{O}_{X,\text{ét}} \to \tau_{V*}\tau^*_V\psi_*\mathscr{O}_{X,\text{ét}}$ for the adjoint pair (τ^*_V, τ_{V*}), and where $\mu_{\Lambda',\Lambda}$ and $\mu^{\mathrm{h}}_{0,\Lambda',\Lambda}$ are defined as in remark 15.5.26(ii). Therefore, both these subdiagrams commute; the left square subdiagram commutes as well, due to the universal property of henselization; taking into account (i), the assertion follows (details left to the reader). ◊

Lastly, we have a natural map

$$A \xrightarrow{\alpha} \mathrm{Equal}\Big(\prod_{i=0}^{n} A^{\mathrm{h}}_{\{i\}} \underset{\beta}{\overset{\gamma}{\rightrightarrows}} \prod_{i,j=0}^{n} A^{\mathrm{h}}_{\{i,j\}}\Big)$$

induced by the maps $\mu^{\mathrm{h}}_{\varnothing,\{i\}}$, where β and γ are deduced from the maps $\mu^{\mathrm{h}}_{\{i\},\{i,j\}}$. From claim 15.5.28 we deduce that α is an isomorphism, and taking into account remark 15.5.26(iii), the theorem follows. □

Remark 15.5.29. Let $\underline{X}$ be any topologically local quasi-affinoid scheme.

(i) As explained in [59, Ch.0, §3.2.2], every sheaf on $\mathscr{Q}(\underline{X})$ admits a natural extension to a sheaf $\mathscr{F}'$ on the topological space $\mathrm{Spa}\,\underline{X}$. Namely, if $U \subset \mathrm{Spa}\,\underline{X}$ is any open subset, one sets

$$\mathscr{F}'(U) := \lim_{V\in(\mathscr{Q}(\underline{X})/U)^o} \mathscr{F}(V)$$

where $\mathscr{Q}(\underline{X})/U$ denotes the full subcategory of $\mathscr{Q}(\underline{X})$ whose objects are the quasi-affinoid open subsets of $\mathrm{Spa}\,\underline{X}$ contained in U. We shall use the notation $\mathscr{O}^{\mathrm{h}}_{\mathrm{Spa}\,\underline{X}}$ and $\mathscr{O}^{\mathrm{h}+}_{\mathrm{Spa}\,\underline{X}}$ also to refer to the sheaves on $\mathrm{Spa}\,\underline{X}$ that extend naturally the respective sheaves on $\mathscr{Q}(\underline{X})$.

(ii) Simple examples show that, in most cases, the presheaves $\mathscr{O}^{\mathrm{loc}}_{\mathrm{Spa}\,\underline{X}}$ and $\mathscr{O}^{\mathrm{loc}+}_{\mathrm{Spa}\,\underline{X}}$ are *not* sheaves on the site $\mathscr{Q}(\underline{X})$. On the other hand, in several situations of interest the presheaves $\mathscr{O}^{\wedge}_{\mathrm{Spa}\,\underline{X}}$ and $\mathscr{O}^{\wedge+}_{\mathrm{Spa}\,\underline{X}}$ are sheaves of topological rings. In Huber's work [98], this sheaf property is proven under two different types of assumptions, corresponding roughly to the two main classes of non-archimedean analytic spaces that are encountered in applications : the "generic fibres" of locally noetherian formal schemes, and the rigid analytic varieties locally of finite type over a complete rank one valued field. In the literature, one finds also some attempts to unify these cases (and the few other known ones) under a single axiomatic framework; hereafter we shall adopt the approach (though, not the terminology) proposed in [71], that is based on the notions introduced in section 15.2. Later, we shall prove the sheaf property also in the case where $\mathscr{O}_X(X)$ admits a perfectoid ring of definition : this is a completely different situation, for which we shall need to develop an *ad hoc* method.

Definition 15.5.30. (i) Let A be any f-adic ring. We say that A is *f-adic analytically noetherian* (resp. *f-adic universally analytically noetherian*), if it admits an analytically noetherian (resp. universally analytically noetherian) ring of definition (see definition 15.2.1(i)).

(ii) Let $\underline{A} := (A, A^+)$ be any affinoid ring. We say that $\underline{A}$ is *analytically noetherian* (resp. *universally analytically noetherian*) if A is f-adic analytically noetherian (resp. f-adic universally analytically noetherian).

(iii) Let $\underline{A} := (A, A^+, U)$ be any quasi-affinoid ring. We say that $\underline{A}$ is *analytically noetherian* (resp. *universally analytically noetherian*) if the same holds for (A, A^+).

(iv) Let $\underline{X}$ be any quasi-affinoid scheme. We say that $\underline{X}$ is *analytically noetherian* (resp. *universally analytically noetherian*) if the same holds for $\Gamma(\underline{X})$.

15.5.31. Let A be any f-adic ring, and $n \in \mathbb{N}$ any integer. According to proposition 8.3.37(i,ii), there exists a unique f-adic topology $\mathscr{T}_n$ on $A[T_1, \dots, T_n]$ such that, if A_0 is any subring of definition of A, then $A_0[T_1, \dots, T_n]$ is a subring of definition of $A[T_1, \dots, T_n]$, and if $I \subset A_0$ is an ideal of adic definition, then $I[T_1, \dots, T_n]$ is an ideal of adic definition for $A_0[T_1, \dots, T_n]$.

Lemma 15.5.32. *With the notation of* (15.5.31)*, the following holds :*

(i) *A is f-adic analytically noetherian (resp. f-adic universally analytically noetherian) if and only if every ring of definition of A is analytically noetherian (resp. universally analytically noetherian).*

(ii) *A is f-adic universally analytically noetherian if and only if $(A[T_1, \dots, T_n], \mathscr{T}_n)$ is f-adic analytically noetherian for every $n \in \mathbb{N}$.*

(iii) *Let $B \subset A$ be any open subring, and endow B with the topology induced from A. Then A is f-adic analytically noetherian (resp. f-adic universally analytically noetherian) if and only if the same holds for B.*

(iv) *Let $f : A \to A'$ be a surjective ring homomorphism, and endow A' with the topology induced by A via f. Then, if A is f-adic analytically noetherian (resp. f-adic universally analytically noetherian), the same holds for A'.*

(v) *If $\underline{A}$ is any f-adic analytically noetherian (resp. f-adic universally analytically noetherian) quasi-affinoid ring,* $\operatorname{Spec} \underline{A}$ *is an analytically noetherian (resp. universally analytically noetherian) quasi-affinoid scheme.*

(vi) *Let $\underline{X}$ be a topologically local and universally analytically noetherian quasi-affinoid scheme, and $\underline{Y} \to \underline{X}$ a morphism of topologically local quasi-affinoid schemes that represents a rational subset of* $\operatorname{Spa} \underline{X}$*. Then $\underline{Y}$ is universally analytically noetherian.*

Proof. (i): Let $A_0, A_1 \subset A$ be two rings of definition, and let us show that A_0 is analytically noetherian if and only if the same holds for A_1. To this aim, suppose first that $A_0 \subset A_1$, and let $I \subset A_0$ be any finitely generated ideal of adic definition, so that IA_1 is an ideal of adic definition for A_1. Lemma 8.3.29(iii) says that the inclusion map $A_0 \to A_1$ identifies the analytic locus U_0 of $\operatorname{Spec} A_0$ with the analytic locus U_1 of $\operatorname{Spec} A_1$, so U_0 is a noetherian scheme if and only if the same holds for U_1. Next, suppose that A_0 is analytically noetherian; let M_1 be any A_1-module of finite type, $x_1, \dots, x_n$ any system of generators for M_1, and set $M_0 := A_0 x_1 + \cdots + A_0 x_n \subset M_1$. By proposition 8.3.18(ii), there exists $n \in \mathbb{N}$ such that $I^n A_1 \subset A_0$, whence $I^n M_1 \subset M_0$. On the other hand, by assumption there exists $r \in \mathbb{N}$ such that $\operatorname{Ann}_{M_0}(I^k) = \operatorname{Ann}_{M_0}(I^r)$ for every $k \geq r$. Thus, let $x \in \operatorname{Ann}_{M_1}(I^k A_1)$ for some $k \geq r$; we get $I^n x \in \operatorname{Ann}_{M_1}(I^k A_1) \cap M_0 = \operatorname{Ann}_{M_0}(I^k)$, so $I^{n+r} x = 0$, as asserted, in this case.

Suppose next that A_1 is analytically noetherian, and let M_0 be any A_0-module of finite type; the short exact sequence of A_0-modules $0 \to A_0 \to A_1 \to A_1/A_0 \to 0$ induces an exact sequence

$$T_1 := \operatorname{Tor}_1^{A_0}(A_1/A_0, M_0) \to M_0 \xrightarrow{\varphi} M_1 := A_1 \otimes_{A_0} M_0$$

and notice that $I^n T_1 = 0$. By assumption, there exists $r \in \mathbb{N}$ such that $\mathrm{Ann}_{M_1}(I^k A_1) = \mathrm{Ann}_{M_1}(I^r A_1)$ for every $k \geq r$. Suppose then that $x \in \mathrm{Ann}_{M_0}(I^k)$ for some $k \geq r$; it follows easily that $I^r x \subset \mathrm{Ker}\,\varphi$, whence $I^{r+n} x = 0$, as required.

Lastly, if A_0 and A_1 are arbitrary rings of definition for A, the same holds for $A_2 := A_0 \cdot A_1$ (corollary 8.3.19(i)). Suppose that A_0 is analytically noetherian; by the foregoing, it follows that the same holds for A_2, and then by the same token it holds for A_1 as well.

The assertion for the universally noetherian case is an immediate consequence.

Assertion (ii) follows directly from (i) : details left to the reader.

(iii) follows from (i) and corollary 8.3.19(ii).

(iv): First, we know that A' is f-adic, by example 8.3.27(iii), and if $A_0 \subset A$ is a ring of definition, $f(A_0)$ is a ring of definition of A'. But if A_0 is analytically noetherian (resp. universally analytically noetherian), the same holds for $f(A_0)$, by lemma 15.2.4(ii). The assertion follows.

(v): Say that $\underline{A} = (A, A^+, U)$, and let $\rho : A \to A_U := \mathscr{O}_{\mathrm{Spec}\,A}(U)$ be the restriction map. By definition, the topology $\mathscr{T}_{\rho(A)}$ on $\rho(A)$ induced by the inclusion into A_U agrees with the quotient topology induced by the surjection $A \to \rho(A)$. From (iv) we deduce that $(\rho(A), \mathscr{T}_{\rho(A)})$ is f-adic analytically noetherian (resp. f-adic universally analytically noetherian). Then, the assertion follows from (iii).

(vi) follows easily from (v) and lemma 15.2.4(ii), after inspecting the proof of theorem 15.5.5. □

Lemma 15.5.33. *Let $\underline{X} := (X, \mathscr{T}_X, A_X^+)$ be any analytically noetherian quasi-affinoid scheme. Then the natural morphism $\Gamma(\underline{X})^\wedge \to \Gamma(\underline{X}^\wedge)$ is an isomorphism (see remark* 15.4.13(ii)*).*

Proof. Let $A_X^\wedge$ be the separated completion of $A_X := \mathscr{O}_X(X)$, and set $Y := \mathrm{Spec}\,A_X^\wedge \times_{\mathrm{Spec}\,A_X} X$; in light of proposition 15.2.10(ii) and corollary 10.3.8, the natural map $A_X^\wedge \to \mathscr{O}_Y(Y)$ is an isomorphism. Then the assertion follows from remark 15.4.13(ii). □

15.5.34. For any topologically local quasi-affinoid scheme $\underline{X}$, we may consider the site

$$(\mathscr{R}(\underline{X}), J_{\mathscr{R}})$$

where $\mathscr{R}(\underline{X})$ is the full subcategory of $\mathscr{Q}(\underline{X})$ whose objects are the rational subsets of $\mathrm{Spa}\,\underline{X}$. The sieves covering a given object R of $\mathscr{R}(\underline{X})$ for the topology $J_{\mathscr{R}}$ are those generated by the families $R_\bullet := (R_\lambda \mid \lambda \in \Lambda)$ of rational subsets of $\mathrm{Spa}\,\underline{X}$ such that $\bigcup_{\lambda \in \Lambda} R_\lambda = R$. For every such $R_\bullet$, after fixing a total ordering on Λ we get an augmented alternating Čech complex $C^\bullet_{\mathrm{alt}}(R_\bullet, \mathscr{O}^\wedge_{\mathrm{Spa}\,\underline{X}})$, whose degree n term, for every $n \geq 0$, is a product of topological rings of the form $\mathscr{O}^\wedge_{\mathrm{Spa}\,\underline{X}}(R_{\lambda_0} \cap \cdots \cap R_{\lambda_n})$, with $\lambda_0 < \cdots < \lambda_n$ ranging over all strictly increasing sequences of elements of Λ (see remark 10.2.7(i). We endow $C^n_{\mathrm{alt}}(R_\bullet, \mathscr{O}^\wedge_{\mathrm{Spa}\,\underline{X}})$ with the corresponding product topology, and we notice that this topology is independent of the chosen ordering for Λ, and the differentials $d^\bullet$ of the Čech complex are continuous maps. With this notation, we have:

Theorem 15.5.35. *For any topologically local and universally analytically noetherian quasi-affinoid scheme $\underline{X} := (X, \mathscr{T}_X, A_X^+)$, and every $R \in \mathrm{Ob}(\mathscr{R}(\underline{X}))$, the following holds :*

(i) *The presheaves $\mathscr{O}^{\wedge+}_{\mathrm{Spa}\,\underline{X}}$ and $\mathscr{O}^\wedge_{\mathrm{Spa}\,\underline{X}}$ are sheaves of topological rings on $(\mathscr{R}(\underline{X}), J_{\mathscr{R}})$.*

(ii) *For every finite covering $\mathfrak{U}$ of R consisting of rational subsets, the differentials of $C^\bullet_{\mathrm{alt}}(\mathfrak{U}, \mathscr{O}^\wedge_{\mathrm{Spa}\,\underline{X}})$ are strict and its cohomology has the discrete topology.*

Proof. (i): Arguing as in the proof of theorem 15.5.27, we reduce to checking the assertion for $\mathscr{O}^\wedge_{\mathrm{Spa}\,\underline{X}}$. We remark :

Claim 15.5.36. For any given $R \in \mathrm{Ob}(\mathscr{R}(\underline{X}))$, let $\varphi_{\underline{Y}/\underline{X}} : \underline{Y} \to \underline{X}$ be any morphism of quasi-affinoid schemes representing the sub-presheaf h_R of $h_{\underline{X}}$; then the morphism $\varphi_{\underline{Y}/\underline{X}}$ induces an equivalence of categories

$$\mathscr{R}(\underline{Y}) \xrightarrow{\sim} \mathscr{R}(\underline{X})/R \qquad V \mapsto \mathrm{Spa}\,\varphi_{\underline{Y}/\underline{X}}(V) \subset R.$$

Moreover, $\underline{Y}$ is still universally analytically noetherian.

Proof of the claim. These assertions follow from lemmata 15.5.10 and 15.5.32(vi). $\Diamond$

Now, say that $\Gamma(\underline{X}) = (A, A^+, X)$ and let $R_\bullet := (R_0, \dots, R_n)$ be the standard covering of $\mathrm{Spa}\,\underline{X}$ attached to a given sequence $f_\bullet := (f_0, \dots, f_n)$ of elements of A; define also the rational subset R_Λ as in remark 15.5.26(i), for every subset $\Lambda \subset \{0, \dots, n\}$. In light of claim 15.5.36 and remark 5.5.2(i), we are reduced to showing that the natural map

$$\mathscr{O}^\wedge_{\mathrm{Spa}\,\underline{X}}(\mathrm{Spa}\,\underline{X}) \to \mathrm{Equal}\Big(\prod_{i=0}^n \mathscr{O}^\wedge_{\mathrm{Spa}\,\underline{X}}(R_i) \rightrightarrows \prod_{i,j=0}^n \mathscr{O}^\wedge_{\mathrm{Spa}\,\underline{X}}(R_{\{i,j\}})\Big) \tag{15.5.37}$$

is an isomorphism of topological rings. To this aim, we pick a subring of definition $A_0 \subset A$, a finitely generated ideal $I_0 \subset A_0$ of adic definition, we set $S := \mathrm{Spec}\,A_0$, we denote by $J \subset A$ the ideal generated by $f_\bullet$, and we consider the system $(\underline{A}_\Lambda \mid \Lambda \subset \{0, \dots, n\})$ of quasi-affinoid rings, and their rings of definition $A_{0,\Lambda} \subset A_\Lambda$ constructed in remark 15.5.26(i). Let also $A^\wedge_\Lambda$ be the separated completion of A_Λ, and notice that the map

$$\omega^\wedge_\Lambda : A^\wedge_\Lambda \to \mathscr{O}^\wedge_{\mathrm{Spa}\,\underline{X}}(R_\Lambda)$$

of remark 15.5.26(ii) is an isomorphism of topological A-algebras for every $\Lambda \subset \{0, \dots, n\}$, by lemma 15.5.33. Define S, V, $\psi : X \to V$ and V_Λ for every subset $\Lambda \subset \{0, \dots, n\}$ as in remark 15.5.26(v); set also $S' := \mathrm{Spec}\,A_0/I_0$ and let $V^\wedge$ be the completion of V along $S' \times_S V$, and $\pi : V^\wedge \to V$ the induced morphism of locally ringed spaces. We define

$$\mathscr{F} := \pi^*\psi_*\mathscr{O}_X \qquad \text{and} \qquad V^\wedge_\Lambda := \pi^{-1}V_\Lambda \qquad \text{for every } \Lambda \subset \{0, \dots, n\}$$

and we regard $\mathscr{F}$ as a presheaf of topological abelian groups, as follows. The natural morphism $\mathscr{O}_V \to \psi_*\mathscr{O}_X$ induces a morphism

$$\varphi : \mathscr{O}_{V^\wedge} \to \mathscr{F}$$

and for every open subset $U \subset V^\wedge$ we endow $\mathscr{F}(U)$ with the unique group topology such that the resulting map $\varphi_U : \mathscr{O}_{V^\wedge}(U) \to \mathscr{F}(U)$ is continuous and open. It follows easily that for every inclusion $U' \subset U$ of open subsets of $V^\wedge$, the restriction map $\mathscr{F}(U) \to \mathscr{F}(U')$ is continuous.

Claim 15.5.38. (i) $\mathscr{F}(V^\wedge_\Lambda)$ is a topological ring for every $\Lambda \subset \{0, \dots, n\}$, and there exists a natural isomorphism of topological A-algebras

$$\omega_\Lambda : A^\wedge_\Lambda \xrightarrow{\sim} \mathscr{F}(V^\wedge_\Lambda).$$

(ii) For every $\Lambda' \subset \Lambda$, let $\rho_{\Lambda',\Lambda} : \mathscr{F}(V^\wedge_{\Lambda'}) \to \mathscr{F}(V^\wedge_\Lambda)$ be the restriction homomorphism associated with the inclusion $V^\wedge_\Lambda \subset V^\wedge_{\Lambda'}$. Then we have a commutative diagram

$$\begin{array}{ccc} A^\wedge_{\Lambda'} & \xrightarrow{\omega_{\Lambda'}} & \mathscr{F}(V^\wedge_{\Lambda'}) \\ {\scriptstyle \mu^\wedge_{\Lambda',\Lambda}}\downarrow & & \downarrow{\scriptstyle \rho_{\Lambda',\Lambda}} \\ A^\wedge_\Lambda & \xrightarrow{\omega_\Lambda} & \mathscr{F}(V^\wedge_\Lambda). \end{array}$$

Proof of the claim. (i): We consider first the case where $\Lambda = \varnothing$, and recall that $V_\varnothing = V$. Notice also that $\psi_*\mathscr{O}_X$ is a quasi-coherent $\mathscr{O}_V$-module, by virtue of [59, Ch.I, Cor.9.2.2]. Since V is a projective S-scheme, corollary 15.2.34(iii) and remark 15.2.38 yield a natural isomorphism

$$\omega_\varnothing : A^\wedge \xrightarrow{\sim} A^\wedge_0 \otimes_{A_0} A \xrightarrow{\sim} A^\wedge_0 \otimes_{A_0} \Gamma(V, \psi_*\mathscr{O}_X) \xrightarrow{\sim} \mathscr{F}(V^\wedge)$$

fitting into a commutative diagram

$$\begin{array}{ccc} A_0^\wedge & \longrightarrow & A^\wedge \\ \downarrow & & \downarrow{\scriptstyle \omega_\varnothing} \\ \mathscr{O}_{V^\wedge}(V^\wedge) & \xrightarrow{\varphi_{V^\wedge}} & \mathscr{F}(V^\wedge) \end{array}$$

whose left vertical arrow is the I_0-adic completion of the natural map $A_0 \to \mathscr{O}_V(V)$, and whose top horizontal arrow is the inclusion map. It follows already that $\omega_\varnothing$ is a continuous map, and it thus remains only to check that $\omega_\varnothing$ is an open map. To this aim, it suffices to check that the same holds for the left vertical arrow $A_0^\wedge \to \mathscr{O}_{V^\wedge}(V^\wedge)$. However, $\mathscr{O}_V(V)$ is an A_0-module of analytically finite type, by theorem 15.2.22, and the support of $\mathscr{O}_V(V)/A_0$ lies in the non-analytic locus of S, by virtue of remark 15.5.26(vi); by remark 15.2.2(iii) we deduce that $I_0^t \cdot \mathscr{O}_V(V) \subset A_0$ for some $t \in \mathbb{N}$, whence $I_0^t \cdot \mathscr{O}_{V^\wedge}(V^\wedge) \subset A_0^\wedge$. Since the topology of $\mathscr{O}_{V^\wedge}(V^\wedge)$ is I_0-adic (corollary 15.2.34(ii)), the assertion follows.

In case Λ is not empty, V_Λ is affine, and a direct inspection yields a natural isomorphism of topological rings $A_{0,\Lambda}^\wedge \xrightarrow{\sim} \mathscr{O}_{V^\wedge}(V_\Lambda^\wedge)$. We may then appeal to proposition 15.2.20(iii) to see that the natural map

(15.5.39) $$A_\Lambda^\wedge \xrightarrow{\sim} A_{0,\Lambda}^\wedge \otimes_{A_{0,\Lambda}} A_\Lambda \xrightarrow{\sim} A_{0,\Lambda}^\wedge \otimes_{A_{0,\Lambda}} \Gamma(V_\Lambda, \psi_*\mathscr{O}_X) \to \mathscr{F}(V_\Lambda^\wedge)$$

is also an isomorphism of rings, fitting into another commutative diagram

(15.5.40) $$\begin{array}{ccc} A_{0,\Lambda}^\wedge & \longrightarrow & A_\Lambda^\wedge \\ \downarrow & & \downarrow \\ \mathscr{O}_{V^\wedge}(V_\Lambda^\wedge) & \xrightarrow{\varphi_{V_\Lambda^\wedge}} & \mathscr{F}(V_\Lambda^\wedge) \end{array}$$

whose top horizontal arrow is again the inclusion map, so $\mathscr{F}(V_\Lambda^\wedge)$ is also a topological ring with the topology defined in the foregoing, and (15.5.39) is even an isomorphism of topological rings, if $\Lambda \neq \varnothing$.

(ii): The abelian groups appearing in the diagram are all endowed with natural $A_{\Lambda'}^\wedge$-module structures, and it is easily seen that all the arrows are both $A_{0,\Lambda'}^\wedge$-linear and $A_{\Lambda'}$-linear maps, whence the assertion. ◊

Next, consider the affine open covering $V_\bullet^\wedge := (V_{\{i\}}^\wedge \mid i = 0, \dots, n)$ of $V^\wedge$, and endow the terms of the augmented alternating Čech complex $C_{\mathrm{alt}}^\bullet(V_\bullet^\wedge, \mathscr{F})$ with the product topologies, as in (15.5.34). Combining claim 15.5.38 and remark 15.5.26(iii) we deduce an isomorphism of complexes of topological abelian groups

(15.5.41) $$C_{\mathrm{alt}}^\bullet(V_\bullet^\wedge, \mathscr{F}) \xrightarrow{\sim} C_{\mathrm{alt}}^\bullet(R_\bullet, \mathscr{O}_{\mathrm{Spa}\,\underline{X}}^\wedge).$$

Hence, to conclude it suffices to show :

Claim 15.5.42. The complex $C_{\mathrm{alt}}^\bullet(V_\bullet^\wedge, \mathscr{F})$ has strict differentials, and its cohomology has the discrete topology.

Proof of the claim. Diagram (15.5.40) implies that $\varphi_{V_\Lambda^\wedge}$ is injective for every $\Lambda \neq \varnothing$; it follows easily that the same holds for $\varphi_{V^\wedge}$. Hence the induced map of alternating Čech complexes

$$C_{\mathrm{alt}}^\bullet(V_\bullet^\wedge, \varphi) : C_{\mathrm{alt}}^\bullet(V_\bullet^\wedge, \mathscr{O}_{V^\wedge}) \to C_{\mathrm{alt}}^\bullet(V_\bullet^\wedge, \mathscr{F})$$

is injective in every degree, and by construction $C_{\mathrm{alt}}^i(V_\bullet^\wedge, \varphi)$ is an open map for every $i \in \mathbb{Z}$, so we are reduced to showing that the differentials d^i of $C_{\mathrm{alt}}^\bullet(V_\bullet^\wedge, \mathscr{O}_{V^\wedge})$ are strict and induce open maps $C_{\mathrm{alt}}^i(V_\bullet^\wedge, \mathscr{O}_{V^\wedge}) \to \mathrm{Ker}\, d^{i+1}$ for every $i \in \mathbb{Z}$. If $i < -1$, there is nothing to show. For

$i=-1$, the assertion follows from remark 5.5.2(i). In the remaining cases, the assertion comes down to the following. For every $i,t\in\mathbb{N}$ there exists $s\in\mathbb{N}$ such that

$$I_0^s\cdot C^{i+1}_{\mathrm{alt}}(V^\wedge_\bullet,\mathscr{O}_{V^\wedge})\cap\operatorname{Ker}d^{i+1}\subset I_0^t\cdot\operatorname{Im}d^i. \qquad (15.5.43)$$

For every $k\in\mathbb{N}$, denote by $d^\bullet_k$ the differentials of the Čech complex $C^\bullet_{\mathrm{alt}}(V^\wedge_\bullet,I_0^k\mathscr{O}_{V^\wedge})$; condition (15.5.43) is equivalent to

$$\operatorname{Ker}d^{i+1}_s\subset\operatorname{Im}d^i_t.$$

Taking into account theorem 15.1.37(i), we are therefore reduced to showing that for every $i,t\in\mathbb{N}$ there exists an integer $s\geq t$ such that the natural map

$$H^{i+1}(V^\wedge,I_0^s\mathscr{O}_{V^\wedge})\to H^{i+1}(V^\wedge,I_0^t\mathscr{O}_{V^\wedge})$$

vanishes. Taking into account corollaries 15.2.30(i) and 15.2.34(i), we are further reduced to showing that for every $i,t\in\mathbb{N}$ there exists $s\in\mathbb{N}$ such that $I^s\cdot H^{i+1}(V,I_0^t\mathscr{O}_V)=0$. Notice that the A_0-module $H^{i+1}(V,I_0^t\mathscr{O}_V)$ is analytically of finite type (theorem 15.2.22 and remark 15.2.38(i)); in light of remark 15.2.2(iii) it then suffices to check that the support of $H^{i+1}(V,I_0^t\mathscr{O}_V)$ lies in the non-analytic locus of S. The latter follows easily from remark 15.5.26(vi) : details left to the reader. ◊

(ii): Arguing as in the proof of (i), we reduce easily to the case where $R=\operatorname{Spa}\underline{X}$. Say that $\mathfrak{U}=(U_i\mid i\in I)$ for some set I, and pick any standard covering $\mathfrak{U}':=(U'_i\mid i\in I')$ of $\operatorname{Spa}\underline{X}$ that refines $\mathfrak{U}$; after fixing total orderings for I and I' we may consider the double complex

$$C^{\bullet\bullet}_{\mathrm{alt}}(\mathfrak{U},\mathfrak{U}'):=\operatorname{Hom}^\bullet_{\mathbb{Z}_X}(R^{\mathrm{alt}}_{\bullet\bullet}(\mathfrak{U},\mathfrak{U}'),\mathscr{O}^\wedge_{\operatorname{Spa}\underline{X}})$$

(notation of remark 10.2.14(ii)) which vanishes in every bidegree (p,q) with either $p<-1$ or $q<-1$. For every $n\in\mathbb{N}$, define I^{n+1}_{alt} and I'^{n+1}_{alt} as in (10.2.4), and for every $\underline{t}\in I^{n+1}_{\mathrm{alt}}$ and every $\underline{t}'\in I'^{n+1}_{\mathrm{alt}}$, set $\mathfrak{U}'_{\underline{t}}:=(U'_{i'}\cap U_{\underline{t}}\mid i'\in I')$ and $\mathfrak{U}_{\underline{t}'}:=(U_i\cap U_{\underline{t}'}\mid i\in I)$. We notice that :

- The complex $C^{\bullet,-1}_{\mathrm{alt}}(\mathfrak{U},\mathfrak{U}')$ coincides with $C^\bullet_{\mathrm{alt}}(\mathfrak{U},\mathscr{O}^\wedge_{\operatorname{Spa}\underline{X}})$.
- The complex $C^{-1,\bullet}_{\mathrm{alt}}(\mathfrak{U},\mathfrak{U}')$ coincides with $C^\bullet_{\mathrm{alt}}(\mathfrak{U}',\mathscr{O}^\wedge_{\operatorname{Spa}\underline{X}})$.
- For every $n\in\mathbb{N}$, the complex $C^{\bullet,n}_{\mathrm{alt}}(\mathfrak{U},\mathfrak{U}')$ is the product $\prod_{\underline{t}'\in I'^{n+1}_{\mathrm{alt}}}C^\bullet_{\mathrm{alt}}(\mathfrak{U}_{\underline{t}'},\mathscr{O}^\wedge_{\operatorname{Spa}\underline{X}})$.
- For every $n\in\mathbb{N}$, the complex $C^{n,\bullet}_{\mathrm{alt}}(\mathfrak{U},\mathfrak{U}')$ is the product $\prod_{\underline{t}\in I^{n+1}_{\mathrm{alt}}}C^\bullet_{\mathrm{alt}}(\mathfrak{U}'_{\underline{t}},\mathscr{O}^\wedge_{\operatorname{Spa}\underline{X}})$.

Since $\mathfrak{U}'_{\underline{t}}$ is a standard covering of $U_{\underline{t}}$, the proof of (i) shows that the differentials of the complex $C^\bullet_{\mathrm{alt}}(\mathfrak{U}'_{\underline{t}},\mathscr{O}^\wedge_{\operatorname{Spa}\underline{X}})$ induce open maps $\delta^{n,\bullet}_v$ as in (8.6.11), for every $n\in\mathbb{N}$. Likewise, the same holds for the induced maps $\delta^{-1,\bullet}_v$. Moreover, since $\mathfrak{U}'$ refines $\mathfrak{U}$, for every $n\in\mathbb{N}$ and every $\underline{t}'\in I'^{n+1}_{\mathrm{alt}}$ there exists $i\in I$ such that $U'_{\underline{t}'}\subset U_i$, in which case remark 10.2.13 yields a homotopy $h^\bullet$ from the identity automorphism of $C^\bullet_{\mathrm{alt}}(\mathfrak{U}_{\underline{t}'},\mathscr{O}^\wedge_{\operatorname{Spa}\underline{X}})$ to the zero map, and a simple inspection shows that h^i is a continuous map, for every $i\in\mathbb{Z}$. Then, lemma 8.6.15(i) says that the differentials of $C^\bullet_{\mathrm{alt}}(\mathfrak{U}_{\underline{t}'},\mathscr{O}^\wedge_{\operatorname{Spa}\underline{X}})$ induce open maps $\delta^{\bullet,n}_h$ as well, for every $n\in\mathbb{N}$. Lastly, we invoke corollary 8.6.12 to conclude. □

15.5.44. Let $\underline{X}:=(X,\mathscr{T}_X,A^+_X)$ be a quasi-affinoid scheme fulfilling the assumptions of theorem 15.5.35, so that $\mathscr{O}^\wedge_{\operatorname{Spa}\underline{X}}$ is a sheaf of topological rings on the site $(\mathscr{R}(\underline{X}),J_{\mathscr{R}})$. Following [59, Ch.0, §3.2.2] (and see also remark 15.5.29(i)), we may extend (uniquely up to unique isomorphism) both $\mathscr{O}^\wedge_{\operatorname{Spa}\underline{X}}$ and $\mathscr{O}^{\wedge+}_{\operatorname{Spa}\underline{X}}$ to sheaves of topological groups on the topological space $\operatorname{Spa}\underline{X}$, and we shall denote these extensions with the same names. Then the pair $(\operatorname{Spa}\underline{X},\mathscr{O}^\wedge_{\operatorname{Spa}\underline{X}})$ is a topologically ringed space. Moreover, lemma 15.5.14 says that $\operatorname{Spa}\underline{X}$ is a locally ringed space, so there exists a unique morphism of locally ringed spaces

$$\sigma_{\underline{X}}:(\operatorname{Spa}\underline{X},\mathscr{O}^\wedge_{\operatorname{Spa}\underline{X}})\to\operatorname{Spec}A \qquad (15.5.45)$$

such that the corresponding map $\mathscr{O}_{\mathrm{Spec}\,A} \to \sigma_{\underline{X}*}\mathscr{O}^{\wedge}_{\mathrm{Spa}\,\underline{X}}$ yields on global sections the natural map $A \to A^{\wedge} := \mathscr{O}^{\wedge}_{\mathrm{Spa}\,\underline{X}}(\mathrm{Spa}\,\underline{X})$ ([58, Ch.I, Prop.1.6.3]). By inspecting *loc.cit.* it is easily seen that $\sigma_{\underline{X}}$ is the composition

$$\mathrm{Spa}\,\underline{X} \subset \mathrm{Cont}\,A \xrightarrow{\sigma_A} \mathrm{Spec}\,A$$

where σ_A is the restriction of the support map (see remark 9.2.4(iii)). Especially, the image of $\sigma_{\underline{X}}$ lies in X, whence a well defined morphism of locally ringed spaces

$$(\mathrm{Spa}\,\underline{X}, \mathscr{O}^{\wedge}_{\mathrm{Spa}\,\underline{X}}) \to (X, \mathscr{O}_X).$$

Corollary 15.5.46. *With the notation of* (15.5.44)*, the morphism $\sigma_{\underline{X}}$ induces an isomorphism*

$$A^{\wedge} \otimes_A H^i(X, \mathscr{O}_X) \xrightarrow{\sim} H^i(\mathrm{Spa}\,\underline{X}, \mathscr{O}^{\wedge}_{\mathrm{Spa}\,\underline{X}}) \qquad \textit{for every } i \in \mathbb{N}.$$

Proof. Let us show first the following special case :

Claim 15.5.47. In the situation of the corollary, suppose moreover that $\underline{X}$ is affinoid. Then $H^i(\mathrm{Spa}\,\underline{X}, \mathscr{O}^{\wedge}_{\mathrm{Spa}\,\underline{X}}) = 0$ for every $i > 0$.

Proof of the claim. In light of theorem 10.2.24(i) and lemma 15.5.21, it suffices to show that for every rational subset R of $\mathrm{Spa}\,\underline{X}$, and every standard covering $\mathfrak{U}$ of R, the alternating Čech complex $C^{\bullet}_{\mathrm{alt}}(\mathfrak{U}, \mathscr{O}^{\wedge}_{\mathrm{Spa}\,\underline{X}})$ is acyclic. However, notice that any such R is represented by an affinoid (topologically local) scheme; then, arguing as in the proof of theorem 15.5.35(i), we are easily reduced to the case where $R = \mathrm{Spa}\,\underline{X}$. Then, after choosing a subring of definition for A we may define the scheme V, its formal completion $V^{\wedge}$, the affine open coverings $V_{\bullet}$ and $V^{\wedge}_{\bullet}$, the projection $\pi : V^{\wedge} \to V$ and the morphism of schemes $\psi : X \to V$ as in the proof of theorem 15.5.35. In light of theorem 15.1.37(i) and of the isomorphism (15.5.41), we are then reduced to showing that $H^i(V^{\wedge}, \pi^*\psi_*\mathscr{O}_X) = 0$ for every $i > 0$. Then, corollary 15.2.34(iii) further reduces to checking that $H^i(V, \psi_*\mathscr{O}_X) = 0$ for every $i > 0$; but since X is affine, the morphism ψ is affine, so that $H^i(V, \psi_*\mathscr{O}_X) \simeq H^i(X, \mathscr{O}_X)$, whence the contention. ◊

Now, say that $X = \mathrm{Spec}\,A \setminus \mathrm{Spec}\,A/J$ for a finitely generated ideal $J \subset A$, and pick a finite system of generators $f_{\bullet} := (f_0, \dots, f_n)$ for J and a subring of definition $A_0 \subset A$. Let also $R_{\bullet}$ be the standard covering of $\mathrm{Spa}\,\underline{X}$ associated with $f_{\bullet}$, and notice that R_i is represented by an affinoid scheme, for every $i = 0, \dots, n$. Set as well $U_i := \mathrm{Spec}\,A[f_i^{-1}]$ for $i = 0, \dots, n$, and notice that $\sigma_{\underline{X}}(R_i) \subset U_i$ for $i = 0, \dots, n$, and $X = U_0 \cup \dots \cup U_n$. There follows a commutative diagram (notation of (10.2.18))

$$\begin{array}{ccc} A_0^{\wedge} \otimes_{A_0} H^i_{\mathrm{alt}}(U_{\bullet}, \mathscr{O}_X) & \xrightarrow{\alpha} & H^i_{\mathrm{alt}}(R_{\bullet}, \mathscr{O}^{\wedge}_{\mathrm{Spa}\,\underline{X}}) \\ \downarrow & & \downarrow \\ A_0^{\wedge} \otimes_{A_0} H^i(X, \mathscr{O}_X) & \longrightarrow & H^i(\mathrm{Spa}\,\underline{X}, \mathscr{O}^{\wedge}_{\mathrm{Spa}\,\underline{X}}) \end{array}$$

where the vertical arrows are isomorphisms, by virtue of claim 15.5.47 and corollary 10.2.21(ii). Since $A^{\wedge} = A_0^{\wedge} \otimes_{A_0} A$, it then suffices to check that α is an isomorphism. On the other hand, as in the proof of theorem 15.5.35, we may associate with the sequence $f_{\bullet}$ a morphism of schemes $\psi : X \to V$, the formal completion $V^{\wedge}$ of V with its natural projection $\pi : V^{\wedge} \to V$, and an affine covering $V_{\bullet} := (V_i \mid i = 0, \dots, n)$ of V such that $\psi^{-1}V_i = U_i$ for $i = 0, \dots, n$; we set $V_i^{\wedge} := \pi^{-1}V_i$ for $i = 0, \dots, n$, and $\mathscr{F} := \pi^*\psi_*\mathscr{O}_X$. There follows a commutative diagram

$$\begin{array}{ccc} A_0^{\wedge} \otimes_{A_0} H^i_{\mathrm{alt}}(V_{\bullet}, \psi_*\mathscr{O}_X) & \xrightarrow{\beta} & H^i_{\mathrm{alt}}(V^{\wedge}_{\bullet}, \mathscr{F}) \\ \downarrow & & \downarrow \\ A_0^{\wedge} \otimes_{A_0} H^i(V, \psi_*\mathscr{O}_X) & \longrightarrow & H^i(V^{\wedge}, \mathscr{F}) \end{array}$$

whose vertical arrows are again isomorphisms, by virtue of proposition 15.2.20(iv), and the same holds for the bottom horizontal arrow, by corollary 15.2.34(iii). To conclude, it suffices to observe that $H^i_{\mathrm{alt}}(U_\bullet, \mathscr{O}_X) = H^i_{\mathrm{alt}}(V_\bullet, \psi_*\mathscr{O}_X)$, and that the isomorphism of Čech complexes (15.5.41) identifies α with β. □

Example 15.5.48. We present two counterexamples that show how theorem 15.5.35 can fail for affinoid schemes that are not analytically noetherian. To begin with, let K be a rank one complete valued field, and fix an element $\pi \neq 0$ in the maximal ideal of K^+. For every $K^+[T]$-module J without π-torsion, let us set

$$A_J := K^+[T] \oplus J \qquad A_{J,K} := A_J \otimes_{K^+} K$$

and endow A_J with the π-adic topology and with the $K^+[T]$-algebra structure such that the inclusion map $K^+[T] \to A$ is a ring homomorphism and J is an ideal with $J^2 = 0$ (*i.e.* $(x, y)\cdot(x', y') := (xx', xy'+x'y)$ for every $(x, y), (x', y') \in A$). Clearly A is an adic topological ring, and we endow $A_{J,K}$ with the f-adic topology such that A_J is a ring of definition. Let also $A^\wedge_J$ and $A^\wedge_{J,K}$ be the completions of A_J and respectively $A_{J,K}$, and $A^{\wedge+}_{J,K}$ the integral closure of $A^\wedge_J$ in $A^\wedge_{J,K}$, and set

$$\underline{X} := \mathsf{Spec}(A^\wedge_{J,K}, A^{\wedge+}_{J,K}).$$

• Take first $J := K^+[T, T^{-1}]/K^+[T]$. We claim that in this case the presheaf $\mathscr{O}^\wedge_{\mathrm{Spa}\,\underline{X}}$ on $\mathscr{R}(\underline{X})$ is not separated. Indeed, consider the standard covering associated with the sequence (π, T); it consists of the rational subsets

$$U_0 := \{v \in \mathrm{Spa}\,\underline{X} \mid v(T) \leq v(\pi)\} \qquad \text{and} \qquad U_1 := \{v \in \mathrm{Spa}\,\underline{X} \mid v(\pi) \leq v(T)\}$$

and $\mathscr{O}^\wedge_{\mathrm{Spa}\,\underline{U}_0}(U_0) = B^\wedge_0 \otimes_{K^+} K$, where $B^\wedge_0$ is the completion of $B_0 := A_J[T/\pi] \subset A_{J,K}$, where the latter is endowed with its π-adic topology. But notice that for every $k, n > 0$ the element T^{-k} of J can be written in B_0 as

$$T^{-k} = T^{-k-n} \cdot (T/\pi)^n \cdot \pi^n \in \pi^n B_0.$$

Since n is arbitrary, it follows that T^{-k} lies in the kernel of the restriction map $\rho_0 : A^\wedge_{J,K} \to \mathscr{O}^\wedge_{\mathrm{Spa}\,\underline{X}}(U_0)$, and since k is arbitrary, we get : $J \subset \mathrm{Ker}\,\rho_0$. Likewise, $\mathscr{O}^\wedge_{\mathrm{Spa}\,\underline{U}_0}(U_1) = B^\wedge_1 \otimes_{K^+} K$, where $B^\wedge_1$ is the completion of $B_1 := A_J[\pi/T] \subset A_J[T^{-1}]$, where the latter is endowed with its π-adic topology. Clearly $A_J[T^{-1}] = K^+[T, T^{-1}]$, whence $J \subset \mathrm{Ker}\,(\rho_1 : A^\wedge_{J,K} \to \mathscr{O}^\wedge_{\mathrm{Spa}\,\underline{X}}(U_1))$ as well, and the claim follows.

• Next, we take J to be the $\mathbb{N}$-graded $K^+[T]$-module such that

$$\mathrm{gr}_n J := T^{-n}K^+[T]/K^+[T] \qquad \text{for every } n \in \mathbb{N}.$$

It follows that $\mathscr{O}^\wedge_{\mathrm{Spa}\,\underline{X}}(U_0) = C_0 \otimes_{K^+} K$, where C_0 is the π-adic completion of $K^+[T/\pi] \oplus J'$, with J' the $\mathbb{N}$-graded $K^+[T/\pi]$-module such that

$$\mathrm{gr}_n J' := \sum_{d=0}^{n-1} K^+T^{-n}(T/\pi)^d \qquad \text{for every } n \in \mathbb{N}.$$

Especially, $\mathscr{O}^\wedge_{\mathrm{Spa}\,\underline{X}}(U_0)$ contains the π-adic completion $J'^\wedge$ of J'. On the other hand, it is clear that $A_J[T^{-1}] = K[T, T^{-1}]$, so the kernel of the restriction map $\rho_{01} : \mathscr{O}^\wedge_{\mathrm{Spa}\,\underline{X}}(U_0) \to \mathscr{O}^\wedge_{\mathrm{Spa}\,\underline{X}}(U_0 \cap U_1)$ contains $J'^\wedge$. Now, consider the series

$$P := \sum_{n\in\mathbb{N}} T^{-n} \qquad \text{where } T^{-n} \in \mathrm{gr}_{2n} J' \text{ for every } n \in \mathbb{N}.$$

Then $T^{-n} = T^{-2n} \cdot (T/\pi)^n \cdot \pi^n \in \pi^n \mathrm{gr}_{2n} J'$, so the series P converges π-adically to a well defined element of $J'^\wedge$, and $\rho_{01}(P) = 0 = \rho_{10}(0)$ (where $\rho_{10} : \mathscr{O}^\wedge_{\mathrm{Spa}\,\underline{X}}(U_1) \to \mathscr{O}^\wedge_{\mathrm{Spa}\,\underline{X}}(U_0 \cap U_1)$ is the other restriction map). We shall show that the series P does not lie in the image of ρ_0, so the presheaf $\mathscr{O}^{\wedge}_{\mathrm{Spa}\,\underline{X}}$ is separated but is not a sheaf. Indeed, recall that the completion $J^{\wedge}$ of J lies in the product of the completions $\mathrm{gr}_n J^{\wedge}$ of its graded summands, and likewise for $J'^{\wedge}$ (remark 8.5.2(iii)); moreover, the induced map $J^{\wedge} \to J'^{\wedge}$ is the restriction of the product of the corresponding maps $\mathrm{gr}_n J^{\wedge} \to \mathrm{gr}_n J'^{\wedge}$. It is easily seen that the latter maps are injective, so if P were in the image of ρ_0, it would be represented by the sequence $(P_n \mid n \in \mathbb{N})$ where $P_{2k+1} = 0$ and $P_{2k} = T^{-k} \in \mathrm{gr}_{2k} J^{\wedge}$ for every $k \in \mathbb{N}$. But then the series $\sum_{n\in\mathbb{N}} P_n$ must converge π-adically in $J^{\wedge}$ (proposition 8.5.3(ii)); the latter means that for every $t \in \mathbb{N}$ there exists $i \in \mathbb{N}$ such that $T^{-k} \in \pi^t \mathrm{gr}_{2k} J$ for every $k \geq i$. But T^{-k} is not divisible by π in $\mathrm{gr}_{2k} J$, for any $k \in \mathbb{N}$, a contradiction.

Remark 15.5.49. The earliest counterexample showing that in some cases the presheaf $\mathscr{O}^{\wedge}_{\mathrm{Spa}\,\underline{X}}$ is not a sheaf, occurs already in [100], where it is attributed to Rost. Constructions similar to that of example 15.5.48, and several others as well, may also be found in [45] and [129], which introduce an interesting general class of "stably uniform" Tate affinoid rings $\underline{A}$ for which the authors can prove that the presheaf $\mathscr{O}^{\wedge}_{\mathrm{Spa}\,\underline{A}}$ is a sheaf.

15.6. **Special loci of quasi-affinoid schemes.** The results of this section will be used in chapter 16, for the study of perfectoid spaces.

Definition 15.6.1. Let $\underline{X} := (X, \mathscr{T}_X, A^+_X)$ be any quasi-affinoid scheme, $V \subset X^+ := \mathrm{Spec}\, A^+_X$ any open subset, and set $A_X := \mathscr{O}_X(X)$. Let moreover $\beta_{\underline{X}} : X \to X^+$ be the restriction of the morphism of schemes $\gamma_{\underline{X}} : \mathrm{Spec}\, A_X \to X^+$ induced by the inclusion map $A^+_X \to A_X$.

(i) We say that $\underline{X}$ *spreads over* V, if $\beta_{\underline{X}}$ restricts to an isomorphism of schemes $\beta^{-1}_{\underline{X}} V \xrightarrow{\sim} V$.

(ii) Let $\underline{Y} := (Y, \mathscr{T}_Y, A^+_Y)$ be another quasi-affinoid scheme, and $\varphi : \underline{Y} \to \underline{X}$ an f-adic morphism of quasi-affinoid schemes. Then the induced ring homomorphism $\varphi^\flat_Y : A_Y \to A_X$ restricts to an f-adic map $\varphi^{\flat+}_Y : A^+_Y \to A^+_X$, and we set

$$\varphi^+ := \mathrm{Spec}\, \varphi^{\flat+}_Y : Y^+ \to X^+.$$

We say that φ *spreads over* V, if $\underline{Y}$ spreads over $(\varphi^+)^{-1}(V)$ (in which case, we also say that $\underline{Y}$ *spreads over* V, if no ambiguity is likely to arise).

(iii) We shall also denote by $\Omega_{\underline{X}} \subset X^+$ the largest open subset such that $\underline{X}$ spreads over $\Omega_{\underline{X}}$.

Remark 15.6.2. (i) Let $\underline{X} := (X, \mathscr{T}_X, A^+_X)$ be any quasi-affinoid scheme; then $\underline{X}$ spreads over the analytic locus of $\mathrm{Spec}\, A^+_X$, by virtue of lemma 8.3.29(iii).

(ii) Keep the notation of definition 15.6.1(ii), let $\psi : \underline{Z} \to \underline{Y}$ be another morphism of quasi-affinoid schemes, and suppose that φ spreads over V and ψ spreads over $(\varphi^+)^{-1}(V)$; then clearly $\varphi \circ \psi$ spreads over V.

Lemma 15.6.3. *With the notation of definition* 15.6.1*, we have :*

(i) *Let $W \subset X$ be an open subset such that $\beta_{\underline{X}}$ restricts to an open immersion $W \to X^+$. Then $\beta_{\underline{X}}(W) \subset \Omega_{\underline{X}}$.*

(ii) *$\gamma_{\underline{X}}$ restricts to an isomorphism of schemes $\gamma^{-1}_{\underline{X}} \Omega_{\underline{X}} \xrightarrow{\sim} \Omega_{\underline{X}}$.*

Proof. (i): First, let us notice the following

Claim 15.6.4. $\mathscr{O}_{X^+}(V)$ is integrally closed in $\mathscr{O}_X(\beta^{-1}_{\underline{X}} V)$, for every open subset $V \subset X^+$.

Proof of the claim. Suppose first that $V = \mathrm{Spec}\, A^+_X[f^{-1}]$ for some $f \in A^+_X$; since $\beta_{\underline{X}*}\mathscr{O}_X$ is a quasi-coherent $\mathscr{O}_{X^+}$-module, we have $\mathscr{O}_X(\beta^{-1}_{\underline{X}} V) = A_X[f^{-1}]$, and since A^+_X is integrally closed in A_X, the assertion holds in this case. For a general open subset V, we may find an affine open covering $V = \bigcup_{i\in I} V_i$ where each V_i is of the form $\mathrm{Spec}\, A^+_X[f_i^{-1}]$ for some $f_i \in A^+_X$, and we get a commutative diagram of rings

$$\begin{array}{ccccc} \mathscr{O}_{X^+}(V) & \longrightarrow & R := \prod_{i\in I} \mathscr{O}_{X^+}(V_i) & \overset{\rho_1}{\underset{\rho_2}{\rightrightarrows}} & \prod_{i,j\in I} \mathscr{O}_{X^+}(V_i \cap V_j) \\ \downarrow & & \downarrow & & \downarrow \\ \mathscr{O}_X(\beta_{\underline{X}}^{-1}V) & \longrightarrow & R' := \prod_{i\in I} \mathscr{O}_X(\beta_{\underline{X}}^{-1}V_i) & \rightrightarrows & \prod_{i,j\in I} \mathscr{O}_X(\beta_{\underline{X}}^{-1}(V_i \cap V_j)). \end{array}$$

Now, if $h \in \mathscr{O}_X(\beta_{\underline{X}}^{-1}V)$ is integral over $\mathscr{O}_{X^+}(V)$, obviously the image h' of h in R' is integral over R; but from the foregoing case, it is easily seen that R is integrally closed in R', so $h' \in R$, and since the vertical arrows are injective maps, we deduce easily that $\rho_1(h') = \rho_2(h')$, whence $h' \in \mathscr{O}_{X^+}(V)$, and the assertion follows. ◊

Now, by assumption, $\beta_{\underline{X}}(W)$ is an open subset of X^+, and $\beta_{\underline{X}}$ restricts to a separated morphism $\beta_{\underline{X}}^{-1}\beta_{\underline{X}}(W) \to \beta_{\underline{X}}(W)$; the latter admits a section whose image is the open subset $W \subset \beta_{\underline{X}}^{-1}\beta_{\underline{X}}(W)$. Taking into account [59, Ch.I, Cor.5.4.6], we see that W is an open and closed subset of $\beta_{\underline{X}}^{-1}\beta_{\underline{X}}(W)$. There follow ring homomorphisms

$$\mathscr{O}_{X^+}(\beta_{\underline{X}}(W)) \to \mathscr{O}_X(\beta_{\underline{X}}^{-1}\beta_{\underline{X}}(W)) \xrightarrow{\rho} \mathscr{O}_X(W)$$

whose composition is an isomorphism, and the kernel of ρ is generated by an idempotent element $e \in \mathscr{O}_X(\beta_{\underline{X}}^{-1}\beta_{\underline{X}}(W))$. However, claim 15.6.4 implies that e lies in the image of $\mathscr{O}_{X^+}(\beta_{\underline{X}}(W))$, so $e = 0$, hence $W = \beta_{\underline{X}}^{-1}\beta_{\underline{X}}(W)$, and the assertion follows immediately.

Claim 15.6.5. Let $f : X \to Y$ and $g : Y \to Z$ be two morphisms of schemes, such that $h := g \circ f$ is an open immersion and the image of f is schematically dense. Then we have :

(i) f is an open immersion.
(ii) If g is separated, then $g^{-1}h(X) = f(X)$.

Proof of the claim. (i): We know from [59, Ch.I, Cor.5.3.13] that the morphism f is an immersion, and since its image is schematically dense, it must be an open immersion.

(ii): Since h is an open immersion, we may replace Z by $h(X)$ and Y by $g^{-1}h(X)$, after which we have $h = \mathbf{1}_X$, and f is a section of g. Since g is separated, [59, Ch.I, Cor.5.4.6] then tells us that f is a closed immersion in this case, and combining with (i), the claim follows. ◊

(ii): Consider now the commutative diagram of schemes

$$\begin{array}{ccc} \beta_{\underline{X}}^{-1}\Omega_{\underline{X}} & \xrightarrow{f} & \gamma_{\underline{X}}^{-1}\Omega_{\underline{X}} \\ & \searrow^{h} & \downarrow{\scriptstyle g} \\ & & \Omega_{\underline{X}} \end{array}$$

where f is the restriction of $\beta^\circ_{\underline{X}}$ and g is the restriction of $\gamma_{\underline{X}}$. Thus, h is the restriction of $\beta_{\underline{X}}$, and it is an isomorphism, by definition of $\Omega_{\underline{X}}$; moreover, clearly f has schematically dense image. By claim 15.6.5, it follows that f is a surjective open immersion, *i.e.* it is an isomorphism, and then the same holds for g as well. □

Proposition 15.6.6. *In the situation of definition* 15.6.1*, suppose that $\underline{X}$ spreads over V. Then:*

(i) *The topological localization $j_{\underline{X}} : \underline{X}_{\rm loc} \to \underline{X}$, the topological henselization $j'_{\underline{X}} : \underline{X}^{\rm h} \to \underline{X}$ and the completion $j''_{\underline{X}} : \underline{X}^\wedge \to \underline{X}$ of $\underline{X}$ spread over V.*
(ii) *Let also $f_0, \dots, f_n$ be a sequence of elements of A_X that generate an ideal I such that $(\beta_{\underline{X}}^{-1}V) \cap \operatorname{Spec} A_X/I = \varnothing$, and set $\underline{B} := \Gamma(\underline{X})(\frac{f_0,\dots,f_n}{f_0})$ (notation of* (15.5.1)*). The induced morphism $\underline{Y} := \mathsf{Spec}\,\underline{B} \to \underline{X}$ spreads over V.*

(iii) *Let $\underline{Y} \to \underline{X}$ and $\underline{Y}' \to \underline{X}$ be two f-adic morphisms of quasi-affinoid schemes that spread over V. The fibre product $\underline{Y} \times_{\underline{X}} \underline{Y}' \to \underline{X}$ spreads over V (see example* 15.4.8*).*

(iv) *Let $U \subset \beta_{\underline{X}}^{-1} V$ be a quasi-compact open subset containing the analytic subset of $\underline{X}$. The induced morphism $\varphi : U \times_X \underline{X} \to \underline{X}$ spreads over V (see example* 15.4.4*).*

Proof. (i): Say that $\underline{X}_{\mathrm{loc}} = (X', \mathscr{T}_{X'}, A^+_{X'})$, and let $j^+_{\underline{X}} : X'^+ \to X^+$ be the morphism induced by the topological localization $A^+_X \to A^+_{X'}$. By inspecting (8.4.8), and taking into account the natural identifications (15.4.11), we get a cartesian diagram of schemes :

$$\begin{array}{ccc} X' & \xrightarrow{j_{\underline{X}}} & X \\ {\scriptstyle \beta_{\underline{X}_{\mathrm{loc}}}}\downarrow & & \downarrow{\scriptstyle \beta_{\underline{X}}} \\ X'^+ & \xrightarrow{j^+_{\underline{X}}} & X^+. \end{array}$$

The assertion for $j_{\underline{X}}$ is an immediate consequence. The same argument applies to $j'_{\underline{X}}$. Next, say that $\underline{X}^\wedge = (X^\wedge, \mathscr{T}_{X^\wedge}, A^+_{X^\wedge})$, and $\Gamma(\underline{X})^\wedge = (A^\wedge_X, A^{\wedge+}_X, X')$. According to corollary 15.4.28(i), the kernel $\mathscr{J}$ of the surjective map $A^\wedge_X \to A_{X^\wedge} := \mathscr{O}_{X^\wedge}(X^\wedge)$ lies in $A^{\wedge+}_X$, and we set $Y := \mathrm{Spec}\, A^{\wedge+}_X / \mathscr{J}$; there follows a commutative diagram of schemes

$$\begin{array}{ccccc} & & X^\wedge & \longrightarrow & X \\ & {\scriptstyle \beta_{\underline{X}^\wedge}} \swarrow & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \beta_{\underline{X}}} \\ X^{\wedge+} & \xrightarrow{\nu} & Y & \longrightarrow & X^+ \end{array}$$

whose square subdiagram is cartesian. With this notation, $\varphi_* \mathscr{O}_{X^\wedge}$ is a quasi-coherent $\mathscr{O}_Y$-algebra, and $A^+_{X^\wedge} = \mathscr{A}(Y)$, where $\mathscr{A}$ is the integral closure of the image of $\mathscr{O}_Y$ in $\varphi_* \mathscr{O}_{X^\wedge}$. Let $V' \subset Y$ and $V'' \subset X^{\wedge+}$ be the preimages of V; under our assumptions, the morphism $\varphi^\flat : \mathscr{O}_Y \to \varphi_* \mathscr{O}_{X^\wedge}$ restricts to an isomorphism $\mathscr{O}_{Y|V'} \xrightarrow{\sim} (\varphi_* \mathscr{O}_{X^\wedge})_{|V'}$, so that $\mathscr{A}_{|V'} = (\varphi_* \mathscr{O}_{X^\wedge})_{|V'}$. The latter implies that the morphism $\gamma_{\underline{X}^\wedge}$ restricts to an isomorphism $\gamma^{-1}_{\underline{X}^\wedge} V'' \xrightarrow{\sim} V''$, and ν restricts to an isomorphism $V'' \xrightarrow{\sim} V'$; lastly, since the image of X in X^+ contains V, it is clear that the image of $X^\wedge$ in Y contains V', and then the image of $X^\wedge$ in $X^{\wedge+}$ contains V'', so $j''_{\underline{X}}$ spreads over V.

(ii): Set $B := A_X[f_0^{-1}]$, $B' := A^+_X[f_1/f_0, \dots, f_n/f_0] \subset B$, say that $\underline{Y} = (Y, \mathscr{T}_Y, B^+)$, and let $Y^+ := \mathrm{Spec}\, B^+$. There follows a commutative diagram of schemes

$$\begin{array}{ccccc} Y & \xrightarrow{i} & \mathrm{Spec}\, B & \xrightarrow{\rho} & \mathrm{Spec}\, B' \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & \mathrm{Spec}\, A_X & \longrightarrow & X^+ \end{array}$$

whose left square subdiagram is cartesian. Denote by $V' \subset \mathrm{Spec}\, B'$ and $V'' \subset Y^+$ the preimages of V, and let $g \in A^+_X$ be any element such that $\mathrm{Spec}\, A^+_X[g^{-1}] \subset V$; according to lemma 15.6.3(ii), the induced map $A^+_X[g^{-1}] \to A_X[g^{-1}]$ is an isomorphism. Especially, we may regard $f_0, \dots, f_n$ as elements of $C := A^+_X[g^{-1}]$, and by assumption $\sum_{i=0}^n f_i C = C$. It follows easily that f_0 is invertible in the subring $B'[g^{-1}] = C[f_1/f_0, \dots, f_n/f_0]$ of $B[g^{-1}]$, *i.e.* $B'[g^{-1}] = B[g^{-1}]$, and since g is arbitrary, it follows that ρ restricts to an isomorphism $\rho^{-1} V' \xrightarrow{\sim} V'$. Recalling now that B^+ is the integral closure of B' in $\mathscr{O}_Y(Y) = B$, we deduce that $\gamma_{\underline{Y}}$ restricts as well to an isomorphism $\gamma^{-1}_{\underline{Y}} V'' \xrightarrow{\sim} V''$. Lastly, since $\beta_{\underline{X}}(X)$ contains V, clearly the image of $\beta_{\underline{Y}}$ contains V''. The assertion follows easily (details left to the reader).

(iii): Say that $\underline{Y} = (Y, \mathscr{T}_Y, A^+_Y)$, $\underline{Y}' = (Y', \mathscr{T}_{Y'}, A^+_{Y'})$, and set $Y^+ := \mathrm{Spec}\, A^+_Y$, $Y'^+ := \mathrm{Spec}\, A^+_{Y'}$. Set also $\underline{Z} := \underline{Y} \times_{\underline{X}} \underline{Y}'$ and $(A_Z, A^+_Z, Z) := \Gamma(\underline{Z})$, so that $Z = Y \times_X Y'$. Denote

$$\varphi : Z \to Z' := Y^+ \times_{X^+} Y'^+ \qquad \text{and} \qquad \pi : Z' \to X^+$$

the induced morphisms. With this notation, $\varphi_*\mathscr{O}_Z$ is a sheaf of quasi-coherent $\mathscr{O}_{Z'}$-algebras, and $A^+_Z = \mathscr{A}(Z')$, where $\mathscr{A}$ is the integral closure of the image of $\mathscr{O}_{Z'}$ in $\varphi_*\mathscr{O}_Z$. Set $V' := \pi^{-1}V$; under the current assumptions it is clear that the associated morphism $\varphi^\flat : \mathscr{O}_{Z'} \to \varphi_*\mathscr{O}_Z$ restricts to an isomorphism $\mathscr{O}_{Z'|V'} \xrightarrow{\sim} (\varphi_*\mathscr{O}_Z)_{|V'}$, whence $\mathscr{A}_{|V'} = (\varphi_*\mathscr{O}_Z)_{|V'}$, which means that the natural morphism $\nu : Z^+ := \mathrm{Spec}\, A^+_Z \to Z'$ restricts to an isomorphism $\nu^{-1}V' \xrightarrow{\sim} V'$; lastly, since the image of φ contains V', it is clear that the image of $\beta_{\underline{Z}}$ contains $\nu^{-1}V'$, whence the assertion.

(iv): Set $\underline{U} := U \times_X \underline{X}$, and let $h : U \to X^+$ be the composition of the open immersion $U \to \beta^{-1}_{\underline{X}}V$ and the restriction $\beta^{-1}_{\underline{X}}V \to X^+$ of $\beta_{\underline{X}}$. Then $h = \varphi^+ \circ \beta_{\underline{U}}$, and clearly $\beta_{\underline{U}}$ has schematically dense image; by claim 15.6.5 we deduce that $\beta_{\underline{U}}$ is an open immersion, whence the contention. □

15.6.7. Let $\underline{X}$ and V be as in definition 15.6.1, and suppose that $\underline{X}$ is topologically local and spreads over V. We denote by

$$\mathscr{R}_V(\underline{X})$$

the site whose underlying category is the full subcategory of $\mathscr{R}(\underline{X})$ whose objects are the rational subsets R such that the sub-presheaf $h_R \subset h_{\mathrm{Spa}\,\underline{X}}$ is represented by a morphism $\underline{Y} \to \underline{X}$ of quasi-affinoid schemes that spreads over V. Notice that, in view of proposition 15.6.6(iii), if $R, R' \in \mathrm{Ob}(\mathscr{R}_V(\underline{X}))$, then also $R \cap R' \in \mathrm{Ob}(\mathscr{R}_V(\underline{X}))$. As usual, a family $(R_i \to R \mid i \in I)$ of morphisms of $\mathscr{R}_V(\underline{X})$ generates a covering sieve of $\mathscr{R}_V(\underline{X})/R$ if and only if $\bigcup_{i\in I} R_i = R$.

In light of proposition 15.6.6(iii), we see that all fibre products are representable in $\mathscr{R}_V(\underline{X})$, and the inclusion functor $\iota : \mathscr{R}_V(\underline{X}) \to \mathscr{R}(\underline{X})$ commutes with fibre products. By lemma 4.2.4, it follows that ι is continuous, *i.e* the induced functor on presheaves $\iota^\wedge : \mathscr{R}(\underline{X})^\wedge \to \mathscr{R}_V(\underline{X})^\wedge$ restricts to a functor on the respective subcategories of sheaves

$$\widetilde{\iota}_* : \mathscr{R}(\underline{X})^\sim \to \mathscr{R}_V(\underline{X})^\sim.$$

We will denote as well by $\widetilde{\iota}_*$ the corresponding functor on abelian sheaves. For every presheaf $\mathscr{F}$ on $\mathscr{R}(\underline{X})$ or $\mathscr{R}_V(\underline{X})$, let us denote as usual by $\mathscr{F}^a$ the sheaf associated to $\mathscr{F}$.

Proposition 15.6.8. *In the situation of* (15.6.7)*, let $\mathscr{G}$ be a presheaf, $\mathscr{A}$ an abelian presheaf, and $\mathscr{F}$ an abelian sheaf on $\mathscr{R}(\underline{X})$. The following holds :*

(i) *The natural morphism $(\iota^\wedge\mathscr{G})^a \to \iota^\wedge(\mathscr{G}^a)$ is an isomorphism.*

(ii) *The natural morphism $\iota_*\mathscr{F} \to R\widetilde{\iota}_*\mathscr{F}$ is an isomorphism in $\mathsf{D}(\mathbb{Z}_{\mathscr{R}_V(\underline{X})}\text{-}\mathbf{Mod})$.*

(iii) *The natural map $\check{H}^i(\mathscr{R}_V(\underline{X}), \iota^\wedge\mathscr{A}) \to \check{H}^i(\mathscr{R}(\underline{X}), \mathscr{A})$ is bijective for every $i \in \mathbb{N}$.*

Proof. (i): Define $\mathscr{F}^+$ as in the proof of theorem 4.1.13; in view of claim 4.1.16 it suffices to check that the the natural morphism $(\iota^\wedge\mathscr{G})^+ \to \iota^\wedge(\mathscr{G}^+)$ is an isomorphism. For the injectivity, let $U \in \mathrm{Ob}(\mathscr{R}_V(\underline{X}))$ be any rational subset, and $s, t \in (\iota^\wedge\mathscr{G})^+(U)$ any two sections whose images agree in $\mathscr{G}^+(U)$. Then we may find a covering family $(U_\lambda \to U \mid \lambda \in \Lambda)$ of $\mathscr{R}_V(\underline{X})/U$ such that s and t are the classes of compatible systems $(s_\lambda \mid \lambda \in \Lambda)$, $(t_\lambda \mid \lambda \in \Lambda)$ with $s_\lambda, t_\lambda \in \mathscr{G}(U_\lambda)$ for every $\lambda \in \Lambda$, and the assumption means that for every $\lambda \in \Lambda$ there exists a covering family $(U_{\lambda,i} \to U_\lambda \mid i \in I_\lambda)$ of $\mathscr{R}(\underline{X})/U_\lambda$ such that $s_{\lambda|U_{\lambda,i}} = t_{\lambda|U_{\lambda,i}}$ for every $\lambda \in \Lambda$ and every $i \in I_\lambda$. But by proposition 15.6.6(ii) and lemma 15.5.21, the covering $U_{\lambda\bullet}$ can be refined by a covering $(U'_{\lambda,i'} \to U_\lambda \mid i' \in I'_\lambda)$ of $\mathscr{R}_V(\underline{X})/U_\lambda$, for every $\lambda \in \Lambda$. Then $s_{\lambda|U'_{\lambda,i'}} = t_{\lambda|U'_{\lambda,i'}}$ for every $\lambda \in \Lambda$ and every $i' \in I'_\lambda$, so that $s = t$. Lastly, let $s \in \mathscr{G}^+(U)$ be any section; by definition we may find a covering family $(U_\lambda \to U \mid \lambda \in \Lambda)$ of $\mathscr{R}(\underline{X})/U$ such that s is represented by a compatible system $(s_\lambda \mid \lambda \in \Lambda)$ with $s_\lambda \in \mathscr{G}(U_\lambda)$ for every $\lambda \in \Lambda$. But again, the covering U_λ can be refined by a covering of $\mathscr{R}_V(\underline{X})$, and we deduce that s is the image of a section of $(\iota^\wedge\mathscr{G})^+(U)$.

(ii): The assertion means that $R^i\widetilde{\iota}_*\mathscr{F} = 0$ for every $i > 0$. However, recall that $R^i\widetilde{\iota}_*\mathscr{F}$ is the sheaf associated to the presheaf that assigns to every $U \in \mathrm{Ob}(\mathscr{R}_V(\underline{X}))$ the abelian group

$H^i(\mathscr{R}(\underline{X})/U, \mathscr{F}_{|U})$. Hence, let us fix such a rational subset U, and let $s \in H^i(\mathscr{R}(\underline{X})/U, \mathscr{F}_{|U})$ be any element; it suffices to exhibit a covering $(U_\lambda \to U \mid \lambda \in \Lambda)$ of U in $\mathscr{R}_V(\underline{X})$ such that the image of s in $H^i(\mathscr{R}(\underline{X})/U_\lambda, \mathscr{F}_{|U_\lambda})$ vanishes for every $\lambda \in \Lambda$. However, notice that there exists a covering $(U'_{\lambda'} \to U \mid \lambda' \in \Lambda')$ of U in $\mathscr{R}(\underline{X})$ such that the image of s in $H^i(\mathscr{R}(\underline{X})/U'_{\lambda'}, \mathscr{F}_{|U'_{\lambda'}})$ vanishes for every $\lambda' \in \Lambda'$ (this is clear, since these cohomology groups are computed by an injective resolution of $\mathscr{F}$, which is exact in degrees > 0). But by proposition 15.6.6(ii) and lemma 15.5.21, the covering $U'_\bullet$ can be refined by a covering $U_\bullet$ consisting of objects of $\mathscr{R}_V(\underline{X})$, and clearly such $U_\bullet$ will do.

(iii): This is an immediate consequence of the fact – already remarked in the foregoing – that every covering family of $\mathscr{R}(\underline{X})$ can be refined by a covering family of $\mathscr{R}_V(\underline{X})$: the details shall be left to the reader. ☐

15.6.9. Consider a quasi-affinoid ring $\underline{A} := (A, A^+, U)$ such that A is an adic (and f-adic) topological ring; let $A_U := \mathscr{O}_U(U)$ and $\underline{U} := (U, \mathscr{T}_U, A_U^+) := \mathsf{Spec}\,\underline{A}$, and set

$$X_A := \mathrm{Spec}\,A \qquad X_A^+ := \mathrm{Spec}\,A_U^+ \qquad X_A^\circ := \mathrm{Spec}\,A_U^\circ.$$

The induced map $A \to A_U$ is f-adic, so its image lies in A_U° (lemma 8.3.24(iii)), and we get a commutative diagram of schemes :

$$\begin{array}{ccccc} U & \xrightarrow{\ i\ } & X_A & & \\ {\scriptstyle j}\downarrow & & \uparrow{\scriptstyle j''} & & \\ \mathrm{Spec}\,A_U & \xrightarrow{\ j'\ } & X_A^\circ & \xrightarrow{\ t\ } & X_A^+. \end{array}$$

Let also $Z \subset X_A^+$ be the support of the A_U^+-module A_U°/A_U^+, and $Z' := X_A^\circ \setminus j' \circ j(U)$.

Lemma 15.6.10. *With the notation of* (15.6.9)*, the following holds :*

(i) *i, j and $j' \circ j$ are open immersions.*
(ii) *The images of j and j' are schematically dense.*
(iii) *The open subset $j' \circ j(U)$ contains the analytic locus of X_A°.*
(iv) *$X_A^+ \setminus \Omega_{\underline{U}}$ is the topological closure of $Z \cup t(Z')$.*

Proof. (ii): The assertion for j' is clear. Next, let $\mathscr{I}$ be the kernel of the induced morphism $j^\flat : \mathscr{O}_{\mathrm{Spec}\,A_U} \to j_*\mathscr{O}_U$; since U is quasi-compact, $j_*\mathscr{O}_U$ is a quasi-coherent $\mathscr{O}_{\mathrm{Spec}\,A_U}$-module, so the same holds for $\mathscr{I}$, and we are reduced to checking that $\Gamma(\mathrm{Spec}\,A_U, \mathscr{I}) = 0$, which is clear, since $j^\flat$ induces an isomorphism on global sections.

(i): For i there is nothing to prove, and the assertion for j follows from [60, Ch.II, Prop.5.1.2]. The assertion for $j' \circ j$ follows from claim 15.6.5(i).

(iii): Notice that j' maps the analytic locus of $\mathrm{Spec}\,A_U$ isomorphically onto that of X_A° (lemma 8.3.29(iii)); likewise, it is easily seen that $j'' \circ j'$ restricts to an isomorphism from the analytic locus of $\mathrm{Spec}\,A_U$ onto that of X_A. Summing up, we conclude that j'' restricts to an isomorphism from the analytic locus of X_A° onto that of X_A, whence the assertion.

(iv): Consider the commutative diagram of schemes

$$\begin{array}{ccc} \beta_{\underline{U}}^{-1}\Omega_{\underline{U}} & \xrightarrow{\ f\ } & t^{-1}\Omega_{\underline{U}} \\ & {\scriptstyle h}\searrow & \downarrow{\scriptstyle g} \\ & & \Omega_{\underline{U}} \end{array}$$

where f is the restriction of $j' \circ j$, and g is the restriction of t. Thus, h is the restriction of $\beta_{\underline{U}}$, and it is an isomorphism, by definition of $\Omega_{\underline{U}}$; moreover, clearly f has schematically dense image. By claim 15.6.5, it follows that f is a surjective open immersion, *i.e.* it is an isomorphism, and then the same holds for g as well. Now, if $x \in \Omega_{\underline{U}}$ is any point, it follows immediately that $x \notin Z$; moreover $g^{-1}(x)$ is the unique point of $t^{-1}(x)$, and this point lies in $j' \circ j(U)$. This shows that $Z \cup t(Z') \subset X^+_A \setminus \Omega_{\underline{U}}$. Conversely, suppose that $x \in X^+_A$ lies neither in $\Omega_{\underline{U}}$ nor the topological closure of Z; then there exists an open neighborhood V of x in X^+_A with $V \cap Z = \varnothing$, and it follows that t restricts to an isomorphism $t^{-1}V \xrightarrow{\sim} V$. In view of (i), the restriction $\beta^{-1}_{\underline{U}}V \to t^{-1}V$ of $j' \circ j$ is still an open immersion, so the same holds for the restriction $\beta^{-1}_{\underline{U}}V \to V$ of $\beta_{\underline{U}}$, and consequently $\beta_{\underline{U}}(U) \cap V = \beta_{\underline{U}}(\beta^{-1}_{\underline{U}}V) \subset \Omega_{\underline{U}}$, according to lemma 15.6.3(i). We conclude that $x \notin \beta_{\underline{U}}(U)$, and since $x \in t(V)$, we finally get $x \in t(Z')$, whence (iv). □

15.7. **Etale coverings of quasi-affinoid schemes.** Let Sch be the category of schemes (in the universe U). For every scheme X, let also $X_{\text{ét}}$ be the full subcategory of the category Sch/X, whose objects are the étale morphisms $Y \to X$ with Y quasi-compact and quasi-separated; notice that every morphism in $X_{\text{ét}}$ is étale. We endow $X_{\text{ét}}$ with its standard *étale topology* $J_{X,\text{ét}}$, whose covering families are all the systems $(f_i : Y_i \to Y \mid i \in I)$ of families of morphisms of X-schemes such that $\bigcup_{i\in I} f_i(Y_i) = Y$. The site $(X_{\text{ét}}, J_{X,\text{ét}})$ is called the *étale site* of X, and is also often denoted simply by $X_{\text{ét}}$, when no ambiguities are likely to arise. It is easily seen that $X_{\text{ét}}$ is a lex-site if and only if X is a quasi-compact and quasi-separated scheme (the details are left to the reader). Recall that we have a fibration $\mathsf{Cov} \to \mathsf{Sch}$ whose fibre category over every $Y \in \mathrm{Ob}(\mathsf{Sch})$ is the category of finite étale Y-schemes (see (13.1)). From the natural functor

$$X_{\text{ét}} \to \mathsf{Sch} \qquad (Y \to X) \mapsto Y$$

we deduce a fibration

$$\mathsf{Cov}_{X_{\text{ét}}} := X_{\text{ét}} \times_{\mathsf{Sch}} \mathsf{Cov} \to X_{\text{ét}}.$$

By faithfully flat descent, it is easily seen that $\mathsf{Cov}_{X_{\text{ét}}}$ is an ind-finite stack over the site $X_{\text{ét}}$.

For every $k \in \mathbb{N}$ let $[k] := \{0, \dots, n\}$, and set as well $[-1] := \varnothing$; we denote by N the full subcategory of the category $\mathbf{Set}$ with $\mathrm{Ob}(\mathsf{N}) := \{[k] \mid k \geq -1\}$. Then, let $\mathsf{N}_{X_{\text{ét}}}$ be the *constant presheaf of categories* on $X_{\text{ét}}$ with value N, *i.e.* the presheaf such that $\mathsf{N}_{X_{\text{ét}}}(Y) := \mathsf{N}$ for every $Y \in \mathrm{Ob}(X_{\text{ét}})$, and $\mathsf{N}_X(f) := \mathbf{1}_\mathsf{N}$ for every morphism f of $X_{\text{ét}}$. We get an $X_{\text{ét}}$-cartesian functor

$$\omega_X : \mathscr{F}ib(\mathsf{N}_{X_{\text{ét}}}) \to \mathsf{Cov}_{X_{\text{ét}}} \qquad (Y, [k]) \mapsto ([k] \times Y \to Y).$$

Lemma 15.7.1. *The functor ω_X is i-covering for $i = 0, 1, 2$.*

Proof. Let $f : Y \to X$ be a finite étale covering, and $x \in X$; in order to check the assertion for $i = 0$, we need to exhibit an étale morphism $g : X' \to X$ and $k \in \mathbb{N}$ from an affine scheme X' such that $x \in g(X')$, with an isomorphism of X'-schemes $X' \times_X Y \xrightarrow{\sim} [k] \times X'$. To this aim, let $\xi : \mathrm{Spec}\, \kappa \to X$ be a geometric point localized at the point x (see definition 4.9.17(i)); we argue by induction on the cardinality c of the set $f^{-1}(\xi)$ (see definition 4.9.17(iv)). If $c = 0$, we have $x \notin f(Y)$, and then there exists an affine open neighborhood X' of x in X with $X' \cap f(Y) = \varnothing$; in this case we may take for g the open inclusion $X' \to X$, and $k := 0$.

Suppose next that $c > 0$, and that the assertion is already known for all schemes X, all $x \in X$, and all finite étale coverings $f : Y \to X$ such that the cardinality of $f^{-1}(\xi)$ is $< c$, for any geometric point ξ localized at x. Notice that the diagonal morphism $\Delta_{Y/X} : Y \to Y \times_X Y$ is a finite étale morphism, hence its image is open and closed in $Y \times_X Y$. Thus, the Y-scheme $Y \times_X Y$ is isomorphic to the disjoint union of Y and a finite étale covering $h : Z \to Y$. The geometric point ξ lifts to a geometric point $\xi_Y : \mathrm{Spec}\, \kappa \to Y$, and it is then easily seen that $h^{-1}(\xi_Y) = c - 1$. By inductive assumption, we find an étale morphism $g_Y : Y' \to Y$ from an affine scheme Y', such that $g_Y(Y')$ contains the support of ξ_Y, and such that the Y'-scheme $Y' \times_Y Z$ is isomorphic to $[c-2] \times Y'$. We may then take $X' := Y'$ and $g := f \circ g_Y$.

Next, to check the assertion for $i = 1$, let $k, k' \geq -1$ be two integers, $g : [k] \times X \to [k'] \times X$ a morphism of X-schemes, and $x \in X$; it suffices to find an affine open neighborhood U of x

in X, and a map of sets $\varphi : [k] \to [k']$ such that the restriction $g_{|U} : [k] \times U \to [k'] \times U$ of g agrees with $\varphi \times U$. To this aim, it suffices to invoke [66, Ch.IV, Cor.17.4.7] : the details are left to the reader. Lastly, it is clear from the definitions that ω_X is a 2-covering functor. $\square$

15.7.2. Denote by $\mathsf{Sch}^{\mathsf{qcqs}}$ the full subcategory of Sch whose objects are the quasi-compact and quasi-separated schemes, and by $\mathsf{Sch}^{\mathsf{qcqs}}_{\text{ét}}$ the full subcategory of $\mathsf{Morph}(\mathsf{Sch}^{\mathsf{qcqs}})$ whose objects are the étale morphisms of schemes. The restriction

$$t : \mathsf{Sch}^{\mathsf{qcqs}}_{\text{ét}} \to \mathsf{Sch}^{\mathsf{qcqs}}$$

of the target functor $\mathsf{t} : \mathsf{Morph}(\mathsf{Sch}^{\mathsf{qcqs}}) \to \mathsf{Sch}^{\mathsf{qcqs}}$ is a fibration, and for every $X \in \mathrm{Ob}(\mathsf{Sch}^{\mathsf{qcqs}})$, the fibre category $t^{-1}X$ is the previously defined category $X_{\text{ét}}$, which we endow with its étale topology $J_{X,\text{ét}}$; we get therefore a well defined fibred site :

$$(\mathsf{Sch}^{\mathsf{qcqs}}_{\text{ét}}, t, J_{\bullet,\text{ét}}).$$

For every category I and every functor $F : I \to \mathsf{Sch}^{\mathsf{qcqs}}$ we let

$$F_{\text{ét}} := I \times_{(\mathsf{Sch}^{\mathsf{qcqs}},F)} (\mathsf{Sch}^{\mathsf{qcqs}}_{\text{ét}}, t, J_{\bullet,\text{ét}})$$

(notation of (4.5.11)). Thus, $F_{\text{ét}}$ is a fibred lex-site over I, whose fibre category over every $i \in \mathrm{Ob}(I)$ is naturally identified with the lex-site $(Fi)_{\text{ét}}$. Moreover, we have a natural functor :

$$F_{\text{ét}} \to \mathsf{Sch} \qquad (i, Y \to Fi) \mapsto Y$$

from which we deduce a fibration over $F_{\text{ét}}$:

$$\mathsf{Cov}_{F_{\text{ét}}} := F_{\text{ét}} \times_{\mathsf{Sch}} \mathsf{Cov} \to F_{\text{ét}}$$

whose restriction to the fibre category $(Fi)_{\text{ét}}$ is naturally identified with the fibration $\mathsf{Cov}_{(Fi)_{\text{ét}}}$, for every $i \in \mathrm{Ob}(I)$. Explicitly, the objects of $\mathsf{Cov}_{F_{\text{ét}}}$ are the data $(i, f : Y \to Fi, g : Z \to Y)$ where f is an étale morphism, g is a finite étale morphism, and $i \in \mathrm{Ob}(I)$. Let $\mathsf{N}_{F_{\text{ét}}}$ be the constant presheaf of categories on $F_{\text{ét}}$ with value N; as in (15.7), we then have a cartesian functor of $F_{\text{ét}}$-fibrations :

$$(15.7.3) \quad \mathscr{F}ib(\mathsf{N}_{F_{\text{ét}}}) \to \mathsf{Cov}_{F_{\text{ét}}} \qquad (i, f : Y \to Fi, [k]) \mapsto (i, f : Y \to Fi, [k] \times Y \to Y).$$

Clearly the restriction of (15.7.3) to each fibre category $(Fi)_{\text{ét}}$ is naturally identified with the functor ω_{Fi} of (15.7); in view of lemma 15.7.1 and proposition 5.7.1(i) we deduce that also (15.7.3) is i-covering for $i = 0, 1, 2$.

15.7.4. Let $\underline{X} := (X, \mathscr{T}_X, A^+_X)$ be a *topologically henselian* quasi-affinoid scheme; we attach to $\underline{X}$ a site $(\mathscr{Q}(\underline{X}), J_{\mathscr{Q}})$ of quasi-affinoid open subsets, as in (15.5.11). We now define as follows a fibred lex-site over the category $\mathscr{Q}(\underline{X})$. For every $U \in \mathrm{Ob}(\mathscr{Q}(\underline{X}))$ choose a topologically henselian quasi-affinoid scheme $\underline{X}^{\mathrm{h}}_U := (X^{\mathrm{h}}_U, \mathscr{T}^{\mathrm{h}}_U, A^{\mathrm{h}+}_U)$ representing the sub-presheaf h'_U of $h'_{\underline{X}}$ (notation of remark 15.5.9(i)). Every inclusion $U \subset U'$ of quasi-affinoid subsets of $\mathrm{Spa}\,\underline{X}$ induces a morphism

$$i_{UU'} : \underline{X}^{\mathrm{h}}_U \to \underline{X}^{\mathrm{h}}_{U'}$$

of quasi-affinoid schemes (cp. (15.5.11)), and for every further inclusion $U' \subset U''$ of quasi-affinoid open subsets of $\mathrm{Spa}\,\underline{X}$ we have $i_{U'U''} \circ i_{UU'} = i_{UU''}$, so we get a well defined functor

$$\underline{X}^{\mathrm{h}}_\bullet : \mathscr{Q}(\underline{X}) \to \mathsf{Sch}^{\mathsf{qcqs}}$$

whence an associated fibred site as in (15.7.2)

$$\pi_{\underline{X}} : \underline{X}^{\mathrm{h}}_{\bullet,\text{ét}} := \mathscr{Q}(\underline{X}) \times_{\mathsf{Sch}^{\mathsf{qcqs}}} (\mathsf{Sch}^{\mathsf{qcqs}}_{\text{ét}}, t, J_{\bullet,\text{ét}}) \to \mathscr{Q}(\underline{X}).$$

Notice that $\underline{X}^{\mathrm{h}}_{\bullet,\text{ét}}$ is a fibred lex-site, since $\underline{X}^{\mathrm{h}}_{U,\text{ét}}$ is quasi-compact and quasi-separated for every $U \in \mathrm{Ob}(\mathscr{Q}(\underline{X}))$. We let $(\underline{X}^{\mathrm{h}}_{\bullet,\text{ét}}, J)$ be the associated total site.

• The identity functor $\mathbf{1}_{\mathscr{Q}(\underline{X})} : \mathscr{Q}(\underline{X}) \to \mathscr{Q}(\underline{X})$ is naturally identified with the fibration $\mathscr{F}ib(\mathsf{F}_{\mathbb{1}}) \to \mathscr{Q}(\underline{X})$, where $\mathsf{F}_{\mathbb{1}} : \mathscr{Q}(\underline{X})^o \to \mathbf{Cat}$ is the constant pseudo-functor with value $\mathbb{1}$. The category $\mathbb{1}$ admits a unique topology $\mathscr{T}_{\mathbb{1}}$, and the resulting pair $(\mathbb{1}, \mathscr{T}_{\mathbb{1}})$ is obviously a lex-site, so we have the fibred lex-site

$$(\mathscr{Q}(\underline{X}), \mathbf{1}_{\mathscr{Q}(\underline{X})}, J^*_\bullet)$$

where J^*_U is the unique topology on the fibre $\mathbf{1}^{-1}_{\mathscr{Q}(\underline{X})}(U) \stackrel{\sim}{\to} \mathbb{1}$, for every $U \in \mathrm{Ob}(\mathscr{Q}(\underline{X}))$, and we let $(\mathscr{Q}(\underline{X}), J^*_{\mathscr{Q}})$ be the resulting total site.

• Denote by

$$e : \mathscr{Q}(\underline{X}) \to \underline{X}^{\mathrm{h}}_{\bullet,\text{ét}}.$$

the functor that assigns to every $U \in \mathrm{Ob}(\mathscr{Q}(\underline{X}))$ the final object X^{h}_U of the fibre category $X^{\mathrm{h}}_{U,\text{ét}}$, and to every inclusion $U \subset U'$ the morphism $i_{UU'}$. It is easily seen that e is a $\mathscr{Q}(\underline{X})$-cartesian functor. Moreover, e restricts to a morphism of sites $X^{\mathrm{h}}_{U,\text{ét}} \to (\mathbb{1}, \mathscr{T}_{\mathbb{1}})$ for every $U \in \mathrm{Ob}(\mathscr{Q}(\underline{X}))$, so e is a morphism of fibred lex-sites, and also a morphism of total sites

$$e : (\underline{X}^{\mathrm{h}}_{\bullet,\text{ét}}, J) \to (\mathscr{Q}(\underline{X}), J^*_{\mathscr{Q}}).$$

• Lastly, for every $U \in \mathrm{Ob}(\mathscr{Q}(\underline{X}))$ we have an inclusion functor $i_U : \underline{X}^{\mathrm{h}}_{U,\text{ét}} \to \underline{X}^{\mathrm{h}}_{\bullet,\text{ét}}$ which identifies $\underline{X}^{\mathrm{h}}_{U,\text{ét}}$ with the fibre category over U. This is a weak morphism of sites

$$i_U : (\underline{X}^{\mathrm{h}}_{\bullet,\text{ét}}, J) \to \underline{X}^{\mathrm{h}}_{U,\text{ét}}$$

and for $U = \mathrm{Spa}\,\underline{X}$ it is even a morphism of sites (example 4.5.8). Let now $\mathscr{E}$ be a stack on the étale site $X_{\text{ét}}$; we consider the fibration :

$$\mathscr{E}_{/\mathscr{Q}} := \mathsf{St}(e)_* \circ \mathsf{St}(i_{\mathrm{Spa}\,\underline{X}})^*\mathscr{E}.$$

Remark 15.7.5. (i) The fibres of the fibration $\mathscr{E}_{/\mathscr{Q}}$ can be described as follows. let $c_X : \mathscr{Q}(\underline{X}) \to \mathsf{Sch}^{\mathsf{qcqs}}$ be the constant functor with value X; we consider as well the fibred lex-site

$$\mathscr{Q}(\underline{X}) \times X_{\text{ét}} = \mathscr{Q}(\underline{X}) \times_{(\mathsf{Sch}^{\mathsf{qcqs}}, c_X)} (\mathsf{Sch}^{\mathsf{qcqs}}_{\text{ét}}, t, J_{\bullet,\text{ét}}) \to \mathscr{Q}(\underline{X})$$

(see (4.5.11)) whose fibre categories are all naturally identified with $X_{\text{ét}}$. Then the rule $U \mapsto (i_{U,\mathrm{Spa}\,\underline{X}} : X^{\mathrm{h}}_U \to X)$ defines a natural transformation $\underline{X}^{\mathrm{h}}_\bullet \Rightarrow c_X$ which induces, after choosing a cleavage for the fibration $\mathsf{Sch}^{\mathsf{qcqs}}_{\text{ét}}$, a morphism of fibred lex-sites over $\mathscr{Q}(\underline{X})$:

$$j_{\underline{X}} : \underline{X}^{\mathrm{h}}_{\bullet,\text{ét}} \to \mathscr{Q}(\underline{X}) \times X_{\text{ét}}$$

(see (4.5.12)) which is also a morphism on the respective total sites

$$j_{\underline{X}} : (\underline{X}^{\mathrm{h}}_{\bullet,\text{ét}}, J) \to (\mathscr{Q}(\underline{X}) \times X_{\text{ét}}, J').$$

Moreover, let $l_{\underline{X}} : X_{\text{ét}} \to \mathscr{Q}(\underline{X}) \times X_{\text{ét}}$ be the inclusion functor that identifies $X_{\text{ét}}$ with the fibre category over the final object of $\mathscr{Q}(\underline{X})$. By example 4.5.8, this functor is a morphism of sites

$$l_{\underline{X}} : (\mathscr{Q}(\underline{X}) \times X_{\text{ét}}, J') \to X_{\text{ét}}$$

and notice that $i_{\mathrm{Spa}\,\underline{X}} = j_{\underline{X}} \circ l_{\underline{X}}$. Furthermore, the projection $p_{\underline{X}} : \mathscr{Q}(\underline{X}) \times X_{\text{ét}} \to X_{\text{ét}}$ is left adjoint to $l_{\underline{X}}$, and from corollary 5.7.2(ii) it follows easily that $p_{\underline{X}}$ is a weak morphism of sites for the topology J'. Combining with proposition 5.4.29(ii,iii), we deduce a pseudo-natural equivalence of pseudo-functors :

$$\mathsf{St}(l_{\underline{X}})^* \stackrel{\sim}{\to} \mathsf{St}(p_{\underline{X}})_*.$$

Especially, we get an equivalence of stacks :

$$\mathsf{St}(i_{\mathrm{Spa}\,\underline{X}})^*\mathscr{E} \stackrel{\sim}{\to} \mathsf{St}(j_{\underline{X}})^*\mathsf{St}(p_{\underline{X}})_*(\mathscr{E}) = \mathsf{St}(j_{\underline{X}})^*(\mathscr{Q}(\underline{X}) \times \mathscr{E}).$$

In light of proposition 5.7.3, we deduce a natural equivalence of stacks on $\underline{X}^{\mathrm{h}}_{U,\mathrm{\acute{e}t}}$:

$$\mathsf{St}(i_U)_* \circ \mathsf{St}(i_{\mathrm{Spa}\,\underline{X}})^*\mathscr{E} \xrightarrow{\sim} \mathscr{E}_U := \mathsf{St}((i_{U,\mathrm{Spa}\,\underline{X}})_{\mathrm{\acute{e}t}})^*\mathscr{E} \qquad \text{for every } U \in \mathrm{Ob}(\mathscr{Q}(\underline{X})).$$

Summing up, we conclude that the fibre category over U of the fibration $\mathscr{E}_{/\mathscr{Q}}$ is naturally equivalent to the fibre category $(\mathscr{E}_U)_{\underline{X}^{\mathrm{h}}_U}$ of the fibration $\mathscr{E}_U$, over the final object $\underline{X}^{\mathrm{h}}_U$ of the site $\underline{X}^{\mathrm{h}}_{U,\mathrm{\acute{e}t}}$.

(ii) By construction, $\mathscr{E}_{/\mathscr{Q}}$ is a stack on the site $(\mathscr{Q}(\underline{X}), J^*_{\mathscr{Q}})$; but in fact we have :

Theorem 15.7.6. *If $\mathscr{E}$ is ind-finite, the fibration $\mathscr{E}_{/\mathscr{Q}}$ is a stack on the site $(\mathscr{Q}(\underline{X}), J_{\mathscr{Q}})$.*

Proof. It suffices to check that the natural functor

$$\mathscr{E}_{/\mathscr{Q}}(U) \to \underset{\mathscr{S}\in J_{\mathscr{Q}}(U)^o}{2\text{-colim}}\, \mathsf{Cart}_{\mathscr{Q}(\underline{X})}(\mathscr{S}, \mathscr{E}_{/\mathscr{Q}}) \qquad \text{for every } U \in \mathrm{Ob}(\mathscr{Q}(\underline{X}))$$

is an equivalence. Since $J_{\mathscr{Q}}(U)$ is cofiltered for the order given by inclusion of sieves, this 2-colimit is represented by the colimit of the same system of categories (example 3.3.13(iv)); then arguing as in the proof of theorem 15.5.27 we may assume that $U = \mathrm{Spa}\,\underline{X}$, and replace $J_{\mathscr{Q}}(\mathrm{Spa}\,\underline{X})$ by its coinitial subset consisting of the sieves generated by the standard coverings. Thus, let $f_\bullet := (f_0, \dots, f_n)$ and $R_\bullet$ be as in remark 15.5.26(i); for every $k \in \mathbb{N}$ let $[k] := \{0, \dots, k\}$, and set as well $[-1] := \varnothing$. We consider the category $\Sigma^+_{2,n}$ whose objects are all the maps $[j] \to [n]$ for $j = -1, 0, 1, 2$; for two such objects $[j] \xrightarrow{\varphi} [n] \xleftarrow{\varphi'} [k]$, the morphisms $\mu : \varphi \to \varphi'$ are the injective non-decreasing maps $\mu : [j] \to [k]$ such that $\varphi' \circ \mu = \varphi$. The unique map $\varphi_\varnothing : [-1] \to [n]$ is clearly the initial object of $\Sigma^+_{2,n}$. Let also $\Sigma_{2,n} \subset \Sigma^+_{2,n}$ be the full subcategory whose objects are the morphisms $[j] \to [n]$ with $j \geq 0$. Consider the functor

$$\Phi^+ : \Sigma^{+o}_{2,n} \to \mathscr{Q}(\underline{X}) \qquad : \qquad ([j] \xrightarrow{\varphi} [n]) \mapsto R_{\varphi([j])}$$

(here $R_\Lambda \subset \mathrm{Spa}\,\underline{X}$ is defined as in remark 15.5.26(i), for every subset $\Lambda \subset [n]$), and let $\Phi : \Sigma^o_{2,n} \to \mathscr{Q}(\underline{X})$ be the restriction of Φ^+. Taking into account the discussion of (3.5.18), we come down to checking that the natural functor

$$\mathscr{E}_{/\mathscr{Q}}(\mathrm{Spa}\,\underline{X}) \to \underset{\Sigma_{2,n}}{2\text{-}\lim}\, \mathscr{E}_{/\mathscr{Q}}(-) \circ \Phi^o$$

is an equivalence (notation of (3.2.4)). Following (4.5.11), we deduce two other fibred lex-sites

$$\mathscr{U}^{\mathrm{h}}_{f_\bullet} := \Sigma^{+o}_{2,n} \times_{\mathscr{Q}(\underline{X})} (\underline{X}^{\mathrm{h}}_{\bullet,\mathrm{\acute{e}t}}, J) \to \Sigma^{+o}_{2,n} \qquad \underline{\Sigma}^{+o}_{2,n} := \Sigma^{+o}_{2,n} \times_{\mathscr{Q}(\underline{X})} (\mathscr{Q}(\underline{X}), \mathbf{1}_{\mathscr{Q}(\underline{X})}, J^*_\bullet)$$

and according to (4.5.11) and (5.7) and proposition 5.4.29(ii), the projections Φ^+ and

$$\pi_{\mathscr{U}} : \mathscr{U}^{\mathrm{h}}_{f_\bullet} \to \underline{X}^{\mathrm{h}}_{\bullet,\mathrm{\acute{e}t}}$$

are cocontinuous weak morphisms of sites for the topologies of the respective total sites. Furthermore, the functor $e \circ \Phi^+ : \Sigma^{+o}_{2,n} \to X^{\mathrm{h}}_{\bullet,\mathrm{\acute{e}t}}$ factors uniquely through a $\Sigma^{+o}_{2,n}$-cartesian functor

$$e_{\mathscr{U}} : \Sigma^{+o}_{2,n} \to \mathscr{U}^{\mathrm{h}}_{f_\bullet}$$

and the projection $\pi_{\mathscr{U}}$. Just like for e, the functor $e_{\mathscr{U}}$ is a morphism of the respective total sites. Set $\mathscr{F} := \mathsf{St}(\pi_{\mathscr{U}})_* \circ \mathsf{St}(i_{\mathrm{Spa}\,\underline{X}})^*\mathscr{E}$; a direct inspection yields an equivalence of stacks

$$\mathscr{G} := \mathsf{St}(e_{\mathscr{U}})_*\mathscr{F} \xrightarrow{\sim} \mathsf{St}(\Phi^+)_*(\mathscr{E}_{/\mathscr{Q}})$$

over the total site of $\underline{\Sigma}^{+o}_{2,n}$, and we are reduced to checking that the natural functor

$$\mathscr{G}(\varphi_\varnothing) \to \underset{\Sigma_{2,n}}{2\text{-}\lim}\, \mathscr{G}(-) \tag{15.7.7}$$

is an equivalence. To this aim, we proceed as in remark 15.7.5(i) : from the constant functor $c_X : \Sigma^{+o}_{2,n} \to \mathsf{Sch}^{\mathsf{qcqs}}$ we deduce the fibred site

$$\Sigma^{+o}_{2,n} \times X_{\mathrm{\acute{e}t}} = \Sigma^{+o}_{2,n} \times_{(\mathsf{Sch}^{\mathsf{qcqs}}, c_X)} (\mathsf{Sch}^{\mathsf{qcqs}}_{\mathrm{\acute{e}t}}, t, J_{\bullet,\mathrm{\acute{e}t}}) \to \Sigma^{+o}_{2,n}$$

whose fibre categories are naturally identified with $X_{\text{ét}}$, endowed with its étale topology. Let $j_{\underline{X}} : \underline{X}^{\text{h}}_{\bullet,\text{ét}} \to \mathscr{Q}(\underline{X}) \times X_{\text{ét}}$ be as in remark 15.7.5(i); we deduce a morphism of fibred lex-sites

$$j_\Sigma := \Sigma^{+o}_{2,n} \times_{\mathscr{Q}(\underline{X})} j_{\underline{X}} : \mathscr{U}^{\text{h}}_{f_\bullet} \to \Sigma^{+o}_{2,n} \times X_{\text{ét}}$$

which as usual induces a morphism on total sites, and by the same token, the projection

$$\Sigma^{+o}_{2,n} \times_{\mathscr{Q}(\underline{X})} p_{\underline{X}} : \Sigma^{+o}_{2,n} \times X_{\text{ét}} \to X_{\text{ét}}$$

is a weak morphism of sites, for the topology of the total site, and is left adjoint to the inclusion functor

$$l_\Sigma := \Sigma^{+o}_{2,n} \times_{\mathscr{Q}(\underline{X})} l_{\underline{X}} : X_{\text{ét}} \to \Sigma^{+o}_{2,n} \times X_{\text{ét}}.$$

By proposition 5.7.3 we have then an equivalence of stacks over the total site of $\mathscr{U}^{\text{h}}_{f_\bullet}$:

$$\mathscr{F} \xrightarrow{\sim} \mathsf{St}(j_\Sigma)^*(\Sigma^{+o}_{2,n} \times \mathscr{E}).$$

Next, let $A_0 \subset A := \mathscr{O}^{\text{h}}_{\text{Spa}\,\underline{X}}(\text{Spa}\,\underline{X})$ be a subring of definition, and $I_0 \subset A_0$ an ideal of adic definition; set $S := \text{Spec}\,A_0$ and $S' := \text{Spec}\,A_0/I_0$. As in remark 15.5.26(iv,v), the sequence $f_\bullet$ induces a morphism of schemes $X \to \mathbb{P}^n_S$, whose schematic image we denote by V, and we let $\psi : X \to V$ be the resulting morphism of S-schemes; moreover, for every non-empty subset $\Lambda \subset [n]$ we have an open subset $V_\Lambda \subset V$ which is naturally identified with $\text{Spec}\,A_0[f_k/f_i \mid (k,i) \in [n] \times \Lambda]$, and we set as well $V_\varnothing := V$. Then, for every $\Lambda \subset [n]$ let V^{h}_Λ be the henselization of V_Λ along its closed subscheme $V'_\Lambda := S' \times_S V_\Lambda$; we have natural identifications

$$X^{\text{h}}_\Lambda \xrightarrow{\sim} X \times_V V^{\text{h}}_\Lambda \qquad \text{for every } \Lambda \subset [n]$$

and we denote by $\psi^{\text{h}}_\Lambda : X^{\text{h}}_\Lambda \to V^{\text{h}}_\Lambda$ the induced projection, and by $q_\Lambda : V^{\text{h}}_\Lambda \to V$ the composition of the henselization map $V^{\text{h}}_\Lambda \to V_\Lambda$ with the open immersion $V_\Lambda \to V$. We consider the functors

$$G : \Sigma^{+o}_{2,n} \to \mathsf{Sch}^{\mathsf{qcqs}} \quad : \quad ([j] \xrightarrow{\varphi} [n]) \mapsto V^{\text{h}}_{\varphi([j])}$$
$$G' : \Sigma^{+o}_{2,n} \to \mathsf{Sch}^{\mathsf{qcqs}} \quad : \quad ([j] \xrightarrow{\varphi} [n]) \mapsto V'_{\varphi([j])}$$

which to every morphism $\mu : \varphi \to \varphi'$ as in the foregoing assign the natural morphisms

$$(15.7.8) \qquad V^{\text{h}}_{\varphi'([k])} \to V^{\text{h}}_{\varphi([j])} \qquad \text{and respectively} \qquad V'_{\varphi'([k])} \to V'_{\varphi([j])}.$$

As in (15.7.2) we deduce fibred lex-sites

$$\mathscr{V}^{\text{h}}_{f_\bullet} := \Sigma^{+o}_{2,n} \times_{(\mathsf{Sch}^{\mathsf{qcqs}},G)} (\mathsf{Sch}^{\mathsf{qcqs}}_{\text{ét}}, t, J_{\bullet,\text{ét}}) \to \Sigma^{+o}_{2,n}$$
$$\mathscr{V}'_{f_\bullet} := \Sigma^{+o}_{2,n} \times_{(\mathsf{Sch}^{\mathsf{qcqs}},G')} (\mathsf{Sch}^{\mathsf{qcqs}}_{\text{ét}}, t, J_{\bullet,\text{ét}}) \to \Sigma^{+o}_{2,n}.$$

The morphisms $(\psi^{\text{h}}_\Lambda \mid \Lambda \subset [n])$, $(q_\Lambda \mid \Lambda \subset [n])$, the closed immersions $(V'_\Lambda \to V^{\text{h}}_\Lambda \mid \Lambda \subset [n])$ and $V' \to V$, and the open immersions $(V'_\Lambda \to V' \mid \Lambda \subset [n])$ define natural transformations $\underline{X}^{\text{h}}_\bullet \circ \Phi^+ \Rightarrow G$, $G \Rightarrow c_V$, $G' \Rightarrow G$, $c_{V'} \Rightarrow c_V$ and respectively $G' \Rightarrow c_{V'}$, where $c_V : \Sigma^{+o}_{2,n} \to \mathsf{Sch}$ is the constant functor with value V, and likewise for $c_{V'}$. After fixing cleavages, these natural transformations induce morphisms of fibred lex-sites

$$\mathscr{U}^{\text{h}}_{f_\bullet} \xrightarrow{\psi_{\mathscr{U}}} \mathscr{V}^{\text{h}}_{f_\bullet} \xleftarrow{\tau_{\mathscr{V}}} \mathscr{V}'_{f_\bullet} \xrightarrow{q'} \Sigma^{+o}_{2,n} \times V'_{\text{ét}} \xrightarrow{\tau_\Sigma} \Sigma^{+o}_{2,n} \times V_{\text{ét}} \xleftarrow{q} \mathscr{V}^{\text{h}}_{f_\bullet}$$

(see (4.5.12)), which are also morphisms of sites for the topologies of the respective total sites (proposition 4.5.9). Define functors $e_{\mathscr{V}} : \Sigma^{+o}_{2,n} \to \mathscr{V}^{\text{h}}_{f_\bullet}$ and $e_{\mathscr{V}'} : \Sigma^{+o}_{2,n} \to \mathscr{V}'_{f_\bullet}$ by the rules :

$$\varphi \mapsto (\varphi, V^{\text{h}}_{\varphi([j])}) \qquad \text{and respectively :} \qquad \varphi \mapsto (\varphi, V'_{\varphi[j]}) \qquad \text{for every } \varphi : [j] \to [n]$$

and which assign to every morphism $\mu : \varphi \to \psi$ as in the foregoing, the morphisms (15.7.8). Just as for e and $e_{\mathscr{U}}$, both $e_{\mathscr{V}}$ and $e_{\mathscr{V}'}$ are morphisms of the respective total sites, and we get a commutative diagram of morphisms of sites :

$$
\begin{array}{ccccc}
V'_{\text{ét}} & \xrightarrow{\tau_{\text{ét}}} & V^{\text{h}}_{\text{ét}} & \xleftarrow{\psi_{\text{ét}}} & X_{\text{ét}} \\
\uparrow{\scriptstyle l_{V'}} & & \uparrow{\scriptstyle l_V} & & \uparrow{\scriptstyle l_\Sigma} \\
\Sigma^{+o}_{2,n} \times V'_{\text{ét}} & \xrightarrow{\tau_\Sigma} & \Sigma^{+o}_{2,n} \times V_{\text{ét}} & \xleftarrow{\psi_\Sigma} & \Sigma^{+o}_{2,n} \times X_{\text{ét}} \\
\uparrow{\scriptstyle q'} & & \uparrow{\scriptstyle q} & & \uparrow{\scriptstyle j_\Sigma} \\
\mathscr{V}'_{f_\bullet} & \xrightarrow{\tau_{\mathscr{V}}} & \mathscr{V}^{\text{h}}_{f_\bullet} & \xleftarrow{\psi_{\mathscr{U}}} & \mathscr{U}^{\text{h}}_{f_\bullet} \\
\downarrow{\scriptstyle e_{\mathscr{V}'}} & & \downarrow{\scriptstyle e_{\mathscr{V}}} & & \downarrow{\scriptstyle e_{\mathscr{U}}} \\
\underline{\Sigma}^{+o}_{2,n} & = & \underline{\Sigma}^{+o}_{2,n} & = & \underline{\Sigma}^{+o}_{2,n}.
\end{array}
\tag{15.7.9}
$$

where ψ_Σ is likewise deduced from ψ. Set $\mathscr{F}' := \mathsf{St}(\psi_{\mathscr{U}})_*\mathscr{F}$. There follows an equivalence :

$$\mathscr{G} \xrightarrow{\sim} \mathsf{St}(e_{\mathscr{V}})_*\mathscr{F}'.$$

We regard the left square of the bottom row and the right square of the central row in (15.7.9) as oriented squares of sites, with orientations given by $\mathbf{1}_{e_{\mathscr{V}'}}$, respectively $\mathbf{1}_{q\circ\psi_{\mathscr{U}}}$. We remark :

Claim 15.7.10. For every ind-finite stack $\mathscr{A}$ on $\mathscr{V}^{\text{h}}_{f_\bullet}$, the base change transformation

$$\Upsilon(\mathsf{St}(\mathbf{1}_{e_{\mathscr{V}'}})^\gamma_*)_{\mathscr{A}} : \mathsf{St}(e_{\mathscr{V}})_*(\mathscr{A}) \to \mathsf{St}(e_{\mathscr{V}'})_*\mathsf{St}(\tau_{\mathscr{V}})^*(\mathscr{A})$$

is a natural equivalence (notation of (2.3.8) and (5.5.23)).

Proof of the claim. This follows by combining corollary 5.7.26 and [74, Th.1' and Cor.1]. Notice that the proof of *loc.cit.* relies on proposition 1 of the same article; for the latter, a complete proof is available in [105, Exp.XX, Prop.6.3.2]. More precisely, in the latter reference, the required results are stated only for stacks in groupoids; however, a direct inspection shows that the proofs work *verbatim* for arbitrary ind-finite stacks. ◊

Notice as well the pseudo-natural equivalences of pseudo-functors :

$$\mathsf{St}(e_{\mathscr{U}})_* \xrightarrow{\sim} \mathsf{St}(e_{\mathscr{V}})_* \circ \mathsf{St}(\psi_{\mathscr{U}})_* \qquad \mathsf{St}(\tau_{\mathscr{V}})^* \circ \mathsf{St}(q)^* \xrightarrow{\sim} \mathsf{St}(q')^* \circ \mathsf{St}(\tau_\Sigma)^*. \tag{15.7.11}$$

Claim 15.7.12. The base change transformation

$$\Upsilon(\mathsf{St}(\mathbf{1}_{q\circ\psi_{\mathscr{U}}})^\gamma_*) : \mathsf{St}(q)^* \circ \mathsf{St}(\psi_\Sigma)_* \to \mathsf{St}(\psi_{\mathscr{U}})_* \circ \mathsf{St}(j_\Sigma)^*$$

is a pseudo-natural equivalence.

Proof of the claim. For every subset $\Lambda \subset [n]$ we consider the oriented square of lex-sites :

$$
\mathscr{D}_\Lambda \quad : \quad
\begin{array}{ccc}
X^{\text{h}}_{\Lambda,\text{ét}} & \xrightarrow{j_\Lambda} & X_{\text{ét}} \\
\downarrow{\scriptstyle \psi_\Lambda} & \not\swarrow \mathbf{1}_{\psi\circ j'_\Lambda} & \downarrow{\scriptstyle \psi} \\
V^{\text{h}}_{\Lambda,\text{ét}} & \xrightarrow{q_\Lambda} & V_{\text{ét}}.
\end{array}
$$

According to corollary 5.7.26, it suffices to check that the base change transformation

$$\Upsilon(\mathsf{St}(\mathbf{1}_{\psi\circ j'_\Lambda})^\gamma_*) : \mathsf{St}(q_\Lambda)^* \circ \mathsf{St}(\psi)_* \to \mathsf{St}(\psi_\Lambda)_* \circ \mathsf{St}(j_\Lambda)^*$$

is a pseudo-natural equivalence for every such Λ. However, V^{h}_Λ is the limit of a cofiltered system of quasi-compact and quasi-separated V-schemes $(V_{\Lambda,i} \mid i \in I)$ with affine transition morphisms, and X^{h}_Λ is the limit of the induced cofiltered system $(X_{\Lambda,i} := X \times_V V_{\Lambda,i} \mid i \in$

I); hence $V^{\mathrm{h}}_{\Lambda,\text{ét}}$ and $X^{\mathrm{h}}_{\Lambda,\text{ét}}$ represent the 2-limits of the induced systems of lex-sites $V_{\bullet,\text{ét}} := (V_{i,\text{ét}} \mid i \in I)$ and $X_{\Lambda,\bullet,\text{ét}} := (X_{\Lambda,i,\text{ét}} \mid i \in I)$. We regard I as a category (see example 1.1.6(iii)), and we may assume that I admits a final object i_0 such that $V_{\Lambda,i_0} = V$; then $V_{\Lambda,\bullet,\text{ét}}$ and $X_{\Lambda,\bullet,\text{ét}}$ yield fibred lex-sites $\underline{V}_{\Lambda,\bullet,\text{ét}}$ and $\underline{X}_{\Lambda,\bullet,\text{ét}}$ over I, and the system of projections $(X_{\Lambda,i} \to V_{\Lambda,i} \mid i \in I)$ induces a morphism of fibred lex-sites $\underline{X}_{\Lambda,\bullet,\text{ét}} \to \underline{V}_{\Lambda,\bullet,\text{ét}}$. Then the assertion follows from corollary 5.7.30. ◊

Now, from proposition 5.6.34 we see that $\Sigma^{+o}_{2,n} \times \mathscr{E}$ is an ind-finite stack, and then the same holds for $\mathscr{F}$ (theorem 5.6.33), and also for $\mathscr{F}'$, again by proposition 5.6.34. Set

$$\mathscr{E}' := \mathsf{St}(\tau_{\text{ét}})^* \circ \mathsf{St}(\psi_{\text{ét}})_*(\mathscr{E}).$$

From (15.7.11) and claims 15.7.10 and 15.7.12, we deduce an equivalence of stacks :

$$\begin{aligned}\mathscr{G} &\xrightarrow{\sim} \mathsf{St}(e_{\mathscr{V}'})_* \circ \mathsf{St}(q')^* \circ \mathsf{St}(\tau_\Sigma)^* \circ \mathsf{St}(\psi_\Sigma)_*(\Sigma^{+o}_{2,n} \times \mathscr{E}) \\ &\xrightarrow{\sim} \mathsf{St}(e_{\mathscr{V}'})_* \circ \mathsf{St}(q')^* \circ \mathsf{St}(l_{V'})^*(\mathscr{E}').\end{aligned}$$

Once again, by corollary 5.7.2(ii) the projection $\Sigma^{+o}_{2,n} \times V'_{\text{ét}} \to V'_{\text{ét}}$ is a weak morphism of sites, for the topology of the total site, and is left adjoint to l_V, so we get an equivalence

$$\mathsf{St}(l_{V'})^*(\mathscr{E}') \xrightarrow{\sim} \Sigma^{+o}_{2,n} \times \mathscr{E}'$$

(proposition 5.4.29(ii,iii)). Similarly, the functor q' admits a left adjoint

$$s : \mathscr{V}'_{f_\bullet} \to \Sigma^{+o}_{2,n} \times V'_{\text{ét}} \qquad ([j] \xrightarrow{\varphi} [n], U \xrightarrow{f} V_{\varphi([j])}) \mapsto (\varphi, U \xrightarrow{f} V_{\varphi([j])} \to V')$$

and as explained in remark 5.7.34, the functor s is cocontinuous and is a weak morphism of sites, for the topologies of the total sites, whence a pseudo-natural equivalence :

$$\mathsf{St}(q')^* \xrightarrow{\sim} \mathsf{St}(s)_*.$$

Summing up, we obtain an equivalence of stacks :

$$\mathscr{G} \xrightarrow{\sim} \mathsf{St}(e_{\mathscr{V}'})_* \circ \mathsf{St}(s)_*(\Sigma^{+o}_{2,n} \times \mathscr{E}'). \tag{15.7.13}$$

Lastly, consider the functor

$$\Psi^+ : \Sigma^{+o}_{2,n} \to V'_{\text{ét}} \qquad ([j] \xrightarrow{\varphi} [n]) \mapsto V'_{\varphi([j])}$$

and let $\Psi : \Sigma^o_{2,n} \to V'_{\text{ét}}$ be the restriction of Ψ^+. By inspecting the definitions, we find that the equivalence (15.7.13) identifies the functor (15.7.7) with the similar natural functor

$$\mathscr{E}'(V') \to \underset{\Sigma_{2,n}}{2\text{-}\lim}\, \mathscr{E}'(-) \circ \Psi^o.$$

The latter is an equivalence, since $\mathscr{E}'$ is a stack on $V'_{\text{ét}}$, whence the theorem. □

15.7.14. Let $f : \underline{X}' \to \underline{X}$ be an adic morphism of topologically henselian quasi-affinoid schemes, and $(U_1, \dots, U_n)$ a finite covering of $\operatorname{Spa} \underline{X}$ consisting of quasi-affinoid open subsets; recall that $U'_i := (\operatorname{Spa} f)^{-1} U_i$ is a quasi-affinoid subset of $\operatorname{Spa} \underline{U}$, for every $i = 1, \dots, n$ (remark 15.5.9(iv)). As in (15.7.4), define the corresponding functors

$$\underline{X}^{\mathrm{h}}_\bullet : \mathscr{Q}(\underline{X}) \to \mathsf{Sch}^{\mathsf{qcqs}} \qquad \underline{X}'^{\mathrm{h}}_\bullet : \mathscr{Q}(\underline{X}') \to \mathsf{Sch}^{\mathsf{qcqs}}$$

and for every $\Lambda \subset \{1, \dots, n\}$, set $U_\Lambda := \bigcap_{i \in \Lambda} U_i$ (with $U_\varnothing := \operatorname{Spa} \underline{X}$) and $U'_\Lambda := (\operatorname{Spa} f)^{-1} U_\Lambda$. To ease notation, let X_Λ (resp. X'_Λ) be the scheme underlying the quasi-affinoid scheme $\underline{X}^{\mathrm{h}}_{U_\Lambda}$ (resp. $\underline{X}'^{\mathrm{h}}_{U'_\Lambda}$). Recall that $\underline{X}'^{\mathrm{h}}_{U'_\Lambda}$ is the topological henselization of $\underline{Z}_\Lambda := \underline{X}' \times_{\underline{X}} \underline{X}^{\mathrm{h}}_{U_\Lambda}$; the composition of the henselization morphism $\underline{X}'^{\mathrm{h}}_{U'_\Lambda} \to \underline{Z}_\Lambda$ and the projection $\underline{Z}_\Lambda \to \underline{X}^{\mathrm{h}}_{U_\Lambda}$ is then a morphism of quasi-affinoid schemes

$$f_\Lambda : \underline{X}'^{\mathrm{h}}_{U'_\Lambda} \to \underline{X}^{\mathrm{h}}_{U_\Lambda}$$

so we have a commutative diagram of schemes :

$$\begin{array}{ccc} X'_\Lambda & \xrightarrow{f_\Lambda} & X_\Lambda \\ {\scriptstyle p'_\Lambda}\downarrow & & \downarrow{\scriptstyle p_\Lambda} \\ X' & \xrightarrow{f} & X \end{array} \qquad \text{for every } \Lambda \subset \{1,\dots,n\}$$

where p_Λ and p'_Λ are the natural projections. Let also $\mathscr{E}$ be an ind-finite stack on $X_{\text{ét}}$, and set

$$\mathscr{E}' := \mathsf{St}(f_{\text{ét}})^*\mathscr{E} \qquad \mathscr{E}_\Lambda := \mathsf{St}(p_{\Lambda,\text{ét}})^*\mathscr{E} \qquad \mathscr{E}'_\Lambda := \mathsf{St}(f_{\Lambda,\text{ét}})^*\mathscr{E}_\Lambda \qquad \text{for every } \Lambda \subset \{1,\dots,n\}.$$

We consider the units of adjunction :

$$\eta : \mathscr{E} \to \mathsf{St}(f_{\text{ét}})_*\mathscr{E}' \qquad \text{and} \qquad \eta_\Lambda : \mathscr{E}_\Lambda \to \mathsf{St}(f_{\Lambda,\text{ét}})_*\mathscr{E}'_\Lambda \qquad \text{for every } \Lambda \subset \{1,\dots,n\}.$$

Lemma 15.7.15. *In the situation of* (15.7.14)*, suppose that the functor*

$$\eta_{\Lambda,X_\Lambda} : \mathscr{E}_{\Lambda,X_\Lambda} \to \mathscr{E}'_{\Lambda,X'_\Lambda}$$

is an equivalence for every non-empty $\Lambda \subset \{1,\dots,n\}$. Then $\eta_X : \mathscr{E}_X \to \mathscr{E}'_{X'}$ is an equivalence.

Proof. Define the category $\Sigma^+_{2,n}$, its full subcategory $\Sigma_{2,n}$, the fibred site $\underline{\Sigma}^+_{2,n}$ and the site $(\mathscr{Q}(\underline{X}), J^*_{\mathscr{Q}})$ as in the proof of theorem 15.7.6. We consider the functor

$$\Phi^+ : \Sigma^{+o}_{2,n} \to \mathscr{Q}(\underline{X}) \qquad : \qquad ([j] \xrightarrow{\varphi} [n]) \mapsto U_{\varphi([j])}$$

and its restriction $\Phi : \Sigma^o_{2,n} \to \mathscr{Q}(\underline{X})$. We attach likewise to the covering $(U'_1,\dots,U'_n)$ of $\operatorname{Spa}\underline{X}'$ the corresponding functors $\Phi'^+ : \Sigma^{+o}_{2,n} \to \mathscr{Q}(\underline{X}')$ and its restriction Φ' to $\Sigma^o_{2,n}$. Then, from $\Psi^+ := \underline{X}^{\mathrm{h}}_\bullet \circ \Phi^+$ and $\Psi'^+ := \underline{X}'^{\mathrm{h}}_\bullet \circ \Phi'^+$ we get fibred sites :

$$\mathscr{U} := \Sigma^{+o}_{2,n} \times_{(\mathsf{Sch},\Psi^+)} (\mathsf{Sch}^{\mathsf{qcqs}}_{\text{ét}}, t, J_{\bullet,\text{ét}}) \qquad \mathscr{U}' := \Sigma^{+o}_{2,n} \times_{(\mathsf{Sch},\Psi'^+)} (\mathsf{Sch}^{\mathsf{qcqs}}_{\text{ét}}, t, J_{\bullet,\text{ét}}).$$

Next, let $e : \Sigma^{+o}_{2,n} \to \mathscr{U}$ be the functor that assigns to every $([j] \xrightarrow{\varphi} [n]) \in \mathrm{Ob}(\Sigma^{+o}_{2,n})$ the final object $X_{\varphi([j])}$ of the site $X_{\varphi([j]),\text{ét}}$; let also $i : X_{\text{ét}} \to \mathscr{U}$ be the inclusion functor which identifies $X_{\text{ét}}$ with the fibre category of the fibration $\mathscr{U} \to \Sigma^{+o}_{2,n}$, over the final object $[-1] \in \mathrm{Ob}(\Sigma^{+o}_{2,n})$. Likewise we define the functor $e' : \Sigma^{+o}_{2,n} \to \mathscr{U}'$ and $i' : X'_{\text{ét}} \to \mathscr{U}'$.

With this notation, Φ^+ is a weak morphism of sites $(\mathscr{Q}(\underline{X}), J^*_{\mathscr{Q}}) \to \Sigma^{+o}_{2,n}$ for the topology of the total site of $\underline{\Sigma}^{+o}_{2,n}$, and likewise for Φ'^+. Also, both i and e are morphisms of sites for the topologies of the total sites, and likewise for i' and e'. Furthermore, the rule : $([j] \xrightarrow{\varphi} [n]) \mapsto f_{\varphi([j])}$ for every $\varphi \in \mathrm{Ob}(\Sigma^{+o}_{2,n})$ defines a natural transformation

$$\underline{X}'^{\mathrm{h}}_\bullet \circ \Phi^+ \Rightarrow \underline{X}^{\mathrm{h}}_\bullet \circ \Phi'^+$$

which, after choosing a cleavage for the fibration $\mathsf{Sch}^{\mathsf{qcqs}}_{\text{ét}} \to \mathsf{Sch}^{\mathsf{qcqs}}$, determines a morphism of fibred sites $\varphi : \mathscr{U}' \to \mathscr{U}$, which is again a morphism of sites for the topologies of the respective total sites. A direct inspection then yields a commutative diagram of morphisms of sites :

$$(15.7.14) \qquad \begin{array}{ccccc} \Sigma^{+o}_{2,n} & \xleftarrow{e'} & \mathscr{U}' & \xrightarrow{i_{\underline{X}'}} & X'_{\text{ét}} \\ \| & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f_{\text{ét}}} \\ \Sigma^{+o}_{2,n} & \xleftarrow{e} & \mathscr{U} & \xrightarrow{i_{\underline{X}}} & X_{\text{ét}}. \end{array}$$

By theorem 15.7.6, we associate with $\mathscr{E}$ a stack $\mathscr{E}_{/\mathscr{Q}}$ on $(\mathscr{Q}(\underline{X}), J_{\mathscr{Q}})$; likewise, since $\mathscr{E}'$ is also ind-finite (theorem 5.6.33), we get the stack $\mathscr{E}'_{/\mathscr{Q}}$ on $(\mathscr{Q}(\underline{X}'), J'_{\mathscr{Q}})$. Then, arguing as in the proof of theorem 15.7.6 we get equivalences of fibrations over $\Sigma^{+o}_{2,n}$:

$$\mathsf{St}(\Phi^+)_*(\mathscr{E}_{/\mathscr{Q}}) \xrightarrow{\sim} \mathscr{E}_{/\Sigma} := \mathsf{St}(e)_* \circ \mathsf{St}(i_{\underline{X}})^*\mathscr{E} \qquad \mathsf{St}(\Phi'^+)_*(\mathscr{E}'_{/\mathscr{Q}}) \xrightarrow{\sim} \mathscr{E}'_{/\Sigma} := \mathsf{St}(e')_* \circ \mathsf{St}(i_{\underline{X}'})^*\mathscr{E}'.$$

On the other hand, from diagram (15.7.14) we deduce an equivalence of stacks

$$\mathsf{St}(g)^* \circ \mathsf{St}(i_{\underline{X}})^*(\mathscr{E}) \stackrel{\sim}{\to} \mathsf{St}(i_{\underline{X}'})^*(\mathscr{E}')$$

whence, by adjunction, a morphism of stacks :

$$\mathsf{St}(i_{\underline{X}})^*(\mathscr{E}) \to \mathsf{St}(g)_* \circ \mathsf{St}(i_{\underline{X}'})^*(\mathscr{E}')$$

and after composing with $\mathsf{St}(e)_*$, we arrive at a morphism of fibrations over $\Sigma^{+o}_{2,n}$:

$$\omega : \mathscr{E}_{/\Sigma} \to \mathscr{E}'_{/\Sigma}.$$

Now, pick cleavages for $\mathscr{E}_{/\Sigma}$ and $\mathscr{E}'_{/\Sigma}$, and let c and c' be the associated pseudo-functors; then ω corresponds to a pseudo-natural transformation $\omega_\bullet : \mathsf{c} \Rightarrow \mathsf{c}'$. Arguing as in remark 15.7.5(i), we see that c_φ is equivalent to $\mathscr{E}_{\varphi([j])}$ for every $([j] \xrightarrow{\varphi} [n]) \in \mathrm{Ob}(\Sigma^{+o}_{2,n})$, and likewise for c'_φ, so we have for every such φ an essentially commutative diagram of categories :

$$\begin{array}{ccc} \mathsf{c}_\varphi & \xrightarrow{\omega_\varphi} & \mathsf{c}'_\varphi \\ \downarrow & & \downarrow \\ \mathscr{E}_{\varphi([j])} & \xrightarrow{\omega^*_\varphi} & \mathscr{E}'_{\varphi([j])} \end{array}$$

whose vertical arrows are equivalences. Lastly, since $\mathscr{E}_{/\mathscr{Q}}$ and $\mathscr{E}'_{/\mathscr{Q}}$ are stacks for the topologies $J_{\mathscr{Q}}$ and respectively $J'_{\mathscr{Q}}$, we know that the natural functors

$$\mathscr{E}_X \to \underset{\Sigma_{2,n}}{2\text{-}\lim}\,\mathsf{c} \qquad \mathscr{E}'_{X'} \to \underset{\Sigma_{2,n}}{2\text{-}\lim}\,\mathsf{c}'$$

are equivalences, hence $\omega_\varnothing : \mathscr{E}_X \to \mathscr{E}'_{X'}$ is identified, up to equivalences of categories and isomorphisms of functors, with the induced functor

$$\underset{\Sigma_{2,n}}{2\text{-}\lim}\,\omega_\bullet : \underset{\Sigma_{2,n}}{2\text{-}\lim}\,\mathsf{c} \to \underset{\Sigma_{2,n}}{2\text{-}\lim}\,\mathsf{c}'.$$

To conclude the proof, it then suffices to check that for every $\varphi : [j] \to [n]$, the functor ω^*_φ is isomorphic to the functor η_{Λ,X_Λ}, with $\Lambda := \varphi([j])$. To this aim, we notice that the right square subdiagram of (15.7.14) can be regarded as an oriented square of links, after choosing adjoints and fixing adjunctions for its four arrows; its orientation (from the bottom left corner to the upper right corner) is given by the identity $\mathbf{1}_{i_{\underline{X}}\circ g} : i_{\underline{X}} \circ g \Rightarrow f_{\text{ét}} \circ i_{\underline{X}'}$. Then, proposition 5.7.3 says that the base change transformation

$$\Upsilon(\mathsf{St}(\mathbf{1}_{i_{\underline{X}}\circ g})^\Upsilon_*) : \mathsf{St}(f_{\text{ét}})^* \circ \mathsf{St}(i_{\underline{X}})_* \to \mathsf{St}(i_{\underline{X}'})_* \circ \mathsf{St}(g)^*$$

is a pseudo-natural equivalence. The assertion then follows, after combining with proposition 2.3.14 : the details shall be left to the reader. □

15.7.16. Resume the situation of (15.7.4), and let also $\underline{X}^\wedge := (X^\wedge, \mathscr{T}^\wedge_X, A^+_{X^\wedge})$ be the completion of $\underline{X}$, and denote by $\pi_X : \underline{X}^\wedge \to \underline{X}$ the *completion morphism* of quasi-affinoid schemes (given by the counit of the adjunction of proposition 15.4.12). The morphism π_X induces a morphism of étale sites $\pi_{X,\text{ét}} : X^\wedge_{\text{ét}} \to X_{\text{ét}}$, and for every stack $\mathscr{E}$ on $X_{\text{ét}}$ we let

$$\mathscr{E}^\wedge := \mathsf{St}(\pi_{X,\text{ét}})^*(\mathscr{E}).$$

Theorem 15.7.17. *With the notation of* (15.7.16)*, if $\mathscr{E}$ is ind-finite, the unit of adjunction*

$$\mathscr{E}_X \to \mathscr{E}^\wedge_{X^\wedge} = \mathsf{St}(\pi_{X,\text{ét}})_*(\mathscr{E}^\wedge)_X$$

is an equivalence of categories.

Proof. To begin with, let $f : \underline{Y} \to \underline{X}$ be any f-adic morphism of quasi-affinoid schemes; since $\pi_X : \underline{X}^\wedge \to \underline{X}$ is also f-adic, the fibre product $\underline{Y}' := \underline{X}^\wedge \times_{\underline{X}} \underline{Y}$ is well defined (example 15.4.8), and we let $\underline{X}^\wedge \xleftarrow{p_1} \underline{Y}' \xrightarrow{p_2} \underline{Y}$ be the natural projections. The morphism $f^\wedge : \underline{Y}^\wedge \to \underline{X}^\wedge$ induced by f, and the completion morphism $\pi_Y : \underline{Y}^\wedge \to \underline{Y}$ determine a unique morphism of quasi-affinoid schemes

$$g : \underline{Y}^\wedge \to \underline{Y}' \qquad \text{such that} \qquad p_1 \circ g = f^\wedge \quad \text{and} \quad p_2 \circ g = \pi_Y.$$

Moreover, since $\underline{Y}^\wedge$ is complete, by adjunction g factors uniquely through a morphism of quasi-affinoid schemes $h : \underline{Y}^\wedge \to \underline{Y}'^\wedge$ and the completion morphism $\pi_{Y'} : \underline{Y}'^\wedge \to \underline{Y}'$.

Claim 15.7.18. In the foregoing situation, h is an isomorphism of quasi-affinoid schemes.

Proof of the claim. It suffices to show that the morphisms $p_2 \circ \pi_{Y'}$ and π_Y satisfy the same universal property. Thus, let $\underline{Z}$ be a complete quasi-affinoid schemes, and $k : \underline{Z} \to \underline{Y}$ any morphism of quasi-affinoid schemes; the composition $f \circ k : \underline{Z} \to \underline{X}$ factors uniquely through a morphism $l : \underline{Z} \to \underline{X}^\wedge$ and π_X, and there exists a unique morphism $l' : \underline{Z} \to \underline{Y}'$ such that $p_1 \circ l' = l$ and $p_2 \circ l' = k$. Again, l' factors uniquely through a morphism $l'^\wedge : \underline{Z} \to \underline{Y}'^\wedge$ and $\pi_{Y'}$. It follows easily that $l'^\wedge$ is the unique morphism such that $p_2 \circ \pi_{Y'} \circ l'^\wedge = k$, whence the contention. ◊

Claim 15.7.19. In order to prove the theorem, we may assume that $\underline{X}$ is an affinoid scheme.

Proof of the claim. Let $U_\bullet := (U_1, \dots, U_n)$ be an open covering of $\mathrm{Spa}\,\underline{X}$ consisting of finitely many affinoid open subsets. (To find such $U_\bullet$, let $A_X := \mathscr{O}_X(X)$, and $J \subset A_X$ a finitely generated ideal such that $\mathrm{Spec}\,A_X/J = \mathrm{Spec}\,A_X \setminus X$; then we may take a finite set of generators $f_\bullet := (f_1, \dots, f_n)$ of J, and let $U_\bullet$ be the standard covering associated with $f_\bullet$, as in (15.5.20).)

For every $\Lambda \subset \{1, \dots, n\}$, define the affinoid open subset $U_\Lambda \subset \mathrm{Spa}\,\underline{X}$ as in (15.7.14), and notice that $U_\Lambda^\wedge := (\mathrm{Spa}\,\pi_X)^{-1} U_\Lambda$ is an affinoid open subset of $\mathrm{Spa}\,\underline{X}^\wedge$. To ease notation, let also $\underline{X}_\Lambda := \underline{X}^{\mathrm{h}}_{U_\Lambda}$ and $\underline{X}'_\Lambda := \underline{X}^{\wedge\mathrm{h}}_{U_\Lambda^\wedge}$, for every such Λ. Let $p_\Lambda : \underline{X}_\Lambda \to \underline{X}$ be the natural projection, and $\pi_\Lambda : \underline{X}'_\Lambda \to \underline{X}_\Lambda$ the morphism of affinoid schemes deduced from π_X, as in (15.7.14), and set $\mathscr{E}_\Lambda := \mathsf{St}(p_{\Lambda,\text{ét}})^* \mathscr{E}$ and $\mathscr{E}'_\Lambda := \mathsf{St}(\pi_{\Lambda,\text{ét}})^* \mathscr{E}_\Lambda$, for every $\Lambda \subset \{1, \dots, n\}$. By lemma 15.7.15, in order to prove the theorem it suffices to show that the unit of adjunction

$$\eta_\Lambda : \mathscr{E}_{\Lambda, X_\Lambda} \to \mathscr{E}'_{\Lambda, X'_\Lambda}$$

is an equivalence for every non-empty $\Lambda \subset \{1, \dots, n\}$. To this aim, let $q_\Lambda : (\underline{X}_\Lambda)^\wedge \to \underline{X}_\Lambda$ and $q'_\Lambda : (\underline{X}'_\Lambda)^\wedge \to \underline{X}'_\Lambda$ be the completion morphisms, and denote by $(X_\Lambda)^\wedge$ (resp. $(X'_\Lambda)^\wedge$) the scheme underlying $(\underline{X}_\Lambda)^\wedge$ ($(\underline{X}'_\Lambda)^\wedge$); since $(\underline{X}'_\Lambda)^\wedge$ is also the completion of $\underline{X}^\wedge \times_{\underline{X}} \underline{X}_\Lambda$ (remark 15.4.10(vi) and corollary 8.4.15(i)), claim 15.7.18 yields an isomorphism of affinoid schemes

$$h : (\underline{X}_\Lambda)^\wedge \xrightarrow{\sim} (\underline{X}'_\Lambda)^\wedge \qquad \text{such that} \qquad \pi_\Lambda \circ q'_\Lambda \circ h = q_\Lambda. \tag{15.7.20}$$

Set $\mathscr{E}_\Lambda^\wedge := \mathsf{St}(q_{\Lambda,\text{ét}})^* \mathscr{E}_\Lambda$ and $\mathscr{E}'^\wedge_\Lambda := \mathsf{St}(q'_{\Lambda,\text{ét}})^* \mathscr{E}'_\Lambda$, and denote

$$\eta_\Lambda^\wedge : \mathscr{E}_{\Lambda, X_\Lambda} \to \mathscr{E}^\wedge_{\Lambda,(X_\Lambda)^\wedge} \qquad \text{and} \qquad \eta'^\wedge_\Lambda : \mathscr{E}'_{\Lambda, X_\Lambda} \to \mathscr{E}'^\wedge_{\Lambda,(X'_\Lambda)^\wedge}$$

the units of adjunction. Now, up to equivalences of categories and isomorphisms of functors, the composition $\eta'^\wedge_\Lambda \circ \eta_\Lambda$ is naturally identified with the counit

$$\mathscr{E}_{\Lambda, X_\Lambda} \to (\mathsf{St}((\pi_\Lambda \circ q'_\Lambda)_{\text{ét}})^* \mathscr{E}_\Lambda)_{(X'_\Lambda)^\wedge}$$

and the latter is an equivalence if and only if the same holds for $\eta^\wedge_\Lambda$, due to (15.7.20). Thus, if both $\eta^\wedge_\Lambda$ and $\eta'^\wedge_\Lambda$ are equivalences, the same will follow for η_Λ. Summing up, the theorem will follow for the quasi-affinoid scheme $\underline{X}$, once we have shown that the theorem holds for the affinoid schemes $\underline{X}_\Lambda$ and $\underline{X}'_\Lambda$, for every non-empty $\Lambda \subset \{1, \dots, n\}$, whence the claim. ◊

For every scheme Y and every finite group G, let G_Y be the group scheme $G \times Y$ over Y; recall that a (right) G-torsor on $Y_{\text{ét}}$ is the datum of a surjective étale morphism of schemes $Z \to Y$, together with a morphism $\rho : Z \times_Y G_Y \to Z$ that induces an action of the group $G_Y(T)$ of T-sections of G_Y on the set $Z(T)$ of T-sections of Z, for every Y-scheme T, and such that ρ and the projection $Z \times_Y G_Y \to Z$ induce an isomorphism

$$Z \times_Y G_Y \xrightarrow{\sim} Z \times_Y Z.$$

A morphism of G_Y-torsors $(Z, \rho) \to (Z', \rho')$ is a morphism of schemes $Z \to Z'$ compatible in the obvious way with the morphisms ρ and ρ'. Clearly the G-torsors over $Y_{\text{ét}}$ form a category :

$$\mathsf{Tors}(Y_{\text{ét}}, G).$$

According to [105, Exp.XX, Prop.6.3.2], in order to prove the theorem, it suffices to check that :

(i) for every sheaf $\mathscr{F}$ on $X_{\text{ét}}$, the natural map $H^0(X_{\text{ét}}, \mathscr{F}) \to H^0(X^\wedge_{\text{ét}}, \widetilde{\pi}^*_{X,\text{ét}}\mathscr{F})$ is bijective
(ii) For every finite group G, and every finite morphism of schemes $Y \to X$, the natural functor $\mathsf{Tors}(Y_{\text{ét}}, G) \to \mathsf{Tors}((X^\wedge \times_X Y)_{\text{ét}}, G)$ is an equivalence of categories.

Moreover, according to [10, Exp.XII, Prop.6.5(i)], condition (i) is equivalent to :

(iii) For every finite morphism of schemes $Y \to X$, the natural map $H^0(Y_{\text{ét}}, \mathbb{Z}/2\mathbb{Z}) \to H^0((X^\wedge \times_X Y)_{\text{ét}}, \mathbb{Z}/2\mathbb{Z})$ is bijective.

However, for every scheme Z, the group $H^0(Z_{\text{ét}}, \mathbb{Z}/2\mathbb{Z})$ is naturally identified with the automorphism group of the trivial $\mathbb{Z}/2\mathbb{Z}$-torsor on $Z_{\text{ét}}$, so (iii) follows from (ii), for $G := \mathbb{Z}/2\mathbb{Z}$.

Now, for every finite morphism of schemes $f : Y \to X$, let $\underline{Y} := Y \times_X \underline{X}$, and recall that the morphism of quasi-affinoid schemes $f : \underline{Y} \to \underline{X}$ is f-adic (see example 15.4.9(v)), hence $\underline{Y}' := \underline{X}^\wedge \times_{\underline{X}} \underline{Y}$ is well defined, and according to claim 15.7.18 we have an isomorphism $h : \underline{Y}^\wedge \xrightarrow{\sim} \underline{Y}'^\wedge$ of quasi-affinoid schemes. Let G be any finite group; we deduce an essentially commutative diagram :

$$\begin{array}{ccc} \mathsf{Tors}(Y_{\text{ét}}, G) & \longrightarrow & \mathsf{Tors}(Y'_{\text{ét}}, G) \\ \downarrow & & \downarrow \\ \mathsf{Tors}(Y^\wedge_{\text{ét}}, G) & \longrightarrow & \mathsf{Tors}(Y'^\wedge_{\text{ét}}, G) \end{array}$$

where $Y', Y^\wedge, Y'^\wedge$ are the schemes underlying respectively $\underline{Y}', \underline{Y}^\wedge, \underline{Y}'^\wedge$. The vertical arrows are induced by $\pi_Y : Y^\wedge \to Y$ and $\pi_{Y'} : Y'^\wedge \to Y'$, and the bottom horizontal arrow is induced by h, so it is an equivalence. Hence, condition (ii) will follow for $f : Y \to X$, if we show that the vertical arrows of the diagram are equivalences. Moreover, by comparing the respective universal properties, we easily obtain a natural isomorphism of affinoid schemes $\underline{Y}' \xrightarrow{\sim} (X^\wedge \times_X Y) \times_{X^\wedge} \underline{X}^\wedge$; since $\underline{X}^\wedge$ is topologically henselian, example 15.4.9(vi) implies that the same holds for $\underline{Y}'$. With claim 15.7.19, we are then reduced to checking that for every topologically henselian affinoid scheme $(X, \mathscr{T}_X, A^+_X)$ and every finite group G, the completion morphism induces an equivalence $\mathsf{Tors}(X_{\text{ét}}, G) \to \mathsf{Tors}(X^\wedge_{\text{ét}}, G)$.

Let $A_0 \subset A := \mathscr{O}_X(X)$ be a subring of definition; let also $(A_i \mid i \in I)$ be the filtered system of finite type A_0-subalgebras of A. For every $i \in I$, endow A_i and $A^+_i := A_i \cap A^+_X$ with the topologies $\mathscr{T}_i$ and $\mathscr{T}^+_i$ induced by the inclusions into A. Let $A^\wedge_0$ be the completion of A_0; then the completions of A_i and A^+_i are respectively $A^\wedge_i := A^\wedge_0 \otimes_{A_0} A_i$ and $A^{\wedge +}_i := A^\wedge_0 \otimes_{A_0} A^+_i$ for every $i \in I$, and the completion of A is $A^\wedge := A^\wedge_0 \otimes_{A_0} A$, so $A^\wedge$ is the colimit of the induced filtered system of $A^\wedge_0$-algebras $(A^\wedge_i \mid i \in I)$. Set as well $X_i := \operatorname{Spec} A_i$ and $X^\wedge_i := \operatorname{Spec} A^\wedge_i$, and let $\mathscr{T}^\wedge_i$ be the topology of $A^\wedge_i$, for every $i \in I$; then $\underline{X}_i := (X_i, \mathscr{T}_i, A^+_i)$ is a topologically henselian affinoid scheme (proposition 8.4.2(iii)), and its completion $\underline{X}^\wedge_i$ is $(X^\wedge_i, \mathscr{T}^\wedge_i, A^{\wedge +}_i)$.

From [65, Ch.IV, Th.8.8.2, Th.8.10.5, Th.11.2.6] it follows easily that the natural functors

$$\text{2-}\operatorname*{colim}_{i\in I} \mathsf{Tors}(X_{i,\text{ét}}, G) \to \mathsf{Tors}(X_{\text{ét}}, G) \qquad \text{2-}\operatorname*{colim}_{i\in I} \mathsf{Tors}(X^\wedge_{i,\text{ét}}, G) \to \mathsf{Tors}(X^\wedge_{\text{ét}}, G)$$

are equivalences of categories. Hence it suffices to check that for every $i \in I$ the functor

$$\mathsf{Tors}(X_{i,\text{ét}}, G) \to \mathsf{Tors}(X^\wedge_{i,\text{ét}}, G)$$

is an equivalence. We may therefore assume from start that A is an A_0-algebra of finite type, say $A = A_0[t_1, \dots, t_n]$ for a finite sequence $t_\bullet := (t_1, \dots, t_n)$ of elements of A. For every $k = 0, \dots, n$, we shall say that the sequence $t_\bullet$ is *adequate up to* k if for every $i = 1, \dots, k$, either one of the following conditions holds :

(a) $t_i \in A^+$
(b) $t_i \in A^\times$ and $1/t_i \in A^+$.

Claim 15.7.21. In order to prove the theorem, we may assume that $t_\bullet$ is adequate up to n.

Proof of the claim. Suppose that the theorem is known for every f-adic ring A generated over some topologically henselian subring of definition A_0 by a sequence of elements $(t_1, \dots, t_n)$ adequate up to n. We argue by descending induction on $k \leq n$, that the theorem then also holds for every f-adic ring generated over any such subring A_0 by a sequence $(t_1, \dots, t_n)$ that is only adequate up to k. If $k = n$, there is nothing to show; let then $k \leq n$ and $t_\bullet$ a sequence adequate up to $k - 1$, and suppose that we have already proved our assertion for all sequences adequate up to k. Let $\underline{A}$ be the affinoid ring (A, A^+); we consider the affinoid rings

$$\underline{B} := \underline{A}\left(\frac{1, t_k}{1}\right) \qquad \underline{C} := \underline{A}\left(\frac{1, t_k}{t_k}\right) \qquad \underline{D} := \underline{B} \otimes_{\underline{A}} \underline{C}$$

(notation of (15.5.1) and example 15.4.5(i)). Explicitly, we have $\underline{B} = (B, B^+)$ and $\underline{C} = (C, C^+)$, where the ring underlying the topological ring B (resp. C) is A (resp. $A[1/t_k]$) and B^+ (resp. C^+) is the integral closure in B (resp. in C) of $A^+[t_k]$ (resp. of $A^+[1/t_k]$); moreover, B (resp. C) admits the ring of definition $B_0 := A_0[t_k]$ (resp. $C_0 := A_0[1/t_k]$). Recall that the topological henselization B^{h} of B is $B_0^{\mathrm{h}} \otimes_{B_0} B$, with B_0^{h} the topological henselization of B_0. Hence the image in B^{h} of the sequence $t_\bullet$ generates the f-adic B_0^{h}-algebra B^{h}, and is adequate up to k. Likewise we see that the image of $t_\bullet$ generates the C_0^{h}-algebra C^{h}, and then the same follows for the image of $t_\bullet$ in the topological ring underlying $\underline{D}^{\mathrm{h}}$: the details are left to the reader. By inductive assumption, the theorem then holds for the affinoid schemes $\mathsf{Spec}\,\underline{B}^{\mathrm{h}}$, $\mathsf{Spec}\,\underline{C}^{\mathrm{h}}$ and $\mathsf{Spec}\,\underline{D}^{\mathrm{h}}$. But notice that $\underline{B}^{\mathrm{h}}$ and $\underline{C}^{\mathrm{h}}$ represent two affinoid open subsets U' and U'' of $\mathrm{Spa}\,\underline{X}$, and $\underline{D}^{\mathrm{h}}$ represents $U' \cap U''$; more precisely, (U', U'') is the standard covering of $\mathrm{Spa}\,\underline{A}$ associated with the sequence $(1, t_k)$ (see (15.5.20)); arguing as in the proof of claim 15.7.19, we deduce that the theorem holds for $\mathsf{Spec}\,\underline{A}$, as required. ◊

Henceforth, we assume that $t_\bullet$ is an adequate sequence up to n; let $S := \{i \leq n \mid t_i \in A^+\}$ and $T := \{1, \dots, n\} \setminus S$, and set $t'_i := t_i$ for every $i \in S$ and $t'_i := 1/t_i$ for every $i \in T$. Moreover, set $t := \prod_{i\in T} t'_i$. The subring $A'_0 := A_0[t'_1, \dots, t'_n]$ is open and bounded in A, hence it is a subring of definition of A, and after replacing A_0 by A'_0 we may assume that $A = A_0[t^{-1}]$ for an element $t \in A_0 \cap A^\times$. Let $I_0 \subset A_0$ be a finite type ideal of adic definition, and denote by B the topological ring whose underlying ring is the same as A, and whose topology is the (t, I_0)-adic topology as defined in [75, Def.5.4.10(ii)] (invoking remark 8.3.2(i), it is easily seen that the (t, I_0)-adic topology is a ring topology on A).

Since $t \in A^\times$, the ideal $t^n I_0$ is open in A for every $n \in \mathbb{N}$, therefore the identity is a continuous map $A \to B$, and its completion is a morphism of topological rings :

$$f : A^\wedge \to B^\wedge.$$

Let also $g : A \to A^\wedge$ and $h : B \to B^\wedge$ be the completion maps, $I_0^\wedge$ the topological closure of the image of I_0 in $A^\wedge$, and $\mathscr{T}$ the $(t, I_0^\wedge)$-adic topology on $A^\wedge$. We notice :

Claim 15.7.22. The map f is also a morphism $(A^\wedge, \mathscr{T}) \to B^\wedge$ of topological rings, and induces an isomorphism of topological rings :

$$f^\wedge : (A^\wedge, \mathscr{T})^\wedge \xrightarrow{\sim} B^\wedge.$$

Proof of the claim. Let $J_0^\wedge$ be the topological closure of the image of I_0 in $B^\wedge$; it is clear that f is a continuous map $(A^\wedge, \mathscr{T}) \to B^\wedge$, since $f(t^n I_0^\wedge) \subset t^n J_0^\wedge$ for every $n \in \mathbb{N}$. In order to show that $f^\wedge$ is an isomorphism, we apply the criterion of theorem 8.2.8(iii) : first, obviously the image of f is dense in $B^\wedge$, since the image of A in $B^\wedge$ is already dense. It remains to check that $\mathscr{T}$ agrees with the topology induced from $B^\wedge$ via $f^\wedge$; we show more precisely :

$$t^n I_0^\wedge = f^{-1}(t^n J_0^\wedge) \qquad \text{for every } n \in \mathbb{N}.$$

Indeed, since both $t^n I_0^\wedge$ and $f^{-1}(t^n J_0^\wedge)$ are open subgroups of $(A^\wedge, \mathscr{T})$, it suffices to check that $g^{-1}(t^n I_0^\wedge) = g^{-1}(f^{-1}(t^n J_0^\wedge))$; but $g^{-1}(t^n I_0^\wedge) = t^n g^{-1}(I_0^\wedge) = t^n I_0$, and $g^{-1}(f^{-1}(t^n J_0^\wedge)) = h^{-1}(t^n J_0^\wedge) = t^n h^{-1}(J_0^\wedge) = t^n I_0$. ◇

Since $h = f \circ g$, it suffices to check that f and h induce equivalences of categories

$$\mathsf{Tors}((\mathrm{Spec}\, A^\wedge)_{\mathrm{\acute{e}t}}, G) \xrightarrow{\sim} \mathsf{Tors}((\mathrm{Spec}\, B^\wedge)_{\mathrm{\acute{e}t}}, G) \xleftarrow{\sim} \mathsf{Tors}((\mathrm{Spec}\, A)_{\mathrm{\acute{e}t}}, G).$$

Now, the pair (A_0, I_0) is henselian by assumption, and since $A_0^\wedge$ is complete for the $I_0^\wedge$-adic topology, also the pair $(A_0^\wedge, I_0^\wedge)$ is henselian; it then follows that the pairs (A_0, tI_0) and $(A_0^\wedge, tI_0^\wedge)$ are henselian [75, Rem.5.1.10(iv)]. By [75, Prop.5.4.53] and claim 15.7.22, we deduce that f and h induce equivalences of categories of finite étale morphisms

$$\mathsf{Cov}(\mathrm{Spec}\, A^\wedge) \xrightarrow{\sim} \mathsf{Cov}(\mathrm{Spec}\, B^\wedge) \xleftarrow{\sim} \mathsf{Cov}(\mathrm{Spec}\, A) \tag{15.7.23}$$

(notation of (13.1)). Now, for any scheme Y, the G-torsors over $Y_{\mathrm{\acute{e}t}}$ are the objects $\varphi : Z \to Y$ of $\mathsf{Cov}(\mathrm{Spec}\, A^\wedge)$ endowed with a (right) G-action and such that φ is a surjective morphism. Such a morphism has an open and closed image in Y, hence it is surjective if and only if it is an epimorphism (in the category $\mathsf{Cov}(Y)$). The equivalences (15.7.23) induce formally equivalences for the corresponding categories of objects endowed with G-action (details left to the reader), and obviously respect epimorphisms, whence the contention. □

Remark 15.7.24. For every finite group G, let $\mathscr{C}_G$ be the category with $\mathrm{Ob}(\mathscr{C}_G) = \{o\}$ and such that the monoid $\mathrm{Hom}_{\mathscr{C}_G}(o, o)$ is isomorphic to G. For every scheme X, let also $\mathscr{C}_{G, X_{\mathrm{\acute{e}t}}}$ be the constant presheaf on $X_{\mathrm{\acute{e}t}}$ with value $\mathscr{C}_G$. By arguing as in the proof of lemma 15.7.1, it is easily seen that the associated stack $\mathscr{C}^a_{G, X_{\mathrm{\acute{e}t}}}$ is naturally equivalent to the stack of G-torsors on $X_{\mathrm{\acute{e}t}}$: the details shall be left to the reader.

15.7.25. For every $U \in \mathrm{Ob}(\mathscr{Q}(\underline{X}))$, choose a complete and separated quasi-affinoid scheme $\underline{X}_U^\wedge := (X_U^\wedge, \mathscr{T}_U^\wedge, A_U^{\wedge +})$ representing the sub-presheaf $h''_U \subset h''_{\underline{X}^\wedge}$ (notation of remark 15.5.9(i)). Any inclusion $U \subset U'$ of quasi-affinoid subsets of $\mathrm{Spa}\, \underline{X}$ induces a morphism $j_{UU'} : \underline{X}_U^\wedge \to \underline{X}_{U'}^\wedge$ of quasi-affinoid schemes (cp. (15.5.11)), and for any further inclusion $U' \subset U''$ we have

$$j_{U'U''} \circ j_{UU'} = j_{UU''}$$

whence a well defined functor

$$\underline{X}_\bullet^\wedge : \mathscr{Q}(\underline{X}) \to \mathsf{Sch} \qquad U \mapsto X_U^\wedge$$

and we consider the fibration over $\mathscr{Q}(\underline{X})$

$$\mathsf{Cov}_{\underline{X}}^\wedge := \mathsf{Fib}(\underline{X}_\bullet^\wedge)^*(\mathsf{Cov})$$

where the fibration $\mathsf{Cov} \to \mathsf{Sch}$ is defined as in (15.7). With this notation, we can now state :

Corollary 15.7.26. *The fibration* $\mathsf{Cov}^{\wedge}_{\underline{X}}$ *is a stack on the site* $(\mathscr{Q}(\underline{X}), J_{\mathscr{Q}})$.

Proof. We have already remarked that the stack $\mathsf{Cov}_{X_{\text{ét}}}$ is ind-finite on $X_{\text{ét}}$, hence the fibration $(\mathsf{Cov}_{X_{\text{ét}}})_{/\mathscr{Q}}$ is a stack on $(\mathscr{Q}(\underline{X}), J_{\mathscr{Q}})$, by theorem 15.7.6. But from theorem 15.7.17 we easily deduce an equivalence of fibrations $(\mathsf{Cov}_{X_{\text{ét}}})_{/\mathscr{Q}} \overset{\sim}{\to} \mathsf{Cov}^{\wedge}_{\underline{X}}$, whence the assertion. □

Remark 15.7.27. A variant of corollary 15.7.26 appears in [114, Th.2.6.9].

15.7.28. We wish to give an explicit description, up to natural equivalence of categories, of the stalks of the fibration $\mathsf{Cov}^{\wedge}_{\underline{X}}$. Hence, let $x \in \mathrm{Spa}\,\underline{X}$ be any point; recall that the stalk of $\mathsf{Cov}^{\wedge}_{\underline{X}}$ over x is the category

$$\mathsf{Cov}^{\wedge}_{\underline{X}}(x) := \underset{U\in\mathscr{Q}(\underline{X},x)^o}{2\text{-colim}}\, \mathsf{Cov}(X^{\wedge}_U)$$

where $\mathscr{Q}(\underline{X}, x)$ denotes the set of all quasi-affinoid open neighborhoods of x in $\mathrm{Spa}\,\underline{X}$, which is cofiltered by inclusion. Pick a finitely generated ideal $J \subset A_X := \mathscr{O}_X(X)$ such that $X = \mathrm{Spec}\,A_X \setminus \mathrm{Spec}\,A_X/J$, and let $f_0, \dots, f_n$ be a finite system of generators of J. Set $U_i := R_{A_X}\big(\frac{f_0,\dots,f_n}{f_i}\big) \cap \mathrm{Spa}\,\underline{X}$ for $i = 0, \dots, n$. Then we have

$$U_0 \cup \cdots \cup U_n = \mathrm{Spa}\,\underline{X}$$

and we may assume that $x \in U_0$. Notice that h''_{U_0} is represented by a quasi-affinoid scheme $\underline{X}^{\wedge}_{U_0} := (X^{\wedge}_{U_0}, \mathscr{T}^{\wedge}_{U_0}, A^{\wedge+}_{U_0})$ with $X^{\wedge}_{U_0} = \mathrm{Spec}\,\mathscr{O}^{\wedge}_{\mathrm{Spa}\,\underline{X}}(U_0)$. Then, since $X^{\wedge}_{U_0}$ is affine, every rational subset of $\mathrm{Spa}\,\underline{X}^{\wedge}_{U_0}$ containing x is likewise represented by a quasi-affinoid scheme whose underlying scheme is affine, and the system of such rational subsets is naturally identified with a final subset of $\mathscr{Q}(\underline{X}, x)$. Taking into account lemma 13.1.6, we deduce a natural equivalence

$$\mathsf{Cov}^{\wedge}_{\underline{X}}(x) \overset{\sim}{\to} \mathsf{Cov}(\mathrm{Spec}\,\mathscr{O}^{\wedge}_{\mathrm{Spa}\,\underline{X},x}).$$

• Next, if x is analytic, combining with lemma 15.5.14(i,v.c), proposition 9.1.17(iii) and [66, Ch.IV, Prop.18.5.15] we arrive at a natural equivalence

$$\mathsf{Cov}^{\wedge}_{\underline{X}}(x) \overset{\sim}{\to} \mathsf{Cov}(\mathrm{Spec}\,\kappa(x^{\wedge})) \overset{\sim}{\to} \mathsf{Cov}(\mathrm{Spec}\,\kappa(x)^{\wedge})$$

where $\kappa(x^{\wedge})$ is the residue field of the natural valuation $|\cdot|^{\wedge}_x$ on $\mathscr{O}^{\wedge}_{\mathrm{Spa}\,\underline{X},x}$, and $\kappa(x)^{\wedge}$ is the completion of $\kappa(x^{\wedge})$ for its valuation topology (see (15.5.12)).

• Lastly, if x is non-analytic, lemma 15.5.14(vi.b) and [75, Th.5.5.7(iii)] yield the natural equivalence

$$\mathsf{Cov}^{\wedge}_{\underline{X}}(x) \overset{\sim}{\to} \mathsf{Cov}(\mathrm{Spec}\,\mathscr{O}^{\wedge}_{\mathrm{Spa}\,\underline{X},x}/A^{\circ\circ}_X\mathscr{O}^{\wedge}_{\mathrm{Spa}\,\underline{X},x}).$$

16. Perfectoid rings and perfectoid spaces

In this chapter we develop a generalization of Scholze's theory of perfectoid rings and perfectoid spaces.

16.1. **Distinguished elements and transversal pairs.** Let p be a prime number, $(A, \mathscr{T})$ a complete and separated topological ring whose topology $\mathscr{T}$ is linear and coarser than the p-adic topology. Remark 9.4.12(ii) endows the topological monoid $\mathbf{E}(A)$ with a natural structure of perfect topological $\mathbb{F}_p$-algebra, such that :

- For every topologically nilpotent ideal $I \subset A$ containing pA, and which is either closed for the topology $\mathscr{T}$, or else finitely generated and closed for the p-adic topology of A, the projection $\pi_I : A \to A/I$ induces an isomorphism of topological rings
$$\mathbf{E}(\pi_I) : \mathbf{E}(A) \overset{\sim}{\to} \mathbf{E}(A/I)$$
where A/I is endowed with the quotient topology induced by $\mathscr{T}$ via π_I.

- Moreover, if $(A', \mathscr{T}')$ is any other topological ring fulfilling the above conditions, and $f : A \to A'$ is any continuous ring homomorphism, the map $\mathbf{E}(f) : \mathbf{E}(A) \to \mathbf{E}(A')$ is a morphism of topological rings.

In light of this canonical identification of $\mathbf{E}(A)$ with $\mathbf{E}(A/I)$, we shall henceforth write slightly abusively

$$\overline{u}_{A/I} : \mathbf{E}(A) \to A/I$$

for the map that would be denoted $\overline{u}_{A/I} \circ \mathbf{E}(\pi_I)$, or equivalently, $\pi_I \circ \overline{u}_A$, with the notation of section 9.4. Another basic construction of this chapter shall be the topological ring :

$$\mathbf{A}(A, \mathscr{T}) := W(\mathbf{E}(A))$$

where $W(-)$ denotes the ring of Witt vectors as in (9.3.11). From remark 9.4.9(i) and lemma 9.3.33(ii,iii) we also know that the topologies of $\mathbf{E}(A)$ and $\mathbf{A}(A, \mathscr{T})$ are linear, complete and separated. Unless we have to deal with several different topologies on A, we shall usually omit mentioning explicitly $\mathscr{T}$, and write simply $\mathbf{A}(A)$. According to proposition 9.3.44(ii), the ring $\mathbf{A}(A)$ is also complete and separated for its p-adic topology, and the same holds for A, by lemma 8.3.12; moreover we see from (9.3.41) that p is a regular element of $\mathbf{A}(A)$, and by remark 9.3.28(i) the ghost component $\boldsymbol{\omega}_0$ descends to an isomorphism of topological rings

$$\mathbf{A}(A) \otimes_{\mathbb{Z}} \mathbb{F}_p \xrightarrow{\sim} \mathbf{E}(A)$$

provided we endow $\mathbf{A}(A) \otimes_{\mathbb{Z}} \mathbb{F}_p$ with the quotient topology induced from $\mathbf{A}(A)$ via the projection $\mathbf{A}(A) \to \mathbf{A}(A) \otimes_{\mathbb{Z}} \mathbb{F}_p$. The map $\boldsymbol{\omega}_0$ also admits a continuous set-theoretic multiplicative splitting, the Teichmüller mapping of (9.3.34), which will be here denoted :

$$\tau_A : \mathbf{E}(A) \to \mathbf{A}(A).$$

Lemma 16.1.1. *With the notation of* (16.1)*, the following holds :*

(i) *There exists a unique ring homomorphism $u_A : \mathbf{A}(A) \to A$ that makes commute the diagram :*

$$\begin{array}{ccc} \mathbf{A}(A) & \xrightarrow{u_A} & A \\ {\scriptstyle \boldsymbol{\omega}_0}\downarrow & & \downarrow{\scriptstyle \pi_{pA}} \\ \mathbf{E}(A) & \xrightarrow{\overline{u}_{A/pA}} & A/pA. \end{array}$$

(ii) *The map u_A is continuous for the topology $\mathscr{T}$ and the topology of $\mathbf{A}(A, \mathscr{T}_A)$.*
(iii) $\mathbf{A}(A)^\times = u_A^{-1}(A^\times)$.
(iv) $u_A \circ \tau_A = \overline{u}_A$.

Proof. We have already remarked that A is complete and separated for its p-adic topology, and the filtration $(J_n := p^n A \mid n \in \mathbb{N})$ of A trivially fulfills the condition of lemma 9.3.4; therefore (i) follows immediately from proposition 9.3.52(iii). More precisely, (9.3.53) translates as the following explicit expression for u_A :

$$u(\underline{a}) = \sum_{n\in\mathbb{N}} p^n \cdot \overline{u}_A(a_n^{p^{-n}}) \qquad \text{for every } \underline{a} := (a_n \mid n \in \mathbb{N}) \in \mathbf{A}(A)$$

from which (ii) follows easily : details left to the reader.

(iii): The inclusion $\mathbf{A}(A)^\times \subset u_A^{-1}(A^\times)$ is obvious. For the converse, let $\alpha \in \mathbf{A}(A)$ be any element such that $u_A(\alpha) \in A^\times$; then $\boldsymbol{\omega}_0(\alpha) \in \overline{u}_A^{-1}(A^\times)$. By remark 9.4.5(iii) we deduce that $\boldsymbol{\omega}_0(\alpha) \in \mathbf{E}(A)^\times$, so it remains only to notice that $\boldsymbol{\omega}_0^{-1}(\mathbf{E}(A)^\times) = \mathbf{A}(A)^\times$, due to proposition 9.3.44(iii).

(iv) is clear from the foregoing explicit formula for u_A. □

Furthermore, $\mathbf{A}(A)$ is endowed with an automorphism $\boldsymbol{\sigma}_A : \mathbf{A}(A) \to \mathbf{A}(A)$ that lifts the Frobenius map of $\mathbf{E}(A)$, *i.e.* such that the horizontal arrows of the diagram :

(16.1.2)
$$\begin{array}{ccc} \mathbf{A}(A) & \xrightarrow{\boldsymbol{\sigma}_A} & \mathbf{A}(A) \\ {\scriptstyle \boldsymbol{\omega}_0}\downarrow\uparrow{\scriptstyle \tau_A} & & {\scriptstyle \boldsymbol{\omega}_0}\downarrow\uparrow{\scriptstyle \tau_A} \\ \mathbf{E}(A) & \xrightarrow{\Phi_{\mathbf{E}(A)}} & \mathbf{E}(A) \end{array}$$

commute with the downward arrows (proposition 9.3.22(iii)). The horizontal arrows commute also with the upward ones, due to (9.3.43).

16.1.3. Keep the notation of (16.1), and recall that τ_A induces a continuous map $\operatorname{Spec}\tau_A : \operatorname{Spec}\mathbf{A}(A) \to \operatorname{Spec}\mathbf{E}(A)$ as in proposition 9.3.66(ii); in light of lemma 16.1.1(iv) and remark 8.1.14(ii) we deduce a continuous and spectral map

$$\operatorname{Spec}\overline{u}_A := (\operatorname{Spec}\tau_A)\circ(\operatorname{Spec}u_A) : \operatorname{Spec}A \to \operatorname{Spec}\mathbf{E}(A) \qquad \mathfrak{p} \mapsto \overline{u}_A^{-1}\mathfrak{p}.$$

Moreover, due to proposition 9.3.66(iv), every morphism $\varphi : A \to A'$ of topological algebras that fulfill the conditions of (16.1) induces a commutative diagram of topological spaces :

(16.1.4)
$$\begin{array}{ccc} \operatorname{Spec}A' & \xrightarrow{\operatorname{Spec}\overline{u}_{A'}} & \operatorname{Spec}\mathbf{E}(A') \\ {\scriptstyle \operatorname{Spec}\varphi}\downarrow & & \downarrow{\scriptstyle \operatorname{Spec}\mathbf{E}(\varphi)} \\ \operatorname{Spec}A & \xrightarrow{\operatorname{Spec}\overline{u}_A} & \operatorname{Spec}\mathbf{E}(A). \end{array}$$

Lemma 16.1.5. *In the situation of* (16.1.3)*, let* $\mathfrak{p} \in \operatorname{Spec}A$ *be any prime ideal, set* $\mathfrak{q} := \operatorname{Spec}\overline{u}_A(\mathfrak{p}) \in \operatorname{Spec}\mathbf{E}(A)$*, and let* $\pi_\mathfrak{p} : A \to \kappa(\mathfrak{p})$*,* $\pi_\mathfrak{q} : \mathbf{E}(A) \to \kappa(\mathfrak{q})$ *be the projections. If* $\mathfrak{p}$ *is a closed subset in the* p*-adic topology of* A*, then* $\overline{u}_A$ *induces a morphism of multiplicative monoids*

$$\overline{u}_\mathfrak{p} : \kappa(\mathfrak{q}) \to \kappa(\mathfrak{p}) \qquad \textit{such that} \qquad \overline{u}_\mathfrak{p}\circ\pi_\mathfrak{q} = \pi_\mathfrak{p}\circ\overline{u}_A.$$

Proof. Since u_A is continuous for the p-adic topologies, $u_A^{-1}\mathfrak{p}$ is a closed subet for the p-adic topology of $\mathbf{A}(A)$, so the assertion follows immediately from proposition 9.3.66(iii). ☐

With the following definition, we single out certain elements of $\mathbf{A}(A)$ that shall play a special role in the study of perfectoid rings.

Definition 16.1.6. Let $(E, \mathscr{T})$ be any topological $\mathbb{F}_p$-algebra.

(i) An element $(a_n \mid n \in \mathbb{N}) \in W(E)$ is called *distinguished* if a_0 is topologically nilpotent in E, and $a_1 \in E^\times$.

(ii) An ideal of $W(E)$ is called *distinguished* if it is generated by a distinguished element.

Remark 16.1.7. Let E be any topological $\mathbb{F}_p$-algebra, $\underline{a} := (a_n \mid n \in \mathbb{N})$ and $\underline{b} := (b_n \mid n \in \mathbb{N})$ two elements of $W(E)$, such that $\underline{a}$ is distinguished and set $\underline{c} := \underline{a}\cdot\underline{b}$.

(i) Suppose that the topology of E is linear, complete and separated. Then $\underline{c}$ is distinguished if and only if $\underline{b}$ is invertible in $W(E)$. Indeed, say that $\underline{c} = (c_n \mid n \in \mathbb{N})$. Then remark 9.3.8 says that $c_0 = a_0b_0$, especially c_0 is topologically nilpotent. Also, $c_1 = a_0^pb_1 + a_1b_0^p$, and notice that $b_0 \in E^\times$, if and only if $\underline{b} \in W(E)^\times$ (proposition 9.3.44(iii)); on the other hand, since the topology of E is linear, $a_0^pb_1$ is topologically nilpotent (remark 8.3.9(v)). Since the topology of E is linear, complete and separated, it follows that c_1 is invertible if and only if the same holds for b_0 (details left to the reader), whence the assertion.

(ii) If E is reduced, then every distinguished ideal of $W(E)$ is a closed subset for the p-adic topology of $W(E)$ and a free $W(E)$-module of rank one (proposition 9.3.47(i,v)).

(iii) Let E be as in (i), and $I, J \subset W(E)$ any two distinguished ideals such that $I \subset J$; then (i) implies that $I = J$. By the same token, we see that every generator of I is distinguished.

Indeed, let $\underline{a}_1$ and $\underline{a}_2$ be any two generators of I; then there exists $\underline{u} := (u_n \mid n \in \mathbb{N}) \in W(E)$ such that $\underline{a}_2 = \underline{u} \cdot \underline{a}_1$. Let $N \subset E$ be the nilpotent ideal, and endow $E_{\mathrm{red}} := E/N$ with the quotient topology induced from E; it follows from (ii) that the image of $\underline{u}$ is invertible in $W(E_{\mathrm{red}})$, therefore the image of u_0 is invertible in E_{red}, so u_0 is invertible in E, and finally $\underline{u}$ is invertible in $W(E)$ (proposition 9.3.44(iii)).

Example 16.1.8. With the notation of example 9.3.78, notice that the only distinguished elements of $W(A, \mathscr{T}_T) = \mathbb{Z}_p\{T^{1/p^\infty}\}$ are of the form pu, where $u \in W(A, \mathscr{T}_T)^\times$ is an arbitrary element; especially, the only distinguished ideal of $\mathbb{Z}_p\{T^{1/p^\infty}\}$ is the principal ideal generated by p. On the other hand, the distinguished elements of $W(A^\wedge, \mathscr{T}_T^\wedge)$ are all those of the form $pu + aT^\lambda$, where a (resp. u) is an arbitrary element (resp. invertible element) of $W(A^\wedge, \mathscr{T}_T^\wedge)$, and $\lambda \in \mathbb{N}[1/p]$ is any strictly positive number.

The following result shall be applied to produce useful distinguished elements in various situations.

Lemma 16.1.9. *Let E be an $\mathbb{F}_p$-algebra, $\mathscr{I} \subset W(E)$ an ideal, $\underline{\alpha} := (\alpha_n \mid n \in \mathbb{N}) \in \mathscr{I}$, and $\beta_\bullet := (\beta_1, \dots, \beta_k)$ a finite system of elements of E. Denote by $J \subset E$ (resp. by $\mathscr{J} \subset W(E)$) the ideal generated by $\beta_\bullet$ (resp. by $\tau_E(\beta_1), \dots, \tau_E(\beta_k)$), and by $u : W(E) \to A := W(E)/\mathscr{I}$ the projection. Suppose that :*

- (a) *α_0 lies in the Jacobson radical of E and $\alpha_1 \in E^\times$.*
- (b) *The Frobenius endomorphism Φ_E of E is surjective.*

(i) *Then the following conditions are equivalent :*

- (c) *$pA \subset u(\mathscr{J})$.*
- (d) *There exists an element $\underline{\alpha}' := (\alpha'_n \mid n \in \mathbb{N}) \in \mathscr{I}$ such that $\alpha'_0 \in J$ and the image of α'_1 is invertible in E/J.*
- (e) *There exists $\underline{\alpha}' := (\alpha'_n \mid n \in \mathbb{N}) \in \mathscr{I}$ such that $\alpha'_0 \in J$ and $\alpha'_1 \in E^\times$.*

(ii) *Suppose furthermore that $\mathscr{I} = \underline{\alpha}W(E)$. Then* (c),(d) *and* (e) *are also equivalent to :*

- (f) *$\alpha_0 \in J$.*
- (g) *$pW(E) + \mathscr{J} = \underline{\alpha}W(E) + \mathscr{J}$.*

Proof. (c)⇒(d): Assumption (c) implies that there exist $\underline{\gamma}_1, \dots, \underline{\gamma}_k \in W(E)$ such that

$$\underline{\alpha}' := p + \underline{\gamma}_1 \cdot \tau_E(\beta_1) + \cdots + \underline{\gamma}_k \cdot \tau_E(\beta_k) \in \mathscr{I}$$

and we claim that (d) holds for this $\underline{\alpha}'$. Indeed, $\alpha'_0 = \sum_{i=1}^k \boldsymbol{\omega}_0(\underline{\gamma}_i) \cdot \beta_i \in J$, and clearly the image of $\underline{\alpha}'$ in $W(E/J)$ equals p, *i.e.* $\underline{\alpha}'_1 \equiv 1 \pmod J$.

(d)⇒(c): Let $\underline{\alpha}'$ be as in (d); then there exists $\lambda \in E$ such that $\lambda \cdot \alpha_1 \in 1 + J$, and assumption (b) implies that $\lambda = \gamma^p$ for some $\gamma \in E$. Recall that $\underline{\alpha}'' := \tau_E(\gamma) \cdot \underline{\alpha}' = (\gamma^{p^n} \cdot \alpha'_n \mid n \in \mathbb{N})$ (proposition 9.3.36(i)); we may then replace $\underline{\alpha}'$ by $\underline{\alpha}''$, after which we may assume that $\alpha'_1 \equiv 1 \pmod J$. Say that $\alpha'_0 = \sum_{i=1}^k c_i\beta_i$ for some $c_1, \dots, c_k \in E$, and set

$$\underline{\delta} := \underline{\alpha}' - (\tau_E(c_1) \cdot \tau_E(\beta_1) + \cdots + \tau_E(c_k) \cdot \tau_E(\beta_k)).$$

Notice that $\delta_0 = 0$, and the images of $\underline{\alpha}'$ and $\underline{\delta}$ agree in $W(E/J)$; in particular, $\delta_1 \equiv 1 \pmod J$. Invoking (b) again, we may find $\gamma \in E$ such that $\gamma^p = \delta_1$, and after replacing $\underline{\alpha}'$ by $\tau_E(\gamma^{p-1}) \cdot \underline{\alpha}'$, and c_i by $\gamma^{p-1}c_i$ for $i = 1, \dots, k$, we may further assume that $\delta_1 \equiv 1 \pmod{\Phi_E(J)}$. In this case, say that $\delta_1 = 1 + \sum_{i=1}^k d_i^p \beta_i^p$ for some $d_1, \dots, d_k \in E$; taking into account remark 9.3.8(ii) and (9.3.41) we see that

$$\underline{\delta} \equiv p + p \cdot (\tau_E(d_1) \cdot \tau_E(\beta_1) + \cdots + \tau_E(d_k) \cdot \tau_E(\beta_k)) \pmod{p^2W(E)}.$$

Summing up, we conclude that there exists $\underline{x} \in W(E)$ such that $p + p^2\underline{x} - \underline{\alpha}' \in \mathscr{J}$. Hence, $p \cdot (1 + p \cdot u(\underline{x})) \in u(\mathscr{J})$. Lastly, we have $1 + p \cdot \underline{x} \in W(E)^\times$, since p is topologically nilpotent in $W(E)$, whence (c).

Obviously, (e)⇒(d), so we suppose that (d) holds, and we show (e). Arguing as in the foregoing, we may assume that $\alpha'_1 \equiv 1 \pmod{J}$. Next, using (b) we find $\gamma \in E$ such that $\gamma^p = \alpha'_1$, and after replacing α' by $\tau_E(\gamma^{p-1}) \cdot \underline{\alpha}'$, we may even assume that $\alpha'_1 \equiv 1 \pmod{\Phi_E(J)}$.

Likewise, we may find $\delta \in E$ such that $\delta^p = \alpha_1^{-1}$, and after replacing $\underline{\alpha}$ by $\tau_E(\delta) \cdot \underline{\alpha}$ we may assume that $\alpha_1 = 1$. Now, write $\alpha'_1 = 1 - x^p$ for some $x \in J$, and set

$$\underline{\alpha}'' := \underline{\alpha}' + \tau_E(x) \cdot \underline{\alpha}.$$

In light of remark 9.3.8(ii), we easily see that $\alpha''_0 \in J$ and $\alpha''_1 - 1 \in \alpha_0 E$. Since α_0 lies in the Jacobson radical of E, it follows that $\alpha''_1 \in E^\times$, whence (e).

Lastly, suppose that $\mathscr{J} = \underline{\alpha} W(E)$. In this case, we already know that (f)⇒(e). Conversely, if (e) holds, remark 9.3.8(ii) shows that there exists $\underline{\beta} := (\beta_n \mid n \in \mathbb{N}) \in W(E)$ with

$$\alpha_0\beta_0 \in J \qquad \text{and} \qquad \alpha_0^p\beta_1 + \alpha_1\beta_0^p \in E^\times.$$

Then, since α_0 is in the Jacobson radical of E, we deduce that $\alpha_1\beta_0^p \in E^\times$, so also $\beta_0 \in E^\times$, and (f) follows. Next, if (c) holds, then $W(E)/(\mathscr{J} + \underline{\alpha}W(E)) = A/u(\mathscr{J})$ is an $\mathbb{F}_p$-algebra, *i.e.* $p \in \mathscr{J} + \underline{\alpha}W(E)$; on the other hand, (b) implies that $pW(E) = \mathrm{Ker}(\boldsymbol{\omega}_0 : W(E) \to E)$ (see (9.3.41)). Hence, $W(E)/(\mathscr{J} + \underline{\alpha}W(E)) = E/(\alpha_0 E + J) = E/J$, since we already know that (c)⇒(f); but we have as well $W(E)/(\mathscr{J} + pW(E)) = E/J$, whence (g). Conversely, if (g) holds, we see that $A/u(\mathscr{J})$ is an $\mathbb{F}_p$-algebra, whence (c). □

Remark 16.1.10. Keep the notation of lemma 16.1.9, and let $S := 1 + pW(E) + \mathscr{J}$ and $R := S^{-1}W(E)$. Suppose that for some $v \in R$ we have

$$p \in v \cdot \underline{\alpha} + \mathscr{J}R \qquad (\text{resp. } \underline{\alpha}_0 \in vp + \mathscr{J}R).$$

Then we claim that $v \in R^\times$. Indeed, it suffices to show that $v \notin \mathfrak{m}$, for every $\mathfrak{m} \in \mathrm{Max}\, R$. However, we have a natural identification : $\mathrm{Max}\, R \xrightarrow{\sim} \mathrm{Max}\, W(E)/(pW(E) + \mathscr{J}) = \mathrm{Max}\, E/J$, and since Φ_E is surjective, it follows easily that the residue field $k(\mathfrak{m})$ is perfect for every such $\mathfrak{m}$, in which case $W(k(\mathfrak{m}))$ is a discrete valuation ring of mixed characteristic $(0, p)$ (proposition 9.3.47(iii)); moreover, the projection $W(E) \to k(\mathfrak{m})$ factors through E, so we get a commutative diagram of rings :

$$\begin{array}{ccc} W(E) & \xrightarrow{W(\pi)} & W(k(\mathfrak{m})) \\ \downarrow & & \downarrow \\ E & \xrightarrow{\pi} & k(\mathfrak{m}) \end{array}$$

whose vertical arrows are the 0-th ghost components. It suffices then to check that $x := W(\pi)(v) \in W(k(\mathfrak{m}))^\times$. Now, if $p \in v \cdot \underline{\alpha} + \mathscr{J}$, we get $p = x \cdot W(\pi)(\underline{\alpha})$ in $W(k(\mathfrak{m}))$, and notice that $\pi(\alpha_0) = 0$ and $\pi(\alpha_1) \in k(\mathfrak{m})^\times$, due to assumption (a) of lemma 16.1.9; hence both p and $W(\pi)(\underline{\alpha})$ generate the maximal ideal of $W(k(\mathfrak{m}))$, whence the assertion in this case. Likewise, if $\underline{\alpha} \in vp + \mathscr{J}$, we get $W(\pi)(\underline{\alpha}) = vp$ in $W(k(\mathfrak{m}))$, and again the assertion follows.

16.1.11. Additionally, often we will be dealing with pairs $(\underline{a}, \mathscr{K})$ consisting of an ideal $\mathscr{K}$ in a given $\mathbb{F}_p$-algebra E, and an element $\underline{a} \in W(E)$ (usually $\underline{a}$ shall be a distinguished element); then we shall say that $(\underline{a}, \mathscr{K})$ is a *transversal pair* if we have

$$W(\mathscr{K}) \cap \underline{a}W(E) = \underline{a}W(\mathscr{K}).$$

This property plays a crucial role in several questions about perfectoid rings, and the following proposition collects a few example of such pairs. One more case is given by theorem 16.3.36(iv).

Proposition 16.1.12. *Let E be any perfect $\mathbb{F}_p$-algebra, and $\underline{a} := (a_n \mid n \in \mathbb{N}) \in W(E)$ any element. The following holds :*

(i) *Suppose that* $a_1 \in E^\times$, *and let* $J_1, J_2 \subset E$ *be any two ideals such that*
   (a) $J_1^{\langle 1\rangle}E = J_1$ *and* $J_2^{\langle 1\rangle}E = J_2$ *(notation of* (9.3.68)*)*
   (b) *there exist* $n, m \in \mathbb{N}$ *such that* $a_0^n J_1 \subset J_2$ *and* $a_0^m J_2 \subset J_1$.
   Then the pair $(\underline{a}, J_1)$ *is transversal if and only if the same holds for the pair* $(\underline{a}, J_2)$.

(ii) *Suppose that* $E = \sum_{n\in\mathbb{N}} a_n E$. *Then the pair* $(\underline{a}, I^{\lfloor r\rfloor}E)$ *is transversal, for every ideal* $I \subset E$ *and every* $r \in \mathbb{R}_+$ *(notation of remark* 9.3.70(i)*).*

(iii) *Let* $J \subset E$ *be any ideal such that* $J^{\langle 1\rangle}E = J$. *Then the pair* (p, J) *is transversal.*

Proof. (iii) is immediate from (9.3.45) and remark 9.3.70(iv).

(i): First, we show that the assertion holds in the special case where $n = 0$ and $m = 1$. To begin with, from lemma 9.3.27(i) and our assumption on a_1 we obtain

$$\underline{a} = \tau_E(a_0) + p\underline{u} \qquad \text{for some } \underline{u} \in W(E)^\times.$$

Notice as well that $\underline{a}$ is a regular element of $W(E)$, by virtue of proposition 9.3.47(i). Now, suppose first that the pair $(\underline{a}, J_1)$ is transversal, and let $x \in W(E)$ be any element such that $\underline{a}\cdot x \in W(J_2)$; then $\tau_E(a_0)\cdot\underline{a}\cdot x \in W(J_1)$ (proposition 9.3.36(i)), so $\tau_A(a_0)\cdot x \in W(J_1)$, by assumption. Hence $p\underline{u}\cdot x = (\underline{a} - \tau_A)\cdot x \in W(J_2)$, so $px \in W(J_2)\cap pW(E) = pW(J_2)$, where the last equality follows from (i.a) and (iii); therefore $x \in W(J_2)$, as p is regular in $W(E)$.

Likewise, suppose that $(\underline{a}, J_2)$ is a transversal pair, and let $x \in W(E)$ be any element such that $\underline{a}\cdot x \in W(J_1)$; then $\underline{a}\cdot x \in W(J_2)$, so $x \in W(J_2)$, and $\tau_E(a_0)\cdot x \in W(J_2)$ (again, by proposition 9.3.36(i)). It follows that $p\underline{u}\cdot x = (\underline{a} - \tau_E(a_0))\cdot x \in W(J_2)$, whence $px \in W(J_2)\cap pW(E) = pW(J_2)$, so $x \in W(J_2)$, as required.

Next, let (J_1, J_2) be any pair of ideals fulfilling conditions (i.a) and (i.b). Notice that we have $(a_0^i J_1)^{\langle 1\rangle}E = a_0^i J_1$ for every $i \in \mathbb{N}$ (lemma 9.3.69(iii)), and the foregoing case implies that assertion (i) holds for the pair $(a_0^i J_1, a_0^{i+1}J_1)$, for every $i \in \mathbb{N}$. It follows that assertion (i) also holds for the pair $(J_1, a_0^n J_1)$; consequently we are reduced to showing assertion (i) for the pair $(J_2, a_0^n J_1)$, and notice that $a_0^{m+n}J_2 \subset a_0^n J_1$. Thus, we may replace J_1 by $a_0^n J_1$, m by $n+m$, and assume from start that $J_1 \subset J_2$ and $a_0^m J_2 \subset J_1$. Now, we have

$$a_0\cdot(J_1 + a_0^i J_2)^{\langle 1\rangle}E = (a_0 J_1 + a_0^{i+1}J_2)^{\langle 1\rangle}E \subset (J_1 + a_0^{i+1}J_2)^{\langle 1\rangle}E \qquad \text{for every } i \in \mathbb{N}$$

so that – again by the foregoing case – we know that assertion (i) holds for the pair

$$((J_1 + a_0^i J_2)^{\langle 1\rangle}E, (J' + a_0^{i+1}J_2)^{\langle 1\rangle}E) \qquad \text{for every } i \in \mathbb{N}$$

and then it also holds for the pair $((J_1 + a_0^m J_2)^{\langle 1\rangle}E, (J_1+J_2)^{\langle 1\rangle}E)$; but $(J_1 + a_0^m J_2)^{\langle 1\rangle}E = J_1$ and $(J_1+J_2)^{\langle 1\rangle}E = J_2$, whence the contention.

(ii): Indeed, let $\underline{b} \in W(A)$ be any element, and suppose that $\underline{a}\cdot\underline{b} \in W(I^{\lfloor r\rfloor}E)$. Define $\mathscr{R}_E$ as in lemma 9.3.85; by lemma 9.3.85(iv) and proposition 9.3.82(iii), we deduce that

$$|\underline{a}|_1\cdot|\underline{b}|_1 = |\underline{a}\cdot\underline{b}|_1 \leq |I|^r \qquad \text{for every } |\cdot| \text{ in } \mathscr{R}_E.$$

But since the system $\{a_n \mid n \in \mathbb{N}\}$ generates E, it is easily seen that $|\underline{a}|_1 = 1$ for every $|\cdot|$ in $\mathscr{R}_E$. We conclude that $|\underline{b}|_1 \leq |I|^r$ for every such $|\cdot|$, and then the assertion follows, again by invoking lemma 9.3.85(iv). □

Corollary 16.1.13. *Let* E *be a perfect* $\mathbb{F}_p$*-algebra,* $b_\bullet := (b_1,\dots,b_n)$ *a finite system of elements of* E, *and* $I \subset E$ *(resp.* $\mathscr{J} \subset W(E)$*) the ideal generated by* $b_\bullet$ *(resp. by* $(\tau_E(b_1)\dots,\tau_E(b_n))$*). Let also* $\mathscr{I} := pW(E) + \mathscr{J}$, *and* $\underline{a} := (a_n \mid n \in \mathbb{N}) \in W(E)$, *and suppose that :*

(a) $E = \sum_{n\in\mathbb{N}} a_n E$.
(b) *E is complete and separated for its I-adic topology.*

We have :

(i) *$W(E)$ is complete and separated for both its $\mathscr{I}$-adic and its $\mathscr{J}$-adic topology.*

(ii) *The $\mathscr{J}$-adic topology of $W(E)$ induces the $\mathscr{J}$-adic topology on $\underline{a}W(E)$, and $\underline{a}W(E)$ is a closed ideal for the $\mathscr{J}$-adic topology of $W(E)$.*

(iii) *Suppose moreover that $a_0 \in I$ and $a_1 \in E^\times$. Then the $\mathscr{I}$-adic topology of $W(E)$ induces the $\mathscr{I}$-adic topology on $\underline{a}W(E)$.*

Proof. (i): By proposition 9.3.77(ii) and lemma 9.3.33(ii), $W(E)$ is complete and separated for its $\mathscr{I}$-adic topology, hence also for its $\mathscr{J}$-adic topology (lemma 8.3.12).

(ii): Taking into account proposition 9.3.77(i) and lemma 9.3.69(iv), we easily see that the $\mathscr{J}$-adic topology on $W(E)$ agrees with the linear topology defined by the cofiltered system of ideals $(W(I^{\lfloor q \rfloor}) \mid q \in \mathbb{Q}_+)$. Then, propositions 16.1.12(ii) and 9.3.47(i) imply that scalar multiplication by $\underline{a}$ is a $W(E)$-linear isomorphism

$$W(E) \xrightarrow{\sim} \underline{a}W(E)$$

that identifies the $\mathscr{J}$-adic topology of $\underline{a}W(E)$ with the one induced by the $\mathscr{J}$-adic topology of $W(E)$. In view of (i), we deduce that $\underline{a}W(E)$ is complete and separated for the subspace topology induced from the $\mathscr{J}$-adic topology of $W(E)$, whence the assertion.

(iii): Notice that I lies in the Jacobson radical of E, due to (b); hence, $\mathscr{I} = \underline{a}W(E) + \mathscr{J}$, by lemma 16.1.9(ii). Now, let $m \in \mathbb{N} \setminus \{0\}$; by virtue of (i) there exists $n \in \mathbb{N}$ such that for every $\underline{w} \in W(E)$ with $\underline{a} \cdot \underline{w} \in \mathscr{J}^n$ we have $\underline{w} \in \mathscr{J}^m$, and clearly we may assume that $n \geq m$. Then, let $\underline{w} \in W(E)$ be any element such that $\underline{a} \cdot \underline{w} \in \mathscr{I}^n$; thus, we have $\underline{a} \cdot \underline{w} = \sum_{i=0}^n \underline{a}^i \cdot \underline{w}_i$ where $\underline{w}_i \in \mathscr{I}^{n-i}$ for $i = 0, \dots, n$. Therefore, $\underline{a} \cdot (\underline{w} - \sum_{i=1}^n \underline{a}^{i-1} \cdot \underline{w}_i) \in \mathscr{J}^n$, so that $\underline{w} - \sum_{i=1}^n \underline{a}^{i-1} \cdot \underline{w}_i \in \mathscr{J}^m$, by the foregoing, and finally $\underline{w} \in \mathscr{I}^{m-1}$, whence the assertion. □

16.2. **P-rings.** In this section we make a preliminary study of an auxiliary class of topological rings containing the perfectoid rings that shall be introduced in section 16.3.

Definition 16.2.1. A complete and separated topological ring $(A, \mathscr{T})$ is called a *P-ring* if there exist a prime integer p and an ideal $I \subset A$ such that the following conditions hold :

(a) I is finitely generated and $\mathscr{T}$ agrees with the I-adic topology.

(b) $pA \subset I^2$ and the Frobenius endomorphism of A/I^2 is a surjection

$$\Phi_{A/I^2} : A/I^2 \to A/I^2 \qquad a \mapsto a^p.$$

Any ideal I of A fulfilling conditions (a) and (b) is called an *ideal of definition* of A. Moreover, we say that I is *special*, if the following holds :

(c) There exists a finite set of generators $a_1, \dots, a_n$ of I such that $pA \subset I^{(p)} := \sum_{i=1}^n Aa_i^p$.

It turns out that if I is a special ideal of definition of the P-ring A, the ideal $I^{(p)}$ depends only on I (and not on the choice of a system of generators for I), so this notation is not abusive. Indeed, we have more generally :

Lemma 16.2.2. *Let A be any ring, p a prime integer, and $(a_1, \dots, a_n), (b_1, \dots, b_m)$ two sequences of elements of A. Suppose that*

$$I := \sum_{i=1}^n Aa_i \subset J := \sum_{i=1}^m Ab_i \qquad \textit{and} \qquad p \in I^{(p)} := \sum_{i=1}^n Aa_i^p.$$

Then :

(i) $I^{(p)} \subset J^{(p)} := \sum_{i=1}^m Ab_i^p$.

(ii) *Especially, if $I = J$, we have $I^{(p)} = J^{(p)}$.*

Proof. It is easily seen that (i)⇒(ii), so we need only check (i). To this aim, it suffices to show that $I_\mathfrak{m}^{(p)} \subset J_\mathfrak{m}^{(p)}$ for every maximal ideal $\mathfrak{m} \subset A$, so we may replace A by $A_\mathfrak{m}$ and assume from start that A is local. Now, suppose first that $J = A$; then it is easily seen that $J^{(p)} = A$ as well, so the assertion is clear in this case. Hence, we may also assume that J is contained in the maximal ideal of A. By assumption, for every $i = 1, \dots, n$ we may write $a_i = \sum_{j=1}^m x_{ij} b_j$ for certain $x_{i1}, \dots, x_{im} \in A$; it follows easily that $a_i^p - \sum_{j=1}^m x_{ij}^p b_j^p \in pJ^p$ (details left to the reader). Summing up, we see that $I^{(p)} \subset J^{(p)} + pJ^p \subset J^{(p)} + I^{(p)} J^{(p)}$, and the assertion follows by Nakayama's lemma. □

Lemma 16.2.3. *Let A be any P-ring, and I any ideal of definition of A. We have :*

(i) *If $A \neq 0$, there exists a unique prime integer p, independent of I, such that condition* (b) *of definition* 16.2.1 *is fulfilled.*
(ii) *I is contained in the Jacobson radical of A.*
(iii) *There exists a special ideal of definition J such that $J^{(p)} = I$.*
(iv) *For p as in* (i)*, the Frobenius endomorphism $\Phi_{A/pA} : A/pA \to A/pA$ is a surjective ring homomorphism, and there exist $\pi \in A$ and $u \in A^\times$ such that $p = u\pi^p$.*
(v) *For p as in* (i)*, the ring A is p-adically complete and separated.*

Proof. (ii) is standard : if $a \in I$, then $1 - ab$ is invertible in A for every $b \in A$, since A is I-adically complete and separated (the inverse is given by the convergent series $\sum_{k \in \mathbb{N}} (ab)^n$); the claim follows.

(i): Let J be another ideal of definition of A, and $p, q \in \mathbb{Z}$ two prime integers with $p \in I^2$ and $q \in J^2$. By (ii), both p and q lie in the Jacobson radical of A; as $A \neq 0$, we get $p = q$.

(iii): Let $(f_\lambda \mid \lambda \in \Lambda)$ be a finite system of generators (indexed by the finite set Λ) for I; by assumption, we may find $g_\lambda \in A$ such that $f_\lambda - g_\lambda^p \in I^2$ for every $\lambda \in \Lambda$. From (ii) and Nakayama's lemma, we deduce that the system $(g_\lambda^p \mid \lambda \in \Lambda)$ generates I, so we can take for J the ideal generated by the system $(g_\lambda \mid \lambda \in \Lambda)$.

(iv): Let $(g_\lambda \mid \lambda \in \Lambda)$ be as in the proof of (iii). For any $a \in A$ we construct inductively a sequence $(b_n \mid n \in \mathbb{N})$ of elements of A, such that

$$a - \sum_{k=0}^{n} b_k^p \in I^{n+1} \qquad \text{and} \qquad b_n \in I^n \qquad \text{for every } n \in \mathbb{N}.$$

Indeed, for $n = 0$, since $\Phi_{A/I}$ is surjective we may find $b_0 \in A$ such that $a - b_0^p \in I$. Next, suppose that $n \geq 0$ and $b_0, \dots, b_n \in A$ are already given such that $c_n := a - \sum_{k=0}^n b_k^p \in I^{n+1}$; we may find a system $(d_{\underline{\lambda}} \mid \underline{\lambda} \in \Lambda^{n+1})$ of elements of A such that

$$c_n = \sum_{\underline{\lambda} \in \Lambda^{n+1}} g_{\underline{\lambda}}^p d_{\underline{\lambda}} \qquad \text{where} \qquad g_{\underline{\lambda}} := \prod_{i=1}^{n+1} g_{\lambda_i} \quad \text{for every } \underline{\lambda} := (\lambda_1, \dots, \lambda_{n+1}) \in \Lambda^{n+1}.$$

In turn, for each $\underline{\lambda} \in \Lambda^{n+1}$ we may write $d_{\underline{\lambda}} = e_{\underline{\lambda}}^p + e'_{\underline{\lambda}}$ for elements $e_{\underline{\lambda}} \in A$ and $e'_{\underline{\lambda}} \in I$, so that

$$c_n = c'_n + c''_n \qquad \text{where} \quad c'_n := \sum_{\underline{\lambda} \in \Lambda^{n+1}} g_{\underline{\lambda}}^p e_{\underline{\lambda}}^p \qquad c''_n := \sum_{\underline{\lambda} \in \Lambda^{n+1}} g_{\underline{\lambda}}^p e'_{\underline{\lambda}} \in I^{n+2}.$$

Thus, $b_{n+1} := \sum_{\underline{\lambda} \in \Lambda^{n+1}} g_{\underline{\lambda}} e_{\underline{\lambda}} \in I^{n+1}$, and since $p \in I^2$, a simple computation shows that $c'_n - b_{n+1}^p \in I^{n+2}$, so finally $a - \sum_{k=0}^{n+1} b_k^p \in I^{n+2}$. Notice that the series $\sum_{k \in \mathbb{N}} b_k^p$ converges to a in the I-adic topology of A, and set $b := \sum_{k \in \mathbb{N}} b_k$; it follows easily that $a - b^p \in pA$, whence the surjectivity of $\Phi_{A/pA}$. Lastly, by assumption we may write $p = \sum_{i=1}^r a_i b_i$ for finitely many elements $a_1, b_1, \dots, a_r, b_r \in I$; by the foregoing, for every $i = 1, \dots, r$ we may further find $f_i, g_i, f'_i, g'_i \in A$ such that $a_i = f_i^p + pf'_i$, $b_i = g_i^p + pg'_i$. Then $f_i^p, g_i^p \in I$, and (ii) implies that f_i and g_i lie in the Jacobson radical $\mathrm{rad}(A)$ of A, for every $i \leq r$. Set $\pi := \sum_{i=1}^r f_i g_i$; summing up, we find $p - \pi^p = pc$ for some $c \in \mathrm{rad}(A)$, so $p \cdot (1 - c) = \pi^p$, and to conclude it suffices to remark that $1 - c \in A^\times$.

(v) is a special case of lemma 8.3.12. □

Remark 16.2.4. Suppose that A is a complete and separated topological $\mathbb{F}_p$-algebra. Then it follows easily from lemma 16.2.3(iv) that A is a P-ring if and only if its Frobenius endomorphism Φ_A is surjective, and there exists a finitely generated ideal $I \subset A$ such that the topology of A agrees with the I-adic topology.

Proposition 16.2.5. *Let A be a ring, p a prime integer, $I \subset A$ a finitely generated ideal such that $pA \subset I$, and denote by $\mathscr{T}_I$ (resp. $\mathscr{T}_p$) the I-adic (resp. p-adic) topology on A. We have :*

(i) *If $(A, \mathscr{T}_I)$ is a P-ring, the same holds for $(A, \mathscr{T}_p)$.*
(ii) *If $(A, \mathscr{T}_p)$ is a P-ring, the same holds for the completion $(A^\wedge_I, \mathscr{T}^\wedge_I)$ of $(A, \mathscr{T}_I)$.*

Proof. (i) follows immediately from lemma 16.2.3(iv,v).

(ii): Let $(f_1, \dots, f_k)$ be a finite system of generators of I; by lemma 16.2.3(iv) there exists $\pi \in A$ such that $pA = \pi^p A$, and the Frobenius endomorphism $\Phi_{A/pA}$ of A/pA is surjective, so for every $i = 1, \dots, k$ we may find $g_i \in A$ such that $f_i - g_i^p \in pA$, and we let $J \subset A$ be the ideal generated by the finite system $(g_1, \dots, g_k, \pi)$. Set $N := (p-1)(k+1)+1$, and notice that $J^N \subset I \subset J^p$. Especially, $\mathscr{T}_I$ agrees with the J-adic topology on A, and moreover $p \in J^2$. Furthermore, $\mathscr{T}^\wedge_I$ agrees with the $JA^\wedge_I$-adic topology on $A^\wedge_I$ (remark 8.3.3(iv) and lemma 8.3.32(iv)), and the natural map $A/J^2 \to A^\wedge_I/J^2A^\wedge_I$ is an isomorphism (remark 8.3.3(ii,iv)); since $\Phi_{A/pA}$ is surjective by assumption, the same holds for Φ_{A/J^2}, and the assertion follows. □

16.2.6. *Henceforth we fix a prime integer p, and we shall assume that the topology of every P-ring that appears throughout the rest of this chapter, is coarser than the p-adic topology*. Let $(A, \mathscr{T})$ be any P-ring and $I \subset A$ an ideal of definition. Notice that if I is special (definition 16.2.1), the Frobenius endomorphism $\Phi_{A/I}$ of A/I factors uniquely as the composition of a morphism of topological rings

$$\overline{\Phi}_{A/I} : A/I \to A/I^{(p)}$$

and the natural projection $A/I^{(p)} \to A/I$.

Lemma 16.2.7. *In the situation of* (16.2.6)*, we have :*

(i) *The maps $u_A : \mathbf{A}(A) \to A$ and $\overline{u}_{A/pA} : \mathbf{E}(A) \to A/pA$ are open and surjective.*
(ii) $\mathrm{Ker}\, u_A$ *contains a distinguished element of* $\mathbf{A}(A)$ *(see definition* 16.1.6*).*
(iii) *If $\underline{\alpha} := (\alpha_n \mid n \in \mathbb{N})$ is any distinguished element in* $\mathrm{Ker}\, u_A$*, we have*

$$\overline{u}_A(\alpha_0) = pu \qquad \textit{for some } u \in A^\times.$$

Proof. (i): Let $I \subset A$ be any ideal of definition of A, so that the topology of A/pA agrees with the I/pA-adic topology; by lemma 16.2.3(iv) we know already that $\Phi_{A/pA}$ is surjective, and then it is easily seen that there exists $n \in \mathbb{N}$ such that $(I/pA)^n \subset \Phi_{A/pA}(I)$. Especially, $\Phi_{A/pA}$ is an open map; then, the assertion for $\overline{u}_{A/pA}$ follows from remark 9.4.9(iii). By virtue of [126, Th.8.4], we deduce the surjectivity of u_A. Next, let $(a_1, \dots, a_r)$ be a finite system of generators for I, and denote by $\overline{a}_i$ the image of a_i in A/pA, for every $i = 1, \dots, k$. Since $\overline{u}_{A/pA}$ is surjective, we may find $\overline{\alpha}_1, \dots, \overline{\alpha}_k \in \mathbf{E}(A)$ such that $\overline{u}_{A/pA}(\overline{\alpha}_i) = \overline{a}_i$ for $i = 1, \dots, k$, and we set $\alpha_i := \tau_A(\overline{\alpha}_i)$ for $i = 1, \dots, k$ (notation of (16.1)). Since I is topologically nilpotent in A, it is easily seen that $\overline{\alpha}_i$ is topologically nilpotent in $\mathbf{E}(A)$, and since τ_A is continuous, we deduce that α_i is topologically nilpotent in $\mathbf{A}(A)$, for every $i = 1, \dots, k$. Let $\mathscr{I} \subset \mathbf{A}(A)$ be the ideal generated by the system $(\alpha_1, \dots, \alpha_k)$; it follows easily that $\mathscr{I}$ is topologically nilpotent as well. On the other hand, by construction we have $a_i - u_A(\alpha_i) \in pA \subset I^2$ for every $i = 1, \dots, k$, so that $u_A(\mathscr{I}) = I$, by Nakayama's lemma. Lastly, let $\mathscr{J} \subset \mathbf{A}(A)$ be any open ideal; then $\mathscr{I}^n \subset \mathscr{J}$ for every sufficiently large $n \in \mathbb{N}$, so $I^n \subset u_A(\mathscr{J})$, and the assertion follows.

(ii): We notice the following :

Claim 16.2.8. There exist $w \in A^\times$ and $t \in \mathbf{E}(A)$ such that $p = w \cdot \overline{u}_A(t)$ in A.

Proof of the claim. By lemma 16.2.3(iv), we may write $p = v \cdot x^p$ for some $v \in A^\times$ and $x \in A$. Since $\overline{u}_{A/pA}$ is surjective, we may find $s \in \mathbf{E}(A)$ such that $x \equiv \overline{u}_A(s) \pmod{pA}$. In view of lemma 9.3.4(i) we deduce that $p \equiv v \cdot \overline{u}_A(s)^p \pmod{p^2A}$, *i.e.* $p = v \cdot \overline{u}_A(t) + p^2 b$ for $t := s^p$ and some $b \in A$; thus $p \cdot (1 - pb) = v \cdot \overline{u}_A(t)$. But since A is p-adically complete and separated (lemma 16.2.3(v)), $1 - pb \in A^\times$ (remark 8.3.10(v)), so we get the sought identity with $w := v \cdot (1 - pb)^{-1}$. ◊

Now, take w and t as in claim 16.2.8. Since u_A is surjective, we may find $\beta := (b_n \mid n \in \mathbb{N}) \in \mathbf{A}(A)$ with $u_A(\beta) = w$. By proposition 9.3.36(i) we have $\beta \cdot \tau_A(t) = (t^{p^n} b_n \mid n \in \mathbb{N})$, and (9.3.42) and remark 9.3.8(ii) show that

$$\gamma := \beta \cdot \tau_A(t) - p = (tb_0, t^p b_1, \dots) - (0, 1, \dots) = (tb_0, t^p b_1 - 1, \dots).$$

But notice that t is topologically nilpotent in $\mathbf{E}(A)$; since the topology of $\mathbf{E}(A)$ is linear, it follows that tb_0 and $t^p b_1$ are topologically nilpotent in $\mathbf{E}(A)$, and since $\mathbf{E}(A)$ is complete and separated, $1 - t^p b_1 \in \mathbf{E}(A)^\times$, so γ is distinguished, and it lies in the kernel of u_A, due to lemma 16.1.1(iv).

(iii): Indeed, from (9.3.41) and lemma 9.3.27(i) we get $\underline{\alpha} = \tau_A(\alpha_0) + p \cdot \underline{\beta}$, where $\beta := (\alpha_{n+1}^{1/p} \mid n \in \mathbb{N})$, and since $\alpha_1 \in \mathbf{E}(A)^\times$, we have $\underline{\beta} \in \mathbf{A}(A)^\times$ (proposition 9.3.44(iii)); now, $\overline{u}_A(\alpha_0) = -p \cdot u_A(\underline{\beta})$, whence the assertion, in light of lemma 16.1.1(iv). □

Proposition 16.2.9. *In the situation of* (16.2.6)*, the following conditions are equivalent :*

(a) $\mathbf{E}(A)$ *is a P-ring.*
(b) *There exists an ideal of definition I of A such that* $\mathrm{Ker}\,\Phi_{A/I}$ *is a finitely generated ideal of A/I.*
(c) *There exists a special ideal of definition J of A such that $\overline{\Phi}_{A/J}$ is an isomorphism.*

Proof. For any ideal of definition I of A, recall that $\mathbf{E}(A/I)$ is the limit of the inverse system $(A_n \mid n \in \mathbb{N})$ with $A_n := A/I$ for every $n \in \mathbb{N}$, and with transition maps $\varphi_n : A_{n+1} \to A_n$ given by the Frobenius endomorphism $\Phi_{A/I}$ of A/I. For every $n \in \mathbb{N}$ we shall denote by

$$\overline{u}_n : \mathbf{E}(A) \to A_n$$

the composition of the canonical identification $\mathbf{E}(A) \xrightarrow{\sim} \mathbf{E}(A/I)$ as in (16.1), with the natural projection $\mathbf{E}(A/I) \to A_n$. Especially $\overline{u}_0 = \overline{u}_{A/I}$; also, each $\overline{u}_n$ is surjective, since the same holds for $\Phi_{A/I}$. Set $\mathscr{J}_0 := \mathrm{Ker}\,\overline{u}_{A/I}$, and let $\Phi_{\mathbf{E}(A)}$ be the Frobenius endomorphism of $\mathbf{E}(A)$; then the family of ideals

$$\mathscr{J}_n := \Phi^n_{\mathbf{E}(A)}(\mathscr{J}_0) = \mathrm{Ker}\,\overline{u}_n \qquad \text{for every } n \in \mathbb{N}$$

is a fundamental system of open neighborhoods of 0 for the topology $\mathscr{T}_{\mathbf{E}(A)}$ of $\mathbf{E}(A)$. Especially, every element of $\mathscr{J}_0$ is topologically nilpotent, and since $\mathbf{E}(A)$ is complete and separated (remark 9.4.9(i)), we easily deduce that $\mathscr{J}_0$ is contained in the Jacobson radical of $\mathbf{E}(A)$ (cp. the proof of lemma 16.2.3(ii)).

(c)⇒(a): Let J be as in (c), and take $I := J$ in the foregoing. Pick a finite system $(a_1, \dots, a_r)$ of generators of J, and let $\pi : A \to A/J^{(p)}$ be the projection. A simple inspection of the definition yields the identity :

$$\overline{u}_{A/J^{(p)}} = \overline{\Phi}_{A/J} \circ \overline{u}_1.$$

Since $\overline{\Phi}_{A/J}$ is an isomorphism, we deduce that $\overline{u}_{A/J^{(p)}}$ is a surjection and we may find

$$\alpha_1, \dots, \alpha_r \in \mathbf{E}(A) \quad \text{with} \quad \overline{u}_{A/J^{(p)}}(\alpha_i) = \pi(a_i) \qquad \text{for every } i = 1, \dots, r.$$

Let $\mathscr{J}'_0 \subset \mathbf{E}(A)$ be the ideal generated by the system $(\alpha_1, \dots, \alpha_r)$, and set

$$\mathscr{J}'_n := \Phi^n_{\mathbf{E}(A)}(\mathscr{J}'_0) \qquad \text{for every } n \in \mathbb{N}.$$

Claim 16.2.10. $\mathscr{J}'_0 = \mathscr{J}_0$, and the $\mathscr{J}_0$-adic topology on $\mathbf{E}(A)$ agrees with $\mathscr{T}_{\mathbf{E}(A)}$.

Proof of the claim. A simple inspection shows that $\mathscr{J}'_0 \subset \mathscr{J}_0$, and consequently $\mathscr{J}'_n \subset \mathscr{J}_n$ for every $n \in \mathbb{N}$. Moreover, $\overline{u}_{A/J^{(p)}}(\mathscr{J}'_0) = J/J^{(p)}$, and therefore

$$\overline{u}_1(\mathscr{J}'_0) = \overline{\Phi}^{-1}_{A/J} \circ \overline{u}_{A/J^{(p)}}(\mathscr{J}'_0) = \overline{\Phi}^{-1}_{A/J}(J/J^{(p)}) = \operatorname{Ker}\varphi_0 = \mathscr{J}_0/\mathscr{J}_1.$$

Especially, the natural map $\mathscr{J}'_0/\mathscr{J}'_1 \to \mathscr{J}_0/\mathscr{J}_1$ is surjective, and since $\Phi_{\mathbf{E}(A)}$ is an automorphism, we deduce that the induced map $\mathscr{J}'_n/\mathscr{J}'_{n+1} \to \mathscr{J}_n/\mathscr{J}_{n+1}$ is surjective for every $n \in \mathbb{N}$. The filtration $(\mathscr{J}_n \mid n \in \mathbb{N})$ is separated on $\mathbf{E}(A)$, so the identity $\mathscr{J}_0 = \mathscr{J}'_0$ will follow from [34, Ch.III, §2, n.8, Cor.2], after we show that $\mathbf{E}(A)$ is complete for the topology $\mathscr{T}$ defined by the filtration $(\mathscr{J}'_n \mid n \in \mathbb{N})$. However, since $\mathscr{J}'_0$ is finitely generated, for every $n \in \mathbb{N}$ there exists $k(n) \in \mathbb{N}$ such that $\mathscr{J}'_{k(n)} \subset \mathscr{J}'^{k(n)}_0 \subset \mathscr{J}'_n$, so the topology $\mathscr{T}$ agrees with the $\mathscr{J}'_0$-adic topology, and $\mathscr{J}'_0$ is topologically nilpotent for the topology $\mathscr{T}_{\mathbf{E}(A)}$, since it is contained in $\mathscr{J}_0$. Then, the assertion follows from lemma 8.3.12. By the same token, we see that the $\mathscr{J}_0$-adic topology agrees with $\mathscr{T}_{\mathbf{E}(A)}$. ◊

The assertion follows immediately from claim 16.2.10 and remark 16.2.4.

(a)⇒(b): Let $I \subset A$ and $\mathscr{I} \subset \mathbf{E}(A)$ be two ideals of definition; then there exists $n \in \mathbb{N}$ such that $\mathscr{J}_n \subset \mathscr{I}$. Clearly $\mathscr{I}' := \Phi^{-n}_{\mathbf{E}(A)}(\mathscr{I})$ is still an ideal of definition, and $\mathscr{J}_0 \subset \mathscr{I}'$. After replacing $\mathscr{I}$ by $\mathscr{I}'$, we may therefore assume that $\mathscr{J}_0 \subset \mathscr{I}$. Write $\overline{u}_{A/I}(\mathscr{I}) = I'/I$ for an ideal $I' \subset A$; we have :

Claim 16.2.11. $\mathscr{I} = \operatorname{Ker}\overline{u}_{A/I'}$ and I' is an ideal of definition of A.

Proof of the claim. We have just remarked that $\operatorname{Ker}\overline{u}_{A/I} \subset \mathscr{I}$, and the stated identity is an easy consequence. Next, since both $\mathscr{I}$ and I are finitely generated, the same holds for I', and obviously $pA \subset I'^2$. Also $\mathscr{I}$ is topologically nilpotent in $\mathbf{E}(A)$, and the topology of A/I is discrete, hence I'/I is a nilpotent ideal of A/I, and it follows easily that the I-adic topology on A agrees with the I'-adic topology. ◊

Due to claim 16.2.11, we may replace I by I', and assume from start that $\mathscr{I} = \mathscr{J}_0$. We may write $\overline{u}_1(\mathscr{J}_0) = J_1/I$ for an ideal $J_1 \subset A$ containing I, and since $\mathscr{J}_0$ and I are finitely generated, the same holds for J_1. It suffices now to notice that

$$J_1/I = \operatorname{Ker}\varphi_0. \qquad (16.2.12)$$

(b)⇒(c): By assumption, there exists a finitely generated ideal $J_1 \subset A$ containing I and such that (16.2.12) holds. Furthermore, by lemma 16.2.3(iii) we may write $I = J'^{(p)}$ for some ideal of definition $J' \subset A$. Set $J := J' + J_1$. From (16.2.12) we get $J_1^{(p)} \subset I$, therefore $J^{(p)} = J'^{(p)} + J_1^{(p)} = I$. We conclude that $\Phi_{A/J}$ induces an isomorphism $A/J_1 \xrightarrow{\sim} A/I = A/J^{(p)}$; but the latter factors through $\overline{\Phi}_{A/J}$ and the projection $A/J_1 \to A/J$, so $\overline{\Phi}_{A/J}$ is an isomorphism as well (and $J = J_1$). Lastly, since $\mathscr{J}_0$ is topologically nilpotent, J_1/I is a nilpotent ideal, so the same holds for J/J', and consequently the J-adic topology agrees with the J'-adic topology; so J is an ideal of definition, whence (b). □

Let A be any P-ring; by lemma 16.2.7(ii), the kernel of u_A contains a distinguished element $\underline{a}$, and we shall see later that the case where actually $\underline{a}$ generates the ideal $\operatorname{Ker}u_A$ is the most interesting for our purposes. For the moment, let us just notice that, if the latter condition holds, the datum $(E := \mathbf{E}(A), \underline{a})$ suffices to recover A : indeed, lemma 16.2.7(i) says that in that case u_A induces an isomorphism of topological rings $W(E)/\underline{a}W(E) \xrightarrow{\sim} A$ (where $W(E)/\underline{a}W(E)$ is endowed with the quotient topology induced from $W(E)$ via the natural projection). It is then convenient to insert hereafter a preliminary study of pairs $(E, \underline{a})$ of this type.

16.2.13. Let $(E, \mathscr{T}_E)$ be a perfect topological $\mathbb{F}_p$-algebra, $\underline{a} := (a_n \mid n \in \mathbb{N}) \in W(E)$ any element, and suppose that $\mathscr{T}_E$ agrees with the a_0-adic topology. Denote by $E^\wedge$ the separated completion of E, and let also E_d (resp. $E^\wedge_d$) be the ring E (resp. $E^\wedge$) endowed with its discrete topology. To ease notation, set

$$A(E) := W(E)/\underline{a}W(E)$$

which we view as a topological ring, with the quotient topology induced from $W(E)$. Define likewise the topological rings $A(E_d)$, $A(E^\wedge)$ and $A(E^\wedge_d)$. The natural diagram of continuous ring homomorphisms

$$\begin{array}{ccc} E_d & \longrightarrow & E^\wedge_d \\ \downarrow & & \downarrow \\ E & \longrightarrow & E^\wedge \end{array}$$

induces a commutative diagram of topological rings

$$\begin{array}{ccc} A(E_d) & \longrightarrow & A(E^\wedge_d) \\ \downarrow & & \downarrow \\ A(E) & \xrightarrow{j_A} & A(E^\wedge). \end{array} \tag{16.2.14}$$

Moreover, notice that there are natural isomorphisms of topological rings

$$A(E) \otimes_{\mathbb{Z}} \mathbb{F}_p \xrightarrow{\sim} E/a_0E \qquad A(E^\wedge) \otimes_{\mathbb{Z}} \mathbb{F}_p \xrightarrow{\sim} E^\wedge/a_0E^\wedge \tag{16.2.15}$$

(where, again the sources and the targets are endowed with the quotient topologies induced from $A(E)$, E, $A(E^\wedge)$, and respectively $E^\wedge$). The composition of the first of these maps with the projection $A(E) \to A(E) \otimes_{\mathbb{Z}} \mathbb{F}_p$ is the continuous ring homomorphism

$$\pi_A : A(E) \to E/a_0E \qquad ((b_n \mid n \in \mathbb{N}) \mod \underline{a}W(E)) \mapsto (b_0 \mod a_0E)$$

and likewise we can describe the corresponding map $\pi^\wedge_A : A(E^\wedge) \to E^\wedge/a_0E^\wedge$. Lastly, denote

$$\beta : \mathbf{E}(A(E)) \to E^\wedge$$

the morphism of topological monoids obtained by composing $\mathbf{E}(\pi_A)$ with the isomorphism of topological rings $\mathbf{E}(E/a_0E) \xrightarrow{\sim} E^\wedge$ provided by corollary 9.4.14.

Proposition 16.2.16. *In the situation of* (16.2.13)*, suppose that $a_0E + a_1E = E$. Then the following holds :*

(i) *The image of $\underline{a}$ is a distinguished element of $W(E^\wedge)$.*
(ii) *All the arrows of* (16.2.14) *are isomorphisms of topological rings.*
(iii) *$A(E)$ is a P-ring, and its topology agrees with the p-adic topology.*
(iv) *β is an isomorphism of topological rings (for the topological ring structure on $A(E)$ provided by* (16.1)*), and we have a commutative diagram of topological rings*

$$\begin{array}{ccc} \mathbf{A}(A(E)) & \xrightarrow{W(\beta)} & W(E^\wedge) \\ {\scriptstyle u_{A(E)}}\downarrow & & \downarrow{\scriptstyle \pi^\wedge_W} \\ A(E) & \xrightarrow{j_A} & A(E^\wedge) \end{array}$$

where $\pi^\wedge_W$ is the natural projection.

Proof. (i): By remark 8.3.10(v), the image of a_0 lies in the Jacobson radical of $E^\wedge$, and since $a_0E + a_1E = E$, it follows easily that $a_1 \in E^\times$ (details left to the reader), whence the assertion.

(ii): Let $\mathscr{I} \subset W(E)$ (resp. $\mathscr{I}^\wedge \subset W(E^\wedge)$) be the ideal generated by the system $(p, \tau_E(a_0))$ (resp. $(p, \tau_{E^\wedge}(a_0))$); proposition 9.3.77(ii) says that the topology of $W(E)$ (resp. of $W(E^\wedge)$)

agrees with the $\mathscr{I}$-adic (resp. $\mathscr{I}^\wedge$-adic) topology, whereas the topologies of both $W(E_d)$ and $W(E_d^\wedge)$ are p-adic. Let $\overline{\mathscr{I}}$ (resp. $\overline{\mathscr{I}}^\wedge$) be the image of $\mathscr{I}$ in $A(E)$ (resp. in $A(E^\wedge)$); it follows easily that the topology of $A(E)$ (resp. of $A(E^\wedge)$) agrees with the $\overline{\mathscr{I}}$-adic (resp. $\overline{\mathscr{I}}^\wedge$-adic) topology, whereas the topologies of both $A(E_d)$ and $A(E_d^\wedge)$ are still p-adic. However, set $\underline{b} := (a_1^{1/p}, a_2^{1/p}, \dots)$, and notice that

$$\underline{a} = \tau_E(a_0) + p \cdot \underline{b} \qquad \text{in } W(E) \tag{16.2.17}$$

((9.3.41) and lemma 9.3.27(i)), which implies that $\overline{\mathscr{I}} = pA(E)$. Likewise, $\overline{\mathscr{I}}^\wedge = pA(E^\wedge)$. Summing up, this proves already that the vertical arrows of (16.2.14) are both isomorphisms.

Next, proposition 9.3.47(v) implies that $A(E)$ is complete and separated for the p-adic topology, hence also for the $\mathscr{I}$-adic topology. Recall as well that the family of ideals

$$\mathrm{Ker}\,(W(E) \to W_n(E/a_0^k E)) \qquad \text{for every } k, n \in \mathbb{N}$$

is a fundamental system of open neighborhoods of $0 \in W(E)$ for the $\mathscr{I}$-adic topology, therefore the natural map

$$A(E) \to \lim_{k,n\in\mathbb{N}} W_n(E/a_0^k E)/\underline{a}W_n(E/a_0^k E)$$

is an isomorphism of topological rings. By the same token, the same description applies to $A(E^\wedge)$, so we conclude that also the horizontal arrows of (16.2.14) are isomorphisms.

(iii): We know already that the topology of $A(E)$ is complete and separated, and agrees with the p-adic topology. On the other hand, notice that $\underline{b}$ is invertible in $W(E^\wedge)$, by (i) and proposition 9.3.44(iii); since the images of $\tau_{E^\wedge}(a_0)$ and $-p \cdot \underline{b}$ agree in $A(E^\wedge)$, we deduce that the p-adic filtration agrees with the $\tau_E(a_0)$-adic filtration on $A(E^\wedge)$, and therefore also on $A(E)$, since we already know that j_A is an isomorphism. Set $J := \tau_E(a_0^{1/p}) \cdot A(E)$; it follows that the topology of $A(E)$ agrees with the J-adic topology, and moreover $p \in J^p \subset J^2$. Lastly, the Frobenius endomorphism is surjective on $A(E)/J^2$, due to the isomorphism (16.2.15), whence the assertion.

(iv): From (iii) and the discussion of (16.1) we see that β is an isomorphism of topological rings; also, under the identifications (16.2.15), the morphism $j_A \otimes_{\mathbb{Z}} \mathbb{F}_p$ corresponds to the natural map $\iota : E/a_0E \xrightarrow{\sim} E^\wedge/a_0E^\wedge$. Therefore we have

$$\pi_A^\wedge \circ j_A \circ u_{A(E)} = \iota \circ \pi_A \circ u_{A(E)} = \iota \circ \pi_A \circ \overline{u}_{A(E)} \circ \boldsymbol{\omega}_0$$

where $\boldsymbol{\omega}_0 : \mathbf{A}(A(E)) \to \mathbf{E}(A(E))$ is the 0-th ghost component map. Then, since $A(E^\wedge)$ is complete and separated for the p-adic topology, proposition 9.3.52(iii) reduces to checking

Claim 16.2.18. $\pi_A^\wedge \circ \pi_W^\wedge \circ W(\beta) = \iota \circ \pi_A \circ \overline{u}_{A(E)} \circ \boldsymbol{\omega}_0$.

Proof of the claim. Notice the identities :

$$\beta \circ \boldsymbol{\omega}_0 = \boldsymbol{\omega}_0^\wedge \circ W(\beta) \qquad \pi_A^\wedge \circ \pi_W^\wedge = \pi_{E^\wedge} \circ \boldsymbol{\omega}_0^\wedge$$

(where $\boldsymbol{\omega}_0^\wedge : W(E^\wedge) \to E^\wedge$ is the 0-th ghost component map, and $\pi_{E^\wedge} : E^\wedge \to E^\wedge/a_0E^\wedge$ is the projection) in light of which, it suffices to show that

$$\pi_{E^\wedge} \circ \beta = \iota \circ \pi_A \circ \overline{u}_{A(E)}.$$

The latter follows by a simple inspection of the definition of β : details left to the reader. $\square$

16.2.19. For any prime p, consider the following categories :

- The category $\mathscr{E}$ whose objects are all the pairs $(E, \mathscr{I})$ where E is both a perfect topological $\mathbb{F}_p$-algebra and a P-ring, and $\mathscr{I} \subset W(E)$ is a distinguished ideal. The morphisms $(E, \mathscr{I}) \to (E', \mathscr{I}')$ are the continuous ring homomorphisms $f : E \to E'$ such that $W(f)(\mathscr{I}) \subset \mathscr{I}'$.

- The category $\mathscr{A}$ of all P-rings A such that $\mathrm{Ker}\, u_A$ is a distinguished ideal of $W(A)$. The morphisms in $\mathscr{A}$ are all the continuous ring homomorphisms.
- The full subcategory $\mathscr{E}_p$ of $\mathscr{E}$ whose objects are all the pairs $(E, \mathscr{I})$ such that the topology of E agrees with the $\boldsymbol{\omega}_0(\mathscr{I})$-adic topology, and the full subcategory $\mathscr{A}_p$ of $\mathscr{A}$ consisting of those objects A whose topology agrees with the p-adic topology.

Then proposition 16.2.16(iii,iv) yields an equivalence of categories

$$(16.2.20) \qquad \mathscr{E}_p \xrightarrow{\sim} \mathscr{A}_p \quad : \quad (E, \mathscr{I}) \mapsto A(E, \mathscr{I}) := W(E)/\mathscr{I}$$

with quasi-inverse given by the rule : $A \mapsto (\mathbf{E}(A), \mathrm{Ker}\, u_A)$. In the following, we aim to extend this equivalence to the whole of $\mathscr{E}$. The first step is :

Proposition 16.2.21. *With the notation of* (16.2.19)*, let* $(E, \mathscr{I})$ *be any object of* $\mathscr{E}$*. Then :*

(i) $A(E, \mathscr{I})$ *is a P-ring.*

(ii) *Especially,* $\mathscr{I}$ *is closed in the topology* $\mathscr{T}_{W(E)}$.

Proof. Clearly, it suffices to check (i). To this aim, let $(a_n \mid n \in \mathbb{N}) \in \mathscr{I}$ be any distinguished element, $I \subset E$ be any ideal of definition, $(x_0, \dots, x_r)$ a finite system of generators for I, denote by $\mathscr{J} \subset W(E)$ the ideal generated by the system $(\tau_E(x_0), \dots, \tau_E(x_r))$, and set $\mathscr{J}' := \mathscr{J} + pW(E)$. By proposition 9.3.77(ii), we know that the topology of $A(E)$ agrees with the $\mathscr{J}'$-adic topology. Now, by assumption a_0 is topologically nilpotent, hence $a_0^k \in I$ for every sufficiently large $k \in \mathbb{N}$; since the Frobenius endomorphism Φ_E is an automorphism of the topological ring E, we may then replace I by $\Phi_E^{-t} I$ for some suitably large $t \in \mathbb{N}$, after which we may assume that $a_0 \in I^{(p)}$, and recall that $I^{(p)}$ is generated by $(x_0^p, \dots, x_r^p)$. Then, lemma 16.1.9(ii) implies that

$$(16.2.22) \qquad p \in \mathscr{J}^2 A(E).$$

and therefore $\mathscr{J}' A(E) = \mathscr{J} A(E)$; especially, the topology of $A(E)$ agrees with the $\mathscr{J}$-adic topology. By corollary 16.1.13(i,ii), $\mathscr{I}$ is a closed subset for the $\mathscr{J}$-adic topology on $W(E)$, and the $\mathscr{J}$-adic topology of $W(E)$ is complete and separated, so the same holds for the $\mathscr{J}$-adic topology of $A(E)$. Moreover, the Frobenius endomorphism of $A(E)/\mathscr{J} A(E)$ is surjective, by virtue of (16.2.15); taking into account (16.2.22), the proposition follows. □

To define a quasi-inverse for our extension of (16.2.20), we shall also need the following:

Proposition 16.2.23. *Let* A *be any P-ring. We have :*

(i) *If* $\mathrm{Ker}\, u_A$ *is the topological closure of a finitely generated ideal,* $\mathbf{E}(A)$ *is a P-ring.*

(ii) *The following conditions are equivalent :*

  (a) $\mathrm{Ker}\, u_A$ *is generated by any distinguished element contained in it.*

  (b) $\mathrm{Ker}\, u_A$ *is a principal ideal.*

  (c) $\mathrm{Ker}\, u_A$ *is the topological closure of a principal ideal.*

Proof. (i): For any topological space X, and any subset $S \subset X$, denote by S^c the topological closure of S in X. By assumption, there exists a finitely generated ideal $\mathscr{I} \subset \mathbf{A}(A)$ such that $\mathscr{I}^c = \mathrm{Ker}\, u_A$. Let now $\boldsymbol{\omega}_0 : \mathbf{A}(A) \to \mathbf{E}(A)$ be the 0-th ghost component; we notice :

Claim 16.2.24. (i) $\boldsymbol{\omega}_0(\mathrm{Ker}\, u_A) = \mathrm{Ker}\, (\overline{u}_{A/pA} : \mathbf{E}(A) \to A/pA)$.

(ii) $(\mathrm{Ker}\, \overline{u}_{A/pA})^c = \mathrm{Ker}\, \overline{u}_{A/(pA)^c}$.

Proof of the claim. (i): Say that $x \in \mathrm{Ker}\, \overline{u}_{A/pA}$, and pick any $y \in \mathbf{A}(A)$ such that $\boldsymbol{\omega}_0(y) = x$; then $u_A(y) = pa$ for some $a \in A$. We pick $z \in \mathbf{A}(A)$ such that $u_A(z) = a$; then $y - pz \in \mathrm{Ker}\, u_A$ and $\boldsymbol{\omega}_0(y - pz) = x$, whence the contention.

(ii): The ideal $K := (\mathrm{Ker}\, \overline{u}_{A/pA})^c$ contains $\mathrm{Ker}\, \overline{u}_{A/pA}$, therefore

$$K = \overline{u}_{A/pA}^{-1}(\overline{u}_{A/pA}(K))$$

so $\overline{u}_{A/pA}(K)$ is a closed ideal of A/pA, by lemma 16.2.7(i), hence $(pA)^c/pA \subset \overline{u}_{A/pA}(K)$ and consequently $K = \overline{u}_{A/pA}^{-1}((pA)^c/pA)$. On the other hand, $\overline{u}_{A/(pA)^c}$ is the composition of $\overline{u}_{A/pA}$ with the projection $A/pA \to A/(pA)^c$, whence the claim. ◊

Claims 8.3.25 and 16.2.24(i) imply that $\boldsymbol{\omega}_0(\mathscr{I})^c = (\mathrm{Ker}\,\overline{u}_{A/pA})^c$. From claim 16.2.24(ii) we deduce that $\mathrm{Ker}\,\overline{u}_{A/(pA)^c}$ is the topological closure in $\mathbf{E}(A)$ of a finitely generated ideal. Next, let $I \subset A$ be any ideal of definition, and $\mathscr{J} \subset \mathbf{E}(A)$ any finitely generated ideal such that $\overline{u}_{A/(pA)^c}(\mathscr{J}) = I/(pA)^c$; it follows easily that $\mathscr{J} + \mathrm{Ker}\,\overline{u}_{A/(pA)^c} = \mathrm{Ker}\,\overline{u}_{A/I}$ (details left to the reader), and then also $\mathrm{Ker}\,\overline{u}_{A/I}$ is the topological closure of a finitely generated ideal of $\mathbf{E}(A)$. Lastly, we have a commutative diagram of continuous and surjective ring homomorphisms

$$\begin{array}{ccc} \mathbf{E}(A) & \xrightarrow{\overline{u}_{A/I}} & A/I \\ {\scriptstyle v}\downarrow & & \| \\ A/I & \xrightarrow{\Phi_{A/I}} & A/I \end{array} \qquad \text{where } v := \overline{u}_{A/I} \circ \Phi^{-1}_{\mathbf{E}(A)}$$

whence $v(\mathrm{Ker}\,\overline{u}_{A/I}) = \mathrm{Ker}\,\Phi_{A/I}$. Since the topology of A/I is discrete, claim 8.3.25 implies that $\mathrm{Ker}\,\Phi_{A/I}$ is a finitely generated ideal, so the assertion follows from proposition 16.2.9.

(ii): Clearly (a)⇒(b), and since $\mathrm{Ker}\,u_A$ is a closed subset of $W(A)$, we have as well (b)⇒(c). Thus, it remains only to check that (c)⇒(a). To this aim, let again $I \subset A$ be any ideal of definition, denote by $J \subset A$ the radical of I, and fix any element $\underline{b} := (b_n \mid n \in \mathbb{N}) \in \mathbf{A}(A)$ such that $\mathrm{Ker}\,u_A$ is the topological closure of $\underline{b}\mathbf{A}(A)$ in $\mathbf{A}(A)$. We notice that A/J is a perfect $\mathbb{F}_p$-algebra, and its quotient topology induced by the projection $\pi : A \to A/J$ is discrete. We have a commutative diagram of continuous and surjective ring homomorphisms :

$$\begin{array}{ccc} \mathbf{A}(A) & \xrightarrow{u_A} & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \pi} \\ W(A/J) & \xrightarrow{\boldsymbol{\omega}_0} & A/J \end{array}$$

whose bottom horizontal arrow is the 0-th ghost map (see (9.3.2)), and with $f := W(\varphi)$, where $\varphi : \mathbf{E}(A) \to A/J$ is the composition of $\overline{u}_{A/pA} : \mathbf{E}(A) \to A/pA$ with the natural projection $A/pA \to A/J$ (details left to the reader).

Claim 16.2.25. $f(\mathrm{Ker}\,u_A) = pW(A/J)$.

Proof of the claim. It is clear that $f(\mathrm{Ker}\,u_A) \subset pW(A/J)$, so we need only show the converse inclusion. Now, according to lemma 16.2.7(ii), we may find a distinguished element $\underline{a} := (a_n \mid n \in \mathbb{N})$ in $\mathrm{Ker}\,u_A$. Since a_0 is topologically nilpotent in $\mathbf{E}(A)$, the element $\overline{u}_{A/pA}(a_0)$ is topologically nilpotent in A/pA, and therefore its image vanishes in A/J; in other words, $a_0 \in \mathrm{Ker}\,\varphi$. However, $\underline{a} = \tau_A(a_0) + p \cdot \underline{u}$, for some $\underline{u} \in \mathbf{A}(A)^\times$, so that

$$f(\underline{a}) = f \circ \tau_A(a_0) + p \cdot f(\underline{u}) = \tau_{A/J} \circ \varphi(a_0) + p \cdot f(\underline{u}) = p \cdot f(\underline{u})$$

(see (9.3.35)) whence the claim. ◊

In light of claims 8.3.25 and 16.2.25 and example 9.3.48(i) we deduce that $f(\underline{b}) \cdot W(A/J)$ is a dense subset of $pW(A/J)$, for the p-adic topology on $W(A/J)$; especially, we have

$$pW(A/J) = p^2W(A/J) + f(\underline{b}) \cdot W(A/J).$$

Then, since p lies in the Jacobson radical of $W(A/J)$, Nakayama's lemma shows that $f(\underline{b}) \cdot W(A/J) = pW(A/J)$, which in turn implies that $\varphi(b_0) = 0$ and $\varphi(b_1)$ is invertible in A/J.

Claim 16.2.26. $\underline{b}$ is a distinguished element of $\mathbf{A}(A)$.

Proof of the claim. First, since J is contained in the Jacobson radical of A (remark 8.3.10(v)) and $\varphi(b_1) \in (A/J)^\times$, we deduce that $\overline{u}_A(b_1) \in A^\times$, and therefore $b_1 \in \mathbf{E}(A)^\times$, by remark 9.4.5(iii). Likewise, we have $\overline{u}_A(b_0) \in JA$, which easily implies that b_0 is topologically nilpotent in $\mathbf{E}(A)$ (details left to the reader). ◊

Now, from (i) and assumption (ii.c) we see that $\mathbf{E}(A)$ is a P-ring; from claim 16.2.26 and proposition 16.2.21(ii) it then follows that $\underline{b}\mathbf{A}(A)$ is a closed ideal of $\mathbf{A}(A)$, and therefore it must coincide with $\mathrm{Ker}\, u_A$. Lastly, let $\underline{a}$ be any other distinguished element of $\mathbf{A}(A)$ contained in $\mathrm{Ker}\, u_A$; taking into account remark 16.1.7(i) and again claim 16.2.26, we conclude that $\underline{a}\mathbf{A}(A) = \underline{b}\mathbf{A}(A)$, whence (ii.a). ☐

16.3. **Perfectoid rings.** We are now ready to introduce our generalizations of Scholze's perfectoid rings that will intervene in the proof of the log regular version of almost purity.

Definition 16.3.1. Let A be any P-ring. We say that A is *perfectoid* if it fulfills any of the three equivalent conditions of proposition 16.2.23(ii).

Example 16.3.2. (i) Let $(A, \mathscr{T})$ be any topological $\mathbb{F}_p$-algebra. Then A is perfectoid if and only if it is perfect, $\mathscr{T}$ is complete and separated, and there exists a finitely generated ideal $I \subset A$ such that $\mathscr{T}$ agrees with the I-adic topology. Indeed, under these assumptions, A is a P-ring (remark 16.2.4), and the map $\overline{u}_A : \mathbf{E}(A) \to A$ is an isomorphism of topological rings that identifies u_A with the 0-th ghost map $\boldsymbol{\omega}_0 : W(A) \to A$, whose kernel is the principal ideal $pW(A)$, by (9.3.45), so A is perfectoid. Conversely, clearly $p \in \mathrm{Ker}\, u_A$, and p is obviously a distinguished element of $\mathbf{A}(A)$, so if A is a perfectoid $\mathbb{F}_p$-algebra we must have $\mathrm{Ker}\, u_A = pW(A)$; moreover, u_A is an open ring homomorphism (lemma 16.2.7(i)), so it induces an isomorphism $\mathbf{E}(A) \xrightarrow{\sim} A$ of topological rings, especially $(A, \mathscr{T})$ is a perfect topological $\mathbb{F}_p$-algebra, as claimed.

(ii) Let E be any perfectoid $\mathbb{F}_p$-algebra, and $\underline{a} := (a_n \mid n \in \mathbb{N})$ any distinguished element of $W(E)$. Then $A(E) := W(E)/\underline{a}W(E)$ is perfectoid for the quotient topology induced by $W(E)$. Indeed, $A(E)$ is a P-ring, by proposition 16.2.21(i); next, lemma 8.3.12 implies that E is complete and separated for the a_0-adic topology, and therefore proposition 16.2.16(ii,iv) applies, and shows that $\mathrm{Ker}\, u_{A(E)}$ is a principal ideal, whence the contention. Moreover, in this case the perfectoid ring E can be recovered from $A(E)$: indeed, we have a natural isomorphism of topological rings

$$\mathbf{E}(A(E)) \xrightarrow{\sim} E$$

obtained as follows. First, since the 0-th ghost component $W(E) \to E$ is an open and surjective map, the same holds for the projection $\alpha : A(E) \to A(E)/pA(E) \xrightarrow{\sim} E/a_0E$, where E/a_0E is endowed with the quotient topology induced by E. Then, α induces an isomorphism $\mathbf{E}(\alpha) : \mathbf{E}(A(E)) \xrightarrow{\sim} \mathbf{E}(E/a_0E)$ of topological rings, by (16.1). But since a_0 is topologically nilpotent, the projection $\beta : \mathbf{E}(E) \to \mathbf{E}(E/a_0E)$ is an isomorphism of topological rings as well (theorem 9.4.10), and the same holds for the map $\overline{u}_E : \mathbf{E}(E) \to E$, since E is perfect. Thus, the sought isomorphism is the composition $\overline{u}_E \circ \beta^{-1} \circ \mathbf{E}(\alpha)$.

Let us also point out the following complement to proposition 16.2.9 :

Corollary 16.3.3. *Let A be any perfectoid ring. Then $\overline{\Phi}_{A/J}$ is an isomorphism for every special ideal of definition J of A.*

Proof. Set $E := \mathbf{E}(A)$; we may assume that there exists a distinguished element $\underline{a} := (a_n \mid n \in \mathbb{N}) \in W(E)$ such that A is the ring $A(E)$ as defined in (16.2.13), and $u_A : W(E) \to A(E)$ is the projection. Moreover, we may find a finite system $\underline{x}_\bullet := (\underline{x}_0, \dots, \underline{x}_r)$ of elements of $W(E)$, such that J is the image in $A(E)$ of the ideal generated by $\underline{x}_\bullet$. We may write

(16.3.4) $$\underline{x}_i = \tau_{\mathbf{E}}(t_i) + p \cdot \underline{y}_i \qquad \text{where } t_i \in E \text{ and } \underline{y}_i \in W(E) \text{ for every } i = 0, \dots, r.$$

In view of lemma 9.3.4(i), we deduce that

$$\underline{x}_i^p \equiv \tau_E(t_i)^p \pmod{p^2 W(E)} \qquad \text{for } i = 0, \dots, r. \tag{16.3.5}$$

Let $\mathscr{J} \subset W(E)$ be the ideal generated by $\tau_E(t_0), \dots, \tau_E(t_r)$; we notice :

Claim 16.3.6. $p \in \mathscr{J}^{(p)} A(E)$ and $\mathscr{J} A(E) = J$.

Proof of the claim. It follows easily from (16.3.5) that there exists $\underline{z} \in W(E)$ with

$$p \cdot u_A(1 + p \cdot \underline{z}) \in \mathscr{J}^{(p)} A(E)$$

(recall that $\mathscr{J}^{(p)}$ is the ideal generated by $\tau_E(t_0)^p, \dots, \tau_E(t_r)^p$: see definition 16.2.1). However, p is topologically nilpotent in $W(E)$, therefore $1+p\cdot\underline{z} \in W(E)^\times$, so $p \in \mathscr{J}^{(p)} A(E)$. The other assertion follows directly from the first one and (16.3.4) : details left to the reader. $\Diamond$

Let $I \subset E$ be the ideal generated by $t_0, \dots, t_r$, and set $E_0 := E/a_0 E$; taking into account lemma 16.1.9(ii) and claim 16.3.6, we deduce that $a_0 \in I^{(p)}$. Also, it is clear that the natural isomorphism (16.2.15) maps the image of J (resp. of $J^{(p)}$) onto IE_0 (resp. $I^{(p)} E_0$). But it also follows that $E_0/IE_0 = E/I$ and $E_0/I^{(p)}E_0 = E/I^{(p)}$; lastly, since E is perfect, the Frobenius endomorphism Φ_E maps I onto $I^{(p)}$, whence the contention. $\square$

Remark 16.3.7. (i) We may now say that the category $\mathscr{A}$ of (16.2.19) is the *category of perfectoid rings*, and we shall denote it henceforth by

$$\mathsf{Perf}.$$

More generally, if A is any perfectoid ring, we shall write

$$A\text{-}\mathsf{Perf}$$

for the category A/Perf of *perfectoid A-algebras*, whose objects are the continuous ring homomorphisms $A \to B$ with B perfectoid. The objects of the category $\mathscr{E}$ of (16.2.19) are the pairs $(E, \mathscr{I})$ consisting of a perfectoid $\mathbb{F}_p$-algebra and a distinguished ideal of $W(E)$. Taking into account example 16.3.2, we see that (16.2.19) generalizes *verbatim* to an equivalence

$$\mathscr{E} \xrightarrow{\sim} \mathsf{Perf} \quad : \quad (E, \mathscr{I}) \mapsto A(E, \mathscr{I}) := W(E)/\mathscr{I}$$

with a natural quasi-inverse given by the functor $\mathbf{E}$, which restricts to equivalences

$$\mathbf{E} : A\text{-}\mathsf{Perf} \xrightarrow{\sim} \mathbf{E}(A)\text{-}\mathsf{Perf} \qquad \text{for every perfectoid ring } A.$$

Hereafter, we shall study the stability of the class of perfectoid rings under some standard operations. The following observations (iii) and (iv) show stability under completion with respect to the J-adic topology corresponding to an ideal $J \subset A$ of finite type with $pA \subset J$.

(ii) Let A be any perfectoid ring, set $\mathbf{E} := \mathbf{E}(A)$, and let $\alpha_\bullet := (\alpha_n \mid n \in \mathbb{N})$ be any distinguished element in $\operatorname{Ker} u_A$. Notice that $\overline{u}_{A/pA} : \mathbf{E} \to A/pA$ induces an isomorphism of topological rings :

$$\omega : \mathbf{E}/\alpha_0 \mathbf{E} \xrightarrow{\sim} A/pA$$

for the quotient topologies induced by the projections $A \to A/pA$ and $\mathbf{E} \to \mathbf{E}/\alpha_0$. Indeed, $\overline{u}_{A/pA}$ is open and surjective (lemma 16.2.7(i)), and $\alpha_\bullet$ generates $\operatorname{Ker} u_A$, so α_0 generates the kernel of $u_{A/pA}$, by virtue of lemma 16.1.1(i), whence the contention.

(iii) The isomorphism ω of (ii) induces a natural bijection :

$$\{\text{ideals } \mathscr{J} \subset \mathbf{E} \mid \alpha_0 \mathbf{E} \subset \mathscr{J}\} \leftrightarrow \{\text{ideals } J \subset A \mid pA \subset J\}.$$

Namely, $\mathscr{J} \subset \mathbf{E}$ and $J \subset A$ correspond under this bijection, if and only if $\omega(\mathscr{J}/\alpha\mathbf{E}) = J/pA$. Then, clearly J is finitely generated if and only if the same holds for $\mathscr{J}$. Let $\mathscr{T}_J$ (resp. $\mathscr{T}_{\mathscr{J}}$) denote the J-adic (resp. $\mathscr{J}$-adic) topology on A (resp. on $\mathbf{E}$); taking into account remark 9.4.9(ii), we deduce that if J is finitely generated, then the topology of $\mathbf{E}(A, \mathscr{T}_J)$ agrees with $\mathscr{T}_{\mathscr{J}}$. Combining with example 16.3.2(i), lemma 8.3.32(iv), example 9.3.48(ii) and remark 9.4.9(v), we conclude that the completion $\mathbf{E}(A,\mathscr{T}_J)^\wedge$ of $\mathbf{E}(A,\mathscr{T}_J)$ is perfectoid.

(iv) In the situation of (iii), suppose that J and $\mathscr{J}$ are finitely generated; notice that, since $\alpha_0 \in \mathscr{J}$, the image of $\alpha_\bullet$ is distinguished in $W(\mathbf{E}(A,\mathscr{T}_J)^\wedge)$, and let $\mathscr{I} := \alpha_\bullet W(\mathbf{E}(A,\mathscr{T}_J)^\wedge)$. We claim that there is a natural isomorphism of topological rings

$$W(\mathbf{E}(A,\mathscr{T}_J)^\wedge)/\mathscr{I} \xrightarrow{\sim} (A_J,\mathscr{T}_J)^\wedge$$

for the quotient topology on $W(\mathbf{E}(A,\mathscr{T}_J)^\wedge)/\mathscr{I}$ induced by $W(\mathbf{E}(A,\mathscr{T}_J)^\wedge)$. Especially, this shows that $(A_J,\mathscr{T}_J)^\wedge$ is perfectoid. For the proof, fix a finite system $(\beta_1,\dots,\beta_k)$ of generators of $\mathscr{J}$, and let $\mathscr{J}_W \subset W(\mathbf{E})$ be the ideal generated by the system $(p,\tau_{\mathbf{E}}(\beta_1),\dots,\tau_{\mathbf{E}}(\beta_k))$; by proposition 9.3.77(ii), the topology $\mathscr{T}_W$ of $W(\mathbf{E},\mathscr{T}_{\mathscr{J}})$ agrees with the $\mathscr{J}_W$-adic topology. By construction we have $u_A(\mathscr{J}_W)/pA = J/pA$, and consequently $u_A(\mathscr{J}_W) = J$, so the quotient topology induced by $\mathscr{T}_W$ on A via u_A agrees with $\mathscr{T}_J$. Lastly, in view of (iii) and proposition 16.2.21(ii) we know that $\mathscr{I}$ is a closed ideal of $W(\mathbf{E}(A,\mathscr{T}_J)^\wedge)$. Taking into account proposition 8.2.13(v) and lemma 9.3.33(iv), the assertion follows.

We can summarize these observations in the following :

Proposition 16.3.8. *Let A be a ring, p a prime integer, $I \subset A$ a finitely generated ideal such that $pA \subset I$, and denote by $\mathscr{T}_I$ (resp. $\mathscr{T}_p$) the I-adic (resp. p-adic) topology on A. We have :*

(i) *If $(A,\mathscr{T}_I)$ is perfectoid, the same holds for $(A,\mathscr{T}_p)$.*
(ii) *If $(A,\mathscr{T}_p)$ is perfectoid, the same holds for the completion $(A_I,\mathscr{T}_I)^\wedge$ of $(A,\mathscr{T}_I)$.*

Proof. (i) follows immediately from proposition 16.2.5(i).

(ii): It suffices to apply remark 16.3.7(iv) to the ideal $I \subset A$ and the corresponding ideal $\mathscr{J} := \overline{u}^{-1}_{A/pA}(I/pA)$ of $\mathbf{E}(A)$. □

The next observation establishes the stability of Perf under completed tensor products :

Proposition 16.3.9. *Let A_0 be a perfectoid ring and A_1, A_2 two perfectoid A_0-algebras. Then:*

(i) *The topological ring $A_3 := A_1 \widehat{\otimes}_{A_0} A_2$ is perfectoid.*
(ii) *There exists a natural isomorphism $\mathbf{E}(A_3) \xrightarrow{\sim} \mathbf{E}(A_1)\widehat{\otimes}_{\mathbf{E}(A_0)}\mathbf{E}(A_2)$ in* Perf.
(iii) *Especially, all finite coproducts are representable in the category* Perf.

Proof. (i): Set $\mathbf{E}_i := \mathbf{E}(A_i)$ for $i = 0,1,2$, and pick a distinguished element $\underline{\alpha} \in W(\mathbf{E}_0)$ that generates the kernel of u_{A_0}; then the image of $\underline{\alpha}$ in $W(\mathbf{E}_i)$ is still distinguished, and we know that $\underline{\alpha}W(\mathbf{E}_i)$ is the kernel of u_{A_i} also for $i = 1,2$. Moreover, it is clear from remark 9.3.56(iii) that the $\mathbb{F}_p$-algebra $\mathbf{E}_3 := \mathbf{E}_1 \widehat{\otimes}_{\mathbf{E}_0} \mathbf{E}_2$ is perfectoid, and then the same holds for the topological ring $A(\mathbf{E}_3) := W(\mathbf{E}_3)/\underline{\alpha}W(\mathbf{E}_3)$, by virtue of example 16.3.2(ii). Lastly, the isomorphism (9.3.58) induces an isomorphism of topological rings

$$A(\mathbf{E}_3) \xrightarrow{\sim} A_1 \widehat{\otimes}_{A_0} A_2$$

whence the assertion.

(ii): Example 16.3.2(ii) also identifies naturally $\mathbf{E}_3$ with $\mathbf{E}(A(\mathbf{E}_3))$, whence the contention.

(iii) is clear, since complete tensor products represent these coproducts, by (8.3.7). □

16.3.10. For any ring A and any ideal $I \subset A$, let $\mathscr{T}_I$ be the I-adic topology on A, and denote $(A_I^\wedge,\mathscr{T}_I^\wedge)$ the separated completion of $(A,\mathscr{T}_I)$. Notice that if $I' \subset A$ is another ideal with $I' \subset I$, the identity map of A yields a continuous map $(A,\mathscr{T}_{I'}) \to (A,\mathscr{T}_I)$, whose completion is a natural continuous ring homomorphism

$$(A_{I'}^\wedge,\mathscr{T}_{I'}^\wedge) \to (A_I^\wedge,\mathscr{T}_I^\wedge). \qquad (16.3.11)$$

Now, suppose that $I, J \subset A$ are any two ideals; we deduce a commutative diagram

(16.3.12)
$$\begin{array}{ccc} (A^\wedge_{I\cap J}, \mathscr{T}^\wedge_{I\cap J}) & \xrightarrow{\varphi_I} & (A^\wedge_I, \mathscr{T}^\wedge_I) \\ {\scriptstyle\varphi_J}\downarrow & & \downarrow{\scriptstyle\beta_I} \\ (A^\wedge_J, \mathscr{T}^\wedge_J) & \xrightarrow{\beta_J} & (A^\wedge_{I+J}, \mathscr{T}^\wedge_{I+J}) \end{array}$$

whose arrows are the continuous ring homomorphisms (16.3.11). Moreover, for every $k \in \mathbb{N}$ let $A^\wedge_{I,k} := A^\wedge_I / J^k A^\wedge_I$, and endow $A^\wedge_{I,k}$ with the quotient topology $\mathscr{T}_{I,k}$ induced by $\mathscr{T}^\wedge_I$ via the natural projection $A^\wedge_I \to A^\wedge_{I,k}$. We set

(16.3.13)
$$(A^\wedge_{I,J}, \mathscr{T}^\wedge_{I,J}) := \lim_{k\in\mathbb{N}} (A^\wedge_{I,k}, \mathscr{T}_{I,k}).$$

For every $k \in \mathbb{N}$, let $\mathscr{T}_{d,k}$ be the discrete topology on $A/(I + J)^k$; the system of continuous projections $((A^\wedge_I, \mathscr{T}^\wedge_I) \to (A/(I + J)^k, \mathscr{T}_d) \mid k \in \mathbb{N})$ factors uniquely through a system of continuous ring homomorphisms $((A^\wedge_{I,k}, \mathscr{T}_{I,k}) \to (A/(I + J)^k, \mathscr{T}_d) \mid k \in \mathbb{N})$, whose limit is a natural continuous ring homomorphism

(16.3.14)
$$(A^\wedge_{I,J}, \mathscr{T}^\wedge_{I,J}) \to (A^\wedge_{I+J}, \mathscr{T}^\wedge_{I+J}).$$

Lemma 16.3.15. *In the situation of* (16.3.10)*, we have :*

(i) (16.3.14) *is an open and surjective map.*

(ii) *Suppose that A is either a noetherian ring or a perfect $\mathbb{F}_p$-algebra, and that both I and J are finitely generated. Then :*

  (a) (16.3.12) *is a cartesian diagram of topological rings and a cocartesian diagram of topological A-modules.*

  (b) (16.3.14) *is an isomorphism of topological rings.*

Proof. For every $n \in \mathbb{N}$, we have a natural diagram of rings

(16.3.16)
$$\begin{array}{ccc} A/(I^n \cap J^n) & \longrightarrow & A/I^n \\ \downarrow & & \downarrow \\ A/J^n & \longrightarrow & A/(I^n + J^n) \end{array}$$

whence a complex of A-modules

$$\Sigma_n \qquad : \qquad 0 \to A/(I^n \cap J^n) \to A/I^n \oplus A/J^n \to A/(I^n + J^n) \to 0$$

and it is easily seen that Σ_n is exact for every $n \in \mathbb{N}$. Therefore, (16.3.16) is a cartesian diagram of discrete topological rings (details left to the reader); let $\mathscr{T}'_{d,n}$ be the discrete topology on $A/(I^n \cap J^n)$ for every $n \in \mathbb{N}$, and set

$$(A', \mathscr{T}') := \lim_{n\in\mathbb{N}} (A/(I^n \cap J^n), \mathscr{T}'_{d,n}).$$

Notice as well that $(I + J)^{2n-1} \subset I^n + J^n \subset (I + J)^n$ for every $n \in \mathbb{N}$, so the $(I + J)$-adic topology on A agrees with the linear topology defined by the descending system of ideals $(I^n + J^n \mid n \in \mathbb{N})$. In light of example 1.5.15(ii), we deduce that the limit of the system of these diagrams is a cartesian diagram of topological rings

(16.3.17)
$$\begin{array}{ccc} (A', \mathscr{T}') & \xrightarrow{\varphi'_I} & (A^\wedge_I, \mathscr{T}^\wedge_I) \\ {\scriptstyle\varphi'_J}\downarrow & & \downarrow{\scriptstyle\beta'_I} \\ (A^\wedge_J, \mathscr{T}^\wedge_J) & \xrightarrow{\beta'_J} & (A^\wedge_{I+J}, \mathscr{T}^\wedge_{I+J}). \end{array}$$

Moreover, since the inverse system $(A/(I^n \cap J^n) \mid n \in \mathbb{N})$ has surjective transition maps, the limit of the system of exact sequences $(\Sigma_n \mid n \in \mathbb{N})$ is still exact (see [163, Lemma 3.5.3]), and therefore (16.3.17) is also a cocartesian diagram of A-modules. Furthermore, by proposition 8.2.13(i,v) we have an induced morphism of exact complexes

$$\begin{array}{ccccccccccc}
\Sigma'_n & & 0 & \longrightarrow & (I^n \cap J^n)^\wedge & \longrightarrow & I^{n\wedge} \oplus J^{n\wedge} & \longrightarrow & (I^n + J^n)^\wedge & \longrightarrow & 0 \\
{\scriptstyle\sigma_n}\downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
\Sigma & & 0 & \longrightarrow & A' & \longrightarrow & A^\wedge_I \oplus A^\wedge_J & \xrightarrow{\beta'_{I,J}} & A^\wedge_{I+J} & \longrightarrow & 0
\end{array}$$

where :

- $(I^n \cap J^n)^\wedge$ is the topological closure of $I^n \cap J^n$ in $(A', \mathscr{T}')$, and likewise for $I^{n\wedge} \oplus J^{n\wedge}$ and $(I^n + J^n)^\wedge$.
- $\mathrm{Coker}\, \sigma_n$ is naturally isomorphic to Σ_n, for every $n \in \mathbb{N}$.
- $\beta'_{I,J}$ is the sum of β'_I and β'_J.

Especially, $\beta'_{I,J}$ is a continuous, open and surjective map, for the product topology $\mathscr{T}^\wedge_I \times \mathscr{T}^\wedge_J$ on $A^\wedge_I \oplus A^\wedge_J$, and it follows easily that (16.3.17) is a cocartesian diagram of topological A-modules.

(i): The system of natural maps $(A^\wedge_I \to A^\wedge_{I,k} \leftarrow A/J^k \mid k \in \mathbb{N})$ yields ring homomorphisms

$$A^\wedge_I \xrightarrow{\psi_I} (A^\wedge_I)^\wedge_J \xleftarrow{\psi_J} A^\wedge_J.$$

There follows a map of abelian groups $\gamma : A^\wedge_I \oplus A^\wedge_J \to (A^\wedge_I)^\wedge_J$ which is continuous for the topology $\mathscr{T}^\wedge_I \times \mathscr{T}^\wedge_J$ and a simple inspection shows that (16.3.14) $\circ\, \gamma$ equals $\beta'_{I,J}$. Now, let $K \subset (A^\wedge_I)^\wedge_J$ be any open ideal; then $K' := \gamma^{-1}K$ is an open subgroup of $A^\wedge_I \oplus A^\wedge_J$ and therefore $\beta'_{I,J}K'$ is an open subgroup of $A^\wedge_{I+J}$ contained in the image K'' of K under (16.3.14), so K'' is open as well, whence (i).

(ii.a): The system of projections $(A/(I \cap J)^n \to A/(I^n \cap J^n) \mid n \in \mathbb{N})$ yields a natural continuous ring homomorphism

$$\nu : (A^\wedge_{I\cap J}, \mathscr{T}^\wedge_{I\cap J}) \to (A', \mathscr{T}')$$

and a simple inspection shows that $\varphi'_I \circ \nu = \varphi_I$ and $\varphi'_J \circ \nu = \varphi_J$. To conclude the proof of (ii.a), it then suffices to check :

Claim 16.3.18. Under the assumptions of (ii), the map ν is an isomorphism of topological rings.

Proof of the claim. Since $(IJ)^{2n} \subset (I \cap J)^{2n} \subset (IJ)^n$ for every $n \in \mathbb{N}$, the $(I \cap J)$-adic and IJ-adic topologies coincide on A. On the other hand, in case A is noetherian, the Artin-Rees lemma ([126, Th.8.5]) implies that for every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $I^m \cap J^n \subset I^n J^n \subset (I \cap J)^n$, from which the claim follows easily.

Thus, suppose A is perfect and both I and J are finitely generated, so the same holds for IJ, and therefore lemma 9.3.69(iv) says that the IJ-adic topology on A agrees with the linear topology defined by the cofiltered system of ideals $((IJ)^{\langle n\rangle}A \mid n \in \mathbb{N})$. Likewise, the linear topology on A defined by the cofiltered system of ideals $(I^n \cap J^n \mid n \in \mathbb{N})$ agrees with the one defined by the cofiltered system of ideals $(I^{\langle n\rangle}A \cap J^{\langle n\rangle}A \mid n \in \mathbb{N})$. So, it suffices to prove that

$$I^{\langle pn\rangle}A \cap J^{\langle pn\rangle}A \subset (IJ)^{\langle n\rangle}A \qquad \text{for every } n \in \mathbb{N}.$$

However, if $x \in I^{\langle pn\rangle}A \cap J^{\langle pn\rangle}A$, we may write $x = x^{1/p} \cdot x^{(p-1)/p} \in I^{\langle n\rangle}J^{\langle (p-1)n/p\rangle}A \subset (IJ)^{\langle n\rangle}A$ whence the contention. ◊

(ii.b): For every $h, k \in \mathbb{N}$, let $A_{h,k} := A/(I^h + J^k)$, and endow $A_{h,k}$ with the discrete topology $\mathscr{T}_{h,k}$; moreover, set

$$(C_k, \mathscr{T}_{C,k}) := \lim_{h\in\mathbb{N}} (A_{h,k}, \mathscr{T}_{h,k}) \qquad \text{for every } k \in \mathbb{N}$$

(where the transition maps $A_{h+1,k} \to A_{h,k}$ are the natural projections). Since I and J are both finitely generated, for every $h, k \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $(I+J)^n \subset I^h + J^k$, so the natural map

$$(16.3.19) \qquad (A^\wedge_{I+J}, \mathscr{T}^\wedge_{I+J}) \to \lim_{h,k\in\mathbb{N}} (A_{h,k}, \mathscr{T}_{h,k}) \xrightarrow{\sim} \lim_{k\in\mathbb{N}} (C_k, \mathscr{T}_{C,k})$$

is an isomorphism of topological rings (see example 1.5.15(ii)). On the other hand, the projection $A \to A/J^k$ extends uniquely to a continuous map $\varphi_k : (A^\wedge_I, \mathscr{T}^\wedge_I) \to (C_k, \mathscr{T}_{C,k})$, and the latter factors uniquely through a continuous ring homomorphism

$$\overline{\varphi}_k : (A^\wedge_{I,k}, \mathscr{T}_{I,k}) \to (C_k, \mathscr{T}_{C,k}) \qquad \text{for every} \in \mathbb{N}.$$

With this notation, a simple inspection shows that (16.3.13) and (16.3.19) identify (16.3.14) with the limit of the system of maps $(\overline{\varphi}_k \mid k \in \mathbb{N})$. However, proposition 8.2.13(i,v) says that φ_k is surjective and its kernel is the topological closure $(J^k A^\wedge_I)^c$ of $J^k A^\wedge_I$ in $(A^\wedge_I, \mathscr{T}^\wedge_I)$, for every $k \in \mathbb{N}$. Thus, in order to show the lemma, it suffices to check that the linear topology on $A^\wedge_I$ defined by the system of ideals $((J^k A^\wedge_I)^c \mid k \in \mathbb{N})$ agrees with the $JA^\wedge_I$-adic topology. In case A is noetherian, this is clear, since in that case $J^k A^\wedge_I$ is already closed in $A^\wedge_I$ ([126, Th.8.11]). For the case where A is perfect, we remark :

Claim 16.3.20. Let $(B, \mathscr{T}_B)$ be a perfect, complete and separated topological $\mathbb{F}_p$-algebra, $I, J \subset B$ two ideals of finite type, such that $\mathscr{T}_B$ agrees with the I-adic topology. For every $\lambda \in \mathbb{N}[1/p]$ let $J^{\langle\lambda\rangle} B^c$ be the topological closure of $J^{\langle\lambda\rangle} B$ in B (notation of (9.3.68)). Then

$$J^{\langle\lambda\rangle} B^c \subset \bigcap_{n\in\mathbb{N}} (J + I^n)^{\langle\lambda\rangle} B \subset J^{\langle\lambda'\rangle} B \qquad \text{for every } \lambda' < \lambda.$$

Proof of the claim. We show first that $J^{\langle\lambda\rangle c} B \subset J^{\langle\lambda'\rangle} B$. Indeed, pick a finite system of generators $(b_1, \dots, b_k)$ (resp. $(b_{k+1}, \dots, b_s)$) for J (resp. for I), and let $x \in J^{\langle\lambda\rangle} B^c$ be any element; we can write

$$x = \sum_{n\in\mathbb{N}} x_n \qquad \text{with} \qquad x_n \in J^{\langle\lambda\rangle} B \cap I^{\langle n\rangle} B \qquad \text{for every } n \in \mathbb{N}.$$

Choose a strictly positive $\varepsilon \in \mathbb{N}[1/p]$ such that $\lambda'' := (1-\varepsilon)\cdot\lambda > \lambda'$, and notice that

$$x_n = x_n^{1-\varepsilon} \cdot x_n^{\varepsilon} \in J^{\langle\lambda''\rangle} \cdot I^{\langle n\varepsilon\rangle} B \qquad \text{for every } n \in \mathbb{N}.$$

By definition, for every $n \in \mathbb{N}$ there exist :

- a finite set $S_n \subset \mathbb{N}[1/p]^{\oplus k}$ with $\mu_1 + \cdots + \mu_k = \lambda''$ for every $\mu := (\mu_1, \dots, \mu_k) \in S_n$
- a system $(a_\mu \mid \mu \in S_n)$ of elements of B such that

$$x_n^{1-\varepsilon} = \sum_{\mu\in S_n} a_\mu b^\mu \qquad \text{where } b^\mu := b_1^{\mu_1} \cdots b_k^{\mu_k} \text{ for every } \mu \in S_n.$$

Now, choose $N \in \mathbb{N}$ such that $\lambda'' - \lambda' \geq kp^{-N}$ and define $\overline{\mu}$ and μ^* as in the proof of proposition 9.3.77, so that $\overline{\mu} \in S := \{\nu \in p^{-N}\mathbb{N}^{\oplus k} \mid \lambda'' \geq \nu_1 + \cdots + \nu_k > \lambda'\}$ for every $n \in \mathbb{N}$ and every $\mu \in S_n$. It follows that

$$x = \sum_{\nu\in S} b^\nu c_\nu \qquad \text{where} \qquad c_\nu := \sum_{n\in\mathbb{N}} x_n^\varepsilon \cdot \sum_{\substack{\mu\in S_n \\ \overline{\mu}=\nu}} a_\mu b^{\mu^*}.$$

Clearly $b^\nu \in J^{\langle\lambda'\rangle} B$, and by lemma 9.3.69(iv) the series c_ν converges in the I-adic topology of B for every $\nu \in S$; also it is easily seen that S is a finite set, whence the contention.

Next, for any $n \in \mathbb{N}$, the ideal $(J + I^n)^{\langle\lambda\rangle} B$ is generated by all monomials of the form

$$b^\mu := b_1^{\mu_1} \cdots b_s^{\mu_s} \qquad \text{such that} \qquad \lambda_1 + n^{-1} \cdot \lambda_2 = \lambda \quad \text{where} \quad \lambda_1 := \sum_{i=1}^{k} \mu_i \quad \lambda_2 := \sum_{i=k+1}^{s} \mu_i$$

and where $\mu := (\mu_1, \dots, \mu_s)$ is any sequence of elements of $\mathbb{N}[1/p]$. Fix also $\lambda'' \in \mathbb{N}[1/p]$ with $\lambda' < \lambda'' < \lambda$. Now, for any such μ, we have either $\lambda_1 \geq \lambda''$, in which case $b^\mu \in J^{\langle\lambda''\rangle}B$, or else $\lambda_2 > n \cdot (\lambda - \lambda'')$. We conclude that

$$\bigcap_{n\in\mathbb{N}} (J + I^n)^{\langle\lambda\rangle}B \subset \bigcap_{n\in\mathbb{N}} (J^{\langle\lambda''\rangle}B + I^{\langle n\cdot(\lambda-\lambda'')\rangle}B) = J^{\langle\lambda''\rangle}B^c$$

where the last identity follows from lemma 9.3.69(iv). But we know already that $J^{\langle\lambda''\rangle}B^c \subset J^{\langle\lambda'\rangle}B$, whence the claim. ◊

To conclude, it suffices now to apply claim 16.3.20 with $B := A_I^\wedge$ and invoke lemma 9.3.69(iv) (details left to the reader). □

Theorem 16.3.21. *In the situation of* (16.3.10)*, let $\mathscr{T}_p$ be the p-adic topology of A, and suppose:*

(a) $(A, \mathscr{T}_p)$ *is perfectoid.*
(b) $p^N \in I \cap J$ *for every sufficiently large* $N \in \mathbb{N}$.
(c) I *and* J *are finitely generated ideals of* A.

Then we have :

(i) (16.3.12) *is a cartesian diagram of rings and a cocartesian diagram of A-modules.*
(ii) (16.3.14) *is an isomorphism of topological rings.*

Proof. (i): Let $\pi \in A$ be as in lemma 16.2.3(iv); quite generally, if $K \subset A$ is any finitely generated ideal containing p^N (for some $N \in \mathbb{N}$), then it is easily seen that the K-adic topology on A agrees with the $(K + \pi A)$-adic topology. In view of our assumptions (b) and (c) we may therefore replace I and J by $I + \pi A$ and respectively $J + \pi A$, and assume from start that $\pi \in I \cap J$. Fix a finite system $(\bar{b}_1, \dots, \bar{b}_n)$ (resp. $(\bar{b}_{n+1}, \dots, \bar{b}_m)$) of generators of the ideal I/pA (resp. of J/pA) of A/pA, and let also $\underline{\alpha} := (\alpha_n \mid n \in \mathbb{N})$ be a distinguished element of $\mathbf{A} := \mathbf{A}(A, \mathscr{T}_p)$ that generates $\operatorname{Ker} u_A$. Pick elements $\beta_1, \dots, \beta_m \in \mathbf{E} := \mathbf{E}(A)$ such that $\overline{u}_{A/pA}(\beta_i) = \bar{b}_i$ for $i = 1, \dots, m$, denote by $\mathscr{I}_\mathbf{E}$ (resp. $\mathscr{J}_\mathbf{E}$) the ideal of $\mathbf{E}$ generated by the system $(\alpha_0, \beta_1, \dots, \beta_n)$ (resp. $(\alpha_0, \beta_{n+1}, \dots, \beta_m)$), and let $(\mathbf{E}^\wedge_\mathscr{I}, \mathscr{T}^\wedge_\mathscr{I})$ (resp. $(\mathbf{E}^\wedge_\mathscr{J}, \mathscr{T}^\wedge_\mathscr{J})$, resp. $(\mathbf{E}^\wedge_{\mathscr{I}\cap\mathscr{J}}, \mathscr{T}^\wedge_{\mathscr{I}\cap\mathscr{J}})$, resp. $(\mathbf{E}^\wedge_{\mathscr{I}+\mathscr{J}}, \mathscr{T}^\wedge_{\mathscr{I}+\mathscr{J}})$) be the $\mathscr{I}_\mathbf{E}$-adic (resp. $\mathscr{J}_\mathbf{E}$-adic, resp. $(\mathscr{I} \cap \mathscr{J})$-adic, resp. $(\mathscr{I} + \mathscr{J})$-adic) completion of $\mathbf{E}$. By lemma 16.3.15(ii), we get a cartesian diagram of topological rings

$$\mathscr{E} \quad : \quad \begin{array}{ccc} (\mathbf{E}^\wedge_{\mathscr{I}\cap\mathscr{J}}, \mathscr{T}^\wedge_{\mathscr{I}\cap\mathscr{J}}) & \longrightarrow & (\mathbf{E}^\wedge_\mathscr{I}, \mathscr{T}^\wedge_\mathscr{I}) \\ \downarrow & & \downarrow \\ (\mathbf{E}^\wedge_\mathscr{J}, \mathscr{T}^\wedge_\mathscr{J}) & \longrightarrow & (\mathbf{E}^\wedge_{\mathscr{I}+\mathscr{J}}, \mathscr{T}^\wedge_{\mathscr{I}+\mathscr{J}}) \end{array}$$

which is also cocartesian as a diagram of topological abelian groups.

Claim 16.3.22. $W(\mathscr{E})$ is still a cartesian diagram of topological rings, and a cocartesian diagram of $\mathbf{A}$-modules.

Proof of the claim. The cartesian property follows from remark 9.3.28(ii). To prove the cocartesian property, it suffices to show that the map of abelian groups deduced from $W(\mathscr{E})$

$$\tau : W(\mathbf{E}^\wedge_\mathscr{I}) \oplus W(\mathbf{E}^\wedge_\mathscr{J}) \to W(\mathbf{E}^\wedge_{\mathscr{I}+\mathscr{J}})$$

is surjective. Endow both target and source of τ with their p-adic filtrations, so that τ becomes a map of filtered abelian groups, and for every $k \in \mathbb{N}$, denote by $\operatorname{gr}^k\tau$ the map induced by τ on the respective k-graded subquotients. Since both target and source of τ are p-adically complete and separated (proposition 9.3.44(iii)), it then suffices to check that $\operatorname{gr}^k\tau$ is surjective for every $k \in \mathbb{N}$ ([34, Ch.III, §2, n.8, Cor.2]). Since multiplication by p is an injective endomorphism for both target and source of τ, we are then further reduced to the case where $k = 0$, in which case

$\mathrm{gr}^0\tau$ is naturally identified with the map $\mathbf{E}^\wedge_{\mathscr{I}} \oplus \mathbf{E}^\wedge_{\mathscr{J}} \to \mathbf{E}^\wedge_{\mathscr{I}+\mathscr{J}}$ deduced from $\mathscr{E}$. But the latter map is indeed surjective, since $\mathscr{E}$ is cocartesian. $\Diamond$

Now, remark 16.3.7(iii) says that the topology of $\mathbf{E}(A, \mathscr{T}_I)$ agrees with the $\mathscr{I}_\mathbf{E}$-adic topology, so the completion $\mathbf{E}^\wedge_I$ of $\mathbf{E}(A, \mathscr{T}_I)$ is isomorphic to $(\mathbf{E}^\wedge_{\mathscr{I}}, \mathscr{T}^\wedge_{\mathscr{I}})$. Likewise, the completion $\mathbf{E}^\wedge_J$ of $\mathbf{E}(A, \mathscr{T}_J)$ is isomorphic to $(\mathbf{E}^\wedge_{\mathscr{J}}, \mathscr{T}^\wedge_{\mathscr{J}})$, and the completion $\mathbf{E}^\wedge_{I+J}$ of $\mathbf{E}(A, \mathscr{T}_{I+J})$ is isomorphic to $(\mathbf{E}^\wedge_{\mathscr{I}+\mathscr{J}}, \mathscr{T}^\wedge_{\mathscr{I}+\mathscr{J}})$.

Next, set $\mathscr{K} := \mathscr{I}\mathscr{J} + \alpha_0\mathbf{E} \subset \mathbf{E}$; since $\mathscr{I}$ and $\mathscr{J}$ are finitely generated and they both contain α_0, it is easily seen that the $(\mathscr{I}\cap\mathscr{J})$-adic topology on $\mathbf{E}$ agrees with the $\mathscr{K}$-adic topology. Likewise, topology $\mathscr{T}_{I\cap J}$ on A agrees with the (IJ)-adic topology. Moreover, $\mathscr{K}$ is a finitely generated ideal, by construction $p \in IJ$, and $\overline{u}_{A/pA}(\mathscr{K}) = (IJ)/pA$, hence remark 16.3.7(iii) also tells us that $(\mathbf{E}^\wedge_{\mathscr{I}\cap\mathscr{J}}, \mathscr{T}^\wedge_{\mathscr{I}\cap\mathscr{J}})$ is isomorphic to the completion $\mathbf{E}^\wedge_{I\cap J}$ of $\mathbf{E}(A, \mathscr{T}_{I\cap J})$.

Notice now that, since the image of A is dense in $A^\wedge_{I\cap J}$, the map φ_I in (16.3.12) is the unique continuous homomorphism of topological A-algebras from $A^\wedge_{I\cap J}$ to $A^\wedge_I$, and the other maps in (16.3.12) enjoy corresponding uniqueness properties. Combining with remark 16.3.7(iv), we conclude that $W(\mathscr{E}) \otimes_\mathbf{A} A$ is naturally identified with (16.3.12), and the image of $\underline{\alpha}$ is still distinguished in each of the Witt rings appearing in $W(\mathscr{E})$. Due to of claim 16.3.22, we already see that (16.3.12) is a cocartesian diagram of A-modules; moreover, it will follow that it is a cartesian diagram of rings, once we know that

$$\mathrm{Tor}_1^\mathbf{A}(W(\mathbf{E}^\wedge_{\mathscr{I}+\mathscr{J}}), A) = 0. \qquad (16.3.23)$$

However $A = \mathbf{A}/\underline{\alpha}\mathbf{A}$, and $\underline{\alpha}$ is a regular element of $W(\mathbf{E}^\wedge_{\mathscr{I}+\mathscr{J}})$ (remark 16.1.7(ii)), so (16.3.23) holds by a standard calculation.

(ii): We know already that (16.3.14) is open and surjective (lemma 16.3.15(i)), so it suffices to show that this map is a ring isomorphism. However, in light of remark 16.3.7(iv) we have a diagram of continuous maps

$$\mathscr{D} \quad : \quad \begin{array}{ccccc} (A^\wedge_I)^\wedge_J & \longrightarrow & A(\mathbf{E}^\wedge_{\mathscr{I}}, \alpha_0\mathbf{E}^\wedge_{\mathscr{I}})^\wedge_J & \longrightarrow & A((\mathbf{E}^\wedge_{\mathscr{I}})^\wedge_{\mathscr{J}}, \alpha_0(\mathbf{E}^\wedge_{\mathscr{I}})^\wedge_{\mathscr{J}}) \\ \downarrow & & & & \downarrow \\ A^\wedge_{I+J} & & \longrightarrow & & A(\mathbf{E}^\wedge_{\mathscr{I}+\mathscr{J}}, \alpha_0\mathbf{E}^\wedge_{\mathscr{I}+\mathscr{J}}) \end{array}$$

whose left vertical arrow equals (16.3.14), and whose remaining arrows are ring isomorphisms. Thus, it suffices to check that $\mathscr{D}$ commutes, *i.e.* that the two continuous ring homomorphisms $f, g : (A^\wedge_I)^\wedge_J \to A(\mathbf{E}^\wedge_{\mathscr{I}+\mathscr{J}}, \alpha_0\mathbf{E}^\wedge_{\mathscr{I}+\mathscr{J}})$ deduced from $\mathscr{D}$ coincide. However, let $i : A \to (A^\wedge_I)^\wedge_J$ denote the completion map; a simple inspection shows that $f \circ i = g \circ i$, and since the topology of $A(\mathbf{E}^\wedge_{\mathscr{I}+\mathscr{J}}, \alpha_0\mathbf{E}^\wedge_{\mathscr{I}+\mathscr{J}})$ is separated and i has dense image, the assertion follows. $\square$

16.3.24. Let A, $\overline{D}$, $\overline{A}$ be three perfectoid rings, and set $\mathbf{E} := \mathbf{E}(A)$, $\overline{\mathbf{E}} := \mathbf{E}(\overline{D})$. Suppose that $\overline{A}$ is a discrete topological $\mathbb{F}_p$-algebra, so that $\mathbf{E}(\overline{A}) = \overline{A}$ (see also corollary 16.3.63(iii)), let $\overline{\varphi}_A : \overline{D} \to \overline{A}$ and $\pi_A : A \to \overline{A}$ be two continuous ring homomorphisms, and suppose as well that π_A is surjective. Define the topological rings D_A and $D_\mathbf{E}$ as the fibre products in the resulting cartesian diagrams of topological rings

$$\begin{array}{ccc} D_A & \xrightarrow{\varphi_A} & A \\ \downarrow & & \downarrow{\scriptstyle \pi_A} \\ \overline{D} & \xrightarrow{\overline{\varphi}_A} & \overline{A} \end{array} \qquad \begin{array}{ccc} D_\mathbf{E} & \xrightarrow{\varphi_\mathbf{E}} & \mathbf{E} \\ \downarrow & & \downarrow{\scriptstyle \pi_\mathbf{E}} \\ \overline{\mathbf{E}} & \xrightarrow{\overline{\varphi}_\mathbf{E}} & \overline{A}. \end{array}$$

where $\pi_\mathbf{E} := \mathbf{E}(\pi_A)$ and $\overline{\varphi}_\mathbf{E} := \mathbf{E}(\overline{\varphi}_A)$.

Proposition 16.3.25. *In the situation of* (16.3.24)*, the rings D_A and $D_{\mathbf{E}}$ are perfectoid, and there exists a natural isomorphism of topological rings*

$$\omega : \mathbf{E}(D_A) \stackrel{\sim}{\to} D_{\mathbf{E}} \qquad \textit{such that} \qquad \varphi_{\mathbf{E}} \circ \omega = \mathbf{E}(\varphi_A).$$

Proof. To begin with, we notice that D_A (resp. $D_{\mathbf{E}}$) is complete and separated, since $\overline{A}$ is separated and both $\overline{D}$ and A (resp. $\overline{\mathbf{E}}$ and $\mathbf{E}$) are complete and separated. Let $(\alpha_n \mid n \in \mathbb{N})$ (resp. $(\alpha'_n \mid n \in \mathbb{N})$) be a distinguished element in $\mathrm{Ker}\, u_A$ (resp. in $\mathrm{Ker}\, u_{\overline{D}}$), and pick a finitely generated ideal of adic definition $J_{\mathbf{E}} \subset \mathrm{Ker}\, \pi_{\mathbf{E}}$ (resp. $J_{\overline{\mathbf{E}}} \subset \mathrm{Ker}\, \overline{\varphi}_{\mathbf{E}}$) for $\mathbf{E}$ (resp. for $\overline{\mathbf{E}}$).

Claim 16.3.26. We may assume that $\alpha_0^{1/p^2}\mathbf{E} \subset J_{\mathbf{E}}$ and $\alpha_0'^{1/p^2}\overline{\mathbf{E}} \subset J_{\overline{\mathbf{E}}}$.

Proof of the claim. For the first stated inclusion, notice that $\alpha_0^{1/p^n} \in \mathrm{Ker}\, \pi_{\mathbf{E}}$ for every $n \in \mathbb{N}$, so $J_{\mathbf{E}} + \alpha_0^{1/p^n}\mathbf{E}$ is still an ideal of adic definition of $\mathbf{E}$ contained in $\mathrm{Ker}\, \pi_{\mathbf{E}}$, for every such n. The same argument applies to $J_{\overline{\mathbf{E}}}$. ◊

By virtue of claim 16.3.26 we may find a system of generators $\beta_\bullet := (\beta_1, \dots, \beta_k)$ for $J_{\mathbf{E}}$ with $\beta_1 = \alpha_0^{1/p^2}$, and we set $b_i := \overline{u}_A(\beta_i)$ for $i = 1, \dots, k$. It follows that $b_\bullet := (b_1, \dots, b_k)$ is a system of generators for an ideal J_A of adic definition of A. Likewise, pick a system of generators $\beta'_\bullet := (\beta'_1, \dots, \beta'_h)$ for $J_{\overline{\mathbf{E}}}$ with $\beta'_1 = \alpha_0'^{1/p^2}$, and set $b'_i := \overline{u}_{\overline{D}}(\beta'_1)$ for $i = 1, \dots, h$. Then $b'_\bullet := (b'_1, \dots, b'_h)$ is a system of generators for an ideal $J_{\overline{D}}$ of adic definition of $\overline{D}$. Notice that $J_A \times J_{\overline{D}} \subset D_A$ (resp. $J_{\mathbf{E}} \times J_{\overline{\mathbf{E}}} \subset D_{\mathbf{E}}$), and the topology of D_A (resp. of $D_{\mathbf{E}}$) is the linear topology defined by the system of ideals $(J_A^n \times J_{\overline{D}}^n \mid n \in \mathbb{N})$ (resp. $(J_{\mathbf{E}}^n \times J_{\overline{\mathbf{E}}}^n \mid n \in \mathbb{N})$). Denote by $K_{\mathbf{E}} \subset D_{\mathbf{E}}$ the ideal generated by $(\beta_\bullet \times \{0\}, \{0\} \times \beta'_\bullet)$; it is easily seen that

$$J_{\mathbf{E}}^{n+1} \times J_{\overline{\mathbf{E}}}^{n+1} \subset K_{\mathbf{E}}^n \subset J_{\mathbf{E}}^n \times J_{\overline{\mathbf{E}}}^n \qquad \text{for every } n \in \mathbb{N}$$

so the topology of $D_{\mathbf{E}}$ agrees with its $K_{\mathbf{E}}$-adic topology, and therefore $D_{\mathbf{E}}$ is a perfectoid $\mathbb{F}_p$-algebra. Likewise, let $K_A \subset D_A$ be the ideal generated by $(b_\bullet \times \{0\}, \{0\} \times b'_\bullet)$; by the same token, we have

$$J_A^{n+1} \times J_{\overline{D}}^{n+1} \subset K_A^n \subset J_A^n \times J_{\overline{D}}^n \qquad \text{for every } n \in \mathbb{N}$$

so the topology of D_A agrees with the K_A-adic topology. Moreover, by construction we have $pA \subset J_A^{p^2}$ and $p\overline{D} \subset J_{\overline{\mathbf{E}}}^{p^2}$ (lemma 16.2.7(iii)); since $p^2 - 1 \geq 2$, we deduce that $pD_A \subset K_A^2$, and we see already that D_A is a P-ring. Next, by remark 9.3.28(ii), we deduce a short exact sequence of $W(D_A)$-modules

$$\mathscr{W} \quad : \quad 0 \to W(D_A) \to W(A) \oplus W(\overline{D}) \to W(\overline{A}) \to 0.$$

On the other hand, by lemma 16.2.7(ii) we may find a distinguished element $\underline{\alpha}'' := (\alpha''_n \mid n \in \mathbb{N})$ in $\mathrm{Ker}\, u_{D_A}$ and to ease notation we set $D'_A := W(D_A)/\underline{\alpha}''W(D_A)$; since the image of $\underline{\alpha}''$ is still distinguished in $W(\overline{A})$, remark 16.1.7(ii) implies that $\mathrm{Tor}_1^{W(D_A)}(D'_A, W(\overline{A})) = 0$, so the sequence

$$\mathscr{W} \otimes_{W(D_A)} D'_A \quad : \quad 0 \to D'_A \to A \oplus \overline{D} \to \overline{A} \to 0$$

is still exact, and consequently u_{D_A} induces a ring isomorphism $D'_A \to D_A$, *i.e.* D_A is perfectoid. Lastly, notice that $u_A \otimes_{\mathbb{Z}} \mathbb{F}_p$ and $u_{\overline{D}}$ induce isomorphisms of topological $\mathbb{F}_p$-algebras

$$D_A/pD_A \stackrel{\sim}{\to} \overline{D}/p\overline{D} \times_{\overline{A}} A/pA \stackrel{\sim}{\to} \overline{D}/\alpha''_0\overline{D} \times_{\overline{\mathbf{E}}} \mathbf{E}/\alpha''_0\mathbf{E} \stackrel{\sim}{\to} D_{\mathbf{E}}/\alpha''_0 D_{\mathbf{E}}.$$

Combining with theorem 9.4.10 we deduce an isomorphism

$$\mathbf{E}(D_A) \stackrel{\sim}{\to} \mathbf{E}(D_A/pD_A) \stackrel{\sim}{\to} \mathbf{E}(D_{\mathbf{E}}/\alpha''_0 D_{\mathbf{E}}) \stackrel{\sim}{\to} D_{\mathbf{E}}$$

which fulfills the stated condition, by a simple inspection. □

Definition 16.3.27. Let A be a perfectoid ring and $\mathscr{J} \subset \mathscr{K} \subset \mathbf{E} := \mathbf{E}(A)$ any two ideals. Let also $\underline{\alpha} := (\alpha_n \mid n \in \mathbb{N}) \in \mathrm{Ker}\, u_A$ be any distinguished element and $\beta \in \mathbf{E}$ any element.

(i) We say that the inclusion of $\mathscr{J}$ in $\mathscr{K}$ is *β-taut* if we have
$$\beta \cdot \Phi_{\mathbf{E}}^{-1}(\mathscr{K}^p) \subset \mathscr{J}.$$

(ii) We say that $\mathscr{J}$ is *β-taut* if the identity map of $\mathscr{J}$ is a β-taut inclusion.

(iii) We say that $\mathscr{J}$ is *strictly β-taut* if it is β^λ-taut for some $\lambda \in \mathbb{Z}[1/p]$ with $0 \leq \lambda < 1$.

(iv) $\mathscr{J}$ is *taut* (resp. *strictly taut*) if it is α_0-taut (resp. strictly α_0-taut), and the inclusion $\mathscr{J} \subset \mathscr{K}$ is *taut* (resp. *strictly taut*), if it is α_0-taut (resp. strictly α_0-taut).

(v) We denote by $\{\mathscr{J}\} \subset A$ the topological closure of the ideal generated by the system $(\overline{u}_A(x) \mid x \in \mathscr{J})$ (notation of (9.3.34)).

Remark 16.3.28. With the notation of definition 16.3.27, the following holds.

(i) It follows easily from remark 16.1.7(i) that the definition of taut and strictly taut ideals does not depend on the choice of $\underline{\alpha}$.

(ii) By inspecting the definition of angular powers (see (9.3.68)), it is easily seen that $\mathscr{J}$ is 1-taut if and only if $\mathscr{J} = \mathscr{J}^{\langle 1 \rangle}\mathbf{E}$.

(iii) It follows from (ii) and lemma 9.3.69(ii.c) that for every ideal $\mathscr{J}$ of $\mathbf{E}$ and every $\lambda \in \mathbb{N}[1/p]$, the ideal $\mathscr{J}^{\langle\lambda\rangle}\mathbf{E}$ is strictly taut. Moreover, say that $\alpha_0 \in \mathscr{J}^{\langle\varepsilon\rangle}\mathbf{E}$ for some $\varepsilon \in \mathbb{N}[1/p]$; then the inclusion $\mathscr{J}^{\langle\lambda+\varepsilon\rangle}\mathbf{E} \subset \mathscr{J}^{\langle\lambda\rangle}\mathbf{E}$ is taut for every $\lambda \in \mathbb{N}[1/p]$, by lemma 9.3.69(ii.b).

(iv) If $\mathscr{J}_1, \mathscr{J}_2 \subset \mathbf{E}$ are two β-taut (resp. strictly β-taut, resp. taut, resp. strictly taut) ideals, then the same holds for $\mathscr{J}_1 \cap \mathscr{J}_2$. Indeed, suppose that both ideals are β-taut; then we have :
$$\begin{aligned}\beta \cdot \Phi_{\mathbf{E}}^{-1}((\mathscr{J}_1 \cap \mathscr{J}_2)^p) &\subset \beta \cdot \Phi_{\mathbf{E}}^{-1}(\mathscr{J}_1^p \cap \mathscr{J}_2^p) \\ &= \beta \cdot (\Phi_{\mathbf{E}}^{-1}(\mathscr{J}_1^p) \cap \Phi_{\mathbf{E}}^{-1}(\mathscr{J}_2^p)) \\ &\subset (\beta \cdot \Phi_{\mathbf{E}}^{-1}(\mathscr{J}_1^p)) \cap (\beta \cdot \Phi_{\mathbf{E}}^{-1}(\mathscr{J}_2^p)) \\ &\subset \mathscr{J}_1 \cap \mathscr{J}_2\end{aligned}$$
whence the claim. In the same vein, if $\mathscr{J}_1 \subset \mathscr{J}_2$ is a β-taut inclusion, and $\mathscr{K}$ is any β-taut ideal of $\mathbf{E}$, then the inclusion $\mathscr{K} \cap \mathscr{J}_1 \subset \mathscr{K} \cap \mathscr{J}_2$ is β-taut as well (details left to the reader); the same holds if β-taut is replaced by strictly β-taut, taut or strictly taut.

(v) If $\mathscr{J}$ is β-taut, then the same holds for the topological closure $\mathscr{J}^c$ of $\mathscr{J}$ in $\mathbf{E}$. Likewise, if $\mathscr{J} \subset \mathscr{K}$ is a β-taut inclusion, then the same holds for the inclusion $\mathscr{J}^c \subset \mathscr{K}^c$ of the respective topological closures; moreover, both $\mathscr{J}$ and $\mathscr{K}$ are β-taut. Then clearly the same assertions hold with β-taut replaced by strictly β-taut, taut, or strictly taut (details left to the reader). In the same vein, notice that
$$\{\mathscr{J}^c\} = \{\mathscr{J}\} \qquad \text{for every ideal } \mathscr{J} \subset \mathbf{E}.$$

(vi) Let $\mathscr{J}_1, \mathscr{J}_2, \mathscr{J}_3 \subset \mathbf{E}$ be three ideals, $n, m \in \mathbb{N}$ any two integers, and suppose that
$$\beta_1 \cdot \Phi_{\mathbf{E}}^{-n}(\mathscr{J}_1^{p^n}) \subset \mathscr{J}_2 \quad \text{and} \quad \beta_2 \cdot \Phi_{\mathbf{E}}^{-m}(\mathscr{J}_2^{p^m}) \subset \mathscr{J}_3 \qquad \text{for some } \beta_1, \beta_2 \in \mathbf{E}.$$
Then it is easily seen that $\beta_1\beta_2 \cdot \Phi_{\mathbf{E}}^{-n-m}(\mathscr{J}_1^{p^{n+m}}) \subset \mathscr{J}_3$. Especially, if $\mathscr{J}$ is β-taut, we have
$$\beta^n \cdot \Phi_{\mathbf{E}}^{-n}(\mathscr{J}^{p^n}) \subset \mathscr{J} \qquad \text{for every } n \in \mathbb{N}. \tag{16.3.29}$$

(vii) For $i = 1, 2$, let $\beta_i \in \mathbf{E}$ be any element, and $\mathscr{J}_i$ a β_i-taut ideal; then it is easily seen that $\mathscr{J}_1\mathscr{J}_2$ is $\beta_1\beta_2$-taut. More generally, if $\mathscr{J} \subset \mathscr{K}$ is a β_1-taut inclusion of ideals of $\mathbf{E}$, and $\mathscr{J}' \subset \mathbf{E}$ is any other β_2-taut ideal, then the inclusion $\mathscr{J}\mathscr{J}' \subset \mathscr{K}\mathscr{J}'$ is $\beta_1\beta_2$-taut.

(viii) Suppose that $\mathscr{J}$ is an open ideal of $\mathbf{E}$. Then the system $(\overline{u}_A(x) \mid x \in \mathscr{J})$ generates an open ideal J of A, and therefore $\{\mathscr{J}\} = J$ is open as well in A. Indeed, let $I \subset A$ be any ideal of definition, pick a finite system $(\overline{a}_1, \dots, \overline{a}_k)$ of generators of I/pA, let $\alpha_1, \dots, \alpha_k$ be elements of $\mathbf{E}$ such that $\overline{u}_{A/pA}(\alpha_i) = \overline{a}_i$ for $i = 1, \dots, k$, and denote by $\mathscr{I} \subset \mathbf{E}$ the ideal generated by the system $(\alpha_1, \dots, \alpha_k)$. Arguing as in the proof of lemma 16.2.7(i), we see that the system

$(\overline{u}_A(\alpha_i) \mid i = 1, \dots, k)$ generates I, and $\mathscr{I}^n \subset \mathscr{J}$ for every sufficiently large $n \in \mathbb{N}$, so that $I^n \subset J$ for every such n, whence the assertion.

16.3.30. Keep the notation of definition 16.3.27, and let $\mathscr{J} \subset \mathscr{K}$ be a taut inclusion of ideals of $\mathbf{E}$. Notice that $\alpha_0 \mathscr{K} \subset \mathscr{J}$, so $\mathscr{K}/\mathscr{J}$ is an $\mathbf{E}/\alpha_0\mathbf{E}$-module, and then it can be viewed as an A/pA-module, via the isomorphism ω of remark 16.3.7(ii). Likewise, from lemma 16.2.7(iii) we easily deduce that $p\{\mathscr{K}\} \subset \{\mathscr{J}\}$, so $\{\mathscr{K}\}/\{\mathscr{J}\}$ is an A/pA-module as well.

Theorem 16.3.31. *Let A be any perfectoid ring, and set $\mathbf{E} := \mathbf{E}(A)$. The following holds :*

(i) *Every taut inclusion $\mathscr{J}_1 \subset \mathscr{J}_2$ of ideals of $\mathbf{E}$ induces an A/pA-linear map*

$$\overline{\tau} : \mathscr{J}_2/\mathscr{J}_1 \to \{\mathscr{J}_2\}/\{\mathscr{J}_1\} \quad : \quad (x \mod \mathscr{J}_1) \mapsto (\overline{u}_A(x) \mod \{\mathscr{J}_1\}).$$

(ii) *If both $\mathscr{J}_1$ and $\mathscr{J}_2$ are closed in the topology of $\mathbf{E}$, the map $\overline{\tau}$ of* (i) *is an isomorphism.*

Proof. Let $\underline{\alpha} := (\alpha_n \mid n \in \mathbb{N}) \in \operatorname{Ker} u_A$ be any distinguished element.

(i): Let $x, y \in \mathscr{J}_2$ be any two elements, and set $\delta := \tau_A(x + y) - \tau_A(x) - \tau_A(y)$. In light of lemma 16.1.1(iv), we have to check that $u_A(\delta)$ lies in $\{\mathscr{J}_1\}$. However, proposition 9.3.62 expresses δ as a series of the form $\sum_{n\in\mathbb{N}} p^n b_n$, where each b_n is in turn a finite sum of terms of the form $c_{n,\sigma}\tau_A(\beta_{n,\sigma})$ (for σ ranging over a certain finite set Σ_n), and with $\beta_{n,\sigma} \in \Phi_{\mathbf{E}}^{-n}(\mathscr{J}_2^{p^n})$ and $c_{n,\sigma} \in \mathbb{Z}_p$ for every $n \in \mathbb{N}$ and every $\sigma \in \Sigma_n$. Thus, we come down to checking that $p^n \cdot \overline{u}_A(\beta_{n,\sigma}) \in \{\mathscr{J}_1\}$ for every $n \in \mathbb{N}$ and $\sigma \in \Sigma_n$. In light of lemma 16.2.7(iii), this holds if and only if $\overline{u}_A(\alpha_0^n \cdot \beta_{n,\sigma}) \in \{\mathscr{J}_1\}$, which in turns will follow, once we know that $\alpha_0^n \cdot \beta_{n,\sigma} \in \mathscr{J}_1$. But the latter is clear from remark 16.3.28(vi) (with $\beta := \alpha_0$: details left to the reader).

Lastly, for any $x \in \mathscr{J}_2$ and $\beta \in \mathbf{E}$ we have $\tau_A(\beta x) = \tau_A(\beta) \cdot \tau_A(x)$, and the isomorphism ω of remark 16.3.7(ii) maps the class of β in $\mathbf{E}/\alpha_0\mathbf{E}$ to the class of $\overline{u}_A(\beta)$ in A/pA; this shows that $\overline{\tau}(\overline{a} \cdot x) = \overline{a} \cdot \overline{\tau}(x)$ for every $\overline{a} \in A/pA$ and every $x \in \mathscr{J}_2/\mathscr{J}_1$, whence (i).

Next, choose a strictly positive $\varepsilon \in \mathbb{N}[1/p]$ such that $\alpha_0 \in \mathscr{J}^{\langle\varepsilon\rangle}\mathbf{E}$, and for every $r \in \mathbb{R}_+$ define the ideal $\mathscr{J}^{\lfloor r\rfloor}\mathbf{E}$ as in remark 9.3.70(i); we point out the following special case of (ii) :

Claim 16.3.32. Let $\mathscr{J}$ be any ideal of definition of $\mathbf{E}$, and $r, r' \in \mathbb{R}_+$ two real numbers such that $\varepsilon \geq r' - r \geq 0$. Then (ii) holds for the taut inclusion $\mathscr{J}^{\lfloor r'\rfloor}\mathbf{E} \subset \mathscr{J}^{\lfloor r\rfloor}\mathbf{E}$.

Proof of the claim. Let us set $W\lfloor s\rfloor := W(\mathscr{J}^{\lfloor s\rfloor}\mathbf{E})$ for every $s \in \mathbb{N}[1/p]$ (notation of remark 9.3.28(iv)); from (9.3.73) we see that τ_A induces a natural isomorphism

$$\mathscr{J}^{\lfloor r\rfloor}\mathbf{E}/\mathscr{J}^{\lfloor r'\rfloor}\mathbf{E} \xrightarrow{\sim} W\lfloor r\rfloor/(pW\lfloor r\rfloor + W\lfloor r'\rfloor)$$

and we are reduced to showing that the restriction of u_A to $W\lfloor r\rfloor$ induces an isomorphism

$$W\lfloor r\rfloor/(pW\lfloor r\rfloor + W\lfloor r'\rfloor) \xrightarrow{\sim} \{\mathscr{J}^{\lfloor r\rfloor}\mathbf{E}\}/\{\mathscr{J}^{\lfloor r'\rfloor}\mathbf{E}\}. \qquad (16.3.33)$$

However, remark 16.3.28(viii) already implies that (16.3.33) is surjective, and its kernel is the image of $W\lfloor r\rfloor \cap \underline{\alpha}W(E)$. In light of proposition 16.1.12(ii) we then come down to showing :

$$\underline{\alpha}W\lfloor r\rfloor \subset pW\lfloor r\rfloor + W\lfloor r'\rfloor.$$

But due to our choice of ε we get $\alpha_0\mathscr{J}^{\lfloor r\rfloor}\mathbf{E} \subset \mathscr{J}^{\lfloor r'\rfloor}\mathbf{E}$, so the latter inclusion follows again from (9.3.73). ◊

Claim 16.3.34. Let B be any ring, $f : M \to M'$ and $g : M' \to M''$ two continuous maps of topological B-modules, and suppose that

(a) the topologies of M and M'' are discrete and that of M' is separated
(b) f has dense image and $g \circ f$ is injective.

Then f is an isomorphism of topological B-modules.

Proof of the claim. By assumption, the topological closure of $f(M)$ in M' equals M'; by claim 8.3.25 it follows that the topological closure of $g \circ f(M)$ in M'' equals the topological closure of $g(M')$. But since the topology of M'' is discrete, this just means that $g \circ f(M) = g(M')$. We may therefore replace M'' by $g(M')$, and assume from start that g is surjective and $g \circ f$ is an isomorphism. Then we may even assume that $M'' = M$ and $g \circ f = \mathbf{1}_M$. Let us endow $\mathrm{Im}\, f$ and $\mathrm{Ker}\, g$ with the topologies induced from the inclusion into M'; then the addition law of M' restricts to a continuous and bijective B-linear map

$$h : \mathrm{Im}\, f \oplus \mathrm{Ker}\, g \to M'$$

(where the direct sum is endowed with the product topology). However, the inverse of h is the map given by the rule : $m' \mapsto (f \circ g(m'), m' - f \circ g(m'))$ for every $m' \in M'$. Clearly, this map is also continuous, so h is an isomorphism of topological B-modules. Now, since M' is separated, the same must hold for $\mathrm{Ker}\, g$; especially, $L := h(\mathrm{Im}\, f \oplus 0)$ is a closed subset of M'. On the other hand, $\mathrm{Im}\, f = L$; since f has dense image, we must then have $L = M'$, *i.e.* $\mathrm{Ker}\, g = 0$, whence the claim. ♢

We may now complete the proof of (ii): fix an ideal of definition $\mathscr{J}$ of $\mathbf{E}$ with $\alpha_0 \in \mathscr{J}$, and for every ideal $\mathscr{K} \subset \mathbf{E}$ set

$$\mathrm{Fil}^n \mathscr{K} := \mathscr{K} \cap \mathscr{J}^{\lfloor n \rfloor}\mathbf{E} \qquad \mathrm{Fil}^n\{\mathscr{K}\} := \{\mathrm{Fil}^n \mathscr{K}\} \qquad \text{for every } n \in \mathbb{N}.$$

If $\mathscr{K}' \subset \mathscr{K}$ is any inclusion of ideals of $\mathbf{E}$, the image of $\mathrm{Fil}^\bullet \mathscr{K}$ in $\mathscr{K}/\mathscr{K}'$ defines a filtration $\mathrm{Fil}^\bullet(\mathscr{K}/\mathscr{K}')$ of $\mathscr{K}/\mathscr{K}'$, and likewise we get a filtration $\mathrm{Fil}^\bullet(\{\mathscr{K}\}/\{\mathscr{K}'\})$ on $\{\mathscr{K}\}/\{\mathscr{K}'\}$. Notice that the inclusion $\mathrm{Fil}^{n+1}\mathscr{K} \subset \mathrm{Fil}^n \mathscr{K}$ is taut for every $n \in \mathbb{N}$ and every taut ideal $\mathscr{K}$, and denote by $\mathrm{gr}^\bullet \mathscr{K}$ (resp. $\mathrm{gr}^\bullet\{\mathscr{K}\}$) the graded A/pA-module associated to the filtration $\mathrm{Fil}^\bullet \mathscr{K}$ (resp. to $\mathrm{Fil}^\bullet\{\mathscr{K}\}$). Likewise we define $\mathrm{gr}^\bullet(\mathscr{K}/\mathscr{K}')$ and $\mathrm{gr}^\bullet(\{\mathscr{K}\}/\{\mathscr{K}'\})$ for an inclusion of ideals $\mathscr{K}' \subset \mathscr{K}$. We remark :

Claim 16.3.35. If $\mathscr{K}$ is taut, the map $\overline{\tau}$ of (i) induces an isomorphism

$$\mathrm{gr}^n \mathscr{K} \xrightarrow{\sim} \mathrm{gr}^n\{\mathscr{K}\} \qquad \text{for every } n \in \mathbb{N}.$$

Proof of the claim. Let us endow any ideal of $\mathbf{E}$ (resp. of A) with the topology induced from the inclusion into $\mathbf{E}$ (resp. into A), and for any inclusion $\mathscr{I} \subset \mathscr{I}'$ (resp. $I \subset I'$) of ideals of $\mathbf{E}$ (resp. of A), let us endow $\mathscr{I}'/\mathscr{I}$ (resp. I'/I) with the corresponding quotient topology. Notice that the inclusions $\mathrm{Fil}^{n+1}\mathscr{K} \subset \mathrm{Fil}^n \mathscr{K}$ and $\mathrm{Fil}^{n+1}\{\mathscr{K}\} \subset \mathrm{Fil}^n\{\mathscr{K}\}$ are both taut (remark 16.3.28(iv)); from (i) we deduce a commutative diagram of A/pA-modules :

$$\begin{array}{ccc} \mathrm{gr}^n \mathscr{K} & \longrightarrow & \mathrm{gr}^n\{\mathscr{K}\} \\ \big\downarrow & & \big\downarrow \\ \mathrm{gr}^n \mathbf{E} & \longrightarrow & \mathrm{gr}^n A \end{array}$$

whose left vertical arrow is injective, and whose bottom horizontal arrow is already known to be an isomorphism, by claim 16.3.32. Moreover, all these maps are continuous, for the topologies that we have just defined on these modules, and furthermore, the top horizontal arrow has dense image. Also, since $\mathscr{J}$ is open, the same holds for $\mathscr{J}^{\lfloor r' \rfloor}\mathbf{E}$, and therefore the two modules on the bottom row have both the discrete topology. Likewise, $\mathscr{K} \cap \mathscr{J}^{\lfloor r' \rfloor}\mathbf{E}$ is open in $\mathscr{K} \cap \mathscr{J}^{\lfloor r' \rfloor}\mathbf{E}$, so also the source of the map on the top row is a discrete A/pA-module. Lastly, since $\{\mathscr{K} \cap \mathscr{J}^{\lfloor r' \rfloor}\mathbf{E}\}$ is a closed ideal, the target of the same map is a separated A/pA-module. Thus, all the conditions of claim 16.3.34 are fulfilled, and it follows that the top horizontal arrow is an isomorphism. ♢

There follows, for every $n \in \mathbb{N}$ a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{gr}^n \mathscr{J}_1 & \longrightarrow & \mathrm{gr}^n \mathscr{J}_2 & \longrightarrow & \mathrm{gr}^n(\mathscr{J}_2/\mathscr{J}_1) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{gr}^n\{\mathscr{J}_1\} & \longrightarrow & \mathrm{gr}^n\{\mathscr{J}_2\} & \longrightarrow & \mathrm{gr}^n(\{\mathscr{J}_2\}/\{\mathscr{J}_1\}) & \longrightarrow & 0 \end{array}$$

and claim 16.3.35 shows that the leftmost and central vertical arrows are both isomorphisms, so the same holds for the rightmost vertical arrow. Now, notice that the filtration $\mathrm{Fil}^\bullet(\mathscr{J}_2/\mathscr{J}_1)$ defines a separated and complete topology on $\mathscr{J}_2/\mathscr{J}_1$, since $\mathscr{J}_2$ and $\mathscr{J}_1$ are both closed ideals in $\mathbf{E}$. The same holds for the topology on $\{\mathscr{J}_2\}/\{\mathscr{J}_1\}$ determined by the filtration $\mathrm{Fil}^\bullet(\{\mathscr{J}_2\}/\{\mathscr{J}_1\})$, since $\{\mathscr{J}_2\}$ and $\{\mathscr{J}_1\}$ are closed ideals in A. Then (ii) follows directly from [34, Ch.III, §2, n.8, Cor.3]. □

Theorem 16.3.36. *Let A be any perfectoid ring, $\mathscr{K}$ a taut ideal of $\mathbf{E} := \mathbf{E}(A)$, and $W(\mathscr{K})^c$ the topological closure of $W(\mathscr{K})$ in $\mathbf{A}(A)$ (notation of remark* 9.3.28(iv)*). We have :*

(i) $\{\mathscr{K}\} = u_A(W(\mathscr{K})^c)$.

(ii) *$\mathscr{K}$ is an open ideal if and only if the same holds for $\{\mathscr{K}\}$.*

(iii) *If $\mathscr{K}' \subset \mathscr{K}$ is another taut ideal, and both $\mathscr{K}$ and $\mathscr{K}'$ are closed, then $\mathscr{K} = \mathscr{K}'$ if and only if $\{\mathscr{K}\} = \{\mathscr{K}'\}$.*

(iv) *Suppose that $\mathscr{K}$ is 1-taut and closed, and let $\underline{\alpha}$ be any distinguished element of* $\mathrm{Ker}\, u_A$. *Then the pair $(\underline{\alpha}, \mathscr{K})$ is transversal (see* (16.1.11)*).*

Proof. Pick any distinguished element $\underline{\alpha} := (\alpha_n \mid n \in \mathbb{N})$ of $\mathrm{Ker}\, u_A$.

(i): First we prove that $u_A(W(\mathscr{K})^c) \subset \{\mathscr{K}\}$. Indeed, let $\underline{a} := (a_n \mid n \in \mathbb{N}) \in W(\mathscr{K})$ be any element; taking into account (9.3.46), we are easily reduced to showing that

$$u_A(p^n \cdot \tau_A(a_n^{p^{-n}})) \in \{\mathscr{K}\} \qquad \text{for every } n \in \mathbb{N}.$$

In light of lemmata 16.2.7(iii) and 16.1.1(iv), it then suffices to check that $\overline{u}_A(\alpha_0^n \cdot a_n^{p^{-n}}) \in \{\mathscr{K}\}$, and the latter is clear from (16.3.29) (applied with $\beta := \alpha_0$). Next, fix an ideal of definition $\mathscr{J}$ of $\mathbf{E}$ containing α_0, and define the filtrations $\mathrm{Fil}^\bullet\mathscr{K}$ on $\mathscr{K}$ and $\mathrm{Fil}^\bullet\{\mathscr{K}\}$ on $\{\mathscr{K}\}$ as in the proof of theorem 16.3.31(ii); by (9.3.29) and remark 16.3.28(v) we may assume that $\mathscr{K}$ is a closed ideal of $\mathbf{E}$, in which case theorem 16.3.31(ii) shows that the map $\overline{u}_A$ induces an isomorphism $\mathrm{gr}^\bullet\mathscr{K} \xrightarrow{\sim} \mathrm{gr}^\bullet\{\mathscr{K}\}$ on the respective associated graded A/pA-modules. Now, let $x \in \{\mathscr{K}\}$ be any element; it follows easily that we may write

$$x = u_A\Big(\sum_{n\in\mathbb{N}} \tau_A(\beta_n)\Big) \qquad \text{where } \beta_n \in \mathrm{Fil}^n\mathscr{K} \text{ for every } n \in \mathbb{N}$$

and the series converges in the topology of $\mathbf{A}(A)$. But clearly $\tau_A(\beta_n) \in W(\mathscr{K})$ for every $n \in \mathbb{N}$, whence the sought converse inclusion.

(ii): In light or remark 16.3.28(viii) we may assume that $\{\mathscr{K}\}$ is open, and we need to show that the same holds for $\mathscr{K}$; then, by lemma 8.3.21(ii.b), we may also assume that $\mathscr{K}$ is a closed ideal. The inclusion of $\mathscr{K}$ into $\mathbf{E}$ is a map of filtered $\mathbf{E}$-modules $\mathrm{Fil}^\bullet\mathscr{K} \to \mathrm{Fil}^\bullet\mathbf{E}$, and likewise we get a map of filtered A-modules $\mathrm{Fil}^\bullet\{\mathscr{K}\} \to \mathrm{Fil}^\bullet A$. The assumption on $\{\mathscr{K}\}$ implies that there exists $n \in \mathbb{N}$ such that the map of associated graded modules $\mathrm{gr}^k\{\mathscr{K}\} \to \mathrm{gr}^k A$ is an isomorphism for every $k \geq n$. Then theorem 16.3.31(ii) implies that the same holds for the corresponding map $\mathrm{gr}^k\mathscr{K} \to \mathrm{gr}^k\mathbf{E}$. However, both $\mathscr{K}$ and $\mathbf{E}$ are complete and separated for their filtrations, so $\mathrm{Fil}^n\mathscr{K} = \mathrm{Fil}^n\mathbf{E} = \mathscr{J}^{\langle n\rangle}$ ([34, Ch.III, §2, n.8, Cor.3]), *i.e.* $\mathscr{K}$ is open.

(iii) is similar : we may assume that $\{\mathscr{K}\} = \{\mathscr{K}'\}$, and we consider the induced map of filtered $\mathbf{E}$-modules $\mathrm{Fil}^\bullet\mathscr{K}' \to \mathrm{Fil}^\bullet\mathscr{K}$. Arguing as in the proof of (ii), we see that the latter induces an isomorphism on the associated graded $\mathbf{E}$-modules, so $\mathscr{K} = \mathscr{K}'$, again by [34, Ch.III, §2, n.8, Cor.3].

(iv): Denote by $(\alpha_0\mathscr{K})^c$ the topological closure of $\underline{\alpha}\mathscr{K}$ in $\mathbf{E}$. We shall show first that

Claim 16.3.37. $W(\mathscr{K}) \cap \mathrm{Ker}\, u_A = W((\alpha_0\mathscr{K})^c) \cap \mathrm{Ker}\, u_A + \underline{\alpha}W(\mathscr{K})$.

Proof of the claim. Indeed, let $\underline{\omega} := (\omega_n \mid n \in \mathbb{N}) \in W(\mathscr{K}) \cap \mathrm{Ker}\, u_A$ be any element; due to remarks 9.3.70(iv) and 16.3.28(ii), we may write

$$\underline{\omega} = \tau_A(\omega_0) + p \cdot \underline{\omega}' \qquad \text{for some } \underline{\omega}' \in W(\mathscr{K})$$

whence $0 = u_A(\underline{\omega}) = \overline{u}_A(\omega_0) + p \cdot u_A(\underline{\omega}')$, and especially, the image of $\overline{u}_A(\omega_0)$ vanishes in $\{\mathscr{K}\}/\{(\alpha_0\mathscr{K})^c\}$ (lemma 16.2.7(iii)). In view of theorem 16.3.31(ii), we deduce that $\omega_0 \in (\alpha_0\mathscr{K})^c$. Write also $\underline{\alpha} = \tau_A(\alpha_0) + p\underline{u}$ for some $\underline{u} \in \mathbf{A}(A)^\times$; we obtain

$$\underline{\omega} = \tau_A(\omega_0) + (\underline{u}^{-1}\underline{\alpha} - \tau_A(\alpha_0)) \cdot \underline{\omega}')$$

and it suffices to notice that $\underline{u}^{-1}\underline{\alpha} \cdot \underline{\omega}' \in \underline{\alpha}W(\mathscr{K}) \subset W(\mathscr{K}) \cap \mathrm{Ker}\, u_A$, and consequently $\tau_A(\omega_0) - \tau_A(\alpha_0) \cdot \underline{\omega}' \in W((\alpha_0\mathscr{K})^c) \cap \mathrm{Ker}\, u_A$. ◊

Notice next that if $\mathscr{K}$ is 1-taut, then the same holds for $\alpha_0\mathscr{K}$, and then also for $(\alpha_0\mathscr{K})^c$ (remark 16.3.28(v)). Thus, we may apply claim 16.3.37 to $(\alpha_0^n\mathscr{K})^c$, and get

$$W((\alpha_0^n\mathscr{K})^c) \cap \mathrm{Ker}\, u_A = W((\alpha_0^{n+1}\mathscr{K})^c) \cap \mathrm{Ker}\, u_A + \underline{\alpha}W((\alpha_0^n\mathscr{K})^c) \qquad \text{for every } n \in \mathbb{N}.$$

Now, let $x_0 \in W(\mathscr{K}) \cap \underline{\alpha}W(\mathbf{E})$ be any element; we may then find inductively identities

$$x_n = \underline{\alpha} \cdot y_n + x_{n+1} \qquad \text{with } x_{n+1} \in W((\alpha^{n+1}\mathscr{K})^c) \cap \mathrm{Ker}\, u_A \text{ and } y_n \in W((\alpha_0^n\mathscr{K})^c)$$

for every $n \in \mathbb{N}$. Therefore :

$$x_0 = \underline{\alpha} \cdot (y_0 + y_1 + \cdots + y_n) + x_{n+1} \qquad \text{for every } n \in \mathbb{N}.$$

Lastly, since $\mathscr{K}$ is 1-taut, the ideal $W(\mathscr{K})$ is closed in the topology of $W(\mathbf{E})$ (see remark 9.3.28(iv)), and since α_0 is topologically nilpotent, for every open ideal I of $W(\mathbf{E})$ there exists $n \in \mathbb{N}$ such that $W((\alpha_0^n\mathscr{K})^c) \subset I$. Consequently, the series $\sum_{n\in\mathbb{N}} y_n$ converges to an element $y \in W(\mathscr{K})$, and $\lim_{n\to+\infty} x_n = 0$, so finally $x = \underline{\alpha} \cdot y$, which concludes the proof. □

Corollary 16.3.38. *Let A be any perfectoid ring, $\mathscr{K} \subset \mathbf{E} := \mathbf{E}(A)$ an ideal such that $\Phi_\mathbf{E}(\mathscr{K}) = \mathscr{K}$, and endow $A/\{\mathscr{K}\}$ (resp. $\mathbf{E}/\mathscr{K}^c$) with the quotient topology induced via the projection $\pi_A : A \to A/\{\mathscr{K}\}$ (resp. $\pi_\mathbf{E} : \mathbf{E} \to \mathbf{E}/\mathscr{K}^c$). Then $A/\{\mathscr{K}\}$ is perfectoid, and there exists a natural isomorphism of topological rings*

$$\omega : \mathbf{E}(A/\{\mathscr{K}\}) \xrightarrow{\sim} \mathbf{E}/\mathscr{K}^c \qquad \textit{such that} \qquad \omega \circ \mathbf{E}(\pi_A) = \pi_\mathbf{E}.$$

Proof. The topological $\mathbb{F}_p$-algebra $\mathbf{E}/\mathscr{K}^c$ is complete and separated, and clearly $\Phi_\mathbf{E}(\mathscr{K}^c) = \mathscr{K}^c$, so $\mathbf{E}/\mathscr{K}^c$ is perfectoid, and we have

$$W(\mathscr{K}^c) = \mathrm{Ker}\,(W(\pi_\mathbf{E}) : W(\mathbf{E}) \to W(\mathbf{E}/\mathscr{K}^c)).$$

It follows that the kernel of the induced projection $\pi'_A : A \to A \otimes_{W(\mathbf{E})} W(\mathbf{E}/\mathscr{K}^c)$ is naturally identified with $u_A(W(\mathscr{K}^c))$. Moreover, since $W(\mathbf{E}/\mathscr{K}^c)$ is separated (lemma 9.3.33(ii)), $W(\mathscr{K}^c)$ is a closed subset of $W(\mathbf{E})$, and taking into account theorem 16.3.36(i) we deduce that $\mathrm{Ker}\,\pi'_A = \{\mathscr{K}\}$, so we get a cocartesian diagram

$$\mathscr{D} \quad : \quad \begin{array}{ccc} W(\mathbf{E}) & \xrightarrow{W(\pi_\mathbf{E})} & W(\mathbf{E}/\mathscr{K}) \\ {\scriptstyle u_A}\downarrow & & \downarrow{\scriptstyle u'} \\ A & \xrightarrow{\pi_A} & A/\{\mathscr{K}\} \end{array}$$

and notice that both u_A and π_A are open and surjective (lemma 16.2.7(i)), so the same holds for u', and therefore $A/\{\mathscr{K}\}$ is perfectoid (proposition 16.2.21(i)). Let also $\pi : A \to A/pA$ be the proejction; taking into account remark 9.4.5(ii), there follows a commutative diagram

$$\mathbf{E}(\mathscr{D} \otimes_{\mathbb{Z}} \mathbb{F}_p) \qquad : \qquad \begin{array}{ccc} \mathbf{E} & \xrightarrow{\pi_{\mathbf{E}}} & \mathbf{E}/\mathscr{K} \\ {\scriptstyle \mathbf{E}(\pi)}\downarrow & & \downarrow{\scriptstyle \mathbf{E}(u')} \\ \mathbf{E}(A/pA) & \xrightarrow{\mathbf{E}(\pi_A \otimes_{\mathbb{Z}} \mathbb{F}_p)} & \mathbf{E}(A/(\{\mathscr{K}\} + pA)) \end{array}$$

whose vertical arrows are both isomorphisms. The assertion follows easily. □

Remark 16.3.39. (i) In the situation of corollary 16.3.38, let $(\alpha_n \mid n \in \mathbb{N})$ be any distinguished element of $\mathrm{Ker}\, u_A$, and suppose that $\mathscr{K} \subset \mathrm{Ann}_{\mathbf{E}}(\alpha_0)$. Then the inclusion $0 \subset \mathscr{K}^c$ is taut, and therefore theorem 16.3.31(ii) yields a natural $W(\mathbf{E})$-linear identification

$$\mathscr{K}^c \xrightarrow{\sim} \{\mathscr{K}\} \qquad \beta \mapsto \overline{u}_A(\beta).$$

Especially, $\{\mathscr{K}\} \subset \mathrm{Ann}_A(p)$.

(ii) Let $\mathscr{I}, \mathscr{K} \subset \mathbf{E}$ be two ideals such the topology of $\mathbf{E}$ agrees with the $\mathscr{I}$-adic topology, and suppose that

$$\mathscr{K}\mathscr{I} = 0 \qquad \text{and} \qquad \Phi_{\mathbf{E}}(\mathscr{K}) = \mathscr{K}.$$

Notice that $\mathscr{K} = \mathscr{K}^{\langle 1 \rangle}$, and $\alpha_0^n \in \mathscr{I}$ for some $n \in \mathbb{N}$, from which it follows easily that $\mathscr{K} \subset \mathrm{Ann}_{\mathbf{E}}(\alpha_0)$. Moreover, $\mathscr{K} \cap \mathscr{I} = 0$; indeed, if $x \in \mathscr{K} \cap \mathscr{I}$, then $x^2 = 0$, hence $x = 0$. Thus, $\mathscr{K} + \mathscr{I}^n = \mathscr{K} \oplus \mathscr{I}^n$ for every $n \in \mathbb{N}$, and therefore

$$\bigcap_{n \in \mathbb{N}} (\mathscr{K} + \mathscr{I}^n) = \mathscr{K} \oplus \bigcap_{n \in \mathbb{N}} \mathscr{I}^n = \mathscr{K}$$

i.e. $\mathscr{K}$ is closed in the topology of $\mathbf{E}$, so corollary 16.3.38 says that $\mathbf{E}/\mathscr{K}$ and $A/\{\mathscr{K}\}$ are perfectoid for their quotient topologies, and there exists an isomorphism of topological rings

$$\omega : \mathbf{E}(A/\{\mathscr{K}\}) \xrightarrow{\sim} \mathbf{E}/\mathscr{K} \qquad \text{such that} \qquad \omega \circ \mathbf{E}(\pi_A) = \pi_{\mathbf{E}}$$

where $\pi_A : A \to A/\{\mathscr{K}\}$ and $\pi_{\mathbf{E}} : \mathbf{E} \to \mathbf{E}/\mathscr{K}$ are the projections.

Corollary 16.3.40. *Let A be any perfectoid ring, and $a_\bullet := (a_1, \dots, a_k)$ a finite system of elements of $\mathbf{E} := \mathbf{E}(A)$. Denote by $J \subset \mathbf{E}$ (resp. $\mathscr{J} \subset W(\mathbf{E})$) the ideal generated by $a_\bullet$ (resp. by $\tau_{\mathbf{E}}(a_1), \dots, \tau_{\mathbf{E}}(a_k)$), and define the ideal $[a_\bullet]^{\langle \lambda \rangle} \subset W(\mathbf{E})$ as in remark* 9.3.70(iii)*, for every $\lambda \in \mathbb{N}[1/p]$. We regard A as a $W(\mathbf{E})$-algebra, via the map u_A. Then we have :*

(i) $[a_\bullet]^{\langle \lambda \rangle} A \subset \{J^{\langle \lambda \rangle}\mathbf{E}\} \subset [a_\bullet]^{\langle \lambda' \rangle} A$ *for every $\lambda' < \lambda$ in $\mathbb{N}[1/p]$.*

(ii) *The following conditions are equivalent :*

(a) *J is open in $\mathbf{E}$*

(b) *$\mathscr{J} A$ is open in A*

(c) *$[a_\bullet]^{\langle \lambda \rangle} A$ is open in A for every $\lambda \in \mathbb{N}[1/p]$*

(d) *There exists a strictly positive $\lambda \in \mathbb{N}[1/p]$ such that $[a_\bullet]^{\langle \lambda \rangle} A$ is open in A*

and if these conditions hold, then $[a_\bullet]^{\langle \lambda \rangle} A = \{J^{\langle \lambda \rangle}\mathbf{E}\}$ for every $\lambda \in \mathbb{N}[1/p]$.

(iii) *If I is any ideal of definition of $\mathbf{E}$, the following holds :*

(a) *$\{I^{\langle \lambda \rangle}\mathbf{E}\}$ is a topologically nilpotent open ideal of A for every $\lambda > 0$ in $\mathbb{N}[1/p]$.*

(b) *$\bigcap_{n \in \mathbb{N}} \{J^{\langle \lambda \rangle}\mathbf{E} + I^{\langle n \rangle}\mathbf{E}\} \subset \{J^{\langle \lambda' \rangle}\mathbf{E}\}$ for every $\lambda' < \lambda$ in $\mathbb{N}[1/p]$.*

Proof. (i): The first inclusion is clear. For the second, notice that $\{J^{\langle \lambda \rangle}\mathbf{E}\} = W(J^{\langle \lambda \rangle})^c \cdot A$ by theorem 16.3.36(i) and remark 16.3.28(iii,v). On the other hand, pick any $\lambda'' \in \mathbb{N}[1/p]$ such that $\lambda' < \lambda'' < \lambda$; we have $W(J^{\langle \lambda \rangle})^c = \prod_{n \in \mathbb{N}} (J^{\langle p^n \lambda \rangle}\mathbf{E})^c$ (see remark 9.3.28(iv) and lemma 9.3.69(ii.a)), and moreover $(J^{\langle p^n \lambda \rangle}\mathbf{E})^c \subset J^{\langle p^n \lambda'' \rangle}\mathbf{E}$ for every $n \in \mathbb{N}$ (claim 16.3.20), hence

$$W(J^{\langle \lambda \rangle})^c \subset W(J^{\langle \lambda'' \rangle}) \subset [a_\bullet]^{\langle \lambda' \rangle}$$

where the last inclusion follows from proposition 9.3.77(i).

(ii): Taking into account lemma 9.3.69(ii.a,iv), it is easily seen that (ii.b)⇔(ii.c)⇔(ii.d). Combining with (i), we conclude that (ii.b) holds if and only if $\{J^{\langle 1\rangle}\mathbf{E}\}$ is open in A. However, from theorem 16.3.36(ii) and remark 16.3.28(iii) we see as well that $\{J^{\langle 1\rangle}\mathbf{E}\}$ is open in A if and only if $J^{\langle 1\rangle}\mathbf{E}$ is open in $\mathbf{E}$. Lastly, $J^{\langle 1\rangle}\mathbf{E}$ is open in $\mathbf{E}$ if and only if the same holds for J, due to lemma 9.3.69(ii.a,iv). Thus (ii.a)⇔(ii.b). Next, we remark :

Claim 16.3.41. For every $\lambda \in \mathbb{N}[1/p]$ and every $n \in \mathbb{N}$, the ideal generated by the system $(\overline{u}_A(x) \mid x \in J^{\langle\lambda\rangle}\mathbf{E})$ is contained in $p^nA + [a_\bullet]^{\langle\lambda\rangle}A$.

Proof of the claim. The claim follows easily from proposition 9.3.62 and lemma 16.1.1(iv) (details left to the reader). ◊

Now, if $\mathscr{J}A$ is open in A, the same holds for $[a_\bullet]^{\langle\lambda\rangle}A$, for every $\lambda \in \mathbb{N}[1/p]$, in which case the latter contains p^n for some sufficiently large $n \in \mathbb{N}$; taking into account claim 16.3.41, we get the second assertion of (ii).

(iii.a): It follows easily from (ii) that $\{I^{\langle\lambda\rangle}\}$ is open, and combining lemma 9.3.69(ii.a,iv) with (i), it is easily seen that $\{I^{\langle\lambda\rangle}\mathbf{E}\}$ is topologically nilpotent : details left to the reader.

(iii.b): Fix $N \in \mathbb{N}$ such that $\lambda' < (1 - p^{-N})\lambda$, let $x \in J^{\langle\lambda\rangle}\mathbf{E}$, $y \in I^{\langle n\rangle}$ be any two elements, and $\alpha \in \mathbb{N}[1/p]$ any rational number ≤ 1; notice that $z_\alpha := \tau_A(x^{1-\alpha} \cdot y^\alpha)$ lies in $\{I^{\langle n/p^N\rangle}\mathbf{E}\}$ if $\alpha \geq p^{-N}$, and lies in $\{J^{\langle\lambda'\rangle}\mathbf{E}\}$ otherwise. However, proposition 9.3.62 says that $\tau_A(x + y)$ can be written as a p-adically convergent series whose terms are of the form $u_\alpha \cdot z_\alpha$ for certain $u_\alpha \in W(\mathbf{E})$; since (ii) implies that $\{I^{\langle n/p^N\rangle}\mathbf{E}\}$ is an open ideal in A, we deduce

$$\bigcap_{n\in\mathbb{N}}\{J^{\langle\lambda\rangle}\mathbf{E} + I^{\langle n\rangle}\mathbf{E}\} \subset \bigcap_{n\in\mathbb{N}}(\{J^{\langle\lambda'\rangle}\mathbf{E}\} + \{I^{\langle n/p^N\rangle}\mathbf{E}\}) = \{J^{\langle\lambda'\rangle}\mathbf{E}\}$$

where the last equality holds because $\{J^{\langle\lambda'\rangle}\mathbf{E}\}$ is a closed ideal, taking into account (iii.a). □

Theorem 16.3.42. *Let $f : A \to A'$ be a continuous ring homomorphism of perfectoid rings; set $C := \operatorname{Im} f$ and $D := \operatorname{Im} \mathbf{E}(f)$. The following holds :*

(i) *f is surjective (resp. bijective, resp. adic, resp. open, resp. open and injective) if and only if the same holds for $\mathbf{E}(f)$.*

(ii) *C is open in A' if and only if D is open in $\mathbf{E}(A')$.*

(iii) *Suppose that C is open in A', then we have :*

(a) *$A'^{\circ\circ} \subset C$.*

(b) *C is perfectoid both for the quotient topology induced by the projection $A \to C$, and for the subspace topology induced by the inclusion map $C \to A'$.*

Proof. Set $\mathbf{E} := \mathbf{E}(A)$ and $\mathbf{E}' := \mathbf{E}(A')$. Also, denote by $\mathscr{T}_D$ and $\mathscr{T}'_D$ (resp. $\mathscr{T}_C$ and $\mathscr{T}'_C$) the topologies on D (resp. on C) induced respectively by the projection $\mathbf{E} \to D$ and the open inclusion $D \to \mathbf{E}'$ (resp. induced by the projection $A \to C$ and the open inclusion $C \to A'$).

Let $\alpha_\bullet := (\alpha_n \mid n \in \mathbb{N})$ be a distinguished element in the kernel of $u_A : W(\mathbf{E}) \to A$, and recall that the image of $\alpha_\bullet$ in $W(\mathbf{E}')$ is a distinguished element in the kernel of $u_{A'}$. We denote by $\mathrm{Fil}^\bullet A$ and $\mathrm{Fil}^\bullet A'$ the p-adic filtrations on A and respectively A'; likewise, let $\mathrm{Fil}^\bullet\mathbf{E}$ and $\mathrm{Fil}^\bullet\mathbf{E}'$ be the α_0-adic filtrations on $\mathbf{E}$ and respectively $\mathbf{E}'$. Moreover, we let $A^\bullet$ be the complex concentrated in degrees 0 and 1, with $A^0 := A$, $A^1 := A'$, and $d^0 := f$. Likewise, we define the complex $\mathbf{E}^\bullet := (\mathbf{E} \xrightarrow{\mathbf{E}(f)} \mathbf{E}')$ in degrees 0 and 1. With the foregoing filtrations, $A^\bullet$ and $\mathbf{E}^\bullet$ are filtered complexes of $\mathbb{Z}$-modules, and we wish to consider the associated spectral sequences, as described in (7.2.12). Explicitly, we have $Z(A^\bullet)^{pq}_r = 0$ whenever $p + q \neq 0, 1$, and :

$$\begin{aligned} Z(A^\bullet)^{p,-p}_r &:= \{a \in \mathrm{Fil}^p A \mid f(a) \in \mathrm{Fil}^{p+r} A'\} \\ Z(A^\bullet)^{p,-p+1}_r &:= \mathrm{Fil}^p A' \end{aligned} \qquad \text{for every } p \in \mathbb{Z} \text{ and } r \in \mathbb{N}$$

and correspondingly for $Z(\mathbf{E}^\bullet)^{pq}_r$. Since $\overline{u}_A(\mathrm{Fil}^p\mathbf{E}) \subset \mathrm{Fil}^pA$ for every $p \in \mathbb{Z}$, and likewise for $\overline{u}_{A'}$, it is then clear that $\overline{u}_A$ and $\overline{u}_{A'}$ restrict to well defined maps

$$\overline{u}^{pq}_r : Z(\mathbf{E}^\bullet)^{pq}_r \to Z(A^\bullet)^{pq}_r \qquad \text{for every } p, q \in \mathbb{Z} \text{ and } r \in \mathbb{N}.$$

Moreover, we have $B(A^\bullet)^{pq}_r = 0$ whenever $p + q \neq 0, 1$, and :

$$\left.\begin{aligned} B(A^\bullet)^{p,-p}_r &:= \mathrm{Fil}^{p+1}A \cap Z(A^\bullet)^{p,-p}_r \\ B(A^\bullet)^{p,-p+1}_r &:= (\mathrm{Fil}^{p+1}A' + f(\mathrm{Fil}^{p-r+1}A)) \cap \mathrm{Fil}^pA' \end{aligned}\right. \qquad \text{for every } p \in \mathbb{Z} \text{ and } r \in \mathbb{N}$$

and correspondingly for $B(\mathbf{E}^\bullet)^{pq}_r$.

Claim 16.3.43. $\overline{u}^{a,b}_r(B(\mathbf{E}^\bullet)^{a,b}_r) \subset B(A^\bullet)^{a,b}_r$ for every $a, b \in \mathbb{Z}$ and $r \in \mathbb{N}$.

Proof of the claim. The assertion is trivial for $a + b \neq 0, 1$. If $b = -a$, the assertion follows directly from our explicit descriptions of $B(\mathbf{E}^\bullet)^{a,-a}_r$ and $B(A^\bullet)^{a,-a}_r$. Next, notice that $B(\mathbf{E}^\bullet)^{a,-a+1}_0 = \mathrm{Fil}^{a+1}\mathbf{E}'$, and likewise for $B(A^\bullet)^{a,-a+1}_0$; the assertion follows then when $r = 0$ and $b = -a + 1$. For the case where $r > 0$ and $b = -a + 1$, lemma 7.2.13 shows that

$$B(\mathbf{E}^\bullet)^{a,-a+1}_r = \mathrm{Fil}^{a+1}\mathbf{E}' + f(Z(\mathbf{E}^\bullet)^{a-r+1,-a+r-1}_{r-1})$$

and likewise for $B(A^\bullet)^{a,-a+1}_r$. Hence, let $x \in \mathrm{Fil}^{p+1}\mathbf{E}'$ and $y \in Z(\mathbf{E}^\bullet)^{a-r+1,-a+r-1}_{r-1}$ and set $z := x + \mathbf{E}(f)(y)$; we need to check that $\overline{u}_{A'}(z) \in \mathrm{Fil}^{a+1}A' + f(Z(A^\bullet)^{a-r+1,-a+r-1}_{r-1})$. However, proposition 9.3.62 says that $\overline{u}_{A'}(z) = \overline{u}_{A'}(x) + \overline{u}_{A'} \circ \mathbf{E}(f)(y) + w$, where w is the limit of a p-adically convergent series whose terms are $\mathbb{Z}_p$-linear combinations of products of the form $p^n \cdot \overline{u}_{A'}(x^{\sigma_0} \cdot \mathbf{E}(f)(y)^{\sigma_1})$, with $n \in \mathbb{N} \setminus \{0\}$, $\sigma_0, \sigma_1 \in \mathbb{Q}_+$, and $\sigma_0 + \sigma_1 = 1$. Now :

$$\overline{u}_{A'} \circ \mathbf{E}(f)(y) = f(\overline{u}_A(y)) \in f(Z(A^\bullet)^{a-r+1,-a+r-1}_{r-1}) \qquad \text{and} \qquad \overline{u}_{A'}(x) \in \mathrm{Fil}^{a+1}A'.$$

Moreover, since $f(\overline{u}_A(y)) \in \mathrm{Fil}^aA'$, we see that p^m divides $\overline{u}_{A'}(x^{\sigma_0} \cdot \mathbf{E}(f)(y)^{\sigma_1}) = \overline{u}_{A'}(x)^{\sigma_0} \cdot f(\overline{u}_A(y))^{\sigma_1}$ in A', for every integer $m \leq \sigma_0 \cdot (a + 1) + \sigma_1 \cdot a = a + \sigma_0$. Thus, $w \in \mathrm{Fil}^{a+1}A'$, whence the contention. ◊

Next, notice that $E(A^\bullet)^{p,-p}_0 = \mathrm{gr}^pA$ and $E(A^\bullet)^{p,-p+1}_0 = \mathrm{gr}^pA'$ for every $p \in \mathbb{Z}$, and likewise for $E(\mathbf{E}^\bullet)^{pq}_0$, whenever $p + q = 0, 1$. Especially, $E(A^\bullet)^{a,b}_r$ is a $\mathbb{Z}/p\mathbb{Z}$-module for every $a, b \in \mathbb{Z}$ and $r \in \mathbb{N}$. Invoking proposition 9.3.62 again, we deduce that the composition

$$Z(\mathbf{E}^\bullet)^{pq}_r \xrightarrow{\overline{u}^{pq}_r} Z(A^\bullet)^{pq}_r \to E(A^\bullet)^{pq}_r$$

is an additive map, and combining with claim 16.3.43, it follows that the latter factors through a well defined group homomorphism

$$v^{pq}_r : E(\mathbf{E}^\bullet)^{pq}_r \to E(A^\bullet)^{pq}_r \qquad \text{for every } p, q \in \mathbb{Z} \text{ and } r \in \mathbb{N}.$$

Lastly, a simple inspection shows that the system of maps $v^{\bullet\bullet}_\bullet$ yields a morphism of spectral sequences $E(\mathbf{E}^\bullet)^{\bullet\bullet}_\bullet \to E(A^\bullet)^{\bullet\bullet}_\bullet$. Since v^{pq}_0 is an isomorphism for every $p, q \in \mathbb{Z}$, it follows that $v^{\bullet\bullet}_\bullet$ is an isomorphism of spectral sequences.

Let us endow $\mathrm{Coker}\, f$ with the filtration induced by $\mathrm{Fil}^\bullet A'$: namely

$$\mathrm{Fil}^p(\mathrm{Coker}\, f) := \mathrm{Im}(\mathrm{Fil}^pA' \to \mathrm{Coker}\, f) \qquad \text{for every } p \in \mathbb{Z}.$$

Likewise, we define a filtration $\mathrm{Fil}^\bullet\mathrm{Coker}\,\mathbf{E}(f)$ on $\mathrm{Coker}\,\mathbf{E}(f)$. Let moreover $\mathscr{T}_{A',p}$ be the p-adic topology on A', and $\mathscr{T}_{\mathbf{E}',\alpha_0}$ the α_0-adic topology on $\mathbf{E}'$.

Claim 16.3.44. C is open in $(A', \mathscr{T}_{A',p})$ (resp. D is open in $(\mathbf{E}', \mathscr{T}_{\mathbf{E}',\alpha_0})$) if and only if there exists $i \in \mathbb{N}$ such that $\mathrm{gr}^i\mathrm{Coker}\, f = 0$ (resp. such that $\mathrm{gr}^i\mathrm{Coker}\,\mathbf{E}(f) = 0$).

Proof of the claim. Clearly, if C is open in $(A', \mathscr{T}_{A',p})$, then we find $i \in \mathbb{N}$ with $\mathrm{gr}^i\mathrm{Coker}\, f = 0$. Conversely, if the latter condition holds, set $M := \mathrm{Fil}^i A' + C \subset A'$; we deduce that $M = pM + C$, whence $M = C$, by [126, Th.8.4] and lemma 16.2.3(v); *i.e.* $\mathrm{Fil}^i A' \subset C$, so C is open in $(A', \mathscr{T}_{A',p})$. Since $\mathbf{E}$ and $\mathbf{E}'$ are complete and separated for their α_0-adic topologies (see (16.2.19)), the same argument shows the assertion for D. ◊

According to (7.2.17), the spectral sequences $E(A^\bullet)^{\bullet\bullet}_{\bullet}$ and $E(\mathbf{E}^\bullet)^{\bullet\bullet}_{\bullet}$ admit natural abutments, and by inspecting the constructions, we deduce natural isomorphisms :

$$E(A^\bullet)^{a,1-a}_\infty \xrightarrow{\sim} \mathrm{gr}^a\mathrm{Coker}\, f \qquad E(\mathbf{E}^\bullet)^{a,1-a}_\infty \xrightarrow{\sim} \mathrm{gr}^a\mathrm{Coker}\,\mathbf{E}(f) \qquad \text{for every } a \in \mathbb{Z}.$$

Moreover, by proposition 7.2.18 they converge in degree 1. Since these spectral sequences are isomorphic, combining with claim 16.3.44 we conclude that C is open in $(A', \mathscr{T}_{A',p})$ if and only if D is open in $(\mathbf{E}', \mathscr{T}_{\mathbf{E}',\alpha_0})$.

(ii): By the foregoing, in order to prove the assertion we may assume that C is open in $(A', \mathscr{T}_{A',p})$ and D is open in $(\mathbf{E}', \mathscr{T}_{\alpha_0,\mathbf{E}'})$. Hence, $\alpha_0^n\mathbf{E}' \subset D$ for some $n \in \mathbb{N}$, and since D and $\mathbf{E}'$ are perfect rings, it follows easily that $\alpha_0\mathbf{E}' \subset D$, *i.e.* $\mathrm{gr}^1\mathrm{Coker}\,\mathbf{E}(f) = 0$. By the foregoing, we deduce that $\mathrm{gr}^1\mathrm{Coker}\, f = 0$ as well, *i.e.* $pA' \subset C$. Endow $\mathbf{E}'/\alpha_0\mathbf{E}'$ and A'/pA' with the quotient topologies $\mathscr{T}_{\mathbf{E}'/\alpha_0\mathbf{E}'}$ and $\mathscr{T}_{A'/pA'}$ induced by $\mathbf{E}'$ and A', and recall that $\overline{u}_{A'/pA'}$ induces an isomorphism of topological rings

$$\omega : (\mathbf{E}'/\alpha_0\mathbf{E}', \mathscr{T}_{\mathbf{E}'/\alpha_0\mathbf{E}'}) \xrightarrow{\sim} (A'/pA', \mathscr{T}_{A'/pA'})$$

(remark 16.3.7(ii)); clearly $\omega(C/pA') = D/\alpha_0\mathbf{E}'$, and especially C/pA' is open in A'/pA' if and only if $D/\alpha_0\mathbf{E}'$ is open in $\mathbf{E}'/\alpha_0\mathbf{E}'$. The contention follows immediately.

(iii.a): In view of (ii), the subring D is open in $\mathbf{E}'$; since D is perfect, it follows easily that $\mathbf{E}'^{\circ\circ} \subset D$. Endow $D/\alpha_0\mathbf{E}'$ (resp. C/pA') with the topology $\mathscr{T}'_{D/\alpha_0\mathbf{E}'}$ (resp. $\mathscr{T}'_{C/pA'}$) induced by $\mathscr{T}'_D$ via the projection $D \to D/\alpha_0\mathbf{E}'$ (resp. by $\mathscr{T}'_C$ via the projection $C \to C/pA'$). Then we have $(D, \mathscr{T}'_D)^{\circ\circ} = \mathbf{E}'^{\circ\circ}$, and notice that

$$(D/\alpha_0\mathbf{E}', \mathscr{T}'_{D/\alpha_0\mathbf{E}'})^{\circ\circ} = (D, \mathscr{T}'_D)^{\circ\circ}/\alpha_0\mathbf{E}' \qquad \text{and} \qquad (C/pA', \mathscr{T}'_{C/pA'})^{\circ\circ} = (C, \mathscr{T}'_C)^{\circ\circ}/pA'.$$

Moreover, according to lemma 8.2.3(i), the topology $\mathscr{T}'_{D/\alpha_0\mathbf{E}'}$ agrees with the topology induced by $\mathscr{T}_{\mathbf{E}'/\alpha_0\mathbf{E}'}$ via the inclusion map $D/\alpha_0\mathbf{E}' \to \mathbf{E}'/\alpha_0\mathbf{E}'$, and likewise for $\mathscr{T}'_{C/pA'}$; it follows that the foregoing isomorphism ω restricts to an isomorphism of topological rings

$$(C/pA', \mathscr{T}'_{C/pA'}) \xrightarrow{\sim} (D/\alpha_0\mathbf{E}', \mathscr{T}'_{D/\alpha_0\mathbf{E}'}).$$

Since $(D/\alpha_0\mathbf{E}', \mathscr{T}'_{D/\alpha_0\mathbf{E}'})^{\circ\circ} = \mathbf{E}'^{\circ\circ}/\alpha_0\mathbf{E}' = (\mathbf{E}'/\alpha_0, \mathscr{T}_{\mathbf{E}'/\alpha_0})^{\circ\circ}$, we get $(C/pA', \mathscr{T}'_{C/pA'})^{\circ\circ} = (A'/pA', \mathscr{T}_{A'/pA'})^{\circ\circ} = A'^{\circ\circ}/pA'$, and finally $A'^{\circ\circ} = (C, \mathscr{T}'_C)^{\circ\circ}$.

(i): If f is bijective, then clearly the same holds for $\mathbf{E}(f)$; conversely, if $\mathbf{E}(f)$ is bijective, then the same holds for

$$W(\mathbf{E})/\alpha_\bullet W(\mathbf{E}) \otimes_{W(\mathbf{E})} W(\mathbf{E}(f)) : W(\mathbf{E})/\alpha_\bullet W(\mathbf{E}) \to W(\mathbf{E}')/\alpha_\bullet W(\mathbf{E}').$$

But the image of $\alpha_\bullet$ is still distinguished in $W(\mathbf{E}')$, and lies in $\mathrm{Ker}\, u_{A'}$, hence the latter map is naturally identified with f; especially, f is bijective.

- If either f or $\mathbf{E}(f)$ is surjective, then $pA' \subset C$ and $\alpha_0\mathbf{E}' \subset D$ by the proof of (ii), and moreover the isomorphism $\mathbf{E}'/\alpha_0\mathbf{E}' \xrightarrow{\sim} A'/pA'$ induced by $\overline{u}_{A/pA}$ restricts to an isomorphism $D/\alpha_0\mathbf{E}' \xrightarrow{\sim} C/pA'$. Then clearly $C = A'$ if and only if $D = \mathbf{E}'$.
- Next, let $\mathscr{J} \subset \mathbf{E}$ be any ideal of definition, $(\beta_1, \dots, \beta_k)$ a finite system of generators for $\mathscr{J}$, and $J \subset A$ the ideal generated by $(\overline{u}_A(\beta_1), \dots, \overline{u}_A(\beta_k))$. By corollary 16.3.40(ii), the ideal JA' is open in A' if and only if $\mathscr{J}\mathbf{E}'$ is open in $\mathbf{E}'$. Moreover, $\mathscr{J}\mathbf{E}'$ (resp. JA') is topologically nilpotent in $\mathbf{E}'$ (resp. in A'), since the same holds for $\mathscr{J}$ (resp. for J); thus, $\mathscr{J}\mathbf{E}'$ is an ideal of adic definition for $\mathbf{E}'$ if and only if JA' is an ideal of adic definition for A', *i.e.* f is adic if and only if $\mathbf{E}(f)$ is adic.

• If either f of $\mathbf{E}(f)$ is open, we know already that C is open in A' and D is open in $\mathbf{E}'$, by (ii). Also, by the foregoing we know already that f is adic if and only if the same holds for $\mathbf{E}(f)$; combining with proposition 8.3.18(v), we conclude that f is open if and only if the same holds for $\mathbf{E}(f)$.

• Lastly, if either f or $\mathbf{E}(f)$ is open and injective, by the foregoing we know already that both f and $\mathbf{E}(f)$ are open, and then $A'^{\circ\circ} \subset C$ and $\mathbf{E}'^{\circ\circ} \subset D$, by (iii.a). Endow $\overline{A}' := A'/A'^{\circ\circ}$, $\overline{\mathbf{E}}' := \mathbf{E}'/\mathbf{E}'^{\circ\circ}$, $\overline{C} := C/A'^{\circ\circ}$ and $\overline{D} := D/\mathbf{E}'^{\circ\circ}$ with their discrete topologies; then the foregoing isomorphism ω induces a ring isomorphism $\overline{\mathbf{E}}' \xrightarrow{\sim} \overline{A}'$, restricting to a ring isomorphism $\overline{D} \xrightarrow{\sim} \overline{C}$. Moreover, we get natural isomorphisms of topological rings :

$$(C, \mathscr{T}'_C) \xrightarrow{\sim} \overline{C} \times_{\overline{A}'} A' \qquad \text{and} \qquad (D, \mathscr{T}'_D) \xrightarrow{\sim} \overline{D} \times_{\overline{\mathbf{E}}'} \mathbf{E}'.$$

Taking into account proposition 16.3.25, it follows that the projection $A \to (C, \mathscr{T}'_C)$ is an isomorphism of topological rings if and only if the same holds for the projection $\mathbf{E} \to (D, \mathscr{T}'_D)$; *i.e.* f is open and injective if and only if the same holds for $\mathbf{E}(f)$.

(iii.b): The map $\mathbf{E}(f)$ is a composition of continuous ring homomorphisms :

$$\mathbf{E} \xrightarrow{g_1} (D, \mathscr{T}_D) \xrightarrow{g_2} (D, \mathscr{T}'_D) \xrightarrow{g_3} \mathbf{E}'$$

where g_1 is open and surjective, g_2 is bijective, and g_3 is an open injective map. Since $\operatorname{Ker} g_1 = \operatorname{Ker} \mathbf{E}(f)$ is a closed ideal of $\mathbf{E}$, we see that $(D, \mathscr{T}_D)$ is a perfect, separated and complete topological ring (proposition 8.2.13(v)) whose topology is I-adic, for an ideal $I \subset D$ of finite type, *i.e.* $(D, \mathscr{T}_D)$ is perfectoid, and the same holds for $(D, \mathscr{T}'_D)$, by virtue of corollaries 8.3.20(i) and 8.3.19(iii). Since the category of perfectoid A-algebras is equivalent to that perfectoid $\mathbf{E}$-algebras (remark 16.3.7(i)) we deduce a corresponding factorization of f :

$$A \xrightarrow{f_1} C_1 \xrightarrow{f_2} C_2 \xrightarrow{f_3} A'$$

where C_1 and C_2 are perfectoid A-algebras with isomorphisms of perfectoid $\mathbf{E}$-algebras

$$\mathbf{E}(C_1) \xrightarrow{\sim} (D, \mathscr{T}_D) \qquad \text{and} \qquad \mathbf{E}(C_2) \xrightarrow{\sim} (D, \mathscr{T}'_D).$$

Moreover, f_1 is open and surjective, f_2 is bijective, and f_3 is open and injective, by (i). But then C_1 is isomorphic to $(C, \mathscr{T}_C)$ and C_2 is isomorphic to $(C, \mathscr{T}'_C)$, whence the contention. □

16.3.45. Let $A_\bullet := (A_\lambda \mid \lambda \in \Lambda)$ be a filtered system of perfectoid rings, with adic transition maps, and for every $\lambda \in \Lambda$ set $\mathbf{E}_\lambda := \mathbf{E}(A_\lambda)$. By virtue of theorem 16.3.42(i), the transition maps of the induced system $\mathbf{E}_\bullet := (\mathbf{E}_\lambda \mid \lambda \in \Lambda)$ are also adic. Endow the colimit A of $A_\bullet$ (resp. $\mathbf{E}$ of $\mathbf{E}_\bullet$) of the unique adic topology $\mathscr{T}_A$ (resp. $\mathscr{T}_\mathbf{E}$) such that the universal co-cone $(\tau_{A,\lambda} : A_\lambda \to A \mid \lambda \in \Lambda)$ (resp. $(\tau_{\mathbf{E},\lambda} : \mathbf{E}_\lambda \to \mathbf{E} \mid \lambda \in \Lambda)$) consists of adic maps. We have :

Corollary 16.3.46. *In the situation of* (16.3.45)*, the separated completion $A^\wedge$ of $(A, \mathscr{T}_A)$ (resp. $\mathbf{E}^\wedge$ of $(\mathbf{E}, \mathscr{T}_\mathbf{E})$) is perfectoid, and there exists a unique isomorphism of topological rings*

$$\omega : \mathbf{E}(A^\wedge) \xrightarrow{\sim} \mathbf{E}^\wedge \qquad \textit{such that} \qquad \omega \circ \mathbf{E}(\tau^\wedge_{A,\lambda}) = \tau^\wedge_{\mathbf{E},\lambda} \qquad \textit{for every } \lambda \in \Lambda.$$

Proof. We may assume that Λ admits an initial element λ_0, and let $\underline{\alpha} := (\alpha_n \mid n \in \mathbb{N})$ be a distinguished element in $\operatorname{Ker}(u_{A_{\lambda_0}})$. It suffices to show that there exists a unique isomorphism of topological rings

$$\omega' : A^\wedge \xrightarrow{\sim} A_{\lambda_0} \otimes_{W(\mathbf{E}_{\lambda_0})} W(\mathbf{E}^\wedge) \quad \text{such that} \quad \omega' \circ \tau^\wedge_{A,\lambda} = A_{\lambda_0} \otimes_{W(\mathbf{E}_{\lambda_0})} W(\tau^\wedge_{\mathbf{E},\lambda}) \text{ for every } \lambda \in \Lambda$$

(under the natural identification $A_\lambda \xrightarrow{\sim} A_0 \otimes_{W(\mathbf{E}_{\lambda_0})} W(\mathbf{E}_\lambda)$). Let $\beta_1, \dots, \beta_k$ be a finite sequence of elements of $\mathbf{E}_{\lambda_0}$ that generates an ideal of adic definition. Let $\mathscr{W}$ be the colimit of the induced system $(W(\mathbf{E}_\lambda) \mid \lambda \in \Lambda)$; denote by $\mathscr{I}_0 \subset W(A_{\lambda_0})$ the ideal generated by the sequence

$(p, \tau_{A_{\lambda_0}}(\beta_1), \dots, \tau_{A_{\lambda_0}}(\beta_k))$, so that the topology of $W(A_{\lambda_0})$ is $\mathscr{I}_0$-adic (proposition 9.3.77(ii)). Let also $\mathscr{W}^\wedge$ be the separated completion of $\mathscr{W}$; by corollary 9.3.79, the natural map

$$\mathscr{W}^\wedge \to W(\mathbf{E}^\wedge)$$

is an isomorphism of topological rings. On the other hand, we have a short exact sequence :

$$0 \to \underline{\alpha}\mathscr{W} \to \mathscr{W} \xrightarrow{u} A \to 0$$

and the topology of A is induced by $\mathscr{W}$ via u. By proposition 8.2.13(iii), it follows that the kernel of the induced map $u^\wedge : \mathscr{W}^\wedge \to A^\wedge$ is the topological closure of the image of $\underline{\alpha}\mathscr{W}$; the latter is also the topological closure of $\underline{\alpha}\mathscr{W}^\wedge$, and moreover it is easily seen that $A^\wedge$ is a P-ring. Hence, $\mathrm{Ker}\, u^\wedge = \underline{\alpha}\mathscr{W}^\wedge$ (proposition 16.2.23(ii)), whence the corollary. □

In the situation of theorem 16.3.36, let $\mathscr{K}$ and $\mathscr{K}'$ be any two closed and taut ideals of $\mathbf{E}$; we would like to show that $\mathscr{K} = \mathscr{K}'$ if and only if $\{\mathscr{K}\} = \{\mathscr{K}'\}$, and taking into account theorem 16.3.36(iii), it would suffice to prove that $\{\mathscr{K} \cap \mathscr{K}'\} = \{\mathscr{K}\} \cap \{\mathscr{K}'\}$. We do not know whether the latter identity always holds, but we have at least the following :

Theorem 16.3.47. *Let A be any perfectoid ring, $\mathscr{K}$ and $\mathscr{K}'$ two taut and closed ideals of $\mathbf{E} := \mathbf{E}(A)$, and $(\alpha_n \mid n \in \mathbb{N})$ any distinguished element of* $\mathrm{Ker}\, u_A$. *Suppose that either one of the following conditions holds :*

(a) $\alpha_0 \in \mathscr{K}'$
(b) *$\mathscr{K}'$ is strictly taut and $\alpha_0^n \subset \mathscr{K}'$ for some sufficiently large $n \in \mathbb{N}$.*

Then $\{\mathscr{K} \cap \mathscr{K}'\} = \{\mathscr{K}\} \cap \{\mathscr{K}'\}$.

Proof. Suppose that (a) holds; in this case, both inclusions $\mathscr{K}' \subset \mathbf{E}$ and $\mathscr{K}' \cap \mathscr{K} \subset \mathscr{K}$ are taut, by remark 16.3.28(iv). There follows a commutative diagram of A/pA-modules

$$\begin{array}{ccc} \mathscr{K}/(\mathscr{K}' \cap \mathscr{K}) & \longrightarrow & \{\mathscr{K}\}/\{\mathscr{K}' \cap \mathscr{K}\} \\ \downarrow & & \downarrow \\ \mathbf{E}/\mathscr{K}' & \longrightarrow & A/\{\mathscr{K}'\} \end{array}$$

both of whose horizontal arrows are isomorphisms, by theorem 16.3.31(ii), and whose left vertical arrow is clearly injective. Then the right vertical arrow is injective as well, whence the contention, in this case.

In case (b) holds, pick $\varepsilon \in \mathbb{N}[1/p]$ with $0 < \varepsilon < 1$ and such that $\mathscr{K}'$ is $\alpha_0^{1-\varepsilon}$-taut. We set

$$\mathscr{I}_n := \{x \in \mathbf{E} \mid \alpha_0^{n\varepsilon} x \in \mathscr{I}\} \qquad \text{for every } n \in \mathbb{N} \text{ and every ideal } \mathscr{I} \subset \mathbf{E}.$$

Claim 16.3.48. With the foregoing notation we have :

(i) $\mathscr{K}'_n$ is a closed ideal of $\mathbf{E}$, for every $n \in \mathbb{N}$.
(ii) The inclusion $\mathscr{K}'_n \subset \mathscr{K}'_{n+1}$ is taut for every $n \in \mathbb{N}$.
(iii) $\mathscr{K}'_0 = \mathscr{K}'$ and $\mathscr{K}'_r = \mathbf{E}$ for every sufficiently large $r \in \mathbb{N}$.

Proof of the claim. Since $\mathscr{K}'$ is closed, it is easily seen that (i) holds. Say that $\alpha_0^n \in \mathscr{K}'$; then clearly $\mathscr{K}'_k = \mathbf{E}$ for every integer $k \geq \varepsilon^{-1}n$. Next, directly from the definition we see that, for any two ideals $\mathscr{I}, \mathscr{J} \subset \mathbf{E}$ and any $n \in \mathbb{N}$ we have

$$\mathscr{J} \cdot \mathscr{I}_n \subset (\mathscr{J}\mathscr{I})_n \qquad \alpha_0^\varepsilon \cdot \mathscr{I}_{n+1} \subset \mathscr{I}_n \qquad (\mathscr{I}_n)^p \subset (\mathscr{I}^p)_{np} \qquad (\Phi_{\mathbf{E}}^{-1}\mathscr{I})_n = \Phi_{\mathbf{E}}^{-1}(\mathscr{I}_{np}).$$

Therefore :

$$\begin{aligned}\alpha_0 \cdot \Phi_{\mathbf{E}}^{-1}(\mathscr{K}'_{n+1})^p &\subset \alpha_0 \cdot \Phi_{\mathbf{E}}^{-1}((\mathscr{K}'^p)_{(n+1)p}) \\ &= \alpha_0 \cdot (\Phi_{\mathbf{E}}^{-1}\mathscr{K}'^p)_{n+1} \\ &\subset \alpha_0^{1-\varepsilon} \cdot (\Phi_{\mathbf{E}}^{-1}\mathscr{K}'^p)_n \\ &\subset (\alpha_0^{1-\varepsilon} \cdot \Phi_{\mathbf{E}}^{-1}\mathscr{K}'^p)_n \\ &\subset \mathscr{K}'_n\end{aligned}$$

as required. ◊

By claim 16.3.48 we have $\mathscr{K}'_r = \mathbf{E}$ for some $r \in \mathbb{N}$, and for every $n \in \mathbb{N}$ we get a commutative diagram of A/pA-modules

$$\begin{array}{ccc}\{\mathscr{K} \cap \mathscr{K}'_{n+1}\}/\{\mathscr{K} \cap \mathscr{K}'_n\} & \longrightarrow & \{\mathscr{K}'_{n+1}\}/\{\mathscr{K}'_n\} \\ \downarrow & & \downarrow \\ (\mathscr{K} \cap \mathscr{K}'_{n+1})/(\mathscr{K} \cap \mathscr{K}'_n) & \longrightarrow & \mathscr{K}'_{n+1}/\mathscr{K}'_n\end{array}$$

whose vertical arrows are isomorphisms, by theorem 16.3.31(ii), and whose bottom horizontal arrow is injective. Then the top horizontal arrow is injective as well, so that

$$\{\mathscr{K} \cap \mathscr{K}'_{n+1}\} \cap \{\mathscr{K}'_n\} \subset \{\mathscr{K} \cap \mathscr{K}'_n\} \qquad \text{for every } n \in \mathbb{N}.$$

We may now prove that $\{\mathscr{K}\} \cap \{\mathscr{K}'_n\} = \{\mathscr{K} \cap \mathscr{K}'_n\}$, by descending induction on n. Indeed, the assertion is obvious in case $n \geq r$. Suppose that the identity has already been shown for some $n > 0$; then we get

$$\{\mathscr{K}\} \cap \{\mathscr{K}'_{n-1}\} = \{\mathscr{K}\} \cap \{\mathscr{K}'_n\} \cap \{\mathscr{K}'_{n-1}\} = \{\mathscr{K} \cap \mathscr{K}'_n\} \cap \{\mathscr{K}'_{n-1}\} \subset \{\mathscr{K} \cap \mathscr{K}'_{n-1}\}$$

and the converse inclusion is obvious, so the sought identity holds for $n-1$. Letting $n = 0$, we get the assertion. □

In the same vein, let us also point out :

Lemma 16.3.49. *In the situation of theorem* 16.3.47*, let* $\lambda_1, \lambda_2 \in \mathbb{N}[1/p]$ *be two rationals such that* $\lambda_1 + \lambda_2 \leq 1$*, and* $\mathscr{J}_1, \mathscr{J}_2 \subset \mathbf{E}$ *two ideals such that* $\mathscr{J}_i$ *is* $\alpha_0^{\lambda_i}$*-taut for* $i = 1, 2$*. Then*

$$(\{\mathscr{J}_1\}\{\mathscr{J}_2\})^c = \{\mathscr{J}_1\mathscr{J}_2\}.$$

Proof. The inclusion $\{\mathscr{J}_1\}\{\mathscr{J}_2\} \subset \{\mathscr{J}_1\mathscr{J}_2\}$ is obvious from the definitions, so it suffices to check that for every $x \in \mathscr{J}_1\mathscr{J}_2$ we have $\overline{u}_A(x) \in (\{\mathscr{J}_1\}\{\mathscr{J}_2\})^c$. However, every such x can be written as a finite sum $\sum_{i=1}^n y_i z_i$, with $y_1, \dots, y_n \in \mathscr{J}_1$ and $z_1, \dots, z_n \in \mathscr{J}_2$; by proposition 9.3.62 we know that $\overline{u}_A(x)$ is the limit of a p-adically convergent series of the form $\sum_{j\in\mathbb{N}} w_j$, where in turn each w_j is a finite $\mathbb{Z}_p$-linear combination of terms of the form

$$w_j^{\underline{\mu}} := p^j \cdot \overline{u}_A((y_1z_1)^{\mu_1} \cdots (y_nz_n)^{\mu_n}) \quad \text{for certain } \mu_1, \dots, \mu_n \in p^{-j}\mathbb{N} \text{ such that } \sum_{i=1}^n \mu_i = 1.$$

Thus, we are reduced to checking that any such term lies in $\{\mathscr{J}_1\}\{\mathscr{J}_2\}$. Now, set $y^{\underline{\mu}} := y_1^{\mu_1} \cdots y_n^{\mu_n}$, and define likewise $z^{\underline{\mu}}$, for every $\underline{\mu} := (\mu_1, \dots, \mu_n) \in p^{-j}\mathbb{N}$; under our assumptions we have $p = \overline{u}_A(\alpha_0^{\lambda_1+\lambda_2}) \cdot a$ for some $a \in A$ (lemma 16.2.7(iii)), hence

$$w_j^{\underline{\mu}} = a^j \cdot \overline{u}_A(\alpha_0^{j\lambda_1} \cdot y^{\underline{\mu}}) \cdot \overline{u}_A(\alpha_0^{j\lambda_2} \cdot z^{\underline{\mu}})$$

and notice that $\alpha_0^{j\lambda_1} \cdot y^{\underline{\mu}} \in \alpha_0^{j\lambda_1} \cdot \Phi_{\mathbf{E}}^{-j}(\mathscr{J}_1^{p^j}) \subset \mathscr{J}_1$ by (16.3.29). Likewise, $\alpha_0^{j\lambda_2} \cdot z^{\underline{\mu}} \in \mathscr{J}_2$, and the assertion follows. □

Corollary 16.3.50. *Let A be a perfectoid ring, $\mathscr{K}, \mathscr{K}' \subset \mathbf{E}(A)$ two taut and closed ideals, and suppose that either $\mathscr{K}$ or $\mathscr{K}'$ fulfills conditions* (a) *and* (b) *of theorem* 16.3.47*. Then we have $\mathscr{K} \subset \mathscr{K}'$ if and only if $\{\mathscr{K}\} \subset \{\mathscr{K}'\}$.*

Proof. As already announced, this follows straightforwardly from remark 16.3.28(iv) and theorems 16.3.47 and 16.3.36(iii). □

Corollary 16.3.51. *In the situation of corollary* 16.3.40*, let $a'_\bullet := (a'_1, \dots, a'_t)$ be another finite system of elements of $\mathbf{E}$, and $J' \subset \mathbf{E}$ the ideal generated by $a'_\bullet$. Define the ideal $[a'_\bullet]^{\langle\beta\rangle} \subset W(\mathbf{E})$ as in remark* 9.3.70(iii)*, for every $\beta \in \mathbb{N}[1/p]$. Moreover, let $\lambda, \mu \in \mathbb{N}[1/p]$ be any two rational numbers. We have :*

(i) *The following conditions are equivalent :*
   (a) $J^{\langle\lambda\rangle}\mathbf{E} \subset J'^{\lfloor\mu\rfloor}\mathbf{E}$.
   (b) $[a_\bullet]^{\langle\lambda\rangle}A \subset [a'_\bullet]^{\langle\mu-\varepsilon\rangle}A$ *for every $\varepsilon \in \mathbb{N}[1/p]$ with $0 < \varepsilon \leq \mu$.*

(ii) *Suppose that either J or J' is open. Then the following conditions are equivalent :*
   (a) $J^{\langle\lambda\rangle}\mathbf{E} \subset J'^{\langle\mu\rangle}\mathbf{E}$.
   (b) $[a_\bullet]^{\langle\lambda\rangle}A \subset [a'_\bullet]^{\langle\mu\rangle}A$.

(iii) *For every $l, m \in \mathbb{N}[1/p]$ such that $l\mu + m\lambda < \lambda\mu$ we have*

$$[a_\bullet]^{\langle\lambda\rangle}A \cap [a'_\bullet]^{\langle\mu\rangle}A \subset [a_\bullet]^{\langle l\rangle} \cdot [a'_\bullet]^{\langle m\rangle}A.$$

Proof. (i): Suppose that (i.a) holds; we deduce :

$$[a_\bullet]^{\langle\lambda\rangle}A \subset \{J^{\langle\lambda\rangle}\mathbf{E}\} \subset \{J'^{\langle\mu-\varepsilon/p\rangle}\mathbf{E}\} \subset [a'_\bullet]^{\langle\mu-\varepsilon\rangle}A \qquad \text{for every } \varepsilon \text{ as in (i.b)}$$

where the first and last inclusions follow from corollary 16.3.40(i).

Conversely, suppose that (i.b) holds; fix a finite system $b_\bullet := (b_1, \dots, b_r)$ of generators for an ideal of definition I of the perfectoid ring $\mathbf{E}$, and for every $n \in \mathbb{N}$, let $b^n_\bullet$ be the finite system consisting of the products of the form $b_{j(1)} \cdots b_{j(n)}$, where j ranges over all the mappings $\{1, \dots, n\} \to \{1, \dots, r\}$. For every $n \in \mathbb{N}$, we may then form the system $(a'_\bullet, b^n_\bullet)$ which is the union of the systems $a'_\bullet$ and $b^n_\bullet$, and for every $\lambda \in \mathbb{N}[1/p]$ consider the ideal $[a'_\bullet, b^n_\bullet]^{\langle\lambda\rangle} \subset W(\mathbf{E})$ attached as in remark 9.3.70(iii) to the system $(a'_\bullet, b^n_\bullet)$. With this notation, corollary 16.3.40(ii) yields :

$$[a_\bullet]^{\langle\lambda\rangle}A \subset [a'_\bullet, b^n_\bullet]^{\langle\mu-\varepsilon/p\rangle}A = \{(J' + I^n)^{\langle\mu-\varepsilon/p\rangle}\mathbf{E}\} \qquad \text{for every } n \in \mathbb{N} \text{ and } \varepsilon \text{ as in (i.a).}$$

However, since the topology of A is coarser than the p-adic topology, claim 16.3.41 implies that $[a_\bullet]^{\langle\lambda\rangle}A$ is a dense subset of $\{J^{\langle\lambda\rangle}\mathbf{E}\}$; but $\{(J' + I^n)^{\langle\mu-\varepsilon/p\rangle}\mathbf{E}\}$ is a closed ideal, so

$$\{J^{\langle\lambda\rangle}\mathbf{E}\} \subset \{(J' + I^n)^{\langle\mu-\varepsilon/p\rangle}\mathbf{E}\} \qquad \text{for every } n \in \mathbb{N} \text{ and } \varepsilon \text{ as in (i.a).} \tag{16.3.52}$$

Now, the topological closure $J^{\langle\lambda\rangle}\mathbf{E}^c$ of $J^{\langle\lambda\rangle}\mathbf{E}$ is (strictly) taut, and $\{J^{\langle\lambda\rangle}\mathbf{E}\} = \{J^{\langle\lambda\rangle}\mathbf{E}^c\}$ (remark 16.3.28(v)); moreover, $(J' + I^n)^{\langle\mu-\varepsilon/p\rangle}\mathbf{E}$ is an open ideal of $W(\mathbf{E})$, hence, from corollary 16.3.50 and (16.3.52) we derive

$$J^{\langle\lambda\rangle}\mathbf{E}^c \subset (J' + I^n)^{\langle\mu-\varepsilon/p\rangle}\mathbf{E} \qquad \text{for every } n \in \mathbb{N} \text{ and } \varepsilon \text{ as in (i.a)}$$

and combining with claim 16.3.20 we conclude that $J^{\langle\lambda\rangle}\mathbf{E} \subset J'^{\langle\mu-\varepsilon\rangle}\mathbf{E}$ for every ε as in (i.a).

(ii): From lemma 9.3.69(ii.a,iv) it follows easily that J is open if and only if the same holds for $J^{\langle\lambda\rangle}\mathbf{E}$, and likewise, J' is open if and only if the same holds for $J'^{\langle\mu\rangle}\mathbf{E}$. Hence, suppose that J is open; if condition (ii.a) holds, then it follows that J' is open as well, and if (ii.b) holds, we use corollary 16.3.40(ii) to deduce first that $[a_\bullet]^{\langle\lambda\rangle}A$ is open, so the same holds for $[a'_\bullet]^{\langle\mu\rangle}A$, and then also for J', again by corollary 16.3.40(ii). In conclusion, for the proof of (ii) we may assume that J' is open. Now, suppose that (ii.a) holds; then we get

$$[a_\bullet]^{\langle\lambda\rangle}A \subset \{J^{\langle\lambda\rangle}\mathbf{E}\} \subset \{J'^{\langle\mu\rangle}\mathbf{E}\} = [a'_\bullet]^{\langle\mu\rangle}A$$

where the first inclusion follows from corollary 16.3.40(i), and the last identity follows from corollary 16.3.40(ii). Conversely, suppose that (ii.b) holds; since the topology of A is coarser that the p-adic topology, claim 16.3.41 implies that $[a_\bullet]^{\langle\lambda\rangle}A$ is a dense subset of $\{J^{\langle\lambda\rangle}\mathbf{E}\}$, and since $\{J'^{\langle\mu\rangle}\mathbf{E}\}$ is a closed ideal, corollary 16.3.40(ii) yields $\{J^{\langle\lambda\rangle}\mathbf{E}\} \subset \{J'^{\langle\mu\rangle}\mathbf{E}\}$. Then, arguing as in the proof of (i), we easily deduce that (ii.a) holds.

(iii): We may assume that $\lambda, \mu > 0$, since otherwise there is nothing to prove. Define $b^n_\bullet$ as in the foregoing, for every $n \in \mathbb{N}$; combining corollary 16.3.40(i) and theorem 16.3.47, we get

$$[a_\bullet, b^n_\bullet]^{\langle\lambda\rangle}A \cap [a'_\bullet, b^n_\bullet]^{\langle\mu\rangle}A = \{(J+I^n)^{\langle\lambda\rangle}\mathbf{E}\} \cap \{(J'+I^n)^{\langle\mu\rangle}\mathbf{E}\} = \{(J+I^n)^{\langle\lambda\rangle}\mathbf{E} \cap (J'+I^n)^{\langle\mu\rangle}\mathbf{E}\}$$

for every $n \in \mathbb{N}$. Now, pick $\alpha \in \mathbb{N}[1/p]$ such that $l/\lambda < 1-\alpha$ and $m/\mu < \alpha$ and set $l' := (1-\alpha)\lambda$, $m' := \alpha\mu$. If $x \in (J+I^n)^{\langle\lambda\rangle}\mathbf{E} \cap (J'+I^n)^{\langle\mu\rangle}\mathbf{E}$, we may write

$$x = x^\alpha \cdot x^{1-\alpha} \in (J+I^n)^{\langle l'\rangle} \cdot (J'+I^n)^{\langle m'\rangle}\mathbf{E}.$$

Next, to ease notation, for any finite sequence $\nu_\bullet := (\nu_1, \dots, \nu_m)$ of elements of $\mathbb{N}[1/p]$ and any sequence $(x_1, \dots, x_m)$ of elements of $\mathbf{E}$, set

$$x^{\nu_\bullet}_\bullet := x_1^{\nu_1} \cdots x_m^{\nu_m} \qquad \text{and} \qquad |\nu_\bullet| := \nu_1 + \cdots + \nu_m.$$

We remark that the ideal $(J+I^n)^{\langle l'\rangle} \cdot (J'+I^n)^{\langle m'\rangle}\mathbf{E}$ is generated by all products

$$a^{\nu_\bullet}_\bullet \cdot a'^{\nu'_\bullet}_\bullet \cdot b^{\beta_\bullet}_\bullet \cdot b^{\beta'_\bullet}_\bullet \qquad \text{where} \qquad |\nu_\bullet| + n^{-1} \cdot |\beta_\bullet| = l' \qquad |\nu'_\bullet| + n^{-1} \cdot |\beta'_\bullet| = m'.$$

Fix $l'', m'' \in \mathbb{N}[1/p]$ such that $l < l'' < l'$ and $m < m'' < m'$, and set $\varepsilon := (l'-l'') + (m'-m'')$; it is easily seen that each of the above products lies either in $J^{\langle l''\rangle} \cdot J'^{\langle m''\rangle}\mathbf{E}$ or else in $I^{\langle n\varepsilon\rangle}\mathbf{E}$. Moreover, we can write $l'' = p^{-N}f$, $m'' := p^{-N}g$ for suitable $f, g, N \in \mathbb{N}$, and summing up, we conclude that, for every $\delta < p^{-N}$, the ideal $[a_\bullet]^{\langle\lambda\rangle}A \cap [a'_\bullet]^{\langle\mu\rangle}A$ is contained in :

$$\bigcap_{n\in\mathbb{N}} \{J^{\langle l''\rangle} \cdot J'^{\langle m''\rangle}\mathbf{E} + I^{\langle n\varepsilon\rangle}\mathbf{E}\} = \bigcap_{n\in\mathbb{N}} \{(J^f J'^g)^{\langle 1/p^N\rangle}\mathbf{E} + I^{\langle n\varepsilon\rangle}\mathbf{E}\} \subset \{(J^f J'^g)^{\langle\delta\rangle}\mathbf{E}\}$$

where the first equality follows from lemma 9.3.69(ii.a,iii) the last inclusion follows from corollary 16.3.40(iii.b). Let us then choose $\delta, \delta' \in \mathbb{N}[1/p]$ with $p^{-N} > \delta > \delta'$ and δ' close enough to p^{-N}, so that $\delta' \cdot f > l$ and $\delta' \cdot g > m$; using corollary 16.3.40(i) we get

$$\{(J^f J'^g)^{\langle\delta\rangle}\mathbf{E}\} \subset [a^f_\bullet a'^g_\bullet]^{\langle\delta'\rangle}A \subset [a_\bullet]^{\langle l\rangle} \cdot [a'_\bullet]^{\langle m\rangle}A$$

whence the contention. □

As an application, we can complement as follows theorem 16.3.21(i) :

Proposition 16.3.53. *In the situation of theorem* 16.3.21*, we have :*

(i) (16.3.12) *is a cartesian diagram of topological rings.*
(ii) *All the topological rings appearing in* (16.3.12) *are perfectoid.*

Proof. We know already that (16.3.17) is a cartesian diagram of topological rings. Moreover, it is easily seen that the I-adic topology on A agrees with the $(I+pA)$-adic topology, and likewise for the J-adic and $(I+J)$-adic topologies, so all the topological rings appearing in (16.3.17) except possibly for $(A', \mathscr{T}')$ are perfectoid (proposition 16.3.8(ii)). By the same token, the IJ-adic topology on A agrees with the $(IJ+pA)$-adic topology, so the proposition will follow, once we have shown the following

Claim 16.3.54. The IJ-adic topology on A agrees with the linear topology $\mathscr{T}$ defined by the cofiltered system of ideals $(I^n \cap J^n \mid n \in \mathbb{N})$.

Proof of the claim. Set $I' := I + pA$ and $J' := J + pA$, and notice that $I^{k+N} \subset I'^{N+k} \subset I^{k+1}$ for every $k \in \mathbb{N}$, and likewise for J. Therefore, $\mathscr{T}$ agrees with the topology defined by the cofiltered system of ideals $(I'^n \cap J'^n \mid n \in \mathbb{N})$, and the IJ-adic topology on A agrees with the $I'J'$-adic topology. Thus, we can replace I, J by I', J', and assume from start that

$p \in I \cap J$. Now, let $(\alpha_n \mid n \in \mathbb{N})$ be any distinguished element in $\mathrm{Ker}\, u_A$. Fix finite systems $\beta_\bullet := (\beta_1, \dots, \beta_k)$ and $\beta'_\bullet := (\beta'_1, \dots, \beta'_r)$ of elements of $\mathbf{E}(A)$ such that $\overline{u}_{A/pA}(\beta_\bullet)$ (resp. $\overline{u}_{A/pA}(\beta'_\bullet)$) is a system of generators for I/pA (resp. for J/pA). In light of lemma 16.2.7(iii), it follows that the sequence $(\overline{u}_A(\alpha_0), \overline{u}_A(\beta_1), \dots, \overline{u}_A(\beta_k))$ is a system of generators for I, and $(\overline{u}_A(\alpha_0), \overline{u}_A(\beta'_1), \dots, \overline{u}_A(\beta'_r))$ is a system of generators for J. On the other hand, lemma 9.3.69(iv) implies that the I-adic (resp. J-adic) topology on A agrees with the linear topology defined by the cofiltered system of ideals $([\alpha_0, \beta_\bullet]^{\langle\lambda\rangle} A \mid \lambda \in \mathbb{N}[1/p])$ (resp. $([\alpha_0, \beta'_\bullet]^{\langle\lambda\rangle} A \mid \lambda \in \mathbb{N}[1/p])$). Then, the claim follows from corollary 16.3.51(iii). □

16.3.55. Let A be a ring, and $a_\bullet := (a_1, \dots, a_k)$ a finite system of elements of A; denote by I the ideal generated by $a_\bullet$, and suppose that

$$p \in I^t \qquad \text{where } t := k(p-1)+1. \tag{16.3.56}$$

Set $J := I^{(p)}$, the ideal generated by the system $(a_1^p, \dots, a_k^p)$, and notice that

$$I^t \subset J. \tag{16.3.57}$$

Therefore J is well defined independently of the choice of $a_\bullet$ (lemma 16.2.2). Denote by $\mathrm{gr}^\bullet_I A$ (resp. $\mathrm{gr}^\bullet_J A$) the graded ring associated with the I-adic (resp. J-adic) filtration on A. Notice that both these two rings are $\mathbb{F}_p$-algebras.

Remark 16.3.58. Let A be any P-ring, I any ideal of definition of A, and $(a_1, \dots, a_k)$ a system of generators of I. By virtue of lemma 16.2.3(iii) and its proof, we may find a sequence of ideals $(I_n \mid n \in \mathbb{N})$ such that

- $I^{(p)}_{n+1} = I_n$ for every $n \in \mathbb{N}$
- I_n admits a system of generators consisting of k elements of A.

Then I_n is still an ideal of definition of A for every $n \in \mathbb{N}$, and $p \in I_n^t$ for every sufficiently large $n \in \mathbb{N}$, where t is defined as in (16.3.55). We may thus regard the following result as a refinement of corollary 16.3.3.

Proposition 16.3.59. *In the situation of* (16.3.55)*, the following holds :*

(i) *The Frobenius endomorphism of* $\mathrm{gr}^\bullet_I A$ *factors through a graded ring homomorphism*

$$\Phi_I : \mathrm{gr}^\bullet_I A \to \mathrm{gr}^\bullet_J A \qquad (x \mod I^{n+1}) \mapsto (x^p \mod J^{n+1}) \qquad \textit{for every } n \in \mathbb{N} \textit{ and } x \in I^n.$$

(ii) *Suppose that* A *is perfectoid for its* p*-adic topology. Then* Φ_I *is an isomorphism.*

Proof. (i): To begin with, we remark :

Claim 16.3.60. For every $n \in \mathbb{N}$ and every $x, y \in I^n$, we have $(x+y)^p - x^p - y^p \in J^{n+1}$.

Proof of the claim. The difference in the claim is a sum of terms that lie in pI^{pn}. Taking into account (16.3.56), we see that $pI^{pn} \subset I^{t+pn}$, so it remains only to check that

$$I^{t+pn} \subset J^{n+1}.$$

We argue by induction on n, and notice that the case where $n = 0$ is (16.3.57). Hence, suppose that $r > 0$, and that the sought inclusion has already been checked for $n := r-1$. The ideal I^{t+pr} is generated by all elements of the type $b := \prod_{j=1}^{t+pr} a_{\varphi(j)}$, where φ is any mapping $\{1, \dots, t+pr\} \to \{1, \dots, k\}$. Since $r \geq 0$, it is easily seen that there exists at least one index $i \in \{1, \dots, k\}$ such that $\varphi^{-1}(i)$ has cardinality $\geq p$. For such index i, we have $b = a_i^p \cdot c$, where $c \in I^{t+pn}$, and the inductive assumption says that $c \in J^{n+1}$, whence $b \in J^{r+1}$, as required. ◊

Let us now check that if $x \in I^n$, then $x^p \in J^n$. Indeed, the assertion is clear if x is a monomial $\prod_{j=1}^n a_{\varphi(j)}$ for some mapping $\varphi : \{1, \dots, n\} \to \{1, \dots, k\}$. However, we may write $x = x_1 + \cdots + x_r$ for some $r \in \mathbb{N}$ and a sequence $x_1, \dots, x_r$ of such monomials. Lastly, claim 16.3.60 and an easy induction on r shows that $x^p - \sum_{j=1}^r x_j^p \in J^{n+1}$, whence the contention.

Thus, the map Φ_I is well defined, and claim 16.3.60 also shows that $\Phi_I(x+y) = \Phi_I(x)+\Phi_I(y)$ for every $n \in \mathbb{N}$ and every $x, y \in \mathrm{gr}^n_I A$. For any such x and y it is also clear that $\Phi_I(xy) = \Phi_I(x) \cdot \Phi_I(y)$, so Φ_I is a ring homomorphism.

(ii): Set $\mathbf{E} := \mathbf{E}(A)$, and let $\underline{\alpha} := (\alpha_n \mid n \in \mathbb{N})$ be a distinguished element in $\mathrm{Ker}\, u_A$; arguing as in the proof of lemma 16.2.7(i), we may assume that for every $i = 1, \dots, k$ we have $a_i = \overline{u}_A(\beta_i)$ for some $\beta_i \in \mathbf{E}$. Denote by $\mathscr{I} \subset \mathbf{E}$ the ideal generated by $(\beta_1, \dots, \beta_k)$, and set $\mathscr{J} := \mathscr{I}^{(p)}$, the ideal generated by $(\beta_1^p, \dots, \beta_k^p)$; from (16.3.56) and lemma 16.1.9(ii) we get

$$\alpha_0 \in \mathscr{I}^t \subset \mathscr{J}. \tag{16.3.61}$$

Let $\mathrm{gr}^\bullet_{\mathscr{I}}$ and $\mathrm{gr}^\bullet_{\mathscr{J}}$ be the graded rings associated with the $\mathscr{I}$-adic and $\mathscr{J}$-adic filtrations on $\mathbf{E}$.

Claim 16.3.62. (i) For every $n \in \mathbb{N}$ we have :

(a) The inclusion $\mathscr{I}^{n+1} \subset \mathscr{I}^n$ is $\alpha_0^{1/p}$-taut and the inclusion $\mathscr{J}^{n+1} \subset \mathscr{J}^n$ is taut.

(b) $\{\mathscr{I}^n\} = I^n$ and $\{\mathscr{J}^n\} = J^n$.

(ii) The map $\overline{u}_A : \mathbf{E}(A) \to A$ induces graded ring isomorphisms

$$\mathrm{gr}^\bullet_{\mathscr{I}} \xrightarrow{\sim} \mathrm{gr}^\bullet_I \qquad \mathrm{gr}^\bullet_{\mathscr{J}} \xrightarrow{\sim} \mathrm{gr}^\bullet_J.$$

Proof of the claim. (i.a): Clearly $\mathscr{J} = \Phi_{\mathbf{E}}(\mathscr{I})$, therefore it suffices to prove the assertion for $\mathscr{J}$; however $\alpha_0 \cdot \Phi_{\mathbf{E}}^{-1}(\mathscr{J}^{np}) = \alpha_0 \mathscr{I}^{np} \subset \mathscr{I}^{t+np}$ by (16.3.61), and arguing as in the proof of claim 16.3.60, we see that $\mathscr{I}^{t+np} \subset \mathscr{J}^{n+1}$, whence the claim.

(i.b): We consider first the assertion for $\mathscr{I}^n$. By construction, it is clear that $a_i \in \{\mathscr{I}\}$ for every $i = 1, \dots, k$, hence $I^n \subset \{\mathscr{I}^n\}$. On the other hand, since A is endowed with its p-adic topology, $\mathbf{E}$ carries the α_0-adic topology (see (16.2.19)), therefore $\mathscr{I}^n$ is open in $\mathbf{E}$, and consequently $\{\mathscr{I}^n\}$ is the ideal generated by the system $(\overline{u}_A(x) \mid x \in \mathscr{I}^n)$ (remark 16.3.28(viii)). Thus, we come down to checking that $\overline{u}_A(x) \in I^n$ for every $x \in \mathscr{I}^n$. This is clear in case x is a monomial of the form $\prod_{j=1}^n \beta_{\varphi(j)}$ for some mapping $\varphi : \{1, \dots, n\} \to \{1, \dots, k\}$. In general, x shall be a finite sum $x_1 + \cdots + x_s$, where $x_1, \dots, x_s$ are monomials of this type. According to proposition 9.3.62, $\tau_A(x)$ can then be expressed as a p-adically convergent series $\sum_{r \in \mathbb{N}} p^r \cdot y_r$, where each y_r is in turn a finite sum of terms of the form $w \cdot \tau_A(z)$, for certain $w \in \mathbb{Z}_p$ and $z \in \Phi_{\mathbf{E}}^{-r}(\mathscr{I}^n)^{p^r}$. In view of (i;A) and remark 16.3.28(vi), we have

$$\alpha_0^{r/p} \cdot \Phi_{\mathbf{E}}^{-r}(\mathscr{I}^n)^{p^r} \subset \mathscr{I}^{n+r} \qquad \text{for every } r, n \in \mathbb{N}.$$

Pick $\pi \in A$ as in lemma 16.2.3(iv); recalling lemma 16.2.7(iii), we deduce that

$$p^r \cdot \overline{u}_A(z) \in \pi^{(p-1)r} \cdot \{\mathscr{I}^{n+r}\} \qquad \text{for every } r \in \mathbb{N} \text{ and every } z \in \Phi_{\mathbf{E}}^{-r}(\mathscr{I}^n)^{p^r}.$$

Since $\{\mathscr{I}^{n+r}\}$ is closed in the p-adic topology of A for every $n, r \in \mathbb{N}$, it follows that $\{\mathscr{I}^n\} \subset I^n + \pi^{(p-1)r}\{\mathscr{I}^{n+r}\}$ for every $r \in \mathbb{N}$. But clearly $\pi^{(p-1)r} \in I^n$ for every large enough $r \in \mathbb{N}$, so $\{\mathscr{I}^n\} \subset I^n$, as required.

To deal with $\mathscr{J}^n$, we repeat the foregoing argument, to get : $\{\mathscr{J}^n\} \subset J^n + \{\mathscr{J}^{n+r}\}$ for every $r \in \mathbb{N}$. However, clearly $\mathscr{J}^{n+r} \subset \mathscr{I}^{n+r}$; taking into account the previous case, we conclude that $\{\mathscr{J}^n\} \subset J^n + I^s$ for every $s \in \mathbb{N}$. But we may find $s \in \mathbb{N}$ large enough, so that $I^s \subset J^n$, so at last $\{\mathscr{J}^n\} \subset J^n$, whence the claim.

(ii) follows immediately from (i) and theorem 16.3.31(ii). ◊

Claim 16.3.62(ii) yields a commutative diagram of graded rings

$$\begin{array}{ccc} \mathrm{gr}^\bullet_{\mathscr{I}} & \longrightarrow & \mathrm{gr}^\bullet_I A \\ \downarrow & & \downarrow{\scriptstyle \Phi_I} \\ \mathrm{gr}^\bullet_{\mathscr{J}} & \longrightarrow & \mathrm{gr}^n_J A \end{array}$$

whose horizontal arrows are isomorphisms, and whose left vertical arrow is induced by $\Phi_{\mathbf{E}}$. Now, since $\mathbf{E}$ is perfect, the left vertical arrow is an isomorphism, so the same holds for the right vertical arrow, as stated. □

Corollary 16.3.63. *Let A be a perfectoid ring, and $\beta \in \mathbf{E}(A)$ any element.*

(i) *A is a reduced ring.*
(ii) *If $\beta \neq 0$, then $\overline{u}_A(\beta) \neq 0$.*
(iii) *If the topology of A is discrete, then A is an $\mathbb{F}_p$-algebra.*

Proof. (i): Suppose that A contains a nilpotent element $x \neq 0$, and let $n > 1$ be the smallest integer such that $x^n = 0$; set $y := x^{n-1}$, so that $y \neq 0$ and $y^p = 0$. By remark 16.3.58 we can find an ideal of definition I of A fulfilling condition (16.3.56); we set $J := I^{(p)}$ and let $\Phi_I : \mathrm{gr}^\bullet_I A \xrightarrow{\sim} \mathrm{gr}^\bullet_J A$ be the isomorphism of proposition 16.3.59(ii). Let also $r \in \mathbb{N}$ be the largest integer such that $y \in I^r$, and denote by $\overline{y} \in \mathrm{gr}^r_I A$ the class of y, so that $\overline{y} \neq 0$; but clearly $\Phi_I(\overline{y}) = 0$, a contradiction.

(ii) and (iii) are immediate consequences of (i). □

Theorem 16.3.64. *In the situation of* (16.3.55)*, let $\mathscr{T}_I$ be the I-adic topology on A, and suppose that $(A, \mathscr{T}_I)$ is a P-ring. Define also the graded ring homomorphism Φ_I as in proposition* 16.3.59(i)*. Then the following conditions are equivalent :*

(a) *$(A, \mathscr{T}_I)$ is perfectoid.*
(b) *The map Φ_I is an isomorphism.*

Proof. Suppose that (a) holds, and let $\mathscr{T}_p$ be the p-adic topology on A; then $(A, \mathscr{T}_p)$ is perfectoid, by proposition 16.3.8(i), in which case proposition 16.3.59(ii) says that (b) holds.

Next, suppose that (b) holds; then proposition 16.2.9 and example 16.3.2(i) say that $\mathbf{E} := \mathbf{E}(A)$ is a perfectoid ring. Moreover, $\overline{u}_{A/pA}$ is surjective (lemma 16.2.7(i)), so for every $i = 1, \dots, k$ we may find $\beta_i \in \mathbf{E}$ such that $\overline{u}_{A/pA}(\beta_i)$ equals the image of a_i in A/pA. Denote by $\mathscr{J} \subset \mathbf{E}$ the ideal generated by the system $(\beta_1, \dots, \beta_k)$; from claim 16.2.10 we see that $\mathscr{J}$ is an ideal of definition of $\mathbf{E}$, and $\overline{u}_{A/pA}$ induces a ring isomorphism

$$\mathbf{E}/\mathscr{J} \xrightarrow{\sim} A/I. \qquad (16.3.65)$$

Denote by $\mathscr{J}_W \subset W(\mathbf{E})$ the ideal generated by $(\tau_{\mathbf{E}}(\beta_1), \dots, \tau_{\mathbf{E}}(\beta_k))$; by construction we have $u_A(\tau_{\mathbf{E}}(\beta_i)) - a_i \in pA$ for every $i = 1, \dots, k$, and notice that $p \in I^t \subset I^2$, so $\mathscr{J}_W A + I^2 = I$ and Nakayama's lemma yields

$$\mathscr{J}_W A = I. \qquad (16.3.66)$$

Furthermore, recall that $\mathscr{I} := \mathrm{Ker}\, u_A$ contains a distinguished element $\underline{\alpha}$ of $W(\mathbf{E})$ (lemma 16.2.7(ii)); on the other hand, (16.3.66) and (16.3.56) imply that $p \in \mathscr{J}^t_W A$, in which case we may apply lemma 16.1.9(i) to get an element $\underline{\alpha}' := (\alpha'_n \mid n \in \mathbb{N}) \in \mathscr{I}$ such that $\alpha'_0 \in \mathscr{J}^t$ and $\alpha'_1 \in \mathbf{E}^\times$. Especially, $\underline{\alpha}'$ is a distinguished element of $W(\mathbf{E})$, and so the ring

$$B := W(\mathbf{E})/\underline{\alpha}' W(\mathbf{E})$$

is perfectoid, for the quotient topology arising from the natural projection $W(\mathbf{E}) \to B$ (example 16.3.2(ii)). By construction, u_A factors through a surjective and continuous ring homomorphism

$$v : B \to A.$$

Moreover, set $I_B := \mathscr{J}_W B \subset B$, and notice that on the one hand, $u_B \circ \tau_{\mathbf{E}}(\alpha'_0) = pt$ for some $t \in B^\times$ (lemma 16.2.7(iii)), and on the other hand we have $u_B \circ \tau_{\mathbf{E}}(\alpha'_0) \in I^t_B$, so

$$p \in I^t_B. \qquad (16.3.67)$$

Combining with proposition 9.3.77(ii), we deduce that I_B is an ideal of definition of B, and taking into account (16.3.66), we also get

$$v(I_B) = I. \tag{16.3.68}$$

Set $J_B := I_B^{(p)}$; from (16.3.67), (16.3.68) and propositions 16.3.59(ii) and 16.3.8(i) we then get a commutative diagram of graded ring homomorphisms

$$\begin{array}{ccc} \mathrm{gr}^\bullet_{I_B} B & \longrightarrow & \mathrm{gr}^\bullet_I A \\ \downarrow & & \downarrow \\ \mathrm{gr}^\bullet_{J_B} B & \longrightarrow & \mathrm{gr}^\bullet_J A \end{array} \tag{16.3.69}$$

both of whose vertical arrows are isomorphisms, and whose horizontal arrows are induced by v. Since B is complete and separated for its I_B-adic topology, and A is complete and separated for its I-adic topology, the theorem will follow from :

Claim 16.3.70. The map v induces a ring isomorphism $v_n : B/I_B^{p^n} \xrightarrow{\sim} A/I^{p^n}$ for every $n \in \mathbb{N}$.

Proof of the claim. We argue by induction on n. For $n = 0$, notice that the projection $W(\mathbf{E}) \to B/I_B$ factors through $\mathbf{E}$, and by construction, the kernel of the resulting surjection is $\mathscr{J}$. Then the assertion follows from (16.3.65). Next, suppose that $n \geq 0$, and that v_n is an isomorphism. In this case, v induces isomorphisms $\mathrm{gr}^r_{I_B} B \xrightarrow{\sim} \mathrm{gr}^r_I A$ for every $r < p^n$. Due to (16.3.69), we then deduce that v also induces isomorphisms $\mathrm{gr}^r_{J_B} B \xrightarrow{\sim} \mathrm{gr}^r_J A$ for every $r < p^n$, from which it follows easily that v induces an isomorphism $B/J_B^{p^n} \xrightarrow{\sim} A/J^{p^n}$. However, clearly $J \subset I^p$ and $J_B \subset I_B^p$, hence $J_B^{p^n} \subset I_B^{p^{n+1}}$ and $J^{p^n} \subset I^{p^{n+1}}$, so finally v_{n+1} is an isomorphism. □

As a corollary of theorem 16.3.64 we obtain the following criterion :

Corollary 16.3.71. *Let A be a ring, $t \in \mathbb{N}$ any integer, $\beta := (b_n \mid n \in \mathbb{N})$ an element of $\mathbf{E}(A)$; suppose that A is complete and separated for its b_0-adic topology $\mathscr{T}_0$, and $p = ab_0$ for some $a \in A$. Let also $J := \bigcup_{n\in\mathbb{N}} b_n A$. Consider the following conditions :*

(a) *A/J is a perfect $\mathbb{F}_p$-algebra.*
(b) *The p-th power map of $A/b_t^2 A$ induces a bijection $b_{t+1}A/b_{t+1}^2 A \xrightarrow{\sim} b_t A/b_t^2 A$.*
(c) $\mathrm{Ann}_A(b_0) = \mathrm{Ann}_A(b_n)$ *for every $n \in \mathbb{N}$.*
(d) *The Frobenius endomorphism Φ_{A/b_iA} of A/b_iA induces an isomorphism*

$$\overline{\Phi}_{A/b_iA} : A/b_{i+1}A \xrightarrow{\sim} A/b_iA \qquad \textit{for every } i \in \mathbb{N}.$$

(e) *$(A, \mathscr{T}_0)$ is a perfectoid ring.*

Then (e) *is equivalent to the conjunction of* (a),(b) *and* (c)*, and also to the conjunction of* (c) *and* (d)*. Moreover, if* (e) *holds, then $A/\mathrm{Ann}_A(b_0)$ is perfectoid for the b_0-adic topology, and the map $\overline{u}_A : \mathbf{E}(A) \to A$ induces an isomorphism of $\mathbf{E}(A)$-modules :*

$$\mathrm{Ann}_{\mathbf{E}(A)}(\beta) \xrightarrow{\sim} \mathrm{Ann}_A(b_0). \tag{16.3.72}$$

Proof. Suppose first that (a),(b),(c) hold. For every $i, j > 0$, consider the commutative diagram

$$\begin{array}{ccc} b_i^{j-1}A/b_i^jA & \xrightarrow{f_{i,j}} & b_i^jA/b_i^{j+1}A \\ \downarrow{\scriptstyle \varphi_{i,j-1}} & & \downarrow{\scriptstyle \varphi_{i,j}} \\ b_{i-1}^{j-1}A/b_{i-1}^jA & \xrightarrow{f_{i-1,j}} & b_{i-1}^jA/b_{i-1}^{j+1}A \end{array}$$

where $\varphi_{i,j}$ is induced by the p-th power map of $A/b_{i-1}^{j+1}A$ for every $i, j \in \mathbb{N}$ with $i > 0$, and $f_{i,j}$ is induced by scalar multiplication by b_i, for every $i, j \in \mathbb{N}$ with $j > 0$. We notice :

Claim 16.3.73. (i) $f_{i,j}$ is an isomorphism for every $i, j \in \mathbb{N}$ with $j > 1$.

(ii) $\varphi_{i,1}$ is an isomorphism for every $i > t$.

Proof of the claim. (i): The surjectivity of $f_{i,j}$ is clear. For the injectivity, suppose that $x = b_i^{j-1}y$ and $b_i x = b_i^{j+1}z$ for some $y, z \in A$. It follows that $b_i^j y = b_i^{j+2}z$, so $b_i^j(y - b_i^2 z) = 0$, and therefore also $b_i(y - b_i^2 z) = 0$, by (c). Since $j > 1$, we deduce that $x = b_i^j z$, whence the assertion.

(ii): We argue by induction on i. The case $i = t+1$ is (b). Suppose then that $i > t$ and that the surjectivity of $\varphi_{i,1}$ is already known. This means that for every $x \in A$ there exist $y, z \in A$ such that $b_{i-1}x = b_{i-1}y^p + b_{i-1}^2 z$, *i.e.* that $b_{i-1}(x - y^p - b_{i-1}z) = 0$. By assumption (c), we deduce that $b_i(x - y^p - b_{i-1}z) = 0$. Set $z' := b_i^{p-1}z$; then $b_i(x - y^p - b_i z') = 0$, *i.e.* $b_i x = b_i y_p + b_i^2 z'$, whence the surjectivity of $\varphi_{i+1,1}$.

The injectivity of $\varphi_{i,1}$ means the following. For every $x \in A$ such that $b_{i-1}x^p \in b_{i-1}^2 A$ we have $b_i x \in b_i^2 A$. Now, let $a \in A$, and suppose that $b_i a^p \in b_i^2 A$; then $b_{i-1}(b_i^{p-1}a^p) = b_i^{2p-1}a^p \in b_{i-1}^2 A$, so that $b_{i+1}^{2p-1}a = b_i b_{i+1}^{p-1} a \in b_i^2 A$. Say that $b_{i+1}^{2p-1}a = b_i^2 y$; then

$$b_{i+2}^{2p^2-p-1}(b_{i+2}a - b_{i+2}^{p+1}y) = 0.$$

By assumption (c), it follows that $b_{i+2}^{p-1}(b_{i+2}a - b_{i+2}^{p+1}y) = 0$ as well, and therefore $b_{i+1}a \in b_{i+1}^2 A$, hence $\varphi_{i+1,1}$ is injective. ◊

From claim 16.3.73 it follows immediately that $\varphi_{i,j}$ is an isomorphism for every $i > t$ and every $j > 0$. Notice now that $(A, \mathscr{T}_0)$ is a P-ring, and $I := b_i A$ is an ideal of definition of A, for every $i > 0$, due to assumptions (a) and (b); moreover, any such I fulfills the conditions of (16.3.55). In light of theorem 16.3.64, it then suffices to show that $\overline{\Phi}_{A/b_iA}$ is bijective for every $i \geq t$. To this aim, from the snake lemma and assumption (a), we are easily reduced to checking that $\Phi_{A/b_{i-1}A}$ induces a bijection $J/b_iA \xrightarrow{\sim} J/b_{i-1}A$ for every $i > t$. Then, since J is the union of its subideals b_kA (for all $k \in \mathbb{N}$), we are further reduced to checking that $\Phi_{A/b_{i-1}A}$ induces a bijection $b_kA/b_iA \xrightarrow{\sim} b_{k-1}A/b_{i-1}A$ for every $k \geq i$. By considering the filtration

$$0 \subset b_{i+1}A/b_iA \subset b_{i+2}A/b_i \subset \cdots \subset b_kA/b_iA$$

an easy induction then reduces to the case where $k = i+1$. Lastly, a similar induction argument, using the filtration $0 \subset b_{i+1}^{p-1}/b_i \subset b_{i+1}^{p-2}/b_i \subset \cdots \subset b_{i+1}/b_i$ further reduces the assertion to the surjectivity of $\varphi_{i,j}$ for every $j = 1, \ldots, p-1$, which is already known, by the foregoing.

Conversely, if (e) holds, then we get both (b) and (d), by virtue of proposition 16.3.59; then, notice that the direct limit of the system of maps $(\overline{\Phi}_{A/b_iA} \mid i \geq t)$ is the Frobenius endomorphism of A/J, whence (a). Next, let $x \in A$ with $b_0 x = 0$; it follows that $(b_n x)^{p^n} = 0$ for every $n \in \mathbb{N}$, and since A is reduced (corollary 16.3.63(i)), we get $b_n x = 0$, whence (c).

Lastly, suppose that (c) and (d) hold; then we get a commutative diagram whose horizontal arrows are restrictions of Φ_{A/b_0A}, and are therefore bijections :

$$\begin{array}{ccc} b_{t+1}^2A/b_1A & \xrightarrow{\sim} & b_t^2A/b_0A \\ {\scriptstyle j_{t+1}}\downarrow & & \downarrow{\scriptstyle j_t} \\ b_{t+1}A/b_1A & \xrightarrow{\sim} & b_tA/b_0A \end{array}$$

and whose vertical arrows are the natural injections; hence, we get an induced isomorphism $\mathrm{Coker}\, j_{t+1} \xrightarrow{\sim} \mathrm{Coker}\, j_t$, whence (b). It has already been remarked that (d) implies (a), so the proof of the stated equivalences is complete.

Next, suppose that A is perfectoid, so all conditions (a),...,(e) hold, and let us notice that

$$\mathrm{Ann}_A(b_0) \cap b_n A = 0 \qquad \text{for every } n \in \mathbb{N}. \tag{16.3.74}$$

Indeed, if $x \in A$ and we have $b_0 \cdot (b_n x) = 0$, then $b_n x = 0$ by (c). This means that $\mathscr{T}_0$ induces the discrete topology on $\mathrm{Ann}_A(b_0)$, so $\overline{A} := A/\mathrm{Ann}_A(b_0)$ is complete and separated for its b_0-adic topology. Consider now the commutative diagram :

$$\begin{array}{ccc} A/b_{t+1}A & \xrightarrow{\overline{\Phi}_{A/b_iA}} & A/b_tA \\ {\scriptstyle f_{1,t+1}}\downarrow & & \downarrow{\scriptstyle f_{1,t}} \\ b_{t+1}A/b_{t+1}^2A & \longrightarrow & b_tA/b_t^2A \end{array}$$

whose left (resp. right) vertical arrow is induced by multiplication by b_{t+1} (resp. by b_t), and whose top (resp. bottom) horizontal arrow is induced by the Frobenius endomorphism of A/b_tA (resp. of A/b_t^2A). Due to (b),(d) and (16.3.74), we see that $\overline{\Phi}_{A/b_tA}$ restricts to a bijection

$$\mathrm{Ann}_A(b_{t+1}) = \mathrm{Ker}\, f_{1,t+1} \xrightarrow{\sim} \mathrm{Ker}\, f_{1,t} = \mathrm{Ann}_A(b_t)$$

and combining with (c) we deduce that the Frobenius endomorphism of A/b_iA restricts to a bijection $\mathrm{Ann}_A(b_0) \xrightarrow{\sim} \mathrm{Ann}_A(b_0)$ for every $i \in \mathbb{N}$. Then, using (d) we conclude that the Frobenius endomorphism of $\overline{A}/b_i\overline{A}$ induces an isomorphism $\overline{\Phi}_{\overline{A}/b_i\overline{A}} : \overline{A}/b_{i+1}\overline{A} \xrightarrow{\sim} \overline{A}/b_i\overline{A}$ for every $i \in \mathbb{N}$. Lastly, it is easily seen that $\mathrm{Ann}_{\overline{A}}(b_i) = 0$ for every $i \in \mathbb{N}$. Summing up, we have shown that the ring $\overline{A}$ fulfills conditions (c) and (d), so it is perfectoid for its b_0-adic topology, by the foregoing. Now, by virtue of (16.3.74), we may naturally identify $\mathrm{Ann}_A(b)$ with its image in A/b_0A, and likewise one easily sees that $\mathrm{Ann}_{\mathbf{E}(A)}(\beta) \cap \beta\mathbf{E}(A) = 0$, hence we may identify $\mathrm{Ann}_{\mathbf{E}(A)}(\beta)$ with its image in $\mathbf{E}(A)/\beta\mathbf{E}(A)$. By claim 16.3.62(ii), we have a commutative diagram of A_0/b_0A-modules

$$\begin{array}{ccc} \mathbf{E}(A)/\beta\mathbf{E}(A) & \longrightarrow & \beta\mathbf{E}(A)/\beta^2\mathbf{E}(A) \\ \downarrow & & \downarrow \\ A/b_0A & \longrightarrow & b_0A_0/b_0^2A_0 \end{array}$$

whose top (resp. bottom) horizontal arrow is induced by multiplication by β (resp. by b_0), and whose vertical arrows are induced by $\overline{u}_A$. To conclude, it suffices to remark that the kernel of the top (resp. bottom) horizontal arrow is $\mathrm{Ann}_{\mathbf{E}(A)}(\beta)$ (resp. $\mathrm{Ann}_A(b_0)$). ☐

Let us point out the following variant of corollary 16.3.71 :

Corollary 16.3.75. *Let A be ring, $b \in A$ an element such that $p \in b^pA$, and such that the b-adic topology $\mathscr{T}$ of A is complete and separated. Then $(A, \mathscr{T})$ is perfectoid if and only if the following two conditions hold :*

(a) *The Frobenius endomorphism Φ_{A/b^pA} of A/b^pA induces a bijection $A/bA \xrightarrow{\sim} A/b^pA$.*
(b) $\mathrm{Ann}_A(b^p) = \mathrm{Ann}_A(b^{p-1})$.

Proof. If A is perfectoid, then $\overline{u}_{A/pA} : \mathbf{E}(A) \to A/pA$ is surjective (lemma 16.2.7(i)), so there exists $(b_n \mid n \in \mathbb{N}) \in \mathbf{E}(A)$ such that $b_0 - b \in pA_0$, and it follows easily that $b_0/b \in A_0^\times$. Thus, we may replace b by b_0, in which case the assumptions of corollary 16.3.71 are fulfilled, and we deduce that both (a) and (b) hold.

Conversely, if (a) holds, then A is a P-ring, and if also (b) holds, arguing as in the foregoing we may again assume that $b = b_0$ for some $(b_n \mid n \in \mathbb{N}) \in \mathbf{E}(A)$; then we have, more generally:

Claim 16.3.76. If (a) holds and there exists $t \in \mathbb{N}$ such that $\mathrm{Ann}_A(b^p) = \mathrm{Ann}_A(b^p/b_t)$, then A is perfectoid.

Proof of the claim. In light of theorem 16.3.64, it suffices to show that the Frobenius endomorphism of A/b_tA induces bijections

$$\Phi_i : \mathrm{gr}_i^{(t+1)} := b_{t+1}^i A/b_{t+1}^{i+1}A \stackrel{\sim}{\to} \mathrm{gr}_i^{(t)} := b_t^i A/b_t^{i+1}A \qquad \text{for every } i \in \mathbb{N}.$$

However, (a) easily implies that Φ_i is an isomorphism for every $i = 0, \dots, p^{t+1} - 1$. We then argue by induction on $i \geq p^{t+1} - 1$. Thus suppose that Φ_i is an isomorphism for such i; we consider the commutative diagram

$$\begin{array}{ccc} \mathrm{gr}_i^{(t+1)} & \xrightarrow{\Phi_i} & \mathrm{gr}_i^{(t)} \\ {\scriptstyle \overline{b}_{t+1}}\downarrow & & \downarrow{\scriptstyle \overline{b}_t} \\ \mathrm{gr}_{i+1}^{(t+1)} & \xrightarrow{\Phi_{i+1}} & \mathrm{gr}_{i+1}^{(t)} \end{array}$$

whose vertical arrows are induced by scalar multiplication by b_{t+1} and respectively b_t. Clearly both vertical arrows are surjections; thus, we are reduced to checking that $\overline{b}_t$ is injective. Hence, say that for some $a \in A$, the class of $b_t^i a$ in $\mathrm{gr}_i^{(t)}$ lies in the kernel of $\overline{b}_t$; this means that there exists $c \in A$ with $b_t^{i+1}a = b_t^{i+2}c$, so that $b_t^{i+1}(a - b_tc) = 0$, and since $i + 1 \geq p^{t+1}$, our assumption easily implies that $b_t^i(a - b_tc) = 0$ as well. So, the class of $b_t^i a$ vanishes in $\mathrm{gr}_i^{(t)}$, as required. □

Proposition 16.3.8 shows that the completion of a perfectoid ring for the topology defined by a p-adically open and finitely generated ideal, is again perfectoid for the p-adic topology. We now wish to prove that the same still holds for a general finitely generated ideal.

16.3.77. Namely, let A be a perfectoid ring, $I \subset A$ (resp. $\mathscr{I} \subset \mathbf{E} := \mathbf{E}(A)$) a finitely generated ideal such that the topology of A (resp. of $\mathbf{E}$) agrees with the I-adic topology $\mathscr{T}_I$ (resp. with the $\mathscr{I}$-adic topology $\mathscr{T}_\mathscr{I}$). Let also $\beta_\bullet := (\beta_1, \dots, \beta_n)$ be any finite system of elements of $\mathbf{E}$ and $\mathscr{J} \subset \mathbf{E}$ (resp. $J \subset A$) the ideal generated by $\beta_\bullet$ (resp. by $\overline{u}_A(\beta_1), \dots, \overline{u}_A(\beta_n)$). Denote by $\mathscr{T}_J$ (resp. $\mathscr{T}_\mathscr{J}$) the J-adic on A (resp. the $\mathscr{J}$-adic topology on $\mathbf{E}$), and let

$$(A_J^\wedge, \mathscr{T}_J^\wedge) \qquad \text{and} \qquad (\mathbf{E}_\mathscr{J}^\wedge, \mathscr{T}_\mathscr{J}^\wedge)$$

respectively the completion of $(A, \mathscr{T}_J)$ and of $(\mathbf{E}, \mathscr{T}_\mathscr{J})$. Lastly, let $\mathscr{T}_I^\wedge$ (resp. $\mathscr{T}_\mathscr{I}^\wedge$) be the $IA_J^\wedge$-adic topology on $A_J^\wedge$ (resp. the $\mathscr{I}\mathbf{E}_\mathscr{J}^\wedge$-adic topology on $\mathbf{E}_\mathscr{J}^\wedge$).

Theorem 16.3.78. *In the situation of* (16.3.77)*, the following holds :*

(i) $(A_J^\wedge, \mathscr{T}_I^\wedge)$ *is perfectoid.*

(ii) *There is a natural isomorphism of topological rings* $\mathbf{E}(A_J^\wedge, \mathscr{T}_I^\wedge) \stackrel{\sim}{\to} (\mathbf{E}_\mathscr{J}^\wedge, \mathscr{T}_\mathscr{I}^\wedge)$.

Proof. Let $J_\mathbf{A} \subset \mathbf{A} := W(\mathbf{E})$ be the ideal generated by $\tau_\mathbf{E}(\beta_1), \dots, \tau_\mathbf{E}(\beta_n)$, set $J'_\mathbf{A} := J_\mathbf{A} + p\mathbf{A}$ and denote by $\mathbf{A}_J^\wedge$ (resp. $\mathbf{A}_{J'}^\wedge$) the $J_\mathbf{A}$-adic (resp. $J'_\mathbf{A}$-adic) completion of $\mathbf{A}$. Moreover, pick any finite system of generators $(\gamma_1, \dots, \gamma_k)$ for $\mathscr{I}$, let $I_\mathbf{A} \subset \mathbf{A}$ be the ideal generated by the system $(p, \tau_\mathbf{E}(\gamma_1), \dots, \tau_\mathbf{E}(\gamma_k))$, and denote by $\mathscr{T}_{I,\mathbf{A}}^\wedge$ indifferently the $I_\mathbf{A}$-adic topologies on $\mathbf{A}_{J'}^\wedge$ and on $\mathbf{A}_J^\wedge$, and by $\mathscr{T}_{J',\mathbf{A}}^\wedge$ the $J'_\mathbf{A}$-adic topology on $\mathbf{A}_{J'}^\wedge$. With this notation, we have natural isomorphisms of topological $\mathbf{A}$-algebras

$$W(\mathbf{E}_\mathscr{J}^\wedge, \mathscr{T}_\mathscr{J}^\wedge) \stackrel{\sim}{\to} (\mathbf{A}_{J'}^\wedge, \mathscr{T}_{J',\mathbf{A}}^\wedge) \qquad \text{and} \qquad W(\mathbf{E}_\mathscr{J}^\wedge, \mathscr{T}_\mathscr{I}^\wedge) \stackrel{\sim}{\to} (\mathbf{A}_{J'}^\wedge, \mathscr{T}_{I,\mathbf{A}}^\wedge) \tag{16.3.79}$$

(proposition 9.3.77(ii) and lemma 9.3.33(iv)). On the other hand, since the $J_\mathbf{A}$-adic topology is finer than the $J'_\mathbf{A}$-adic topology, the completion map $\mathbf{A} \to \mathbf{A}_{J'}^\wedge$ factors through a unique map of $\mathbf{A}$-algebras

$$\mathbf{A}_J^\wedge \to \mathbf{A}_{J'}^\wedge. \tag{16.3.80}$$

Claim 16.3.81. The map (16.3.80) is a ring isomorphism.

Proof of the claim. Quite generally, consider any abelian group M, endowed with two topologies $\mathscr{T}$ and $\mathscr{T}'$ defined by cofiltered systems of subgroups $(M_\lambda \mid \lambda \in \Lambda)$ and respectively $(M'_\mu \mid \mu \in \Lambda')$. Endow each quotient $Q_\lambda := M/M_\lambda$ with the quotient topology $\mathscr{T}'_\lambda$ induced by $\mathscr{T}'$ via the projection map $M \to Q_\lambda$, and suppose moreover that $\mathscr{T}$ is finer than $\mathscr{T}'$ and $(Q_\lambda, \mathscr{T}'_\lambda)$ is separated and complete for every $\lambda \in \Lambda$. Then it follows that the natural map

$$(M, \mathscr{T})^\wedge \to (M, \mathscr{T}')^\wedge$$

(that extends the completion map $M \to (M, \mathscr{T}')^\wedge$), is an isomorphism of abelian groups. Indeed, clearly the family $(M_\lambda + M'_\mu \mid \lambda \in \Lambda, \mu \in \Lambda')$ is still a fundamental system of open neighborhoods of zero in M for the topology $\mathscr{T}'$, so we have natural isomorphisms

$$(M, \mathscr{T}')^\wedge \xrightarrow{\sim} \lim_{(\lambda,\mu)\in\Lambda\times\Lambda'} M/(M_\lambda + M'_\mu) \xrightarrow{\sim} \lim_{\lambda\in\Lambda} \lim_{\mu\in\Lambda'} M/(M_\lambda + M'_\mu) \xrightarrow{\sim} \lim_{\lambda\in\Lambda} (Q_\lambda, \mathscr{T}')^\wedge$$

(where the second isomorphism follows from example 1.5.15(ii)) but $(Q_\lambda, \mathscr{T}')^\wedge = Q_\lambda$ under the current assumptions, whence the assertion.

Recall now that the family $(W(\mathscr{J}^{\langle\lambda\rangle}) \mid \lambda \in \mathbb{N}[1/p])$ is a fundamental system of open neighborhoods of zero in for the $J_{\mathbf{A}}$-adic topology on $\mathbf{A}$ (proposition 9.3.77(i) and lemma 9.3.69(iv)). In view of the foregoing, the claim will then follow, once we have shown that the quotients $\mathbf{A}/W(\mathscr{J}^{\langle\lambda\rangle})$ are complete and separated for the $J'_{\mathbf{A}}$-adic topology. Moreover, it is clear that $J'_{\mathbf{A}}$-adic topology on $\mathbf{A}/W(\mathscr{J}^{\langle\lambda\rangle})$ agrees with the p-adic topology, so – taking into account proposition 9.3.44(ii) – we come down to checking that $W(\mathscr{J}^{\langle\lambda\rangle})$ is a closed ideal for the p-adic topology $\mathscr{T}_{p,\mathbf{A}}$ on $\mathbf{A}$. But notice that if $\mathscr{T}_d$ denotes the discrete topology on $\mathbf{E}$, then $W(\mathbf{E}, \mathscr{T}_d) = (\mathbf{A}, \mathscr{T}_{p,\mathbf{A}})$, so the assertion is a special case of remark 9.3.28(iv). ◊

Claim 16.3.82. $(\mathbf{E}^\wedge_{\mathscr{J}}, \mathscr{T}^\wedge_{\mathscr{J}})$ is perfectoid.

Proof of the claim. Since $\mathbf{E}$ is a perfect topological ring, the same holds for $(\mathbf{E}, \mathscr{T}_{\mathscr{J}})$, hence also for $(\mathbf{E}^\wedge_{\mathscr{J}}, \mathscr{T}^\wedge_{\mathscr{J}})$ (remark 9.4.9(v) and example 9.3.48(ii)), and by the same token, $(\mathbf{E}^\wedge_{\mathscr{J}}, \mathscr{T}^\wedge_{\mathscr{J}})$ is a perfect topological $\mathbb{F}_p$-algebra. By example 16.3.2(i), it then remains only to check that the topology $\mathscr{T}^\wedge_{\mathscr{J}}$ is complete and separated. However, since $\mathbf{E}$ is complete and separated for the $\mathscr{I}$-adic topology, $\mathbf{E}^\wedge_{\mathscr{J}}$ is complete and separated for the $(\mathscr{I} + \mathscr{J})$-adic topology (lemma 16.3.15(ii.b)) and then the assertion follows from lemma 8.3.12. ◊

Let $\underline{\alpha} := (\alpha_n \mid n \in \mathbb{N}) \in \mathrm{Ker}\, u_A$ be any distinguished element; clearly $\underline{\alpha}$ is still distinguished in $W(\mathbf{E}^\wedge_{\mathscr{J}}, \mathscr{T}^\wedge_{\mathscr{J}})$, hence $W(\mathbf{E}^\wedge_{\mathscr{J}}, \mathscr{T}^\wedge_{\mathscr{J}}) \otimes_{\mathbf{A}} A$ is perfectoid (example 16.3.2(ii)). From (16.3.79) and claim 16.3.81 we deduce that both (i) and (ii) will follow, once we have shown

Claim 16.3.83. There is a natural isomorphism of topological rings :

$$(\mathbf{A}^\wedge_J, \mathscr{T}^\wedge_{I,\mathbf{A}}) \otimes_{\mathbf{A}} A \xrightarrow{\sim} (A^\wedge_J, \mathscr{T}^\wedge_I).$$

Proof of the claim. By construction, the $J_{\mathbf{A}}$-adic topology induces the J-adic topology on A, via the projection $u_A : \mathbf{A} \to A$. Therefore, u_A extends to a surjective map

$$u^\wedge_A : \mathbf{A}^\wedge_J \to A^\wedge_J$$

whose kernel is the topological closure of $\underline{\alpha}\mathbf{A}$ in the $J_{\mathbf{A}}$-adic topology of $\mathbf{A}^\wedge_J$ (proposition 8.2.13(i,v) and lemma 8.3.32(iv)). On the other hand, by proposition 9.3.77(ii) and lemma 16.2.7(i), the $I_{\mathbf{A}}$-adic topology on $\mathbf{A}$ induces the I-adic topology on A via u_A, so the topology $\mathscr{T}^\wedge_{I,\mathbf{A}}$ induces the topology $\mathscr{T}^\wedge_I$ on $A^\wedge_J$, via $u^\wedge_A$. Thus, we are reduced to checking that $\underline{\alpha}\mathbf{A}^\wedge_J$ is a closed ideal in the $J_{\mathbf{A}}$-adic topology of $\mathbf{A}^\wedge_J$. By claim 16.3.81, it then further suffices to show that $\underline{\alpha}\mathbf{A}^\wedge_{J'}$ is closed for the topology $\mathscr{T}^\wedge_{J',\mathbf{A}}$. But in light of (16.3.79), the latter assertion follows from corollary 16.1.13(ii). □

As an application we get the following generalization of theorem 16.3.21(ii) :

Corollary 16.3.84. *Let $(A, \mathscr{T})$ be a perfectoid ring, $\beta_\bullet := (\beta_1, \dots, \beta_k)$ and $\gamma_\bullet := (\gamma_1, \dots, \gamma_l)$ two finite sequences of elements of $\mathbf{E} := \mathbf{E}(A)$, and denote by I (resp. J) the ideal of A generated by the system $(\overline{u}_A(\beta_1), \dots, \overline{u}_A(\beta_k))$ (resp. by $\overline{u}_A(\gamma_1), \dots, \overline{u}_A(\gamma_l)$). Then the resulting map* (16.3.14) *is an isomorphism.*

Proof. Let A' be the I-adic completion of A, and for every $k \in \mathbb{N}$, denote by $(J^k)^c$ the topological closure of the ideal J^kA' relative to the IA'-adic topology; arguing as in the proof of lemma 16.3.15(ii.b), we are easily reduced to showing that the JA'-adic topology on A' agrees with the linear topology defined by the system of ideals $((J^k)^c \mid k \in \mathbb{N})$. However, let K be an ideal of definition of A, and and endow A' with its KA'-adic topology $\mathscr{T}'$; by theorem 16.3.78, the topological ring $(A', \mathscr{T}')$ is also perfectoid. Moreover, for every $i \leq l$, denote by $\gamma'_i \in \mathbf{E}' := \mathbf{E}(A')$ the image of γ_i under the homomorphism $\mathbf{E} \to \mathbf{E}'$ induced by the completion map $A \to A'$; then clearly JA' is the ideal generated by the system $(\overline{u}_{A'}(\gamma'_1), \dots, \overline{u}_{A'}(\gamma'_l))$. Hence, we may replace A by A', after which we may assume that A is complete and separated for its I-adic topology. To conclude, it then suffices to show :

Claim 16.3.85. In the situation of the corollary, suppose moreover that A is complete and separated for the I-adic topology. Then the J-adic topology of A agrees with the linear topology induced by the system of ideals $((J^k)^c \mid k \in \mathbb{N})$.

Proof of the claim. From lemma 9.3.69(iv) we know already that the J-adic topology on A agrees with the linear topology defined by the system of ideals $([\gamma_\bullet]^{\langle\lambda\rangle}A \mid \lambda \in \mathbb{N}[1/p])$ so it suffices to check that

$$([\gamma_\bullet]^{\langle\lambda\rangle}A)^c \subset [\gamma_\bullet]^{\langle\lambda'\rangle}A \qquad \text{for every } \lambda, \lambda' \text{ in } \mathbb{N}[1/p] \text{ with } \lambda' < \lambda.$$

Now, any element of $([\gamma_\bullet]^{\langle\lambda\rangle}A)^c$ can be written as an I-adically convergent series $\sum_{n\in\mathbb{N}} x_n$, where $x_n \in [\gamma_\bullet]^{\langle\lambda\rangle}A \cap [\beta_\bullet]^{\langle n\rangle}A$ for every $n \in \mathbb{N}$ (again by lemma 9.3.69(iv)). Fix $\lambda'', \varepsilon \in \mathbb{N}[1/p]$ with $\lambda' < \lambda'' < \lambda$, and $0 < \varepsilon < 1 - \lambda''/\lambda$; by corollary 16.3.51(iii) we get

$$x_n \in [\gamma_\bullet]^{\langle\lambda''\rangle} \cdot [\beta_\bullet]^{\langle n\varepsilon\rangle}A \qquad \text{for every } n \in \mathbb{N}.$$

Now we argue as in the proof of claim 16.3.20 : by definition, for every $n \in \mathbb{N}$ there exist

- a finite set $S_n \subset \mathbb{N}[1/p]^{\oplus l}$ with $\mu_1 + \cdots + \mu_l = \lambda''$ for every $\mu := (\mu_1, \dots, \mu_l) \in S_n$
- a system $(a_\mu \mid \mu \in S_n)$ of elements of $[\beta_\bullet]^{\langle n\varepsilon\rangle}A$ such that

$$x_n = \sum_{\mu\in S_n} a_\mu \cdot \overline{u}_A(\gamma_\bullet^\mu) \qquad \text{where } \gamma_\bullet^\mu := \gamma_1^{\mu_1} \cdots \gamma_l^{\mu_l} \text{ for every } \mu \in S_n.$$

Choose $N \in \mathbb{N}$ such that $\lambda'' - \lambda' > lp^{-N}$, and define $\overline{\mu}$ and μ^* as in the proof of proposition 9.3.77, so that $\overline{\mu} \in S := \{\nu \in p^{-N}\mathbb{N}^{\oplus l} \mid \lambda'' \geq \nu_1 + \cdots + \nu_l > \lambda'\}$ for every $n \in \mathbb{N}$ and every $\mu \in S_n$. It follows that

$$x = \sum_{\nu\in S} c_\nu \cdot \overline{u}_A(\gamma_\bullet^\nu) \qquad \text{where} \qquad c_\nu := \sum_{n\in\mathbb{N}} \sum_{\substack{\mu\in S_n \\ \overline{\mu}=\nu}} a_\mu \cdot \overline{u}_A(\gamma_\bullet^{\mu^*}).$$

Clearly $\overline{u}_A(\gamma_\bullet^\nu) \in [\gamma_\bullet]^{\langle\lambda'\rangle}A$, and by lemma 9.3.69(iv) the series c_ν converges in the I-adic topology of A for every $\nu \in S$; also it is easily seen that S is a finite set, whence the contention. □

16.4. **Homological theory of perfectoid rings.** This section gathers some results on the homological properties of perfectoid rings. The first result is a simplification of the criterion of theorem 16.3.64, valid in the case where a regular sequence generates a special ideal of definition of the P-ring A.

Theorem 16.4.1. *Let A be a P-ring, and $(a_1, \dots, a_k)$ a regular sequence of elements of A that generates a special ideal of definition I of A. Then the following conditions are equivalent :*

(a) *The Frobenius endomorphism of A/pA induces an isomorphism $A/I \xrightarrow{\sim} A/I^{(p)}$.*
(b) *A is perfectoid.*

Proof. We know already that (b)⇒(a), by corollary 16.3.3. Suppose that (a) holds. Then $\mathbf{E} := \mathbf{E}(A)$ is perfectoid (proposition 16.2.9 and example 16.3.2(i)). Arguing as in the proof of theorem 16.3.64, we may then find elements $\beta_1, \dots, \beta_k$ in $\mathbf{E}$, which generate an ideal of definition $\mathscr{J}$ of $\mathbf{E}$, and such that $\overline{u}_{A/pA}(\beta_i)$ equals the image of a_i in A/pA, for $i = 1, \dots, k$; moreover, $\overline{u}_{A/pA}$ induces an isomorphism (16.3.65).

Next, since $p \in I^{(p)}$, we may find – again as in the proof of theorem 16.3.64 – a distinguished element $\underline{\alpha}' \in \mathrm{Ker}\, u_A$ such that $\alpha'_0 \in \Phi_{\mathbf{E}}(\mathscr{J})$. We let $B := W(\mathbf{E})/\underline{\alpha}'W(\mathbf{E})$, and we notice, as in *loc. cit.* that B is perfectoid for the quotient topology inherited from $W(\mathbf{E})$, and u_A factors through a surjective ring homomorphism $v : B \to A$. We define the ideal $I_B \subset B$ as in the proof of theorem 16.3.64, so (16.3.68) holds as well in the current situation, and arguing as in *loc. cit.* we also see that $p \in I_B^{(p)}$. Set $B_0 := B/I_B$ and $A_0 := A/I$; there follows a commutative diagram

$$\begin{array}{ccc} \mathrm{Sym}^\bullet_{B_0}(I_B/I_B^2) & \longrightarrow & \mathrm{gr}^\bullet_{I_B} B \\ \downarrow & & \downarrow \\ \mathrm{Sym}^\bullet_{A_0}(I/I^2) & \longrightarrow & \mathrm{gr}^\bullet_I A \end{array} \qquad (16.4.2)$$

where $\mathrm{gr}^\bullet_I A$ is the graded ring associated to the I-adic filtration on A, and likewise for $\mathrm{gr}^\bullet_{I_B} B$. The top horizontal arrow of this diagram is surjective, and by assumption, the bottom horizontal arrow is an isomorphism. Furthermore, we notice :

Claim 16.4.3. The maps $\overline{u}_{A/pA}$, $\overline{u}_{B/pB}$ and v induce ring isomorphisms

$$\mathbf{E}/\mathscr{J}^2 \xrightarrow{\sim} B/I_B^2 \xrightarrow{\sim} A/I^2.$$

Proof of the claim. We have a commutative diagram

$$\begin{array}{ccc} \mathbf{E}/\mathscr{J} & \longrightarrow & A/I \\ {\scriptstyle \overline{\Phi}_{\mathbf{E}/\mathscr{J}}} \downarrow & & \downarrow {\scriptstyle \overline{\Phi}_{A/I}} \\ \mathbf{E}/\mathscr{J}^{(p)} & \longrightarrow & A/I^{(p)} \end{array}$$

whose horizontal arrows are induced by $\overline{u}_{A/pA}$. Now, the vertical arrows are isomorphisms, and we have already remarked that the top horizontal arrow is also an isomorphism. Thus, the same holds for the bottom horizontal arrow; since $\overline{u}_{A/pA}(\mathscr{J}^2) = I^2/pA$, we deduce already that $\overline{u}_{A/pA}$ induces an isomorphism $\mathbf{E}/\mathscr{J}^2 \xrightarrow{\sim} A/I^2$. Similarly, since B is perfectoid, $\overline{u}_{B/pB}$ induces an isomorphism $\mathbf{E}/\alpha'_0\mathbf{E} \xrightarrow{\sim} B/pB$; however, $\alpha'_0 \in \mathscr{J}^2$, $p \in I_B^2$, and $\overline{u}_{B/pB}(\mathscr{J}^2) = I_B^2/pB$, so $\overline{u}_{B/pB}$ induces an isomorphism $\mathbf{E}/\mathscr{J}^2 \xrightarrow{\sim} B/I_B^2$. By construction, it is then clear that v induces an isomorphism $B/I_B^2 \xrightarrow{\sim} A/I^2$. ◊

From claim 16.4.3 we see that v induces an isomorphism $B_0 \xrightarrow{\sim} A_0$, and we get as well isomorphisms of $\mathbf{E}$-modules

$$\mathscr{J}/\mathscr{J}^2 \xrightarrow{\sim} I_B/I_B^2 \xrightarrow{\sim} I/I^2.$$

Thus, the left vertical arrow of (16.4.2) is an isomorphism; therefore the same holds for the top horizontal arrow, and hence also for the right vertical arrow. Since B is complete and separated for its I_B-adic topology, and A is complete and separated for its I-adic topology, we conclude that v is an isomorphism of topological rings, whence (b). □

16.4.4. For every $n, r \in \mathbb{N}$, set (see also example 4.8.55(iii))

$$P_r := \mathbb{N}[1/p]^{\oplus r} \qquad R_{r,n} := \mathbb{Z}[P_r] = \mathbb{Z}[T_1^{1/p^\infty}, \dots, T_r^{1/p^\infty}]$$

so, actually $R_{r,n}$ is independent of n, but we also consider the ring homomorphism

$$v_{r,n} : R_{r,n} \to R_{r,0} \qquad T_i^a \mapsto T_i^{a/p^n} \qquad \text{for every } a \in \mathbb{N}[1/p] \text{ and } i = 1, \dots, r$$

for every $r, n \in \mathbb{N}$. Let also $T_\bullet P_r \subset P_r$ be the ideal generated by $\mathbb{N}^{\oplus r}$, and set

$$I_{r,n} := (T_\bullet P_r) \cdot R_{r,n} \qquad I_{r,n}^{\lceil s \rceil} := (T_\bullet P_r)^{\lceil s \rceil} \cdot R_{r,n} \qquad \text{for every } s \in \mathbb{R}_+$$

(notation of remark 9.3.70(i)). Thus $I_{r,n}^{\lceil s \rceil}$ is an ideal of $R_{r,n}$, for every $s \in \mathbb{R}_+$ and every $r, n \in \mathbb{N}$. Next, let A be any ring, $u_0 : R_{r,0} \to A$ any ring homomorphism, and for every $n \in \mathbb{N}$ set $u_n := u_0 \circ v_{r,n} : R_{r,n} \to A$, denote by $\mathbf{f}^{(n)}$ the sequence $(u_n(T_1), \dots, u_n(T_r))$ of elements of A; for $n = 0$, we shall also write just $\mathbf{f}$ instead of $\mathbf{f}^{(0)}$ and denote by $J \subset A$ the ideal generated by $\mathbf{f}$. We may then state :

Lemma 16.4.5. *In the situation of* (16.4.4)*, suppose furthermore that*

$$\mathrm{Tor}_i^{R_{r,0}}(R_{r,0}/I_{r,0}^{\lceil s \rceil}, A) = 0 \qquad \textit{for every } i > 0 \textit{ and every } s \in \mathbb{R}_+.$$

Then the following holds :

(i) *The ring A satisfies condition* $(\mathrm{a})_{\mathbf{f}^{(n)}}^{\mathrm{un}}$ *of* (7.8.21) *with step $\leq r$, for every $n \in \mathbb{N}$.*
(ii) $\mathrm{Ann}_A(J^k) = \mathrm{Ann}_A(I_{r,0}^{\lceil 0 \rceil} \cdot A)$ *for every $k > 0$.*
(iii) $\mathrm{Ann}_A(J) \cap I_{r,0}^{\lceil 0 \rceil} \cdot A = 0$.

Proof. (i): In case $r = 0$, there is nothing to prove, so we assume henceforth that $r > 0$. For every $n \in \mathbb{N}$, endow A with the $R_{r,n}$-module structure induced by u_n.

Claim 16.4.6. $\mathrm{Tor}_i^{R_{r,n}}(R_{r,n}/I_{r,n}^{\lceil s \rceil}, A) = 0$ for every $i > 0$ and every $s \in \mathbb{R}_+$.

Proof of the claim. The base change $v_{r,n} : R_{r,n} \to R_{r,0}$ yields a spectral sequence

$$E_{ij}^2 := \mathrm{Tor}_i^{R_{r,0}}(\mathrm{Tor}_j^{R_{r,n}}(R_{r,n}/I_{r,n}^{\lceil s \rceil}, R_{r,0}), A) \Rightarrow \mathrm{Tor}_i^{R_{r,n}}(R_{r,n}/I_{r,n}^{\lceil s \rceil}, A).$$

But since $v_{r,n}$ is an isomorphism and $v_{r,n}(I_{r,n}^{\lceil s \rceil}) = I_{r,0}^{\lceil s/p^n \rceil}$, we see that $E_{ij}^2 = 0$ for every $j > 0$, and $E_{i0}^2 = \mathrm{Tor}_i^{R_{r,0}}(R_{r,0}/I_{r,0}^{\lceil s/p^n \rceil}, A)$. Thus, $E_{i0}^2 = 0$ for $i > 0$, whence the claim. $\diamondsuit$

In view of claim 16.4.6, we may replace u_0 by u_n, and reduce to checking condition $(\mathrm{a})_{\mathbf{f}}^{\mathrm{un}}$ for A. To this aim, define A_r, I_r and the ring homomorphism $\beta_{\mathbf{f}} : A_r \to A$ as in (7.8.21); endow also $R_{r,0}$ with the A_r-module structure induced by the inclusion map $A_r \to R_{r,0}$, and consider the change of rings spectral sequence

$$E(n)_{pq}^2 := \mathrm{Tor}_p^{R_{r,0}}(\mathrm{Tor}_q^{A_r}(A_r/I_r^n, R_{r,0}), A) \Rightarrow \mathrm{Tor}_{p+q}^{A_r}(A_r/I_r^n, A) \qquad \text{for every } n \in \mathbb{N}.$$

It is easily seen that $R_{r,0}$ is a free A_r-module, hence $E(n)_{pq}^2 = 0$ whenever $q > 0$, and $E(n)_{p0}^2 = R_{r,0}/I_{r,0}^n$. There follows a natural isomorphism

$$\mathrm{Tor}_p^{R_{r,0}}(R_{r,0}/I_{r,0}^n, A) \xrightarrow{\sim} \mathrm{Tor}_p^{A_r}(A_r/I_r^n, A) \qquad \text{for every } p, n \in \mathbb{N}.$$

Now, recall that $I_{r,0}^{n+r} \subset I_{r,0}^{\lceil n+r-1 \rceil} \subset I_{r,0}^n$ for every $n \in \mathbb{N}$ (lemma 9.3.69(iv)); then our assumption implies immediately that the inverse system $(\mathrm{Tor}_p^{A_r}(A_r/I_r^n, A) \mid n \in \mathbb{N})$ is uniformly essentially zero, with step $\leq r$.

(iii): We have just seen that condition $(\mathrm{a})_{\mathbf{f}^{(n)}}^{\mathrm{un}}$ holds for every n, with step bounded by r. For every $n \in \mathbb{N}$, let $J_n \subset A$ be the ideal generated by $\mathbf{f}^{(n)}$; according to remark 7.8.34(ii), it follows that there exists a sequence of integers $(d(i) \mid i \in \mathbb{N})$ such that inverse system

$(H_{i+1}(\mathbf{f}^{(n)}, J_n^k) \mid k \in \mathbb{N})$ is uniformly essentially zero, with step bounded by $d(i)$, for every $i, n \in \mathbb{N}$. Especially, for $d := d(r)$ we get

$$\mathrm{Ann}_A(J_n) \cap J_n^d = 0 \qquad \text{for every } n \in \mathbb{N}.$$

By a simple induction argument, we deduce that $\mathrm{Ann}_A(J_n^{k+1}) \cap J_n^d = 0$ for every $n, k \in \mathbb{N}$. But clearly, for every $n \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that $J_n^{k+1} \subset J$, whence

$$\mathrm{Ann}_A(J) \cap J_n^d = 0 \qquad \text{for every } n \in \mathbb{N}$$

which yields easily the contention.

(ii) is an easy consequence of (iii). □

Proposition 16.4.7. *Let $f : A \to B$ be any homomorphism of perfect $\mathbb{F}_p$-algebras, $I \subset A$ any ideal. Then the natural morphism*

$$A/I^{\lceil s \rceil}A \overset{\mathbf{L}}{\otimes}_A B \to A/I^{\lceil s \rceil}A \otimes_A B[0]$$

is an isomorphism in $\mathsf{D}(A\text{-}\mathbf{Mod})$, for every $s \in \mathbb{R}_+$ (notation of remark 9.3.70(i)).

Proof. We reduce easily to the case where I is finitely generated, and $f = \mathbf{E}^*(f_0)$ for a map $f_0 : A_0 \to B_0$ of $\mathbb{F}_p$-algebras of finite type (where A_0 and B_0 are endowed with their discrete topologies : notation of (9.4.7)). Then we may also assume that $I = I_0 A$ for some finitely generated ideal $I_0 \subset A_0$, and it suffices to show :

Claim 16.4.8. In the foregoing situation, for every $\lambda, \lambda' \in \mathbb{N}[1/p]$ with $\lambda' < \lambda$, the inclusion $I^{\langle\lambda\rangle}A \subset I^{\langle\lambda'\rangle}A$ induces the zero map

$$\mathrm{Tor}_i^A(A/I^{\langle\lambda\rangle}A, B) \to \mathrm{Tor}_i^A(A/I^{\langle\lambda'\rangle}A, B) \qquad \text{for every } i > 0.$$

Proof of the claim. Under the current assumptions, A is the colimit of the system of $\mathbb{F}_p$-algebras $(A_n \mid n \in \mathbb{N})$ such that $A_n := A_0$ for every $n \in \mathbb{N}$, with transition maps $\varphi_n : A_n \to A_{n+1}$ given by the Frobenius map Φ_{A_0}, and likewise for B. Moreover, say that $\lambda = ap^k$ and $\lambda' = bp^k$ for some $a, b, k \in \mathbb{N}$, and for every $n \in \mathbb{N}$ define ideals of A_n by the rules :

$$I_n := \begin{cases} 0 & \text{for } n < k \\ I_0^{ap^{n-k}} & \text{for } n \geq k \end{cases} \qquad \text{and} \qquad I'_n := \begin{cases} 0 & \text{for } n < k \\ I_0^{bp^{n-k}} & \text{for } n \geq k. \end{cases}$$

Then it is easily seen that $\varphi_n(I_n) \subset I_{n+1}$, and likewise for I'_n, for every $n \in \mathbb{N}$. Furthermore, the colimit of the system $(I_n \mid n \in \mathbb{N})$ (resp. $(I'_n \mid n \in \mathbb{N})$) is $I^{\langle\lambda\rangle}A$ (resp. $I^{\langle\lambda'\rangle}A$). By claim 7.11.37(i), we are therefore reduced to showing that the map induced by the inclusion $I_n \subset I'_n$

$$\mathrm{Tor}_i^{A_0}(A_0/I_n, B_0) \to \mathrm{Tor}_i^{A_0}(A_0/I'_n, B_0)$$

vanishes, for every sufficiently large $n \in \mathbb{N}$. Since $a > b$, the latter assertion follows from claim 7.9.18(ii) (applied with $M := A$ and $N := B$). □

16.4.9. We are going now to apply the general setup of (16.4.4) to the case where A is a perfectoid ring, and

$$\mathbf{f} := (\overline{u}_A(\beta_1), \dots, \overline{u}_A(\beta_r))$$

for a given sequence $\beta_\bullet := (\beta_1, \dots, \beta_r)$ of elements of $\mathbf{E} := \mathbf{E}(A)$. To ease notation, we shall simply write P instead of P_r and R instead of $R_{r,0}$. Also, we shall write $I, I^{\lceil s \rceil} \subset R$ instead of $I_{r,0}$, and respectively $I_{r,0}^{\lceil s \rceil}$, for every $s \in \mathbb{R}_+$. Likewise, we shall write u instead of u_0; hence $u : R \to A$ is the unique ring homomorphism such that $u(T_i) = \overline{u}_A(\beta_i)$ for $i = 1, \dots, r$.

Proposition 16.4.10. (i) *With the notation of* (16.4.9)*, the natural morphism*

$$R/I^{\lceil s\rceil} \overset{\mathbf{L}}{\otimes}_R A \to A/I^{\lceil s\rceil}A[0]$$

is an isomorphism in $\mathsf{D}(R\text{-}\mathbf{Mod})$*, for every* $s \in \mathbb{R}_+$*.*

(ii) *In particular, A satisfies condition* $(\mathrm{a})^{\mathrm{un}}_{\mathbf{f}^{(n)}}$ *of* (7.8.21) *with step* $\leq r$*, for every* $n \in \mathbb{N}$*.*

Proof. In view of lemma 16.4.5(i), it suffices to check (i). Notice that $R_0 := R \otimes_{\mathbb{Z}} \mathbb{F}_p$ is a perfect $\mathbb{F}_p$-algebra, and $I^{\lceil s\rceil}R_0 = I_0^{\lceil s\rceil}R_0$, where $I_0 \subset R_0$ is the ideal generated by $(T_1, \dots, T_r)$. In light of proposition 16.4.7, we deduce

$$\text{(16.4.11)} \qquad \mathrm{Tor}_i^{R_0}(R_0/I^{\lceil s\rceil}R_0, \mathbf{E}) = 0 \qquad \text{for every } s \in \mathbb{R}_+ \text{ and every } i > 0.$$

On the other hand, we have a standard 2-spectral sequence

$$E^2_{pq} := \mathrm{Tor}_p^{R_0}(\mathrm{Tor}_q^R(R/I^{\lceil s\rceil}, R_0), \mathbf{E}) \Rightarrow \mathrm{Tor}^R_{p+q}(R/I^{\lceil s\rceil}, \mathbf{E})$$

([163, Th.5.6.6]). However, by construction $I^{\lceil s\rceil}$ is a direct summand of the free $\mathbb{Z}$-module R, so $R/I^{\lceil s\rceil}$ is also a free $\mathbb{Z}$-module, and then a standard calculation shows that $E^2_{pq} = 0$ whenever $q > 0$. From (16.4.11) we see as well that $E^2_{p0} = 0$ for every $p > 0$, so we conclude

$$\mathrm{Tor}_i^R(R/I^{\lceil s\rceil}, \mathbf{E}) = 0 \qquad \text{for every } s \in \mathbb{R}_+ \text{ and every } i > 0.$$

Next, an easy induction argument using the short exact sequences of $W(\mathbf{E})$-modules

$$0 \to \mathbf{E} \simeq p^n W_{n+1}(\mathbf{E}) \to W_{n+1}(\mathbf{E}) \to W_n(\mathbf{E}) \to 0 \qquad \text{for every } n \in \mathbb{N}$$

shows more generally that

$$\text{(16.4.12)} \qquad \mathrm{Tor}_i^R(R/I^{\lceil s\rceil}, W_n(\mathbf{E})) = 0 \qquad \text{for every } s \in \mathbb{R}_+, \text{ every } i > 0 \text{ and every } n \in \mathbb{N}.$$

For every $n \in \mathbb{N}$, set $J_n := \boldsymbol{p}_P^{-n}(T_\bullet P)$, where $\boldsymbol{p}_P$ denotes the p-Frobenius automorphism of P (see definition 4.8.40(ii)), and consider the projection

$$\pi_n^{(a,b)} : R/J_n^a R \to R/J_n^b R \qquad \text{for every } a, b, n \in \mathbb{N} \text{ with } a \geq b.$$

Claim 16.4.13. With the foregoing notation, we have :

(i) $\mathrm{Tor}_i^R(\pi_n^{(a,b)}, W_m(\mathbf{E})) = 0$ for every $a, b, i, m, n \in \mathbb{N}$ with $a - b \geq r$ and $i > 0$.

(ii) For every $a, n \in \mathbb{N}$ there exists a complex of free R-modules of finite type $L^\bullet_{a,n}$ with a quasi-isomorphism $L^\bullet_{a,n} \to R/J_n^a[0]$ of complexes of R-modules.

(iii) $\mathrm{Tor}_i^R(\pi_n^{(a,b)}, W(\mathbf{E})) = 0$ for every $a, b, i, n \in \mathbb{N}$ with $a - b \geq 2r$ and $i > 0$.

Proof of the claim. If $r = 0$ there is nothing to prove, so we may assume that $r > 0$.

(i): Notice that J_n is generated by a system of r elements of P, for every $n \in \mathbb{N}$; in view of lemma 9.3.69(ii.c,iv), we deduce

$$J_n^a \subset J_n^{\langle a\rangle} \subset (T_\bullet P)^{\lceil (b+r-1)p^{-n}\rceil} \subset J_n^{\langle b+r-1\rangle} \subset J_n^b$$

so the assertion follows from (16.4.12).

(ii): The ideal J_nR is generated by the system $(T_1^{1/p^n}, \dots, T_r^{1/p^n})$ of elements of the subring $S_n := \mathbb{Z}[T_1^{1/p^n}, \dots, T_r^{1/p^n}]$; since S_n is noetherian, we may find a complex $L'^\bullet_{a,n}$ of free S_n-modules of finite type, with a quasi-isomorphism $L'^\bullet_{a,n} \to S_n/J_n^a S_n[0]$, and since the inclusion map $S_n \to R$ is flat, the complex $L^\bullet_{a,n} := L'^\bullet_{a,n} \otimes_{S_n} R$ will do.

(iii): For given $a, n \in \mathbb{N}$, let $L^\bullet_{a,n}$ be as in (ii); it is easily seen that the natural map

$$\lim_{m\in\mathbb{N}} L^\bullet_{a,n} \otimes_R W_m(\mathbf{E}) \to L^\bullet_{a,n} \otimes_R W(\mathbf{E})$$

is an isomorphism in $\mathsf{C}(R\text{-}\mathbf{Mod})$. To ease notation, set $\mathscr{T}^a_{i,n,m} := \mathrm{Tor}^R_i(R/J^a_nR, W_m(\mathbf{E}))$ for every $i, m, n \in \mathbb{N}$; in light of [163, Th.3.5.8] we deduce natural short exact sequences

$$0 \to R^1 \lim_{m\in\mathbb{N}} \mathscr{T}^a_{i-1,n,m} \to \mathrm{Tor}^R_i(R/J^a_nR, W(\mathbf{E})) \to \lim_{m\in\mathbb{N}} \mathscr{T}^a_{i,n,m} \to 0$$

for every $i, a, n \in \mathbb{N}$. Then the assertion follows from (i). $\diamondsuit$

Now, for any $s \in \mathbb{R}_+$, let S_s be the set of all pairs $(a, n) \in \mathbb{N} \times \mathbb{N}$ such that $ap^{-n} > s$; we define a partial ordering on S_s by the rule : $(a, n) \geq (b, m)$ if and only if $n \geq m$ and $ap^m \leq bp^n$. With this notation, notice that $J^b_m \subset J^a_n$ whenever $(a, n) \geq (b, m)$, and $I^{\lceil s\rceil}$ is the union of its filtered system of subideals $(J^a_nR \mid (a, n) \in S_s)$. Taking into account claim 16.4.13(iii) we deduce that

$$\text{(16.4.14)} \qquad \mathrm{Tor}^R_i(R/I^{\lceil s\rceil}, W(\mathbf{E})) = 0 \qquad \text{for every } s \in \mathbb{R}_+ \text{ and every } i > 0.$$

Next, from remark 16.1.7(ii) we get a short exact sequence of R-modules

$$0 \to W(\mathbf{E}) \xrightarrow{\underline{\alpha}} W(\mathbf{E}) \to A \to 0$$

where $\underline{\alpha}$ is any distinguished element in $\mathrm{Ker}\, u_A$. Then, by the induced long exact Tor-sequence, (16.4.14) implies already that $\mathrm{Tor}^R_i(R/I^{\lceil s\rceil}, A)$ vanishes for every $i > 1$. By the same token, we also see that $\mathrm{Tor}^R_1(R/I^{\lceil s\rceil}, A)$ is isomorphic to the kernel of scalar multiplication by $\underline{\alpha}$ on the R-module $W(\mathbf{E})/I^{\lceil s\rceil}W(\mathbf{E})$; to conclude the proof, it then suffices to remark :

Claim 16.4.15. $\underline{\alpha}W(\mathbf{E}) \cap I^{\lceil s\rceil}W(\mathbf{E}) = \underline{\alpha}I^{\lceil s\rceil}W(\mathbf{E})$.

Proof of the claim. Denote by $\mathscr{I} \subset \mathbf{E}$ the ideal generated by $\beta_\bullet$; from proposition 9.3.77(i) we easily deduce that $I^{\lceil s\rceil}W(\mathbf{E}) = \bigcup_{s'>s} W(\mathscr{I}^{\lfloor s'\rfloor})$, and in view of proposition 16.1.12(ii) we get

$$\underline{\alpha}W(\mathbf{E}) \cap I^{\lceil s\rceil}W(\mathbf{E}) = \bigcup_{s'>s}(\underline{\alpha}W(\mathbf{E}) \cap W(\mathscr{I}^{\lfloor s'\rfloor})) = \bigcup_{s'>s} \underline{\alpha}W(\mathscr{I}^{\lfloor s'\rfloor}) = \underline{\alpha}I^{\lceil s\rceil}W(\mathbf{E})$$

as stated. $\square$

Corollary 16.4.16. *In the situation of* (16.4.9)*, set $J := IA$, and denote by $A^\wedge_J$ the J-adic completion of A. Let also $C_\bullet$ be any bounded above complex of flat A-modules such that for every $i \in \mathbb{Z}$ there exists $c(i) \in \mathbb{N}$ with $J^{c(i)} \cdot H_iC_\bullet = 0$. Then the natural morphism*

$$C_\bullet \to A^\wedge_J \otimes_A C_\bullet$$

is an isomorphism in $\mathsf{D}(A\text{-}\mathbf{Mod})$.

Proof. It is an immediate consequence of proposition 16.4.10(ii) and corollary 8.6.34. $\square$

Proposition 16.4.17. *In the situation of* (16.4.9)*, suppose that the sequence $\beta_\bullet$ generates an open ideal of* $\mathbf{E}$*. Then the following conditions are equivalent :*

(a) *The sequence $\beta_\bullet$ is completely secant in* $\mathbf{E}$*.*
(b) *The sequence $b_\bullet := (\overline{u}_A(\beta_1), \dots, \overline{u}_A(\beta_r))$ is completely secant in A.*

Proof. Set as usual $J := IA$, and let $A^\wedge_J$ be the J-adic completion of A. In view of corollary 16.4.16, we see that the sequence $b_\bullet$ is completely secant in A if and only if its image is completely secant in $A^\wedge_J$. Moreover, if $\mathscr{J} \subset \mathbf{E}$ denotes the ideal generated by $\beta_\bullet$, we have a natural ring isomorphism $\mathbf{E}(A^\wedge_J) \xrightarrow{\sim} \mathbf{E}^\wedge_{\mathscr{J}}$ (theorem 16.3.78(ii)), and by the same token the sequence $\beta_\bullet$ is completely secant in the perfectoid ring $\mathbf{E}$ if and only if the same holds for its image in $\mathbf{E}^\wedge_{\mathscr{J}}$. Thus, we may replace A by $A^\wedge_J$, and assume from start that A is J-adically complete and separated. Next, let $(\alpha_n \mid n \in \mathbb{N})$ be any distinguished element in $\mathrm{Ker}\, u_A$; since $\mathscr{J}$ is open, we have $\alpha_0^n \in \mathscr{J}$ for some integer $n \in \mathbb{N}$, and then we may find $k \in \mathbb{N}$ such that α_0 is contained in the ideal generated by the sequence $\beta'_\bullet := (\beta_1^{1/p^k}, \dots, \beta_r^{1/p^k})$. In light of corollary 7.8.8, we may then replace $\beta_\bullet$ with $\beta'_\bullet$, and assume that $\alpha_0 \in \mathscr{J}$. Notice now that $\{\mathscr{J}^n\} = J^n$ for every

$\in \mathbb{N}$ (remark 16.3.28(viii)), and the inclusion $\mathscr{J}^{n+1} \subset \mathscr{J}^n$ is taut for every $n \in \mathbb{N}$, so theorem 16.3.31(ii) yields a natural isomorphism

$$\overline{\tau} : \bigoplus_{n \in \mathbb{N}} \mathscr{J}^n / \mathscr{J}^{n+1} \xrightarrow{\sim} \bigoplus_{n \in \mathbb{N}} J^n / J^{n+1}.$$

Moreover, a simple inspection shows that $\overline{\tau}$ is an isomorphism of A/pA-algebras; it follows immediately that the sequence $\beta_\bullet$ is $\mathbf{E}$-quasi-regular if and only if the sequence $b_\bullet$ is A-quasi-regular (see remark 7.10.44(i)). To conclude, it suffices now to invoke proposition 7.10.47. $\square$

Proposition 16.4.18. *Let $f : A \to A'$ be any continuous map of perfectoid rings, I (resp. $\mathscr{I}$) a finite type ideal of definition of A (resp. of $\mathbf{E}(A)$), and denote by $\mathscr{T}_I$ (resp. $\mathscr{T}_\mathscr{I}$) the IA'-adic topology of A' (resp. the $\mathscr{I}\mathbf{E}(A')$-adic topology of $\mathbf{E}(A')$). Then $f : A \to (A', \mathscr{T}_I)$ is adically flat if and only if the same holds for $\mathbf{E}(f) : \mathbf{E}(A) \to (\mathbf{E}(A'), \mathscr{T}_\mathscr{I})$ (see definition* 8.3.23(iii)*).*

Proof. Set $\mathbf{E} := \mathbf{E}(A)$, $\mathbf{E}' := \mathbf{E}(A')$, and let $\underline{\alpha} := (\alpha_n \mid n \in \mathbb{N})$ be a distinguished element of $\mathrm{Ker}(u_A)$; by lemma 8.3.12 and proposition 16.3.8, the topological ring $(A', \mathscr{T}_I)$ is perfectoid, so we may assume that f is adic, in which case the same holds for $\mathbf{E}(f) : \mathbf{E} \to \mathbf{E}'$, by theorem 16.3.42(i). Let also $\beta_\bullet := (\beta_1, \dots, \beta_n)$ be a sequence of elements of $\mathbf{E}$ that generates $\mathscr{I}$; we may assume that $\alpha_0 \in \mathscr{I}$, and by corollary 16.3.40(i,ii) we may also assume that I is generated by $\mathbf{f} := (\overline{u}_A(\beta_1), \dots, \overline{u}_A(\beta_k))$. Set moreover $\gamma_\bullet := (p, \tau_A(\beta_1), \dots, \tau_A(\beta_k))$ and $\delta_\bullet := (\underline{\alpha}, \tau_A(\beta_1), \dots, \tau_A(\beta_k))$ (notation of (16.1)), and denote by ∂_γ (resp. ∂_δ) the $W(\mathbf{E})$-linear form $W(\mathbf{E})^{\oplus k+1} \to W(\mathbf{E})$ such that $\partial_\gamma(w_0, \dots, w_k) := pw_0 + \sum_{i=1}^k \tau_A(\beta_i)w_i$ (resp. $\partial_\gamma(w_0, \dots, w_k) := \underline{\alpha}w_0 + \sum_{i=1}^k \tau_A(\beta_i)w_i$) for every $(w_0, \dots, w_k) \in W(\mathbf{E})^{\oplus k+1}$. Let $\mathscr{J} \subset W(\mathbf{E})$ be the ideal generated by the sequence $\gamma_\bullet$, and set $S := 1 + \mathscr{J}$; according to remark 16.1.10, there exists a $S^{-1}W(\mathbf{E})$-linear automorphism f of $S^{-1}W(\mathbf{E})^{\oplus k+1}$ with $S^{-1}\partial_\delta = S^{-1}\partial_\gamma \circ f$; the latter induces an isomorphism of differential graded $S^{-1}W(\mathbf{E})$-algebras :

$$\mathbf{K}_\bullet(\gamma_\bullet) \xrightarrow{\sim} \mathbf{K}_\bullet(\delta_\bullet) \tag{16.4.19}$$

(notation of (7.8)). Recall furthermore that p and $\underline{\alpha}$ are regular elements of $W(\mathbf{E})$ (proposition 9.3.47(i)); there follow natural identifications in $\mathsf{D}(W(\mathbf{E}))$:

$$\mathbf{K}_\bullet(p) \xrightarrow{\sim} W(\mathbf{E})/pW(\mathbf{E})[0] \xrightarrow{\sim} E[0] \qquad \mathbf{K}_\bullet(\underline{\alpha}) \xrightarrow{\sim} W(\mathbf{E})/\underline{\alpha}W(\mathbf{E})[0] \xrightarrow{\sim} A[0]$$

whence $W(\mathbf{E})$-linear isomorphisms :

$$H_i(\gamma_\bullet, W(\mathbf{E})) \xrightarrow{\sim} H_i(\mathbf{f}, A) \qquad H_i(\delta_\bullet, W(\mathbf{E})) \xrightarrow{\sim} H_i(\beta_\bullet, \mathbf{E}) \qquad \text{for every } i \in \mathbb{N}.$$

Notice that $H_i(\gamma_\bullet, W(\mathbf{E}))$ is a $W(\mathbf{E})/\mathscr{J}$-module for every $i \in \mathbb{N}$ (lemma 7.8.2(i)), hence the localization map $H_i(\gamma_\bullet, W(\mathbf{E})) \to H_i(\gamma_\bullet, S^{-1}W(\mathbf{E}))$ is an isomorphism, and likewise for $H_i(\delta_\bullet, W(\mathbf{E}))$, since $\delta_\bullet$ generates $\mathscr{J}$ (lemma 16.1.9(ii)). Combining with (16.4.19), we then deduce a commutative diagram :

$$\begin{array}{ccc} H_i(\gamma_\bullet, W(\mathbf{E})) & \xrightarrow{\sim} & H_i(\delta_\bullet, W(\mathbf{E})) \\ \downarrow & & \downarrow \\ H_i(\mathbf{f}, A) & \xrightarrow{\omega_{i,\mathbf{f}}} & H_i(\beta_\bullet, \mathbf{E}) \end{array} \qquad \text{for every } i \in \mathbb{N}$$

all whose arrows are isomorphisms of $W(\mathbf{E})$-modules. Next, denote by $\beta'_\bullet$ (resp. $\mathbf{f}'$) the image of the sequence $\beta_\bullet$ in $\mathbf{E}'$ (resp. of the sequence $\mathbf{f}$ in A'); we get likewise an isomorphism of $W(\mathbf{E}')$-modules $\omega_{i,\mathbf{f}'} : H_i(\mathbf{f}', A') \xrightarrow{\sim} H_i(\beta'_\bullet, \mathbf{E}')$ for every $i \in \mathbb{N}$. Lastly, since $\alpha_0 \in \mathscr{I}$, the map $\overline{u}_{A'} : \mathbf{E}' \to A'$ induces an isomorphism of $W(\mathbf{E}')$-algebras $\mathbf{E}'/\mathscr{I}\mathbf{E}' \xrightarrow{\sim} A'/IA'$, and notice that

$H_i(\mathbf{f}', A')$ (resp. $H_i(\beta'_\bullet, \mathbf{E}')$) is an A'/IA'-module (resp. an $\mathbf{E}'/\mathscr{I}\mathbf{E}'$-module) for every $i \in \mathbb{N}$. By inspecting the constructions, we then obtain a commutative diagram of $W(\mathbf{E}')$-modules :

$$\begin{array}{ccc} A'/IA' \otimes_A H_i(\mathbf{f}, A) & \xrightarrow{A'/IA' \otimes_A \omega_{i,\mathbf{f}}} & \mathbf{E}'/\mathscr{I}\mathbf{E}' \otimes_{\mathbf{E}} H_i(\beta_\bullet, \mathbf{E}) \\ \downarrow & & \downarrow \\ H_i(\mathbf{f}', A') & \xrightarrow{\omega_{i,\mathbf{f}'}} & H_i(\beta'_\bullet, \mathbf{E}'). \end{array}$$

The proposition now follows, by combining with proposition 14.2.49(i,iii). □

Remark 16.4.20. (i) In the situation of proposition 16.4.18, we also have that $f : A \to (A', \mathscr{T}_I)$ is adically faithfully flat if and only if the same holds for $\mathbf{E}(f) : \mathbf{E}(A) \to (\mathbf{E}(A'), \mathscr{T}_{\mathscr{I}})$. Indeed, notice that the closed immersion $\mathrm{Spec}\, A/I \to \mathrm{Spec}\, A/I^n$ is a homeomorphism for every $n \in \mathbb{N}$, and likewise for the map $\mathrm{Spec}\, \mathbf{E}(A)/\mathscr{I} \to \mathrm{Spec}\, \mathbf{E}(A)/\mathscr{I}^n$. However, f is adically faithfully flat if and only if it is adically flat and the induced map $h_n : \mathrm{Spec}\, A'/I^nA' \to \mathrm{Spec}\, A'/I^n$ is surjective for every $n \in \mathbb{N}$, and the latter condition then holds if and only if h_1 is surjective. Likewise, $\mathbf{E}(f)$ is adically faithfully flat if and only if it is adically flat and the induced map k_1 : $\mathrm{Spec}\, \mathbf{E}(A')/\mathscr{I}\mathbf{E}(A') \to \mathrm{Spec}\, \mathbf{E}(A)/\mathscr{I}$ is surjective. Thus, we come down to checking that h_1 is surjective if and only the same holds for k_1, but this is clear, because arguing as in the proof of proposition 16.4.18 we may assume that $\overline{u}_A$ and $\overline{u}_{A'}$ induce isomorphisms $\mathbf{E}(A)/\mathscr{I} \xrightarrow{\sim} A/I$ and $\mathbf{E}(A')/\mathscr{I}\mathbf{E}(A') \xrightarrow{\sim} A'/IA'$, and these isomorphisms identify h_1 with k_1.

(ii) For every perfectoid ring A, let A-$\mathsf{Perf}_{\mathrm{ad.fl}}$ be the full subcategory of A-Perf whose objects are the adic and adically flat maps $A \to B$ of perfectoid rings. Proposition 16.4.18 shows that the functor $\mathbf{E}$ of remark 16.3.7(i) restricts to an equivalence of categories

$$A\text{-}\mathsf{Perf}_{\mathrm{ad.fl}} \xrightarrow{\sim} \mathbf{E}(A)\text{-}\mathsf{Perf}_{\mathrm{ad.fl}}.$$

(iii) Let $\mathscr{I} \subset \mathbf{E}(A)$ be an ideal of adic definition, set $\mathbf{E}_0 := \mathbf{E}(A)/\mathscr{I}$, and denote by $\mathscr{A}(\mathbf{E}_0)$ the category whose objects are all the flat ring homomorphisms $f : \mathbf{E}_0 \to C$ such that the following commutative diagram is cocartesian :

(16.4.21)
$$\begin{array}{ccc} \mathbf{E}_0 & \xrightarrow{f} & C \\ {\scriptstyle\Phi_{\mathbf{E}_0}}\downarrow & & \downarrow{\scriptstyle\Phi_C} \\ \mathbf{E}_0 & \xrightarrow{f} & C \end{array}$$

where the vertical arrows are the Frobenius endomorphisms. Now, let $g : \mathbf{E}(A) \to E$ be any object of $\mathbf{E}(A)$-$\mathsf{Perf}_{\mathrm{ad.fl}}$; we get a commutative diagram :

$$\begin{array}{ccccc} \mathbf{E}_0 & \xrightarrow{\overline{\Phi}_{\mathbf{E}(A)}} & \mathbf{E}(A)/\mathscr{I}^p & \longrightarrow & \mathbf{E}_0 \\ \downarrow & & \downarrow & & \downarrow \\ E/\mathscr{I}E & \xrightarrow{\overline{\Phi}_E} & E/\mathscr{I}^pE & \longrightarrow & E/\mathscr{I}E \end{array}$$

where $\overline{\Phi}_{\mathbf{E}(A)}$ (resp. $\overline{\Phi}_E$) is induced by the Frobenius endomorphism of $\mathbf{E}(A)$ (resp. of E). Since $\mathbf{E}(A)$ and E are perfect $\mathbb{F}_p$-algebras, we easily see that both square subdiagrams of this diagram are cocartesian, whence an object $\mathbf{E}_0 \otimes_{\mathbf{E}(A)} g : \mathbf{E}_0 \to E/\mathscr{I}E$ of $\mathscr{A}(\mathbf{E}_0)$. We have thus a well defined functor :

(16.4.22)
$$\mathbf{E}(A)\text{-}\mathsf{Perf}_{\mathrm{ad.fl}} \to \mathscr{A}(\mathbf{E}_0).$$

Proposition 16.4.23. *The functor* (16.4.22) *is an equivalence of categories.*

Proof. We construct an explicit quasi-inverse as follows. Let $f : \mathbf{E}_0 \to C$ be any object of $\mathscr{A}(\mathbf{E}_0)$, and set $\mathbf{E}_n := \mathbf{E}(A)/\mathscr{I}^{n+1}$ for every $n \in \mathbb{N}$; we exhibit inductively a compatible system of flat ring homomorphisms $(f_n : \mathbf{E}_n \to C_n \mid n \in \mathbb{N})$ with isomorphisms

$$\varphi_n : C_n/\mathscr{I}^n C_n \xrightarrow{\sim} C_{n-1} \quad \text{such that} \quad \varphi_n \circ (\mathbf{E}_{n-1} \otimes_{\mathbf{E}_n} f_n) = f_{n-1} \quad \text{for every } n > 0$$

and such that the following commutative diagram is cocartesian :

$$\mathscr{D}_n \quad : \quad \begin{array}{ccc} \mathbf{E}_n & \xrightarrow{f_n} & C_n \\ {\scriptstyle \Phi_{\mathbf{E}_n}}\downarrow & & \downarrow{\scriptstyle \Phi_{C_n}} \\ \mathbf{E}_n & \xrightarrow{f_n} & C_n. \end{array}$$

Indeed, set $C_0 := C$, and $f_0 := f$. Suppose next that $n \in \mathbb{N}$, and that f_n and φ_n have already been given; according to [75, Lemma 6.5.13(i)], we have $\mathbb{L}_{C_n/\mathbf{E}_n} = 0$ in $\mathsf{D}(C\text{-}\mathbf{Mod})$, so the existence of a flat ring homomorphism $f_{n+1} : \mathbf{E}_{n+1} \to C_{n+1}$ with the sought isomorphism φ_{n+1} is ensured by [75, Cor.3.2.11(i)]. Diagram $\mathscr{D}_{n+1}$ induces a map of $\mathbf{E}_{n+1}$-algebras ψ_{n+1} : $\mathbf{E}_{n+1,\Phi} \otimes_{\mathbf{E}_{n+1}} C_{n+1} \to C_{n+1}$, where $\mathbf{E}_{n+1,\Phi}$ denotes the ring $\mathbf{E}_{n+1}$, regarded as a $\mathbf{E}_{n+1}$-algebra via the map $\Phi_{\mathbf{E}_{n+1}}$. Since (16.4.21) is cocartesian, $\mathbf{E}_1 \otimes_{\mathbf{E}_{n+1}} \psi_{n+1}$ is an isomorphism, hence the same for $\mathscr{I}^k/\mathscr{I}^{k+1} \otimes_{\mathbf{E}_{n+1}} \psi_{n+1}$, for every $k = 0, \dots, n+1$; but since both C_{n+1} and $D_{n+1} := \mathbf{E}_{n+1,\Phi} \otimes_{\mathbf{E}_{n+1}} C_{n+1}$ are flat $\mathbf{E}_{n+1}$-algebras, the latter map is naturally identified with the map $\mathscr{I}^k D_{n+1}/\mathscr{I}^{k+1} D_{n+1} \to \mathscr{I}^k C_{n+1}/\mathscr{I}^{k+1} C_{n+1}$ induced by ψ_{n+1} ([126, Th.22.3]). It follows easily that ψ_{n+1} is an isomorphism, *i.e.* $\mathscr{D}_{n+1}$ is cocartesian, as required, and this completes the construction of C_n. Let $E := \lim_{n\in\mathbb{N}} C_n$; according to [59, Ch.0, Prop.7.2.7] (and its proof), E is complete and separated for its $\mathscr{I}$-adic topology, and the natural map $E/\mathscr{I}^{n+1}E \to C_n$ is an isomorphism for every $n \in \mathbb{N}$. Moreover, the limit of the system $(f_n \mid n \in \mathbb{N})$ yields an adic ring homomorphism $g : \mathbf{E} \to E$. Lastly, for every $n \in \mathbb{N}$ we consider the commutative diagram

$$\begin{array}{ccccc} \mathbf{E}_n & \xrightarrow{\pi_{\mathbf{E},n}} & \mathbf{E}/\Phi_{\mathbf{E}}^{-1}(\mathscr{I}^{n+1}) & \xrightarrow{\overline{\Phi}_{\mathbf{E}_n}} & \mathbf{E}_n \\ {\scriptstyle f_n}\downarrow & & \downarrow & & \downarrow{\scriptstyle f_n} \\ C_n & \xrightarrow{\pi_{E,n}} & E/\Phi_{\mathbf{E}}^{-1}(\mathscr{I}^{n+1})E & \xrightarrow{\overline{\Phi}_{C_n}} & C_n \end{array}$$

where $\pi_{\mathbf{E},n}$ and $\pi_{E,n}$ are the natural projections, and $\overline{\Phi}_{\mathbf{E}_n}$ (resp. $\overline{\Phi}_{C_n}$) is induced by the Frobenius endomorphism of $\mathbf{E}_n$ (resp. of C_n), so that the composition of the top (resp. bottom) horizontal arrows is $\Phi_{\mathbf{E}_n}$ (resp. Φ_{C_n}). Since $\mathscr{D}_n$ is cocartesian, and since $\pi_{\mathbf{E},n}$ is surjective, it is easily seen that the right square subdiagram is cocartesian; but $\overline{\Phi}_{\mathbf{E}_n}$ is an isomorphism, so the same holds for $\overline{\Phi}_{C_n}$. On the other hand, $(\Phi_{\mathbf{E}}^{-1}(\mathscr{I}^{n+1})E \mid n \in \mathbb{N})$ is a fundamental system of open ideals of E; therefore, the limit of the system of maps $(\overline{\Phi}_{C_n} \mid n \in \mathbb{N})$ is an automorphism of E; clearly the latter is the Frobenius automorphism, so E is a perfectoid $\mathbb{F}_p$-algebra.

In order to check the functoriality of the rule : $(f : \mathbf{E}_0 \to C) \mapsto (g : \mathbf{E}(A) \to E)$, consider any other system $(f'_n : \mathbf{E}_n \to C'_n \mid n \in \mathbb{N})$ of ring homomorphisms, with isomorphisms

$$\varphi'_n : C'_n/\mathscr{I}^n C'_n \xrightarrow{\sim} C'_{n-1} \quad \text{such that} \quad \varphi'_n \circ (\mathbf{E}_{n-1} \otimes_{\mathbf{E}_n} f'_n) = f'_{n-1} \quad \text{for every } n > 0$$

and let $h_0 : C = C_0 \to C'_0$ be a given map of $\mathbf{E}_0$-algebras. We construct inductively a system $h_\bullet := (h_n : C_n \to C'_n \mid n \in \mathbb{N})$ where h_n is a map of $\mathbf{E}_n$-algebras, and $\varphi'_{n+1} \circ (\mathbf{E}_n \otimes_{\mathbf{E}_{n+1}} h_{n+1}) = h_n \circ \varphi_{n+1}$ for every $n \in \mathbb{N}$. The map h_0 is already given. Suppose now that $n \in \mathbb{N}$, and that h_n has already been exhibited. Notice, since f_{n+1} is flat, the projection $C_{n+1} \to C_n$ induces $\mathbf{E}_n$-linear isomorphisms

$$\mathscr{I}\mathbf{E}_{n+1} \otimes_{\mathbf{E}_{n+1}} C_n \xrightarrow{\sim} \mathscr{I}\mathbf{E}_{n+1} \otimes_{\mathbf{E}_{n+1}} C_{n+1} \xrightarrow{\sim} \mathscr{I} C_{n+1}$$

whence a $\mathbf{E}_n$-linear map

$$l_{n+1} : \mathscr{I}C_{n+1} \stackrel{\sim}{\to} \mathscr{I}\mathbf{E}_{n+1} \otimes_{\mathbf{E}_{n+1}} C_n \to \mathscr{I}\mathbf{E}_{n+1} \otimes_{\mathbf{E}_{n+1}} C'_n \stackrel{\sim}{\to} \mathscr{I}\mathbf{E}_{n+1} \otimes_{\mathbf{E}_{n+1}} C'_{n+1} \to \mathscr{I}C'_{n+1}.$$

Then, since $\mathbb{L}_{C_n/\mathbf{E}_n} = 0$, by [75, Prop.3.2.16] there exists a unique ring homomorphism $h_{n+1} : C_{n+1} \to C'_{n+1}$ that makes commute the diagram :

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{I}C_{n+1} & \longrightarrow & C_{n+1} & \longrightarrow & C_n & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle l_{n+1}} & & \downarrow{\scriptstyle h_{n+1}} & & \downarrow{\scriptstyle h_n} & & \\ 0 & \longrightarrow & \mathscr{I}C'_{n+1} & \longrightarrow & C'_{n+1} & \longrightarrow & C'_n & \longrightarrow & 0 \end{array}$$

and this completes the construction of the system $h_\bullet$. The sought functoriality follows immediately. Furthermore, for any object $g' : \mathbf{E}(A) \to E'$ of $\mathbf{E}(A)\text{-}\mathsf{Perf}_{\mathrm{ad.fl}}$, set $C_0 := E'/\mathscr{I}E'$, and let $(\mathbf{E}_n \to C_n \mid n \in \mathbb{N})$ be the system of ring homomorphisms associated to $\mathbf{E}_0 \otimes_{\mathbf{E}(A)} g'$ by the foregoing inductive procedure; by the above, we obtain a compatible system of maps $(h'_n : C_n \to E'/\mathscr{I}^{n+1}E' \mid n \in \mathbb{N})$, and h'_0 is an isomorphism. By a simple induction, we deduce that h'_n is an isomorphism for every $n \in \mathbb{N}$, whence an isomorphism of $\mathbf{E}(A)$-algebras $\lim_{n\in\mathbb{N}} C_n \stackrel{\sim}{\to} E'$; this shows that our functor is indeed a quasi-inverse for (16.4.22). $\square$

16.4.24. Let A be a perfectoid ring, set $\mathbf{E} := \mathbf{E}(A)$, and denote by $\underline{\alpha} := (\alpha_n \mid n \in \mathbb{N})$ any distinguished element of $\mathrm{Ker}\, u_A$. Set $I_1 := \mathrm{Ann}_{\mathbf{E}}(\alpha_0)$, let $I_2 \subset \mathbf{E}$ be the radical of the ideal $\alpha_0\mathbf{E}$, and $I_3 := I_1 + I_2$; let also $\mathbf{E}_i := \mathbf{E}/I_i$ for $i = 1, 2, 3$. We claim that the resulting diagram of ring homomorphisms

$$\mathscr{E} \quad : \quad \begin{array}{ccc} \mathbf{E} & \longrightarrow & \mathbf{E}_1 \\ \downarrow & & \downarrow \\ \mathbf{E}_2 & \longrightarrow & \mathbf{E}_3 \end{array}$$

is cartesian. Indeed, it is clear the induced map $\mathbf{E} \to \mathbf{E}_1 \times_{\mathbf{E}_3} \mathbf{E}_2$ is surjective; for the injectivity, suppose that $x \in I_1 \cap I_2$; then $x^n \in \alpha_0 A$ for some $n \in \mathbb{N}$, and $\alpha_0 x = 0$, so that $x^{n+1} = 0$, and therefore $x = 0$, since $\mathbf{E}$ is perfect. Notice also that the $\alpha_0\mathbf{E}$-adic topology agrees with the discrete topology on $\mathbf{E}_2$ and $\mathbf{E}_3$, and by arguing as in remark 16.3.39(ii) we see that I_1 is a closed ideal for the α_0-adic topology of $\mathbf{E}$. Moreover, since $\mathbf{E}$ is perfect, it is easily seen that I_1 is a radical ideal; then it follows that the same holds for I_3, since we have already observed that the latter equals $I_1 \oplus I_2$. Summing up, we deduce that $\mathbf{E}_i$, endowed with its α_0-adic topology, is a perfectoid ring for $i = 1, 2, 3$. Consequently, the same holds for the rings

$$A_i := A \otimes_{\mathbb{W}(\mathbf{E})} W(\mathbf{E}_i) \qquad i = 1, 2, 3$$

and $\mathbf{E}(A_i) = \mathbf{E}_i$, provided we endow each A_i with its p-adic topology (proposition 16.2.21(i)). Furthermore, arguing as in the proof of claim 16.3.22 we easily see that the resulting diagram of ring homomorphisms :

$$A(\mathscr{E}) \quad : \quad \begin{array}{ccc} A & \longrightarrow & A_1 \\ \downarrow & & \downarrow \\ A_2 & \longrightarrow & A_3 \end{array}$$

is still cartesian. Set as well

$$X_{\mathbf{E}} := \mathrm{Spec}\, \mathbf{E} \quad X_A := \mathrm{Spec}\, A \quad \text{and} \quad X_{\mathbf{E}_k} := \mathrm{Spec}\, \mathbf{E}_k \quad X_{A_k} := \mathrm{Spec}\, A_k \quad \text{for } k = 1, 2, 3$$

and denote the resulting closed immersions by

$$i_{\mathbf{E}_k} : X_{\mathbf{E}_k} \to X_{\mathbf{E}} \qquad i_{A_k} : X_{A_k} \to X_A \qquad \text{for } k = 1, 2, 3.$$

According to (16.1.3), we get moreover a continuous and spectral map

$$\varphi := (\mathrm{Spec}\, \overline{u}_A) : X_A \to X_{\mathbf{E}}$$

Remark 16.4.25. In the situation of (16.4.24), we notice that :

(i) The image of α_0 is regular on $\mathbf{E}_1$, hence the induced isomorphism $\mathbf{E} \xrightarrow{\sim} \mathbf{E}_1 \times_{\mathbf{E}_3} \mathbf{E}_2$ identifies I_1 with $0 \times_{\mathbf{E}_3} \mathbf{E}_2$, *i.e.* with $\mathrm{Ker}\,(\mathbf{E}_2 \to \mathbf{E}_3)$.

(ii) Therefore, the image of $\overline{u}_A(\alpha_0)$ is regular in A_1 (proposition 16.4.17), and consequently, also p is regular in A_1.

(iii) For $i = 2, 3$, the image of $\underline{\alpha}$ in $W(\mathbf{E}_i)$ equals pu, for some $u \in W(\mathbf{E}_i)^\times$, so u_{A_2} and u_{A_3} induce isomorphisms of topological rings

$$\mathbf{E}_2 \xrightarrow{\sim} A_2 \qquad \mathbf{E}_3 \xrightarrow{\sim} A_3$$

that identify the projection $\mathbf{E}_2 \to \mathbf{E}_1$ in $\mathscr{E}$ with the map $A_2 \to A_3$ in $A(\mathscr{E})$.

(iv) Consequently, the isomorphism $A \xrightarrow{\sim} A_1 \times_{A_3} A_2$ induced by $A(\mathscr{E})$ identifies $\mathrm{Ann}_A(p)$ with $0 \times_{A_3} A_2$, *i.e.* with $\mathrm{Ker}\,(A_2 \to A_3)$, and summing up, we get a natural $W(\mathbf{E})$-linear identification

$$\mathrm{Ann}_{\mathbf{E}}(\alpha_0) \xrightarrow{\sim} \mathrm{Ann}_A(p) \qquad : \qquad \beta \mapsto \overline{u}_A(\beta).$$

(v) Let I be an ideal of adic definition of A, and $J \subset A$ another ideal such that $IJ = 0$. Since A is reduced (corollary 16.3.63(i)), we deduce that $I^n \cap J = 0$ for every $n \in \mathbb{N}$, whence $J + I^n = J \oplus I^n$ for every such n. It follows that

$$\bigcap_{n\in\mathbb{N}} (J + I^n) = \bigcap_{n\in\mathbb{N}} (J \oplus I^n) = J \oplus \bigcap_{n\in\mathbb{N}} I^n = J$$

which means that J is closed in the topology of A.

Proposition 16.4.26. *With the notation of* (16.4.24)*, the following holds :*

(i) *The map φ is surjective.*

(ii) *For every quasi-coherent $\mathscr{O}_{X_A}$-module $\mathscr{F}$ and every integer $i > 0$ we have*

$$R^i\varphi_*\mathscr{F} = 0.$$

Proof. (i): Consider any $\mathfrak{q} \in X_{\mathbf{E}}$, set

$$J := \sum_{x\in\mathfrak{q}} \overline{u}_A(x) \cdot A \qquad S := \{\overline{u}_A(x) \mid x \in \mathbf{E} \setminus \mathfrak{q}\}$$

and notice that S is a multiplicative subset of A. Let $\mathfrak{p} \in X_A$ be any prime ideal; it is easily seen that $\varphi(\mathfrak{p}) = \mathfrak{q}$ if and only if $S \cap \mathfrak{p} = \varnothing$ and $J \subset \mathfrak{p}$. Thus, we come down to showing that $JA_S \neq A_S$. Suppose this fails; then there exist $y \in \mathbf{E} \setminus \mathfrak{q}$ and sequences $x_\bullet := (x_1, \dots, x_k)$, $a_\bullet := (a_1, \dots, a_k)$ of elements of $\mathfrak{q}$ and respectively A, such that

$$\overline{u}_A(y) = \sum_{i=1}^k \overline{u}_A(x_i) \cdot a_i.$$

Especially, $[y]^{\langle 1\rangle}A \subset [x_\bullet]^{\langle 1\rangle}A$. Set $\mathscr{J} := \sum_{i=1}^k \mathbf{E}x_i \subset \mathbf{E}$; by corollary 16.3.51(i) we deduce

$$y \in \mathscr{J}^{\langle 1-1/p\rangle}\mathbf{E} \subset \mathfrak{q}$$

a contradiction.

(ii): Let $f \in \mathbf{E}$ any element, and set $[f] := \overline{u}_A(f)$; in light of (9.3.67) we get

$$\varphi^{-1}(\mathrm{Spec}\, E_f) = \mathrm{Spec}\, A_{[f]}. \tag{16.4.27}$$

Now, recall that $R^i\varphi_*\mathscr{F}$ is naturally isomorphic to the sheaf associated to the presheaf

$$U \mapsto H^i(\varphi^{-1}U, \mathscr{F}) \qquad \text{for every open subset } U \subset X_{\mathbf{E}}.$$

Then the assertion follows immediately from (16.4.27). □

16.4.28. In the situation of (16.4.24), let $\underline{\alpha} := (\alpha_n \mid n \in \mathbb{N})$ be any distinguished element of $\mathrm{Ker}\, u_A$; notice the commutative diagram of topological spaces

$$\begin{array}{ccc} X_A^0 := \mathrm{Spec}\, A/pA & \xrightarrow{\varphi_0} & X_{\mathbf{E}}^0 := \mathrm{Spec}\, \mathbf{E}/\alpha_0\mathbf{E} \\ \downarrow{\scriptstyle \iota_A} & & \downarrow{\scriptstyle \iota_{\mathbf{E}}} \\ X_A & \xrightarrow{\varphi} & X_{\mathbf{E}} \end{array}$$

whose vertical arrows are the inclusion maps, and with $\varphi_0 := \mathrm{Spec}\, \omega$, where $\omega : \mathbf{E}/\alpha_0\mathbf{E} \xrightarrow{\sim} A/pA$ is the ring isomorphism deduced from $u_A \otimes_{\mathbb{Z}} \mathbb{F}_p$ as in remark 16.3.7(ii). Denote also

$$\mathscr{O}^{\mathrm{t}}_{X_{\mathbf{E}}} \subset \mathscr{O}^{\mathrm{t}}_{X_{\mathbf{E}}} \qquad \text{and} \qquad \mathscr{O}^{\mathrm{t}}_{X_A} \subset \mathscr{O}^{\mathrm{t}}_{X_A}$$

respectively the subsheaf of α_0-torsion sections and the subsheaf of p-torsion sections. Notice that $\mathscr{O}^{\mathrm{t}}_{X_{\mathbf{E}}}$ is independent of the choice of $\underline{\alpha}$ (remark 16.1.7(i)); more precisely, corollary 16.3.63(i) implies that

$$\mathscr{O}^{\mathrm{t}}_{X_{\mathbf{E}}} = \underline{\Gamma}_{X^0_{\mathbf{E}}} \mathscr{O}_{X_{\mathbf{E}}} = \mathrm{Ker}\,(\mathscr{O}_{X_{\mathbf{E}}} \to i_{\mathbf{E}_1 *}\mathscr{O}_{X_{\mathbf{E}_1}})$$
$$\mathscr{O}^{\mathrm{t}}_{X_A} = \underline{\Gamma}_{X^0_A} \mathscr{O}_{X_A} = \mathrm{Ker}\,(\mathscr{O}_{X_A} \to i_{A_1 *}\mathscr{O}_{X_{A_1}}).$$

Lemma 16.4.29. *In the situation of* (16.4.28)*, the following holds :*

(i) *The corresponding isomorphism of sheaves* $\varphi_0^\flat : \mathscr{O}_{X^0_{\mathbf{E}}} \xrightarrow{\sim} \varphi_{0*}\mathscr{O}_{X^0_A}$ *lifts to a map of sheaves of monoids*
$$\varphi^\flat : \mathscr{O}_{X_{\mathbf{E}}} \to \varphi_*\mathscr{O}_{X_A}.$$

(ii) $\varphi^\flat$ *restricts to an isomorphism of sheaves of abelian groups*
$$\mathscr{O}^{\mathrm{t}}_{X_{\mathbf{E}}} \xrightarrow{\sim} \varphi_*\mathscr{O}^{\mathrm{t}}_{X_A}.$$

(iii) *For every open subset* $U \subset X_{\mathbf{E}}$ *we have*
$$\varphi^\flat_U(x+y) = \varphi^\flat_U(x) + \varphi^\flat_U(y) \qquad \text{for every } x \in \mathscr{O}^{\mathrm{t}}_{X_{\mathbf{E}}}(U) \text{ and } y \in \mathscr{O}_{X_{\mathbf{E}}}(U).$$

(iv) *For every closed subset* $W_A \subset X^0_A$*, the* $\mathscr{O}_{X_A}$*-module* $\underline{\Gamma}_{W_A}\mathscr{O}_{X_A}$ *is quasi-coherent.*

Proof. (i): For every $f \in \mathbf{E}$, set $D(f) := \mathrm{Spec}\, \mathbf{E}_f$ and $[f] := \overline{u}_A(f)$, and let

$$\varphi^\flat_f := (\overline{u}_A)_f : \mathbf{E}_f \to A_{[f]}$$

(notation of (4.8.33)). Now, let $g \in \mathbf{E}$ be any other element, and suppose that $D(g) \subset D(f)$; there follows a commutative diagram

$$\begin{array}{ccccc} \mathbf{E} & \longrightarrow & \mathbf{E}_f & \longrightarrow & \mathbf{E}_g \\ \downarrow{\scriptstyle \overline{u}_A} & & \downarrow{\scriptstyle \varphi^\flat_f} & & \downarrow{\scriptstyle \varphi^\flat_g} \\ A & \longrightarrow & A_{[f]} & \longrightarrow & A_{[g]} \end{array}$$

whose horizontal arrows are the localization maps. Since all localizations are epimorphisms in the category of monoids, we deduce that the diagram commutes for every such f, g. Especially, $\varphi^\flat_f$ depends only on $D(f)$ (and not on f). Then the rule $D(f) \mapsto \varphi^\flat_f$ extends to a morphism of sheaves of monoids on $X_{\mathbf{E}}$ (see [59, Ch.0, §3.2.3]), and by construction the resulting diagram

$$\begin{array}{ccc} \mathscr{O}_{X_{\mathbf{E}}} & \xrightarrow{\varphi^\flat} & \varphi_*\mathscr{O}_{X_A} \\ \downarrow & & \downarrow \\ \iota_{\mathbf{E}*}\mathscr{O}_{X^0_{\mathbf{E}}} & \xrightarrow{\iota_{\mathbf{E}*}(\varphi^\flat_0)} & \iota_{\mathbf{E}*}\varphi_{0*}\mathscr{O}_{X^0_A} \end{array}$$

commutes.

(iii): It suffices to check the stated identity for U of the form $\mathrm{Spec}\,\mathbf{E}_f$, where $f \in \mathbf{E}$ is any element. Then, in light of (16.4.30), we easily reduce to the case where $x, y \in \mathrm{Im}\,(\mathbf{E} \to \mathbf{E}_f)$. Then, we are further reduced to checking that $\overline{u}_A(x+y) = \overline{u}_A(x) + \overline{u}_A(y)$ for every $x, y \in \mathbf{E}$ such that $\alpha_0 \cdot x = 0$. The latter follows easily from proposition 9.3.62 and lemma 16.2.7(iii) : details left to the reader.

(ii): Lemma 16.2.7(iii) easily implies that $\varphi^\flat$ send $\mathscr{O}_{X_\mathbf{E}}$ into $\varphi_*\mathscr{O}_{X_A}$. In light of (iii), it then suffices to check that the map $\varphi^\flat_f$ restricts to an isomorphism $\mathrm{Ann}_{\mathbf{E}_f}(\alpha_0) \xrightarrow{\sim} \mathrm{Ann}_{A_{[f]}}(p)$ for every $f \in \mathbf{E}$. However

$$\mathrm{Ann}_{\mathbf{E}_f}(\alpha_0) = \mathrm{Ann}_{\mathbf{E}}(\alpha_0)_f \qquad \text{and} \qquad \mathrm{Ann}_{A_{[f]}}(p) = \mathrm{Ann}_A(p)_{[f]} \tag{16.4.30}$$

so the assertion follows from remark 16.4.25(iv).

(iv): Set $W_\mathbf{E} := \varphi_0(W_A)$; from (ii) we see that $\varphi^\flat$ induces an isomorphism

$$\underline{\Gamma}_{W_A}\mathscr{O}_{X_A} = \underline{\Gamma}_{W_A} \circ \underline{\Gamma}_{X^0_A}\mathscr{O}_{X_A} \xrightarrow{\sim} \underline{\Gamma}_{W_A}(\iota_{A*}\varphi_0^{-1}\iota_\mathbf{E}^{-1}\underline{\Gamma}_{X^0_\mathbf{E}}\mathscr{O}_{X_\mathbf{E}}) = \iota_{A*}\varphi_0^{-1}\iota_\mathbf{E}^{-1}\underline{\Gamma}_{W_\mathbf{E}}\mathscr{O}_{X_\mathbf{E}}$$

of $\mathscr{O}_{X_A}$-modules (lemma 10.4.13(v)). Since the functor ι_{A*} sends quasi-coherent $\mathscr{O}_{X^0_A}$-modules to quasi-coherent $\mathscr{O}_{X_A}$-modules ([59, Ch.I, Cor.9.2.2]), we are then reduced to checking that $\iota_\mathbf{E}^{-1}\underline{\Gamma}_{W_\mathbf{E}}\mathscr{O}_{X_\mathbf{E}}$ is a quasi-coherent $\mathscr{O}_{X^0_\mathbf{E}}$-module. However, it was already remarked in (16.4.28) that $\underline{\Gamma}_{W_\mathbf{E}}\mathscr{O}_{X_\mathbf{E}} = \underline{\Gamma}_{W_\mathbf{E}}\mathscr{O}^{\mathrm{t}}_{X_\mathbf{E}}$ is an $\iota_{\mathbf{E}*}\mathscr{O}_{X^0_\mathbf{E}}$-module, so we are further reduced to checking that $\underline{\Gamma}_{W_\mathbf{E}}\mathscr{O}^{\mathrm{t}}_{X_\mathbf{E}}$ is a quasi-coherent $\mathscr{O}_{X_\mathbf{E}}$-module.

Thus, let $f \in \mathbf{E}$ be any element, and $s \in \underline{\Gamma}_{W_\mathbf{E}}\mathscr{O}^{\mathrm{t}}_{X_\mathbf{E}}(D(f))$ any section; pick also a subset $S \subset \mathbf{E}$ such that $U_\mathbf{E} := X_\mathbf{E} \setminus W_\mathbf{E} = \bigcup_{g \in S} D(g)$. By assumption, $s_{|D(fg)} = 0$ for every $g \in S$, and there exist $n \in \mathbb{N}$ and $t \in \mathbf{E}$ such that $s = f^{-n}t$; it follows that for every $g \in S$ there exists an integer $m_g \in \mathbb{N}$ such that $(fg)^{m_g}t = 0$ in $\mathbf{E}$. Since $\mathbf{E}$ is perfect, we deduce that $fgt = 0$ for every such g, hence $(ft)_{|U_\mathbf{E}} = 0$, so finally $s \in (\Gamma_{W_\mathbf{E}}\mathscr{O}^{\mathrm{t}}_{X_\mathbf{E}})_f$, whence the contention. □

Remark 16.4.31. (i) Keep the notation of lemma 16.4.29. Notice that the rule $A \mapsto (\varphi, \varphi^\flat)$ is functorial in A. Namely, if $g : A' \to A''$ is any continuous ring homomorphism of perfectoid rings, and $(\varphi', \varphi'^\flat)$ $(\varphi'', \varphi''^\flat)$ the pairs attached to A' and respectively A'' as in lemma 16.4.29, then we have commutative diagrams of topological spaces and of sheaves of monoids :

$$\begin{array}{ccc} X_{A''} & \xrightarrow{\varphi''} & X_{\mathbf{E}''} \\ {\scriptstyle \psi_A}\downarrow & & \downarrow{\scriptstyle \psi_\mathbf{E}} \\ X_{A'} & \xrightarrow{\varphi'} & X_{\mathbf{E}'} \end{array} \qquad \begin{array}{ccc} \mathscr{O}_{X_{\mathbf{E}'}} & \xrightarrow{\psi^\flat_E} & \psi_{\mathbf{E}*}\mathscr{O}_{X_{\mathbf{E}''}} \\ {\scriptstyle \varphi'^\flat}\downarrow & & \downarrow{\scriptstyle \psi_{\mathbf{E}*}\varphi''^\flat} \\ \varphi'_*\mathscr{O}_{X_{A'}} & \xrightarrow{\varphi'_*\psi^\flat_A} & \psi_{\mathbf{E}*}\varphi''_*\mathscr{O}_{X_{A''}} = \varphi'_*\psi_{A*}\mathscr{O}_{X_{A''}} \end{array}$$

where $X_{A'} := \mathrm{Spec}\,A'$, $X_{\mathbf{E}'} := \mathrm{Spec}\,\mathbf{E}(A')$ and similarly for $X_{A''}$ and $X_{\mathbf{E}''}$, and $\psi_A := \mathrm{Spec}\,g$, $\psi_\mathbf{E} := \mathrm{Spec}\,\mathbf{E}(g)$. The verification shall be left to the reader.

(ii) The short exact sequence of $\mathscr{O}_{X_\mathbf{E}}$-modules

$$\mathscr{E} \quad : \quad 0 \to \mathscr{O}^{\mathrm{t}}_{X_\mathbf{E}} \to \mathscr{O}_{X_\mathbf{E}} \to i_{\mathbf{E}_1*}\mathscr{O}_{X_{\mathbf{E}_1}} \to 0$$

induces a commutative ladder for every open subset $\Omega \subset X_\mathbf{E}$:

$$\begin{array}{ccccccccc} 0 & \to & H^0(\Omega, \mathscr{O}^{\mathrm{t}}_{X_\mathbf{E}}) & \to & H^0(\Omega, \mathscr{O}_{X_\mathbf{E}}) & \to & H^0(\Omega, i_{\mathbf{E}_1*}\mathscr{O}_{X_{\mathbf{E}_1}}) & \to & H^1(\Omega, \mathscr{O}^{\mathrm{t}}_{X_\mathbf{E}}) \\ & & {\scriptstyle \beta^{\mathrm{t}}}\downarrow & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \gamma} \\ 0 & \to & H^0(\varphi^{-1}\Omega, \mathscr{O}^{\mathrm{t}}_{X_A}) & \to & H^0(\varphi^{-1}\Omega, \mathscr{O}_{X_A}) & \to & H^0(\varphi^{-1}\Omega, i_{A_1*}\mathscr{O}_{X_{A_1}}) & \to & H^1(\varphi^{-1}\Omega, \mathscr{O}^{\mathrm{t}}_{X_A}) \end{array}$$

whose rows are the initial terms of the long exact cohomology sequence attached to $\mathscr{E}$, and where β is induced by $\varphi^\flat$. The map β_1 is induced by $i_{\mathbf{E}_1*}\varphi_1^\flat$, where $\varphi_1^\flat : \mathscr{O}_{X_{\mathbf{E}_1}} \to \varphi_1^*\mathscr{O}_{X_{A_1}}$ is the morphism associated with the perfectoid ring A_1 and the induced continuous map $\varphi_1 : X_{\mathbf{E}_1} \to X_{A_1}$ as in lemma 16.4.29(i). Lastly, β^{t} and γ are deduced from the restriction $\mathscr{O}^{\mathrm{t}}_{X_\mathbf{E}} \to \varphi_*\mathscr{O}^{\mathrm{t}}_{X_A}$

of $\varphi^\flat$, which is an isomorphism of abelian sheaves (lemma 16.4.29(ii,iii)); furthermore, these subsheaves are supported on the closed subsets $X^0_{\mathbf{E}}$ and respectively X^0_A, and φ restricts to a homeomorphism $\varphi_0 : X^0_{\mathbf{E}} \xrightarrow{\sim} X^0_A$, so β^{t} and γ are isomorphisms of abelian groups. The commutativity of the left square subdiagram is clear, and that of the central one follows from (i). To show the commutativity of the right square subdiagram, pick any $\overline{s} \in H^0(\Omega, i_{\mathbf{E}1*}\mathscr{O}_{X_{\mathbf{E}}})$, and find a covering $\Omega = \bigcup_{i\in I} \Omega_i$ consisting of open subsets, such that $\overline{s}_{|\Omega_i}$ is the image of some $s_i \in H^0(\Omega_i, \mathscr{O}_{X_{\mathbf{E}}})$, for every $i \in I$. Set $\Omega_{ij} := \Omega_i \cap \Omega_j$ for every $i, j \in I$, and let $s_{ij} := s_{i|\Omega_{ij}} - s_{j|\Omega_{ij}}$, and notice that $s_{ij} \in H^0(\Omega_{ij}, \mathscr{O}^{\mathrm{t}}_{X_{\mathbf{E}}})$ for every such i, j. Then the image of $\overline{s}$ in $H^1(\Omega, \mathscr{O}^{\mathrm{t}}_{X_{\mathbf{E}}})$ is the class of the Čech cocycle $(s_{ij} \mid i, j \in I)$, and the latter maps to the class of the cocycle $(\varphi^\flat_{\Omega_{ij}}(s_{ij}) \mid i, j \in I)$ in $H^1(\varphi^{-1}\Omega, \mathscr{O}^{\mathrm{t}}_{X_A})$. On the other hand, set $\overline{t} := \beta(\overline{s})$; for every $i \in I$, the restriction of $\overline{t}$ to $\varphi^{-1}\Omega_i$ is the image of $\varphi^\flat_{\Omega_i}(s_i) \in H^0(\Omega_i, \mathscr{O}_{X_A})$, hence the image of $\overline{t}$ in $H^1(\varphi^{-1}\Omega, \mathscr{O}^{\mathrm{t}}_{X_A})$ is the class of the cocycle $(t_{ij} \mid i \in I)$ with $t_{ij} := t_{i|\varphi^{-1}\Omega_{ij}} - t_{j|\varphi^{-1}\Omega_{ij}}$ for every $i, j \in I$. Lemma 16.4.29(iii) implies that $t_{ij} = \varphi^\flat_{\Omega_{ij}}(s_{ij})$ for every $i, j \in I$, whence the assertion.

The following result shows that if A is a perfectoid ring, proposition 8.3.30 holds even in case the open subset U of $\operatorname{Spec} A$ is not quasi-compact.

Lemma 16.4.32. *Let A be a perfectoid ring, and $U \subset X_A := \operatorname{Spec} A$ an open subset containing the analytic locus of X_A. We have :*

- (i) *There exists a unique topology $\mathscr{T}_U$ on $A_U := \mathscr{O}_{X_A}(U)$ such that $(A_U, \mathscr{T}_U)$ is f-adic and the restriction map $\rho_U : A \to A_U$ is open.*
- (ii) *$(A_U, \mathscr{T}_U)$ is complete and separated.*
- (iii) *Let $f : A \to B$ be any morphism of f-adic topological rings, such that the image of $\operatorname{Spec} f$ lies in U. Then the resulting map $f_U : (A_U, \mathscr{T}_U) \to B$ is continuous.*

Proof. (i): We pick a system of elements $(\beta_1, \dots, \beta_r)$ of $\mathbf{E}(A)$ such that the system $(f_i := \overline{u}_A(\beta_i) \mid i = 1, \dots, r)$ generates an ideal I of adic definition of A. Like in the proof of proposition 8.3.30(i), there exists a unique topology $\mathscr{T}_U$ on A_U such that $(A_U, \mathscr{T}_U)$ is a topological group for its additive group structure, and we need to check the following. For every $a \in A_U$ and every $n \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that

$$a \cdot \rho_U(I^k) \subset \rho_U(I^n).$$

Now, for $i = 1, \dots, r$, let $a_i \in \operatorname{Spec} A_{f_i}$ be the image of a under the restriction map $\rho_i : A_U \to A_{f_i}$; we may find $m \in \mathbb{N}$ and $b_1, \dots, b_r \in A$ such that $a_i = f_i^{-m} b_i$, hence $f_i^m a - b_i \in \operatorname{Ker} \rho_i$ for $i = 1, \dots, r$. Pick a family $(g_\lambda \mid \lambda \in \Lambda)$ of elements of A such that $U = \bigcup_{\lambda\in\Lambda} V_\lambda$, where $V_\lambda := \operatorname{Spec} A_{g_\lambda}$ for every $\lambda \in \Lambda$. Notice that $(f_i^m a - b_i)_{|V_\lambda}$ lies in the kernel of the restriction map $\mathscr{O}_U(V_\lambda) \to \mathscr{O}_U(V_\lambda \cap \operatorname{Spec} A_{f_i})$, for every $i = 1, \dots, r$ and every $\lambda \in \Lambda$. Thus, for every such λ and i there exist $t_\lambda, s_\lambda \in \mathbb{N}$ and $c_{i,\lambda} \in A$ such that

$$g_\lambda^{s_\lambda} \cdot (f_i^m a - b_i)_{|V_\lambda} = \rho_\lambda(c_{i,\lambda}) \qquad \text{and} \qquad f_i^{t_\lambda} c_{i,\lambda} = 0 \qquad \text{in } A$$

where $\rho_\lambda : A \to A_{g_\lambda}$ is the localization map. From corollary 16.3.63(i) we then deduce that $f_i c_{i,\lambda} = 0$, so

$$(f_i^{m+1} a - f_i b_i)_{|V_\lambda} = 0 \qquad \text{for every } \lambda \in \Lambda \text{ and every } i = 1, \dots, r$$

i.e. $f_i^{m+1} a = \rho_U(f_i b_i)$ for every $i = 1, \dots, r$. From this, we may argue as in the proof of proposition 8.3.30(i), to derive the assertion.

(ii): It suffices to show that $\operatorname{Ker} \rho_U$ is a closed ideal of A. However, clearly $I \cdot \operatorname{Ker} \rho_U = 0$, so the assertion follows from remark 16.4.25(v).

(iii) is proven as proposition 8.3.30(iii). □

We wish next to show that the formula for the Teichmüller representative of a sum obtained in proposition 9.3.62, is still valid in f-adic rings of the type considered in lemma 16.4.32.

16.4.33. Indeed, resume the situation of (16.4.28), and let $U \subset X_{\mathbf{E}}$ be an open subset containing $X_{\mathbf{E}} \setminus \operatorname{Spec} \mathbf{E}/\mathbf{E}^{\circ\circ}$; from (16.4.27) we easily see that $X_A \setminus \operatorname{Spec} A/A^{\circ\circ} \subset \varphi^{-1}U$, so

$$\mathbf{E}_U := \mathscr{O}_{X_{\mathbf{E}}}(U) \qquad \text{and} \qquad A_U := \mathscr{O}_{X_A}(\varphi^{-1}U)$$

admit the f-adic topologies $\mathscr{T}_U$ and respectively $\mathscr{T}_{\varphi^{-1}U}$ furnished by lemma 16.4.32.

Proposition 16.4.34. *With the notation of* (16.4.33)*, the following holds :*

(i) *The map* $\varphi^\flat_U : (\mathbf{E}_U, \mathscr{T}_U) \to (A_U, \mathscr{T}_{\varphi^{-1}U})$ *of lemma* 16.4.29(i) *is continuous.*

(ii) *For every* $\underline{x} := (x_0, \dots, x_k) \in \mathbf{E}_U^{k+1}$ *we have the identity*

$$\varphi^\flat_U(x_0 + \cdots + x_k) = \sum_{n\in\mathbb{N}} p^n \cdot \sum_{\sigma\in\Sigma^{(k)}_n} c_\sigma \cdot \varphi^\flat_U(\underline{x}^\sigma)$$

where $\Sigma^{(k)}_n$ *is defined as in* (9.3.61) *for every* $n \in \mathbb{N}$*, and* $c_\sigma \in \mathbb{Z}_p$ *is provided by proposition* 9.3.62*, for every* $\sigma \in \bigcup_{n\in\mathbb{N}} \Sigma^{(k)}_n$*; the convergence of the series is relative to the topology* $\mathscr{T}_{\varphi^{-1}U}$*.*

Proof. (Recall that $\underline{x}^\sigma := x_0^{\sigma_0} \cdots x_k^{\sigma_k}$ for every $\sigma := (\sigma_0, \dots, \sigma_k) \in \mathbb{N}[1/p]^{k+1}$).

(ii): Let $(\alpha_n \mid n \in \mathbb{N})$ be any distinguished element in $\operatorname{Ker} u_A$; since α_0 is topologically nilpotent in $\mathbf{E}$, its image is topologically nilpotent in $\mathbf{E}_U$, hence there exists $c \in \mathbb{N}$ such that $\alpha_0^c \cdot x_i \in \operatorname{Im}(\mathbf{E} \to \mathbf{E}_U)$ for $i = 0, \dots, k$. In light of lemma 16.2.7(iii), we deduce that

$$y_n(\underline{x}) := \sum_{\sigma\in\Sigma^{(k)}_n} p^n \cdot c_\sigma \cdot \varphi^\flat_U(x^\sigma) \in p^{n-c} \cdot \operatorname{Im}(A \to A_U) \qquad \text{for every integer } n \geq c.$$

Taking into account lemma 16.4.32(ii), it follows that the series $\sum_{n\in\mathbb{N}} y_n(\underline{x})$ converges for the topology $\mathscr{T}_{\varphi^{-1}U}$ to a well determined element $y \in A_U$, and it remains only to check that $y = \varphi^\flat_U(x_0 + \cdots + x_k)$. To this aim, pick a family $(g_\lambda \mid \lambda \in \Lambda)$ of elements of $\mathbf{E}$, such that $U = \bigcup_{\lambda\in\Lambda} V_\lambda$, where $V_\lambda := \operatorname{Spec} \mathbf{E}_{g_\lambda}$ for every $\lambda \in \Lambda$. Then, for every such λ we may find $n_\lambda \in \mathbb{N}$ such that $(g_\lambda^{n_\lambda} \cdot x_i)_{|V_\lambda} \in \operatorname{Im}(\mathbf{E} \to \mathbf{E}_{g_\lambda})$ for $i = 0, \dots, k$, and it suffices to check that $([g_\lambda^{n_\lambda}] \cdot \varphi^\flat_U(x_0 + \cdots + x_k))_{|\varphi^{-1}V_\lambda} = ([g_\lambda^{n_\lambda}] \cdot y)_{|\varphi^{-1}V_\lambda}$, where $[g_\lambda^{n_\lambda}] := \overline{u}_A(g_\lambda^{n_\lambda})$ for every such λ. However, we have

$$[g_\lambda^{n_\lambda}] \cdot \varphi^\flat_U(x_0 + \cdots + x_k) = \varphi^\flat_U(g_\lambda^{n_\lambda} \cdot x_0 + \cdots + g_\lambda^{n_\lambda} \cdot x_k) \qquad \text{and} \qquad [g_\lambda^{n_\lambda}] \cdot y = \sum_{n\in\mathbb{N}} y_n(g_\lambda^{n_\lambda} \cdot \underline{x})$$

where $g_\lambda^{n_\lambda} \cdot \underline{x} := (g_\lambda^{n_\lambda} \cdot x_0, \dots, g_\lambda^{n_\lambda} \cdot x_k)$. Hence, we may replace $\underline{x}$ by $g_\lambda^{n_\lambda} \cdot \underline{x}$, after which we may assume that $x_i \in \operatorname{Im}(\rho_U : \mathbf{E} \to \mathbf{E}_U)$ for $i = 0, \dots, k$. In this case, say that $x_i = \rho_U(z_i)$ for $i = 0, \dots, k$ and for certain $z_0, \dots, z_k \in \mathbf{E}$; we have

$$\varphi^\flat_U(x_0 + \cdots + x_k) = \rho_{\varphi^{-1}U} \circ \varphi^\flat_{X_{\mathbf{E}}}(z_0 + \cdots + z_k) = \rho_{\varphi^{-1}U} \circ \overline{u}_A(z_0 + \cdots + z_k)$$

where $\rho_{\varphi^{-1}U} : A \to A_U$ is the corresponding restriction map; on the other hand, since $\rho_{\varphi^{-1}U}$ is continuous for the topology $\mathscr{T}_{\varphi^{-1}U}$, we have as well

$$\begin{aligned} y &= \sum_{n\in\mathbb{N}} p^n \cdot \sum_{\sigma\in\Sigma^{(k)}_n} c_\sigma \cdot \rho_{\varphi^{-1}U} \circ \varphi^\flat_{X_{\mathbf{E}}}(\underline{z}^\sigma) = \rho_{\varphi^{-1}U}\Big(\sum_{n\in\mathbb{N}} p^n \cdot \sum_{\sigma\in\Sigma^{(k)}_n} c_\sigma \cdot \varphi^\flat_{X_{\mathbf{E}}}(\underline{z}^\sigma)\Big) \\ &= \rho_{\varphi^{-1}U}\Big(\sum_{n\in\mathbb{N}} p^n \cdot \sum_{\sigma\in\Sigma^{(k)}_n} c_\sigma \cdot \overline{u}_A(\underline{z}^\sigma)\Big) = \rho_{\varphi^{-1}U} \circ u_A\Big(\sum_{n\in\mathbb{N}} p^n \cdot \sum_{\sigma\in\Sigma^{(k)}_n} c_\sigma \cdot \tau_A(\underline{z}^\sigma)\Big) \end{aligned}$$

so the sought identity follows from proposition 9.3.62.

(i): Fix $x_0 \in \mathbf{E}_U$; the proof of (ii) shows that for every $k \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $y_r(x_0 + x) \in p^k \cdot \operatorname{Im}(A \to A_U)$ for every $r \geq n$ and every $x \in \operatorname{Im}(\mathbf{E} \to \mathbf{E}_U)$. On the other hand, clearly the mappings y_i are continuous for every $i \in \mathbb{N}$, relative to the topologies $\mathscr{T}_U$ and $\mathscr{T}_{\varphi^{-1}U}$. The assertion follows immediately. □

16.4.35. Keep the notation of (16.4.28), let $U_{\mathbf{E}} \subset X_{\mathbf{E}}$ be any open subset, and set $U_A := \varphi^{-1}U_{\mathbf{E}}$, $Z_{\mathbf{E}} := X_{\mathbf{E}} \setminus U_{\mathbf{E}}$, $Z_A := X_A \setminus U_A$. Denote by $j_{\mathbf{E}} : U_{\mathbf{E}} \to X_{\mathbf{E}}$ and $j_A : U_A \to X_A$ the open immersions; the map $\varphi^\flat$ of lemma 16.4.29(i) induces a morphism of sheaves of monoids

$$\psi : j_{\mathbf{E}*}\mathscr{O}_{U_{\mathbf{E}}} \xrightarrow{j_{\mathbf{E}*}j^*_{\mathbf{E}}(\varphi^\flat)} j_{\mathbf{E}*}j^*_{\mathbf{E}}\varphi_*\mathscr{O}_{X_A} \xrightarrow{\sim} \varphi_*j_{A*}\mathscr{O}_{U_A}.$$

Moreover, let

$$\overline{\mathscr{O}}_{X_{\mathbf{E}}} := \mathrm{Im}\,(\mathscr{O}_{X_{\mathbf{E}}} \to j_{\mathbf{E}*}\mathscr{O}_{U_{\mathbf{E}}}) \qquad \overline{\mathscr{O}}_{X_A} := \mathrm{Im}\,(\mathscr{O}_{X_A} \to j_{A*}\mathscr{O}_{U_A}).$$

Denote by $\pi_{\mathbf{E}} : j_{\mathbf{E}*}\mathscr{O}_{U_{\mathbf{E}}} \to R^1\underline{\Gamma}_{Z_{\mathbf{E}}}\mathscr{O}_{X_{\mathbf{E}}}$ and $\pi_A : j_{A*}\mathscr{O}_{U_A} \to R^1\underline{\Gamma}_{Z_A}\mathscr{O}_{X_A}$ the projections, set $q := \overline{u}_A(\alpha_0)$, and for every $c \in \mathbb{N}[1/p]$ define

$$\alpha_0^{-c}\mathscr{O}_{X_{\mathbf{E}}} := \mathrm{Ker}\,\alpha_0^c \cdot \pi_{\mathbf{E}} \qquad p^{-c}\mathscr{O}_{X_A} := \mathrm{Ker}\,q^c \cdot \pi_A$$
$$\mathscr{Q}^c_{U_{\mathbf{E}}} := \pi_{\mathbf{E}}(\alpha_0^{-c}\mathscr{O}_{X_{\mathbf{E}}}) \qquad \mathscr{Q}^c_{U_A} := \pi_A(p^{-c}\mathscr{O}_{X_A}).$$

Also, set

$$\mathscr{R}^c := \mathrm{Coker}\,(\varphi_*\overline{\mathscr{O}}_{X_A} \to \varphi_*(p^{-c}\mathscr{O}_{X_A})) \qquad \text{and let} \qquad \lambda : \mathscr{R}^c \to \varphi_*\mathscr{Q}^c_{U_A}$$

be the natural morphism of $\mathscr{O}_{X_A}$-modules; lemma 16.2.7(iii) easily implies that ψ restricts to a morphism of sheaves of sets

$$\psi^c : \alpha_0^{-c}\mathscr{O}_{X_{\mathbf{E}}} \to \varphi_*(p^{-c}\mathscr{O}_{X_A}) \qquad \text{for every } c \in \mathbb{N}[1/p].$$

Proposition 16.4.36. *With the notation of* (16.4.35)*, suppose moreover that :*

(a) $\mathrm{Spec}\,\mathbf{E}[\alpha_0^{-1}] \subset U_{\mathbf{E}}$.
(b) $U_{\mathbf{E}}$ *is quasi-compact.*
(c) $c \leq p/(p-1)$.

Then we have :

(i) *The map ψ^c descends to a morphism of sheaves of sets*

$$\overline{\psi}^c : \mathscr{Q}^c_{U_{\mathbf{E}}} \to \mathscr{R}^c \xrightarrow{\lambda} \varphi_*\mathscr{Q}^c_{U_A}.$$

(ii) *There exist unique morphisms of sheaves of sets*

$$\overline{\boldsymbol{p}}^c_{\mathbf{E}} : \mathscr{Q}^{c/p}_{U_{\mathbf{E}}} \to \mathscr{Q}^c_{U_{\mathbf{E}}} \qquad \textit{and} \qquad \overline{\boldsymbol{p}}^c_A : \mathscr{Q}^{c/p}_{U_A} \to \mathscr{Q}^c_{U_A}$$

fitting into commutative diagrams

$$\begin{array}{ccc} \alpha_0^{-c/p}\mathscr{O}_{X_{\mathbf{E}}} & \longrightarrow & \alpha_0^{-c}\mathscr{O}_{X_{\mathbf{E}}} \\ \downarrow & & \downarrow \\ \mathscr{Q}^{c/p}_{U_{\mathbf{E}}} & \xrightarrow{\overline{\boldsymbol{p}}^c_{\mathbf{E}}} & \mathscr{Q}^c_{U_{\mathbf{E}}} \end{array} \qquad \begin{array}{ccc} p^{-c/p}\mathscr{O}_{X_A} & \longrightarrow & p^{-c}\mathscr{O}_{X_A} \\ \downarrow & & \downarrow \\ \mathscr{Q}^{c/p}_{U_A} & \xrightarrow{\overline{\boldsymbol{p}}^c_A} & \mathscr{Q}^c_{U_A} \end{array}$$

whose vertical arrows are the projections, and whose top horizontal arrows are the restrictions of the p-Frobenius endomorphisms of the sheaves of monoids $j_{\mathbf{E}*}\mathscr{O}_{U_{\mathbf{E}}}$ and respectively $j_{A*}\mathscr{O}_{U_A}$ (with composition law given by multiplication).

(iii) *Moreover, we have a commutative diagram of sheaves of sets :*

$$\begin{array}{ccc} \mathscr{Q}^{c/p}_{U_{\mathbf{E}}} & \xrightarrow{\overline{\boldsymbol{p}}^c_{\mathbf{E}}} & \mathscr{Q}^c_{U_{\mathbf{E}}} \\ {\scriptstyle \overline{\psi}^{c/p}}\downarrow & & \downarrow{\scriptstyle \overline{\psi}^c} \\ \varphi_*\mathscr{Q}^{c/p}_{U_A} & \xrightarrow{\varphi_*(\overline{\boldsymbol{p}}^c_A)} & \varphi_*\mathscr{Q}^c_{U_A}. \end{array}$$

(iv) *If $c \leq 1$, the map $\overline{\psi}^c$ is an isomorphism of sheaves of abelian groups.*

Proof. (i): Let $f \in \mathbf{E}$ be any element, and set $V := \mathrm{Spec}\,\mathbf{E}_f$; we easily come down to showing

Claim 16.4.37. If $c \leq p/(p-1)$, we have

$$\psi^c_V(x+y) - \psi^c_V(x) \in \overline{\mathscr{O}}_{X_A}(\varphi^{-1}V) \qquad \text{for every } x \in \alpha_0^{-c}\mathscr{O}_{X_\mathbf{E}}(V) \text{ and } y \in \overline{\mathscr{O}}_{X_\mathbf{E}}(V).$$

Proof of the claim. Since $U_\mathbf{E}$ is quasi-compact, $j_{\mathbf{E}*}\mathscr{O}_{X_\mathbf{E}}$ is a quasi-coherent $\mathscr{O}_{X_\mathbf{E}}$-module ([59, Ch.I, Cor.9.2.2]), hence the same holds for $\alpha_0^{-c}\mathscr{O}_{X_\mathbf{E}}$, and consequently

$$\alpha_0^{-c}\mathscr{O}_{X_\mathbf{E}}(V) = \alpha_0^{-c}\mathscr{O}_{X_\mathbf{E}}(X_\mathbf{E})_f.$$

We may therefore find $n \in \mathbb{N}$ such that $f^n x$ (resp. $f^n y$) is the restriction to V of a global section $x' \in \alpha_0^{-c}\mathscr{O}_{X_\mathbf{E}}(X_\mathbf{E})$ (resp. of an element $y' \in \mathbf{E}$), and it suffices to show that

$$\psi^c_{X_\mathbf{E}}(x'+y') - \psi^c_{X_\mathbf{E}}(x') \in \mathrm{Im}\,(A \to j_{A*}\mathscr{O}_{U_A}(X_A) = \mathscr{O}_{X_A}(U_A)).$$

Hence, we may replace V by $X_\mathbf{E}$, and assume from start that

$$x \in \mathscr{O}_{X_\mathbf{E}}(U_\mathbf{E}) \qquad \text{and} \qquad \alpha_0^c \cdot x, y \in \mathrm{Im}\,(\mathbf{E} \to \mathscr{O}_{X_\mathbf{E}}(U_\mathbf{E}))$$

and we have to show that $\varphi^\flat_{U_\mathbf{E}}(x+y) - \varphi^\flat_{U_\mathbf{E}}(x) \in \mathrm{Im}\,(A \to \mathscr{O}_{X_A}(U_A))$. Notice that the assertion is independent of the topology $\mathscr{T}$ of A such that $(A, \mathscr{T})$ is perfectoid; by virtue of proposition 16.3.8(i) we may then assume that $\mathscr{T}$ is the p-adic topology on A, and therefore the topology of $\mathbf{E}$ agrees with the α_0-adic topology. In this case, assumption (a), proposition 16.4.34(ii) and lemma 16.2.7(iii) imply that $\varphi^\flat_{U_\mathbf{E}}(x+y) - \varphi^\flat_{U_\mathbf{E}}(x) - \varphi^\flat_{U_\mathbf{E}}(y)$ is a p-adically convergent series $\sum_{n>0} z_n$, where z_n is a $\mathbb{Z}_p$-linear combination of terms of the form $\varphi^\flat_{U_\mathbf{E}}(\alpha_0^n \cdot x^\sigma \cdot y^{1-\sigma})$, with $1-\sigma, \sigma \in p^{-n}\mathbb{N} \setminus p^{1-n}\mathbb{N}$, for every $n \in \mathbb{N}$. Hence, it suffices to check that $n \geq p \cdot \sigma/(p-1)$ for every such σ and every integer $n > 0$. We leave the easy verification to the reader. ◊

(ii): The uniqueness of $\overline{\boldsymbol{p}}^c_{U_A}$ is clear. For the existence, let V be as in the foregoing; we come down to showing that

$$(x+y)^p - x^p \in \overline{\mathscr{O}}_{X_A}(V) \qquad \text{for every } x \in p^{-c/p}\mathscr{O}_{X_A}(V) \text{ and every } y \in \overline{\mathscr{O}}_{X_A}(V).$$

However, $(x+y)^p - x^p = y^p + \sum_{i=1}^{p-1} \binom{p}{i} \cdot x^i y^{p-i}$, and $px^i \in \overline{\mathscr{O}}_{X_A}(V)$ for every $i = 1, \dots, p-1$, since $c \leq p/(p-1)$, whence the contention. Assertion (iii) shall be left to the reader.

(iv): Let V be as in the foregoing; in order to prove that $\overline{\psi}^c$ is a morphism of abelian sheaves when $c \leq 1$, we come down to showing that

$$\psi^1_V(x+y) - \psi^1_V(x) - \psi^1_V(y) \in \overline{\mathscr{O}}_{X_A}(\varphi^{-1}V) \qquad \text{for every } x, y \in \alpha_0^{-1}\mathscr{O}_{X_\mathbf{E}}(V)$$

and arguing as in the proof of claim 16.4.37, we reduce to the case where $V = X_\mathbf{E}$ and x, y are sections of $\mathscr{O}_{X_\mathbf{E}}(U_\mathbf{E})$ such that $\alpha_0 x, \alpha_0 y \in \mathrm{Im}\,(\mathbf{E} \to \mathscr{O}_{X_\mathbf{E}}(U_\mathbf{E}))$. Then, it suffices to check that

$$\varphi^\flat_{U_\mathbf{E}}(\alpha_0^n \cdot y^\sigma \cdot x^{1-\sigma}) \in \mathrm{Im}(A \to \mathscr{O}_{X_A}(U_A)) \qquad \text{whenever } n > 0 \text{ and } 1-\sigma, \sigma \in \mathbb{N}[1/p]$$

which is clear. Next notice that, by remark 16.4.31, the pair $(\varphi, \varphi^\flat)$ restricts to isomorphisms of schemes

$$(\varphi_2, \varphi_2^\flat) : X_{A_2} \xrightarrow{\sim} X_{\mathbf{E}_2} \qquad (\varphi_3, \varphi_3^\flat) : X_{A_3} \xrightarrow{\sim} X_{\mathbf{E}_3} \qquad \text{such that} \qquad \varphi_2(Z_A) = Z_\mathbf{E}.$$

Claim 16.4.38. Assertion (iv) holds for $U_\mathbf{E} = \mathrm{Spec}\,\mathbf{E}[\alpha_0^{-1}]$.

Proof of the claim. We have commutative diagrams

$$\begin{array}{ccc} U_{\mathbf{E}_1} := \mathrm{Spec}\,\mathbf{E}_1[\alpha_0^{-1}] & \longrightarrow & U_\mathbf{E} \\ {\scriptstyle j_{\mathbf{E}_1}}\downarrow & & \downarrow{\scriptstyle j_\mathbf{E}} \\ X_{\mathbf{E}_1} & \xrightarrow{i_{\mathbf{E}_1}} & X_\mathbf{E} \end{array} \qquad \begin{array}{ccc} U_{A_1} := \mathrm{Spec}\,A_1[p^{-1}] & \longrightarrow & U_A \\ {\scriptstyle j_{A_1}}\downarrow & & \downarrow{\scriptstyle j_A} \\ X_{A_1} & \xrightarrow{i_{A_1}} & X_A \end{array}$$

whose top horizontal arrows are isomorphisms of schemes, whence commutative diagrams of abelian sheaves

$$\begin{array}{ccc} \overline{\mathscr{O}}_{X_{\mathbf{E}}} & \xrightarrow{\sim} & i_{\mathbf{E}_1*}\mathscr{O}_{X_{\mathbf{E}_1}} \\ \downarrow & & \downarrow \\ j_{\mathbf{E}*}\mathscr{O}_{U_{\mathbf{E}}} & \xrightarrow{\sim} & i_{\mathbf{E}_1*}\circ j_{\mathbf{E}_1*}\mathscr{O}_{U_{\mathbf{E}_1}} \end{array} \qquad \begin{array}{ccc} \overline{\mathscr{O}}_{X_A} & \xrightarrow{\sim} & i_{A_1*}\mathscr{O}_{X_{A_1}} \\ \downarrow & & \downarrow \\ j_{A*}\mathscr{O}_{U_A} & \xrightarrow{\sim} & i_{A_1*}\circ j_{A_1*}\mathscr{O}_{U_{A_1}} \end{array}$$

whose left (resp. right) vertical arrows are the inclusion maps (resp. are induced by the inclusion maps $\mathscr{O}_{X_{\mathbf{E}_1}} \to j_{\mathbf{E}_1*}\mathscr{O}_{U_{\mathbf{E}_1}}$ and $\mathscr{O}_{X_{A_1}} \to j_{A_1*}\mathscr{O}_{U_{A_1}}$). Moreover, denote by $\varphi_1 : X_{A_1} \to X_{\mathbf{E}_1}$ the continuous map associated with A_1, as in (16.4.24), and by

$$\psi_1 : j_{\mathbf{E}_1*}\mathscr{O}_{U_{\mathbf{E}_1}} \to \varphi_{1*}j_{A_1*}\mathscr{O}_{U_{A_1}} \qquad (\text{resp. } \overline{\psi}{}^c_1 : \mathscr{Q}^c_{U_{\mathbf{E}_1}} \to \varphi_{1*}\mathscr{Q}^c_{U_{A_1}})$$

the corresponding morphism of sheaves of monoids as in (16.4.35) (resp. morphism of sheaves of sets, as in (i)); then $i_{\mathbf{E}_1}\circ\varphi_1 = \varphi\circ i_{A_1}$ (remark 16.4.31), and the foregoing isomorphisms identify ψ with $i_{\mathbf{E}_1*}\psi_1$. So finally, $\overline{\psi}{}^c$ is identified with $i_{\mathbf{E}_1*}\overline{\psi}{}^c_1$ for every $c \leq p/(p-1)$, and we may replace from start A by A_1, after which, we may assume that α_0 (resp. p) is regular in $\mathbf{E}$ (resp. in A). In this case, for every $c \leq 1$ we have a commutative diagram of sheaves :

$$\begin{array}{ccc} \mathscr{Q}^c_{U_{\mathbf{E}}} & \xrightarrow{\overline{\psi}{}^c_1} & \varphi_*\mathscr{Q}^c_{U_A} \\ \downarrow & & \downarrow \\ \mathscr{O}_{X_{\mathbf{E}}}/\alpha_0^c\mathscr{O}_{X_{\mathbf{E}}} & \longrightarrow & \varphi_*(\mathscr{O}_{X_A}/p^c\mathscr{O}_{X_A}) \end{array}$$

whose left (resp. right) vertical arrow is the isomorphism induced by scalar multiplication by α_0^c (resp. by q^c) and whose bottom horizontal arrow is the isomorphism induced by $\varphi_0^\flat$ (notation of (16.4.28)). The claim follows. ◊

Claim 16.4.39. We may assume that the natural maps $\mathscr{O}_{X_{\mathbf{E}}} \to \overline{\mathscr{O}}_{X_{\mathbf{E}}}$ and $\mathscr{O}_{X_A} \to \overline{\mathscr{O}}_{X_A}$ are isomorphisms.

Proof of the claim. Assumption (a) implies that $Z_{\mathbf{E}} \subset X_{\mathbf{E}_2}$ and $Z_A \subset X_{A_2}$, therefore

$$\mathscr{J}_{\mathbf{E}} := \mathrm{Ker}\,(\mathscr{O}_{X_{\mathbf{E}}} \to \overline{\mathscr{O}}_{X_{\mathbf{E}}}) = \underline{\Gamma}_{Z_{\mathbf{E}}}\mathscr{O}^{\mathrm{t}}_{X_{\mathbf{E}}} \qquad \mathscr{J}_A := \mathrm{Ker}\,(\mathscr{O}_{X_A} \to \overline{\mathscr{O}}_{X_A}) = \underline{\Gamma}_{Z_A}\mathscr{O}^{\mathrm{t}}_{X_A}$$

(notation of (16.4.28)), and lemma 16.4.29(ii,iii) implies that $\varphi^\flat$ restricts to an isomorphism of abelian sheaves

$$\mathscr{J}_{\mathbf{E}} \xrightarrow{\sim} \varphi_*\mathscr{J}_A. \tag{16.4.40}$$

Notice that $J := \Gamma(X_{\mathbf{E}}, \mathscr{J}_{\mathbf{E}})$ is an ideal of both $\mathbf{E}$ and $\mathbf{E}_2$, and moreover $\Phi_{\mathbf{E}}(J) = J$. Indeed, J consists of all $x \in \mathbf{E}$ whose images vanish in $\mathbf{E}_f$, for every $f \in \mathbf{E}$ such that $\mathrm{Spec}\,\mathbf{E}_f \subset U_{\mathbf{E}}$; since $\mathbf{E}_f$ is still a perfect $\mathbb{F}_p$-algebra, it follows easily that $x \in J$ if and only if $x^p \in J$, whence the assertion. Thus, both $\overline{\mathbf{E}} := \mathbf{E}/J$ and $\overline{\mathbf{E}}_2 := \mathbf{E}_2/J$ are still perfect $\mathbb{F}_p$-algebras, and are even perfectoid, with the quotient topologies induced by the projections $\mathbf{E} \to \overline{\mathbf{E}}$ and $\mathbf{E}_2 \to \overline{\mathbf{E}}_2$. Moreover, it is easily seen that the diagram $\mathscr{E}$ of (16.4.24) induces a natural identification

$$\overline{\mathbf{E}} \xrightarrow{\sim} \mathbf{E}_1 \times_{\mathbf{E}_3} \overline{\mathbf{E}}_2.$$

Let $h_{\mathbf{E}} : X_{\overline{\mathbf{E}}} := \mathrm{Spec}\,\overline{\mathbf{E}} \to X_{\mathbf{E}}$ be the closed immersion, $j_{\overline{\mathbf{E}}} : U_{\overline{\mathbf{E}}} := U_{\mathbf{E}} \cap X_{\overline{\mathbf{E}}} \to X_{\overline{\mathbf{E}}}$ the open immersion, and consider the corresponding quasi-coherent $\mathscr{O}_{X_{\overline{\mathbf{E}}}}$-module $\mathscr{Q}^c_{U_{\overline{\mathbf{E}}}}$ as in (16.4.35). Since $U_{\mathbf{E}}$ is quasi-compact, $\mathscr{J}_{\mathbf{E}}$ is a quasi-coherent $\mathscr{O}_{X_{\mathbf{E}}}$-module (see (10.4.16)), so

$\mathscr{J}_{\mathbf{E}} = \mathrm{Ker}\,(\mathscr{O}_{X_{\mathbf{E}}} \to h_{\mathbf{E}*}\mathscr{O}_{X_{\overline{\mathbf{E}}}})$; it follows that the induced map $\mathscr{O}_{X_{\overline{\mathbf{E}}}} \to j_{\overline{\mathbf{E}}*}\mathscr{O}_{U_{\overline{\mathbf{E}}}}$ is a monomorphism, and $h_{\mathbf{E}}$ restricts to an isomorphism of open subschemes $U_{\overline{\mathbf{E}}} \xrightarrow{\sim} U_{\mathbf{E}}$, so the resulting map $j_{\mathbf{E}*}\mathscr{O}_{U_{\mathbf{E}}} \to h_{\mathbf{E}*}j_{\overline{\mathbf{E}}*}\mathscr{O}_{U_{\overline{\mathbf{E}}}}$ is an isomorphism, and we get a natural identification

$$\omega_{\mathbf{E}} : \mathscr{Q}^c_{U_{\mathbf{E}}} \xrightarrow{\sim} h_{\mathbf{E}*}\mathscr{Q}^c_{U_{\overline{\mathbf{E}}}}.$$

Next, consider the cartesian diagram $A(\mathscr{E})$ of (16.4.24), and recall that the bottom horizontal arrow $A_2 \to A_3$ of $A(\mathscr{E})$ is naturally isomorphic to the bottom horizontal arrow $\mathbf{E}_2 \to \mathbf{E}_3$ of $\mathscr{E}$ (remark 16.4.24(iii)). There follows a commutative diagram

$$\begin{array}{ccccc} A & \longrightarrow & \overline{A} & \xrightarrow{\overline{\pi}_1} & A_1 \\ \downarrow & & \downarrow{\scriptstyle \overline{\pi}_2} & & \downarrow \\ \mathbf{E}_2 & \longrightarrow & \overline{\mathbf{E}}_2 & \longrightarrow & \mathbf{E}_3 \end{array}$$

whose two square subdiagrams are cartesian. Especially, J is naturally identified with the kernel of the projection $A \to \overline{A}$; moreover, we have an isomorphism of topological rings

$$\mathbf{E}(\overline{A}) \xrightarrow{\sim} \overline{\mathbf{E}}$$

that identifies $\mathbf{E}(\overline{\pi}_1)$ and $\mathbf{E}(\overline{\pi}_2)$ with the projections $\overline{\mathbf{E}} \to \mathbf{E}_1$ and $\overline{\mathbf{E}} \to \overline{\mathbf{E}}_2$, and furthermore, $\overline{A}$ is perfectoid (proposition 16.3.25). Let also

$$h_A : X_{\overline{A}} := \mathrm{Spec}\,\overline{A} \to X_A \qquad \text{and} \qquad j_{\overline{A}} : U_{\overline{A}} := U_A \cap X_{\overline{A}} \to X_{\overline{A}}$$

be respectively the closed and the open immersion, and consider the corresponding quasi-coherent $\mathscr{O}_{X_{\overline{A}}}$-module $\mathscr{Q}^c_{U_{\overline{A}}}$ defined as in (16.4.35); in light of (16.4.40), we see that $\mathscr{J}_A = \mathrm{Ker}\,(\mathscr{O}_{X_A} \to h_{A*}\mathscr{O}_{X_{\overline{A}}})$, and h_A restricts to an isomorphism of open subschemes $U_{\overline{A}} \xrightarrow{\sim} U_A$, so the resulting map $j_{A*}\mathscr{O}_{U_A} \to h_{A*}j_{\overline{A}*}\mathscr{O}_{U_{\overline{A}}}$ is an isomorphism, and we get a natural identification

$$\omega_A : \mathscr{Q}^c_{U_A} \xrightarrow{\sim} h_{A*}\mathscr{Q}^c_{U_{\overline{A}}}.$$

Lastly, let $\overline{\varphi} : X_{\overline{A}} \to X_{\overline{\mathbf{E}}}$ be the continuous map provided by (16.4.24), and $\overline{\psi}^c_{\overline{A}} : \mathscr{Q}^c_{U_{\overline{\mathbf{E}}}} \to \overline{\varphi}_*\mathscr{Q}^c_{U_{\overline{A}}}$ the morphism of abelian sheaves attached to $\overline{A}$ and $U_{\overline{\mathbf{E}}}$ as in (i); a direct inspection yields a commutative diagram

$$\begin{array}{ccc} \mathscr{Q}^c_{U_{\mathbf{E}}} & \xrightarrow{\overline{\psi}^c} & \varphi_*\mathscr{Q}^c_{U_A} \\ {\scriptstyle \omega_{\mathbf{E}}}\downarrow & & \downarrow{\scriptstyle \varphi_*\omega_A} \\ h_{\mathbf{E}*}\mathscr{Q}^c_{U_{\overline{\mathbf{E}}}} & \xrightarrow{h_{\mathbf{E}*}\overline{\psi}^c_{\overline{A}}} & h_{\mathbf{E}*}\overline{\varphi}_*\mathscr{Q}^c_{U_{\overline{A}}} = \varphi_* h_{A*}\mathscr{Q}^c_{U_{\overline{A}}} \end{array}$$

whence the claim. ◇

Henceforth we assume that $\mathscr{O}_{X_{\mathbf{E}}} = \overline{\mathscr{O}}_{X_{\mathbf{E}}}$ an $\mathscr{O}_{X_A} = \overline{\mathscr{O}}_{X_A}$. Set $U'_{\mathbf{E}} := \mathrm{Spec}\,\mathbf{E}[\alpha_0^{-1}]$, so that $U'_A := \varphi^{-1}U'_{\mathbf{E}}$ is the open subset $\mathrm{Spec}\,A[p^{-1}]$ of X_A, and let $j'_{\mathbf{E}} : U'_{\mathbf{E}} \to X_{\mathbf{E}}$ and $j'_A : U'_A \to X_A$ be the open immersions. Set also $Z'_{\mathbf{E}} := X_{\mathbf{E}} \setminus U'_{\mathbf{E}}$ and $Z'_A := X_A \setminus U'_A$. Notice that

$$i_{\mathbf{E}_1*}\mathscr{O}_{X_{\mathbf{E}_1}} = \mathrm{Im}(\mathscr{O}_{X_{\mathbf{E}}} \to j'_{\mathbf{E}*}\mathscr{O}_{U'_{\mathbf{E}}}) \qquad \text{and} \qquad i_{A_1*}\mathscr{O}_{X_{A_1}} = \mathrm{Im}(\mathscr{O}_{X_A} \to j'_{A*}\mathscr{O}_{U'_A})$$

and denote by $\overline{\psi}'^c : \mathscr{Q}^c_{U'_{\mathbf{E}}} \to \mathscr{Q}^c_{U'_A}$ the map of abelian sheaves associated as in (i) with A and $U'_{\mathbf{E}}$. Since $U'_{\mathbf{E}} \subset U_{\mathbf{E}}$, the natural maps $j_{\mathbf{E}*}\mathscr{O}_{U_{\mathbf{E}}} \to j'_{\mathbf{E}*}\mathscr{O}_{U'_{\mathbf{E}}}$ and $j_{A*}\mathscr{O}_{U_A} \to j'_{A*}\mathscr{O}_{U'_A}$ induce morphisms $\rho_{\mathbf{E}} : R^1\underline{\Gamma}_{Z_{\mathbf{E}}}\mathscr{O}_{X_{\mathbf{E}}} \to R^1\underline{\Gamma}_{Z'_{\mathbf{E}}}\mathscr{O}_{X_{\mathbf{E}}}$ and $\rho_A : R^1\underline{\Gamma}_{Z_A}\mathscr{O}_{X_A} \to R^1\underline{\Gamma}_{Z'_A}\mathscr{O}_{X_A}$, and notice that the image of $\rho_{\mathbf{E}}$ (resp. of ρ_A) lies in $\underline{\Gamma}_{Z_{\mathbf{E}}} R^1\underline{\Gamma}_{Z'_{\mathbf{E}}}\mathscr{O}_{X_{\mathbf{E}}}$ (resp. in $\underline{\Gamma}_{Z_A} R^1\underline{\Gamma}_{Z'_A}\mathscr{O}_{X_A}$); there follows a commutative diagram

$$\begin{array}{ccc} \mathscr{Q}^c_{U_{\mathbf{E}}} & \xrightarrow{\quad\overline{\psi}^c\quad} & \varphi_*\mathscr{Q}^c_{U_A} \\ {\scriptstyle \rho^c_{\mathbf{E}}}\downarrow & & \downarrow{\scriptstyle \varphi_*(\rho^c_A)} \\ \underline{\Gamma}_{Z_{\mathbf{E}}}\mathscr{Q}^c_{U'_{\mathbf{E}}} & \xrightarrow{\underline{\Gamma}_{Z_{\mathbf{E}}}\overline{\psi}'^c} & \underline{\Gamma}_{Z_{\mathbf{E}}}\varphi_*\mathscr{Q}^c_{U'_A} = \varphi_*\underline{\Gamma}_{Z_A}\mathscr{Q}^c_{U'_A} \end{array}$$

where $\rho^c_{\mathbf{E}}$ (resp. ρ^c_A) is the restriction of $\rho_{\mathbf{E}}$ (resp. of ρ_A). From the snake lemma, we obtain natural identifications

$$\mathrm{Ker}\,\rho^c_{\mathbf{E}} \xrightarrow{\sim} \mathrm{Ker}\,(\alpha_0^{-c}\mathscr{O}_{X_{\mathbf{E}}} \to j'_{\mathbf{E}*}\mathscr{O}_{U'_{\mathbf{E}}})/\mathrm{Ker}\,(\mathscr{O}_{X_{\mathbf{E}}} \to i_{\mathbf{E}_1*}\mathscr{O}_{X_{\mathbf{E}_1}}) = (\underline{\Gamma}_{X_{\mathbf{E}_2}}\alpha_0^{-c}\mathscr{O}_{X_{\mathbf{E}}})/\mathscr{O}^{\mathrm{t}}_{X_{\mathbf{E}}}$$

$$\mathrm{Ker}\,\rho^c_A \xrightarrow{\sim} \mathrm{Ker}\,(p^{-c}\mathscr{O}_{X_A} \to j'_{A*}\mathscr{O}_{U'_A})/\mathrm{Ker}\,(\mathscr{O}_{X_A} \to i_{A_1*}\mathscr{O}_{X_{A_1}}) = (\underline{\Gamma}_{X_{A_2}}p^{-c}\mathscr{O}_{X_A})/\mathscr{O}^{\mathrm{t}}_{X_A}$$

(notation of (16.4.28)).

Claim 16.4.41. The map $\overline{\psi}^c$ restricts to an isomorphism of abelian sheaves

$$\mathrm{Ker}\,\rho^c_{\mathbf{E}} \xrightarrow{\sim} \mathrm{Ker}\,\varphi_*(\rho^c_A).$$

Proof of the claim. Notice that

$$\underline{\Gamma}_{X_{\mathbf{E}_2}}\alpha_0^{-c}\mathscr{O}_{X_{\mathbf{E}}} = j_{\mathbf{E}*}j^*_{\mathbf{E}}\mathscr{O}^{\mathrm{t}}_{X_{\mathbf{E}}} \qquad \text{and} \qquad \underline{\Gamma}_{X_{A_2}}p^{-c}\mathscr{O}_{X_A} = j_{A*}j^*_A\mathscr{O}^{\mathrm{t}}_{X_A}.$$

In view of lemma 16.4.29(ii), we deduce that $\varphi^\flat$ induces an isomorphism of abelian sheaves

$$\mathrm{Ker}\,\rho^c_{\mathbf{E}} \xrightarrow{\sim} \mathscr{R}' := \varphi_*(\underline{\Gamma}_{X_{\mathbf{E}_2}}\alpha_0^{-c}\mathscr{O}_{X_{\mathbf{E}}})/\varphi_*(\mathscr{O}^{\mathrm{t}}_{X_A}).$$

Now, proposition 16.4.26(ii) implies that φ_* is an exact functor on the category of quasi-coherent $\mathscr{O}_{X_A}$-modules, and since both U_A and U'_A are quasi-compact (see (16.1.3)), the $\mathscr{O}_{X_A}$-modules $j_{A*}j^*_A\mathscr{O}^{\mathrm{t}}_{X_A}$ and $\mathscr{O}^{\mathrm{t}}_{X_A}$ are quasi-coherent ([59, Ch.I, Cor.9.2.2]) so λ restricts to an isomorphism $\mathscr{R}' \xrightarrow{\sim} \varphi_*\mathrm{Ker}\,\rho^c_A = \mathrm{Ker}\,\varphi_*(\rho^c_A)$, whence the claim. ◊

Now, claim 16.4.38 implies that $\underline{\Gamma}_{Z_{\mathbf{E}}}\overline{\psi}'^c$ is an isomorphism, and in light of claim 16.4.41 we see that $\overline{\psi}^c$ is a monomorphism, so we are reduced to showing that $\underline{\Gamma}_{Z_{\mathbf{E}}}\overline{\psi}'^c$ induces an epimorphism $\mathrm{Im}\,\rho^c_{\mathbf{E}} \to \mathrm{Im}\,\varphi_*(\rho^c_A)$. To this aim, let $x \in X_{\mathbf{E}}$ be any point, and $s_x \in \mathrm{Im}\,(\varphi_*\rho^c_A)_x = \varphi_*(\mathrm{Im}\,\rho^c_A)_x$ any element; by proposition 16.4.26(ii), the functor φ_* is exact on quasi-coherent $\mathscr{O}_{X_A}$-modules, so both maps

$$\varphi_*(p^{-c}\mathscr{O}_{X_A})_x \to \varphi_*(\mathscr{Q}^c_{U_A})_x \to \varphi_*(\mathrm{Im}\,\rho^c_A)_x$$

are surjective, hence we may find an open neighborhood $\Omega \subset X_{\mathbf{E}}$ of x such that s_x lifts to a section $s \in \mathscr{O}_{X_A}(U_A \cap \varphi^{-1}\Omega)$ and $q^c \cdot s$ extends to a section of $\mathscr{O}_{X_A}(\varphi^{-1}\Omega)$ (where q is as in (16.4.35)). Denote by $t_x \in (\underline{\Gamma}_{Z_{\mathbf{E}}}\mathscr{Q}^c_{U'_{\mathbf{E}}})_x$ the preimage of s_x. Notice that

$$\underline{\Gamma}_{Z_{\mathbf{E}}}R^1\underline{\Gamma}_{Z'_{\mathbf{E}}}\mathscr{O}_{X_{\mathbf{E}}} = (j_{\mathbf{E}*}j^*_{\mathbf{E}}(i_{\mathbf{E}_1*}\mathscr{O}_{X_{\mathbf{E}_1}}))/i_{\mathbf{E}_1*}\mathscr{O}_{X_{\mathbf{E}_1}}$$

so we may replace Ω by a smaller open neighborhood of x, and assume that t_x is the image of a section $\overline{t} \in i_{\mathbf{E}_1*}\mathscr{O}_{X_{\mathbf{E}_1}}(U_{\mathbf{E}} \cap \Omega)$. Denote by $\overline{s} \in i_{A_1*}\mathscr{O}_{X_{A_1}}(U_A \cap \varphi^{-1}\Omega)$ the image of s, and set

$$\overline{u} := i_{\mathbf{E}_1*}\varphi^\flat_1(\overline{t}) - \overline{s} \in j_{A*}j^*_A i_{A_1*}\mathscr{O}_{X_{A_1}}(\varphi^{-1}\Omega).$$

By construction, the image of $\overline{u}_x$ vanishes in $(\varphi_*R^1\underline{\Gamma}_{Z'_A}\mathscr{O}_{X_A})_x$, thus $\overline{u}_x \in \varphi_*(i_{A_1*}\mathscr{O}_{X_{A_1}})_x$. By proposition 16.4.26(ii), the natural map $\varphi_*(\mathscr{O}_{X_A})_x \to \varphi_*(i_{A_1*}\mathscr{O}_{X_{A_1}})_x$ is surjective, so we may replace Ω by a smaller open neighborhood of x, and assume that $\overline{u}$ is the image of a local section $u \in \mathscr{O}_{X_A}(\varphi^{-1}\Omega)$. However, s and $s' := s + u_{|U_A \cap \varphi^{-1}\Omega}$ have the same image in $\varphi_*(\mathscr{Q}^c_{U_A})_x$, so we may replace s by s', and assume that $i_{\mathbf{E}_1*}\varphi^\flat_1(\overline{t}) = \overline{s}$. In this situation, remark 16.4.31(ii) implies that $\overline{t}$ lifts to a section $t \in \mathscr{O}_{X_{\mathbf{E}}}(U_{\mathbf{E}} \cap \Omega)$ such that $\varphi^\flat(t) = s$, and we denote by $v \in R^1\underline{\Gamma}_{Z_\mathbf{E}}\mathscr{O}_{X_\mathbf{E}}(\Omega)$ the image of t. Clearly, the image of v in $(\underline{\Gamma}_{Z_\mathbf{E}}R^1\underline{\Gamma}_{Z'_\mathbf{E}}\mathscr{O}_{X_\mathbf{E}})_x$ agrees with t_x, hence it remains only to check that $v_x \in \mathscr{Q}^c_{U_\mathbf{E},x}$. But notice that $\alpha_0^c \cdot t_x = 0$, so $\alpha_0^c \cdot v_x \in (j_{\mathbf{E}*}j^*_\mathbf{E}\mathscr{O}^\mathrm{t}_{X_\mathbf{E}})_x$. Pick any $\varepsilon \in \mathbb{N}[1/p]$ such that $0 < \varepsilon \leq \min(p/(p-1)-c, 1)$; by virtue of corollary 16.3.63(i), it follows that $\alpha_0^{c+\varepsilon} \cdot v_x = 0$, *i.e.* $\alpha_0^c \cdot v_x \in \mathscr{Q}^\varepsilon_{U_\mathbf{E},x}$. Lastly, from (i) we get

$$\overline{\psi}^\varepsilon(\alpha_0^c v_x) = q^c \cdot \overline{\psi}^{c+\varepsilon}(v_x)$$

and $\overline{\psi}^{c+\varepsilon}(v_x)$ is the image of s in $\varphi_*(\mathscr{Q}^{c+\varepsilon}_{U_A})_x$. By construction, the image of $q^c \cdot s$ vanishes in $\varphi_*(\mathscr{Q}^{c+\varepsilon}_{U_A})_x$, and therefore $\alpha_0^c \cdot v_x = 0$, since we have already remarked that $\overline{\psi}^\varepsilon$ is a monomorphism. The assertion follows. □

16.5. **Perfectoid quasi-affinoid rings.** We wish now to merge the theory of perfectoid rings with that of affinoid rings of section 15.3. First, let us make the following :

Definition 16.5.1. (i) We say that a quasi-affinoid ring (A, A^+, U) is *perfectoid* if A is a perfectoid ring.

(ii) We say that a quasi-affinoid scheme $\underline{X}$ is *perfectoid*, if there exists a perfectoid quasi-affinoid ring $\underline{A}$ and an isomorphism $\underline{X} \xrightarrow{\sim} \mathsf{Spec}\,\underline{A}$.

(iii) We denote by

$$\mathsf{q.Afd.Ring}_\mathrm{perf} \qquad \text{and} \qquad \mathsf{q.Afd.Sch}_\mathrm{perf}$$

the subcategories of q.Afd.Ring and respectively q.Afd.Sch whose objects are the perfectoid quasi-affinoid rings (resp. schemes), and whose morphisms are the f-adic morphisms of quasi-affinoid rings (resp. schemes).

16.5.2. Clearly we have a well defined functor

$$\mathsf{q.Afd.Ring}_\mathrm{perf} \to \mathsf{q.Afd.Sch}^o_\mathrm{perf} \qquad \underline{A} \mapsto \mathsf{Spec}\,\underline{A}$$

and we will construct a right adjoint. To this aim, we need a few preliminaries : we consider any perfectoid quasi-affinoid ring $\underline{A} := (A, A^+, U_A)$ and set $X_A := \mathrm{Spec}\,A$, $Z_A := X_A \setminus U_A$. Let also $\mathbf{E} := \mathbf{E}(A)$, define the continuous map $\varphi : X_A \to X_\mathbf{E} := \mathrm{Spec}\,\mathbf{E}$ as in (16.4.24), and set $Z_\mathbf{E} := \varphi(Z_A)$ and $U_\mathbf{E} := X_\mathbf{E} \setminus Z_\mathbf{E}$. Then it is easily seen that $U_\mathbf{E}$ contains the analytic locus of $X_\mathbf{E}$, so *any choice of ring of integral elements* $\mathbf{E}^+ \subset \mathbf{E}$ will give another perfectoid quasi-affinoid ring $\underline{\mathbf{E}} := (\mathbf{E}, \mathbf{E}^+, U_\mathbf{E})$, as well as the attached perfectoid quasi-affinoid schemes

$$(U_A, \mathscr{T}_{U_A}, A^+_U) := \mathsf{Spec}\,\underline{A} \qquad (U_\mathbf{E}, \mathscr{T}_{U_\mathbf{E}}, \mathbf{E}^+_U) := \mathsf{Spec}\,\underline{\mathbf{E}}.$$

More precisely, let $\underline{\alpha} := (\alpha_n \mid n \in \mathbb{N})$ be any distinguished element in $\mathrm{Ker}\,u_A$; since φ maps $\mathrm{Spec}\,A/pA$ bijectively onto $\mathrm{Spec}\,\mathbf{E}/\alpha_0\mathbf{E}$, and maps $\mathrm{Spec}\,A[p^{-1}]$ onto $\mathrm{Spec}\,\mathbf{E}[\alpha_0^{-1}]$, it is easily seen that $U_\mathbf{E}$ is the unique open subset of $X_\mathbf{E}$ containing the analytic locus, and such that

$$\varphi^{-1}(U_\mathbf{E}) = U_A. \tag{16.5.3}$$

Recall that, according to lemma 16.4.32, the topologies $\mathscr{T}_{U_A}$ on $A_U := \mathscr{O}_{X_A}(U_A)$ and $\mathscr{T}_{U_\mathbf{E}}$ on $\mathbf{E}_U := \mathscr{O}_{X_\mathbf{E}}(U_\mathbf{E})$ are f-adic, complete and separated, and the restriction maps $A \to A_U$ and $\mathbf{E} \to \mathbf{E}_U$ are open. Especially the image $\overline{A}$ of A (resp. $\overline{\mathbf{E}}$ of $\mathbf{E}$) is open in A_U (resp. in $\mathbf{E}_U$), and we endow this subring with the quotient topology induced by the projection $\pi_A : A \to \overline{A}$ (resp. $\pi_\mathbf{E} : \mathbf{E} \to \overline{\mathbf{E}}$). Moreover, we have a continuous map of (multiplicative) monoids

$$\varphi^\flat_U : \mathbf{E}_U \to A_U$$

(proposition 16.4.34(i)). Let also

$$j^\circ_\mathbf{E} : \overline{\mathbf{E}} \to \mathbf{E}^\circ_U \qquad \text{and} \qquad j^\circ_A : \overline{A} \to A^\circ_U$$

be the inclusion maps, and we endow A°_U (resp. $\mathbf{E}^\circ_U$) with the topology induced by the inclusion into A_U (resp. into $\mathbf{E}_U$), so that j°_A and $j^\circ_\mathbf{E}$ are open maps.

Proposition 16.5.4. *In the situation of* (16.5.2)*, we have :*

(i) *For every $x \in A_U$, the following conditions are equivalent :*
   (a) *x is power bounded in A_U.*
   (b) *The subset $\{x^{p^n} \mid n \in \mathbb{N}\}$ is bounded in A_U.*
   (c) *The image of x in $R^1\Gamma_{Z_A}\mathscr{O}_{X_A}$ is annihilated by $A^{\circ\circ}$.*
   (d) *$x \cdot A_U^{\circ\circ} \subset A_U^{\circ\circ}$.*
   (e) *$v(x) \leq 1$ for every analytic rank one valuation $v \in \mathrm{Spa}\,\underline{A}$.*

(ii) *$\overline{A}$ and $\overline{\mathbf{E}}$ are perfectoid rings, and there exists a unique isomorphism*

$$\overline{\omega} : \overline{\mathbf{E}} \xrightarrow{\sim} \mathbf{E}(\overline{A}) \qquad \textit{such that} \qquad \overline{\omega} \circ \pi_{\mathbf{E}} = \mathbf{E}(\pi_A).$$

(iii) *The maps π_A and $\pi_{\mathbf{E}}$ restrict to bijections $\mathbf{E}^{\circ\circ} \xrightarrow{\sim} \mathbf{E}_U^{\circ\circ}$ and $A^{\circ\circ} \xrightarrow{\sim} A_U^{\circ\circ}$.*

(iv) *The map $\varphi_U^\flat$ restricts to a continuous map of topological monoids*

$$\varphi_U^{\flat\circ} : \mathbf{E}_U^\circ \to A_U^\circ$$

which induces an isomorphism of perfect $\mathbb{F}_p$-algebras

$$\overline{\varphi}_U^{\flat\circ} : \mathbf{E}_U^\circ/\mathbf{E}_U^{\circ\circ} \xrightarrow{\sim} A_U^\circ/A_U^{\circ\circ}.$$

Proof. We consider first the following special case :

Claim 16.5.5. In the situation of (16.5.2), let $\beta \in \mathbf{E}$ be any regular element such that $\alpha_0 \in \beta\mathbf{E}$, set $b := \overline{u}_A(\beta) \in A$, and suppose moreover that bA is an ideal of adic definition for A, and U_A is the analytic locus of X_A. Then conditions (i.a)–(i.d) are equivalent in this case, and

$$A^{\circ\circ} = A_U^{\circ\circ} = \bigcup_{\lambda \in \mathbb{N}[1/p]\setminus\{0\}} b^\lambda A \qquad \text{with } b^\lambda := \overline{u}_A(\beta^\lambda) \text{ for every } \lambda \in \mathbb{N}[1/p].$$

Proof of the claim. First, notice that under the assumptions of the claim, $p \in bA$ (lemma 16.2.7(iii)) and b^λ is a regular element of A for every $\lambda \in \mathbb{N}[1/p]$ (proposition 16.4.17). Especially, the localization map $A \to A[b^{-1}]$ is injective, and $U_A = \mathrm{Spec}\,A[b^{-1}]$, so that $A_U = A[b^{-1}]$, $R^1\Gamma_{Z_A}\mathscr{O}_{X_A} = A[b^{-1}]/A$, and condition (i.c) implies :

(i.f) $b^\lambda x \in A$ for every $\lambda \in \mathbb{N}[1/p] \setminus \{0\}$.

We show first the equivalence of (i.a), (i.b) and (i.f). Indeed, obviously (i.f)⇒(i.a)⇒(i.b), so we may assume that (i.b) holds, and we set

$$\rho := \inf\{\gamma \in \mathbb{N}[1/p] \mid b^\gamma x^{p^n} \in A \text{ for every } n \in \mathbb{N}\}.$$

Notice that (i.b) implies that $\rho < +\infty$. We claim that $\rho = 0$. Indeed suppose, by way of contradiction, that $\rho > 0$ and pick any $\gamma \in \mathbb{N}[1/p]$ such that $\rho > \gamma > \max(\rho/p, \rho - 1/p)$. Then $b^{1/p}b^\gamma x^{p^n} \in A$, and hence $b^\gamma x^{p^n} \in b^{-1/p}A$ for every $n \in \mathbb{N}$; on the other hand, we also see that $b^{p\gamma}x^{p^n} \in A$ for every $n \in \mathbb{N}$. Now, by corollary 16.3.3, the Frobenius endomorphism of A/pA induces an isomorphism $A/b^{1/p}A \to A/bA$; since b is regular, there follows a bijective map

$$b^{-1/p}A/A \xrightarrow{\sim} b^{-1}A/A \qquad (x \mod A) \mapsto (x^p \mod A).$$

We deduce that $b^\gamma x^{p^n} \in A$ for every $n \in \mathbb{N}$, *i.e.* $\gamma \leq \rho$, which is absurd. This shows that (i.f) holds, whence the assertion. Next, say that $x \in A[b^{-1}]^{\circ\circ}$; since bA is open in $A[b^{-1}]$, we must have $x^{p^n} \in bA$ for some $n \in \mathbb{N}$, so if set $y := b^{-1/p^n}x$, we get $y^{p^n} \in A$, and therefore the subset $\{y^{p^n} \mid n \in \mathbb{N}\}$ is bounded in A_U. Pick any integer $r > n$; by the foregoing, it follows that $b^{1/p^r}y \in A$, and finally $x \in b^{1/p^n - 1/p^r}A$. This also implies the equivalence of (i.c),(i.d) and (i.f), so the proof of the claim is complete. ◊

For the general case, pick a finite system $\beta_\bullet := (\beta_0, \dots, \beta_k)$ of elements of $\mathbf{E}$ with $\beta_0 = \alpha_0$, and such that $\beta_\bullet$ generates an ideal $\mathscr{I}$ of adic definition for $\mathbf{E}$. Then the ideal $I \subset A$ generated by $(\overline{u}_A(\beta_0), \dots, \overline{u}_A(\beta_k))$ is of adic definition for A (see remark 16.3.7(iii)). We notice :

Claim 16.5.6. The ideal $A^{\circ\circ}$ is generated by $(\overline{u}_A(\beta_i^\lambda) \mid i = 0, \dots, k, \ \lambda \in \mathbb{N}[1/p] \setminus \{0\})$.

Proof of the claim. Clearly $\overline{u}_A(\beta_i^\lambda) \in A^{\circ\circ}$ for every $i = 0, \dots, k$ and every strictly positive $\lambda \in \mathbb{N}[1/p]$. Conversely, let $c \in A^{\circ\circ}$ be any element; then $c^{p^n} \in I$ for every sufficiently large $n \in \mathbb{N}$. By remark 16.3.7(iii) we may find $\gamma \in \mathbf{E}$ such that $\overline{u}_A(\gamma) - c \in pA$, and the image of γ^{p^n} in $\mathbf{E}/\alpha_0\mathbf{E}$ lies in $\mathscr{I}/\alpha_0\mathbf{E}$, so $\gamma^{p^n} \in \mathscr{I}$, hence $\gamma \in \mathscr{I}^{\langle 1/p^n\rangle}\mathbf{E}$, and therefore $c \in \mathscr{I}^{\langle 1/p^n\rangle}A + pA = \mathscr{I}^{\langle 1/p^n\rangle}A$ (notation of remark 9.3.70(i)), whence the claim. ◊

(i.c)⇒(i.a): By assumption, $\overline{u}_A(\beta_i^\lambda) \cdot x$ lies in $\overline{A}$, for every $i = 0, \dots, k$ and every strictly positive $\lambda \in \mathbb{N}[1/p]$; let $n > 0$ be any integer, and pick $t \in \mathbb{N}$ such that $p^t \geq n$; then $\overline{u}_A(\beta_i^{n/p^t}) \cdot x^n = (\overline{u}_A(\beta^{1/p^t}) \cdot x)^n$ lies as well in $\overline{A}$, and we conclude that $\overline{u}_A(\beta_i) \cdot x^n \in \overline{A}$ for every $n \in \mathbb{N}$, whence the assertion.

Obviously, (i.a)⇒(i.b). To show that (i.b)⇒(i.c), let $\mathscr{T}_{A,p}$ be the p-adic topology on A, and $\mathscr{T}_{U,p}$ the unique f-adic topology on A_U such that the restriction map $(A, \mathscr{T}_{A,p}) \to (A_U, \mathscr{T}_{U,p})$ is open (lemma 16.4.32(i)). Suppose first that the topology of A agrees with $\mathscr{T}_{A,p}$, and let

$$A_1 := A/\mathrm{Ann}_A(p) \qquad Z' := \mathrm{Spec}\, A/pA \qquad U' := X_A \setminus Z'.$$

Denote as well by $x_{|U'} \in \mathscr{O}_{X_A}(U') = A[1/p]$ the image of x. We recall that p is regular in A_1 (remark 16.4.25(ii)), and we endow A_1 with its p-adic topology $\mathscr{T}_{A_1,p}$, and $A[1/p]$ with the unique f-adic topology $\mathscr{T}_{U',p}$ such that the inclusion map $(A_1, \mathscr{T}_{A_1,p}) \to (A[1/p], \mathscr{T}_{U',p})$ is open, so that the subset $\{x_{|U'}^{p^n} \mid n \in \mathbb{N}\}$ is bounded in $A[1/p]$. Set $q := \overline{u}_A(\alpha_0)$; by claim 16.5.5 and lemma 16.2.7(iii), it then follows that $q^\lambda \cdot x_{|U'}$ lies in A_1 for every $\lambda \in \mathbb{N}[1/p] \setminus \{0\}$. Denote also by $\overline{x} \in R^1\Gamma_Z\mathscr{O}_{X_A}$ the image of x; we deduce that the image of $q^\lambda \cdot \overline{x}$ vanishes in $R^1\Gamma_{Z'}\mathscr{O}_{X_A}$, for every such λ. To conclude in this case, it suffices to remark :

Claim 16.5.7. The kernel of the natural map $R^1\Gamma_{Z_A}\mathscr{O}_{X_A} \to R^1\Gamma_{Z'}\mathscr{O}_{X_A}$ is annihilated by q^λ for every $\lambda \in \mathbb{N}[1/p] \setminus \{0\}$.

Proof of the claim. By the snake lemma, this kernel is a quotient of $\Gamma_{Z'}j_{A*}\mathscr{O}_{U_A}$, where $j_A : U_A \to X_A$ is the open immersion. However, $\Gamma_{Z'}j_*\mathscr{O}_{U_A} = H^0(U_A, \underline{\Gamma}_{Z'}\mathscr{O}_{X_A})$. Since $\underline{\Gamma}_{Z'}\mathscr{O}_{X_A}$ is a quasi-coherent $\mathscr{O}_{X_A}$-module (lemma 10.4.17(i)), we are reduced to checking that $\mathrm{Ann}_A(p)$ is annihilated by q^λ for every $\lambda \in \mathbb{N}[1/p] \setminus \{0\}$. The latter holds by corollary 16.3.63(i). ◊

Next, let A be any perfectoid ring; let $j_{\mathbf{E}} : U_{\mathbf{E}} \to X_{\mathbf{E}}$ be the open immersion, and for every $c \in \mathbb{N}[1/p]$ consider the $\mathscr{O}_{X_{\mathbf{E}}}$-module $\mathscr{Q}^c_{U_{\mathbf{E}}}$ and the $\mathscr{O}_{X_A}$-module $\mathscr{Q}^c_{U_A}$ as in (16.4.35). We set

$$\mathscr{Q}^0_{U_{\mathbf{E}}} := \bigcap_{c>0} \mathscr{Q}^c_{U_{\mathbf{E}}} \qquad \text{and} \qquad \mathscr{Q}^0_{U_A} := \bigcap_{c>0} \mathscr{Q}^c_{U_A}.$$

Condition (i.b) implies that the set $\{x^{p^n} \mid n \in \mathbb{N}\}$ is bounded also for the topology $\mathscr{T}_{U,p}$ of A_U, and since $(A, \mathscr{T}_{A,p})$ is perfectoid (proposition 16.3.8(i)), the foregoing case shows that $\overline{x}$ lies in $\mathscr{Q}^0_{U_A}(X_A)$. By proposition 16.4.36(iv), the map $\varphi^\flat$ of lemma 16.4.29(i) induces an isomorphism

$$\overline{\psi}^0 : \mathscr{Q}^0_{U_{\mathbf{E}}} \xrightarrow{\sim} \varphi_*\mathscr{Q}^0_{U_A}$$

of abelian sheaves, fitting into a commutative diagram

$$\text{(16.5.8)} \qquad \begin{array}{ccc} \mathscr{Q}^0_{U_{\mathbf{E}}} & \xrightarrow{\overline{\psi}^0} & \varphi_*\mathscr{Q}^0_{U_A} \\ {\scriptstyle \overline{\boldsymbol{p}}_{\mathbf{E}}}\downarrow & & \downarrow{\scriptstyle \varphi_*(\overline{\boldsymbol{p}}_A)} \\ \mathscr{Q}^0_{U_{\mathbf{E}}} & \xrightarrow{\overline{\psi}^0} & \varphi_*\mathscr{Q}^0_{U_A} \end{array}$$

where $\overline{\boldsymbol{p}}_{\mathbf{E}}$ (resp. $\overline{\boldsymbol{p}}_A$) is induced by the p-Frobenius endomorphism of the sheaf of monoids $j_{\mathbf{E}*}\mathscr{O}_{U_{\mathbf{E}}}$ (resp. $j_{A*}\mathscr{O}_{U_A}$). With this notation, condition (i.b) says that for some $n \in \mathbb{N}$ we have

$$\text{(16.5.9)} \qquad \overline{u}_A(\beta_i^n) \cdot \overline{\boldsymbol{p}}_A^r(\overline{x}) = 0 \qquad \text{for every } i = 0, \dots, k \text{ and every } r \in \mathbb{N}.$$

Let $\overline{\gamma} \in \mathscr{Q}^0_{U_{\mathbf{E}}}(X_{\mathbf{E}})$ be the preimage of $\overline{x}$; then $\beta_i^\lambda \cdot \overline{\gamma}$ is the preimage of the class of $\overline{u}_A(\beta_i^\lambda) \cdot x$ in $\mathscr{Q}^0_{U_A}(X_A)$, for every $i = 0, \dots, k$ and every $\lambda \in \mathbb{N}[1/p]$. By virtue of claim 16.5.6, we are then reduced to checking that $\beta_i^\lambda \cdot \overline{\gamma} = 0$ for every such i and λ, and (16.5.9) yields already

$$\beta_i^n \cdot \overline{\boldsymbol{p}}^r_{\mathbf{E}}(\overline{\gamma}) = 0 \qquad \text{for every } i = 0, \dots, k \text{ and every } r \in \mathbb{N}.$$

Now, we may write $\lambda = t \cdot p^{-r}$ for some integers $t \geq n$ and $r \in \mathbb{N}$; we deduce

$$\overline{\boldsymbol{p}}^r_{\mathbf{E}}(\beta_i^\lambda \cdot \overline{\gamma}) = \beta_i^t \cdot \overline{\boldsymbol{p}}^r_{\mathbf{E}}(\overline{\gamma}_{\mathbf{E}}) = 0.$$

Lastly, since $\mathbf{E}$ is perfect, it is easily seen that $\overline{\boldsymbol{p}}_{\mathbf{E}}$ is an isomorphism, whence the assertion.

(ii): This was already remarked in the proof of claim 16.4.39. Let us add the following :

Claim 16.5.10. The map $\pi_{\mathbf{E}}$ and π_A restrict to bijections $\mathbf{E}^{\circ\circ} \xrightarrow{\sim} \overline{\mathbf{E}}^{\circ\circ}$ and $A^{\circ\circ} \xrightarrow{\sim} \overline{A}^{\circ\circ}$.

Proof of the claim. First, notice that the image of $\beta_\bullet$ in $\overline{\mathbf{E}}$ generates an ideal of adic definition for $\overline{\mathbf{E}}$; in view of (ii) and claim 16.5.6, we deduce immediately that $\overline{A}^{\circ\circ} = \pi_A(A^{\circ\circ})$ and $\overline{\mathbf{E}}^{\circ\circ} = \pi_{\mathbf{E}}(\mathbf{E}^{\circ\circ})$. Next, if I is any ideal of adic definition for A, clearly we have $I \cdot \mathrm{Ker}\,\pi_A = 0$, whence $I \cap \mathrm{Ker}\,\pi_A = 0$, since A is reduced (corollary 16.3.63); but $A^{\circ\circ}$ is the union of all ideals of adic definition for A, so we conclude that $A^{\circ\circ} \cap \mathrm{Ker}\,\pi_A = 0$. Likewise, we see that $\mathbf{E}^{\circ\circ} \cap \mathrm{Ker}\,\pi_{\mathbf{E}} = 0$, whence the claim. ◊

(iii): Let $x \in A_U^{\circ\circ}$ be any element; denote by $\overline{x} \in R^1\Gamma_{Z_A}\mathscr{O}_{X_A}$ the image of x; from (i) we see that $\overline{x} \in Q_A := \mathscr{Q}^0_{U_A}(X_A)$, and since $\overline{A}$ is open in A_U, there exists $n \in \mathbb{N}$ such that $x^{p^n} \in \overline{A}$, therefore $\overline{\boldsymbol{p}}^n_A(\overline{x}) = 0$. On the other hand, we have already remarked that the maps $\overline{\psi}^0$ and $\overline{\boldsymbol{p}}_{\mathbf{E}}$ appearing in (16.5.8) are isomorphisms, so the same holds for $\overline{\boldsymbol{p}}_A$, and consequently $\overline{x} = 0$, *i.e.* $x \in \overline{A}$; then the assertion follows from claim 16.5.10. Next, from claim 16.5.6 it is easily seen that the isomorphism $\mathbf{E}/\alpha_0\mathbf{E} \xrightarrow{\sim} A/pA$ of remark 16.3.7(ii) restricts to a natural identification

$$\mathbf{E}^{\circ\circ}/\alpha_0\mathbf{E} \xrightarrow{\sim} A^{\circ\circ}/pA \tag{16.5.11}$$

therefore $Z_{\mathbf{E}} \subset X_{\mathbf{E}} \setminus \mathrm{Spec}\,\mathbf{E}/\mathbf{E}^{\circ\circ}$, so the foregoing applies as well to the perfectoid ring $\mathbf{E}$ and its closed subset $Z_{\mathbf{E}}$, and the proof of (iii) is thus complete.

Now, taking into account (iii), we easily see that (i.d)⇒(i.c). Conversely, if (i.c) holds, we get $x \cdot A^{\circ\circ} \cdot A^{\circ\circ} \subset A_U^{\circ\circ}$, but claim 16.5.6 implies that $A^{\circ\circ} \cdot A^{\circ\circ} = A^{\circ\circ}$, whence (i.d). Lastly, from remark 15.4.15(iv) it is clear that (i.a)⇒(i.e). Conversely, suppose (i.e) holds; then we have as well $v(xa) < 1$ for every $a \in A_U^{\circ\circ}$ and every rank one analytic valuation $v \in \mathrm{Spa}\,\underline{A}$. Taking into account lemma 15.3.14(v), it follows that $v(xa) < 1$ for every $a \in A_U^{\circ\circ}$ and every $v \in (\mathrm{Spa}\,\underline{A})_{\mathrm{a}}$. On the other hand, we have $v(xa) = 0$ for every $v \in (\mathrm{Spa}\,\underline{A})_{\mathrm{na}}$, and we conclude that $xa \in A_U^{\circ\circ}$, by corollary 15.4.27(ii). This shows that (i.e)⇒(i.d), and completes the proof of (i).

(iv): Set also $Q_{\mathbf{E}} := \mathscr{Q}^0_{U_{\mathbf{E}}}(X_{\mathbf{E}})$; in light of (16.5.11), it is easily seen that the isomorphism $\overline{\psi}^0$ restricts to a natural identification

$$\mathrm{Ann}_{Q_{\mathbf{E}}}(\mathbf{E}^{\circ\circ}) \xrightarrow{\sim} \mathrm{Ann}_{Q_A}(A^{\circ\circ})$$

and taking into account (i) we deduce that $\varphi_U^\flat(\mathbf{E}_U^\circ) \subset A_U^\circ$, so the sought map $\varphi_U^{\flat\circ}$ is well defined, and it is continuous, by proposition 16.4.34(i); combining with (iii), we also see that $\varphi_U^{\flat\circ}$ induces bijections

$$\mathbf{E}_U^\circ/\overline{\mathbf{E}} \xrightarrow{\sim} A_U^\circ/\overline{A} \qquad \overline{\mathbf{E}}/\mathbf{E}_U^{\circ\circ} \xrightarrow{\sim} \overline{A}/A_U^{\circ\circ}.$$

On the other hand, we remark :

Claim 16.5.12. (i) For every $x, y \in \mathbf{E}_U^\circ$ we have $\varphi_U^\flat(x+y) - \varphi_U^\flat(x) - \varphi_U^\flat(y) \in A_U^{\circ\circ}$.

(ii) $\varphi_U^{\flat\circ}(\mathbf{E}_U^{\circ\circ}) \subset A_U^{\circ\circ}$.

Proof of the claim. (i): By proposition 16.4.34(ii), the difference in the claim can be written as a p-adically convergent series $\sum_{n\in\mathbb{N}\setminus\{0\}} p^n \cdot c_n$, where each summand c_n is a finite $\mathbb{Z}_p$-linear combination of terms of the form $\varphi^\flat_U(x^\lambda y^{1-\lambda})$, where $\lambda, 1-\lambda \in \mathbb{N}[1/p]$. It is easily seen that $x^\lambda y^{1-\lambda} \in \mathbf{E}^\circ_U$ for every such λ, and obviously $p \in A^{\circ\circ}_U$, whence the contention.

(ii) is obvious, since $\varphi^{\flat\circ}_U$ is a continuous morphism of multiplicative monoids. ◊

From claim 16.5.12 it follows that $\varphi^{\flat\circ}_U$ descends to a map $\overline{\varphi}^{\flat\circ}_U : \mathbf{E}^\circ_U/\mathbf{E}^{\circ\circ}_U \to A^\circ_U/A^{\circ\circ}_U$ of abelian groups, which must then be bijective, by the foregoing. By remark 8.3.10(iv), both the source and target of $\overline{\varphi}^{\flat\circ}_U$ are endowed with natural quotient ring structures, and it is easily seen that $\mathbf{E}^\circ_U/\mathbf{E}^{\circ\circ}_U$ is a perfect $\mathbb{F}_p$-algebra (details left to the reader); to conclude the proof it then suffices to notice that, by construction, $\overline{\varphi}^{\flat\circ}_U$ is also a morphism of multiplicative monoids. □

Theorem 16.5.13. *In the situation of* (16.5.2)*, the following holds :*

(i) $\mathbf{E}^\circ_U$ *and* A°_U *are both perfectoid, and there exists an isomorphism of topological rings*
$$\omega^\circ : \mathbf{E}^\circ_U \xrightarrow{\sim} \mathbf{E}(A^\circ_U) \qquad \textit{such that} \qquad \omega^\circ \circ j^\circ_{\mathbf{E}} = \mathbf{E}(j^\circ_A).$$

(ii) *The rule* $D \mapsto \mathbf{E}(D)$ *establishes a bijection from the set* $\mathscr{S}_A$ *of open integrally closed subrings of* A°_U *to the set* $\mathscr{S}_{\mathbf{E}}$ *of open integrally closed subrings of* $\mathbf{E}^\circ_U$*.*

(iii) *Moreover, every open integrally closed subring of* A°_U *is perfectoid (for the topology induced by* A°_U*).*

Proof. (i): First, notice that $\mathbf{E}^{\circ\circ}_U$ (resp. $\mathbf{A}^{\circ\circ}_U$) is a bounded open ideal of $\mathbf{E}^\circ_U$ (resp. of A°_U), by virtue of proposition 16.5.4(iii); it follows that $\mathbf{E}^\circ_U$ and A°_U are bounded, so their topologies are both adic and f-adic (corollary 8.3.19(iii)). Moreover, since $\mathbf{E}_U$ and A_U are complete and separated (lemma 16.4.32(ii)), the same holds for $\mathbf{E}^\circ_U$ and A°_U. Furthermore, since $\mathbf{E}_U$ is a perfect $\mathbb{F}_p$-algebra, proposition 16.5.4(i) easily implies that the same holds for $\mathbf{E}^\circ_U$, so the latter is perfectoid (example 16.3.2(i)). Next, let $I \subset A$ be any ideal of definition; proposition 16.5.4(iii) implies that $I_U := IA^\circ_U \subset A^{\circ\circ}$, so I_U is a finitely generated ideal of adic definition for A°_U (corollary 8.3.19(iii)), and clearly $p \in I^2_U$. Furthermore, $I_U \subset \overline{A}$ (by claim 16.5.10), and the Frobenius endomorphism Φ_{A_U/I_U} of A_U/I_U induces surjective endomorphisms of both $\overline{A}/I_U$ and $A_U/\overline{A}$ (proposition 16.5.4(iv)); thus, Φ_{A_U/I_U} is surjective, which shows that A_U is a P-ring. Next, endow A°_U/pA°_U with the quotient topology induced by the projection $\pi_U : A^\circ_U \to A^\circ_U/pA^\circ_U$, and let $\varphi^{\flat\circ}_U : \mathbf{E}^\circ_U \to A^\circ_U$ and $\overline{\omega} : \overline{\mathbf{E}} \xrightarrow{\sim} \mathbf{E}(\overline{A})$ be respectively the continuous morphism of topological monoids and the isomorphism of topological rings provided by proposition 16.5.4(iii,iv); in light of proposition 16.4.34(ii) it is easily seen that $\pi_U \circ \varphi^{\flat\circ}_U : \mathbf{E}^\circ_U \to A^\circ_U/pA^\circ_U$ is a continuous ring homomorphism fitting into a commutative diagram

$$\begin{array}{ccc} \overline{\mathbf{E}} & \xrightarrow{\overline{u}_{\overline{A}/p\overline{A}} \circ \overline{\omega}} & \overline{A}/p\overline{A} \\ {\scriptstyle j^\circ_{\mathbf{E}}}\downarrow & & \downarrow{\scriptstyle j^\circ_A \otimes_{\mathbb{Z}} \mathbb{F}_p} \\ \mathbf{E}^\circ_U & \xrightarrow{\pi_U \circ \varphi^{\flat\circ}_U} & A^\circ_U/pA^\circ_U. \end{array}$$

By proposition 9.3.52(iii), there follows a continuous ring homomorphism $v_U : W(\mathbf{E}^\circ_U) \to A^\circ_U$ fitting into a commutative diagram

$$\begin{array}{ccc} W(\overline{\mathbf{E}}) & \xrightarrow{u_{\overline{A}} \circ W(\overline{\omega})} & \overline{A} \\ {\scriptstyle W(j^\circ_{\mathbf{E}})}\downarrow & & \downarrow{\scriptstyle j^\circ_A} \\ W(\mathbf{E}^\circ_U) & \xrightarrow{v_U} & A^\circ_U. \end{array}$$

Since both $u_{\overline{A}}$ and j°_A are open, the same holds for v_U, and since $\overline{A}$ is perfectoid (proposition 16.5.4(iii)), the kernel of $u_{\overline{A}} \circ W(\overline{\omega})$ is a distinguished ideal, so it remains only to check

Claim 16.5.14. v_U induces a ring isomorphism

$$\overline{v}_U : W(\mathbf{E}_U^\circ) \otimes_{W(\overline{\mathbf{E}})} \overline{A} \xrightarrow{\sim} A_U^\circ.$$

Proof of the claim. Let $\Phi_{\mathbf{E}_U^\circ}$ be the Frobenius endomorphism of $\mathbf{E}_U^\circ$, and notice that

$$\Phi_{\mathbf{E}_U^\circ}(\mathbf{E}_U^{\circ\circ}) = \mathbf{E}_U^{\circ\circ}.$$

Moreover, $W(\mathbf{E}_U^{\circ\circ})$ is a closed ideal of both $W(\overline{\mathbf{E}})$ and $W(\mathbf{E}_U^\circ)$ (remark 9.3.28(iv)); there follows a commutative ladder with exact rows :

$$\mathscr{L} \quad : \quad \begin{array}{ccccccccc} 0 & \longrightarrow & W(\mathbf{E}_U^{\circ\circ}) & \longrightarrow & W(\overline{\mathbf{E}}) & \longrightarrow & W(\overline{\mathbf{E}}/\mathbf{E}_U^{\circ\circ}) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & W(\mathbf{E}_U^{\circ\circ}) & \longrightarrow & W(\mathbf{E}_U^\circ) & \longrightarrow & W(\mathbf{E}_U^\circ/\mathbf{E}_U^{\circ\circ}) & \longrightarrow & 0. \end{array}$$

Let $\underline{\alpha}$ be any distinguished element of $\mathrm{Ker}\, u_A$; since the image of $\underline{\alpha}$ is a regular element in both $W(\mathbf{E}_U^\circ/\mathbf{E}_U^{\circ\circ})$ and $W(\overline{\mathbf{E}}/\mathbf{E}_U^{\circ\circ})$, we have

$$\mathrm{Tor}_1^{W(\mathbf{E})}(W(\mathbf{E}_U^\circ/\mathbf{E}_U^{\circ\circ}), A) = 0 \qquad \mathrm{Tor}_1^{W(\mathbf{E})}(W(\overline{\mathbf{E}}/\mathbf{E}_U^{\circ\circ}), A) = 0$$

so the rows of the ladder $\mathscr{L} \otimes_{W(\overline{\mathbf{E}})} \overline{A}$ are still exact. Furthermore, since the topologies of $\overline{\mathbf{E}}/\mathbf{E}_U^{\circ\circ}$ and $\mathbf{E}_U^\circ/\mathbf{E}_U^{\circ\circ}$ are discrete, the isomorphism $\overline{\varphi}_U^{\flat\circ}$ of proposition 16.5.4(iv) induces identifications

$$W(\overline{\mathbf{E}}/\mathbf{E}_U^{\circ\circ}) \otimes_{W(\overline{\mathbf{E}})} \overline{A} \xrightarrow{\sim} \overline{A}/A_U^{\circ\circ} \qquad W(\mathbf{E}_U^\circ/\mathbf{E}_U^{\circ\circ}) \otimes_{W(\overline{\mathbf{E}})} \overline{A} \xrightarrow{\sim} A_U^\circ/A_U^{\circ\circ}.$$

Summing up, we obtain the commutative diagram with exact rows :

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_U^{\circ\circ} & \longrightarrow & \overline{A} & \longrightarrow & \overline{A}/A_U^{\circ\circ} & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle w} & & \downarrow & & \\ 0 & \longrightarrow & A_U^{\circ\circ} & \longrightarrow & W(\mathbf{E}_U^\circ) \otimes_{W(\overline{\mathbf{E}})} \overline{A} & \longrightarrow & A_U^\circ/A_U^{\circ\circ} & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \overline{v}_U} & & \| & & \\ 0 & \longrightarrow & A_U^{\circ\circ} & \longrightarrow & A_U^\circ & \longrightarrow & A_U^\circ/A_U^{\circ\circ} & \longrightarrow & 0 \end{array}$$

and to conclude, it suffices to check that $\overline{v}_U \circ w = j_A^\circ$. To this aim, it suffices to show that $\overline{v}_U \circ w \circ u_{\overline{A}} = j_A^\circ \circ u_{\overline{A}} : W(\mathbf{E}(\overline{A})) \to A_U^\circ$. However, by construction we have

$$\overline{v}_U \circ w \circ u_{\overline{A}} = v_U \circ W(j_{\mathbf{E}}^\circ \circ \overline{\omega}^{-1}) \qquad \text{and} \qquad j_A^\circ \circ u_{\overline{A}} \circ W(\overline{\omega}) = v_U \circ W(j_{\mathbf{E}}^\circ)$$

whence the claim. ◊

(ii): Let $D \subset A_U^\circ$ be any open integrally closed subring; it is easily seen that $A_U^{\circ\circ} \subset D$, so $A_U^{\circ\circ}$ is an ideal of D (remark 8.3.10(iv)) and we set $\overline{D} := D/A_U^{\circ\circ} \subset C_A := A_U^\circ/A_U^{\circ\circ}$.

Claim 16.5.15. With the foregoing notation, the rule : $D \mapsto \overline{D}$ establishes a bijection from the set of open integrally closed subrings of A_U° to the set of integrally closed subrings of C_A.

Proof of the claim. Let us check that $\overline{D}$ is integrally closed in C_A. Indeed, let $\overline{x} \in C_A$, and $P(T) \in D[T]$ a monic polynomial such that $P(\overline{x}) = 0$ in C_A; if $x \in A_U^\circ$ is any representative for the class $\overline{x}$, we get $P(x) \in A_U^{\circ\circ}$. Set $Q(T) := P(T) - P(x)$; then $Q(T) \in D[T]$ and $Q(x) = 0$, so $x \in D$, and therefore $\overline{x} \in \overline{D}$, which shows the contention. Conversely, if $\overline{D} \subset C_A$ is any integrally closed subring, let us show that the preimage $D \subset A_U^\circ$ of $\overline{D}$ is integrally closed in A_U°. Indeed, say that $x \in A_U^\circ$ is integral over D; then the class $\overline{x} \in C_A$ of x is integral over $\overline{D}$, hence $\overline{x} \in \overline{D}$, and the claim follows. ◊

Claim 16.5.15 also implies that if $D \subset \mathbf{E}_U^\circ$ is any open and integrally closed subring, then $\mathbf{E}_U^{\circ\circ} \subset D$, and the rule $D \mapsto D/\mathbf{E}_U^{\circ\circ}$ establishes a bijection between the set of all open integrally closed subrings of $\mathbf{E}_U^\circ$ and the set of all integrally closed subrings of $C_\mathbf{E} := \mathbf{E}_U^\circ/\mathbf{E}_U^{\circ\circ}$. However, proposition 16.5.4(iv) yields a natural ring isomorphism $\overline{\varphi}_U^{\flat\circ} : C_A \xrightarrow{\sim} C_\mathbf{E}$, whence a bijection $\mathscr{S}_A \xrightarrow{\sim} \mathscr{S}_\mathbf{E}$, and it remains to check that this bijection is realized by the rule $D \mapsto \mathbf{E}(D)$. To this aim, notice that every integrally closed subring of C_A is a perfect $\mathbb{F}_p$-algebra; then the assertion follows easily from proposition 16.3.25, which also gives (iii). □

16.5.16. In the situation of (16.5.2), pick a finitely generated ideal $J_\mathbf{E} \subset \mathbf{E}$ with $\alpha_0 \in J_\mathbf{E}$ and $Z_\mathbf{E} = \operatorname{Spec} \mathbf{E}/J_\mathbf{E}$. From (16.5.3), it follows easily that $Z_A = \operatorname{Spec} A/J_\mathbf{E}^{\langle 1\rangle} A$ (notation of remark 9.3.70(i)). Also, we easily see that $J_\mathbf{E}^{\lceil 0\rceil}$ is the radical of $J_\mathbf{E}$; on the other hand, $\overline{u}_A$ induces an isomorphism $\mathbf{E}/J_\mathbf{E}^{\lceil 0\rceil} \xrightarrow{\sim} A/J_\mathbf{E}^{\lceil 0\rceil} A$, so $A/J_\mathbf{E}^{\lceil 0\rceil} A$ is reduced, *i.e.* $J_\mathbf{E}^{\lceil 0\rceil} A$ is the radical of $J_\mathbf{E}^{\langle 1\rangle} A$.

Corollary 16.5.17. *With the notation of* (16.5.16)*, suppose moreover that* $A = A^+$*. Then :*

$$J_\mathbf{E}^{\lceil 0\rceil} A_U^+ = J_\mathbf{E}^{\lceil 0\rceil}\overline{A}.$$

Proof. Notice that it suffices to show that $J_\mathbf{E}^{\lceil 0\rceil} A_U^+ \subset \overline{A}$. Indeed, suppose that the latter holds; since we have as well $(J_\mathbf{E}^{\lceil 0\rceil})^2 = J_\mathbf{E}^{\lceil 0\rceil}$, we deduce that $J_\mathbf{E}^{\lceil 0\rceil} A_U^+ \subset J_\mathbf{E}^{\lceil 0\rceil}\overline{A}$, and the converse inclusion is obvious. Notice that $\mathbf{E}_U^+$ is the integral closure of $\overline{\mathbf{E}}$ in $\mathbf{E}_U$, so that theorem 16.8.2(ii) implies that the assertion holds for $\mathbf{E}_U^+$, *i.e.*

$$J_\mathbf{E}^{\lceil 0\rceil} \mathbf{E}_U^+ \subset \overline{\mathbf{E}}. \tag{16.5.18}$$

Next, recall that proposition 16.5.4(iv) yields a ring isomorphism $\overline{\varphi}_U^{\flat\circ} : \mathbf{E}_U^\circ/\mathbf{E}^{\circ\circ} \xrightarrow{\sim} A_U^\circ/A_U^{\circ\circ}$ restricting to an isomorphism $\overline{\mathbf{E}}/\mathbf{E}_U^{\circ\circ} \xrightarrow{\sim} \overline{A}/A_U^{\circ\circ}$. Moreover, claim 16.5.15 shows that the subring $\mathbf{E}_U^+/\mathbf{E}_U^{\circ\circ}$ is the integral closure of $\overline{\mathbf{E}}/\mathbf{E}_U^{\circ\circ}$ in $\mathbf{E}_U^\circ/\mathbf{E}^{\circ\circ}$ and likewise, $A_U^+/A_U^{\circ\circ}$ is the integral closure of $\overline{A}/A_U^{\circ\circ}$ in $A_U/A_U^{\circ\circ}$. Hence, $A_U^+/A_U^{\circ\circ} = \overline{\varphi}_U^{\flat\circ}(\mathbf{E}_U^+/\mathbf{E}_U^{\circ\circ})$, and taking into account (16.5.18), the assertion follows easily. □

16.5.19. Keep the notation of (16.5.2). By virtue of lemma 15.6.10(i,iii) we may identify naturally U_A with an open subset of $X_A^\circ := \operatorname{Spec} A_U^\circ$, and we have a well defined quasi-affinoid ring $\underline{A}_U^\circ := (A_U^\circ, A_U^+, U_A)$, which is perfectoid, by theorem 16.5.13(i). Set

$$(U_A, \mathscr{T}_{U_A}^\circ, A_U^+) := \mathsf{Spec}\, \underline{A}_U^\circ.$$

since, by definition, the continuous maps $A \to A_U^\circ \to A_U$ are both f-adic, we see that the topology $\mathscr{T}_{U_A}^\circ$ agrees with $\mathscr{T}_{U_A}$; *i.e.* we have a natural identification

$$\mathsf{Spec}\, \underline{A}_U^\circ \xrightarrow{\sim} \mathsf{Spec}\, \underline{A}.$$

Lastly, it is clear that $\underline{A}_U^\circ$ depends only on $\mathsf{Spec}\, \underline{A}$, and we claim that the rule $\mathsf{Spec}\, \underline{A} \mapsto \underline{A}_U^\circ$ extends to a well defined functor

$$\Gamma^\circ : \mathsf{q.Afd.Sch}_{\mathrm{perf}} \to \mathsf{q.Afd.Ring}^o_{\mathrm{perf}}$$

with a natural isomorphism $\varepsilon_{\underline{X}} : \mathsf{Spec} \circ \Gamma^\circ(\underline{X}) \xrightarrow{\sim} \underline{X}$ for every object $\underline{X}$ of $\mathsf{q.Afd.Sch}_{\mathrm{perf}}$. Indeed, if $\underline{X} := (X, \mathscr{T}_X, A_X^+)$ and $\underline{Y} := (Y, \mathscr{T}_Y, A_Y^+)$ are any two perfectoid quasi-affinoid schemes, and $\psi : \underline{X} \to \underline{Y}$ any f-adic morphism of quasi-affinoid schemes, then the corresponding map $\psi^\flat : \mathscr{O}_Y(Y) \to \mathscr{O}_X(X)$ induces a morphism $\psi^{\flat\circ} : (\mathscr{O}_Y(Y)^\circ, A_Y^+) \to (\mathscr{O}_X(X)^\circ, A_X^+)$ of affinoid rings (lemma 8.3.24(iii.a)), which is obviously also f-adic, and the resulting diagram

$$\begin{array}{ccc}
\operatorname{Spec} \mathscr{O}_X(X) & \xrightarrow{\operatorname{Spec}\psi^\flat} & \operatorname{Spec} \mathscr{O}_Y(Y) \\
{\scriptstyle j_X}\downarrow & & \downarrow{\scriptstyle j_Y} \\
\operatorname{Spec} \mathscr{O}_X(X)^\circ & \xrightarrow{\operatorname{Spec}\psi^{\flat\circ}} & \operatorname{Spec} \mathscr{O}_Y(Y)^\circ
\end{array}$$

shows that $\mathrm{Spec}\,\psi^{\flat\circ}$ restricts to a morphism $j_X(X) \to j_Y(Y)$, whence a well defined morphism of perfectoid quasi-affinoid rings :

$$\Gamma^\circ(\psi) : \Gamma^\circ(\underline{Y}) \to \Gamma^\circ(\underline{X}).$$

Moreover, if $\underline{A} := (A, A^+, U)$ is any perfectoid quasi-affinoid ring, the natural map $A \to \mathscr{O}_U(U)^\circ$ induces a morphism of quasi-affinoid rings $\eta_{\underline{A}} : \underline{A} \to \Gamma^\circ \circ \mathsf{Spec}(\underline{A})$ which is f-adic, and therefore it is a morphism of perfectoid quasi-affinoid rings. Lastly, it is easily seen that the pair $(\varepsilon_\bullet, \eta_\bullet)$ fulfills the triangular conditions of (1.1.13), so Γ° is right adjoint to Spec.

16.5.20. We also want to upgrade the functor $\mathbf{E}$ to well defined functors

$$\mathbf{E} : \mathsf{q.Afd.Ring}_{\mathrm{perf}} \to \mathsf{q.Afd.Ring}_{\mathrm{perf}} \qquad \mathbf{E} : \mathsf{q.Afd.Sch}_{\mathrm{perf}} \to \mathsf{q.Afd.Sch}_{\mathrm{perf}}.$$

Namely, let $\underline{A} := (A, A^+, U)$ be any perfectoid quasi-affinoid ring, and set $\mathbf{E} := \mathbf{E}(A)$; by applying theorem 16.5.13(ii) to the perfectoid quasi-affinoid ring $(A, A^+, \mathrm{Spec}\,A)$, we see that $\mathbf{E}^+ := \mathbf{E}(A^+)$ is a subring of integral elements of $\mathbf{E}$, so we get a perfectoid quasi-affinoid ring

$$\mathbf{E}(\underline{A}) := (\mathbf{E}, \mathbf{E}^+, \mathbf{E}(U))$$

where $\mathbf{E}(U) \subset \mathrm{Spec}\,\mathbf{E}$ is defined as in (16.5.2) : namely, it is the unique open subset of $\mathrm{Spec}\,\mathbf{E}$ containing the analytic locus and such that $\varphi^{-1}\mathbf{E}(U) = U$, where $\varphi : \mathrm{Spec}\,A \to \mathrm{Spec}\,\mathbf{E}$ is the continuous map given by (16.4.24). Moreover, if $\underline{\alpha}$ is any distinguished element in $\mathrm{Ker}\,u_A$, endow $A_0 := A/pA$ and $\mathbf{E}_0 := \mathbf{E}/\alpha_0\mathbf{E}$ with the ring topologies induced by A and respectively $\mathbf{E}$; from remark 16.3.7(ii) we see as well that the map $u_A : A \to \mathbf{E}$ induces a natural isomorphism of quasi-affinoid rings (notation of example 15.4.7(i)) :

$$A_0 \otimes_A \underline{A} \xrightarrow{\sim} \mathbf{E}_0 \otimes_{\mathbf{E}} \mathbf{E}(\underline{A}). \tag{16.5.21}$$

Furthermore, for every morphism $f : \underline{A} \to \underline{B} := (B, B^+, V)$ of perfectoid quasi-affinoid rings, it follows easily from (16.1.4) that the morphism of schemes $\mathrm{Spec}\,\mathbf{E}(f) : \mathrm{Spec}\,\mathbf{E}(B) \to \mathrm{Spec}\,\mathbf{E}$ maps $\mathbf{E}(V)$ into $\mathbf{E}(U)$. Moreover, the ring homomorphism $\mathbf{E}(f) : \mathbf{E} \to \mathbf{E}(B)$ is adic (theorem 16.3.42(i)), whence a well defined morphism of perfectoid quasi-affinoid rings

$$\mathbf{E}(f) : \mathbf{E}(\underline{A}) \to \mathbf{E}(\underline{B}).$$

Let also $\underline{B}\text{-}\mathsf{q.Afd.Ring}_{\mathrm{perf}} := \underline{B}/\mathsf{q.Afd.Ring}_{\mathrm{perf}}$ for every perfectoid quasi-affinoid ring $\underline{B}$; just as in remark 16.3.7(i), this new functor $\mathbf{E}$ restricts to an equivalence

$$\underline{A}\text{-}\mathsf{q.Afd.Ring}_{\mathrm{perf}} \xrightarrow{\sim} \mathbf{E}(\underline{A})\text{-}\mathsf{q.Afd.Ring}_{\mathrm{perf}}$$

which admits a natural quasi-inverse functor

$$\mathbf{A} : \mathbf{E}(\underline{A})\text{-}\mathsf{q.Afd.Ring}_{\mathrm{perf}} \xrightarrow{\sim} \underline{A}\text{-}\mathsf{q.Afd.Ring}_{\mathrm{perf}} \qquad (E, E^+, V) \mapsto (\mathbf{A}(E), \mathbf{A}(E^+), \mathbf{A}(V))$$

with $\mathbf{A}(E) := W(E) \otimes_{W(A)} A$ and correspondingly for $\mathbf{A}(E^+)$, and where $\mathbf{A}(V) := \varphi^{-1}V$. Likewise, for every perfectoid quasi-affinoid scheme $\underline{U}$ we define

$$\mathbf{E}(\underline{U}) := \mathsf{Spec} \circ \mathbf{E} \circ \Gamma^\circ(\underline{U}) \qquad \text{and} \qquad \underline{U}\text{-}\mathsf{q.Afd.Sch}_{\mathrm{perf}} := \mathsf{q.Afd.Sch}_{\mathrm{perf}}/\underline{U}.$$

Then, the resulting functor $\mathbf{E}$ on quasi-affinoid schemes restricts to an equivalence

$$\underline{U}\text{-}\mathsf{q.Afd.Sch}_{\mathrm{perf}} \xrightarrow{\sim} \mathbf{E}(\underline{U})\text{-}\mathsf{q.Afd.Sch}_{\mathrm{perf}}$$

with quasi-inverse given by the rule : $\underline{V} \mapsto \mathsf{Spec} \circ \mathbf{A} \circ \Gamma^\circ(\underline{V})$ (details left to the reader). Lastly, say that $\underline{U} = \Gamma^\circ(\underline{A})$, and set $\underline{U}_0 := \mathsf{Spec}\,(A_0 \otimes_A \underline{A})$ and $\mathbf{E}(\underline{U})_0 := \mathsf{Spec}\,(\mathbf{E}_0 \otimes_{\mathbf{E}} \underline{\mathbf{E}})$; then we deduce from (16.5.21) a natural isomorphism of quasi-affinoid schemes

$$\underline{U}_0 \times_{\underline{U}} \underline{V} \xrightarrow{\sim} \mathbf{E}(\underline{U})_0 \times_{\mathbf{E}(\underline{U})} \mathbf{E}(\underline{V}) \qquad \text{for every } \underline{V} \in \mathrm{Ob}(\underline{U}\text{-}\mathsf{q.Afd.Sch}_{\mathrm{perf}}). \tag{16.5.22}$$

(see example 15.4.8).

16.5.23. Next, we wish to extend the results of section 16.4 to the perfectoid quasi-affinoid case. Consider any perfectoid quasi-affinoid ring $\underline{A} := (A, A^+, U_A)$, and let $(\mathbf{E}, \mathbf{E}^+, U_{\mathbf{E}}) := \mathbf{E}(\underline{A})$ (notation of (16.5.20)). Let also $\overline{A}$ and $\overline{\mathbf{E}}$ be the images of the restriction maps $A \to A_U := \mathscr{O}_{U_A}(U_A)$ and respectively $\mathbf{E} \to \mathbf{E}_U := \mathscr{O}_{U_{\mathbf{E}}}(U_{\mathbf{E}})$. We endow $\overline{A}$ and $\overline{\mathbf{E}}$ with the quotient topologies induced by the projections $A \to \overline{A}$ and respectively $\mathbf{E} \to \overline{\mathbf{E}}$. Pick any finitely generated ideal of definition I of $\mathbf{E}$; we get a descending filtration on $\mathbf{E}_U$ and A_U by the rules :

$$\mathrm{Fil}^s\mathbf{E}_U := I^{\langle s\rangle}\overline{\mathbf{E}} \qquad \mathrm{Fil}^s A_U := I^{\langle s\rangle}\overline{A}$$

$$\mathrm{Fil}^{-s}\mathbf{E}_U := \{x \in \mathbf{E}_U \mid I^{\langle s\rangle}x \subset \overline{\mathbf{E}}\} \qquad \mathrm{Fil}^{-s}A_U := \{x \in A_U \mid I^{\langle s\rangle}x \subset \overline{A}\}$$

for every $s \in \mathbb{N}[1/p]$. As in example 9.1.9(i) we associate with I an *angular order function*

$$\nu : \mathbf{E}_U \to \mathbb{R} \cup \{+\infty\} \qquad x \mapsto \sup\{s \in \mathbb{Z}[1/p] \mid x \in \mathrm{Fil}^s\mathbf{E}_U\}.$$

Notice that, since the topology of $\mathbf{E}_U$ is separated, we have $\nu(x) = +\infty$ if and only if $x = 0$. Then, we fix $\rho \in]0, 1[$, and set

$$|x|_I := \rho^{\nu(x)} \qquad \text{for every } x \in \mathbf{E}_U$$

(with the convention that $\rho^{+\infty} := 0$).

Lemma 16.5.24. *With the notation of* (16.5.23)*, we have :*

(i) *The mapping* $|\cdot|_I$ *is the asymptotic Samuel function on* $\mathbf{E}_U$ *associated with* I *(see example* 9.1.9(i)*). Especially, it is a real-valued power-multiplicative norm on* $\mathbf{E}_U$.

(ii) $\varphi^\flat_U(\mathrm{Fil}^s\mathbf{E}_U) \subset \mathrm{Fil}^s A_U$ *for every* $s \in \mathbb{Z}[1/p]$ *(notation of proposition* 16.4.34(i)*).*

Proof. (i): Let $|\cdot|^*_I$ be the asymptotic Samuel function attached to I (and the real number ρ), let $x \in \mathbf{E}_U$ be any element, and $s \in \mathbb{Z}[1/p]$ any rational number; consider the following conditions:

(a) $|x|_I \leq \rho^s$.
(b) For every $t < s$ in $\mathbb{Z}[1/p]$ we have $x \in \mathrm{Fil}^t\mathbf{E}_U$.
(c) For every $t < s$ in $\mathbb{Z}[1/p]$ we may find $n \in \mathbb{N}$ such that $p^n t \in \mathbb{Z}$ and $x^{p^n} \in I^{p^n t}$, where the integral power $I^{p^n t}$ is defined as in example 9.1.9(i).
(d) $|x|^*_I \leq \rho^s$.

It is easily seen that each of these conditions is equivalent to the following one, whence the assertion.

(ii): Suppose first that $s \geq 0$; in this case, by definition we have $\varphi^\flat_U(I^{\langle s\rangle}) \subset \mathrm{Fil}^s A_U$. On the other hand, proposition 16.4.34(ii) implies that $\varphi^\flat_U(\mathrm{Fil}^s\mathbf{E}_U)$ lies in the topological closure of the $\overline{A}$-submodule of A_U generated by $\varphi^\flat_U(I^{\langle s\rangle})$, and since $\mathrm{Fil}^s A_U$ is an open $\overline{A}$-submodule of A_U, the assertion follows.

Lastly, suppose that $s < 0$, and let $x \in \mathrm{Fil}^s\mathbf{E}_U$ be any element; the condition means that $I^{\langle s\rangle}x \subset \overline{\mathbf{E}}$, whence $\varphi^\flat_U(x) \cdot \varphi^\flat_U(I^{\langle s\rangle}) \subset \overline{A}$, and finally $\varphi^\flat_U(x) \in \mathrm{Fil}^s A_U$, as stated. □

16.5.25. The norm $|\cdot|_I$ in turns induces a mapping

$$|\cdot|_1 : W(\mathbf{E}_U) \to \mathbb{R}_+ \cup \{+\infty\} \qquad (a_n \mid n \in \mathbb{N}) \mapsto \sup_{n\in\mathbb{N}} |a_n|_I^{p^{-n}}$$

as in (9.3.80), such that the subset

$$W(\mathbf{E}_U, 1) := \{\underline{a} \in W(\mathbf{E}_U) \mid |\underline{a}|_1 \in \mathbb{R}_+\}$$

is a subring of $W(\mathbf{E}_U)$, and the restriction of $|\cdot|_1$ is a real-valued norm on $W(\mathbf{E}_U, 1)$ (proposition 9.3.82(ii)). We endow $W(\mathbf{E}_U, 1)$ with the topology defined by the norm $|\cdot|_1$. Then clearly $W(\mathbf{E}_U, 1)$ is a separated topological ring, and it is independent of the choice of I (see example 9.1.9(iv)). Furthermore, for every $s \in \mathbb{Z}[1/p]$ we set

$$W\langle s\rangle := \{(x_n \mid n \in \mathbb{N}) \in W(\mathbf{E}_U) \mid x_n \in \mathrm{Fil}^{p^n s}\mathbf{E}_U \text{ for every } n \in \mathbb{N}\} \subset W(\mathbf{E}_U, 1).$$

Recall that the f-adic topologies of $\mathbf{E}_U$ and A_U agree with the $\mathbb{Z}$-linear topologies defined by $\mathrm{Fil}^\bullet\mathbf{E}_U$ and respectively $\mathrm{Fil}^\bullet A_U$ (lemma 9.3.69(iv) and corollary 16.3.40(ii)). Moreover, $\mathrm{Fil}^s A_U$ and $\mathrm{Fil}^s\mathbf{E}_U$ are bounded subsets of A_U and respectively $\mathbf{E}_U$, for every $s \in \mathbb{Z}[1/p]$. It follows easily that for every $\underline{a} := (a_n \mid n \in \mathbb{N}) \in W(\mathbf{E}_U, 1)$, the series

$$\sum_{n\in\mathbb{N}} p^n \cdot \varphi^\flat_U(a_n^{1/p^n})$$

converges to a unique element $u_U(\underline{a})$ of A_U. Moreover, just as in definition 16.3.27(v), for any $\overline{\mathbf{E}}$-submodule $\mathscr{K}$ of $\mathbf{E}_U$ we denote by $\{\mathscr{K}\}$ the topological closure in A_U of the $\overline{A}$-submodule generated by $(\varphi^\flat_U(x) \mid x \in \mathscr{K})$. We also define filtrations on $\mathscr{K}$ and $\{\mathscr{K}\}$ by the rules :

$$\mathrm{Fil}^s\mathscr{K} := \mathscr{K} \cap \mathrm{Fil}^s\mathbf{E}_U \qquad \text{and} \qquad \mathrm{Fil}^s\{\mathscr{K}\} := \{\mathrm{Fil}^s\mathscr{K}\} \qquad \text{for every } s \in \mathbb{Z}[1/p].$$

Proposition 16.5.26. *With the notation of* (16.5.25)*, we have :*

(i) *The resulting mapping* $u_U : W(\mathbf{E}_U, 1) \to A_U$ *is a morphism of topological rings.*
(ii) $u_U(W\langle s\rangle) = \{\mathrm{Fil}^s\mathbf{E}_U\} \subset \mathrm{Fil}^s A_U$ *for every* $s \in \mathbb{Z}[1/p]$.
(iii) $pW(\mathbf{E}_U) \cap W\langle s\rangle = pW\langle s\rangle$ *for every* $s \in \mathbb{Z}[1/p]$.
(iv) *The topological ring* $W(\mathbf{E}_U, 1)$ *is complete and separated.*

Proof. (i): The continuity of u_U follows from the continuity of $\varphi^\flat_U$ (proposition 16.4.34(i)) and a simple inspection of the definition. Next, we remark :

Claim 16.5.27. For every $a \in \mathbf{E}_U$ and every $\underline{x} \in W(\mathbf{E}_U, 1)$ we have

$$\tau_{\mathbf{E}_U}(a) \cdot \underline{x} \in W(\mathbf{E}_U, 1) \qquad \text{and} \qquad u_U(\tau_{\mathbf{E}_U}(a) \cdot \underline{x}) = \varphi^\flat_U(a) \cdot u_U(\underline{x}).$$

Proof of the claim. Say that $\underline{x} = (x_n \mid n \in \mathbb{N})$; by proposition 9.3.36(i) we have $\tau_{\mathbf{E}_U}(a) \cdot \underline{x} = (a^{p^n}x_n \mid n \in \mathbb{N})$, whence the first assertion. By the same token, since $\varphi^\flat_U$ is a continuous morphism of multiplicative monoids, we may compute

$$u_U(\tau_{\mathbf{E}_U}(a) \cdot \underline{x}) = \sum_{n\in\mathbb{N}} p^n \cdot \varphi^\flat_U(ax_n^{1/p^n}) = \varphi^\flat_U(a) \cdot \sum_{n\in\mathbb{N}} p^n \cdot \varphi^\flat_U(x_n^{1/p^n}) = \varphi^\flat_U(a) \cdot u_U(\underline{x})$$

as stated. ◊

To conclude, we have to check that

$$u_U(\underline{x} + \underline{y}) = u_U(\underline{x}) + u_U(\underline{y}) \qquad \text{and} \qquad u_U(\underline{x} \cdot \underline{y}) = u_U(\underline{x}) \cdot u_U(\underline{y})$$

for every $\underline{x} := (x_n \mid n \in \mathbb{N}), \underline{y} := (y_n \mid n \in \mathbb{N}) \in W(\mathbf{E}_U, 1)$. However, pick $s \in \mathbb{Z}[1/p]$ such that $\underline{x}, \underline{y} \in W\langle s\rangle$, and notice that u_U restricts to a map $W\langle s\rangle \to \mathrm{Fil}^s A_U$; set $\underline{x}^{(k)} := \sum_{n=0}^k p^n \cdot \tau_{\mathbf{E}_U}(x_n)$ and define likewise $\underline{y}^{(k)}$ for every $k \in \mathbb{N}$. The sequences $(\underline{x}^{(k)} \mid n \in \mathbb{N})$ and $(\underline{y}^{(k)} \mid n \in \mathbb{N})$ lie in $W\langle s\rangle$ and converge p-adically to $\underline{x}$ and respectively $\underline{y}$; moreover, the p-adic topology is separated on $\mathrm{Fil}^s A_U$, so it suffices to check the sought identities with $\underline{x}$ and $\underline{y}$ replaced by $\underline{x}^{(k)}$ and $\underline{y}^{(k)}$, for every $k \in \mathbb{N}$. We may thus assume that there exists $k \in \mathbb{N}$ such that $x_n = y_n = 0$ for every $n > k$. Then, we argue as in the proof of proposition 16.4.34(ii) : we pick a family $(g_\lambda \mid \lambda \in \Lambda)$ of elements of $\mathbf{E}$ such that $U_\mathbf{E} = \bigcup_{\lambda\in\Lambda} \mathrm{Spec}\,\mathbf{E}[g_\lambda^{-1}]$, and we set

$$h_\lambda := \overline{u}_A(g_\lambda) \in A \qquad \text{and} \qquad t_\lambda := \tau_\mathbf{E}(g_\lambda) \in W(\mathbf{E}) \qquad \text{for every } \lambda \in \Lambda$$

so that $U_A = \bigcup_{\lambda\in\Lambda} \mathrm{Spec}\,A[h_\lambda^{-1}]$, and it suffices to check that the sought identities hold in $A[h_\lambda^{-1}] = \overline{A}[h_\lambda^{-1}]$, for every $\lambda \in \Lambda$. Now, for every $\lambda \in \Lambda$ we may find $n_\lambda \in \mathbb{N}$ such that $g_\lambda^{p^i n_\lambda} \cdot x_i, g_\lambda^{p^i n_\lambda} \cdot y_i \in \overline{\mathbf{E}}$ for every $i = 0, \dots, k$, and therefore

$$\underline{x}_\lambda := t_\lambda^{n_\lambda} \cdot \underline{x},\ \underline{y}_\lambda := t_\lambda^{n_\lambda} \cdot \underline{y} \in W(\overline{\mathbf{E}})$$

(proposition 9.3.36(i)). Lastly, we consider the commutative diagram

$$\begin{array}{ccccc} W(\overline{\mathbf{E}}) & \xrightarrow{u_{\overline{A}}} & \overline{A} & \longrightarrow & \overline{A}[h_\lambda^{-1}] \\ \downarrow & & \downarrow & & \| \\ W(\mathbf{E}_U, 1) & \xrightarrow{u_U} & A_U & \longrightarrow & A[h_\lambda^{-1}] \end{array}$$

whose two vertical arrows are the natural inclusions, and whose unmarked horizontal arrows are the localization maps; taking into account claim 16.5.27, we may compute for every $\lambda \in \Lambda$:

$$\begin{aligned} \varphi_U^\flat(t_\lambda^{n_\lambda}) \cdot u_U(\underline{x} + \underline{y}) &= u_U(\underline{x}_\lambda + \underline{y}_\lambda) \\ &= u_{\overline{A}}(\underline{x}_\lambda + \underline{y}_\lambda) \\ &= u_{\overline{A}}(\underline{x}_\lambda) + u_{\overline{A}}(\underline{y}_\lambda) \\ &= u_U(\underline{x}_\lambda) + u_U(\underline{y}_\lambda) \\ &= \varphi_U^\flat(t_\lambda^{n_\lambda}) \cdot (u_U(\underline{x}) + u_U(\underline{y})) \end{aligned}$$

as required. Likewise one shows the other sought identity.

(ii): Notice first that if $s \geq 0$, then $W\langle s\rangle$ is the ideal $W(I^{\langle s\rangle}\overline{\mathbf{E}})$ of $W(\overline{\mathbf{E}})$ (notation of remark 9.3.28(iv)); since $I^{\langle s\rangle}\overline{\mathbf{E}}$ is a taut and open ideal of $\overline{\mathbf{E}}$, the ideal $W(I^{\langle s\rangle}\overline{\mathbf{E}})$ is closed in $W(\overline{\mathbf{E}})$, and from theorem 16.3.36(i) it follows that

$$u_U(W\langle s\rangle) = u_{\overline{A}}(W(I^{\langle s\rangle}\overline{\mathbf{E}})) = \{\mathrm{Fil}^s\mathbf{E}_U\}$$

as required. Especially $u_U(W\langle s\rangle)$ is an open $\overline{A}$-submodule of A_U for every $s \geq 0$; but then obviously the same holds also more generally for every $s \in \mathbb{Z}[1/p]$. Next, for a general $s \in \mathbb{Z}[1/p]$ notice that every element of $W\langle s\rangle$ can be written as a p-adically convergent series $\sum_{n\in\mathbb{N}} p^n \cdot \tau_{\mathbf{E}_U}(x_n)$, for a unique sequence $(x_n \mid n \in \mathbb{N})$ of elements of $\mathrm{Fil}^s\mathbf{E}_U$; to any such element, the map u_U assigns the p-adically convergent series $\sum_{n\in\mathbb{N}} p^n \cdot \varphi_U^\flat(x_n)$, which clearly lies in $\{\mathrm{Fil}^s\mathbf{E}_U\}$, from which we see that $u_U(W\langle s\rangle)$ is dense in $\{\mathrm{Fil}^s\mathbf{E}_U\}$. However, we have just seen that $u_U(W\langle s\rangle)$ is a closed subset of A_U for every such s, whence the assertion.

(iii): The intersection $pW(\mathbf{E}_U) \cap W\langle s\rangle$ consists of all elements $\underline{x} := (x_n \mid n \in \mathbb{N})$ such that $x_0 = 0$ and $x_n \in \mathrm{Fil}^{sp^n}\mathbf{E}_U$ for every $n \geq 1$. For such $\underline{x}$, we have $y_n := x_{n+1}^{1/p} \in \mathrm{Fil}^{sp^n}\mathbf{E}_U$ for every $n \in \mathbb{N}$, whence $\underline{y} := (y_n \mid n \in \mathbb{N}) \in W\langle s\rangle$ and $p \cdot \underline{y} = \underline{x}$, whence the contention.

(iv): Since $W(\overline{\mathbf{E}})$ is an open subring of $W(\mathbf{E}_U, 1)$, it suffices to show that $W(\overline{\mathbf{E}})$ is complete and separated for the topology $\mathscr{T}$ induced by the norm $|\cdot|_1$. To this aim, let $x_1, \dots, x_n$ be any finite system of generators of $I\overline{\mathbf{E}}$, and denote by $\mathscr{I} \subset W(\overline{\mathbf{E}})$ the ideal generated by $\tau_{\overline{\mathbf{E}}}(x_1), \dots, \tau_{\overline{\mathbf{E}}}(x_n)$. Taking into account proposition 9.3.77(i) and lemma 9.3.69(iv), it is easily seen that $\mathscr{T}$ agrees with the $\mathscr{I}$-adic topology on $W(\overline{\mathbf{E}})$. However, let also $\mathscr{T}_{\overline{\mathbf{E}}}$ be the $I\overline{\mathbf{E}}$-adic topology on $\overline{\mathbf{E}}$ and $\mathscr{T}_{W(\overline{\mathbf{E}})}$ the topology of $W(\overline{\mathbf{E}}, \mathscr{T}_E)$ (as in definition 9.3.14); then $\mathscr{T}_{W(\overline{\mathbf{E}})}$ agrees with the $(pW(\overline{\mathbf{E}}) + \mathscr{I})$-adic topology (proposition 9.3.77(ii)). On the other hand, since $\mathscr{T}_{\overline{\mathbf{E}}}$ is complete and separated, the same holds for $\mathscr{T}_{W(\overline{\mathbf{E}})}$ (lemma 9.3.33(ii)). To conclude, we may now appeal to lemma 8.3.12. □

16.5.28. In the situation of (16.5.23), let $\mathscr{K}, \mathscr{J} \subset \mathbf{E}_U$ be any two $\overline{\mathbf{E}}$-submodules; we let $\mathscr{K}\mathscr{J} \subset \mathbf{E}_U$ be the $\overline{\mathbf{E}}$-submodule generated by the system $(xy \mid x \in \mathscr{K}, y \in \mathscr{J})$. Especially, we have well defined $\overline{\mathbf{E}}$-submodules $\mathscr{K}^n$ of $\mathbf{E}_U$, defined inductively by the rule : $\mathscr{K}^0 := \overline{\mathbf{E}}$ and $\mathscr{K}^{n+1} := \mathscr{K}^n\mathscr{K}$ for every $n \in \mathbb{N}$. Next, suppose that $\mathscr{J} \subset \mathscr{K}$, and let $\beta \in \mathbf{E}$ be any element; as in definition 16.3.27, we shall say that the inclusion of $\mathscr{J}$ in $\mathscr{K}$ is *β-taut* if

$$\beta \cdot \Phi_{\mathbf{E}_U}^{-1}(\mathscr{K}^p) \subset \mathscr{J}$$

where $\Phi_{\mathbf{E}_U}$ is the Frobenius automorphism of $\mathbf{E}_U$. For $\beta = \alpha_0$ (where $(\alpha_n \mid n \in \mathbb{N})$ is a fixed distinguished element in $\mathrm{Ker}\, u_A$), we just say that the inclusion is *taut*. Likewise, we say that $\mathscr{K}$ is *β-taut* (resp. *taut*) if the identity map of $\mathscr{K}$ is a β-taut (resp. taut) inclusion. Just as in (16.3.30), for any taut inclusion $\mathscr{J} \subset \mathscr{K}$, the quotients $\mathscr{K}/\mathscr{J}$ and $\{\mathscr{K}\}/\{\mathscr{J}\}$ are both $\overline{A}/p\overline{A}$-modules. Also, just as in remark 16.3.28(v), if the inclusion $\mathscr{J} \subset \mathscr{K}$ is taut, then both $\mathscr{J}$ and $\mathscr{K}$ are taut. With this terminology, we have the following extension of theorem 16.3.31:

Lemma 16.5.29. *In the situation of* (16.5.25)*, the following holds :*

(i) *Every taut inclusion $\mathscr{K}_1 \subset \mathscr{K}_2$ of $\overline{\mathbf{E}}$-submodules of $\mathbf{E}_U$ induces an $\overline{A}/p\overline{A}$-linear map*

$$\mathscr{K}_2/\mathscr{K}_1 \to \{\mathscr{K}_2\}/\{\mathscr{K}_1\} \quad : \quad (x \mod \mathscr{K}_1) \mapsto (\varphi^\flat_U(x) \mod \mathscr{K}_1).$$

(ii) *If $\mathscr{K}_1$ and $\mathscr{K}_2$ are topologically closed in $\mathbf{E}_U$, the map of* (i) *is an isomorphism.*

(iii) *For every taut $\overline{\mathbf{E}}$-submodule $\mathscr{K}$ of $\mathbf{E}_U$ and every $t \in \mathbb{Z}[1/p]$ we have*

$$\{\mathscr{K}\} \cap \{\mathrm{Fil}^t \mathbf{E}_U\} = \mathrm{Fil}^t\{\mathscr{K}\}.$$

Proof. (i): The proof is the same as that of theorem 16.3.31(i) : the only difference is that instead of using proposition 9.3.62 one must appeal to the more general proposition 16.4.34(ii).

Next, we prove the following special case of (ii) :

Claim 16.5.30. Pick any strictly positive $\varepsilon \in \mathbb{N}[1/p]$ such that $\alpha_0 \in I^{\langle\varepsilon\rangle}\mathbf{E}$. Then we have :

(i) For every $s, t \in \mathbb{Z}[1/p]$ such that $t - \varepsilon \leq s \leq t$, the inclusion $\mathrm{Fil}^t\mathbf{E}_U \subset \mathrm{Fil}^s\mathbf{E}_U$ is taut, and the map of (i) is an isomorphism

$$\tau_{s,t} : \mathrm{Fil}^s\mathbf{E}_U/\mathrm{Fil}^t\mathbf{E}_U \xrightarrow{\sim} \{\mathrm{Fil}^s\mathbf{E}_U\}/\{\mathrm{Fil}^t\mathbf{E}_U\}.$$

(ii) Moreover, the inclusion $\{\mathrm{Fil}^s\mathbf{E}_U\} \subset \mathrm{Fil}^s A_U$ induces an injection

$$\mu_{s,t} : \{\mathrm{Fil}^s\mathbf{E}_U\}/\{\mathrm{Fil}^t\mathbf{E}_U\} \to \mathrm{Fil}^s A_U/\mathrm{Fil}^t A_U.$$

Proof of the claim. (i): First notice that, since $\overline{A}$ is perfectoid and $\mathbf{E}(\overline{A}) = \overline{\mathbf{E}}$, the assertion in case $s \geq 0$ is a special case of theorem 16.3.31(ii). Thus, we assume henceforth that $s < 0$. Next, suppose that $t > 0$; in this case, we know already that $\tau_{0,t}$ is an isomorphism, and therefore the 5-lemma implies that $\tau_{s,t}$ is an isomorphism if and only if the same holds for $\tau_{s,0}$ (details left to the reader). Hence, we may further assume that $t \leq 0$ as well. Then, notice that the map $\tau_{s,t}$ has dense image, for the quotient topology on the target; however, $\{\mathrm{Fil}^t\mathbf{E}_U\}$ is an open $\mathbf{E}_U$-submodule of $\{\mathrm{Fil}^t\mathbf{E}_U\}$, so this quotient topology is discrete, and thus $\tau_{s,t}$ is surjective. To conclude, it then suffices to check that $\mu_{s,t} \circ \tau_{s,t}$ is injective. Hence, let $x \in \mathrm{Fil}^s\mathbf{E}_U$ be any element, and suppose that $\varphi^\flat_U(x) \in \mathrm{Fil}^t A_U$; we need to show that $x \in \mathrm{Fil}^t\mathbf{E}_U$, *i.e.* that for every $b \in I^{\langle -t\rangle}$ we have $z := bx \in \overline{\mathbf{E}}$. However, for such z we have

$$\varphi^\flat_U(z) = \varphi^\flat_U(b) \cdot \varphi^\flat_U(x) \in I^{\langle -t\rangle} \cdot \mathrm{Fil}^t A_U \subset \overline{A}.$$

Notice that $z \in \mathrm{Fil}^{s-t}\mathbf{E}_U$, and let $\overline{z} \in \mathrm{Fil}^{s-t}\mathbf{E}_U/\mathrm{Fil}^0\mathbf{E}_U = \mathrm{Fil}^{s-t}\mathbf{E}_U/\overline{\mathbf{E}}$ be the class of z. Since $s - t \geq -\varepsilon$, and since $\alpha_0 \in I^{\langle\varepsilon\rangle}\mathbf{E}$, we see that $\mathrm{Fil}^{s-t}\mathbf{E}_U/\overline{\mathbf{E}} \subset \mathrm{Ann}_{\mathbf{E}_U/\overline{\mathbf{E}}}(\alpha_0)$; on the other hand, proposition 16.4.36(iv) implies that $\varphi^\flat_U$ induces an isomorphism

$$\mathrm{Ann}_{\mathbf{E}_U/\overline{\mathbf{E}}}(\alpha_0) \xrightarrow{\sim} \mathrm{Ann}_{A_U/\overline{A}}(p)$$

whence the contention. Assertion (ii) is an immediate consequence. ◊

(iii): Let $\varepsilon, s, t \in \mathbb{Z}[1/p]$ be as in claim 16.5.30(i), and notice that the composition

$$M := \frac{\mathrm{Fil}^s\mathscr{K}}{\mathrm{Fil}^t\mathscr{K}} \xrightarrow{\beta} M' := \frac{\mathrm{Fil}^s\{\mathscr{K}\}}{\mathrm{Fil}^t\{\mathscr{K}\}} \xrightarrow{\gamma} \frac{\{\mathscr{K}\} \cap \{\mathrm{Fil}^s\mathbf{E}_U\}}{\{\mathscr{K}\} \cap \{\mathrm{Fil}^t\mathbf{E}_U\}} \to M'' := \frac{\{\mathrm{Fil}^s\mathbf{E}_U\}}{\{\mathrm{Fil}^t\mathbf{E}_U\}}$$

factors through an injective map $M \to \mathrm{Fil}^s\mathbf{E}_U/\mathrm{Fil}^t\mathbf{E}_U$ and the isomorphism $\tau_{s,t}$. Hence, β is injective; moreover, β has dense image, and the quotient topologies on M and M'' are discrete, so β is an isomorphism, by claim 16.3.34. We deduce that γ is injective.

Claim 16.5.31. Assertion (iii) holds if there exists $s \in \mathbb{Z}[1/p]$ such that $\mathscr{K} \subset \mathrm{Fil}^s\mathbf{E}_U$.

Proof of the claim. Notice that $\mathscr{K} \cap \mathrm{Fil}^r\mathbf{E}_U$ is still taut for every $r \in \mathbb{Z}[1/p]$. Now, if $t < s$ there is nothing to prove. In case $t \geq s$, let $n \in \mathbb{N}$ be the unique integer such that $t - n\varepsilon - s \in [0, \varepsilon[$; arguing by induction on n, we are easily reduced to the case where $n = 0$, *i.e.* we may assume that $t - \varepsilon < s \leq t$. In this case, clearly γ is an isomorphism, and the sought identity follows immediately. ◊

Now, for a general $\mathscr{K}$, clearly $\{\mathscr{K}\}$ is the topological closure of $\bigcup_{s\in\mathbb{Z}[1/p]}\{\mathscr{K} \cap \mathrm{Fil}^s\mathbf{E}_U\}$ in A_U, and since $\{\mathrm{Fil}^t\mathbf{E}_U\}$ is open in A_U, we deduce that $\{\mathscr{K}\} \cap \{\mathrm{Fil}^t\mathbf{E}_U\}$ is the topological closure of $\bigcup_{s\in\mathbb{Z}[1/p]}(\{\mathscr{K} \cap \mathrm{Fil}^s\mathbf{E}_U\} \cap \{\mathrm{Fil}^t\mathbf{E}_U\})$. But for every $s \leq t$, we have $\{\mathscr{K} \cap \mathrm{Fil}^s\mathbf{E}_U\} \cap \{\mathrm{Fil}^t\mathbf{E}_U\} = \{\mathscr{K} \cap \mathrm{Fil}^t\mathbf{E}_U\}$, by virtue of claim 16.5.31, whence (iii).

We may now complete the proof of (ii) arguing as in the proof of theorem 16.3.31(ii) : namely, we may assume that $\alpha_0 \in I$, and for every $\overline{\mathbf{E}}$-submodule $\mathscr{K}$ of $\mathbf{E}_U$ we set

$$\mathrm{gr}^n\mathscr{K} := \mathrm{Fil}^n\mathscr{K}/\mathrm{Fil}^{n+1}\mathscr{K} \qquad \mathrm{gr}^n\{\mathscr{K}\} := \mathrm{Fil}^n\{\mathscr{K}\}/\mathrm{Fil}^{n+1}\{\mathscr{K}\} \qquad \text{for every } n \in \mathbb{Z}.$$

Moreover, for every $n \in \mathbb{Z}$ we let $\mathrm{Fil}^n(\mathscr{K}_2/\mathscr{K}_1)$ (resp. $\mathrm{Fil}^n(\{\mathscr{K}_2\}/\{\mathscr{K}_1\})$) be the image of $\mathrm{Fil}^n\mathscr{K}_2$ in $\mathscr{K}_2/\mathscr{K}_1$ (resp. of $\mathrm{Fil}^b\{\mathscr{K}_2\}$ in $\{\mathscr{K}_2\}/\{\mathscr{K}_1\}$) and we denote again by $\mathrm{gr}^\bullet(\mathscr{K}_2/\mathscr{K}_1)$ and $\mathrm{gr}^\bullet(\{\mathscr{K}_2\}/\{\mathscr{K}_1\})$ the respective associated graded modules. There follows for every $n \in \mathbb{Z}$ a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{gr}^n\mathscr{K}_1 & \longrightarrow & \mathrm{gr}^n\mathscr{K}_2 & \longrightarrow & \mathrm{gr}^n(\mathscr{K}_2/\mathscr{K}_1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{gr}^n\{\mathscr{K}_1\} & \longrightarrow & \mathrm{gr}^n\{\mathscr{K}_2\} & \longrightarrow & \mathrm{gr}^n(\{\mathscr{K}_2\}/\{\mathscr{K}_1\}) & \longrightarrow & 0
\end{array}$$

and since both $\mathscr{K}_1$ and $\mathscr{K}_2$ are taut, we know already from the proof of (iii) that the leftmost and central vertical arrows are isomorphisms, hence the same holds for the rightmost vertical arrow. Now, notice that the filtration $\mathrm{Fil}^\bullet(\mathscr{K}_2/\mathscr{K}_1)$ defines a separated and complete topology on $\mathscr{K}_2/\mathscr{K}_1$, since $\mathscr{K}_2$ and $\mathscr{K}_1$ are both closed $\overline{\mathbf{E}}$-submodules of $\mathbf{E}$. The same holds for the topology on $\{\mathscr{K}_2\}/\{\mathscr{K}_1\}$ determined by the filtration $\mathrm{Fil}^\bullet(\{\mathscr{K}_2\}/\{\mathscr{K}_1\})$, since $\{\mathscr{K}_2\}$ and $\{\mathscr{K}_1\}$ are closed $\overline{A}$-submodules in A. Then (ii) follows directly from [34, Ch.III, §2, n.8, Cor.3]. □

Theorem 16.5.32. *In the situation of* (16.5.25)*, we have :*

(i) *The continuous ring homomorphism u_U is surjective and open.*
(ii) $\mathrm{Ker}\, u_U = \underline{\alpha}W(\mathbf{E}_U, 1)$.

Proof. (ii): Clearly $\underline{\alpha} \in \mathrm{Ker} u_U$; thus, it suffices to check that u_U induces an isomorphism

$$\overline{u}_s : W\langle s\rangle/\underline{\alpha}W\langle s\rangle \xrightarrow{\sim} \{\mathrm{Fil}^s\mathbf{E}_U\} \qquad \text{for every } s \in \mathbb{Z}[1/p]$$

and by virtue of proposition 16.5.26(ii) we know already that $\overline{u}_s$ is surjective for every such s. Now, if $s \geq 0$, we have $W\langle s\rangle = W(I^{\langle s\rangle}\overline{\mathbf{E}}) \subset \mathbb{W}(\overline{\mathbf{E}})$, and the assertion follows from theorem 16.3.36(iv). For the general case, we may assume that $\alpha_0 \in I$, and then an easy induction argument then reduces to checking :

Claim 16.5.33. Suppose that $\alpha_0 \in I$. Then, if $\overline{u}_{s+1}$ is injective, the same holds for $\overline{u}_s$.

Proof of the claim. We consider the commutative diagram

(16.5.34)
$$\begin{array}{ccc}
W\langle s+1\rangle/\underline{\alpha}W\langle s+1\rangle & \xrightarrow{\ j\ } & W\langle s\rangle/\underline{\alpha}W\langle s\rangle \\
{\scriptstyle \overline{u}_{s+1}}\downarrow & & \downarrow{\scriptstyle \overline{u}_s} \\
\{\mathrm{Fil}^{s+1}\mathbf{E}_U\} & \xrightarrow{\ i\ } & \{\mathrm{Fil}^s\mathbf{E}_U\}
\end{array}$$

where i is the inclusion map, and j is induced by the inclusion $W\langle s+1\rangle \subset W\langle s\rangle$. Since $\overline{u}_{s+1}$ is injective by assumption, the same holds for j. Next, let us write $\alpha = \tau_{\mathbf{E}}(\alpha_0) + p \cdot \underline{u}$ for some invertible element $\underline{u}$ of $W(\mathbf{E})$; from proposition 9.3.36(ii) and our assumption on α_0, we deduce that $\tau_{\mathbf{E}}(\alpha_0) \cdot W\langle s\rangle \subset W\langle s+1\rangle$, hence the cokernel of j is annihilated by p. Taking into account proposition 16.5.26(iii), it follows that $\mathrm{Coker}\, j$ is naturally identified with the cokernel of the natural map $j' : (\mathrm{Fil}^{s+1}\mathbf{E}_U)/\alpha_0(\mathrm{Fil}^{s+1}\mathbf{E}_U) \to (\mathrm{Fil}^s\mathbf{E}_U)/\alpha_0(\mathrm{Fil}^s\mathbf{E}_U)$. But by the same token we see that $\alpha_0(\mathrm{Fil}^s\mathbf{E}_U) \subset \mathrm{Fil}^{s+1}\mathbf{E}_U$, so $\mathrm{Coker}\, j' = \mathrm{Fil}^s\mathbf{E}_U/\mathrm{Fil}^{s+1}\mathbf{E}_U$. Moreover, a simple inspection of the definitions shows that the map $\mathrm{Coker}\, j' \to \mathrm{Coker}\, i$ resulting from (16.5.34) is the same as the isomorphism $\tau_{s,s+1}$ of claim 16.5.30(i). Then the assertion follows from the 5-lemma. $\diamondsuit$

(i): By proposition 16.5.26(ii) we know already that u_U is an open map. Next, we remark :

Claim 16.5.35. To prove that u_U is surjective, we may assume that the topology of $\mathbf{E}$ is α_0-adic.

Proof of the claim. Notice that the image R of u_U contains $u_U(W(\overline{\mathbf{E}})) = u_{\overline{A}}(W(\overline{\mathbf{E}})) = \overline{A}$ (lemma 16.2.7(i)). Especially, R is an open subring of A_U; by the same token, R contains the $\overline{A}$-subalgebra R' of A_U generated by $\varphi_U^\flat(\mathbf{E}_U)$, and proposition 16.5.26(ii) implies that R' is dense in R. However, R' is also open in A_U, so $R = R'$. Furthermore, a simple inspection shows that the definition of $\varphi_U^\flat$ depends only on the ring A and the open subset U (and is independent of the topology of A or of $\mathbf{E}$). This shows that the image of u_U is independent of the topology of $\mathbf{E}_U$. Lastly, let $\mathscr{T}_{\alpha_0}$ be the α_0-adic topology of $\mathbf{E}$; it follows easily from example 16.3.2(i) and lemma 8.3.12 that the topological ring $(\mathbf{E}, \mathscr{T}_{\alpha_0})$ is perfectoid, and $U_{\mathbf{E}}$ contains the analytic locus of $\mathrm{Spec}\,\mathbf{E}$, relative to the topology $\mathscr{T}_{\alpha_0}$, whence the claim. $\diamondsuit$

Henceforth, we assume that the topology of $\mathbf{E}_U$ is α_0-adic. Set $J_1 := \mathrm{Ann}_{\overline{\mathbf{E}}}(\alpha_0)$, let $J_2 \subset \overline{\mathbf{E}}$ be the radical of the ideal $\alpha_0\overline{\mathbf{E}}$, and $J_3 := J_1 + J_2$; let $\overline{X}_{\mathbf{E}} := \mathrm{Spec}\,\overline{\mathbf{E}}$, $\overline{X}_A := \mathrm{Spec}\,\overline{A}$, and for $i = 1,2,3$ set $\overline{\mathbf{E}}_i := \overline{\mathbf{E}}/J_i$, $\overline{A}_i := W(\overline{\mathbf{E}}_i) \otimes_{W(\overline{A})} \overline{A}$, $\overline{X}_{\mathbf{E},i} := \mathrm{Spec}\,\overline{\mathbf{E}}_i$ and $\overline{X}_{A,i} := \mathrm{Spec}\,\overline{A}_i$. Lastly, let $j_{\mathbf{E},i} : \overline{X}_{\mathbf{E},i} \to \overline{X}_{\mathbf{E}}$ and $j_{A,i} : \overline{X}_{A,i} \to \overline{X}_A$ be the induced closed immersions and set $\mathbf{E}_{U,i} := j_{\mathbf{E},i*}\mathscr{O}_{\overline{X}_{\mathbf{E},i}}(U_{\mathbf{E}})$, $A_{U,i} := j_{A,i*}\mathscr{O}_{\overline{X}_{A,i}}(U_A)$ for $i = 1,2,3$. According to (16.4.24) we have cartesian diagrams of quasi-coherent $\mathscr{O}_{\overline{X}_{\mathbf{E}}}$-modules

$$\begin{array}{ccc} \mathscr{O}_{\overline{X}_{\mathbf{E}}} & \longrightarrow & j_{E,1*}\mathscr{O}_{\overline{X}_{\mathbf{E},1}} \\ \downarrow & & \downarrow \\ j_{\mathbf{E},2*}\mathscr{O}_{\overline{X}_{\mathbf{E},2}} & \longrightarrow & j_{\mathbf{E},3*}\mathscr{O}_{\overline{X}_{\mathbf{E},3}} \end{array} \qquad \begin{array}{ccc} \mathscr{O}_{\overline{X}_A} & \longrightarrow & j_{A,1*}\mathscr{O}_{\overline{X}_{A,1}} \\ \downarrow & & \downarrow \\ j_{A,2*}\mathscr{O}_{\overline{X}_{A,2}} & \longrightarrow & j_{A,3*}\mathscr{O}_{\overline{X}_{A,3}} \end{array}$$

whence cartesian diagrams of ring homomorphisms

$$(16.5.36) \qquad \begin{array}{ccc} \mathbf{E}_U & \longrightarrow & \mathbf{E}_{U,1} \\ \downarrow & & \downarrow \\ \mathbf{E}_{U,2} & \longrightarrow & \mathbf{E}_{U,3} \end{array} \qquad \begin{array}{ccc} A_U & \longrightarrow & A_{U,1} \\ \downarrow & & \downarrow \\ A_{U,2} & \longrightarrow & A_{U,3}. \end{array}$$

We endow each $\mathbf{E}_{U,i}$ with its α_0-adic topology $\mathscr{T}_i$; then $\mathscr{T}_i$ is the discrete topology for $i = 2,3$, and the topological ring $(\mathbf{E}_{U,i}, \mathscr{T}_i)$ is perfectoid for $i = 1,2,3$ (see (16.4.24)). We also define the power-multiplicative norms $|\cdot|_{I,i} : \mathbf{E}_{U,i} \to \mathbb{R}_+$ associated with the ideal $I\mathbf{E}_{U,i}$ as in (16.5.23), and we get consequently the normed topological rings $W(\mathbf{E}_{U,i}, 1)$ as in (16.5.25).

Claim 16.5.37. The induced diagrams of rings

$$\mathscr{W}(\mathbf{E}) \ : \ \begin{array}{ccc} W(\mathbf{E}_U) & \longrightarrow & W(\mathbf{E}_{U,1}) \\ \downarrow & & \downarrow \\ W(\mathbf{E}_{U,2}) & \longrightarrow & W(\mathbf{E}_{U,3}) \end{array} \qquad \mathscr{W}(\mathbf{E},1) \ : \ \begin{array}{ccc} W(\mathbf{E}_U,1) & \longrightarrow & W(\mathbf{E}_{U,1},1) \\ \downarrow & & \downarrow \\ W(\mathbf{E}_{U,2},1) & \longrightarrow & W(\mathbf{E}_{U,3},1) \end{array}$$

are cartesian.

Proof of the claim. The assertion is clear for the left diagram. Next, set

$$\mathbf{E}_U^+ := \{x \in \mathbf{E}_U \mid |x|_I \leq 1\}$$

and define likewise the subring $\mathbf{E}_{U,i}^+ \subset \mathbf{E}_{U,i}$ for $i = 1, 2$. For every $x \in \mathbf{E}_U$ and $i = 1, 2$, denote by $x^{(i)} \in \mathbf{E}_{U,i}$ the image of x; since we already know that diagram $\mathscr{W}(\mathbf{E})$ is cartesian, the assertion for $\mathscr{W}(\mathbf{E}, 1)$ comes down to checking that if $x \in \mathbf{E}_U$ and $|x^{(i)}|_{I,i} \leq \rho^{-s}$ for $i = 1, 2$ and some $s \in \mathbb{N}[1/p]$, then $|x|_I \leq \rho^{-s}$. The latter is equivalent in turn to the following. Suppose that for every strictly positive $\varepsilon \in \mathbb{N}[1/p]$ and every $b \in I^{\langle s+\varepsilon\rangle}$ we have $b^{(i)}x^{(i)} \in \overline{\mathbf{E}}_i$; then $z := bx \in \mathbf{E}_U^+$. Now, for such x and b pick $y_i \in \overline{\mathbf{E}}$ with $y_i^{(i)} = b^{(i)}x^{(i)}$ for $i = 1, 2$; then $(z - y_1)(z - y_2) \in J_1 \cap J_2 = 0$, and especially, z is integral over $\overline{\mathbf{E}}$. But $\overline{\mathbf{E}} \subset \mathbf{E}_U^+$, and $\mathbf{E}_U^+$ is integrally closed in $\mathbf{E}_U$ (remark 9.1.4(vi)), whence the claim. ◊

Claim 16.5.38. The diagram of rings $\mathscr{W}(\mathbf{E}, 1) \otimes_{W(\mathbf{E})} A$ is cartesian.

Proof of the claim. Consider the complex in $\mathsf{C}^{[0,1]}(W(\mathbf{E})\text{-}\mathbf{Mod})$

$$K^\bullet \quad : \quad 0 \to W(\mathbf{E}_{U,1}, 1) \oplus W(\mathbf{E}_{U,2}, 1) \to W(\mathbf{E}_{U,3}, 1) \to 0$$

whose differential is deduced from $\mathscr{W}(\mathbf{E}, 1)$, so that $H^0K^\bullet = W(\mathbf{E}_U, 1)$. Taking into account proposition 9.3.47(i), we have a short exact sequence

$$0 \to K^\bullet \xrightarrow{\underline{\alpha}\cdot\mathbf{1}_{K^\bullet}} K^\bullet \to K^\bullet/\underline{\alpha}K^\bullet \to 0$$

and by the same token, scalar multiplication is injective on $H^0K^\bullet$, so we get a long exact cohomology sequence

$$0 \to H^0(K^\bullet)/\underline{\alpha} \cdot H^0(K^\bullet) \xrightarrow{f_{\underline{\alpha}}} H^0(K^\bullet/\underline{\alpha}K^\bullet) \to H^1K^\bullet \xrightarrow{\underline{\alpha}} H^1K^\bullet$$

and the claim comes down to checking that $f_{\underline{\alpha}}$ is an isomorphism, or equivalently, that scalar multiplication by $\underline{\alpha}$ is injective on $H^1K^\bullet$. However, notice that $H^1K^\bullet$ is a quotient of the $W(\mathbf{E})$-module $W(\mathbf{E}_{U,3}, 1)$, and the latter is annihilated by $\tau_{\mathbf{E}}(\alpha_0)$. Since $\underline{\alpha} = \tau_{\mathbf{E}}(\alpha_0) + p\underline{u}$ for an invertible element $\underline{u}$ of $W(\mathbf{E})$, we are reduced to checking that scalar multiplication by p is injective on $H^1K^\bullet$, *i.e.* we may assume that $\underline{\alpha} = p$, and then it suffices to show that f_p is an isomorphism. However, it follows easily from proposition 16.5.26(iii) that

$$pW(\mathbf{E}_{U,i}) \cap W(\mathbf{E}_{U,i}, 1) = pW(\mathbf{E}_{U,i}, 1) \qquad \text{for } i = 1, 2, 3 \tag{16.5.39}$$

and likewise for $W(\mathbf{E}_U, 1)$. Therefore, $H^0(K^\bullet)/p \cdot H^0(K^\bullet) = \mathbf{E}_U$, and $K^\bullet/pK^\bullet$ is the complex

$$0 \to \mathbf{E}_{U,1} \oplus \mathbf{E}_{U,2} \to \mathbf{E}_{U,2} \to 0$$

deduced from the cartesian diagram (16.5.36). The claim follows. ◊

From claim 16.5.38 we deduce a commutative diagram of rings

$$\begin{array}{ccccccc}
A_U & & \longrightarrow & & & & A_{U,i} \\
& \nwarrow^{u_U} & & & & \nearrow^{u_{U,1}} & \\
\Big\downarrow & & W(\mathbf{E}_U, 1) \otimes_{W(\mathbf{E})} A & \longrightarrow & W(\mathbf{E}_{U,1}, 1) \otimes_{W(\mathbf{E})} A & & \Big\downarrow \\
& & \Big\downarrow & & \Big\downarrow & & \\
& & W(\mathbf{E}_{U,2}, 1) \otimes_{W(\mathbf{E})} A & \longrightarrow & W(\mathbf{E}_{U,3}, 1) \otimes_{W(\mathbf{E})} A & & \\
& \swarrow_{u_{U,2}} & & & & \searrow_{u_{U,3}} & \\
A_{U,2} & & \longrightarrow & & & & A_{U,3}
\end{array}$$

whose two square subdiagrams are both cartesian. Recall that the topologies of $\mathbf{E}_{U,2}$ and $\mathbf{E}_{U,3}$ are discrete; taking into account theorem 16.5.13(i) it follows that $\mathbf{E}_{U,i} = \mathbf{E}^{\circ}_{U,i} = A^{\circ}_{U,i} = A_{U,i}$ and $W(\mathbf{E}_{U,i}, 1) = W(\mathbf{E}_{U,i})$ for $i = 2, 3$, which easily implies that $u_{U,2}$ and $u_{U,3}$ are isomorphisms. Thus, to conclude the proof it suffices to show that $u_{U,1}$ is an isomorphism. Hence, we may replace $\mathbf{E}$ by $\overline{\mathbf{E}}_1$ and assume from start that α_0 is a regular element of $\mathbf{E}$ and the topology of $\mathbf{E}$ is α_0-adic. Let R be the image of u_U; under the current assumptions, p is a regular element of A (remark 16.4.25(ii)), and therefore

$$A \subset R \subset A_U \subset A[1/p]$$

whence $R[1/p] = A[1/p]$. Moreover, we notice :

Claim 16.5.40. The inclusion $R \subset A$ induces an injective map $R/pR \to A_U/pA_U$.

Proof of the claim. From (ii) and (16.5.39) we get a natural identification

$$R/pR \xrightarrow{\sim} W(\mathbf{E}_U, 1)/(\underline{\alpha}W(\mathbf{E}_U, 1) + pW(\mathbf{E}_U, 1)) \xrightarrow{\sim} \mathbf{E}_U/\alpha_0\mathbf{E}_U \qquad (16.5.41)$$

and from the short exact sequence of $\mathscr{O}_{U_{\mathbf{E}}}$-modules

$$0 \to \mathscr{O}_{U_{\mathbf{E}}} \xrightarrow{\alpha_0 \cdot} \mathscr{O}_{U_{\mathbf{E}}} \to \mathscr{O}_{U_{\mathbf{E}}}/\alpha_0\mathscr{O}_{U_{\mathbf{E}}} \to 0$$

we see that the natural map

$$\mathbf{E}_U/\alpha_0\mathbf{E}_U \to \mathbf{E}^0_U := H^0(U_{\mathbf{E}} \cap \mathrm{Spec}\,\mathbf{E}/\alpha_0\mathbf{E}, \mathscr{O}_{U_{\mathbf{E}}}/\alpha_0\mathscr{O}_{U_{\mathbf{E}}})$$

is injective. Likewise, we see that the same holds for the natural map

$$A_U/pA_U \to A^0_U := H^0(U_A \cap \mathrm{Spec}\,A/pA, \mathscr{O}_{U_A}/p\mathscr{O}_{U_A}).$$

On the other hand, recall that the ring homomorphism $u_A : W(\mathbf{E}) \to A$ induces an isomorphism of schemes $\varphi_0 : \mathrm{Spec}\,A/pA \xrightarrow{\sim} \mathrm{Spec}\,\mathbf{E}/\alpha_0\mathbf{E}$ (see (16.4.28)), whence a ring isomorphism $\varphi^\flat_{0,U} : \mathbf{E}^0_U \xrightarrow{\sim} A^0_U$ and a simple inspection of the definitions shows that the resulting diagram

$$\begin{array}{ccc} \mathbf{E}_U/\alpha_0\mathbf{E}_U & \xrightarrow{u_U \otimes_{W(\mathbf{E})} A/pA} & A_U/pA_U \\ \downarrow & & \downarrow \\ \mathbf{E}^0_U & \xrightarrow{\varphi^\flat_{0,U}} & A^0_U \end{array}$$

commutes. But it is easily seen that the isomorphism (16.5.41) identifies $u_U \otimes_{W(\mathbf{E})} A/pA$ with the map $R/pR \to A_U/pA_U$ induced by the inclusion $R \subset A$, whence the claim. ◊

Now, let $a \in A_U$ be any element, and pick $n \in \mathbb{N}$ such that $p^n a \in A$; then there exists $\underline{x} \in W(\mathbf{E})$ such that $u_U(\underline{x}) = u_A(\underline{x}) = p^n a$. To conclude, we shall show, by induction on $i = 0, \dots, n$ that $p^{n-i}a \in R$. Indeed, we have just shown that $p^n a \in R$. Suppose that $i \geq 0$ and $p^{n-i}a \in R$; if $i = n$, we are done, and if $i < n$, the image of $p^{n-i}a$ vanishes in A_U/pA_U, hence also in R/pR, *i.e.* there exists $b \in R$ with $pb = p^{n-i}a$; since p is regular in R, it follows that $b = p^{n-i-1}a$, as required. □

Corollary 16.5.42. *In the situation of* (16.5.25)*, we have :*

(i) $\{\mathrm{Fil}^s\mathbf{E}_U\} = \mathrm{Fil}^s A_U$ *for every* $s \in \mathbb{Z}[1/p]$.

(ii) *The map* $\varphi^\flat_U$ *induces a ring isomorphism* $\mathbf{E}_U/\alpha_0\mathbf{E}_U \xrightarrow{\sim} A_U/pA_U$.

Proof. (i): Let us first remark that

$$\{\mathrm{Fil}^{s-n}\mathbf{E}_U\} \cap \mathrm{Fil}^s A_U = \{\mathrm{Fil}^s\mathbf{E}_U\} \qquad \text{for every } n \in \mathbb{N} \text{ and every } s \in \mathbb{Z}[1/p].$$

Indeed, the case where $n = 1$ follows immediately from claim 16.5.30(ii), and then the general case follows by a simple induction on $n \in \mathbb{N}$: details left to the reader. But theorem 16.5.32(i) shows that $A_U = \bigcup_{n\in\mathbb{N}}\{\mathrm{Fil}^{s-n}\mathbf{E}_U\}$, whence the assertion.

(ii) follows directly from theorem 16.5.32(i) and (16.5.41). □

As an application, we point out the following criterion :

Corollary 16.5.43. *Let A be any perfectoid ring, $U_{\mathbf{E}} \subset V_{\mathbf{E}} \subset X_{\mathbf{E}} := \mathrm{Spec}\,\mathbf{E}(A)$ two constructible open subsets that contain the analytic locus. Set also $U_A := \varphi^{-1}U_{\mathbf{E}}$ and $V_A := \varphi^{-1}V_{\mathbf{E}}$, where $\varphi : X_A := \mathrm{Spec}\,A \to X_{\mathbf{E}}$ is the continuous map of* (16.4.24). *Then we have :*

(i) *$U_{\mathbf{E}}$ is an affine scheme if and only if the same holds for U_A.*
(ii) *More generally, the inclusion map $U_{\mathbf{E}} \to V_{\mathbf{E}}$ is an affine morphism of schemes if and only if the same holds for the inclusion map $U_A \to V_A$.*

Proof. (i): Set $\mathbf{E}_U := \mathscr{O}_{X_{\mathbf{E}}}(U_{\mathbf{E}})$ and $A_U := \mathscr{O}_{X_A}(U_A)$; by lemma 15.6.10(i), we get commutative diagrams of schemes

$$\begin{array}{ccc} U_{\mathbf{E}} & \xrightarrow{j_{\mathbf{E}}} & \mathrm{Spec}\,\mathbf{E}_U \\ & {\scriptstyle i_{\mathbf{E}}}\searrow & \downarrow{\scriptstyle j'_{\mathbf{E}}} \\ & & X_{\mathbf{E}} \end{array} \qquad \begin{array}{ccc} U_A & \xrightarrow{j_A} & \mathrm{Spec}\,A_U \\ & {\scriptstyle i_A}\searrow & \downarrow{\scriptstyle j'_A} \\ & & X_A \end{array}$$

where $i_{\mathbf{E}}$, i_A, $j_{\mathbf{E}}$ and j_A are open immersions. We have to show that $j_{\mathbf{E}}$ is an isomorphism if and only if the same holds for j_A. To this aim, set $Z_{\mathbf{E}} := X_{\mathbf{E}} \setminus U_{\mathbf{E}}$ and $Z_A := X_A \setminus U_A$; equivalently, we have to check that $j'^{-1}_{\mathbf{E}} Z_{\mathbf{E}} = \varnothing$ if and only if $j'^{-1}_A Z_A = \varnothing$. Now, since $U_{\mathbf{E}}$ contains the analytic locus of $X_{\mathbf{E}}$, we may assume that $Z_{\mathbf{E}} \subset X^0_{\mathbf{E}} := \mathrm{Spec}\,\mathbf{E}/\alpha_0\mathbf{E}$, where $(\alpha_n \mid n \in \mathbb{N})$ is any fixed distinguished element of $\mathrm{Ker}\,u_A$, and likewise $Z_A \subset X^0_A := \mathrm{Spec}\,A/pA$. On the other hand, by corollary 16.5.42(ii) we have a commutative diagram of schemes

$$\begin{array}{ccc} \mathrm{Spec}\,A_U/pA_U & \longrightarrow & \mathrm{Spec}\,\mathbf{E}_U/\alpha_O\mathbf{E}_U \\ \downarrow & & \downarrow \\ X^0_A & \xrightarrow{\varphi_0} & X^0_{\mathbf{E}} \end{array}$$

whose horizontal arrows are isomorphisms induced by φ and by the map $\varphi^\flat_U : \mathbf{E}_U \to A_U$ of proposition 16.4.34(i), and whose vertical arrows are the restrictions of $j'_{\mathbf{E}}$ and j'_A. Lastly, by construction we have $\varphi_0^{-1} Z_{\mathbf{E}} = Z_A$, whence the contention.

(ii) is analogous : the open immersion $U_{\mathbf{E}} \to V_{\mathbf{E}}$ is affine if and only if for every affine open subset $W \subset V_{\mathbf{E}}$, the intersection $W \cap U_{\mathbf{E}}$ is still affine. The latter condition means that

$$\mathrm{Spec}\,\mathscr{O}_{X_{\mathbf{E}}}(W \cap U_{\mathbf{E}}) \times_W Z_{\mathbf{E}} = \varnothing \qquad \text{for every such } W$$

which in turns is equivalent to the condition :

$$j'^{-1}_{\mathbf{E}}(Z_{\mathbf{E}} \cap V_{\mathbf{E}}) = \varnothing. \tag{16.5.44}$$

Likewise, we easily see that the inclusion $U_A \subset V_A$ is affine if and only if $j'^{-1}_A(Z_A \cap V_A) = \varnothing$, and since $Z_{\mathbf{E}} \subset X^0_{\mathbf{E}}$, this latter condition is equivalent to (16.5.44) : details left to the reader. □

Lemma 16.5.45. *In the situation of* (16.5.23)*, let $\mathscr{K}$ be any $\overline{\mathbf{E}}$-submodule of $\mathbf{E}_U$, and set :*

$$W(\mathscr{K}) := \{(x_n \mid n \in \mathbb{N}) \mid x_n \in \mathscr{K}^{p^n} \text{ for every } n \in \mathbb{N}\} \qquad W(\mathscr{K}, 1) := W(\mathscr{K}) \cap W(\mathbf{E}_U, 1).$$

(notation of (16.5.28)*). Denote also by $(\mathscr{K}^n)^c$ the topological closure of $\mathscr{K}^n$ in $\mathbf{E}_U$, for every $n \in \mathbb{N}$, and by $W(\mathscr{K}, 1)^c$ the topological closure of $W(\mathscr{K}, 1)$ in $W(\mathbf{E}_U, 1)$. We have :*

(i) *$W(\mathscr{K})$ and $W(\mathscr{K}, 1)$ are $W(\overline{\mathbf{E}})$-submodules of respectively $W(\mathbf{E}_U)$ and $W(\mathbf{E}_U, 1)$.*
(ii) *$W(\mathscr{K}, 1)^c = M := \{(x_n \mid n \in \mathbb{N}) \in W(\mathbf{E}_U, 1) \mid x_n \in (\mathscr{K}^{p^n})^c \text{ for every } n \in \mathbb{N}\}$.*
(iii) *If $\mathscr{K}$ is taut, $u_U(W(\mathscr{K}, 1)^c) = \{\mathscr{K}\}$.*
(iv) *If $\mathscr{K}$ is 1-taut and topologically closed in $\mathbf{E}_U$, then u_U induces an isomorphism*

$$W(\mathscr{K}, 1)/\underline{\alpha}W(\mathscr{K}, 1) \xrightarrow{\sim} \{\mathscr{K}\}.$$

Proof. (i): This is analogous to remark 9.3.28 : details left to the reader.

(ii): Let $\underline{x} := (x_n \mid n \in \mathbb{N}) \in M$ be any element, and fix any $\varepsilon \in \mathbb{R}_{>0}$. We shall exhibit inductively a sequence $\underline{y} := (y_n \mid n \in \mathbb{N})$ of elements of $\mathbf{E}_U$ such that

$$|\underline{y}|_1 \leq \varepsilon \qquad \text{and} \qquad \underline{x} + \underline{y} \in W(\mathscr{K}, 1).$$

Indeed, let $n \in \mathbb{N}$, and suppose that we have already exhibited $y_i \in (\mathscr{K}^{p^i})^c$ for $i = 0, \dots, n-1$ such that $\underline{y}^{(n)} := \sum_{i=0}^{n-1} p^i \cdot \tau_{\mathbf{E}_U}(y_i)$ has norm $< \varepsilon$ and such that if we set $(z_i^{(n)} \mid i \in \mathbb{N}) := \underline{x} + \underline{y}^{(n)}$, we have $z_i^{(n)} \in \mathscr{K}^{p^i}$ for every $i = 0, \dots, n-1$. It follows easily that $z_i^{(n)} \in (\mathscr{K}^{p^i})^c$ for every $i \in \mathbb{N}$. We may then pick $y_n \in (\mathscr{K}^{p^n})^c$ such that $|y_n|_I < \varepsilon^{p^n}$ and $z_n^{(n)} + y_n \in \mathscr{K}^{p^n}$. Set $\underline{y}^{(n+1)} := \underline{y}^{(n)} + p^n \cdot \tau_{\mathbf{E}_U}(y_n)$ and $(z_i^{(n+1)} \mid i \in \mathbb{N}) := \underline{x} + \underline{y}^{(n+1)}$. In light of lemma 9.3.27(i) and claim 9.3.31 it is easily seen that $z_i^{(n+1)} = z_i^{(n)}$ for every $i = 0, \dots, n-1$, and $z_n^{(n+1)} = z_n^{(n)} + y_n$. Then, it is clear that the element $\underline{y} := \sum_{n\in\mathbb{N}} p^n \cdot \tau_{\mathbf{E}_U}(y_n)$ will do.

This shows that $M \subset W(\mathscr{K}, 1)^c$. The converse inclusion is obvious, since $\prod_{n\in\mathbb{N}}(\mathscr{K}^{p^n})^c$ is a closed subset of $W(\mathbf{E}_U)$ that contains $W(\mathscr{K}, 1)$ and the topology of $W(\mathbf{E}_U, 1)$ is finer than the one induced by $W(\mathbf{E}_U)$.

(iii): Recall that $\varphi_U^\flat(\alpha_0) = \overline{u}_{\overline{A}}(\alpha_0) = pv$ for an element $v \in \overline{A}^\times$. Then, since $\mathscr{K}$ is taut, we see that $p^n \cdot \varphi_U^\flat(x^{1/p^n}) = v^{-n}\varphi_U^\flat(\alpha_0^n x^{1/p^n}) \in \{\mathscr{K}\}$ for every $x \in \mathscr{K}^{p^n}$. It follows easily that $u_U(\underline{x}) \in \{\mathscr{K}\}$ for every $\underline{x} \in W(\mathscr{K})$, and thus $u_U(W(\mathscr{K}, 1)^c) \subset \{\mathscr{K}\}$.

For the converse inclusion, pick a strictly positive $\varepsilon \in \mathbb{N}[1/p]$ such that $\alpha_0 \in I^{\langle\varepsilon\rangle}$; it was observed in the proof of lemma 16.5.29(iii) that $\varphi_U^\flat$ induces an $\overline{A}/p\overline{A}$-linear isomorphism

$$(\mathrm{Fil}^s\mathscr{K})/(\mathrm{Fil}^{s+\varepsilon}\mathscr{K}) \xrightarrow{\sim} (\mathrm{Fil}^s\{\mathscr{K}\})/(\mathrm{Fil}^{s+\varepsilon}\{\mathscr{K}\}) \qquad \text{for every } s \in \mathbb{Z}[1/p]$$

and moreover $\mathrm{Fil}^s\{\mathscr{K}\} = \{\mathscr{K}\} \cap \{\mathrm{Fil}^s\mathbf{E}_U\} = \{\mathscr{K}\} \cap \mathrm{Fil}^s A_U$ for every $s \in A_U$ (lemma 16.5.29(iii) and corollary 16.5.42), hence the filtration $\mathrm{Fil}^\bullet\{\mathscr{K}\}$ is exhaustive and separated on $\{\mathscr{K}\}$. Thus, say that $x \in \mathrm{Fil}^s\{\mathscr{K}\}$ for some $s \in \mathbb{Z}[1/p]$; it follows easily that we may find a sequence $(y_n \mid n \in \mathbb{N})$ of elements of $\mathbf{E}_U$ such that $y_n \in \mathrm{Fil}^{s+n\varepsilon}\mathscr{K}$ for every $n \in \mathbb{N}$, and the series $\sum_{n\in\mathbb{N}} \varphi_U^\flat(y_n)$ converges to x in the topology of A_U. Clearly $\tau_{\mathbf{E}_U}(y_n) \in W(\mathscr{K})$ and $|\tau_{\mathbf{E}_U}(y_n)|_1 \leq \rho^{s+n\varepsilon}$ for every $n \in \mathbb{N}$, so the series $\sum_{n\in\mathbb{N}} \tau_{\mathbf{E}_U}(y_n)$ converges in $W(\mathbf{E}_U, 1)$ to an element $\underline{z} \in W(\mathscr{K})^c$, and finally $u_U(\underline{z}) = \sum_{n\in\mathbb{N}} u_U \circ \tau_{\mathbf{E}_U}(y_n) = x$.

(iv): The assumptions on $\mathscr{K}$ imply that $\mathscr{K}^{p^n}$ is topologically closed in $\mathbf{E}_U$, for every $n \in \mathbb{N}$; combining with (ii) and (iii), it follows already that the induced map $W(\mathscr{K}, 1)/\underline{\alpha}W(\mathscr{K}, 1) \to \{\mathscr{K}\}$ is surjective. Next, say that $\alpha_0 \in I^{\langle\varepsilon\rangle}$ for some element $\varepsilon > 0$ of $\mathbb{Z}[1/p]$; notice that $\mathrm{Fil}^s\mathscr{K}$ is still 1-taut and topologically closed in $\mathbf{E}_U$ for every $s \in \mathbb{Z}[1/p]$; taking into account lemma 16.5.29(iii), we may therefore reduce to the case where $\mathscr{K} \subset \mathrm{Fil}^{-n\varepsilon}\mathbf{E}_U$ for some $n \in \mathbb{N}$. We argue by induction on n : indeed, the assertion for $n = 0$ is already known, by theorem 16.3.36(iv). Thus, suppose that $n > 0$, and that the assertion is already known for $\mathscr{K} \cap \mathrm{Fil}^{(1-n)\varepsilon}\mathbf{E}_U$; let $\underline{x} := (x_n \mid n \in \mathbb{N}) \in W(\mathscr{K}, 1)$ be an element in the kernel of u_U. This means that $\sum_{n\in\mathbb{N}} p^i \cdot \varphi_U^\flat(x_n^{1/p^n}) = 0$ in A_U, and we need to check that $\underline{x} \in \underline{\alpha}W(\mathscr{K}, 1)$. Now, since $\mathscr{K}$ is 1-taut, we have $\underline{z} := (x_{n+1}^{1/p} \mid n \in \mathbb{N}) \in W(\mathscr{K}, 1)$ as well, so that

$$\varphi_U^\flat(x_0) = -\sum_{n\in\mathbb{N}} p^{n+1} \cdot \varphi_U^\flat(x_{n+1}^{1/p^{n+1}}) \in \mathrm{Fil}^{(1-n)\varepsilon}A_U.$$

In view of claim 16.5.30(ii), we deduce that $x_0 \in \mathrm{Fil}^{(1-n)\varepsilon}\mathscr{K}$. Moreover, $\underline{x} = \tau_{\mathbf{E}_U}(x_0) + p \cdot \underline{z}$, and recall that $\underline{\alpha} = \tau_{\mathbf{E}}(\alpha_0) + p \cdot \underline{u}$ for an invertible element $\underline{u} \in W(\mathbf{E})$; we conclude that $\underline{x}$ is the sum of

$$\tau_{\mathbf{E}_U}(x_0) - \underline{u}^{-1} \cdot \tau_{\mathbf{E}}(\alpha_0) \cdot \underline{z} \in W(\mathrm{Fil}^{(1-n)\varepsilon}\mathscr{K}, 1) \qquad \text{and} \qquad \underline{u}^{-1} \cdot \underline{\alpha} \cdot \underline{z} \in \underline{\alpha}W(\mathscr{K}, 1).$$

Especially, $\tau_{\mathbf{E}_U}(x_0) - \underline{u}^{-1} \cdot \tau_{\mathbf{E}}(\alpha_0) \cdot \underline{z}$ lies as well in the kernel of u_U; by inductive assumption it is therefore an element of $\underline{\alpha}W(\mathrm{Fil}^{(1-n)\varepsilon}\mathscr{K}, 1)$, and the proof is concluded. □

Proposition 16.5.46. *Let A be any complete and separated topological ring, whose topology is linear and coarser than the p-adic topology, and $v \in \mathrm{Cont}(A)$ any valuation. We have :*

(i) *The mapping $v \circ \overline{u}_A$ is a continuous valuation of the topological ring $\mathbf{E} := \mathbf{E}(A)$ (see remark* 9.4.12(ii)*).*

(ii) *The rule : $v \mapsto v \circ \overline{u}_A$ defines continuous maps*

$$\mathrm{Cont}(\overline{u}_A) : \mathrm{Cont}(A) \to \mathrm{Cont}(\mathbf{E}) \qquad \mathrm{Cont}^+(\overline{u}_A) : \mathrm{Cont}^+(A) \to \mathrm{Cont}^+(\mathbf{E}).$$

(iii) *The diagrams of continuous maps*

$$\begin{array}{ccc} \mathrm{Cont}(A) & \xrightarrow{\mathrm{Cont}(\overline{u}_A)} & \mathrm{Cont}(\mathbf{E}) \\ \downarrow & & \downarrow \\ \mathrm{Spec}\,A & \xrightarrow{\mathrm{Spec}\,\overline{u}_A} & \mathrm{Spec}\,\mathbf{E} \end{array} \qquad \begin{array}{ccc} \mathrm{Cont}^+(A) & \xrightarrow{\mathrm{Cont}^+(\overline{u}_A)} & \mathrm{Cont}^+(\mathbf{E}) \\ \downarrow & & \downarrow \\ \mathrm{Spec}\,A & \xrightarrow{\mathrm{Spec}\,\overline{u}_A} & \mathrm{Spec}\,\mathbf{E} \end{array}$$

commute, where $\mathrm{Spec}\,\overline{u}_A$ is defined as in (16.1) *and the vertical arrows in the left (resp. right) diagram are the restrictions of the support maps σ_A and $\sigma_{\mathbf{E}}$ (resp. of the center maps σ_A^+ and $\sigma_{\mathbf{E}}^+$) : see remark* 9.2.4(iii,v)*.*

(iv) *If $v' \in \mathrm{Cont}(A)$ is a primary (resp. secondary) specialization of v, then $\mathrm{Cont}(\overline{u}_A)(v')$ is a primary (resp. secondary) specialization of $\mathrm{Cont}(\overline{u}_A)(v)$.*

(v) *The map $\mathrm{Cont}(\overline{u}_A)$ restricts to a surjection from the set of secondary specializations of v onto the set of secondary specializations of $\mathrm{Cont}(\overline{u}_A)(v)$.*

(vi) *If A is a P-ring, $\mathrm{Cont}(\overline{u}_A)$ also restricts to a surjection from the set of secondary generizations of v onto the set of secondary generizations of $\mathrm{Cont}(\overline{u}_A)(v)$.*

Proof. (i): Since both v and $\overline{u}_A$ are continuous morphisms of pointed monoids, the same holds for $v \circ \overline{u}_A$. Thus, it remains only to check that

$$v(\overline{u}_A(a_\bullet + b_\bullet)) \leq \max(v(a_0), v(b_0)) \qquad \text{for every } a_\bullet, b_\bullet \in \mathbf{E}.$$

However, recall that

$$\overline{u}_A(a_\bullet + b_\bullet) = \lim_{n \to +\infty} (a_n + b_n)^{p^n}$$

(remark 9.4.12(ii)). Since the pointed monoid $\Gamma_{v\circ}$ is separated for its topology $\mathscr{T}_{\Gamma_v}$ (notation of definition 15.3.12(i)), it follows that $v(\overline{u}_A(a_\bullet + b_\bullet))$ is the unique limit point of the sequence $(v(a_n + b_n)^{p^n} \mid n \in \mathbb{N})$. However :

$$v(a_n + b_n)^{p^n} \leq \max(v(a_n)^{p^n}, v(b_n)^{p^n}) = \max(v(a_0), v(b_0))$$

whence the assertion.

(ii): Let $a_\bullet, b_\bullet \in \mathbf{E}$ be any two elements, and set $U := \{v \in \mathrm{Cont}(\mathbf{E}) \mid v(a_\bullet) \leq v(b_\bullet) \neq 0\}$; directly from the definition we find :

$$\mathrm{Cont}(\overline{u}_A)^{-1}U = \{w \in \mathrm{Cont}(A) \mid w(a_0) \leq w(b_0) \neq 0\}$$

whence the assertion.

(iii) and (iv) are immediate from the definitions.

(v): Set $w := \mathrm{Cont}(\overline{u}_A)(v)$ and notice that $\mathfrak{p} := \sigma_A(v)$ is a closed subset in the p-adic topology of A; we remark the following :

Claim 16.5.47. Denote by $\overline{v}$ (resp. $\overline{w}$) the residual valuation of v (resp. of w). Then we have :

(i) The map $\overline{u}_{\mathfrak{p}} : \kappa(w) \to \kappa(v)$ of lemma 16.1.5 restricts to a local morphism of multiplicative monoids $\overline{u}_v^+ : (\kappa(w), \overline{w})^+ \to (\kappa(v), \overline{v})^+$.

(ii) Let also $\overline{\kappa}(v)$ (resp. $\overline{\kappa}(w)$) be the residue field of $\kappa(v)^+$ (resp. of $\kappa(w)^+$). There exists a unique ring homomorphism $\overline{\varphi}_v : \overline{\kappa}(w) \to \overline{\kappa}(v)$ fitting into a commutative diagram

$$\begin{array}{ccc} \kappa(w)^+ & \xrightarrow{\overline{u}_v^+} & \kappa(v)^+ \\ \downarrow & & \downarrow \\ \overline{\kappa}(w) & \xrightarrow{\overline{\varphi}_v} & \overline{\kappa}(v) \end{array}$$

whose vertical arrows are the projections.

Proof of the claim. (i): Indeed, a simple inspection of the definitions yields a commutative diagram of monoids

$$\begin{array}{ccc} \kappa(w) & \xrightarrow{\overline{u}_{\mathfrak{p}}} & \kappa(v) \\ & {\scriptstyle \overline{w}}\searrow \quad \swarrow{\scriptstyle \overline{v}} & \\ & \Gamma_v & \end{array}$$

whence the assertion : details left to the reader.

(ii): Let $\overline{\pi}_v : \kappa(v)^+ \to \overline{\kappa}(v)$ be the projection; since $\overline{u}_v^+$ is local, we are easily reduced to checking that the composition $\psi := \overline{\pi}_v \circ \overline{u}_v^+$ is a ring homomorphism. However, ψ is obviously a morphism of multiplicative monoids, so it suffices to show that

$$\overline{v}(\overline{u}_v^+(x_1 + x_2) - \overline{u}_v^+(x_1) - \overline{u}_v^+(x_2)) < 1 \qquad \text{for every } x_1, x_2 \in \kappa(w)^+.$$

However, let $\pi_w : \mathbf{E} \to \kappa(w)$ and $\pi_v : A \to \kappa(v)$ be the projections; we may write

$$x_1 = \pi_w(\beta_1)/\pi_w(\gamma) \qquad \text{and} \qquad x_2 = \pi_w(\beta_2)/\pi_w(\gamma) \qquad \text{for some } \beta_1, \beta_2, \gamma \in \mathbf{E}$$

with $w(\beta_1), w(\beta_2) \leq w(\gamma) \neq 0$. Set $x_3 := x_1 + x_2$ and $\beta_3 := \beta_1 + \beta_2$; then

$$\overline{u}_v^+(x_i) = \pi_v \circ \overline{u}_A(\beta_i)/\pi_v \circ \overline{u}_A(\gamma) \qquad \text{for } i = 1, 2, 3.$$

Let $y := \overline{u}_A(\beta_3) - \overline{u}_A(\beta_1) - \overline{u}_A(\beta_2)$; we are then reduced to checking that $v(y) < v(\overline{u}_A(\gamma))$. However, proposition 9.3.62 says that $y = \sum_{n \in \mathbb{N}} p^{n+1} z_n$ where each z_n is a finite $\mathbb{Z}_p$-linear combination of terms of the form $\overline{u}_A(\beta_1^\lambda \beta_2^{1-\lambda})$ with $\lambda, 1 - \lambda \in \mathbb{N}[1/p]$. Clearly $w(\beta_1^\lambda \beta_2^{1-\lambda}) \leq w(\gamma)$ for every such λ, so that

$$v(p^{n+1} z_n) \leq v(p)^{n+1} \cdot v(\overline{u}_A(\gamma)) \qquad \text{for every } n \in \mathbb{N}.$$

Since v is a continuous valuation and $v(p)$ is final in Γ_v, the contention follows easily : details left to the reader. ◊

Now we argue as in the proof of lemma 9.2.25(iii). Namely, let w' be a secondary specialization of w; then $\kappa(w') = \kappa(w)$, and $\kappa(w')^+ \subset \kappa(w)^+$. The image W of $\kappa(w')^+$ in $\overline{\kappa}(w)$ is a valuation ring, and by corollary 9.1.25 there exists a valuation ring V of $\overline{\kappa}(v)$ that dominates $\overline{\varphi}_v(W)$. The preimage of V in $\kappa(v)^+$ is a valuation ring of $\kappa(v)$, and the corresponding valuation v' is a secondary specialization of v. Lastly, v' is continuous, by remark 15.3.13(iv), and by construction we have $\mathrm{Cont}(\overline{u}_A)(v') = w'$.

(vi): Clearly $\overline{u}_A$ induces an injective morphism of ordered groups $i : \Gamma_w \to \Gamma_v$, and $\mathrm{Spec}\, i$ is surjective, by remark 9.1.2(v). Now, let w' be a secondary generization of $w := \mathrm{Cont}(\overline{u}_A)(v)$, so that $w' = w_\Delta$ for a convex subgroup $\Delta \subset \Gamma_w$, and we may write $\Delta = i^{-1}\Delta'$ for some convex subgroup $\Delta' \subset \Gamma_v$. If $\Delta' \neq \Gamma_{v'}$, the projection $\Gamma_{v\circ} \to (\Gamma_v/\Delta')_\circ$ is continuous (remark 15.3.13(iii)), so $v' := v_{\Delta'}$ is a secondary specialization of v in $\mathrm{Cont}(A)$ with $\mathrm{Cont}(\overline{u}_A)(v') = w'$. Hence, we may assume that w' is a trivial valuation. It follows easily that $\mathbf{E}^{\circ\circ} \subset \mathrm{Ker}\, w'$. On the other hand, we may find finitely many elements $\beta_1, \dots, \beta_k \in \mathbf{E}^{\circ\circ}$ such that the system $\overline{u}_A(\beta_1), \dots, \overline{u}_A(\beta_k)$ generates an open ideal of A (see the proof of lemma 16.2.7(i)). Summing up, we see that v' is an analytic valuation of A, especially $v' \in \mathrm{Cont}(A)$, and the proof is concluded. $\square$

Remark 16.5.48. Keep the notation of proposition 16.5.46; for every $v \in \mathrm{Cont}(A)$ we have the commutative diagram of topological monoids

$$\begin{array}{ccc} \mathbf{E} & \xrightarrow{\overline{u}_A} & A \\ {\scriptstyle \mathbf{E}(v)}\downarrow & & \downarrow{\scriptstyle v} \\ \mathbf{E}(\Gamma_{v\circ}) & \xrightarrow{\overline{u}_{\Gamma_{v\circ}}} & \Gamma_{v\circ} \end{array}$$

(where $\Gamma_{v\circ}$ is endowed with its natural topology $\mathscr{T}_{\Gamma_v}$ as in definition 15.3.12(i)) and under the natural identification $\mathbf{E}(\Gamma_{v\circ}) \xrightarrow{\sim} \mathbf{E}(\Gamma_v)_\circ$, the map $\overline{u}_{\Gamma_{v\circ}}$ corresponds to $(\overline{u}_{\Gamma_v})_\circ$. Then $\mathbf{E}(\Gamma_v)$ inherits from Γ_v a unique ordering such that $\overline{u}_{\Gamma_v}$ is an injective morphism of ordered abelian groups. Moreover, since the p-Frobenius endomorphism of Γ_v is injective, it is easily seen that the topology of $\mathbf{E}(\Gamma_{v\circ}, \mathscr{T}_{\Gamma_v})$ is the same as the topology $\mathscr{T}_{\mathbf{E}(\Gamma_v)}$ provided by definition 15.3.12(i) (details left to the reader). In view of proposition 16.5.46(i), it follows that $\mathbf{E}(v)$ is a continuous valuation on $\mathbf{E}(v)$, and we have

$$\mathrm{Cont}(\overline{u}_A)(v) = \mathbf{E}(v) \qquad \text{in } \mathrm{Cont}(\mathbf{E}).$$

Proposition 16.5.49. *Let A be any perfectoid ring. We have :*

(i) *A is a valuation ring if and only if the same holds for $\mathbf{E} := \mathbf{E}(A)$.*

(ii) *Suppose that A is a valuation ring, and let*

$$v_A : A \to \Gamma_{A\circ} := (\mathrm{Frac}\, A)/A^\times \qquad \text{and} \qquad v_{\mathbf{E}} : \mathbf{E} \to \Gamma_{\mathbf{E}\circ} := (\mathrm{Frac}\, \mathbf{E})/\mathbf{E}^\times$$

be the valuations of A and $\mathbf{E}$ (see remark 9.1.13(iv)). Then :

(a) *$v_A \in \mathrm{Cont}(A)$ and $v_{\mathbf{E}} \in \mathrm{Cont}(\mathbf{E})$.*

(b) *The map $\overline{u}_A$ induces an isomorphism of ordered abelian groups :*

$$\gamma_A : \Gamma_{\mathbf{E}} \xrightarrow{\sim} \Gamma_A$$

fitting into a commutative diagram

$$\begin{array}{ccc} \mathbf{E} & \xrightarrow{\overline{u}_A} & A \\ {\scriptstyle v_{\mathbf{E}}}\downarrow & & \downarrow{\scriptstyle v_A} \\ \Gamma_{\mathbf{E}\circ} & \xrightarrow{\gamma_{A\circ}} & \Gamma_{A\circ}. \end{array}$$

(c) *More precisely, if $(\alpha_n \mid n \in \mathbb{N})$ is any distinguished element of $\mathrm{Ker}\, u_A$, the map $\overline{u}_A$ induces a group isomorphism*

$$\mathrm{Frac}(\mathbf{E})^\times/(1 + \alpha_0 \mathbf{E}) \xrightarrow{\sim} \mathrm{Frac}(A)^\times/(1 + pA).$$

(iii) *Let V be a valuation ring, and $\mathscr{T}_V$ a linear, complete, separated and f-adic topology on V that is coarser than the p-adic topology. Suppose that :*

(a) *The value group of V is p-divisible.*

(b) *The Frobenius endomorphism of V/pV is surjective.*

Then $(V, \mathscr{T}_V)$ is a perfectoid ring.

Proof. (i): We remark :

Claim 16.5.50. If $\mathbf{E}$ is a valuation ring, then for every $a \in A$ there exist $t \in A^\times$ and $e \in \mathbf{E}$ such that $a = t \cdot \overline{u}_A(e)$.

Proof of the claim. Fix any distinguished element $(\alpha_n \mid n \in \mathbb{N})$ of $\mathrm{Ker}\, u_A$, and recall that $p = w \cdot \overline{u}_A(\alpha_0)$ for some $w \in A^\times$ (lemma 16.2.7(iii)). Since the p-adic topology is separated on A, there exists $n \in \mathbb{N}$ such that $a \in p^n A \setminus p^{n+1}A$, so that $a = p^n b$ for some $b \in A \setminus pA$. Then there exists $\beta \in \mathbf{E}$ such that $b - \overline{u}_A(\beta) = pz$ for some $z \in A$ (lemma 16.2.7(i)), and notice that $\beta \notin \alpha_0\mathbf{E}$, since otherwise we would have $\overline{u}_A(\beta) \in pA$, whence $b \in pA$, a contradiction. Thus, $\alpha_0 = \beta \cdot \gamma$ for some element γ of the maximal ideal of $\mathbf{E}$, and we may write

$$x = p^n b = p^n \cdot (\overline{u}_A(\beta) + pz) = w^n \cdot \overline{u}_A(\alpha_0^n \beta) \cdot (1 + \overline{u}_A(\gamma) \cdot wz).$$

We need to check that $u := 1 + \overline{u}_A(\gamma) \cdot wz$ is invertible in A; to this aim, since p lies in the Jacobson radical of A, it suffices to show that the image of u is invertible in A/pA, and we are further reduced to checking that the image $\overline{\gamma}$ of $\overline{u}_A(\gamma)$ lies in the Jacobson radical of A/pA. However, u_A induces a ring isomorphism $\omega : \mathbf{E}/\alpha_0\mathbf{E} \xrightarrow{\sim} A/pA$ (remark 16.3.7(ii)), and $\omega^{-1}(\overline{\gamma})$ is the class of γ, whence the contention. ◊

Now, suppose that $\mathbf{E}$ is a valuation ring; directly from claim 16.5.50 and corollary 16.3.63(ii) we deduce that A is a domain. Then, let $a_1, a_2 \in A$ be any two elements, and write $a_i = t_i \cdot \overline{u}_A(e_i)$ with $t_i \in A^\times$ and $e_i \in \mathbf{E}$ for $i = 1, 2$; we may assume that $e_1 \in e_2\mathbf{E}$, whence $a_1 \in a_2 A$, which easily implies that A is a valuation ring.

Conversely, suppose that A is a valuation ring; if the field of fractions K of A has characteristic 0, we know already from lemma 9.4.16 that $\mathbf{E}$ is a valuation ring. If the characteristic of K equals p, the ring $\mathbf{E}$ is isomorphic to A, whence the assertion.

(ii.a): In light of proposition 9.1.16(i,ii) we see that either the topology $\mathscr{T}_A$ of A is discrete, or else it agrees with the valuation topology of A. In either case, it is clear that $v_A \in \mathrm{Cont}(A)$. The same applies to $\mathbf{E}$, since the latter is perfectoid as well.

(ii.b): From claim 16.5.50 we immediately see that γ_A is surjective, and the injectivity follows from remark 16.5.48.

(ii.c): By (ii.b) and the snake lemma, we are reduced to showing that $\overline{u}_A$ induces a group isomorphism

$$\mathbf{E}^\times/(1 + \alpha_0\mathbf{E}) \xrightarrow{\sim} A^\times/(1 + pA).$$

However, $\overline{u}_A$ induces a ring isomorphism $\mathbf{E}/\alpha_0\mathbf{E} \xrightarrow{\sim} A/pA$ (remark 16.3.7(ii)), whence a group isomorphism $(\mathbf{E}/\alpha_0\mathbf{E})^\times \xrightarrow{\sim} (A/pA)^\times$, and it remains only to observe that the natural maps

$$\mathbf{E}^\times/(1 + \alpha_0\mathbf{E}) \to (\mathbf{E}/\alpha_0\mathbf{E})^\times \qquad A^\times/(1 + pA) \to (A/pA)^\times$$

are isomorphisms, since $\alpha_0\mathbf{E}$ (resp. pA) lies in the Jacobson radical of $\mathbf{E}$ (resp. of A).

(iii): In light of proposition 9.1.16(i), we see that $\mathscr{T}_V$ is either the discrete topology or the valuation topology on V. If $\mathscr{T}_V$ is discrete and coarser than the p-adic topology, then V must be an $\mathbb{F}_p$-algebra, so in this case the assertion is a special case of example 16.3.2(i). If $\mathscr{T}_V$ is the valuation topology, let us endow $K := \mathrm{Frac}\, V$ with its valuation topology $\mathscr{T}_K$; then $(K, \mathscr{T}_K)$ is a Tate ring, by proposition 9.1.16(ii), hence V admits an ideal of adic definition of the form $I := aV$, for a non-zero topologically nilpotent element $a \in V$ (corollary 8.3.20(ii)). Since the value group of V is p-divisible, and p is topologically nilpotent in V, we may assume that $p \in I^p$, in which case, assumption (iii.b) implies that the Frobenius endomorphism of V/pV induces an isomorphism $V/I \xrightarrow{\sim} V/I^p$. By the same token, we easily see that $(V, \mathscr{T}_V)$ is a P-ring, so the assertion follows from theorem 16.4.1. □

Remark 16.5.51. Let A, A' be two perfectoid valuation rings with valuations $v : A \to \Gamma_{A^\circ}$, $v' : A' \to \Gamma_{A'^\circ}$, and $f : A \to A'$ a continuous ring homomorphism; according to proposition 16.5.49(i), both $\mathbf{E} := \mathbf{E}(A)$ and $\mathbf{E}' := \mathbf{E}(A')$ are valuation rings, and we denote by $v_\mathbf{E} : \mathbf{E} \to \Gamma_{\mathbf{E}^\circ}$, $v_{\mathbf{E}'} : \mathbf{E}' \to \Gamma_{\mathbf{E}'^\circ}$ their respective valuations. Then it follows easily from proposition 16.5.49(ii.c) that f is injective if and only if the same holds for $\mathbf{E}(f) : \mathbf{E} \to \mathbf{E}'$. Moreover, if

f is injective there exist unique group homomorphisms $\varphi_{\mathbf{E}} : \Gamma_{\mathbf{E}} \to \Gamma_{\mathbf{E}'}$, $\varphi_A : \Gamma_A \to \Gamma_{A'}$ fitting into a commutative diagram :

$$\Gamma_{\mathbf{E}^\circ} \xrightarrow{\varphi_{\mathbf{E}^\circ}} \Gamma_{\mathbf{E}'^\circ}$$

$$\begin{array}{ccccccc} \Gamma_{\mathbf{E}^\circ} & \xleftarrow{v_{\mathbf{E}}} & \mathbf{E} & \xrightarrow{\mathbf{E}(f)} & \mathbf{E}' & \xrightarrow{v_{\mathbf{E}'}} & \Gamma_{\mathbf{E}'^\circ} \\ \downarrow{\scriptstyle \gamma_{A^\circ}} & & \downarrow{\scriptstyle \overline{u}_A} & & \downarrow{\scriptstyle \overline{u}_{A'}} & & \downarrow{\scriptstyle \gamma_{A'^\circ}} \\ \Gamma_{A^\circ} & \xleftarrow{v_A} & A & \xrightarrow{f} & A' & \xrightarrow{v_{A'}} & \Gamma_{A'^\circ} \end{array}$$

$$\Gamma_{A^\circ} \xrightarrow{\varphi_{A^\circ}} \Gamma_{A'^\circ}$$

where γ_A and $\gamma_{A'}$ are the group isomorphisms provided by proposition 16.5.49(ii.b) : details left to the reader.

Theorem 16.5.52. *Let A be any perfectoid ring, and set $\mathbf{E} := \mathbf{E}(A)$. We have :*

(i) *The induced map* $\mathrm{Cont}(\overline{u}_A)$ *of proposition* 16.5.46(ii) *is a homeomorphism.*

(ii) *For every rational subset R of* $\mathrm{Cont}(\mathbf{E})$*, the preimage* $\mathrm{Cont}(\overline{u}_A)^{-1}R$ *is a rational subset of* $\mathrm{Cont}(A)$ *(see* (15.3.16)*).*

(iii) $\mathrm{Cont}(\overline{u}_A)$ *restricts to bijections*

$$\mathrm{Cont}(A)_{\mathrm{a}} \xrightarrow{\sim} \mathrm{Cont}(\mathbf{E})_{\mathrm{a}} \qquad \textit{and} \qquad \mathrm{Cont}(A)_{\mathrm{na}} \xrightarrow{\sim} \mathrm{Cont}(\mathbf{E})_{\mathrm{na}}.$$

(iv) *Let $w \in \mathrm{Cont}(A)$ be any valuation, and set $v := \mathrm{Cont}(\overline{u}_A)(w)$. Then $\overline{u}_A$ induces group isomorphisms*

$$\gamma_w : \Gamma_v \xrightarrow{\sim} \Gamma_w \qquad \textit{and} \qquad c\Gamma_v \xrightarrow{\sim} c\Gamma_w.$$

Proof. (ii): If A is perfectoid, both $\mathrm{Cont}(A)$ and $\mathrm{Cont}(\mathbf{E})$ are spectral spaces, and their rational subsets form a basis of constructible open subsets. Thus, let $(e_i \mid i = 0, \dots, n)$ be any finite system of elements of $\mathbf{E}$ that generate an open ideal, and set $R := R_{\mathbf{E}}(\frac{e_1}{e_0}, \dots, \frac{e_n}{e_0}) \cap \mathrm{Cont}(\mathbf{E})$. Let also $a_i := \overline{u}_A(e_i)$ for every $i = 0, \dots, n$. Then

$$\mathrm{Cont}(\overline{u}_A)^{-1}R = R_A\Big(\frac{a_1}{a_0}, \dots, \frac{a_n}{a_0}\Big) \cap \mathrm{Cont}(A).$$

However, the system $(a_i \mid i = 0, \dots, n)$ generates an open ideal of A (corollary 16.3.40(ii)), whence the assertion.

(iii): Let $\underline{\alpha} := (\alpha_n \mid n \in \mathbb{N})$ be any distinguished element of $\mathrm{Ker}\, u_A$; since α_0 is topologically nilpotent, it lies in every open prime ideal of $\mathbf{E}(A/pA)$. Likewise, pA is contained in every open prime ideal of A, and due to lemmata 16.2.7(i,iii) and 16.1.1(iv), the map $\overline{u}_A$ induces an isomorphism

$$\omega : \mathbf{E}/\alpha_0\mathbf{E} \xrightarrow{\sim} A/pA$$

of topological rings, so that $\mathrm{Cont}(\overline{u}_A)$ agrees with $\mathrm{Cont}(\omega)$ on the closed subset $\mathrm{Cont}(\mathbf{E}/\alpha_0\mathbf{E})$ of $\mathrm{Cont}(\mathbf{E})$, and induces a homeomorphism

$$\mathrm{Cont}(\mathbf{E}/\alpha_0\mathbf{E}) \xrightarrow{\sim} \mathrm{Cont}(A/pA).$$

It is also clear that $\mathrm{Cont}(\omega)$ maps $\mathrm{Cont}(\mathbf{E}/\alpha_0\mathbf{E})_{\mathrm{na}} = \mathrm{Cont}(\mathbf{E})_{\mathrm{na}}$ homeomorphically onto $\mathrm{Cont}(A/pA)_{\mathrm{na}} = \mathrm{Cont}(A)_{\mathrm{na}}$, so we need only to check that the restriction

$$C(A) := \mathrm{Cont}(A) \cap R_A\Big(\frac{p}{p}\Big) \to C(\mathbf{E}) := \mathrm{Cont}(\mathbf{E}) \cap R_{\mathbf{E}}\Big(\frac{\alpha_0}{\alpha_0}\Big)$$

of $\mathrm{Cont}(\overline{u}_A)$ is a bijection. Set $C^+(A) := C(A) \cap \mathrm{Spv}^+ A$ and $C^+(\mathbf{E}) := C(\mathbf{E}) \cap \mathrm{Spv}^+\mathbf{E}$. Clearly $\mathrm{Cont}(\overline{u}_A)$ restricts to a mapping $C^+(A) \to C^+(\mathbf{E})$, and we shall first exhibit an inverse mapping

$$\sigma : C^+(\mathbf{E}) \to C^+(A)$$

for $\mathrm{Cont}(\overline{u}_A)_{|C^+(A)}$. To this aim, let $v \in C^+(\mathbf{E})$ be any valuation; then v factors through a ring homomorphism $\pi : \mathbf{E} \to V := \kappa(v)^+$ and the residual valuation $\overline{v} : V \to \Gamma_{v\circ}$ of v. Since $\mathbf{E}$ is a perfect $\mathbb{F}_p$-algebra, the field $\kappa(v)$ is perfect, hence V is a perfect $\mathbb{F}_p$-algebra as well : indeed, it is clear that $x^p \in V$ if and only if $x \in V$, for every $x \in \kappa(v)$. Let us endow V with its valuation topology $\mathscr{T}_V$, so that $\overline{v}$ is continuous for the topology $\mathscr{T}_V$ (remark 15.3.13(i)).

Claim 16.5.53. The completion $(V^\wedge, \mathscr{T}_V^\wedge)$ of $(V, \mathscr{T}_V)$ is a perfectoid valuation ring.

Proof of the claim. We know that $V^\wedge$ is a valuation ring, by proposition 9.1.16(iii), and its topology $\mathscr{T}_V^\wedge$ is adic with a principal ideal of adic definition, since the same holds for $\mathscr{T}_V$ (lemma 15.3.14(ii)). Moreover, it is easily seen that the Frobenius endomorphism Φ_V of V is a homeomorphism for the topology $\mathscr{T}_V$, hence it induces a homeomorphism $\Phi_V^\wedge$ on $V^\wedge$; but clearly $\Phi_V^\wedge$ is just the Frobenius endomorphism of $V^\wedge$, so the latter is a perfect topological $\mathbb{F}_p$-algebra, and the assertion follows (see example 16.3.2(i)). ◊

Let $\pi^\wedge : \mathbf{E} \to V^\wedge$ be the composition of π with the completion map $V \to V^\wedge$; since $\pi^\wedge$ is a continuous map of perfectoid $\mathbb{F}_p$-algebras (remark 15.3.13(i)), the image of $\underline{\alpha}$ under the map $W(\pi^\wedge)$ is still distinguished in $W(V^\wedge)$, so claim 16.5.53 and proposition 16.5.49(i,ii) say that

$$A_V := W(V^\wedge) \otimes_{W(\mathbf{E})} A$$

is a perfectoid valuation ring whose valuation v_{A_V}, is continuous, and the map $\overline{u}_{A_V}$ induces an isomorphism of ordered abelian groups from the value group of $V^\wedge$ onto that of A_V

$$\gamma_{V^\wedge} : \Gamma_{V^\wedge} \xrightarrow{\sim} \Gamma_{A_V} \qquad \text{such that} \qquad v_{A_V} \circ \overline{u}_{A_V} = \gamma_{V^\wedge\circ} \circ \overline{v}^\wedge$$

where $\overline{v}^\wedge$ is the valuation of $V^\wedge$. Moreover, we get a continuous map of perfectoid rings

$$\varphi := W(\pi^\wedge) \otimes_{W(\mathbf{E})} A : A \to A_V$$

and $w := v_{A_V} \circ \varphi$ is a continuous valuation of A such that

$$\mathrm{Cont}(\overline{u}_A)(w) = w \circ \overline{u}_A = v_{A_V} \circ \overline{u}_{V^\wedge} \circ \pi^\wedge = \gamma_{V^\wedge\circ} \circ \overline{v}^\wedge \circ \pi^\wedge = \gamma_{V^\wedge\circ} \circ v.$$

Since $v(\alpha_0) \neq 0$, we have $w(p) \neq 0$ as well, so we obtain the sought map σ, by setting $\sigma(v) := w$, and we already see that σ is a right inverse for $\mathrm{Cont}(\overline{u}_A)_{|C^+(A)}$. We notice that since $v(\mathbf{E}) \cap \Gamma_v$ generates the group Γ_v, the subset $w \circ \overline{u}_A(\mathbf{E}) \cap \Gamma_{A_V}$ generates the group Γ_{A_V}, so that $\overline{u}_A$ induces an isomorphism

$$\Gamma_v \xrightarrow{\sim} \Gamma_{\sigma(v)} \qquad \text{for every } v \in C^+(\mathbf{E}). \tag{16.5.54}$$

Next, we check that σ is a left inverse as well. To this aim, let $v' \in C^+(A)$ be any valuation; then v' factors through a ring homomorphism $\pi' : A \to V' := \kappa(v')^+$ and the residual valuation $\overline{v}' : V' \to \Gamma_{v'\circ}$. By lemma 15.3.14(ii), the map π' is continuous for the valuation topology $\mathscr{T}_{V'}$ of V', and $\mathscr{T}_{V'}$ is adic with a principal ideal of adic definition. Since $pV' \neq 0$ and p is topologically nilpotent in A, the ideal pV' is open and topologically nilpotent in V', so $\mathscr{T}_{V'}$ agrees with the p-adic topology on V'. Let $(V'^\wedge, \mathscr{T}_{V'}^\wedge)$ be the completion of $(V', \mathscr{T}_{V'})$; then $V'^\wedge$ is a valuation ring whose valuation $\overline{v}'^\wedge : V'^\wedge \to \Gamma_{v'\circ}$ extends $\overline{v}'$ (proposition 9.1.16(iii,v)), and we let $\pi'^\wedge : A \to V'^\wedge$ be the composition of π' with the completion map $V' \to V'^\wedge$. Then $V'_{\mathbf{E}} := \mathbf{E}(V'^\wedge)$ is a valuation ring whose valuation is

$$\overline{v}'_{\mathbf{E}} := \overline{v}'^\wedge \circ \overline{u}_{V'^\wedge} : V'_{\mathbf{E}} \to \Gamma_{v'\circ}$$

(lemma 9.4.16) and $\mathbf{E}(\pi'^{\wedge}) : \mathbf{E} \to V'_{\mathbf{E}}$ is a continuous ring homomorphism. Since $\pi'^{\wedge} \circ \overline{u}_A = \overline{u}_{V'^{\wedge}} \circ \mathbf{E}(\pi'^{\wedge})$, we deduce that

$$v'_{\mathbf{E}} := \overline{v}'_{\mathbf{E}} \circ \mathbf{E}(\pi'^{\wedge}) = \mathrm{Cont}(\overline{u}_A)(v') \qquad \text{in } \mathrm{Cont}(\mathbf{E}). \tag{16.5.55}$$

Claim 16.5.56. (i) The topology of $V'_{\mathbf{E}}$ agrees with the valuation topology $\mathscr{T}_{V'_{\mathbf{E}}}$.

(ii) $(V'_{\mathbf{E}}, \mathscr{T}_{V'_{\mathbf{E}}})$ is a perfectoid ring.

Proof of the claim. (i): Since the quotient topology induced by $\mathscr{T}_{V'}$ on V'/pV' is discrete, the topology of $V'_{\mathbf{E}}$ is linear, complete and separated (remark 9.4.9(i)), so it suffices to check that this topology is not discrete (proposition 9.1.16(i)). To this aim, since α_0 is topologically nilpotent in $\mathbf{E}$, it suffices to check that $\nu := \mathbf{E}(\pi'^{\wedge})(\alpha_0) \neq 0$. However :

$$\overline{u}_{V'^{\wedge}}(\nu) = \pi'^{\wedge} \circ \overline{u}_A(\alpha_0) = \pi'^{\wedge}(pu) \qquad \text{for some } u \in A^{\times}$$

(lemma 16.2.7(iii)), and since $v'(p) \neq 0$, we must have $\pi'^{\wedge}(p) \neq 0$, whence the claim.

(ii): We have just seen that ν is a non-zero topologically nilpotent element of $V'_{\mathbf{E}}$, so $\mathscr{T}_{V'_{\mathbf{E}}}$ agrees with the ν-adic topology on $V'_{\mathbf{E}}$; then the assertion follows from example 16.3.2(i). ◊

Arguing as in the foregoing we see that the image of $\underline{\alpha}$ in $W(V'_{\mathbf{E}})$ is still distinguished, so

$$A_{V'} := W(V'_{\mathbf{E}}) \otimes_{W(\mathbf{E})} A$$

is a perfectoid valuation ring whose value group is naturally identified with the value group $\Gamma_{v'_{\mathbf{E}}}$ of $v'_{\mathbf{E}}$ (proposition 16.5.49(i,ii.b) and claim 16.5.56(ii)); by the same token, we get a natural isomorphism

$$\beta : \mathbf{E}(A_{V'}) \xrightarrow{\sim} V'_{\mathbf{E}} \qquad \text{such that} \qquad \pi_{V'_{\mathbf{E}}} \circ W(\beta) = u_{A_{V'}}$$

where $\pi_{V'_{\mathbf{E}}} : W(V'_{\mathbf{E}}) \to A_{V'}$ is the projection (proposition 16.2.16(iv)). Moreover, in light of the commutative diagram

$$\begin{array}{ccc} W(\mathbf{E}) & \xrightarrow{u_A} & A \\ {\scriptstyle W(\mathbf{E}(\pi'^{\wedge}))}\downarrow & & \downarrow{\scriptstyle \pi'^{\wedge}} \\ W(V'_{\mathbf{E}}) & \xrightarrow{u_{V'^{\wedge}}} & V'^{\wedge} \end{array}$$

we get a unique homomorphism of A-algebras

$$\psi : A_{V'} \to V'^{\wedge} \qquad \text{such that} \qquad u_{V'^{\wedge}} = \psi \circ \pi_{V'_{\mathbf{E}}}.$$

There follows a diagram :

$$\begin{array}{ccccc} & & \overset{\psi}{\frown} & & \\ A_{V'} & \xleftarrow{\overline{u}_{A_{V'}} \circ \beta^{-1}} & V'_{\mathbf{E}} & \xrightarrow{\overline{u}_{V'^{\wedge}}} & V'^{\wedge} \\ {\scriptstyle v'_A}\downarrow & & \downarrow{\scriptstyle \overline{v}'_{\mathbf{E}}} & & \downarrow{\scriptstyle \overline{v}'^{\wedge}} \\ \Gamma_{v'^{\circ}_{\mathbf{E}}} & = & \Gamma_{v'^{\circ}_{\mathbf{E}}} & \xrightarrow{j} & \Gamma_{v'^{\circ}} \end{array}$$

whose two square subdiagrams commute, where v'_A is the valuation of $A_{V'}$ and j the inclusion map of $\Gamma_{v'^{\circ}_{\mathbf{E}}}$ into $\Gamma_{v'^{\circ}}$. Let also $\varphi' : A \to A_{V'}$ be the structure map of the A-algebra $A_{V'}$.

Claim 16.5.57. (i) The top subdiagram commutes as well.

(ii) The external subdiagram also commutes, *i.e.* $\overline{v}'^{\wedge} \circ \psi = j \circ v'_A$.

(iii) The topology of $A_{V'}$ agrees with its valuation topology.

(iv) ψ is an isomorphism of topological rings and $\Gamma_{v'_{\mathbf{E}}} = \Gamma_{v'}$.

Proof of the claim. (i): Indeed, taking into account lemma 16.1.1(iv) we may compute :

$$\begin{aligned}\psi\circ\overline{u}_{A_{V'}} &= \psi\circ u_{A_{V'}}\circ\tau_{\mathbf{E}(A_{V'})}\\ &= \psi\circ\pi_{V'_{\mathbf{E}}}\circ W(\beta)\circ\tau_{\mathbf{E}(A_{V'})}\\ &= u_{V'}\circ W(\beta)\circ\tau_{\mathbf{E}(A_{V'})}\\ &= u_{V'}\circ\tau_{V'_{\mathbf{E}}}\circ\beta\\ &= \overline{u}_{V'}\circ\beta.\end{aligned}$$

(ii): Let $x\in A_{V'}$ be any element; since the left square subdiagram commutes, we may find $y\in V'_{\mathbf{E}}$ such that $\overline{u}_{A_{V'}}\circ\beta^{-1}(y)\cdot u = x$ for some $u\in A^{\times}_{V'}$, and therefore $\psi(x)=\overline{u}_{V'^{\wedge}}(y)\cdot\psi(u)$. Obviously, $\psi(u)$ is invertible in $V'^{\wedge}$, so that $\overline{v}'^{\wedge}\circ\psi(x)=\overline{v}'^{\wedge}\circ\overline{u}_{V'^{\wedge}}(y)=j\circ\overline{v}'_{\mathbf{E}}(y)=j\circ v'_A(x)$, as stated.

(iii): In light of proposition 9.1.16(i), it suffices to check that the topology of $A_{V'}$ is not discrete. However, notice that by construction φ' is a continuous map; thus, if the topology of $A_{V'}$ were discrete, $\mathrm{Ker}\,\varphi'$ would be an open ideal, and then the same would hold for $\mathrm{Ker}\,\pi'^{\wedge}$. But this is absurd, since v' is analytic.

(iv): As an immediate consequence of (ii), we already see that ψ is injective. Then it is clear that φ' factors uniquely through π' and an injective map $V'\to A_{V'}$ whose composition with ψ equals the completion map. Thus, the image of ψ is a dense subring of $V'^{\wedge}$; combining with (ii), we deduce already the stated equality of value groups. Then by (iii) the topology of $A_{V'}$ agrees with the topology induced by $V'^{\wedge}$ via ψ. Since $A_{V'}$ is complete and separated, theorem 8.2.8(iii) implies that ψ is an isomorphism. ♢

From claim 16.5.57(ii) we get the following equalities in $\mathrm{Cont}(A)$:

$$\begin{aligned}\sigma\circ\mathrm{Cont}(\overline{u}_A)(v') &= \sigma(\overline{v}'^{\wedge}\circ\overline{u}_{V'^{\wedge}}\circ\mathbf{E}(\pi'^{\wedge}_{\mathbf{E}}))\\ &= \sigma(\overline{v}'_{\mathbf{E}}\circ\mathbf{E}(\pi'^{\wedge}))\\ &= v'_A\circ\varphi'\\ &= \overline{v}'^{\wedge}\circ\psi\circ\varphi'\\ &= \overline{v}'^{\wedge}\circ\pi'^{\wedge}\\ &= v'\end{aligned}$$

as required. Let $\kappa(v'_{\mathbf{E}})^{\wedge+}$ be the completion of $\kappa(v'_{\mathbf{E}})^+$ for its valuation topology, $\pi'_{\mathbf{E}}:\mathbf{E}\to\kappa(v'_{\mathbf{E}})^+$ the natural ring homomorphism (whose composition with the valuation $\kappa(v'_{\mathbf{E}})^+\to\Gamma_{v'^{\circ}}$ yields $v'_{\mathbf{E}}$), and $\pi'^{\wedge}_{\mathbf{E}}:\mathbf{E}\to\kappa(v'_{\mathbf{E}})^{\wedge+}$ the completion of $\pi'_{\mathbf{E}}$; we notice as well :

Claim 16.5.58. There exists a ring isomorphism

$$\omega^{\wedge}_{\mathbf{E}}:\kappa(v'_{\mathbf{E}})^{\wedge+}\xrightarrow{\sim}V'_{\mathbf{E}}\qquad\text{such that}\qquad\omega^{\wedge}_{\mathbf{E}}\circ\pi'^{\wedge}_{\mathbf{E}}=\mathbf{E}(\pi'^{\wedge}).$$

Proof of the claim. The map $\mathbf{E}(\pi'^{\wedge})$ factors through $\pi'_{\mathbf{E}}$ and a unique injective ring homomorphism $\omega_{\mathbf{E}}:\kappa(v'_{\mathbf{E}})^+\to V'_{\mathbf{E}}$ such that $\overline{v}'_{\mathbf{E}}\circ\omega_{\mathbf{E}}:\kappa(v'_{\mathbf{E}})^+\to\Gamma_{v'^{\circ}}$ is the residual valuation. Thus, the valuation topology of $\kappa(v'_{\mathbf{E}})^+$ agrees with the topology induced from $V'_{\mathbf{E}}$ via $\omega_{\mathbf{E}}$, so the latter extends to a continuous ring homomorphism $\omega^{\wedge}_{\mathbf{E}}:\kappa(v'_{\mathbf{E}})^{\wedge+}\to V'_{\mathbf{E}}$ such that $\overline{v}'_{\mathbf{E}}\circ\omega^{\wedge}_{\mathbf{E}}$ is the valuation of $\kappa(v'_{\mathbf{E}})^{\wedge+}$. The map $\omega^{\wedge}_{\mathbf{E}}$ is injective (proposition 8.2.13(i)), and clearly $\omega^{\wedge}_{\mathbf{E}}\circ\pi'^{\wedge}_{\mathbf{E}}=\mathbf{E}(\pi'^{\wedge})$, so it remains only to check that $\omega^{\wedge}_{\mathbf{E}}$ is bijective. To this aim, set $V'':=W(\kappa(v'_{\mathbf{E}})^{\wedge+})\otimes_{W(\mathbf{E})}A$; by the foregoing we know already that V'' is a perfectoid valuation ring with value group $\Gamma_{v'}$, and it suffices to show that

$$\omega^{\wedge}_A:=W(\omega^{\wedge}_{\mathbf{E}})\otimes_{W(\mathbf{E})}A:V''\to A_{V'}$$

is an isomorphism. However, set $\pi'^{\wedge}_A := W(\pi'^{\wedge}_{\mathbf{E}}) \otimes_{W(\mathbf{E})} A$; by remark 16.5.51, we know already that $\omega^{\wedge}_A$ is injective, and moreover we have a commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{\pi'^{\wedge}_A} & V'' & \xrightarrow{v''} & \Gamma_{v'^{\circ}} \\ & \searrow^{\varphi'} & \downarrow{\scriptstyle \omega^{\wedge}_A} & & \| \\ & & A_{V'} & \xrightarrow{v'_A} & \Gamma_{v'^{\circ}} \end{array}$$

where v'' is the valuation of V''. It follows that the valuation topology of V'' is induced by the valuation topology of $A_{V'}$ via $\omega^{\wedge}_A$. Furthermore, we deduce that $v'' \circ \pi'^{\wedge}_A = v'_A \circ \varphi' = v'$, so $\pi'^{\wedge}_A$ factors through $\pi' : A \to V'$ and an injective map $V' \to V''$ whose composition with $\omega^{\wedge}_A$ has dense image, by virtue of claim 16.5.57(iv). Since V'' is complete, it follows that $\omega^{\wedge}_A$ is surjective, whence the claim. $\Diamond$

Next, let A^+ be the smallest subring of integral elements of A (remark 15.4.2(iv)); by theorem 16.5.13(ii,iii), the ring A^+ is perfectoid, and $\mathbf{E}(A^+)$ is naturally identified with the smallest ring $\mathbf{E}^+$ of integral elements of $\mathbf{E}$. Moreover, we get a commutative diagram

$$\begin{array}{ccc} \mathrm{Cont}(A) & \xrightarrow{\mathrm{Cont}(\overline{u}_A)} & \mathrm{Cont}(\mathbf{E}) \\ {\scriptstyle \mathrm{Cont}(j_A)}\downarrow & & \downarrow{\scriptstyle \mathrm{Cont}(j_{\mathbf{E}})} \\ \mathrm{Cont}(A^+) & \xrightarrow{\mathrm{Cont}(\overline{u}_{A^+})} & \mathrm{Cont}(\mathbf{E}^+) \end{array}$$

where $j_A : A^+ \to A$ and $j_{\mathbf{E}} : \mathbf{E}^+ \to \mathbf{E}$ are the inclusion maps. Since A^+ is the integral closure in A of the $\mathbb{Z}$-subalgebra generated by $A^{\circ\circ}$, it is easily seen that $\mathrm{Cont}(A^+) = \mathrm{Cont}^+(A^+)$, and likewise for $\mathrm{Cont}(\mathbf{E}^+)$. Therefore, $C(A^+) = C^+(A^+)$ and $C(\mathbf{E}^+) = C^+(\mathbf{E}^+)$, and the foregoing shows that $\mathrm{Cont}(\overline{u}_{A^+})$ restricts to a bijection

$$C(\mathbf{E}^+) \xrightarrow{\sim} C(A^+).$$

Lastly, proposition 15.3.18(ii) says that $\mathrm{Cont}(j_A)$ restricts to a bijection from the analytic valuations of A to those of A^+, and likewise for $\mathrm{Cont}(j_{\mathbf{E}})$, whence (iii).

(iv): Let $\pi_A : A \to A/pA$ and $\pi_{\mathbf{E}} : \mathbf{E} \to \mathbf{E}/\alpha_0\mathbf{E}$ be the projections; if $w(p) = 0$, then $w = \mathrm{Cont}(\pi_A)(w')$ for a unique $w' \in \mathrm{Cont}(A/pA)$, and $v = \mathrm{Cont}(\pi_{\mathbf{E}})(v')$, where $v' := \mathrm{Cont}(\omega)(w')$. Since π_A and $\pi_{\mathbf{E}}$ are surjective, obviously $\Gamma_w = \Gamma_{w'}$ and $\Gamma_v = \Gamma_{v'}$, and likewise for the characteristic subgroups; since ω is a ring isomorphism, it induces a group isomorphism $\Gamma_{v'} \xrightarrow{\sim} \Gamma_{w'}$ that identifies the respective characteristic subgroups, whence the assertion, in this case. Next, suppose that $w \in C(A)$, and set $w' := \mathrm{Cont}(j_A)(w)$, $v' := \mathrm{Cont}(\overline{u}_{A^+})(w')$; then $w' \in C^+(A^+)$, and from (iii) and (16.5.54) we see that $\overline{u}_{A^+}$ induces an isomorphism $\gamma_{w'} : \Gamma_{w'} \xrightarrow{\sim} \Gamma_{v'}$. Moreover, we have a commutative diagram of groups

$$\begin{array}{ccc} \Gamma_{w'} & \xrightarrow{\gamma_{w'}} & \Gamma_{v'} \\ {\scriptstyle i_w}\downarrow & & \downarrow{\scriptstyle i_v} \\ \Gamma_w & \xrightarrow{\gamma_w} & \Gamma_v \end{array}$$

where i_w and i_v are the injective maps induced by j_A and $j_{\mathbf{E}}$. In order to prove that γ_w is an isomorphism, it suffices therefore to check that i_w and i_v are surjective. To this aim, it suffices to show that $S_{A^+} := w(A^+) \cap \Gamma_w$ generates Γ_w and $S_{\mathbf{E}^+} := v(A^+) \cap \Gamma_v$ generates Γ_v. However, notice that $pA \subset A^+$ and $\alpha_0\mathbf{E} \subset \mathbf{E}^+$; since $w(p) \neq 0$ and $v(\alpha_0) \neq 0$, it follows easily that S_{A^+} and $S_A := w(A) \cap \Gamma_w$ generate the same subgroup of Γ_w, and likewise for $S_{\mathbf{E}^+}$. To conclude, it suffices to remark that S_A generates Γ_w, and likewise for Γ_v.

Next, obviously γ_w restricts to an injective group homomorphism $c\Gamma_v \to c\Gamma_w$. To show that this map is surjective, consider any $a \in A$ such that $w(a) > 1$; since $\overline{u}_{A/pA}$ is surjective (lemma

16.2.7(i)), we may write $a = \overline{u}_A(\beta) + pb$ for some $\beta \in \mathbf{E}$ and $b \in A$, and since $pb \in A^{\circ\circ}$, we have $w(pb) < 1$, so that $w(a) = v(\beta)$, whence the contention.

(i): We argue as in the proof of proposition 15.3.19 : by (ii) and (iii) we know already that $\mathrm{Cont}(\overline{u}_A)$ is a spectral and bijective map of spectral spaces, so it suffices to check that $\mathrm{Cont}(\overline{u}_A)$ is a closed map, and by proposition 8.1.47(i) and corollary 8.1.50(i) we are further reduced to showing that $\mathrm{Cont}(\overline{u}_A)$ is specializing. Hence, let $w \in \mathrm{Cont}(A)$ be any valuation, and $v' \in \mathrm{Cont}(A)$ any specialization of $v := \mathrm{Cont}(\overline{u}_A)(w)$; we may find a valuation v'' of $\mathbf{E}$ that is both a secondary specialization of v and a primary generization of v', and $v'' \in \mathrm{Cont}(\mathbf{E})$, by remark 15.3.13(iv). By proposition 16.5.46(v), there exists a secondary specialization w'' of w such that $\mathrm{Cont}(\overline{u}_A)(w'') = v''$. By (iv) we know that w'' and v'' have the same value groups and characteristic subgroups, so we may find a primary specialization w' of w such that $\mathrm{Cont}(\overline{u}_A)(w') = v'$, whence the assertion. □

16.5.59. Let $\underline{X}$ be any perfectoid quasi-affinoid scheme, and with the notation of (16.5.19) and (16.5.20), let us set

$$\underline{A} := (A^\circ_X, A^+_X, X) := \Gamma^\circ(\underline{X}) \qquad \text{and} \qquad \underline{\mathbf{E}} := (\mathbf{E}^\circ_X, \mathbf{E}^+_X, X_{\mathbf{E}}) := \mathbf{E}(\underline{A}).$$

Especially, $\mathbf{E}^\circ_X = \mathbf{E}(A^\circ_X)$ and $\mathbf{E}^+_X = \mathbf{E}(A^+_X)$; it is then clear that $\mathrm{Cont}(\overline{u}_{A^\circ_X})$ maps the subset $\mathrm{Spa}\,(A^\circ_X, A^+_X)$ into $\mathrm{Spa}\,(\mathbf{E}^\circ_X, \mathbf{E}^+_X)$. Conversely, let $v : A^\circ_X \to \Gamma_v$ be any continuous valuation such that $v \circ \overline{u}_{A^\circ_X} \in \mathrm{Spa}\,(\mathbf{E}^\circ_X, \mathbf{E}^+_X)$, and $\underline{\alpha}$ any distinguished element of $\mathrm{Ker}\, u_{A^\circ_X}$; for every $a \in A^+_X$ there exists $b \in A^+_X$ and $e \in \mathbf{E}^+_X$ such that $a = \overline{u}_{A^\circ_X}(e) + \alpha_0 b$, and since $\alpha_0 b \in A^{\circ\circ}_X$, we deduce that $v(a) \leq 1$, so $v \in \mathrm{Spa}\,(A^\circ_X, A^+_X)$. Combining with proposition 16.5.46(iii), we deduce that $\mathrm{Cont}(\overline{u}_{A^\circ_X})$ restricts to a homeomorphism

$$\mathrm{Spa}\,(\overline{u}_{\underline{A}}) : \mathrm{Spa}\,\underline{A} \xrightarrow{\sim} \mathrm{Spa}\,\underline{\mathbf{E}}$$

and taking into account the natural identifications $\mathrm{Spa}\,\underline{A} \xrightarrow{\sim} \mathrm{Spa}\,\underline{X}$ and $\mathrm{Spa}\,\underline{\mathbf{E}} \xrightarrow{\sim} \mathrm{Spa}\,\mathbf{E}(\underline{X})$, we get as well an induced homeomorphism

$$\mathrm{Spa}(\overline{u}_{\underline{X}}) : \mathrm{Spa}\,\underline{X} \xrightarrow{\sim} \mathrm{Spa}\,\mathbf{E}(\underline{X}).$$

Moreover, it is easily seen that every morphism $f : \underline{Y} \to \underline{X}$ of perfectoid quasi-affinoid schemes yields a commutative diagram

$$\begin{array}{ccc} \mathrm{Spa}\,\underline{Y} & \xrightarrow{\mathrm{Spa}(\overline{u}_{\underline{Y}})} & \mathrm{Spa}\,\mathbf{E}(\underline{Y}) \\ {\scriptstyle \mathrm{Spa}\,f}\downarrow & & \downarrow{\scriptstyle \mathrm{Spa}\,\mathbf{E}(f)} \\ \mathrm{Spa}\,\underline{X} & \xrightarrow{\mathrm{Spa}(\overline{u}_{\underline{X}})} & \mathrm{Spa}\,\mathbf{E}(\underline{X}). \end{array} \tag{16.5.60}$$

Corollary 16.5.61. *In the situation of* (16.5.59)*, let $x \in \mathrm{Spa}\,\underline{X}$ be any element and set $y := \mathrm{Spa}\,(\overline{u}_{\underline{X}})(x)$. Denote also by $\pi^{\wedge+}_x : A^+_X \to \kappa(x)^{\wedge+}$ and $\pi^{\wedge+}_y : \mathbf{E}^+_X \to \kappa(y)^{\wedge+}$ the natural maps (notation of* (15.5.12)*), and endow $\kappa(x)^{\wedge+}$ and $\kappa(y)^{\wedge+}$ with the unique ring topologies $\mathscr{T}^\wedge_x$ and $\mathscr{T}^\wedge_y$ such that $\pi^{\wedge+}_x$ and $\pi^{\wedge+}_y$ are adic ring homomorphisms. We have :*

(i) *The topological rings $(\kappa(x)^{\wedge+}, \mathscr{T}^\wedge_x)$ and $(\kappa(y)^{\wedge+}, \mathscr{T}^\wedge_y)$ are perfectoid valuation rings.*
(ii) *There exists a unique isomorphism of topological rings :*

$$\omega^{\wedge+} : \mathbf{E}(\kappa(x)^{\wedge+}) \xrightarrow{\sim} \kappa(y)^{\wedge+} \qquad \textit{such that} \qquad \omega^{\wedge+} \circ \mathbf{E}(\pi^{\wedge+}_x) = \pi^{\wedge+}_y.$$

Proof. Suppose first that $p \in \mathrm{Ker}\,\pi^+_x$; in this case x corresponds to a continuous ring homomorphism $\overline{\pi}_x : A^\circ_X/pA^\circ_X \to \kappa(x)$ and y corresponds to the continuous ring homomorphism $\overline{\pi}_y := \overline{\pi}_x \circ \varphi : \mathbf{E}^\circ_X/\alpha_0\mathbf{E}^\circ_X \to \kappa(x)$, where α_0 is as in (16.5.59) and $\varphi : \mathbf{E}^\circ_X/\alpha_0\mathbf{E}^\circ_X \xrightarrow{\sim} A^\circ_X/pA^\circ_X$

is the isomorphism of topological rings induced by $\overline{u}_{A^\circ_X}$. There follows a unique isomorphism of valued fields $\omega : \kappa(y) \xrightarrow{\sim} \kappa(x)$ fitting into the commutative diagram

$$\begin{array}{ccc} \mathbf{E}^\circ_X/\alpha_0\mathbf{E}^\circ_X & \xrightarrow{\ \varphi\ } & A^\circ_X/pA^\circ_X \\ {\scriptstyle\overline{\pi}_y}\downarrow & & \downarrow{\scriptstyle\overline{\pi}_x} \\ \kappa(y) & \xrightarrow{\ \ \omega\ \ } & \kappa(x). \end{array}$$

Let us now endow $\kappa(x)^+$ and $\kappa(y)^+$ with the unique ring topologies such that the natural maps $\pi^+_x : A^+_X \to \kappa(x)^+$ and $\pi^+_y : \mathbf{E}^+_X \to \kappa(y)^+$ are adic; then clearly ω restricts to an isomorphism $\omega^+ : \kappa(y)^+ \xrightarrow{\sim} \kappa(x)^+$ of topological rings, and assertion (ii) follows straightforwardly in this case. Moreover, since $\mathbf{E}$ is a perfect $\mathbb{F}_p$-algebra, the same holds for its quotient $\mathbf{E}/\mathrm{Ker}\,\pi_y$, and then also for $\kappa(y) = \mathrm{Frac}\,(\mathbf{E}/\mathrm{Ker}\,\pi_y)$, and for $\kappa(y)^\wedge$ as well (example 9.3.48(ii)). By remark 9.4.9(v), it follows easily that $\kappa(y)^{\wedge+}$ is a perfect adic and f-adic topological ring (details left to the reader), and hence it is perfectoid (example 16.3.2(i)); the same then holds for $\kappa(x)^{\wedge+}$.

Next, if $\pi^+_x(p) \neq 0$, the point x is analytic, and the same then holds for y. In this case, it is easily seen that the topologies $\mathscr{T}^\wedge_x$ and $\mathscr{T}^\wedge_y$ are separated and not discrete, so they coincide with the respective valuation topologies (proposition 9.1.16(i)). Recall that the inclusion map $i : A^+_X \to A^\circ_X$ induces a homeomorphism $\mathrm{Cont}(i)_\mathrm{a} : \mathrm{Cont}(A^\circ_X)_\mathrm{a} \xrightarrow{\sim} \mathrm{Cont}(A^+_X)_\mathrm{a}$ and likewise for $\mathrm{Cont}(\mathbf{E}^\circ_X)$ (proposition 15.3.18(ii)); moreover, if $x' := \mathrm{Cont}(i)(x)$, we have a unique isomorphism of valued fields $\kappa(x') \xrightarrow{\sim} \kappa(x)$ fitting into the commutative diagram

$$\begin{array}{ccccc} A^+_X & & \xrightarrow{\qquad i \qquad} & & A^\circ_X \\ {\scriptstyle\pi^+_{x'}}\downarrow & & & & \downarrow{\scriptstyle\pi^\circ_x} \\ \kappa(x')^+ & \hookrightarrow & \kappa(x') & \xrightarrow{\ \sim\ } & \kappa(x) \end{array}$$

where π°_x and $\pi^+_{x'}$ are the natural maps, and likewise for y and its image $y' \in \mathrm{Cont}(\mathbf{E}^+_X)$. Hence we may regard x and y as elements of $\mathrm{Cont}(A^+_X)$ and respectively $\mathrm{Cont}(\mathbf{E}^+_X)$, and in this case assertions (i) and (ii) follow from claims 16.5.57(iv) and 16.5.58. ☐

16.6. **Graded perfectoid rings.** This section is mainly dedicated to the construction and investigation of certain *angular Rees algebras* that shall be useful in the study of blowing up morphisms of perfectoid formal schemes.

Proposition 16.6.1. *Let $(\Gamma, +, 0)$ be a monoid, $(A, \underline{B})$ a Γ-graded structure on the topological ring $(A, \mathscr{T})$, and $i_0 : \mathrm{gr}_0 B \to A$ the inclusion map. Suppose that A is a P-ring. Then we have :*

(i) *If the p-Frobenius endomorphism $\boldsymbol{p}_\Gamma$ of Γ is injective, there exists a finitely generated graded ideal J of B such that JA is an ideal of definition of A.*
(ii) *Suppose that i_0 is c-adic and $\boldsymbol{p}_\Gamma$ is injective. Then $\mathrm{gr}_0 B$ is a P-ring, and if A is perfectoid, the same holds for $\mathrm{gr}_0 B$.*
(iii) *If A is a perfectoid ring, $\mathrm{Ker}\, u_A$ is generated by a distinguished element in $\mathbf{A}(\mathrm{gr}_0 B)$.*

Proof. (i): By lemma 16.2.3(iv), we may write $p = b^p u$ for some $u \in A^\times$ and $b \in A$, and by the same token, $u = x^p + py$ for some $x, y \in A$, so that $p = a^p + p^2 w$, with $a := bx$ and $w := u^{-1} y$. Let now $(a_\gamma \mid \gamma \in \Gamma)$ be the sequence attached to a, as in remark 8.5.2(iii), and c (resp. d) the limit of the Cauchy net $(c_S \mid S \subset \Gamma)$ (resp. $(d_S \mid S \subset \Gamma)$) provided by lemma 8.5.6(i), where S ranges over the finite subsets of Γ, and $a^p_S = c_S + p \cdot d_S$ for every such S (with a_S defined as in remark 8.5.2(iii)). In light of proposition 8.5.3(ii), we deduce that $a^p = c + pd$, whence $p = c + pd + p^2 w$. Moreover, since the p-Frobenius map of Γ is injective, we have

$\pi'_{p\gamma}(c_S) = a^p_\gamma$ for every $S \subset \Gamma$ and every $\gamma \in S$; therefore $\pi'_{p\gamma}(c) = a^p_\gamma$ for every $\gamma \in \Gamma$, and especially, $\pi'_0(c) = a^p_0$. Consequently

$$p = a^p_0 + p \cdot d_0 + p^2 w_0$$

where $(d_\gamma \mid \gamma \in \Gamma)$ and $(w_\gamma \mid \gamma \in \Gamma)$ are the sequences attached to d and respectively w. Next, notice that a is topologically nilpotent in A, hence a_γ is topologically nilpotent in B, for every $\gamma \in \Gamma$ (lemma 8.5.6(ii)), and then the same holds for d_0. Thus $v := 1 - d_0 - pw_0$ is invertible in A, with inverse given by the convergent series $\sum_{n\in\mathbb{N}}(d_0 + pw_0)^n$, and since $\mathrm{gr}_0 B$ is closed in A, we conclude that $v \in (\mathrm{gr}_0 B)^\times$. Summing up, we arrive at the identity

$$pv = a^p_0 \qquad \text{with } v \in (\mathrm{gr}_0 B)^\times \text{ and } a_0 \in \mathrm{gr}_0 B.$$

Lastly, by proposition 8.5.11(i.a) we know already that A admits an ideal of adic definition of the form $J'A$, for some graded ideal $J' \subset B$, and we set $J := J' + a_0 B$; then it is easily seen that JA is an ideal of definition for the P-ring A.

(ii): We remark :

Claim 16.6.2. In the situation of the proposition, suppose that $\boldsymbol{p}_\Gamma$ is injective. Then we have :

(i) The Frobenius endomorphism of B/pB restricts to surjective maps

$$\mathrm{gr}_\gamma B \otimes_\mathbb{Z} \mathbb{F}_p \to \mathrm{gr}_{p\gamma} B \otimes_\mathbb{Z} \mathbb{F}_p \qquad \text{for every } \gamma \in \Gamma.$$

(ii) $\mathrm{gr}_\gamma B = 0$ for every $\gamma \in \Gamma \setminus \mathbf{E}(\Gamma)$.

Proof of the claim. (i): Say that $b \in \mathrm{gr}_\gamma B$; by lemma 16.2.3(iv) there exists $x \in A$ such that $x^p - b \in pA$. Let $(x_\gamma \mid \gamma \in \Gamma)$ be the system attached to x as in remark 8.5.2(iii); it was remarked in the proof of (i) that the system $(x^p_\gamma \mid \gamma \in \mathbb{N})$ is a Cauchy net in B, and if $y \in A$ denotes the limit of this net, then $x^p - y \in pA$, so that $y - b \in pA$ as well. Consequently,

$$b = \pi'_\gamma(b) \equiv \pi'_\gamma(y) = x^p_{p^{-1}\gamma} \pmod{p \cdot \mathrm{gr}_\gamma B}$$

if $\gamma \in p\Gamma$, and otherwise $b \in p \cdot \mathrm{gr}_\gamma B$. Assertion (i) follows already. Next, using part (i) of the claim it is easily seen that, if $\gamma \in \Gamma \setminus \mathbf{E}(\Gamma)$, we have $\mathrm{gr}_\gamma B \otimes_\mathbb{Z} \mathbb{F}_p = 0$, whence (ii), since the p-adic topology is separated on B. ◊

Claim 16.6.2 says especially that the Frobenius endomorphism of $\mathrm{gr}_0 B \otimes_\mathbb{Z} \mathbb{F}_p$ is surjective. On the other hand, we already know by proposition 8.5.11(iii) that the topology of $\mathrm{gr}_0 B$ is adic with a finitely generated ideal of adic definition I. Moreover, the proof of (i) shows that there exists $a_0 \in \mathrm{gr}_0 B$ such that $p = a^p_0 u$ for some $u \in (\mathrm{gr}_0 B)^\times$; then, it is easily seen that $\mathrm{gr}_0 B$ is a P-ring with ideal of definition $I + a_0 \cdot \mathrm{gr}_0 B$.

Lastly, suppose that A is perfectoid, and to ease notation set $B_0 := \mathrm{gr}_0 B$. By the foregoing, we know already that B_0 is a P-ring, and then i_0 is adic, by lemma 8.3.24(i.c). Pick any ideal of definition I_0 of B_0, and say that I_0 is generated by k elements, for some $k \in \mathbb{N}$; by remark 16.3.58, we can assume that $p \in I_0^{k(p-1)+1}$, and therefore $I := I_0 A$ is an open ideal of A that fulfills the same condition. Set $J_0 := I_0^{(p)}$ and $J := I^{(p)} = J_0 A$; it follows that the Frobenius endomorphism of A/pA induces an isomorphism $\Phi_I : \mathrm{gr}^\bullet_I A \to \mathrm{gr}_J A$ (propositions 16.3.8(i) and 16.3.59(ii)).

Claim 16.6.3. The natural maps

$$\mathrm{gr}^n_I B := I^n_0 B / I^{n+1}_0 B \to \mathrm{gr}^n_I A \qquad \mathrm{gr}^n_J B := J^n_0 B / J^{n+1}_0 B \to \mathrm{gr}^n_J A$$

are bijective for every $n \in \mathbb{N}$.

Proof of the claim. By claim 8.3.26 we have $(I^n B)^c = I^n A$, so that $I^n = I^n B + I^{n+1}$, whence the surjectivity of the first map. For the injectivity, we need to show that $I^{n+1}_0 B = I^n_0 B \cap I^{n+1}$. Hence, let $x \in I^n_0 B \cap I^{n+1}$; then $x = \sum_{i=1}^r x_r a_r$ for some $r \in \mathbb{N}$ and certain $x_i \in I^{n+1}_0$ and

$a_i \in A$ ($i = 1, \dots, r$). Say that $x \in \mathrm{gr}_S B := \bigoplus_{\gamma \in S} \mathrm{gr}_\gamma B$ for some finite subset $S \subset \Gamma$, and let $\pi'_S : A \to \mathrm{gr}_S B$ be the induced projection (the sum of the canonical γ-projections as in remark 8.5.2(iii), for γ ranging over the subset S). Then $x = \pi'_S(x) = \sum_{i=0}^r x_i \cdot \pi'_S(a)$, whence the contention. The same argument applies to the second map. $\diamondsuit$

From claim 16.6.3 we conclude that the map $\Phi_I : \mathrm{gr}^\bullet_I B \to \mathrm{gr}^\bullet_J B$ is also an isomorphism. But then, since $\boldsymbol{p}_\Gamma$ is injective, clearly the direct summand $\mathrm{gr}_0 \Phi_I := \Phi_{I_0} : \mathrm{gr}^\bullet_{I_0} B_0 \to \mathrm{gr}^\bullet_{J_0} B_0$ is still an isomorphism, and therefore B_0 is perfectoid, by theorem 16.3.64.

(iii): By proposition 16.3.8(i), we may assume that $\mathscr{T}$ is the p-adic topology of A. Let $j_A : A \to B'$ be as in remark 8.5.2(iii); clearly $p^n B' \cap B_0 = p^n B_0$ and $p^n A \subset j_A^{-1}(p^n B')$, so $B_0 \cap p^n A = p^n B_0$ for every $n \in \mathbb{N}$, and therefore the topology of B_0 agrees with its p-adic topology. It follows that i_0 is an adic map, and therefore B_0 is even perfectoid, by (ii); then any distinguished element of $\mathrm{Ker}\, u_{B_0}$ generates $\mathrm{Ker}\, u_A$, by proposition 16.2.23(ii). $\square$

16.6.4. Let $(\Gamma, 0, +)$ be a p-perfect monoid, $(A, \underline{B})$ a perfectoid ring with Γ-graded structure, and set $(\mathbf{E}, \underline{B}_{\mathbf{E}}) := \mathbf{E}(A, \underline{B})$. Let also $\beta_1, \dots, \beta_r$ (resp. $\gamma_1, \dots, \gamma_r$) be a finite system of elements of $\mathbf{E}$ (resp. of Γ), such that $\beta_i \in \mathrm{gr}_{\gamma_i} \underline{B}_{\mathbf{E}}$ for every $i = 1, \dots, r$. Set $\mathbf{f} := (\overline{u}_A(\beta_1), \dots, \overline{u}_A(\beta_r))$ and define the ring $R_{r,0}$ and its ideals $I_{r,0}^{\lceil s \rceil}$ for every $s \in \mathbb{R}_+$, as in (16.4.4); we endow $\underline{B}$ with the $R_{r,0}$-module structure induced by the unique ring homomorphism $u : R_{r,0} \to \underline{B}$ such that $u(T_i^{1/p^n}) := \overline{u}_A(\beta_i^{1/p^n})$ for every $i = 1, \dots, r$ and every $n \in \mathbb{N}$.

Proposition 16.6.5. *In the situation of* (16.6.4)*, suppose moreover that the translation map* $\Gamma \to \Gamma \ : \ \gamma \mapsto \gamma + \gamma_i$ *is injective for every* $i = 1, \dots, r$. *Then we have :*

(i) $\mathrm{Tor}_i^{R_{r,0}}(R_{r,0}/I_{r,0}^{\lceil s \rceil}, \underline{B}) = 0$ *for every* $i > 0$ *and every* $s \in \mathbb{R}_+$.

(ii) *The ring* $\underline{B}$ *satisfies condition* $(\mathrm{a})^{\mathrm{un}}_{\mathbf{f}}$ *of* (7.8.21).

Proof. (i): Denote by $\Gamma_0 \subset \Gamma$ the submonoid generated by $(\gamma_1/p^n, \dots, \gamma_r/p^n \mid n \in \mathbb{N})$, and let $\underline{B}_0 := \Gamma_0 \times_\Gamma \underline{B}$. Recall that $R_{r,0} := \mathbb{Z}[P_r]$, where $P_r := \mathbb{N}[1/p]^{\oplus r}$, and let $\varphi : P_r \to \Gamma_0$ be the unique morphism of monoids such that $\varphi(e_i) := \gamma_i$ for $i = 1, \dots, r$, where $(e_1, \dots, e_r)$ is the standard minimal system of generators of $\mathbb{N}^{\oplus r}$. With this notation, $R_{r,0}$ is a P_r-graded $\mathbb{Z}$-algebra, and we set $\underline{S} := (R_{r,0})_{/\Gamma_0}$ (see definition 7.6.1(v)). Then clearly u induces a morphism of Γ_0-graded $\mathbb{Z}$-algebras $\underline{S} \to \underline{B}_0$, and $\underline{B}$ can be regarded naturally as a Γ-graded $\underline{S}_{/\Gamma}$-module. We remark, quite generally :

Claim 16.6.6. Let $(\Delta, 0, +)$ be a monoid, $\Delta_0 \subset \Delta$ a submonoid such that the translation maps $\Delta \to \Delta : \delta \mapsto \delta + \delta_0$ are injective for every $\delta_0 \in \Delta_0$. Let also $\underline{S} = (S, \mathrm{gr}_\bullet S)$ be a Δ_0-graded $\mathbb{Z}$-algebra, N a Δ_0-graded $\underline{S}$-module, and M a Δ-graded $\underline{S}_{/\Delta}$-module. Then we have :

(i) The product $M' := \prod_{\delta \in \Delta} \mathrm{gr}_\delta M$ carries a natural S-module structure such that the natural map $M \to M'$ is S-linear.

(ii) The induced map $\tau_i : \mathrm{Tor}_i^S(N, M) \to \mathrm{Tor}_i^S(N, M')$ is injective, for every $i \in \mathbb{N}$.

Proof of the claim. (i): We define a $\mathbb{Z}$-bilinear map

$$\mu_{\delta_0} : \mathrm{gr}_{\delta_0} S \times M' \to M' \qquad \text{for every } \delta_0 \in \Delta_0$$

by the rule : $(s, (m_\delta \mid \delta \in \Delta)) \mapsto (m'_\delta \mid \delta \in \Delta)$, where $m'_\delta := s \cdot m_\gamma$ if there exists a (necessarily unique) $\gamma \in \Delta$ with $\gamma + \delta_0 = \delta$, and otherwise $m'_\delta := 0$. The sum of the maps μ_{δ_0} yields a well defined $\mathbb{Z}$-bilinear map

$$\mu : S \times M' \to M'$$

and it is easily seen that μ defines an S-module structure on M' with the sought property.

(ii): In view of remark 7.6.18(i), we may find a resolution $L_\bullet \to N$ by free S-modules such that L_i is a Δ_0-graded $\underline{S}$-module, and the differential $d_{i+1} : L_{i+1} \to L_i$ is a morphism of Δ_0-graded $\underline{S}$-modules, for every $i \in \mathbb{N}$. Then, for every $i \in \mathbb{N}$ the tensor product $P_i := L_i \otimes_S M$

carries a natural structure of Δ-graded $\underline{S}_{/\Delta}$-module : namely

$$\mathrm{gr}_\delta P_i := \mathrm{Im}\Big(\bigoplus_{(\delta_0,\gamma)\in D(\delta)} \mathrm{gr}_{\delta_0} L_i \otimes_{\mathbb{Z}} \mathrm{gr}_\gamma M \to P_i\Big) \qquad \text{for every } \delta \in \Delta$$

where $D(\delta)$ is the set of all pairs $(\delta_0, \gamma) \in \Delta_0 \times \Delta$ such that $\gamma + \delta_0 = \delta$. With this definition, it is then easily seen that the differential $d^P_{i+1} := d_{i+1} \otimes_S \mathbf{1}_M$ of $P_\bullet$ is a morphism of Δ-graded $\underline{S}_{/\Delta}$-modules, whence an induced Δ-graded $\underline{S}_{/\Delta}$-module structure on $\mathrm{Tor}^S_i(N, M)$, for every $i \in \mathbb{N}$. Moreover, we get a $\mathbb{Z}$-bilinear map

$$\psi : L_i \times M' \to P'_i := \prod_{\delta\in\Delta} \mathrm{gr}_\delta P_i \qquad \text{for every } i \in \mathbb{N}$$

namely, the unique one whose restriction $\mathrm{gr}_{\delta_0} L_i \times M' \to P'_i$ for every $\delta_0 \in \Delta_0$ is given by the rule : $(l, (m_\delta \mid \delta \in \Delta)) \mapsto (l \otimes m'_\delta \mid \delta \in \Delta)$, where $m'_\delta := m_\gamma$ if $\gamma + \delta_0 = \delta$, and $m'_\delta := 0$ if there is no such γ. Then it is easily seen that ψ is even S-bilinear, so it induces an S-linear map

$$L_i \otimes_S M' \to P'_i \qquad \text{for every } i \in \mathbb{N}.$$

Furthermore, the resulting diagram of S-linear maps

$$\begin{array}{ccc} L_{i+1} \otimes_S M' & \longrightarrow & P'_{i+1} \\ {\scriptstyle d_{i+1}\otimes_S \mathbf{1}_{M^\wedge}} \downarrow & & \downarrow {\scriptstyle \prod_{\delta\in\Delta} \mathrm{gr}_\delta d^P_{i+1}} \\ L_i \otimes_S M' & \longrightarrow & P'_i \end{array}$$

commutes for every $i \in \mathbb{N}$, and we deduce an S-linear map

$$\mathrm{Tor}^S_i(N, M') \to \prod_{\delta\in\Delta} \mathrm{gr}_\delta \mathrm{Tor}^S_i(N, M) \qquad \text{for every } i \in \mathbb{N}$$

whose composition with τ_i is the natural injection. The claim follows. ◊

Now, set $B' := \prod_{\gamma\in\Gamma} \mathrm{gr}_\gamma B$, endow B' with the linear topology defined as in remark 8.5.2(ii), and recall that the inclusion map $j : B \to B'$ factors through the inclusion map $B \to A$ and a continuous map $j_A : A \to B'$ (remark 8.5.2(iii)). Endow B' with the S-module structure provided by claim 16.6.6(i); a direct inspection of the proof of *loc.cit.* shows that scalar multiplication by any homogeneous element s of S defines a continuous endomorphism of B', and on the other hand we have $j(s \cdot b) = s \cdot j(b)$ for every such s and every $b \in B$; since j is also a continuous map, we deduce that the same identity holds more generally for every $b \in A$, *i.e.* j_A is an S-linear map. Lastly, notice that the ideal $I^{[s]}_{r,0}$ of $\underline{S}$ is Γ_0-graded, so $R_{r,0}/I^{[s]}_{r,0}$ is a Γ_0-graded $\underline{S}$-module for every $s \in \mathbb{R}_+$. By claim 16.6.6, it follows that the composition of the two natural maps

$$\mathrm{Tor}^{R_{r,0}}_i(R_{r,0}/I^{\lceil s\rceil}_{r,0}, \underline{B}) \to \mathrm{Tor}^{R_{r,0}}_i(R_{r,0}/I^{\lceil s\rceil}_{r,0}, A) \to \mathrm{Tor}^{R_{r,0}}_i(R_{r,0}/I^{\lceil s\rceil}_{r,0}, B')$$

is injective for every $i \in \mathbb{N}$; on the other hand, the middle term vanishes for $i > 0$, due to proposition 16.4.10(i), whence the contention.

(ii) follows from (i) and lemma 16.4.5(i). □

16.6.7. We consider now a perfectoid quasi-affinoid ring (A, A^+, U_A), and let $(\mathbf{E}, \mathbf{E}^+, U_{\mathbf{E}}) := \mathbf{E}(A, A^+, U_A)$ (see (16.5.20)). Set as well $X_A := \mathrm{Spec}\, A$, $X_{\mathbf{E}} := \mathrm{Spec}\, \mathbf{E}$, and define as usual the f-adic rings $A_U := \mathscr{O}_{X_A}(U_A)$ and $\mathbf{E}_U := \mathscr{O}_{X_{\mathbf{E}}}(U_{\mathbf{E}})$, so that the restriction maps $A \to A_U$ and $\mathbf{E} \to \mathbf{E}_U$ are open, hence the image $\overline{A}$ of A (resp. $\overline{\mathbf{E}}$ of $\mathbf{E}$) is open in A_U (resp. in $\mathbf{E}_U$), and we endow it with the topology induced by the projection $A \to \overline{A}$ (resp. $\mathbf{E} \to \overline{\mathbf{E}}$). Suppose we are given a family $(\mathscr{K}_\gamma \mid \gamma \in \Gamma)$ consisting of topologically closed 1-taut bounded $\overline{\mathbf{E}}$-submodules of $\mathbf{E}_U$, indexed by a p-perfect monoid $(\Gamma, +, 0)$, and such that (notation of (16.5.28)) :

(a) $1 \in \mathscr{K}_0$
(b) $\mathscr{K}_{p\gamma} = \mathscr{K}_\gamma^p$ for every $\gamma \in \Gamma$
(c) $\mathscr{K}_\gamma \cdot \mathscr{K}_\mu \subset \mathscr{K}_{\gamma+\mu}$ for every $\gamma, \mu \in \Gamma$

(notice that, in terms of the descending filtration $\mathrm{Fil}^\bullet \mathbf{E}_U$ defined in (16.5.23), a subset of $\mathbf{E}_U$ is bounded if and only if it lies in $\mathrm{Fil}^s \mathbf{E}_U$, for some $s \in \mathbb{Z}[1/p]$). Set

$$\mathscr{E} := \bigoplus_{\gamma\in\Gamma} \mathscr{K}_\gamma \qquad \mathscr{A} := \bigoplus_{\gamma\in\Gamma} \{\mathscr{K}_\gamma\}.$$

It is easily seen that the multiplication of $\mathbf{E}_U$ (resp. of A_U) induces a Γ-graded $\overline{\mathbf{E}}$-algebra structure (resp. $\overline{A}$-algebra structure) on $\mathscr{E}$ (resp. on $\mathscr{A}$). Choose ideals of definition J for A and $\mathscr{J}$ for $\mathbf{E}$, and endow $\mathscr{E}$ (resp. $\mathscr{A}$) with its $\mathscr{J}$-adic (resp. J-adic) topology. Let also $\mathscr{E}^\wedge$ (resp. $\mathscr{A}^\wedge$) be the separated completion of $\mathscr{E}$ (resp. of $\mathscr{A}$).

Lemma 16.6.8. *The pairs*

$$(\mathscr{E}^\wedge, \mathscr{E}) \qquad \textit{and} \qquad (\mathscr{A}^\wedge, \mathscr{A})$$

are topological rings with Γ*-graded structures.*

Proof. Indeed, the assertion comes down to checking that $\mathscr{K}_\gamma$ (resp. $\{\mathscr{K}_\gamma\}$) is a closed subset of $\mathscr{E}^\wedge$ (resp. of $\mathscr{A}^\wedge$) for every $\gamma \in \Gamma$, and notice that the topological closure of $\mathscr{K}_\gamma$ in $\mathscr{E}^\wedge$ (resp. of $\{\mathscr{K}_\gamma\}$ in $\mathscr{A}^\wedge$) is the $\mathscr{J}$-adic (resp. J-adic) completion of $\mathscr{K}_\gamma$ (resp. of $\{\mathscr{K}_\gamma\}$). Now, by assumption $\mathscr{K}_\gamma$ is bounded in $\mathbf{E}_U$, so $\{\mathscr{K}_\gamma\}$ is bounded in A_U, for every $\gamma \in \Gamma$; it follows that the topology $\mathscr{T}_\gamma$ on $\mathscr{K}_\gamma$ (resp. on $\{\mathscr{K}_\gamma\}$) induced by the inclusion into $\mathbf{E}_U$ (resp. into A_U) is coarser than the $\mathscr{J}$-adic (resp. J-adic) topology; on the other hand, $\mathscr{T}_\delta$ is complete and separated, since the same holds for the topology of $\mathbf{E}_U$ (resp. of A_U). Then lemma 8.3.12 implies that $\mathscr{K}_\gamma$ (resp. $\{\mathscr{K}_\gamma\}$) is complete and separated for its $\mathscr{J}$-adic (resp. J-adic) topology, whence the contention. □

Proposition 16.6.9. *With the notation of* (16.6.7)*, the following holds :*

(i) *The rings* $\mathscr{E}^\wedge$*, and* $\mathscr{A}^\wedge$ *are perfectoid.*
(ii) *There is a natural isomorphism of topological rings with* Γ*-graded structures*

$$\mathbf{E}(\mathscr{A}^\wedge, \mathscr{A}) \xrightarrow{\sim} (\mathscr{E}^\wedge, \mathscr{E}).$$

Proof. Taking into account remark 9.4.9(v) and example 9.3.48(ii), it is easily seen that $\mathscr{E}^\wedge$ is a perfect topological $\mathbf{E}$-algebra, hence it is perfectoid (see example 16.3.2(i)). Set

$$(\mathscr{W}^\wedge, \mathscr{W}) := W(\mathscr{E}^\wedge, \mathscr{E}) \qquad (\overline{\mathscr{W}}^\wedge, \overline{\mathscr{W}}) := (\mathscr{W}^\wedge, \mathscr{W}) \otimes_{W(\mathbf{E})} A.$$

To prove both assertions, it will therefore suffice to exhibit isomorphisms

$$(\overline{\mathscr{W}}^\wedge, \overline{\mathscr{W}}) \xrightarrow{\sim} (\mathscr{A}^\wedge, \mathscr{A}) \tag{16.6.10}$$

of topological A-algebras with Γ-graded structures. We shall first exhibit a continuous map

$$u_\mathscr{A} : (\mathscr{W}^\wedge, \mathscr{W}) \to (\mathscr{A}^\wedge, \mathscr{A}).$$

To this aim, define $W(\mathscr{K}_\gamma)$ as in lemma 16.5.45, and notice that condition (b) of (16.6.7) yields:

$$\mathrm{gr}_\gamma \mathscr{W} = W(\mathscr{K}_\gamma) \qquad \text{for every } \gamma \in \Gamma$$

and the multiplication law of $\mathscr{W}$ is the unique bilinear map $\mathscr{W} \times \mathscr{W} \to \mathscr{W}$ whose restriction

$$W(\mathscr{K}_\gamma) \times W(\mathscr{K}_\mu) \to W(\mathscr{K}_{\delta+\gamma})$$

is induced by the multiplication law of $W(\mathbf{E}_U)$, for every $\gamma, \mu \in \Gamma$. Also, since $\mathscr{K}_\gamma$ is closed and 1-taut, lemma 16.5.45(iv) says that u_U restricts to a well defined surjective map

$$\mathrm{gr}_\gamma u_\mathscr{A} : W(\mathscr{K}_\gamma) \to \mathrm{gr}_\gamma \mathscr{A} \qquad \text{for every } \gamma \in \Gamma$$

whose kernel is $\underline{\alpha}W(\mathscr{K}_\gamma)$, for any distinguished element $\underline{\alpha}$ of $\mathrm{Ker}\, u_A$. Summing up, we conclude that the direct sum of the system of maps $(\mathrm{gr}_\gamma u_{\mathscr{A}} \mid \gamma \in \Gamma)$ is a well defined surjective ring homomorphism $\mathrm{gr}_\bullet u_{\mathscr{A}}$ that makes commute the diagram

$$\begin{array}{ccc} W(\mathbf{E}) & \xrightarrow{u_A} & A \\ \downarrow & & \downarrow \\ \mathscr{W} & \xrightarrow{\mathrm{gr}_\bullet u_{\mathscr{A}}} & \mathscr{A} \end{array} \tag{16.6.11}$$

and with kernel equal to $\underline{\alpha}\mathscr{W}$. Moreover, it follows easily from propositions 9.3.77(ii) and 8.5.11(i.b) that the left vertical map of (16.6.11) is an adic ring homomorphism, and the same obviously holds for the right vertical map. Since u_A is an open map, we deduce that $\mathrm{gr}_\bullet u_{\mathscr{A}}$ is continuous and open as well; especially, it extends to a well defined continuous surjective ring homomorphism $u_{\mathscr{A}}$ as sought, whose kernel is the topological closure of $\underline{\alpha}\mathscr{W}$ in $\mathscr{W}^\wedge$ (proposition 8.2.13). However, recall that $\underline{\alpha}\mathscr{W}^\wedge$ is a closed ideal of $\mathscr{W}^\wedge$ (proposition 16.2.21(ii)), and obviously $\underline{\alpha}\mathscr{W}$ is dense in it, hence the kernel of $u_{\mathscr{A}}$ is $\underline{\alpha}\mathscr{W}^\wedge$, as required. Lastly, since $\mathrm{gr}_\bullet u_{\mathscr{A}}$ is an open map, the same holds for $u_{\mathscr{A}}$, and the proof is concluded. □

Example 16.6.12. In the situation of (16.6.7), let $\varphi^\flat_U : \mathbf{E}_U \to A_U$ be the continuous morphism of topological monoids furnished by proposition 16.4.34(i). Let also $\beta_\bullet := (\beta_\lambda \mid \lambda \in \Lambda)$ be any bounded system of elements of $\mathbf{E}_U$. We consider the unique morphism of monoids

$$\varphi_{\mathbf{E}} : P := \mathbb{N}[1/p]^{(\Lambda)} \to \mathbf{E}_U \qquad e_\lambda \mapsto \beta_\lambda \qquad \text{for every } \lambda \in \Lambda$$

where $e_\bullet := (e_\lambda \mid \lambda \in \Lambda)$ is the standard minimal set of generators of $\mathbb{N}^{(\Lambda)}$. Let also $I \subset P$ be the ideal generated by $e_\bullet$. The composition $\varphi_A : \varphi^\flat_U \circ \varphi_{\mathbf{E}} : P \to A$ is also a morphism of monoids, and we let $I^{\langle\delta\rangle}\overline{\mathbf{E}}$ (resp. $I^{\langle\delta\rangle}\overline{A}$) be the $\overline{\mathbf{E}}$-submodule of $\mathbf{E}_U$ generated by $\varphi_{\mathbf{E}}(I^{\langle\delta\rangle})$ (resp. the $\overline{A}$-submodule of A_U generated by $\varphi_A(I^{\langle\delta\rangle})$), for every $\delta \in \Delta := \mathbb{N}[1/p]$ (notation of (9.3.68)). We denote by $\mathrm{gr}_\delta\mathscr{E}$ (resp. $\mathrm{gr}_\delta\mathscr{A}$) the topological closure of $I^{\langle\delta\rangle}\overline{\mathbf{E}}$ in $\mathbf{E}_U$ (resp. of $I^{\langle\delta\rangle}\overline{A}$ in A_U) for every $\delta \in \Delta$ (notation of remark 9.3.70(i)). We set

$$\mathscr{E}(\beta_\bullet) := \bigoplus_{\delta\in\Delta} \mathrm{gr}_\delta\mathscr{E} \qquad \mathscr{A}(\beta_\bullet) := \bigoplus_{\delta\in\Delta} \mathrm{gr}_\delta\mathscr{A}.$$

Hence, $\mathscr{E}(\beta_\bullet)$ is a Δ-graded $\mathbf{E}$-algebra, and $\mathscr{A}(\beta_\bullet)$ is a Δ-graded A-algebras. We endow $\mathscr{E}(\beta_\bullet)$ with the $\mathscr{J}$-adic topology and $\mathscr{A}(\beta_\bullet)$ with the J-adic topology, and let $\mathscr{E}(\beta_\bullet)^\wedge$ and $\mathscr{A}(\beta_\bullet)^\wedge$ be the respective separated completion. It is easily seen that $\mathrm{gr}_\delta\mathscr{E}$ is 1-taut and bounded in $\mathbf{E}_U$, and moreover $\mathrm{gr}_{p\delta}\mathscr{E} = (\mathrm{gr}_\delta\mathscr{E})^p$ for every $\delta \in \Delta$; furthermore, $\mathrm{gr}_\delta\mathscr{A} = \{\mathrm{gr}_\delta\mathscr{E}\}$ for every such δ. Then proposition 16.6.9 shows that $(\mathscr{E}(\beta_\bullet)^\wedge, \mathscr{E}(\beta_\bullet))$, and $(\mathscr{A}(\beta_\bullet)^\wedge, \mathscr{A}(\beta_\bullet))$ are perfectoid rings with Δ-graded structures, and we have a natural isomorphism of topological rings with Δ-graded structures :

$$\mathbf{E}(\mathscr{A}(\beta_\bullet)^\wedge, \mathscr{A}(\beta_\bullet)) \xrightarrow{\sim} (\mathscr{E}(\beta_\bullet)^\wedge, \mathscr{E}(\beta_\bullet)).$$

Example 16.6.13. In the situation of (16.5.23), let $\Gamma := \mathbb{Z}[1/p]$; the family $(\mathrm{Fil}^\gamma\mathbf{E}_U \mid \gamma \in \Gamma)$ fulfills conditions (a)–(c) of (16.6.7), and taking into account corollary 16.5.42 and proposition 16.6.9, we get perfectoid rings with Γ-graded structures $(\mathscr{E}^\wedge_U, \mathscr{E}_U)$ and $(\mathscr{A}^\wedge_U, \mathscr{A}_U)$, by setting

$$\mathscr{E}_U := \bigoplus_{\gamma\in\Gamma} \mathrm{Fil}^\gamma\mathbf{E}_U \qquad \mathscr{A}_U := \bigoplus_{\gamma\in\Gamma} \mathrm{Fil}^\gamma A_U$$

as well as a natural isomorphism of topological rings with Γ-graded structures

$$\mathbf{E}(\mathscr{A}^\wedge_U, \mathscr{A}_U) \xrightarrow{\sim} (\mathscr{E}^\wedge_U, \mathscr{E}_U). \tag{16.6.14}$$

Moreover, by inspecting the proof of proposition 16.6.9 it is easily seen that – under the identification (16.6.14) – the restriction $\mathrm{gr}_\gamma\mathscr{E}_U \to \mathrm{gr}_\gamma\mathscr{A}_U$ of the map $\overline{u}_{\mathscr{A}^\wedge_U} : \mathscr{E}^\wedge_U \to \mathscr{A}^\wedge_U$ agrees with the restriction $\mathrm{Fil}^\gamma\mathbf{E}_U \to \mathrm{Fil}^\gamma A_U$ of the map $\varphi^\flat_U : \mathbf{E}_U \to A_U$, for every $\gamma \in \Gamma$.

16.6.15. Let Γ be a p-perfect monoid, $(A^\wedge, A)$ a perfectoid ring with Γ-graded structure, set $(\mathbf{E}^\wedge, \mathbf{E}) := \mathbf{E}(A^\wedge, A)$ and let $\underline{\alpha} \in W(\mathrm{gr}_0\mathbf{E})$ be a distinguished element of $\mathrm{Ker}\, u_A$ (see proposition 16.6.1(iii)). We denote by $\overline{u}_A : \mathbf{E} \to A$ the restriction of the map $u_{A^\wedge} : \mathbf{E}^\wedge \to A^\wedge$. For every integer $n > 0$ let

$$\lambda_n := \sum_{i=0}^{n-1} p^{-i} \qquad S_n := \{(i,j) \in \mathbb{N}[1/p]^{\oplus 2} \mid \lambda_n \leq i < n \quad \text{and} \quad 1 > j > 1 - i/n\}$$

and for every $\gamma \in \Gamma$ and every $e \in \mathrm{gr}_\gamma\mathbf{E}$, define the $\mathrm{gr}_0 A$-module

$$\mathscr{I}(e, n, \gamma) := \sum_{(i,j)\in S_n} \mathrm{gr}_{(1-j)\gamma} A \cdot \overline{u}_A(\alpha_0^i \cdot e^j).$$

Proposition 16.6.16. *With the notation of* (16.6.15)*, let $\gamma \in \Gamma$ and $a \in \mathrm{gr}_\gamma A$ be any elements. Then, for every integer $n > 0$ there exists $e \in \mathrm{gr}_\gamma\mathbf{E}$ such that*

$$a - \overline{u}_A(e) \in \mathscr{I}(e, n, \gamma) + p^n \cdot \mathrm{gr}_\gamma A.$$

Proof. Notice that the assertion is independent of the topology $\mathscr{T}_{A^\wedge}$ of $A^\wedge$, so we may assume that $\mathscr{T}_{A^\wedge}$ is the p-adic topology of $A^\wedge$ (proposition 16.3.8(i)). We argue by induction on $n \in \mathbb{N}$. For $n = 1$, we have $S_1 = \varnothing$, and the assertion means that there exists $e \in \mathrm{gr}_\gamma\mathbf{E}$ such that $a - \overline{u}_A(e) \in p \cdot \mathrm{gr}_\gamma A$. Hence, let $\pi : A \to A/pA$ be the projection; it suffices to remark :

Claim 16.6.17. The map $\pi \circ \overline{u}_A : \mathbf{E} \to A/pA$ is surjective.

Proof of the claim. The assertion follows easily from claim 16.6.2(i). ◊

Thus, suppose that the proposition has already been proven for some $n \geq 1$, for every $\gamma \in \Gamma$ and every $a \in \mathrm{gr}_\gamma A$. We notice :

Claim 16.6.18. (i) For every $\delta \in \Gamma$, every $b \in \mathrm{gr}_\delta A$ and every $k \in \mathbb{N}$ there exist a sequence $(\beta_0, \dots, \beta_k)$ of elements of $\mathrm{gr}_\delta\mathbf{E}$, and an element $d \in \mathrm{gr}_\delta A$ such that

$$b = \sum_{i=0}^{k} p^i \cdot \overline{u}_A(\beta_i) + p^{k+1} d. \qquad (16.6.19)$$

(ii) There exist elements $c \in \mathrm{gr}_\gamma A$, $e_n \in \mathrm{gr}_\gamma\mathbf{E}$, a finite set Λ, a system of positive integers $(k_\lambda \mid \lambda \in \Lambda)$, a mapping $\Lambda \to S_n : \lambda \mapsto (i_\lambda, j_\lambda)$, a system $(f_\lambda \mid \lambda \in \Lambda)$ with $f_\lambda \in \mathrm{gr}_{(1-j_\lambda)\gamma}\mathbf{E}$ for every $\lambda \in \Lambda$, such that

$$a = \overline{u}_A(e_n) + \sum_{\lambda\in\Lambda} k_\lambda \cdot \overline{u}_A(f_\lambda \cdot \alpha_0^{i_\lambda} \cdot e_n^{j_\lambda}) + p^n c. \qquad (16.6.20)$$

(iii) There exists $f \in \mathrm{gr}_0\mathbf{E}$ such that $p^n - \overline{u}_A(\alpha_0 \cdot f)^n \in p^{n+1}\mathrm{gr}_0 A$ for every $n \in \mathbb{N}$.

Proof of the claim. (i): We argue by induction on k, and notice that the assertion for $k = 0$ follows from the case $n = 1$ of the proposition, which is already known. Thus, suppose that $k \in \mathbb{N}$, and that we have already found a sequence $(\beta_0, \dots, \beta_k)$ such that (16.6.19) holds. Then we may also find $\beta_{k+1} \in \mathrm{gr}_\delta\mathbf{E}$ and $d' \in \mathrm{gr}_\delta A$ such that $d = \overline{u}_A(\beta_{k+1}) + pd'$, and replacing this expression for d in (16.6.19) we get the sought identity for $k + 1$.

(ii): By inductive assumption, there exist a finite subset $T \subset S_n$ and for every $(i,j) \in T$ an element $b_{(i,j)} \in \mathrm{gr}_{(1-j)\gamma} A$ such that

$$a = \overline{u}_A(e_n) + \sum_{(i,j)\in T} b_{(i,j)} \cdot \overline{u}_A(\alpha_0^i \cdot e_n^j) + p^n d \qquad \text{for some } e_n \in \mathrm{gr}_\gamma\mathbf{E} \text{ and } d \in \mathrm{gr}_\gamma A.$$

By (i), we may then find, for every $(i,j) \in T$, a system $(f_{(i,j),0}, \dots, f_{(i,j),n-1})$ of elements of $\mathrm{gr}_{(1-j)\gamma}\mathbf{E}$ and an element $c_{(i,j)} \in \mathrm{gr}_{(1-j)\gamma}A$ such that $b_{(i,j)} = \sum_{i=0}^{n-1} p^i \cdot \overline{u}_A(f_{(i,j),i}) + p^n c_{(i,j)}$. Hence, the sought identity holds with

$$c := d + \sum_{(i,j)\in T} c_{(i,j)} \cdot \overline{u}_A(\alpha_0^i \cdot e_n^j) \qquad \Lambda := T \times \{0, \dots, n-1\} \qquad k_{(t,s)} := p^s \quad \text{for every } (t,s) \in \Lambda$$

and with the mapping $\Lambda \to S_n : (t,k) \mapsto t$.

(iii): By lemma 16.2.7(iii), there exists $x \in A^\wedge$ such that $p = \overline{u}_A(\alpha_0) \cdot x$. Let $\pi_0^\wedge : A^\wedge \to \mathrm{gr}_0 A$ be the canonical 0-projection (see remark 8.5.2(iii)); it follows that $p = \overline{u}_A(\alpha_0) \cdot \pi_0^\wedge(x)$, so we may assume that $x \in \mathrm{gr}_0 A$. By (i), we have $y_0 := x - \overline{u}_A(f) \in p \cdot \mathrm{gr}_0 A$ for some $f \in \mathbf{E}$, and we check, by induction on n, that such f will do. Indeed, for $n = 0$ the assertion is trivial, and for $n = 1$ we have

$$p = \overline{u}_A(\alpha_0) \cdot (y_0 + \overline{u}_A(f)) = \overline{u}_A(\alpha_0 \cdot f) + y_0 \cdot \overline{u}_A(\alpha_0)$$

and $y_1 := y_0 \cdot \overline{u}_A(\alpha_0) = p y_0 \cdot \pi_0^\wedge(x^{-1}) \in p^2 \cdot \mathrm{gr}_0 A$, again by lemma 16.2.7(iii). Suppose now that $n \geq 1$, and that we have already shown that $y_n := p^n - \overline{u}_A(\alpha_0 \cdot f)^n \in p^{n+1}\mathrm{gr}_0 A$. We compute

$$\begin{aligned} p y_n &= p^{n+1} - p \cdot \overline{u}_A(\alpha_0 \cdot f)^n \\ &= p^{n+1} - (y_1 + \overline{u}_A(\alpha_0 \cdot f)) \cdot \overline{u}_A(\alpha_0 \cdot f)^n \\ &= p^{n+1} - \overline{u}_A(\alpha_0 \cdot f)^{n+1} - y_1 \cdot \overline{u}_A(\alpha_0 \cdot f)^n \end{aligned}$$

Since both $p y_n$ and $y_1 \cdot \overline{u}_A(\alpha_0 \cdot f)^n = p^n y_1 \cdot \overline{u}_A(f)^n \cdot \pi_0^\wedge(x^{-n})$ lie in $p^{n+2}\mathrm{gr}_0 A$, the assertion follows for $n+1$. ◊

Pick any identity (16.6.20) as provided by claim 16.6.18(ii), and any $f \in \mathrm{gr}_0\mathbf{E}$ such that claim 16.6.18(iii) holds; by claim 16.6.18(i) we may also find $g \in \mathrm{gr}_\gamma \mathbf{E}$ such that $c - \overline{u}_A(g) \in p \cdot \mathrm{gr}_\gamma A$, and notice that, after replacing the set Λ by $\bigcup_{\lambda\in\Lambda} \{\lambda\} \times \{1, \dots, k_\lambda\}$, we may also assume that $k_\lambda = 1$ for every $\lambda \in \Lambda$, so that

$$a = \overline{u}_A(e_n) + \sum_{\lambda\in\Lambda} \overline{u}_A(f_\lambda \cdot \alpha_0^{i_\lambda} \cdot e_n^{j_\lambda}) + \overline{u}_A(g \cdot \alpha_0^n \cdot f^n) + p^{n+1} d \qquad \text{for some } d \in \mathrm{gr}_\gamma A.$$

We let $S'_n := \{(i,j) \in \mathbb{N}[1/p]^{\oplus 2} \mid \lambda_{n+1} \leq i < n+1, \ nj + i \geq 1 + n \ \text{ and } \ j < 1\}$ and we set

$$\delta := \sum_{\lambda\in\Lambda} f_\lambda \cdot \alpha_0^{i_\lambda} \cdot e_n^{j_\lambda} \qquad \text{and} \qquad e_{n+1} := e_n + \delta + g \cdot \alpha_0^n \cdot f^n.$$

Claim 16.6.21. There exists a finite subset Λ' with a mapping $\Lambda' \to S'_n : \lambda \mapsto (i'_\lambda, j'_\lambda)$, and for every $\lambda \in \Lambda'$ an element $z_\lambda \in \mathrm{gr}_{(1-j'_\lambda)\gamma} A$ such that

$$a = \overline{u}_A(e_{n+1}) + \sum_{\lambda\in\Lambda'} z_\lambda \cdot \overline{u}_A(\alpha_0^{i'_\lambda} \cdot e_n^{j'_\lambda}) + p^{n+1} d' \qquad \text{for some } d' \in \mathrm{gr}_\gamma A.$$

Proof of the claim. For every $r = 1, \dots, n$, let Λ'_r be the subset of all sequences $\underline{\sigma} := (\sigma_0, \sigma_1, \sigma_\lambda \mid \lambda \in \Lambda)$ of elements of $p^{-r}\mathbb{N} \setminus p^{1-r}\mathbb{N}$ such that $\sigma_0 + \sigma_1 + \sum_{\lambda\in\Lambda} \sigma_\lambda = 1$, and set $\Lambda' := \bigcup_{r=1}^n \Lambda'_r$. Let also $x := a - \overline{u}_A(e_{n+1})$; according to proposition 9.3.62, there exist a mapping $\Lambda' \to \mathbb{Z}_p : \underline{\sigma} \mapsto z'_{\underline{\sigma}}$ and an element $d' \in \mathrm{gr}_\gamma A$ such that

$$\begin{aligned} x &= \sum_{r=1}^n p^r \cdot \sum_{\underline{\sigma}\in\Lambda'_r} z'_{\underline{\sigma}} \cdot \overline{u}_A\Big(e_n^{\sigma_0} \cdot g^{\sigma_1} \cdot f^{n\sigma_1} \cdot \alpha_0^{n\sigma_1} \cdot \prod_{\lambda\in\Lambda} f_\lambda^{\sigma_\lambda} \cdot \alpha_0^{\sigma_\lambda i_\lambda} \cdot e_n^{\sigma_\lambda j_\lambda}\Big) + p^{n+1} d' \\ &= \sum_{r=1}^n p^r \cdot \sum_{\underline{\sigma}\in\Lambda'_r} z'_{\underline{\sigma}} \cdot \overline{u}_A(f'_{\underline{\sigma}} \cdot \alpha_0^{i''_{\underline{\sigma}}} \cdot e_n^{j'_{\underline{\sigma}}}) + p^{n+1} d' \end{aligned}$$

where

$$f'_{\underline{\sigma}} := g^{\sigma_1} \cdot f^{n\sigma_1} \cdot \prod_{\lambda\in\Lambda} f_\lambda^{\sigma_\lambda} \qquad i''_{\underline{\sigma}} := n\sigma_1 + \sum_{\lambda\in\Lambda} \sigma_\lambda i_\lambda \qquad j'_{\underline{\sigma}} := \sigma_0 + \sum_{\lambda\in\Lambda} \sigma_\lambda j_\lambda \qquad \text{for every } \underline{\sigma} \in \Lambda'.$$

Let us remark that

$$(16.6.22)\quad j'_{\underline{\sigma}} + \frac{i''_{\underline{\sigma}}}{n} = \sigma_0 + \sigma_1 + \sum_{\lambda\in\Lambda} \sigma_\lambda \cdot \Big(j_\lambda + \frac{i_\lambda}{n}\Big) \geq \sigma_0 + \sigma_1 + \sum_{\lambda\in\Lambda} \sigma_\lambda = 1 \quad \text{for every } \underline{\sigma} \in \Lambda'.$$

Moreover, we notice that

$$(16.6.23)\qquad 1 > j'_{\underline{\sigma}} \qquad \text{and} \qquad n > i''_{\underline{\sigma}} \geq \frac{\lambda_n}{p^r} \qquad \text{for every } r = 1, \dots, n \text{ and every } \underline{\sigma} \in \Lambda'_r.$$

Indeed, the first inequality is immediate, since $\sigma_0 < 1$; the second inequality is clear if $\sigma_\lambda \neq 0$ for at least one index $\lambda \in \Lambda$, since in this case $\sigma_\lambda \geq p^{-r}$ and $i''_{\underline{\sigma}} < n\sigma_1 + n\sum_{\lambda\in\Lambda} \sigma_\lambda < n$. If $\sigma_\lambda = 0$ for every $\lambda \in \Lambda$, we must have $1 > \sigma_1 > 0$, since otherwise we would get $\sigma_0, \sigma_1 \in \mathbb{N}$, which is absurd; hence, in this case $1 > \sigma_1 \geq p^{-r}$, so $n > i''_{\underline{\sigma}} \geq n/p^r \geq \lambda_n/p^r$.

Next, for every $r = 1, \dots, n$ and every $\underline{\sigma} \in \Lambda'_r$, we may find $z_{\underline{\sigma}} \in \mathrm{gr}_{(1-j'_{\underline{\sigma}})\gamma} A$ such that

$$p^r \cdot z'_{\underline{\sigma}} \cdot \overline{u}_A(f'_{\underline{\sigma}}) = z_{\underline{\sigma}} \cdot \overline{u}_A(\alpha_0^r)$$

(cp. the proof of claim 16.6.18(iii)). Hence, if we set

$$i'_{\underline{\sigma}} := i''_{\underline{\sigma}} + r \qquad \text{for every } r = 1, \dots, n \text{ and every } \underline{\sigma} \in \Lambda'_r$$

we get an expression for a of the sought type, and it remains only to check that $(i'_{\underline{\sigma}}, j'_{\underline{\sigma}}) \in S'_n$ for every $\underline{\sigma} \in \Lambda'$. However, from (16.6.23) we see that $i'_{\underline{\sigma}} \geq 1 + \lambda_n/p = \lambda_{n+1}$ if $\underline{\sigma} \in \Lambda'_1$, and $i'_{\underline{\sigma}} \geq 2 + \lambda_n/p^2 > \lambda_{n+1}$ if $\underline{\sigma} \in \Lambda'_r$ for some $r > 1$. Lastly, from (16.6.22) we get $i''_{\underline{\sigma}} + j'_{\underline{\sigma}} \cdot n \geq n$ for every $\underline{\sigma} \in \Lambda'$, whence $i'_{\underline{\sigma}} + j'_{\underline{\sigma}} \cdot n \geq n + 1$, and the proof is complete. ◊

Let I (resp. J) be the ideal of $\mathbf{E}$ generated by the system (α_0^n, e_n) (resp. by $(\alpha_0^n, e_n + \delta)$) and set $q := \min(j + i/n \mid (i, j) \in \Lambda)$; then $\delta \in \mathrm{i.c.}(I, \mathbf{E}, q)$, and the radical of I equals the radical of J. Since $q > 1$, lemma 9.3.74(ii) shows therefore that

$$I^{\langle 1\rangle}\mathbf{E} = J^{\langle 1\rangle}\mathbf{E}$$

and consequently $e_n^t \in J^{\langle t\rangle}\mathbf{E}$ for every $t \in \mathbb{N}[1/p]$. Notice that (α_0^n, e_{n+1}) is another system of generators for J. Since the topology of A is p-adic, it follows that for every $t \in \mathbb{N}[1/p]$ there exist a finite subset $T_t \subset \mathbb{N}[1/p]$ with $0 \leq \sigma \leq t$ for every $\sigma \in T_t$, and a system $(h'_\sigma \mid \sigma \in T_t)$ of elements of $A^\wedge$ such that

$$\overline{u}_A(e_n^t) = \sum_{\sigma\in T_t} h'_\sigma \cdot \overline{u}_A(\alpha_0^{n\sigma} \cdot e_{n+1}^{t-\sigma})$$

(corollary 16.3.40(ii)); after replacing h'_σ with its canonical $\sigma\gamma$-projection, we may also assume that $h'_\sigma \in \mathrm{gr}_{\sigma\gamma} A$ for every $\sigma \in T_t$. Now, pick an expression for a as in claim 16.6.21; to conclude the proof, it suffices to show that $\overline{u}_A(\alpha_0^{i'_\lambda} \cdot e_n^{j'_\lambda}) \in \mathscr{I}(e_{n+1}, n+1, \gamma) + p^{n+1} \cdot \mathrm{gr}_{j'_\lambda\gamma} A$ for every $\lambda \in \Lambda'$. To ease notation, set $(s, t) := (i'_\lambda, j'_\lambda)$; we see that

$$\overline{u}_A(\alpha_0^s \cdot e_n^t) = \sum_{\sigma\in T_t} h'_\sigma \cdot \overline{u}_A(\alpha_0^{s+n\sigma} \cdot e_{n+1}^{t-\sigma}).$$

Now, since $(s, t) \in S'_n$, we have $s + n\sigma \geq s \geq \lambda_{n+1}$ for every $\sigma \in T'_t$. Moreover :

$$s + n\sigma + (t - \sigma) \cdot (n + 1) = s + nt + t - \sigma \geq n + 1 + t - \sigma.$$

Recall also that $s + nt \geq n + 1$; therefore, if $s + n\sigma < n + 1$, we must have $t - \sigma > 0$, so the pair $(s + n\sigma, t - \sigma)$ lies in S_{n+1}; lastly, if $s + n\sigma \geq n + 1$, the term $\overline{u}_A(\alpha_0^{s+n\sigma} \cdot e_{n+1}^{t-\sigma})$ lies in $p^{n+1} \cdot \mathrm{gr}_{(t-\sigma)\gamma} A$, whence $h'_\sigma \cdot \overline{u}_A(\alpha_0^{s+n\sigma} \cdot e_{n+1}^{t-\sigma}) \in p^{n+1} \cdot \mathrm{gr}_{t\gamma} A$ and the assertion follows. □

Corollary 16.6.24. *Let A be a perfectoid ring, and $a \in A$ such that $p^k \in Aa$ for some $k \in \mathbb{N}$. Then there exist $t \in A^\times$, $s \in A$, and $e \in \mathbf{E}(A)$ such that $a = t \cdot \overline{u}_A(e)$ and $p = s \cdot \overline{u}_A(e^{1/p^k})$.*

Proof. In view of proposition 16.3.8(ii), we may assume that the topology of A is p-adic, and then the topology of $\mathbf{E} := \mathbf{E}(A)$ is α_0-adic, where $\underline{\alpha} := (\alpha_n \mid n \in \mathbb{N}) \in W(\mathbf{E})$ is a distinguished element of $\mathrm{Ker}(u_A)$ (remark 16.3.7(iii)). Also, the case $k = 0$ is trivial, so we may assume that $k > 0$. We apply proposition 16.6.16 to A (graded with by the trivial monoid $\Gamma = \{0\}$), its element a, and with $n := p^{4k}$. With the notation of (16.6.15), notice that for every $(i,j) \in S_n$ we have either $i \in [p^k, n[$, in which case $i \geq k+1$ and $\overline{u}_A(\alpha_0^i) \in p^{k+1}A$ (lemma 16.2.7(iii)), or else $i \in [\lambda_n, p^k[$, in which case $j > 1 - p^{-3k}$. Consequently, we get $e \in \mathbf{E}$ and $b, c \in A$ with :

$$a - \overline{u}_A(e) \in \overline{u}_A(\alpha_0^{1+p^{-k}} e^{1-p^{-3k}})b + p^{k+1}c.$$

By assumption, $p^k = ad$ for some $d \in A$, so that $a - p^{k+1}c = a(1 - cdp)$, and notice that $x := 1 - cdp \in A^\times$; hence $a = x^{-1} \cdot \overline{u}_A(e^{1-p^{-3k}}) \cdot (\overline{u}_A(\alpha_0^{1+p^{-k}}) + \overline{u}_A(e^{p^{-3k}}))$, and in particular :

$$p^k \in \overline{u}_A(e^{1-p^{-3k}})A.$$

Notice also that $p^k A = \overline{u}_A(\alpha_0^k)$; combining with corollary 16.3.51(ii), it follows that

$$\alpha_0^k \in e^{1-p^{-3k}}\mathbf{E} \qquad \text{and therefore} \qquad \alpha_0^{kp^{-2k}} \in e^{p^{-2k}-p^{-5k}}\mathbf{E}.$$

Clearly $r := p^{-k} - kp^{-2k} > 0$, and $p^{-2k} - p^{-5k} - p^{-3k} \geq 0$, so that :

$$\alpha_0^{1+p^{-k}} e^{1-p^{-3k}} \in \alpha_0^{1+r} e$$

and finally we get, for some $y \in \mathbf{E}$:

$$ax = \overline{u}_A(e) \cdot z \qquad \text{where} \qquad z := 1 + \overline{u}_A(\alpha_0^{1+r}y).$$

Clearly $z \in A^\times$, so the first sought identity holds with $t := z/x$. It follows that $\overline{u}_A(\alpha_0^k) \in \overline{u}_A(e)A$, so that $\alpha_0^k \in e\mathbf{E}$, again by corollary 16.3.51(ii); thus, $\alpha_0 \in e^{1/p^k}\mathbf{E}$, whence the second identity. ☐

Remark 16.6.25. (i) One can show by a direct computation that the element e_n appearing in the proof of proposition 16.6.16 is integral over the ideal generated by e_{n+1} and α_0^n; as a consequence, the proof of *loc. cit.* can be made in principle completely effective, so that one can extract from it an algorithm that exhibits an element e with the sought properties, for any given $a \in \mathrm{gr}_\gamma A$.

(ii) In [149], Scholze proves theorem 16.5.52 (for his perfectoid rings, that are a special case of our perfectoid quasi-affinoid rings) by means of his "approximation lemma" (see [149, Lemma 6.5]). With proposition 16.6.16, we can give an alternative proof of theorem 16.5.52 that generalizes to quasi-affinoid perfectoid rings, along the lines of Scholze's original argument. The first step is the following extension of proposition 16.5.46 to quasi-affinoid perfectoid rings.

Lemma 16.6.26. *In the situation of* (16.5.23)*, let v be any continuous valuation on A_U. Then :*

(i) *The mapping $v \circ \varphi_U^\flat$ is a continuous valuation of $\mathbf{E}_U$.*

(ii) *The rule : $v \mapsto v \circ \varphi_U^\flat$ defines a continuous map*

$$\mathrm{Cont}(\varphi_U^\flat) : \mathrm{Cont}(A_U) \to \mathrm{Cont}(\mathbf{E}_U).$$

Proof. (i): Since both v and $\varphi_U^\flat$ are morphisms of pointed monoids, the same holds for $v \circ \varphi_U^\flat$. Thus, it remains only to check that

$$v(\varphi_U^\flat(x+y)) \leq M(x,y) := \max(v \circ \varphi_U^\flat(x), v \circ \varphi_U^\flat(y)) \qquad \text{for every } x, y \in \mathbf{E}_U.$$

However, from proposition 16.4.34(ii) we know that $\varphi_U^\flat(x+y)$ is the sum of $\varphi_U^\flat(x) + \varphi_U^\flat(y)$ and a p-adically convergent series, whose n-th term can be written as $p^n \cdot z_n$, where z_n is in turn a finite $\mathbb{Z}_p$-linear combination of elements of the form $\varphi^\flat_U(x^\sigma y^{1-\sigma})$, with σ ranging over the elements of $\Sigma_n := p^{-n}\mathbb{N} \cap [0,1]$. It follows that

$$v(z_n) \leq \max(\sigma \cdot v(\varphi^\flat_U(x)) + (1-\sigma) \cdot v(\varphi^\flat_U(y)) \mid \sigma \in \Sigma_n) \leq M(x,y).$$

On the other hand, $v(p)$ is cofinal in the value group of v, since the latter is continuous (see (15.3.1)). The assertion is an immediate consequence.

The proof of (ii) is similar to that of proposition 16.5.46(ii) : details left to the reader. □

Next, we remark the following corollary of proposition 16.6.16, that generalizes and improves slightly Scholze's "approximation lemma" in [149, Lemma 6.5]. The same improvement appears in [114, Cor.3.6.7].

Corollary 16.6.27. *Resume the notation of* (16.5.23). *For every* $\gamma \in \mathbb{Z}[1/p]$, *every* $a \in \mathrm{Fil}^\gamma A_U$, *every* $m \in \mathbb{N}$ *and every* $\beta \in \mathbb{N}[1/p]$ *with* $\beta < p/(p-1)$ *there exists* $e \in \mathrm{Fil}^\gamma \mathbf{E}_U$ *such that*

$$v(a - \varphi^\flat_U(e)) \leq v(p)^\beta \cdot \max(v(\varphi^\flat_U(e)), v(p)^m) \qquad \textit{for every } v \in \mathrm{Cont}(A_U).$$

Proof. (Notice that the value group Γ_v is p-divisible, due to theorem 16.5.52(iv), but even discounting *loc.cit.*, the term $v(p)^\beta$ is still well defined as an element of $(\mathbb{Q} \otimes_\mathbb{Z} \Gamma_v)_\circ$, so the above inequality makes sense in any case.) Fix v, m, β as in the corollary, and a distinguished element $\underline{\alpha} := (\alpha_n \mid n \in \mathbb{N}) \in \mathrm{Ker}\, u_A$, and define the sequence of rational numbers $(\lambda_{n+1} \mid n \in \mathbb{N})$ as in (16.6.15), as well as the subset $S_n \subset \mathbb{N}[1/p]^{\oplus 2}$ and the $\overline{A}$-submodule $\mathscr{I}(e,n,\gamma)$ of $\mathrm{Fil}^\gamma A_U$, for every $e \in \mathrm{Fil}^\gamma \mathbf{E}_U$ and every integer $n > 0$. Notice that

$$\lim_{n \to +\infty} \lambda_n = p/(p-1). \tag{16.6.28}$$

We apply proposition 16.6.16 to the perfectoid rings with $\mathbb{Z}[1/p]$-graded structure $\mathscr{E}_U$ and $\mathscr{A}_U$ given by example 16.6.13 : for every integer $n > 0$ we then get $e_n \in \mathrm{Fil}^\gamma \mathbf{E}_U$, $d_n \in \mathrm{Fil}^\gamma A_U$, a finite subset $T_n \subset S_n$, and for every $(i,j) \in T_n$ an element $b_{i,j,n} \in \mathrm{Fil}^{(1-j)\gamma} A_U$ such that

$$a = \varphi^\flat_U(e_n) + \sum_{(i,j) \in T_n} b_{i,j,n} \cdot \varphi^\flat_U(\alpha_0^i \cdot e_n^j) + p^n d_n.$$

Pick also $\varepsilon \in \mathbb{N}[1/p] \setminus \{0\}$ such that $\alpha_0 \in \mathrm{Fil}^\varepsilon \mathbf{E}_U$, and $k \in \mathbb{N}$ large enough so that $\gamma + k\varepsilon > 0$.

Claim 16.6.29. We have $v(p^n d_n) \leq v(p)^{\beta+m}$ for every sufficiently large $n \in \mathbb{N}$.

Proof of the claim. We have $p^k d_n \in \mathrm{Fil}^{\gamma+k\varepsilon} A_U$, and notice that $\mathrm{Fil}^\delta A_U \subset A_U^{\circ\circ}$ for every $\delta > 0$. It follows that $v(p^k d_n) < 1$, so the claim holds for every $n \geq k + m + \beta$. ◊

Recall that $pu = \varphi^\flat_U(\alpha_0)$ for some $u \in \overline{A}^\times$; suppose first that there exists $n_0 \in \mathbb{N}$ such that $v(\varphi^\flat_U(e_n)) \leq v(p)^m$ for every $n \geq n_0$. Then we have

$$v(b_{i,j,n} \cdot \varphi^\flat_U(\alpha_0^i \cdot e_n^j)) \leq v(b_{i,j,n}) \cdot v(u)^i \cdot v(p)^{i+mj} \qquad \text{for every } n \geq n_0 \text{ and every } (i,j) \in T_n$$

and on the other hand $b_{i,j,n} \cdot \varphi^\flat_u(\alpha_0^{(1-j)k}) \in \mathrm{Fil}^{(1-j)(\gamma+k\varepsilon)} \subset A_U^{\circ\circ}$, and $v(u)^i \cdot v(x) < 1$ for every $x \in A_U^{\circ\circ}$, whence

$$v(u)^i \cdot v(b_{i,j,n} \cdot \varphi^\flat_u(\alpha_0^{(1-j)k})) < 1 \qquad \text{for every } n \in \mathbb{N} \text{ and every } (i,j) \in T_n.$$

By the same token, we have $v(u)^{-(1-j)k} \cdot v(p)^\delta < 1$ for every $\delta > 0$, whence

$$v(p)^\delta \leq v(\varphi^\flat_U(\alpha_0^{(1-j)k})) \qquad \text{for every } \delta > (1-j)k.$$

Summing up we obtain, for every $(i,j) \in T_n$, the inequality

$$v(b_{i,j,n} \cdot \varphi^\flat_U(\alpha_0^i \cdot e_n^j)) \leq v(p)^{\beta+m} \qquad \text{whenever} \qquad i + mj - (1-j)k > m + \beta.$$

The latter condition is the same as the inequality $i > (1-j)(m+k)+\beta$. However, notice that $1-j < i/n$, so this inequality holds provided $i > i \cdot (m+k)/n + \beta$, *i.e.* when

$$i \cdot (1-(m+k)/n) > \beta. \qquad (16.6.30)$$

Lastly, from (16.6.28) we see that $\beta < \lambda_n$ for every sufficiently large n, and since $i \geq \lambda_n$, we may find n large enough so that (16.6.30) is verified for every $(i,j) \in T_n$; taking into account claim 16.6.29, we get the corollary, in this case.

Thus, we may assume that there exists an infinite subset $\Sigma \subset \mathbb{N}$ such that $v(p)^m < v(\varphi_U^\flat(e_s))$ for every $s \in \Sigma$, and taking into account claim 16.6.29, we are reduced to checking that

$$v(b_{i,j,s} \cdot \varphi_U^\flat(\alpha_0^i \cdot e_s^j)) \leq v(p)^\beta \cdot v(\varphi_U^\flat(e_s)) \qquad \text{for some } s \in \Sigma \text{ and every } (i,j) \in T_s.$$

However, arguing as in the foregoing we easily see that

$$v(\varphi_U^\flat(\alpha_0^i)) \leq v(p)^\delta \qquad \text{for every } \delta < i.$$

Hence, after replacing β by some $\beta' \in \mathbb{N}[1/p]$ with $p/(p-1) > \beta' > \beta$, we reduce to checking that there exists $s \in \Sigma$ such that

$$v(b_{i,j,s}) \cdot v(p)^{i-\beta} \leq v(p)^{(1-j)m} \qquad \text{for every } (i,j) \in T_s.$$

On the other hand, we have $b_{i,j,n} \cdot \varphi_U^\flat(\alpha_0^{(1-j)k}) \in A_U^{\circ\circ}$, and $v(u)^{(1-j)k} \cdot v(x) < 1$ for every $x \in A_U^{\circ\circ}$, whence

$$v(b_{i,j,n}) \cdot v(p)^{(1-j)k} < 1 \qquad \text{for every } n \in \mathbb{N} \text{ and every } (i,j) \in T_n.$$

Thus, it suffices to show that there exists $s \in \Sigma$ such that

$$i - \beta > (1-j)k + (1-j)m = (1-j)(m+k) \qquad \text{for every } (i,j) \in T_s.$$

This condition is equivalent to (16.6.30), so we may argue as in the foregoing to conclude. □

16.6.31. Now, in the situation of corollary 16.6.27, let

$$C(f_\bullet) := \mathrm{Cont}(A_U) \cap R_{A_U}\big(\tfrac{f_1}{f_0}, \cdots, \tfrac{f_n}{f_0}\big) \qquad \text{and} \qquad S(f_\bullet) := C(f_\bullet) \cap \mathrm{Spa}\,\underline{A}$$

for a sequence $f_\bullet := (f_0, \dots, f_n)$ of elements of A_U that generates an open ideal. There exist $a_0, \dots, a_n \in A_U$ and an integer $m > 0$ such that $p^{m-1} = \sum_{i=0}^n a_i f_i$, and after replacing each a_i by $p^n a_i$ for a suitable $n \in \mathbb{N}$, and m by $m+n$, we may even assume that $a_0, \dots, a_n \in \overline{A}$ (recall that $\overline{A}$ is the image of A in A_U, and $\overline{\mathbf{E}}$ is the image of $\mathbf{E}$ in $\mathbf{E}_U$) whence

$$v(p)^m \leq \max(v(pa_i) \cdot v(f_i) \mid i = 0, \dots, n) \leq \max(v(f_i) \mid i = 0, \dots, n) \qquad (16.6.32)$$

for every $v \in \mathrm{Cont}(A_U)$. It follows that

$$v(f_0) \geq v(p)^m \qquad \text{for every } v \in C(f_\bullet). \qquad (16.6.33)$$

By corollary 16.6.27 we may find $e_0, \dots, e_n \in \mathbf{E}_U$ such that for every $i = 1, \dots, n$ we have :

$$v(f_i - \varphi_U^\flat(e_i)) \leq v(p) \cdot \max(v(\varphi_U^\flat(e_i)), v(p)^m) \qquad \text{for every } v \in \mathrm{Cont}(A_U). \qquad (16.6.34)$$

We may now state the announced generalization of theorem 16.5.52 :

Proposition 16.6.35. *With the notation of* (16.6.31)*, we have :*

(i) *The map* $\mathrm{Cont}(\varphi_U^\flat)$ *of lemma* 16.6.26(ii) *is a homeomorphism.*
(ii) *The system* $e_\bullet := (e_0, \dots, e_n)$ *generates an open ideal of* $\mathbf{E}_U$.
(iii) $C(f_\bullet) = \mathrm{Cont}(\varphi_U^\flat)^{-1} C(e_\bullet)$*, where* $C(e_\bullet) := R_{\mathbf{E}_U}(\frac{e_1}{e_0}, \cdots, \frac{e_n}{e_0}) \cap \mathrm{Cont}(\mathbf{E}_U)$.
(iv) $\varphi_U^\flat(e_0) \in \mathscr{O}^{\mathrm{loc}}_{\mathrm{Spa}\,\underline{A}}(S(f_\bullet))^\times$ *and* $f_i/f_0 - \varphi_U^\flat(e_i)/\varphi_U^\flat(e_0) \in \mathscr{O}^{\mathrm{loc}}_{\mathrm{Spa}\,\underline{A}}(S(f_\bullet))^{\circ\circ}$ *for* $i = 1, \dots, n$ *(notation of* (15.5.11)*).*

Proof. (ii): Set $\varphi_U^\flat(e_\bullet) := (\varphi_U^\flat(e_0), \dots, \varphi_U^\flat(e_n))$; to begin with, we remark :

Claim 16.6.36. (ii) holds if and only if the system $\varphi_U^\flat(e_\bullet)$ generates an open ideal of A_U.

Proof of the claim. Let $b_1, \dots, b_r$ be a finite system of elements of $\overline{\mathbf{E}}$ that generates an ideal of definition. Then we may find $k \in \mathbb{N}$ large enough so that $b_i^k e_j \in \overline{\mathbf{E}}$ for every $i = 1, \dots, r$ and every $j = 1, \dots, n$. Clearly the system $b_\bullet^k e_\bullet := (b_i^k e_j \mid i \leq r, \ j \leq n)$ generates an open ideal of $\mathbf{E}_U$ if and only if the same holds for $e_\bullet$; likewise; $\varphi^\flat(e_\bullet)$ generates an open ideal of A_U if and only if the same holds for $\varphi_U^\flat(b_\bullet^k e_\bullet)$. So we may replace $e_\bullet$ by $b_\bullet^k e_\bullet$, and assume from start that $e_i \in \overline{\mathbf{E}}$ for $i = 1, \dots, n$, in which case $\varphi_U^\flat(e_i) = \overline{u}_{\overline{A}}(e_i) \in \overline{A}$ for $i = 1, \dots, n$. Now, let $I_\mathbf{E} \subset \overline{\mathbf{E}}$ be the ideal generated by $e_\bullet$; then $I_\mathbf{E}$ is open in $\overline{\mathbf{E}}$ if and only if $Z := \operatorname{Spec} \overline{\mathbf{E}}/I_\mathbf{E}$ lies in the non-analytic locus X of $\operatorname{Spec} \overline{\mathbf{E}}$; likewise, $I_\mathbf{E}\mathbf{E}_U$ is open in $\mathbf{E}_U$ if and only if $Z' := \operatorname{Spec} \mathbf{E}_U/I_\mathbf{E}\mathbf{E}_U$ lies in the non-analytic locus X' of $\operatorname{Spec} \mathbf{E}_U$ (lemma 8.3.29(v)). On the other hand, let $j : \overline{\mathbf{E}} \to \mathbf{E}_U$ be the inclusion map; then $Z' = (\operatorname{Spec} j)^{-1}Z$, and lemma 8.3.29(iii) implies that Z lies in X if and only if Z' lies in X'. Summing up, we see that $I_\mathbf{E}$ is open in $\overline{\mathbf{E}}$ if and only if $I_\mathbf{E}\mathbf{E}_U$ is open in $\mathbf{E}_U$. Likewise, let $I_A \subset \overline{A}$ be the ideal generated by $\varphi_U^\flat(e_\bullet)$; then I_A is open in $\overline{A}$ if and only if $I_A A_U$ is open in A_U. Thus, we are further reduced to showing that $e_\bullet$ generates an open ideal of $\overline{\mathbf{E}}$ if and only if $\varphi_U^\flat(e_\bullet)$ generates an open ideal of $\overline{A}$. The latter assertion follows from corollary 16.3.40(ii). $\diamondsuit$

By claim 16.6.36, it suffices to show that the system $\varphi_U^\flat(e_\bullet)$ generates an open ideal of A_U. To this aim, we apply the criterion of lemma 15.3.25, which reduces to checking that for every $v \in \operatorname{Cont}(A_U)_\mathrm{a}$ there exists $i \in \{0, \dots, n\}$ such that $v(\varphi_U^\flat(e_i)) \neq 0$. However, from (16.6.34) we deduce that for every $i = 0, \dots, n$ and every such v we have :

- either $v(f_i) = v(\varphi_U^\flat(e_i))$
- or else $v(p) > 0$ and $v(f_i), v(\varphi_U^\flat(e_i)) < v(p)^m$.

Now, since the system $f_\bullet$ generates an open ideal of A_U, lemma 15.3.25 implies that $v(f_i) \neq 0$ for some $i \leq n$. If $v(p) = 0$ we deduce that $v(\varphi_U^\flat(e_i)) \neq 0$, as required.

Thus, suppose $v(p) > 0$; from (16.6.32) we know that $v(f_i) \geq v(p)^m > 0$ for some $i \leq n$, whence $v(\varphi_U^\flat(e_i)) = v(f_i) \neq 0$, again as needed.

(iii): In view of (16.6.33) and the foregoing, it follows that

$$v(f_0) = v(\varphi_U^\flat(e_0)) \geq v(p)^m \qquad \text{for every } v \in C(f_\bullet). \tag{16.6.37}$$

Now, let $v \in C(f_\bullet)$ and $i \in \{0, \dots, n\}$; if $v(f_i) \geq v(p)^m$, the foregoing also implies that

$$v(\varphi_U^\flat(e_i)) = v(f_i) \leq v(f_0) = v(\varphi_U^\flat(e_0)).$$

If $v(f_i) < v(p)^m$, likewise we get $v(\varphi_U^\flat(e_i)) < v(p)^m \leq v(\varphi_U^\flat(e_0))$, so $v \in \operatorname{Cont}(\varphi_U^\flat)^{-1}C(e_\bullet)$.

Lastly, let $v \in \operatorname{Cont}(A_U) \setminus C(f_\bullet)$; if $v(p) = 0$, we know that $v(f_i) = v(\varphi_U^\flat(e_i))$ for every $i = 0, \dots, n$, whence $v \notin \operatorname{Cont}(\varphi_U^\flat)^{-1}C(e_\bullet)$. Otherwise, (16.6.32) says that $v(f_j) \geq v(p)^m$ for some $j \leq n$, whence $v(\varphi_U^\flat(e_j)) \geq v(p)^m$. If now $v(f_0) < v(p)^m$, we get $v(\varphi_U^\flat(e_0)) < v(p)^m$, so $v \notin \operatorname{Cont}(\varphi_U^\flat)^{-1}C(e_\bullet)$. Hence, suppose $v(\varphi_U^\flat(e_0)) = v(f_0) \geq v(p)^m > 0$. Then there exists $k \leq n$ such that $v(f_k) > v(f_0)$, whence $v(f_k) = v(\varphi_U^\flat(e_k))$, and again we conclude that $v \notin \operatorname{Cont}(\varphi_U^\flat)^{-1}C(e_\bullet)$.

(i): First, let us show that the continuous map $\operatorname{Cont}(\varphi_U^\flat)$ is bijective. Indeed, a simple inspection yields a commutative diagram of continuous maps

$$\begin{array}{ccc}
\operatorname{Cont}(A_U) & \xrightarrow{\operatorname{Cont}(\varphi_U^\flat)} & \operatorname{Cont}(\mathbf{E}_U) \\
\downarrow & & \downarrow \\
\operatorname{Cont}(\overline{A}) & \xrightarrow{\operatorname{Cont}(\overline{u}_{\overline{A}})} & \operatorname{Cont}(\overline{\mathbf{E}})
\end{array}$$

where the vertical arrows are induced by the open inclusion maps $\overline{A} \to A_U$ and $\overline{\mathbf{E}} \to \mathbf{E}_U$ (notation of (16.5.2)). Therefore, combining theorem 16.5.52(iii) and proposition 15.3.18(ii) we already see that $\operatorname{Cont}(\varphi_U^\flat)$ restricts to a bijection $\operatorname{Cont}(A_U)_\mathrm{a} \xrightarrow{\sim} \operatorname{Cont}(\mathbf{E}_U)_\mathrm{a}$. On the other hand, proposition 16.5.4(iv) implies easily that $\mathrm{Cont}(\varphi^\flat_U)$ restricts as well to a bijection $\mathrm{Cont}(A_U)_{\mathrm{na}} \stackrel{\sim}{\to} \mathrm{Cont}(\mathbf{E}_U)_{\mathrm{na}}$, whence the contention. Lastly, (ii) and (iii) imply easily that the topology of $\mathrm{Cont}(A_U)$ is induced by the topology of $\mathrm{Cont}(\mathbf{E}_U)$, via the map $\mathrm{Cont}(\varphi^\flat_U)$; the assertion follows.

(iv): Since f_0 is invertible in $\mathscr{O}^{\mathrm{loc}}_{\mathrm{Spa}\,\underline{A}}(S(f_\bullet))$, combining (16.6.37) and proposition 15.4.30 we see that the same holds for $\varphi^\flat_U(e_0)$. For the second assertion, in view of corollary 15.4.27(ii) it suffices to show :

Claim 16.6.38. $v(f_i/f_0 - \varphi^\flat_U(e_i)/\varphi^\flat_U(e_0)) < 1$ for every $i = 1, \dots, n$ and every $v \in C(f_\bullet)$.

Proof of the claim. From (16.6.34) we easily deduce that

$$v(1/f_0 - 1/\varphi^\flat_U(e_0)) \leq v(p/f_0) \qquad \text{for every } v \in C(f_\bullet)$$

whence $v(f_i/f_0 - f_i/\varphi^\flat_U(e_0)) \leq v(pf_i/f_0) < 1$ for every $v \in C(f_\bullet)$ and every $i = 1, \dots, n$. We are then reduced to checking that $v(f_i/\varphi^\flat_U(e_0) - \varphi^\flat_U(e_i)/\varphi^\flat_U(e_0)) < 1$, or equivalently, that $v(f_i - \varphi^\flat_U(e_i)) < v(\varphi^\flat_U(e_0))$ for every such v and i. Now, if $v(f_i) < v(p)^m$, we know already that $v(\varphi^\flat_U(e_i)) < v(p)^m$, whence $v(f_i - \varphi^\flat_U(e_i)) < v(p)^m$, and the assertion follows from (16.6.37) in this case. Otherwise, we have $v(f_i) = v(\varphi^\flat_U(e_i))$, whence $v(\varphi^\flat_U(e_i)) \leq v(f_0) = v(\varphi^\flat_U(e_0))$, and thus $v(f_i - v(\varphi^\flat_U(e_i))) \leq v(p) \cdot \max(\varphi^\flat_U(e_0), v(p)^m) = v(p \cdot \varphi^\flat_U(e_0)) < v(\varphi^\flat_U(e_0))$. ☐

16.6.39. Let Γ, Δ be two monoids and $(A, \underline{B})$ a topological ring with Δ-graded structure. Then we may form the complete topological ring with $\Delta \oplus \Gamma$-graded structure

$$(A, \underline{B})[\Gamma]^\wedge$$

by combining the constructions of example 8.5.7(i) and remark 8.5.2(iv). Explicitly, a direct inspection shows that $(A, \underline{B})[\Gamma]^\wedge$ is the pair $(A[\Gamma]^\wedge, \underline{B}^c[\Gamma])$, where $\underline{B}^c$ is the Δ-graded ring such that $\mathrm{gr}_\delta B^c$ is the topological closure of $\mathrm{gr}_\delta B$ in the completion $A^\wedge$ of A, and $A[\Gamma]^\wedge$ is the completion of $A[\Gamma]$, for the unique topology on the latter ring such that the inclusion map $A \to A[\Gamma]$ is adic. Notice that if A is complete and separated, then $B^c = B$, hence in this case $(A, \underline{B})[\Gamma]^\wedge = (A[\Gamma]^\wedge, \underline{B}[\Gamma])$.

Lemma 16.6.40. *In the situation of* (16.6.39)*, suppose furthermore that A is perfectoid, and Γ, Δ are both p-perfect, and let*

$$A \xrightarrow{j_A} A[\Gamma]^\wedge \xleftarrow{i_A} \Gamma \qquad \textit{and} \qquad \mathbf{E}(A) \xrightarrow{j_{\mathbf{E}}} \mathbf{E}(A)[\Gamma]^\wedge \xleftarrow{i_{\mathbf{E}}} \Gamma$$

be the natural inclusion maps. Then we have :

(i) *$A[\Gamma]^\wedge$ is perfectoid.*

(ii) *There exists a natural isomorphism of topological rings with $\Delta \oplus \Gamma$-graded structures*

$$\omega : \mathbf{E}(A, \underline{B})[\Gamma]^\wedge \stackrel{\sim}{\to} \mathbf{E}((A, \underline{B})[\Gamma]^\wedge)$$

fitting into a commutative diagram

$$\begin{array}{ccccc} \mathbf{E}(A) & \xrightarrow{j_{\mathbf{E}}} & \mathbf{E}(A)[\Gamma]^\wedge & \xleftarrow{i_{\mathbf{E}}} & \Gamma \\ & \searrow^{\mathbf{E}(j_A)} & \downarrow{\scriptstyle\omega} & \swarrow_{\mathbf{E}(i_A)} & \\ & & \mathbf{E}(A[\Gamma]^\wedge). & & \end{array}$$

Proof. (Here Γ is identified with the source $\mathbf{E}(\Gamma)$ of the morphism $\mathbf{E}(i_A)$, via $\overline{u}_\Gamma$.)

(i): Indeed, by remark 16.3.58 we may find an ideal of definition $I \subset A$ fulfilling the conditions of (16.3.55), and we let $J := I^{(p)}$; by proposition 16.3.59(ii), the corresponding map $\Phi_I : \mathrm{gr}^\bullet_I A \to \mathrm{gr}^\bullet_J A$ is a ring isomorphism. Set $I' := IA[\Gamma]^\wedge$ and $J' := JA[\Gamma]^\wedge$; by construction, the topology of $A[\Gamma]^\wedge$ agrees with the I'-adic topology, and the natural map

$(A/I^2)[\Gamma] \to A[\Gamma]^\wedge/I'^2$ is an isomorphism (remark 8.3.3(ii) and lemma 8.3.32(iv)). Since Γ is p-perfect, it follows easily that the Frobenius endomorphism of $A[\Gamma]^\wedge/I'^2$ is surjective, so $A[\Gamma]^\wedge$ is a P-ring. On the other hand, notice the natural identifications :

$$\mathrm{gr}^\bullet_{I'}(A[\Gamma]^\wedge) \xrightarrow{\sim} (\mathrm{gr}^\bullet_I A)[\Gamma] \qquad \mathrm{gr}^\bullet_{J'}(A[\Gamma]^\wedge) \xrightarrow{\sim} (\mathrm{gr}^\bullet_J A)[\Gamma]$$

which identify the ring homomorphism $\Phi_{I'}$ of proposition 16.3.59(i) with the map induced by Φ_I and the p-Frobenius endomorphism of Γ. Especially, $\Phi_{I'}$ is an isomorphism, so the assertion follows from theorem 16.3.64.

(ii): According to proposition 16.6.1(i), we may find a Δ-graded ideal I_B of B such that $I_A := I_B A$ is an ideal of definition of A; then notice that the topological closure $I_B^\wedge$ of I_B in A lies in I_A and $I_B^\wedge \cap B$ is the topological closure I_B^c of I_B in B, therefore $I_B^\wedge = I_A$ and $I_B^c = B \cap I_A$, which says especially that I_B^c is open in B. Set $A_0 := A/I_A$ and $B_0 := B/I_B^c$; since I_B^c is a Δ-graded ideal as well (remark 8.5.2(i)), we may form the quotient of $(A, \underline{B})[\Gamma]^\wedge$ by the open ideal $I_B^c[\Gamma]$ of $B[\Gamma]$ as in example 8.5.4(), and get a natural isomorphism :

$$(A, \underline{B})[\Gamma]^\wedge \otimes_B B_0 \xrightarrow{\sim} (A_0[\Gamma], B_0[\Gamma]). \tag{16.6.41}$$

However, the projections $(A, \underline{B}) \to (A_0, \underline{B}_0)$ and $(A, \underline{B})[\Gamma]^\wedge \to (A, \underline{B})[\Gamma]^\wedge \otimes_B B_0$ induce morphisms (see (9.4.24))

$$\mathbf{E}(A, \underline{B}) \to \mathbf{E}(A_0, \underline{B}_0) \qquad \mathbf{E}((A, \underline{B})[\Gamma]^\wedge) \to \mathbf{E}((A, \underline{B})[\Gamma]^\wedge \otimes_B B_0). \tag{16.6.42}$$

On the other hand, the projections $A \to A_0$ and $A[\Gamma]^\wedge \to A_0[\Gamma]$ induce isomorphisms

$$\mathbf{E}(A) \xrightarrow{\sim} \mathbf{E}_0 := \mathbf{E}(A_0) \qquad \text{and} \qquad \mathbf{E}(A[\Gamma]^\wedge) \xrightarrow{\sim} \mathbf{E}(A_0[\Gamma])$$

of topological rings (see (16.1)). Denote

$$A_0 \xrightarrow{j_{A_0}} A_0[\Gamma]^\wedge \xleftarrow{i_{A_0}} \Gamma \qquad \text{and} \qquad \mathbf{E}_0 \xrightarrow{j_{\mathbf{E}_0}} \mathbf{E}_0[\Gamma]^\wedge \xleftarrow{i_{\mathbf{E}_0}} \Gamma$$

the natural inclusion maps; taking into account example 8.5.7(iii), it follows that (16.6.42) are isomorphisms of topological rings with graded structures, and in view of (16.6.41) it suffices to exhibit an isomorphism $\omega_0 : (\mathbf{E}_0[\Gamma], \mathbf{E}(\underline{B}_0)[\Gamma])^\wedge \xrightarrow{\sim} \mathbf{E}(A_0[\Gamma], B_0[\Gamma])$ such that

$$\omega_0 \circ j_{\mathbf{E}_0} = \mathbf{E}(j_{A_0}) \qquad \text{and} \qquad \omega_0 \circ i_{\mathbf{E}_0} = \mathbf{E}(i_{A_0})$$

(where again, Γ is identified with the source $\mathbf{E}(\Gamma)$ of $\mathbf{E}(i_{A_0})$ via $\overline{u}_\Gamma$). To this aim, set

$$J := \mathrm{Ker}\,(\overline{u}_{A_0}[\Gamma] : \mathbf{E}_0[\Gamma] \to A_0[\Gamma])$$

and notice that, since A_0 is a discrete topological ring, the topology of $\mathbf{E}_0[\Gamma]$ agrees with the J-adic topology (remark 9.4.9(ii)); moreover, J is finitely generated, since A is perfectoid (details left to the reader). By corollary 9.4.14 and remark 9.4.9(v) we deduce a natural isomorphism of topological rings

$$\omega_0 : \mathbf{E}_0[\Gamma]^\wedge \xrightarrow{\sim} \mathbf{E}(A_0[\Gamma]) \qquad \text{such that} \qquad \overline{u}_{A_0[\Gamma]} \circ \omega_0 = \overline{u}_{A_0}[\Gamma]^\wedge$$

where $\overline{u}_{A_0}[\Gamma]^\wedge : \mathbf{E}_0[\Gamma]^\wedge \to A_0[\Gamma]$ is the completion of $\overline{u}_{A_0}[\Gamma]$, and a simple inspection shows that ω_0 is a morphism $(\mathbf{E}_0[\Gamma], \mathbf{E}(\underline{B}_0)[\Gamma])^\wedge \to \mathbf{E}(A_0[\Gamma], \underline{B}_0[\Gamma])$, which then must be an isomorphism, again by example 8.5.7(iii). Let us check that $\omega_0 \circ i_{\mathbf{E}_0} = \mathbf{E}(i_{A_0})$. Indeed, we have

$$\overline{u}_{A_0[\Gamma]} \circ \omega_0 \circ i_{\mathbf{E}_0} = \overline{u}_{A_0}[\Gamma]^\wedge \circ i_{\mathbf{E}_0} = i_{A_0} = \overline{u}_{A_0[\Gamma]} \circ \mathbf{E}(i_{A_0})$$

whence the sought identity, by adjunction. A similar argument proves the remaining sought identity. □

We conclude this section with a sample of other graded perfectoid rings that are analogous to certain constructions found in [149], and might be interesting for other purposes, but shall not be needed in the rest of this treatise.

Example 16.6.43. (i) Let $(\Delta, 0, +)$ be a monoid, $(A, \underline{B})$ a topological ring with Δ-graded structure, and consider two morphisms of Δ-graded monoids

$$\Gamma_2 \xleftarrow{\varphi} \Gamma_1 \xrightarrow{\psi} B^*$$

where B^* is the Δ-graded monoid defined as in (9.4.21). Hence, $B[\Gamma_2]$ is a $\Delta \oplus \Gamma_2$-graded $\mathbb{Z}$-algebra, and we may form the Δ-graded $\mathbb{Z}$-algebra $B[\Gamma_2]_{/\Delta}$ corresponding to the morphism

$$\eta : \Delta \oplus \Gamma_2 \to \Delta \qquad (\delta, \gamma) \mapsto \delta + |\gamma| \qquad \text{for every } \delta \in \Delta \text{ and } \gamma \in \Gamma_2$$

where $|\cdot| : \Gamma_2 \to \Delta$ is the Δ-grading of Γ_2. It follows easily that

$$I := \mathrm{Ker}\,(B[\Gamma_2] \to \Gamma_2 \otimes_{\Gamma_1} B)$$

is a Δ-graded ideal of $B[\Gamma_2]_{/\Delta}$, and by combining the constructions of examples 8.5.4(i) and 8.5.7(i,ii) we may define the topological ring with Δ-graded structure

$$\Gamma_2 \otimes_{\Gamma_1} (A, \underline{B}) := (A, \underline{B})[\Gamma_2]_{/\Delta}/I.$$

Explicitly, this is the pair $(A', \underline{B}')$, such that A' (resp. B') is the maximal separated quotient of $\Gamma_2 \otimes_{\Gamma_1} A$ (resp. of $\Gamma_2 \otimes_{\Gamma_1} B$), where the latter is endowed with the unique linear topology such that the natural map

$$A \to \Gamma_2 \otimes_{\Gamma_1} A \qquad (\text{resp. } B \to \Gamma_2 \otimes_{\Gamma_1} B)$$

is adic. Then, $\mathrm{gr}_\delta B'$ is the topological closure of the image of $\mathrm{gr}_\delta(B[\Gamma_2]_{/\Delta})$ in A', for each $\delta \in \Delta$. By construction, we have a natural morphism of topological rings with Δ-graded structures (resp. of monoids)

$$j_{(A,\underline{B})} : (A, \underline{B}) \to \Gamma_2 \otimes_{\Gamma_1} (A, \underline{B}) \qquad (\text{resp. } i_{(A,\underline{B})} : \Gamma_2 \to \Gamma_2 \otimes_{\Gamma_1} A\,).$$

(ii) Suppose now that Δ is p-perfect, and the topology of A is complete, separated and coarser than the p-adic topology. Then the topological ring with Δ-graded structure $\mathbf{E}(A, \underline{B})$ is well defined as in (9.4.24). Moreover, from the morphisms φ and ψ we derive two morphisms of monoids

$$\mathbf{E}(\Gamma_2) \xleftarrow{\varphi_{\mathbf{E}}} \mathbf{E}(\Gamma_1) \xrightarrow{\psi_{\mathbf{E}}} \mathbf{E}(B^*) \qquad \text{where } \varphi_{\mathbf{E}} := \mathbf{E}(\varphi) \text{ and } \psi_{\mathbf{E}} := \mathbf{E}(\psi)$$

so the considerations of (i) can be repeated on $\mathbf{E}(A, \underline{B})$, and we get a topological ring with $\mathbf{E}(\Delta)$-graded structure

$$\mathbf{E}(\Gamma_2) \otimes_{\mathbf{E}(\Gamma_1)} \mathbf{E}(A, \underline{B}).$$

Proposition 16.6.44. *In the situation of example* 16.6.43*, suppose moreover that A is perfectoid, and Δ, Γ_1, Γ_2 are p-perfect. We have :*

(i) *The completion $(\Gamma_2 \otimes_{\Gamma_1} A)^\wedge$ of $\Gamma_2 \otimes_{\Gamma_1} A$ is perfectoid.*

(ii) *There exists a natural isomorphism of topological rings with Δ-graded structures :*

$$\omega : \mathbf{E}((\Gamma_2 \otimes_{\Gamma_1} (A, \underline{B}))^\wedge) \xrightarrow{\sim} (\Gamma_2 \otimes_{\Gamma_1} \mathbf{E}(A, \underline{B}))^\wedge$$

(notation of (9.4.24)*) fitting into the commutative diagrams*

$$\begin{array}{ccc} \mathbf{E}(A, \underline{B}) & \xrightarrow{\mathbf{E}(j_{(A,\underline{B})})} & \mathbf{E}((\Gamma_2 \otimes_{\Gamma_1} (A, \underline{B}))^\wedge) \\ & \searrow^{j_{\mathbf{E}(A,\underline{B})}} & \downarrow \omega \\ & & (\Gamma_2 \otimes_{\Gamma_1} \mathbf{E}(A, \underline{B}))^\wedge \end{array} \qquad \begin{array}{ccc} \Gamma_2 & \xrightarrow{\mathbf{E}(i_{(A,\underline{B})})} & \mathbf{E}((\Gamma_2 \otimes_{\Gamma_1} A)^\wedge) \\ & \searrow^{i_{\mathbf{E}(A,\underline{B})}} & \downarrow \omega \\ & & (\Gamma_2 \otimes_{\Gamma_1} \mathbf{E}(A))^\wedge. \end{array}$$

Proof. (Here we identify Γ_1 and Γ_2 with $\mathbf{E}(\Gamma_1)$ and $\mathbf{E}(\Gamma_2)$, via the isomorphisms $\overline{u}_{\Gamma_1}$ and $\overline{u}_{\Gamma_2}$.) Let us form as well the topological ring with Δ-graded structure

$$(A', \underline{D}) := ((A, \underline{B})[\Gamma_2]^\wedge)_{/\Delta}.$$

Explicitly, A' is the completion of $A[\Gamma_2]$, where the latter is endowed with the unique linear topology such that the natural map $A \to A[\Gamma_2]$ is adic; then $\mathrm{gr}_\delta D$ is the topological closure in A' of $\mathrm{gr}_\delta(B[\Gamma_2]_{/\Delta})$, for every $\delta \in \Delta$. Here, as in example 16.6.43(i), the Δ-graded $\mathbb{Z}$-algebra $B[\Gamma_2]_{/\Delta}$ is obtained from the $\Delta \oplus \Gamma_2$-graded $\mathbb{Z}$-algebra $B[\Gamma_2]$ via the morphism η. With this notation, from proposition 8.2.13(i,v) and a simple inspection we get a natural identification

(16.6.45) $$(\Gamma_2 \otimes_{\Gamma_1} (A, \underline{B}))^\wedge \xrightarrow{\sim} (A', \underline{D})/ID.$$

Set $(\mathbf{E}, \underline{B}_\mathbf{E}) := \mathbf{E}(A, \underline{B})$; from lemma 16.6.40 and proposition 9.4.22(iii), we know already that A' is perfectoid, and there exists an isomorphism

(16.6.46) $$(\mathbf{E}', \underline{D}_\mathbf{E}) := \mathbf{E}(A', \underline{D}) \xrightarrow{\sim} ((\mathbf{E}, \underline{B}_\mathbf{E})[\Gamma_2]^\wedge)_{/\Delta}$$

of topological rings with Δ-graded structures. Set

$$\mathscr{I} := \mathrm{Ker}\,(B_\mathbf{E}[\Gamma_2] \to \Gamma_2 \otimes_{\Gamma_1} B_\mathbf{E}).$$

Claim 16.6.47. (i) $\Phi_{\mathbf{E}'}(\mathscr{I}\mathbf{E}') = \mathscr{I}\mathbf{E}'$.

(ii) $\{\mathscr{I}\mathbf{E}'\} = (IA')^c$.

Proof of the claim. Let $\overline{\psi}_\mathbf{E} := \mathbf{E}(i) \circ \psi_\mathbf{E} : \Gamma_1 \to \mathbf{E}$ where $i : B^* \to A$ is the natural map. Clearly, the ideals IA' and $\mathscr{I}\mathbf{E}'$ are generated respectively by the systems

$$\mathscr{S}_A := (\varphi(\gamma) - \psi(\gamma) \mid \gamma \in \Gamma_1) \qquad \text{and} \qquad \mathscr{S}_\mathbf{E} := (\varphi_\mathbf{E}(\gamma) - \overline{\psi}_\mathbf{E}(\gamma) \mid \gamma \in \Gamma_1).$$

Since Γ_1 is perfect, assertion (i) follows already, and for (ii) we come down to showing that the topological closure of the ideal generated by the system $\overline{u}_{A'}(\mathscr{S}_\mathbf{E})$ equals the topological closure of the ideal generated by $\mathscr{S}_A$. Now, proposition 9.3.62 says that each generator $\overline{u}_{A'}(\varphi_\mathbf{E}(\gamma) - \overline{\psi}_\mathbf{E}(\gamma))$ can be written as a p-adically convergent series $\sum_{n\in\mathbb{N}} p^n c_{\gamma,n}$ where

(16.6.48) $$c_{\gamma,0} = \overline{u}_{A'}(\varphi_\mathbf{E}(\gamma)) - \overline{u}_{A'}(\overline{\psi}_\mathbf{E}(\gamma))$$

and for every $n > 0$, the term $c_{\gamma,n}$ is a finite $\mathbb{Z}_p$-linear combination of elements of the form

$$\overline{u}_{A'}(\varphi_\mathbf{E}(\gamma)^s \cdot \overline{\psi}_\mathbf{E}(\gamma)^{s'}) - \overline{u}_{A'}(\varphi_\mathbf{E}(\gamma)^{s'} \cdot \overline{\psi}_\mathbf{E}(\gamma)^s)$$

where $s, s' \in p^{-n}\mathbb{N}$ and $s + s' = 1$. However, lemma 16.6.40(ii) implies that

$$\overline{u}_{A'}(\varphi_\mathbf{E}(\gamma)) = \varphi(\gamma) \qquad \text{and} \qquad \overline{u}_{A'}(\overline{\psi}_\mathbf{E}(\gamma)) = \psi(\gamma) \qquad \text{for every } \gamma \in \Gamma_1.$$

Thus $\{\mathscr{I}\mathbf{E}'\}$ lies in the topological closure of the ideal generated by the elements

$$t_{\gamma,s} := \varphi(\gamma)^s \cdot \psi(\gamma)^{s'} - \varphi(\gamma)^{s'} \cdot \psi(\gamma)^s \qquad \text{for every } \gamma \in \Gamma_1 \text{ and } s, s' \in \mathbb{N}[1/p] \text{ with } s + s' = 1.$$

To ease notation, set $\lambda := \varphi(\gamma)$ and $\mu := \psi(\gamma)$; we notice that

$$(\lambda^s - \mu^s) \cdot (\mu^{1-s} + \lambda^{1-s}) - (\lambda - \mu) = t_{\gamma,s}$$

and since $\lambda^s - \mu^s, \lambda - \mu \in \mathscr{S}_A$, we conclude that $t_{\gamma,s} \in IA'$. By the same token, from (16.6.48) we get $c_0 = \lambda - \mu$, so finally $(IA')^c = \{\mathscr{I}\mathbf{E}'\} + p(IA')^c$. From this, an easy induction yields

$$(IA')^c = \{\mathscr{I}\mathbf{E}'\} + p^n(IA')^c \qquad \text{for every } n \in \mathbb{N}$$

whence the claim, since $\{\mathscr{I}\mathbf{E}'\}$ is a closed ideal. ◊

From claim 16.6.47 and corollary 16.3.38 we deduce that $A'/(IA')^c$ is perfectoid and there exists an isomorphism of topological rings $\omega' : \mathbf{E}(A'/(IA')^c) \xrightarrow{\sim} \mathbf{E}'/(\mathscr{I}\mathbf{E}')^c$ such that

$$\omega' \circ \mathbf{E}(\pi_{A'}) = \pi_{\mathbf{E}'}$$

where $\pi_{A'} : A' \to A'/(IA')^c$ and $\pi_{\mathbf{E}'} : \mathbf{E}' \to \mathbf{E}'/(\mathscr{I}\mathbf{E}')^c$ are the projections. This completes the proof of (i), and we also see that the projection $(\mathbf{E}', \underline{D}_\mathbf{E}) \to (\mathbf{E}', \underline{D}_\mathbf{E})/\mathscr{I}\underline{D}_\mathbf{E}$ factors through a morphism

(16.6.49) $$\mathbf{E}((A', \underline{D})/ID) \to (\mathbf{E}', \underline{D}_\mathbf{E})/\mathscr{I}\underline{D}_\mathbf{E}$$

which must be an isomorphism, by example 8.5.7(iii). On the other hand, from (16.6.46) we get a natural identification

$$(16.6.50)\qquad (\mathbf{E}', \underline{D}_{\mathbf{E}})/\mathscr{I}\underline{D}_{\mathbf{E}} \xrightarrow{\sim} (\Gamma_2 \otimes_{\Gamma_1} \mathbf{E}(A, \underline{B}))^{\wedge}.$$

Lastly, combining (16.6.49), (16.6.50) and (16.6.45) we get the sought isomorphism ω. The commutativity of the diagrams in (ii) follows by direct inspection, taking into account the corresponding properties of the isomorphism (16.6.46) stated in lemma 16.6.40(ii) : details left to the reader. □

16.6.51. Let Λ be any (small) set and $(\Delta, +, 0, \leq)$ any ordered abelian group; we set

$$\Delta^+ = \{\delta \in \Delta \mid \delta \geq 0\} \qquad P := \Delta^{+(\Lambda)} \qquad Q := \Delta^{+(\Lambda\times\Lambda)}$$

(cp. definition 9.1.1(i)). Denote by $I \subset P$ the ideal generated by the canonical system of generators $e_\bullet := (e_\lambda \mid \lambda \in \Lambda)$ for P, *i.e.* $e_\lambda := (\delta_{\lambda\mu} \mid \mu \in \Lambda)$, where $\delta_{\lambda\mu} := 1$ for $\lambda = \mu$, and $\delta_{\lambda\mu} := 0$ otherwise, for every $\lambda, \mu \in \Lambda$. Likewise, we let

$$(f_\lambda, f'_\lambda \mid \lambda \in \Lambda) \qquad (\text{resp. } (e_{\lambda,\lambda'} \mid \lambda, \lambda' \in \Lambda))$$

be the canonical system of generators of $P \times P$ (resp. of Q).

- With this notation, we define Δ-gradings on P, $P \times P$ and Q by the rules :

$$|\delta e_\lambda|_P := \delta \qquad |\delta f_\lambda + \delta' f'_\lambda|_{P\times P} := \delta \qquad |\delta e_{\lambda,\lambda'}|_Q := \delta$$

for every $\lambda, \lambda' \in \Lambda$ and every $\delta, \delta' \in \Delta$ (see definition 4.8.8(i)), and we set

$$I^{\langle\delta\rangle} := \{x \in P \mid |x|_P \geq \delta\} \qquad \text{for every } \delta \in \Delta.$$

Notice that, in case $\Delta = \mathbb{Z}[1/p]$, the ideals $I^{\langle\delta\rangle}$ of P are the same as the ones introduced in (9.3.68), so the notation does not conflict with *loc.cit.* Lastly, we define

$$\Gamma_I^+ := \bigcup_{\delta\in\Delta^+} I^{\langle\delta\rangle} \times \{\delta\} \qquad \Gamma_I := \bigcup_{\delta\in\Delta} I^{\langle\delta\rangle} \times \{\delta\}$$

and notice that Γ_I^+ (resp. Γ_I) is a submonoid of $P \times \Delta^+$ (resp. of $P \times \Delta$), so it inherits a natural Δ-grading, given by the projection on the factor Δ. We have morphisms of Δ-graded monoids

$$(16.6.52)\qquad Q \underset{\psi'}{\overset{\psi}{\rightrightarrows}} P \times P \xrightarrow{\varphi} \Gamma_I^+$$

such that

$$\varphi(\delta f_\lambda + \delta' f'_\lambda) := ((\delta + \delta')e_\lambda, \delta) \qquad \psi(\delta e_{\lambda,\lambda'}) := \delta(f_\lambda + f'_{\lambda'}) \qquad \psi'(\delta e_{\lambda,\lambda'}) := \delta(f'_\lambda + f_{\lambda'})$$

for every $\lambda, \lambda' \in \Lambda$ and every $\delta, \delta' \in \Delta$.

Lemma 16.6.53. *With the notation of* (16.6.51)*, we have :*

(i) (16.6.52) *is a presentation for* Γ_I^+, i.e. φ *identifies* Γ_I^+ *with the coequalizer of* ψ *and* ψ'.

(ii) *Define a* Δ*-grading of* $P \times \Delta^+$ *by the rule :* $(x, \delta) \mapsto |x|_P - \delta$ *for every* $x \in P$ *and* $\delta \in \Delta^+$. *Then the mapping*

$$P \times \Delta^+ \to \Gamma_I \qquad (x, \delta) \mapsto (x, |x|_P - \delta)$$

is an isomorphism of Δ*-graded monoids.*

Proof. (i): Indeed, say that

$$g := \sum_{\lambda\in\Lambda}(r_\lambda f_\lambda + r'_\lambda f'_\lambda) \qquad \text{and} \qquad h := \sum_{\lambda\in\Lambda}(s_\lambda f_\lambda + s'_\lambda f'_\lambda)$$

are two elements of $P \times P$ whose images agree in Γ_I^+, so that $r_\lambda, r'_\lambda, s_\lambda, s'_\lambda \in \Delta^+$ and

$$r_\lambda + r'_\lambda = s_\lambda + s'_\lambda \qquad \text{for every } \lambda \in \Lambda \qquad \text{and} \qquad \sum_{\lambda\in\Lambda} r_\lambda = \sum_{\lambda\in\Lambda} s_\lambda. \tag{16.6.54}$$

Let $\sim$ be the equivalence relation defined by the pair (ψ, ψ'); we have to check that $g \sim h$. Now, consider any $\lambda \in \Lambda$, and suppose that $r_\lambda \leq s_\lambda$; then we may write $g = g' + r_\lambda f_\lambda$, $h = h' + r_\lambda f_\lambda$ for elements $g', h' \in P \times P$ such that $\varphi(g') = \varphi(h')$; it then suffices to check that $g' \sim h'$, and we are reduced to the case where $r_\lambda = 0$. In case $r_\lambda > s_\lambda$, we argue symmetrically, to reduce to the case where $s_\lambda = 0$. Likewise, we can reduce to the case where either $r'_\lambda = 0$ or $s'_\lambda = 0$. Repeating the argument for every $\lambda \in \Lambda$, and taking (16.6.54) into account, we may suppose from start that

$$g = \sum_{\lambda\in A} r_\lambda f_\lambda + \sum_{\mu\in B} r_\mu f'_\mu \qquad h = \sum_{\lambda\in A} r_\lambda f'_\lambda + \sum_{\mu\in B} r_\mu f_\mu$$

for some finite subsets $A, B \in \Lambda$ with $A \cap B = \varnothing$, and a system $(r_\lambda \mid \lambda \in A \cup B)$ of elements of $\Delta^+ \setminus \{0\}$ such that

$$\sum_{\mu\in B} r_\mu = \sum_{\lambda\in A} r_\lambda. \tag{16.6.55}$$

Next, we argue by induction on the cardinality c of $A \cup B$. If $c = 0$, there is nothing to prove. Suppose then that $c > 0$ and that the assertion is already known whenever the cardinality of $A \cup B$ is strictly smaller than c. Notice that $A \neq \varnothing$, since otherwise (16.6.55) would imply that $B = \varnothing$, contradicting the assumption that $c > 0$; likewise, we must have $B \neq \varnothing$. Thus, pick any $\lambda \in A$ and $\mu \in B$, set $A' := A \setminus \{\lambda\}$, $B' := B \setminus \{\mu\}$, and suppose first that $r_\lambda \leq r_\mu$; we set

$$g' := \sum_{\lambda\in A'} r_\lambda f_\lambda + \sum_{\mu\in B'} r_\mu f'_\mu + (r_\mu - r_\lambda)\cdot f'_\mu \qquad h' := \sum_{\lambda\in A'} r_\lambda f'_\lambda + \sum_{\mu\in B'} r_\mu f_\mu + (r_\mu - r_\lambda)\cdot f_\mu$$

so that $g = g' + r_\lambda \cdot (f_\lambda + f'_\mu)$ and $h = h' + r_\lambda \cdot (f'_\lambda + f_\mu)$. However, $r_\lambda \cdot (f_\lambda + f'_\mu) \sim r_\lambda \cdot (f'_\lambda + f_\mu)$, so we are reduced to checking that $g' \sim h'$. The latter is known by our inductive assumption. A symmetric argument will do, in case $r_\lambda > r_\mu$.

(ii) shall be left to the reader. □

16.6.56. In the situation of (16.6.51), take $\Delta := \mathbb{Z}[1/p]$ with its standard ordering, so that $\Delta^+ = \mathbb{N}[1/p]$. In this case, it is easily seen that P, Γ_I^+ and Γ_I are p-perfect monoids. In this paragraph, it will be convenient to switch to a multiplicative notation for the composition laws of these monoids : in other words, we shall hereafter replace $(P, +, 0)$ by $(\exp P, \cdot, 1)$, and likewise for Γ_I^+ and Γ_I (notation of (6.1)). Correspondingly, the product $\delta \cdot x$ of an element $\delta \in \Delta^+$ and an element $x \in P$ shall be replaced by the exponential x^δ.

Let also A be any perfectoid ring, set $\mathbf{E} := \mathbf{E}(A)$ and denote by $\underline{A}$ the topological ring with Δ-graded structure (A, A, Δ), such that $\mathrm{gr}_0 A = A$ and $\mathrm{gr}_\delta A = 0$ for every $\delta \neq 0$ in Δ; define likewise the topological ring with Δ-graded structure $\underline{\mathbf{E}} := (\mathbf{E}, \mathbf{E}, \Delta)$. Choose an ideal of definition J for A, and set $\mathscr{J} := \overline{u}_{A/pA}^{-1}(J/pA)$, which is an ideal of definition for $\mathbf{E}$. Let $\beta_\bullet := (\beta_\lambda \mid \lambda \in \Lambda)$ be any system of elements of $\mathbf{E}$; then $\beta_\bullet$ determines a unique morphism of monoids

$$\varphi_{\mathbf{E}} : P \to \mathbf{E} \qquad \text{such that} \qquad e_\lambda^\delta \mapsto \beta_\lambda^\delta \qquad \text{for every } \lambda \in \Lambda \text{ and } \delta \in \Delta$$

and composing with the map $\overline{u}_A : \mathbf{E} \to A$ we get also a morphism $\varphi_A : P \to A$. Moreover, notice that $I^{\langle 0\rangle} = P$, so we have as well a morphism $i_0 : P \to \Gamma_I^+$, and we can define the rings

$$\mathscr{E}_+ := \Gamma_I^+ \otimes_P \mathbf{E} \subset \mathscr{E} := \Gamma_I \otimes_P \mathbf{E} \qquad \text{and} \qquad \mathscr{A}_+ := \Gamma_I^+ \otimes_P A \subset \mathscr{A} := \Gamma_I \otimes_P A$$

as well as the topological rings with Δ-graded structures

$$(\mathscr{E}_+^\wedge, \underline{\mathscr{E}}^c_+) := (\Gamma_I^+ \otimes_P \underline{\mathbf{E}})^\wedge \quad \subset \quad (\mathscr{E}^\wedge, \underline{\mathscr{E}}^c_+) := (\Gamma_I \otimes_P \underline{\mathbf{E}})^\wedge$$
$$(\mathscr{A}_+^\wedge, \underline{\mathscr{A}}^c_+) := (\Gamma_I^+ \otimes_P \underline{A})^\wedge \quad \subset \quad (\mathscr{A}, \underline{\mathscr{A}}^c) := (\Gamma_I \otimes_P \underline{A})^\wedge$$

(notation of (6.1.33) and example 16.6.43). We endow $\mathscr{E}$ and $\mathscr{E}_+$ with their $\mathscr{J}$-adic topologies, and $\mathscr{A}$, $\mathscr{A}_+$ with their J-adic topologies; then $\mathscr{E}_+^\wedge$ (resp. $\mathscr{E}^\wedge$) is the completion of $\mathscr{E}_+$ (resp. of $\mathscr{E}$) and $\mathscr{A}_+^\wedge$ (resp. $\mathscr{A}^\wedge$) is the completion of $\mathscr{A}_+$ (resp. of $\mathscr{A}$). Furthermore, the rings $\mathscr{A}$ and $\mathscr{E}$ inherit natural Δ-gradings from Γ_I, such that

$$I^{\langle\delta\rangle} \otimes_P \mathbf{E} = \mathrm{gr}_\delta \mathscr{E} \qquad I^{\langle\delta\rangle} \otimes_P A = \mathrm{gr}_\delta \mathscr{A} \qquad \text{for every } \delta \in \Delta$$

and by restricting to Δ^+, we get corresponding Δ^+-gradings on $\mathscr{A}_+$ and $\mathscr{E}_+$. Then, a direct inspection shows that $\mathrm{gr}_\delta \mathscr{E}^c$ (resp. $\mathrm{gr}_\delta \mathscr{A}^c$) is the separated completion of $\mathrm{gr}_\delta \mathscr{E}$ (resp. of $\mathrm{gr}_\delta \mathscr{A}$), for every $\delta \in \Delta$. Let $i_\Gamma : \Gamma_I^+ \to \Gamma_I$ be the inclusion map; as in example 16.6.43(i), we point out the commutative diagram of monoids and of topological rings with Δ-graded structures :

$$\begin{array}{ccccccccc}
\underline{\mathbf{E}} & \xrightarrow{j^+_{\mathbf{E}}} & (\mathscr{E}_+^\wedge, \underline{\mathscr{E}}^c_+) & \xleftarrow{i^+_{\mathbf{E}}} & \Gamma_I^+ & \xrightarrow{i^+_A} & (\mathscr{A}_+^\wedge, \underline{\mathscr{A}}^c_+) & \xleftarrow{j^+_A} & \underline{A} \\
& \searrow^{j_{\mathbf{E}}} & \downarrow{\scriptstyle (i_\Gamma \otimes_P \mathbf{E})^\wedge} & & \downarrow{\scriptstyle i_\Gamma} & & \downarrow{\scriptstyle (i_\Gamma \otimes_P A)^\wedge} & \swarrow_{j_A} & \\
& & (\mathscr{E}^\wedge, \underline{\mathscr{E}}^c) & \xleftarrow{i_{\mathbf{E}}} & \Gamma_I & \xrightarrow{i_A} & (\mathscr{A}^\wedge, \underline{\mathscr{A}}^c). & &
\end{array}$$

Remark 16.6.57. In the situation of (16.6.56), suppose furthermore that the system $\beta_\bullet$ generates an open ideal of $\mathbf{E}$. Then, combining corollaries 8.6.40(ii) and 16.3.40(ii) together with proposition 16.6.5(ii), we deduce that $\Gamma_I^+ \otimes_P \mathbf{E}$ and $\Gamma_I \otimes_P \mathbf{E}$ are already $\mathscr{J}$-adically complete and separated, and $\Gamma_I^+ \otimes_P A$ and $\Gamma_I \otimes_P A$ are J-adically complete and separated.

Corollary 16.6.58. *With the notation of* (16.6.56)*, the rings $\mathscr{E}_+^\wedge$, $\mathscr{E}^\wedge$, $\mathscr{A}_+^\wedge$ and $\mathscr{A}^\wedge$ are perfectoid, and we have natural isomorphisms of topological rings with Δ-graded structures*

$$\omega_+ : \mathbf{E}(\mathscr{A}_+^\wedge, \underline{\mathscr{A}}^c_+) \xrightarrow{\sim} (\mathscr{E}_+^\wedge, \underline{\mathscr{E}}^c_+) \qquad \omega : \mathbf{E}(\mathscr{A}^\wedge, \underline{\mathscr{A}}^c) \xrightarrow{\sim} (\mathscr{E}^\wedge, \underline{\mathscr{E}}^c)$$

fitting into the commutative diagrams

$$\begin{array}{ccccc}
\underline{\mathbf{E}} & \xrightarrow{\mathbf{E}(j^+_A)} & \mathbf{E}(\mathscr{A}_+^\wedge, \underline{\mathscr{A}}^c_+) & \xleftarrow{\mathbf{E}(i^+_A)} & \Gamma_I^+ \\
& \searrow^{j^+_{\mathbf{E}}} & \downarrow{\scriptstyle \omega_+} & \swarrow_{i^+_{\mathbf{E}}} & \\
& & (\mathscr{E}_+^\wedge, \underline{\mathscr{E}}^c_+) & &
\end{array}
\qquad
\begin{array}{ccccc}
\underline{\mathbf{E}} & \xrightarrow{\mathbf{E}(j_A)} & \mathbf{E}(\mathscr{A}^\wedge, \underline{\mathscr{A}}^c) & \xleftarrow{\mathbf{E}(i_A)} & \Gamma_I \\
& \searrow^{j_{\mathbf{E}}} & \downarrow{\scriptstyle \omega} & \swarrow_{i_{\mathbf{E}}} & \\
& & (\mathscr{E}^\wedge, \underline{\mathscr{E}}^c). & &
\end{array}$$

Proof. Clearly we have $\mathbf{E}(\underline{A}) = \underline{\mathbf{E}}$. The corollary then becomes a special case of proposition 16.6.44, after we have shown

Claim 16.6.59. $\varphi_{\mathbf{E}} \circ \overline{u}_P = \mathbf{E}(\overline{u}_A \circ \varphi_{\mathbf{E}})$.

Proof of the claim. Clearly $\varphi_{\mathbf{E}} \circ \overline{u}_P = \overline{u}_{\mathbf{E}} \circ \mathbf{E}(\varphi_{\mathbf{E}})$, so we are reduced to checking the identity $\overline{u}_{\mathbf{E}} = \mathbf{E}(\overline{u}_A)$, and the latter holds by remark 9.4.5(ii). ☐

16.6.60. From lemma 16.6.53(i) we deduce a corresponding coequalizer presentation for $\mathscr{E}_+$ in the category of Δ-graded $\mathbb{Z}$-algebras :

$$\mathbb{Z}[Q] \rightrightarrows \mathbb{Z}[P \times P] \otimes_{\mathbb{Z}[P]} \mathbf{E} = \mathbf{E}[P] \longrightarrow \mathbb{Z}[\Gamma_I^+] \otimes_{\mathbb{Z}[P]} \mathbf{E} = \mathscr{E}_+$$

where $\mathbf{E}$ and $\mathbb{Z}[P \times P]$ are regarded as $\mathbb{Z}[P]$-algebras via $\varphi_{\mathbf{E}}$ and respectively via the morphism of monoids $P \to P \times P$ given by the rule $e^\delta_\lambda \mapsto f'^\delta_\lambda$ for every $\lambda \in \Lambda$ and every $\delta \in \Delta^+$. Likewise we may present $\mathscr{A}_+$, and there follow natural isomorphisms

$$\mathscr{E}_+ \xrightarrow{\sim} \mathbf{E}[P]/\mathscr{I}_{\mathbf{E}} \qquad \mathscr{A}_+ \xrightarrow{\sim} A[P]/\mathscr{I}_A$$

where $\mathscr{I}_{\mathbf{E}} \subset \mathbf{E}[P]$ is the ideal generated by the system

$$(\beta^r_\lambda \cdot e^r_\mu - \beta^r_\mu \cdot e^r_\lambda \mid \lambda, \mu \in \Lambda;\ r \in \mathbb{N}[1/p]).$$

and $\mathscr{I}_A \subset A[P]$ is the ideal generated by the system

$$(\overline{u}_A(\beta^r_\lambda) \cdot e^r_\mu - \overline{u}_A(\beta^r_\mu) \cdot e^r_\lambda \mid \lambda, \mu \in \Lambda;\ r \in \mathbb{N}[1/p]).$$

Especially, notice that $\mathscr{I}_{\mathbf{E}}$ (resp. $\mathscr{I}_A$) is a graded ideal of the Δ-graded ring $\mathbf{E}[P]_{/\Delta}$ (resp. $A[P]_{/\Delta}$), arising via the grading $|\cdot|_P : P \to \Delta$, and

$$\mathrm{gr}_\delta \mathscr{E}_+ = (\mathrm{gr}_\delta \mathbf{E}[P]_{/\Delta})/\mathrm{gr}_\delta \mathscr{I}_{\mathbf{E}} = \mathbf{E}[\mathrm{gr}_\delta P]/\mathrm{gr}_\delta \mathscr{I}_{\mathbf{E}} \qquad \text{for every } \delta \in \Delta.$$

We can then form the topological rings with Δ-graded structure

$$(\mathscr{E}^s_+, \underline{\mathscr{E}}^s_+) := (\underline{\mathbf{E}}[P]_{/\Delta})/\mathscr{I}_{\mathbf{E}} \qquad (\mathscr{A}^s_+, \underline{\mathscr{A}}^s_+) := (\underline{A}[P]_{/\Delta})/\mathscr{I}_A$$

that are the maximal separated quotients $\mathscr{E}^s_+$ of $\mathscr{E}_+$ and $\mathscr{A}^s_+$ of $\mathscr{A}_+$, endowed with the Δ-graded $\mathbb{Z}$-algebra structures

$$\underline{\mathscr{E}}^s_+ := \bigoplus_{\delta \in \Delta^+} (I^{\langle \delta \rangle} \otimes_P \mathbf{E})^s \qquad \text{and} \qquad \underline{\mathscr{A}}^s_+ := \bigoplus_{\delta \in \Delta^+} (I^{\langle \delta \rangle} \otimes_P A)^s$$

where $(I^{\langle \delta \rangle} \otimes_P \mathbf{E})^s$ denotes the maximal separated quotient of $I^{\langle \delta \rangle} \otimes_P \mathbf{E}$, for every $\delta \in \Delta$, and likewise for $(I^{\langle \delta \rangle} \otimes_P A)^s$. It is now straightforward that the completion maps $\mathscr{E}^s_+ \to \mathscr{E}^\wedge_+$ and $\mathscr{A}^s_+ \to \mathscr{A}^\wedge_+$ factor through natural isomorphisms

$$(\mathscr{E}^s_+, \underline{\mathscr{E}}^s_+)^\wedge \xrightarrow{\sim} (\mathscr{E}^\wedge_+, \underline{\mathscr{E}}^c_+) \qquad \text{and} \qquad (\mathscr{A}^s_+, \underline{\mathscr{A}}^s_+)^\wedge \xrightarrow{\sim} (\mathscr{A}^\wedge_+, \underline{\mathscr{A}}^c_+).$$

16.7. **Perfectoid spaces.** Some basic verifications concerning perfectoid spaces will employ a certain amount of formal algebraic geometry, that we collect hereafter.

16.7.1. Quite generally, let R be any $\mathbb{Q}_+$-graded ring. We associate with R a projective scheme

$$\mathrm{Proj}\, R$$

as follows. For every $\gamma \in \mathbb{Q}_{>0}$, let $R^{(\gamma)} := \bigoplus_{k \in \mathbb{N}} R_{k\gamma}$, which we regard as an $\mathbb{N}$-graded ring, whose k-graded direct summand is $R_{k\gamma}$, for every $k \in \mathbb{N}$. Set $Y^{(\gamma)} := \mathrm{Proj}\, R^{(\gamma)}$ for every such γ. If $n > 0$ is any integer, the discussion of (10.6.11) yields a natural isomorphism

$$Y^{(\gamma)} \xrightarrow{\Omega^{(\gamma)}_n} Y^{(n\gamma)}$$

of R_0-schemes, and it is easily seen that

$$\Omega^{(n\gamma)}_p \circ \Omega^{(\gamma)}_n = \Omega^{(\gamma)}_{pn} \qquad \text{for every integers } n, p > 0.$$

Thus, the rule $\gamma \mapsto Y^{(\gamma)}$ yields a well defined filtered system of isomorphisms, and we let $\mathrm{Proj}\, R$ be any choice of an R_0-scheme representing the colimit of this system; *e.g.* $\mathrm{Proj}\, R := Y^{(\gamma_0)}$, for some fixed $\gamma_0 \in \mathbb{Q}_{>0}$, together with the cocone $(\Omega^{(\gamma)} : Y^{(\gamma)} \to \mathrm{Proj}\, R \mid \gamma \in \mathbb{Q}_{>0})$ resulting from the filtered system of morphisms $\Omega^{(\bullet)}_\bullet$.

16.7.2. To ease notation, we shall let $Y := \mathrm{Proj}\, R$. For every $\gamma \in \mathbb{Q}_{>0}$, every $n \in \mathbb{N}$, and every $f \in R_{n\gamma}$ we have the open subset $D_+(f) \subset Y^{(\gamma)}$ defined as in (10.6.1), and we denote also $D_+(f) \subset Y$ the image of this open subset under $\Omega^{(\gamma)}$. It is easily seen that the resulting open subset of Y is independent of the choice of γ. We also set $\mathscr{O}_Y(0) := \mathscr{O}_Y$ and

$$\mathscr{O}_Y(\gamma) := \Omega^{(|\gamma|)}_* \mathscr{O}_{Y^{(|\gamma|)}}(\gamma/|\gamma|) \qquad \text{for every } \gamma \in \mathbb{Q} \setminus \{0\}$$

and we notice that – according to (10.6.16) – the quasi-coherent $\mathscr{O}_Y$-module $\mathscr{O}_Y(\gamma)$ restricts to an invertible $\mathscr{O}_Y$-module on the open subset

$$U_\gamma(R) := \bigcup_{f \in R_{|\gamma|}} D_+(f) \qquad \text{for every } \gamma \in \mathbb{Q} \setminus \{0\}.$$

Remark 16.7.3. Let $\Gamma \subset \mathbb{Q}_+$ be any submonoid, and suppose that $R = \mathbb{Q}_+ \times_\Gamma R'$ for a Γ-graded ring R'. Then it is easily seen that $\mathscr{O}_Y(\gamma) = 0$ and $U_\gamma(R) = \varnothing$ for every $\gamma \in \mathbb{Q} \setminus \Gamma^{\mathrm{gp}}$.

16.7.4. In view of the isomorphisms (10.6.14), we get natural identifications

$$\mathscr{O}_Y(n\gamma) \xrightarrow{\sim} \Omega_*^{(\gamma)}\mathscr{O}_{Y^{(\gamma)}}(n) \qquad \text{for every } n \in \mathbb{Z} \text{ and every } \gamma \in \mathbb{Q}_{>0}. \tag{16.7.5}$$

For every $\gamma, \gamma' \in \mathbb{Q}$ there exists as well a natural morphism

$$\mathscr{O}_Y(\gamma) \otimes_{\mathscr{O}_Y} \mathscr{O}_Y(\gamma') \to \mathscr{O}_Y(\gamma + \gamma') \tag{16.7.6}$$

that generalizes (10.6.21). Namely, pick a common divisor, *i.e.* a rational number $\delta \in \mathbb{Q}_{>0}$ and integers $p, q \in \mathbb{N}$ such that $p\delta = \gamma$ and $q\delta = \gamma'$; we have first a natural identification

$$\mathscr{O}_Y(\gamma) \otimes_{\mathscr{O}_Y} \mathscr{O}_Y(\gamma') \xrightarrow{\sim} \Omega_*^{(\delta)}\mathscr{O}_{Y^{(\delta)}}(p) \otimes_{\mathscr{O}_Y} \Omega_*^{(\delta)}\mathscr{O}_{Y^{(\delta)}}(q) \xrightarrow{\sim} \Omega_*^{(\delta)}(\mathscr{O}_{Y^{(\delta)}}(p) \otimes_{\mathscr{O}_Y^{(\delta)}} \mathscr{O}_{Y^{(\delta)}}(q))$$

which we combine with the map

$$\Omega_*^{(\delta)}(\mathscr{O}_{Y^{(\delta)}}(p) \otimes_{\mathscr{O}_Y^{(\delta)}} \mathscr{O}_{Y^{(\delta)}}(q)) \to \Omega_*^{(\delta)}\mathscr{O}_{Y^{(\delta)}}(p+q) \xrightarrow{\sim} \mathscr{O}_Y(\gamma + \gamma')$$

deduced from (10.6.21) and the inverse of the isomorphism (16.7.5), to obtain (16.7.6), and it is easily seen that the resulting map is independent of the choice of δ : details left to the reader. Especially, (16.7.6) restricts to an isomorphism on $U_{\gcd(\gamma,\gamma')}(R)$, where $\gcd(\gamma, \gamma')$ is the greatest common divisor of γ and γ' (here we let $\gcd(0,0) := 0$; actually, it is not difficult to show that (16.7.6) is an isomorphism on the open subset $U_\gamma(R) \cup U_{\gamma'}(R)$). Furthermore, let

$$\pi^{(\gamma)} : Y^{(\gamma)} \to \operatorname{Spec} R_0 \qquad \text{and} \qquad \pi : Y \to \operatorname{Spec} R_0$$

be the structure morphisms; the morphism (10.6.22) induces a natural map

$$\pi^* R_\gamma^\sim \xrightarrow{\sim} \Omega_*^{(\gamma)}\pi^{(\gamma)*}R_1^{(\gamma)} \to \Omega_*^{(\gamma)}\mathscr{O}_{Y^{(\gamma)}}(1) = \mathscr{O}_Y(\gamma) \qquad \text{for every } \gamma \in \Gamma \setminus \{0\}$$

which restricts to an epimorphism on $U_\gamma(R)$.

16.7.7. Let R' be another $\mathbb{Q}_+$-graded ring, and $\varphi : R \to R'$ a morphism of $\mathbb{Q}_+$-graded rings; set $Y' := \operatorname{Proj} R'$, $Y'^{(\gamma)} := \operatorname{Proj} R'^{(\gamma)}$, and denote by $\Omega'^{(\gamma)} : Y^{(\gamma)} \to Y'$ and $\Omega_n'^{(\gamma)} : Y^{(\gamma)} \to Y'^{(n\gamma)}$ the induced morphism, for every $\gamma \in \mathbb{Q}_{>0}$ and every integer $n > 0$. The restriction of φ

$$\varphi^{(\gamma)} : R^{(\gamma)} \to R'^{(\gamma)} \qquad \text{for every } \gamma \in \mathbb{Q}_{>0}$$

is a morphism of $\mathbb{N}$-graded rings, whence a morphism $\operatorname{Proj} \varphi^{(\gamma)} : G(\varphi^{(\gamma)}) \to Y^{(\gamma)}$ (notation of (10.6.5)). Moreover, it is easily seen that

$$\Omega_n'^{(\gamma)} G(\varphi^{(\gamma)}) = G(\varphi^{(n\gamma)}) \qquad \text{for every } \gamma \in \mathbb{Q}_{>0} \text{ and every integer } n > 0.$$

Furthermore, the resulting diagram of schemes

$$\begin{array}{ccc} G(\varphi^{(\gamma)}) & \xrightarrow{\operatorname{Proj} \varphi^{(\gamma)}} & Y^{(\gamma)} \\ {\scriptstyle \Omega_n'^{(\gamma)}} \downarrow & & \downarrow {\scriptstyle \Omega_n^{(\gamma)}} \\ G(\varphi^{(n\gamma)}) & \xrightarrow{\operatorname{Proj} \varphi^{(n\gamma)}} & Y^{(n\gamma)} \end{array}$$

commutes for every such γ and n. Thus, we may set

$$G(\varphi) := \Omega^{(\gamma)}(G(\varphi^{(\gamma)})) \qquad \text{for any } \gamma \in \mathbb{Q}_{>0}$$

and the colimit of the system $(\operatorname{Proj} \varphi^{(\gamma)} \mid \gamma \in \mathbb{Q}_+)$ is a well defined morphism

$$\operatorname{Proj} \varphi : G(\varphi) \to Y.$$

Remark 16.7.8. (i) Set $R_+ := \bigoplus_{\gamma\in\mathbb{Q}_{>0}} R_\gamma$, and define likewise R'_+. Let also $f' \in R'_+$ be any homogeneous element; we claim that $D_+(f') \subset G(\varphi)$ if and only if f' lies in the radical J of the ideal of R' generated by $\varphi(R_+)$. Indeed, say that $f' \in R'_\gamma$ for some $\gamma \in \mathbb{Q}_{>0}$; if $D_+(f') \subset G(\varphi)$, then $D_+(f') \subset G(\varphi^{(\gamma)})$ as well, so f' lies in the radical of the ideal of $R'^{(\gamma)}$ generated by $\varphi^{(\gamma)}(R_+^{(\gamma)})$ (see the discussion of (10.6.5)), and therefore f' lies in J as well. Conversely, if $f' \in J$, we may find $n, k \in \mathbb{N}$, and elements $\nu(i) \in \mathbb{Q}_{>0}$, $a_i \in R_{\nu(i)}$, $f'_i \in R'_{\gamma-\nu(i)}$ for $i = 1, \dots, k$, such that $f'^n = \sum_{i=1}^k f_i \cdot \varphi(a_i)$. Then, pick any common divisor δ of $\gamma, \nu(1), \dots, \nu(k)$ in $\mathbb{Q}_{>0}$; it follows that f' lies in the radical of the ideal of $R'^{(\delta)}$ generated by $\varphi^{(\delta)}(R_+^{(\delta)})$, whence $D_+(f') \subset G(\varphi^{(\delta)})$, and finally $D_+(f') \in G(\varphi)$.

(ii) Especially, if $\varphi(R_+)$ generates the ideal R'_+ of R', then $G(\varphi) = Y'$.

16.7.9. We shall apply the foregoing constructions to the situation contemplated in (16.4.4) : we let (A, u_0) be a pair consisting of a ring A, and a ring homomorphism

$$u_0 : R_{r,0} := \mathbb{Z}[T_1^{1/p^\infty}, \dots, T_r^{1/p^\infty}] \to A.$$

Also, we denote by $T_\bullet P_r \subset P_r := \mathbb{N}[1/p]^{\oplus r}$ the ideal generated by $\mathbb{N}^{\oplus r}$, we set $\Gamma := \mathbb{N}[1/p]$, and we consider the Γ-graded *angular Rees algebra*

$$\mathsf{R}\langle A\rangle := \bigoplus_{\gamma\in\Gamma} T_\bullet^{\langle\gamma\rangle} A.$$

Hence, for every $\gamma \in \Gamma$, the direct summand $\mathsf{R}\langle A\rangle_\gamma$ is the ideal of A generated by all products of the form $u_0(T_1^{a_1} \cdots T_r^{a_r})$, where $(a_1, \dots, a_r) \in P_r$ is any sequence of exponents such that $a_1 + \cdots + a_r = \gamma$. The *angular blowing up* of the ideal $T_\bullet A_0$ is the morphism

$$Y := \mathrm{Proj}\,(\mathsf{R}\langle A\rangle_{/\mathbb{Q}_+}) \to X := \mathrm{Spec}\, A$$

(notation of definition 7.6.1(v)). Notice that $U_\gamma(\mathsf{R}\langle A\rangle_{/\mathbb{Q}_+}) = Y$ for every $\gamma \in \Gamma \setminus \{0\}$.

Theorem 16.7.10. *In the situation of* (16.7.9)*, suppose furthermore that the ring A fulfills the condition of lemma* 16.4.5*. Then, for every $s \in \mathbb{R}_+$ we have :*

(i) *The natural maps $T_\bullet^{\lceil s\rceil} A \to T_\bullet^{\lceil s\rceil} \cdot H^0(Y, \mathscr{O}_Y) \to H^0(Y, T_\bullet^{\lceil s\rceil}\mathscr{O}_Y)$ are isomorphisms.*
(ii) $H^p(Y, T_\bullet^{\lceil s\rceil}\mathscr{O}_Y) = 0$ *for every $p > 0$.*
(iii) *The ring $\mathsf{R}\langle A\rangle$ fulfills the condition of lemma* 16.4.5.
(iv) *For every $t \in \mathbb{R}_+$ the natural map*

$$T_\bullet^{\lceil s\rceil} A \otimes_A T_\bullet^{\lceil t\rceil} A \to T_\bullet^{\lceil t+s\rceil} A$$

is an isomorphism.

Proof. Notice first that $T_\bullet^{\lceil s\rceil} \cdot H^0(Y, \mathscr{O}_Y)$ and $H^0(Y, T_\bullet^{\lceil s\rceil}\mathscr{O}_Y)$ are A-submodules of $H^0(Y, \mathscr{O}_Y)$; hence, the map $T_\bullet^{\lceil s\rceil} \cdot H^0(Y, \mathscr{O}_Y) \to H^0(Y, T_\bullet^{\lceil s\rceil}\mathscr{O}_Y)$ is injective, and in order to show (i), it suffices to prove that the composition $T_\bullet^{\lceil s\rceil} A \to H^0(Y, T_\bullet^{\lceil s\rceil}\mathscr{O}_Y)$ is an isomorphism.

Now, for every $k \in \mathbb{N}$ we set $\Gamma_k := \{n/p^k \mid n \in \mathbb{N}\} \subset \Gamma$, and consider the Γ_k-graded ring

$$S_k := \bigoplus_{\gamma\in\Gamma_k} I_k^{p^k\gamma} A \qquad \text{and the } \mathbb{Q}_+\text{-graded ring} \qquad S'_k := (S_k)_{/\mathbb{Q}_+}$$

where $I_k \subset R_{r,0}$ denotes the ideal generated by $(T_1^{1/p^k}, \dots, T_k^{1/p^k})$. The natural inclusion map is a morphism of $\mathbb{Q}_+$-graded rings $S'_k \to S'_{k+1}$ for every $k \in \mathbb{N}$, and the colimit of the resulting system $(S'_k \mid k \in \mathbb{N})$ is $\mathsf{R}(A)_{/\mathbb{Q}_+}$. Set as well

$$Y_k := \mathrm{Proj}\, S'_k \qquad \text{for every } k \in \mathbb{N}.$$

There follows a system of affine morphisms of X-schemes $(Y_k \mid k \in \mathbb{N})$, and in view of [65, Ch.IV, Prop.8.2.3] it is easily seen that its limit is Y. For every $k \in \mathbb{N}$, let $\pi_k : Y \to Y_k$ be the natural projection. Notice that the direct summand $S'_{k,\gamma}$ of S' is an ideal of A, and

$$\bigcup_{k\in\mathbb{N}} S'_{k,\gamma} = T^{\langle\gamma\rangle}_\bullet A \qquad \text{for every } \gamma \in \Gamma.$$

We deduce natural identifications as in remark 10.6.39(iv) for every $\gamma \in \Gamma$ and every $k \in \mathbb{N}$

$$\mathscr{O}_{Y_k}(\gamma) \xrightarrow{\sim} S'_{k,\gamma}\mathscr{O}_{Y_k} \qquad \operatorname*{colim}_{i\in\mathbb{N}} \pi_i^{-1}\mathscr{O}_{Y_i}(\gamma) \xrightarrow{\sim} \mathscr{O}_Y(\gamma) \xrightarrow{\sim} T^{\langle\gamma\rangle}_\bullet \mathscr{O}_Y$$

whence, by proposition 10.1.10(ii), a natural isomorphism of A-modules :

$$H^p(Y, T^{\langle\gamma\rangle}_\bullet \mathscr{O}_Y) \xrightarrow{\sim} \operatorname*{colim}_{i\in\mathbb{N}} H^p(Y_i, \mathscr{O}_{Y_i}(\gamma)) \qquad \text{for every } p \in \mathbb{N} \text{ and every } \gamma \in \Gamma. \tag{16.7.11}$$

By the same token, we have as well a natural isomorphism

$$H^p(Y, T^{[s]}_\bullet \mathscr{O}_Y) \xrightarrow{\sim} \operatorname*{colim}_{\gamma>s} H^p(Y, T^{\langle\gamma\rangle}_\bullet \mathscr{O}_Y) \qquad \text{for every } p \in \mathbb{N} \text{ and every } s \in \mathbb{R}_+. \tag{16.7.12}$$

Claim 16.7.13. There exists an integer $c > 0$ such that for every $k \in \mathbb{N}$ and every $\gamma \in \Gamma_k$, the inclusion map $i : S'_{k,\gamma+c/p^k}\mathscr{O}_{Y_k} \to S'_{k,\gamma}\mathscr{O}_{Y_k}$ induces the zero map in cohomology :

$$0 = H^p(Y_k, i) : H^p(Y_k, \mathscr{O}_{Y_k}(\gamma + c/p^k)) \to H^p(Y_k, \mathscr{O}_{Y_k}(\gamma)) \qquad \text{for every } p > 0$$

and moreover the kernel and cokernel of the natural morphism of inverse systems

$$(S'_{k,n/p^k} \to H^0(Y_k, \mathscr{O}_{Y_k}(n/p^k)) \mid n \in \mathbb{N})$$

is uniformly essentially zero with step $\leq c$.

Proof of the claim. Recall that Y_k is the colimit of the filtered system $(Y^{(\gamma)}_k \mid \gamma \in \Gamma_k)$ defined as in (16.7.1). By inspecting the construction, it is easily seen that $Z_k := Y^{(1/p^k)}_k$ is the blowing up morphism of the ideal of $\mathscr{O}_X$ generated by the sequence $\mathbf{f}^{(k)} := (u_0(T_1^{1/p^k}), \dots, u_0(T_k^{1/p^k}))$. Say now that $\gamma = a/p^k$; according to (16.7.5), we have a natural identification

$$\mathscr{O}_{Y_k}(\gamma + d/p^k) \xrightarrow{\sim} \Omega^{(1/p^k)}_* \mathscr{O}_{Z_k}(a + d) \qquad \text{for every } d \in \mathbb{N}.$$

On the other hand, lemma 16.4.5(i) implies that the ring A satisfies condition $(\mathrm{a})^{\mathrm{un}}_{\mathbf{f}^{(k)}}$ with step $\leq r$. Then, by theorem 10.6.50 and remark 10.6.57, there exists a constant $c \in \mathbb{N}$ independent of k and a, such that the inclusion map $\mathscr{O}_{Z_k}(a) \to \mathscr{O}_{Z_k}(a+c)$ induces the zero map in cohomology

$$H^p(Z_k, \mathscr{O}_{Z_k}(a + c)) \to H^p(Z_k, \mathscr{O}_{Z_k}(a)) \qquad \text{for every } p > 0$$

and moreover, the kernel and cokernel of the morphism of inverse systems

$$(I^n_k A \to H^0(Z_k, \mathscr{O}_{Z_k}(n)) \mid n \in \mathbb{N})$$

are uniformly essentially zero with step $\leq c$. Both assertions of the claim are an immediate consequence. ◊

Now, in view of (16.7.12), in order to prove (ii), it suffices to show that for every $\gamma > s$ there exists $\gamma' \in \mathbb{Q}_+$ with $s < \gamma' < \gamma$, and such that the natural map

$$H^p(Y, T^{\langle\gamma\rangle}_\bullet \mathscr{O}_Y) \to H^p(Y, T^{\langle\gamma'\rangle}_\bullet \mathscr{O}_Y)$$

vanishes. To this aim, pick any $k \in \mathbb{N}$ such that $c/p^k < \gamma - s$, where $c \in \mathbb{N}$ is as in claim 16.7.13; by virtue of (16.7.11), we get the sought vanishing with $\gamma' := \gamma - c/p^k$. A similar argument proves also assertion (i) : details left to the reader.

(iii): It suffices to check that for every $\gamma' < \gamma$ in $\mathbb{N}[1/p]$ the inclusion $I^{\langle\gamma\rangle}_{r,0} \subset I^{\langle\gamma'\rangle}_{r,0}$ induces the zero morphism

$$\mathrm{Tor}^{R_{r,0}}_i(R_{r,0}/I^{\langle\gamma\rangle}_{r,0}, \mathsf{R}\langle A\rangle) \to \mathrm{Tor}^{R_{r,0}}_i(R_{r,0}/I^{\langle\gamma'\rangle}_{r,0}, \mathsf{R}\langle A\rangle) \qquad \text{for every } i > 0.$$

However, for every $k \in \mathbb{N}$ set $A_{r,k} := \mathbb{Z}[T_1^{1/p^k}, \dots, T_r^{1/p^k}]$ and let $J_{r,k} \subset A_{r,k}$ be the ideal generated by $(T_1^{1/p^k}, \dots, T_r^{1/p^k})$; we regard A as an $A_{r,k}$-algebra for every $k \in \mathbb{N}$, via restriction of scalars along the inclusion map $A_{r,k} \to R_{r,0}$, and we let $R_k(A)$ be the Rees A-algebra associated with the $J_{r,k}A$-adic filtration of A. Now, say that $\gamma = c/p^n$ and $\gamma' = c'/p^n$ for some $c, c', n \in \mathbb{N}$; in view of claim 7.11.37(i) it suffices to show that for every $i > 0$ there exists $k_0 \in \mathbb{N}$ such that the inclusion $J_{r,n+k}^{cp^k} \subset J_{r,n+k}^{c'p^k}$ induces the zero morphism

$$\mathrm{Tor}_i^{A_{r,n+k}}(A_{r,n+k}/J_{r,n+k}^{cp^k}, R_{n+k}(A)) \to \mathrm{Tor}_i^{A_{r,n+k}}(A_{r,n+k}/J_{r,n+k}^{c'p^k}, R_{n+k}(A))$$

for every $k \geq k_0$. However, for every $k \in \mathbb{N}$ let $\mathbf{f}_\bullet^{(k)} := (u_0(T_1^{1/p^k}), \dots, u_0(T_r^{1/p^k}))$, and denote by $\mathbf{g}_\bullet^{(k)}$ the image of $\mathbf{f}_\bullet^{(k)}$ under the natural ring homomorphism $A \to R_k(A)$. By lemma 16.4.5(i) we know that A fulfills condition $(\mathrm{a})^{\mathrm{un}}_{\mathbf{f}_\bullet^{(k)}}$ with step $\leq r$; then corollary 7.9.22(ii) and remark 7.9.27 imply that $R_k(A)$ fulfills condition $(\mathrm{a})^{\mathrm{un}}_{\mathbf{g}_\bullet^{(k)}}$ with a step independent of k; explicitly, the latter means that the inverse system $(\mathrm{Tor}_i^{A_{r,k}}(A_{r,k}/J_{r,k}^t, R_k(A)) \mid t \in \mathbb{N})$ is uniformly essentially zero for every $k \in \mathbb{N}$ and every $i > 0$, with a step independent of k. The assertion is an immediate consequence.

(iv): It suffices to show that the natural map

$$T_\bullet^{\lceil s \rceil} A \otimes_A T_\bullet^{\langle t \rangle} A \to T_\bullet^{\lceil t+s \rceil} A \qquad (16.7.14)$$

is an isomorphism for every $t \in \mathbb{R}_+$. However, (iii) easily implies that the natural map

$$T_\bullet^{\lceil s \rceil} R_{r,0} \otimes_{R_{r,0}} T_\bullet^{\langle t \rangle} A \to T_\bullet^{\lceil t+s \rceil} A$$

is an isomorphism, and the latter factors through (16.7.14) and a surjection $T_\bullet^{\lceil s \rceil} R_{r,0} \otimes_{R_{r,0}} T_\bullet^{\langle t \rangle} A \to T_\bullet^{\lceil s \rceil} A \otimes_A T_\bullet^{\langle t \rangle} A$, whence the contention. □

16.7.15. In the situation of (16.7.9), suppose that A fulfills the condition of lemma 16.4.5, and is endowed with a $\mathbb{Q}_+$-grading which makes it an A_0-subalgebra of $A_0[\mathbb{Q}_+]$ (where $A_0 \subset A$ is the subring of homogeneous elements of degree zero), and moreover suppose that the image of u_0 lies in A_0. In this case, the foregoing considerations apply to A_0 as well, and we get a well defined Γ-graded subring $\mathsf{R}\langle A_0 \rangle$ of $\mathsf{R}\langle A \rangle$, as well the angular blowing up morphism

$$Y_0 := \mathrm{Proj}\,(\mathsf{R}\langle A_0 \rangle_{/\mathbb{Q}_+}) \to X_0 := \mathrm{Spec}\, A_0.$$

Notice also that $\mathsf{R}\langle A \rangle$ carries a natural $(\Gamma \times \mathbb{Q}_+)$-graded ring structure : namely,

$$\mathrm{gr}_{(\gamma,t)}\mathsf{R}\langle A \rangle := T_\bullet^{\langle \gamma \rangle} A_t \qquad \text{for every } (\gamma, t) \in \Gamma \times \mathbb{Q}_+$$

which is an ideal in A_0, for every such (γ, t), and we set

$$\mathsf{R}\langle A_t \rangle := \bigoplus_{\gamma \in \Gamma} T_\bullet^{\langle \gamma \rangle} A_t = A_t \mathsf{R}\langle A_0 \rangle \qquad \text{for every } t \in \mathbb{Q}_+.$$

Corollary 16.7.16. *In the situation of* (16.7.15)*, for every $s \in \mathbb{R}_+$ and every $t \in \mathbb{Q}_+$ we have :*

(i) *The natural maps $T_\bullet^{\lceil s \rceil} A_t \to T_\bullet^{\lceil s \rceil} A_t \cdot H^0(Y_0, \mathscr{O}_{Y_0}) \to H^0(Y_0, T_\bullet^{\lceil s \rceil} A_t \mathscr{O}_{Y_0})$ are isomorphisms.*

(ii) *$H^p(Y_0, T_\bullet^{\lceil s \rceil} A_t \mathscr{O}_{Y_0}) = 0$ for every $p > 0$.*

Proof. Let $i : A_0 \to A$ and $\mathsf{R}\langle i \rangle : \mathsf{R}\langle A_0 \rangle \to \mathsf{R}\langle A \rangle$ be the inclusion maps; since $i(\mathsf{R}\langle A_0 \rangle_+)$ generates the ideal $\mathsf{R}\langle A \rangle_+$ of $\mathsf{R}\langle A \rangle$, then i induces a morphism of schemes $\mathrm{Proj}\, \mathsf{R}\langle i \rangle : Y \to Y_0$

(remark 16.7.8(ii)) fitting into the commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\mathrm{Proj}\,\mathsf{R}\langle i\rangle} & Y_0 \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi_0} \\ X & \xrightarrow{\mathrm{Spec}\, i} & X_0. \end{array}$$

To ease notation, we let $\varphi_X := \mathrm{Spec}\, i$ and $\varphi_Y := \mathrm{Proj}\,\mathsf{R}\langle i\rangle$. Notice that

$$T_\bullet^{\lceil s\rceil}\mathsf{R}\langle A\rangle_{(f)} = T_\bullet^{\lceil s\rceil}\Big(\bigoplus_{t\in\mathbb{Q}_+}\mathsf{R}\langle A_t\rangle\Big)_{(f)} = \bigoplus_{t\in\mathbb{Q}_+} T_\bullet^{\lceil s\rceil}A_t\mathsf{R}\langle A_0\rangle_{(f)}$$

for every homogeneous element $f\in\mathsf{R}\langle A_0\rangle$, which shows that

$$\varphi_{Y*}T_\bullet^{\lceil s\rceil}\mathscr{O}_Y = \bigoplus_{t\in\mathbb{Q}_+} T_\bullet^{\lceil s\rceil}A_t\mathscr{O}_{Y_0}.$$

Since φ_Y is an affine morphism, we deduce

$$H^p(Y, T_\bullet^{\lceil s\rceil}\mathscr{O}_Y) = H^p(Y_0, \varphi_{Y*}T_\bullet^{\lceil s\rceil}\mathscr{O}_Y) = \bigoplus_{t\in\mathbb{Q}_+} H^p(Y_0, T_\bullet^{\lceil s\rceil}A_t\mathscr{O}_{Y_0})$$

and combining with theorem 16.7.10 we get the corollary. □

Example 16.7.17. (i) According to proposition 16.4.10(i), theorem 16.7.10 applies especially to the case where A is perfectoid. In this case there exist unique $\beta_1,\dots,\beta_r\in\mathbf{E}:=\mathbf{E}(A)$ such that $u_0(T_i^{1/p^k}) = \overline{u}_A(\beta_i^{1/p^k})$ for every $i=1,\dots,r$ and every $k\in\mathbb{N}$, so we may regard as well $\mathbf{E}$ as an $R_{r,0}$-algebra via the unique ring homomorphism

$$R_{r,0}\to\mathbf{E} \qquad : \qquad T_i\mapsto\beta_i \qquad \text{for every } i=1,\dots,r.$$

(ii) More generally, proposition 16.6.5 shows that theorem 16.7.10 applies to the case where A is the graded subring of a perfectoid ring $(A^\wedge, A)$ with Δ-graded structure, where Δ is an integral p-perfect monoid, provided the elements $\beta_1,\dots,\beta_r$ as in (i) are homogeneous elements of the Δ-graded subring of $\mathbf{E}(A)$.

(iii) A ring A as in (ii) is given by example 16.6.12 : we take a perfectoid ring A_0, a second finite family $\beta'_\bullet := (\beta'_1,\dots,\beta'_{r'})$ of elements of $\mathbf{E}_0 := \mathbf{E}(A_0)$, we set $I := \beta'_\bullet\mathbf{E}_0$, and let A_t be the topological closure in A_0 of $I^{\langle t\rangle}A_0$ for every $t\in\Delta := \mathbb{N}[1/p]$. Then, clearly corollary 16.7.16 shall apply to the $\mathbb{Q}_+$-graded ring $A_{/\mathbb{Q}_+}$.

(iv) Let A_0, $\mathbf{E}_0$, $\beta'_\bullet$ and A be as in (iii), and suppose that also the sequence $\beta_\bullet$ lies in $\mathbf{E}(A_0)$. Suppose moreover that I is an ideal of adic definition for $\mathbf{E}_0$. In this case, the ideal $I^{\langle t\rangle}A_0$ is also open in A_0 (corollary 16.3.40(ii)), and therefore it coincides with A_t for every $t\in\mathbb{N}[1/p]$. Especially, $\mathrm{gr}_{(\gamma,t)}\mathsf{R}\langle A\rangle = T_\bullet^{\langle\gamma\rangle}I^{\langle t\rangle}A_0$ for every $(\gamma,t)\in\Gamma\times\mathbb{N}[1/p]$.

(v) In the situation of (iv), suppose furthermore that the ideal $T_\bullet\mathbf{E}_0\subset\mathbf{E}_0$ generated by $\beta_\bullet$ is open. Notice that $T_\bullet^{\lceil 0\rceil}\mathbf{E}_0$ is the radical of $T_\bullet\mathbf{E}_0$ (details left to the reader). It follows that for every $s\in\mathbb{N}[1/p]\setminus\{0\}$ there exists $s'>0$ such that $I^{\langle s\rangle}\mathbf{E}_0\subset T_\bullet^{\langle s'\rangle}\mathbf{E}_0$, whence $I^{\langle s\rangle}A_0\subset T_\bullet^{\langle s'\rangle}A_0$ (corollary 16.3.51(ii)). Therefore :

$$I^{\langle t+s\rangle}A_0\subset I^{\langle t\rangle}T_\bullet^{\lceil 0\rceil}A_0\subset I^{\langle t\rangle}A_0 \qquad \text{for every } t,s\in\mathbb{N}[1/p] \text{ with } s>0.$$

Corollary 16.7.18. *In the situation of* (16.7.9)*, suppose that A is perfectoid, and let I be any finitely generated ideal of adic definition of $\mathbf{E} := \mathbf{E}(A)$. With the notation of example* 16.7.17(v)*, for every $t\in\mathbb{R}_+$ we have :*

(i) *The natural maps $I^{\lceil t\rceil}A\to I^{\lceil t\rceil}\cdot H^0(Y,\mathscr{O}_Y)\to H^0(Y, I^{\lceil t\rceil}\mathscr{O}_Y)$ are isomorphisms.*

(ii) $H^p(Y, I^{\lceil t\rceil}\mathscr{O}_Y)=0$ *for every $p>0$.*

Proof. With the notation of example 16.7.17(v), notice that

$$H^p(Y, I^{\lceil t\rceil}T_\bullet^{\lceil 0\rceil}\mathscr{O}_Y) = \operatorname*{colim}_{s>t} H^p(Y, I^{\langle s\rangle}T_\bullet^{\lceil 0\rceil}\mathscr{O}_Y).$$

On the other hand, example 16.7.17(v) implies that $I^{\lceil t\rceil}A = I^{\lceil t\rceil}T_\bullet^{\lceil 0\rceil}A$ for every $t \in \mathbb{R}_+$. To conclude, it suffices to invoke corollary 16.7.16 as in example 16.7.17(iii,iv) : *i.e.* with A_0 and Y_0 replaced by the current A and respectively Y, and with A replaced by the $\mathbb{Q}_+$-graded ring $A'_{/\mathbb{Q}_+}$, where $A'_s := I^{\langle s\rangle}A$ for every $s \in \mathbb{N}[1/p]$. □

Definition 16.7.19. We say that a topological ring $(A, \mathscr{T})$ is a *formal P-ring* (resp. a *formally perfectoid ring*) if it is adic with a finitely generated ideal of adic definition, and the separated completion of $(A, \mathscr{T})$ is a P-ring (resp. a perfectoid ring).

Remark 16.7.20. Let $(A, \mathscr{T})$ be any topological ring.

(i) Suppose that $(A, \mathscr{T})$ is a P-ring. Then, any ideal of definition of the P-ring A is also an ideal of adic definition of the adic ring A. Remark 16.2.4 implies that the converse holds if and only if A is an $\mathbb{F}_p$-algebra. Indeed, if I is any ideal of adic definition of A, then any power I^n is also an ideal of adic definition, but if p does not vanish in A, then one such power will not contain p, whence the contention.

(ii) Let $\varphi : (A, \mathscr{T}) \to (A', \mathscr{T}')$ be an adic morphism of topological rings, and suppose that φ is a weakly étale ring homomorphism. Then [75, Lemma 3.1.2(i)] implies that φ is adically weakly étale (see definition 8.3.23(iii)).

Theorem 16.7.21. *Let $f : A \to B$ be a c-adically weakly étale morphism of topological rings.*

- (i) *If A and B are f-adic and A is a formal P-ring (resp. a formally perfectoid ring), the same holds for B.*
- (ii) *If f is c-adically faithfully flat, the following holds :*
  - (a) *If B is f-adic and A and B are complete and separated, A is f-adic and f is adic.*
  - (b) *If B is a formal P-ring (resp. a formally perfectoid ring), the same holds for A.*

Proof. (i): In light of lemma 8.3.32(iii,iv), we may assume that A and B are complete and separated, in which case A is a P-ring (resp. a perfectoid ring), f is adically weakly étale (lemma 8.3.24(i.c)), and we have to show that B is a P-ring (resp. is perfectoid).

Suppose first that A is a P-ring, and let I be any ideal of definition of A. It suffices to check that IB is an ideal of definition for B. However, the natural map $B/I^2B \to B^\wedge/I^2B^\wedge$ is an isomorphism, by remark 8.3.3(ii), so we come down to checking that the Frobenius endomorphism of B/I^2B is surjective. However, the induced ring homomorphism $A/I^2 \to B/I^2B$ is weakly étale by assumption; since the Frobenius endomorphism of A/I^2 is surjective by assumption, the assertion then follows easily from [75, Th.3.5.13(ii)].

Next, suppose that A is perfectoid. Due to remark 16.3.58, we may assume that $p \in I^t$, where $t \in \mathbb{N}$ is defined as in (16.3.55), in terms of the length of a system of generators for I. Set $J := I^{(p)}$; taking into account (i) and theorem 16.3.64, we are then reduced to checking that the morphism of graded rings

$$\Phi_{IB} : \mathrm{gr}^\bullet_{IB}B \to \mathrm{gr}^\bullet_{JB}B$$

is an isomorphism. However, since $f \otimes_A A/I^n$ is flat for every $n \in \mathbb{N}$, we get induced graded ring isomorphisms

$$\varphi_I : B \otimes_A \mathrm{gr}^\bullet_I A \xrightarrow{\sim} \mathrm{gr}^\bullet_{IB}B \qquad \varphi_J : B \otimes_A \mathrm{gr}^\bullet_J A \xrightarrow{\sim} \mathrm{gr}^\bullet_{JB}B$$

and φ_I identifies the map

$$\mathrm{gr}^\bullet_I f : \mathrm{gr}^\bullet_I A \to \mathrm{gr}^\bullet_{IB}B$$

with the base change $f \otimes_A \mathrm{gr}^\bullet_I A$. Especially, since $f \otimes_A A/I$ is weakly étale, the same holds for $\mathrm{gr}^\bullet_I f$ ([75, Lemma 3.1.2(i)]). Moreover, denote by $(\mathrm{gr}^\bullet_I A)_{(\Phi)}$ the $\mathrm{gr}^\bullet_I A$-algebra whose underlying ring is $\mathrm{gr}^\bullet_I A$ and whose structure map is the Frobenius endomorphism $\Phi_{\mathrm{gr}^\bullet_I A}$, and define likewise $(\mathrm{gr}^\bullet_{IB} B)_{(\Phi)}$; by [75, Th.3.5.13(ii)], the maps φ_J and $\mathrm{gr}^\bullet_J f$ induce an isomorphism

$$\psi_J : B \otimes_A (\mathrm{gr}^\bullet_J A)_{(\Phi)} \xrightarrow{\sim} \mathrm{gr}^\bullet_{JB} B \otimes_{\mathrm{gr}^\bullet_J A} (\mathrm{gr}^\bullet_J A)_{(\Phi)} \xrightarrow{\sim} (\mathrm{gr}^\bullet_{JB} B)_{(\Phi)} \qquad b \otimes a \mapsto b^p \cdot \mathrm{gr}_I f(a).$$

Summing up, we get a commutative diagram of ring homomorphisms

$$(16.7.22)\qquad \begin{array}{ccc} B \otimes_A \mathrm{gr}^\bullet_I A & \xrightarrow{B\otimes_A \Phi_I} & B \otimes_A (\mathrm{gr}^\bullet_J A)_{(\Phi)} \\ {\scriptstyle\varphi_I}\downarrow & & \downarrow{\scriptstyle\psi_J} \\ \mathrm{gr}^\bullet_{IB} B & \xrightarrow{\Phi_{IB}} & (\mathrm{gr}^\bullet_{JB} B)_{(\Phi)} \end{array}$$

whose vertical arrows are isomorphisms. Lastly, since A is a formally perfectoid ring, Φ_I is an isomorphism as well (theorem 16.3.64), whence the assertion.

(ii.a): Let J be any finitely generated ideal of adic definition of B, and $(I_\lambda \mid \lambda \in \Lambda)$ a cofiltered system of ideals that defines the linear topology of A. By assumption, $(I_\lambda B)^c$ is open in B, hence $I_\lambda B = (I_\lambda B)^c$ for every $\lambda \in \Lambda$ (lemma 8.3.21(ii.b)); then the induced map

$$f_\lambda : A_\lambda := A/I_\lambda \to B_\lambda := B/I_\lambda B$$

is weakly étale and faithfully flat for every $\lambda \in \Lambda$. Also by assumption, we may find $\mu \in \Lambda$ such that $I_\mu B \subset J^2$. Hence, JB_μ is contained in the nilradical $\mathrm{nil}(B_\mu)$ of B_μ, and it follows easily from claim 14.3.27(i) that

$$\mathrm{nil}(B_\mu) = \mathrm{nil}(A_\mu) \cdot B_\mu.$$

Therefore, we may find a finitely generated ideal $K' \subset \mathrm{nil}(A_\mu)$ such that $JB_\mu \subset K'B_\mu$. Pick a finitely generated ideal $K \subset A$ such that $KA_\mu = K'$. It follows easily that $J \subset KB + J^2$. However, B is complete and J is topologically nilpotent in B, so J lies in the Jacobson radical of B, and consequently $J \subset KB$, by Nakayama's lemma. On the other hand, by construction, we may find $n \in \mathbb{N}$ such that $K'^n = 0$, whence $K^n \subset I_\mu$, and therefore $K^n B \subset J^2$. Summing up, we conclude that KB is an ideal of adic definition of B. For every $n \in \mathbb{N}$, pick $\nu(n) \in \Lambda$ such that $I_{\nu(n)} B \subset J^n$; then $I_{\nu(n)} B_\lambda \subset K^n B_\lambda$ for every $\lambda \in \Lambda$, so that $I_{\nu(n)} A_\lambda \subset K^n A_\lambda$, as f_λ is faithfully flat. We deduce that the topological closure $(K^n)^c$ of K^n contains $I_{\nu(n)}$, and especially, it is an open ideal of A, for every $n \in \mathbb{N}$. On the other hand, for every $\lambda \in \Lambda$ we may find $k(\lambda) \in \mathbb{N}$ such that $J^{2k(\lambda)} \subset I_\lambda B$, so that $K^{n \cdot k(\lambda)} B_\nu \subset I_\lambda B_\nu$ for every $\nu \geq \lambda$. It follows that $K^{n \cdot k(\lambda)} A_\nu \subset I_\lambda A_\nu$, again since f_ν is faithfully flat; then, since I_λ is closed in A, we deduce that $K^{n \cdot k(\lambda)} \subset I_\lambda$, for every $\lambda \in \Lambda$. Summing up, we see that the topology of A is c-adic, hence f-adic, by lemma 8.3.21(ii.a). Then f is adic, by lemma 8.3.24(i.c).

(ii.b): In light of lemma 8.3.32(iii), we may replace f by $f^\wedge$, and assume from start that A and B are complete and B is a P-ring (resp. is perfectoid), and we need to show that the same holds for A. However, (ii.a) already says that A is f-adic, and f is adic. Suppose first that B is a P-ring, and pick any finitely generated ideal of adic definition for A; without loss of generality we may assume that J is an ideal of definition for the P-ring B, and that $IB \subset J$. Set $A_n := A/I^{n+1}$ and $B_n := B/I^{n+1}B$ for every $n \in \mathbb{N}$, and notice that JB_0 is nilpotent ideal of B_0; arguing as in the proof of (ii.a), we deduce that there exists a finitely generated ideal

$$(16.7.23)\qquad K_0 \subset \mathrm{nil}(A_0)$$

such that $JB_0 \subset K_0 B_0$. Then, the preimage K of K_0 in A is a finitely generated ideal such that $JB \subset KB$; hence $p^2 \in K^2 B_n$ for every $n \in \mathbb{N}$. Since the induced map $A_n \to B_n$ is faithfully flat, it follows that $p \in K^2 A_n$ for every $n \in \mathbb{N}$, so p lies in the topological closure of K^2 in A; but $(K^2)^c = K^2$, since K is open in A by construction. Lastly, (16.7.23) implies that $K^n \subset I$

for some sufficiently large $n \in \mathbb{N}$; thus, K is also an ideal of adic definition of A, and we are reduced to checking that the Frobenius endomorphism Φ_{A/J^2} of A/J^2 is surjective. However, by lemma 16.2.3(iv) the Frobenius endomorphism Φ_{B/J^2B} is surjective; by [75, Th.3.5.13(ii)], it follows that $\Phi_{A/J^2} \otimes_{A/J^2} B/J^2B$ is also surjective. But the induced map $A/J^2 \to B/J^2B$ is faithfully flat, whence the contention.

Lastly, suppose that B is perfectoid; by the foregoing, we know already that A is a P-ring, and by remark 16.3.58 we may find an ideal of definition I of A such that $p \in I^t$, where $t \in \mathbb{N}$ is defined as in (16.3.55). We set $J := I^{(p)}$; arguing as in the proof of (i), we get a commutative diagram (16.7.22), whose vertical arrows are isomorphisms. Now, notice that, since f is adic, IB is an ideal of definition of B, so the bottom horizontal arrow of (16.7.22) is an isomorphism; hence the same holds for $B \otimes_A \Phi_I = (B/IB) \otimes_{A/I} \Phi_I$. But by assumption, f induces a faithfully flat map $A/I \to B/IB$, so Φ_I is an isomorphism, and therefore A is perfectoid, by theorem 16.3.64. □

16.7.24. Let $\underline{U} := (U, \mathscr{T}_A, A_U^+)$ be any perfectoid quasi-affinoid scheme, and set $\underline{U}_{\mathbf{E}} := \mathbf{E}(\underline{U})$ (notation of (16.5.20)), so that $\underline{U}_{\mathbf{E}} = (\mathbf{E}(U), \mathscr{T}_{\mathbf{E}}, \mathbf{E}_U^+)$, where $\mathbf{E}_U^+ := \mathbf{E}(A_U^+)$. Let as well

$$A_U := \mathscr{O}_U(U) \qquad \mathbf{E}_U := \mathscr{O}_{\mathbf{E}(U)}(\mathbf{E}(U))$$

and pick a perfectoid subring of definition A of A_U *such that A_U^+ is the integral closure of $A \cap A_U^+$ in A_U and such that the induced map $U \to \operatorname{Spec} A$ is an open immersion* (by theorem 16.5.13(i) and lemma 15.6.10(i), the ring A_U° is one such subring of definition), and recall that $\mathbf{E} := \mathbf{E}(A)$ is a ring of definition of $\mathbf{E}_U$. We denote as usual by $\varphi_U^\flat : \mathbf{E}_U \to A_U$ the continuous map of monoids provided by proposition 16.4.34(i), and recall that $\varphi_U^\flat$ restricts to a continuous map $\overline{u}_A : \mathbf{E} \to A$. Now, let $e_0, \dots, e_n$ be a finite system of elements of $\mathbf{E}_U$ that generates an open ideal; according to claim 16.6.36, the system $(a_i := \varphi_U^\flat(e_i) \mid i = 0, \dots, n)$ generates an open ideal of A_U, so we may consider the rational subset

$$R := R_{A_U}\Big(\frac{a_1}{a_0}, \dots, \frac{a_n}{a_0}\Big) \cap \operatorname{Spa} \underline{U}$$

and the corresponding topological rings $\mathscr{O}^\wedge_{\operatorname{Spa} U}(R)$ and $\mathscr{O}^{\wedge +}_{\operatorname{Spa} U}(R)$, defined as in (15.5.11). Recall the construction of $\mathscr{O}^\wedge_{\operatorname{Spa} U}(R)$: first, there is a natural f-adic topology on the localization $A_R := A_U[1/a_0]$ such that the localization map $A_U \to A_R$ is f-adic, and the subring

$$B_R := A[a_i/a_0 \mid i = 1, \dots, n] \subset A_R$$

is a ring of definition of A_R; hence A_R induces on B_R the unique linear topology such that the natural map $A \to B_R$ is adic. We let A_R^+ be the integral closure of $A_U^+[a_i/a_0 \mid i = 1, \dots, n]$ in A_R, set $U_R := U \cap \operatorname{Spec} A_R$, and define

$$\underline{A}_R := (A_R, A_R^+, U_R) \qquad \text{and} \qquad \underline{U}_R^\wedge := (\mathsf{Spec}\, \underline{A}_R)^\wedge.$$

Then (up to natural isomorphism) we have

$$\Gamma(\underline{U}_R^\wedge) = (\mathscr{O}^\wedge_{\operatorname{Spa} U}(R), \mathscr{O}^{\wedge +}_{\operatorname{Spa} U}(R), U_R^\wedge) \qquad \text{where} \qquad U_R^\wedge := U_R \times_{\operatorname{Spec} A_R} \operatorname{Spec} \mathscr{O}^\wedge_{\operatorname{Spa} U}(R).$$

Notice that A_R contains also the subrings

$$C_R := A\Big[\varphi_U^\flat(e_i^{1/p^k})/\varphi_U^\flat(e_0^{1/p^k}) \mid (i,k) \in \{1, \dots, n\} \times \mathbb{N}\Big]$$
$$D_R := A_U^+\Big[\varphi_U^\flat(e_i^{1/p^k})/\varphi_U^\flat(e_0^{1/p^k}) \mid (i,k) \in \{1, \dots, n\} \times \mathbb{N}\Big]$$

and endow C_R and D_R with the topologies induced by the inclusion into A_R. Let also $C_R^+ := C_R \cap A_R^+$. The first observation is the following :

Proposition 16.7.25. *With the notation of* (16.7.24)*, the following holds :*

(i) *The natural morphism $\underline{A}_R^\wedge \to \Gamma(\underline{U}_R^\wedge)$ is an isomorphism.*

(ii) *The datum* $\underline{C}_R := (C_R, C_R^+, U_R)$ *is a quasi-affinoid ring, and the natural morphism* $\mathsf{Spec}\,\underline{A}_R \to \mathsf{Spec}\,\underline{C}_R$ *is an isomorphism.*
(iii) $\underline{C}_R^\wedge$ *is a perfectoid quasi-affinoid ring, and* $\underline{U}_R^\wedge$ *is a perfectoid quasi-affinoid scheme.*
(iv) *Suppose moreover that* $\beta_{\underline{U}}(U_R) \subset \Omega_{\underline{U}}$ *(notation of definition* 15.6.1*). Then the datum* $\underline{D}_R := (D_R, D_R, U_R)$ *is a quasi-affinoid ring, and* $\underline{D}_R^\wedge$ *is perfectoid.*

Proof. (ii): We consider the natural morphisms of schemes

$$U_R \xrightarrow{f} \mathrm{Spec}\,A_R \xrightarrow{g} \mathrm{Spec}\,C_R \xrightarrow{h} \mathrm{Spec}\,A$$

whose composition is also the composition of the open immersion $U_R \to U$ with the natural morphism $i : U \to \mathrm{Spec}\,A$. By assumption, i is an open immersion, and its image contains the analytic locus of $\mathrm{Spec}\,A$ (lemma 8.3.29(iii)). It follows that $h \circ g \circ f$ is an open immersion; moreover, by construction, both f and g have schematically dense images. From claim 15.6.5, we deduce that $g \circ f$ is an open immersion, and $f(U_R) = (h \circ g)^{-1} i(U_R) = (h \circ g)^{-1} i(U)$. Now, U contains the analytic locus of $\mathrm{Spec}\,A$, and the natural map $A \to A_R$ is f-adic; taking into account lemma 8.3.29(iv), it follows that $f(U_R)$ contains the analytic locus of $\mathrm{Spec}\,A_R$. Lastly, g identifies the analytic locus of $\mathrm{Spec}\,A_R$ with that of $\mathrm{Spec}\,C_R$ (lemma 8.3.29(iii)), so $g \circ f(U_R)$ contains the analytic locus of $\mathrm{Spec}\,C_R$. Summing up, this shows that $\underline{C}_R$ is a quasi-affinoid ring. Then, a simple inspection shows that the inclusion map $C_R \to A_R$ induces an isomorphism $\mathsf{Spec}\,\underline{A}_R \to \mathsf{Spec}\,\underline{C}_R$.

(i): To begin with, we remark :

Claim 16.7.26. C_R and D_R are subrings of definition of A_R.

Proof of the claim. Clearly C_R and D_R are open in A_R, and $D_R \subset C_R$; by virtue of proposition 8.3.18(ii), we have then only to check that C_R is bounded in A_R. To this aim, since B_R is open and bounded in A_R, it suffices to show that there exists an open ideal J of B_R such that $J \cdot C_R \subset B_R$. However, pick any finite system $f_1, \dots, f_r$ of generators for an ideal of definition of $\mathbf{E}$; since the system $e_0, \dots, e_n$ generates an open ideal of $\mathbf{E}_U$, we may find an integer $k \in \mathbb{N}$ large enough so that the $f_i^r e_j \in \mathbf{E}$ for every $i = 1, \dots, r$ and every $j = 0, \dots, n$, and then it is clear that the system $(f_i^r e_j \mid i \leq r,\ j \leq n)$ generates an open ideal of $\mathbf{E}$. By claim 16.6.36, it follows that the system $(\varphi_U^\flat(f_i^r e_j) \mid i \leq r,\ j \leq n)$ generates an open ideal J_0 of A, and therefore $J := J_0 B_R$ is open in B_R. To conclude, it suffices to show that $\varphi_U^\flat(f_i^r e_j e_s^\nu e_0^{-\nu}) \in B_R$ for every $i = 1, \dots, r$ every $j, s = 0, \dots, n$, and every $\nu \in \mathbb{N}[1/p]$. Moreover, we may easily reduce to the case where $0 < \nu < 1$. Then we see that

$$f_i^r e_j e_s^\nu e_0^{-\nu} = \frac{e_j}{e_0} \cdot (f_i^{r\nu} e_s^\nu) \cdot (f_i^{r(1-\nu)} e_0^{1-\nu})$$

where both $f_i^{r\nu} e_s^\nu$ and $f_i^{r(1-\nu)} e_0^{1-\nu}$ lie in $\mathbf{E}$, since the latter is a perfect $\mathbb{F}_p$-algebra. The assertion follows. ◊

Pick a finite system $\beta_\bullet := (\beta_1, \dots, \beta_s)$ of elements of $\mathbf{E}$ that generates an ideal of adic definition, and set $f_i := \overline{u}_A(\beta_i)$ for $i = 1, \dots, s$, so that the system $\mathbf{f}_\bullet := (f_1, \dots, f_s)$ generates an ideal of adic definition of A. By claim 16.7.26, it then follows that the system $\mathbf{f}_\bullet$ also generates an ideal of adic definition J for C_R. Denote by $C_R^\wedge$ the separated completion of C_R; we notice :

Claim 16.7.27. The ring C_R fulfills condition $(\mathrm{a})_{\mathbf{f}}^{\mathrm{un}}$ of (7.8.21), and $C_R^\wedge$ is perfectoid.

Proof of the claim. Set $\Delta := \mathbb{N}[1/p]$; recall that example 16.6.12 attaches to the sequence $e_\bullet := (e_0, \dots, e_n)$ of elements of $\mathbf{E}_U$, two perfectoid rings with Δ-graded structure

$$(\mathscr{E}(e_\bullet)^\wedge, \mathscr{E}(e_\bullet)) \qquad \text{and} \qquad (\mathscr{A}(e_\bullet)^\wedge, \mathscr{A}(e_\bullet)).$$

Moreover, $\mathrm{gr}_0\mathscr{E}(e_\bullet) = \mathbf{E}$, $\mathrm{gr}_0\mathscr{A}(e_\bullet) = A$, and we have a natural isomorphism

$$\mathbf{E}(\mathscr{A}(e_\bullet)^\wedge, \mathscr{A}(e_\bullet)) \xrightarrow{\sim} (\mathscr{E}(e_\bullet)^\wedge, \mathscr{E}(e_\bullet)).$$

Furthermore, since $e_\bullet$ generates an open ideal of $\mathbf{E}_U$, the $\mathbf{E}$-module $\mathrm{gr}_\delta\mathscr{E}(e_\bullet)$ (resp. the A-module $\mathrm{gr}_\delta\mathscr{A}(e_\bullet)$) is the submodule of $\mathbf{E}_U$ (resp. of A_U) generated by the elements of the form $e_0^{\nu_0}\cdots e_n^{\nu_n}$ (resp. $\varphi_U^\flat(e_0^{\nu_0}\cdots e_n^{\nu_n})$), where $(\nu_0,\dots,\nu_n)$ ranges over all elements of $\Delta^{\oplus n+1}$ such that $\nu_0+\cdots+\nu_n=\delta$ (see remark 16.3.28(viii)). Especially, we may regard $\beta_\bullet$ as a sequence of elements of $\mathrm{gr}_0\mathscr{E}(e_\bullet)$, whose image under the map $\overline{u}_{\mathscr{A}(e_\bullet)} : \mathscr{E}(e_\bullet) \to \mathscr{A}(e_\bullet)$ is (naturally identified with) the sequence $\mathbf{f}_\bullet$. Next, let us endow $\mathscr{A}' := \mathscr{A}(e_\bullet)[a_0^{-1}]$ with its $J\mathscr{A}'$-adic topology; since we are inverting an element $a_0 \in \mathrm{gr}_1\mathscr{A}(e_\bullet)$, the A-algebra $\mathscr{A}'$ is also Δ-graded, and a simple inspection yields a natural isomorphism of $C_R \xrightarrow{\sim} \mathrm{gr}_0\mathscr{A}'$ of topological rings, which identifies the sequences $\mathbf{f}_\bullet$ in these two rings. Lastly, by virtue of proposition 16.6.5(ii), we deduce that $\mathscr{A}(e_\bullet)$ satisfies condition $(\mathrm{a})^{\mathrm{un}}_{\mathbf{f}}$ of (7.8.21), and then the same holds for $\mathrm{gr}_0\mathscr{A}'$, whence the first assertion.

In order to show that $C_R^\wedge$ is perfectoid, notice that $J^n\mathscr{A}' = \bigoplus_{\delta\in\Delta} J^n\mathrm{gr}_\delta\mathscr{A}'$ for every $n \in \mathbb{N}$. It follows easily that the maximal separated quotient $\mathscr{A}''$ of $\mathscr{A}'$ is also a Δ-graded ring (so that the projection $\mathscr{A}' \to \mathscr{A}''$ is a homomorphism of Δ-graded rings), and $\mathrm{gr}_0\mathscr{A}''$ is the maximal separated quotient of $\mathrm{gr}_0\mathscr{A}'$: details left to the reader. Hence $(\mathscr{A}'', \mathscr{A}'')$ is a topological ring with Δ-pre-graded structure, and therefore $(\mathscr{C}^\wedge, \mathscr{C}) := (\mathscr{A}'', \mathscr{A}'')^\wedge$ is a topological ring with Δ-graded structure (proposition 8.5.3(iii)), and the induced map $C_R^\wedge \to \mathrm{gr}_0\mathscr{C}$ is an isomorphism of topological rings. On the other hand, since $\mathscr{A}(e_\bullet)^\wedge$ is perfectoid, and since the localization map $\mathscr{A}(e_\bullet) \to \mathscr{A}'$ is an adic ring homomorphism, theorem 16.7.21(i) implies that $\mathscr{C}^\wedge$ is perfectoid as well. To conclude, it suffices now to invoke proposition 16.6.1(ii). ◊

As the morphism $i : U \to \mathrm{Spec}\, A$ is an open immersion whose image contains the analytic locus of $\mathrm{Spec}\, A$, we may find an open ideal $I \subset A$ such that $\mathrm{Spec}\, A \setminus U = \mathrm{Spec}\, A/I$, and we pick a finite system $\mathbf{g}_\bullet := (g_1,\dots,g_r)$ of generators of I. On the other hand, i factors through the natural open immersion $j : U \to \mathrm{Spec}\, A_U$ and the morphism $\varphi : \mathrm{Spec}\, A_U \to \mathrm{Spec}\, A$ induced by the inclusion map $A \to A_U$; by claim 15.6.5(ii), it follows that $\varphi^{-1}i(U) = j(U)$, whence $\mathrm{Spec}\, A_U/IA_U = \mathrm{Spec}\, A_U \setminus j(U)$, and therefore

$$Z := \mathrm{Spec}\, A_R/IA_R = \mathrm{Spec}\, A_R \setminus U_R.$$

In view of claim 16.7.26, remark 15.4.13(ii) and proposition 8.3.33(iii), we are then reduced to showing that

$$\mathrm{depth}_Z A_R \otimes_{C_R} C_R^\wedge > 1.$$

However, set $Q^\bullet := \mathrm{Cone}(C_R[0] \to C_R^\wedge[0])$; clearly we have $\mathrm{depth}_{IA_U} A_U > 1$, whence $\mathrm{depth}_Z A_R > 1$, and thus it suffices to check that

$$R\Gamma_Z(A_R \otimes_{C_R} Q^\bullet) = 0.$$

In light of proposition 10.4.32(i), we are then reduced to showing that the Koszul complex $\mathbf{K}^\bullet(\mathbf{g}_\bullet, A_R \otimes_{C_R} Q^\bullet)$ is acyclic. We remark :

Claim 16.7.28. The natural map $A_R \overset{\mathbf{L}}{\otimes}_{C_R} Q^\bullet \to A_R \otimes_{C_R} Q^\bullet$ is an isomorphism in $\mathsf{D}(C_R\text{-}\mathbf{Mod})$.

Proof of the claim. We have a natural morphism of distinguished triangles

$$\begin{array}{ccccccc}
A_R[0] & \longrightarrow & A_R[0] \overset{\mathbf{L}}{\otimes}_{C_R} C_R^\wedge[0] & \longrightarrow & A_R \overset{\mathbf{L}}{\otimes}_{C_R} Q^\bullet & \longrightarrow & A_R[1] \\
\| & & \downarrow & & \downarrow & & \| \\
A_R[0] & \longrightarrow & A_R \otimes_{C_R} C_R^\wedge[0] & \longrightarrow & A_R[0] \otimes_{C_R} Q^\bullet & \longrightarrow & A_R[1]
\end{array}$$

in light of which, we are reduced to checking that $T_i := \mathrm{Tor}_i^{C_R}(A_R, C_R^\wedge) = 0$ for every $i > 0$. However, clearly the natural map $T_i \to \mathrm{Tor}_i^{C_R}(A_R/C_R, C_R^\wedge)$ is an isomorphism for $i > 1$, and is injective for $i = 1$, and moreover $A_R/C_R = \bigcup_{n\in\mathbb{N}} \mathrm{Ann}_{A_R/C_R}(J^n)$. Hence, the assertion follows from claims 8.6.36 and 16.7.27. $\diamondsuit$

Taking into account lemma 7.8.2(v) and claim 16.7.28, we are further reduced to showing that $\mathbf{K}_\bullet(\mathbf{g}_\bullet) \overset{\mathbf{L}}{\otimes}_{C_R} A_R \overset{\mathbf{L}}{\otimes}_{C_R} Q^\bullet$ is acyclic, and to this aim it suffices to check that the same holds for the complex $\mathbf{K}_\bullet(\mathbf{g}_\bullet) \otimes_{C_R} Q^\bullet$. The latter follows easily from lemma 7.8.2(ii), claim 16.7.27 and corollary 8.6.34(i).

(iii): From claim 16.7.27 we know already that $\underline{C}_R^\wedge$ is perfectoid, and from (i) and (ii) we get isomorphisms of quasi-affinoid schemes

$$\underline{U}_R^\wedge \overset{\sim}{\to} \mathsf{Spec}\,(\underline{A}_R^\wedge) \overset{\sim}{\to} \mathsf{Spec}\,(\underline{C}_R^\wedge)$$

whence the contention.

(iv): We consider the natural morphisms of schemes :

$$U_R \xrightarrow{f} \mathrm{Spec}\, A_R \xrightarrow{g} \mathrm{Spec}\, D_R \to \mathrm{Spec}\, A_U^+$$

whose composition is the same as the restriction of $\beta_{\underline{U}}$ to U_R, hence it is an open immersion. Notice also that $\beta_{\underline{U}}$ is in turn the composition of the open immersion $i : U \to \mathrm{Spec}\, A$ and the natural morphism $\mathrm{Spec}\, A \to \mathrm{Spec}\, A_U^+$; the latter identifies the analytic locus of $\mathrm{Spec}\, A$ with that of $\mathrm{Spec}\, A_U^+$ (lemma 8.3.29(iii)), hence the image of $\beta_{\underline{U}}$ contains the analytic locus of $\mathrm{Spec}\, A_U^+$. We may then argue as in the proof of (ii) to conclude that $g \circ f$ is an open immersion and $g \circ f(U_R)$ contains the analytic locus of $\mathrm{Spec}\, D_R$. Together with claim 16.7.26, this already shows that $\underline{D}_R$ is a quasi-affinoid ring. Moreover, we also know that A_U^+ is a perfectoid ring (theorem 16.5.13(iii)), and our assumptions imply that $\underline{A}' := (A_U^+, A_U^+, U)$ is another perfectoid quasi-affinoid ring with $\mathsf{Spec}\, \underline{A}' = \underline{U}$. We may then argue as in the proof of claim 16.7.27 to see that the separated completion $D_R^\wedge$ of D_R is perfectoid, and the proof is complete. $\square$

Remark 16.7.29. (i) Let $\underline{X}$ be any perfectoid quasi-affinoid scheme; taking into account proposition 16.6.35(iii) we deduce that the continuous map $\mathrm{Spa}(\overline{u}_{\underline{X}})$ of (16.5.59) induces a natural isomorphism of sites (notation of (15.5.34)) :

$$\mathbf{E} : \mathscr{R}(\underline{X}) \overset{\sim}{\to} \mathscr{R}(\mathbf{E}(\underline{X})) \qquad R \mapsto \mathrm{Spa}(\overline{u}_{\underline{X}})(R).$$

(ii) Moreover, proposition 16.7.25(iii) says that for every $R \in \mathrm{Ob}(\mathscr{R}(\underline{X}))$ the sub-presheaf h''_R of $h''_{\underline{X}}$ is represented by a perfectoid quasi-affinoid scheme $\underline{Y}$ (notation of remark 15.5.9(i)), and taking into account (16.5.60) together with the equivalence of categories

$$\underline{X}\text{-}\mathsf{q.Afd.Sch}_{\mathrm{perf}} \overset{\sim}{\to} \mathbf{E}(\underline{X})\text{-}\mathsf{q.Afd.Sch}_{\mathrm{perf}}$$

described in (16.5.20), we deduce easily that $\mathbf{E}(\underline{Y})$ represents the sub-presheaf $h''_{\mathbf{E}(R)}$ of $h''_{\mathbf{E}(\underline{X})}$ (details left to the reader).

Theorem 16.7.30. *For every perfectoid quasi-affinoid scheme $\underline{X}$, the following holds :*

(i) *The presheaves $\mathscr{O}^\wedge_{\mathrm{Spa}\underline{X}}$, $\mathscr{O}^{\wedge\circ}_{\mathrm{Spa}\underline{X}}$ and $\mathscr{O}^{\wedge+}_{\mathrm{Spa}\underline{X}}$ are sheaves of topological rings on the site $\mathscr{R}(\underline{X})$ (notation of (15.5.34)).*

(ii) *For every rational subset R of $\mathrm{Spa}\,\underline{X}$ and every finite covering $\mathfrak{U}$ of R consisting of rational subsets, the augmented alternating Čech complex $C^\bullet_{\mathrm{alt}}(\mathfrak{U}, \mathscr{O}^\wedge_{\mathrm{Spa}\,\underline{X}})$ has strict differentials, and its cohomology has the discrete topology.*

Proof. (i): By virtue of corollary 15.4.27(i) and proposition 16.5.4(i), if $\mathscr{O}^\wedge_{\mathrm{Spa}\,\underline{X}}$ is a sheaf, the same holds for $\mathscr{O}^{\wedge+}_{\mathrm{Spa}\,\underline{X}}$ and $\mathscr{O}^{\wedge\circ}_{\mathrm{Spa}\,\underline{X}}$, so it suffices to check the assertion for $\mathscr{O}^\wedge_{\mathrm{Spa}\,\underline{X}}$. Now, say that $\underline{X} = (X, \mathscr{T}_X, A_X^+)$. Notice first that, for every rational subset R of $\mathrm{Spa}\,\underline{X}$, and every morphism $\varphi_{\underline{Y}/\underline{X}} : \underline{Y} \to \underline{X}$ representing the sub-presheaf h_R of $h_{\underline{X}}$, the quasi-affinoid scheme

$\underline{Y}^\wedge$ is also perfectoid (proposition 16.7.25(iii)). Moreover, lemmata 15.4.17(iv) and 15.5.10 and proposition 16.7.25(i) imply that $\varphi_{\underline{Y}/\underline{X}}$ induces equivalences of categories

$$\mathscr{R}(\underline{Y}^\wedge) \stackrel{\sim}{\to} \mathscr{R}(\underline{Y}) \stackrel{\sim}{\to} \mathscr{R}(\underline{X})/R. \tag{16.7.31}$$

Hence, let $A_X := \mathscr{O}_X(X)$ and $A := A_X^\circ$, so that $\Gamma^\circ(\underline{X}) = (A, A_X^+, X)$ (notation of (16.5.19)); according to lemma 15.5.21, every open covering of $\operatorname{Spa}\underline{X}$ can be refined by the standard covering $R_\bullet := (R_0, \dots, R_n)$ attached to a given sequence $a_\bullet := (a_0, \dots, a_n)$ of elements of A that generates an (open) ideal $J \subset A$ such that $\operatorname{Spec} A/J \subset \operatorname{Spec} A \setminus X$; for every subset $\Lambda \subset \{0, \dots, n\}$ we define the rational subset $R_\Lambda \subset R$ and the quasi-affinoid ring $\underline{A}_{X,\Lambda} := (A_{X,\Lambda}, A^+_{X,\Lambda}, X_\Lambda)$ as in remark 15.5.26(i), so that $R_\Lambda = \operatorname{Spa}\underline{A}_{X,\Lambda}$ for every such Λ. As in the proof of theorem 15.5.27, we are then reduced to checking that the natural map

$$\mathscr{O}^\wedge_{\operatorname{Spa}\underline{X}}(\operatorname{Spa}\underline{X}) \to \operatorname{Equal}\Big(\prod_{i=0}^n \mathscr{O}^\wedge_{\operatorname{Spa}\underline{X}}(R_i) \rightrightarrows \prod_{i,j=0}^n \mathscr{O}^\wedge_{\operatorname{Spa}\underline{X}}(R_{\{i,j\}})\Big) \tag{16.7.32}$$

is an isomorphism of topological rings. To this aim, set $\mathbf{E} := \mathbf{E}(A)$; by proposition 16.6.35, we may assume that there exists a sequence $e_\bullet := (e_0, \dots, e_n)$ of elements of $\mathbf{E}$ such that $a_i = \overline{u}_A(e_i)$ for $i = 0, \dots, n$. Then, for every subset $\Lambda \subset \{0, \dots, n\}$, we consider the subring

$$C_{R,\Lambda} := A\Big[\overline{u}_A(e_j^{1/p^k})/\overline{u}_A(e_i^{1/p^k}) \mid (j, i, k) \in \{0, \dots, n\} \times \Lambda \times \mathbb{N}\Big] \subset A_{X,\Lambda}.$$

Claim 16.7.33. For every $\Lambda \subset \{0, \dots, n\}$ the following holds :

(i) $C_{R,\Lambda}$ is a ring of definition of $A_{X,\Lambda}$.
(ii) The natural map $\omega^\wedge_\Lambda : A^\wedge_{X,\Lambda} \to \mathscr{O}^\wedge_{\operatorname{Spa}\underline{X}}(R_\Lambda)$ of remark 15.5.26(ii) is an isomorphism of topological A-algebras.

Proof of the claim. Both assertions are clear when $\Lambda = \{i\}$, for any $i = 0, \dots, n$, due to proposition 16.7.25(i) and claim 16.7.26. We reduce to this case as follows. Let r be the cardinality of Λ, and set

$$\Sigma := \{e_{i_1} \cdots e_{i_r} \mid 0 \leq i_1 \leq i_2 \leq \dots \leq i_r \leq n\}.$$

It is easily seen that the family $(\overline{u}_A(e) \mid e \in \Sigma)$ generates an open ideal of A whose radical equals the radical of J. Set also $e_\Lambda := \prod_{i\in\Lambda} e_i \in \Sigma$, and notice that

$$R_\Lambda = R_A\Big(\frac{\overline{u}_A(e)}{\overline{u}_A(e_\Lambda)} \mid e \in \Sigma\Big) \qquad C_{R,\Lambda} = A\Big[\overline{u}_A(e^{1/p^k})/\overline{u}_A(e_\Lambda^{1/p^k}) \mid e \in \Sigma,\ k \in \mathbb{N}\Big].$$

After replacing e_0 by e_Λ, and the system $(e_0, \dots, e_n)$ by Σ, we are therefore reduced to the case where $R = R_0$, to which proposition 16.7.25(i) and claim 16.7.26 apply. ◊

In view of claim 16.7.33, we may now proceed as in remark 15.5.26(iv) : we set

$$B := A[T_0^{1/p^\infty}, \dots, T_n^{1/p^\infty}]$$

and consider the homomorphism of $\mathbb{N}[1/p]$-graded A-algebras

$$u : B \to A_X[Y^{1/p^\infty}] \qquad T_i^{1/p^k} \mapsto \overline{u}_A(e_i^{1/p^k}) \cdot Y^{1/p^k} \qquad \text{for } i = 0, \dots, n \text{ and every } k \in \mathbb{N}.$$

Set also $S := \operatorname{Spec} A_X^\circ$; according to (16.7.7), the map u induces a well defined morphism of S-schemes

$$\operatorname{Proj}(u) : X \to \operatorname{Proj} B.$$

Recall also that the inclusion map $A[T_0, \dots, T_n] \to B$ induces an affine morphism of schemes $\psi : \operatorname{Proj} B \to \mathbb{P}^n_S$. Let $\mathbb{P}^n_S = \Omega_0 \cup \dots \cup \Omega_n$ be the standard affine covering by complements of hyperplanes, as in remark 15.5.26(iv), set $\Omega'_i := \psi^{-1}\Omega_i$ and $U_i := (\operatorname{Proj} u)^{-1}\Omega'_i$ for $i = 0, \dots, n$; there follows an affine open covering of $\operatorname{Proj} B$ and an open covering of X :

$$\operatorname{Proj} B = \Omega'_0 \cup \dots \cup \Omega'_n \qquad X = U_0 \cup \dots \cup U_n.$$

Set also $U_\Lambda := \bigcap_{i\in\Lambda} U_i$ for every non-empty $\Lambda \subset \{0,\dots,n\}$; by remark 15.5.26(i) we have

$$\mathscr{O}_X(U_\Lambda) = A_{X,\Lambda} \qquad \text{for every non-empty } \Lambda \subset \{0,\dots,n\}. \tag{16.7.34}$$

Let V be the schematic image of $\mathrm{Proj}\, u$ (see remark 15.5.26(v)); a simple inspection shows that

$$V = \mathrm{Proj}\, \mathscr{A}(e_\bullet)$$

where $\mathscr{A}(e_\bullet)$ is the $\mathbb{N}[1/p]$-graded algebra associated with the sequence $e_\bullet$ via example 16.6.12 (cp. the proof of claim 16.7.27). Especially, we see that

$$V_\Lambda := V \cap \bigcap_{i\in\Lambda} \Omega'_i = \mathrm{Spec}\, C_{R,\Lambda} \qquad \text{for every non-empty subset } \Lambda \subset \{0,\dots,n\}.$$

Fix a finitely generated ideal I of adic definition for $\mathbf{E}$, and for every $t \in \mathbb{R}_+$, denote by $C_t^\bullet$ the augmented alternating Čech complex of $I^{\lceil t\rceil}\mathscr{O}_V$, relative to the open covering $(V_i \mid i = 0,\dots,n)$. Notice that if $\Lambda \neq \varnothing$, the scheme V_Λ is affine, hence we have $I^{\lceil t\rceil} C_{R,\Lambda} = H^0(V_\Lambda, I^{\lceil t\rceil}\mathscr{O}_V)$. Moreover, by corollary 16.7.18 (and theorem 10.2.28(ii)), the complex $C_t^\bullet$ is acyclic, and the natural map $I^{\lceil t\rceil}A \to C_t^{-1}$ is an isomorphism, for every $t \in \mathbb{R}_+$. It follows that

$$I^{\lceil s\rceil} C_t^i = C_{t+s}^i \qquad \text{for every } i \in \mathbb{Z} \text{ and every } s,t \in \mathbb{R}_+$$

and we endow C_t^i with the I-adic topology, so that $(C_{s+t}^i \mid s \in \mathbb{R}_+)$ is a fundamental system of open submodules of C_t^i, for every $i \in \mathbb{Z}$ and every $t \in \mathbb{R}_+$.

Furthermore, denote by $A_X^\bullet$ the augmented alternating Čech complex of $\mathscr{O}_X$ relative to the open covering $(U_i \mid i = 0,\dots,n)$, and for every $i \in \mathbb{Z}$ endow A_X^i with the topology deduced from the topologies of the rings $A_{X,\Lambda}$, via the identifications (16.7.34), as in (15.5.34). There follows a natural monomorphism of complexes of topological A-modules

$$C_t^\bullet \to A_X^\bullet \qquad \text{for every } t \in \mathbb{R}_+$$

and the image of C_t^i is open in A_X^i, for every $i \in \mathbb{Z}$. Denote by $(A_X^{\wedge\bullet}, d_X^\bullet)$ and $(C_t^{\wedge\bullet}, d_t^\bullet)$ the complexes obtained by termwise completion of $A_X^\bullet$ and $C_t^\bullet$; for every $t \in \mathbb{R}_+$ the induced map

$$C_t^{\wedge\bullet} \to A_X^{\wedge\bullet}$$

is still a monomorphism. By the foregoing, we have a natural isomorphism of complexes

$$C_t^{\wedge\bullet} \xrightarrow{\sim} \lim_{s\in\mathbb{R}_+} C_t^\bullet / C_{s+t}^\bullet \qquad \text{for every } t \in \mathbb{R}_+$$

and moreover $C_t^\bullet / C_{s+t}^\bullet$ is acyclic for every $s,t \in \mathbb{R}_+$. By virtue of [163, Th.3.5.8], it follows that $C_t^{\wedge\bullet}$ is acyclic for every $t \in \mathbb{R}_+$. Next, notice that the induced morphism

$$A_X^\bullet / C_t^\bullet \to A_X^{\wedge\bullet} / C_t^{\wedge\bullet}$$

is an isomorphism of complexes, for every $t \in \mathbb{R}_+$; we conclude that

$$H^n A_X^{\wedge\bullet} = H^n(A_X^{\wedge\bullet}/C_t^{\wedge\bullet}) = H^n(A_X^\bullet/C_t^\bullet) = H^n A_X^\bullet \qquad \text{for every } n \in \mathbb{Z}.$$

On the other hand, claim 16.7.33(ii) and remark 15.5.26(iii) yield an isomorphism of complexes of topological A-modules

$$A_X^{\wedge\bullet} \xrightarrow{\sim} C^\bullet_{\mathrm{alt}}(R_\bullet, \mathscr{O}^\wedge_{\mathrm{Spa}\,\underline{X}}). \tag{16.7.35}$$

Since $H^0 A_X^\bullet = 0$, we conclude that (16.7.32) is a ring isomorphism. Lastly, let E be the equalizer appearing in (16.7.32), and endow it with the topology induced by the inclusion into $\prod_{i=0}^n \mathscr{O}^\wedge_{\mathrm{Spa}\,\underline{X}}(R_i)$; we have to check that the resulting map $\mathscr{O}^\wedge_{\mathrm{Spa}\,\underline{X}}(\mathrm{Spa}\,\underline{X}) \to E$ is open. In view of the isomorphism (16.7.35), the assertion will follow from the following :

Claim 16.7.36. The differentials of the complex $(A_X^{\wedge\bullet}, d_X^\bullet)$ are strict and its cohomology has the discrete topology.

Proof of the claim. For every $i \in \mathbb{Z}$, the family $(\mathrm{Ker}\, d^i_t \mid t \in \mathbb{R}_+)$ yields a fundamental system of open submodules of $\mathrm{Ker}\, d^i_X$. But we have already noticed that $C^{\wedge\bullet}_t$ is acyclic, so this system is the image of $(C^{\wedge i}_t \mid t \in \mathbb{R}_+)$, which is a fundamental system of open subgroups of $A^{\wedge i}_X$, whence the contention. ◊

(ii): We may argue as in the proof of theorem 15.5.35(ii) : we choose a standard covering $\mathfrak{U}'$ that refines $\mathfrak{U}$ and we apply lemma 8.6.15(i) and corollary 8.6.12 to the double complex $C^\bullet_{\mathrm{alt}}(\mathfrak{U}, \mathfrak{U}')$ (details left to the reader). □

16.7.37. Let $\underline{X} := (X, \mathscr{T}_X, A^+_X)$ be any perfectoid quasi-affinoid scheme; in light of theorem 16.7.30, we may now argue as in (15.5.44) to first extend $\mathscr{O}^\wedge_{\mathrm{Spa}\,\underline{X}}$ to a sheaf of topological rings on the topological space $\mathrm{Spa}\,\underline{X}$, and then show that the stalks of the latter sheaf are local rings. There follows a well defined morphism of locally ringed spaces

$$\sigma_{\underline{X}} : (\mathrm{Spa}\,\underline{X}, \mathscr{O}^\wedge_{\mathrm{Spa}\,\underline{X}}) \to (X, \mathscr{O}_X)$$

whose induced map on global sections is the identity map $\mathscr{O}_X(X) \to \mathscr{O}^\wedge_{\mathrm{Spa}\,\underline{X}}(\mathrm{Spa}\,\underline{X})$, and whose underlying continuous map $\mathrm{Spa}\,\underline{X} \to X$ is the restriction of the support map of remark 9.2.4(iii). Also corollary 15.5.46 extends to the perfectoid case :

Corollary 16.7.38. *With the notation of* (16.7.37)*, the morphism $\sigma_{\underline{X}}$ induces an isomorphism*

$$H^i(X, \mathscr{O}_X) \xrightarrow{\sim} H^i(\mathrm{Spa}\,\underline{X}, \mathscr{O}^\wedge_{\mathrm{Spa}\,\underline{X}}) \qquad \textit{for every } i \in \mathbb{N}.$$

Proof. We proceed as in the proof of corollary 15.5.46 : first we show the following

Claim 16.7.39. In the situation of the corollary, suppose moreover that $\underline{X}$ is affinoid. Then $H^i(\mathrm{Spa}\,\underline{X}, \mathscr{O}^\wedge_{\mathrm{Spa}\,\underline{X}}) = 0$ for every $i > 0$.

Proof of the claim. Arguing as in the proof of claim 15.5.47, we reduce to checking that for every standard covering $\mathfrak{U}$ of $\mathrm{Spa}\,X$ the Čech cohomology $H^i := H^i_{\mathrm{alt}}(\mathfrak{U}, \mathscr{O}^\wedge_{\mathrm{Spa}\,\underline{X}})$ vanishes for every $i > 0$; but the proof of theorem 16.7.30(i) shows that H^i is naturally isomorphic to the Čech cohomology $H^i_{\mathrm{alt}}(U_\bullet, X)$, relative to a certain affine open covering $U_\bullet$ of X. Then the assertion follows from theorem 10.2.28. ◊

Now, set $A_X := \mathscr{O}_X(X)$, say that $X = \mathrm{Spec}\, A_X \setminus \mathrm{Spec}\, A_X/J$ for some finitely generated ideal $J \subset A_X$, and pick a finite system of generators $f_\bullet := (f_0, \dots, f_n)$ for J. Let $R_\bullet := (R_0, \dots, R_n)$ be the standard covering of $\mathrm{Spa}\,\underline{X}$ associated with $f_\bullet$, and set as well $U_i := \mathrm{Spec}\, A_X[f_i^{-1}]$ for every $i = 0, \dots, n$, so that $U_\bullet := (U_0, \dots, U_n)$ is an affine open covering of X, and moreover $R_i = \sigma^{-1}_{\underline{X}}(U_i)$ for every $i = 0, \dots, n$. There follows a commutative diagram

$$\begin{array}{ccc} H^i_{\mathrm{alt}}(U_\bullet, \mathscr{O}_X) & \longrightarrow & H^i_{\mathrm{alt}}(R_\bullet, \mathscr{O}^\wedge_{\mathrm{Spa}\,\underline{X}}) \\ \downarrow & & \downarrow \\ H^i(X, \mathscr{O}_X) & \longrightarrow & H^i(\mathrm{Spa}\,\underline{X}, \mathscr{O}^\wedge_{\mathrm{Spa}\,\underline{X}}) \end{array}$$

where the vertical arrows are isomorphisms, by virtue of claim 16.7.39 and corollary 10.2.21(ii). We are then reduced to checking that the top horizontal arrow is an isomorphism; but as already pointed out, the latter assertion was shown in the proof of theorem 16.7.30(i). □

16.7.40. Let again $\underline{X}$ be as in (16.7.37); set $A_X := \mathscr{O}_X(X)$, and let $V \subset X^+ := \mathrm{Spec}\, A^+_X$ be any quasi-compact open subset containing the analytic locus of X^+, and such that $\underline{X}$ spreads over V (see definition 15.6.1(i)). Let also $J_V \subset A^+_X$ be the unique radical ideal such that $\mathrm{Spec}\, A^+_X/J_V = X^+ \setminus V$. We consider the sheaf of ideals

$$\mathscr{J}_V \subset \mathscr{O}^{\wedge+}_{\mathrm{Spa}\,\underline{X}} \qquad \text{on } \mathscr{R}(\underline{X})$$

defined as the sheaf associated to the presheaf given by the rule : $R \mapsto J_V \mathscr{O}^{\wedge+}_{\mathrm{Spa}\,\underline{X}}(R)$ for every rational subset $R \subset \mathrm{Spa}\,\underline{X}$.

Theorem 16.7.41. *In the situation of* (16.7.40)*, we have :*

(i) $\mathscr{J}_V(R) = J_V \mathscr{O}^{\wedge+}_{\mathrm{Spa}\,\underline{X}}(R)$ *for every* $R \in \mathrm{Ob}(\mathscr{R}_V(\underline{X}))$ *(notation of* (15.6.7)*).*

(ii) $H^i(X, \mathscr{J}_V) = 0$ *for every* $i > 0$.

Proof. To begin with, we remark :

Claim 16.7.42. Let $R \in \mathrm{Ob}(\mathscr{R}_V(\underline{X}))$, and $\underline{Y}$ any topologically local quasi-affinoid scheme representing the sub-presheaf h_R of $h_{\mathrm{Spa}\,\underline{X}}$. Say that $\underline{Y}^\wedge := (Y^\wedge, \mathscr{T}^\wedge_Y, A^{\wedge+}_Y)$. We have :

(i) The completion $\underline{Y}^\wedge$ spreads over the preimage W of V in $Y^{\wedge+} := \mathrm{Spec}\, A^{\wedge+}_Y$.

(ii) Let $J_W \subset A^{\wedge+}_Y$ be the unique radical ideal such that $\mathrm{Spec}\, A^{\wedge+}_Y / J_W = Y^{\wedge+} \setminus W$. Then $J_W = J_V A^{\wedge+}_Y$.

Proof of the claim. (i): Since $\underline{Y}$ spreads over V, the assertion follows from proposition 15.6.6(i).

(ii): We may find a finitely generated open ideal $J_{\mathbf{E}} \subset \mathbf{E}^+$ such that $J^{\lceil 0 \rceil}_{\mathbf{E}} A^+_X = J_V$ (see (16.5.16)); then we have $\mathrm{Spec}\, A^{\wedge+}_Y / J_V A^{\wedge+}_Y = Y^{\wedge+} \setminus W$, and on the other hand, arguing as in (16.5.16) we see that $J^{\lceil 0 \rceil}_{\mathbf{E}} A^{\wedge+}_Y$ is a radical ideal of $A^{\wedge+}_Y$, whence the claim. ◊

Claim 16.7.43. In order to prove the theorem it suffices to show that for every $\underline{X}$ and V as in (16.7.40) we have $\check{H}^0(\mathscr{R}(\underline{X}), J_V \mathscr{O}^{\wedge+}_{\mathrm{Spa}\,\underline{X}}) = J_V$ and $\check{H}^i(\mathscr{R}(\underline{X}), J_V \mathscr{O}^{\wedge+}_{\mathrm{Spa}\,\underline{X}}) = 0$ for every $i > 0$.

Proof of the claim. Indeed, in light of proposition 15.6.8(i,ii), both (i) and (ii) will follow once we show that the presheaf $J_V \mathscr{O}^{\wedge+}_{\mathrm{Spa}\,\underline{X}}$ is a sheaf on the site $\mathscr{R}_V(\underline{X})$, and $H^i(\mathscr{R}_V(\underline{X}), J_V \mathscr{O}^{\wedge+}_{\mathrm{Spa}\,\underline{X}}) = 0$ for every $i > 0$. Next, in view of theorem 10.2.24, in order to prove the latter assertions it suffices to show that for every $R \in \mathrm{Ob}(\mathscr{R}_V(\underline{X}))$ we have $\check{H}^0(\mathscr{R}_V(\underline{X})/R, J_V \mathscr{O}^{\wedge+}_{\mathrm{Spa}\,\underline{X}}) = J_V \mathscr{O}^{\wedge+}_{\mathrm{Spa}\,\underline{X}}(R)$ and $\check{H}^i(\mathscr{R}_V(\underline{X})/R, J_V \mathscr{O}^{\wedge+}_{\mathrm{Spa}\,\underline{X}}) = 0$ for every $i > 0$. Proposition 15.6.8(iii) then further reduces to showing both identities, with $\mathscr{R}_V(\underline{X})$ replaced by $\mathscr{R}(\underline{X})$. Lastly, for such a rational subset R, define $\underline{Y}$ and W as in claim 16.7.42(i); in light of claim 16.7.42(ii), and taking into account the equivalences (16.7.31), we may then replace $\underline{X}$ by $\underline{Y}^\wedge$, and V by W, and reduce to prove the foregoing identities for $R = \mathrm{Spa}\,\underline{X}$, whence the claim. ◊

We consider first the case where the induced morphism $\beta_{\underline{X}} : X \to X^+ := \mathrm{Spec}\, A^+_X$ is an open immersion, so that $\Omega_{\underline{X}}$ is the image of $\beta_{\underline{X}}$ (notation of definition 15.6.1), and we take $V := \Omega_{\underline{X}}$. By theorem 16.5.13(iii) we have the perfectoid quasi-affinoid ring

$$\Gamma^+(\underline{X}) := (A^+_X, A^+_X, V)$$

and clearly $\mathsf{Spec}\,(\Gamma^+(\underline{X}))$ is isomorphic to $\underline{X}$. In view of lemma 15.5.21, we deduce that every open covering of $\mathrm{Spa}\,\underline{X}$ can be refined by the standard covering $\mathfrak{U}_\bullet := (\mathfrak{U}_0, \dots, \mathfrak{U}_n)$ associated to some sequence $(a_0, \dots, a_n)$ of elements of A^+_X that generates an ideal K with $\mathrm{Spec}\, A^+_X/K \subset X^+ \setminus V$, and notice that $\mathfrak{U}_\bullet$ is a covering family of the site $\mathscr{R}_V(\underline{X})$, due to proposition 15.6.6(ii). We shall show more precisely that the augmented alternating Čech complex $C^\bullet_{\mathrm{alt}}(\mathfrak{U}_\bullet, J_V \mathscr{O}^{\wedge+}_{\mathrm{Spa}\,\underline{X}})$ is acyclic. To this aim, set $\mathbf{E}^+ := \mathbf{E}(A^+_X)$; in view of proposition 16.6.35, we may assume that there exists a sequence $e_\bullet := (e_0, \dots, e_n)$ of elements of $\mathbf{E}^+$ such that $a_i = \overline{u}_{A^+_X}(e_i)$ for $i = 0, \dots, n$, and we let $K_{\mathbf{E}} \subset \mathbf{E}^+$ be the open ideal generated by $e_\bullet$ (proposition 16.6.35(ii)). Notice that $J^{\lceil 0 \rceil}_{\mathbf{E}}$ and $K^{\lceil 0 \rceil}_{\mathbf{E}}$ are the radicals of $J_{\mathbf{E}}$ and respectively $K_{\mathbf{E}}$; we deduce that

$$J^{\lceil 0 \rceil}_{\mathbf{E}} = J^{\lceil 0 \rceil}_{\mathbf{E}} J^{\lceil 0 \rceil}_{\mathbf{E}} \subset K^{\lceil 0 \rceil}_{\mathbf{E}} J^{\lceil 0 \rceil}_{\mathbf{E}} \subset J^{\lceil 0 \rceil}_{\mathbf{E}}. \tag{16.7.44}$$

We consider the $\mathbb{N}[1/p]$-graded A^+_X-algebra $\mathscr{A}$ associated as in example 16.6.12 to the sequence $e_\bullet$, *i.e.* such that

$$\mathscr{A}_0 := A^+_X \qquad \text{and} \qquad \mathscr{A}_\gamma := [e_\bullet]^{\langle \gamma \rangle} A^+_X \qquad \text{for every } \gamma \in \mathbb{N}[1/p] \setminus \{0\}$$

(with multiplication induced by that of A_X^+). We have then a ring homomorphism as in (16.7.9)

$$\varphi : \mathbb{Z}[T_0^{1/p^\infty}, \dots, T_n^{1/p^\infty}] \to \mathscr{A} \qquad T_i^{1/p^r} \mapsto \overline{u}_{A_X^+}(e_i^{1/p^r}) \qquad \text{for } i = 0, \dots, n \text{ and every } r \in \mathbb{N}$$

whose image lies in $\mathscr{A}_0 = A_X^+$, whence corresponding angular Rees algebras $\mathsf{R}\langle A_X^+\rangle$ and $\mathsf{R}\langle\mathscr{A}\rangle$. Set $Y_0 := \mathrm{Proj}\,\mathsf{R}\langle A_X^+\rangle$; the map φ induces a morphism of schemes

$$\mathrm{Proj}\,\varphi : Y_0 \to \mathbb{P}^n := \mathrm{Proj}\,\mathbb{Z}[T_0^{1/p^\infty}, \dots, T_n^{1/p^\infty}].$$

Let $\mathbb{P}^n = \Omega_0 \cup \dots \cup \Omega_n$ be the standard affine open covering by complements of hyperplanes, set $U_i := (\mathrm{Proj}\,\varphi)^{-1}\Omega_i$ for $i = 0, \dots, n$, and let

$$C^\bullet := C^\bullet_{\mathrm{alt}}(U_\bullet, \mathscr{O}_{Y_0})$$

be the augmented alternating Čech complex relative to the covering $U_\bullet := (U_0, \dots, U_n)$. Taking into account (16.7.44) and corollary 16.7.16 (as well as theorem 10.2.28(ii)), we deduce that

$$C^\bullet_{\mathrm{alt}}(U_\bullet, J_V\mathscr{O}_{Y_0}) = J_V C^\bullet$$

and moreover this second augmented alternating Čech complex is acyclic. Furthermore, let $I_{\mathbf{E}}$ be any finitely generated ideal of adic definition for $\mathbf{E}^+$; corollary 16.7.18 shows likewise that

$$C^\bullet_{\mathrm{alt}}(U_\bullet, I_{\mathbf{E}}^{[t]}\mathscr{O}_{Y_0}) = I_{\mathbf{E}}^{[t]}C^\bullet \qquad \text{for every } t \in \mathbb{R}_+$$

and also these complexes are acyclic. We may describe the terms of $C^\bullet$ as in the proof of theorem 16.7.30 : for every $i = 0, \dots, n$, the A_X^+-module C^i is a direct sum $\bigoplus_{\Lambda\subset\{0,\dots,n\}} \mathscr{O}_{Y_0}(U_\Lambda)$, where Λ ranges over the set of subsets of cardinality equal to $i+1$, and $U_\Lambda := \bigcap_{j\in\Lambda} U_j$ for every such Λ. Then

$$\mathscr{O}_{Y_0}(U_\Lambda) = A_X^+[\overline{u}_{A_X^+}(e_i^{1/p^\infty})/\overline{u}_{A_X^+}(e_j^{1/p^\infty}) \mid (i,j) \in \{0,\dots,n\}\times\Lambda] \subset A_X^+[a_j^{-1} \mid j \in \Lambda]$$

for every non-empty $\Lambda \subset \{0, \dots, n\}$. If we set likewise $\mathfrak{U}_\Lambda := \bigcap_{j\in\Lambda}\mathfrak{U}_j$ for every such Λ, we see that $\mathscr{O}_{Y_0}(U_\Lambda) \subset \mathscr{O}^{\wedge+}_{\mathrm{Spa}\,\underline{X}}(\mathfrak{U}_\Lambda)$. Set as usual $A_X := \mathscr{O}_X(X)$; in light of claim 16.7.33 we see that there exists for every $\Lambda \neq \varnothing$ a (unique) f-adic topology on $A_X[a_j^{-1} \mid j \in \Lambda]$ for which the localization map $A_X \to A_X[a_j^{-1} \mid j \in \Lambda]$ is f-adic and $\mathscr{O}_{Y_0}(U_\Lambda)$ is a subring of definition. Especially, the induced topology on $\mathscr{O}_{Y_0}(U_\Lambda)$ is adic, and the natural map $A_X^+ \to \mathscr{O}_{Y_0}(U_\Lambda)$ is an adic ring homomorphism. Let us endow C^i with the corresponding product topology, for every $i = 0, \dots, n$; then $(I_{\mathbf{E}}^{[t]}C^i \mid t \in \mathbb{R}_+)$ is a fundamental system of open submodules of C^i, and hence also of J_VC^i, for every such i. Moreover, we know that the natural maps $J_VA_X^+ \to J_VC^{-1}$ and $I_{\mathbf{E}}^{[t]}A_X^+ \to I_{\mathbf{E}}^{[t]}C^{-1}$ are isomorphisms (again, by corollaries 16.7.16(i) and 16.7.18(i)), so $(I_{\mathbf{E}}^{[t]}C^{-1} \mid t \in \mathbb{R}_+)$ is a fundamental system of open submodules of J_VC^{-1}. Taking into account [163, Th.3.5.8], we conclude that the termwise completion of $J_VC^\bullet$ is still acyclic. Furthermore, the latter is isomorphic to $J_VC^{\wedge\bullet}$, where $C^{\wedge\bullet}$ is the termwise completion of $C^\bullet$: indeed, $J_VC^{\wedge i}$ is dense and open in the completion of J_VC^i for every $i = 0, \dots, n$, so the assertion is clear.

Claim 16.7.45. Set $W_\Lambda := U_\Lambda \cap \mathrm{Spec}\,\mathscr{O}_{Y_0}(U_\Lambda)[a_j^{-1} \mid j \in \Lambda]$ for every non-empty $\Lambda \subset \{0, \dots, n\}$. The datum $(\mathscr{O}_{Y_0}(U_\Lambda), \mathscr{O}_{Y_0}(U_\Lambda), W_\Lambda)$ is a quasi-affinoid ring, and its completion is a perfectoid quasi-affinoid ring

$$\underline{R}_\Lambda := (\mathscr{O}_{Y_0}(U_\Lambda)^\wedge, \mathscr{O}_{Y_0}(U_\Lambda)^\wedge, W_\Lambda^\wedge := W_\Lambda \times_{U_\Lambda} \mathrm{Spec}\,\mathscr{O}_{Y_0}(U_\Lambda)^\wedge).$$

Proof of the claim. Arguing as in the proof of claim 16.7.33, we may assume that $\Lambda = \{i\}$, for some $i \in \{0, \dots, n\}$, in which case it suffices to invoke proposition 16.7.25(iv). ◊

With the notation of claim 16.7.45, we see that

$$\Gamma(\mathsf{Spec}\,\underline{R}_\Lambda) = (\mathscr{O}_{Y_0}(U_\Lambda)^\wedge[a_j^{-1} \mid j \in \Lambda], \mathscr{O}_{Y_0}(U_\Lambda)^{\wedge+}, W_\Lambda^\wedge)$$

where $\mathscr{O}_{Y_0}(U_\Lambda)^\wedge[a_j^{-1} \mid j \in \Lambda]$ carries the unique f-adic topology such that the localization map $\mathscr{O}_{Y_0}(U_\Lambda)^\wedge \to \mathscr{O}_{Y_0}(U_\Lambda)^\wedge[a_j^{-1} \mid j \in \Lambda]$ is open (proposition 8.3.30(ii)), and $\mathscr{O}_{Y_0}(U_\Lambda)^{\wedge+}$ is the integral closure of $\mathscr{O}_{Y_0}(U_\Lambda)^\wedge$ in $\mathscr{O}_{Y_0}(U_\Lambda)^\wedge[a_j^{-1} \mid j \in \Lambda]$. In other words, $\mathscr{O}_{Y_0}(U_\Lambda)^\wedge[a_j^{-1} \mid j \in \Lambda]$ is the completion of $\mathscr{O}_{Y_0}(U_\Lambda)[a_j^{-1} \mid j \in \Lambda] = A_X^+[a_j^{-1} \mid j \in \Lambda]$, where the latter is endowed with the unique f-adic topology such that the localization map $\mathscr{O}_{Y_0}(U_\Lambda) \to A_X^+[a_j^{-1} \mid j \in \Lambda]$ is open (proposition 8.3.33). It follows that $\mathscr{O}_{Y_0}(U_\Lambda)^{\wedge+}$ is also the completion of the integral closure of $\mathscr{O}_{Y_0}(U_\Lambda)$ in $A_X^+[a_j^{-1} \mid j \in \Lambda]$ (lemma 8.3.6), which is the same as the completion of the integral closure of $A_X^+[a_i/a_j \mid (i,j) \in \{0,\dots,n\} \times \Lambda]$ in $A_X^+[a_j^{-1} \mid j \in \Lambda]$. This means that

$$\Gamma(\mathsf{Spec}\,\underline{R}_\Lambda) = (\mathscr{O}^\wedge_{\mathrm{Spa}\,\underline{X}}(\mathfrak{U}_\Lambda), \mathscr{O}^{\wedge+}_{\mathrm{Spa}\,\underline{X}}(\mathfrak{U}_\Lambda), W^\wedge_\Lambda).$$

We have already observed that $J_V\mathscr{O}^{\wedge+}_{\mathrm{Spa}\,\underline{X}}(\mathfrak{U}_\Lambda)$ is a radical ideal of $\mathscr{O}^{\wedge+}_{\mathrm{Spa}\,\underline{X}}(\mathfrak{U}_\Lambda)$; combining with corollary 16.5.17 we conclude that

$$J_V\mathscr{O}^{\wedge+}_{\mathrm{Spa}\,\underline{X}}(\mathfrak{U}_\Lambda) = J_V\mathscr{O}_{Y_0}(U_\Lambda)^\wedge \qquad \text{for every non-empty } \Lambda \subset \{0,\dots,n\}$$

which implies that the natural map of complexes $J_VC^{\wedge\bullet} \to C^\bullet_{\mathrm{alt}}(\mathfrak{U}_\bullet, J_V\mathscr{O}^{\wedge+}_{\mathrm{Spa}\,\underline{X}})$ is an isomorphism, so the proof of the theorem is complete in this case.

For the general case, let us set

$$(X_{\mathbf{E}}, \mathscr{T}_{\mathbf{E}}, \mathbf{E}_X^+) := \mathbf{E}(\underline{X}) \qquad X' := \beta_{\underline{X}}^{-1}(V) \qquad A_{X'} := \mathscr{O}_{X'}(X') \qquad \underline{X}' := X' \times_X \underline{X}$$

(see example 15.4.4). Say that $\underline{X}' = (X', \mathscr{T}_{X'}, A^+_{X'})$; then $\underline{X}'$ spreads over the preimage V' of V in $X'^+ := \mathrm{Spec}\,A^+_{X'}$ (proposition 15.6.6(iv)); on the other hand, by construction the image of $\beta_{\underline{X}'} : X' \to X'^+$ lies in V', so $\beta_{\underline{X}'}$ is an open immersion and $V' = \Omega_{\underline{X}'}$. Therefore we have as well the perfectoid quasi-affinoid ring $\Gamma(\underline{X}') := (A_X^+, A_X^+, V')$, and $\mathsf{Spec}\,\Gamma^+(\underline{X}')$ is isomorphic to $\underline{X}'$; especially, the latter is a perfectoid quasi-affinoid scheme of the type considered in the foregoing case. Next, since the natural morphism $\pi : \underline{X}' \to \underline{X}$ is f-adic, proposition 15.4.22(iii.a) says that $\mathrm{Spa}\,\pi : \mathrm{Spa}\,\underline{X}' \to \mathrm{Spa}\,\underline{X}$ induces a functor

$$\mathscr{R}(\underline{X}) \to \mathscr{R}(\underline{X}') \qquad R \mapsto (\mathrm{Spa}\,\pi)^{-1}R$$

which restricts to a functor

$$\mathscr{R}_V(\underline{X}) \to \mathscr{R}_{V'}(\underline{X}').$$

Indeed, let R be any rational subset of $\mathrm{Spa}\,\underline{X}$, and say that the corresponding sub-presheaf h_R of $h_{\mathrm{Spa}\,\underline{X}}$ is represented by an f-adic morphism $\underline{Y} \to \underline{X}$ of quasi-affinoid schemes. Then $(\mathrm{Spa}\,\pi)^{-1}R$ is represented by the induced morphism $\underline{Y}' := (\underline{Y} \times_{\underline{X}} \underline{X}')_{\mathrm{loc}} \to \underline{X}'$ (see remark 15.5.9(iii)). Now, if $\underline{Y}$ spreads over V, proposition 15.6.6(i,iii,iv) implies that $\underline{Y}'$ spreads over V', whence the contention. Furthermore, let $J_{V'} \subset A^+_{X'}$ be the unique radical ideal such that $\mathrm{Spec}\,A^+_{X'}/J_{V'} = X'^+ \setminus V'$; we have :

Claim 16.7.46. For every $R \in \mathrm{Ob}(\mathscr{R}_V(\underline{X}))$, the induced map

$$J_V\mathscr{O}^{\wedge+}_{\mathrm{Spa}\,\underline{X}}(R) \to J_{V'}\mathscr{O}^{\wedge+}_{\mathrm{Spa}\,\underline{X}'}((\mathrm{Spa}\,\pi)^{-1}R) \tag{16.7.47}$$

is an isomorphism.

Proof of the claim. We consider first the case where $R = \mathrm{Spa}\,\underline{X}$: arguing as in the proof of claim 16.7.42(ii) we easily see that $J_{V'} = J_VA^+_{X'}$. Therefore, let $\overline{A}{}^+_X$ be the image of A^+_X in $A^+_{X'}$; combining with corollary 16.5.17, it follows that

$$J_{V'} = J_V\overline{A}{}^+_X$$

which already shows that the natural ring homomorphism $\rho : A^+_X \to A^+_{X'}$ restricts to a surjection $J_V \to J_{V'}$. Next, notice that the support of $\mathrm{Ker}\,\rho$ lies in the closed subset $X^+ \setminus V$, and therefore $J_V \cap \mathrm{Ker}\,\rho$ is contained in the nilradical of A^+_X; but A^+_X is reduced (corollary 16.3.63(i)), whence the claim, in this case. Next, let $R \in \mathrm{Ob}(\mathscr{R}_V(\underline{X}))$ be a general rational subset; to ease notation, set $R' := (\mathrm{Spa}\,\underline{\pi})^{-1}R$, and let $\underline{Z}$ be any topologically local quasi-affinoid scheme representing the sub-presheaf $h_{R'}$ of $h_{\mathrm{Spa}\,\underline{X}'}$. Define $\underline{Y}^\wedge$, $Y^\wedge$, $Y^{\wedge+}$, $W \subset Y^{\wedge+}$ and J_W as in claim 16.7.42. Let also $Y' := \beta^{-1}_{\underline{Y}^\wedge}(W)$ and $\underline{Y}' := Y' \times_{Y^\wedge} \underline{Y}^\wedge$. Say that $\underline{Y}' = (Y', \mathscr{T}_{Y'}, A^+_{Y'})$, and let moreover W' be the preimage of W in $Y'^+ := \mathrm{Spec}\, A^+_{Y'}$, and $J_{W'} \subset A^+_{Y'}$ the unique radical ideal such that $\mathrm{Spec}\, A^+_{Y'}/J_{W'} = Y'^+ \setminus W'$; arguing as in the foregoing, we see that $\underline{Y}'$ is a perfectoid quasi-affinoid scheme that spreads over W', and a direct inspection of the constructions shows that $\underline{Y}'$ is the completion of $\underline{Z}$. Thus, the image of $\mathrm{Spa}\,\underline{Y}^\wedge$ (resp. of $\mathrm{Spa}\,\underline{Y}'$) in $\mathrm{Spa}\,\underline{X}$ (resp. in $\mathrm{Spa}\,\underline{X}'$) agrees with R (resp. with R'), and taking into account claim 16.7.42(ii) we conclude that (16.7.47) is naturally identified with the map

$$J_W \to J_{W'}$$

induced by the projection $\underline{Y}' \to \underline{Y}^\wedge$. By the previous case, we know already that this map is bijective, whence the claim. ◊

We may now conclude the proof of the theorem : indeed, we have already remarked that every covering family of $\mathscr{R}(\underline{X})$ can be refined by a standard covering $\mathfrak{U}_\bullet := (\mathfrak{U}_0, \dots, \mathfrak{U}_n)$ which is also a covering family of $\mathscr{R}_V(\underline{X})$. It follows that $\mathfrak{U}'_\bullet := ((\mathrm{Spa}\,\pi)^{-1}\mathfrak{U}_i \mid i = 0, \dots, n)$ is a standard covering of $\mathrm{Spa}\,\underline{X}'$ which is also a covering family of $\mathscr{R}_{V'}(\underline{X}')$. By the previous case, we know already that $\check{H}^0(\mathfrak{U}'_\bullet, J_{V'}\mathscr{O}^{\wedge+}_{\mathrm{Spa}\,\underline{X}'}) = J_{V'}$ and $\check{H}^i(\mathfrak{U}'_\bullet, J_{V'}\mathscr{O}^{\wedge+}_{\mathrm{Spa}\,\underline{X}'}) = 0$ for every $i > 0$. In view of claim 16.7.46, it follows that $\check{H}^0(\mathfrak{U}_\bullet, J_V\mathscr{O}^{\wedge+}_{\mathrm{Spa}\,\underline{X}'}) = J_V$ and $\check{H}^i(\mathfrak{U}_\bullet, J_V\mathscr{O}^{\wedge+}_{\mathrm{Spa}\,\underline{X}'}) = 0$ for every $i > 0$, and taking into account claim 16.7.43, the theorem follows. □

16.8. **Almost purity.** To begin with, let p be a prime integer; we consider a perfect $\mathbb{F}_p$-algebra V and a radical ideal $\mathfrak{m} \subset V$. Since V is perfect, $(V, \mathfrak{m})$ is clearly a basic setup, in the sense of [75, §2.1.1]. Our first observation is the following :

Lemma 16.8.1. *In the situation of* (16.8)*, let $A \subset B$ be an inclusion of perfect V-algebras, and $\mathfrak{m}_0 \subset \mathfrak{m}$ a subideal whose radical is $\mathfrak{m}$. We set*

$$A' := \{x \in B \mid \mathfrak{m} \cdot x \subset A\} \qquad B' := \{x \in B \mid \text{there exists } n \in \mathbb{N} \text{ with } \mathfrak{m}_0^n \cdot x \subset A\}.$$

Then A' and B' are perfect V-algebras, and A' is integrally closed in B'.

Proof. It is easily seen that A' and B' are V-subalgebras of B, and obviously $A \subset A' \subset B'$. Next, let $x \in B$, and say that $x^p \in A'$; thus, $\mathfrak{m} \cdot x^p \subset A$, and therefore $(ax)^p \in A$ for every $a \in \mathfrak{m}$. Since A is perfect, it follows that $ax \in A$ for every such a, so that $x \in A'$; this show that A' is perfect. Likewise one checks that B' is perfect.

Next, let $x \in B'$, and suppose that $x^n + a_1x^{n-1} + \cdots + a_n = 0$ for some $a_1, \dots, a_n \in A'$. Then $A'[x] = A' + A'x + \cdots + A'x^{n-1} \subset B'$. By assumption, there exists $k \in \mathbb{N}$ with $\mathfrak{m}_0^k \cdot x \subset A$, and $\mathfrak{m}_0 A' \subset \mathfrak{m}A' \subset A$; it follows easily that $\mathfrak{m}_0^{nk}A'[x] \subset A$. Now, let $a \in \mathfrak{m}$; by assumption, there exists $M \in \mathbb{N}$ such that $a^{p^M} \in \mathfrak{m}_0^{nk}$, whence $(ax)^{p^M} \in A$, and since A is perfect, we conclude that $ax \in A$ for every such a, *i.e.* $x \in A'$. This shows that A' is integrally closed in B'. □

We may now state the following generalization of [75, Th.3.5.28] :

Theorem 16.8.2. *In the situation of* (16.8)*, set $S := \mathrm{Spec}\, V$ and $Z := \mathrm{Spec}\, V/\mathfrak{m}$. Then :*

(i) *The pair (S, Z) is almost pure, relative to the basic setup* $(V, \mathfrak{m})$.
(ii) *If Z is constructible in S, the pair (S, Z) is normal, for the basic setup* $(V, \mathfrak{m})$.

Proof. (ii): Denote by $j : U \to S$ the open immersion; the kernel $\mathscr{K}$ of the natural map $\mathscr{O}_S \to j_*\mathscr{O}_U$ is the largest $\mathscr{O}_S$-submodule such that $\mathfrak{m}\mathscr{K} = 0$, so the induced morphism $\mathscr{O}^a_S \to j_*\mathscr{O}^a_U$ is a monomorphism. It remains to check that the image of $\mathscr{O}^a_S$ is integrally closed in $j_*\mathscr{O}^a_U$, and since Z is constructible, we are reduced to showing that V^a is integrally closed in $\mathscr{O}_S(U)^a$ ([59, Ch.I, Prop.9.2.1]). The latter follows from lemma 16.8.1 and [75, Lemma 8.2.28].

(i): Let $(Z_\lambda \mid \lambda \in \Lambda)$ be the cofiltered system of closed constructible subsets of S containing Z, and for every $\lambda \in \Lambda$ let $\mathfrak{m}_\lambda \subset \mathfrak{m}$ be the unique radical ideal such that $\mathrm{Spec}\, V/\mathfrak{m}_\lambda = Z_\lambda$. Set also $U := S \setminus Z$ and $U_\lambda := S \setminus Z_\lambda$ for every $\lambda \in \Lambda$. Then $\mathfrak{m}_\lambda = \mathfrak{m}_\lambda^2$ and notice that both $\mathfrak{m}_\lambda$ (for every $\lambda \in \Lambda$) and $\mathfrak{m}$ trivially fulfill condition $(\mathbf{B})$ of [75, §2.1.6], since V is perfect. We then get an essentially commutative diagram

$$\begin{array}{ccc} (\mathscr{O}_S, \mathfrak{m}\mathscr{O}_S)^a\text{-}\mathbf{\acute{E}t}_{\mathrm{afr}} & \xrightarrow{\ \rho\ } & \mathscr{O}_U\text{-}\mathbf{\acute{E}t} \\ \downarrow & & \downarrow \\ (\mathscr{O}_S, \mathfrak{m}_\lambda\mathscr{O}_S)^a\text{-}\mathbf{\acute{E}t}_{\mathrm{afr}} & \xrightarrow{\ \rho_\lambda\ } & \mathscr{O}_{U_\lambda}\text{-}\mathbf{\acute{E}t} \end{array} \qquad \text{for every } \lambda \in \Lambda.$$

However, on the one hand, proposition 14.2.3 and corollary 14.2.5(ii,iv,v) imply that the category $(\mathscr{O}_S, \mathfrak{m}\mathscr{O}_S)^a\text{-}\mathbf{\acute{E}t}_{\mathrm{afr}}$ is equivalent to the 2-limit of the system $((\mathscr{O}_S, \mathfrak{m}_\lambda\mathscr{O}_S)^a\text{-}\mathbf{\acute{E}t}_{\mathrm{afr}} \mid \lambda \in \Lambda)$; on the other hand, it is clear that $\mathscr{O}_U\text{-}\mathbf{\acute{E}t}$ is equivalent to the 2-limit of the system $(\mathscr{O}_{U_\lambda}\text{-}\mathbf{\acute{E}t} \mid \lambda \in \Lambda)$. Thus, we are reduced to checking that ρ_λ is an equivalence for every $\lambda \in \Lambda$, and we may therefore assume from start that Z is constructible. In view of (ii) and lemma 14.4.5(ii), we are then reduced to showing that for every finite étale $\mathscr{O}_U$-algebra $\mathscr{B}$ there exists an étale $\mathscr{O}_S^a$-algebra $\mathscr{A}$ of almost finite rank such that $\rho(\mathscr{A})$ is isomorphic to $\mathscr{B}$. Now, by applying proposition 14.3.30 with $X := S$ and $\mathfrak{m}_0 = \mathfrak{m} := V$ (so, we consider here the "classical limit" almost ring structure) we obtain a quasi-coherent $\mathscr{O}_S$-algebra $\mathscr{R}_0$ which is finitely presented as an $\mathscr{O}_S$-module, with an isomorphism $\mathscr{B} \xrightarrow{\sim} \mathscr{R}_{0|U}$ of $\mathscr{O}_U$-algebras.

Let $\Phi_S : S \to S$ be the Frobenius endomorphism, and consider the directed system of quasi-coherent $\mathscr{O}_S$-algebras $(\mathscr{R}_n \mid n \in \mathbb{N})$ with $\mathscr{R}_n := \Phi_S^{-n}\mathscr{R}_0$ for every $n \in \mathbb{N}$; explicitly, $\mathscr{R}_n(U') = \mathscr{R}_0(U')$ for every open subset $U' \subset S$, and the $\mathscr{O}_S(U')$-algebra structure on $\mathscr{R}_n(U')$ is given by the composition of the Frobenius endomorphism $\mathscr{O}_S(U') \to \mathscr{O}_S(U')$ with the structure morphism $\mathscr{O}_S(U') \to \mathscr{R}_0(U')$. The transition maps $\varphi_n : \mathscr{R}_n \to \mathscr{R}_{n+1}$ are given on each such U' by the Frobenius endomorphism $\mathscr{R}_0(U') \to \mathscr{R}_0(U')$, for every $n \in \mathbb{N}$. We let

$$\mathscr{R} := \operatorname*{colim}_{n \in \mathbb{N}} \mathscr{R}_n.$$

Since V is perfect, [75, Th.3.5.13] (applied again in the "classical limit" case with $\mathfrak{m} = V$) implies that $\varphi_{n|U}$ is an isomorphism for every $n \in \mathbb{N}$, so the natural morphism $\mathscr{R}_0 \to \mathscr{R}$ induces an isomorphism of $\mathscr{O}_U$-algebras

$$\mathscr{B} \xrightarrow{\sim} \mathscr{R}_{|U}.$$

Claim 16.8.3. There exists a subideal $\mathfrak{m}_0 \subset \mathfrak{m}$ whose radical is $\mathfrak{m}$, and with

$$\mathfrak{m}_0 \cdot \mathrm{Ker}\, \varphi_0 = \mathfrak{m}_0 \cdot \mathrm{Coker}\, \varphi_0 = 0.$$

Proof of the claim. Since Z is constructible, there exist finitely many elements $x_1, \dots, x_n \in \mathfrak{m}$ such that $U = \bigcup_{i=1}^n \mathrm{Spec}\, V[x_i^{-1}]$. Notice that both $\mathscr{R}_0$ and $\mathscr{R}_1$ are finitely presented $\mathscr{O}_S$-modules : indeed, this holds for assumption for $\mathscr{R}_0$, and then it follows also for $\mathscr{R}_1$, since Φ_S is an automorphism of S. Thus, since moreover $\varphi_{0|U}$ is an isomorphism, after replacing each x_i by a power x_i^N for some suitable $N \in \mathbb{N}$, we may assume already that $x_i \cdot \mathrm{Coker}\, \varphi_0 = 0$ for every $i = 1, \dots, n$. Next, let $R_i := \mathscr{R}_i(S)$ for $i = 0, 1$, and denote by $f : R_0 \to R_1$ the Frobenius map; the natural map

$$V[x_i^{-1}] \otimes_V \mathrm{Hom}_V(R_1, R_0) \to \mathrm{Hom}_{V[x_i^{-1}]}(R_1[x_i^{-1}], R_0[x_i^{-1}])$$

is bijective ([75, Lemma 2.4.29(i.a)]) and $V[x_i^{-1}] \otimes_V f$ is an isomorphism, hence we may find for every $i = 1, \dots, n$ a V-linear map $g_i : R_1 \to R_0$ such that $V[x_i^{-1}] \otimes_V (g \circ f) = \mathbf{1}_{R_0}$. Then there exists $N \in \mathbb{N}$ such that $x_i^N \cdot (g \circ f) = x_i^N \cdot \mathbf{1}_{R_0}$ for every $i = 1, \dots, n$, whence $x_i^N \cdot \mathrm{Ker}\, f = 0$ for every $i = 1, \dots, n$. The claim follows. ◊

Let $\mathfrak{m}_0$ be as in claim 16.8.3, and for every $n \in \mathbb{N}$ set $\mathfrak{m}_n := \Phi_V^{-n}(\mathfrak{m}_0)$, where $\Phi_V : V \to V$ denotes the Frobenius automorphism; then $\mathfrak{m}_n \cdot \mathrm{Ker}\,\varphi_n = \mathfrak{m}_n \cdot \mathrm{Coker}\,\varphi_n = 0$ for every $n \in \mathbb{N}$. Notice that

$$\mathfrak{m}_n^2 \subset \mathfrak{m}_n \cdot \mathfrak{m}_{n+1} \cdots \mathfrak{m}_{n+k} \qquad \text{for every } n, k \in \mathbb{N}$$

and therefore $\mathfrak{m}_n^2$ annihilates the kernel and cokernel of the transition map $\mathscr{R}_n \to \mathscr{R}_{n+k}$, for every $n, k \in \mathbb{N}$; we conclude that

$$\mathfrak{m}_n^2 \cdot \mathrm{Ker}\,(\mathscr{R}_n \to \mathscr{R}) = \mathfrak{m}_n^2 \cdot \mathrm{Coker}\,(\mathscr{R}_n \to \mathscr{R}) = 0 \qquad \text{for every } n \in \mathbb{N}. \tag{16.8.4}$$

Especially, $\mathscr{A} := \mathscr{R}^a$ is an almost finitely presented and uniformly almost finitely generated $\mathscr{O}_S^a$-module ([75, Cor.2.3.13]). Next, pick again elements $x_1, \dots, x_n \in \mathfrak{m}$ such that U is the union of the affine open subsets $U_i := \mathrm{Spec}\, V[x_i^{-1}]$ for $i = 1, \dots, n$; since $\mathscr{B}$ is a projective $\mathscr{O}_U$-module of finite rank, for every $i = 1, \dots, n$ there exist $N_i \in \mathbb{N}$ and $\mathscr{O}_U$-linear morphisms

$$\mathscr{B}_{|U_i} \xrightarrow{f_i} \mathscr{O}_{U_i}^{\oplus N_i} \xrightarrow{g_i} \mathscr{B}_{|U_i} \qquad \text{such that} \qquad g_i \circ f_i = \mathbf{1}_{\mathscr{B}|U_i}.$$

Then, arguing as in the proof of claim 16.8.3 we find for every $i = 1, \dots, n$, an integer $M_i \in \mathbb{N}$ and $\mathscr{O}_S$-linear morphisms

$$\mathscr{R}_0 \xrightarrow{f_{0,i}} \mathscr{O}_S^{\oplus N_i} \xrightarrow{g_{0,i}} \mathscr{R}_0 \qquad \text{such that} \qquad g_{0,i} \circ f_{0,i} = x_i^{M_i} \cdot \mathbf{1}_{\mathscr{R}_0}$$

whence, for every $k \in \mathbb{N}$, and every $i = 1, \dots, n$, a pair of $\mathscr{O}_S$-linear morphisms

$$\mathscr{R}_k \xrightarrow{f_{k,i}} \mathscr{O}_S^{\oplus N_i} \xrightarrow{g_{k,i}} \mathscr{R}_k \qquad \text{such that} \qquad g_{k,i} \circ f_{k,i} = x_i^{M_i/p^k} \cdot \mathbf{1}_{\mathscr{R}_0}.$$

It follows easily that $x_i^{M_i/p^k} \cdot \mathrm{Tor}_j^{\mathscr{O}_S}(\mathscr{R}_k, \mathscr{N}) = 0$ for every $j > 0$, every $i = 1, \dots, n$, every $k \in \mathbb{N}$ and every quasi-coherent $\mathscr{O}_S$-module $\mathscr{N}$. After taking colimits, we conclude that $\mathscr{A}$ is a flat $\mathscr{O}_S^a$-module; then $\mathscr{A}$ is also an almost projective $\mathscr{O}_S$-module of finite rank, in light of [75, Prop.2.4.18(ii) and Rem.4.3.10(i)]. Next, let e be the diagonal idempotent of the étale $\mathscr{O}_U$-algebra $\mathscr{B}$ (see remark 14.3.10(i)). Pick a finite system of elements $\varepsilon_1, \dots, \varepsilon_k \in \mathfrak{m}$ such that $\mathfrak{m}$ is the radical of the ideal $\sum_{i=1}^k V\varepsilon_i$; we may then find $N \in \mathbb{N}$ and for every $i = 1, \dots, k$ an element $e_i \in R_0 \otimes_V R_0$ such that

$$(\varepsilon_i^N \cdot e_i)_{|\mathrm{Spec}\, V[\varepsilon^{-1}]} = (\varepsilon_i^N \cdot e)_{|\mathrm{Spec}\, V[\varepsilon^{-1}]} \qquad \text{and} \qquad \mu_0(e_i) = \varepsilon_i^N \tag{16.8.5}$$

where $\mu_n : \mathscr{R}_n \otimes_{\mathscr{O}_S} \mathscr{R}_n \to \mathscr{R}_n$ is the multiplication map, for every $n \in \mathbb{N}$. Next, let $a_1, \dots, a_n$ be a finite system of generators for the V-module R_0; we may find $N' \in \mathbb{N}$ such that

$$\varepsilon_i^{N'} \cdot (a_j \otimes 1 - 1 \otimes a_j) \cdot e_i = 0 \qquad \text{for every } i = 1, \dots, k \text{ and every } j = 1, \dots, n$$

and after replacing e_i by $\varepsilon_i^{N'} e_i$ and N by $N + N'$ we may therefore assume that (16.8.5) still holds, and additionally :

$$e_i \cdot \mathrm{Ker}\,\mu_0 = 0 \qquad \text{for every } i = 1, \dots, k.$$

The same $e_1, \dots, e_k$ can also be regarded as elements of $R_n \otimes_V R_n$, and we deduce that

$$\mu_n(e_i) = \varepsilon_i^{N/p^n} \qquad \text{and} \qquad e_i \cdot \mathrm{Ker}\,\mu_n = 0 \qquad \text{for } i = 1, \dots, k \text{ and every } n \in \mathbb{N}. \tag{16.8.6}$$

Set $R := \mathscr{R}(S)$ and denote by $e_{n,i} \in R \otimes_V R$ the image of e_i under the natural morphism $\mathscr{R}_n \to \mathscr{R}$, for $i = 1, \dots, k$. Say that $\varepsilon_i^{N''} \in \mathfrak{m}_0$ for $i = 1, \dots, k$ (where $\mathfrak{m}_0$ is as in claim 16.8.3); then $\varepsilon_i^{N''/p^n} \in \mathfrak{m}_n$ for every $n \in \mathbb{N}$ and $i = 1, \dots, k$. Recall that the kernel of the multiplication map $\mu : \mathscr{R} \otimes_{\mathscr{O}_S} \mathscr{R} \to \mathscr{R}$ is generated, as a $\mathscr{R} \otimes_{\mathscr{O}_S} \mathscr{R}$-module, by the system $(a \otimes 1 - 1 \otimes a \mid a \in R)$; in light of (16.8.4) and (16.8.6) we deduce that

$$\varepsilon_i^{2N''/p^n} \cdot e_{n,i} \cdot \mathrm{Ker}\,\mu = 0 \qquad \text{for } i = 1, \dots, k$$

where $\mu : \mathscr{R} \otimes_{\mathscr{O}_S} \mathscr{R} \to \mathscr{R}$ is the multiplication map. After replacing $e_{n,i}$ by $\varepsilon_i^{2N''/p^n} \cdot e_{n,i}$ and N by $N + 2N''$, we then obtain $\mathscr{R} \otimes_{\mathscr{O}_S} \mathscr{R}$-linear morphisms

$$\nu_{n,i} : \mathscr{R} \to \mathscr{R} \otimes_{\mathscr{O}_S} \mathscr{R} \qquad x \mapsto (x \otimes 1) \cdot e_{n,i} \qquad \text{for every } n \in \mathbb{N} \text{ and } i = 1, \dots, k$$

such that $\mu \circ \nu_{n,i} = \varepsilon_i^{N/p^n} \cdot \mathbf{1}_{\mathscr{R}}$. Set $R := \mathscr{R}(S)$ for every $n \in \mathbb{N}$; it follows easily that $\varepsilon_i^{N/p^n} \cdot \mathrm{Ext}^j_{R \otimes_V R}(R, M) = 0$ for every $i = 1, \dots, k$, every $n \in \mathbb{N}$, every $j > 0$ and every $R \otimes_V R$-module M, so $\mathscr{A}$ is an almost projective $\mathscr{A} \otimes_{\mathscr{O}_S} \mathscr{A}$-module, and the proof is concluded. ☐

Notice next that, in the situation of (16.8), the ring V is reduced, hence every flat almost finitely presented V^a-module has almost finite rank (lemma 14.3.4(ii)). We can generalize this observation as follows :

Proposition 16.8.7. *Let* $(V, \mathfrak{m})$ *be a basic setup as in* (16.8)*, and consider ideals*

$$I \subset J \subset \mathfrak{m}$$

with J finitely generated. Let P be any flat almost finitely presented $(V/I)^a$-module. We have :

(i) *P has almost finite rank.*

(ii) *If* $\mathrm{Spec}\, V/\mathfrak{m}$ *is a constructible subset of* $S := \mathrm{Spec}\, V$*, then P has finite rank.*

Proof. In light of corollary 14.2.5(iv) we may assume that $\mathfrak{m}$ is the radical of a finitely generated subideal $\mathfrak{m}_0$ containing J, in which case $\mathrm{Spec}\, V/\mathfrak{m}$ is constructible in S, and it suffices to show assertion (ii). Then we may even replace J by $\mathfrak{m}_0$, and assume that $J^{\lceil 0 \rceil} V = \mathfrak{m}$ (notation of remark 9.3.70(i)).

Claim 16.8.8. We may assume that J lies in the Jacobson radical of V.

Proof of the claim. Indeed, let $x_1, \dots, x_k$ be a finite system of generators of J; the induced map

$$V \to (1 + J)^{-1} V \times \prod_{i=1}^{k} V[x_i^{-1}]$$

is faithfully flat, and clearly $V[x_i^{-1}] \otimes_V P$ is a projective $V[x_i^{-1}] \otimes_V V/I$-module of finite rank for every $i = 1, \dots, k$, so we are reduced to checking that $(1 + J)^{-1} V \otimes_V P$ is an almost projective $(1 + J)^{-1} V \otimes_V (V/I)^a$-module of finite rank, whence the claim. ◊

Henceforth we assume that J lies in the Jacobson radical of V. Now, suppose that we have $\Lambda^r_{(V/J)^a}(P/JP) = 0$ for some $r \in \mathbb{N}$; then $J \cdot \Lambda^r_{(V/I)^a} P = \Lambda^r_{(V/I)^a} P$, and since J is tight and $\Lambda^i_{(V/I)} P$ is almost finitely generated, we deduce that $\Lambda^r_{(V/I)^a} P = 0$, by [75, Lemma 5.1.7]. Thus, we are reduced to checking the assertion in case $I = J$. Next, notice that there exists $n \in \mathbb{N}$ such that $(J^{\langle 1 \rangle})^n V = J^{\langle n \rangle} V \subset J$ (lemma 9.3.69(ii.a,iv)), so $J^{\langle 1 \rangle} V$ is also a tight ideal contained in the Jacobson radical of V, and by the same token we are reduced to checking that $Q := P/J^{\langle 1 \rangle} P$ is a $(V/J^{\langle 1 \rangle})^a$-module of finite rank. To this aim, let us consider the S-scheme

$$Y := \mathrm{Proj}\, \mathsf{R} \qquad \text{where} \qquad \mathsf{R} := \bigoplus_{\gamma \in \mathbb{N}[1/p]} J^{\langle \gamma \rangle} V$$

as in (16.7.9), as well as the closed immersion

$$i : Y_0 := Y \times_S \mathrm{Spec}\, V/J^{\langle 1 \rangle} V = \mathrm{Proj}\, \mathsf{R}/J^{\langle 1 \rangle} \mathsf{R} \to Y.$$

Let also $\mathscr{T}$ be the discrete topology on V, and notice that $(V, \mathscr{T})$ is a perfectoid ring. Notice furthermore that $(J^{\langle 1 \rangle} V)^a = (J^{\lceil 1 \rceil} V)^a$; in light of example 16.7.9(i), theorem 16.7.10, and the short exact sequence of quasi-coherent $\mathscr{O}_Y$-modules

$$0 \to J^{\langle 1 \rangle} \mathscr{O}_Y \to \mathscr{O}_Y \to i_* \mathscr{O}_{Y_0} \to 0$$

we then deduce an isomorphism of V^a-algebras

$$(V/J^{\langle 1\rangle}V)^a \xrightarrow{\sim} H^0(Y_0, \mathscr{O}_{Y_0})^a$$

(details left to the reader). Say that J is generated by a finite system $x_1, \dots, x_k$; then Y admits a closed immersion $Y \to \mathbb{P}^{k-1}_S := \mathrm{Proj}\, V[T_1^{1/p^\infty}, \dots, T_k^{1/p^\infty}]$ (see example 4.8.55(iii)), whence an affine covering $Y = U_1 \dots, U_k$ consisting of preimages of complements of hyperplanes of $\mathbb{P}^k_S$, as usual. Set $U_{0,i} := U_i \times_Y Y_0$ for $i = 1, \dots, k$; we get an injective ring homomorphism

$$H^0(Y_0, \mathscr{O}_{Y_0}) \to A := \prod_{i=1}^k H^0(U_{0,i}, \mathscr{O}_{Y_0})$$

and on the other hand, natural identifications of V-algebras

$$A_i := \mathrm{gr}_0 \mathsf{R}[x_i^{-1}] \xrightarrow{\sim} H^0(U_i, \mathscr{O}_Y) \qquad A_i/J^{\langle 1\rangle}A_i \xrightarrow{\sim} H^0(U_{0,i}, \mathscr{O}_{Y_0}) \qquad i = 1, \dots, k.$$

Since Q is a flat $(V/J^{\langle 1\rangle}V)^a$-module, the same holds for $\Lambda^r_{(V/J^{\langle 1\rangle}V)^a}Q$, for every $r \in \mathbb{N}$; hence the induced map

$$\Lambda^r_{(V/J^{\langle 1\rangle}V)^a}Q \to A^a \otimes_{V^a} \Lambda^r_{(V/J^{\langle 1\rangle}V)^a}Q = \Lambda^r_{A^a}(A^a \otimes_{V^a} Q)$$

is a monomorphism, and we are reduced to checking that $A_i^a \otimes_{V^a} Q$ is an $(A_i/J^{\langle 1\rangle}A_i)^a$-module of finite rank for every $i = 1, \dots, k$. Now, let (B_i, JB_i) be the henselization of the pair (A_i, JA_i). The ideal $\mathfrak{m}B_i$ is generated by the system $(x_i^{1/p^n} \mid n \in \mathbb{N})$ consisting of regular elements of B_i; by [75, Th.2.1.12(ii.b)] we deduce that $\mathfrak{m}B_i$ is a B_i-module of homological dimension ≤ 1. Lastly, since $B_i/J^{\langle 1\rangle}B_i = A_i/J^{\langle 1\rangle}A_i$, we may apply [75, Th.5.5.7(i)] to find a flat almost finitely presented B_i^a-module Q' with an isomorphism $Q'/J^{\langle 1\rangle}Q' \xrightarrow{\sim} A_i^a \otimes_{V^a} Q$; since the localization $B_i \to B_i[x_i^{-1}]$ is a monomorphism, and since $B_i[x_i^{-1}] \otimes_{B_i} Q'$ is a projective $B_i[x_i^{-1}]$-module of finite rank, the B_i^a-module Q' has finite rank as well. But then obviously the same holds for the $(A_i/J^{\langle 1\rangle}A)^a$-module $A_i^a \otimes_{V^a} Q$, as required. □

16.8.9. Let A be a perfectoid ring, $\mathbf{E} := \mathbf{E}(A)$, and $\underline{\alpha} \in W(\mathbf{E})$ a distinguished element in the kernel of $u_A : W(\mathbf{E}) \to A$. Recall that $\overline{u}_A : \mathbf{E} \to A$ induces an isomorphism of topological rings $\omega : \mathbf{E}/\alpha_0\mathbf{E} \xrightarrow{\sim} A/pA$ (remark 16.3.7(ii)); hence, for every ideal $I \subset A$ containing pA, there exists a unique ideal $I_\mathbf{E} \subset \mathbf{E}$ containing $\alpha_0\mathbf{E}$ such that $\omega(I_\mathbf{E}/\alpha_0\mathbf{E}) = I/pA$.

Definition 16.8.10. In the situation of (16.8.9), let $(A, \mathfrak{m})$ and $(\mathbf{E}, \mathfrak{n})$ be two basic setups. We say that $(A, \mathfrak{m})$ and $(\mathbf{E}, \mathfrak{n})$ are *compatible* if there exists an ideal of definition $I \subset A$ such that ω induces a bijection $(\mathfrak{n} + I_\mathbf{E})/I_\mathbf{E} \xrightarrow{\sim} (\mathfrak{m} + I)/I$.

Remark 16.8.11. With the notation of definition 16.8.10, let M, N be two A/I-modules, and $h : M \to N$ an A-linear map. By assumption, $\overline{u}_A$ induces a ring isomorphism $\omega_I : \mathbf{E}/I_\mathbf{E} \xrightarrow{\sim} A/I$, and the compatibility of $(A, \mathfrak{m})$ and $(\mathbf{E}, \mathfrak{n})$ means that ω_I is an isomorphism

$$(\mathbf{E}/I_\mathbf{E}, (\mathfrak{n} + I_\mathbf{E})/I_\mathbf{E}) \xrightarrow{\sim} (A/I, (\mathfrak{m} + I)/I)$$

in the category $\mathscr{B}$ of basic setups as in [75, §3.5]. Hence h^a is an isomorphism of $(A, \mathfrak{m})^a$-modules if and only if $\omega_I^*(h^a) : \omega_I^*(M^a) \to \omega_I^*(N^a)$ is an isomorphism of $(\mathbf{E}, \mathfrak{n})^a$-modules.

Lemma 16.8.12. *In the situation of* (16.8.9)*, consider two compatible basic setups* $(A, \mathfrak{m})$ *and* $(\mathbf{E}, \mathfrak{n})$*, and let also* $f : B \to B'$ *be any continuous map in* A-Perf*. Then the morphism* $f^a : B^a \to B'^a$ *is an isomorphism in* $(A, \mathfrak{m})^a$-Alg *if and only if* $\mathbf{E}(f)^a : \mathbf{E}(B)^a \to \mathbf{E}(B')^a$ *is an isomorphism in* $(\mathbf{E}, \mathfrak{n})^a$-$\mathsf{Alg}$.

Proof. By assumption, there exists an ideal of definition $I \subset A$ such that ω induces a bijection $(\mathfrak{m}+I)/I \stackrel{\sim}{\to} (\mathfrak{n}+I_{\mathbf{E}})/I_{\mathbf{E}}$, and arguing as in the proof of lemma 16.2.7(i), we find a sequence $\beta_1,\dots,\beta_k$ of elements of $\mathbf{E}$ such that I is generated by $\overline{u}_A(\beta_1),\dots,\overline{u}_A(\beta_k)$; then, after replacing each β_i by β_i^{1/p^n} for some large $n\in\mathbb{N}$, we may also assume that $p\in I^t$, with $t:=k(p-1)+1$. In light of lemma 16.1.9(ii), it follows easily that $I_{\mathbf{E}}$ is generated by $\beta_1,\dots,\beta_k$, and $\alpha_0\in I^t_{\mathbf{E}}$. Denote by $\mathscr{T}$ and $\mathscr{T}'$ the I-adic topologies of B and B'; by lemma 8.3.12 and proposition 16.3.8, the topological rings $(B,\mathscr{T})$ and $(B',\mathscr{T}')$ are still perfectoid. Now, let $\mathrm{gr}^\bullet B$ (resp. $\mathrm{gr}^\bullet B'$) be the graded ring associated with the I-adic filtration on B (resp. on B'); likewise, let $\mathrm{gr}^\bullet\mathbf{E}(B)$ (resp. $\mathrm{gr}^\bullet\mathbf{E}(B')$) be the graded ring associated with the $I_{\mathbf{E}}$-adic filtration on $\mathbf{E}(B)$ (resp. on $\mathbf{E}(B')$). By claim 16.3.62(ii), the maps $\overline{u}_B$ and $\overline{u}_{B'}$ induce graded ring isomorphisms

$$\vartheta:\mathrm{gr}^\bullet\mathbf{E}(B)\stackrel{\sim}{\to}\mathrm{gr}^\bullet B\qquad \vartheta':\mathrm{gr}^\bullet\mathbf{E}(B')\stackrel{\sim}{\to}\mathrm{gr}^\bullet B'.$$

On the other hand, f and $\mathbf{E}(f)$ induce graded ring homomorphisms

$$\mathrm{gr}^\bullet(f):\mathrm{gr}^\bullet B\to\mathrm{gr}^\bullet B'\qquad \mathrm{gr}^\bullet\mathbf{E}(f):\mathrm{gr}^\bullet\mathbf{E}(B)\to\mathrm{gr}^\bullet\mathbf{E}(B')$$

such that :

$$\mathrm{gr}^\bullet(f)\circ\vartheta=\vartheta'\circ\mathrm{gr}^\bullet\mathbf{E}(f). \tag{16.8.13}$$

Now, if f^a is an isomorphism, then the same holds for $\mathrm{gr}^\bullet(f)^a$; conversely, if $\mathrm{gr}^\bullet(f)^a$ is an isomorphism, then we easily see that $f^a_n := (A/I^n\otimes_A f)^a : (B/I^nB)^a \to (B'/I^nB')^a$ is an isomorphism for every $n\in\mathbb{N}$, and therefore $f^a=\lim_{n\in\mathbb{N}} f^a_n$ is an isomorphism (recall that the functor $(-)^a$ commutes with limits, since it is a right adjoint). Likewise we see that $\mathbf{E}(f)^a$ is an isomorphism if and only if the same holds for $\mathrm{gr}^\bullet\mathbf{E}(f)^a$. Thus, we are reduced to checking that $\mathrm{gr}^\bullet(f)^a$ is an isomorphism of $(A,\mathfrak{m})^a$-algebras if and only if $\mathrm{gr}^\bullet\mathbf{E}(f)^a$ is an isomorphism of $(\mathbf{E},\mathfrak{n})^a$-algebras. However, notice that $\mathrm{gr}^\bullet B$ and $\mathrm{gr}^\bullet B'$ are A/I-modules, and let $\omega_I:\mathbf{E}/I_{\mathbf{E}}\stackrel{\sim}{\to}A/I$ be the ring isomorphism deduced from $\overline{u}_A$; we may then regard ϑ as an isomorphism $\mathrm{gr}^\bullet\mathbf{E}(B)\stackrel{\sim}{\to}\omega_I^*(\mathrm{gr}^\bullet B)$ of $\mathbf{E}/I_{\mathbf{E}}$-modules, and likewise for ϑ'. Also, $\mathrm{gr}^\bullet(f)$ is an isomorphism of $(A,\mathfrak{m})^a$-algebras if and only if $\omega_I^*(\mathrm{gr}^\bullet(f))$ is an isomorphism of $(\mathbf{E},\mathfrak{n})^a$-algebras, by remark 16.8.11. Taking into account (16.8.13), the assertion follows. □

16.8.14. Let now A be a perfectoid ring, and $\mathfrak{m}\subset A$ an open radical ideal. It follows easily that $A^{\circ\circ}\subset\mathfrak{m}$; moreover, if $\mathfrak{m}_{\mathbf{E}}\subset\mathbf{E}:=\mathbf{E}(A)$ is associated with $\mathfrak{m}$ as in (16.8.9), then $\overline{u}_A$ induces a ring isomorphism $\mathbf{E}/\mathfrak{m}_{\mathbf{E}}\stackrel{\sim}{\to}A/\mathfrak{m}$.

Proposition 16.8.15. *In the situation of* (16.8.14)*, let also $f:B\to B'$ be any continuous map in A-*Perf*. The following holds :*

- (i) $(A,\mathfrak{m})$ *and* $(\mathbf{E},\mathfrak{m}_{\mathbf{E}})$ *are basic setups fulfilling condition* **(B)** *of* [75, §2.1.6].
- (ii) *More precisely, the multiplication maps* $\widetilde{\mathfrak{m}}:=\mathfrak{m}\otimes_A\mathfrak{m}\to\mathfrak{m}$ *and* $\widetilde{\mathfrak{m}}_{\mathbf{E}}:=\mathfrak{m}_{\mathbf{E}}\otimes_{\mathbf{E}}\mathfrak{m}_{\mathbf{E}}\to\mathfrak{m}_{\mathbf{E}}$ *are isomorphisms.*
- (iii) *The morphism* $f^a:B^a\to B'^a$ *is an isomorphism in* $(A,\mathfrak{m})^a$-Alg *if and only if* $\mathbf{E}(f)^a:\mathbf{E}(B)^a\to\mathbf{E}(B')^a$ *is an isomorphism in* $(\mathbf{E},\mathfrak{m}_{\mathbf{E}})^a$-$\mathsf{Alg}$.
- (iv) $\mathfrak{m}B$ *is a radical ideal of* B.

Proof. (i): Let I be any ideal of definition of A. Since $A/\mathfrak{m}$ is reduced, the same holds for $\mathbf{E}/\mathfrak{m}_{\mathbf{E}}$, so $\mathfrak{m}_{\mathbf{E}}$ is a radical ideal of $\mathbf{E}$, hence $\mathfrak{m}^2_{\mathbf{E}}=\mathfrak{m}_{\mathbf{E}}$, since $\mathbf{E}$ is perfect. Now, let $\underline{\alpha}\in\mathrm{Ker}\,u_A$ be any distinguished element; by remark 16.3.7(ii), the map $\overline{u}_A$ induces a ring isomorphism $\mathbf{E}/\alpha_0\mathbf{E}\stackrel{\sim}{\to}A/pA$, and it follows easily that $\mathfrak{m}^2+pA=\mathfrak{m}$. On the other hand, we have $pA\subset I^2\subset\mathfrak{m}^2$, so $\mathfrak{m}^2=\mathfrak{m}$. Next, it is clear that $\mathfrak{m}_{\mathbf{E}}$ fulfills condition **(B)**, in particular $\mathfrak{m}_{\mathbf{E}}$ is generated by its p-th powers, therefore the same follows for the ideal $\mathfrak{m}_{\mathbf{E}}/\alpha_0\mathbf{E}$ of $\mathbf{E}/\alpha_0\mathbf{E}$, and then also for the ideal $\mathfrak{m}/pA$ of A/pA. Combining with lemma 16.2.3(iv), we deduce that $\mathfrak{m}$ is generated by its p-powers. If $l \neq p$ is any prime integer, we have $l \in A^\times$, hence $\mathfrak{m}/l\mathfrak{m} = 0$, and taking into account [75, Claim 2.1.9], it follows that $\mathfrak{m}$ fulfills condition (**B**).

(ii): Let us write $\mathfrak{m}$ as the union of a filtered system $(\mathfrak{m}_\lambda \mid \lambda \in \Lambda)$ of open radical subideals such that $\mathrm{Spec}\, A/\mathfrak{m}_\lambda$ is constructible in $\mathrm{Spec}\, A$ for every $\lambda \in \Lambda$; clearly it suffices to show that the corresponding map $\mathfrak{m}_\lambda \otimes_A \mathfrak{m}_\lambda \to \mathfrak{m}_\lambda$ is an isomorphism for every $\lambda \in \Lambda$. We may therefore assume that $\mathfrak{m}$ is the radical of an open ideal generated by finitely many elements $a_1, \dots, a_r$ of A; moreover, we may also assume that there exist $\alpha_1, \dots, \alpha_r \in \mathbf{E}$ with $a_i = \overline{u}_A(\alpha_i)$ for $i = 1, \dots, r$ (see remark 16.3.7(iii)). In this situation, we have a ring homomorphism $u_0 : R_{r,0} \to A$ such that $u_0(T_i^{1/p^k}) = \overline{u}_A(\alpha_i^{1/p^k})$ for $i = 0, \dots, r$ (notation of (16.7.9)), and it is easily seen that $\mathfrak{m} = T^{\lceil 0 \rceil} A$. Then the assertion follows from proposition 16.4.10(i) and theorem 16.7.10(iv).

(iii): It is clear that the basic setups $(A, \mathfrak{m})$ and $(\mathbf{E}, \mathfrak{m}_\mathbf{E})$ are compatible, in the sense of definition 16.8.10; then the assertion follows from lemma 16.8.12.

(iv): The map $\overline{u}_B$ induces a ring isomorphism :

$$\mathbf{E}(B)/\mathfrak{m}_\mathbf{E}\mathbf{E}(B) \xrightarrow{\sim} B/\mathfrak{m}B$$

and we need to check that $B/\mathfrak{m}B$ is reduced; so we are reduced to showing that the same holds for $\mathbf{E}(B)/\mathfrak{m}_\mathbf{E}\mathbf{E}(B)$, *i.e.* that $\mathfrak{m}_\mathbf{E}\mathbf{E}(B)$ is a radical ideal. But since $\mathbf{E}(B)$ is perfect, this is equivalent to proving that $\Phi_{\mathbf{E}(B)}(\mathfrak{m}_\mathbf{E} B) = \mathfrak{m}_\mathbf{E} B$ (where $\Phi_{\mathbf{E}(B)}$ denotes the Frobenius automorphism of $\mathbf{E}(B)$); the latter is clear, since we have already shown that $\mathfrak{m}_\mathbf{E}$ is a radical ideal of $\mathbf{E}$. □

16.8.16. Let A be a perfectoid ring, $\mathbf{E} := \mathbf{E}(A)$, and $\mathfrak{m} \subset A$ an open radical ideal; let also B be any perfectoid A-algebra, and $\mathfrak{m}_\mathbf{E} \subset \mathbf{E}$ the unique open ideal such that $\overline{u}_A : \mathbf{E} \to A$ induces a ring isomorphism $\mathbf{E}/\mathfrak{m}_\mathbf{E} \xrightarrow{\sim} A/\mathfrak{m}$ (see (16.8.9)). According to proposition 16.8.15(i), we have a well defined $(A, \mathfrak{m})^a$-algebra B^a and an $(\mathbf{E}, \mathfrak{m}_\mathbf{E})^a$-algebra $\mathbf{E}(B)^a$. We denote by

$$\eta_B : B \to B^a_* \qquad \text{and} \qquad \eta_{\mathbf{E}(B)} : \mathbf{E}(B) \to \mathbf{E}(B)^a_*$$

the units of adjunction, and endow B^a_* (resp. $\mathbf{E}(B)^a_*$) with the unique ring topology such that η_B (resp. $\eta_{\mathbf{E}(B)}$) is an adic morphism.

Corollary 16.8.17. *With the notation of* (16.8.16)*, the topological rings B^a_* and $\mathbf{E}(B)^a_*$ are perfectoid, and there exists an isomorphism of $\mathbf{E}$-algebras*

$$\omega : \mathbf{E}(B)^a_* \xrightarrow{\sim} \mathbf{E}(B^a_*) \qquad \textit{such that} \qquad \omega \circ \eta_{\mathbf{E}(B)} = \mathbf{E}(\eta_B).$$

Proof. We consider first the case where the structure map $A \to B$ is adic, and $Z := \mathrm{Spec}\, A/\mathfrak{m}$ is constructible in $X := \mathrm{Spec}\, A$. Then, set $U := \mathrm{Spec}\, B \times_X (X \setminus Z)$, $B_U := \mathscr{O}_U(U)$, and let $\overline{B} \subset B_U$ be the image of the restriction map $\rho : B \to B_U$. Notice that B is reduced, by corollary 16.3.63(i); it follows easily that

$$\mathfrak{m} \cdot \mathrm{Ker}\, \rho = 0 \tag{16.8.18}$$

(the details are left to the reader). We remark :

Claim 16.8.19. If Z is constructible in X, we have a natural identification of B-algebras :

$$B^a_* \xrightarrow{\sim} C := \{x \in B_U \mid \mathfrak{m} \cdot x \subset \overline{B}\}.$$

Proof of the claim. Let $a_1, \dots, a_n \in A$ be a finite system of elements that generate an ideal whose radical equals $\mathfrak{m}$; then :

$$B_U = \mathrm{Equal}\left(\prod_{i=1}^n B[a_i^{-1}] \rightrightarrows \prod_{i,j=1}^n B[a_i^{-1}, a_j^{-1}]\right).$$

Now, obviously $B[a_i^{-1}]^a_* = B[a_i^{-1}]$, and likewise for $B[a_i^{-1}, a_j^{-1}]$, for every $i, j = 1, \dots, n$; since the functor of almost elements is left exact, we deduce that $B^a_{U*} = B_U$. Therefore, ρ induces a map of B-algebras $\rho^a_* : B^a_* \to B_U$, and notice that ρ^a is a monomorphism, due to (16.8.18), so the same holds for ρ_*^a (proposition 1.3.18(i)). Obviously $\overline{B}^a = B^a$, hence $\mathfrak{m}\cdot\rho_*^a(B_*^a) = \mathfrak{m}\overline{B}$, *i.e.* the image of ρ_*^a lies in the subring C. Moreover, ρ induces an isomorphism $C^a \stackrel{\sim}{\to} B^a$, whence by adjunction, a unique map of B-algebras $C \to B_*^a$ whose composition with ρ_*^a agrees with the inclusion map $C \to B_U$. This shows that ρ_*^a maps B_*^a onto C, whence the claim. ◊

Recall that B_U carries a natural f-adic topology for which $\overline{B}$ is a subring of definition (lemma 16.4.32(i)), and $\mathfrak{m}\overline{B}$ is an open ideal of $\overline{B}$, since the structure map $A \to B$ is adic; from claim 16.8.19 it then follows that B_*^a is a bounded subring of B_U, and therefore its topology as defined in (16.8.16) agrees with the topology induced by the inclusion into B_U (proposition 8.3.18(i,ii)). From proposition 16.5.4(iii) we also see that $(B_*^a)^{\circ\circ} = B_U^{\circ\circ} = B^{\circ\circ}$.

Claim 16.8.20. If the structure map $A \to B$ is adic, and Z is constructible in X, we have :

(i) $B_*^a/B_U^{\circ\circ}$ is a perfect $\mathbb{F}_p$-algebra and it is integrally closed in $B_U^\circ/B_U^{\circ\circ}$.

(ii) B_*^a is perfectoid and integrally closed in B_U°.

Proof of the claim. (i): Notice that $A/A^{\circ\circ}$, $\overline{B}/B^{\circ\circ}$ and $B_U^\circ/B_U^{\circ\circ}$ are perfect $\mathbb{F}_p$-algebras; also, $\mathfrak{m}/A^{\circ\circ}$ is a radical ideal of $A/A^{\circ\circ}$, and since Z is constructible, there exists a finitely generated ideal $\mathfrak{m}_0 \subset A/A^{\circ\circ}$ whose radical is $\mathfrak{m}/A^{\circ\circ}$. With this notation, $B_U^\circ/B_U^{\circ\circ}$ is an $A/A^{\circ\circ}$-algebra, and for every $x \in B_U^\circ/B_U^{\circ\circ}$ there exists $k \in \mathbb{N}$ such that $\mathfrak{m}_0^k \cdot x \subset \overline{B}/B^{\circ\circ}$. Then the assertion follows from lemma 16.8.1.

(ii) follows from (i), claim 16.5.15, and theorem 16.5.13(iii). ◊

Next, set $Z_{\mathbf{E}} := \operatorname{Spec}\mathbf{E}/\mathfrak{m}_{\mathbf{E}}$, $X_{\mathbf{E}} := \operatorname{Spec}\mathbf{E}$, and $U_{\mathbf{E}} := \operatorname{Spec}\mathbf{E}(B)\times_{X_{\mathbf{E}}}(X_{\mathbf{E}}\setminus Z_{\mathbf{E}})$. The foregoing argument applies also to $\mathbf{E}(B)_*^a$, and shows that the latter is a perfectoid subring of $\mathbf{E}_U := \mathscr{O}_{U_{\mathbf{E}}}(U_{\mathbf{E}})$; moreover, we get cartesian diagrams of topological rings :

$$\begin{array}{ccc} B_*^a & \xrightarrow{i_B} & B_U^\circ \\ \downarrow & & \downarrow \\ B_*^a/B_U^{\circ\circ} & \longrightarrow & B_U^\circ/B_U^{\circ\circ} \end{array} \qquad \begin{array}{ccc} \mathbf{E}(B)_*^a & \xrightarrow{\iota_{\mathbf{E}}} & \mathbf{E}_U^\circ \\ \downarrow & & \downarrow \\ \mathbf{E}(B)_*^a/\mathbf{E}_U^{\circ\circ} & \longrightarrow & \mathbf{E}_U^\circ/\mathbf{E}_U^{\circ\circ}. \end{array}$$

Furthermore, according to proposition 16.5.4(iv), the morphism of topological monoids $\varphi_U^{\flat\circ} : \mathbf{E}_U^\circ \to B_U^\circ$ induces a ring isomorphism $\overline{\varphi}_U^{\flat\circ} : \mathbf{E}_U^\circ/\mathbf{E}_U^{\circ\circ} \stackrel{\sim}{\to} B_U^\circ/B_U^{\circ\circ}$, which in turn restricts to a ring isomorphism $\overline{\mathbf{E}}(B)/\mathbf{E}_U^{\circ\circ} \stackrel{\sim}{\to} \overline{B}/B_U^{\circ\circ}$, where $\overline{\mathbf{E}}(B)$ denotes the image of $\mathbf{E}(B)$ into $\mathbf{E}_U$. Notice also that claim 16.8.19 yields natural identifications :

$$B_*^a/B_U^{\circ\circ} \stackrel{\sim}{\to} \{x \in B_U^\circ/B_U^{\circ\circ} \mid \mathfrak{m}\cdot x \subset \overline{B}/B_U^{\circ\circ}\}$$
$$\mathbf{E}(B)_*^a/\mathbf{E}_U^{\circ\circ} \stackrel{\sim}{\to} \{x \in \mathbf{E}_U^\circ/\mathbf{E}_U^{\circ\circ} \mid \mathfrak{m}_{\mathbf{E}}\cdot x \subset \overline{\mathbf{E}}(B)/\mathbf{E}_U^{\circ\circ}\}.$$

But clearly $\overline{\varphi}_U^{\flat\circ}(\mathfrak{m}_{\mathbf{E}}\cdot x) = \mathfrak{m}\cdot\overline{\varphi}_U^{\flat\circ}(x)$ for every $x \in \mathbf{E}_U^\circ/\mathbf{E}_U^{\circ\circ}$, so $\overline{\varphi}_U^{\flat\circ}$ restricts as well to a ring isomorphism

$$\mathbf{E}(B)_*^a/\mathbf{E}_U^{\circ\circ} \stackrel{\sim}{\to} B_*^a/B_U^{\circ\circ}.$$

Summing up, and invoking proposition 16.3.25, we obtain a unique isomorphism of topological rings $\omega : \mathbf{E}(B)_*^a \stackrel{\sim}{\to} \mathbf{E}(B_*^a)$ fitting into the commutative diagram :

$$\begin{array}{ccc} \mathbf{E}(B)_*^a & \xrightarrow{\iota_{\mathbf{E}}} & \mathbf{E}_U^\circ \\ {\scriptstyle\omega}\downarrow & & \downarrow{\scriptstyle\omega^\circ} \\ \mathbf{E}(B_*^a) & \xrightarrow{\mathbf{E}(\iota_B)} & \mathbf{E}(B_U^\circ) \end{array}$$

where ω° is the isomorphism provided by theorem 16.5.13(i). Lastly, denote by $\rho^\circ : B \to B_U^\circ$ and $\rho_{\mathbf{E}}^\circ : \mathbf{E}(B) \to \mathbf{E}_U^\circ$ the continuous ring homomorphisms induced by the restriction maps;

the isomorphism ω° is characterized by the identity :

$$\omega^\circ \circ \rho^\circ_{\mathbf{E}} = \mathbf{E}(\rho^\circ).$$

Since ρ° and $\rho^\circ_{\mathbf{E}}$ factor through η_B and respectively $\eta_{\mathbf{E}(B)}$, we conclude that $\omega \circ \eta_{\mathbf{E}(B)} = \mathbf{E}(\eta_B)$, as stated. This completes the proof of the corollary in case Z is constructible.

Next, we consider the case where Z can be an arbitrary closed subset of X, and the structure map $A \to B$ is still adic. Let us write $\mathfrak{m}$ as the filtered union of a system $(\mathfrak{m}_\lambda \mid \lambda \in \Lambda)$ of open radical subideals such that $\mathrm{Spec}\, A/\mathfrak{m}_\lambda$ is constructible in X for every $\lambda \in \Lambda$; correspondingly, we obtain a filtered system $(\mathfrak{m}_{\mathbf{E},\lambda} \mid \lambda \in \Lambda)$ of ideals in $\mathbf{E}$ whose union is $\mathfrak{m}_{\mathbf{E}}$. For every such λ let B_λ (resp. $\mathbf{E}(B)_\lambda$) be the image of B (resp. of $\mathbf{E}(B)$) in the category $(A, \mathfrak{m}_\lambda)^a$-Alg (resp. $(\mathbf{E}, \mathfrak{m}_{\mathbf{E},\lambda})^a$-Alg); we endow $B^a_{\lambda*}$ (resp. $\mathbf{E}(B)^a_{\lambda*}$) with the unique adic topology such that the unit of adjunction $\eta_{B,\lambda} : B \to B^a_{\lambda*}$ (resp. $\eta_{\mathbf{E}(B),\lambda} : \mathbf{E}(B) \to \mathbf{E}(B)^a_{\lambda*}$) is adic. After replacing Λ by a cofinal subset, we may assume that Λ admits an initial element λ_0. The localization functors $(A, \mathfrak{m})^a$-Alg $\to (A, \mathfrak{m}_\lambda)^a$-Alg $\to (A, \mathfrak{m}_\mu)^a$-Alg induce natural maps of B-algebras

$$f_\lambda : B^a_* \to B^a_{\lambda*} \qquad \text{and} \qquad f_{\lambda\mu} : B^a_{\lambda*} \to B^a_{\mu*} \qquad \text{for every } \lambda, \mu \in \Lambda \text{ with } \lambda \geq \mu$$

which are clearly continuous. Moreover, clearly the images f^a_λ and $f^a_{\lambda\mu}$ of f_λ and $f_{\lambda\mu}$ in $(A, \mathfrak{m}_{\lambda_0})^a$-Alg are isomorphism, hence

$$\mathfrak{m}_{\lambda_0}\cdot\mathrm{Ker}\, f_\lambda = \mathfrak{m}_{\lambda_0}\cdot\mathrm{Coker}\, f_\lambda = 0 \qquad \mathfrak{m}_{\lambda_0}\cdot\mathrm{Ker}\, f_{\lambda\mu} = \mathfrak{m}_{\lambda_0}\cdot\mathrm{Coker}\, f_{\lambda\mu} = 0 \qquad \text{for every } \lambda \geq \mu.$$

Claim 16.8.21. The maps f_λ and $f_{\lambda\mu}$ induce bijections

$$\mathfrak{m}_{\lambda_0} B^a_* \stackrel{\sim}{\to} \mathfrak{m}_{\lambda_0} B^a_{\lambda*} \stackrel{\sim}{\to} \mathfrak{m}_{\lambda_0} B^a_{\mu*} \qquad \text{for every } \lambda \geq \mu.$$

Proof of the claim. By the foregoing, we see already that f_λ (resp. $f_{\lambda\mu}$) maps $\mathfrak{m}_{\lambda_0} B^a_*$ onto $\mathfrak{m}_{\lambda_0} B^a_{\lambda*}$ (resp. $\mathfrak{m}_{\lambda_0} B^a_{\lambda*}$ onto $\mathfrak{m}_{\lambda_0} B^a_{\mu*}$). Moreover, by the previous case we also know that $B^a_{\lambda*}$ is perfectoid, and under our current assumptions $\mathfrak{m}_{\lambda_0} B^a_{\lambda*}$ is an open ideal of $B^a_{\lambda*}$, hence $B^a_{\lambda*}/\mathfrak{m}_{\lambda_0} B^a_{\lambda*} = B^a_{\lambda*} \widehat{\otimes}_A A/\mathfrak{m}_{\lambda_0}$ is a discrete perfectoid $\mathbb{F}_p$-algebra (proposition 16.3.9(i)); especially, the latter is a reduced ring (corollary 16.3.63(i)). Now, let $x \in \mathfrak{m}_{\lambda_0} B^a_{\lambda*} \cap \mathrm{Ker}\, f_{\lambda\mu}$; then $\mathfrak{m}_{\lambda_0} \cdot x = 0$, whence $x^2 = 0$, and finally $x = 0$, which shows that $f_{\lambda\mu}$ is injective on $\mathfrak{m}_{\lambda_0} B^a_{\lambda*}$ for every $\lambda \in \Lambda$. Next, from corollary 14.2.5(i) we know that B^a_* is the limit of the cofiltered system of B-algebras $(B^a_{\lambda*} \mid \lambda \in \Lambda)$, hence the natural map

$$\mathfrak{m}_{\lambda_0} B^a_* \to L := \lim_{\lambda \in \Lambda} \mathfrak{m}_{\lambda_0} B^a_{\lambda*}$$

is injective. But we have also just seen that all the projections $L \to \mathfrak{m}_{\lambda_0} B^a_{\lambda*}$ are isomorphisms, so the maps $\mathfrak{m}_{\lambda_0} B^a_* \to \mathfrak{m}_{\lambda_0} B^a_{\lambda*}$ are injective as well. $\Diamond$

From claim 16.8.21 and [163, Lemma 3.5.3] we deduce that the natural map

$$B^a_*/\mathfrak{m}_{\lambda_0} B^a_* \to \lim_{\lambda \in \Lambda} B^a_{\lambda*}/\mathfrak{m}_{\lambda_0} B^a_{\lambda*}$$

is a ring isomorphism; since we have already observed that each quotient $B^a_{\lambda*}/\mathfrak{m}_{\lambda_0} B^a_{\lambda*}$ is a perfect $\mathbb{F}_p$-algebra, it follows easily that the same holds for $B^a_*/\mathfrak{m}_{\lambda_0} B^a_*$; especially, the latter is a discrete perfectoid ring. We then get for every $\lambda \in \Lambda$ a cartesian diagram of rings

$$\mathscr{D}_\lambda \quad : \quad \begin{array}{ccc} B^a_* & \xrightarrow{f_\lambda} & B^a_{\lambda*} \\ \downarrow & & \downarrow \\ B^a_*/\mathfrak{m}_{\lambda_0} B^a_* & \longrightarrow & B^a_{\lambda*}/\mathfrak{m}_{\lambda_0} B^a_{\lambda*} \end{array}$$

which is also cartesian in the category of topological rings, by lemma 8.3.11, and proposition 16.3.25 implies that B^a_* is perfectoid. The same argument applies as well to the perfectoid

$\mathbf{E}$-algebra $\mathbf{E}(B)$, and it yields a cartesian diagram of perfectoid rings :

$$\mathscr{D}_{\mathbf{E},\lambda} \quad : \quad \begin{array}{ccc} \mathbf{E}(B)^a_* & \xrightarrow{f_{\mathbf{E},\lambda}} & \mathbf{E}(B)^a_{\lambda *} \\ \downarrow & & \downarrow \\ \mathbf{E}(B)^a_*/\mathfrak{m}_{\mathbf{E},\lambda_0}\mathbf{E}(B)^a_* & \longrightarrow & \mathbf{E}(B)^a_{\lambda *}/\mathfrak{m}_{\mathbf{E},\lambda}\mathbf{E}(B)^a_{\lambda *}. \end{array}$$

By the same token, the system $(f_{\mathbf{E},\lambda} \mid \lambda \in \Lambda)$ induces a ring isomorphism

$$\mathbf{E}(B)^a_*/\mathfrak{m}_{\mathbf{E},\lambda_0}\mathbf{E}(B)^a_* \to \lim_{\lambda \in \Lambda} \mathbf{E}(B)^a_{\lambda *}/\mathfrak{m}_{\mathbf{E},\lambda_0}\mathbf{E}(B)^a_{\lambda *}.$$

Furthermore, by the previous case we already have an isomorphism of topological rings

$$\omega_\lambda : \mathbf{E}(B)^a_{\lambda *} \xrightarrow{\sim} \mathbf{E}(B^a_{\lambda *}) \qquad \text{such that} \qquad \omega_\lambda \circ \eta_{\mathbf{E}(B),\lambda} = \mathbf{E}(\eta_{B,\lambda}) \qquad \text{for every } \lambda \in \Lambda.$$

We may then identify naturally the bottom row of $\mathscr{D}_\lambda$ with that of $\mathscr{D}_{\mathbf{E},\lambda}$, and by invoking again proposition 16.3.25 we obtain a natural isomorphism of topological rings $\omega : \mathbf{E}(B)^a_* \xrightarrow{\sim} \mathbf{E}(B^a_*)$ fitting into the commutative diagrams

$$\begin{array}{ccc} \mathbf{E}(B)^a_* & \xrightarrow{f_{\mathbf{E},\lambda}} & \mathbf{E}(B)^a_{\lambda *} \\ {\scriptstyle\omega}\downarrow & & \downarrow{\scriptstyle\omega_\lambda} \\ \mathbf{E}(B^a_*) & \xrightarrow{\mathbf{E}(f_\lambda)} & \mathbf{E}(B^a_{\lambda *}) \end{array} \qquad \text{for every } \lambda \in \Lambda.$$

Then the sought identity $\omega \circ \eta_{\mathbf{E}(B)} = \mathbf{E}(\eta_B)$ follows from the corresponding ones for the isomorphisms ω_λ, after taking limits (details left to the reader). This completes the proof, in case the topologies of A and B are p-adic.

For the general case, let $\mathscr{T}_{A,p}$ (resp. $\mathscr{T}_{B,p}$) be the p-adic topology on A (resp. on B); then $(A, \mathscr{T}_{A,p})$ and $(B, \mathscr{T}_{B,p})$ are perfectoid topological rings as well (proposition 16.3.8(i)), and the rings underlying $\mathbf{E}((B, \mathscr{T}_{B,p})^a_*)$ and $\mathbf{E}(B, \mathscr{T}_{B,p})^a_*$ coincide with the rings underlying $\mathbf{E}(B^a_*)$ and respectively $\mathbf{E}(B)^a_*$, so the previous case already yields a ring isomorphism ω as sought. It remains to check that B^a_* (resp. $\mathbf{E}(B)^a_*$) is perfectoid for the topology such that η_B (resp. $\eta_{\mathbf{E}(B)}$) is adic, and that ω is also an isomorphism of topological rings for these topologies. However, let $I \subset B$ be any finitely generated ideal of definition; the natural map $B^a \to \lim_{n \in \mathbb{N}}(B/I^n)^a$ is an isomorphism, hence the same holds for the induced map $B^a_* \to \lim_{n \in \mathbb{N}}(B/I^n)^a_*$, and especially, B^a_* is complete and separated for the linear topology for which $(J_n := \mathrm{Ker}\, B^a_* \to (B/I^n)^a_* \mid n \in \mathbb{N})$ is a fundamental system of open submodules (corollary 8.2.16(i)). Clearly $I^n B^a_* \subset J_n$ for every $n \in \mathbb{N}$, so B^a_* is also complete and separated for its I-adic topology (lemma 8.3.12). Together with proposition 16.3.8(ii), this proves that B^a_* is perfectoid, and the same applies to $\mathbf{E}(B)^a_*$. Lastly, since η_B is adic, the same holds for $\mathbf{E}(\eta_B)$ (corollary 16.3.8), but then the identity $\omega \circ \eta_{\mathbf{E}(B)} = \mathbf{E}(\eta_B)$ implies easily that ω is an isomorphism of topological rings. □

16.8.22. In the situation of (16.8.16), denote also by $\varepsilon_B : B^a_{!!} \to B$ and $\varepsilon_{\mathbf{E}(B)} : \mathbf{E}(B)^a_{!!} \to \mathbf{E}(B)$ the counits of adjunction, and endow $B^a_{!!}$ (resp. $\mathbf{E}(B)^a_{!!}$) with the unique topology such that the natural map $A \to B^a_{!!}$ (resp. $\mathbf{E} \to \mathbf{E}(B)^a_{!!}$) is adic.

Corollary 16.8.23. *With the notation of* (16.8.22)*, the topological rings $B^a_{!!}$ and $\mathbf{E}(B)^a_{!!}$ are perfectoid and there exists an isomorphism of $\mathbf{E}$-algebras*

$$\omega : \mathbf{E}(B)^a_{!!} \xrightarrow{\sim} \mathbf{E}(B^a_{!!}) \qquad \textit{such that} \qquad \mathbf{E}(\varepsilon_B) \circ \omega = \varepsilon_{\mathbf{E}(B)}.$$

Proof. Notice that, by virtue of proposition 16.8.15(ii) we have cartesian diagrams of rings

$$\mathscr{D}\quad:\quad\begin{array}{ccc} B^a_{!!} & \xrightarrow{\varepsilon_B} & B \\ \downarrow & & \downarrow \\ A/\mathfrak{m} & \longrightarrow & B/\mathfrak{m}B. \end{array}\qquad\qquad \mathscr{D}_{\mathbf{E}}\quad:\quad\begin{array}{ccc} \mathbf{E}(B)^a_{!!} & \xrightarrow{\varepsilon_{\mathbf{E}(B)}} & \mathbf{E}(B) \\ \downarrow & & \downarrow \\ \mathbf{E}/\mathfrak{m}_{\mathbf{E}} & \longrightarrow & \mathbf{E}(B)/\mathfrak{m}_{\mathbf{E}}\mathbf{E}(B). \end{array}$$

Moreover, clearly we may assume that the structure map $A \to B$ is an adic ring homomorphism, in which case the same holds for the map $\mathbf{E} \to \mathbf{E}(B)$ (theorem 16.3.42(i)), and the bottom rows of $\mathscr{D}$ and $\mathscr{D}_{\mathbf{E}}$ consist of discrete perfectoid $\mathbb{F}_p$-algebras (cp. the proof of claim 16.8.21). Then $\mathscr{D}$ and $\mathscr{D}_{\mathbf{E}}$ are cartesian in the category of topological rings as well (lemma 8.3.11), and the assertion follows from proposition 16.3.25. $\square$

Lemma 16.8.24. *In the situation of* (16.8.16)*, let* $f : B \to C$ *be a morphism of perfectoid* A*-algebras,* $(b_\lambda \mid \lambda \in \Lambda)$ *a system of elements of* B*, and for every* $n \in \mathbb{N}$ *denote by* $I_n \subset B$ *the ideal generated by the system* $(b^n_\lambda \mid \lambda \in \Lambda)$*. Suppose that* $p^k B \subset I_1$ *for some* $k \in \mathbb{N}$*, and that* $(f \otimes_B B/I_1)^a : (B/I_1)^a \to (C/I_1C)^a$ *is a flat morphism of* $(A, \mathfrak{m})^a$*-algebras. Then the same holds for* $(f \otimes_B B/I_n)^a$*, for every* $n \in \mathbb{N}$*.*

Proof. Set $J_n := p^n B + I_n$ for every $n \in \mathbb{N}$; since $p^k B \subset I_1$, it is easily seen that for every $n \in \mathbb{N}$ there exists $n' \in \mathbb{N}$ such that $J_{n'} \subset I_n \subset J_n$ (details left to the reader). Hence, it suffices to show that $(f \otimes_B B/J_n)^a$ is flat for every $n \in \mathbb{N}$. Next, let $\underline{\alpha} \in \operatorname{Ker} u_B$ be a distinguished element, and pick for every $\lambda \in \Lambda$ elements $e_\lambda \in \mathbf{E}(B)$ and $c_\lambda \in B$ such that

$$b_\lambda = \overline{u}_B(e_\lambda) + \overline{u}_B(\alpha_0) \cdot c_\lambda.$$

For every $n \in \mathbb{N}$, denote by $I'_n \subset B$ the ideal generated by $(\overline{u}_B(e^n_\lambda) \mid \lambda \in \Lambda)$, and set $J'_n := \overline{u}_B(\alpha^n_0) \cdot B + I'_n$; it follows easily that

$$J_{2n-1} \subset J'_n \qquad \text{and} \qquad J'_{2n-1} \subset J_n \qquad \text{for every } n \in \mathbb{N}$$

so $(f \otimes_B B/J'_1)^a$ is flat, and we are further reduced to checking that $(f \otimes_B B/J'_n)^a$ is flat for every $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, denote by $I_{\mathbf{E},n} \subset \mathbf{E}(B)$ the ideal generated by $(e^n_\lambda \mid \lambda \in \Lambda)$, and set $J_{\mathbf{E},n} := \alpha^n_0 \mathbf{E}(B) + I_{\mathbf{E},n}$; notice the natural identifications

$$B/J'_1 = \mathbf{E}(B)/J_{\mathbf{E},1} \qquad \text{and} \qquad C/J'_1C = \mathbf{E}(C)/J_{\mathbf{E},1}\mathbf{E}(C). \tag{16.8.25}$$

Denote also by Φ_{B/J'_1} and Φ_{C/J'_1C} the Frobenius endomorphism of B/J'_1 and respectively C/J'_1C. We remark :

Claim 16.8.26. The induced diagram of $\mathbf{E}$-algebras

$$\begin{array}{ccc} B/J'_1 & \xrightarrow{f\otimes_B B/J'_1} & C/J'_1C \\ {\scriptstyle \Phi_{B/J'_1}}\downarrow & & \downarrow{\scriptstyle \Phi_{C/J'_1C}} \\ B/J'_1 & \xrightarrow{f\otimes_B B/J'_1} & C/J'_1C \end{array}$$

is cocartesian.

Proof of the claim. Indeed, since $\mathbf{E}(B)$ is perfect, it is easily seen that Φ_{B/J'_1} is surjective, and its kernel is the ideal generated by the system $\{\alpha^{1/p}_0\} \cup \{e^{1/p}_\lambda \mid \lambda \in \Lambda\}$, under the identifications (16.8.25); likewise, Φ_{C/J'_1C} is surjective and its kernel is generated by the image in C/J'_1C of the same system, whence the claim. $\Diamond$

For every $n \in \mathbb{N}$, set $\mathfrak{n}_n := \{\underline{a} \in W_n\mathbf{E} \mid a_0, \dots, a_{n-1} \in \mathfrak{m}\}$; since $\mathfrak{m}_{\mathbf{E}}$ fulfills condition $(\mathbf{B})$ of [75, §2.1.6], proposition 14.7.11 says that $\mathfrak{n}_n$ is the unique ideal $\mathfrak{n}_n \subset W_n\mathbf{E}$ with $\mathfrak{n}^2_n = \mathfrak{n}_n$

and $\overline{\omega}_i(\mathfrak{n}_n) = \mathfrak{m}_{\mathbf{E}}$ for $i = 0, \dots, n-1$. From claim 16.8.26 and theorem 14.7.26(i) it follows that the induced morphism of $(W_n\mathbf{E}, \mathfrak{n}_n)^a$-algebras

$$W_n(f \otimes_B B/J'_1)^a : W_n(\mathbf{E}(B)/J_{\mathbf{E},1})^a \to W_n(\mathbf{E}(C)/J_{\mathbf{E},1}\mathbf{E}(C))^a$$

is flat for every $n \in \mathbb{N}$. Moreover, the Frobenius automorphisms of $\mathbf{E}(B)$ and $\mathbf{E}(C)$ induce isomorphisms $\mathbf{E}(B)/J_{\mathbf{E},1} \xrightarrow{\sim} \mathbf{E}(B)/J_{\mathbf{E},p^k}$ and $\mathbf{E}(C)/J_{\mathbf{E},1}\mathbf{E}(C) \xrightarrow{\sim} \mathbf{E}(C)/J_{\mathbf{E},p^k}\mathbf{E}(C)$ for every $k \in \mathbb{N}$, whence a commutative diagram of $W_n\mathbf{E}$-algebras

$$\begin{array}{ccc} W_n(\mathbf{E}(B)/J_{\mathbf{E},1}) & \xrightarrow{W_n(f\otimes_B \mathbf{E}(B)/J_{\mathbf{E},1})} & W_n(\mathbf{E}(C)/J_{\mathbf{E},1}\mathbf{E}(C)) \\ \downarrow & & \downarrow \\ W_n(\mathbf{E}(B)/J_{\mathbf{E},p^k}) & \xrightarrow{W_n(f\otimes_B \mathbf{E}(B)/J_{\mathbf{E},p^k})} & W_n(\mathbf{E}(C)/J_{\mathbf{E},p^k}\mathbf{E}(C)) \end{array} \qquad \text{for every } n,k \in \mathbb{N}$$

whose vertical arrows are isomorphisms, and where the rings on the bottom row are regarded as $W_n\mathbf{E}$-algebras via restriction of scalars along the automorphism $W_n(\Phi_{\mathbf{E}}) : W_n\mathbf{E} \to W_n\mathbf{E}$. Since we have

$$W_n(\Phi_{\mathbf{E}})(\mathfrak{n}_n) = \mathfrak{n}_n \qquad \text{for every } n \in \mathbb{N}$$

it follows that $W_n(f \otimes_B \mathbf{E}(B)/J_{\mathbf{E},p^k})^a$ is a flat morphism of $(W_n\mathbf{E}, \mathfrak{n}_n)^a$-algebras, for every $n, k \in \mathbb{N}$. We remark, quite generally :

Claim 16.8.27. Let R be any ring such that the Frobenius endomorphism $\Phi_{R/pR}$ of R/pR is surjective, $(x_\lambda \mid \lambda \in \Lambda)$ any set of elements of R, and $I \subset R$ the ideal generated by this set. Then, for every $n \in \mathbb{N}$, the kernel of the projection $\pi_n : W_nR \to W_nR/I$ is generated by

$$(V_R^i(x_\lambda) \mid \lambda \in \Lambda, \ i = 0, \dots, n-1).$$

Proof of the claim. Arguing by induction on $n \in \mathbb{N}$, we are reduced to checking that the ideal $V_nI := V_nR \cap \operatorname{Ker} \pi_{n+1}$ is generated by $(V^n x_\lambda \mid \lambda \in \Lambda)$, for every $n \in \mathbb{N}$. However, according to claim 9.3.31 we have an isomorphism of $W_{n+1}R$-modules : $I \xrightarrow{\sim} V_nI$, where I is regarded as a $W_{n+1}R$-module via restriction of scalars along the ring homomorphism $\overline{\omega}_n : W_{n+1}R \to R$. Thus, we come down to showing that $\overline{\omega}_n$ is a surjection for every $n \in \mathbb{N}$. We argue by induction on n : the assertion is trivial for $n = 0$, so suppose that $\overline{\omega}_n$ is already known to be surjective for some $n \geq 0$; since $\Phi_{R/pR}$ is surjective, the same holds for $\Phi^n_{R/pR}$, and taking into account (9.3.3) together with our inductive assumption, we easily deduce that $\overline{\omega}_{n+1}$ is surjective as well. ◊

In light of claim 16.8.27 we see that

$$W_n(\mathbf{E}(B)/J_{\mathbf{E},p^k}) = W(\mathbf{E}(B))/(p^nW(\mathbf{E}(B)) + \mathscr{I}_k)$$

where $\mathscr{I}_k$ is the ideal generated by $(p^i \cdot \tau_{\mathbf{E}(B)}(e_\lambda^{p^{k-i}}) \mid \lambda \in \Lambda, \ i = 0, \dots, k)$. Taking into account lemma 16.1.1(iv), we deduce that

$$A \otimes_{W(\mathbf{E})} W_n(\mathbf{E}(B)/J_{\mathbf{E},p^k}) = B/(p^nB + I''_k) \qquad \text{for every } n,k \in \mathbb{N}$$

where I''_k is the ideal generated by $(p^i \cdot \overline{u}_B(e_\lambda^{p^{k-i}}) \mid \lambda \in \Lambda, \ i = 0, \dots, k)$. Notice that

$$J'_{p^{2k-1}} \subset I''_{2k-1} \subset J'_k \qquad \text{for every } k \in \mathbb{N}. \tag{16.8.28}$$

We have as well $A \otimes_{W(\mathbf{E})} W_n(f \otimes_{\mathbf{E}(B)} \mathbf{E}(B)/J_{\mathbf{E},p^k}) = f \otimes_B B/(p^nB + I''_k)$, and furthermore, the ring homomorphism $W_n\mathbf{E} \to A/p^nA$ induced by u_A maps $\mathfrak{n}_n$ onto $\mathfrak{m}/p^nA$; therefore, $(f \otimes_B B/(p^nB + I''_k))^a$ is a flat morphism of $(A, \mathfrak{m})^a$-algebras, for every $n, k \in \mathbb{N}$. Combining with (16.8.28), the assertion follows. □

Proposition 16.8.29. *In the situation of* (16.8.16)*, let $f^a : B^a \to C$ be an étale morphism of $(A, \mathfrak{m})^a$-algebras such that C is an almost finitely presented B^a-module, and endow C_* with the unique ring topology such that the induced map $B \to C_*$ is adic. Then C_* is perfectoid.*

Proof. Let $I \subset B$ be any ideal of definition, and $I_{\mathbf{E}} \subset \mathbf{E}(B)$ the corresponding ideal such that $\overline{u}_A$ induces an isomorphism $\mathbf{E}(B)/I_{\mathbf{E}} \xrightarrow{\sim} B/I$ (see remark 16.3.7(iii)); then $f^a \otimes_B B/I : \mathbf{E}(B)/I_{\mathbf{E}} \to C/IC$ is an étale morphism of $(\mathbf{E}, \mathfrak{m}_{\mathbf{E}})^a$-algebras. By [75, Th.5.3.27], the latter lifts uniquely to an étale morphism $g : \mathbf{E}(B)^a \to C'$ of $(\mathbf{E}, \mathfrak{m}_{\mathbf{E}})^a$-algebras such that C' is almost finitely presented as an $\mathbf{E}(B)^a$-module. Since $\mathbf{E}(B)$ is perfect, [75, Th.3.5.13] implies that the Frobenius morphism

$$(\Phi_{\mathbf{E}}, \Phi_{C'}) : ((\mathbf{E}(B), \mathfrak{m}_{\mathbf{E}}), C') \to ((\mathbf{E}(B), \mathfrak{m}_{\mathbf{E}}), C')$$

is cartesian in the fibred category $\mathscr{B}^a$-Alg of [75, §3.5.3]. Since the functor $(-)_*$ on $\mathscr{B}^a$-algebras is cartesian (see [75, §3.5.4]), it follows that the morphism

$$(\Phi_{\mathbf{E}}, \Phi_{C'_*}) : ((\mathbf{E}(B), \mathfrak{m}_{\mathbf{E}}), C'_*) \to ((\mathbf{E}(B), \mathfrak{m}_{\mathbf{E}}), C'_*)$$

is cartesian as well, *i.e.* C'_* is a perfect $\mathbb{F}_p$-algebra. Let us endow C'_* with its $I_{\mathbf{E}}$-adic topology. We remark :

Claim 16.8.30. C'_* is perfectoid.

Proof of the claim. Since $\mathbf{E}(B)^a$ is complete and separated for its $I_{\mathbf{E}}$-adic topology, [75, Claim 5.3.25], implies that the natural morphism $C' \to \lim_{n\in\mathbb{N}} C'/I^n_{\mathbf{E}}C'$ is an isomorphism, so the same holds for the induced map $C'_* \to \lim_{n\in\mathbb{N}}(C'/I^n_{\mathbf{E}}C')_*$. Especially, C'_* is complete and separated for the linear topology defined by the system of ideals $(J_n := \mathrm{Ker}\,(C'_* \to (C'/I^n_{\mathbf{E}}C')_* \mid n \in \mathbb{N})$ (corollary 8.2.16(i)). Clearly $I^n_{\mathbf{E}}C'_* \subset J_n$ for every $n \in \mathbb{N}$, so C'_* is also complete and separated for its $I_{\mathbf{E}}$-adic topology (lemma 8.3.12), whence the claim. ◊

By claim 16.8.30 it follows that there exists a perfectoid A-algebra D with an isomorphism of perfectoid $\mathbf{E}$-algebras $C'_* \xrightarrow{\sim} \mathbf{E}(D)$. We consider the composition

$$h : \mathbf{E}(B) \xrightarrow{\eta_{\mathbf{E}(B)}} \mathbf{E}(B)^a_* \xrightarrow{g_*} C'_* \xrightarrow{\sim} \mathbf{E}(D)$$

where $\eta_{\mathbf{E}(B)}$ is the unit of adjunction; since h is an adic ring homomorphism, there exists a unique adic map of A-algebras $k : B \to D$ with $\mathbf{E}(k) = h$ (theorem 16.3.42(i)). By construction, we have also an isomorphism $(C/IC)^a \xrightarrow{\sim} (D/ID)^a$ of $(B/I)^a$-algebras; especially, $(k \otimes_B B/I)^a$ is a flat ring homomorphism, and therefore the same holds for $(k \otimes_B B/I^n)^a : (B/I^n)^a \to (D/I^nD)^a$, for every $n \in \mathbb{N}$ (lemma 16.8.24). By virtue of [75, Cor.3.2.11(ii,iii)] and remark 14.1.91, it follows that there exists a unique isomorphism of $(B/I^n)^a$-algebras

$$\omega_n : C/I^nC \xrightarrow{\sim} (D/I^nD)^a \qquad \text{for every } n \in \mathbb{N}.$$

Notice that the uniqueness of the morphisms ω_n implies that $\omega_{n+1} \otimes_B B/I^n = \omega_n$ for every $n \in \mathbb{N}$. However, D is complete and separated for its I-adic topology, since h is adic; the same holds for C, since the latter is an almost projective B-module of almost finite presentation ([75, Claim 5.3.25]). Hence, the limit of the inverse system $(\omega_n \mid n \in \mathbb{N})$ is an isomorphism $\omega : C \xrightarrow{\sim} D^a$ of B^a-algebras, whence the isomorphism $\omega_* : C_* \xrightarrow{\sim} D^a_*$ of B-algebras, and it suffices to invoke corollary 16.8.17 to conclude. □

16.8.31. Let now $\underline{A} := (A, A^+, U)$ be any perfectoid quasi-affinoid ring, and set

$$\underline{U} := \mathsf{Spec}\,\underline{A} \qquad \underline{U}_{\mathbf{E}} := \mathbf{E}(\underline{U}) \qquad X_A := \mathrm{Spec}\,A \qquad Z_A := X_A \setminus U$$

(notation of (16.5.20)). We let $\mathfrak{m} \subset A$ be the unique radical ideal such that $\mathrm{Spec}\,A/\mathfrak{m} = Z_A$. Also, set $\mathbf{E} := \mathbf{E}(A)$, $X_{\mathbf{E}} := \mathrm{Spec}\,\mathbf{E}$, and let $\mathfrak{m}_{\mathbf{E}} \subset \mathbf{E}$ be the unique ideal such that $\overline{u}_A$ induces a ring isomorphism $\mathbf{E}/\mathfrak{m}_{\mathbf{E}} \xrightarrow{\sim} A/\mathfrak{m}$ (see remark 16.3.7(iii)). According to proposition 16.8.15(i), both pairs $(A, \mathfrak{m})$ and $(\mathbf{E}, \mathfrak{m}_{\mathbf{E}})$ are basic setups.

Theorem 16.8.32. *With the notation of* (16.8.31)*, the pair* (X_A, Z_A) *is normal and almost pure, relative to the basic setup* $(A, \mathfrak{m})$.

Proof. Let $j_A : U \to X_A$ be the open immersion; since $j_{A*}\mathscr{O}_U$ is a quasi-coherent $\mathscr{O}_{X_A}$-algebra ([59, Ch.I, Prop.9.4.2(i)]), in order to check that the pair (X_A, Z_A) is normal, it suffices to show that the induced map $A \to \mathscr{O}_{X_A}(U)$ is almost injective and the image of A^a is integrally closed in $\mathscr{O}_{X_A}(U)^a$. These assertions follow from claim 16.8.20(ii).

Next, pick any ideal of definition $I \subset A$, set $\mathbf{E} := \mathbf{E}(A)$, let $I_\mathbf{E} \subset \mathbf{E}$ be the unique ideal of definition such that $\overline{u}_A$ induces a ring isomorphism $\mathbf{E}/I_\mathbf{E} \xrightarrow{\sim} A/I$, and consider the following diagram of categories :

$$\mathscr{D}_A \quad : \quad \begin{array}{ccccc} \mathsf{Cov}(U) & \xleftarrow{j_A^*} & (A,\mathfrak{m})^a\text{-}\mathbf{\acute{E}t}_{\mathrm{afr}} & \xrightarrow{i_A^*} & (A/I,\mathfrak{m}/I)^a\text{-}\mathbf{\acute{E}t}_{\mathrm{afr}} \\ \uparrow{\scriptstyle w_A} & & & & \| \\ \mathsf{Cov}(U_\mathbf{E}) & \xleftarrow{j_\mathbf{E}^*} & (\mathbf{E},\mathfrak{m}_\mathbf{E})^a\text{-}\mathbf{\acute{E}t}_{\mathrm{afr}} & \xrightarrow{i_\mathbf{E}^*} & (\mathbf{E}/I_\mathbf{E},\mathfrak{m}_\mathbf{E}/I_\mathbf{E})^a\text{-}\mathbf{\acute{E}t}_{\mathrm{afr}} \end{array}$$

where j_A^* (resp. i_A^*) is induced by the open (resp. closed) immersion $j_A : U \to X_A$ (resp. $i_A : Z_A \to X_A$) and likewise for $j_\mathbf{E}^*$ and $i_\mathbf{E}^*$. We need to show that j_A^* is an equivalence. However, according to [75, Th.5.5.7(iii)], both i_A^* and $i_\mathbf{E}^*$ are equivalences, and the same holds for $j_\mathbf{E}^*$, by virtue of theorem 16.8.2(i). Then, choose an arbitrary quasi-inverse functor $s_\mathbf{E}$ for $j_\mathbf{E}^*$ and t_A for i_A^*; we let the dotted arrow be the functor $w_A := j_A^* \circ t_A \circ i_\mathbf{E}^* \circ s_\mathbf{E} : \mathsf{Cov}(U_\mathbf{E}) \to \mathsf{Cov}(U)$, so that $\mathscr{D}_A$ is essentially commutative, and we are reduced to checking that w_A is an equivalence.

Claim 16.8.33. We may assume that $\underline{A} = \Gamma^\circ(\underline{U})$ (notation of (16.5.19)).

Proof of the claim. Say that $\Gamma^\circ(\underline{U}) = (A_U^\circ, A_U^+, U)$, set $X_U^\circ := \mathrm{Spec}\, A_U^\circ$, and denote by $\mathfrak{m}_U \subset A_U^\circ$ the unique radical ideal such that $\mathrm{Spec}\, A_U^\circ/\mathfrak{m}_U = X_U^\circ \setminus U$. Likewise, say that $\Gamma^\circ(\underline{U}_\mathbf{E}) = (\mathbf{E}_U^\circ, \mathbf{E}_U^+, U_\mathbf{E})$, set $X_\mathbf{E}^\circ := \mathrm{Spec}\, \mathbf{E}_U^\circ$ and denote by $\mathfrak{m}_{\mathbf{E},U} \subset \mathbf{E}_U^\circ$ the unique radical ideal such that $\mathrm{Spec}\, \mathbf{E}_U^\circ/\mathfrak{m}_{\mathbf{E},U} = X_\mathbf{E}^\circ \setminus U_\mathbf{E}$. It follows easily that

$$\mathfrak{m}_U \subset \mathfrak{m}A_U^\circ \qquad \mathfrak{m}_{U,\mathbf{E}} \subset \mathfrak{m}_\mathbf{E}\mathbf{E}_U^\circ.$$

Moreover, set $I_U := IA_U^\circ$ and $I_{\mathbf{E},U} := I_\mathbf{E}\mathbf{E}_U^\circ$; the map $\overline{u}_{A_U^\circ} : \mathbf{E}_U^\circ \to A_U^\circ$ induces a ring isomorphism $\mathbf{E}_U^\circ/I_{\mathbf{E},U} \xrightarrow{\sim} A_U^\circ/I_U$. Also, as in the foregoing, the projection $A_U^\circ \to A_U^\circ/I_U$ and the open immersion $j_{\mathbf{E},U} : U_\mathbf{E} \to X_\mathbf{E}^\circ$ induce equivalences of categories

$$i_U^* : (A_U^\circ, \mathfrak{m}_U)^a\text{-}\mathbf{\acute{E}t}_{\mathrm{afr}} \xrightarrow{\sim} (A_U^\circ/I_U, \mathfrak{m}_U/I_U)^a\text{-}\mathbf{\acute{E}t}_{\mathrm{afr}} \qquad j_{\mathbf{E},U}^* : (\mathbf{E}_U^\circ, \mathfrak{m}_{\mathbf{E},U})^a\text{-}\mathbf{\acute{E}t}_{\mathrm{afr}} \xrightarrow{\sim} \mathsf{Cov}(U_\mathbf{E}).$$

After choosing quasi-inverse functors t_U for i_U^* and $s_{\mathbf{E},U}$ for $j_{\mathbf{E},U}^*$, there follows an essentially commutative diagram of functors

$$\begin{array}{ccccc} (A,\mathfrak{m})^a\text{-}\mathbf{\acute{E}t}_{\mathrm{afr}} & \xleftarrow{t_A} & (A/I,\mathfrak{m}/I)^a\text{-}\mathbf{\acute{E}t}_{\mathrm{afr}} & \xleftarrow{i_\mathbf{E}^*} & (\mathbf{E},\mathfrak{m}_\mathbf{E})^a\text{-}\mathbf{\acute{E}t}_{\mathrm{afr}} \\ \downarrow{\scriptstyle j_A^*} & & & & \uparrow{\scriptstyle s_\mathbf{E}} \\ \mathsf{Cov}(U) & & \downarrow & & \mathsf{Cov}(U_\mathbf{E}) \\ \uparrow{\scriptstyle j_U^*} & & & & \downarrow{\scriptstyle s_{\mathbf{E},U}} \\ (A_U^\circ,\mathfrak{m}_U)^a\text{-}\mathbf{\acute{E}t}_{\mathrm{afr}} & \xleftarrow{t_U} & (A_U^\circ/I_U,\mathfrak{m}_U/I_U)^a\text{-}\mathbf{\acute{E}t}_{\mathrm{afr}} & \xleftarrow{i_{\mathbf{E},U}^*} & (\mathbf{E}_U^\circ,\mathfrak{m}_{\mathbf{E},U})^a\text{-}\mathbf{\acute{E}t}_{\mathrm{afr}} \end{array}$$

whose unmarked vertical arrows are induced by the natural ring homomorphisms $A \to A_U^\circ$, $A/I \to A_U^\circ/I_U$ and $\mathbf{E} \to \mathbf{E}_U^\circ$, and where j_U^* is induced by the open immersion $j_U : U \to X_U^\circ$. The claim follows easily. ◊

The restriction to the site $\mathscr{R}(\underline{U})$ of the stack $\mathsf{Cov}_{\underline{U}}^\wedge$ of corollary 15.7.26 along the inclusion $\mathscr{R}(\underline{U}) \subset \mathscr{Q}(\underline{U})$ is a stack that we denote again

$$\pi_{\underline{U}} : \mathsf{Cov}_{\underline{U}}^\wedge \to \mathscr{R}(\underline{U}).$$

Recall the construction : for every rational subset $R \subset \mathrm{Spa}\,\underline{U}$ we choose a complete and separated quasi-affinoid scheme $\underline{U}^\wedge_R := (U^\wedge_R, \mathscr{T}^\wedge_R, A^{\wedge+}_R)$ representing the sub-presheaf h''_R of $h''_{\underline{U}}$; then $\pi^{-1}_{\underline{U}}(R) := \mathsf{Cov}(U^\wedge_R)$. We shall consider similarly the quasi-affinoid schemes

$$\underline{X}_A := \mathsf{Spec}\,(A, A^+, X_A) \qquad \text{and} \qquad \underline{Y}_A := \mathsf{Spec}\,A/I \otimes_A (A, A^+, X_A)$$

(notation of example 15.4.7(i)) and the fibrations

$$\pi_{\underline{X}^a_A} : \mathsf{Cov}^\wedge_{(\underline{X}_A,\mathfrak{m})^a} \to \mathscr{R}(\underline{U}) \qquad \pi_{\underline{Y}^a_A} : \mathsf{Cov}^\wedge_{(\underline{Y}_A,\mathfrak{m})^a} \to \mathscr{R}(\underline{U})$$

defined as follows. For any $R \in \mathrm{Ob}(\mathscr{R}(\underline{U}))$, we know that $(A^{\wedge\circ}_R, A^{\wedge+}_R, U^\wedge_R) := \Gamma^\circ(\underline{U}^\wedge_R)$ is a perfectoid quasi-affinoid ring (proposition 16.7.25(iii) and (16.5.19)), and we set $X^\circ_R := \mathrm{Spec}\,A^{\wedge\circ}_R$; then, let $\mathfrak{m}_R \subset A^{\wedge\circ}_R$ be the unique radical ideal such that $X^\circ_R \setminus U^\wedge_R = \mathrm{Spec}\,A^{\wedge\circ}_R/\mathfrak{m}_R$. Set also $I_R := IA^{\wedge\circ}_R$; by proposition 16.8.15(i), the pair $(A^{\wedge\circ}_R, \mathfrak{m}_R)$ is a basic setup, and we let

$$\pi^{-1}_{\underline{X}^a_A}(R) := (A^{\wedge\circ}_R, \mathfrak{m}_R)^a\text{-}\mathbf{\acute{E}t}_{\mathrm{afp}} \qquad \pi^{-1}_{\underline{Y}^a_A}(R) := (A^{\wedge\circ}_R/I_R, \mathfrak{m}_R/I_R)^a\text{-}\mathbf{\acute{E}t}_{\mathrm{afp}}.$$

If $R' \subset R$ is another rational subset, the induced inclusion of sub-presheaves $h''_{R'} \subset h''_R$ corresponds to a morphism $\underline{U}^\wedge_{R'} \to \underline{U}^\wedge_R$ of quasi-affinoid schemes, which in turn yields a morphism $\Gamma^\circ(\underline{U}^\wedge_R) \to \Gamma^\circ(\underline{U}^\wedge_{R'})$ of perfectoid quasi-affinoid rings, and it follows easily that

$$\mathfrak{m}_{R'} \subset \mathfrak{m}_R A^{\wedge\circ}_{R'}$$

whence, finally, induced functors

$$j^*_{R'R} : \pi^{-1}_{\underline{X}^a_A}(R) \to \pi^{-1}_{\underline{X}^a_A}(R') \qquad j'^*_{R'R} : \pi^{-1}_{\underline{Y}^a_A}(R) \to \pi^{-1}_{\underline{Y}^a_A}(R').$$

Moreover, for every further inclusion of rational subset $R'' \subset R'$ we have a natural isomorphisms of functors

$$j^*_{R''R'} \circ j^*_{R'R} \xrightarrow{\sim} j^*_{R''R} \qquad j'^*_{R''R'} \circ j'^*_{R'R} \xrightarrow{\sim} j'^*_{R''R}$$

and the systems of such functors and isomorphisms of functors amount to well defined pseudo-functors $\mathscr{R}(\underline{U})^o \to \mathbf{Cat}$, whence the sought fibrations $\pi_{\underline{X}^a_A}$ and $\pi_{\underline{Y}^a_A}$. Furthermore, the system of open immersions $j_R : U^\wedge_R \to X^\circ_R$ and the projections $A^{\wedge\circ}_R \to A^{\wedge\circ}_R/I_R$ induce pseudo-natural transformations

$$(A^{\wedge\circ}_R/I_R, \mathfrak{m}_R/I_R)^a\text{-}\mathbf{\acute{E}t}_{\mathrm{afp}} \xleftarrow{i^*_R} (A^{\wedge\circ}_R, \mathfrak{m}_R)^a\text{-}\mathbf{\acute{E}t}_{\mathrm{afp}} \xrightarrow{j^*_R} \mathsf{Cov}(U^\wedge_R) \qquad \text{for every } R \in \mathrm{Ob}(\mathscr{R}(\underline{U}))$$

whence morphisms of fibrations

$$\mathsf{Cov}^\wedge_{(\underline{Y}_A,\mathfrak{m})^a} \xleftarrow{i^*_{\underline{U}}} \mathsf{Cov}^\wedge_{(\underline{X}_A,\mathfrak{m})^a} \xrightarrow{j^*_{\underline{U}}} \mathsf{Cov}^\wedge_{\underline{U}}.$$

Likewise, by repeating the foregoing with the quasi-affinoid schemes

$$\underline{X}_{\mathbf{E}} := \mathsf{Spec}\,(\mathbf{E}, \mathbf{E}^+, X_{\mathbf{E}}) \qquad \underline{Y}_{\mathbf{E}} := \mathsf{Spec}\,\mathbf{E}/I_{\mathbf{E}} \otimes_{\mathbf{E}} (\mathbf{E}, \mathbf{E}^+, X_{\mathbf{E}})$$

we get the following fibrations and morphisms of fibrations over the site $\mathscr{R}(\underline{U}_{\mathbf{E}})$:

$$\mathsf{Cov}^\wedge_{(\underline{Y}_{\mathbf{E}},\mathfrak{m}_{\mathbf{E}})^a} \xleftarrow{i^*_{\underline{U}_{\mathbf{E}}}} \mathsf{Cov}^\wedge_{(\underline{X}_{\mathbf{E}},\mathfrak{m}_{\mathbf{E}})^a} \xrightarrow{j^*_{\underline{U}_{\mathbf{E}}}} \mathsf{Cov}^\wedge_{\underline{U}_{\mathbf{E}}}. \tag{16.8.34}$$

However, in view of the isomorphism of sites $\mathbf{E} : \mathscr{R}(\underline{U}) \xrightarrow{\sim} \mathscr{R}(\underline{U}_{\mathbf{E}})$ of remark 16.7.29(i), we may also regard (16.8.34) as fibrations and morphisms of fibrations over the site $\mathscr{R}(\underline{U})$. Moreover, by combining remark 16.7.29(ii) and (16.5.22), we obtain natural isomorphisms

$$\underline{Y}_A \times_{\underline{X}_A} \underline{U}^\wedge_R \xrightarrow{\sim} \underline{Y}_{\mathbf{E}} \times_{\underline{X}_{\mathbf{E}}} \underline{U}^\wedge_{\mathbf{E}(R)} \qquad \text{for every } R \in \mathrm{Ob}(\mathscr{R}(\underline{U})) \tag{16.8.35}$$

and a simple inspection shows that the resulting ring isomorphism $A^\circ_R/I_R \xrightarrow{\sim} A^\circ_{\mathbf{E}(R)}/I_{\mathbf{E}(R)}$ maps $\mathfrak{m}_R/I_R$ isomorphically onto $\mathfrak{m}_{\mathbf{E}(R)}/I_{\mathbf{E}(R)}$. Summing up, we get a natural fibrewise equivalence of fibrations over $\mathscr{R}(\underline{U})$:

$$\mathsf{Cov}^\wedge_{(\underline{Y}_{\mathbf{E}},\mathfrak{m}_{\mathbf{E}})^a} \xrightarrow{\sim} \mathsf{Cov}^\wedge_{(\underline{Y}_A,\mathfrak{m})^a}. \tag{16.8.36}$$

Furthermore, both $i^*_{\underline{U}}$ and $i^*_{\underline{U}_{\mathbf{E}}}$ are fibrewise equivalences, by virtue of [75, Th.5.5.7(iii)], and the same holds for $j^*_{\underline{U}_{\mathbf{E}}}$, by theorem 16.8.2(i). We may then find morphisms of fibrations

$$s_{\underline{U}_{\mathbf{E}}} : \mathsf{Cov}^{\wedge}_{\underline{U}_{\mathbf{E}}} \to \mathsf{Cov}^{\wedge}_{(\underline{X}_{\mathbf{E}},\mathfrak{m}_{\mathbf{E}})^a} \qquad \text{and} \qquad t_{\underline{U}} : \mathsf{Cov}^{\wedge}_{(\underline{Y}_A,\mathfrak{m})^a} \to \mathsf{Cov}^{\wedge}_{(\underline{X}_A,\mathfrak{m})^a}$$

that are quasi-inverse functors for $j^*_{\underline{U}_{\mathbf{E}}}$ and $i^*_{\underline{U}}$ (corollary 3.1.28(i)). The composition of these functors yields a morphism of stacks

$$w_{\underline{U}} : \mathsf{Cov}^{\wedge}_{\underline{U}_{\mathbf{E}}} \xrightarrow{s_{\underline{U}_{\mathbf{E}}}} \mathsf{Cov}^{\wedge}_{(\underline{X}_{\mathbf{E}},\mathfrak{m}_{\mathbf{E}})^a} \xrightarrow{i^*_{\underline{U}_{\mathbf{E}}}} \mathsf{Cov}^{\wedge}_{(\underline{Y}_{\mathbf{E}},\mathfrak{m}_{\mathbf{E}})^a} \xrightarrow{\sim} \mathsf{Cov}_{(\underline{Y}_A,\mathfrak{m})^a} \xrightarrow{t_{\underline{U}}} \mathsf{Cov}^{\wedge}_{(\underline{X}_A,\mathfrak{m})^a} \xrightarrow{j^*_{\underline{U}}} \mathsf{Cov}^{\wedge}_{\underline{U}}$$

such that $(w_{\underline{U}})_{\mathrm{Spa}\,\underline{U}} : \mathsf{Cov}(U_{\mathbf{E}}) \to \mathsf{Cov}(U)$ is isomorphic to w_A.

For every $x \in \mathrm{Spa}\,\underline{U}$, let $x_{\mathbf{E}} := \mathrm{Spa}(u_{\underline{X}})(x) \in \mathrm{Spa}\,\underline{U}_{\mathbf{E}}$; by propositions 5.2.9 and 5.6.37, we are then reduced to checking that the induced functor on stalks

$$w_{\underline{U}}(x) : \mathsf{Cov}^{\wedge}_{\underline{U}_{\mathbf{E}}}(x_{\mathbf{E}}) \to \mathsf{Cov}^{\wedge}_{\underline{U}}(x)$$

is an equivalence, for every such x. To this aim, suppose first that x is analytic; in this case, the completed residue field $\kappa(x)^\wedge$ is a Tate valued field (see (15.5.12)), so that the analytic locus of $\mathrm{Spec}\,\kappa(x)^{\wedge+}$ is $\{\eta_x\}$, where η_x is the generic point. We have a well defined quasi-affinoid ring

$$\underline{\kappa}(x)^{\wedge\circ} := (\kappa(x)^{\wedge\circ}, \kappa(x)^{\wedge+}, \{\eta_x\})$$

and the natural map $A \to \kappa(x)^{\wedge\circ}$ yields a morphism of quasi-affinoid rings $\pi^{\wedge}_x : \underline{A} \to \underline{\kappa}(x)^{\wedge\circ}$. Likewise we get a morphism of quasi-affinoid rings

$$\pi^{\wedge}_{x_{\mathbf{E}}} : \mathbf{E}(\underline{A}) \to \underline{\kappa}(x_{\mathbf{E}})^{\wedge\circ} := (\kappa(x_{\mathbf{E}})^{\wedge\circ}, \kappa(x_{\mathbf{E}})^{\wedge+}, \{\eta_{x_{\mathbf{E}}}\}).$$

Moreover, by virtue of corollary 16.5.61, both $\underline{\kappa}(x)^{\wedge\circ}$ and $\underline{\kappa}(x_{\mathbf{E}})^{\wedge\circ}$ are perfectoid, and there exists a unique isomorphism of quasi-affinoid rings

$$\omega : \mathbf{E}(\underline{\kappa}(x)^{\wedge\circ}) \xrightarrow{\sim} \underline{\kappa}(x_{\mathbf{E}})^{\wedge\circ} \qquad \text{such that} \qquad \omega \circ \mathbf{E}(\pi^{\wedge}_x) = \pi^{\wedge}_{x_{\mathbf{E}}}. \tag{16.8.37}$$

We may then repeat the foregoing constructions with $\underline{A}$ replaced by $\underline{\kappa}(x)^{\wedge\circ}$ and $\underline{\kappa}(x_{\mathbf{E}})^{\wedge\circ}$: we get first the quasi-affinoid schemes

$$\underline{U}(x) := \mathsf{Spec}\,\underline{\kappa}(x)^{\wedge\circ} \qquad \underline{U}_{\mathbf{E}}(x_{\mathbf{E}}) := \mathsf{Spec}\,\underline{\kappa}(x_{\mathbf{E}})^{\wedge\circ}.$$

Then we set $\underline{A}(x) := (\kappa(x)^{\wedge\circ}, \kappa(x)^{\wedge+}, \mathrm{Spec}\,\kappa(x)^{\wedge\circ})$, and

$$\underline{X}_A(x) := \mathsf{Spec}\,\underline{A}(x) \qquad \underline{Y}_A(x) := \mathsf{Spec}\,(A/I \otimes_A \underline{A}(x))$$

and define likewise $\underline{X}_{\mathbf{E}}(x_{\mathbf{E}})$ and $\underline{Y}_{\mathbf{E}}(x_{\mathbf{E}})$. Also, let $\mathfrak{m}_x \subset \kappa(x)^{\wedge\circ}$ and $\mathfrak{m}_{\mathbf{E},x} \subset \kappa(x_{\mathbf{E}})^{\wedge\circ}$ be the maximal ideals. To these quasi-affinoid schemes we attach as in the foregoing fibrations over the sites $\mathscr{R}(\underline{U}(x))$ and $\mathscr{R}(\underline{U}_{\mathbf{E}}(x_{\mathbf{E}}))$. However, notice that the points of $\mathrm{Spa}\,\underline{U}(x)$ are (the equivalence classes of) the continuous valuations $v : \kappa(x)^\circ \to \Gamma_{v\circ}$ with $\mathrm{Ker}\,v = 0$ and $v(\kappa(x)^+) \subset \Gamma^+_{v\circ}$; it is easily seen that there is only one such valuation, *i.e.* the one corresponding to the point x of $\mathrm{Spa}\,\underline{U}$. Likewise, we have $\mathrm{Spa}\,\underline{U}_{\mathbf{E}} = \{x_{\mathbf{E}}\}$; hence, these fibrations amount to categories, or equivalently, they can be identified naturally with their stalks over the unique point x of $\mathrm{Spa}\,\underline{U}$ and the unique point $x_{\mathbf{E}}$ of $\mathrm{Spa}\,\underline{U}_{\mathbf{E}}$. Then, a simple inspection yields an essentially commutative diagram of categories :

$$\mathscr{D}'_A \quad : \quad \begin{array}{ccccc} \mathsf{Cov}^{\wedge}_{(\underline{Y}_A,\mathfrak{m})^a}(x) & \xleftarrow{i^*_{\underline{U}}(x)} & \mathsf{Cov}^{\wedge}_{(\underline{X}_A,\mathfrak{m})^a}(x) & \xrightarrow{j^*_{\underline{U}}(x)} & \mathsf{Cov}^{\wedge}_{\underline{U}}(x) \\ \downarrow & & \downarrow & & \downarrow \\ \mathsf{Cov}^{\wedge}_{(\underline{Y}_A(x),\mathfrak{m}_x)^a} & \xleftarrow{i^*_{\underline{U}(x)}} & \mathsf{Cov}^{\wedge}_{(\underline{X}_A(x),\mathfrak{m}_x)^a} & \xrightarrow{j^*_{\underline{U}(x)}} & \mathsf{Cov}^{\wedge}_{\underline{U}(x)} \end{array}$$

whose right-most vertical arrow is an equivalence, by virtue of (15.7.28). Likewise we have an essentially commutative diagram $\mathscr{D}'_{\mathbf{E}}$ for the corresponding fibrations over the site $\mathscr{R}(\underline{U}_{\mathbf{E}})$; moreover, in light of (16.8.37) we have as well an essentially commutative diagram

$$\begin{array}{ccc} \mathsf{Cov}^{\wedge}_{(\underline{Y}_{\mathbf{E}},\mathfrak{m}_{\mathbf{E}})^a}(x_{\mathbf{E}}) & \longrightarrow & \mathsf{Cov}^{\wedge}_{(\underline{Y}_A,\mathfrak{m})^a}(x) \\ \downarrow & & \downarrow \\ \mathsf{Cov}^{\wedge}_{(\underline{Y}_{\mathbf{E}}(x_{\mathbf{E}}),\mathfrak{m}_{\mathbf{E},x})^a} & \longrightarrow & \mathsf{Cov}^{\wedge}_{(\underline{Y}_A(x),\mathfrak{m}_x)^a} \end{array}$$

whose vertical arrows are the same as the left-most vertical arrows of $\mathscr{D}'_A$ and $\mathscr{D}'_{\mathbf{E}}$, and whose horizontal arrows are induced by the equivalences (16.8.36). Summing up, we are reduced to checking the following :

Claim 16.8.38. The functor $j^*_{\underline{U}(x)}$ is an equivalence.

Proof of the claim. By unwinding the definitions, we see that $\mathsf{Cov}^{\wedge}_{\underline{U}(x)}$ is the category of étale $\kappa(x)^{\wedge}$-algebras, $\mathsf{Cov}^{\wedge}_{(\underline{X}_A(x),\mathfrak{m}_x)^a}$ is the category of étale $(\kappa(x)^{\wedge\circ},\mathfrak{m}_x)^a$-algebras of almost finite rank, and $j^*_{\underline{U}(x)}$ is the natural restriction functor. In light of lemma 14.4.5(ii) and remark 14.4.3(i), it follows already that $j^*_{\underline{U}(x)}$ is fully faithful. Next, let E be any finite separable field extension of $\kappa(x)^{\wedge}$, and E° the integral closure of $\kappa(x)^{\wedge\circ}$ in E; it suffices to check that $E^{\circ a}$ is an étale $\kappa(x)^{\wedge\circ a}$-algebra of finite rank. However, notice that $\kappa(x)^{\wedge\circ}$ is a valuation ring of rank one with field of fractions $\kappa(x)^{\wedge}$; moreover, it is perfectoid for the ring topology such that the natural map $A \to \kappa(x)^{\wedge\circ}$ is adic (corollary 16.5.61(i)), and therefore it is deeply ramified, by [75, Prop.6.6.6] and lemma 16.2.3(iv). Let also E^{s} be any separable closure of E, and $E^{\mathrm{s}\circ}$ the integral closure of E° in E^{s}; by [75, Prop.6.6.2 and Rem.6.1.12(iv)] it follows that $(\Omega_{E^{\mathrm{s}\circ}/\kappa(x)^{\wedge\circ}})^a = 0$, and combining with [75, Th.6.3.23] we deduce that $(\Omega_{E^{\circ}/\kappa(x)^{\wedge\circ}})^a = 0$ (cp. the proof of [75, Prop.6.6.2]). By invoking again [75, Th.6.3.23], it follows that $\mathscr{D}_{E^{\circ}/\kappa(x)^{\wedge\circ}} = E^{\circ a}$. Then the assertion follows from [75, Prop.6.3.8 and Lemma 4.1.27]. ◊

Lastly, in case x is non-analytic, set

$$Y(x) := \mathrm{Spec}\,(A/I \otimes_A \mathscr{O}^{\wedge}_{\mathrm{Spa}\,\underline{U},x}) \qquad Y_{\mathbf{E}}(x_{\mathbf{E}}) := \mathrm{Spec}\,(\mathbf{E}/I_{\mathbf{E}} \otimes_{\mathbf{E}} \mathscr{O}^{\wedge}_{\mathrm{Spa}\,\underline{U}_{\mathbf{E}},x_{\mathbf{E}}})$$

According to (15.7.28) we have natural equivalences of categories

$$\Psi : \mathsf{Cov}^{\wedge}_{\underline{U}}(x) \xrightarrow{\sim} \mathsf{Cov}(Y(x)) \qquad \Psi_{\mathbf{E}} : \mathsf{Cov}^{\wedge}_{\underline{U}}(x_{\mathbf{E}}) \xrightarrow{\sim} \mathsf{Cov}(Y_{\mathbf{E}}(x_{\mathbf{E}})).$$

Recall the construction of these equivalences : an object of $\mathsf{Cov}_{\underline{U}}(x)$ is a pair (R,φ) where R is a rational subset of $\mathrm{Spa}\,\underline{U}$ containing x, and $\varphi : V \to U^{\wedge}_R$ is a finite étale morphism of schemes; to such an object, the equivalence Ψ assigns the finite étale morphism $Y(x) \times_{U^{\wedge}_R} \varphi$. Similarly we may describe $\Psi_{\mathbf{E}}$. We get likewise well defined functors

$$\mathsf{Cov}^{\wedge}_{(X_A,\mathfrak{m}_A)^a}(x) \xrightarrow{\Psi'} \mathsf{Cov}(Y(x)) \xleftarrow{\Psi''} \mathsf{Cov}^{\wedge}_{(\underline{Y}_A,\mathfrak{m}_A)^a}(x).$$

Namely, an object of $\mathsf{Cov}^{\wedge}_{(X_A,\mathfrak{m}_A)^a}(x)$ is a pair $(R, f : (A^{\wedge\circ}_R,\mathfrak{m}_R)^a \to B^a)$ where R is as in the foregoing and f is an étale covering of almost rings, and notice that $\mathfrak{m}_R\mathscr{O}^{\wedge}_{\mathrm{Spa}\,\underline{U},x} = \mathscr{O}^{\wedge}_{\mathrm{Spa}\,\underline{U},x}$, so $f_x := A/I \otimes_A \mathscr{O}^{\wedge}_{\mathrm{Spa}\,\underline{U},x} \otimes_{A^{\wedge\circ}_R} f$ is a homomorphism of (usual) rings; then the functor Ψ' assigns to the pair (R,f) the finite étale covering $\mathrm{Spec}\,f_x$ of $Y(x)$. Similarly we define Ψ'', and it is easily seen that the resulting diagram of categories :

$$\mathscr{D}''_A \quad : \quad \begin{array}{ccccc} \mathsf{Cov}^{\wedge}_{(\underline{Y}_A,\mathfrak{m})^a}(x) & \xleftarrow{i^*_{\underline{U}}(x)} & \mathsf{Cov}^{\wedge}_{(\underline{X}_A,\mathfrak{m})^a}(x) & \xrightarrow{j^*_{\underline{U}}(x)} & \mathsf{Cov}^{\wedge}_{\underline{U}}(x) \\ & \searrow^{\Psi''} & \downarrow{\Psi'} & \swarrow_{\Psi} & \\ & & \mathsf{Cov}(Y(x)) & & \end{array}$$

is essentially commutative. Correspondingly, we have an essentially commutative diagram $\mathscr{D}''_{\mathbf{E}}$ for the stalks over $x_{\mathbf{E}}$, and notice that all the arrows of $\mathscr{D}''_{\mathbf{E}}$ are equivalences (details left to the reader). On the other hand, the system of isomorphisms (16.8.35) induces an isomorphism of schemes

$$Y_A(x) \xrightarrow{\sim} Y_{\mathbf{E}}(x_{\mathbf{E}})$$

whence an essentially commutative diagram

$$\begin{array}{ccc} \mathsf{Cov}^\wedge_{(\underline{Y}_{\mathbf{E}},\mathfrak{m}_{\mathbf{E}})^a}(x_{\mathbf{E}}) & \xrightarrow{\Psi''_{\mathbf{E}}} & \mathsf{Cov}(Y_{\mathbf{E}}(x_{\mathbf{E}})) \\ \downarrow & & \downarrow \\ \mathsf{Cov}^\wedge_{(\underline{Y}_A,\mathfrak{m})^a}(x) & \xrightarrow{\Psi''} & \mathsf{Cov}(Y(x)) \end{array}$$

whose left vertical arrow is induced by the equivalence (16.8.36), and the right vertical arrow is an equivalence as well. We conclude that Ψ'' is an equivalence; then the same holds for $j^*_{\underline{U}}(x)$, and finally also for $w_{\underline{U}}(x)$, as sought. The proof is complete. □

Remark 16.8.39. Theorem 16.8.32 extends results of [149] (for perfectoid algebras over perfectoid fields), as well as [114, Th.5.5.9] (for perfectoid Tate rings A with $p \in A^\times \cap A^{\circ\circ}$), and [151, Th.7.4.5].

16.8.40. *Almost purity in the formally perfectoid case.* We show now how to extend theorem 16.8.32 to the case of formally perfectoid rings. We begin with some preliminary observations:

Lemma 16.8.41. *Let A be an adic topological ring that admits a finitely generated ideal I of adic definition, $A^\wedge$ the completion of A, and $\mathfrak{m} \subset A$ an open ideal. The following holds :*

(i) $\mathfrak{m}$ *is a radical ideal of A if and only if $\mathfrak{m}A^\wedge$ is a radical ideal of $A^\wedge$.*
(ii) $\mathfrak{m} = \mathfrak{m}^2$ *if and only if* $\mathfrak{m}A^\wedge = \mathfrak{m}^2 A^\wedge$.
(iii) $\mathfrak{m}$ *fulfills condition* (**B**) *of* [75, §2.1.6] *if and only if the same holds for* $\mathfrak{m}A^\wedge$.

Proof. By assumption, $I^n \subset \mathfrak{m}$ for some integer $n > 0$; after replacing I by I^n, we may then assume that $I \subset \mathfrak{m}$.

(i): Let $\mathfrak{n}$ be the radical ideal of $\mathfrak{m}$; since $I \subset \mathfrak{m}$, it is easily seen that $\mathfrak{n}/I$ is the radical ideal of $\mathfrak{m}/I$ in the quotient A/I. Since the completion map $A \to A^\wedge$ induces an isomorphism $A/I \xrightarrow{\sim} A^\wedge/IA^\wedge$, it follows that $\mathfrak{n}A^\wedge/IA^\wedge$ is the radical ideal of $\mathfrak{m}A^\wedge/IA^\wedge$; again, this easily implies that $\mathfrak{n}A^\wedge$ is the radical ideal of $\mathfrak{m}A^\wedge$, whence the assertion.

(ii): We have $\mathfrak{m} = \mathfrak{m}^2 \Leftrightarrow \mathfrak{m}/I^2 = \mathfrak{m}^2/I^2$, and since the completion map induces an isomorphism $A/I^2 \xrightarrow{\sim} A^\wedge/I^2A^\wedge$, we have $\mathfrak{m}/I^2 = \mathfrak{m}^2/I^2 \Leftrightarrow \mathfrak{m}A^\wedge/I^2A^\wedge = \mathfrak{m}^2/I^2A^\wedge$. Again, the latter condition holds if and only if $\mathfrak{m}A^\wedge = \mathfrak{m}^2A^\wedge$, whence the assertion.

(iii): In view of (ii), it is clear that if $\mathfrak{m}$ fulfills condition (**B**), then so does $\mathfrak{m}A^\wedge$. Conversely, suppose that $\mathfrak{m}A^\wedge$ fulfills condition (**B**); by (ii), it follows already that $\mathfrak{m} = \mathfrak{m}^2$, and it remains to check that for every integer $k > 0$, the ideal $\mathfrak{m}$ is generated by the system $x^k_\bullet := (x^k \mid x \in \mathfrak{m})$. However, let $a_1, \dots, a_n$ be a finite system of generators of I, and choose $N \in \mathbb{N}$ such that I^N lies in the ideal generated by $a^k_\bullet := (a^k_1, \dots, a^k_n)$; by assumption $\mathfrak{m}A^\wedge$ is generated by the system $b^k_\bullet := (b^k \mid b \in \mathfrak{m}A^\wedge)$, hence the same holds for $\mathfrak{m}A^\wedge/I^NA^\wedge$. The completion map induces an isomorphism $\mathfrak{m}/I^N \xrightarrow{\sim} \mathfrak{m}A^\wedge/I^NA^\wedge$ and maps $x^k_\bullet$ onto the family $b^k_\bullet$, so that $\mathfrak{m}/I^N$ is generated by $x^k_\bullet$, and therefore the same holds for $\mathfrak{m}$, since the family $x^k_\bullet$ contains $a^k_\bullet$. □

Proposition 16.8.42. *Let A be a formally perfectoid ring, $I \subset A$ a finitely generated ideal of adic definition, $\mathfrak{m} \subset A$ a radical ideal with $I \subset \mathfrak{m}$, and $U := \mathrm{Spec}\, A \setminus \mathrm{Spec}\, A/\mathfrak{m}$. We have :*

(i) $(A, \mathfrak{m})$ *is a basic setup fulfilling condition* (**B**) *(see* [75, §2.1.1, §2.1.6]*).*
(ii) *For every subideal $J \subset I$, every almost finitely generated almost projective $(A/J)^a$-module P has almost finite rank, for the almost structure induced by* $(A, \mathfrak{m})$.

(iii) *In the situation of* (ii)*, if moreover U is quasi-compact, then P has finite rank.*

Proof. (i) follows from lemma 16.8.41 and proposition 16.8.15(i).

(ii): After replacing I by $I + pA$, we may assume that $p \in I$. Set $X := \mathrm{Spec}\, A/J$ and $U' := X \setminus \mathrm{Spec}\, A/I$. Let $S \subset A$ be the multiplicative subset $1 + I$, and $B := S^{-1}(A/J)$. Let also $\mathscr{P}$ be the quasi-coherent $\mathscr{O}_X$-module associated with the A/J-module $P_!$. Then $X = U' \cup \mathrm{Spec}\, B$, and U' is quasi-compact, so that $\mathscr{P}_{|U'}$ is a locally free $\mathscr{O}_{U'}$-module of finite rank. Thus, it suffices to check that $P_B := P \otimes_A B$ is almost projective B^a-module of almost finite rank. Now, the ideal $I_B := I/J$ of B is obviously tight, and by construction $I_B \subset \mathrm{rad}(B)$; moreover, the localization $A/J \to B$ induces a ring isomorphism $A/I \xrightarrow{\sim} B/I_B$, whence an isomorphism of $(A/I)^a$-modules $P_0 := P/IP \xrightarrow{\sim} P_B/I_B P_B$. By lemma 14.3.17(ii) we are then reduced to checking that P_0 is an $(A/I)^a$-module of almost finite rank. To this aim, let $\mathbf{E} := \mathbf{E}(A)$, and pick a distinguished element $\underline{\alpha} \in \mathrm{Ker}\, u_A$; by remark 16.3.7(ii), the map $\overline{u}_A$ induces a ring isomorphism $\omega : \mathbf{E}/\alpha_0\mathbf{E} \xrightarrow{\sim} A/pA$, and we let $\mathfrak{m}_\mathbf{E}$ and $I_\mathbf{E}$ be the unique ideals of $\mathbf{E}$ with

$$\alpha_0 \in \mathfrak{m}_\mathbf{E} \qquad \alpha_0 \in I \qquad \mathfrak{m}/pA \xrightarrow{\sim} \mathfrak{m}_\mathbf{E}/\alpha_0\mathbf{E} \qquad I/pA \xrightarrow{\sim} I_\mathbf{E}/\alpha_0\mathbf{E}.$$

In particular, $I_\mathbf{E}$ is finitely generated, and $(\mathbf{E}, \mathfrak{m}_\mathbf{E})$ is a basic setup, by proposition 16.8.15(i). Via the isomorphism ω, we may then regard P_0 as an almost finitely generated almost projective $(\mathbf{E}/I_\mathbf{E})^a$-module (for the almost structure given by $(\mathbf{E}, \mathfrak{m}_\mathbf{E})$), and clearly it suffices to show that P_0 is of almost finite rank, when regarded as an $(\mathbf{E}/I_\mathbf{E})^a$-module. The latter follows from proposition 16.8.7(i).

(iii): Arguing as in the proof of (ii), we reduce to showing that if U is quasi-compact, then P_0 is of finite rank, when regarded as a $(\mathbf{E}/I_\mathbf{E})^a$-module, and according to proposition 16.8.7(ii), the latter will follow, once we show that $Z := \mathrm{Spec}\, \mathbf{E}/\mathfrak{m}_\mathbf{E}$ is constructible in $\mathrm{Spec}\, \mathbf{E}$. However, $(\mathrm{Spec}\, A/pA) \setminus (\mathrm{Spec}\, A/\mathfrak{m}) = U \cap \mathrm{Spec}\, A/pA$ is quasi-compact, hence the same holds for $W := (\mathrm{Spec}\, \mathbf{E}/\alpha_0\mathbf{E}) \setminus Z$. Thus, $(\mathrm{Spec}\, \mathbf{E}) \setminus Z = W \cup \mathrm{Spec}\mathbf{E}[\alpha_0^{-1}]$ is constructible in $\mathrm{Spec}\, \mathbf{E}$, and then the same follows for Z. ☐

16.8.43. Let A be a formally perfectoid ring, $I \subset A$ a finitely generated ideal of adic definition, and $\mathfrak{m} \subset A$ a radical ideal with $I \subset \mathfrak{m}$. Set $X := \mathrm{Spec}\, A$, $Z := \mathrm{Spec}\, A/\mathfrak{m}$, and $U := X \setminus Z$. According to proposition 16.8.42(i), we may consider the almost structure associated with the basic setup $(A, \mathfrak{m})$. We then have :

Theorem 16.8.44. *In the situation of* (16.8.43)*, let also $r \in \mathbb{N}$. The following holds :*

(i) *The pair (X, Z) is almost pure.*

(ii) *Let $\mathscr{M}$ be any almost projective $\mathscr{O}_X^a$-module of almost finite rank. Then $\mathscr{M}$ is a faithfully flat $\mathscr{O}_X^a$-module (resp. is an $\mathscr{O}_X^a$-module of finite rank $\leq r$) if and only if the same holds for the $\mathscr{O}_U$-module $\mathscr{M}_{|U}$.*

Proof. (i): Suppose first that Z is constructible in X. Let A^h and $A^\wedge$ be respectively the topological henselization and the completion of A, and set $X^\mathrm{h} := \mathrm{Spec}\, A^\mathrm{h}$ and $X^\wedge := \mathrm{Spec}\, A^\wedge$. Let also $U^\mathrm{h} := X^\mathrm{h} \times_X U$ and $U^\wedge := X^\wedge \times_X U$. In light of theorem 16.8.32, it suffices to check that both square subdiagrams of the following induced diagram are 2-cartesian :

$$\begin{array}{ccccc}
\mathscr{O}_X^a\text{-}\mathbf{\acute{E}t}_\mathrm{afr} & \longrightarrow & \mathscr{O}_{X^\mathrm{h}}^a\text{-}\mathbf{\acute{E}t}_\mathrm{afr} & \longrightarrow & \mathscr{O}_{X^\wedge}^a\text{-}\mathbf{\acute{E}t}_\mathrm{afr} \\
\downarrow & & \downarrow & & \downarrow \\
\mathscr{O}_U\text{-}\mathbf{\acute{E}t}_\mathrm{fr} & \longrightarrow & \mathscr{O}_{U^\mathrm{h}}\text{-}\mathbf{\acute{E}t}_\mathrm{fr} & \longrightarrow & \mathscr{O}_{U^\wedge}\text{-}\mathbf{\acute{E}t}_\mathrm{fr}.
\end{array}$$

Now, by [75, Prop.5.5.6(i)] it follows that the natural functor

$$\mathscr{O}_X^a\text{-}\mathbf{\acute{E}t}_{\mathrm{afr}} \to \mathscr{O}_{X^{\mathrm{h}}}^a\text{-}\mathbf{\acute{E}t}_{\mathrm{afr}} \overset{2}{\underset{\mathscr{O}_{U^{\mathrm{h}}}\text{-}\mathbf{\acute{E}t}_{\mathrm{fr}}}{\times}} \mathscr{O}_U\text{-}\mathbf{\acute{E}t}_{\mathrm{fr}}$$

is fully faithful. Thus, let $(\mathscr{E}^{\mathrm{h}}, \mathscr{E}_U, \xi)$ be the datum of an object $\mathscr{E}^{\mathrm{h}}$ of $\mathscr{O}_{X^{\mathrm{h}}}^a\text{-}\mathbf{\acute{E}t}_{\mathrm{afr}}$, an object $\mathscr{E}_U$ of $\mathscr{O}_U\text{-}\mathbf{\acute{E}t}_{\mathrm{fr}}$, and an isomorphism ξ in $\mathscr{O}_{U^{\mathrm{h}}}\text{-}\mathbf{\acute{E}t}_{\mathrm{fr}}$ between the images of $\mathscr{E}^{\mathrm{h}}$ and $\mathscr{E}_U$. By *loc.cit.*, there exists a quasi-coherent $\mathscr{O}_X^a$-algebra $\mathscr{E}$ whose restriction to X^{h} and to U is isomorphic to $\mathscr{E}^{\mathrm{h}}$ and respectively $\mathscr{E}_U$, and in view of proposition 16.8.42(iii), it remains to check that $\mathscr{E}$ is étale and almost finitely presented as an $\mathscr{O}_X^a$-module. Since the induced morphism $U \sqcup X^{\mathrm{h}} \to X$ is quasi-compact and faithfully flat, the assertion follows from lemmata 14.3.5(i) and 14.3.26(i). This shows that the left square subdiagram of the foregoing diagram is 2-cartesian.

Next, since both of the natural maps $A/I \to A^{\mathrm{h}}/IA^{\mathrm{h}}$ and $A/I \to A^\wedge/IA^\wedge$ are isomorphisms, [75, Th.5.5.7(iii)] implies that the top horizontal arrow of the right square subdiagram is an equivalence. To conclude the proof in this case, it then suffices to show :

Claim 16.8.45. The base change functor $\mathscr{O}_{U^{\mathrm{h}}}\text{-}\mathbf{\acute{E}t}_{\mathrm{fr}} \to \mathscr{O}_{U^\wedge}\text{-}\mathbf{\acute{E}t}_{\mathrm{fr}}$ is an equivalence.

Proof of the claim. According to proposition 8.3.30(i) there exists a unique f-adic topology on $B := \mathscr{O}_{U^{\mathrm{h}}}(U^{\mathrm{h}})$ (resp. on $C := \mathscr{O}_{U^\wedge}(U^\wedge)$) such that the restriction map $A^{\mathrm{h}} \to B$ (resp. $A^\wedge \to C$) is open. It follows easily that the induced map $B \to C$ is continuous and adic, since the same holds for the completion map $A \to A^\wedge$. Set $A^{\mathrm{h}+} := A^{\mathrm{h}\circ} \subset A^{\mathrm{h}}$ and $A^{\wedge+} := A^{\wedge\circ} \subset A^\wedge$; in view of lemma 8.3.24(iii.a), the integral closure in B (resp. in C) of the image of $A^{\mathrm{h}+}$ (resp. of $A^{\wedge+}$) is a subring of integral elements, that we denote B^+ (resp. C^+). Recall also that the natural morphism of schemes $\mathrm{Spec}\, B \to X^{\mathrm{h}}$ (resp. $X^\wedge \to \mathrm{Spec}\, C$) induces an isomorphism $U_B := U^{\mathrm{h}} \times_{X^{\mathrm{h}}} \mathrm{Spec}\, B \xrightarrow{\sim} U^{\mathrm{h}}$ (resp. $U_C := U^\wedge \times_{X^\wedge} \mathrm{Spec}\, C \xrightarrow{\sim} U^\wedge$); combining with lemma 15.4.6, we deduce that the resulting diagram

$$\begin{array}{ccc} \mathsf{Spec}(C, C^+, U_C) & \longrightarrow & \mathsf{Spec}(B, B^+, U_B) \\ {\scriptstyle\varphi}\downarrow & & \downarrow \\ \mathsf{Spec}(A^\wedge, A^{\wedge+}, U^\wedge) & \longrightarrow & \mathsf{Spec}(A^{\mathrm{h}}, A^{\mathrm{h}+}, U^{\mathrm{h}}) \end{array}$$

is cartesian in the category of quasi-affinoid schemes. Furthermore, the quasi-affinoid scheme $\mathsf{Spec}(C, C^+, U_C)$ is complete, by virtue of lemma 16.4.32(ii). By arguing with universal properties, it follows easily that the morphism φ identifies $\mathsf{Spec}(C, C^+, U_C)$ with the completion of $\mathsf{Spec}(A^\wedge, A^{\wedge+}, U^\wedge)$: the details are left to the reader. Then the claim follows from theorem 15.7.17 and proposition 16.8.42(ii). $\diamondsuit$

Lastly, we consider the case where Z is an arbitrary closed subset in X. Then, let $(I_\lambda \mid \lambda \in \Lambda)$ be the filtered system of all finitely generated ideals of A containing I and contained in $\mathfrak{m}$. For every $\lambda \in \Lambda$, let also $\mathfrak{m}_\lambda$ be the radical of I_λ; then $Z_\lambda := \mathrm{Spec}\, A/\mathfrak{m}_\lambda = \mathrm{Spec}\, A/I_\lambda$ and $U_\lambda := X \setminus Z_\lambda$ are constructible subsets of X, and $(A, \mathfrak{m}_\lambda)$ is a basic setup fulfilling condition (*B*), for every such λ (proposition 16.8.42(i)). Both natural functors :

$$(A, \mathfrak{m})^a\text{-}\mathbf{\acute{E}t}_{\mathrm{afr}} \to 2\text{-}\lim_{\lambda \in \Lambda} (A, \mathfrak{m}_\lambda)^a\text{-}\mathbf{\acute{E}t}_{\mathrm{afr}} \qquad \mathscr{O}_U\text{-}\mathbf{\acute{E}t}_{\mathrm{fr}} \to 2\text{-}\lim_{\lambda \in \Lambda} \mathscr{O}_{U_\lambda}\text{-}\mathbf{\acute{E}t}_{\mathrm{fr}}$$

are equivalences of categories (corollary 14.2.5(ii,iv,v)), hence we are reduced to prove the theorem for the pair (X, Z_λ), relative to the almost structure $(A, \mathfrak{m}_\lambda)$, for every $\lambda \in \Lambda$. The latter is already known by the previous case, so the proof is concluded.

(ii): Again, we assume first that Z is constructible, in which case $\mathscr{M}$ has finite rank (proposition 16.8.42(iii)). Let A_{loc} be the topological localization of A (see (8.4.8)); set also $X_{\mathrm{loc}} := \mathrm{Spec}\, A_{\mathrm{loc}}$. Then A_{loc} is a formally perfectoid ring, and the localization $A \to A_{\mathrm{loc}}$ induces an isomorphism of the respective completions $A^\wedge \stackrel{\sim}{\to} A^\wedge_{\mathrm{loc}}$ (corollary 8.4.15(i)), and an isomorphism of A-schemes $\pi : U_{\mathrm{loc}} := X_{\mathrm{loc}} \times_X U \stackrel{\sim}{\to} U$. Set $M := \mathscr{M}(X)$, $M_{\mathrm{loc}} := A_{\mathrm{loc}} \otimes_A M$, and let $\mathscr{M}_{\mathrm{loc}}$ be the almost projective $\mathscr{O}^a_{X_{\mathrm{loc}}}$-module of finite rank corresponding to M_{loc}. Then π induces a natural identification of $\mathscr{M}_{|U}$ with $(\mathscr{M}_{\mathrm{loc}})_{|U_{\mathrm{loc}}}$, so that the latter is a faithfully flat $\mathscr{O}_{U_{\mathrm{loc}}}$-module. We notice :

Claim 16.8.46. In the situation of (16.8.31), let $\mathscr{F}$ be any almost projective $\mathscr{O}^a_{X_A}$-module of finite rank. Then $\mathscr{F}$ is a faithfully flat $\mathscr{O}^a_{X_A}$-module (resp. is an $\mathscr{O}^a_{X_A}$-module of finite rank $\leq r$) if and only if the same holds for the $\mathscr{O}_U$-module $\mathscr{F}_{|U}$.

Proof of the claim. Since the pair (X, Z) is normal (theorem 16.8.32), the assertion follows immediately from lemma 14.3.18. ◊

From claim 16.8.46, it follows that $A^\wedge_{\mathrm{loc}} \otimes_{A_{\mathrm{loc}}} M_{\mathrm{loc}}$ is a faithfully flat $(A^\wedge_{\mathrm{loc}})^a$-module of finite rank; consequently $M_{\mathrm{loc}}/IM_{\mathrm{loc}}$ is a faithfully flat $A^a_{\mathrm{loc}}/IA^a_{\mathrm{loc}}$-module of finite rank. By construction, $I^a_{\mathrm{loc}} \subset \mathrm{rad}(A^a_{\mathrm{loc}})$ is a tight ideal of A^a_{loc} (relative to the basic setup $(A, \mathfrak{m})$), hence M_{loc} is a faithfully flat A^a_{loc}-module (lemma 14.3.17(i)). Lastly, let Y be the disjoint union of U and $\mathrm{Spec}\, A_{\mathrm{loc}}$; we have an obvious faithfully flat morphism of A-schemes $f : Y \to X$, and the foregoing implies that $f^*\mathscr{M}$ is a faithfully flat $\mathscr{O}^a_Y$-module, so $\mathscr{M}$ is a faithfully flat $\mathscr{O}^a_X$-module, by lemma 14.3.5(i).

Next, for a general Z, define the filtered system of subideals $(\mathfrak{m}_\lambda \mid \lambda \in \Lambda)$ of $\mathfrak{m}$, and the corresponding filtered system of open constructible subsets $(U_\lambda \mid \lambda \in \Lambda)$ of U, as in the proof of (i). Then if $\mathscr{M}_{|U}$ is a faithfully flat $\mathscr{O}_U$-module (resp. if the rank of $\mathscr{M}$ is $\leq r$), obviously the same holds for the $\mathscr{O}_{U_\lambda}$-module $\mathscr{M}_{|U_\lambda}$. For every $\lambda \in \Lambda$, denote by M_λ the image of M in the category of $(A, \mathfrak{m}_\lambda)^a$-modules; by the foregoing case, M_λ is a faithfully flat $(A, \mathfrak{m}_\lambda)^a$-module (resp. has rank $\leq r$) for every such λ. Then M is a faithfully flat $(A, \mathfrak{m})^a$-module (resp. has rank $\leq r$), by corollary 14.2.5(iii,iv). □

Remark 16.8.47. A ring A is Witt-perfect (relative to a given prime integer p) in the sense of [52, Def.1.8] if and only if the p-adic completion of A is a P-ring in our sense (for its p-adic topology). In the case where $\mathfrak{m}$ is the radical of $I := Ap$, with p regular in A, and A integrally closed in $A[1/p]$, theorem 16.8.44 is essentially the purity theorem in [52, Th.2.9]. This includes cases studied by Faltings using the technique of normalized length.

16.9. **Perfectoid Tate rings and perfectoid Abhyankar's lemma.** In this section we introduce some special classes of perfectoid quasi-affinoid rings that will play an important role in the proof of Hochster's direct summand conjecture. The main results are theorems 16.9.17 and 16.9.42, that generalize important results from Y.André's papers [5] and [6]. The following definition of perfectoid Tate ring is equivalent to the one found in [70].

Definition 16.9.1. Let $(A, \mathscr{T}_A)$ be a Tate ring (see definition 8.3.8(vi)).

(i) We say that A is a *perfectoid Tate ring* (resp. *formally perfectoid Tate ring*) if it has a subring of definition A_0 that is perfectoid (resp. formally perfectoid), for the topology induced by $\mathscr{T}_A$ via the inclusion map $A_0 \to A$.

(ii) We say that A is *uniform*, if A° is a bounded subset of A (see definition 8.3.8(ii)). We let

$$\mathsf{Tate} \qquad \text{and} \qquad \mathsf{u.Tate}$$

be the full subcategories of $\mathbb{Z}$-TopAlg (see definition 8.3.1(iii)) whose objects are the Tate rings and respectively the uniform Tate rings.

Lemma 16.9.2. (i) *Let $(A, \mathscr{T}_A)$ be a Tate ring. The following conditions are equivalent :*

(a) *A is uniform.*

(b) *The completion $A^\wedge$ of A is uniform.*

(c) *$\mathscr{T}_A$ agrees with the topology given by a power-multiplicative semi-norm $\|\cdot\| : A \to \mathbb{R}$.*

(ii) *Moreover, if the power-multiplicative semi-norm* $\|\cdot\|$ *defines the topology* $\mathscr{T}_A$*, we have :*

$$A^\circ = \{a \in A \mid \|a\| \leq 1\} \qquad \textit{and} \qquad A^{\circ\circ} = \{a \in A \mid \|a\| < 1\}.$$

(iii) *The inclusion functor* $\mathsf{u.Tate} \to \mathsf{Tate}$ *admits a left adjoint*

$$\mathsf{u} : \mathsf{Tate} \to \mathsf{u.Tate} \qquad A \mapsto \mathsf{u}(A).$$

We call $\mathsf{u}(A)$ *the* uniformization *of the Tate ring* A.

Proof. (i.a)⇒(i.b): Indeed, if A° is a bounded subset of A, then its image B in $A^\wedge$ is bounded in the latter ring; but then the same holds for the topological closure B^c of B in $A^\wedge$ (remark 8.3.9(iii)). However, $B^c = (A^\wedge)^\circ$ (corollary 8.4.15(ii) and proposition 8.2.13(iii)) whence the contention.

(i.b)⇒(i.a): Let $j : A \to A^\wedge$ be the completion map; we have $j^{-1}((A^\wedge)^\circ) = A^\circ$, by corollaries 8.4.15(ii) and 8.2.17(ii). Since the topology of A is induced by that of $A^\wedge$ via j, the assertion follows easily.

(i.a)⇒(i.c): Fix $\rho \in]0,1[$ and pick $t \in A^\times \cap A^{\circ\circ}$. Notice that A° is a subring of definition of A (proposition 8.3.18(ii)). Denote by $\nu : A \to \mathbb{Z} \cup \{+\infty\}$ the order function associated with the ideal tA° of A° (see example 9.1.9(i)), and set $|a| := \rho^{\nu(a)}$ for every $a \in A$. Moreover, let $\|\cdot\| : A \to \mathbb{R}$ be the asymptotic Samuel function of tA°, *i.e.* $\|a\| := \lim\limits_{n\to+\infty} |a^n|^{1/n}$ for every $a \in A$. Notice that $\nu(t^k a) = k + \nu(a)$ for every $k \in \mathbb{Z}$ and $a \in A$, whence :

$$\|t^k a\| = \rho^k \cdot \|a\| \qquad \text{for every } k \in \mathbb{Z} \text{ and } a \in A. \tag{16.9.3}$$

By construction, $\|a\| \leq 1$ for every $a \in A^\circ$. On the other hand, if $\|a\| > r > 1$, then there exists $k \in \mathbb{N}$ such that $|a^n| > r^n$ for every $n \geq k$, and it follows easily that $a \notin A^\circ$. Combining with (16.9.3), we conclude that $t^k A^\circ = \{a \in A \mid \|a\| \leq \rho^k\}$ for every $k \in \mathbb{N}$, whence (i.c).

(i.c)⇒(i.a): Clearly $\{a \in A \mid \|a\| < 1\} = A^{\circ\circ}$. Pick $t \in A^\times \cap A^{\circ\circ}$, so that $r := \|t\| \in]0,1[$; then $t^k A^{\circ\circ}$ is an open ideal of A° for every $k \in \mathbb{N}$, and it lies in the subset $\{a \in A \mid \|a\| < r^k\}$, whence (i.a).

(ii) follows by simple inspection of the definitions.

(iii): Let $A_0 \subset A$ be a subring of definition, $t \in A^\times \cap A^{\circ\circ}$, and $\nu : A \to \mathbb{Z} \cup \{+\infty\}$ the order function associated with the ideal tA_0 of A_0. For given $\rho \in]0,1[$, the map $|\cdot|_A : A \to \mathbb{R}$ such that $|a|_A := \rho^{\nu(a)}$ for every $a \in A$ is a semi-norm, and the topology defined by $|\cdot|_A$ agrees with $\mathscr{T}_A$. Then, let $\|\cdot\|_A : A \to \mathbb{R}$ be the asymptotic Samuel function of tA_0. Recall that $\|\cdot\|_A$ is a multiplicative semi-norm, and let $\mathscr{T}_A^{\mathsf{u}}$ be the topology on A defined by $\|\cdot\|_A$. Lemma 9.1.8(iii) implies that the identity map of A is a continuous ring homomorphism $(A, \mathscr{T}_A) \to (A, \mathscr{T}_A^{\mathsf{u}})$; also, the discussion of example 9.1.9 shows that $\mathscr{T}_A^{\mathsf{u}}$ depends only on the original topology $\mathscr{T}_A$ (and is independent of all auxiliary choices). Clearly $A^+ := \{a \in A \mid \|a\|_A \leq 1\}$ is an open subring of $(A, \mathscr{T}_A^{\mathsf{u}})$ and arguing as in the foregoing we see that (16.9.3) holds as well for the semi-norm $\|\cdot\|_A$, so that $t^k A^+ = \{a \in A \mid \|a\| \leq \rho^k\}$ for every $k \in \mathbb{Z}$; especially, $\mathscr{T}_A^{\mathsf{u}}$ induces the t-adic topology on tA^+, and $(A, \mathscr{T}_A^{\mathsf{u}})^\circ = A^+$, so $(A, \mathscr{T}_A^{\mathsf{u}})$ is a uniform Tate ring. Now, let $(B, \mathscr{T}_B)$ be any uniform Tate ring, and $f : (A, \mathscr{T}_A) \to (B, \mathscr{T}_B)$ a continuous ring homomorphism. By lemma 8.3.29(iv), the map f is f-adic, hence it restricts to an adic ring homomorphism $f_0 : A_0 \to B^\circ$ (lemma 8.3.24(iii.a,iii.b)), and the proof of (i) shows that $\mathscr{T}_B$ agrees with the topology defined by the asymptotic Samuel function $\|\cdot\|_B : B \to \mathbb{R}$ of the ideal $f(t) \cdot B^\circ$. With this notation, it follows easily that f is a morphism of normed rings $(A, |\cdot|_A) \to (B, \|\cdot\|_B)$ (see (9.1.6)), and then lemma 9.1.8(iv) shows that f is also a morphism of uniform semi-normed rings $(A, \|\cdot\|_A) \to (B, \|\cdot\|_B)$. In view of (i), we deduce that $f : (A, \mathscr{T}_A^{\mathsf{u}}) \to (B, \mathscr{T}_B)$ is a continuous ring homomorphism of uniform Tate rings. Thus the sought functor is obtained by the rule : $(A, \mathscr{T}_A) \mapsto (A, \mathscr{T}_A^{\mathsf{u}})$. □

Example 16.9.4. Let A be a Tate ring, $A_0 \subset A$ a subring of definition, and suppose there exists an integer $n > 1$ such that $\{x \in A \mid x^n \in A_0\} \subset A_0$. Then it follows easily that $A^{\circ\circ} \subset A_0$, and since $A^{\circ\circ}$ is an ideal of A°, we deduce that A is uniform.

16.9.5. Let $(A, \mathscr{T}_A)$ be a perfectoid Tate ring, and $A_0 \subset A$ a perfectoid ring of definition, so that $A = A_0[t^{-1}]$, for any $t \in A^\times \cap A_0^{\circ\circ}$; moreover, tA_0 is an ideal of adic definition for A_0 (corollary 8.3.20(iv)). Let also A^+ be the integral closure of A_0 in A; it follows that $U := \mathrm{Spec}\, A$ is the analytic locus of $\mathrm{Spec}\, A_0$, and

$$\underline{U} := (U, \mathscr{T}_A, A^+)$$

is a perfectoid affinoid scheme (see definition 16.5.1(i)). Then, according to (16.5.20), theorem 16.5.13(ii) and corollary 16.5.43(i), we get a perfectoid affinoid scheme

$$(U_{\mathbf{E}}, \mathscr{T}_{\mathbf{E}}, \mathbf{E}_U^+) := \mathbf{E}(\underline{U}) \qquad \text{with } \mathbf{E}_U := \mathscr{O}_{U_{\mathbf{E}}}(U_{\mathbf{E}})$$

so that $\mathbf{E}_U$ is a complete f-adic ring, and its open subring $\mathbf{E}_U^+$ is perfectoid and naturally isomorphic to $\mathbf{E}(A^+)$. Let $(\alpha_n \mid n \in \mathbb{N})$ be a distinguished element in $\mathrm{Ker}\, u_{A^\circ}$; then the isomorphism of topological rings $\mathbf{E}_U^\circ/\alpha_0\mathbf{E}_U^\circ \xrightarrow{\sim} A^\circ/pA^\circ$ of remark 16.3.7(ii) induces an isomorphism

$$\overline{\omega} : \mathbf{E}_U^\circ/\mathbf{E}_U^{\circ\circ} \xrightarrow{\sim} A^\circ/A^{\circ\circ}.$$

Recall also that the composition $\mathbf{E}_U^\circ \to A$ of $\overline{u}_{A^\circ} : \mathbf{E}_U^\circ \to A^\circ$ with the localization $A^\circ \to A$, factors through a continuous morphism of monoids

$$\varphi_U^\flat : \mathbf{E}_U \to A$$

(lemma 16.4.29(i) and proposition 16.4.34(i)). Let $\pi_A : A^\circ \to A^\circ/A^{\circ\circ}$ and $\pi_{\mathbf{E}} : \mathbf{E}_U^\circ \to \mathbf{E}_U^\circ/\mathbf{E}_U^{\circ\circ}$ be the projections; denote by $\mathscr{D}_A$ and $\mathscr{D}_{\mathbf{E}}$ the set of perfectoid subrings of adic definition of A and respectively $\mathbf{E}_U$, and by $\mathscr{P}_A$ the set of perfect subrings of $A^\circ/A^{\circ\circ}$.

Lemma 16.9.6. *In the situation of* (16.9.5)*, the following holds :*

(i) $(A, \mathscr{T}_A)$ *is a uniform Tate ring.*

(ii) $(\mathbf{E}_U, \mathscr{T}_{\mathbf{E}})$ *is a perfectoid Tate ring, and* $U_{\mathbf{E}}$ *is the analytic locus of* $\mathrm{Spec}\, \mathbf{E}_U^\circ$.

(iii) *The rules :* $B \mapsto \pi_A^{-1}B$ *and* $B \mapsto (\overline{\omega} \circ \pi_{\mathbf{E}})^{-1}B$ *establish natural bijections*

$$\mathscr{D}_{\mathbf{E}} \xleftarrow{\sim} \mathscr{P}_A \xrightarrow{\sim} \mathscr{D}_A.$$

Proof. (i) follows immediately from theorem 16.5.13(i).

(ii): It is easily seen that tA° is an ideal of adic definition for A°, and $A^\circ[t^{-1}] = A$, hence U is the analytic locus of $\mathrm{Spec}\, A^\circ$, and therefore $U_{\mathbf{E}}$ is the analytic locus of $\mathrm{Spec}\, \mathbf{E}_U^\circ$. Pick $m \in \mathbb{N}$ such that $p^m/t \in A^\circ$; by applying corollary 16.6.27 to the quasi-affinoid ring (A°, A°, U), we deduce that there exists $t' \in \mathbf{E}_U^\circ$ such that

$$v(t - \overline{u}_{A^\circ}(t')) \leq v(p) \cdot \max(v(\overline{u}_{A^\circ}(t')), v(p^m)) \qquad \text{for every } v \in \mathrm{Cont}(A^\circ)$$

where $\overline{u}_{A^\circ} : \mathbf{E}_U^\circ \to A^\circ$ is the natural morphism of topological monoids as in (9.4). It follows that $v(t) = v(\overline{u}_{A^\circ}(t'))$ for every $v \in \mathrm{Cont}(A^\circ)$. We have $v(t) < 1$ for every $v \in \mathrm{Cont}(A^\circ)$ and $v(t) \neq 0$ for every $v \in \mathrm{Cont}(A)$, since $t \in A^{\circ\circ} \cap A^\times$ (theorem 15.3.15(i)); hence $w(t') < 1$ for every $w \in \mathrm{Cont}(\mathbf{E}_U^\circ)$ (theorem 16.5.52(i)), and therefore $t' \in \mathbf{E}_U^{\circ\circ}$ (corollary 15.4.27(ii)). Also, combining with (16.5.59) we deduce that $w(t') \neq 0$ for every $w \in \mathrm{Spa}\, \mathbf{E}(\underline{U})$, hence $t' \in \mathbf{E}_U^\times$ (proposition 15.4.30).

(iii): For every $A' \in \mathscr{D}_A$, the inclusion map $A' \to A^\circ$ induces an open injective continuous ring homomorphism $\mathbf{E}' := \mathbf{E}(A') \to \mathbf{E}_U^\circ$, and moreover we have $A^{\circ\circ} \subset A'$ and $\mathbf{E}_U^{\circ\circ} \subset \mathbf{E}'$, by theorem 16.3.42(i,iii.a). As in (16.9.5), we deduce a ring isomorphism $\mathbf{E}'/\mathbf{E}_U^{\circ\circ} \xrightarrow{\sim} A'/A^{\circ\circ}$, which shows that $A'/A^{\circ\circ} \in \mathscr{P}_A$. Conversely, let $B \in \mathscr{P}_A$; according to proposition 16.3.25, the open subring $\pi_A^{-1}B = B \times_{A^\circ/A^{\circ\circ}} A^\circ$ of A° is perfectoid for the topology induced by the inclusion into A°, and then $\pi_A^{-1}B \in \mathscr{D}_A$. Likewise one proves that the map $\mathscr{P}_A \to \mathscr{D}_\mathbf{E}$ is a bijection. □

16.9.7. Let now $(A, \mathscr{T}_A)$ be a perfectoid Tate ring, $A_0 \subset A$ a perfectoid subring of definition, $(B, \mathscr{T}_B)$ an f-adic ring, and $f : A \to B$ a continuous ring homomorphism. Define the perfectoid affinoid schemes $\underline{U}$ and $(U_\mathbf{E}, \mathscr{T}_\mathbf{E}, \mathbf{E}_U^+)$ and the f-adic ring $\mathbf{E}_U$ as in (16.9.5); by lemma 16.9.6(ii) we find $t' \in \mathbf{E}_U^\times \cap \mathbf{E}_U^{\circ\circ}$, and we let $t := \overline{u}_{A^\circ}(t')$, which is a topologically nilpotent unit of A. Then more generally, the element $t^\gamma := \varphi_U^\flat(t'^\gamma) \in A$ is well defined for every $\gamma \in \mathbb{Z}[1/p]$. Notice that $s := f(t)$ is a topologically nilpotent unit of B, hence B is a Tate ring; especially, the non-analytic loci of both $\mathrm{Spec}\,A$ and $\mathrm{Spec}\,B$ are empty, so that f is an f-adic ring homomorphism (lemma 8.3.29(iv)). By lemma 8.3.24(iii.c) there exists a subring of definition $B_0 \subset B$ such that f restricts to a continuous ring homomorphism $f_0 : A_0 \to B_0$. Then sB_0 is an ideal of adic definition of B_0, and $B = B_0[s^{-1}]$ (corollary 8.3.20(ii.b)). Let $s^\gamma := f(t^\gamma)$ for every $\gamma \in \mathbb{Z}[1/p]$. We fix $\rho \in]0, 1[$ and consider the map

$$|\cdot|_B : B \to \mathbb{R} \cup \{+\infty\} \qquad b \mapsto \inf\{\rho^\gamma \mid b \in s^\gamma B_0\}$$

(*cp.* (16.5.23)). Let also $\|\cdot\|_B : B \to \mathbb{R} \cup \{+\infty\}$ be the asymptotic Samuel function of sB_0, as defined in example 9.1.9(i). Recall furthermore that A° is perfectoid for the topology induced by A (theorem 16.5.13(i)), and the pair $(A^\circ, A^{\circ\circ})$ is a basic setup verifying condition (**B**) of [75, §2.1.6] (proposition 16.8.15(i)). Hence, the $(A^\circ, A^{\circ\circ})^a$-algebra B^a is well defined, and in view of proposition 16.8.15(ii) we see that :

$$B_*^a = \mathrm{Hom}_{A^\circ}(A^{\circ\circ}, B) = \mathrm{Hom}_A(A \otimes_{A^\circ} A^{\circ\circ}, B) = \mathrm{Hom}_A(A, B) = B.$$

Lemma 16.9.8. (i) *For every* $b \in B$ *we have :* $\lim\limits_{n\to+\infty} |b^n|_B^{1/n} = \|b\|_B$.

(ii) $\|s^\gamma b\|_B = \rho^\gamma \cdot \|b\|_B$ *for every* $\gamma \in \mathbb{Z}[1/p]$ *and every* $b \in B$.

(iii) $B^{\circ\circ} = \{b \in B \mid \|b\|_B < 1\}$.

(iv) $(B^\circ)_*^a = \{b \in B \mid \|b\|_B \leq 1\}$.

Proof. (i): Recall that $\|b\|_B := \lim\limits_{n\to+\infty} \rho^{\nu(b^n)/n}$, where $\nu(b) := \inf\{k \in \mathbb{Z} \mid b \in s^k B_0\}$. For every $b \in B$, set $\mu(b) := \sup\{\gamma \in \mathbb{Z}[1/p] \mid b \in s^\gamma B_0\}$, so that $|b|_B = \rho^{\mu(b)}$. Clearly we have

$$\nu(b^n) < \mu(b^n) < \nu(b^n) + 1 \qquad \text{for every } b \in B \text{ and } n \in \mathbb{N}.$$

It follows that $\lim_{n\to+\infty}(|b^n|_B/\rho^{\nu(b^n)})^{1/n} = 1$, whence the contention.

(ii): It suffices to observe that $|s^\gamma b|_B = \rho^\gamma \cdot |b|_B$ for every such b and γ, and apply (i).

(iii): It suffices to remark that $\|b\|_B < 1$ if and only if $x^n \in sB_0$ for some $n \in \mathbb{N}$.

(iv): Notice that $(B^\circ)_*^a \subset B_*^a = B$. On the other hand, we have $(B^\circ)_*^a = \mathrm{Hom}_{A^\circ}(A^{\circ\circ}, B^\circ)$. It then follows that $(B^\circ)_*^a = \{b \in B \mid bA^{\circ\circ} \subset B^\circ\} = \{b \in B \mid bA^{\circ\circ} \subset B^{\circ\circ}\}$. Especially, $s^\gamma b \in B^{\circ\circ}$ for every $b \in (B^\circ)_*^a$ and $\gamma \in \mathbb{N}[1/p] \setminus \{0\}$, whence $\rho^\gamma \cdot \|b\|_B < 1$ for every such γ, in view of (ii) and (iii); *i.e.* $\|b\|_B \leq 1$. Conversely, if $\|b\|_B \leq 1$, we get $\|f(a)b\|_B \leq \|b\|_B \cdot \|f(a)\|_B < 1$ for every $a \in A^{\circ\circ}$, due to (iii); then $f(a)b \in B^{\circ\circ}$ for every such a, again due to (iii), and finally, $b \in (B^\circ)_*^a$. □

16.9.9. Keep the situation of (16.9.7), and set $C := (B^\circ)_*^a$; lemma 16.9.8(ii,iv) implies that

$$s^\gamma C = \{b \in B \mid \|b\|_B \leq \rho^\gamma\} \qquad \text{for every } \gamma \in \mathbb{Z}[1/p].$$

In light of lemma 16.9.2(iii), we may now conclude that C is the subring of power bounded elements of the uniformization $\mathsf{u}(B)$ of $(B, \mathscr{T}_B)$, and the topology of $\mathsf{u}(B)$ is the f-adic topology such that $C \subset B$ is a subring of definition, and sC is an ideal of adic definition. Notice that $f : (A, \mathscr{T}_A) \to \mathsf{u}(B)$ is still an f-adic ring homomorphism. Moreover, from lemmata 16.9.2(ii) and 16.9.8(iii) we see that

$$\mathsf{u}(B)^{\circ\circ} = B^{\circ\circ}. \qquad (16.9.10)$$

Definition 16.9.11. (i) With the notation of (16.9.7), we say that B is a *pre-perfectoid* A*-algebra*, if $\mathsf{u}(B)$ is a formally perfectoid Tate ring.

Remark 16.9.12. (i) With the notation of (16.9.9), endow C with its sC-adic topology $\mathscr{T}_C$; from corollary 8.4.15(ii) and theorem 16.5.13(i) we see that B is a pre-perfectoid A-algebra if and only if $(C, \mathscr{T}_C)^\wedge$ is a perfectoid ring.

(ii) Let $D \subset C$ be any subring that is p-root closed in C (see definition 9.8.24(i)). From (16.9.10) we see that :

$$D \text{ is open in } (B, \mathscr{T}_B) \Leftrightarrow B^{\circ\circ} \subset D \Leftrightarrow \mathsf{u}(B)^{\circ\circ} \subset D \Leftrightarrow D \text{ is open in } \mathsf{u}(B).$$

Suppose then that D is open in $(B, \mathscr{T}_B)$, and denote by $\mathscr{T}_D$ the topology of D induced by $\mathscr{T}_C$; then we claim that B is a pre-perfectoid A-algebra if and only if $(D, \mathscr{T}_D)^\wedge$ perfectoid. Indeed, suppose that B is a pre-perfectoid A-algebra; since $B^{\circ\circ} \subset D$, we see that $D/B^{\circ\circ}$ is a subring of $C/B^{\circ\circ}$, and in view of (16.9.10) we get natural identifications :

$$C/B^{\circ\circ} = C/\mathsf{u}(B)^{\circ\circ} = \overline{C} := (C, \mathscr{T}_C)^\wedge/(\mathsf{u}(B)^\wedge)^{\circ\circ}.$$

Then, (i) implies that $C/B^{\circ\circ}$ is a perfect ring, and $D/B^{\circ\circ}$ is a perfect subring, since D is p-root closed in C. In light of (i), lemma 16.9.6(iii) and proposition 8.2.13(i,ii), we conclude that $(D, \mathscr{T}_D)^\wedge = D/B^{\circ\circ} \times_{\overline{C}} (C, \mathscr{T}_C)^\wedge$ is a perfectoid subring of definition of $\mathsf{u}(B)^\wedge$. Conversely, suppose that $(D, \mathscr{T}_D)^\wedge$ is perfectoid; we know already that D is open in $\mathsf{u}(B)$, and then $(D, \mathscr{T}_D)^\wedge$ is open in $\mathsf{u}(B)^\wedge$, and its topology is induced by the inclusion into $\mathsf{u}(B)^\wedge$ (proposition 8.2.13(i,ii)); hence, B is a pre-perfectoid A-algebra.

(iii) Lastly, let $D \subset B$ be an open subring, endow D with the topology induced by B, and suppose that D is formally perfectoid (definition 16.7.19); then D is p-root closed in B. Indeed, under these assumptions, the topology of $D^\wedge$ is adic, so $D^\wedge$ is a perfectoid subring of definition of $B^\wedge$, hence $(B^\wedge)^{\circ\circ} \subset D^\wedge$, by virtue of lemma 16.9.6(iii); especially, $s^\gamma \in D$ for every $\gamma \in \mathbb{N}[1/p]$, and $s^{-\gamma}p \in D$ for every sufficiently small $\gamma \in \mathbb{N}[1/p]$ (corollary 8.2.17(ii)). It then suffices to invoke the criterion of theorem 16.4.1, together with lemma 9.8.26(i).

Example 16.9.13. Let us return to the situation of (16.6.31) : we consider a quasi-affinoid perfectoid ring $\underline{A} := (A, A^+, U_A)$ and a sequence $f_\bullet := (f_0, \dots, f_n)$ of elements of $A_U := \mathscr{O}_{U_A}(U_A)$ that generates an open ideal (for the f-adic topology on A_U as in lemma 16.4.32(i)).

(i) Let $(\mathbf{E}, \mathbf{E}^+, U_\mathbf{E}) := \mathbf{E}(\underline{A})$ (notation of (16.5.20)), and $\mathbf{E}_U := \mathscr{O}_{U_\mathbf{E}}(U_\mathbf{E})$. As recalled in (16.9.5), the map $\overline{u}_A : \mathbf{E} \to A$ extends to a continuous morphism of topological monoids $\varphi^\flat_U : \mathbf{E}_U \to A_U$; we may then find $m \in \mathbb{N}$ and $e_0, \dots, e_n \in \mathbf{E}_U$ verifying the inequalities (16.6.33) and (16.6.34) for every $i = 1, \dots, n$. Let also $A_0 \subset A_U$ be any subring of definition, and $I_0 \subset A_0$ an ideal of adic definition; recall that the I_0-adic completion $B(f_\bullet)^\wedge$ of $B(f_\bullet) := A_0[f_1/f_0, \dots, f_n/f_0] \subset A_U[1/f_0]$ is a subring of definition of $\mathscr{O}^\wedge_{\mathrm{Spa}\,\underline{A}}(S(f_\bullet))$, with $S(f_\bullet) := R_{A_U}\big(\frac{f_1}{f_0}, \cdots, \frac{f_n}{f_0}\big) \cap \mathrm{Spa}\,\underline{A}$. Moreover, the sequence $\varphi^\flat_U(e_0), \dots, \varphi^\flat_U(e_n)$ generates an open ideal of A_U as well, according to proposition 16.6.35(ii) and claim 16.6.36, and $S(f_\bullet) = R_{A_U}\big(\frac{\varphi^\flat_U(e_1)}{\varphi^\flat_U(e_0)}, \cdots, \frac{\varphi^\flat_U(e_n)}{\varphi^\flat_U(e_0)}\big) \cap \mathrm{Spa}\,\underline{A}$, by proposition 16.6.35(iii). It follows that the I_0-adic completion $B(e_\bullet)^\wedge$ of $B(e_\bullet) := A_0\big[\frac{\varphi^\flat_U(e_1)}{\varphi^\flat_U(e_0)}, \dots, \frac{\varphi^\flat_U(e_n)}{\varphi^\flat_U(e_0)}\big]$ is another subring of definition of $\mathscr{O}^\wedge_{\mathrm{Spa}\,\underline{A}}(S(f_\bullet))$. Therefore, the p-root closures $D(f_\bullet)$ and $D(e_\bullet)$ of $B(f_\bullet)^\wedge$ and $B(e_\bullet)^\wedge$ in $\mathscr{O}^\wedge_{\mathrm{Spa}\,\underline{A}}(S(f_\bullet))$ both contain $\mathscr{O}^\wedge_{\mathrm{Spa}\,\underline{A}}(S(f_\bullet))^{\circ\circ}$. Combining with proposition 16.6.35(iv), we conclude that $D(f_\bullet) = D(e_\bullet)$.

(ii) In the situation of (i), notice that if $f_0, \dots, f_n \in A$, we may take $e_0, \dots, e_n \in \mathbf{E}$, and by construction we shall then have $f_i - \overline{u}_A(e_i) \in pA$ for $i = 0, \dots, n$.

Theorem 16.9.14. *In the situation of* (16.9.7)*, pick* $\gamma \in \mathbb{N}[1/p] \setminus \{0\}$ *with* $pB_0 \subset s^{p\gamma}B_0$*, and suppose that the Frobenius endomorphism of* $B_0/s^{p\gamma}B_0$ *is surjective. Then* B *is pre-perfectoid.*

Proof. We exhibit an increasing chain of subrings

$$B_0 \subset B_1 \subset B_2 \subset \cdots$$

of the p-root closure of B_0 in B, verifying the following conditions for every $i \in \mathbb{N}$:

(a) The Frobenius endomorphism Φ_i of $B_i/s^{p\gamma}B_i$ is surjective.
(b) Let $\overline{\Phi}_i : B_i/s^{\gamma}B_i \to B_i/s^{p\gamma}B_i$ be the ring homomorphism induced by Φ_i. Then the kernel of the induced map $j_i : B_i/s^{\gamma}B_i \to B_{i+1}/s^{\gamma}B_{i+1}$ contains $\mathrm{Ker}\,\overline{\Phi}_i$.

To this aim, we argue by induction on $i \in \mathbb{N}$. Hence, suppose that for a given $i \in \mathbb{N}$ we have already exhibited a subring B_i fulfilling condition (a); we need to exhibit B_{i+1} so that condition (b) holds as well for B_i, and moreover such that condition (a) holds for B_{i+1}. Now, since (a) holds for B_i, for every $x \in \mathrm{Ker}\,\overline{\Phi}_i$ we may find a sequence $(a(x,n) \mid n \in \mathbb{N})$ of elements of B_i, such that the image of $a(x,0)$ in $B_i/s^{\gamma}B_i$ agrees with x, and :

$$a(x,n+1)^p - a(x,n) \in s^{p\gamma}B_i \qquad \text{for every } n \in \mathbb{N}.$$

It follows easily that $a(x,n)^{p^{n+1}} \in s^{p\gamma}B_i$, and therefore $b(x,n) := a(x,n) \cdot s^{-\gamma/p^n}$ lies in the p-root closure of B_i in B, for every $n \in \mathbb{N}$. Let us then set

$$B_{i+1} := B_i[b(x,n) \mid (x,n) \in \mathrm{Ker}(\overline{\Phi}_i) \times \mathbb{N}] \subset B.$$

Since the image of $b(x,0) \cdot s^{\gamma}$ in $B_i/s^{\gamma}B_i$ agrees with the image of x, we see that $j_i(x) = 0$ for every $x \in \mathrm{Ker}\,\overline{\Phi}_i$, hence condition (b) holds for B_i. Lastly, notice that

$$b(x,n+1)^p - b(x,n) \in B_i \qquad \text{for every } n \in \mathbb{N}.$$

Since $B_i/s^{p\gamma}B_i \subset \mathrm{Im}\,\overline{\Phi}_{i+1}$, it follows easily that condition (a) holds for B_{i+1}, as required.

Set $B_\infty := \bigcup_{i\in\mathbb{N}} B_i = B_\infty$; by construction, the Frobenius endomorphism of B_∞ induces an isomorphism $B_\infty/s^{\gamma}B_\infty \xrightarrow{\sim} B_\infty/s^{p\gamma}B_\infty$, hence B_∞ is the p-root closure of B_0 in B, by lemma 9.8.26(i). Next, let $B_\infty^\wedge$ be the completion of B_∞ for its s-adic topology; the Frobenius endomorphism of $B_\infty^\wedge$ induces an isomorphism $B_\infty^\wedge/s^{\gamma}B_\infty^\wedge \xrightarrow{\sim} B_\infty^\wedge/s^{p\gamma}B_\infty^\wedge$. Clearly $B_\infty^\wedge$ is a P-ring; by theorem 16.4.1, the ring $B_\infty^\wedge$ is then perfectoid, and the assertion follows from remark 16.9.12(ii). □

Corollary 16.9.15. *In the situation of* (16.9.7)*, let $I \subset B$ be any ideal, and endow $\overline{B} := B/I$ with the f-adic topology $\mathscr{T}_{\overline{B}}$ induced by $\mathscr{T}_B$ via the projection $B \to \overline{B}$ (see example* 8.3.27(iii)*). Then, if B is a pre-perfectoid A-algebra, the same holds for $\overline{B}$.*

Proof. Let $\mathscr{T}_B^{\mathsf{u}}$ be the topology of $\mathsf{u}(B)$, and denote by $\mathscr{T}_{\overline{B}}^{\mathsf{u}}$ the f-adic topology induced by $\mathscr{T}_B^{\mathsf{u}}$ via the projection $B \to \overline{B}$. If C is any uniform Tate ring, and $f : (\overline{B}, \mathscr{T}_B) \to C$ any continuous ring homomorphism, it is easily seen that f factors through the identity map $\mathbf{1}_{\overline{B}} : (\overline{B}, \mathscr{T}_B) \to (\overline{B}, \mathscr{T}_{\overline{B}}^{\mathsf{u}})$, which is continuous, and the continuous ring homomorphism $f : (\overline{B}, \mathscr{T}_{\overline{B}}^{\mathsf{u}}) \to C$. By the universal property of uniformization, this implies that $\mathbf{1}_{\overline{B}}$ induces an isomorphism of uniform Tate rings

$$\mathsf{u}(\overline{B}, \mathscr{T}_B) \xrightarrow{\sim} \mathsf{u}(\overline{B}, \mathscr{T}_{\overline{B}}^{\mathsf{u}})$$

(whose underlying ring homomorphism is of course again the identity map of $\overline{B}$). Thus, it suffices to check that $(\overline{B}, \mathscr{T}_{\overline{B}}^{\mathsf{u}})$ is pre-perfectoid. However, since B is pre-perfectoid, $C := \mathsf{u}(B)^\circ$ is a perfectoid ring, hence for every $\gamma \in \mathbb{N}[1/p]\setminus\{0\}$ such that $pC \subset s^{p\gamma}C$, the Frobenius endomorphism of $C/s^{p\gamma}C$ is surjective. By construction, the image $\overline{C}$ of C in $\overline{B}$ is a subring of definition for the topology $\mathscr{T}_{\overline{B}}^{\mathsf{u}}$, and it follows that the Frobenius endomorphism of $\overline{C}/s^{p\gamma}\overline{C}$ is again surjective; then the assertion follows from theorem 16.9.14. □

16.9.16. Let A be a formally perfectoid Tate ring, $A_0 \subset A$ a formally perfectoid subring of definition, $t \in A_0^{\circ\circ} \cap A^\times$, and $P \in A_0[X] \setminus A_0$ a monic polynomial. We let $\overline{P}$ be the image of P in $\overline{A}_0 := A_0/A_0^{\circ\circ}$, we define $B_0 := A_0[X^{1/p^\infty}] := \bigcup_{n\in\mathbb{N}} A_0[X^{1/p^n}]$, $B := A \otimes_{A_0} B_0$, and set

$$C_{0,n} := A_0[X^{1/p^n}]/(P) \quad \text{for every } n \in \mathbb{N} \qquad C_0 := B_0/PB_0 \qquad \text{and} \qquad C := B/PB.$$

Clearly the natural map of A_0-algebras $C_{0,m} \to C_{0,n}$ is injective for every $n, m \in \mathbb{N}$ with $n \geq m$, and $C_{0,n}$ is a free A_0-module of rank $p^n \cdot \deg_X(P)$ for every such n. Hence $C_0 = \bigcup_{n\in\mathbb{N}} C_{0,n}$ is a faithfully flat A_0-algebra, and the localization $C_0 \to C$ is injective. Lastly, let $D \subset C$ be the p-root closure of C_0 in C (see definition 9.8.24(ii)), and endow D with its t-adic topology.

The following result and its proof are a generalization and refinement of [5, Th.2.5.2] (modulo powers of p, as in [5, Rem.2.6.1(4)]); see also [23, Th.2.3]. We also borrowed from a set of lecture notes of B.Bhatt the idea of stating the theorem for a general monic polynomial P : see [24, Th.9.4.3] (whereas André's original theorem would be the special case with $P = X - a$, for any $a \in A_0$).

Theorem 16.9.17. (i) *In the situation of* (16.9.16)*, D is a faithfully flat A_0-algebra.*

(ii) *Moreover, D is a formally perfectoid ring for its t-adic topology.*

(iii) *Furthermore, there exists an isomorphism of $\overline{A}_0$-algebras (notation of* (9.8.11)*) :*

$$(\overline{A}_0[X]/(\overline{P}))^{\mathrm{perf}} \xrightarrow{\sim} \overline{A}_0 \otimes_{A_0} D.$$

Proof. Clearly $D[1/t] = C$ is a faithfully flat A-algebra, and the image of t is regular in D; by virtue of [75, Lemma 5.2.1], in order to show (i), it then suffices to check that D/tD is a faithfully flat A_0/tA_0-algebra. Recall that the completion $A_0^\wedge$ is a subring of definition for the completion $A^\wedge$ of A (proposition 8.3.33(ii)); set $C_0' := A_0^\wedge \otimes_{A_0} C_0$, $C' := A^\wedge \otimes_A C$, and let D' be the p-root closure of C_0' in C'. Endow also C_0 (resp. C_0') with its t-adic topology, and C (resp. C') with the unique f-adic topology such that C_0 (resp. C_0') is a subring of definition; let $\mathscr{T}_D$ (resp. $\mathscr{T}_{D'}$) be the topology induced by C on D (resp. by C' on D'). Taking into account proposition 8.3.33, it is easily seen that the natural maps $C_0 \to C_0'$ and $C \to C'$ induce isomorphisms on the respective completions; by virtue of lemma 9.8.26(ii), the same then holds for the induced continuous ring homomorphism $(D, \mathscr{T}_D) \to (D', \mathscr{T}_{D'})$.

Claim 16.9.18. Let $D^\wedge$ be the completion of $(D, \mathscr{T}_D)$. Then the completion map $u : D \to D^\wedge$ induces an isomorphism $D/t^k D \xrightarrow{\sim} D^\wedge/t^k D^\wedge$ for every $k \in \mathbb{N}$.

Proof of the claim. Let also $C_0^\wedge$ be the completion of C_0; since $t^k C_0$ is an open A_0-submodule of D, the map u induces an isomorphism of A_0-modules $\overline{u} : D/t^k C_0 \xrightarrow{\sim} D^\wedge/t^k C_0^\wedge$. Hence, $\overline{u}(t^k D/t^k C_0) = t^k \cdot \overline{u}(D/t^k C_0) = t^k D^\wedge/t^k C_0^\wedge$, whence the assertion. ◇

By applying claim 16.9.18 to both $D^\wedge$ and the completion $D'^\wedge$ of $(D', \mathscr{T}_{D'})$, we reduce to checking that D'/tD' is a faithfully flat A_0/tA_0-algebra, and that D' is formally perfectoid for its t-adic topology. Thus, we may replace A by $A^\wedge$, and assume that *A is a perfectoid Tate ring*.

(ii): By construction, the Frobenius endomorphism of C_0 is a surjection; hence C is a pre-perfectoid A-algebra, by theorem 16.9.14. Then the assertion follows from remark 16.9.12(ii).

(i): Define the affinoid schemes $\underline{U} := (U, \mathscr{T}_A, A^+)$ and $\underline{U}_{\mathbf{E}} := (U_{\mathbf{E}}, \mathscr{T}_{\mathbf{E}}, \mathbf{E}_U^+)$ as in (16.9.5); recall that $\mathbf{E}_0 := \mathbf{E}(A_0)$ is a subring of definition of the Tate perfectoid ring $\mathbf{E}_U := \mathscr{O}_{U_{\mathbf{E}}}(U_{\mathbf{E}})$ (lemma 16.9.6(ii,iii)). We may also assume that $t := \overline{u}_{A_0}(t') \in A$ for some $t' \in \mathbf{E}_U^\times \cap \mathbf{E}_0^{\circ\circ}$.

We endow B_0 with its t-adic topology, and B with the f-adic topology such that B_0 is a subring of definition. We have then a short exact sequence of flat A_0-modules

$$\Sigma \qquad : \qquad 0 \to PB_0 \to B_0 \xrightarrow{\pi_0} C_0 \to 0.$$

As in (16.9.7), we let $t^\gamma := \varphi_U^\flat(t'^\gamma) \in A$ for every $\gamma \in \mathbb{Z}[1/p]$. With this notation, the sequence $A_0/t^\gamma A_0 \otimes_{A_0} \Sigma$ is still short exact, for every $\gamma \in \mathbb{N}[1/p]$, which means that $t^\gamma B_0 \cap PB_0 = t^\gamma PB_0$

for every such γ. Especially, the topology of B_0 induces the t-adic topology on the ideal PB_0, and after taking completions, we still get a short exact sequence :

$$\Sigma^\wedge \qquad : \qquad 0 \to PB_0^\wedge \to B_0^\wedge \xrightarrow{\pi_0^\wedge} C_0^\wedge \to 0.$$

Recall that the completion $C^\wedge$ of C is naturally identified with $C_0^\wedge[t^{-1}]$, and $C_0^\wedge$ is a subring of definition for $C^\wedge$ (proposition 8.3.33(ii,iii)). Likewise, $B_0^\wedge$ is a subring of definition of $B^\wedge = B_0^\wedge[t^{-1}]$. We consider the affinoid ring $\underline{B}^\wedge := (B^\wedge, B^{\wedge\circ})$, and the rational subset

$$R_n := R_{B^\wedge}\left(\frac{P}{t^n}\right) \cap \operatorname{Spa} \underline{B}^\wedge \qquad \text{for every } n \in \mathbb{N}.$$

Endow the subring $B_n := B_0^\wedge[P/t^n] \subset B^\wedge$ with its t-adic topology, and let $\mathscr{T}_n$ be the f-adic topology on $B^\wedge$ such that B_n is a subring of definition; then the completion $B_n^\wedge$ of B_n is a subring of definition of $\mathscr{O}^\wedge_{\operatorname{Spa} \underline{B}^\wedge}(R_n) = (B^\wedge, \mathscr{T}_n)^\wedge$, for every $n \in \mathbb{N}$. Notice that the projection $\pi_0^\wedge$ extends to a continuous open ring homomorphism $\pi^\wedge : B^\wedge \to C^\wedge$, and for every $n \in \mathbb{N}$, the image of the induced continuous map

$$\operatorname{Spa} \pi^\wedge : \operatorname{Spa}(C^\wedge, C^{\wedge\circ}) \to \operatorname{Spa} \underline{B}^\wedge$$

lies in R_n. It follows that $\pi^\wedge$ factors uniquely through a continuous ring homomorphism

$$\psi_n : \mathscr{O}^\wedge_{\operatorname{Spa} \underline{B}^\wedge}(R_n) \to C^\wedge \qquad \text{for every } n \in \mathbb{N}$$

(see remark 15.5.9(i)). Moreover, we have $R_{n+1} \subset R_n$ for every $n \in \mathbb{N}$, and clearly the composition of ψ_{n+1} with the restriction map $\mathscr{O}^\wedge_{\operatorname{Spa} \underline{B}^\wedge}(R_n) \to \mathscr{O}^\wedge_{\operatorname{Spa} \underline{B}^\wedge}(R_{n+1})$ agrees with ψ_n. Furthermore, a direct inspection shows that $\psi_n(B_n^\wedge) = C_0^\wedge$ for every $n \in \mathbb{N}$. Summing up, we deduce a commutative diagram of rings :

$$(16.9.19) \qquad \begin{array}{ccc} B' := \operatorname*{colim}_{n \in \mathbb{N}} B_n^\wedge & \longrightarrow & \mathscr{O} := \operatorname*{colim}_{n \in \mathbb{N}} \mathscr{O}^\wedge_{\operatorname{Spa} \underline{B}^\wedge}(R_n) \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ C_0^\wedge & \longrightarrow & C^\wedge \end{array}$$

whose vertical (resp. horizontal) arrows are surjections (resp. injections).

Claim 16.9.20. The diagram (16.9.19) is cartesian.

Proof of the claim. The assertion means that the induced map $\operatorname{Ker} \alpha \to \operatorname{Ker} \beta$ is bijective. However, this map is clearly injective, so it remains to check its surjectivity. To this aim, notice that for every $n \in \mathbb{N}$ and every $g_n \in \mathscr{O}^\wedge_{\operatorname{Spa} \underline{B}^\wedge}(R_n)$ there exists $N \in \mathbb{N}$ such that

$$t^N g_n = \sum_{i=0}^{\infty} b_i \cdot \left(\frac{P}{t^n}\right)^i \qquad \text{with } b_i \in B_0^\wedge \text{ for every } i \in \mathbb{N} \text{ and } \lim_{i \to +\infty} b_i = 0.$$

With this notation, it follows easily that $g_n \in \operatorname{Ker} \psi_n$ if and only if $b_0 = 0$. Suppose then that $\psi_n(g_n) = 0$; the image g_{n+N} of g_n in $\mathscr{O}^\wedge_{\operatorname{Spa} \underline{B}^\wedge}(R_{n+N})$ can be written as :

$$g_{n+N} = \sum_{i=1}^{\infty} t^{N(i-1)} b_i \cdot \left(\frac{P}{t^{n+N}}\right)^i$$

and we have $t^{N(i-1)} b_i \in B_0^\wedge$ for every $i \geq 1$, and $\lim_{i \to +\infty} t^{N(i-1)} b_i = 0$. Hence $g_{n+N} \in B^\wedge_{n+N}$, and clearly $g_{n+N} \in \operatorname{Ker} \psi_{n+N}$, whence the contention. ◊

Let $\mathscr{D}$ be the p-root closure of B' in $\mathscr{O}$, and endow $\mathscr{D}$ with its t-adic topology; it follows easily from claim 16.9.20 that $\mathscr{D} = \beta^{-1} D$. Moreover, the induced ring homomorphism $\beta_{|\mathscr{D}} : \mathscr{D} \to D$

is surjective, and its kernel is $\mathrm{Ker}\,\beta$. The latter is clearly t-divisible, hence $\beta_{|\mathscr{D}}$ induces an isomorphism of A_0-algebras

$$\mathscr{D}/t^n\mathscr{D} \xrightarrow{\sim} D/t^nD \qquad \text{for every } n \in \mathbb{N}.$$

Thus, it suffices to check that the natural ring homomorphism $A_0 \to \mathscr{D}$ is adically faithfully flat. To this aim, let also $\mathscr{D}_n$ be the p-root closure of $B_n^\wedge$ in $\mathscr{O}^\wedge_{\mathrm{Spa}\,\underline{B}^\wedge}(R_n)$, for every $n \in \mathbb{N}$, and endow $\mathscr{D}_n$ with its t-adic topology; in view of lemma 9.8.26(iii.b) we are reduced to checking that $\mathscr{D}_n$ is an adically faithfully flat A_0-algebra for every $n \in \mathbb{N}$.

Now, notice that $B_0^\wedge$ is a perfectoid ring, by virtue of lemma 16.6.40, and we have a natural identification of topological rings :

$$\mathbf{E}(B_0^\wedge) \xrightarrow{\sim} \mathbf{E}_0[X^{1/p^\infty}]^\wedge \qquad \text{such that} \qquad \overline{u}_{B_0^\wedge}(X^\gamma) = X^\gamma \qquad \text{for every } \gamma \in \mathbb{N}[1/p]$$

where $\mathbf{E}_0[X^{1/p^\infty}]^\wedge$ denotes the t'-adic completion of $\mathbf{E}_0[X^{1/p^\infty}]$. Next, let $(\alpha_n \mid n \in \mathbb{N})$ be a distinguished element in the kernel of $u_{A_0} : W(\mathbf{E}_0) \to A_0$; after replacing t' by t'^γ for some sufficiently small $\gamma \in \mathbb{N}[1/p] \setminus \{0\}$, we may assume that $\alpha_0\mathbf{E}_0 \subset t'\mathbf{E}_0$, in which case the isomorphism ω of remark 16.3.7(ii) induces ring isomorphisms

$$\mathbf{E}_0/t'\mathbf{E}_0 \xrightarrow{\sim} A_0/tA_0 \qquad \mathbf{E}(B_0^\wedge)/t'\mathbf{E}(B_0^\wedge) \xrightarrow{\sim} B_0^\wedge/tB_0^\wedge. \tag{16.9.21}$$

On the other hand, set $U_B := \mathrm{Spec}\,B^\wedge$; then $(B_0^\wedge, B_0^\wedge, U_B)$ is a perfectoid quasi-affinoid ring, and by example 16.9.13 we may then find $e_n \in \mathbf{E}(B_0^\wedge)$ such that the $\mathscr{D}_n$ is the p-root closure in $\mathscr{O}^\wedge_{\mathrm{Spa}\,\underline{B}^\wedge}(R_n)$ of the t-adic completion of $B_0^\wedge[\overline{u}_{B_0^\wedge}(e_n)/t^n]$, and moreover – after replacing again t' by t'^γ for some small $\gamma \in \mathbb{N}[1/p] \setminus \{0\}$ – we may assume that $P - \overline{u}_{B_0^\wedge}(e_n) \in tB_0^\wedge$. Hence, there exist a monic polynomial $Q_n \in \mathbf{E}_0[X]$ and $g_n \in \mathbf{E}_0[X^{1/p^\infty}]^\wedge$ with

$$e_n = Q_n + t'g_n. \tag{16.9.22}$$

According to proposition 16.7.25(iii), the subring

$$D_n := B_0^\wedge[\overline{u}_{B_0^\wedge}(e_n^{1/p^k})/t^{n/p^k} \mid k \in \mathbb{N}] \subset B^\wedge$$

is formally perfectoid for the topology $\mathscr{T}'_n$ induced by $\mathscr{T}_n$. Then $\mathscr{T}'_n$ is the t-adic topology, and combining with remark 16.9.12(iii) and lemma 9.8.26(ii), we deduce that

$$(D_n, \mathscr{T}'_n)^\wedge = \mathscr{D}_n.$$

Thus, we are further reduced to checking that the A_0-algebra $D_{n,k} := B_0^\wedge[\overline{u}_{B_0^\wedge}(e_n^{1/p^k})/t^{n/p^k}]$ is adically faithfully flat for every $n, k \in \mathbb{N}$ with $n > 0$ and $n/p^k \leq 1$. Since t is a regular element of $D_{n,k}$, the local flatness criterion ([126, Th.22.3]) further reduces to showing that $D_{n,k}/(t^{n/p^k})$ is a faithfully flat $A_0/(t^{n/p^k})$-algebra, for every such n and k. To this aim, notice that (16.9.21) induces an isomorphism $E_{n,k} := \mathbf{E}_0/(t'^{n/p^k}) \xrightarrow{\sim} A_0/(t^{n/p^k})$, and $B_0^\wedge/(t^{n/p^k})$ is isomorphic to the A_0-algebra $E_{n,k}[X^{1/p^\infty}]$ (as $n/p^k \leq 1$). Also, the image of $\overline{u}_{B_0^\wedge}(e_n^{1/p^k})$ in $E_{n,k}[X^{1/p^\infty}]$ is the regular element Q_n^{1/p^k}, due to (16.9.22). Thus, the sequence $(t^{n/p^k}, \overline{u}_{B_0^\wedge}(e_n^{1/p^k}))$ is regular in the ring $B_0^\wedge$; in particular, this sequence is completely secant (proposition 7.8.7), so the natural map

$$B_0^\wedge[Y]/(\overline{u}_{B_0^\wedge}(e_n^{1/p^k}) - t^{n/p^k}Y) \to D_{n,k}$$

is an isomorphism of A_0-algebras (lemma 7.8.17). We are then furter reduced to checking that for every $n, k, r \in \mathbb{N}$ with $n > 0$, $r \geq k$ and $n/p^k \leq 1$, the $E_{n,k}$-algebra

$$E_{n,k}[X^{1/p^r}, Y]/(Q_n^{1/p^k})$$

is faithfully flat. The latter follows from [65, Ch.IV, Prop.11.3.7].

(iii): The proof of (i) yields an isomorphism of $\overline{A}_0$-algebras :

$$\overline{A}_0 \otimes_{A_0} D \xrightarrow{\sim} \operatorname*{colim}_{n \in \mathbb{N}} \overline{A}_0 \otimes_{A_0} D_n$$

with $D_n = B_0^\wedge[Y^{1/p^\infty}]/((t^nY)^{1/p^k} - \overline{u}_{B_0^\wedge}(e_n^{1/p^k}) \mid k \in \mathbb{N})$ for every $n \in \mathbb{N}$, and the transition map $D_n \to D_{n+1}$ is the map of $B_0^\wedge$-algebras such that $Y^{1/p^k} \mapsto (tY)^{1/p^k}$ for every $k, n \in \mathbb{N}$. Set $\overline{B}_0 := \overline{A}_0 \otimes_{A_0} B_0$; it follows that $\overline{A}_0 \otimes_{A_0} D_n = \overline{B}_0[Y^{1/p^\infty}]/(\overline{P}^{1/p^k} \mid k \in \mathbb{N})$ for every $n \in \mathbb{N}$, and the induced map $\overline{A}_0 \otimes_{A_0} D_n \to \overline{A}_0 \otimes_{A_0} D_{n+1}$ is the map of $\overline{B}_0$-algebras such that $Y^{1/p^k} \mapsto 0$ for every $n, k \in \mathbb{N}$. Therefore, we get an isomorphism $\overline{A}_0 \otimes_{A_0} D \xrightarrow{\sim} \overline{B}_0/(\overline{P}^{1/p^k} \mid k \in \mathbb{N})$, whence the assertion. ☐

16.9.23. Let (A, a) be a pair consisting of a ring A and an element $a \in A$ such that $p^n A \subset aA$ for some $n \in \mathbb{N}$, and denote by $\mathscr{C}(A, a)$ the category of all formally perfectoid A-algebras B whose topology is a-adic, and *whose a-torsion is bounded*, *i.e.* there exists $k \in \mathbb{N}$ with $\mathrm{Ann}_B(a^k) = \mathrm{Ann}_B(a^{k+1})$. The morphisms of $\mathscr{C}(A, a)$ are all the maps of A-algebras. Moreover, let $P \in A[X]$ be a non-constant monic polynomial; we associate with P the functor

$$F_P : \mathscr{C}(A, a) \to \mathbf{Set} \qquad B \mapsto \{(\beta_n \mid n \in \mathbb{N}) \in \mathbf{E}(B) \mid P(\beta_0) = 0\}.$$

Every morphism $f : B \to B'$ of $\mathscr{C}(A, a)$ induces a map $\mathbf{E}(f) : \mathbf{E}(B) \to \mathbf{E}(B')$, and we let $F_P(f) : F_P B \to F_P B'$ be the restriction of $\mathbf{E}(f)$.

Lemma 16.9.24. *With the notation of* (16.9.23)*, let $B \in \mathrm{Ob}(\mathscr{C}(A, a))$, set $B_0 := B/\mathrm{Ann}_B(a)$, $B_1 := B/B^{\circ\circ}$, and $B_2 := B_0 \otimes_B B_1$. Then we have :*

(i) *B_1 and B_2 are perfect $\mathbb{F}_p$-algebras, and B_0 is formally perfectoid for its a-adic topology.*

(ii) *The image of a is regular in B_0, and B_0 is p-root closed in $B_0[a^{-1}]$.*

(iii) *The resulting commutative diagram of rings :*

$$\begin{array}{ccc} B & \longrightarrow & B_0 \\ \downarrow & & \downarrow \\ B_1 & \longrightarrow & B_2 \end{array}$$

is cartesian (and cocartesian).

(iv) *$B_0, B_1, B_2 \in \mathrm{Ob}(\mathscr{C}(A, a))$, and $F_P B = F_P B_0 \times_{F_P B_2} F_P B_1$.*

Proof. We begin with the following observation :

Claim 16.9.25. Let M be any A-module; endow M with its a-adic topology, and denote by $M^\wedge$ the completion of M. Then, $\mathrm{Ann}_{M^\wedge}(a^k)$ is the topological closure of the image of $\mathrm{Ann}_M(a^k)$ in $M^\wedge$, for every $k \in \mathbb{N}$.

Proof of the claim. Set $N := a^k M$, and consider the induced exact sequence :

$$0 \to \mathrm{Ann}_M(a^k) \to M \xrightarrow{f} N \to 0.$$

Endow N with its a-adic topology, and $\mathrm{Ann}_M(a^k)$ with the topology induced by M; after taking a-adic completions, we get the exact sequence :

$$0 \to \mathrm{Ann}_M(a^k)^\wedge \to M^\wedge \xrightarrow{f^\wedge} N^\wedge \to 0$$

and $\mathrm{Ann}_M(a^k)^\wedge$ is the topological closure in $M^\wedge$ of the image of $\mathrm{Ann}_M(a^k)$ in $M^\wedge$ (proposition 8.2.13(i,iii,v)). Moreover, $N^\wedge = a^k M^\wedge$, and under this identification, $f^\wedge$ corresponds to the scalar multiplication by a^k on $M^\wedge$ (remark 8.3.3(iv)), whence the assertion. ◊

Now, notice that $\mathrm{Ann}_B(a^i) \cap a^k B = a^k \cdot \mathrm{Ann}_B(a^{k+i})$ for every $i, k \in \mathbb{N}$; if $\mathrm{Ann}_B(a^k) = \mathrm{Ann}_B(a^{k+1})$, it follows that $\mathrm{Ann}_B(a^i) \cap a^k B = a^k \cdot \mathrm{Ann}_B(a^k) = 0$ for every $i > 0$. In particular, the a-adic topology of B induces the discrete topology on $\mathrm{Ann}_B(a^i)$, for every $i \in \mathbb{N}$, and taking into account claim 16.9.25, we see that the natural map $\mathrm{Ann}_B(a^i) \to \mathrm{Ann}_{B^\wedge}(a^i)$ is an isomorphism, for every $i \in \mathbb{N}$. Next, according to corollary 16.6.24, there exists $\beta := (b_k \mid k \in \mathbb{N}) \in \mathbf{E}(B^\wedge)$ such that $a/b_0 \in (B^\wedge)^\times$ and $p \in b_n B^\wedge$; then the a-adic topology of $B^\wedge$ agrees with its b_k-adic topology, for every $k \in \mathbb{N}$, and by corollary 16.3.71, $B_0^\wedge := B^\wedge/\mathrm{Ann}_{B^\wedge}(b_n)$ is perfectoid for its b_n-adic topology; moreover, $B^\wedge$ is reduced (corollary 16.3.63(i)), so $\mathrm{Ann}_{B^\wedge}(b_n) = \mathrm{Ann}_{B^\wedge}(b_0)$. By claim 16.9.25 and proposition 8.2.13(i,v), $B_0^\wedge$ is then naturally identified with the a-adic completion of B_0, *i.e.* B_0 is formally perfectoid for its a-adic topology. Recall as well that $(B^\wedge)^{\circ\circ} = (B^{\circ\circ})^\wedge$, and $B^{\circ\circ}$ is a radical ideal of B (remark 8.3.10(iv) and corollary 8.4.15(ii)); since $B^\wedge$ is perfectoid, it follows easily that B_1 is a perfect $\mathbb{F}_p$-algebra. Also, obviously the a-adic topology is discrete on B_2; in light of proposition 16.3.9(i) we deduce that B_2 is a perfect $\mathbb{F}_p$-algebra, and this completes the proof of (i).

(ii): The ring $B^\wedge$ is reduced (corollary 16.3.63(i)), so $\mathrm{Ann}_{B^\wedge}(a) = \mathrm{Ann}_{B^\wedge}(a^k)$ for every $k > 0$, and in view of the foregoing, we deduce that $\mathrm{Ann}_B(a) = \mathrm{Ann}_B(a^k)$ for every $k > 0$, so the image of a is regular in B_0. Let us endow $B_0[a^{-1}]$ with the f-adic topology such that B_0 is a subring of definition, and $aB_0 \subset B_0$ is an ideal of adic definition; then, let C be the p-root closure of B_0 in $B_0[a^{-1}]$, endow B_0 and C with the topologies induced by $B_0[a^{-1}]$, and denote by $B_0^\wedge$, $C^\wedge$ and $B_0[a^{-1}]^\wedge$ the respective completions. Notice that the quotient topology on C/B_0 is discrete, so the completion map $C/B_0 \to (C/B_0)^\wedge$ is an isomorphism; we need to check that $C = B_0$, and taking into account proposition 8.2.13(i,v), we are then reduced to showing that $B_0^\wedge = C^\wedge$. To this aim, let $(b_k \mid k \in \mathbb{N})$ be as in the proof of (i); we have already noticed that the topology of $B^\wedge$ is b_{n+1}-adic, so the same holds for the topology of $B_0^\wedge$, and clearly $B_0^\wedge[a^{-1}] = B_0^\wedge[b_{n+1}^{-1}]$. On the other hand, $C^\wedge$ is the p-root closure of $B_0^\wedge$ in $B_0[a^{-1}]^\wedge$ (lemma 9.8.26(ii)), and $B_0[a^{-1}]^\wedge = B_0^\wedge[a^{-1}]$ (proposition 8.3.33(iii)), so finally $C^\wedge$ is the p-root closure of $B_0^\wedge$ in $B_0^\wedge[b_{n+1}^{-1}]$. Since $B_0^\wedge$ is perfectoid, the assertion then follows from lemma 9.8.26(i) and theorem 16.4.1.

(iii): By the foregoing, we see that the natural map

$$\mathrm{Ann}_B(a) \cap B^{\circ\circ} \to J := \mathrm{Ann}_{B^\wedge}(b_0) \cap (B^\wedge)^{\circ\circ}$$

is injective. However $J = 0$: indeed, if $x \in J$, we have $x^n = b_0 c$ for some $n \in \mathbb{N}$ and some $c \in B^\wedge$, and on the other hand $b_0 x = 0$, whence $x^{n+1} = 0$, so $x = 0$, as $B^\wedge$ is reduced (corollary 16.3.63(i)). So finally $\mathrm{Ann}_B(a) \cap B^{\circ\circ} = 0$, and the assertion follows easily.

(iv): Since B_1 and B_2 are perfect, they are reduced, so their a-torsion is bounded. The same holds trivially for the a-torsion of B_0, due to (ii). This shows that $B_0, B_1, B_2 \in \mathrm{Ob}(\mathscr{C}(A,a))$. The remaining identity follows easily from the definitions. □

Proposition 16.9.26. *In the situation of* (16.9.23)*, suppose that* $A \in \mathrm{Ob}(\mathscr{C}(A,a))$*. Then the functor* F_P *is represented by a faithfully flat* A*-algebra* $D_P \in \mathrm{Ob}(\mathscr{C}(A,a))$.

Proof. Let J be the radical of the ideal aA, and $I := \mathrm{Ann}_A(a)$; set $A_1 := A/J$ and $A_2 := A/(I+J)$. By lemma 16.9.24 we get a cartesian diagram of rings :

$$\begin{array}{ccc} A & \longrightarrow & A_0 \\ \big\downarrow & & \big\downarrow \\ A_1 & \longrightarrow & A_2 \end{array}$$

whose arrows are the natural projections, where A_1 and A_2 are perfect $\mathbb{F}_p$-algebras, and A_0 is formally perfectoid for its a-adic topology; moreover, a is regular in A_0. Let $\overline{P}$ denote the image of P in $A_2[X]$; according to theorem 16.9.17, the polynomial P acquires a root in a faithfully flat A_0-algebra D_0 that is formally perfectoid for its a-adic topology, and such that we have an isomorphism $A_2 \otimes_{A_0} D_0 \xrightarrow{\sim} D_2 := (A_2[X]/(\overline{P}))^{\mathrm{perf}}$ of $\overline{A}_0$-algebras. Let then also $Q \in A_1[X]$ be the image of P, and $D_1 := (A_1[X]/(Q))^{\mathrm{perf}}$; we notice :

Claim 16.9.27. (i) The natural map $A_2 \otimes_{A_1} D_1 \to D_2$ is an isomorphism.

(ii) D_1 is a faithfully flat A_1-algebra.

Proof of the claim. (i): The functor $(-)^{\mathrm{perf}}$ of (9.8.11) commutes with all colimits, since it is a left adjoint; in particular, the natural map $A_2^{\mathrm{perf}} \otimes_{A_1^{\mathrm{perf}}} D_1 \to D_2$ is an isomorphism. But $A_2 = A_2^{\mathrm{perf}}$ and $A_1 = A_1^{\mathrm{perf}}$, whence the assertion.

(ii): Let $j_0 : A_1 \to D_{1,0}$ be the structure map, and say that $Q = X^k + a_1 X^{k-1} + \cdots + a_k$ with $a_1, \dots, a_k \in A_1$; it suffices to show that the composition $j_n : A_1 \to D_{1,n}$ of j_0 with $\varphi_{n-1} \circ \cdots \circ \varphi_0 : D_{1,0} \to D_{1,n}$ is faithfully flat for every $n \in \mathbb{N}$. But since A_1 is a perfect $\mathbb{F}_p$-algebra, j_n identifies $D_{1,n}$ with the A_1-algebra $A_1[X]/(Q_n)$, where $Q_n := X^k + a_1^{1/p^n} X^{k-1} + \cdots + a_k^{1/p^n}$, whence the assertion. ◊

In view of claim 16.9.27 and [75, Prop.3.4.21] the A-algebra $D_P := D_0 \times_{D_2} D_1$ is flat, and the projections $D_P \to D_i$ induce isomorphisms $A_i \otimes_A D_P \xrightarrow{\sim} D_i$ of A_i-algebras, for $i = 0, 1, 2$. Hence, $\mathrm{Ann}_{D_P}(a^n) = D_P \otimes_A \mathrm{Ann}_A(a^n)$ for every $n \in \mathbb{N}$, and since the a-torsion of A is bounded, the same then holds for the a-torsion of D_P. We endow D_P with the topology $\mathscr{T}$ of the fibre product of the formally perfectoid ring D_0 with the discrete topological $\mathbb{F}_p$-algebra D_1, over the discrete topological $\mathbb{F}_p$-algebra D_2. It is easily seen that $\mathscr{T}$ coincides with the a-adic topology, and the completion $(D, \mathscr{T})^\wedge$ of $(D, \mathscr{T})$ is the fibre product $D_0^\wedge \times_{D_2} D_1$, where $D_0^\wedge$ denotes the completion of D_0. Hence, D_P is formally perfectoid, by proposition 16.3.25; this shows that $D_P \in \mathrm{Ob}(\mathscr{C}(A, a))$. In order to check that D_P is faithfully flat, let M be any non-zero A-module, and set $M_i := A_i \otimes_A M$ for $i = 0, 1$; it suffices to show that $D_P \otimes_A M \neq 0$ ([126, Th.7.2]), and according to [75, Lemma 3.4.18], we have either $M_0 \neq 0$ or $M_1 \neq 0$, so either $D_0 \otimes_A M \neq 0$ or $D_1 \otimes_A M \neq 0$, since D_i is a faithfully flat A_i-algebra for $i = 0, 1$, whence the assertion.

It remains to check that D_P represents the functor F_P. To this aim, recall first that D_0 is the p-root closure of $C_0 := A_0[X^{1/p^\infty}]/(P)$ in $C_0[a^{-1}]$; hence the image of $\delta := (X^{1/p^n} \mid n \in \mathbb{N})$ in $\mathbf{E}(D_0)$ is an element $\delta_0 \in F_P D_0$. Likewise, the image of δ in $\mathbf{E}(D_1)$ is an element $\delta_1 \in F_P D_1$, and clearly the images of δ_0 and δ_1 coincide in $F_P D_2$, so the pair (δ_0, δ_1) is a well defined element $\delta_P \in F_P D_P$ (lemma 16.9.24(iv)). Now, let B be any object of $\mathscr{C}(A, a)$, and define B_0, B_1, B_2 as in lemma 16.9.24; we easily get a natural identification :

$$\mathrm{Hom}_{A\text{-}\mathsf{Alg}}(D_P, B) \xrightarrow{\sim} \mathrm{Hom}_{A_0\text{-}\mathsf{Alg}}(D_0, B_0) \times_{\mathrm{Hom}_{A_2\text{-}\mathsf{Alg}}(D_2, B_2)} \mathrm{Hom}_{A_1\text{-}\mathsf{Alg}}(D_1, B_1)$$

(detail left to the reader). Taking into account lemma 16.9.24(iv), we are then reduced to checking that for $i = 0, 1$, and every $\beta_i \in F_P B_i$, there exists a unique map of A_i-algebras $f_i : D_i \to B_i$ such that $F_P(f_i)(\delta_i) = \beta_i$. This is clear for $i = 1$, since B_1 is perfect and D_1 is the perfection of $A_1[X]/(Q)$. Lastly, say that $\beta_0 = (b_n \mid n \in \mathbb{N})$, so that $b_{n+1}^p = b_n$ for every $n \in \mathbb{N}$. Then there is a unique map of A_0-algebras $g : A_0[X^{1/p^\infty}] \to B_0$ such that $g(X^{1/p^n}) = b_n$ for every $n \in \mathbb{N}$; since $P(b_0) = 0$, the map g factors uniquely through a map $h : C_0 \to B_0$. But according to lemma 16.9.25(ii), B_0 is p-root closed in $B_0[a^{-1}]$; in view of remark 9.8.25(ii), the map h then extends to a map $f_0 : D_0 \to B_0$, which is necessarily the unique map whose restriction to C_0 agrees with h, since $A_0[a^{-1}] \otimes_{A_0} h = A_0[a^{-1}] \otimes_{A_0} f_0$, and since the localization $B_0 \to B_0[a^{-1}]$ is injective. □

Remark 16.9.28. (i) In the situation of proposition 16.9.26, let $f : (A, a) \to (A', a')$ be any morphism of $\mathscr{C}(A, a)$, and denote by $f(P)$ the image of P in $A'[X]$. Then we can consider as well the category $\mathscr{C}(A', a')$, the corresponding functor $F_{f(P)} : \mathscr{C}(A', a) \to \mathbf{Set}$, and the representing A'-algebra $D_{f(P)}$. Let also $\delta \in F_P D_P$ and $\delta' \in F_{f(P)} D_{f(P)}$ be the respective universal elements. Notice that $A' \otimes_A D_P$ is formally perfectoid for its a'-adic topology (proposition 16.3.9(i)), and $\mathrm{Ann}_{A' \otimes_A D_P}(a'^n) = D_P \otimes_A \mathrm{Ann}_{A'}(a'^n)$ for every $n \in \mathbb{N}$, since D_P is a faithfully flat A-algebra; hence $A' \otimes_A D_P \in \mathrm{Ob}(\mathscr{C}(A', a'))$, and by comparing the respective universal properties, it follows easily that there exists a unique isomorphism of A'-algebras

$$\omega : D_{f(P)} \xrightarrow{\sim} A' \otimes_A D_P \qquad \text{such that} \qquad (F_{f(P)}\omega)(\delta') = 1 \otimes \delta.$$

(ii) We may also consider the following variant of the situation of (16.9.23). Let A be a ring, $I \subset A$ an ideal of finite type, and suppose that A is perfectoid for its I-adic topology; let also $P \in A[X]$ be a non-constant monic polynomial. We associate with P the functor

$$G_P : A\text{-Perf} \to \mathbf{Set} \qquad B \mapsto \{(\beta_n \mid n \in \mathbb{N}) \in \mathbf{E}(B) \mid P(\beta_0) = 0\}$$

defined just as F_P. Then we claim that G_P is represented by an adic and adically faithfully flat map $A \to D_P^\wedge$ of perfectoid rings (see definition 8.3.23(i,iii)). Indeed, recall that there exists $a \in A^{\circ\circ}$ such that $pA = a^p A$ (lemma 16.2.3(iv)), and A is perfectoid for its a-adic topology (which is just the p-adic topology : proposition 16.3.8(i)), hence we may consider the category $\mathscr{C}(A, a)$ and the functor $F_P : \mathscr{C}(A, a) \to \mathbf{Set}$; moreover, $A \in \mathrm{Ob}(\mathscr{C}(A, a))$ (corollary 16.3.75) so F_P is represented by a faithfully flat A-algebra $D_P \in \mathrm{Ob}(\mathscr{C}(A, a))$, and its I-adic completion $D_P^\wedge$ is perfectoid, by proposition 16.3.8(ii). Now, if $A \to B$ is any continuous map of perfectoid rings, then $B \in \mathrm{Ob}(\mathscr{C}(A, a))$, by proposition 16.3.8(i) and corollary 16.3.75, and on the other hand, every map of A-algebras $D_P \to B$ extends uniquely to a continuous map $D_P^\wedge \to B$; it follows easily that $D_P^\wedge$ represents G_P.

The following result strenghtens [25, Th.7.12].

Proposition 16.9.29. *For every perfectoid $\mathbb{F}_p$-algebra E there exists an adic and adically faithfully flat map $E \to D$ of perfectoid rings such that the following holds. For every non-constant monic polynomial $P \in W(D)[X]$ and every distinguished ideal $\mathscr{I} \subset W(D)$ there exists $\underline{w} \in W(D)$ such that $P(\underline{w}) \in \mathscr{I}$.*

Proof. Let $I \subset E$ be an adic ideal of definition; let $\mathscr{F}$ be the set of all non-constant monic polynomials in $W(E)[X]$, and $\mathscr{G}$ the set of all distinguished ideals of $W(E)$. For every $(P, \mathscr{I}) \in \mathscr{F} \times \mathscr{G}$, let $A_\mathscr{I} := W(E)/\mathscr{I}$, define the functor $G_P : A_\mathscr{I} \to \mathbf{Set}$ as in remark 16.9.28(ii) (so here we identify P with its image in $A_\mathscr{I}[X]$), and denote by $D_{P,\mathscr{I}}^\wedge$ the adic and adically faithfully flat perfectoid $A_\mathscr{I}$-algebra that represents G_P. We have a natural identification $E \xrightarrow{\sim} \mathbf{E}(A_\mathscr{I})$ for every $\mathscr{I} \in \mathscr{G}$ (example 16.3.2(ii)), so $\mathbf{E}(D_{P,\mathscr{I}}^\wedge)$ is naturally an E-algebra, for every $(P, \mathscr{I}) \in \mathscr{F} \times \mathscr{G}$; moreover, the structure map $E \to \mathbf{E}(D_{P,\mathscr{I}}^\wedge)$ is adic and adically faithfully flat, by remark 16.4.20(i). We denote by $\mathscr{E}(A)$ the coproduct of the family of E-algebras $(\mathbf{E}(D_{P,\mathscr{I}}^\wedge) \mid (P, \mathscr{I}) \in \mathscr{F} \times \mathscr{G})$, and we let $\mathscr{E}(A)^\wedge$ be the I-adic completion of $\mathscr{E}(A)$. Since each $\mathbf{E}(D_{P,\mathscr{I}}^\wedge)$ is a perfect ring, it is easily seen that the same holds for $\mathscr{E}(A)$, and then the induced map $E \to \mathscr{E}(A)^\wedge$ is an adic and adically flat ring homomorphism of perfectoid $\mathbb{F}_p$-algebras (remark 9.4.9(v)). Next, by transfinite induction we define, for every ordinal, a perfectoid E-algebra as follows : $\mathscr{E}_0(A) := E$, $\mathscr{E}_{\alpha+1}(A) := \mathscr{E}(\mathscr{E}_\alpha(A))^\wedge$ for every successor ordinal $\alpha + 1$, and $\mathscr{E}_\alpha(A)$ is the I-adic completion of $\mathrm{colim}_{\beta<\alpha}\, \mathscr{E}_\beta(A)$, for every limit ordinal α. We observe :

Claim 16.9.30. Let $(\Lambda, \leq)$ be a partially ordered set such that the following holds : for every finite or countable subset $S \subset \Lambda$ there exists $\lambda \in \Lambda$ such that $\lambda \geq \mu$ for every $\mu \in S$. We have :

(i) Let R be a ring, $J \subset R$ an ideal, $M_\bullet := (M_\lambda \mid \lambda \in \Lambda)$ a system of J-adically complete and separated R-modules. Then the colimit M of $M_\bullet$ is J-adically complete and separated.

(ii) Let $R_\bullet := (R_\lambda \mid \lambda)$ be a system of rings, and R the colimit of $R_\bullet$. Then the natural map $\tau : \mathrm{colim}_{\lambda\in\Lambda}\, W(R_\lambda) \to W(R)$ is a ring isomorphism.

Proof of the claim. (i): To check that M is separated, let $x \in M$ such that $x \in J^n M$ for every $n \in \mathbb{N}$; then x is represented by an element $y \in M_\lambda$, for some $\lambda \in \Lambda$, and the condition means that for every $n \in \mathbb{N}$ there exists $\lambda(n) \in \Lambda$ such that $\lambda(n) \geq \lambda$, and the image of y in $M_{\lambda(n)}$ lies in $I^n M_{\lambda(n)}$. By assumption, there exists $\mu \in \Lambda$ such that $\mu \geq \lambda(n)$ for every $n \in \mathbb{N}$, so the image y_μ of y in M_μ lies in $\bigcap_{n\in\mathbb{N}} J^n M_\mu$. Since M_μ is separated, we get $y_\mu = 0$, whence $x = 0$, as required. Next, let $x_\bullet := (x_n \mid n \in \mathbb{N})$ be a Cauchy sequence in M; after replacing $x_\bullet$ by a subsequence, we may assume that $x_{n+1} - x_n \in J^n M$ for every $n \in \mathbb{N}$. Each x_n is represented by an element $y_n \in M_{\mu(n)}$ for some $\mu(n) \in \Lambda$; then we find $\mu \in \Lambda$ such that $\mu \geq \mu(n)$ for every $n \in \mathbb{N}$, and after replacing each y_n by its image in M_μ, we may assume that $x_\bullet$ is the image of a sequence $y_\bullet$ in M_μ. For every $n \in \mathbb{N}$ and every $\lambda \in \Lambda$ with $\lambda \geq \mu$, let us denote by $y_{n,\lambda}$ the image of y_n in M_λ. With this notation, for every $n \in \mathbb{N}$ we can then find $\lambda(n) \in \Lambda$ such that $\lambda(n) \geq \mu$, and such that $y_{n+1,\lambda(n)} - y_{n,\lambda(n)} \in J^n M_{\lambda(n)}$. Pick $\lambda \in \Lambda$ such that $\lambda \geq \lambda(n)$ for every $n \in \mathbb{N}$; then $y_{n+1,\lambda} - y_{n,\lambda} \in J^n M_\lambda$ for every $n \in \mathbb{N}$, so the sequence $(y_{n,\lambda} \mid n \in \mathbb{N})$ is Cauchy in M_λ, and therefore it admits a limit y. Clearly, the image of y in M is a limit for $x_\bullet$.

(ii): Let $\lambda \in \Lambda$, and $\underline{w} := (w_n \mid n \in \mathbb{N}) \in W(R_\lambda)$ whose image in $\mathscr{W} := \mathrm{colim}_{\lambda \in \Lambda} W(R_\lambda)$ lies in $\mathrm{Ker}(\tau)$; then for each $n \in \mathbb{N}$ there exists $\lambda(n) \in \Lambda$ such that $\lambda(n) \geq \lambda$, and the image of w_n vanishes in $R_{\lambda(n)}$. Pick $\mu \in \Lambda$ such that $\mu \geq \lambda(n)$ for every $n \in \mathbb{N}$; then clearly the image of $\underline{w}$ vanishes in $W(R_\mu)$, so also in $\mathscr{W}$. This shows that τ is injective. Lastly, let $\underline{u} := (u_n \mid n \in \mathbb{N}) \in W(R)$; then for every $n \in \mathbb{N}$ there exists $\lambda(n) \in \Lambda$ and an element in $R_{\lambda(n)}$ whose image in R agrees with u_n. Pick again $\lambda \in \Lambda$ with $\lambda \geq \lambda(n)$ for every $n \in \mathbb{N}$; then clearly $\underline{u}$ is the image of an element of $W(R_\lambda)$, so τ is surjective. ◊

Let ω_1 be the smallest uncountable ordinal; we claim that $D := \mathscr{E}_{\omega_1}(A)$ will do. Indeed, let $P \in W(D)[X]$ be a non-constant monic polynomial, and $\underline{\alpha} := (\alpha_n \mid n \in \mathbb{N}) \in W(D)$ a distinguished element; according to claim 16.9.30(i), D represents the direct limit of the system of E-algebras $(\mathscr{E}_\beta(A) \mid \beta < \omega_1)$, so we may assume that $P \in \mathscr{E}_\beta(A)[X]$ for some $\beta < \omega_1$. Moreover, $W(D) = \mathrm{colim}_{\beta<\omega_1} W(\mathscr{E}_\beta(A))$, by claim 16.9.30(ii), so that we may also assume that $\underline{\alpha} \in W(\mathscr{E}_\beta(A))$. Since the topology of D is I-adic, we have $\alpha_0^k \in ID$ for some $k \in \mathbb{N}$, and $\alpha_1 \in D^\times$; then, up to replacing β by a larger ordinal, we may assume as well that $\alpha_0^k \in I\mathscr{E}_\beta(A)$ and $\alpha_1 \in \mathscr{E}_\beta(A)^\times$, so that $\underline{\alpha}$ is distinguished in $W(\mathscr{E}_\beta(A))$. By construction, there exists therefore $\underline{w} \in W(\mathscr{E}_{\beta+1}(A))$ such that $P(\underline{w}) \in \underline{\alpha} W(\mathscr{E}_{\beta+1}(A))$, and since $\beta + 1 < \omega_1$, the proposition follows. □

16.9.31. Let A be a perfectoid Tate ring, $A_0 \subset A$ a perfectoid subring of definition, and $(A_0, \mathfrak{m})$ a basic setup in the sense of [75, §2.1.1], such that $\mathfrak{m}$ is the filtered union of a countable system of principal subideals. If $\mathfrak{m}$ is an open radical ideal of A_0, then we have attached to $\mathfrak{m}$ an open radical ideal $\mathfrak{m}_\mathbf{E}$ of $\mathbf{E}_0 := \mathbf{E}(A_0/pA_0)$, so that $(\mathbf{E}_0, \mathfrak{m}_\mathbf{E})$ is a basic setup as well (see proposition 16.8.15). One can show that here $\mathfrak{m}$ is still a radical ideal; however, it is not necessarily open, but nevertheless we can construct a useful basic setup $(\mathbf{E}_0, \mathfrak{m}_\mathbf{E})$ as follows. By assumption, $\mathfrak{m}$ is the filtered union of a countable system of principal ideals; hence, we may find a generating system $(a_n \mid n \in \mathbb{N})$ for $\mathfrak{m}$ such that for every $n \in \mathbb{N}$ there exists an element $c_n \in A$ with $a_n = c_n a_{n+1}^p$. For every $x \in A_0$, let $\overline{x} \in A_0/pA_0$ be the class of x; since A_0 is perfectoid, for every $n \in \mathbb{N}$ there exists $(\overline{x}_{n,k} \mid k \in \mathbb{N}) \in \mathbf{E}_0$ with $\overline{x}_{n,0} = \overline{c}_n$ (recall that $\overline{x}_{n,k+1}^p = \overline{x}_{n,k}$ for every $k \in \mathbb{N}$). Set

$$\alpha_{n,k} := \overline{a}_{n+k} \cdot \overline{x}_{n,k} \cdot \overline{x}_{n+1,k-1} \cdots \overline{x}_{n+k-1,1} \qquad \text{for every } n \in \mathbb{N}.$$

Thus, $\alpha_{n,0} = \overline{a}_n$ and it is easily seen that $\alpha_{n,\bullet} := (\alpha_{n,k} \mid k \in \mathbb{N}) \in \mathbf{E}_0$ for every $n \in \mathbb{N}$; moreover, $\alpha_{n,\bullet} \in \alpha_{n+1,\bullet}^p \mathbf{E}_0$ for every $n \in \mathbb{N}$. Let then $\mathfrak{m}_\mathbf{E} \subset \mathbf{E}_0$ be the ideal generated by the system $(\alpha_{n,\bullet} \mid n \in \mathbb{N})$; it follows that $(\mathbf{E}_0, \mathfrak{m}_\mathbf{E})$ is a basic setup.

Remark 16.9.32. (i) In the situation of (16.9.31), we have :

$$\bigcap_{n\in\mathbb{N}} (\overline{u}_{A_0}(\mathfrak{m}_\mathbf{E}) \cdot A_0 + p^n A_0) = \bigcap_{n\in\mathbb{N}} (\mathfrak{m} + p^n A_0). \tag{16.9.33}$$

Indeed, on the one hand, $\overline{u}_{A_0}(\alpha_{n,\bullet})$ is the limit of the p-adically convergent sequence

$$((a_{n+k} x_{n,k} \cdots x_{n+k-1,1})^{p^k} \mid k \in \mathbb{N})$$

hence it lies in the p-adic topological closure of $\mathfrak{m}$. On the other hand, by construction for every $x \in \mathfrak{m}$ there exists $y \in \overline{u}_{A_0}(\mathfrak{m}_E)$ such that $x - y \in pA_0$; then $x^{p^n} - y^{p^n} \in p^n A_0$ for every $n \in \mathbb{N}$ (lemma 9.3.4(i)), and since $\mathfrak{m}$ is generated by $(x^{p^n} \mid x \in \mathfrak{m})$, we deduce the converse inclusion.

(ii) Quite generally, let $(V, \mathfrak{m})$ be a basic setup in the sense of [75, §2.1.1], such that $\mathfrak{m}$ is a filtered countable union of principal subideals; then it is easily seen that we may write $\mathfrak{m} = \bigcup_{n \in \mathbb{N}} V a_n$ for a system $(a_n \mid n \in \mathbb{N})$ of elements of A such that $Va_n \subset \mathfrak{m} \cdot a_{n+1}$ for every $n \in \mathbb{N}$: the details are left to the reader. Hence, for every $n \in \mathbb{N}$ let $c_n \in \mathfrak{m}$ such that $a_n = c_n a_{n+1}$. We consider the inductive system $L_\bullet := (L_n \mid n \in \mathbb{N})$ with $L_n := V$ for every $n \in \mathbb{N}$, with V-linear transition map $\varphi_n : L_n \to L_{n+1}$ such that $\varphi_n(1) := c_n$ for every $n \in \mathbb{N}$. The inductive limit L of $L_\bullet$ is the set of equivalence classes $[x, n]$ of all pairs (x, n) with $n \in \mathbb{N}$ and $x \in L_n$. We have an obvious V-linear surjection

$$\psi : L \to \mathfrak{m} \qquad [x, n] \mapsto xa_n.$$

Notice that if $[x, n] \in \operatorname{Ker} \psi$, then $a_n \cdot [x, n] = [a_n x, n] = 0$; on the other hand, for every $k \geq n$ there exists $y \in V$ such that $[x, n] = [y, k]$, whence $a_k \cdot [x, n] = a_k \cdot [y, k] = 0$ as well. This shows that $\mathfrak{m} \cdot \operatorname{Ker} \psi = 0$. Therefore, $\mathfrak{m} \otimes_V \psi : \mathfrak{m} \otimes_V L \to \widetilde{\mathfrak{m}} := \mathfrak{m} \otimes_V \mathfrak{m}$ is an isomorphism. But by construction, L is a flat V-module, hence the natural map $\mathfrak{m} \otimes_V L \to \mathfrak{m}L$ is an isomorphism; lastly, for every $[x, n] \in L$ we have $[x, n] = c_n[x, n+1]$, so $\mathfrak{m}L = L$. Summing up, $\mathfrak{m} \otimes_V \psi$ is naturally identified with the isomorphism :

$$L \xrightarrow{\sim} \widetilde{\mathfrak{m}} \qquad [x, n] \mapsto x \otimes a_n.$$

(iii) Furthermore, the functor $\mathbf{E}$ induces a functor on almost algebras

$$\mathbf{E}^a : (A_0, \mathfrak{m})^a\text{-}\mathsf{Alg} \to (\mathbf{E}_0, \mathfrak{m}_\mathbf{E})^a\text{-}\mathsf{Alg} \qquad R^a \mapsto \mathbf{E}(R/pR)^a.$$

Indeed, let $f : R \to S$ be a homomorphism of A_0-algebras such that $f^a : R^a \to S^a$ is an isomorphism of $(A_0, \mathfrak{m})^a$-algebras, and set $f_0 := f \otimes_\mathbb{Z} \mathbb{Z}/p\mathbb{Z}$; we need to check that $\mathbf{E}(f_0)^a$: $\mathbf{E}(R/pR)^a \to \mathbf{E}(S/pS)^a$ is an isomorphism of $(\mathbf{E}_0, \mathfrak{m}_\mathbf{E})^a$-algebras. However, $\mathbf{E}(f_0)$ is the limit of the system of rings $(R_n \mid n \in \mathbb{N})$, with $R_n := R/pR$ for every $n \in \mathbb{N}$, and with transition map $R_{n+1} \to R_n$ given by the Frobenius endomorphism $\Phi_{R/pR}$. Now, if $g : A_0 \to R$ is the structure map of the A_0-algebra R, then R_0 is naturally an $\mathbf{E}_0$-algebra, via the composition $g' : \mathbf{E}_0 \to R_0$ of the projection $\overline{u}_{A_0/pA_0} : \mathbf{E}_0 \to A_0/pA_0$ and $g \otimes_\mathbb{Z} \mathbb{Z}/p\mathbb{Z} : A_0/pA_0 \to R_0$. Let $\Phi_{\mathbf{E}_0}$ be the Frobenius automorphism of $\mathbf{E}_0$; then for every $n \in \mathbb{N}$, the ring R_n is an $\mathbf{E}_0$-algebra with structure morphism $g' \circ \Phi_{\mathbf{E}_0}^{-n} : \mathbf{E}_0 \to R_n$, *i.e.* $R_n = (R_0)_{(\Phi_{\mathbf{E}_0}^{-n})}$, and $\Phi_{R/pR} : R_{n+1} \to R_n$ is a homomorphism of $\mathbf{E}_0$-algebras, for these $\mathbf{E}_0$-algebra structures. Likewise, $\mathbf{E}(S/pS)$ is the limit of a system of $\mathbf{E}_0$-algebras $(S_n \mid n \in \mathbb{N})$, and f induces a system of maps of $\mathbf{E}_0$-algebras $f_n := f_0 : R_n \to S_n$, whose inverse limit is $\mathbf{E}(f_0)$. Since f^a is an isomorphism, and since $\Phi_{\mathbf{E}_0} : (\mathbf{E}_0, \mathfrak{m}_\mathbf{E}) \xrightarrow{\sim} (\mathbf{E}_0, \mathfrak{m}_\mathbf{E})$ is an isomorphism of basic setups, it is easily seen that $f_n^a : R_n^a \to S_n^a$ is an isomorphism of $(\mathbf{E}_0, \mathfrak{m}_\mathbf{E})^a$-algebras, for every $n \in \mathbb{N}$; on the other hand, the functor $(-)^a$ commutes with limits, since it has a left adjoint, whence the contention.

16.9.34. Let A be a Tate ring, $A_0 \subset A$ a subring of definition, $p \in \mathbb{N}$ a prime integer, and $A_1 \subset A$ the p-root closure of A_0 in A (see definition 9.8.24(ii)). Moreover, suppose there exists $b \in A_0^{\circ\circ} \cap A^\times$ with $pA_0 \subset b^p A_0$, and let $A_0^\wedge$ and $A_1^\wedge$ be the b-adic completions of A_0 and A_1. Notice that the Frobenius endomorphism Φ_{A_i} of $A_i/b^p A_i$ induces a ring homomorphism

$$\overline{\Phi}_{A_i} : A_i/bA_i \to A_i/b^p A_i \qquad \text{for } i = 0, 1.$$

Furthermore, let $(A_0, \mathfrak{m})$ be a basic setup, such that $\mathfrak{m}$ is the filtered union of a countable system of principal subideals, and denote by $\overline{\mathfrak{m}}$ the image of $\mathfrak{m}$ in $\overline{A}_0 := A_0/pA_0$. Then $\mathfrak{m}$ is generated by the system $(x^p \mid x \in \mathfrak{m})$ ([75, Prop.2.17]), and therefore the Frobenius endomorphism Φ of $\overline{A}_0$ is a morphism of basic setups in the sense of [75, §3.5]

$$\Phi : (\overline{A}_0, \overline{\mathfrak{m}}) \to (\overline{A}_0, \overline{\mathfrak{m}}).$$

As detailed in [75, §3.5.7], the latter induces a pull-back functor

$$(\overline{A}_0, \overline{\mathfrak{m}})^a\text{-}\mathsf{Alg} \to (\overline{A}_0, \overline{\mathfrak{m}})^a\text{-}\mathsf{Alg} \qquad B \mapsto B_{(\Phi)}.$$

Definition 16.9.35. In the situation of (16.9.34), we say that the basic setup $(A_0, \mathfrak{m})$ is *almost perfectoid* if A_0 is complete and separated, and the Frobenius endomorphism of A_0/b^pA_0 induces an isomorphism of $(A_0, \mathfrak{m})^a$-algebras $\overline{\Phi}^a_{A_0} : (A_0/bA_0)^a \xrightarrow{\sim} (A_0/b^pA_0)^a_{(\Phi)}$.

Remark 16.9.36. The following proposition 16.9.37(i) implies in particular that definition 16.9.35 is independent of the choice of the element $b \in A_0$ such that $p \in b^pA_0$: if $b' \in A_0$ is another such element, and if the b-adic topology on A_0 agrees with the b'-adic topology, then $\overline{\Phi}^a_{A_0}$ is an isomorphism if and only if the same holds for the corresponding morphism of $(A_0, \mathfrak{m})^a$-algebras $(A_0/b'A_0)^a \xrightarrow{\sim} (A_0/b'^pA_0)^a_{(\Phi)}$. See also lemma 17.5.10.

Proposition 16.9.37. (i) *In the situation of* (16.9.34)*, the following conditions are equivalent :*

(a) *The basic setup* $(A_0^\wedge, \mathfrak{m}A_0^\wedge)$ *is almost perfectoid.*

(b) *The basic setup* $(A_1^\wedge, \mathfrak{m}A_1^\wedge)$ *is almost perfectoid, and the inclusion map* $j : A_0 \to A_1$ *induces isomorphisms of* $(A_0, \mathfrak{m})^a$*-algebras* $j^a : A_0^a \xrightarrow{\sim} A_1^a$*.*

(c) *The topological closure* $\overline{C} \subset A_0$ *of the subring* $C := \mathbb{Z}[a^p \mid a \in A_0]$ *contains* $\mathfrak{m}$*, and* j *induces an isomorphism of* $(A_0, \mathfrak{m})^a$*-algebras* $j^a : A_0^a \xrightarrow{\sim} A_1^a$*.*

(ii) *If* A_0 *is integrally closed (resp.* p*-root closed) in* A*, then* A^a_{0*} *is integrally closed (resp.* p*-root closed) in* $A^a_{0*}[b^{-1}]$*.*

(iii) *If* $(A_0, \mathfrak{m})$ *is an almost perfectoid basic setup, the same holds for* $(A^a_{0*}, \mathfrak{m}A^a_{0*})$*.*

(iv) *If* $(A_0, \mathfrak{m})$ *is an almost perfectoid basic setup, and the unit of adjunction* $A_0 \to (A_0)^a_*$ *is an isomorphism, then* $A_0 = A_1$*.*

Proof. By simple inspection we see that (i.b)⇒(i.a). Conversely, suppose that (i.a) holds, and let also $j^\wedge : A_0^\wedge \to A_1^\wedge$ be the b-adic completion of j; in order to show (i.b), it suffices to check :

Claim 16.9.38. $\mathrm{Ker}\, \overline{\Phi}^a_{A_0} = 0 \Leftrightarrow j^a$ is an isomorphism $\Leftrightarrow (j^\wedge)^a$ is an isomorphism.

Proof of the claim. Suppose first that $\mathrm{Ker}\, \overline{\Phi}^a_{A_0} = 0$. We can construct A_1 as the increasing union of the ascending chain $(A_{0,n} \mid n \in \mathbb{N})$ of subrings of A defined inductively as follows: $A_{0,0} := A_0$, and $A_{0,n} := A_{0,n-1}[\Sigma_n]$, with $\Sigma_n := \{x \in A \mid x^p \in A_{0,n-1}\}$, for every $n > 0$. Thus, we are reduced to showing that the inclusion $A_0 \to A_{0,n}$ induces an isomorphism $A_0^a \xrightarrow{\sim} A^a_{0,n}$ for every $n \in \mathbb{N}$. However, if for some $n > 0$ the inclusion $j_{n-1} : A_{0,n-1} \to A_{0,n}$ induces an isomorphism $j^a_{n-1} : A^a_{0,n-1} \xrightarrow{\sim} A^a_{0,n}$, then the same holds for $j_n : A_{0,n} \to A_{0,n+1}$: indeed, let $x \in \Sigma_{n+1}$; since $x^p \in A_{0,n}$, by assumption we have $(ax)^p \in A_{0,n-1}$ for every $a \in \mathfrak{m}$, whence $ax \in \Sigma_n \subset A_{0,n}$. Thus, we are further reduced to checking that j_0 induces an isomorphism $j_0^a : A_0^a \xrightarrow{\sim} A^a_{0,1}$, *i.e.* that $(\mathrm{Coker}\, j_0)^a = 0$. We argue as in the proof of lemma 9.8.26(i) : let $x \in A_1 \setminus A_0$, and denote by $m \in \mathbb{N}$ the smallest integer such that $b^m x \cdot \mathfrak{m} \not\subset A_0$. Hence, $y := b^{mp}x^p \in A_0$, and $(b^{m+1}x)^p = b^p y^p \in b^pA_0$. By assumption, for every $a \in \mathfrak{m}$ we then have $z \in A_0$ such that $ab^{m+1}x = bz$, whence $ab^m x \in A_0$, contradicting the choice of m.

Conversely, suppose that j^a is an isomorphism, and let $x \in A_0$ be an element whose class $\overline{x} \in A_0/bA_0$ lies in $\mathrm{Ker}\, \overline{\Phi}_{A_0}$; then $x^p \in b^pA_0$, so $(x/b)^p \in A_0$, and our assumption implies that $\mathfrak{m} \cdot x \in bA_0$, *i.e.* $\mathfrak{m} \cdot \overline{x} = 0$.

Next, if j^a is an isomorphism, the same holds for $j^a_k := A_0^a/b^kA_0 \otimes_{A_0^a} j^a : A_0^a/b^kA_0^a \to A_1^a/b^kA_1^a$, for every $k \in \mathbb{N}$, and then also for the limit of the inverse system $(j^a_k \mid k \in \mathbb{N})$. But the functor $(-)^a$ commutes with limits, since it has a left adjoint, so $(j^\wedge)^a$ is an isomorphism.

Conversely, if $(j^\wedge)^a$ is an isomorphism, the induced map $A_0^\wedge[b^{-1}]/A_0^\wedge \to A_1^\wedge[b^{-1}]/A_1^\wedge$ is an almost isomorphism of A_0-modules. But proposition 8.3.33(iii) easily implies that the natural map $A_0[b^{-1}]/A_0 \to A_0^\wedge[b^{-1}]/A_0^\wedge$ is bijective, and similarly for A_1; hence j induces isomorphisms $A_0^a[b^{-1}]/A_0^a \xrightarrow{\sim} A_1^a[b^{-1}]/A_1^a$ and $A_0[b^{-1}] \xrightarrow{\sim} A_1[b^{-1}]$. By the 5-lemma, it follows easily that j^a is an isomorphism. ◊

Next, let $\mathfrak{m}_{\overline{C}} \subset \overline{C}$ be the ideal generated by $(a^p \mid a \in \mathfrak{m})$. Since $\mathfrak{m}$ is the filtered union of a countable system of principal subideals, the same holds for $\overline{\mathfrak{m}}_C$; by the same token, since $\mathfrak{m} = \mathfrak{m}^2$, for every $a \in \mathfrak{m}$ there exist $b, c \in \mathfrak{m}$ such that $a = bc$, whence $a^p = b^p c^p$, which shows that $\mathfrak{m}_{\overline{C}} = \mathfrak{m}_{\overline{C}}^2$. Hence, $(\overline{C}, \mathfrak{m}_{\overline{C}})$ is a basic setup; moreover, we have already remarked that $\mathfrak{m} = \mathfrak{m}_{\overline{C}} A_0$, hence the inclusion $\overline{C} \to A_0$ is a morphism of basic setups $(\overline{C}, \mathfrak{m}_{\overline{C}}) \to (A_0, \mathfrak{m})$.

Notice then that the $\overline{C}$-module $A_0/\overline{C}$ is the limit of the inverse system

$$(M_n := A_0/(b^n A_0 + C) \mid n \in \mathbb{N}).$$

Hence, $\mathfrak{m} \subset \overline{C}$ if and only if the $(\overline{C}, \mathfrak{m}_{\overline{C}})^a$-module $(A_0/\overline{C})^a$ vanishes, if and only if $M_n^a = 0$ for every $n \in \mathbb{N}$. Now, for given $n \in \mathbb{N}$, consider the b^p-adic filtration on M_n, and let $\mathrm{gr}_\bullet M_n$ be the associated graded $\overline{C}$-module; clearly $\mathrm{gr}_i M_n$ is a quotient of M_p, for every $i \in \mathbb{N}$. Hence, $\mathfrak{m} \subset \overline{C}$ if and only if $M_p^a = 0$; the latter in turns holds if and only if $\mathrm{Coker}\,\overline{\Phi}_{A_0}^a = 0$. This already shows that (a),(b)⇒(c). Conversely, if (c) holds, in order to deduce (a), it suffices to check that $\mathrm{Ker}\,\overline{\Phi}_{A_0}^a = 0$; since j^a is an isomorphism, the latter holds by claim 16.9.38.

(ii): Notice first that since $b \cdot \mathbf{1}_{A_0}$ is an injective map, the same holds for $(b \cdot \mathbf{1}_{A_0})_*^a = b \cdot \mathbf{1}_{A_{0*}^a}$, since the functor $(-)_*^a$ commutes with all limits. Thus, A_{0*}^a is naturally a subring of $A_{0*}^a[b^{-1}]$. Now, let $x \in A_{0*}^a[b^{-1}]$ be integral over A_{0*}^a; hence there exist $n, k \in \mathbb{N}$ and $y, a_1, \dots, a_k \in A_{0*}^a$ such that $x = y/b^n$ and

$$(y/b^n)^k + a_1 \cdot (y/b^n)^{k-1} + \cdots + a_k = 0.$$

Recall that $A_{0*}^a = \mathrm{Hom}_{A_0}(\widetilde{\mathfrak{m}}, A_0)$, with $\widetilde{\mathfrak{m}} := \mathfrak{m} \otimes_{A_0} \mathfrak{m}$. Then we get $(y(t)/b^n)^k + a_1(t) \cdot (y(t)/b^n)^{k-1} + \cdots + a_k(t) = 0$ for every $t \in \widetilde{\mathfrak{m}}$, and if A_0 is integrally closed in A, it follows that $y(t)/b^n \in A_0$ for every such t, *i.e.* $x \in A_{0*}^a$. One shows likewise that if A_0 is p-root closed in A, then A_{0*}^a is p-root closed in $A_{0*}^a[b^{-1}]$.

(iii): By assumption, $b \cdot \mathbf{1}_{A_0}$ is an injective map, so the same holds for $b \cdot \mathbf{1}_{A_{0*}^a}$, because the functor $(-)_*^a$ commutes with limits. This shows that A_{0*}^a has no b-torsion; next, we need to check that A_{0*}^a is b-adically complete and separated. This follows from the more general :

Claim 16.9.39. Let $(V, \mathfrak{m}_V)$ be a basic setup, A a V-algebra, $I \subset A$ an ideal of finite type, M an A-module whose I-adic topology is complete and separated. Then the same holds for the I-adic topology of M_*^a.

Proof of the claim. Both functors $(-)^a$ and $(-)_*$ commute with limits, since they are right adjoints, hence M_*^a is the limit of the inverse system $((M/I^n M)_*^a \mid n \in \mathbb{N})$. For every $n \in \mathbb{N}$, let M_n be the image of the natural map $\pi_n : M_*^a \to (M/I^n M)_*^a$; it follows that M_*^a is also the limit of the induced inverse system $(M_n \mid n \in \mathbb{N})$, and therefore M_*^a is complete and separated for the linear topology that admits the fundamental system of open submodules $(\mathrm{Ker}\,\pi_n = (I^n M)_*^a \mid n \in \mathbb{N})$ (corollary 8.2.16(i)). Since $I^n M_*^a \subset (I^n M)_*^a$ for every $n \in \mathbb{N}$, the claim now follows from lemma 8.3.12. ◊

It remains to check that the Frobenius endomorphism of $R := A_{0*}^a$ induces an isomorphism $(R/bR)^a \xrightarrow{\sim} (R/b^p R)_{(\Phi)}^a$; the latter is clear, since the same holds by assumption for the Frobenius endomorphism of A_0, and the natural map $A_0 \to A_{0*}^a$ is an almost isomorphism.

(iv): Under the stated assumptions, j induces an isomorphism $A_0 \xrightarrow{\sim} (A_0)_*^a \xrightarrow{\sim} (A_1)_*^a$, by (i), and on the other hand, $(A_1)_*^a$ is p-root closed in $A_{0*}^a[b^{-1}] = A$, by (ii), hence A_0 is p-root closed in A, *i.e.* $A_0 = A_1$. □

16.9.40. Let A be a perfectoid Tate ring, $A_0 \subset A$ a perfectoid subring of definition, and $b \in A_0^{\circ\circ} \cap A^\times$, so that the topology of A_0 is b-adic. Let also $g_\bullet := (g_n \mid n \in \mathbb{N}) \in \mathbf{E}_0 := \mathbf{E}(A_0)$, and set $g := g_0 = \overline{u}_{A_0}(g_\bullet)$. Hence, g^γ is well defined in A_0 for every $\gamma \in \mathbb{N}[1/p]$, and set

$$\mathfrak{m} := \bigcup_{n\in\mathbb{N}} g_n A_0 \qquad A'_0 := \mathrm{Im}(A_0 \to A_0[1/g]) \qquad \text{and} \qquad A_1 := \{a \in A_0[1/g] \mid \mathfrak{m}\cdot a \subset A'_0\}.$$

Clearly $(A_0, \mathfrak{m})$ is a basic setup fulfilling the conditions of (16.9.34).

Lemma 16.9.41. *With the notation of* (16.9.40)*, let A_0^a be the $(A_0, \mathfrak{m})^a$-algebra associated with A_0. Then the natural map $A_0 \to A_1$ factors through the unit of adjunction $A_0 \to A_{0*}^a$ and an isomorphism of A_0-algebras $A_{0*}^a \xrightarrow{\sim} A_1$.*

Proof. We claim that the induced surjection $\pi : A_0 \to A'_0$ induces an isomorphism of $(A_0, \mathfrak{m})^a$-algebras $\pi^a : A_0^a \xrightarrow{\sim} A_0'^a$. Indeed, if $a \in \mathrm{Ker}\,\pi$, there exists $n \in \mathbb{N}$ such that $g^n a = 0$ in A_0; then $(g^{n/p^k}a)^{p^k} = 0$ in A_0, and consequently $g^{n/p^k}a = 0$ in A_0 for every $k \in \mathbb{N}$, due to corollary 16.3.63(i), whence the contention. Since the natural map $A_0 \to A_1$ factors through π and the inclusion $i : A'_0 \to A_1$, and since π^a is an isomorphism we are reduced to checking that i factors through the unit of adjunction $A'_0 \to A_{0*}'^a$ and an isomorphism $A_{0*}'^a \xrightarrow{\sim} A_1$.

Next, we claim that the surjection $\mathfrak{m} \to \mathfrak{m}A'_0$ is bijective. Indeed, say that $x \in \mathfrak{m}$ and $\pi(x) = 0$; we may write $x = g^\gamma y$ for some $\gamma \in \mathbb{N}[1/p] \setminus \{0\}$, and then clearly $\pi(y) = 0$. But by the foregoing, it follows that $g^\gamma y = 0$, whence the contention.

Set $\widetilde{\mathfrak{m}} := \mathfrak{m} \otimes_{A_0} \mathfrak{m}$; by the foregoing, the isomorphism π^a induces identifications :

$$A_{0*}^a = \mathrm{Hom}_{A_0}(\widetilde{\mathfrak{m}}, A_0) \xrightarrow{\sim} A_{0*}'^a = \mathrm{Hom}_{A_0}(\widetilde{\mathfrak{m}}, A'_0) = \mathrm{Hom}_{A'_0}(\mathfrak{m}A'_0 \otimes_{A'_0} \mathfrak{m}A'_0, A'_0).$$

Notice that the multiplication map $\mathfrak{m}A'_0 \otimes_{A'_0} \mathfrak{m}A'_0 \to \mathfrak{m}A'_0$ is the colimit of the system of isomorphisms $(g^\gamma A'_0 \otimes_{A'_0} g^\gamma A'_0 \xrightarrow{\sim} g^{2\gamma}A'_0 \mid \gamma \in \mathbb{N}[1/p]\setminus\{0\})$, and it is therefore an isomorphism. So, finally, we get a natural identification

$$A_{0*}^a \xrightarrow{\sim} \mathrm{Hom}_{A'_0}(\mathfrak{m}A'_0, A'_0) = A_1$$

from which the lemma follows easily : details left to the reader. □

The following result generalizes the "perfectoid Abhyankar's lemma" of [6].

Theorem 16.9.42. *In the situation of* (16.9.40)*, let B be a finite étale $A_0[1/g]$-algebra, B_1 the integral closure of A_1 in B, and endow A_1 and B_1 with their b-adic topologies. For the almost structure given by $(A_0, \mathfrak{m})$, we have :*

- (i) *The basic setup $(B_1, \mathfrak{m}B_1)$ is almost perfectoid, and B_1 is p-root closed in $B_1[1/b]$.*
- (ii) *$(B_1/b^n B_1)^a$ is an étale $(A_0/b^n A_0)^a$-algebra of finite rank, for every $n \in \mathbb{N}$.*
- (iii) *If B is a faithfully flat $A_0[1/g]$-algebra, then $(B_1/b^n B_1)^a$ is a faithfully flat $(A_0/b^n A_0)^a$-algebra, for every $n \in \mathbb{N}$.*
- (iv) *A_1 is complete and separated, and is integrally closed in $A_0[1/g]$.*
- (v) *The unit of adjunction $B_1 \to B_{1*}^a$ is an isomorphism.*

Proof. Arguing as in (16.9.7), we may assume that $pA_0 \subset b^p A_0$, and $b = \overline{u}_{A_0}(b_\bullet)$ for some $b_\bullet \in \mathbf{E}(A_0)$, so that b^γ is well defined in A for every $\gamma \in \mathbb{Z}[1/p]$. We let $I := A_0 b + A_0 g$, and $U := \mathrm{Spec}\,A_0 \setminus \mathrm{Spec}\,A/I$; consider the quasi-affinoid ring $\underline{A}_0 := (A_0, A_0, U)$, and the rational subsets of $X := \mathrm{Spa}\,\underline{A}_0$

$$R_n := R_{A_0}\big(\tfrac{b^n}{g}\big) \cap X \qquad R'_n := R_{A_0}\big(\tfrac{g}{b^n}\big) \cap X \qquad \text{for every } n \in \mathbb{N}$$

(definition 15.4.14(iv)). Set $\mathsf{A}_n := \mathscr{O}_X^{\wedge+}(R_n)$ and $\mathsf{A}'_n := \mathscr{O}_X^{\wedge+}(R'_n)$ for every $n \in \mathbb{N}$; since $R_n \subset R_{n+1}$ for every $n \in \mathbb{N}$, we deduce a well defined inverse system of A_0-algebras $(\mathsf{A}_n \mid n \in \mathbb{N})$.

Claim 16.9.43. (i) The natural map $A_0 \to \mathscr{O}^{\wedge+}_X(X)$ induces an isomorphism $A^a_0 \xrightarrow{\sim} \mathscr{O}^{\wedge+}_X(X)^a$ of $(A_0, \mathfrak{m})^a$-algebras.

(ii) For every $i \in \mathbb{N}$, the following holds :

(a) The induced inverse system $(\mathsf{A}_n/b^i\mathsf{A}_n \mid n \in \mathbb{N})$ is almost essentially constant, relative to the basic setup $(A_0, \mathfrak{m})$ (see definition 14.2.11(iii)).

(b) The natural cone $(\rho_n : A_0 \to \mathsf{A}_n \mid n \in \mathbb{N})$ induces a universal cone $((A_0/b^iA_0)^a \to (\mathsf{A}_n/b^i\mathsf{A}_n)^a \mid n \in \mathbb{N})$ in the category of $(A_0, \mathfrak{m})^a$-algebras.

Proof of the claim. (i): By construction, $\mathscr{O}^{\wedge+}_X(X)$ is the integral closure of the image of A_0 in $A_U := \mathscr{O}_U(U)$. On the other hand, A^a_{0*} is integrally closed in A°_U (claim 16.8.20(ii)), and recall that A°_U is integrally closed in A_U (remark 8.3.10(iv)). It follows that the unit of adjunction $\eta : A_0 \to A^a_{0*}$ factors through the inclusion map $\mathscr{O}^{\wedge+}_X(X) \to A^a_{0*}$. Since η^a is an isomorphism, the assertion follows.

(ii): In light of (i) and theorem 16.7.41, the following sequence of $(A_0, \mathfrak{m})^a$-modules is short exact for every $n \in \mathbb{N}$:

$$0 \to A^a_0 \to \mathsf{A}^a_n \oplus \mathsf{A}'^a_n \to \mathscr{O}^+_X(R_n \cap R'_n)^a \to 0.$$

Moreover, clearly the image of b is a regular element in $\mathscr{O}^+_X(R_n \cap R'_n)$; by the snake lemma, we easily deduce, for every $n, i \in \mathbb{N}$, a short exact sequence of $(A_0, \mathfrak{m})^a$-modules

$$0 \to A^a_0/b^iA^a_0 \xrightarrow{r_n} A_0/b^iA_0 \otimes_{A_0} (\mathsf{A}_n \oplus \mathsf{A}'_n)^a \xrightarrow{s_n} A_0/b^iA_0 \otimes_{A_0} \mathscr{O}^+_X(R_n \cap R'_n)^a \to 0.$$

Let us show that for every $\gamma \in \mathbb{N}[1/p] \setminus \{0\}$, every $n \in \mathbb{N}$ with $n\gamma \geq i$ and every $a \in A_0$ with $\rho_n(a) \in b^i\mathsf{A}_n$, we have $g^{2\gamma}a \in b^iA_0$. Indeed, let $\overline{a} \in A_0/b^iA_0$ be the image of a; then $r_n(\overline{a}) = (0, \overline{x})$ for some $\overline{x} \in \mathsf{A}'_n$, and we notice that $g^\gamma \cdot \overline{x} = b^{n\gamma} \cdot (g/b^n)^\gamma \cdot \overline{x}$, whence $g^\gamma \cdot \overline{x} = 0$, since $(g/b^n)^\gamma \in \mathsf{A}'_n$. It follows that $r_n(g^\gamma \cdot \overline{a}) = 0$, whence $g^{2\gamma} \cdot \overline{a} = 0$, as required. Lastly, let us check that for every $\gamma \in \mathbb{N}[1/p] \setminus \{0\}$, every $n \in \mathbb{N}$ with $n\gamma \geq i$ and every $\overline{x} \in \mathsf{A}_n/b^i\mathsf{A}_n$ there exists $\overline{a} \in A_0/b^iA_0$ with $\rho_n(\overline{a}) = g^{2\gamma} \cdot \overline{x}$. To this aim, set $\overline{y} := s_n(\overline{x}, 0)$; again, we notice that $g^\gamma \cdot \overline{y} = b^{n\gamma} \cdot (g/b^n)^\gamma \cdot \overline{y} = 0$, so that $(g^\gamma\overline{x}, 0) \in \operatorname{Ker} s_n = \operatorname{Im} r_n$, whence the contention. This completes the proof of (ii.a). Assertion (ii.b) follows from the proof of (ii.a) and lemma 14.2.14(iii). ◊

Recall that $\mathscr{O}^\wedge_X(R_n) = \mathsf{A}_n[1/g]$, and set $\mathsf{B}_n := \mathsf{A}_n[1/g] \otimes_{A_0} B$ for every $n \in \mathbb{N}$; recall as well that A_n is perfectoid for its b-adic topology (proposition 16.7.25(iii) and theorem 16.5.13(iii)). Since the functor $(-)^a$ commutes with limits, from claim 16.9.43 we deduce that also the cone $(\rho_n \mid n \in \mathbb{N})$ is universal, *i.e.* we have an isomorphism of $(A_0, \mathfrak{m})^a$-algebras :

$$A^a_0 \xrightarrow{\sim} \lim_{i\in\mathbb{N}} \lim_{n\in\mathbb{N}} \mathsf{A}^a_n/b^i\mathsf{A}^a_n \xrightarrow{\sim} \lim_{n\in\mathbb{N}} \mathsf{A}^a_n. \tag{16.9.44}$$

Claim 16.9.45. $\mathfrak{m}\mathsf{A}_n$ is an open radical ideal of A_n, for every $n \in \mathbb{N}$.

Proof of the claim. Clearly $\mathfrak{m}\mathsf{A}_n$ is an open ideal of A_n, since $(b^n/g)^\gamma \in \mathsf{A}_n$ for every $\gamma \in \mathbb{N}[1/p]$. Moreover, set $J := \bigcup_{\gamma\in\mathbb{N}[1/p]\setminus\{0\}} b^\gamma\mathsf{A}_n$; by corollary 16.3.71, the quotient A_n/J is a perfect $\mathbb{F}_p$-algebra; it follows easily that $\mathfrak{m}/J$ is a radical ideal of A_n/J, whence the claim. ◊

Say that B is a projective $A_0[1/g]$-module of rank $\leq r$; from claim 16.9.45 and theorem 16.8.44, we deduce that for every $n \in \mathbb{N}$ there exists an étale A^a_n-algebra C_n of rank $\leq r$, unique up to unique isomorphism, with an isomorphism of $\mathsf{A}_n[1/g]$-algebras $C_n[1/g] \xrightarrow{\sim} \mathsf{B}_n$. From the uniqueness property of C_n, there follows a unique isomorphism of A^a_n-algebras :

$$\mathsf{A}^a_n \otimes_{\mathsf{A}^a_{n+1}} C_{n+1} \xrightarrow{\sim} C_n \qquad \text{for every } n \in \mathbb{N}. \tag{16.9.46}$$

Denote by D (resp. by E) the inverse limit of the resulting system $(C_n \mid n \in \mathbb{N})$ of A^a_0-algebras (resp. $(\mathsf{B}_n \mid n \in \mathbb{N})$ of A_0-algebras), and for every $i \in \mathbb{N}$ let as well D_i be the limit of the induced system $(C_n/b^iC_n \mid n \in \mathbb{N})$ of $A^a_0/b^iA^a_0$-algebras. By proposition 14.2.32(ii,iii),

$(C_n/b^iC_n \mid n \in \mathbb{N})$ is almost essentially constant, and the projection $D_i \to C_n/b^iC_n$ induces an isomorphism of $A_0^a/b^iA_0^a$-algebras :

$$\mathsf{A}_n^a \otimes_{A_0^a} D_i \xrightarrow{\sim} C_n/b^iC_n \qquad \text{for every } n \in \mathbb{N}. \tag{16.9.47}$$

Notice that b and g are regular elements of A_n, and then they are also regular elements of C_{n*}, since C_n is a flat A_n^a-algebra; in view of the induced isomorphism

$$D_* \xrightarrow{\sim} \lim_{n\in\mathbb{N}} C_{n*} \tag{16.9.48}$$

(proposition 1.3.25(iii)) it follows easily that b and g are regular elements of D_*.

Claim 16.9.49. (i) For every $n \in \mathbb{N}$, the ring C_{n*} is perfectoid for its b-adic topology.

(ii) D_* is p-root closed in $D_*[1/b]$ and is integrally closed in $D_*[1/g]$.

(iii) The A_0^a/b^iA_0-algebra D_i is étale of rank $\leq r$ for every $i \in \mathbb{N}$.

(iv) If B is a faithfully flat $A_0[1/g]$-algebra, then D_i is a faithfully flat A_0^a/b^iA_0-algebra for every $i \in \mathbb{N}$.

(v) B and D_* are integrally closed in E.

(vi) The basic setup $(D_*, \mathfrak{m}D_*)$ is almost perfectoid for the b-adic topology of D_*, and the projection $D \to D_i$ factors through an isomorphism $D/b^iD \xrightarrow{\sim} D_i$ for every $i \in \mathbb{N}$.

Proof of the claim. (i) follows from proposition 16.8.29 and claim 16.9.45, since we have already recalled that A_n is perfectoid for its b-adic topology. Together with corollary 16.3.75 and lemma 9.8.26(i) we deduce that C_{n*} is p-root closed in $C_{n*}[1/b]$, for every $n \in \mathbb{N}$. Then lemma 9.8.26(iii.a) and the isomorphism (16.9.48) imply that D_* is p-root closed in the limit L of the system $(C_{n,*}[1/b] \mid n \in \mathbb{N})$. However, the induced map $D_* \to L$ factors through an injective map $D_*[1/b] \to L$, so D_* is also p-root closed in $D_*[1/b]$.

Similarly, C_{n*} is integrally closed in $C_{n*}[1/g] = B_n$ for every $n \in \mathbb{N}$, due to claim 16.8.20(ii); it follows easily that D_* is integrally closed in E. But the induced map $D_* \to E$ factors through an injective map $D_*[1/g] \to E$, so D_* is also integrally closed in $D_*[1/g]$.

(iii) follows from theorem 14.2.39(iii,iv).

(iv): If B is a faithfully flat $A_0[1/g]$-algebra, C_n is a faithfully flat A_n^a-algebra for every $n \in \mathbb{N}$ (theorem 16.8.44(ii)); then the assertion follows from theorem 14.2.39(ii)).

(v): We have already observed that D_* is integrally closed in E. Next, by construction, $\mathsf{B}_n = B \otimes_{A_0[1/g]} \mathsf{A}_n[1/g]$ for every $n \in \mathbb{N}$. Since B is a projective $A_0[1/g]$-module of finite rank, there follows a natural identification :

$$E \xrightarrow{\sim} B \otimes_{A_0[1/g]} \lim_{n\in\mathbb{N}} \mathsf{A}_n[1/g].$$

Since B is an étale $A_0[1/g]$-algebra, by proposition 14.3.25 we are then reduced to showing that $A_0[1/g]$ is integrally closed in the limit L of the system of A_0-algebras $(\mathsf{A}_n[1/g] \mid n \in \mathbb{N})$. However, invoking again claim 16.8.20(ii) we see that A_{n*}^a is integrally closed in $\mathsf{A}_n[1/g]$ for every $n \in \mathbb{N}$, hence the limit L' of the system $(\mathsf{A}_{n*}^a \mid n \in \mathbb{N})$ is integrally closed in L, and therefore $L'[1/g]$ is integrally closed in L. But in view of (16.9.44), we have a natural identification of A_0-algebras : $A_0[1/g] \xrightarrow{\sim} A_{0*}^a[1/g] \xrightarrow{\sim} L'[1/g]$, whence the assertion.

(vi): Since C_n is the limit of the inverse system $(C_n/b^iC_n \mid i \in \mathbb{N})$ for every $n \in \mathbb{N}$, we obtain a natural isomorphism of A_0^a-algebras :

$$D \xrightarrow{\sim} \lim_{i\in\mathbb{N}} D_i.$$

On the other hand, in view of (16.9.47) we have a natural isomorphism of A_0^a-algebras

$$(D_{i+1} \otimes_{A_0^a} A_0^a/b^iA_0^a) \otimes_{A_0^a} \mathsf{A}_n^a \xrightarrow{\sim} C_n/b^iC_n \qquad \text{for every } n, i \in \mathbb{N}$$

whence, again by theorem 14.2.39(i), an induced isomorphism of A_0^a-algebras :

$$D_{i+1} \otimes_{A_0^a} A_0^a/b^i A_0^a \xrightarrow{\sim} D_i \qquad \text{for every } i \in \mathbb{N}. \tag{16.9.50}$$

For every $i \in \mathbb{N}$, let $\pi_i : D \to D_i$ be the projection; notice that $I := \operatorname{Ker} \pi_1 = bD + \operatorname{Ker} \pi_{i+1}$ for every $i \in \mathbb{N}$, in light of the isomorphism $D_{i+1}/bD_{i+1} \xrightarrow{\sim} D_1$ deduced from (16.9.50). By [75, Lemma 5.3.5(i)], it follows that $I = \bigcap_{i\in\mathbb{N}}(bD + \overline{I^{i+1}}) = \overline{bD}$ (where $\overline{J}$ denotes the closure of J, for every ideal $J \subset D$: see [75, Def.5.3.1(iii)]). By [75, Lemma 5.3.8(iii)] we deduce that $I = bD$, and more generally $b^i D = \operatorname{Ker} \pi_i$ for every $i \in \mathbb{N}$. On the other hand, π_i is an epimorphism for every $i \in \mathbb{N}$, by lemma 14.2.24(i), so it induces an isomorphism $D/b^i D \xrightarrow{\sim} D_i$, for every $i \in \mathbb{N}$. Hence, the natural morphism $D \to \lim_{i\in\mathbb{N}} D/b^i D$ is an isomorphism of A_0^a-algebras, so we have an induced isomorphism of A_1-algebras $D_* \xrightarrow{\sim} \lim_{i\in\mathbb{N}}(D/b^i D)_*$ (lemma 16.9.41). As we have already noticed, b is regular in D_*, so for every $i \in \mathbb{N}$ we get a short exact sequence of A_0^a-modules $0 \to D \xrightarrow{b^i \cdot \mathbf{1}_{D_*}} D \to D/b^i D \to 0$ and since the functor $(-)_*$ is left exact, we deduce an injective A_1-linear map

$$D_*/b^i D_* \to (D/b^i D)_* \qquad \text{for every } i \in \mathbb{N}$$

from which it follows easily that also the induced map $D_* \to \lim_{i\in\mathbb{N}} D_*/b^i D_*$ is an isomorphism, hence D_* is b-adically complete and separated. In order to show that $(D_*, \mathfrak{m}D_*)$ is almost perfectoid, it remains only to check that the Frobenius endomorphism of D_*/pD_* induces an isomorphism $D/bD \xrightarrow{\sim} (D/b^p D)_{(\Phi)}$ of A_0^a/bA_0^a-algebras (where Φ denotes the Frobenius endomorphism of A_0/pA_0). But we have already seen that $D/b^i D$ is the inverse limit of the system $(C_n/b^i C_n \mid n \in \mathbb{N})$, for every $i \in \mathbb{N}$, hence we are reduced to checking that the Frobenius endomorphism of C_{n*}/pC_{n*} induces an isomorphism $C_n/bC_n \xrightarrow{\sim} (C_n/b^p C_n)_{(\Phi)}$ of A_0^a/bA_0^a-algebras, for every $n \in \mathbb{N}$. The latter is clear, since we know already that C_{n*} is perfectoid (corollary 16.3.3). ◊

Next, recall that for every $n \in \mathbb{N}$ we have A_0^a-algebras $\mathsf{A}_{n,0}, \dots, \mathsf{A}_{n,r}$, and an isomorphism of A_0^a-algebras $\mathsf{A}_n^a \xrightarrow{\sim} \mathsf{A}_{n,0} \times \cdots \times \mathsf{A}_{n,r}$ such that $C_{n,k} := \mathsf{A}_{n,k} \otimes_{\mathsf{A}_n^a} C_n$ is an $\mathsf{A}_{n,k}$-module of constant rank k, for every $k = 0, \dots, r$ ([75, Prop.4.3.27]). For every $n \in \mathbb{N}$ and $k = 0, \dots, r$ consider the map of sets :

$$\chi^{(n,k)} : C_{n,k*} \to \mathsf{A}_{n,k*}^a[X] \qquad c \mapsto \chi_c(X)$$

which associates with every $c \in C_{n,k*}$ the characteristic polynomial $\chi_c(X)$ of the $\mathsf{A}_{n,k}^a$-linear endomorphism $c \cdot \mathbf{1}_{C_{n,k}}$ of $C_{n,k}$ (see [75, §4.4.29]); by [75, Prop.4.4.30] we have $\chi_c(c) = 0$ for every such c. By virtue of (16.9.46), we see that the restriction map $\mathsf{A}_{n+1} \to \mathsf{A}_n$ induces isomorphisms of A_0^a-algebras

$$\mathsf{A}_n^a \otimes_{\mathsf{A}_{n+1}^a} \mathsf{A}_{n+1,k} \xrightarrow{\sim} \mathsf{A}_{n,k} \qquad \text{for every } k = 0, \dots, r$$

and from the compatibility of traces with ring extensions ([75, Prop.4.1.8(ii)]), we deduce for every $n \in \mathbb{N}$ and $k = 0, \dots, r$ a commutative diagram :

$$\begin{array}{ccc} C_{n+1,k*} & \xrightarrow{\chi^{(n+1,k)}} & \mathsf{A}_{n+1,k*}^a[X] \\ \downarrow & & \downarrow \\ C_{n,k*} & \xrightarrow{\chi^{(n,k)}} & \mathsf{A}_{n,k*}^a[X] \end{array}$$

whose vertical arrows are induced by the induced projection $\mathsf{A}_{n+1,k} \to \mathsf{A}_{n,k}$. For every $k = 0, \dots, r$, let $\mathsf{A}^{(k)}$ be the limit of the induced inverse system of A_0^a-algebras $(\mathsf{A}_{n,k} \mid n \in \mathbb{N})$, and set $D^{(k)} := \mathsf{A}^{(k)} \otimes_{A_0^a} D$; in view of (16.9.44), there follows an isomorphism of A_0^a-algebras $D \xrightarrow{\sim} D^{(0)} \times \cdots \times D^{(r)}$. Notice as well that $\chi_c(X)$ is a monic polynomial of degree k, for every

$n \in \mathbb{N}$, every $k = 0, \dots, r$ and and every $c \in C_{n,k*}$; in light of (16.9.48), it follows easily that the limit of the system of maps $(\chi^{(n,k)} \mid n \in \mathbb{N})$ is a well defined map of sets

$$\chi^{(k)} : D_*^{(k)} \to \mathsf{A}_*^{(k)}[X] \qquad d \mapsto \chi_d(X) \qquad \text{for every } k = 0, \dots, r.$$

Moreover, $\chi_d(X)$ is a monic polynomial with $\chi_d(d) = 0$ for every $k = 0, \dots, r$ and every $d \in D_*^{(k)}$. Taking into account lemma 16.9.41, it follows that D_* is an integral A_1-algebra. Since both D_* and B are integrally closed subrings of E (claim 16.9.49(v)), and since $A_1 \subset A_0[1/g]$, we deduce that $D_* \subset B$, and hence D_* is also the integral closure of A_1 in B, *i.e.* $D_* = B_1$. Then assertions (i), (ii) and (iii) of the theorem follow from claim 16.9.49.

(iv): By construction, A_1 is an open bounded subring of A, hence it is complete and separated for the topology induced by A, which agrees with the b-adic topology (proposition 8.3.18(i,ii)). In order to show that A_1 is integrally closed in $A_0[1/g]$, consider the case where $B = A_0[1/g]$; then $C_n = A_{0,n}$ for every $n \in \mathbb{N}$, so that $B_1 = D_* = A_{0*}^a = A_1$, whence the assertion. □

Remark 16.9.51. (i) In the situation of theorem 16.9.42, let $g'_\bullet := (g'_n \mid n \in \mathbb{N}) \in \mathbf{E}_0$ be another element, and $g' := g'_0 = \overline{u}_{A_0}(g'_\bullet)$. Let also $A''_0 := \mathrm{Im}(A_0 \to A_0[1/(gg')])$ and set

$$\mathfrak{m}' := \bigcup_{n\in\mathbb{N}} g'_n A_0 \qquad \mathfrak{m}'' := \mathfrak{m} \cdot \mathfrak{m}' \qquad A_2 := \{a \in A_0[1/(gg')] \mid \mathfrak{m}'' \cdot a \subset A''_0\}.$$

Clearly $(A_0, \mathfrak{m}')$ and $(A_0, \mathfrak{m}'')$ are two other basic setups; for every A_0-module M, let $(M, \mathfrak{m})^a \in (A_0, \mathfrak{m})^a$-**Mod** (resp. $(M, \mathfrak{m}')^a \in (A_0, \mathfrak{m}')^a$-**Mod**, resp. $(M, \mathfrak{m}'')^a \in (A_0, \mathfrak{m}'')^a$-**Mod**) be the image of M. We may apply theorem 16.9.42 with $(A_0, \mathfrak{m})$ and B replaced respectively by $(A_0, \mathfrak{m}'')$, and $B[1/g']$: hence, let B_2 be the integral closure of A_2 in $B[1/g']$; then $B_2 = (B_2, \mathfrak{m}'')_*^a$ is an almost perfectoid A_0-algebra relative to $(A_0, \mathfrak{m}'')$, and $(B_2/b^n B_2, \mathfrak{m}'')^a$ is an étale $(A_0/b^n A_0, \mathfrak{m}'')^a$-algebra of finite rank, for every $n \in \mathbb{N}$. The localizations $A_0[1/g] \to A_0[1/(gg')]$ and $B \to B[1/g']$ clearly restrict to ring homomorphisms $A_1 \to A_2$ and $B_1 \to B_2$.

Now, $(A_0, \mathfrak{m}'')^a$ is integrally closed in $(A_0[1/(gg')], \mathfrak{m}'')^a$ (theorem 16.9.42(iv) and lemma 16.9.41), hence the same holds for $(A_0[1/g], \mathfrak{m}'')^a$; since B is a finite étale $A_0[1/g]$-algebra, it follows that $(B, \mathfrak{m}'')^a$ is integrally closed in $(B[1/g'], \mathfrak{m}'')^a$ (proposition 14.3.25). Taking into account theorem 16.9.42(v), remark 14.2.7(ii) and [75, Lemma 8.2.28] we deduce :

$$\begin{aligned}
B_2 = (B_2, \mathfrak{m}'')_*^a &= \text{i.c.}((A_0, \mathfrak{m}'')^a, (B[1/g'], \mathfrak{m}'')^a)_* \\
&= \text{i.c.}((A_0, \mathfrak{m}'')^a, (B, \mathfrak{m}'')^a)_* \\
&= (\text{i.c.}((A_0, \mathfrak{m}'')_*^a, (B, \mathfrak{m}'')_*^a), \mathfrak{m}'')_*^a \\
&= (\text{i.c.}((A_1, \mathfrak{m}')_*^a, (B, \mathfrak{m}')_*^a), \mathfrak{m}'')_*^a \\
&= ((\text{i.c.}(A_1, B), \mathfrak{m}')_*^a, \mathfrak{m}'')_*^a \\
&= ((B_1, \mathfrak{m}')_*^a, \mathfrak{m}'')_*^a \\
&= (B_1, \mathfrak{m}'')_*^a \\
&= (B_1, \mathfrak{m}')_*^a.
\end{aligned}$$

(ii) As a special case of (i), we may take $g'_\bullet := b_\bullet$ such that $b := b_0 \in A^\times \cap A_0^{\circ\circ}$. Then, endow B_1 with its b-adic topology, so that B_1 is a subring of definition of the Tate ring $B_1[1/b]$, and by theorem 16.9.42(i) it is p-root closed in $B_1[1/b]$, hence $B_1[1/b]^{\circ\circ} \subset B_1$ (example 16.9.4). It follows easily that $(B_1[1/b]^\circ, \mathfrak{m}')^a = (B_1, \mathfrak{m}')^a$, so we have inclusions of rings :

$$B_1 \subset B^\circ \subset (B_1, \mathfrak{m}')_*^a = (B_1[1/b]^\circ, \mathfrak{m}')_*^a \subset B_1[1/b].$$

On the other hand, it is easily seen that $(B_1, \mathfrak{m}')_*^a \subset B_1[1/b]^\circ$, so $B_2 = (B_1, \mathfrak{m}')_*^a = B_1[1/b]^\circ$.

(iii) In the situation of theorem 16.9.42, suppose that A is an $\mathbb{F}_p$-algebra; then $A_0 = \mathbf{E}_0$ is a perfect $\mathbb{F}_p$-algebra, and it follows easily that the same holds for A_1. Then, set $X := \operatorname{Spec} A_1$

and $Z := \mathrm{Spec}\, A_1/gA_1$; by theorem 16.8.2, the pair (X, Z) is normal and almost pure, hence B_1^a is an étale A_1^a-algebra of finite rank (proposition 14.4.8).

(iv) If A is not an $\mathbb{F}_p$-algebra, we know neither whether A_1 is perfectoid, nor whether B_1^a is an étale A_1^a-algebra. However, by theorem 16.9.42(ii) and lemma 14.3.15(ii), for every $n \in \mathbb{N}$ we have a well defined map

$$\chi_{B_1/b^nB_1} : (B_1/b^nB_1)^a_* \to (A_0/b^nA_0)^a_*[T] \qquad b \mapsto \det((1+bT)\cdot \mathbf{1}_{B_1^a/b^nB_1^a})$$

and clearly the inverse system of maps $(\chi_{B_1/b^nB_1} \mid n \in \mathbb{N})$ amounts to a well defined map

$$\chi_{B_1} : B_1 \to A_1[T].$$

By the same token, we may associate to the finite étale $A_0[1/g]$-algebra B the map

$$\chi_B : B \to A_0[1/g, T] \qquad b \mapsto \det((1+bT)\cdot \mathbf{1}_B)$$

and we claim that

$$\chi_B(b) = \chi_{B_1}(b) \qquad \text{for every } b \in B_1.$$

For the proof, define the inverse system of A_0-algebras $\mathsf{A}_\bullet := (\mathsf{A}_n \mid n \in \mathbb{N})$, and for every $n \in \mathbb{N}$ the étale A_n^a-algebra C_n of rank $\leq r$, as in the proof of theorem 16.9.42; recall that A_1 (resp. B_1) is the limit of $\mathsf{A}_\bullet$ (resp. of $(C_{n*} \mid n \in \mathbb{N})$). It follows easily that χ_{B_1} is also the inverse limit of the system of similar maps $(\chi_{C_n} \mid n \in \mathbb{N})$. On the other hand, $A_0[1/g]$ is integrally closed in the limit L of the induced inverse system $(\mathsf{A}_n[1/g] \mid n \in \mathbb{N})$, and B is the integral closure of $A_0[1/g]$ in the inverse system E of the induced inverse system $(C_n[1/g] \mid n \in \mathbb{N})$; also, the natural map $E \to B \otimes_{A_0[1/g]} L$ is an isomorphism (see the proof of claim 16.9.49). Thus, both χ_B and χ_{B_1} are restrictions of the corresponding map $\chi_E : E \to L$, whence the assertion.

(v) Furthermore, set $R := B_1 \otimes_{A_1} B_1$, let $R^\wedge$ be the b-adic completion of R, and

$$e_{B/A_0[1/g]} \in R[1/g] = B \otimes_{A_0[1/g]} B$$

the diagonal idempotent of the $A_0[1/g]$-algebra B; we notice :

Corollary 16.9.52. *In the situation of theorem* 16.9.42*, we have :*

(i) *The following conditions are equivalent:*

- (a) $g^\gamma \cdot e_{B/A_0[1/g]} \in R' := \mathrm{Im}(R \to R[1/g])$ *for every* $\gamma \in \mathbb{N}[1/p] \setminus \{0\}$.
- (b) *For every* $\gamma \in \mathbb{N}[1/p] \setminus \{0\}$ *there exists* $N_\gamma \in \mathbb{N}$ *such that* $b^{N_\gamma} g^\gamma \cdot e_{B/A_0[1/g]} \in R'$.
- (c) B_1^a *is an étale* A_1^a*-algebra.*
- (d) B_1^a *is a weakly unramified* A_1^a*-algebra.*
- (e) B_1^a *is an almost finitely generated* A_1^a*-module.*
- (f) B_1^a *is an almost finite projective* A_1^a*-module.*
- (g) $B_1^a[1/b]$ *is a weakly unramified* $A_1^a[1/b]$*-algebra.*
- (h) $B_1^a[1/b]$ *is an almost finitely generated* $A_1^a[1/b]$*-module.*

(ii) *There exists* $n \in \mathbb{N}$ *such that* g^n *annihilates the kernel and cokernel of the completion map* $R \to R^\wedge$*. Also,* $\mathrm{Ann}_{R^\wedge}(g)^a = 0$.

Proof. Obviously (c)⇒(a),(d); also (d)⇒(g), (a)⇒(b) and (f)⇒(e)⇒(h).

(a)⇒(c),(f): The assumption implies that $e_{B/A_0[1/g]}$ lies in the image of the induced map

$$R^a_* \to B \otimes_{A_0[1/g]} B.$$

Then the assertion follows from proposition 14.3.21 and theorem 16.9.42(iv)).

(f)⇒(c): We consider the commutative diagram of A_1^a-algebras :

(16.9.53)
$$\begin{array}{ccc} R^a & \xrightarrow{\ \varphi\ } & \lim_{n\in\mathbb{N}} R^a/b^nR^a \\ {\scriptstyle \mu^a}\downarrow & & \downarrow \\ B_1^a & \longrightarrow & \lim_{n\in\mathbb{N}} B_1^a/b^nB_1^a \end{array}$$

where μ^a is the multiplication law, and the right vertical arrow is the inverse limit of the system of multiplication laws $(R^a/b^nR \to B_1^a/b^nB_1^a \mid n \in \mathbb{N})$. According to theorem 16.9.42(i), B_1 is b-adically complete and separated; then by lemma 14.2.24(iii) the morphism φ is an isomorphism. By theorem 16.9.42(ii), for each $n \in \mathbb{N}$ we have a diagonal idempotent $e_n \in (R/b^nR)_*$, and it is easily seen that the system $(e_n \mid n \in \mathbb{N})$ corresponds, under the isomorphism φ, to an idempotent element $e_{B_1^a/A_1^a} \in R^a_*$ such that $\mu(_{B_1^a/A_1^a}) = 1$ and $x \cdot e_{B_1^a/A_1^a} = 0$ for every $x \in (\mathrm{Ker}\,\mu^a)_*$. Then the assertion follows from [75, Prop.3.1.4].

(d)⇒(a): Let $J := \bigcup_{n\in\mathbb{N}} \mathrm{Ann}_R(g^n)$ and $\overline{R} := R/J$. By construction, g is a regular element of B_1, so that $JB_1 = 0$, for the R-algebra structure on B_1 given by the multiplication map $\mu : R \to B_1$. On the other hand, by assumption μ^a is a flat morphism of A_1^a-algebras; hence $\mu^a \otimes_R \overline{R} : \overline{R} \to B_1$ is still flat. Also, the localisation $R \to R[1/g]$ factors through a monomorphism $\overline{R} \to R[1/g]$ of A_1^a-algebras, and since B is an étale $A_0[1/g]$-algebra, $B = B_1[1/g]$ is a projective $R[1/g]$-module of finite rank. By [75, Prop.2.4.19] it follows that B_1 is an almost finite projective $\overline{R}$-module, and therefore $\mathrm{Ker}\,(\mu^a \otimes_R \overline{R})$ is generated by an idempotent almost element $d \in \overline{R}_*$ ([75, Rem.3.1.8]). For every $\gamma \in \mathbb{N}[1/p] \setminus \{0\}$, we then find $e_\gamma \in R$ whose image in $(R/J)_*$ agrees with $g^\gamma \cdot (1-d)$. It follows easily that the image of e_γ in $R[1/g]$ agrees with $g^\gamma e_{B/A_0[1/g]}$, whence the contention.

(g)⇒(b): The assumption means that $\mu^a \otimes_{A_1} A_1[1/b] : R[1/b]^a \to B_1[1/b]^a$ is a flat morphism, hence the same holds for $\mu^a \otimes_R \overline{R}[1/b]$, and arguing as in the foregoing we find for every $\gamma \in \mathbb{N}[1/p] \setminus \{0\}$ an integer $N_\gamma \in \mathbb{N}$ and $e_\gamma \in R$ whose image in $R[1/(gb)]$ agrees with the image of $g^\gamma b^{N_\gamma} e_{B/A_0[1/g]}$. Notice that, since $B_1[1/g] = B$ is a flat $A_1[1/g]$-algebra, and since b is regular in A_1, it follows that b is regular also in $R[1/g]$, hence the localization $R[1/g] \to R[1/(gb)]$ is injective, and we deduce that the image of e_γ agrees with $g^\gamma b^{N_\gamma} e_{B/A_0[1/g]}$ *already in* $R[1/g]$, as required.

(b)⇒(a): Define the inverse system of A_0-algebras $(\mathsf{A}_n \mid n \in \mathbb{N})$ as in the proof of theorem 16.9.42, and for every $n \in \mathbb{N}$ let C_n be the unique étale A_n^a-algebra up to unique isomorphism, such that $C_n[1/g]$ is isomorphic to $\mathsf{A}_n[1/g] \otimes_{A_0} B$. By claim 16.9.43(ii), for every $i \in \mathbb{N}$ the induced inverse system $(\mathsf{A}_n/b^i\mathsf{A}_n \mid n \in \mathbb{N})$ is almost essentially constant relative to the basic setup $(A_0, \mathfrak{m})$. In view of (16.9.46) the same follows for the inverse system $(C_n^a/b^iC_n^a \mid n \in \mathbb{N})$, for every $i \in \mathbb{N}$; but the proof of theorem 16.9.42 also shows that the inverse limit of the latter system is naturally isomorphic to $B_1^a/b^iB_1^a$, for every $i \in \mathbb{N}$. Thus, set $R_n := C_n \otimes_{\mathsf{A}_n^a} C_n$ for every $n \in \mathbb{N}$; we deduce that for every $i \in \mathbb{N}$ and there exists $n(i,\gamma) \in \mathbb{N}$ such that g^γ annihilates the kernel of the induced morphism

$$R^a/b^iR^a \to R_{n(i,\gamma)}/b^iR_{n(i,\gamma)}.$$

Notice also that, since g is regular in A_n, and C_n is a flat A_n^a-algebra, then g is regular in C_n, and the localization $R_n \to R_n[1/g]$ is a monomorphism of A_n^a-modules, for every $n \in \mathbb{N}$. Moreover, the image of $e_{B/A_0[1/g]}$ in $R_n[1/g]$ is the diagonal idempotent of the étale $\mathsf{A}_n[1/g]$-algebra $C_n[1/g]$, and the latter agrees with the image of the diagonal idempotent $e_n \in R_{n*}$ of the étale A_n^a-algebra C_n. Summing up, we conclude that for every $\gamma \in \mathbb{N}[1/p] \setminus \{0\}$, the image of e_γ vanishes in $R_{n(N_\gamma,\beta)}/b^{N_\gamma}R_{n(N_\gamma,\gamma)}$, and hence $g^{2\gamma}e_\gamma$ vanishes in $R/b^{N_\gamma}R$, *i.e.* $g^{2\gamma}e_\gamma = b^{N_\gamma}e'_\gamma$ for some $e'_\gamma \in R$. Since b is regular in $R[1/g]$, it follows that the image of e'_γ in $R[1/g]$ agrees with $g^{3\gamma}e_{B/A_0[1/g]}$, as required.

(h)⇒(b): By assumption, for every $\gamma \in \mathbb{N}[1/p] \setminus \{0\}$ there exists $n(\gamma) \in \mathbb{N}$ and an A_1-linear map $f_\gamma : A_1^{\oplus n(\gamma)} \to B_1$ such that

$$g^\gamma \cdot \mathrm{Coker}(A_1[1/b] \otimes_{A_1} f_\gamma) = 0. \qquad (16.9.54)$$

Then, let E_γ be the fibre product in the cartesian diagram of A_1-modules :

$$\mathscr{D}_\gamma \quad : \quad \begin{array}{ccc} E_\gamma & \xrightarrow{f'_\gamma} & B_1 \\ \downarrow & & \downarrow {\scriptstyle g^\gamma \mathbf{1}_{B_1}} \\ A_1^{\oplus n(\gamma)} & \xrightarrow{f_\gamma} & B_1. \end{array}$$

By virtue of (16.9.54), the image of $A_1[1/b] \otimes_{A_1} g^\gamma \mathbf{1}_{B_1}$ lies in the image of $A_1[1/b] \otimes_{A_1} f_\gamma$; on the other hand, the diagram $A_1[1/b] \otimes_{A_1} \mathscr{D}_\gamma$ is still cartesian, so the map $A_1[1/b] \otimes_{A_1} f'_\gamma$ *is surjective*. Recall that b is regular in both A_1 and B_1, therefore also in E_γ; then endow $A_1[1/b]$ (resp. $B_1[1/b]$) with the f-adic topology $\mathscr{T}_{A_1}$ (resp. $\mathscr{T}_{B_1}$) such that A_1 (resp. B_1) is a subring of definition, with ideal of definition bA_1 (resp. $b_1 B_1$); since A_1 and B_1 are b-adically complete and separated (theorem 16.9.42(i)), then $\mathscr{T}_{A_1}$ and $\mathscr{T}_{B_1}$ are complete and separated (proposition 8.3.33). Endow also $A_1[1/b]^{\oplus n(\gamma)}$ with the product topology $\mathscr{T}_{A_1} \times \cdots \times \mathscr{T}_{A_1}$; it is easily seen that all the maps of $A_1[1/b] \otimes_{A_1} \mathscr{D}_\gamma$ are continuous for these topologies, hence $A_1[1/b] \otimes_{A_1} E_\gamma$ is a closed subset of $A_1[1/b] \otimes_{A_1} (A_1^{\oplus n(\gamma)} \oplus B_1)$, and is therefore complete and separated for the topology induced by the inclusion in this A_1-module; moreover, since $A_1^{\oplus n(\gamma)}$ is open in $A_1[1/b]^{\oplus n(\gamma)}$ and B_1 is open in $B_1[1/b]$, we see that E_γ *is open in* $A_1[1/b] \otimes_{A_1} E_\gamma$. By the Banach open mapping theorem ([100, Lemma 2.4]), there exists therefore $N_\gamma \in \mathbb{N}$ such that $b^{N_\gamma} B_1 \subset \mathrm{Im}(f'_\gamma)$. It follows easily that

$$g^\gamma b^{N_\gamma} B_1 \subset \mathrm{Im}(f_\gamma). \tag{16.9.55}$$

By virtue of lemma 14.2.24(iv), we deduce that

$$g^\gamma b^{N_\gamma} \mathrm{Coker}\, \varphi = 0 \tag{16.9.56}$$

where φ is as in (16.9.53). On the other hand, since $A_1[1/(gb)] \otimes_{A_1} f_\gamma : A[1/g]^{\oplus n(\gamma)} \to B[1/b]$ is surjective, and since $B[1/b]$ is a projective $A[1/g]$-module, there exists an $A[1/g]$-linear map $h : B[1/b] \to A[1/g]^{\oplus n(\gamma)}$ such that $f_\gamma \circ h = \mathbf{1}_{B[1/b]}$. Let $\varepsilon_1, \dots, \varepsilon_{n(\gamma)}$ be the canonical basis of the free $A[1/g]$-module $A[1/g]^{\oplus n(\gamma)}$, and pick $k \in \mathbb{N}$ such that $h \circ f_\gamma(b^k g^k \varepsilon_i) \in A_1^{\oplus n(\gamma)}$ for every $i = 1, \dots, n(\gamma)$. In view of (16.9.55), we deduce an A_1-linear map :

$$h' : B_1 \to A_1^{\oplus n(\gamma)} \qquad x \mapsto b^{N_\gamma + k} g^{k+\gamma} h(x)$$

and by construction we have $f_\gamma \circ h' = b^{N_\gamma + k} g^{k+\gamma} \mathbf{1}_{B_1}$. Then lemma 14.2.24(iv) implies that

$$b^{N_\gamma + k} g^{k+\gamma} \mathrm{Ker}\, \varphi = 0. \tag{16.9.57}$$

Now, let $e_\bullet := (e_n \mid n \in \mathbb{N}) \in \lim_{n \in \mathbb{N}} (R^a / b^n R^a)_*$ be the compatible system of diagonal idempotents of the étale $(A_0/b^n A_0)^a$-algebras $B_1^a / b^n B_1^a$, for every $n \in \mathbb{N}$; by (16.9.56) there exists $e' \in R$ such that $\varphi(e') = g^{2\gamma} b^{N_\gamma} e_\bullet$. Then we have

$$\varphi(x \cdot e') = 0 \qquad \text{for every } x \in \mathrm{Ker}\, \mu.$$

Combining with (16.9.57), it follows that $b^{N_\gamma + k} g^{k+\gamma} x \cdot e' = 0$ in R, for every $x \in \mathrm{Ker}\, \mu$. But we have already observed that b is regular in $R[1/g]$; we then easily conclude that the image of e' in $R[1/g]$ agrees with $g^{2\gamma} b^{N_\gamma} e_{B_1/A_0[1/g]}$.

(ii): Pick $n \in \mathbb{N}$ such that $g^n e_{B/A_0[1/g]}$ lies in the image of the localization $B_1 \otimes_{A_1} B_1 \to B \otimes_{A_0[1/g]} B$. Arguing as in the proof of claim 14.3.23, we then find $k \in \mathbb{N}$ and A_1-linear maps $B_1 \to A_1^{\oplus k} \to B_1$ whose composition is $g^k \cdot \mathbf{1}_{B_1}$. Recall now that $R^\wedge$ is the limit of the inverse system $(B_1 \otimes_{A_1} B_1 / b^i B_1 \mid i \in \mathbb{N})$; then the first assertion follows from lemma 14.2.24(iv).

In order to show the second assertion of (ii), let C_n be the étale A_n^a-algebra of finite rank as in the proof of theorem 16.9.42, for every $n \in \mathbb{N}$. Since A_n is b-adically complete and separated, the almost projective A_n^a-module of finite rank $S_n := C_n \otimes_{\mathsf{A}_n} C_n$ is b-adically complete and separated as well, for every $n \in \mathbb{N}$, by [75, Claim 5.3.25], and moreover $\mathrm{Ann}_{S_n}(g) = 0$, since $\mathrm{Ann}_{\mathsf{A}_n}(g) = 0$; then we are easily reduced to checking :

Claim 16.9.58. $(R^\wedge)^a$ represents the limit of the inverse system of A_0-algebras $(S_n \mid n \in \mathbb{N})$.

Proof of the claim. It suffices to show that the $A_0^a/b^k A_0^a$-algebra $(B_1/b^k B_1 \otimes_{A_1/b^k A_1} B_1/b^k B_1)^a$ represents the limit of the induced inverse system $(C_n/b^k C_n \otimes_{\mathsf{A}_n^a/b^k \mathsf{A}_n^a} C_n/b^k C_n \mid n \in \mathbb{N})$, for every $k \in \mathbb{N}$. However, the proof of theorem 16.9.42 shows that the inverse system $(C_n/b^k C_n \mid n \in \mathbb{N})$ is almost essentially constant for every $k \in \mathbb{N}$, and the same holds fo the inverse system $(\mathsf{A}_n^a/b^k \mathsf{A}_n^a \mid n \in \mathbb{N})$, by claim 16.9.43(ii). Then the assertion follows from proposition 14.2.38. $\square$

We conclude with a discussion of some further constructions that we do not need (though, see remark 17.5.17), but may be useful for other questions, and are related to some recent work of Y.André and others as well.

16.9.59. Let A be a perfectoid Tate ring, $A_0 \subset A$ a perfectoid subring of definition, $p \in \mathbb{N}$ a prime integer, and $b \in A_0^{\circ\circ} \cap A^\times$ with $pA_0 \subset b^p A_0$. We consider the categories :

$$A_0\text{-}\mathsf{cp.Adic} \qquad A_0\text{-}\mathsf{Tate} \qquad A_0\text{-}\mathsf{Perf.Adic} \qquad A_0\text{-}\mathsf{Perf.Tate}$$

such that :

- the objects of A_0-$\mathsf{cp.Adic}$ are the adic ring homomorphisms $A_0 \to B_0$ of complete separated topological rings, and the morphisms are the maps of A_0-algebras
- A_0-Tate is the full subcategory of A_0-$\mathsf{cp.Adic}$ whose objects are the A_0-algebras B_0 such that the localisation $B_0 \to B_0[b^{-1}]$ is injective
- A_0-$\mathsf{Perf.Adic}$ (resp. A_0-$\mathsf{Perf.Tate}$) is the full subcategory of A_0-$\mathsf{cp.Adic}$ (resp. of A_0-Tate) whose objects are the perfectoid A_0-algebras.

Notice that every morphism in these categories is adic. We wish to exhibit right adjoints for the inclusion functors

(16.9.60) $$A_0\text{-}\mathsf{Perf.Adic} \to A_0\text{-}\mathsf{cp.Adic} \qquad A_0\text{-}\mathsf{Perf.Tate} \to A_0\text{-}\mathsf{Tate}.$$

Thus, consider any $B_0 \in \mathrm{Ob}(A_0\text{-}\mathsf{cp.Adic})$, set $\overline{A}_0 := A_0/pA_0$, $\overline{B}_0 := B_0/pB_0$, and let $\overline{\beta}_\bullet := (\overline{\beta}_n \mid n \in \mathbb{N}) \in \mathbf{E}_0 := \mathbf{E}(\overline{A}_0)$ be any element such that $\overline{\beta}_0 \in \overline{A}_0$ is the class of b. Endow $\mathbf{E}(\overline{B}_0)$ with its $\overline{\beta}_\bullet$-adic topology $\mathscr{T}_{\mathbf{E}(\overline{B}_0)}$; let also $\mathscr{T}_{\overline{B}_0}$ be the b-adic topology of $\overline{B}_0$. Then the topology of $\mathbf{E}(\overline{B}_0, \mathscr{T}_{\overline{B}_0})$ is complete, separated, adic, and coarser than $\mathscr{T}_{\mathbf{E}(\overline{B}_0)}$ (remark 9.4.9(ii)). By lemma 8.3.12, it follows that $\mathscr{T}_{\mathbf{E}(\overline{B}_0)}$ is complete and separated. Then the A_0-algebra

$$B_0^\natural := A_0 \otimes_{W(\mathbf{E}_0)} W(\mathbf{E}(\overline{B}_0), \mathscr{T}_{\mathbf{E}(\overline{B}_0)})$$

is a perfectoid ring for its b-adic topology (example 16.3.2(ii)). The structure map $A_0 \to B_0$ of B_0, and the map $u_{B_0} : W(\mathbf{E}(\overline{B}_0)) \to B_0$ induce an adic map

$$\varepsilon_{B_0} : B_0^\natural \to B_0 \qquad 1 \otimes (x_n \mid n \in \mathbb{N}) \mapsto \sum_{n\in\mathbb{N}} p^n \cdot \overline{u}_A(x_n^{p^{-n}})$$

and notice that ε_{B_0} is an isomorphism if B_0 is perfectoid. Next, let $\beta_\bullet := (\beta_n \mid n \in \mathbb{N}) \in \mathbf{E}(A_0)$ be the preimage of $\overline{\beta}_\bullet$ under the isomorphism $\mathbf{E}(A_0) \xrightarrow{\sim} \mathbf{E}_0$ induced by the projection $A_0 \to \overline{A}_0$. Hence $\beta_0 - b \in pA_0$, so that $\beta_0/b \in A_0^\times$; if $B_0 \in \mathrm{Ob}(A_0\text{-}\mathsf{Tate})$, then the image of b is regular in B_0, so the same holds for the image of β_0, and finally, the image of $\beta_\bullet$ is a regular element of $\mathbf{E}(\overline{B}_0)$ (proposition 16.4.17); by the same token, the image of β_0 in $B_0^\natural$ is then regular, and hence the same holds for the image of b. Hence, in this case $B_0^\natural$ is an object of A_0-$\mathsf{Perf.Tate}$. Moreover, every morphism $g : B_0 \to C_0$ of A_0-Tate induces an adic map of A_0-algebras

$$g^\natural : B_0^\natural \to C_0^\natural \qquad 1 \otimes (\overline{x}_n \mid n \in \mathbb{N}) \mapsto 1 \otimes (\mathbf{E}(\overline{g})(\overline{x}_n) \mid n \in \mathbb{N})$$

with $\overline{g} := g \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} : \overline{B}_0 \to \overline{C}_0$. We have therefore well defined functors

$$(-)^\natural : A_0\text{-}\mathsf{cp.Adic} \to A_0\text{-}\mathsf{Perf.Adic} \qquad (-)^\natural : A_0\text{-}\mathsf{Tate} \to A_0\text{-}\mathsf{Perf.Tate}.$$

Now, let $C \in \mathrm{Ob}(A_0\text{-}\mathsf{Perf})$; to every adic map $f : C \to B_0^\natural$ let us attach $f^* := \varepsilon_{B_0} \circ f : C \to B_0$, and to every adic map $h : C \to B_0$ let us attach $h^* := h^\natural \circ \varepsilon_C^{-1} : C \to B_0^\natural$. By a simple inspection we get $h^{**} = h$. In order to check that $f^{**} = f$, it suffices to show that $\varepsilon_{B_0^\natural} = \varepsilon_{B_0}^\natural$. Thus, let $\overline{x}_\bullet := (x_n \mid n \in \mathbb{N}) \in W(\mathbf{E}(\overline{B}_0^\natural))$ be any element; via the natural identification $\mathbf{E}(B_0^\natural) \xrightarrow{\sim} \mathbf{E}(\overline{B}_0^\natural)$, we may view $\overline{x}_\bullet$ as a system $x_{\bullet\bullet} := (x_{n,k} \mid n,k \in \mathbb{N})$ of elements of $B_0^\natural$ such that $x_{n,k+1}^p = x_{n,k}$ for every $n, k \in \mathbb{N}$; then we have :

$$\varepsilon_{B_0^\natural}(1 \otimes x_{\bullet\bullet}) = \sum_{n\in\mathbb{N}} p^n x_{n,n} \qquad \varepsilon_{B_0}^\natural(1 \otimes \overline{x}_\bullet) = 1 \otimes (\mathbf{E}(\overline{\varepsilon}_{B_0})(\overline{x}_n) \mid n \in \mathbb{N}).$$

But we have $\sum_{n\in\mathbb{N}} p^n \cdot \tau(\mathbf{E}(\overline{\varepsilon}_{B_0})(\overline{x}_n)^{p^{-n}}) = (\mathbf{E}(\overline{\varepsilon}_{B_0})(\overline{x}_n) \mid n \in \mathbb{N})$ in $W(\mathbf{E}(B_0^\natural))$, by virtue of (9.3.46), where τ denotes the Teichmüller representative. Hence, let $\overline{y}_\bullet := (\overline{y}_k \mid k \in \mathbb{N}) \in \mathbf{E}(\overline{B}_0^\natural)$ be any element, which we identify with a unique element $(y_k \mid k \in \mathbb{N}) \in \mathbf{E}(B_0^\natural)$; we come down to showing that $1 \otimes \tau(\overline{y}_\bullet) = y_0 = \overline{u}_{B_0^\natural}(y_\bullet)$ in $B_0^\natural$. But we have $1 \otimes \tau(\overline{y}_\bullet) = u_{B_0^\natural}(\tau(\overline{y}_\bullet))$, so the assertion follows from lemma 16.1.1(iv). This shows that the rule $f \mapsto f^*$ yields the sought adjunctions between (16.9.60) and the functors $(-)^\natural$.

16.9.61. In the situation of (16.9.59), let also $(A_0, \mathfrak{m})$ be a basic setup, such that $\mathfrak{m}$ is the filtered union of a countable system of principal subideals, and consider as well the categories

$$(A_0, \mathfrak{m})^a\text{-}\mathsf{Perf.Adic} \qquad (A_0, \mathfrak{m})^a\text{-}\mathsf{Perf.Tate}$$

with the same objects as A_0-Perf.Adic (resp. as A_0-Perf.Tate), and whose morphisms $B_0 \to C_0$ are the morphisms $B_0^a \to C_0^a$ of $(A_0, \mathfrak{m})^a$-algebras. We have obvious forgetful functors :

$$\text{(16.9.62)} \qquad \begin{aligned} A_0\text{-}\mathsf{Perf.Adic} &\to (A_0, \mathfrak{m})^a\text{-}\mathsf{Perf.Adic} \\ A_0\text{-}\mathsf{Perf.Tate} &\to (A_0, \mathfrak{m})^a\text{-}\mathsf{Perf.Tate} \end{aligned} \qquad B_0 \mapsto B_0^a.$$

As explained in (16.9.31), we may attach to $\mathfrak{m}$ an ideal $\mathfrak{m}_\mathbf{E} \subset \mathbf{E}_0$, and remark 16.9.32(i) implies that $(\mathbf{E}_0, \mathfrak{m}_\mathbf{E})$ is a basic setup compatible with $(A_0, \mathfrak{m})$, in the sense of definition 16.8.10. For every $B_0 \in \mathrm{Ob}(A_0\text{-}\mathsf{cp.Adic})$, we then let B_0^a (resp. $\mathbf{E}(B_0)^a$) be the $(A_0, \mathfrak{m})^a$-algebra attached to B (resp. the $(\mathbf{E}_0, \mathfrak{m}_\mathbf{E})^a$-algebras attached to $\mathbf{E}(B_0)$).

Theorem 16.9.63. *Let $B_0 \in \mathrm{Ob}(A_0\text{-}\mathsf{cp.Adic})$, and endow $C := (B_0)^a_{!!}$ with its b-adic topology, and $D := \mathbf{E}(B_0)^a_{!!}$ with its $\overline{\beta}_\bullet$-adic topology, as in* (16.9.59). *We have :*

(i) $(B_0)^a_* \in \mathrm{Ob}(A_0\text{-}\mathsf{cp.Adic})$, *and if* $B_0 \in \mathrm{Ob}(A_0\text{-}\mathsf{Tate})$, *then* $(B_0)^a_* \in \mathrm{Ob}(A_0\text{-}\mathsf{Tate})$.
(ii) *There exists a natural isomorphism* $\mathbf{E}(B^a_{0*}) \xrightarrow{\sim} (\mathbf{E}(B_0))^a_*$ *of* $\mathbf{E}_0$*-algebras.*
(iii) *Suppose moreover that the basic setup* $(B_0, \mathfrak{m}B_0)$ *is almost perfectoid; then we have :*
  (a) $\varepsilon^a_{B_0} : (B_0^\natural)^a \to B_0^a$ *is an isomorphism.*
  (b) *The completions* $C^\wedge$ *and* $D^\wedge$ *of* C *and* D *are perfectoid.*
  (c) $C^\wedge/\mathrm{Ann}_{C^\wedge}(b) \in \mathrm{Ob}(A_0\text{-}\mathsf{Perf.Tate})$.
  (d) *There exists a natural isomorphism* $\mathbf{E}(C^\wedge) \xrightarrow{\sim} D^\wedge$.
(iv) *The natural morphisms* $(C^\wedge)^a \to B_0^a$ *and* $(D^\wedge)^a \to \mathbf{E}(B_0)^a$ *are isomorphisms.*
(v) *If* B_0 *is perfectoid, the same holds for* $C^\wedge$ *and* $D^\wedge$.
(vi) *The functors* (16.9.62) *admit both left and right adjoints.*

Proof. (i): If $b \cdot \mathbf{1}_{B_0}$ is an injective map, the same holds for $b \cdot \mathbf{1}_{B^a_{0*}}$, because the functor $(-)^a_*$ commutes with limits; the assertion then follows from claim 16.9.39.

(ii): By remark 16.9.32(iii), the natural map $B_0 \to B^a_{0*}$ induces an isomorphism $\mathbf{E}(B_0)^a_* \xrightarrow{\sim} \mathbf{E}(B^a_{0*})^a_*$; in light of (i), we may then replace B_0 by B^a_{0*}, and assume from start that $B_0 = B^a_{0*}$, in which case we show that the natural map $\mathbf{E}(B_0) \to \mathbf{E}(B_0)^a_*$ is an isomorphism. To this aim, consider the system of sets and maps of sets $S_{\bullet\bullet} := (S_{ij}, \varphi^v_{ij}, \varphi^h_{ij} \mid i, j \in \mathbb{N})$ with $S_{ij} := (B_0/b^j B_0)^a_*$ for every $i, j \in \mathbb{N}$, and where :

- $\varphi^h_{ij} : S_{i+1,j} \to S_{ij}$ is given by the rule : $x \mapsto x^p$ for every $x \in (B_0/b^j B_0)^a_*$
- $\varphi^v_{ij} : S_{i,j+1} \to S_{ij}$ is induced by the projection $B_0/b^{j+1}B_0 \to B_0/b^j B_0$.

We compute the inverse limit L of the system $S_{\bullet\bullet}$ in two different ways :

$$L = \lim_{i\in\mathbb{N}} \lim_{j\in\mathbb{N}} S_{ij} \xrightarrow{\sim} \lim_{j\in\mathbb{N}} \lim_{i\in\mathbb{N}} S_{ij}.$$

Notice that, for fixed $i \in \mathbb{N}$, the system $S_{i\bullet} := (S_{ij}, \varphi^v_{ij} \mid j \in \mathbb{N})$ consists of rings and ring homomorphisms, and since the functor $(-)^a_*$ commutes with limits, the limit of $S_{i\bullet}$ represents $L_i := B^a_{0*} = B$; then, the system of maps $(\varphi^h_{ij} \mid j \in \mathbb{N})$ is a natural transformation $S_{i+1\bullet} \to S_{i\bullet}$ whose limit $\varphi^h_i : L_{i+1} \to L_i$ is obviously the map given by the rule : $x \mapsto x^p$ for every $x \in B$. Thus $L := \lim_{i\in\mathbb{N}}(S_{i\bullet}, \varphi^h_i)$ represents the set underlying $\mathbf{E}(B)$. Next, for every $j \in \mathbb{N}$, let L'_j be the limit of the system of sets $S_{\bullet j} := (S_{ij}, \varphi^h_{ij} \mid i \in \mathbb{N})$; again, the system $(\varphi^v_{ij} \mid i \in \mathbb{N})$ is a natural transformation $S_{\bullet j+1} \to S_{\bullet j}$, inducing a map $\varphi^v_j : L'_{j+1} \to L_j$ and L is also $\lim_{j\in\mathbb{N}}(L'_j, \varphi^v_j)$. Notice now that L'_0 is the set underlying $\mathbf{E}(B_0/bB_0)^a_*$, since the functor $(-)^a_*$ commutes with limits. We are therefore reduced to checking that φ^v_j is bijective for every $j \in \mathbb{N}$. To this aim, it suffices to exhibit a system of maps $(s_{ij} : S_{i+1,j} \to S_{i,j+1} \mid i \in \mathbb{N})$ such that

$$s_{ij} \circ \varphi^v_{i+1,j} = \varphi^h_{i,j+1} \qquad \varphi^v_{ij} \circ s_{ij} = \varphi^h_{ij} \qquad \text{for every } i, j \in \mathbb{N}$$

(details left to the reader). This in turn is achieved by the following more general :

Claim 16.9.64. Let $(V, \mathfrak{m}_V)$ be a basic setup such that $\widetilde{\mathfrak{m}}_V := \mathfrak{m}_V \otimes_V \mathfrak{m}_V$ is a flat V-module, R a V-algebra, $I \subset R$ an ideal with $pI = 0$ and $x^p = 0$ for every $x \in I$. Denote by $\pi : R \to R/I$ the projection. Then there exists a map s that makes commute the diagram of sets :

$$\begin{array}{ccc} R^a_* & \xrightarrow{\Phi} & R^a_* \\ {\scriptstyle \pi^a_*}\downarrow & \overset{s}{\nearrow} & \downarrow{\scriptstyle \pi^a_*} \\ (R/I)^a_* & \xrightarrow{\overline{\Phi}} & (R/I)^a_* \end{array}$$

where Φ is the map given by the rule : $x \mapsto x^p$ for every $x \in R^a_*$, and likewise for $\overline{\Phi}$.

Proof of the claim. For every V-algebra S, denote by $\sigma_S : R/I \otimes_V S \to R \otimes_V S$ the map defined as follows. For every $\overline{x} \in R/I \otimes_V S$, pick $x \in R \otimes_V S$ such that $\pi(x) = \overline{x}$, and set $\sigma_S(x) := x^p$; it is easily seen that this map is independent of the choice of representatives, and moreover the rule : $S \mapsto \sigma_S$ yields a homogeneous polynomial law $\sigma : R/I \rightsquigarrow R$ of degree p (see definition 9.5.1(ii)). Thus, σ factors through the universal homogeneous polynomial law $\boldsymbol{\lambda}^p_{R/I} : R/I \rightsquigarrow \Gamma^p_V(R/I)$ and a unique R-linear map $\sigma^* : \Gamma^p_V(R/I) \to R$ (see (9.5.19)). On the other hand, since $\widetilde{\mathfrak{m}}_V$ is a flat V-module, we have a V-linear isomorphism $\omega : \widetilde{\mathfrak{m}}_V \xrightarrow{\sim} \Gamma^p_V(\widetilde{\mathfrak{m}}_V)$ ([75, (2.1.11)]); then the map given by the rule :

$$\mathrm{Hom}_V(\widetilde{\mathfrak{m}}_V, R/I) \to \mathrm{Hom}_V(\widetilde{\mathfrak{m}}_V, R) \qquad \varphi \mapsto (\widetilde{\mathfrak{m}}_V \xrightarrow{\omega} \Gamma^p_V(\widetilde{\mathfrak{m}}_V) \xrightarrow{\Gamma^p_V(\varphi)} \Gamma^p_V(R/I) \xrightarrow{\sigma^*} R)$$

fulfills the required conditions : details left to the reader. ◊

(iii.a): Let B_1 be the p-root closure of B_0 in $B_0[b^{-1}]$, endow B_1 with its b-adic topology, and let $B_1^\wedge$ be the completion of B_1. Since $b \cdot \mathbf{1}_{B_1}$ is injective, the same holds for $(b \cdot \mathbf{1}_{B_1})^\wedge = b \cdot \mathbf{1}_{B_1^\wedge}$ (proposition 8.2.13(i)). Then, according to claim 16.9.38 and proposition 16.9.37(i), the basic setup $(B_1^\wedge, \mathfrak{m}B_1^\wedge)$ is almost perfectoid and the induced map $i : B_0 \to B_1^\wedge$ is an almost isomorphism. Moreover, $B_1^\wedge$ is p-root closed in $B_1^\wedge[b^{-1}]$, by lemma 9.8.26(ii) and proposition 8.3.33(iii). We consider the commutative diagram

$$\begin{array}{ccc} B_0^\natural & \xrightarrow{\varepsilon_{B_0}} & B_0 \\ {\scriptstyle i^\natural}\downarrow & & \downarrow{\scriptstyle i} \\ (B_1^\wedge)^\natural & \xrightarrow{\varepsilon_{B_1^\wedge}} & B_1^\wedge. \end{array}$$

By remark 16.9.32(iii), i induces an isomorphism $\mathbf{E}(i)^a : \mathbf{E}(B_0)^a \to \mathbf{E}(B_1^\wedge)^a$ of $(\mathbf{E}, \mathfrak{m}_\mathbf{E})^a$-algebras. Let $\mathfrak{n} \subset A_0$ be the ideal generated by $\overline{u}_{A_0}(\mathfrak{m}_\mathbf{E})$; it is easily seen that $(A_0, \mathfrak{n})$ is a basic setup, and then $i^\natural := A_0 \otimes_{W(\mathbf{E}_0)} W(\mathbf{E}(i))$ is an almost isomorphism relative to the almost structure furnished by $(A_0, \mathfrak{n})$. Now, $i^\natural$ is the inverse limit of the system of maps $(i_n := i^\natural \otimes_\mathbb{Z} \mathbb{Z}/p^n \mid n \in \mathbb{N})$, and in view of (16.9.33) we deduce that i_n is an almost isomorphism also relative to the original almost structure $(A_0, \mathfrak{m})$, for every $n \in \mathbb{N}$, and finally, the same holds for $i^\natural$. Summing up, we are therefore reduced to checking that $\varepsilon^a_{B_1^\wedge}$ is an isomorphism of $(A_0, \mathfrak{m})^a$-algebras, so we may assume from start that B_0 is p-root closed. Denote by $\mathrm{gr}_\bullet B_1$ the graded ring associated with the b-adic filtration on B_0, and likewise define $\mathrm{gr}_\bullet B_0^\natural$; since both B_0 and $B_0^\natural$ are b-adically complete and separated, we are further reduced to showing that the induced map $\mathrm{gr}_\bullet \varepsilon_{B_0} : \mathrm{gr}_\bullet B_0^\natural \to \mathrm{gr}_\bullet B_0$ is an almost isomorphism, and since b is regular in both B_0 and $B_0^\natural$, it suffices to prove that the same holds for

$$\mathrm{gr}_0 \varepsilon_{B_0} : B_0^\natural / bB_0^\natural \to B_0/bB_0.$$

Claim 16.9.65. Define $\beta_\bullet \in \mathbf{E}(A_0)$ as in (16.9.59). Then we have :

(i) The Frobenius endomorphism of $B_0/\beta_n B_0$ induces an injective map

$$B_0/\beta_{n+1}B_0 \to B_0/\beta_n B_0 \qquad \text{for every } n \in \mathbb{N}.$$

(ii) The kernel of the projection $\mathbf{E}(B_0) \to B_0/bB_0$ is $\beta_\bullet \mathbf{E}(B_0)$.

(iii) We have a natural identification $B_0^\natural/bB_0^\natural \xrightarrow{\sim} \mathbf{E}(B_0)/\beta_\bullet \mathbf{E}(B_0)$.

Proof of the claim. (i) follows straightforwardly from the assumption that B_0 is p-root closed in $B_0[b^{-1}]$, and (ii) follows easily from (i) : the details are left to the reader.

(iii): Let $\alpha_\bullet \in W(\mathbf{E}_0)$ be a distinguished element in the kernel of $u_A : W(\mathbf{E}_0) \to A_0$; we have $p \in b^p$ by assumption, and $\alpha_0 \in \beta_\bullet^p \mathbf{E}_0$, by lemma 16.1.9(ii). But the natural isomorphism

$$\mathbf{E}(B_0)/\alpha_0\mathbf{E}(B_0) \xrightarrow{\sim} \mathbf{E}_0/\alpha_0\mathbf{E}_0 \otimes_{\mathbf{E}_0} \mathbf{E}(B_0) \xrightarrow{\sim} A_0 \otimes_{W(\mathbf{E}_0)} \mathbf{E}(B_0) \xrightarrow{\sim} B_0^\natural/pB_0^\natural$$

maps the class of $\beta_\bullet$ to the class of b, whence the contention. ◊

By claim 16.9.65(ii), the projection $\overline{u}_{B_0/pB_0} : \mathbf{E}(B_0) \xrightarrow{\sim} \mathbf{E}(B_0/pB_0) \to B_0/bB_0$ induces an injective map

$$j : \mathbf{E}(B_0)/\beta_\bullet \mathbf{E}(B_0) \to B_0/bB_0$$

and claim 16.9.65(iii) yields a natural identification of j with $\mathrm{gr}_0\varepsilon_{B_0}$. Thus, it remains only to check that $\overline{u}^a_{B_0/pB_0}$ is an epimorphism of $(\mathbf{E}_0, \mathfrak{m}_\mathbf{E})^a$-algebras. However, recall that $\mathbf{E}(B_0)$ is the limit of the system of rings $(R_n \mid n \in \mathbb{N})$, with $R_n := B_0/bB_0$ for every $n \in \mathbb{N}$, and with transition maps given by the Frobenius endomorphism Φ_{B_0/bB_0}. Thus, $\overline{u}_{B_0/pB_0}$ is the inverse limit of the system of maps $(\Phi^n_{B_0/bB_0} : R_n \to B_0/bB_0 \mid n \in \mathbb{N})$, each of which is almost surjective, since $(B_0, \mathfrak{m}B_0)$ is almost perfectoid (as usual, we regard $(\Phi^n_{B_0/bB_0})^a$ as a morphism of $(\mathbf{E}_0, \mathfrak{m}_\mathbf{E})^a$-algebras $R^a_{n,(\Phi^{-n}_{\mathbf{E}_0})} \to B_0^a/bB_0^a$: cp. remark 16.9.32(iii)). To conclude, it suffices now to invoke lemma 14.2.24(i).

(iii.b): We consider first the assertion for $C^\wedge$: set $\overline{A}_0 := A_0/bA_0$, and $\overline{\mathfrak{m}} := \mathfrak{m}\overline{A}_0$; by assumption, the Frobenius endomorphism of B_0/b^pB_0 induces an isomorphism $\overline{\Phi}^a_{B_0} : (B_0/bB_0)^a \xrightarrow{\sim} (B_0/b^pB_0)^a_{(\Phi)}$ of $(\overline{A}_0, \overline{\mathfrak{m}})^a$-algebras, whence the isomorphism of $\overline{A}_0$-algebras

$$(\overline{\Phi}_{B_0})^a_{!!} : (B_0/bB_0)^a_{!!} \xrightarrow{\sim} ((B_0/b^pB_0)^a_{(\Phi)})_{!!}$$

where $(-)_{!!}$ denotes the left adjoint of the localization functor $(-)^a : \overline{A}_0\text{-}\mathsf{Alg} \to (\overline{A}_0, \overline{\mathfrak{m}})^a\text{-}\mathsf{Alg}$. On the other hand, by a direct inspection of the construction of these left adjoints, one gets natural identifications

$$C/bC \xrightarrow{\sim} (B_0/bB_0)^a_{!!} \qquad C/b^pC \xrightarrow{\sim} ((B_0/b^pB_0)^a_{(\Phi)})_{!!}$$

where, however, the construction of $C := (B_0)^a_{!!}$ refers to the left adjoint of the corresponding localization $(-)^a : A_0\text{-}\mathsf{Alg} \to (A_0, \mathfrak{m})^a\text{-}\mathsf{Alg}$. It is then easily seen that, under these identifications, the map $(\overline{\Phi}_{B_0})^a_{!!}$ corresponds to the ring homomorphism $\overline{\Phi}_C : C/bC \to C/b^pC$ induced by the Frobenius endomorphism of C/b^pC. The latter is therefore an isomorphism, and so the same holds for the corresponding ring homomorphism $\overline{\Phi}_{C^\wedge} : C^\wedge/bC^\wedge \to C^\wedge/b^pC^\wedge$. Then, according to corollary 16.3.75, in order to check that $C^\wedge$ is perfectoid, it remains only to show that $\mathrm{Ann}_{C^\wedge}(b^p) = \mathrm{Ann}_{C^\wedge}(b^{p-1})$, and in light of remark 8.6.39(iii), we are reduced to proving that $\mathrm{Ann}_C(b^p) = \mathrm{Ann}_C(b^{p-1})$. Let $\mu : \widetilde{\mathfrak{m}} \to A_0$ be the A_0-linear map such that $\mu(x \otimes y) := xy$ for every $x, y \in \mathfrak{m}$; we notice :

Claim 16.9.66. $\mathrm{Ann}_C(b^n)$ is a quotient of $\mathrm{Ker}(\widetilde{\mathfrak{m}}/b^n\widetilde{\mathfrak{m}} \xrightarrow{A_0/b^nA_0 \otimes_{A_0} \mu} A_0/b^nA_0)$, for all $n \in \mathbb{N}$.

Proof of the claim. We consider the commutative diagram of A_0-modules with exact rows :

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{n} & \longrightarrow & A_0 \oplus (\widetilde{\mathfrak{m}} \otimes_{A_0} B_0) & \longrightarrow & (B_0)^a_{!!} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathfrak{n} & \longrightarrow & A_0 \oplus (\widetilde{\mathfrak{m}} \otimes_{A_0} B_0) & \longrightarrow & (B_0)^a_{!!} & \longrightarrow & 0
\end{array}$$

whose vertical arrows are scalar multiplication by b^n, and where $\mathfrak{n}$ denotes the image of the map $\widetilde{\mathfrak{m}} \to A_0 \oplus (\widetilde{\mathfrak{m}} \otimes_{A_0} B_0)$ given by the rule : $z \mapsto (\mu(z), z \otimes 1)$ for every $z \in \widetilde{\mathfrak{m}}$. By assumption, b is a regular element in both A_0 and B_0, and $\widetilde{\mathfrak{m}}$ is a flat A_0-module, by [75, Prop.2.1.7(i)]. Hence, the central vertical arrow is injective, and the assertion follows easily from the snake lemma. ◊

In view of claim 16.9.66, we are reduced to showing that $b \cdot \mathrm{Ker}(A_0/b^nA_0 \otimes_{A_0} \mu) = 0$ for every $n \in \mathbb{N}$. To this aim, write $\mathfrak{m} = \bigcup_{n\in\mathbb{N}} A_0a_i$ for a system $(a_i \mid i \in \mathbb{N})$ of elements of A_0 such that for every $i \in \mathbb{N}$ we have $a_i = c_ia^p_{i+1}$ for some $c_i \in A_0$ (see (16.9.31)); by remark 16.9.32(ii), the A_0-module $\widetilde{\mathfrak{m}}$ is then isomorphic to the inductive limit L of a system $(F_i \mid i \in \mathbb{N})$ of free rank one A_0-modules $F_i = A_0e_i$, with transition maps $f_i : F_i \to F_{i+1}$ such that $f_i(e_i) = c_ia^{p-1}_{i+1}e_{i+1}$ for every $i \in \mathbb{N}$. Hence L is the set of equivalence classes $[x, i]$ of pairs (x, i) with $i \in \mathbb{N}$ and $x \in F_i$, and with this notation, μ corresponds to the A_0-linear map $L \to A_0$ given by the rule : $[ye_i, i] \mapsto ya_i$ for every $i \in \mathbb{N}$ and $y \in A_0$. Hence, let $[ye_i, i] \in L$ and suppose that $ya_i \in b^nA_0$; we need to show that $[ye_i, i] \in b^{n-1}L$. Pick $k \in \mathbb{N}$ with $p^k \geq n$; by a simple induction, we see that :

$$[ye_i, i] = y_k[e_{i+k}, i + k] \qquad \text{with} \qquad y_k := yc_ic^p_{i+1} \cdots c^{p^{k-1}}_{i+k-1}a^{p^k-1}_{i+k}.$$

We are then reduced to checking that $y_k \in b^{n-1}A_0$, and by theorem 16.3.64, the latter is equivalent to the condition : $y^{p^k}_k \in b^{(n-1)p^k}A_0$. But we have :

$$y^{p^k}_k = y^{p^k}c_ic^p_{i+1} \cdots c^{p^{k-1}}_{i+k-1}a^{p^k-1}_i = yc_ic^p_{i+1} \cdots c^{p^{k-1}}_{i+k-1} \cdot (ya_i)^{p^k-1} \in b^{n(p^k-1)}A_0 \subset b^{(n-1)p^k}A_0.$$

Next, we consider the assertion concerning $D^\wedge$: the discussion of (16.9.59) already shows that $\mathbf{E}(B_0)$ is a perfectoid ring, when endowed with its $\overline{\beta}_\bullet$-adic topology, and moreover the image of $\overline{\beta}_\bullet$ is regular in $\mathbf{E}(B_0)$. Hence, the assertion for $D^\wedge$ follows from the foregoing case, after replacing the basic setup $(A_0, \mathfrak{m})$ by $(\mathbf{E}, \mathfrak{m}_\mathbf{E})$.

(iii.c): Arguing as in the proof of corollary 16.3.75, we may assume that $b = b_0$ for some $(b_n \mid n \in \mathbb{N}) \in \mathbf{E}(A_0)$; notice also that $C^\wedge$ is reduced, due to (iii.b) and corollary 16.3.63(i), hence $\mathrm{Ann}_{C^\wedge}(b) = \bigcup_{n\in\mathbb{N}} \mathrm{Ann}_{C^\wedge}(b^n)$, so scalar multiplication by b is an injective endomorphism of $C^\wedge/\mathrm{Ann}_{C^\wedge}(b)$. Then the assertion follows from corollary 16.3.71.

(iv): The counit of adjunction for the pair of functors $((-)_{!!}, (-)^a)$ is an almost isomorphism $\varepsilon_{B_0} : C \to B_0$, hence the same for its b-adic completion (cp. the proof of claim 16.9.38). Likewise, recall that $\mathbf{E}(B_0)$ is $\overline{\beta}_\bullet$-adically complete and separated (see (16.9.59)); then the same argument shows that the counit of adjunction $\varepsilon_{\mathbf{E}(B_0)} : D \to \mathbf{E}(B_0)$ induces, after taking $\overline{\beta}_\bullet$-adic completions, an almost isomorphism $D^\wedge \to \mathbf{E}(B_0)$.

(v): We may assume that $b = \beta_0$, where $\beta_\bullet \in \mathbf{E}(A_0)$ is as in (16.9.59); also, arguing as in the proof of (iii.b), we see that $\overline{\Phi}_{C^\wedge}$ is an isomorphism, and then claim 16.3.76 and remark 8.6.39(iii) reduce to showing that $\mathrm{Ann}_C(b^p) = \mathrm{Ann}_C(b^p/\beta_1) = \mathrm{Ann}_C(\beta_1^{p^2-1})$. But, arguing as in the proof of claim 16.9.66 we obtain, for every $n \in \mathbb{N}$, an exact sequence of A_0-modules

$$\widetilde{\mathfrak{m}} \otimes_{A_0} \mathrm{Ann}_{B_0}(b^n) \to \mathrm{Ann}_C(b^n) \to M$$

where M is a quotient of $\mathrm{Ker}(A_0/b^n A_0 \otimes_{A_0} \mu)$. The proof of (iii.b) then shows that $bM = 0$, and on the other hand $\beta_1 \cdot \mathrm{Ann}_{B_0}(b^n) = 0$, by corollary 16.3.71; thus $\beta_1^{p+1} \cdot \mathrm{Ann}_C(b^n) = 0$, and since $p^2 - 1 \geq p + 1$, the assertion follows.

(vi): Let $X \in \mathrm{Ob}(A_0\text{-}\mathsf{Perf.Tate})$ and $Y \in \mathrm{Ob}((A_0, \mathfrak{m})^a\text{-}\mathsf{Perf.Tate})$, and set $Z := (Y^a_{!!})^\wedge$; we know already from (i) and (iii.c) that $Y^a_* \in \mathrm{Ob}(A_0\text{-}\mathsf{Tate})$ and $Z/\mathrm{Ann}_Z(b) \in \mathrm{Ob}(A_0\text{-}\mathsf{Perf.Tate})$. Then, in view of the discussion of (16.9.59) we get natural bijections

$$\mathrm{Hom}_{(A_0,\mathfrak{m})^a\text{-}\mathsf{Perf.Tate}}(X^a, Y^a) \xrightarrow{\sim} \mathrm{Hom}_{A_0\text{-}\mathsf{Tate}}(X, Y^a_*) \xrightarrow{\sim} \mathrm{Hom}_{A_0\text{-}\mathsf{Perf.Tate}}(X, (Y^a_*)^\natural)$$
$$\mathrm{Hom}_{(A_0,\mathfrak{m})^a\text{-}\mathsf{Perf.Tate}}(Y^a, X^a) \xrightarrow{\sim} \mathrm{Hom}_{A_0\text{-}\mathsf{-Alg}}(Y^a_{!!}, X) \xrightarrow{\sim} \mathrm{Hom}_{A_0\text{-}\mathsf{Perf.Tate}}(Z/\mathrm{Ann}_Z(b), X)$$

so the rule $Y \mapsto (Y^a_*)^\natural$ (resp. $Y \mapsto Z/\mathrm{Ann}_Z(b)$) yields the sought right (resp. left) adjoint for the functor A_0-$\mathsf{Perf.Tate} \to (A_0, \mathfrak{m})^a$-$\mathsf{Perf.Tate}$. The same argument shows that the rule $Y \mapsto (Y^a_*)^\natural$ also provides a right adjoint for the forgetful functor A_0-$\mathsf{Perf.Adic} \to (A_0, \mathfrak{m})^a$-$\mathsf{Perf.Adic}$, and using (v) we easily see that the rule $Y \mapsto Z$ yields a left adjoint for the same functor.

(iii.d): To begin with, let us notice :

Claim 16.9.67. The functors (16.9.62) induce equivalences between $(A_0, \mathfrak{m})^a$-$\mathsf{Perf.Adic}$ (resp. $(A_0, \mathfrak{m})^a$-$\mathsf{Perf.Tate}$) and the localization of A_0-$\mathsf{Perf.Adic}$ (resp. A_0-$\mathsf{Perf.Tate}$) obtained by inverting all almost isomorphisms.

Proof of the claim. Recall that $\varepsilon_{B_0} : B_0^\natural \to B_0$ is an isomorphism if B_0 is perfectoid (see (16.9.59)); we deduce for every $Y \in \mathrm{Ob}(A_0\text{-}\mathsf{Perf.Adic})$ an isomorphism $((Y^a_*)^\natural)^a \xrightarrow{\sim} (Y^a_*)^a \xrightarrow{\sim} Y^a$ in the category $(A_0, \mathfrak{m})^a$-$\mathsf{Perf.Adic}$. Combining with (vi) and proposition 1.1.20(iii), we deduce that the right adjoints to both of the forgetful functors (16.9.62) are fully faithful. Then the claim follows from proposition 1.6.13. ◊

From (vi) and remark 16.9.32(iii) we get a diagram of categories :

$$\begin{array}{ccc} A_0\text{-}\mathsf{Perf.Adic} & \xrightarrow{\mathbf{E}(-)} & \mathbf{E}_0\text{-}\mathsf{Perf.Adic} \\ \uparrow\downarrow (-)^a & & (-)^a \downarrow\uparrow \\ (A_0, \mathfrak{m})^a\text{-}\mathsf{Perf.Adic} & \xrightarrow{\mathbf{E}(-)} & (\mathbf{E}_0, \mathfrak{m}_\mathbf{E})^a\text{-}\mathsf{Perf.Adic} \end{array}$$

whose upward arrows are left adjoints to the downward arrows, and whose top horizontal arrow is an equivalence; in light of claim 16.9.67, it follows easily that the bottom horizontal arrow is an equivalence as well. Clearly the horizontal arrows commute with the downward arrows; then the horizontal arrows also commute with the upward arrows, up to isomorphism of functors (example 1.2.7(ii)). Hence, if $R \in \mathrm{Ob}(A_0\text{-}\mathsf{Perf.Adic})$, we have a natural ring isomorphism

$$\omega_R : \mathbf{E}((R^a_{!!})^\wedge) \stackrel{\sim}{\to} (\mathbf{E}(R)^a_{!!})^\wedge.$$

Next, if $(B_0, \mathfrak{m}B_0)$ is almost perfectoid, from (iv) and remark 16.9.32(iii) we deduce an almost isomorphism $\mathbf{E}((B_0)^a_{!!})^\wedge) \to \mathbf{E}(B_0)$, whence a ring isomorphism

$$\tau_{B_0} : (\mathbf{E}((B^a_{0!!})^\wedge)^a_{!!})^\wedge \stackrel{\sim}{\to} (\mathbf{E}(B_0)^a_{!!})^\wedge.$$

Lastly, $R := (B^a_{0!!})^\wedge$ is perfectoid, by (v), whence the isomorphism

$$\tau_{B_0} \circ \omega_R : \mathbf{E}((R^a_{!!})^\wedge) \stackrel{\sim}{\to} (\mathbf{E}(B_0)^a_{!!})^\wedge.$$

Thus, we are reduced to checking that the natural map $(((B^a_{0!!})^\wedge)^a_{!!})^\wedge \to (B^a_{0!!})^\wedge$ is an isomorphism. This in turn will follow, once we have shown that the natural map $((B^a_{0!!})^\wedge)^a_{!!} \to B^a_{0!!}$ is an isomorphism. But the latter is a straightforward consequence of (iv). □

17. Applications

17.1. **Model algebras.** We let $(K, |\cdot|)$ be a valued field of characteristic 0 with value group Γ_K of rank one, such that the residue field κ of K^+ has characteristic $p > 0$.

Definition 17.1.1. The category MA_K of *model K^+-algebras* consists of all the pairs (A, Γ), where $(\Gamma, +)$ is an integral monoid, and A is a Γ-graded K^+-algebra A, fulfilling the following conditions :

(MA1) $\mathrm{gr}_\gamma A$ is a torsion-free K^+-module, with $\dim_K \mathrm{gr}_\gamma A \otimes_{K^+} K = 1$, for every $\gamma \in \Gamma$.
(MA2) $\mathrm{gr}_\alpha A \cdot \mathrm{gr}_\beta A \neq 0$ for every $\alpha, \beta \in \Gamma$, and $(\mathrm{gr}_\gamma A)^n = \mathrm{gr}_{n\gamma} A$ for every $n \in \mathbb{N}$ and $\gamma \in \Gamma$.
(MA3) $\mathrm{gr}_0 A = K^+$.
(MA4) Γ is saturated and Γ^{gp} is a $\mathbb{Z}[1/p]$-module.

The morphisms $(A, \Gamma) \to (A', \Gamma')$ in MA_K are the pairs (f, φ), where $\varphi : \Gamma \to \Gamma'$ is a morphism of monoids, and $f : A \to \Gamma \times_{\Gamma'} A'$ is a morphism of Γ-graded K^+-algebras (see (7.6.19)).

Example 17.1.2. Let M be an integral monoid, $N \to M$ an exact and injective morphism of monoids, $N \to K^+ \setminus \{0\}$ a morphism of monoids, and suppose that :

- M is *divisible*, *i.e.* the k-Frobenius endomorphism of M is surjective for every $k > 0$
- $M^{\mathrm{gp}}/N^{\mathrm{gp}}$ is a $\mathbb{Q}$-vector space.

Let $(\Gamma, +)$ be the image of M in $M^{\mathrm{gp}}/N^{\mathrm{gp}}$, and denote by I the nilradical of the K^+-algebra $A := M \otimes_N K^+$; then A is a Γ-graded K^+-algebra, and I is a Γ-graded ideal (proposition 7.6.28(ii)). We claim that $I \otimes_{K^+} K = 0$ and $(A/I, \Gamma)$ is a model K^+-algebra. Indeed, (MA4) is immediate, and it is easily seen that $\mathrm{gr}_{n\gamma} A = (\mathrm{gr}_\gamma A)^n$ for every $\gamma \in \Gamma$ and $n \in \mathbb{N}$. Moreover, since the map $N \to M$ is exact, the kernel of the map $M \to M^{\mathrm{gp}}/N^{\mathrm{gp}}$ equals N, so $(A/I)_0 = K^+$, *i.e.* (MA3) holds as well. The remaining assertions can be checked after tensoring with K : namely, we have to show that the Γ-graded K-algebra $A_K := M \otimes_N K$ is reduced, with $\dim_K \mathrm{gr}_\gamma A_K = 1$ for every $\gamma \in \Gamma$, and $\mathrm{gr}_\alpha A_K \cdot \mathrm{gr}_\beta A_K \neq 0$ for every $\alpha, \beta \in \Gamma$. However, the morphism $N \to K^+$ extends uniquely to a group homomorphism $N^{\mathrm{gp}} \to K^\times$, and $A_K = (N^{-1}M) \otimes_{N^{\mathrm{gp}}} K$; hence, we may assume that N is a group, in which case Γ is the set-theoretic quotient of M by the translation action of N (lemma 4.8.31(iii)). In this case, choose a representative $\gamma^* \in M$ for every $\gamma \in \Gamma$; it follows easily that $(\gamma^* \otimes 1 \mid \gamma \in \Gamma)$ is a basis of the K-vector space A_K, whence (MA1), and $(\alpha^* \otimes 1) \cdot (\beta^* \otimes 1)$ is a non-zero multiple of $(\alpha+\beta)^* \otimes 1$, which yields (MA2). Lastly, since the nilradical I_K of A_K is Γ-graded (proposition 7.6.28(ii)), we also deduce that $I_K = 0$, as sought.

Remark 17.1.3. (i) The category MA_K admits a tensor product, defined by the rule :

$$(A, \Gamma) \otimes (A', \Gamma') := (A \otimes_{K^+} A', \Gamma \oplus \Gamma').$$

(ii) Let (A, Γ) be any model K^+-algebra, and suppose that $\Gamma = \Gamma_1 \oplus \Gamma_2$ is a given decomposition of Γ as direct sum of monoids. Then it is easily seen that Γ_1 and Γ_2 are integral and saturated, and they fulfill axiom (MA4). There follows a morphism of model K^+-algebras :

$$(\Gamma_1 \times_\Gamma A, \Gamma_1) \otimes (\Gamma_2 \times_\Gamma A, \Gamma_2) \to (A, \Gamma) \tag{17.1.4}$$

(iii) In the situation of (ii), set $A_K := A \otimes_{K^+} K$. Then it is easily seen that (17.1.4) induces an isomorphism of K-algebras

$$(\Gamma_1 \times_\Gamma A_K) \otimes_K (\Gamma_2 \times_\Gamma A_K) \xrightarrow{\sim} A_K.$$

In general, (17.1.4) need not be an isomorphism. However, (17.1.4) is an isomorphism, if either Γ_1 or Γ_2 is a torsion abelian group. Indeed, in any case the induced maps

$$\mathrm{gr}_\alpha A \otimes_{K^+} \mathrm{gr}_\beta A \to \mathrm{gr}_{\alpha+\beta} A \qquad a_1 \otimes a_2 \mapsto a_1 a_2$$

will be injective for every $\alpha \in \Gamma_1$ and $\beta \in \Gamma_2$. To check surjectivity, let $a \in \mathrm{gr}_{\alpha+\beta} A$ be any element, and say that $n\beta = 0$; then $a^n \in \mathrm{gr}_{n\alpha} A$, and by (MA2) we know that there exist $a_1 \in \mathrm{gr}_\alpha A$ and $x \in K^+$ such that $a^n = a_1^n x$. It follows that $a = a_1 a_2$ for some $a_2 \in \mathrm{gr}_\beta A_K$ such that $a_2^n \in K^+$. Then $a_2 \in \mathrm{gr}_\beta A$, whence the claim.

(iv) Let $(E, |\cdot|_E)$ be any valued field extension of $(K, |\cdot|)$, such that $|\cdot|_E$ is also a valuation of rank one. Then we have an obvious base change functor :

$$\mathsf{MA}_K \to \mathsf{MA}_E \quad : \quad (A, \Gamma) \mapsto (A \otimes_{K^+} E^+, \Gamma).$$

17.1.5. Let (A, Γ) be any model K^+-algebra. For any subset $\Delta \subset \Gamma$, set

$$A_\Delta := \Delta \times_\Gamma A \qquad A_{\Delta,K} := A_\Delta \otimes_{K^+} K \qquad \Delta[1/p] := \bigcup_{n\in\mathbb{N}} \{\gamma \in \Gamma \mid p^n\gamma \in \Delta\}.$$

Clearly, if Δ is a submonoid of Γ, then A_Δ is a K^+-subalgebra of A, and (A_Δ, Δ) is a subobject of (A, Γ), provided Δ satisfies (MA4). On the other hand, if Δ is saturated, it is easily seen that the same holds for $\Delta[1/p]$, and then the latter does satisfy (MA4).

Also, if Δ is an ideal of Γ, then clearly A_Δ is an ideal of A.

Set $K^* := K^+ \setminus \{0\}$ and let $A^*_\gamma := \mathrm{gr}_\gamma A \setminus \{0\}$ for every $\gamma \in \Gamma$; for any submonoid $\Delta \subset \Gamma$, we deduce a sequence of morphisms of integral monoids :

$$1 \to K^* \xrightarrow{\varphi_\Delta} A^*_\Delta \to \Delta \to 1 \qquad \text{where } A^*_\Delta := \bigoplus_{\gamma\in\Delta} A^*_\gamma \tag{17.1.6}$$

(and where the direct sum is formed in the category of K^*-modules), such that $(17.1.6)^{\mathrm{gp}}$ is a short exact complex of abelian groups; then (MA2) implies that φ_Δ is saturated (proposition 6.2.31). Also, (MA1) implies that each A^*_γ is a filtered union of free cyclic K^*-modules, hence φ_Δ is also integral. Furthermore, (MA1) and (MA2) imply that $A^{*\mathrm{gp}}_\Delta$ is Δ^{gp}-graded, and

$$\mathrm{gr}_\gamma A^{*\mathrm{gp}}_\Delta = A^*_\gamma \otimes_{K^*} K^\times \qquad \text{for every } \gamma \in \Delta. \tag{17.1.7}$$

(details left to the reader). We shall just write A^* and A_K instead of A^*_Γ, and respectively $A_{\Gamma,K}$.

17.1.8. Let now (A, Γ) be any model K^+-algebra; set $S := \mathrm{Spec}\, K^+$, $X := \mathrm{Spec}\, A$, and let $\overline{x}$ be any geometric point of X, localized on the closed subset $Z := X \times_S \mathrm{Spec}\, \kappa$. Let $\mathscr{T}_{A,p}$ be the p-adic topology on A. Suppose furthermore, that K^+ is deeply ramified (see [75, Def.6.6.1]), and let $(K^+, \mathfrak{m}_K)$ be the standard setup associated with K^+ (see [75, §6.1.15]); then we have the corresponding sheaf $\mathscr{O}^a_{X(\overline{x})}$ of K^{+a}-algebras on $X(\overline{x})$, and we may state the following *almost purity* theorem :

Theorem 17.1.9. *With the notation of* (17.1.8)*, the following holds :*

(i) $(A, \mathscr{T}_{A,p})$ *is a formally perfectoid ring.*
(ii) *The pair* $(X(\overline{x}), Z(\overline{x}))$ *is almost pure.*

Proof. (i): Let $\pi \in K^+$ be any element such that $\pi^p K^+ = pK^+$; denote by $A^\wedge$ the completion of $(A, \mathscr{T}_{A,p})$, and set $I := \pi A^\wedge$. Then the topology of $A^\wedge$ agrees with the I-adic topology (remark 8.3.3(iv)), and the image of π is a regular element in $A^\wedge$, by proposition 8.2.13(i) (the details shall be left to the reader). Moreover we have $pA^\wedge = I^{(p)}$ (notation of definition 16.2.1); by theorem 16.4.1, it then suffices to check that the Frobenius endomorphism $\Phi_{A/pA}$ of $A^\wedge/pA^\wedge = A/pA$ induces an isomorphism $A/\pi A \xrightarrow{\sim} A/pA$. We are then further reduced to showing that $\Phi_{A/pA}$ restricts to a bijection

$$\mathrm{gr}_\gamma A/\pi \cdot \mathrm{gr}_\gamma A \xrightarrow{\sim} \mathrm{gr}_{p\gamma} A/p \cdot \mathrm{gr}_{p\gamma} A \qquad \text{for every } \gamma \in \Gamma.$$

However, since Γ_K is of rank one, there exists a sequence $(a_n \mid n \in \mathbb{N})$ of elements of $\mathrm{gr}_\gamma A$ such that $K^+ a_n \subset K^+ a_{n+1}$ for every $n \in \mathbb{N}$ and $\mathrm{gr}_\gamma A = \bigcup_{n\in\mathbb{N}} K^+ a_n$. Then $\mathrm{gr}_{p\gamma} A = \bigcup_{n\in\mathbb{N}} K^+ a_n^p$. By [75, Prop.6.6.6] the Frobenius endomorphism of K^+ induces an isomorphism $K^+/\pi K^+ \xrightarrow{\sim} K^+/pK^+$. It follows that $\Phi_{A/pA}$ induces a bijection $(K^+ a_n)/\pi(K^+ a_n) \xrightarrow{\sim} (K^+ a_n^p)/p(K^+ a_n^p)$ for every $n \in \mathbb{N}$, whence the contention.

(ii) follows from (i), and theorems 16.7.21(i) and 16.8.44. □

17.1.10. The inclusion map $A^* \to A$ is a morphism of (multiplicative) monoids, hence induces a log structure on $X := \operatorname{Spec} A$ (see (12.1.15)). We shall denote

$$\mathbb{S}(A, \Gamma) := (X, (A^*_X)^{\log})$$

the resulting log scheme. Clearly, every morphism $(f, \varphi) : (A, \Gamma) \to (A', \Gamma')$ of model algebras induces a morphism of log schemes

$$\mathbb{S}(f, \varphi) : \mathbb{S}(A', \Gamma') \to \mathbb{S}(A, \Gamma).$$

Especially, by lemma 12.1.13(iv), the map φ_Γ yields a saturated morphism

$$\mathbb{S}(A, \Gamma) \to \mathbb{S}(K^+) := \mathbb{S}(K^+, \{1\}). \tag{17.1.11}$$

Also, if $(K, |\cdot|) \to (E, |\cdot|_E)$ is an extension of rank one valued fields, the inclusion $A^* \subset (A \otimes_{K^+} E^+)^*$ induces a morphism of log schemes

$$\mathbb{S}(A \otimes_{K^+} E^+, \Gamma) \to \mathbb{S}(A, \Gamma)$$

for any model K^+-algebra (A, Γ), and we remark that the resulting diagram of log schemes

$$\begin{array}{ccc} \mathbb{S}(A \otimes_{K^+} E^+, \Gamma) & \longrightarrow & \mathbb{S}(A, \Gamma) \\ \big\downarrow & & \big\downarrow \\ \mathbb{S}(E^+) & \longrightarrow & \mathbb{S}(K^+) \end{array}$$

is cartesian. Indeed, since (12.1.16) is right exact, it suffices to check that the natural map

$$A^* \otimes_{K^*} E^* \to (A \otimes_{K^+} E^+)^*$$

is an isomorphism, which is clear.

Remark 17.1.12. (i) Let (A, Γ) be any model K^+-algebra, and suppose that $\Delta_0 \subset \Gamma$ is a fine and saturated submonoid, such that Δ_0^{gp} is torsion-free. In this case, Δ_0^{gp} is a free abelian group of finite rank, hence $(17.1.6)^{\mathrm{gp}}$ admits a splitting $\sigma : \Delta_0^{\mathrm{gp}} \to A^{*\mathrm{gp}}$. Then, using (17.1.7) and (MA1), it is easily seen that the rule $\gamma \mapsto \sigma(\gamma)$ extends to an isomorphism of Δ_0-graded K-algebras

$$K[\Delta_0] \xrightarrow{\sim} A_{\Delta_0, K}.$$

(ii) In the situation of (i), suppose furthermore, that K is algebraically closed. Pick any $x \in K^\times$ such that $\gamma := |x| \neq 1$, and let $\langle\gamma\rangle \subset \Gamma_K$ be the subgroup generated by γ; we define a group homomorphism $\langle\gamma\rangle \to K^\times$ by the rule : $\gamma^k \mapsto x^k$ for every $k \in \mathbb{Z}$. Since $K^\times$ is divisible, the latter map extends to a group homomorphism

$$\Gamma_K \to K^\times \tag{17.1.13}$$

and since Γ_K is a group of rank one, it is easily seen that (17.1.13) is a right inverse for the valuation map $|\cdot| : K^\times \to \Gamma_K$, whence a decomposition :

$$K^\times \stackrel{\sim}{\to} (K^+)^\times \oplus \Gamma_K.$$

On the other hand, set $A^*_{\Delta_0,K} := A^*_{\Delta_0} \otimes_{K^*} K^\times$; from (i) we deduce an isomorphism of Δ_0-graded monoids :

$$A^*_{\Delta_0,K} \stackrel{\sim}{\to} \Delta_0 \oplus K^\times.$$

Combining these two isomorphisms, we deduce a surjection $\tau : A^*_{\Delta_0,K} \to (K^+)^\times$ which is a left inverse to the inclusion $(K^+)^\times \to A^*_0$. Then, for every $\gamma \in \Delta_0$, let us set $C_\gamma := A^*_\gamma \cap \mathrm{Ker}\,\tau$; there follows a (non-canonical) isomorphism of Δ-graded monoids :

$$A^*_{\Delta_0} \stackrel{\sim}{\to} (K^+)^\times \oplus C \qquad \text{where} \quad C := \bigoplus_{\gamma\in\Delta_0} C_\gamma \subset A^*_{\Delta_0} \quad \text{and} \quad C^{\mathrm{gp}} \simeq \Delta_0^{\mathrm{gp}} \oplus \Gamma_K$$

(details left to the reader).

(iii) Suppose that Γ^{gp} is torsion-free, and K is still algebraically closed. Let us set :

$$\Delta_n := \{\gamma \in \Gamma \mid p^n\gamma \in \Delta_0\} \qquad \text{for every } n \in \mathbb{N}.$$

It is easily seen that Δ_n is still fine and saturated, and since $K^\times$ is divisible, we may extend inductively the splitting σ of (i) to a system of homomorphisms

$$\sigma_n : \Delta_n^{\mathrm{gp}} \to A^{*\mathrm{gp}} \qquad \text{such that} \quad \sigma_{n+1|\Delta_n^{\mathrm{gp}}} = \sigma_n \quad \text{for every } n \in \mathbb{N}$$

whence a compatible system of isomorphisms

$$K[\Delta_n] \stackrel{\sim}{\to} A_{\Delta_n,K} \qquad \text{for every } n \in \mathbb{N}. \tag{17.1.14}$$

Proceeding as in (ii), we deduce a compatible system of isomorphisms of Δ_n-graded monoids

$$A^*_{\Delta_n} \stackrel{\sim}{\to} (K^+)^\times \oplus C^{(n)} \qquad \text{such that } C^{(n)} \subset C^{(n+1)} \text{ for every } n \in \mathbb{N}.$$

In this situation, notice that the p-Frobenius automorphism of Γ^{gp} restricts to an isomorphism

$$\Delta_{n+1} \stackrel{\sim}{\to} \Delta_n \qquad \text{for every } n \in \mathbb{N}.$$

Likewise, since Γ_K is p-divisible, taking p-th powers induces isomorphisms

$$C^{(n+1)} \stackrel{\sim}{\to} C^{(n)} \qquad \text{for every } n \in \mathbb{N}.$$

Definition 17.1.15. Let (B, Δ) be model K^+-algebra. We say that (B, Δ) is *small*, if the following conditions hold :

(a) $\mathrm{gr}_\gamma B$ is a finitely generated K^+-module for every $\gamma \in \Delta$.
(b) $\Delta = \Delta_0[1/p]$ for some fine and saturated submonoid Δ_0.
(c) $\Delta_0 \times_\Delta B$ is a finitely generated K^+-algebra.

Remark 17.1.16. Notice that condition (a) of definition 17.1.15 and axiom (MA1) imply that $\mathrm{gr}_\gamma B$ is a free K^+-module of rank one, for every small model algebra (B, Δ) and every $\gamma \in \Delta$. Moreover, actually conditions (b) and (c) (together with axioms (MA1) and (MA2)) imply condition (a). Indeed, (c), (MA1) and proposition 7.6.11(ii) imply that $\mathrm{gr}_\gamma B$ is a free K^+-module of rank one, for every $\gamma \in \Delta_0$. Now, in case $n\gamma \in \Delta_0$, (MA2) implies that $\mathrm{gr}_{n\gamma} B$ is generated by an element of the form $z = x_1 \cdots x_n$, for certain $x_1, \dots, x_n \in \mathrm{gr}_\gamma B$. Suppose that $x_1 = ay$ for some $a \in K^+$ and $y \in \mathrm{gr}_\gamma B$; then z is divisible by a in $\mathrm{gr}_{n\gamma} B$, so $a \in (K^+)^\times$, *i.e.*

x_1 generates $\mathrm{gr}_\gamma B$. In view of (b), for every $\gamma \in \Delta$ we may find $k \geq 0$ such that $p^k\gamma \in \Delta_0$, so (a) follows.

Remark 17.1.17. Let (B, Δ) be any small model algebra, so that conditions (b) and (c) of definition 17.1.15 are satisfied for some submonoid $\Delta_0 \subset \Delta$.

(i) Choose a decomposition $\Delta_0^\times = G \oplus H$, where G is a free abelian group, and H is the torsion subgroup of Δ_0; we have an isomorphism

$$\Delta_0 \simeq \Lambda \oplus H \qquad \text{with } \Lambda := \Delta_0^\sharp \oplus G$$

(lemma 6.2.10), which induces a decomposition :

$$\Delta \simeq \Lambda[1/p] \oplus H$$

inducing, in turn, an isomorphism of model K^+-algebras :

$$(B, \Delta) \xrightarrow{\sim} (B_{\Lambda[1/p]}, \Lambda[1/p]) \otimes (B_H, H)$$

(remark 17.1.3(iii)). Notice that both B_Λ and B_H are finitely generated K^+-algebras (proposition 7.6.11(i)). In other words, every small model algebra can be written as the tensor product of two small model algebras (B', Δ') and (B'', Δ''), such that Δ'^{gp} is torsion-free, and Δ'' is a finite abelian group whose order is not divisible by p.

(ii) For every $n \in \mathbb{N}$, set $\Delta_n := \{\gamma \in \Delta \mid p^n\gamma \in \Delta_0\}$. Then $B_n := \Delta_n \times_\Delta B$ is a finitely generated K^+-algebra for every $n \in \mathbb{N}$. Indeed, in view of (i), it suffices to check the assertion in case Δ^{gp} is a torsion-free abelian group. Now, pick a system $x_1, \dots, x_k$ of homogeneous generators of the K^+-algebra B_0; by (MA2), for every $i = 1, \dots, k$ there exist a homogeneous element $y_i \in B_n$ and $u_i \in (K^+)^\times$ such that $u_i y_i^{p^n} = x_i$. Let $z \in B_n$ be any homogeneous element; then $z^{p^n} = v x_1^{t_1} \cdots x_k^{t_k}$ for some $v \in K$ and $t_1, \dots, t_k \in \mathbb{N}$. Set $y := y_1^{t_1} \cdots y_k^{t_k}$ and $u := u_1^{t_1} \cdots u_k^{t_k} \in K^\times$; then $y^{p^n} uv = z^{p^n}$, and since Δ^{gp} is torsion-free, we deduce that $yw = z$ for some $w \in K^+$, *i.e.* the system $y_1, \dots, y_k$ generates the K^+-algebra B_n.

(iii) Suppose that Δ is a finite group whose order is not divisible by p. Then we claim that B is a finite étale K^+-algebra. Indeed, in view of remark 17.1.3(iii), it suffices to verify the assertion for Δ a cyclic finite group, say of order n, with $(n, p) = 1$; in the latter case, (MA2) implies that $B \simeq K^+[X]/(X^n - u)$ for some $u \in (K^+)^\times$, whence the contention.

(iv) Let $F \subset \Delta$ be any face. Then (B_F, F) is a small model algebra as well. Indeed, by (i), it suffices to consider the case where Δ^{gp} is a torsion-free abelian group. In this case, set $F_0 := F \cap \Delta_0$; it is easily seen that $F = F_0[1/p]$, and F_0 is a fine and saturated monoid, by lemma 6.1.20(ii) and corollary 6.2.33(ii). Moreover, $B_{F_0} = F_0 \times_F B_{\Delta_0}$ is a finitely generated K^+-algebra, by proposition 7.6.11(i), whence the contention.

(v) Suppose moreover, that K is algebraically closed, and let (B, Δ) be any small model K^+-algebra. Then we have a (non-canonical) isomorphism :

$$K[\Delta] \xrightarrow{\sim} B_K.$$

To exhibit such an isomorphism, we may – in light of (i) – assume that Δ^{gp} is either torsion-free, or a finite group of order not divisible by p. In the latter case, the assertion follows easily from (iii). In case Δ^{gp} is torsion-free, the sought isomorphism is the colimit of the system of isomorphisms (17.1.14).

Lemma 17.1.18. *Let (A, Γ) be a model K^+-algebra, and denote by $\mathscr{F}(\Gamma)$ the filtered family of all fine and saturated submonoids of Γ. We have :*

(i) $A = \mathrm{colim}_{\Delta \in \mathscr{F}(\Gamma)} A_\Delta$.

(ii) *Suppose that Γ^{gp} is a torsion-free abelian group. Then A is a normal domain.*

Proof. (i) is an immediate consequence of corollary 6.4.1(ii).

(ii): We show first the following :

Claim 17.1.19. Suppose that $\Gamma^{\rm gp}$ is a torsion-free abelian group. Then A_K is a normal domain.

Proof of the claim. In view of (i), it suffices to show that $A_{\Delta,K}$ is a normal domain, when $\Delta \subset \Gamma$ is fine and saturated. The latter assertion follows from remark 17.1.12(i) and theorem 6.4.18(iii). ◊

Let A^ν be the integral closure of A in A_K; in view of claim 17.1.19 we are reduced to showing that $A = A^\nu$. By proposition 7.6.29(ii), A^ν is Γ-graded. Suppose now that $x \in A^\nu$; we need to show that $x \in A$, and we may assume that $x \in \mathrm{gr}_\gamma A^\nu$ for some $\gamma \in \Gamma$. Hence, let

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0 \qquad \text{with } a_1, \dots, a_n \in A$$

be an integral equation for x over A. If we replace each a_i by its homogeneous component in degree $i\gamma$, we still obtain an integral equation for x, so we may assume that $a_i \in \mathrm{gr}_{i\gamma} A$ for every $i = 1, \dots, n$. Then, (MA2) implies that, for every $i = 1, \dots, n$ there exist $u_i \in K^+$ and $b_i \in \mathrm{gr}_\gamma A$, such that $|u_i| = 1$ and $a_i = u_i b_i^i$. Also, from (MA1) we deduce that there exists $b \in \mathrm{gr}_\gamma A$ such that $b_1, \dots, b_n \in K^+ b$. Set $y := b^{-1}x \in \mathrm{gr}_0 A \otimes_{K^+} K$; clearly y is integral over the subring $\mathrm{gr}A_0$. Lastly, (MA3) shows that $y \in K^+$, whence $x \in \mathrm{gr}_\gamma A$, which proves the contention. □

Proposition 17.1.20. *Let* (A, Γ) *be a model* K^+*-algebra, and suppose that* Γ_K *is divisible. Then* (A, Γ) *is the filtered union of its small model* K^+*-subalgebras.*

Proof. In view of lemma 17.1.18(i), we see that (A, Γ) is the colimit of the filtered system of its subobjects $(A_{\Delta[1/p]}, \Delta[1/p])$, for Δ ranging over the fine and saturated submonoids of Γ. We may then assume from start that $\Gamma = \Gamma_0[1/p]$ for some fine and saturated submonoid Γ_0.

Next, in view of remark 17.1.17(i), we may consider separately the cases where $\Gamma_0^{\rm gp}$ is a torsion-free abelian group, and where $\Gamma = \Gamma_0$ is a finite abelian group.

Suppose first that $\Gamma_0^{\rm gp}$ is torsion-free, and let $\underline{\gamma} := (\gamma_1, \dots, \gamma_n)$ be a finite system of generators for Γ_0. For every $i = 1, \dots, n$, choose $a_i \in A^*_{\gamma_i}$, and let $B(\underline{\gamma}, \underline{a}) \subset A$ be the K^+-subalgebra generated by $\underline{a} := (a_1, \dots, a_n)$. Clearly the grading of A induces a Γ_0-grading on $B(\underline{\gamma}, \underline{a})$, and $\mathrm{gr}_\beta B(\underline{\gamma}, \underline{a})$ is a finitely generated K^+-module for every $\beta \in \Gamma_0$ (proposition 7.6.11(ii)); by virtue of (MA1), we know that $\mathrm{gr}_\beta B(\underline{\gamma}, \underline{a})$ is then even a free rank one K^+-module, for every $\beta \in \Gamma_0$. Furthermore, $\underline{a}$ generates a fine submonoid of A^*, so – by proposition 6.6.35(ii) – there exists an integer $k > 0$ such that

$$(\mathrm{gr}_{k\beta} B(\underline{\gamma}, \underline{a}))^n = \mathrm{gr}_{nk\beta} B(\underline{\gamma}, \underline{a}) \qquad \text{for every } \beta \in \Gamma_0 \text{ and every integer } n > 0.$$

Now, for any $\beta \in \Gamma$, let $t > 0$ be an integer such that $p^t\beta \in \Gamma_0$, and pick a generator b of the K^+-module $\mathrm{gr}_{kp^t\beta} B(\underline{\gamma}, \underline{a})$; in light of (MA2) we may find $x \in K^+$ and $c \in \mathrm{gr}_\beta A$, such that $b = c^{kp^t} x$. Since Γ_K is divisible, we may write $x = y^{kp^t} u$ for some $y, u \in K^+$ with $|u| = 1$.

It is easily seen that the K^+-submodule of $\mathrm{gr}_\beta A$ generated by cy does not depend on the choices of c, y, t and k; hence we denote $\mathrm{gr}_\beta C(\underline{\gamma}, \underline{a})$ this submodule; a simple inspection shows that the resulting Γ-graded K^+-module

$$C(\underline{\gamma}, \underline{a}) := \bigoplus_{\beta \in \Gamma} \mathrm{gr}_\beta C(\underline{\gamma}, \underline{a})$$

is actually a Γ-graded K^+-subalgebra of A, for which (MA2) holds, and therefore $(C(\underline{\gamma}, \underline{a}), \Gamma)$ is a small model algebra. Lastly, it is clear that the family of all such $(C(\underline{\gamma}, \underline{a}), \Gamma)$, for $\underline{\gamma}$ ranging over all finite sets of generators of Γ, and $\underline{a}$ ranging over all the finite sequences of elements of A as above, form a cofiltered system of subobjects, whose colimit is (A, Γ). This concludes the proof of the proposition in this case.

Next, suppose that $\Gamma = \Gamma_0$ is a finite abelian group. In view of remark 17.1.3(iii), we are reduced to the case where Γ is cyclic, say $\Gamma = \mathbb{Z}/n\mathbb{Z}$. Fix any generator γ of Γ, and for every

$a \in \mathrm{gr}_\gamma A$, let $C(a)$ be the K^+-subalgebra of A generated by a; it is easily seen that the colimit of the filtered family $((C(a), \Gamma) \mid a \in \mathrm{gr}_\gamma A)$ equals (A, Γ). (Details left to the reader.) □

17.1.21. Suppose now that K is algebraically closed, let (B, Δ) be a small model K^+-algebra, with Δ^{gp} torsion-free, and write $\Delta = \Delta_0[1/p]$ for some fine and saturated submonoid Δ_0 such that the K^+-algebra $B_0 := B_{\Delta_0}$ is finitely generated. Let us set

$$\Delta_n := \{\gamma \in \Delta \mid p^n\gamma \in \Delta_0\} \qquad B_n := B_{\Delta_n} \qquad Y_n := \mathrm{Spec}\, B_n \qquad \text{for every } n \in \mathbb{N}.$$

We wish to construct a ladder of log schemes

$$(17.1.22)\qquad \begin{array}{ccccccccc} \cdots \xrightarrow{g_{n+1}} & (Y_{n+1}, \underline{M}_{n+1}) & \xrightarrow{g_n} & (Y_n, \underline{M}_n) & \xrightarrow{g_{n-1}} \cdots \xrightarrow{g_0} & (Y_0, \underline{M}_0) \\ & \downarrow{\scriptstyle \varphi_{n+1}} & & \downarrow{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \varphi_0} \\ \cdots \xrightarrow{h_{n+1}} & (S, \underline{N}_{n+1}) & \xrightarrow{h_n} & (S, \underline{N}_n) & \xrightarrow{h_{n-1}} \cdots \xrightarrow{h_0} & (S, \underline{N}_0) \end{array}$$

such that, for every $n \in \mathbb{N}$:

- φ_n is smooth and saturated, and $\underline{M}_n$ admits a chart, given by a Δ_n-graded fine monoid $P^{(n)}$, such that $P_0^{(n)}$ is sharp, and the inclusion map $P_0^{(n)} \to P^{(n)}$ is flat and saturated, and gives a chart for φ_n
- the morphism of schemes underlying h_n (resp. g_n) is the identity of S (resp. is induced by the inclusion $B_n \subset B_{n+1}$)
- the morphism g_n admits a chart, given by an injective map $P^{(n)} \to \Delta_n \times_{\Delta_{n+1}} P^{(n+1)}$ of Δ_n-graded monoids, whose restriction $P_0^{(n)} \to P_0^{(n+1)}$ gives a chart for h_n.

This will be achieved in several steps, as follows :

• First, let $b_1, \dots, b_k$ be a finite system of generators for B_0. By remark 17.1.12(ii), we have a decomposition $B_0^* = (K^+)^\times \oplus C$ for some submonoid $C \subset B_0^*$, and we may suppose that each b_i lies in C. Let $P^{(0)} \subset B_0^*$ be the submonoid generated by $b_1, \dots, b_k$. The restriction of the surjection $B_0^* \to \Delta_0$ is then still a surjection $\pi : P^{(0)} \to \Delta_0$. Notice that, for every $x \in \mathrm{Ker}\,\pi^{\mathrm{gp}}$, we have either $x \in K^+$ or $x^{-1} \in K^+$; especially, we may find a finite system $x_1, \dots, x_n$ of generators for $\mathrm{Ker}\,\pi^{\mathrm{gp}}$, such that $x_i \in K^+$ for every $i = 1, \dots, n$. Now, let $\Sigma \subset P^{(0)\mathrm{gp}}$ be the submonoid generated by $x_1, \dots, x_n$; after replacing $P^{(0)}$ by $P^{(0)} \cdot \Sigma$, we may assume that $\mathrm{Ker}\,\pi^{\mathrm{gp}} = P_0^{(0)\mathrm{gp}}$, where $P_0^{(0)} := \mathrm{Ker}\,\pi^{\mathrm{gp}} \cap P^{(0)} \subset K^+$ is a fine submonoid.

• Next, by theorem 6.6.45, we may find a finitely generated submonoid $\Sigma' \subset P_0^{(0)\mathrm{gp}} \cap K^+$ such that the induced morphism $P_0^{(0)} \cdot \Sigma' \to P^{(0)} \cdot \Sigma'$ is flat. Clearly

$$P_0^{(0)} \cdot \Sigma' = \mathrm{Ker}\,\pi^{\mathrm{gp}} \cap (P^{(0)} \cdot \Sigma')$$

hence, we may replace $P^{(0)}$ by $P^{(0)} \cdot \Sigma'$, and assume that the morphism $P_0^{(0)} \to P^{(0)}$ is also flat.

• We claim that the map $P_0^{(0)} \to P^{(0)}$ is also saturated. We shall apply the criterion of proposition 6.2.31 : indeed, for given $\gamma \in \Delta_0$ and integer $n > 0$, let x (resp. y) be a generator of the $P_0^{(0)}$-module $P_\gamma^{(0)}$ (resp. $P_{n\gamma}^{(0)}$). Then x (resp. y) is also a generator of the K^+-module $\mathrm{gr}_\gamma B$ (resp. $\mathrm{gr}_{n\gamma} B$), and (MA2) implies that there exists $u \in (K^+)^\times$ such that $uy = x^n$; but since $x, y \in C$, we must have $u = 1$ therefore $(P_\gamma^{(0)})^n = P_{n\gamma}^{(0)}$, as required.

• Next, we claim that the induced map of Δ_0-graded K^+-algebras

$$(17.1.23)\qquad P^{(0)} \otimes_{P_0^{(0)}} K^+ \to B_0$$

is an isomorphism. Indeed, since $P^{(0)}$ contains a set of generators for the K^+-algebra B, the map (17.1.23) is obviously surjective. However, for every $\gamma \in \Delta_0$, the $P_0^{(0)}$-module $P_\gamma^{(0)}$ is free of rank one (remark 6.2.5(iv)), therefore $(P^{(0)} \otimes_{P_0^{(0)}} K^+)_\gamma$ is a free K^+-module of rank one. It follows easily that (17.1.23) is also injective.

• Now, denote by $\underline{N}_0$ (resp. $\underline{M}_0$) the fine log structure on S (resp. on X_0) deduced from the inclusion map $P_0^{(0)} \to K^+$ (resp. $P^{(0)} \to B_0$). By lemma 12.1.13(iv), the inclusion $P_0^{(0)} \to P^{(0)}$ yields a chart for a morphism $\varphi_0 : (Y_0, \underline{M}_0) \to (S, \underline{N}_0)$ that is saturated, as sought. Lastly, since (17.1.23) is an isomorphism, theorem 12.3.37 shows that φ_0 is also smooth.

• Next, according to remark 17.1.12(iii), we have a compatible system of decompositions

$$B_n^* = (K^+)^\times \oplus C^{(n)} \qquad \text{for every } n \in \mathbb{N}$$

with $C^{(0)} = C$, and such that the p-Frobenius induces an isomorphism $\tau_n : C^{(n+1)} \xrightarrow{\sim} C^{(n)}$ for every $n \in \mathbb{N}$. Hence, define inductively an increasing sequence of submonoids

$$P^{(0)} \subset P^{(1)} \subset P^{(2)} \subset \cdots \subset B^* \quad \text{by letting} \quad P^{(n+1)} := \tau_n^{-1} P^{(n)} \text{ for every } n \in \mathbb{N}.$$

Clearly, the grading of B^* restricts to a Δ_n-grading on $P^{(n)}$, and induces a Δ-grading on P. We deduce isomorphisms of Δ_{n+1}-graded monoids :

$$P^{(n+1)} \xrightarrow{\sim} \Delta_{n+1} \times_{\Delta_n} P^{(n)} \qquad \text{for every } n \in \mathbb{N}. \tag{17.1.24}$$

Especially, the induced maps of monoids $P_0^{(n)} \to P^{(n)}$ are still flat and saturated, and $P_0^{(n)}$ is still sharp (remark 17.1.26). The inclusion maps $P^{(n)} \to B_n$ and $P_0^{(n)} \to K^+$ determine an isomorphism of Δ_n-graded K^+-algebras

$$P^{(n)} \otimes_{P_0^{(n)}} K^+ \xrightarrow{\sim} B_n \qquad \text{for every } n \in \mathbb{N} \tag{17.1.25}$$

as well as fine log structures $\underline{M}_n$ on Y_n, and $\underline{N}_n$ on S, whence a morphism of log schemes $\varphi_n : (Y_n, \underline{M}_n) \to (S, \underline{N}_n)$ which is again smooth and saturated. This completes the construction of (17.1.22).

Remark 17.1.26. (i) With the notation of (17.1.21), notice that, by construction, $P^{(n)\mathrm{gp}} \subset C^{(n)\mathrm{gp}}$; especially, since Δ^{gp} is torsion-free, the same holds for $P^{(n)\mathrm{gp}}$. Likewise, the construction shows that the induced map $P_0^{(n)\mathrm{gp}} \to \Gamma_K$ is always injective; especially, $P_0^{(n)\mathrm{gp}}$ is a free abelian group of finite rank, and $P_0^{(n)}$ is sharp. It follows that the $P_0^{(n)}$-module $P_\gamma^{(n)}$ admits a unique generator g_γ, for every $\gamma \in \Delta_n$, and the saturation condition for the inclusion $P_0^{(n)} \to P^{(n)}$ translates as the system of identities :

$$g_\gamma^k = g_{k\gamma} \qquad \text{for every } k \in \mathbb{N} \text{ and every } \gamma \in \Delta_n.$$

(ii) Moreover, set $P := \bigcup_{n \in \mathbb{N}} P^{(n)}$. Notice that the colimit of the maps (17.1.25) is an isomorphism of Δ-graded K^+-algebras

$$P \otimes_{P_0} K^+ \xrightarrow{\sim} B. \tag{17.1.27}$$

Furthermore, since the natural map $P^{(n)}/P_0^{(n)} \to \Delta$ is an isomorphism for every $n \in \mathbb{N}$, we deduce that the grading of P induces an isomorphism :

$$P/P_0 \xrightarrow{\sim} \Delta. \tag{17.1.28}$$

(iii) We also obtain a commutative diagram of log schemes :

$$\begin{array}{ccc} \mathbb{S}(B_n, \Delta_n) & \longrightarrow & (Y_n, \underline{M}_n) \\ \downarrow & & \downarrow{\scriptstyle \varphi_n} \\ \mathbb{S}(K^+) & \longrightarrow & (S, \underline{N}_n) \end{array} \qquad \text{for every } n \in \mathbb{N} \tag{17.1.29}$$

whose left vertical arrow is the saturated morphism (17.1.11), and whose bottom (resp. top) arrow is the identity of S (resp. of Y_n) on the underlying schemes, and is induced by the inclusion map $P_n^{(0)} \to K^*$ (resp. $P^{(n)} \to B_n^*$). It is easily seen that (17.1.29) is cartesian : indeed, since the functor (12.1.16) is right exact, it suffices to check that the natural map

$$P^{(n)} \otimes_{P_0^{(n)}} K^* \to B_n^*$$

is an isomorphism, which is clear from (17.1.25).

(iv) Notice that the inclusion map $i_0 : P_0^{(0)} \to P^{(0)}$ is also local; then corollary 6.4.7 yields a decomposition

$$\vartheta_0 : P^{(0)} \xrightarrow{\sim} P^{(0)\times} \oplus P^{(0)\sharp}$$

such that $\vartheta_0 \circ i_0 = \iota_0 \circ \lambda_0^\sharp$, where $\iota_0 : P^{(0)\sharp} \to P^{(0)\times} \oplus P^{(0)\sharp}$ is the natural inclusion map. By means of the isomorphisms (17.1.24), we may then inductively construct decompositions

$$\vartheta_n : P^{(n)} \xrightarrow{\sim} P^{(n)\times} \oplus P^{(n)\sharp} \qquad \text{for every } n \in \mathbb{N}$$

fitting into a commutative diagram

$$\begin{array}{ccccc} P_0^{(n)} & \xrightarrow{i_n} & P^{(n)} & \xrightarrow{j_n} & P^{(n+1)} \\ \downarrow{\scriptstyle i_n^\sharp} & & \downarrow{\scriptstyle \vartheta_n} & & \downarrow{\scriptstyle \vartheta_{n+1}} \\ P^{(n)\sharp} & \xrightarrow{\iota_n} & P^{(n)\times} \oplus P^{(n)\sharp} & \xrightarrow{j_n^\times \oplus j_n^\sharp} & P^{(n+1)\times} \oplus P^{(n+1)\sharp} \end{array}$$

where i_n, j_n and ι_n are the natural inclusion maps. Now, set

$$B'_n := K^+[P^{(n)\times}] \qquad B''_n := P^{(n)\sharp} \otimes_{P_0^{(n)}} K^+ \qquad \text{for every } n \in \mathbb{N}$$

and let Δ'_n (resp. Δ''_n) be the image of $P^{(n)\times}$ (resp. of $\vartheta_n^{-1} P^{(n)\sharp}$) in Δ_n; since the grading of $P^{(n)}$ induces an isomorphism $P^{(n)}/P_0^{(n)} \xrightarrow{\sim} \Delta_n$, we see that $\Delta_n = \Delta'_n \oplus \Delta''_n$, and by construction, for every $n \in \mathbb{N}$ we have isomorphisms

$$(B_n, \Delta_n) \xrightarrow{\sim} (B'_n, \Delta'_n) \otimes (B''_n, \Delta''_n)$$

of model K^+-algebras, which identify the inclusions $B_n \to B_{n+1}$ with the tensor product of the induced inclusion maps $B'_n \to B'_{n+1}$ and $B''_n \to B''_{n+1}$. Furthermore, set $Y'_n := \operatorname{Spec} B'_n$ and $Y''_n := \operatorname{Spec} B''_n$ for every $n \in \mathbb{N}$. The induced map of monoids $P^{(n)\sharp} \to B''_n$ determines a fine log structure $\underline{M}''_n$ on Y''_n, the map $i_n^\sharp$ gives a chart for a smooth and saturated morphism of log schemes

$$\varphi''_n : (Y''_n, \underline{M}''_n) \to (S, \underline{N}_n)$$

and there follows an isomorphism of $(S, \underline{N}_n)$-schemes (lemma 12.1.4)

$$(17.1.30) \qquad (Y_n, \underline{M}_n) \xrightarrow{\sim} Y'_n \times_S (Y''_n, \underline{M}''_n) \qquad \text{for every } n \in \mathbb{N}.$$

(v) Starting with (17.1.31), we shall consider the strict henselization B^{sh} of B at a given geometric point $\overline{x}$ of $\operatorname{Spec} B \otimes_{K^+} \kappa$. In this situation, let $x \in Y := \operatorname{Spec} B$ be the support of $\overline{x}$, and $\mathfrak{p}_x \subset B$ the corresponding prime ideal. Let also P be as in (ii), denote by $\beta : P \to B$ the natural map deduced from (17.1.27), and set $\mathfrak{p} := \beta^{-1}\mathfrak{p}_x$. Moreover, set $\mathfrak{p}_n := \mathfrak{p} \cap P^{(n)}$ and $Q^{(n)} := P^{(n)}_{\mathfrak{p}_n}$ for every $n \in \mathbb{N}$; clearly $\mathfrak{p}_n \subset \mathfrak{p}_{n+1}$ (so the isomorphism (17.1.24) maps $\mathfrak{p}_{n+1}$ onto $\mathfrak{p}_n$), and therefore

$$Q^{(n)} \subset Q^{(n+1)} \qquad \text{for every } n \in \mathbb{N}.$$

Furthermore, the image of $Q^{(n)}$ in Δ_n^{gp} is a localization Γ_n of Δ_n, especially it is still saturated, and the maps (17.1.24) extend to isomorphisms of Γ_{n+1}-graded monoids

$$Q^{(n+1)} \xrightarrow{\sim} \Gamma_{n+1} \times_{\Gamma_n} Q^{(n)} \qquad \text{for every } n \in \mathbb{N}.$$

It is also clear that the p-Frobenius of Γ_{n+1} factors through an isomorphism $\Gamma_{n+1} \xrightarrow{\sim} \Gamma_n$, for every $n \in \mathbb{N}$. Set $F_n := P^{(n)} \setminus \mathfrak{p}_n$ for every $n \in \mathbb{N}$; notice that, by construction, $F_n \cap P_0^{(n)} = \{1\}$, hence $F_n \cap P_\gamma^{(n)}$ is either empty or else it contains exactly one element, namely the generator g_γ

of $P_\gamma^{(n)}$, by virtue of (i). It follows that $Q_0^{(n)} = P_0^{(n)}$ for every $n \in \mathbb{N}$, and therefore the inclusion map $Q_0^{(n)} \to Q^{(n)}$ is still flat and saturated (lemma 6.2.12(ii)). Let $\Gamma := \bigcup_{n\in\mathbb{N}} \Gamma_n$; we conclude that $Q := P_\mathfrak{p}$ is a Γ-graded monoid with $Q_0 = P_0$, and we may define

$$B_\mathfrak{p} := Q \otimes_{Q_0} K^+.$$

The foregoing shows that $(B_\mathfrak{p}, \Gamma)$ is still a small model algebra, and we also obtain a ladder of log schemes with the properties listed in (17.1.21) : namely, set $B_{\mathfrak{p},n} := B_{\mathfrak{p},\Gamma_n} = Q^{(n)} \otimes_{Q_0^{(n)}} K^+$ for every $n \in \mathbb{N}$, and endow $Y_{\mathfrak{p},n} := \operatorname{Spec} B_{\mathfrak{p},n}$ with the log structure $\underline{M}_{\mathfrak{p},n}$ determined by the induced map $Q^{(n)} \to B_{\mathfrak{p},n}$. Clearly the geometric point $\overline{x}$ lifts uniquely to a geometric point $\overline{x}_\mathfrak{p}$ of $Y_\mathfrak{p} := \operatorname{Spec} B_\mathfrak{p}$, and the localization map $B \to B_\mathfrak{p}$ induces an isomorphism

$$Y_\mathfrak{p}(\overline{x}_\mathfrak{p}) \xrightarrow{\sim} Y(\overline{x}).$$

Hence, for the study of the scheme $Y(\overline{x})$, it shall be usually possible to replace the original small algebra B by its localization $B_\mathfrak{p}$ thus constructed. In so doing, we gain one more property: indeed, notice that the new chart $Q^{(n)} \to B_{\mathfrak{p},n}$ is local at the geometric point $\overline{x}$, for every $n \in \mathbb{N}$. Now, choose a compatible system of decompositions

$$(Y_{\mathfrak{p},n}, \underline{M}_{\mathfrak{p},n}) \xrightarrow{\sim} Y'_{\mathfrak{p},n} \times_S (Y''_{\mathfrak{p},n}, \underline{M}''_{\mathfrak{p},n})$$

as in (iv), and denote by $\overline{x}_{\mathfrak{p},n}$ (resp. $\overline{x}''_{\mathfrak{p},n}$) the image of $\overline{x}_\mathfrak{p}$ in $Y_{\mathfrak{p},n}$ (resp. in $Y''_{\mathfrak{p},n}$), for every $n \in \mathbb{N}$. By inspecting the construction, it is easily seen that the chart $Q^{(n)\sharp} \to B''_{\mathfrak{p},n} := Q^{(n)\sharp} \otimes_{Q_0^{(n)}} K^+$ is also local at $\overline{x}''_{\mathfrak{p},n}$.

17.1.31. Let us return to the situation of (17.1.21). Fix a geometric point $\overline{x}$ of $Y := \operatorname{Spec} B$, localized on $Y \times_S \operatorname{Spec} \kappa$, and for every $n \in \mathbb{N}$, let $\overline{x}_n$ be the image of $\overline{x}$ in Y_n. Set

$$B^{\mathrm{sh}} := \mathscr{O}_{Y(\overline{x}),\overline{x}} \qquad \text{and} \qquad B_n^{\mathrm{sh}} := \mathscr{O}_{Y_n(\overline{x}_n),\overline{x}_n} \quad \text{for every } n \in \mathbb{N}.$$

For the next step, we shall apply the method of (14.5.60), to construct a normalized length function for B^{sh}-modules. To this aim, we need an auxiliary model K^+-algebra, defined as follows. First, notice that the grading $P \to \Delta$ extends to a morphism of monoids

$$\pi_\mathbb{Q} : P_\mathbb{Q} \to \Delta_\mathbb{Q}$$

(notation of (6.3.20)); *i.e.* $P_\mathbb{Q}$ is a $\Delta_\mathbb{Q}$-graded monoid. Since $K^\times$ is a divisible group, the inclusion map $P_0 \to K^*$ extends to a group homomorphism $P_{\mathbb{Q},0}^{\mathrm{gp}} \to K^\times$, and it is easily seen that the latter restricts to a morphism of monoids

$$P_{\mathbb{Q},0} \to K^*.$$

So finally, we may set

$$A := P_\mathbb{Q} \otimes_{P_{\mathbb{Q},0}} K^+.$$

Taking into account (17.1.27), we deduce an isomorphism of Δ-graded K^+-algebras

$$B \xrightarrow{\sim} A_\Delta. \tag{17.1.32}$$

Moreover, arguing as in remark 17.1.12(iii), we get an isomorphism

$$K[\Delta_\mathbb{Q}] \xrightarrow{\sim} A_K \tag{17.1.33}$$

fitting into a commutative diagram of K-algebras

$$\begin{array}{ccc} K[\Delta] & \xrightarrow{\sim} & B_K \\ \downarrow & & \downarrow \\ K[\Delta_\mathbb{Q}] & \xrightarrow{\sim} & A_K \end{array}$$

whose top horizontal arrow is the isomorphism of remark 17.1.17(v), and whose right (resp. left) vertical arrow is deduced from (17.1.32) (resp. is induced by the natural inclusion $\Delta \subset \Delta_{\mathbb{Q}}$).

Lemma 17.1.34. *With the notation of* (17.1.31)*, we have :*

(i) *$P_{\mathbb{Q},\gamma}$ is a free $P_{\mathbb{Q},0}$-module of rank one, for every $\gamma \in \Delta_{\mathbb{Q}}$.*
(ii) *$P_{\mathbb{Q},0}$ is a sharp monoid, and the inclusion map $P_{\mathbb{Q},0} \to P_{\mathbb{Q}}$ is flat and saturated.*

Proof. (i): We can write $P_{\mathbb{Q}}$ as the colimit of the system of monoids

$$(P_{[n]};\ \mu_{P,n,m} : P_{[n]} \to P_{[nm]} \mid n, m \in \mathbb{N})$$

such that $P_{[n]} := P^{(0)}$ for every $n \in \mathbb{N}$, and $\mu_{P,n,m}$ is the m-Frobenius map of $P^{(0)}$, for every $n, m \in \mathbb{N}$. Likewise, we may write $\Delta_{\mathbb{Q}}$ as colimit of a system of Frobenius endomorphisms $(\Delta_{0,[n]};\ \mu_{\Delta,n,m} \mid n, m \in \mathbb{N})$, and $\pi_{\mathbb{Q}}$ is the colimit of the corresponding system of maps $(\pi_{[n]} : P_{[n]} \to \Delta_{0,[n]} \mid n \in \mathbb{N})$ where $\pi_{[n]} := \pi$ for every $n \in \mathbb{N}$. Hence, for any $\gamma \in \Delta_0$ there exists $n \in \mathbb{N}$ such that γ is the image of some $\gamma_n \in \Delta_{0,[n]}$, and

$$P_{\mathbb{Q},\gamma} = \operatorname*{colim}_{k \in \mathbb{N}} P_{[nk],k\gamma_n} = \operatorname*{colim}_{k \in \mathbb{N}} P_{k\gamma_n}.$$

However, we know that $P_{k\gamma_n}$ is free of rank one, generated by a unique element $g_{k\gamma_n}$; moreover, the transition maps $\mu_{P,n,k}$ send g_{γ_n} onto $g_{k\gamma_n}$, for every $k \in \mathbb{N}$ (remark 17.1.26). This implies that $P_{\mathbb{Q},\gamma}$ is generated by the image of g_{γ_n}, whence (i).

(ii): The flatness follows from (i). Since the inclusion map $P_0^{(0)} \to P^{(0)}$ is saturated, it is easily seen that the same holds for the inclusion $P_{\mathbb{Q},0} \to P_{\mathbb{Q}}$. Lastly, the sharpness of $P_{\mathbb{Q},0}$ is likewise deduced from the sharpness of $P_0^{(0)}$ (details left to the reader). ☐

17.1.35. Lemma 17.1.34(ii) implies that the pair $(A, \Delta_{\mathbb{Q}})$ fulfills axiom (MA2), and we have thus our sought auxiliary model K^+-algebra. Since $P_{\mathbb{Q},0}$ is sharp, the $P_{\mathbb{Q},0}$-module $P_{\mathbb{Q},\gamma}$ admits a unique generator g_γ, for every $\gamma \in \Delta_{\mathbb{Q}}$. The image $g_\gamma \otimes 1$ of g_γ in A is a generator of the direct summand A_γ, which is a free K^+-module of rank one; hence, there exists a unique $a_\gamma \in K$ such that $\gamma \otimes a_\gamma$ gets mapped to $g_\gamma \otimes 1$, under the isomorphism (17.1.33).

After choosing an order-preserving isomorphism (14.5.63), we may define a function

$$f_A : \Delta_{\mathbb{Q}} \to \mathbb{R} \quad : \quad \gamma \mapsto \log |a_\gamma|.$$

The inclusion $A_\gamma \cdot A_\delta \subset A_{\gamma+\delta}$ translates as the inequality

$$f_A(\gamma) + f_A(\delta) \geq f_A(\gamma + \delta) \qquad \text{for every } \gamma, \delta \in \Delta_{\mathbb{Q}}.$$

Likewise, the saturation condition of axiom (MA2) translates as the identity

$$f_A(n\gamma) = n \cdot f_A(\gamma) \qquad \text{for every } \gamma \in \Delta_{\mathbb{Q}} \text{ and every } n \in \mathbb{N}.$$

We also fix a (Banach) norm

$$\Delta^{\mathrm{gp}}_{\mathbb{R}} \to \mathbb{R} \qquad \gamma \mapsto \|\gamma\|.$$

We can then state the following

Lemma 17.1.36. *With the notation of* (17.1.35)*, we have :*

(i) *There exists a Δ_0^{gp}-rational subdivision Θ of the convex polyhedral cone $(\Delta^{\mathrm{gp}}_{\mathbb{R}}, \Delta_{\mathbb{R}})$, such that*

$$f_A(\gamma + \delta) = f_A(\gamma) + f_A(\delta) \qquad \textit{for every } \sigma \in \Theta \textit{ and every } \gamma, \delta \in \sigma \cap \Delta_{\mathbb{Q}}.$$

(ii) *Especially, the function f_A is of* Lipschitz type, i.e. *there exists a real constant $C_A > 0$ such that*

$$|f_A(\gamma) - f_A(\gamma')| \leq C_A \cdot \|\gamma - \gamma'\| \qquad \textit{for every } \gamma, \gamma' \in \Delta^{\mathrm{gp}}_{\mathbb{Q}}.$$

Proof. (Here, $\Delta^{\mathrm{gp}}_{\mathbb{R}} := \Delta^{\mathrm{gp}}_0 \otimes_{\mathbb{Z}} \mathbb{R}$ and $\Delta_{\mathbb{R}} \subset \Delta^{\mathrm{gp}}_{\mathbb{R}}$ is the cone spanned by Δ_0.) Combining proposition 6.3.28 and lemma 6.3.33, we find a Δ^{gp}_0-rational subdivision Θ of $\Delta_{\mathbb{R}}$ such that

$$P_{\mathbb{Q},\gamma+\delta} = P_{\mathbb{Q},\gamma} + P_{\mathbb{Q},\delta} \qquad \text{for every } \sigma \in \Theta \text{ and every } \gamma, \delta \in \sigma \cap \Delta_{\mathbb{Q}}.$$

This means that $g_\gamma \cdot g_\delta = g_{\gamma+\delta}$ for every $\sigma \in \Theta$ and every $\gamma, \delta \in \sigma \cap \Delta_{\mathbb{Q}}$, whence (i).

(ii): Clearly, for every $\sigma \in \Theta$ we may find a constant C_σ such that the stated inequality holds – with C_A replaced by C_σ – for every $\gamma, \gamma' \in \sigma \cap \Delta_{\mathbb{Q}}$. It is easily seen that $C_A := \max(C_\sigma \mid \sigma \in \Theta)$ will do (details left to the reader). □

17.1.37. Fix $n \in \mathbb{N}$, pick a subdivision Θ of $(\Delta^{\mathrm{gp}}_{\mathbb{R}}, \Delta_{\mathbb{R}})$ as in lemma 17.1.36, and let $\Theta^s \subset \Theta$ be the subset of all $\sigma \in \Theta$ that span $\Delta^{\mathrm{gp}}_{\mathbb{R}}$. Also, for every $\gamma \in \Delta^{\mathrm{gp}}_{\mathbb{Q}}$, denote by $[\gamma]$ the class of γ in $\Delta^{\mathrm{gp}}_{\mathbb{Q}}/\Delta^{\mathrm{gp}}_n$. For any subset $\Sigma \subset \Delta^{\mathrm{gp}}_{\mathbb{R}}$ we let

$$A_\Sigma := A_{\Sigma \cap \Delta_{\mathbb{Q}}}$$

(where the right-hand side is defined as in (17.1.5)). In light of lemma 6.3.26 we obtain a decomposition of $B_n = A_{\Delta_n} = A_{[0]}$ as sum of K^+-subalgebras :

$$B_n = \sum_{\sigma \in \Theta^s} A_{[0] \cap \sigma}$$

and a corresponding decomposition of the B_n-module $A_{[\gamma]}$ as sum of $A_{[0]\cap\sigma}$-modules

$$A_{[\gamma]} = \sum_{\sigma \in \Theta^s} A_{[\gamma] \cap \sigma} \qquad \text{for every } \gamma \in \Delta^{\mathrm{gp}}_{\mathbb{Q}}.$$

Lemma 17.1.38. *With the notation of* (17.1.37) *and* (17.1.21)*, there exists $N \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ and every $\gamma \in \Delta^{\mathrm{gp}}_{\mathbb{Q}}$, the B_n-module $A_{[\gamma]}$ admits a system of generators of cardinality at most N.*

Proof. According to proposition 6.3.22(ii), for every $\sigma \in \Theta^s$ and every $\gamma \in \Delta^{\mathrm{gp}}_{\mathbb{Q}}$, the subset $\Sigma_{\sigma,\gamma} := \Delta^{\mathrm{gp}}_n \cap (\sigma - \gamma)$ is a finitely generated $(\Delta^{\mathrm{gp}}_n \cap \sigma)$-module; hence, let us fix a finite system of generators $G_{\sigma,\gamma}$ for $\Sigma_{\sigma,\gamma}$. In light of lemma 17.1.36(i) we see that the finite set

$$G'_{\sigma,\gamma} := \{g_{\gamma+\delta} \otimes 1 \mid \delta \in G_{\sigma,\gamma}\} \subset A$$

generates the $A_{[0]\cap\sigma}$-module $A_{[\gamma]\cap\sigma}$. Consequently, the finite set $\bigcup_{\sigma\in\Theta^s} G'_{\sigma,\gamma}$ generates the B_n-module $A_{[\gamma]}$. On the other hand, since $\mathscr{S}_{\Delta^{\mathrm{gp}}_n,\sigma}$ is a finitely generated Δ^{gp}_n-module (proposition 6.3.35(ii)) it is clear that the cardinality of $G_{\sigma,\gamma}$ is bounded by a constant N_σ that is independent of γ, and then $A_{[\gamma]}$ is generated by at most $N_n := \sum_{\sigma\in\Theta^s} N_\sigma$ elements. It remains to show that the estimate for N_n is independent of n. However, notice that, for every $\sigma \in \Theta^s$, the automorphism of $\Delta^{\mathrm{gp}}_{\mathbb{Q}}$ given by multiplication by p^n, induces a natural bijection

$$\mathscr{S}_{\Delta^{\mathrm{gp}}_n,\sigma} \xrightarrow{\sim} \mathscr{S}_{\Delta^{\mathrm{gp}}_0,\sigma}$$

that sends each Δ_n-module $\Delta^{\mathrm{gp}}_n \cap (\sigma - v)$ onto the Δ_0-module

$$\Delta^{\mathrm{gp}}_0 \cap (\sigma - p^n v) = \Delta_0 \otimes_{\Delta_n} (\Delta^{\mathrm{gp}}_n \cap (\sigma - v))$$

(where the extension of scalars $\Delta_n \to \Delta_0$ is the isomorphism given by the rule : $\gamma \mapsto p^n\gamma$ for every $\gamma \in \Delta_n \cap \sigma$). Thus, we see that $N := N_0$ will already do. □

17.1.39. Let us choose N as in lemma 17.1.38, and endow the set $\mathscr{M}_N(B_n^a)$ with the uniform structure defined in (14.5.85) (relative to the standard setup attached to K^+); we consider the mapping

$$\Delta_{\mathbb{Q}}^{\mathrm{gp}} \to \mathscr{M}_N(B_n^a) \qquad \gamma \mapsto A_{[\gamma]}^a.$$

Lemma 17.1.40. *Keep the notation of* (6.3.34) *and* (14.5.85)*, and let also* $C_A > 0$ *be the constant appearing in lemma* 17.1.36(ii)*. Then :*

$$(A_{[\gamma]}^a, A_{[\gamma']}^a) \in E_{C_A \cdot \|\gamma - \gamma'\| \cdot 2}$$

for every $\Sigma \in \mathscr{S}_{\Delta_n^{\mathrm{gp}}, \Delta_{\mathbb{R}}}$ *and every* $\gamma, \gamma' \in \Omega(\Delta_{\mathbb{R}}, \Sigma) \cap \Delta_{\mathbb{Q}}^{\mathrm{gp}}$.

Proof. By inspecting the definitions, we get an isomorphism of B_n-modules

$$A_{[\gamma]} \xrightarrow{\sim} \bigoplus_{\delta \in \Sigma} \{\delta \otimes a \in K[\Delta_n^{\mathrm{gp}}] \mid \log|a| \geq f_A(\gamma + \delta)\} \tag{17.1.41}$$

for every $\Sigma \in \mathscr{S}_{\Delta_n^{\mathrm{gp}}, \Delta_{\mathbb{R}}}$ and every $\gamma \in \Omega(\Delta_{\mathbb{R}}, \Sigma) \cap \Delta_{\mathbb{Q}}^{\mathrm{gp}}$. Hence, suppose more generally that Σ, Σ' are two elements of $\mathscr{S}_{\Delta_n^{\mathrm{gp}}, \Delta_{\mathbb{R}}}$ such that $\Sigma \subset \Sigma'$, and let $\gamma \in \Omega(\Delta_{\mathbb{R}}, \Sigma) \cap \Delta_{\mathbb{Q}}^{\mathrm{gp}}$, $\gamma' \in \Omega(\Delta_{\mathbb{R}}, \Sigma') \cap \Delta_{\mathbb{Q}}^{\mathrm{gp}}$ be any two elements; from lemma 17.1.36(ii) we get, for any $b \in K^+$ with $\log|b| \geq C_A \|\gamma' - \gamma\|$, a B_n-linear map

$$\tau_{b,\gamma'-\gamma} : A_{[\gamma]} \to A_{[\gamma']}$$

which, under the identifications (17.1.41), corresponds to the map given by the rule : $\delta \otimes a \mapsto \delta \otimes ba$ for every $\delta \in \Sigma$ and every $a \in K$ such that $\log|a| \geq f(\gamma + \delta)$. In case $\Sigma = \Sigma'$, also $\tau_{b,\gamma-\gamma'}$ is well defined, and clearly

$$\tau_{b,\gamma'-\gamma} \circ \tau_{b,\gamma-\gamma'} = b^2 \cdot \mathbf{1}_{A_{[\gamma']}} \qquad \tau_{b,\gamma-\gamma'} \circ \tau_{b,\gamma'-\gamma} = b^2 \cdot \mathbf{1}_{A_{[\gamma]}}$$

whence the contention. ☐

Remark 17.1.42. Let $\gamma_1, \gamma_2, \gamma'_1, \gamma'_2 \in \Delta_{\mathbb{Q}}^{\mathrm{gp}}$ be four elements, and set $\gamma_3 := \gamma_1 + \gamma_2$, $\gamma'_3 := \gamma'_1 + \gamma'_2$; suppose that

$$\Delta_n^{\mathrm{gp}} \cap (\Delta_{\mathbb{R}} - \gamma_i) \subset \Delta_n^{\mathrm{gp}} \cap (\Delta_{\mathbb{R}} - \gamma'_i) \qquad \text{for } i = 1, 2, 3.$$

With the notation of the proof of lemma 17.1.40, for every $b_1, b_2 \in K^+$ such that

$$\log|b_i| \geq C_A \|\gamma'_i - \gamma_i\| \qquad (i = 1, 2)$$

we obtain a commutative diagram of B_n-linear maps

$$\begin{array}{ccc} A_{[\gamma_1]} \otimes_{B_n} A_{[\gamma_2]} & \longrightarrow & A_{[\gamma_3]} \\ {\scriptstyle \tau_{b_1,\gamma'_1-\gamma_1} \otimes \tau_{b_2,\gamma'_2-\gamma_2}} \downarrow & & \downarrow {\scriptstyle \tau_{b_1 b_2,\gamma'_3-\gamma_3}} \\ A_{[\gamma'_1]} \otimes_{B_n} A_{[\gamma'_2]} & \longrightarrow & A_{[\gamma'_3]} \end{array}$$

whose horizontal arrows are the restrictions of the multiplication map of the B_n-algebra B : the detailed verification shall be left as an exercise for the reader.

Theorem 17.1.43. *In the situation of* (17.1.31)*, the* K^+*-algebra* B^{sh} *is ind-measurable.*

Proof. First, notice that, for every $k \geq 0$, the inclusion map $B_n \to B_{n+k}$ induces a radicial morphism $B_n \otimes_{K^+} \kappa \to B_{n+k} \otimes_{K^+} \kappa$, and therefore also an isomorphism of B_{n+k}-algebras

$$B_n^{\mathrm{sh}} \otimes_{B_n} B_{n+k} \xrightarrow{\sim} B_{n+k}^{\mathrm{sh}} \tag{17.1.44}$$

([66, Ch.IV, Prop.18.8.10]). Notice as well, that B_n^{sh} is a measurable K^+-algebra (see definition 14.5.3(ii)), hence we have a well defined normalized length function λ_n on isomorphism classes of B_n^{sh}-modules, characterized as in theorem 14.5.43. Thus, our task is to exhibit a sequence $(d_n \mid n \in \mathbb{N})$ of normalizing factors fulfilling conditions (a) and (b) of definition 14.5.64, for the directed system $(B_n^{\rm sh} \mid n \in \mathbb{N})$, whose colimit is $B^{\rm sh}$. To this aim, let M be any object of $B_n^{\rm sh}\text{-}\mathbf{Mod}_{{\rm coh},\{s\}}$; in view of (17.1.44), we need to estimate

$$\lambda_{k+n}(B_{k+n} \otimes_{B_n} M) = \lambda_{k+n}\left(\bigoplus_{[\gamma]\in\Delta^{\rm gp}_{k+n}/\Delta^{\rm gp}_n} A_{[\gamma]} \otimes_{B_n} M\right).$$

To this aim, let us introduce the function

$$l_M : \Delta^{\rm gp}_{\mathbb{Q}} \to \mathbb{R} \qquad \gamma \mapsto \lambda_n(A_{[\gamma]} \otimes_{B_n} M).$$

Claim 17.1.45. Let Σ be as in lemma 17.1.40, set $J := \mathrm{Ann}_{B_0^{\rm sh}}(M/\mathfrak{m}M)$, and suppose that M admits a generating set of cardinality k. Then the restriction of l_M to $\Omega(\Delta_{\mathbb{R}},\Sigma) \cap \Delta^{\rm gp}_{\mathbb{Q}}$ is *of Lipschitz type*, *i.e.* there exists a real constant $C'_A > 0$ independent of n, Σ and M such that

$$|l_M(\gamma) - l_M(\gamma')| \leq C'_A \cdot k \cdot \mathrm{length}_{B_n^{\rm sh}}(B_n^{\rm sh}/JB_n^{\rm sh}) \cdot \|\gamma - \gamma'\|$$

for every $\gamma, \gamma' \in \Omega(\Delta_{\mathbb{R}},\Sigma) \cap \Delta^{\rm gp}_{\mathbb{Q}}$.

Proof of the claim. Let $I \subset B_0^{\rm sh}$ be any finitely generated $\mathfrak{m}_{B_n^{\rm sh}}$-primary ideal contained in the annihilator of M. Also, let N' be the cardinality of a finite set of generators for the $B_n^{\rm sh}$-module M, and N the constant provided by lemma 17.1.38; clearly $A_{[\gamma]} \otimes_{B_n} M$ admits a system of generators of cardinality $\leq NN'$, for every $\gamma \in \Delta^{\rm gp}_{\mathbb{Q}}$. Furthermore, an argument as in the proof of [75, Lemma 2.3.7(iv)] shows that the induced mapping

$$\mathscr{M}_N(B_n) \to \mathscr{M}_{NN'}(B_n^{\rm sh}/IB_n^{\rm sh}) \qquad L \mapsto L \otimes_{B_0} M$$

is of Lipschitz type; more precisely, it sends the entourage E_r into E_{2r}, for every $r \in \mathbb{R}_{>0}$ (details left to the reader). Then the claim follows by combining lemmata 14.5.86 and 17.1.40. ◊

From proposition 6.3.35(iv), we see that, for every $\Sigma \in \mathscr{S}_{\Delta_n^{\rm gp},\Delta_{\mathbb{R}}}$, the subset $\Omega(\Delta_{\mathbb{R}},\Sigma) \cap \Delta^{\rm gp}_{\mathbb{Q}}$ is dense in $\Omega(\Delta_{\mathbb{R}},\Sigma)$; taking into account claim 17.1.45, it follows that l_M extends (uniquely) to a function $l_{M,\mathbb{R}} : \Delta^{\rm gp}_{\mathbb{R}} \to \mathbb{R}$, whose restriction to every $\Omega(\Delta_{\mathbb{R}},\Sigma)$ is continuous and still satisfies the same Lipschitz type estimate. Notice that $l_{M,\mathbb{R}}$ descends to a function on the torus

$$\bar{l}_M : \mathbf{T}_n := \Delta^{\rm gp}_{\mathbb{R}}/\Delta^{\rm gp}_n \to \mathbb{R}.$$

Hence, set $r := \mathrm{rk}_{\mathbb{Z}}\Delta^{\rm gp}_0$, let us fix a basis $e_1,\dots,e_r$ for the free $\mathbb{Z}$-module $\Delta^{\rm gp}_0$, and define

$$\Omega_n := \left\{\sum_{i=1}^r t_i e_i \mid -\frac{1}{2p^n} \leq t_i < \frac{1}{2p^n} \text{ for } i = 1,\dots,r\right\}$$

so $\Omega_n \subset \Delta^{\rm gp}_{\mathbb{R}}$ is a *fundamental domain* for the lattice $\Delta^{\rm gp}_n$; in view of theorem 14.5.43(ii.b), we reach the following identity :

$$\lambda_{k+n}(B_{k+n} \otimes_{B_n} M) = \frac{1}{[\kappa(B^{\rm sh}_{k+n}) : \kappa(B^{\rm sh}_n)]} \cdot \sum_{\gamma\in\Delta^{\rm gp}_{k+n}\cap\Omega_n} l_M(\gamma).$$

Now, set

$$d_n := p^{nr} \cdot [\kappa(B^{\rm sh}_n) : \kappa(B^{\rm sh}_0)]^{-1} \qquad \text{for every } n \in \mathbb{N}. \tag{17.1.46}$$

We claim that $(d_n \mid n \in \mathbb{N})$ is a suitable sequence of normalizing factors for $B^{\rm sh}$. Indeed, recall that $\Omega(\Delta_{\mathbb{R}},\Sigma)$ is linearly constructible for every $\Sigma \in \mathscr{S}_{\Delta_0^{\rm gp},\Delta_{\mathbb{R}}}$ (proposition 6.3.35(iii)), especially, the bounded function l_M is continuous outside a subset of $\Delta^{\rm gp}_{\mathbb{R}}$ of measure zero, hence it is integrable on every bounded measurable subset of $\Delta_{\mathbb{R}}$. It follows that $\bar{l}_M$ is integrable relative to the invariant measure $d\mu_n$ on $\mathbf{T}_n$ of total volume equal to 1; lastly, a simple inspection (comparison of Riemann and Lebesgue integrals) yields

$$\lambda(B \otimes_{B_n} M) := \lim_{k \to +\infty} d_{k+n}^{-1} \cdot \lambda_{k+n}(B_{k+n} \otimes_{B_n} M) = d_n^{-1} \int_{\mathbf{T}_n} \bar{l}_M d\mu_n$$

which shows that condition (a) of definition 14.5.64 holds for this choice of normalizing factors.

In order to check condition (b), set $\Delta_{\mathbb{R}}(\rho) := \{v \in \Delta_{\mathbb{R}} \mid \|v\| \leq \rho\}$ for every $\rho > 0$. Notice that the automorphism of $\Delta_{\mathbb{R}}^{\mathrm{gp}}$ given by multiplication by p^n restricts to a bijection

$$\Omega(\Delta_{\mathbb{R}}, \Delta_n) \stackrel{\sim}{\to} \Omega(\Delta_{\mathbb{R}}, \Delta_0) \qquad \text{for every } n \in \mathbb{N}.$$

Then, by claim 6.3.40, we see that there exists ρ_0 such that

$$(17.1.47) \qquad \Delta_{\mathbb{R}}(p^{-n}\rho_0) \subset \Omega_n \cap \Omega(\Delta_{\mathbb{R}}, \Delta_n) \qquad \text{for all } n \in \mathbb{N}.$$

Especially, for every $\rho \leq \rho_0$ we may regard $\Delta_{\mathbb{R}}(p^{-n}\rho)$ as a measurable subset of $\mathbf{T}_n$, whose measure $\mathrm{Vol}(\rho)$ is strictly positive and independent of n.

Claim 17.1.48. Let $I \subset B_0^{\mathrm{sh}}$ be any finitely generated $\mathfrak{m}_{B_0^{\mathrm{sh}}}$-primary ideal. Then there exists a real constant $C_I > 0$ such that

$$d_n^{-1} \cdot \mathrm{length}_{B_n^{\mathrm{sh}}}(B_n^{\mathrm{sh}}/(I + \mathfrak{m})B_n^{\mathrm{sh}}) \leq C_I \qquad \text{for every } n \in \mathbb{N}.$$

Proof of the claim. On the one hand, we may write B_n^{sh} as the direct sum of the B_0^{sh}-modules $A_{[\gamma]\cap\Delta_{\mathbb{Q}}} \otimes_{B_0} B_0^{\mathrm{sh}}$, for γ ranging over the elements of $\Delta_n^{\mathrm{gp}}/\Delta_0^{\mathrm{gp}}$. There are p^{rn} such direct summands, and each of them admits a generating system of cardinality $\leq N$, where N is the constant provided by lemma 17.1.38. It follows that

$$\mathrm{length}_{B_0^{\mathrm{sh}}}(B_n^{\mathrm{sh}}/(I + \mathfrak{m})B_n^{\mathrm{sh}}) \leq Np^{rn} \cdot \mathrm{length}_{B_0^{\mathrm{sh}}}(B_0^{\mathrm{sh}}/(I + \mathfrak{m}B_0^{\mathrm{sh}})).$$

On the other hand, say that $l := \mathrm{length}_{B_n^{\mathrm{sh}}}(B_n^{\mathrm{sh}}/(I+\mathfrak{m})B_n^{\mathrm{sh}})$; this means that may find a filtration of $B_n^{\mathrm{sh}}/(I + \mathfrak{m})B_n^{\mathrm{sh}}$ of length l, consisting of B_n^{sh}-submodules, whose graded subquotients are all isomorphic to $\kappa(B_n^{\mathrm{sh}})$. Given such a filtration, we easily obtain a filtration of $B_n^{\mathrm{sh}}/(I+\mathfrak{m})B_n^{\mathrm{sh}}$ of length $l \cdot [\kappa(B_n^{\mathrm{sh}}) : \kappa(B_0^{\mathrm{sh}})]$, consisting of B_0^{sh}-submodules, whose graded subquotients are all isomorphic to $\kappa(B_0^{\mathrm{sh}})$. Therefore

$$l \cdot [\kappa(B_n^{\mathrm{sh}}) : \kappa(B_0^{\mathrm{sh}})] = \mathrm{length}_{B_0^{\mathrm{sh}}}(B_n^{\mathrm{sh}}/(I + \mathfrak{m})B_n^{\mathrm{sh}})$$

whence the claim. ◊

Now, fix $\varepsilon > 0$, and suppose there is given a finitely generated $\mathfrak{m}_{B_0^{\mathrm{sh}}}$-primary ideal $I \subset B_0^{\mathrm{sh}}$, and a surjection of finitely presented $B_n^{\mathrm{sh}}/IB_n^{\mathrm{sh}}$-modules $M \to M'$, generated by k elements, such that

$$d_n^{-1} \cdot (\lambda_n(M) - \lambda_n(M')) > \varepsilon.$$

Set $C(I,k) := 2kC'_A C_I$, where C'_A and C_I are as in claims 17.1.45 and 17.1.48; since

$$l_M(0) - l_{M'}(0) = \lambda_n(M) - \lambda_n(M')$$

we may estimate :

$$\begin{aligned}|\lambda(B^{\mathrm{sh}} \otimes_{B_n^{\mathrm{sh}}} M) - \lambda(B^{\mathrm{sh}} \otimes_{B_n^{\mathrm{sh}}} M')| &\geq d_n^{-1} \cdot \int_{\Delta_{\mathbb{R}}(p^{-n}\rho)} (\bar{l}_M - \bar{l}_{M'}) d\mu_n \\ &\geq \mathrm{Vol}(\rho) \cdot (d_n^{-1}(l_M(0) - l_{M'}(0)) - C(I,k) \cdot p^{-n}\rho) \\ &\geq \mathrm{Vol}(\rho) \cdot (\varepsilon - C(I,k) \cdot \rho)\end{aligned}$$

for every $\rho \leq \rho_0$. Therefore, if we set

$$\rho(k, \varepsilon, I) := \min\{\rho_0, 2^{-1}C(I,k)^{-1}\varepsilon\}$$

it is easily seen that the sought condition (b) holds with $\delta(k, \varepsilon, I) := 2^{-1} \cdot \mathrm{Vol}(\rho(k, \varepsilon, I)) \cdot \varepsilon$. □

17.2. **Almost purity : the log regular case.** In this section, we prove an almost purity theorem for certain towers of regular log schemes.

17.2.1. Let $\underline{M}_0$ be a log structure on the Zariski site of a local scheme X_0, such that $(X_0, \underline{M}_0)$ is regular at the closed point $x_0 \in X_0$, and say that $X_0 = \operatorname{Spec} B_0$ for a local ring B_0 which is necessarily noetherian, normal and Cohen-Macaulay (corollary 12.5.13). Let also $\beta_0 : P \to B_0$ be a chart for $\underline{M}_0$ that is sharp at x_0. Especially, P is a fine and saturated monoid, and $A := B_0/\mathfrak{m}_P B_0$ is a regular local ring; we denote by $\mathfrak{m}_A$ (resp. $\mathfrak{m}_{B_0}$) the maximal ideal of A (resp. of B_0). We assume furthermore that :

(a) The characteristic p of the residue field $\kappa(x_0)$ of A is positive.
(b) The Frobenius endomorphism Φ_{B_0} of B_0/pB_0 is a finite ring homomorphism.

Notice that (b) implies that B_0/pB_0 is excellent (theorem 9.7.26(i)) and also that $\Omega^1_{A/\mathbb{Z}} \otimes_A \kappa(x_0)$ is a finite dimensional $\kappa(x_0)$-vector space. Moreover, $(X_0, \underline{M}_0)$ is a regular log scheme, by theorem 12.5.31.

Example 17.2.2. Let (B, β) be a pair consisting of

(a) a local ring B whose residue field κ_B has positive characteristic
(b) a local morphism of monoids $\beta : P \to B$ from a fine, sharp and saturated monoid P.

Denote by $\mathfrak{m}_B$ the maximal ideal of B, by $\underline{M}$ the logarithmic structure on the Zariski site of $X := \operatorname{Spec} B$ induced by β, and suppose that $(X, \underline{M})$ is regular at the closed point of X.

(i) If moreover, B is complete for the $\mathfrak{m}_B$-adic topology and κ_B is perfect, then the Frobenius endomorphism of B/pB is a finite map. Indeed, in this case, for a suitable $d \in \mathbb{N}$ we have a surjective ring homomorphism

$$R[[P \times \mathbb{N}^{\oplus d}]] \to B$$

where R is either κ_B, or a discrete valuation ring with an isomorphism $R/pR \xrightarrow{\sim} \kappa_B$ (remark 12.5.12). The assertion is an immediate consequence.

(ii) It follows that for any such pair (B, β) we may find a local ring B_0 and a local and flat ring homomorphism $f : B \to B_0$ such that the log scheme

$$(X_0, \underline{M}_0) := \operatorname{Spec} B_0 \times_X (X, \underline{M})$$

and its chart $\beta_0 := \beta \circ f : P \to B_0$ fulfill the conditions of (17.2.1). Indeed, according to [61, Ch.0, Prop.10.3.1] there exists a flat and local B-algebra C such that $\kappa_B \otimes_B C$ is a perfect field, and we take B_0 to be the completion of C. By proposition 12.5.30(ii), the resulting log scheme $(X_0, \underline{M}_0)$ is regular at the closed point of X_0, so the assertion follows from (i).

17.2.3. In the situation of (17.2.1), set

$$P^{(n)} := \{\gamma \in P_{\mathbb{Q}} \mid \gamma^{p^n} \in P\} \qquad \text{for every } n \in \mathbb{N}.$$

The p^n-Frobenius map of $P^{(n)}$ identifies $P^{(n)}$ with its submonoid P; in other words, the inclusion map $P \to P^{(n)}$ is naturally identified with the p^n-Frobenius endomorphism of P. Fix a sequence $(f_1, \dots, f_r)$ of elements of B_0 whose image in A is maximal in the sense of remark 9.6.35(iii), in which case we say that $(f_1, \dots, f_r)$ is *maximal in* B_0. Notice that $\dim_{\kappa(x_0)} \mathfrak{m}_A/\mathfrak{m}_A^2 = \dim A$, since A is regular; it follows that

$$r = \dim A + \dim_{\kappa(x_0)} \Omega^1_{\kappa(x_0)/\mathbb{Z}}$$

by virtue of the short exact sequence (9.6.5), if $p \in \mathfrak{m}_A^2$, and otherwise, by virtue of proposition 9.6.17. Denote also by $\overline{f}_i \in A$ the image of f_i, for every $i = 1, \dots, r$. According to corollary 9.6.34, the ring

$$A_n := A[T_1, \dots, T_r]/(T_1^{p^n} - \overline{f}_1, \dots, T_r^{p^n} - \overline{f}_r)$$

is regular. For every $n \in \mathbb{N}$, we set :

$$B'_n := P^{(n)} \otimes_P B_0 \qquad B''_n := B_0[T_1, \dots, T_r]/(T_1^{p^n} - f_1, \dots, T_r^{p^n} - f_r) \qquad B_n := B'_n \otimes_{B_0} B''_n.$$

Lemma 17.2.4. *The induced maps*

$$B_n \to B_{n+1} \qquad \textit{and} \qquad B_n/pB_n \to B_{n+1}/pB_{n+1}$$

are injective, for every $n \in \mathbb{N}$.

Proof. The natural map $B_n \to B_{n+1}$ factors as the composition

$$B'_n \otimes_{B_0} B''_n \to B'_{n+1} \otimes_{B_0} B''_n \to B'_{n+1} \otimes_{B_0} B''_{n+1} \qquad \text{for every } n \in \mathbb{N}.$$

Notice that $B_0 = B''_0$, and B''_{n+1} is a free B''_n-module (of rank p^r) for every $n \in \mathbb{N}$; we are then reduced to checking that the maps $B'_n \to B'_{n+1}$ and $B'_n/pB'_n \to B'_{n+1}/pB'_{n+1}$ are injective for every $n \in \mathbb{N}$. However, set $G := P^{(n+1)}/P^{(n)}$, and notice that $P^{(n+1)}$ is a G-graded monoid, with $P_0^{(n+1)} = P^{(n)}$, hence B'_{n+1} is a G-graded B'_n-algebra with $(B'_{n+1})_0 = B'_n$ for every $n \in \mathbb{N}$. The assertion follows. □

17.2.5. In view of lemma 17.2.4, we may set

$$B'' := \bigcup_{n\in\mathbb{N}} B''_n \qquad B := \bigcup_{n\in\mathbb{N}} B_n \qquad P^{(\infty)} := \bigcup_{n\in\mathbb{N}} P^{(n)}$$

and clearly

$$B = P^{(\infty)} \otimes_P B'' \qquad \text{for every } n \in \mathbb{N}$$

from which we see that B is naturally a $P^{(\infty)}/P^{(n)}$-graded B_n-algebra, for every $n \in \mathbb{N}$. Also, the induced morphism $\operatorname{Spec} B_{n+1}/pB_{n+1} \to \operatorname{Spec} B_n/pB_n$ is radicial and surjective, so $\operatorname{Spec} B_n/pB_n$ is a local scheme for every $n \in \mathbb{N}$; on the other hand, the map $B_0 \to B_n$ is finite, so every point of $\operatorname{Spec} B_n$ specializes to a point of $\operatorname{Spec} B_n/pB_n$. We conclude that B_n is a local ring, and we denote by $\mathfrak{m}_{B_n}$ its maximal ideal, for every $n \in \mathbb{N}$. Let $\underline{M}_n$ be the log structure on the Zariski site of $X_n := \operatorname{Spec} B_n$ deduced from the natural map $\beta_n : P^{(n)} \to B_n$; notice that $B_n/\mathfrak{m}_{P^{(n)}}B_n = A_n$ is a regular local ring of dimension equal to $\dim A$. Since we have as well $\dim P^{(n)} = \dim P$, it follows that $(X_n, \underline{M}_n)$ is regular at the closed point $x_n \in X_n$. Then theorem 12.5.31 shows that $(X_n, \underline{M}_n)$ is a regular log scheme. Thus, we have obtained a tower of finite morphisms of regular log schemes

$$(17.2.6) \qquad \cdots \to (X_{n+1}, \underline{M}_{n+1}) \to (X_n, \underline{M}_n) \to \cdots \to (X_0, \underline{M}_0)$$

which we call the *maximal tower* associated with the chart $\beta_0 : P \to B_0$ and the maximal sequence $(f_1, \dots, f_r)$. The limit of the tower (17.2.6) is a log scheme $(X, \underline{M})$ whose log structure admits a chart $P^{(\infty)} \to B$ which is sharp at the closed point.

Remark 17.2.7. Keep the notation of (17.2.5), and let $s \in \mathbb{N}$ be any integer; from remark 9.6.35(iii) it is easily seen that the sequence $((X_{n+s}, \underline{M}_{n+s}) \mid n \in \mathbb{N})$, obtained by removing from (17.2.6) the first s terms, is the maximal tower associated with the chart β_n and the maximal sequence $(f_1^{1/p^s}, \dots, f_r^{1/p^s})$.

17.2.8. Let now $y \in (X_0, \underline{M}_0)_{\mathrm{tr}}$ be any point, and $n \in \mathbb{N}$ any integer. The chart β_0 extends uniquely to a morphism of monoids $\beta_0^{\mathrm{gp}} : P^{\mathrm{gp}} \to B_{0,y} := \mathscr{O}_{X_0,y}$, and we have a natural isomorphism

$$B_n \otimes_{B_0} B_{0,y} \xrightarrow{\sim} P^{(n)\mathrm{gp}} \otimes_{P^{\mathrm{gp}}} B_{0,y} \otimes_{B_0} B''_n.$$

Especially, $B_n \otimes_{B_0} B_{0,y}$ is a free $B_{0,y}$-module of rank p^{nd}, where

$$d := \dim P + r = \dim P + \dim A + \dim_{\kappa(x_0)} \Omega^1_{\kappa(x_0)/\mathbb{Z}}.$$

However, since $(X_0, \underline{M}_0)$ is regular, we have $\dim A + \dim P = \dim B_0$. Summing up, we find

$$d = \dim B_0 + \dim_{\kappa(x_0)} \Omega^1_{\kappa(x_0)/\mathbb{Z}}.$$

Next, let $z \in \operatorname{Spec} B_0/pB_0 \subset X_0$ be any point, and $\mathfrak{p}_z \subset B_0$ the corresponding prime ideal; as in the foregoing, the chart β_0 extends uniquely to a morphism $\beta_\mathfrak{p} : P_\mathfrak{p} \to B_{0,z} := \mathscr{O}_{X_0,z}$, where $\mathfrak{p} := \beta_0^{-1}\mathfrak{p}_z$. As already remarked, the point z lifts uniquely to a point $z_n \in X_n$, and on the other hand, there exists a unique prime ideal $\mathfrak{p}^{(n)} \subset P^{(n)}$ containing $\mathfrak{p}$, and the inclusion map $j_\mathfrak{p} : P_\mathfrak{p} \to P^{(n)}_{\mathfrak{p}^{(n)}}$ is naturally identified with the p^n-Frobenius endomorphism of $P_\mathfrak{p}$. By lemma 6.2.10, there exists an isomorphism of monoids $P_\mathfrak{p} \xrightarrow{\sim} G \times Q$, with $G := P_\mathfrak{p}^\times$ and $Q := P_\mathfrak{p}^\sharp$; we may then find a corresponding decomposition $P^{(n)}_{\mathfrak{p}^{(n)}} = G^{(n)} \times Q^{(n)}$ that identifies $j_\mathfrak{p}$ with the product of maps of monoids $G \to G^{(n)}$ and $Q \to Q^{(n)}$. Summing up, there follows an isomorphism of B_0-algebras

$$\mathscr{O}_{X_n,z_n} \xrightarrow{\sim} P^{(n)}_{\mathfrak{p}^{(n)}} \otimes_{P_\mathfrak{p}} B_{0,z} \otimes_{B_0} B''_n \xrightarrow{\sim} (Q^{(n)} \otimes_Q B_{0,z}) \otimes_{B_{0,z}} (G^{(n)} \otimes_G B_{0,z}) \otimes_{B_0} B''_n.$$

Fix a basis $g_1, \dots, g_s$ of the free $\mathbb{Z}$-module G, and set $f_{r+i} := \beta_\mathfrak{p}(g_i)$ for $i = 1, \cdots, s$; clearly

$$(G^{(n)} \otimes_G B_{0,z}) \otimes_{B_0} B''_n = B_{0,z}[T_1, \dots, T_{r+s}]/(T_1^{p^n} - f_1, \dots, T_{r+s}^{p^n} - f_{r+s}).$$

On the other hand, set $A_z := B_{0,z}/\mathfrak{m}_Q B_{0,z}$; we have

$$\begin{aligned} \dim P &= \dim Q + s && \text{(corollary 6.4.12(i))} \\ \dim B_{0,z} &= \dim Q + \dim A_z && \text{(since } (X_0, \underline{M}_0) \text{ is regular)} \\ \dim B_{0,z} &= d - \dim_{\kappa(z)} \Omega^1_{\kappa(z)/\mathbb{Z}} && \text{(proposition 9.7.20).} \end{aligned}$$

Therefore $r + s = \dim A_z + \dim_{\kappa(z)} \Omega^1_{\kappa(z)/\mathbb{Z}}$. However, $A_{z_n} := \mathscr{O}_{X_n,z_n}/\mathfrak{m}_{Q^{(n)}}\mathscr{O}_{X_n,z_n}$ is a regular local ring, since $(X_n, \underline{M}_n)$ is regular, and by inspecting the construction we see that

$$A_{z_n} = A_z[T_1, \dots, T_{r+s}]/(T_1^{p^n} - f_1, \dots, T_{r+s}^{p^n} - f_{r+s})$$

hence the image of the system $(f_1, \dots, f_{r+s})$ yields a basis of either $\Omega^1_{A_z/\mathbb{Z}} \otimes_{A_z} \kappa(z)$ or $\boldsymbol{\Omega}_{A_z}$, depending on whether or not $p \in \mathfrak{m}^2_{A_z}$ (corollary 9.6.34). In conclusion, we see that the induced sequence of morphisms of log schemes

$$\cdots \to (X_{n+1}(z_n), \underline{M}_{n+1}(z_n)) \to (X_n(z_n), \underline{M}_n(z_n)) \to \cdots \to (X_0(z), \underline{M}_0(z))$$

(notation of (12.7.11)) is the maximal tower associated with the induced chart $Q \to \mathscr{O}_{X_0,z}$ and the maximal sequence $(f_1, \dots, f_{r+s})$.

Remark 17.2.9. (i) Let $\overline{x}_0$ be any geometric point of X_0, localized at x_0, and denote by B_0^{sh} the strict henselization of B_0 at the point $\overline{x}_0$. In light of [75, Th.3.5.13(ii)], we see that the log scheme $(X_0(\overline{x}_0), \underline{M}_0(\overline{x}_0))$ still fulfills the conditions of (17.2.1). Moreover, the image in B_0^{sh} of the maximal sequence $(f_1, \dots, f_n)$ is still maximal (remark 9.6.35(iv)). It follows easily that the induced tower of regular log schemes

$$\cdots \to X_0(\overline{x}_0) \times_{X_0} (X_{n+1}, \underline{M}_{n+1}) \to X_0(\overline{x}_0) \times_{X_0} (X_n, \underline{M}_n) \to \cdots \to (X_0(\overline{x}_0), \underline{M}_0(\overline{x}_0))$$

is (naturally isomorphic to) the maximal tower associated with the induced chart $P_0 \to B_0^{\mathrm{sh}}$ and the maximal sequence $(f_1, \dots, f_n)$ of B_0^{sh}.

(ii) In the same vein, let $I \subset B_0$ be any proper ideal, $B_{0,I}^\wedge$ the I-adic completion of the local ring B_0, and set $X_0^\wedge := \operatorname{Spec} B_{0,I}^\wedge$; according to claim 9.7.21(i), the log scheme $(X_0^\wedge, \underline{M}_0^\wedge) := X_0^\wedge \times_{X_0} (X_0, \underline{M}_0)$ still fulfills the conditions of (17.2.1), and the image in $B_{0,I}^\wedge$ of the maximal sequence $(f_1, \dots, f_n)$ of B_0 is still maximal (remark 9.6.35(iv)). It follows that the induced tower of log schemes

$$\cdots \to X_0^\wedge \times_{X_0} (X_{n+1}, \underline{M}_{n+1}) \to X_0^\wedge \times_{X_0} (X_n, \underline{M}_n) \to \cdots \to (X_0^\wedge, \underline{M}_0^\wedge)$$

is (naturally isomorphic to) the maximal tower associated with the induced chart $P_0 \to B^{\wedge}_{0,I}$ and the maximal sequence $(f_1, \dots, f_n)$ of $B^{\wedge}_{0,I}$.

17.2.10. Let $\Phi_{B_n} : B_n/pB_n \to B_n/pB_n$ be the Frobenius endomorphism of B_n/pB_n; taking into account lemma 17.2.4, we see that $\Phi_{B_{n+1}}$ factors through a unique ring homomorphism

$$\Psi_{B_{n+1}} : B_{n+1}/pB_{n+1} \to B_n/pB_n \qquad \text{for every } n \in \mathbb{N}.$$

Lemma 17.2.11. *The map $\Psi_{B_{n+1}}$ is surjective for every $n \in \mathbb{N}$.*

Proof. Let us start out with the following general :

Claim 17.2.12. Let $\varphi : R \to S$ be an injective, finite and radicial ring homomorphism. Then φ is an isomorphism if and only if $\Omega^1_{S/R} = 0$.

Proof of the claim. We may assume that $\Omega^1_{S/R} = 0$, and we show that φ is an isomorphism. To this aim, it suffices to show that

$$\varphi \otimes_R R/\mathfrak{m} : R' := R/\mathfrak{m} \to S' := S/\mathfrak{m}S$$

is an isomorphism for every maximal ideal $\mathfrak{m} \subset R$. However, $\Omega^1_{S'/R'} = \Omega^1_{S/R} \otimes_R R' = 0$, so we may replace R by R' and S by S', and assume from start that R is a field. In this case, S is a local unramified R-algebra, so S must be a finite separable field extension of R, by [66, Ch.IV, Cor.17.4.2]. But S is also a radicial extension of R, so $S = R$. ◊

Claim 17.2.13. Let $p > 0$ be a prime integer, R a local $\mathbb{F}_p$-algebra whose Frobenius endomorphism Φ_R is a finite map, k_R the residue field of R, and $\Sigma \subset R$ any subset. Set $R_0 := R/\Sigma R$, and let $(g_1, \dots, g_n)$ be a finite sequence of elements of R such that $dg_1, \dots, dg_n$ is a system of generators for the k_R-vector space $\Omega^1_{R_0/\mathbb{Z}} \otimes_R k_R$. Then $R = R^p[\Sigma, g_1, \dots, g_n]$.

Proof of the claim. Set $S := R^p[\Sigma, g_1, \dots, g_n]$. The inclusion map $S \to R$ is clearly radicial, and it is finite, since Φ_R is finite; hence claim 17.2.12 reduces to checking that $\Omega^1_{R/S}$ vanishes. By Nakayama's lemma, it then suffices to show that $\Omega^1_{R/S} \otimes_R k_R = 0$. However, let M (resp. N) be the R-submodule of $\Omega := \Omega^1_{R/\mathbb{Z}}$ generated by $\{da \mid a \in \Sigma\}$ (resp. by $\{dg_1, \dots, dg_n\}$); then $\Omega^1_{R/S} = \Omega/(M + N)$ ([63, Ch.0, Th.20.5.7(i)]) and on the other hand, the induced map $(\Omega/M) \otimes_R R_0 \to \Omega^1_{R_0/\mathbb{Z}}$ is an isomorphism ([63, Ch.0, Th.20.5.12(i)]). We deduce a right exact sequence of R-modules

$$N \xrightarrow{j} \Omega^1_{R_0/\mathbb{Z}} \otimes_R k_R \to \Omega^1_{R/S} \otimes_R k_R \to 0.$$

But our choice of the sequence $(g_1, \dots, g_n)$ implies that j is surjective, whence the contention. ◊

Finally, notice that the image of $\Psi_{B_{n+1}}$ is the subring $(B_n/pB_n)^p[\beta_n(P^{(n)}), f_1^{1/p^n}, \dots, f_r^{1/p^n}]$. For every $n > 0$, consider the exact sequence

$$\Omega^1_{A/\mathbb{Z}} \otimes_A \kappa(x_n) \xrightarrow{j} \Omega^1_{A_n/\mathbb{Z}} \otimes_{A_n} \kappa(x_n) \to \Omega^1_{A_n/A} \otimes_{A_n} \kappa(x_n) \to 0$$

([63, Ch.0, Th.20.5.7(i)]); the image of j is generated by the image of the generating system $df_\bullet := (df_1, \dots, df_r)$ of the $\kappa(x_0)$-vector space $\Omega^1_{A/\mathbb{Z}} \otimes_A \kappa(x_0)$. However, since f_i admits a p-th root in A_n for every $i = 1, \dots, r$, it is easily seen that the image of $df_\bullet$ vanishes in $\Omega^1_{A_n/\mathbb{Z}}$. Hence $\Omega^1_{A_n/\mathbb{Z}} \otimes_{A_n} \kappa(x_n) = \Omega^1_{A_n/A} \otimes_{A_n} \kappa(x_n)$ is generated by the image of the system $(df_1^{1/p^n}, \dots, df_r^{1/p^n})$. Therefore, to prove the lemma, it suffices to apply claim 17.2.13 to $R := B_n/pB_n$, $\Sigma := \beta_n(\mathfrak{m}_{P^{(n)}})$ and the sequence $(f_1^{1/p^n}, \dots, f_r^{1/p^n})$. □

Theorem 17.2.14. *With the notation of* (17.2.5)*, we have :*

(i) *If B_0 is an $\mathbb{F}_p$-algebra, Ψ_{B_n} is an isomorphism for every $n \in \mathbb{N}$.*

(ii) *If B_0 is not an $\mathbb{F}_p$-algebra, then there exist $\pi \in B_1$ and $u \in B_1^\times$ such that $\pi^p = pu$.*
(iii) *For every $\pi \in B_1$ as in* (ii) *and every integer $n > 0$, we have* $\mathrm{Ker}\,\Psi_{B_n} = \pi B_n/pB_n$.
(iv) *Let $\mathscr{T}_{p,B}$ be the p-adic topology of B. Then $(B, \mathscr{T}_{p,B})$ is a formally perfectoid ring.*

Proof. (i) is immediate from lemma 17.2.11, since in that case Φ_{B_n} is the Frobenius endomorphism of B_n, so it is injective, since the latter is a domain.

(iii): Suppose we have found π as in (ii), and let $x \in B_n$ whose image in B_n/pB_n lies in $\mathrm{Ker}\,\Psi_{B_n}$; this means that $x^p = py$ holds in B_n for some $y \in B_n$; hence $(x/\pi)^p \in B_n$, and since B_n is a normal ring (corollary 12.5.13), we deduce that $x/\pi \in B_n$, *i.e.* $x \in \pi B_n$.

(ii): Assume first that $p \in \mathfrak{m}_A^2$. Then we may write

$$p = \sum_{i=1}^{t} b_i b_i' + \sum_{j=1}^{s} b_j'' \cdot \beta_0(x_j)$$

for certain $b_1, b_1', \dots, b_t, b_t' \in \mathfrak{m}_{B_0}$, $b_1'', \dots, b_s'' \in B_0$, and $x_1, \dots, x_s \in \mathfrak{m}_P$. In light of lemma 17.2.11, we may then write

$$b_i = c_i^p + pd_i \quad b_i' = c_i'^p + pd_i' \qquad \text{where } c_i, c_i', d_i, d_i' \in \mathfrak{m}_{B_1} \text{ for every } i = 1, \dots, t$$

and likewise, $b_j'' = c_j''^p + pd_j''$ where $c_j'' \in B_1$ for $j = 1, \dots, s$. Moreover, by construction, β_0 extends to a morphism of monoids $\beta_1 : P^{(1)} \to B_1$, and we may write $x_j = y_j^p$ with $y_j \in \mathfrak{m}_{P^{(1)}}$ for $j = 1, \dots, s$. A simple computation then yields

$$(17.2.15) \qquad p \cdot (1 + e) = \sum_{i=1}^{t} c_i^p c_i'^p + \sum_{j=1}^{s} c_j''^p \cdot \beta_1(y_j)^p \qquad \text{for some } e \in \mathfrak{m}_{B_1}.$$

However, the right-hand side of (17.2.15) can be written in the form $g^p + ph$ for some $g, h \in \mathfrak{m}_{B_1}$ (details left to the reader); clearly $1 + e - h \in B_1^\times$, whence the contention, in this case.

Next, suppose that $p \notin \mathfrak{m}_A^2$. In this case, recall that

$$\dim_{\kappa(x_0)^{1/p}} \boldsymbol{\Omega}_A = 1 + \dim_{\kappa(x_0)} \Omega^1_{A/\mathbb{Z}} \otimes_A \kappa(x_0).$$

Therefore, after reordering the sequence $(f_1, \dots, f_r)$, we may assume that $df_1, \dots, df_{r-1}$ is a basis of the $\kappa(x_0)$-vector space $\Omega^1_{A/\mathbb{Z}} \otimes_A \kappa(x_0)$. Set

$$B' := (P^{(1)} \otimes_P B_0)[T_1, \dots, T_{r-1}]/(T_1^p - f_1, \dots, T_{r-1}^p - f_{r-1}).$$

Clearly B_1 is a faithfully flat B'-algebra, hence B'/pB' is a B_0/pB_0-subalgebra of B_1/pB_1, so the natural map $B_0/pB_0 \to B'/pB'$ is injective (lemma 17.2.4), and just as in (17.2.5), we deduce that the Frobenius endomorphism of B'/pB' factors through a ring homomorphism $\Psi_{B'} : B'/pB' \to B_0/pB_0$, and arguing as in the foregoing, we also see that B' is a local ring. Moreover, by applying claim 17.2.13 with $R := B'/pB'$, $\Sigma := \beta_0(\mathfrak{m}_P)$ and the sequence $(f_1, \dots, f_{r-1})$ we conclude – as in the proof of lemma 17.2.11 – that $\Psi_{B'}$ is surjective. Hence, denote by $\mathfrak{m}_{B'}$ the maximal ideal of B'; it follows that there exist $g \in \mathfrak{m}_{B'}$ and $h \in B'$ such that $f_r = g^p + ph$ in B'. Set

$$A' := B'/\mathfrak{m}_{P^{(1)}} B' = A[T_1, \dots, T_{r-1}]/(T_1^p - f_1, \dots, T_{r-1}^p - f_{r-1}).$$

Then A' is a regular local ring, by corollary 9.6.34, applied to the sequence $(\overline{f}_1, \dots, \overline{f}_{r-1})$ consisting of the images of the elements f_i in A. By the same criterion – applied to the similar sequence $(\overline{f}_1, \dots, \overline{f}_r)$ – we see that $A'[T]/(T^p - f_r)$ is regular as well. So once again the same corollary – applied to the element $\overline{f}_r$ of A' – says that the element $\boldsymbol{d}(f_r)$ of $\boldsymbol{\Omega}_{A'}$ does not vanish. However

$$\boldsymbol{d}(f_r) = \boldsymbol{d}(g^p) + \boldsymbol{d}(ph) = pg^{p-1}\boldsymbol{d}(g) + h\boldsymbol{d}(p) + p\boldsymbol{d}(h) = h\boldsymbol{d}(p) \qquad \text{in } \boldsymbol{\Omega}_{A'}$$

(see (9.6.16)). We conclude that $h \in B'^{\times}$. Lastly, notice that f_r admits a p-th root $f_r^{1/p}$ in B_1, hence $ph = f_r - g^p = (f_r^{1/p} - g)^p + pe$ in B_1 for some $e \in \mathfrak{m}_{B_1}$ (details left to the reader). Since $h - e \in B_1^{\times}$, we are done.

(iv): If B_0 is an $\mathbb{F}_p$-algebra, then $\mathscr{T}_{p,B}$ is the discrete topology, and B is a perfect $\mathbb{F}_p$-algebra, whence the assertion in this case. Suppose then that B_0 is not an $\mathbb{F}_p$-algebra, in which case the topology of the completion $B^\wedge$ of $(B, \mathscr{T}_{p,B})$ is still p-adic (remark 8.3.3(iv)). Pick $\pi \in B_1$ as in (ii), set $I := \pi B^\wedge$, and notice that π is a regular element of $B^\wedge$, by proposition 8.2.13(i) : the details shall be left to the reader. In light of (iii), it is easily seen that $B^\wedge$ is a P-ring, I is a special ideal of definition, and the Frobenius endomorphism of $B^\wedge : pB^\wedge = B/pB$ induces an isomorphism $B^\wedge/I \xrightarrow{\sim} B^\wedge/I^{(p)}$. The assertion then follows from theorem 16.4.1. □

17.2.16. Let $\beta : P^{(\infty)} \to B$ be the chart of the log structure $\underline{M}$ on X from (17.2.5), and notice that the inclusion map $P^{(n)} \to P^{(\infty)}$ induces bijections $\operatorname{Spec} P^{(\infty)} \xrightarrow{\sim} \operatorname{Spec} P^{(n)}$ for every $n \in \mathbb{N}$ (lemma 6.4.59(i)); especially, $\operatorname{Spec} P^{(\infty)}$ is a finite set. Let $I \subset B$ be any non-zero ideal; we say that I is a *branch ideal*, if there exist radical ideals $J \subset B$ and $\mathfrak{r} \subset P^{(\infty)}$ such that

$$(17.2.17) \qquad p \in J \qquad \text{and} \qquad I = J \cap \mathfrak{r}B.$$

Remark 17.2.18. (i) Notice that, in case B_0 is an $\mathbb{F}_p$-algebra, a branch ideal is just any non-zero radical ideal of B.

(ii) Let $I \subset B$ be a branch ideal, and pick radical ideals $J \subset B$ and $\mathfrak{r} \subset P^{(\infty)}$ such that (17.2.17) holds. Set $\mathfrak{r}^{(n)} := \mathfrak{r} \cap P^{(n)}$ for every $n \in \mathbb{N}$; from corollary 12.5.20(ii) we know that $\mathfrak{r}^{(n)}B_n$ is a radical ideal of B_n, hence $\mathfrak{r}B$ is a radical ideal of B, and then the same holds for I.

(iii) In the situation of (ii), let $\mathfrak{Z} \subset \operatorname{Spec} P^{(\infty)}$ be a subset such that $\mathfrak{r} = \bigcap_{\mathfrak{p} \in \mathfrak{Z}} \mathfrak{p}$ (lemma 6.1.16), and set $\mathfrak{p}^{(n)} := \mathfrak{p} \cap P^{(n)}$ for every $\mathfrak{p} \in \mathfrak{Z}$ and every $n \in \mathbb{N}$, so $\mathfrak{r}^{(n)} = \bigcap_{\mathfrak{p} \in \mathfrak{Z}} \mathfrak{p}^{(n)}$ for every $n \in \mathbb{N}$. In view of proposition 12.5.16 and lemmata 12.5.1 and 6.1.37, we see that $\mathfrak{r}^{(n)}B = \bigcap_{\mathfrak{p} \in \mathfrak{Z}} \mathfrak{p}^{(n)}B_n$, and therefore

$$(17.2.19) \qquad \mathfrak{r}B = \bigcap_{\mathfrak{p} \in \mathfrak{Z}} \mathfrak{p}B.$$

(iv) Furthermore, any branch ideal $I \subset B$ is the radical of I_0B, for some ideal $I_0 \subset B_0$. Indeed, pick J and $\mathfrak{r}$ such that (17.2.17) holds; since the projection $\operatorname{Spec} B/pB \to (X_0)_{/p}$ is radicial and surjective, it is clear that J is the radical of J_0B for some ideal $J_0 \subset B_0$, and on the other hand, $\mathfrak{r}B$ is the radical of $\mathfrak{r}^{(0)}B$.

Proposition 17.2.20. *With the notation of* (17.2.16)*, let $I \subset B$ be any branch ideal. We have :*

(i) *$I^2 = I$, and I fulfills condition* $(\mathbf{B})$ *of* [75, §2.1.6].

(ii) *Let $I_1, I_2 \subset B$ be any two branch ideals. Then $I_1 \cap I_2$ and $I_1 + I_2$ are also branch ideals, and $I_1I_2 = I_1 \cap I_2$.*

Proof. (i): By assumption, $I = J \cap \mathfrak{r}B$ for radical ideals $J \subset B$ and $\mathfrak{r} \subset P^{(\infty)}$ with $p \in J$.

Claim 17.2.21. $(\mathfrak{r}B)^2 = \mathfrak{r}B$ and $J^2 = J$, and the ideals $\mathfrak{r}B$ and J fulfill condition $(\mathbf{B})$.

Proof of the claim. Since the p-Frobenius endomorphism of $P^{(\infty)}$ is an automorphism, the first stated identity is clear. Next, set $\overline{J} := J/pB$. By lemma 17.2.11, the Frobenius endomorphism of B/pB is surjective; since $\overline{J}$ is a radical ideal, we deduce that $\overline{J}^2 = \overline{J}$. On the other hand, from theorem 17.2.14(ii) we see that $p \in J^2$, from which also the second identity follows easily.

Next, it is clear that $\mathfrak{r}B$ fulfills condition $(\mathbf{B})$, and for J we apply [75, Claim 2.1.9] which reduces checking that J/pJ is generated by the p-th powers of its elements. However, the foregoing already shows that J/pB is generated by the p-th powers of its elements; combining with theorem 17.2.14(ii), the contention follows easily. ◊

Taking into account remark 17.2.18(i), we see that claim 17.2.21 already implies the assertion, in case B_0 is an $\mathbb{F}_p$-algebra. Thus, we may assume that B_0 is not an $\mathbb{F}_p$-algebra, in which case – again by claim 17.2.21 – it suffices to show that $J \cap \mathfrak{r}B = \mathfrak{r} \cdot J$, or equivalently, that :

$$\mathrm{Tor}_1^B(B/\mathfrak{r}B, B/J) = 0.$$

Now, the chart β yields a ring homomorphism $C := \mathbb{Z}[P^{(\infty)}] \to B$, and we have a change of rings spectral sequence for Tor-functors (see [163, Th.5.6.6]) :

$$E^2_{pq} := \mathrm{Tor}_p^B(\mathrm{Tor}_q^C(C/\mathfrak{r}C, B), B/J) \Rightarrow \mathrm{Tor}_{p+q}^C(C/\mathfrak{r}C, B/J).$$

Claim 17.2.22. $\mathrm{Tor}_q^C(C/\mathfrak{r}C, B) = 0$ for every $q > 0$.

Proof of the claim. Set $\mathfrak{r}^{(n)} := \mathfrak{r} \cap P^{(n)}$ and $C_n := \mathbb{Z}[P^{(n)}]$ for every $n \in \mathbb{N}$. We have natural isomorphisms :

$$\mathrm{Tor}_q^C(C/\mathfrak{r}C, B) \xrightarrow{\sim} \operatorname*{colim}_{n\in\mathbb{N}} \mathrm{Tor}_q^{C_n}(C_n/\mathfrak{r}_n C, B_n) \qquad \text{for every } q, n \in \mathbb{N}$$

so we are reduced to showing that $\mathrm{Tor}_q^{C_n}(C_n/\mathfrak{r}_n C, B_n) = 0$ for every $q, n \in \mathbb{N}$ with $q > 0$. The latter follows from propositions 12.5.16(ii) and 6.1.40(i). ◊

From claim 17.2.22 we see that $E^2_{pq} = 0$ for every $q > 0$ and every $p \in \mathbb{N}$. Thus, we deduce a natural isomorphism :

$$\mathrm{Tor}_p^B(B/\mathfrak{r}B, B/J) \xrightarrow{\sim} \mathrm{Tor}_p^C(C/\mathfrak{r}C, B/J) \qquad \text{for every } p \in \mathbb{N}.$$

By the same token, the projection $C \to \overline{C} := C/pC$ induces a change of rings spectral sequence

$$F^2_{ij} := \mathrm{Tor}_i^{\overline{C}}(\mathrm{Tor}_j^C(C/\mathfrak{r}C, \overline{C}), B/J) \Rightarrow \mathrm{Tor}_{i+j}^C(C/\mathfrak{r}C, B/J)$$

(notice that B/J is a $\overline{C}$-algebra, since $p \in J$) and $\mathrm{Tor}_j^C(C/\mathfrak{r}C, \overline{C}) = 0$ for every $j > 0$, since $C/\mathfrak{r}C$ has no p-torsion. Hence $F^2_{ij} = 0$ whenever $j > 0$, and we get an isomorphism :

$$\mathrm{Tor}_i^{\overline{C}}(\overline{C}/\mathfrak{r}\overline{C}, B/J) \xrightarrow{\sim} \mathrm{Tor}_i^C(C/\mathfrak{r}C, B/J) \qquad \text{for every } i \in \mathbb{N}.$$

Notice now that $\overline{C}$ is a perfect $\mathbb{F}_p$-algebra, since $P^{(\infty)}$ is p-divisbile, and with $I_{\overline{C}} := \mathfrak{r}\overline{C}$ we have $\overline{C}/\mathfrak{r}\overline{C} = \overline{C}/I_{\overline{C}}^{[0]}\overline{C}$. Then $\mathrm{Tor}_i^{\overline{C}}(\overline{C}/\mathfrak{r}\overline{C}, B/J) = 0$ for every $i > 0$, by proposition 16.4.7, whence the contention.

(ii): It is clear that $I_1 \cap I_2$ is a branch ideal; we have inclusions

$$(I_1 \cap I_2)^2 \subset I_1 I_2 \subset I_1 \cap I_2$$

and from (i) we know that the first ideal in this chain coincides with the third, so $I_1 I_2 = I_1 \cap I_2$.

Next, for $k = 1, 2$ let us write $I_k = J_k \cap \mathbf{r}_k B$, where $J_k \subset B$ and $\mathfrak{r}_k \subset P^{(\infty)}$ are radical ideals and $p \in J_k$. Set $Z_k := \operatorname{Spec} B/J_k$, $Z'_k := \operatorname{Spec} B/\mathfrak{r}_k B$ for $k = 1, 2$; clearly

$$Z := \operatorname{Spec} B/(I_1 + I_2) = (Z_1 \cup Z'_1) \cap (Z_2 \cup Z'_2).$$

Now let $J \subset B$ be the largest ideal such that $\operatorname{Spec} B/J = (Z_1 \cap Z_2) \cup (Z_1 \cap Z'_2) \cup (Z'_1 \cap Z_2)$. It follows easily that

$$Z = (\operatorname{Spec} B/J) \cup (\operatorname{Spec} B/(\mathfrak{r}_1 \cup \mathfrak{r}_2)B).$$

Now, J and $\mathfrak{r}_1 \cup \mathfrak{r}_2$ are radical ideals, and $p \in J$, so we deduce that the radical of $I_1 + I_2$ is a branch ideal, and it remains only to show that $I_1 + I_2$ is a radical ideal. However, notice the inclusions

$$(\mathfrak{r}_1 J_1 + \mathfrak{r}_2 J_2)^2 \subset I := (J_1 + J_2)(J_1 + \mathfrak{r}_2 B)(\mathfrak{r}_1 B + J_2)(\mathfrak{r}_1\mathfrak{r}_2 B) \subset (\mathfrak{r}_1 J_1 + \mathfrak{r}_2 J_2).$$

Especially, $\mathfrak{r}_1 J_1 + \mathfrak{r}_2 J_2$ is contained in the radical I' of I, Since we already know that $\mathfrak{r}_k J_k = I_k$ for $k = 1, 2$, it suffices to check that each of the four factors of the middle term in this chain of inclusions is a branch ideal : indeed, in this case the same holds for I, and therefore $I = I'$, so finally $I = I_1 + I_2$.

• First, set $J := J_1 + J_2$, and let $\overline{J} \subset B/pB$ be the image of J; since $p \in J$, in order to see that J is a branch ideal, it suffices to check that J is a radical ideal; then it suffices to prove that the same holds for $\overline{J}$, and the latter assertion will follow, if we show that $\Phi_B^{-1}\overline{J} = \overline{J}$, where $\Phi_B : B/pB \to B/pB$ is the Frobenius endomorphism. Now, say that $x \in B/pB$ and $x^p \in \overline{J}$; then $x^p = x_1 + x_2$ for some $x_i \in \overline{J}_i := J_i/pB$ ($i = 1, 2$), and since Φ_B is surjective and $\overline{J}_i$ is a radical ideal, we can write $x_i = y_i^p$ for some $y_i \in \overline{J}_i$. Consequently $x^p = (y_1 + y_2)^p$, so $x - y_1 - y_2$ is nilpotent; but clearly $\overline{J}$ contains the nilradical of B/pB, whence the claim.

• Next we consider $J' := J_1 + \mathfrak{r}_2 B$; as in the previous case, since $p \in J$, we see that J' is a branch ideal, provided J'/pB is a radical ideal of B/pB. However, say that $x \in B/pB$ and $x^p = y + z$, for some $y \in \overline{J}_1$ and $z \in \mathfrak{r}_2 B/pB$; arguing as in the foregoing case, we may write $y = u^p$ for some $u \in \overline{J}_1$. Likewise, since both Φ_B and the Frobenius endomorphism of $P^{(\infty)}$ are surjective, we see that $z = w^p$ for some $w \in \mathfrak{r}_2 B/pB$, so $x - u - w$ is nilpotent, and especially, it lies in $\overline{J}_1$, whence the claim. The same argument applies as well to the factor $\mathfrak{r}_1 B + J_2$.

• Lastly, since $\mathfrak{r}_1$ and $\mathfrak{r}_2$ are both radical ideals of $P^{(\infty)}$, the same holds for $\mathfrak{r}_1\mathfrak{r}_2$, so $\mathfrak{r}_1\mathfrak{r}_2 B$ is a branch ideal. □

Proposition 17.2.20(i) says that the pair (B, I) is a basic setup, in the sense of [75, (2.1.1)], so we may consider the associated categories of almost modules and almost algebras.

17.2.23. Henceforth we shall restrict to the case where B_0 is not an $\mathbb{F}_p$-algebra, so B_0 is a local integral domain whose field of fractions has characteristic zero, and whose residue field has characteristic p. For every $n \in \mathbb{N}$, set

$$(X_n)_{/p} := \mathrm{Spec}\, B_n/pB_n \qquad V_n := X_n \setminus (X_n)_{/p} \qquad V_{n,\mathrm{tr}} := (X_n, \underline{M})_{\mathrm{tr}} \cap V_n.$$

We consider now a finite morphism $\varphi_0 : Y_0 \to X_0$ with Y_0 a normal scheme, such that φ_0 maps each connected component of Y_0 onto X_0, and such that the restriction $\varphi_0^{-1}V_{0,\mathrm{tr}} \to V_{0,\mathrm{tr}}$ is a finite étale covering. For every $n \in \mathbb{N}$, we let Y_n be the normalization of X_n in $\varphi^{-1}V_{0,\mathrm{tr}} \times_{X_0} X_n$, and denote by $\varphi_n : Y_n \to X_n$ the resulting finite morphism ([126, Lemma 1, p.262]), by

$$\varphi : Y \to X$$

the limit of the system $(\varphi_n \mid n \in \mathbb{N})$, and by $W_n \subset X_n$ the *étale locus of* φ_n, *i.e.* the largest open subset such that the restriction $\varphi_n^{-1}W_n \to W_n$ of φ_n is an étale covering (lemma 13.1.7(iii) and claim 13.1.8). Let also $(X_{n,\mathfrak{p}} \mid \mathfrak{p} \in \mathrm{Spec}\, P^{(n)})$ be the logarithmic stratification of $(X_n, \underline{M}_n)$ (see (12.5.35)) and set $V_{n,\mathfrak{p}} := X_{n,\mathfrak{p}} \cap V_n$ for every $n \in \mathbb{N}$ and every $\mathfrak{p} \in \mathrm{Spec}\, P^{(n)}$. We call $(V_{n,\mathfrak{p}} \mid \mathfrak{p} \in \mathrm{Spec}\, P^{(n)})$ the *logarithmic stratification* of V_n.

Lemma 17.2.24. *With the notation of* (17.2.23)*, the following holds :*

(i) $V_n \setminus W_n$ *is a union of strata of the logarithmic stratification of* V_n*, for every* $n \in \mathbb{N}$.

(ii) $X_n \times_{X_m} W_m = W_n$ *for every sufficiently large* $m \in \mathbb{N}$ *and every* $n \geq m$.

(iii) *For* m *as in* (ii)*, set* $W := X \times_{X_m} W_m$*, and let* $I \subset B$ *be the largest ideal such that* $\mathrm{Spec}\, B/I = X \setminus W$*. Then* I *is a branch ideal of* B.

Proof. (i): Since every $X_{n,\mathfrak{p}}$ is irreducible (corollary 12.5.37(iii)), the same holds for each $V_{n,\mathfrak{p}}$. Hence, suppose that $V_n \setminus W_n$ contains a point z of $V_{n,\mathfrak{p}}$, for some $\mathfrak{p} \in \mathrm{Spec}\, P^{(n)}$; we have to show that in this case $V_{n,\mathfrak{p}} \cap W_n = \varnothing$, and since W_n is an open subset of X_n, it suffices to check that the generic point η of $V_{n,\mathfrak{p}}$ does not lie in W_n. However, let $\overline{\eta}$ be any geometric point localized at η; then η lies in W_n if and only if the induced morphism $\varphi_n \times_{X_n} X_n(\overline{\eta})$ is étale (claim 13.1.9). Let also $\overline{z}$ be a geometric point localized at z, and denote by $\underline{M}_{n,\overline{z}}$ (resp. $\underline{M}_{n,\overline{\eta}}$) the stalk at $\overline{z}$ (resp. $\overline{\eta}$) of the logarithmic structure of $(X_n(\overline{z}), \underline{M}_n(\overline{z}))$ (resp. of $(X_n(\overline{\eta}), \underline{M}_n(\overline{\eta}))$). Any choice of a strict specialization morphism $s : X_n(\overline{\eta}) \to X_n(\overline{z})$ induces a strict specialization map $\overline{\sigma} : \underline{M}_{n,\overline{z}} \to \underline{M}_{n,\overline{\eta}}$ (see (4.9.23)) that extends the specialization map $\sigma : \underline{M}_{n,z} \to \underline{M}_{n,\eta}$. Since the natural maps $\underline{M}^\sharp_{n,z} \to \underline{M}^\sharp_{n,\overline{z}}$ and $\underline{M}^\sharp_{n,\eta} \to \underline{M}^\sharp_{n,\overline{\eta}}$ are isomorphisms (see (12.1.9)), and $\sigma^{\sharp\mathrm{gp}}$ is an isomorphism (since z and η lie in the same stratum), the same holds for $\overline{\sigma}^{\sharp\mathrm{gp}}$, and therefore the induced map $\underline{M}^{\mathrm{gp}\vee}_{n,\overline{\eta}} \to \underline{M}^{\mathrm{gp}\vee}_{n,\overline{z}}$ is bijective. Pick any geometric point ξ of $X_n(\overline{z})$ localized at the maximal point, and lift ξ to a geometric point ξ_η of $X_n(\overline{\eta})$; in light of (13.3.54), we conclude that s induces an isomorphism

$$\pi_1((X_n(\overline{\eta}), \underline{M}_n(\overline{\eta}))_{\mathrm{ét}}, \xi_\eta) \xrightarrow{\sim} \pi_1((X_n(\overline{z}), \underline{M}_n(\overline{z}))_{\mathrm{ét}}, \xi).$$

Therefore, $\varphi_n \times_{X_n} X_n(\overline{\eta})$ is étale if and only if it is a trivial covering, if and only if the same holds for $\varphi_n \times_{X_n} X_n(\overline{z})$, if and only if $z \in W_n$, whence the assertion.

(ii): For every $n \in \mathbb{N}$, set

$$\mathfrak{Z}_n := \{\mathfrak{p} \in \mathrm{Spec}\, P^{(n)} \mid V_{n,\mathfrak{p}} \neq \varnothing \quad \text{and} \quad V_{n,\mathfrak{p}} \cap W_n = \varnothing\}. \tag{17.2.25}$$

The continuous map $\omega_n : \mathrm{Spec}\, P^{(n+1)} \to \mathrm{Spec}\, P^{(n)}$ induced by the inclusion $P^{(n)} \to P^{(n+1)}$ sends $\mathfrak{Z}_{n+1}$ into $\mathfrak{Z}_n$, for every $n \in \mathbb{N}$. On the other hand, ω_n is also a bijection of finite sets (lemmata 6.1.20(iii) and 6.4.59(i)); we conclude that ω_n restricts to a bijection $\mathfrak{Z}_{n+1} \xrightarrow{\sim} \mathfrak{Z}_n$ for every sufficiently large integer $n \in \mathbb{N}$. For every $n, m \in \mathbb{N}$ with $n \geq m$, let $g_{n,m} : X_n \to X_m$ be the transition morphism in the maximal tower (17.2.6), in light of (i), it follows that

$$g^{-1}_{n,m}(V_m \cap W_m) = V_n \cap W_n \qquad \text{for every sufficiently large } m \in \mathbb{N} \text{ and every } n \geq m.$$

On the other hand, the restriction $(X_n)_{/p} \to (X_m)_{/p}$ of $g_{n,m}$ is radicial and surjective for every such $n, m \in \mathbb{N}$, so the underlying continuous map is a homeomorphism; since the underlying topological spaces are noetherian, we see as well that

$$g^{-1}_{n,m}((X_m)_{/p} \cap W_m) = (X_n)_{/p} \cap W_n \qquad \text{for every sufficiently large } m \in \mathbb{N} \text{ and every } n \geq m.$$

Summing up, the assertion follows.

(iii): The assertion follows easily from (i) and (17.2.19). □

17.2.26. Let $I \subset B$ be a fixed branch ideal, and consider a pair (X, Z) with $Z \subset \mathrm{Spec}\, B/I$ and such that Z is constructible in X. Clearly such a pair is normal (see definition 14.4.1(ii)), and we aim to show that (X, Z) is almost pure for the almost structure given by the basic setup (B, I) supplied by proposition 17.2.20. This will be achieved in several steps.

To begin with, set $U := X \setminus Z$, and let $\mathscr{A}$ be any étale almost finitely presented $\mathscr{O}^a_U$-algebra. Set also $U_I := X \setminus \mathrm{Spec}\, B/I$; then $\mathscr{A}_{|U_I}$ is a finite étale $\mathscr{O}_{U_I}$-algebra, $U_I = X \times_{X_0} U_{I,0}$ for some open subset $U_{I,0} \subset X_0$ (remark 17.2.18(iv)), and a simple inspection shows that

$$V_{0,\mathrm{tr}} \subset U_{I,0}.$$

Let $\psi : U_I \to U_{I,0}$ be the natural projection; by [66, Ch.IV, Prop.17.7.8(ii)] and [65, Ch.IV, Th.8.8.2(ii)], and by virtue of remark 17.2.7, we may assume – after replacing $(X_0, \underline{M}_0)$ by $(X_n, \underline{M}_n)$ for some sufficiently large $n \in \mathbb{N}$ – that there exists a coherent étale $\mathscr{O}_{U_{I,0}}$-algebra $\mathscr{A}_0$ with an isomorphism $\psi^*\mathscr{A}_0 \xrightarrow{\sim} \mathscr{A}_{|U_I}$ of $\mathscr{O}_{U_I}$-algebras. Denote by Y_0 the normalization of X_0 in $\mathrm{Spec}\, \mathscr{A}_0(U_{I,0})$; the resulting morphism $\varphi_0 : Y_0 \to X_0$ is finite (lemma 9.7.4(i)) and induces an isomorphism $(\varphi_{0*}\mathscr{O}_{Y_0})_{|U_{I,0}} \xrightarrow{\sim} \mathscr{A}_0$. Especially, φ_0 is a morphism of the type contemplated in (17.2.23), and the resulting morphism $\varphi : Y \to X$ induces an isomorphism $(\varphi_*\mathscr{O}_Y)_{|U_I} \xrightarrow{\sim} \mathscr{A}_{|U_I}$. However, we have as well $\mathscr{A} = (\mathscr{A}_{|U_I})^\nu$ (lemma 14.4.5(i)), and since Y is normal, there follows an isomorphism of $\mathscr{O}^a_U$-algebras $(\varphi_*\mathscr{O}^a_Y)_{|U} \xrightarrow{\sim} \mathscr{A}$. Then proposition 14.4.8 and [75, Lemma 8.2.28] show that (X, Z) is almost pure if and only if the resulting $\mathscr{O}^a_X$-algebra $\varphi_*\mathscr{O}^a_Y$ is weakly unramified, for every such φ_0.

17.2.27. Keep the notation of (17.2.26), and *suppose now that B_0 is strictly henselian*; especially, the N-torsion subgroup μ_N of $B_0^\times$ has cardinality equal to N, for every $N > 0$ such that $(N, p) = 1$. Let $k > 0$ be any integer and write $k = p^s \cdot q$, with $s, q \in \mathbb{N}$ and $(q, p) = 1$; set $Q := P$, let $\nu : P \to Q$ be the k-Frobenius map, define

$$C'_0 := Q \otimes_P B_0 \qquad C_0 := C'_0 \otimes_{B_0} B''_s$$

and endow $X'_0 := \operatorname{Spec} C_0$ with the log structure $\underline{M}'_0$ deduced from the natural map $Q \to C_0$. Notice that C_0 is strictly henselian and the Frobenius endomorphism of C_0/pC_0 is still a finite map (claim 9.7.21(i)); also, since the induced map $A \to C'_0/\mathfrak{m}_Q C'_0$ is an isomorphism, the image of the sequence $(f_1, \dots, f_r)$ in C_0 is still maximal, in the sense of (17.2.3), and therefore $C_0/\mathfrak{m}_Q C_0$ is still a regular local ring, by corollary 9.6.34. Arguing as in (17.2.5), we deduce that $(X'_0, \underline{M}'_0)$ is a regular log scheme, and we may consider the maximal tower

$$((X'_n, \underline{M}'_n) \mid n \in \mathbb{N})$$

associated with the chart $Q \to C_0$ and the maximal sequence $(f_1^{1/p^s}, \dots, f_r^{1/p^s})$ (see remark 9.6.35(iii)). So, $X'_n = \operatorname{Spec} C_n$ for a finite C_0-algebra C_n, and $\underline{M}'_n$ is given by a chart $Q^{(n)} \to C_n$, where $Q^{(n)}$ is a submonoid of $Q_\mathbb{Q}$ containing Q, for every $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, we set $V'_{n,\mathrm{tr}} := V_{0,\mathrm{tr}} \times_{X_0} X'_n$, we let Y'_n be the normalization of X'_n in $Y_0 \times_{X_0} V'_{n,\mathrm{tr}}$, we denote by

$$\varphi'_n : Y'_n \to X'_n \qquad \text{and} \qquad h_n : (X'_n, \underline{M}'_n) \to (X_n, \underline{M}_n)$$

the resulting finite morphisms, and by $\varphi' : Y' \to X'$ (resp. $h : (X', \underline{M}') \to (X, \underline{M})$) the limit of the system of morphisms $(\varphi'_n \mid n \in \mathbb{N})$ (resp. the limit of the system $(h_n \mid n \in \mathbb{N})$). Recall also that the induced morphism $V'_{0,\mathrm{tr}} \to V_{0,\mathrm{tr}}$ is a torsor for the finite abelian group

$$G := \operatorname{Hom}_\mathbb{Z}(Q^{\mathrm{gp}}, \mu_k)$$

(see (13.3.32)). Set

$$C := \bigcup_{n \in \mathbb{N}} C_n \qquad Q^{(\infty)} := \bigcup_{n \in \mathbb{N}} Q^{(n)}.$$

With this notation, we have

$$Q^{(\infty)} = Q \otimes_P P^{(\infty)} \qquad \text{and} \qquad C = Q^{(\infty)} \otimes_P B''.$$

Let also write $Y = \operatorname{Spec} D$ (resp. $Y' = \operatorname{Spec} D'$) for a B-algebra D (resp. for a C-algebra D'). The morphism $h : (X', \underline{M}') \to (X, \underline{M})$ shall be called the *standard covering* of $(X, \underline{M})$ of degree k.

Lemma 17.2.28. *In the situation of* (17.2.27)*, set $U' := X' \setminus h^{-1}Z$. The induced diagram*

$$\begin{array}{ccc} \mathscr{O}_X^a\text{-}\mathbf{\acute{E}t}_{\mathrm{fr}} & \xrightarrow{\rho} & \mathscr{O}_U^a\text{-}\mathbf{\acute{E}t}_{\mathrm{fr}} \\ \downarrow & & \downarrow \\ \mathscr{O}_{X'}^a\text{-}\mathbf{\acute{E}t}_{\mathrm{fr}} & \xrightarrow{\rho'} & \mathscr{O}_{U'}^a\text{-}\mathbf{\acute{E}t}_{\mathrm{fr}} \end{array}$$

is 2-cartesian (notation of definition 14.4.2*).*

Proof. A simple inspection shows that h_0 factors through a morphism $(X'_0, \underline{M}'_0) \to (X_s, \underline{M}_s)$, and indeed we have natural isomorphisms of $(X', \underline{M}')$-schemes

$$(X'_n, \underline{M}'_n) \xrightarrow{\sim} (X'_0, \underline{M}'_0) \times_{(X_s, \underline{M}_s)} (X_{n+s}, \underline{M}_{n+s}) \qquad \text{for every } n \in \mathbb{N}$$

that identify the transition morphisms $(X'_{n+1}, \underline{M}'_{n+1}) \to (X'_n, \underline{M}'_n)$ with the base change of the corresponding morphisms for the tower $((X_{n+s}, \underline{M}_{n+s}) \mid n \in \mathbb{N})$. In view of remark 17.2.7, we may then replace $(X_0, \underline{M}_0)$ by $(X_s, \underline{M}_s)$, and P by $P^{(s)}$, and assume that $(k, p) = 1$, so also $(o(G), p) = 1$.

Suppose now that $(\varphi'_*\mathscr{O}^a_{Y'})_{|U'}$ is in the essential image of the restriction functor ρ'; by proposition 14.4.8, this means that D'^a is an étale almost finitely presented $(C, IC)^a$-algebra. Then, taking into account the discussion of (17.2.26), and recalling that ρ and ρ' are fully faithful (lemma 14.4.5(ii)), we are reduced to checking that D^a is an étale almost finitely presented $(B, I)^a$-algebra.

However, the action of G on $V'_{0,\mathrm{tr}}$ is inherited by C and D', and clearly $C^G = B$ and $D'^G = D$; by corollary 14.6.28(ii), it suffices therefore to show that the action of G on D'^a is horizontal. Denote

$$G \to \mathrm{Aut}(C) \quad : \quad \chi \mapsto \rho_\chi$$

the action of G, let $\chi \in G$ be any element, and $J_\chi \subset C$ the ideal generated by all elements of the form $\rho_\chi(c) - c$, for c ranging over all elements of C. By the discussion in (13.3.32) we get

$$\rho_\chi((q \otimes y) \otimes b) = \chi(q) \cdot (q \otimes y) \otimes b \qquad \text{for every } q \in Q, y \in P^{(\infty)} \text{ and } b \in B''.$$

Hence, J_χ is the ideal generated by all elements of the form $(1 - \chi(q)) \cdot q \otimes y \otimes 1$, for all $q \otimes y \in Q^{(\infty)}$. However, since $\chi(q) \in \mu_k$ and $(k, p) = 1$, it is easily seen that $1 - \chi(q)$ either vanishes, or else it is invertible in B_0. Thus, denote by $\mathfrak{q}_\chi \subset Q^{(\infty)}$ the ideal generated by all elements of the form $q \otimes y$, with $\chi(q) \neq 1$; it follows that

$$J_\chi = \mathfrak{q}_\chi \cdot C.$$

Claim 17.2.29. $\mathfrak{q}_\chi$ is a radical ideal.

Proof of the claim. Indeed, say that $(q \otimes y)^n \in \mathfrak{q}_\chi$, so there exist elements $q_1 \otimes y_1$ and x_1 of $Q^{(\infty)}$ such that $(q \otimes y)^n = (q_1 \otimes y_1) \cdot x_1$ and $\chi(q_1) \neq 1$. We may assume that $n = p^t$ for some integer $t \in \mathbb{N}$, and since $Q^{(\infty)}$ is uniquely p-divisible, we may write $q_1 \otimes y_1 = (q_2 \otimes y_2)^n$ and $x_1 = x_2^n$ for some elements $q_2 \otimes y_2$ and x_2 of $Q^{(\infty)}$, and then $q \otimes y = (q_2 \otimes y_2) \cdot x_2$; however, clearly $\chi(q_2) \neq 1$, whence the claim. ◊

In view of claim 17.2.29, we may write $\mathfrak{q}_\chi = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s$, for certain $\mathfrak{q}_1, \dots, \mathfrak{q}_s \in \mathrm{Spec}\, Q^{(\infty)}$ (lemma 6.1.16). Set $\mathfrak{Q} := \{\mathfrak{q}_1, \dots, \mathfrak{q}_s\}$; we also know that $(\mathfrak{q} \cap Q^{(n)}) \cdot C_n$ is a prime ideal of C_n, for every $\mathfrak{q} \in \mathfrak{Q}$ and every $n \in \mathbb{N}$ (corollary 12.5.20(i)), therefore $\mathfrak{q}C$ is a prime ideal of C, for every $\mathfrak{q} \in \mathfrak{Q}$. Especially, the induced map

$$C/J_\chi \to \prod_{\mathfrak{q} \in \mathfrak{Q}} C/\mathfrak{q}C \qquad (17.2.30)$$

is injective. Since, by assumption, D'^a is a flat C^a-algebra, it follows that the map of almost modules $D'^a \otimes_C$ (17.2.30) is a monomorphism. Summing up, we are reduced to checking that χ acts trivially on $(D'/\mathfrak{q}D')^a$, for every $\mathfrak{q} \in \mathfrak{Q}$. However, set

$$X'_\mathfrak{q} := \mathrm{Spec}\, C/\mathfrak{q}C \quad \text{and} \quad U'_\mathfrak{q} := X'_\mathfrak{q} \times_X U_I \qquad \text{for every } \mathfrak{q} \in \mathfrak{Q}.$$

Suppose first that $U'_\mathfrak{q} = \varnothing$; in that case, set $\mathfrak{p} := \mathfrak{q} \cap P^{(\infty)}$, and notice that $\mathfrak{q}$ is the radical of the ideal $\mathfrak{p} \cdot Q^{(\infty)}$, so $\mathfrak{q}C$ is the radical of $\mathfrak{p}C$, and therefore $\mathrm{Spec}\,(C/\mathfrak{p}C) \times_X U_I = \varnothing$. However, the induced morphism $\mathrm{Spec}\, C/\mathfrak{p}C \to \mathrm{Spec}\, B/\mathfrak{p}B$ is surjective, so $\mathrm{Spec}\,(B/\mathfrak{p}B) \cap U_I = \varnothing$; since $\mathfrak{p}B$ is a prime ideal of B, the latter means that $I \subset \mathfrak{p}B$, whence $(B/\mathfrak{p}B)^a = 0$, so $(D'/\mathfrak{q}D')^a$ vanishes as well, and the assertion is trivial. If $U'_\mathfrak{q} \neq \varnothing$, set $Y'_\mathfrak{q} := Y' \times_{X'} X'_\mathfrak{q}$, let $\varphi'_\mathfrak{q} : Y'_\mathfrak{q} \to X'_\mathfrak{q}$ be the induced morphism, and define

$$\mathscr{D} := \varphi_*\mathscr{O}_Y \qquad \mathscr{D}' := \varphi'_*\mathscr{O}_{Y'} \qquad \mathscr{D}'_\mathfrak{q} := \varphi'_{\mathfrak{q}*}\mathscr{O}_{Y'_\mathfrak{q}}.$$

On the one hand, by assumption $\mathscr{D}'^a_\mathfrak{q}$ is a flat $\mathscr{O}^a_{X'_\mathfrak{q}}$-algebra; on the other hand, $X'_\mathfrak{q}$ is reduced and irreducible, hence the restriction map $\Gamma(X'_\mathfrak{q}, \mathscr{O}_{X'_\mathfrak{q}}) \to \Gamma(U'_\mathfrak{q}, \mathscr{O}_{X'_\mathfrak{q}})$ is injective. Consequently, the restriction map

$$(D'/\mathfrak{q}D')^a = \Gamma(X'_\mathfrak{q}, \mathscr{D}'^a_\mathfrak{q}) \to \Gamma(U'_\mathfrak{q}, \mathscr{D}'^a_\mathfrak{q})$$

is a monomorphism; so, we are reduced to checking that χ acts trivially on $\Gamma(U'_{\mathfrak q}, \mathscr{D}'^a_{\mathfrak q})$. To this aim, we remark that the open subset $U'_I := X' \times_X U_I$ is stable under the action of G, and the restriction $\mathscr{D}'_{|U'_I}$ of $\mathscr{D}'$ to the open subset U'_I is isomorphic to $(h^*\mathscr{D})_{|U'_I}$ ([66, Ch.IV, Prop.17.5.8(iii)]), so the action of G on $\mathscr{D}'_{|U'_I}$ is horizontal, and the assertion follows easily. □

17.2.31. In the situation of (17.2.27), define $\mathfrak{Z}_n$ as in (17.2.25), for every $n \in \mathbb{N}$. As already remarked, in view of lemma 17.2.24(i,ii) we may assume that the map $\operatorname{Spec} P^{(n+1)} \to \operatorname{Spec} P^{(n)}$ sends $\mathfrak{Z}_{n+1}$ bijectively onto $\mathfrak{Z}_n$, for every $n \in \mathbb{N}$. Set $V'_0 := V_0 \times_{X_0} X'_0$, and let

$$(V'_{0,\mathfrak p} \mid \mathfrak p \in \operatorname{Spec} Q) \qquad \text{and} \qquad (V_{0,\mathfrak p} \mid \mathfrak p \in \operatorname{Spec} P)$$

be the logarithmic stratifications of V'_0 and respectively V_0, defined as in (17.2.23); notice that the map $\nu^* : \operatorname{Spec} Q \to \operatorname{Spec} P$ induced by ν is bijective (lemma 6.4.59(i)), and clearly

$$V'_{0,\mathfrak p} = h_0^{-1} V_{0,\nu^*(\mathfrak p)} \qquad \text{for every } \mathfrak p \in \operatorname{Spec} Q.$$

Let $\mathfrak{Z}'_0 := \nu^{*-1}\mathfrak{Z}_0$, and for every $\mathfrak p \in \mathfrak{Z}'_0$, let $\eta'_{\mathfrak p}$ (resp. $\eta_{\mathfrak p}$) denote the generic point of $V'_{0,\mathfrak p}$ (resp. of $V_{0,\nu^*(\mathfrak p)}$), pick geometric points $\overline{\eta}'_{\mathfrak p}$ and $\xi'_{\mathfrak p}$ localized respectively at $\eta'_{\mathfrak p}$ and at a point of $V'_{0,\mathrm{tr}}(\overline{\eta}'_{\mathfrak p}) := V'_{0,\mathrm{tr}} \times_{X'_0} X'_0(\overline{\eta}'_{\mathfrak p})$. Denote by $\overline{\eta}_{\mathfrak p}$ the image of $\overline{\eta}'_{\mathfrak p}$ in $V_{0,\nu^*(\mathfrak p)}$, and by $\xi_{\mathfrak p}$ the image of $\xi'_{\mathfrak p}$ in $V_{0,\mathrm{tr}}(\overline{\eta}_{\mathfrak p}) := V_{0,\mathrm{tr}} \times_{X_0} X_0(\overline{\eta}_{\mathfrak p})$. According to (13.3.31) there follows, for every integer $N > 0$, a commutative diagram of groups

$$\begin{array}{ccc} \pi_1(V'_{0,\mathrm{tr}}(\overline{\eta}'_{\mathfrak p})_{\mathrm{ét}}, \xi'_{\mathfrak p}) & \longrightarrow & \pi_1(V_{0,\mathrm{tr}}(\overline{\eta}_{\mathfrak p})_{\mathrm{ét}}, \xi_{\mathfrak p}) \\ \downarrow & & \downarrow \\ \underline{M}'^{\mathrm{gp}\vee}_{0,\eta'_{\mathfrak p}} \otimes_{\mathbb Z} \mu_N & \longrightarrow & \underline{M}^{\mathrm{gp}\vee}_{0,\eta_{\mathfrak p}} \otimes_{\mathbb Z} \mu_N \end{array}$$

(where μ_N is the N-torsion subgroup of $\kappa(\xi_{\mathfrak p})^\times$) whose top arrow is induced by the natural morphism $V'_{0,\mathrm{tr}}(\overline{\eta}'_{\mathfrak p}) \to V_{0,\mathrm{tr}}(\overline{\eta}_{\mathfrak p})$, and whose bottom arrow is induced by $\nu^{\mathrm{gp}\vee} : Q^{\mathrm{gp}\vee} \to P^{\mathrm{gp}\vee}$, *i.e.* by the k-Frobenius map of $P^{\mathrm{gp}\vee}$, for every $\mathfrak p \in \mathfrak{Z}'_0$. Now, the restriction

$$\varphi_{\mathfrak p} : Y_0(\overline{\eta}_{\mathfrak p}) := Y_0 \times_{X_0} X_0(\overline{\eta}_{\mathfrak p}) \to X_0(\overline{\eta}_{\mathfrak p})$$

of φ is a tamely ramified covering, hence the action of $\pi_1(V_{0,\mathrm{tr}}(\overline{\eta}_{\mathfrak p})_{\mathrm{ét}}, \xi_{\mathfrak p})$ on $F_{\mathfrak p} := \varphi_{\mathfrak p}^{-1}(\xi_{\mathfrak p})$ factors through a group homomorphism

$$\rho_{\mathfrak p} : \underline{M}^{\mathrm{gp}\vee}_{0,\eta_{\mathfrak p}} \otimes_{\mathbb Z} \mu_N \to \operatorname{Aut}(F_{\mathfrak p})$$

for some sufficiently large $N \in \mathbb{N}$ (theorem 13.3.45). We may then find $k \in \mathbb{N}$ such that the image of $k \cdot P^{\mathrm{gp}\vee}$ in $\underline{M}^{\mathrm{gp}\vee}_{0,\eta_{\mathfrak p}}$ lies in the kernel of $\rho_{\mathfrak p}$, for every $\mathfrak p \in \mathfrak{Z}'_0$. Especially, for this choice of k, the image of $\underline{M}'^{\mathrm{gp}\vee}_{0,\eta'_{\mathfrak p}} \otimes_{\mathbb Z} \mu_N$ acts trivially on $\operatorname{Aut}(F_{\mathfrak p})$ via $\rho_{\mathfrak p}$, for every such $\mathfrak p$. Consequently, $\pi_1(V'_{0,\mathrm{tr}}(\overline{\eta}'_{\mathfrak p})_{\mathrm{ét}}, \xi'_{\mathfrak p})$ acts trivially on the fibres $\varphi_0'^{-1}(\xi'_{\mathfrak p})$ (by virtue of (3.6.22)); after applying lemma 17.2.24(i) to the morphism φ'_0, we conclude that the étale locus of φ'_0 contains the whole of V'_0.

Theorem 17.2.32. *In the situation of* (17.2.26)*, the pair* (X, Z) *is almost pure, relative to the basic setup* (B, I)*.*

Proof. Fix a geometric point $\overline{x}$ of X localized at the closed point x of X, and let B^{sh} denote the strict henselization of B at x. In view of proposition 14.4.10, in order to prove that (X, Z) is almost pure, it suffices to show that the pair $(X(\overline{x}), Z(\overline{x}))$ is almost pure relative to the basic setup (B, I), or – equivalently – relative to the basic setup $(B^{\mathrm{sh}}, IB^{\mathrm{sh}})$. However, it is easily seen that IB^{sh} is also a branch ideal of B^{sh}, if the latter is endowed with the chart $P^{(\infty)} \to B^{\mathrm{sh}}$ deduced from the given chart of $\underline{M}$. Taking into account remark 17.2.9(i), we may then replace B by B^{sh}, and assume that X_0 and X are strictly local.

Now, consider a finite morphism $\varphi_0 : Y_0 \to X_0$ and the resulting morphism $\varphi : Y \to X$ as in (17.2.23). The discussion of (17.2.26) shows that (X, Z) is almost pure if and only if $\varphi_* \mathscr{O}^a_Y$ is an étale $\mathscr{O}^a_X$-algebra, for every such φ_0. Set $V := \mathrm{Spec}\, B[1/p]$; in view of (17.2.31) and lemma 17.2.28, we may then assume that $V_0 \subset U_0$, and therefore $(\varphi_* \mathscr{O}_Y)_{|V}$ is a finite étale $\mathscr{O}_X$-algebra. On the other hand, let $I' \subset B$ be the radical of the ideal $I + pB$, and set $Z' := Z \setminus V$; clearly, I' is a branch ideal. It then suffices to show that $\varphi_* \mathscr{O}^a_Y$ is an étale $\mathscr{O}^a_X$-algebra, for the almost structure given by the new setup (B, I'). Thus, we may assume that $p \in I$, in which case the assertion follows from theorems 17.2.14(iii) and 16.8.44. □

Theorem 17.2.33. *In the situation of* (17.2.5)*, the ring B is ind-measurable.*

Proof. (See remark 14.5.69(i) for the definition of ind-measurable ring.) One argues as in the proof of theorem 17.1.43, with some simplifications. We have to exhibit a sequence $(d_n \mid n \in \mathbb{N})$ of normalizing factors fulfilling conditions (a) and (b) of definition 14.5.64, where the λ_n occurring in *loc.cit.* is meant to be the usual length function for finitely generated B_n-modules supported at the closed point x_n of X_n. Now, fix $n \in \mathbb{N}$, set

$$\mathbf{T}_\mathbb{R} := P^{\mathrm{gp}}_\mathbb{R} / P^{(n)\mathrm{gp}}$$

and endow $\mathbf{T}_\mathbb{R}$ with its invariant measure $d\mu_n$ of total volume equal to 1. For every $\gamma \in P^{\mathrm{gp}}_\mathbb{R}$, let $[\gamma] \in \mathbf{T}_\mathbb{R}$ be the equivalence class of γ; notice that the $P^{(n)}$-module

$$S_{[\gamma]} := \gamma P^{(n)\mathrm{gp}} \cap P_\mathbb{Q} \qquad (17.2.34)$$

is finitely generated (proposition 6.3.22(ii)) and depends only on the class $[\gamma]$, and for any given finitely generated B_n-module M supported at x_n, consider the function

$$l_M : \mathbf{T}_\mathbb{R} \to \mathbb{N} \qquad [\gamma] \mapsto \lambda_n(S_{[\gamma]} \otimes_{P^{(n)}} M).$$

Let $e_1, \dots, e_r$ be a basis of the free $\mathbb{Z}$-module P^{gp}, and define $\Omega_n \subset P^{\mathrm{gp}}_\mathbb{R}$ as in the proof of theorem 17.1.43, so that Ω_n is a fundamental domain for the lattice $P^{(n)\mathrm{gp}}$, and 0 lies in the interior of Ω_n. Denote also by $\Sigma \subset \mathbf{T}_\mathbb{R}$ the image of $P_\mathbb{R} \cap \Omega_n$.

Claim 17.2.35. There is a partition of $\mathbf{T}_\mathbb{R}$ into finitely many measurable subsets $\Theta_1, \dots, \Theta_t$, independent of M, such that :

(i) For every $\gamma, \lambda \in \Omega_n$, the classes $[\gamma], [\lambda] \in \mathbf{T}_\mathbb{R}$ lie in the same Θ_i if and only if $\gamma^{-1} S_{[\gamma]} = \lambda^{-1} S_{[\lambda]}$.
(ii) Especially, l_M restricts to a constant function on each Θ_i.
(iii) Let $\Theta \in \{\Theta_1, \dots, \Theta_t\}$ be the subset containing $[0] \in \mathbf{T}_\mathbb{R}$; then $\Theta \cap \Sigma$ has measure > 0.

Proof of the claim. According to proposition 6.3.35(i,iii), the set $\mathscr{S} := \{\gamma^{-1} S_{[\gamma]} \mid \gamma \in \Omega_n\}$ is finite, and for every non-empty $S \in \mathscr{S}$, the set $\{\gamma \in \Omega_n \mid \gamma^{-1} S_{[\gamma]} = S\}$ is the intersection of Ω_n with a $\mathbb{Q}$-linearly constructible subset. It follows that the same must hold also in case $S = \varnothing$. The image in $\mathbf{T}_\mathbb{R}$ of any such $\mathbb{Q}$-linearly constructible subset is obviously measurable, whence (i). Moreover, in view of our choice of Ω, assertion (ii) follows easily from claim 6.3.40. ◊

Let $m \geq n$ be any integer; since B''_m is a free B''_n-module of rank $p^{r(m-n)}$ (notation of (17.2.3)), we may compute :

$$\lambda_m(B_m \otimes_{B_n} M) = \frac{p^{r(m-n)}}{[\kappa(x_m) : \kappa(x_n)]} \cdot \sum_{[\gamma] \in P^{(m)\mathrm{gp}}/P^{(n)\mathrm{gp}}} l_M([\gamma]).$$

However, lemma 17.2.11 easily implies that $\kappa(x_{n+1})^p = \kappa(x_n)$ for every $n \in \mathbb{N}$, whence

$$[\kappa(x_{n+1}) : \kappa(x_n)] = p^{e_n} \qquad \text{where } e_n := \Omega^1_{\kappa(x_n)/\mathbb{Z}}$$

by virtue of [63, Ch.0, Th.21.4.5]. But by the same token, the field $\kappa(x_m)$ is isomorphic to $\kappa(x_n)$ for every $m \geq n$, so e_n is actually independent of n, and we get

$$[\kappa(x_m) : \kappa(x_n)] = p^{e_0(m-n)} \qquad \text{for every } m \geq n.$$

Therefore, set

$$d_n := p^{n \cdot \dim B_0} \qquad \text{for every } n \in \mathbb{N}.$$

We claim that $(d_n \mid n \in \mathbb{N})$ is a suitable sequence of normalizing factors for B. Indeed, claim 17.2.35(ii) says that l_M is a measurable function on $\mathbf{T}_{\mathbb{R}}$, and the foregoing, together with the discussion of (17.2.8) implies that :

$$\lambda(B \otimes_{B_n} M) := \lim_{m \to +\infty} d_m^{-1} \cdot \lambda_m(B_m \otimes_{B_n} M) = d_n^{-1} \int_{\mathbf{T}_{\mathbb{R}}} l_M d\mu_n$$

(recall that $\dim P = \mathrm{rk}_{\mathbb{Z}} P^{\mathrm{gp}}$, by corollary 6.4.12(i)), so condition (a) holds for this choice of factors. Next, fix $\varepsilon > 0$, let $N \to N'$ be a surjection of finitely generated B_n-modules supported at x_n, and suppose that $d_n^{-1}(\lambda_n(N) - \lambda_n(N')) \geq \varepsilon$. Since $\lambda_n(N) = l_N(0)$ (and likewise for N'), we deduce that

$$\lambda(B \otimes_{B_n} N) - \lambda(B \otimes_{B_n} N') \geq \varepsilon \cdot \int_{\Theta} d\mu_n$$

where $\Theta \in \{\Theta_1, \dots, \Theta_t\}$ is the unique subset of $\mathbf{T}_{\mathbb{R}}$ such that $[0] \in \Theta$, so the volume of Θ is > 0, by claim 17.2.35(iii). This shows that condition (b) holds as well, and concludes the proof of the theorem. □

17.3. **The direct summand conjecture.** In the following paragraphs we prove a generalization of the direct summand conjecture for finite injective extensions of log-regular rings.

To begin with, let A be a ring, and consider a short exact sequence of A-modules

$$\Sigma \qquad : \qquad 0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0.$$

We say that Σ is *universally exact* if the sequence $X \otimes_A \Sigma$ is short exact for every A-module X.

Lemma 17.3.1. (i) *In the situation of* (17.3)*, the following conditions are equivalent :*

(a) *The sequence Σ is universally exact.*
(b) *The sequence $X \otimes_A \Sigma$ is short exact for every finitely presented A-module X.*
(c) *The sequence $\Sigma^\vee := \mathrm{Hom}_{\mathbb{Z}}(\Sigma, \mathbb{Q}/\mathbb{Z})$ splits.*
(d) *$\Sigma^\vee$ is universally exact.*
(e) *The sequence $\mathrm{Hom}_A(X, \Sigma)$ is exact for every finitely presented A-module X.*
(f) *Σ is the colimit of a filtered system of split short exact sequences of A-modules.*

(ii) *If moreover M'' is finitely presented, then conditions* (a)–(f) *are also equivalent to :*

(g) *Σ is a split short exact sequence of A-modules.*

(iii) *If furthermore, A is local and noetherian, and M, M', M'' are finitely generated, then conditions* (a)–(g) *are also equivalent to :*

(h) *The sequence $X \otimes_A \Sigma$ is short exact for every A-module X of finite length.*

Proof. Obviously (f)⇒(a)⇒(b). To see that (b)⇒(a), write any given A-module X as the filtered colimit of a system $(X_\lambda \mid \lambda \in \Lambda)$ of finitely presented A-modules; then $H^i(X \otimes_A \Sigma)$ is the colimit of the induced filtered system of A-modules $(H^i(X_\lambda \otimes_A \Sigma) \mid \lambda \in \Lambda)$, for every $i \in \mathbb{Z}$. But (b) says that each $X_\lambda \otimes_A \Sigma$ is exact, whence (a).

(a)⇒(c): Clearly $\Sigma^\vee$ is short exact, since $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module. For every A-module X, recall that $X^\vee := \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Q}/\mathbb{Z})$ is naturally an A-module (see (14.2.26)); notice that for every pair of A-modules X, Y we have a natural isomorphism of A-modules :

$$\omega_{X,Y} : \mathrm{Hom}_A(X^\vee, Y^\vee) \xrightarrow{\sim} (X^\vee \otimes_A Y)^\vee \qquad \varphi \mapsto (h \otimes y \mapsto \varphi(h)(y)).$$

There follows a commutative diagram :

$$\begin{array}{ccc} \mathrm{Hom}_A(M'^{\vee}, M^{\vee}) & \xrightarrow{\mathrm{Hom}_A(M'^{\vee}, f^{\vee})} & \mathrm{Hom}_A(M'^{\vee}, M'^{\vee}) \\ {\scriptstyle \omega_{M',M}} \downarrow & & \downarrow {\scriptstyle \omega_{M',M'}} \\ (M'^{\vee} \otimes_A M)^{\vee} & \xrightarrow{(M'^{\vee} \otimes_A f)^{\vee}} & (M'^{\vee} \otimes_A M')^{\vee}. \end{array}$$

By assumption, $M'^{\vee} \otimes_A f$ is injective, hence $(M'^{\vee} \otimes_A f)^{\vee}$ is surjective, so the same holds for $\mathrm{Hom}_A(M'^{\vee}, f^{\vee})$, and (c) follows easily.

Obviously (c)⇒(d). In order to show that (d)⇒(e), let X be any finitely presented A-module; by assumption, $X \otimes_A g^{\vee}$ is an injection. On the other hand, for every pair of A-modules X, Y we have a natural A-linear map

$$\sigma_{X,Y} : X \otimes_A Y^{\vee} \to \mathrm{Hom}_A(X, Y)^{\vee} \qquad x \otimes t \mapsto (h \mapsto t \circ h(x))$$

and it is easily seen that $\sigma_{X,Y}$ is an isomorphism if X is finitely presented (details left to the reader). By naturality of $\sigma_{X,\bullet}$, it follows that $\mathrm{Hom}_A(X, g^{\vee})^{\vee}$ is injective, hence $\mathrm{Hom}_A(X, g^{\vee})$ is surjective, whence (e).

(e)⇒(f): Write M'' as the colimit of a filtered system $(M''_\lambda \mid \lambda \in \Lambda)$ of finitely presented A-modules, and let $(j_\lambda : M_\lambda \to M \mid \lambda \in \Lambda)$ be the universal co-cone; for every λ, set $\Sigma_\lambda := j^*_\lambda \Sigma$. Then Σ is the colimit of the resulting system of short exact sequences $(\Sigma_\lambda \mid \lambda \in \Lambda)$, and we are reduced to checking that each Σ_λ is split. However, (e) implies that for every λ there exists an A-linear map $h_\lambda : M''_\lambda \to M$ such that $g \circ h_\lambda = j_\lambda$; the assertion follows straightforwardly.

Next, if M'' is finitely presented, (e) implies that $\mathrm{Hom}_A(M'', g)$ is surjective, whence (g).

(iii): Let $(A, \mathfrak{m})$ be a local and noetherian ring, and let X be any finitely presented A-module; if (h) holds, then for every $n \in \mathbb{N}$ the map $f_n := X/\mathfrak{m}^n X \otimes_A f$ is injective. Let $t \in \mathrm{Ker}\,(X \otimes_A f) \setminus \{0\}$; since the $\mathfrak{m}$-adic topology of $X \otimes_A M'$ is separated ([126, Th.8.10(i)]), there exists $n \in \mathbb{N}$ such that the image $\bar{t}$ of t in $X/\mathfrak{m}^n X \otimes_A M'$ does not vanish, so $f_n(\bar{t}) \neq 0$ by assumption, and therefore $f(t) \neq 0$, a contradiction. So $X \otimes_A f$ is injective for every such X, whence (b). □

17.3.2. Let $\mathscr{M}$ be a regular log-structure on either the Zariski or the étale site of a noetherian and affine scheme X; say that $X = \mathrm{Spec}\, A$ for a noetherian ring A. Let also $f : A \to B$ be a finite injective ring homomorphism; we want to show that there exists an A-linear map $g : B \to A$ such that $g \circ f = \mathbf{1}_A$, or equivalently, that the short exact sequence of A-modules

$$\Sigma_f \qquad : \qquad 0 \to A \xrightarrow{f} B \to Q := \mathrm{Coker}\, f \to 0$$

splits. The latter holds if and only if the class $[\Sigma_f] \in \mathrm{Ext}^1_A(Q, A)$ of Σ_f vanishes. Let us recall:

Lemma 17.3.3. (i) *Let A be a noetherian ring, and M, N two A-modules, with M of finite type; then every flat A-algebra A' induces an isomorphism :*

$$A' \otimes_A \mathrm{Ext}^1_A(M, N) \xrightarrow{\sim} \mathrm{Ext}^1_{A'}(A' \otimes_A M, A' \otimes_A N).$$

(ii) *In the situation of* (17.3)*, suppose that A is noetherian and M'' is of finite type. Then the following conditions are equivalent :*

(a) *Σ is a split short exact sequence of A-modules.*
(b) *For every $\mathfrak{p} \in \mathrm{Spec}\, A$ there exists a faithfully flat $A_\mathfrak{p}$-algebra A' such that $A' \otimes_A \Sigma$ is a split short exact sequence of A'-module.*

Proof. (i): Pick a resolution $P^\bullet \to M$ by projective A-modules of finite type; then $\mathrm{Ext}^1_A(M, N)$ is computed by $H^1\mathrm{Hom}_A(P^\bullet, N)$, and likewise $\mathrm{Ext}^1_{A'}(A' \otimes_A M, A' \otimes_A N)$ is computed by

$H^1\mathrm{Hom}_{A'}(A'\otimes_A P^\bullet, A'\otimes_A N)$. But since A is noetherian and P^i is an A-module of finite type for every i, the induced morphism of complexes

$$A'\otimes_A \mathrm{Hom}_A(P^\bullet, N) \to \mathrm{Hom}_{A'}(A'\otimes_A P^\bullet, A'\otimes_A N)$$

is an isomorphism, whence the assertion.

(ii): It is easily seen that the class $[A'\otimes_A \Sigma] \in \mathrm{Ext}^1_{A'}(A'\otimes_A Q, A')$ corresponds – under the isomorphism of (i) – to $1\otimes[\Sigma] \in A'\otimes_A \mathrm{Ext}^1_A(Q, A')$. The assertion follows immediately. □

In view of lemma 17.3.3(ii) we are reduced to checking that $f_\mathfrak{p} : A_\mathfrak{p} \to B_\mathfrak{p}$ admits an $A_\mathfrak{p}$-linear section for every $\mathfrak{p} \in \mathrm{Spec}\, A$; hence *we may assume that* $(A, \mathfrak{m})$ *is a local noetherian ring*. Likewise, let A^{sh} be a strict henselization of A (at the closed point $\mathfrak{m}$); since the structure map $A \to A^{\mathrm{sh}}$ is faithfully flat, lemma 17.3.3(ii) implies that $[\Sigma_f] = 0$ if and only if $[A^{\mathrm{sh}}\otimes_A \Sigma_f] = 0$ in $\mathrm{Ext}^1_{A^{\mathrm{sh}}}(A^{\mathrm{sh}}\otimes_A Q, A^{\mathrm{sh}})$; also, A^{sh} is noetherian, by [66, Ch.IV, Prop.18.8.8(iv)]. Thus, it suffices to check that $A^{\mathrm{sh}}\otimes_A f$ admits an A^{sh}-linear section, and therefore *we may assume that* $\mathscr{M}$ *admits a chart* $\beta : P \to A$ *that is sharp at the point* $\mathfrak{m}$.

17.3.4. Arguing likewise, we may replace A by its $\mathfrak{m}$-adic completion, and suppose therefore that A is a complete local noetherian ring. Moreover, A is normal and Cohen-Macaulay, by corollary 12.5.13. Furthermore, the proof of theorem 12.5.10 show that there exists $r \in \mathbb{N}$ and a surjective ring homomorphism

$$\varphi : V[[Q]] \to A \qquad \text{where } Q := P \times \mathbb{N}^{\oplus r}$$

and where :

- $(V, \mathfrak{m}_V)$ is a *coefficient ring* of A, *i.e.* either a field, or a complete discrete valuation ring with $A = \varphi(V) + \mathfrak{m}$ and such that $\mathfrak{m}_V = pV$ for a prime integer p
- The kernel of φ is trivial if V is a field, and otherwise it is principal, generated by a power series $\sum_{q\in Q} a_q \cdot q$ with $a_0 \in \mathfrak{m}_V \setminus \mathfrak{m}_V^2$
- The chart β is the composition of φ with the obvious inclusion map $P \to Q \to V[[Q]]$.

Let k' be an algebraically closed field extension of the residue field k of A; by [126, Th.29.1] there exists a flat ring homomorphism $\psi : V \to V'$, where $V' = k'$ if $V = k$, and otherwise V' is a complete discrete valuation ring with maximal ideal pV' and residue field k'. The induced ring extension $\psi[Q] : V[Q] \to V'[Q]$ is again faithfully flat, hence it induces a faithfully flat ring homomorphism $V[[Q]] \to V'[[Q]]$ ([126, Th.22.4]). Set $A' := V'[[Q]] \otimes_{V[[Q]]} A$; the induced map $A \to A'$ is faithfully flat, and A' is the quotient of $V'[[Q]]$ by the principal ideal generated by the image of $\mathrm{Ker}\,\varphi$. Hence, A' is a complete local noetherian ring, and by theorems 12.5.10 and 12.5.31, the induced map of monoids $Q \to A'$ is a chart for a regular log structure on $\mathrm{Spec}\, A'$. By lemma 17.3.3(ii), we may then replace A, B, P by respectively $A', A'\otimes_A B, Q$, and *assume that* k *is algebraically closed and* $\mathfrak{m} = \mathfrak{m}_P A$ (with $\mathfrak{m}_P = P \setminus \{0\}$: see definition 6.1.10(i)).

17.3.5. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_k$ be the finitely many minimal prime ideals of B; since A is reduced, $f(A)\cap\mathfrak{p}_1\cap\cdots\cap\mathfrak{p}_k = 0$, hence the composition of f with the natural map $B \to B/\mathfrak{p}_1\times\cdots\times B/\mathfrak{p}_k$ is still finite and injective, and its kernel contains the ideal $\prod_{i=1}^k f^{-1}\mathfrak{p}_i$. Then there exists $i \in \{1, \dots, k\}$ such that $f^{-1}\mathfrak{p}_i = 0$, so that the composition $\overline{f}_i : A \to B/\mathfrak{p}_i$ of f with the projection $B \to B/\mathfrak{p}_i$ is still injective; clearly it suffices to check that $\overline{f}_i$ admits an A-linear section, so *we may assume that* B *is a domain*.

Let K be the field of fractions of A, set $B_K := K\otimes_A B$, let $d := \dim_K B_K$, and denote by $\mathrm{Tr}_{B_K/K} : B_K \to K$ the trace form; since A is integrally closed in K, the composition of $\mathrm{Tr}_{B_K/K}$ with the localization $B \to B_K$ factors through an A-linear map (lemma 14.3.15(i))

$$\mathrm{Tr}_{B/A} : B \to A.$$

Then $\mathrm{Tr}_{B/A} \circ f = d \cdot \mathbf{1}_A$. In particular, if A is a $\mathbb{Q}$-algebra, the map $d^{-1} \cdot \mathrm{Tr}_{B/A}$ is a section of f as sought. Thus, *we may assume that k has characteristic* $p > 0$. Let us set

$$P^{(n)} := \{\gamma \in P_{\mathbb{Q}} \mid \gamma^{p^n} \in P\} \qquad P^{(\infty)} := \bigcup_{n\in\mathbb{N}} P^{(n)} \qquad \text{and} \qquad \Gamma := (P^{(\infty)}/P)^{\mathrm{gp}}.$$

Then $P^{(\infty)}$ is naturally a Γ-graded monoid. More precisely, for every $\gamma \in (P^{(\infty)})^{\mathrm{gp}}$, denote by $[\gamma] \in \Gamma$ the class of γ; then we have :

$$\mathrm{gr}_{[\gamma]} P^{(\infty)} = P^{(\infty)} \cap \gamma P^{\mathrm{gp}} \qquad \text{for every } \gamma \in (P^{(\infty)})^{\mathrm{gp}}.$$

Theorem 17.3.6. *In the situation of* (17.3.2)*, suppose that A is an $\mathbb{F}_p$-algebra. Then f admits an A-linear section.*

Proof. After the foregoing reductions, we may assume that $A = k[[P]]$ for a sharp, fine and saturated monoid P, and that B is a domain. For every $n \in \mathbb{N}$, set

$$A_n := k[[P^{(n)}]] \qquad B_n := A_n \otimes_A B \qquad \text{and} \qquad A_\infty := \bigcup_{n\in\mathbb{N}} A_n.$$

Notice that A_0 is a direct summand of the A_0-module A_n, hence the inclusion map $i_n : A \to A_n$ has a section $\sigma_n : A_n \to A$ for every $n \in \mathbb{N}$; we are then reduced to checking that the induced map $f_n : A_n \to B_n$ has an A_n-linear section $s_n : B_n \to A_n$, for some $n \in \mathbb{N}$: indeed, in this case let $j_n : B \to B_n$ be the induced inclusion map, and set $s := \sigma_n \circ s_n \circ j_n : B \to A$. We get:

$$s \circ f = \sigma_n \circ s_n \circ j_n \circ f = \sigma_n \circ s_n \circ f_n \circ i_n = \sigma_n \circ i_n = \mathbf{1}_A$$

as required. On the other hand, let K_n be the field of fractions of A_n, and denote by B'_n the maximal reduced quotient of B_n, for every $n \in \mathbb{N}$; since A_n is reduced, the induced map $f'_n : A_n \to B'_n$ is still injective, so it suffices to exhibit a section for f'_n. Moreover, $E_n := K_n \otimes_{A_n} B'_n$ is a finite field extension of K_n (since K_n is a purely inseparable field extension of K), and the inclusion map $K_n \to K_{n+1}$ induces a field extension $E_n \to E_{n+1}$ for every such n. Notice also that $K_\infty := \bigcup_{n\in\mathbb{N}} K_n$ is a perfect field; hence $E_\infty := \bigcup_{n\in\mathbb{N}} E_n$ is the maximal reduced quotient of $K_\infty \otimes_A B$, and is a finite *separable* field extensions of K_∞. It follows easily that for $n \in \mathbb{N}$ sufficiently large, E_n is already a *separable and finite* field extension of K_n. Summing up, we may replace A and B by A_n and B'_n for n large enough, and *assume that f is generically étale, i.e.* there exists $g \in A \setminus \{0\}$ such that $A[1/g] \otimes_A f$ is a finite, injective and étale map. Then, notice that A_∞ is a perfect $\mathbb{F}_p$-algebra, and let $\mathfrak{n} \subset A_\infty$ be the radical of the ideal $A_\infty g$; let also C_∞ be the integral closure of A_∞ in $A_\infty[1/g] \otimes_A B$. By theorems 16.8.2 and 16.8.44(ii), and proposition 14.4.8, the $(A_\infty, \mathfrak{n})^a$-algebra C^a_∞ is étale, faithfully flat and of almost finite rank. Now, let X be any A-module of finite length; by virtue of lemma 17.3.1(ii,iii), it suffices to check that the map $X \otimes_A f : X \to X \otimes_A B$ is injective. However, since A is a direct summand of the A-module A_∞, the induced map $h : X \to X \otimes_A A_\infty$ is injective, and since C^a_∞ is a faithfully flat $(A_\infty, \mathfrak{n})^a$-algebra, the kernel of the induced map $l : X \otimes_A A_\infty \to X \otimes_A C_\infty$ is annihilated by $\mathfrak{n}$. Clearly $l \circ h$ factors through $X \otimes_A f$; hence :

$$\mathfrak{n} \cdot \mathrm{Ker}(X \otimes_A f) = 0$$

(we regard $\mathrm{Ker}(X \otimes_A f)$ as a subset of $\mathrm{Ker}\, l$, via h). Now, as $P^{(\infty)}$ is Γ-graded, A_∞ is naturally a Γ-graded A-algebra, and $X \otimes_A A_\infty$ is a Γ-graded $(A_\infty, \mathrm{gr}_\bullet A_\infty)$-module (see definition 7.6.1(ii)); by definition :

$$\mathrm{gr}_{[\gamma]} A_\infty := A \otimes_{\mathbb{Z}[P]} \mathbb{Z}[\mathrm{gr}_{[\gamma]} P^{(\infty)}] \qquad \text{for every } \gamma \in P^{(\infty)}.$$

According to proposition 6.4.14(ii), there exist $\lambda_1, \ldots, \lambda_k \in P^\vee := \mathrm{Hom}_{\mathbf{Mnd}}(P, \mathbb{N})$ such that $\mathbb{R}\lambda_1, \ldots, \mathbb{R}\lambda_k$ are the extremal rays of $P^\wedge_{\mathbb{R}}$ (see (6.3.15)).

Claim 17.3.7. If $\gamma \in P^{(\infty)}$, and $0 \leq \lambda_i(\gamma) < 1$ for every $i = 1, \ldots, k$, then $\mathrm{gr}_{[\gamma]} P^{(\infty)} = \gamma P$.

Proof of the claim. By proposition 6.4.14(iv), we have

$$P^{(\infty)} = \{\gamma \in (P^{(\infty)})^{\mathrm{gp}} \mid \lambda_i(\gamma) \geq 0 \text{ for every } i = 1, \dots, k\}.$$

Hence $\mathrm{gr}_{[\gamma]} P^{(\infty)}$ is the subset of all $\gamma \cdot \beta$, with $\beta \in P^{\mathrm{gp}}$ and $\lambda_i(\gamma \cdot \beta) \geq 0$ for $i = 1, \dots, k$. But $\lambda_i(\beta) \in \mathbb{Z}$ for every such i and β; our assumption on $\lambda_i(\gamma)$ then implies that $\lambda_i(\beta) \in \mathbb{N}$ for every such i and β, whence $\beta \in P$ (again, by proposition 6.4.14(iv)), and the claim follows. $\diamondsuit$

We have $g = \sum_{\gamma \in P} g_\gamma \cdot \gamma$ for a system $(g_\gamma \mid \gamma \in P)$ of elements of k. Pick $\beta \in P$ with $g_\beta \neq 0$, and $N \in \mathbb{N}$ large enough, so that $\lambda_i(\beta^{1/p^N}) = p^{-N}\lambda_i(\beta) < 1$ for $i = 1, \dots, k$; by claim 17.3.7, we then have $\mathrm{gr}_{[\beta^{1/p^N}]} P^{(\infty)} = \beta^{1/p^N} P$. Clearly $g^{1/p^N} = \sum_{\gamma \in P} g_\gamma^{1/p^N} \cdot \gamma^{1/p^N}$; let us also write $g^{1/p^N} = \sum_{[\gamma] \in \Gamma} g_{[\gamma]}$, with $g_{[\gamma]} \in \mathrm{gr}_{[\gamma]} A_\infty$ for every $[\gamma] \in \Gamma$. It follows that $g_{[\beta^{1/p^N}]} = g_\beta^{1/p^N} \cdot \beta^{1/p^N} \cdot (1 + a)$ for some $a \in \mathfrak{m}$. Let now $x \in \mathrm{Ker}\, X \otimes_A f$; we have $x \in \mathrm{gr}_0(X \otimes_A A_\infty)$, and $g^{1/p^N} x = 0$; then $g_{[\gamma]} x = 0$ for every $[\gamma] \in \Gamma$, and especially :

$$x \otimes g_\beta^{1/p^N} \cdot \beta^{1/p^N} = 0 \quad \text{in} \quad \mathrm{gr}_{[\beta^{1/p^N}]}(X \otimes_A A_\infty) = X \otimes_A A_{[\beta^{1/p^N}]} = X \otimes_{\mathbb{Z}[P]} \mathbb{Z}[\beta^{1/p^N} P]$$

and since $g_\beta \neq 0$, we have $x \otimes \beta^{1/p^N} = 0$ in $X \otimes_{\mathbb{Z}[P]} \mathbb{Z}[\beta^{1/p^N} P]$, so finally $x = 0$, as required. $\square$

Remark 17.3.8. Theorem 17.3.6 also has a proof along the lines of [93, Rem.6.2], which proves the direct summand conjecture for regular $\mathbb{F}_p$-algebras. This alternative argument still relies on claim 17.3.7, but avoids lemma 17.3.1 and the reduction to the case where f is generically étale.

17.3.9. By theorem 17.3.6 and the foregoing reductions, we may assume that

$$A = V[[P]]/(\vartheta)$$

for a fine, sharp and saturated monoid $P \neq 0$, a complete discrete valuation ring $(V, \mathfrak{m}_V)$ with $\mathfrak{m}_V = pV$, whose residue field $k := V/pV$ is algebraically closed, and a power series $\vartheta := \sum_{\gamma \in P} a_\gamma \cdot \gamma$ with $a_0 \in pV \setminus p^2 V$. Then $\mathfrak{m}_P A = \mathfrak{m}$, the maximal ideal of A. We set

$$A_n := A \otimes_{\mathbb{Z}[P]} \mathbb{Z}[P^{(n)}] \qquad \text{and} \qquad B_n := A_n \otimes_A B \qquad \text{for every } n \in \mathbb{N}.$$

Notice that the induced map $A_n \to A_{n+1}$ is injective for every $n \in \mathbb{N}$, since $\mathbb{Z}[P^{(n)}]$ is a direct summand of the $\mathbb{Z}[P^{(n)}]$-module $\mathbb{Z}[P^{(n+1)}]$; hence, let as well $A_\infty := \bigcup_{n \in \mathbb{N}} A_n$. Moreover, notice that $A/\mathfrak{m}_P A = V/a_0 V = k$, and $\Omega^1_{k/\mathbb{Z}} = 0$, since k is perfect; then, the integer r of (17.2.3) vanishes in the current situation, hence theorem 17.2.14(iv) tells us that A_∞ *is a formally perfectoid ring, for its p-adic topology*, and clearly it is also naturally Γ-graded.

Let $\lambda_1, \dots, \lambda_k \in P^\vee := \mathrm{Hom}_{\mathbf{Mnd}}(P, \mathbb{N})$ such that $\mathbb{R}\lambda_1, \dots, \mathbb{R}\lambda_k$ are the extremal rays of $P^\vee_{\mathbb{R}}$ (see the proof of theorem 17.3.6), and $\beta_1, \dots, \beta_l \in P$ a finite system of generators of the $\mathbb{Q}_+$-module $\mathbb{Q}_+ \otimes_{\mathbb{N}} P$ (see (3.7.17)); set

$$\lambda := \lambda_1 + \cdots + \lambda_k \qquad \beta := \beta_1 \cdots \beta_l \qquad \text{and} \qquad L := \max(\lambda(\beta_1), \dots, \lambda(\beta_l)).$$

Lemma 17.3.10. *In the situation of* (17.3.9)*, let $J \subset A_\infty$ be a Γ-graded ideal, $g \in A_\infty$, and let $e, n \in \mathbb{N}$ such that*

$$g \in \mathfrak{m}^e A_\infty \setminus \mathfrak{m}^{e+1} A_\infty \qquad g \in J^{p^n} \qquad \textit{and} \qquad 1 \geq \varepsilon(e, n) := (eL + \lambda(\beta))/p^n.$$

Then there exists $\gamma \in P^{(\infty)}$ such that :

$$\mathrm{gr}_{[\gamma]} J = \mathrm{gr}_{[\gamma]} A_\infty \qquad \textit{and} \qquad \lambda(\gamma) < \varepsilon(e, n).$$

Proof. We may write $g = g_1 + \cdots + g_r$ for a finite sequence $g_1, \dots, g_r \in J^{p^n}$ such that each g_i is in turn of the form $g_i = \prod_{j=1}^{p^n} a_{ij} \otimes \gamma_{ij}$ with :

$$a_{i1}, \dots, a_{i,p^n} \in A \qquad \gamma_{i1}, \dots, \gamma_{i,p^n} \in P^{(\infty)} \qquad a_{ij} \otimes \gamma_{ij} \in \mathrm{gr}_{[\gamma_{ij}]} J \quad \text{for every } j = 1, \dots, p^n.$$

It is clear that there exists $e' \leq e$ and $i \in \{1, \dots, r\}$ such that $g_i \in \mathfrak{m}^{e'} A_\infty \setminus \mathfrak{m}^{e'+1} A_\infty$. Up to reordering the p^n factors of g_i, we may then assume that there exists $c \in \{0, \dots, p^n\}$ such that $\lambda(\gamma_{ij}) < \varepsilon(e, n)$ for $j = 1, \dots, c$, and $\lambda(\gamma_{ij}) \geq \varepsilon(e, n)$ for $j = c+1, \dots, p^n$. Suppose now that the lemma fails; then we must have $a_{ij} \in \mathfrak{m}$ for every $i = 1, \dots, c$, and therefore

$$\prod_{j=c+1}^{p^n} a_{ij} \otimes \gamma_{ij} \notin \mathfrak{m}^{e'-c+1} A_\infty. \tag{17.3.11}$$

For every $j = c+1, \dots, p^n$, write $\gamma_{ij} = \prod_{s=1}^{l} \beta_s^{d_{ijs}}$ for certain $d_{ij1}, \dots, d_{ijl} \in \mathbb{Q}_+$; then there exist unique $d'_{is} \in \mathbb{N}$ and $d''_{is} \in [0, 1[$ with

$$\sum_{j=c+1}^{p^n} d_{ijs} = d'_{is} + d''_{is} \qquad \text{for every } s = 1, \dots, l$$

and (17.3.11) then implies that $\sum_{s=1}^{l} d'_{is} \leq e' - c$, whence :

$$\sum_{j=c+1}^{p^n} \lambda(\gamma_{ij}) < (e' - c) \cdot L + \lambda(\beta') \leq p^n \cdot \varepsilon(e, n) - cL.$$

However, $\sum_{j=c+1}^{p^n} \lambda(\gamma_{ij}) \geq (p^n - c) \cdot \varepsilon(e, n)$, and since $\varepsilon(e, n) \leq 1 \leq L$, this is absurd. □

Theorem 17.3.12. *In the situation of* (17.3.2)*, the map f admits an A-linear section.*

Proof. In view of the foregoing reductions, we may assume that A and A_∞ are as in (17.3.9) and B is a domain; recall also that $B[1/g]$ is an étale and faithfully flat $A[1/g]$-algebra. Let :

$$A'_n := A_\infty[X^{1/p^n}]/(X - g) \qquad \text{for every } n \in \mathbb{N} \qquad \text{and} \qquad A'_\infty := \bigcup_{n \in \mathbb{N}} A'_n.$$

Let also D be the p-root closure of A'_∞ in $A'_\infty[1/p]$, and endow D with its p-adic topology; by theorem 16.9.17, we know that D is a faithfully flat A_∞-algebra, and that D is a formally perfectoid ring. Denote by $D^\wedge$ the p-adic completion of D, and set $B' := D^\wedge \otimes_A B[1/g]$. Hence, B' is a finite étale $D^\wedge[1/g]$-algebra. Let now $\mathfrak{n} \subset D^\wedge$ be the ideal generated by $(g^{1/p^n} \mid n \in \mathbb{N})$; then $(D^\wedge, \mathfrak{n})$ is a basic setup, and we set $D_1^\wedge := (D^\wedge, \mathfrak{n})_*^a \subset D^\wedge[1/g]$ (see lemma 16.9.41); lastly, let B'_1 be the integral closure of $D_1^\wedge$ in B'. By theorem 16.9.42(ii,iii), for every $n \in \mathbb{N}$, the $(D/p^n D)^a$-algebra $(B'_1/p^n B'_1)^a$ is faithfully flat and étale of finite rank, for the almost structure given by the basic setup $(D_1^\wedge, \mathfrak{n})$. After these preliminaries, let X be any A-module of finite length; by lemma 17.3.1(ii,iii), it suffices to show that the map $X \otimes_A f$ is injective. However, let $x \in \mathrm{Ker}(X \otimes_A f)$, and pick $m \in \mathbb{N}$ large enough, so that $p^m X = 0$. Since A_∞ is a Γ-graded A-algebra with $\mathrm{gr}_0 A_\infty = A$, the induced map $\varphi : X \to X_\infty := X \otimes_A A_\infty/p^m A_\infty$ is injective, and X_∞ is naturally a Γ-graded $(A_\infty, \mathrm{gr}_\bullet)$-module, with $\mathrm{gr}_0 X_\infty = X$. Especially, the annihilator J of $\varphi(x) = x \otimes 1 \in \mathrm{gr}_0 X_\infty$ is a Γ-graded ideal of A_∞. Next, since D is a faithfully flat A_∞-algebra, the induced map $\varphi' : X_\infty \to X_\infty \otimes_{A_\infty} D/p^m D$ is again injective, and $\mathrm{Ann}_D(\varphi'(x \otimes 1)) = JD$. Lastly, since $(B'_1/p^m B'_1)^a$ is a faithfully flat $(D/p^m D)^a$-algebra, the kernel of the induced map $\varphi'' : X_\infty \otimes_{A_\infty} D/p^m D \to X_\infty \otimes_{A_\infty} B'_1/p^m B'_1$ is annihilated by $\mathfrak{n}$. Clearly $\varphi'' \circ \varphi' \circ \varphi$ factors through $X \otimes_A f$, hence $\varphi'(x \otimes 1) \in \mathrm{Ker}\,\varphi''$, and consequently :

$$\mathfrak{n} \subset JD.$$

Thus, $g^{1/p^n} \in JD \cap A_\infty = J$ for every $n \in \mathbb{N}$ ([126, Th.7.5(ii)]), *i.e.* $g \in J^{p^n}$ for every $n \in \mathbb{N}$. Notice that the $\mathfrak{m}$-adic topology is separated on A_∞; hence, let $e \in \mathbb{N}$ such that $g \in \mathfrak{m}^e A_\infty \setminus \mathfrak{m}^{e+1} A_\infty$; by lemma 17.3.10, for every $n \in \mathbb{N}$ there exists $\gamma_n \in P^{(\infty)}$ such that $\lambda(\gamma_n) < \varepsilon(e, n)$ and $\mathrm{gr}_{[\gamma_n]} J = \mathrm{gr}_{[\gamma_n]} A_\infty$. By claim 17.3.7, it follows that for every sufficiently large $n \in \mathbb{N}$, we have $\mathrm{gr}_{[\gamma_n]} J = A \otimes_{\mathbb{Z}[P]} \mathbb{Z}[\gamma_n + P]$, a free A-module of rank one, with generator

$1 \otimes \gamma_n$. Finally, we get $x \otimes \gamma_n = 0$ in $\mathrm{gr}_{[\gamma_n]} X_\infty = X \otimes_{\mathbb{Z}[P]} \mathbb{Z}[\gamma_n + P]$ for every such n; thus, $x = 0$, and the proof is concluded. □

17.4. **Diagonal idempotents of generically étale maps.** Let $f : A \to B$ be a finite and injective ring homomorphism of noetherian rings; set $X := \mathrm{Spec}\, A$, denote by $\mathscr{B}$ the quasi-coherent $\mathscr{O}_X$-algebra associated with B, and by $\mu : \mathscr{B} \otimes_{\mathscr{O}_X} \mathscr{B} \to \mathscr{B}$ the multiplication morphism. For every affine open subset $U \subset X$ such that $\mathscr{B}_{|U}$ is an étale $\mathscr{O}_U$-algebra, we have a unique *diagonal idempotent* $e_{f,U} \in \mathscr{B}(U) \otimes_{\mathscr{O}_X(U)} \mathscr{B}(U)$ such that :

$$\mu_U(e_{f,U}) = 1 \quad \text{and} \quad x \cdot e_{f,U} = 0 \quad \text{for every } x \in \mathrm{Ker}\,(\mathscr{B}(U) \otimes_{\mathscr{O}_X(U)} \mathscr{B}(U) \xrightarrow{\mu_U} \mathscr{B}(U)).$$

We suppose moreover that f is *generically étale*, *i.e.* that the union $U_f \subset X$ of all such affine open subsets is dense in X; the uniqueness property of the diagonal idempotents then implies that there exists a unique section

$$e_f \in \Gamma(U_f, \mathscr{B} \otimes_{\mathscr{O}_X} \mathscr{B})$$

such that $e_{f|U} = e_{f,U}$ for every affine open subset $U \subset U_f$. Moreover, $\mathscr{B}_{|U_f}$ is a locally free $\mathscr{O}_{U_f}$-module of finite rank, hence we have a well defined $\mathscr{O}_{U_f}$-linear *trace map*

$$\mathrm{tr}_{\mathscr{B}/\mathscr{O}_X} : \mathscr{B}_{|U_f} \to \mathscr{O}_{U_f}$$

that assigns to every affine open subset $U \subset U_f$ and every $b \in \mathscr{B}(U)$ the trace of the $\mathscr{O}_X(U)$-linear endomorphism $b \cdot \mathbf{1}_{\mathscr{B}(U)}$. The corresponding *trace form*

$$t_{\mathscr{B}/\mathscr{O}_X} := \mathrm{tr}_{\mathscr{B}/\mathscr{O}_X} \circ \mu : (\mathscr{B} \otimes_{\mathscr{O}_X} \mathscr{B})_{|U_f} \to \mathscr{O}_{U_f}$$

is a perfect pairing ([75, Th.4.1.14]), hence it induces an $\mathscr{O}_{U_f}$-linear isomorphism

$$\omega_{\mathscr{B}/\mathscr{O}_X} : \mathscr{B}_{|U_f} \xrightarrow{\sim} (\mathscr{B}^\vee)_{|U_f} \qquad \text{where} \qquad \mathscr{B}^\vee := \mathscr{H}om_{\mathscr{O}_X}(\mathscr{B}, \mathscr{O}_X).$$

Remark 17.4.1. Let $f : A \to B$ be as in (17.4), $p \in \mathbb{N}$ a prime integer, and suppose that A is an $\mathbb{F}_p$-algebra. Let also $h \in A$ such that the localization $f_h : A_h \to B_h$ of f is étale. It is easily seen that the natural map (notation of (9.8.11))

$$B^{\mathrm{perf}} \otimes_{A^{\mathrm{perf}}} B^{\mathrm{perf}} \to (B \otimes_A B)^{\mathrm{perf}}$$

is bijective. By assumption, $h^n e_f$ is in the image of the localization $B \otimes_A B \to B_h \otimes_{A_h} B_h$, for some $n \in \mathbb{N}$. The image e_f^{perf} of e_f in $(B \otimes_A B)_h^{\mathrm{perf}}$ is the diagonal idempotent of the étale map $A_h^{\mathrm{perf}} \otimes_A f : A_h^{\mathrm{perf}} \to B_h^{\mathrm{perf}} = A_h^{\mathrm{perf}} \otimes_A B$. Since $(e_f^{\mathrm{perf}})^p = e_f^{\mathrm{perf}}$, we conclude that $h^{n/p^k} e_f^{\mathrm{perf}}$ lies in the image of the localization $(B \otimes_A B)^{\mathrm{perf}} \to (B \otimes_A B)_h^{\mathrm{perf}}$, for every $k \in \mathbb{N}$.

Definition 17.4.2. (i) A *log ring* (A, P, β) is the datum of a ring A, an integral saturated monoid P whose associated abelian group P^{gp} is *torsion-free*, and a morphism of monoids $\beta : P \to A$ from P to the multiplicative monoid $(A, \cdot)$, furnishing a chart for a *log structure* $\mathscr{P}$ on the Zariski site of $\mathrm{Spec}\, A$ (see definition 12.1.17(i)), which we call *the log structure of* (A, P, β).

(ii) We say that the log ring (A, P, β) is *regular* if A is noetherian, P is fine and saturated, and the log structure of (A, P, β) is *regular*. In this case, P^{gp} is a free abelian group of finite rank, and A is a product of finitely many normal domains (corollary 12.5.13).

17.4.3. Let (A, P, β) be any log ring, $f : A \to B$ a ring homomorphism as in (17.4), and set $X := \mathrm{Spec}\, A$. We consider the monoid $P^{(\infty)}$ associated with P as in (17.3.5), and the abelian group $\Gamma := (P^{(\infty)}/P)^{\mathrm{gp}}$. We will likewise use the monoid $P_{\mathbb{Q}} := \mathbb{Q}_+ \otimes_{\mathbb{N}} P$, and the abelian group $\Gamma_{\mathbb{Q}} := (P_{\mathbb{Q}}/P)^{\mathrm{gp}}$, and for every $\gamma \in P_{\mathbb{Q}}$, we let $[\gamma] \in \Gamma_{\mathbb{Q}}$ be the class of γ. Then the ring

$$A_\infty := A \otimes_{\mathbb{Z}[P]} \mathbb{Z}[P_{\mathbb{Q}}]$$

is naturally a $\Gamma_{\mathbb{Q}}$-graded A-algebra, with graded summands :

$$A_{[\gamma]} := A \otimes_{\mathbb{Z}[P]} \mathbb{Z}[\mathrm{gr}_{[\gamma]} P_{\mathbb{Q}}] \qquad \text{for every } \gamma \in P_{\mathbb{Q}}.$$

Likewise, set $B_\infty := A_\infty \otimes_A B$; both B_∞ and its dual A_∞-module $B^\vee_\infty := \mathrm{Hom}_{A_\infty}(B_\infty, A_\infty)$ inherit natural $\Gamma_\mathbb{Q}$-graded structures whose graded summands are respectively

$$B_{[\gamma]} := A_{[\gamma]} \otimes_A B \qquad \text{and} \qquad B^\vee_{[\gamma]} := \mathrm{Hom}_A(B, A_{[\gamma]}) \qquad \text{for every } \gamma \in P_\mathbb{Q}.$$

Let $\mathscr{O}_{X_\infty}$ and $\mathscr{B}_\infty$ be the quasi-coherent $\mathscr{O}_X$-algebras associated with A_∞ and B_∞. Hence, $\mathscr{B}_{\infty|U_f}$ is an étale $\mathscr{O}_{X_\infty|U_f}$-algebra, whose trace morphism induces the isomorphism

$$\omega_{\mathscr{B}_\infty/\mathscr{O}_{X_\infty}} := \mathscr{O}_{X_\infty|U_f} \otimes_{\mathscr{O}_{U_f}} \omega_{\mathscr{B}/\mathscr{O}_X} : \mathscr{B}_{\infty|U_f} \xrightarrow{\sim} \mathscr{B}^\vee_{\infty|U_f}.$$

Also, the corresponding diagonal idempotent $e_{f,\infty} \in \Gamma(U_f, \mathscr{B}_\infty \otimes_{\mathscr{O}_{X_\infty}} \mathscr{B}_\infty)$ is the image of e_f, under the natural morphism

$$\mathscr{B} \otimes_{\mathscr{O}_X} \mathscr{B} \to \mathscr{O}_{X_\infty} \otimes_{\mathscr{O}_X} \mathscr{B} \otimes_{\mathscr{O}_X} \mathscr{B} \xrightarrow{\sim} \mathscr{B}_\infty \otimes_{\mathscr{O}_{X_\infty}} \mathscr{B}_\infty. \tag{17.4.4}$$

More precisely, since U_f is quasi-compact, the A_∞-module $\Gamma(U_f, \mathscr{B}_\infty \otimes_{\mathscr{O}_{X_\infty}} \mathscr{B}_\infty)$ is also $\Gamma_\mathbb{Q}$-graded, and on the other hand, $\mathrm{gr}_0 P_\mathbb{Q} = P$, since P is saturated; then $\mathrm{gr}_0 B_\infty = B$, and the map (17.4.4) identifies $\Gamma(U_f, \mathscr{B} \otimes_{\mathscr{O}_X} \mathscr{B})$ with $\mathrm{gr}_0\Gamma(U_f, \mathscr{B}_\infty \otimes_{\mathscr{O}_{X_\infty}} \mathscr{B}_\infty)$.

17.4.5. Let $\underline{f} := (f, P, \beta)$ be the datum of a log ring (A, P, β) and a ring homomorphism $f : A \to B$ as in (17.4); with the notation of (17.4.3), we attach to $\underline{f}$ the composition

$$B^\vee_\infty \otimes_{A_\infty} B^\vee_\infty \xrightarrow{\rho} \Gamma(U_f, \mathscr{B}^\vee_\infty \otimes_{\mathscr{O}_{X_\infty}} \mathscr{B}^\vee_\infty) \xrightarrow{\alpha} \Gamma(U_f, \mathscr{B}_\infty \otimes_{\mathscr{O}_{X_\infty}} \mathscr{B}_\infty) \tag{17.4.6}$$

where ρ is the restriction map of the quasi-coherent $\mathscr{O}_X$-module $\mathscr{B}^\vee_\infty \otimes_{\mathscr{O}_{X_\infty}} \mathscr{B}^\vee_\infty$, and α is induced by the isomorphism $\omega^{-1}_{\mathscr{B}_\infty/\mathscr{O}_{X_\infty}} \otimes_{\mathscr{O}_{X_\infty|U_\infty}} \omega^{-1}_{\mathscr{B}_\infty/\mathscr{O}_{X_\infty}}$. Set as well :

$$\mathsf{R}(B, \Delta) := \bigoplus_{[\gamma]\in\Delta} B^\vee_{[\gamma]} \otimes_A B^\vee_{[1/\gamma]} \qquad \text{for every subset } \Delta \subset \Gamma_\mathbb{Q}.$$

Since (17.4.6) is a morphism of graded A_∞-modules, it restricts to an A-linear map

$$\vartheta_{\underline{f}} : \mathsf{R}(B, \Gamma_\mathbb{Q}) \to \mathrm{gr}_0\Gamma(U_f, \mathscr{B}_\infty \otimes_{\mathscr{O}_{X_\infty}} \mathscr{B}_\infty) \xrightarrow{\sim} \Gamma(U_f, \mathscr{B} \otimes_{\mathscr{O}_X} \mathscr{B}).$$

Remark 17.4.7. (i) In the situation of (17.4.5), let $g : A \to A'$ be a ring homomorphism; set $B' := A' \otimes_A B$, let $f' : A' \to B'$ be the induced map, and suppose that $\underline{f}' := (f', P, g \circ \beta)$ is also a datum as in (17.4.5). Set $X' := \mathrm{Spec}\, A'$, and let $h : X' \to X$ be the morphism of schemes induced by g; also, let $\mathscr{B}'$ be the quasi-coherent $\mathscr{O}_{X'}$-algebra arising from B'. For every $[\gamma] \in \Gamma_\mathbb{Q}$ we get a natural A-linear map

$$B^\vee_{[\gamma]} \to \mathrm{Hom}_A(B, A'_{[\gamma]}) \xrightarrow{\sim} B'^\vee_{[\gamma]} := \mathrm{Hom}_{A'}(B', A'_{[\gamma]}) \tag{17.4.8}$$

and a direct inspection of the definitions yields a commutative diagram :

$$\begin{array}{ccccc}
\mathsf{R}(B, \Gamma_\mathbb{Q}) & & \xrightarrow{\quad\vartheta_{\underline{f}}\quad} & & \Gamma(U_f, \mathscr{B} \otimes_{\mathscr{O}_X} \mathscr{B}) \\
\downarrow & & & & \downarrow \\
\mathsf{R}(B', \Gamma_\mathbb{Q}) & \xrightarrow{\vartheta_{\underline{f}'}} & \Gamma(U_{f'}, \mathscr{B}' \otimes_{\mathscr{O}_{X'}} \mathscr{B}') & \xrightarrow{\rho} & \Gamma(h^{-1}U_f, \mathscr{B}' \otimes_{\mathscr{O}_{X'}} \mathscr{B}')
\end{array}$$

whose left (resp. right) vertical arrow is induced by the maps (17.4.8) (resp. by the natural morphisms of quasi-coherent $\mathscr{O}_X$-algebras $\mathscr{O}_X \to h_*\mathscr{O}_{X'}$ and $\mathscr{B} \to h_*\mathscr{B}'$), and where ρ is the restriction map. Also, it is clear that the right vertical arrow maps e_f to $\rho(e_{f'})$.

(ii) Let R be any ring, S and S' two finite R-algebras whose underlying R-modules are projective (of finite rank). Then all the trace maps $\mathrm{tr}_{S/R}, \mathrm{tr}_{S'/R}$ and $\mathrm{tr}_{S\otimes_R S'/R}$ are well defined, and [75, Lemma 4.1.3] implies that :

$$\mathrm{tr}_{S\otimes_R S'/R} = \mathrm{tr}_{S/R} \otimes_R \mathrm{tr}_{S'/R} : S \otimes_R S' \to R.$$

We deduce the following alternative description of $\vartheta_{\underline{f}}$. First, we have a natural A_∞-linear map

$$B_\infty^\vee \otimes_{A_\infty} B_\infty^\vee \to (B_\infty \otimes_{A_\infty} B_\infty)^\vee := \mathrm{Hom}_A(B \otimes_A B, A_\infty) \qquad (17.4.9)$$

given by the rule : $\varphi \otimes \varphi' \mapsto (b \otimes b' \mapsto \varphi(b) \cdot \varphi'(b'))$ for every $\varphi, \varphi' \in B_\infty^\vee$ and every $b, b' \in B$. Next, the trace map of the $\mathscr{O}_{X_\infty}$-algebra $\mathscr{B}_\infty \otimes_{\mathscr{O}_X} \mathscr{B}_\infty$ induces an $\mathscr{O}_{X_\infty}$-linear map

$$\omega_{\mathscr{B}_\infty \otimes_{\mathscr{O}_{X_\infty}} \mathscr{B}_\infty / \mathscr{O}_{X_\infty}} : \mathscr{B}_\infty \otimes_{\mathscr{O}_X} \mathscr{B}_\infty \to (\mathscr{B}_\infty \otimes_{\mathscr{O}_X} \mathscr{B}_\infty)^\vee$$

whose restriction to the open subset U_f is an isomorphism. Then it is easily seen that (17.4.6) equals the composition of (17.4.9) with the map

$$(B_\infty \otimes_{A_\infty} B_\infty)^\vee \xrightarrow{\rho'} \Gamma(U_f, (\mathscr{B}_\infty \otimes_{\mathscr{O}_X} \mathscr{B}_\infty)^\vee) \xrightarrow{\alpha'} \Gamma(U_f, \mathscr{B}_\infty \otimes_{\mathscr{O}_X} \mathscr{B}_\infty)$$

where ρ' is the restriction map of the quasi-coherent $\mathscr{O}_X$-module $(\mathscr{B}_\infty \otimes_{\mathscr{O}_X} \mathscr{B}_\infty)^\vee$, and α' is induced by the isomorphism $(\omega_{\mathscr{B}_\infty \otimes_{\mathscr{O}_{X_\infty}} \mathscr{B}_\infty / \mathscr{O}_{X_\infty}})_{|U_f}$. Moreover, $(B_\infty \otimes_{A_\infty} B_\infty)^\vee$ is naturally a $\Gamma_\mathbb{Q}$-graded A_∞-module, with a natural identification :

$$\mathrm{gr}_0(B_\infty \otimes_{A_\infty} B_\infty)^\vee \xrightarrow{\sim} (B \otimes_A B)^\vee := \mathrm{Hom}_A(B \otimes_A B, A)$$

and (17.4.9) is a morphism of $\Gamma_\mathbb{Q}$-graded modules. It follows that $\vartheta_{\underline{f}}$ is the composition

$$\mathsf{R}(B, \Gamma_\mathbb{Q}) \xrightarrow{\xi_{\underline{f}}} (B \otimes_A B)^\vee \xrightarrow{\xi'_{\underline{f}}} \Gamma(U_f, \mathscr{B} \otimes_{\mathscr{O}_X} \mathscr{B})$$

where $\xi_{\underline{f}}$ is induced by the restriction of (17.4.9), and $\xi'_{\underline{f}}$ is the restriction of $\alpha' \circ \rho'$. In turn, $\xi'_{\underline{f}}$ can be described as the composition of the restriction map $(B \otimes_A B)^\vee \to \Gamma(U_f, (\mathscr{B} \otimes_{\mathscr{O}_X} \mathscr{B})^\vee)$ of the quasi-coherent $\mathscr{O}_X$-module $(\mathscr{B} \otimes_{\mathscr{O}_X} \mathscr{B})^\vee$ arising from $(B \otimes_A B)^\vee$, and the isomorphism $\Gamma(U_f, (\mathscr{B} \otimes_{\mathscr{O}_X} \mathscr{B})^\vee) \xrightarrow{\sim} \Gamma(U_f, \mathscr{B} \otimes_{\mathscr{O}_X} \mathscr{B})$ induced by the isomorphism $(\mathscr{B} \otimes_{\mathscr{O}_X} \mathscr{B})^\vee_{|U_f} \xrightarrow{\sim} (\mathscr{B} \otimes_{\mathscr{O}_X} \mathscr{B})_{U_f}$ deduced as usual from the trace map of $\mathscr{B} \otimes_{\mathscr{O}_X} \mathscr{B}$.

(iii) With the notation of (ii), notice as well that $\xi'_{\underline{f}}$ is injective if A is reduced. Indeed, if $\varphi : B \otimes_A B \to A$ restricts on U_f to the zero section of $(\mathscr{B} \otimes_{\mathscr{O}_X} \mathscr{B})^\vee$, then for every $b, b' \in B$ the element $\varphi(b \otimes b') \in A$ restricts on U_f to the zero section of $\mathscr{O}_X$; since A is reduced and U_f is dense in X, this means that $\varphi(b \otimes b') = 0$ for every such b, b', whence the contention.

Lemma 17.4.10. (i) *In the situation of* (17.4.5)*, let $\Delta \subset \Gamma_\mathbb{Q}$ be any subset. Then there exists a finite subset $\Sigma \subset \Delta$ such that $\xi_{\underline{f}}(\mathsf{R}(B, \Delta)) = \xi_{\underline{f}}(\mathsf{R}(B, \Sigma))$ (where $\xi_{\underline{f}}$ is defined as in remark* 17.4.7(ii)*). Moreover, Σ depends only on P and Δ (and neither on β nor f).*

(ii) *Suppose that $P = P_1 \times P_2$, where $P_2 = \mathbb{N}^{\oplus r} \times \mathbb{Z}^{\oplus s}$, for some $r, s \in \mathbb{N}$. For $i = 1, 2$, let $\Delta_i \subset \Gamma_{i,\mathbb{Q}} := (P_{i,\mathbb{Q}}/P_i)^{\mathrm{gp}}$ be any subset, so that $\Delta_1 \times \Delta_2 \subset \Gamma_\mathbb{Q} = \Gamma_{1,\mathbb{Q}} \oplus \Gamma_{2,\mathbb{Q}}$. Then*

$$\xi_{\underline{f}}(\mathsf{R}(B, \Delta_1 \times \Delta_2)) = \xi_{\underline{f}}(\mathsf{R}(B, \Delta_1 \times \{0\})).$$

Proof. (i): On the one hand, the multiplication law of A_∞ induces by restriction A-linear maps

$$\mu_{[\gamma]} : A_{[\gamma]} \otimes_A A_{[1/\gamma]} \to \mathrm{gr}_0 A_\infty = A \qquad (a \otimes \gamma\lambda) \otimes (a' \otimes \lambda'/\gamma) \mapsto aa' \cdot \beta(\lambda\lambda')$$

(notice that if $\gamma, \lambda, \lambda' \in P_\mathbb{Q}^{\mathrm{gp}}$ and we have both $\gamma\lambda \in \mathrm{gr}_{[\gamma]} P_\mathbb{Q}$ and $\lambda'/\gamma \in \mathrm{gr}_{[1/\gamma]} P_\mathbb{Q}$, then $\lambda\lambda' \in \mathrm{gr}_0 P_\mathbb{Q} = P$). On the other hand, we have an obvious isomorphism of P-modules :

$$Q_\gamma := P^{\mathrm{gp}} \cap \gamma^{-1} P_\mathbb{Q} \xrightarrow{\sim} \mathrm{gr}_{[\gamma]} P_\mathbb{Q} \qquad \lambda \mapsto \gamma\lambda \qquad \text{for every } \gamma \in P_\mathbb{Q}^{\mathrm{gp}}.$$

Especially, if $Q_\gamma = Q_\delta$ for some $\gamma, \delta \in P_\gamma^{\mathrm{gp}}$, we deduce an A-linear isomorphism

$$\omega_{\gamma,\delta} : A_{[\gamma]} \xrightarrow{\sim} A_{[\delta]} \qquad a \otimes \gamma\lambda \mapsto a \otimes \delta\lambda \qquad \text{for every } a \in A \text{ and } \gamma\lambda \in \mathrm{gr}_{[\gamma]} P_\mathbb{Q}$$

which induces an A-linear isomorphism

$$\omega_{\gamma,\delta *} : B_{[\gamma]}^\vee \xrightarrow{\sim} B_{[\delta]}^\vee \qquad (\varphi : B \to A_{[\gamma]}) \mapsto (\omega_{\gamma,\delta} \circ \varphi : B \to A_{[\delta]}).$$

Furthermore, if we have both $Q_\gamma = Q_\delta$ and $Q_{1/\gamma} = Q_{1/\delta}$ for some $\gamma, \delta \in P^{\mathrm{gp}}_\gamma$, a direct inspection yields a commutative diagram :

(17.4.11)
$$\begin{array}{ccc} A_{[\gamma]} \otimes_A A_{[1/\gamma]} & \xrightarrow{\omega_{\gamma,\delta}\otimes_A \omega_{1/\gamma,1/\delta}} & A_{[\delta]} \otimes_A A_{[1/\delta]} \\ & {\scriptstyle \mu_{[\gamma]}} \searrow \quad \swarrow {\scriptstyle \mu_{[\delta]}} & \\ & A. & \end{array}$$

For such γ and δ, there follows a commutative diagram :

$$\begin{array}{ccc} B^\vee_{[\gamma]} \otimes_A B^\vee_{[1/\gamma]} & \xrightarrow{\omega_{\gamma,\delta *}\otimes_A \omega_{1/\gamma,1/\delta *}} & B^\vee_{[\delta]} \otimes_A B^\vee_{[1/\delta]} \\ & \searrow \quad \swarrow & \\ & (B \otimes_A B)^\vee & \end{array}$$

whose downward arrows are the restrictions of $\xi_{\underline{f}}$: the details shall be left to the reader. Pick a section of the projection $P^{\mathrm{gp}}_{\mathbb{Q}} \to \Gamma_{\mathbb{Q}}$:

$$\Gamma_{\mathbb{Q}} \to P^{\mathrm{gp}}_{\mathbb{Q}} \subset P^{\mathrm{gp}}_{\mathbb{R}} \qquad [\gamma] \mapsto [\gamma]^*$$

whose image is contained in a *bounded* subset of the finite dimensional $\mathbb{R}$-vector space $P^{\mathrm{gp}}_{\mathbb{R}}$. Summing up, we see that if $Q_{[\gamma]^*} = Q_{[\delta]^*}$ and $Q_{[1/\gamma]^*} = Q_{[1/\delta]^*}$, then $\xi_{\underline{f}}(B^\vee_{[\gamma]} \otimes B^\vee_{[1/\gamma]}) = \xi_{\underline{f}}(B^\vee_{[\delta]} \otimes B^\vee_{[1/\delta]})$. To conclude, it suffices now to invoke proposition 6.3.35(i).

(ii): For every $(\gamma_1, \gamma_2) \in P^{\mathrm{gp}}_{\mathbb{Q}}$, let $[\gamma_i] \in \Gamma_{i,\mathbb{Q}}$ be the class of γ_i for $i = 1, 2$, and $[\gamma_1, \gamma_2] \in \Gamma_{\mathbb{Q}}$ the class of (γ_1, γ_2); also, for each $x \in \mathbb{Q}$, denote by $x^\dagger$ the smallest element of $(x + \mathbb{Z}) \cap \mathbb{Q}_+$. It is easily seen that

$$\mathrm{gr}_{[\gamma_1,\gamma_2]} P_{\mathbb{Q}} = (\mathrm{gr}_{[\gamma_1]} P_{1,\mathbb{Q}}) \times (\gamma_2^\dagger P_2) \qquad \text{where } \gamma_2^\dagger := (\gamma_{2,1}^\dagger, \dots, \gamma_{2,r}^\dagger)$$

whence a natural identification :

$$B^\vee_{[\gamma_1,0]} \otimes_{\mathbb{Z}[P]} \mathbb{Z}[\gamma_2^\dagger P_2] \xrightarrow{\sim} B^\vee_{[\gamma_1,\gamma_2]}. \tag{17.4.12}$$

Notice that for every $x \in \mathbb{Q}$ the rational number $x^\dagger + (-x)^\dagger$ equals 0 if $x \in \mathbb{Z}$, and otherwise it equals 1. Especially :

$$\gamma_2^\dagger + (-\gamma_2)^\dagger \in P_2 \qquad \text{for every } \gamma_2 \in P^{\mathrm{gp}}_{2,\mathbb{Q}}.$$

Therefore the multiplication law of $\mathbb{Z}[P_{2,\mathbb{Q}}]$ induces by restriction an A-linear map :

$$\mu_{\gamma_2} : \mathbb{Z}[\gamma_2^\dagger P_2] \otimes_{\mathbb{Z}[P]} \mathbb{Z}[(-\gamma_2)^\dagger P_2] \to \mathbb{Z}[P_2].$$

We then get a commutative diagram

$$\begin{array}{ccc} (B^\vee_{[\gamma_1,0]} \otimes_{\mathbb{Z}[P_2]} \mathbb{Z}[\gamma_2^\dagger P_2]) \otimes_A (B^\vee_{[1/\gamma_1,0]} \otimes_{\mathbb{Z}[P_2]} \mathbb{Z}[(-\gamma_2)^\dagger P_2]) & \longrightarrow & B^\vee_{[\gamma_1,\gamma_2]} \otimes_A B^\vee_{[1/\gamma_1,-\gamma_2]} \\ \Big\downarrow {\scriptstyle \xi_{\underline{f},[\gamma_1,0]} \otimes_{\mathbb{Z}[P_2]} \mathbb{Z}[\gamma_2^\dagger P_2] \otimes_{\mathbb{Z}[P_2]} \mathbb{Z}[(-\gamma_2)^\dagger P_2]} & & \Big\downarrow {\scriptstyle \xi_{\underline{f},[\gamma_1,\gamma_2]}} \\ (B \otimes_A B)^\vee \otimes_{\mathbb{Z}[P_2]} \mathbb{Z}[\gamma_2^\dagger P_2] \otimes_{\mathbb{Z}[P_2]} \mathbb{Z}[(-\gamma_2)^\dagger P_2] & \longrightarrow & (B \otimes_A B)^\vee \end{array}$$

where $\xi_{\underline{f},[\gamma_1,0]} : B^\vee_{[\gamma_1,0]} \otimes_A B^\vee_{[1/\gamma_1,0]} \to (B \otimes_A B)^\vee$ is the restriction of $\xi_{\underline{f}}$, and likewise for $\xi_{\underline{f},[\gamma_1,\gamma_2]}$. Also, the top horizontal arrow is the isomorphism induced by (17.4.12), and the bottom horizontal arrow is $(B \otimes_A B)^\vee \otimes_{\mathbb{Z}[P_2]} \mu_{\gamma_2}$. Summing up, we conclude that :

$$\xi_{\underline{f}}(B^\vee_{[\gamma_1,\gamma_2]} \otimes_A B^\vee_{[1/\gamma_1,-\gamma_2]}) \subset \xi_{\underline{f}}(B^\vee_{[\gamma_1,0]} \otimes_A B^\vee_{[1/\gamma_1,0]}) \qquad \text{for every } [\gamma_1, \gamma_2] \in P^{\mathrm{gp}}_{\mathbb{Q}}$$

whence the assertion. □

Example 17.4.13. Let V be a noetherian regular ring, P a fine and saturated monoid such that P^{gp} is torsion-free, $\psi : V[P] \to A$ a smooth ring homomorphism, and $\beta : P \to A$ the composition of ψ with the natural inclusion map $\alpha : P \to V[P]$. Set $S := \mathrm{Spec}\, V$, $X_0 := \mathrm{Spec}\, V[P]$ and $X := \mathrm{Spec}\, A$; then α is a chart for a log structure $\mathscr{P}_0$ on the Zariski site of X_0, and the induced morphism of log schemes $(X_0, \mathscr{P}_0) \to (S, \mathscr{O}_S^\times)$ is smooth, by virtue of proposition 12.3.34. Since $(S, \mathscr{O}_S^\times)$ is trivially a regular log scheme, it follows that the same holds for $(X_0, \mathscr{P}_0)$, by theorem 12.5.28; then, again by theorem 12.5.28 and corollary 12.3.27, the log scheme $(X, \mathscr{P}) := X \times_{X_0} (X_0, \mathscr{P}_0)$ is regular, and β is a chart for $\mathscr{P}$. Summing up, this shows that the datum (A, P, β) is a regular log ring.

Lemma 17.4.14. *Let A be a domain, B an A-algebra of finite type, C a B-algebra of finite type, M a B-module of finite type, and N a C-module of finite type. Set also $K := \mathrm{Frac}(A)$.*

(i) *There exists $f \in A \setminus \{0\}$ such that the following holds for every A-algebra A'. Set $B' := A' \otimes_A B$, $C' := A' \otimes_A C$, $M' := B' \otimes_B M$ and $N' := C' \otimes_C N$; then the natural map*

$$A' \otimes_A \mathrm{Hom}_B(M, N)_f \to \mathrm{Hom}_{B'}(M', N')_f$$

is an isomorphism.

(ii) *For every B-linear map $\varphi : M \to N$ there exists $f \in A \setminus \{0\}$ such that $(\mathrm{Coker}\, \varphi)_f$ is a free A-module.*

(iii) *In the situation of* (ii)*, let $x \in N$. Then $x \in \mathrm{Im}\,(K \otimes_A \varphi)$ if and only if there exists a dense subset $U \subset \mathrm{Spec}\, A$ such that $1 \otimes x \in \mathrm{Im}(\kappa(\mathfrak{p}) \otimes_A \varphi)$ for every $\mathfrak{p} \in U$.*

Proof. (i): Pick $n \in \mathbb{N}$ and a B-linear surjection $\psi : B^{\oplus n} \to M$; let also $M' \subset B^{\oplus n}$ be a finitely generated B-submodule such that $K \otimes_A M' = \mathrm{Ker}\,(K \otimes_A \psi)$. Set $M'' := B^{\oplus n}/M'$, and denote by $\overline{\psi} : M'' \to M$ the B-linear surjection induced by ψ. By [65, Ch.IV, Lemma 8.9.4.1], there exists $f \in A \setminus \{0\}$ such that the A_f-module M''_f is flat, and by construction $\mathrm{Ker}(\overline{\psi}_f : M''_f \to M_f)$ is a torsion A-module; then $\mathrm{Ker}\, \overline{\psi}_f = 0$, hence M_f is a finitely presented B_f-module. We may thus replace A, B, C, M, N by A_f, B_f, C_f, M_f, N_f, and assume that M is a B-module of finite presentation. Now, let

$$B^{\oplus m} \xrightarrow{\varphi} B^{\oplus n} \to M$$

be a finite presentation of M; there follow short exact sequences of B-modules :

$$\begin{aligned} \Sigma \quad &: \quad 0 \to \mathrm{Hom}_B(M, N) \to N^{\oplus n} \to T := \mathrm{Im}(\varphi^\vee \otimes_B N) \to 0 \\ \Sigma' \quad &: \quad 0 \to T \to N^{\oplus m} \to T' := \mathrm{Coker}(\varphi^\vee \otimes_B N) \to 0 \end{aligned}$$

where $\varphi^\vee : B^{\oplus n} \to B^{\oplus m}$ is the transpose of φ. Notice that T and T' are C-modules of finite type. By invoking again [65, Ch.IV, Lemma 8.9.4.1], we find $f \in A \setminus \{0\}$ such that T_f and T'_f are flat A_f-modules; after replacing again A, B, C, M, N and φ by their respective localizations, we may therefore assume that T and T' are flat A-modules. In this case, the induced sequences $A' \otimes_A \Sigma$ and $A' \otimes_A \Sigma'$ are again short exact, for every A-algebra A', whence a left exact sequence

$$0 \to A' \otimes_A \mathrm{Hom}_B(M, N) \to N'^{\oplus n} \xrightarrow{\varphi'^\vee \otimes_{B'} N'} N'^{\oplus m} \qquad \text{with } \varphi' := \varphi \otimes_B B'.$$

But we have a natural identification : $\mathrm{Ker}(\varphi'^\vee \otimes_{B'} N') \xrightarrow{\sim} \mathrm{Hom}_{B'}(M', N')$, and the resulting isomorphism $A' \otimes_A \mathrm{Hom}_B(M, N) \xrightarrow{\sim} \mathrm{Hom}_{B'}(M', N')$ is the natural map of (i).

(ii): Arguing as in the proof of (i), we reduce easily to the case where B and C are finitely presented A-algebras, and M (resp. N) is a finitely presented B-module (resp. C-module). Then we may find a $\mathbb{Z}$-subalgebra of finite type $A_0 \subset A$, an A_0-algebra of finite type B_0, a B_0-algebra of finite type C_0, a B_0-module M_0 of finite type, a C_0-module of finite type N_0, and a B_0-linear map $\varphi_0 : M_0 \to N_0$ with isomorphisms

$$A \otimes_{A_0} B_0 \xrightarrow{\sim} B \qquad A \otimes_{A_0} C_0 \xrightarrow{\sim} C \qquad B \otimes_{B_0} M_0 \xrightarrow{\sim} M \qquad C \otimes_{C_0} N_0 \xrightarrow{\sim} N.$$

which identify $B \otimes_{B_0} \varphi_0$ with φ. It suffices then to exhibit $f \in A_0$ such that $(\mathrm{Coker}\,\varphi_0)_f$ is a free $A_{0,f}$-module; the latter follows easily from [126, Th.24.1].

(iii): If $x \in \mathrm{Im}\,(K \otimes_A \varphi)$, we have $x \in \mathrm{Im}\,\varphi_f$ for some $f \in A \setminus \{0\}$, and then $x \otimes 1 \in \mathrm{Im}(\kappa(\mathfrak{p}) \otimes_A \varphi)$ for every $\mathfrak{p} \in \mathrm{Spec}\,A_f$. Conversely, suppose that $U \subset \mathrm{Spec}\,A$ is a dense subset such that $x \otimes 1 \in \mathrm{Im}(\kappa(\mathfrak{p}) \otimes_A \varphi)$ for every $\mathfrak{p} \in U$. We pick $f \in A \setminus \{0\}$ as in (ii), so that $(\mathrm{Coker}\,\varphi)_f$ is a free A-module. Clearly $U \cap \mathrm{Spec}\,A_f$ is still dense in $\mathrm{Spec}\,A_f$, hence we may replace A, B, C, M, N and φ by A_f, B_f, C_f, M_f, N_f and φ_f, and assume that $\mathrm{Coker}\,\varphi$ is a free A-module, say with basis $(e_i \mid i \in I)$. Suppose now that $x \notin \mathrm{Im}\,(K \otimes_A \varphi)$, so that the image $\overline{x} \in \mathrm{Coker}\,\varphi$ of x does not vanish. Then there exists a finite subset $I_0 \subset I$ and a system $(f_i \mid i \in I_0)$ of non-zero elements of A, with $\overline{x} = \sum_{i \in I_0} f_i e_i$. Pick $j \in I_0$, and any $\mathfrak{p} \in U \cap \mathrm{Spec}\,A_{f_j}$; then $\mathrm{Coker}\,(\kappa(\mathfrak{p}) \otimes_A \varphi)$ is a free $\kappa(\mathfrak{p})$-vector space with basis $(1 \otimes e_i \mid i \in I)$, and $1 \otimes x = \sum_{i \in I_0} \overline{f}_i \otimes e_i$, where $\overline{f}_i \in \kappa(\mathfrak{p})$ denotes the image of f_i, for every $i \in I$. Especially, $\overline{f}_j \neq 0$, whence $1 \otimes x \neq 0$, a contradiction. □

17.4.15. Let $(P, +, 0)$ be a fine and saturated monoid such that P^{gp} is torsion-free, and fix a Banach norm $\|\cdot\|$ on the finite dimensional $\mathbb{R}$-vector space $P^{\mathrm{gp}}_{\mathbb{R}} := \mathbb{R} \otimes_{\mathbb{Z}} P^{\mathrm{gp}}$. Let also

$$P^{\mathrm{gp}}_{\mathbb{R}}(\rho) := \{\gamma \in P^{\mathrm{gp}}_{\mathbb{R}} \mid \|\gamma\| \leq \rho\} \qquad \text{for all } \rho \in \mathbb{R}_+.$$

Define also $P^{(\infty)}$ and $P_{\mathbb{Q}}$ as in (17.4.3), as well as the polyhedral cone $P_{\mathbb{R}} := \mathbb{R}_+ \otimes_{\mathbb{Z}} P \subset P^{\mathrm{gp}}_{\mathbb{R}}$. Moreover, for every $m \in \mathbb{N} \setminus \{0\}$, let

$$P_{(m)} := \{\gamma \in P_{\mathbb{Q}} \mid m\gamma \in P^{(\infty)}\}.$$

Lemma 17.4.16. *With the notation of* (17.4.15)*, there exists an integer $m > 0$ with $(p, m) = 1$ such that the following holds. For every $\delta > 0$ there exists a real number $\varepsilon > 0$ such that for every $\beta_1, \beta_2 \in P^{\mathrm{gp}}_{\mathbb{R}}$ with $\|\beta_1 + \beta_2\| < \varepsilon$ we may find $\beta'_1, \beta'_2 \in P^{\mathrm{gp}}_{(m)}$ with :*

(i) $\beta'_1 + \beta'_2 = 0$
(ii) $\|\beta'_i - \beta_i\| < \delta$ *for* $i = 1, 2$
(iii) $(P_{\mathbb{R}} - \beta_i) \cap P^{\mathrm{gp}} \subset (P_{\mathbb{R}} - \beta'_i) \cap P^{\mathrm{gp}}$ *for* $i = 1, 2$.

Proof. Since $P^{(\infty)\mathrm{gp}}$ is p-divisible, $P^{\mathrm{gp}}_{(m)} = P^{\mathrm{gp}}_{(pm)}$ for every integer $m > 0$, so we may always arrange that $(p, m) = 1$, if all the other conditions are already fulfilled. Moreover, let $\rho > 0$ such that every class of $P^{\mathrm{gp}}_{\mathbb{R}}/P^{\mathrm{gp}}$ admits a representative $\beta \in P^{\mathrm{gp}}_{\mathbb{R}}$ with $\|\beta\| \leq \rho$. Then it is easily seen that, after replacing β_i by $\beta_i + (-1)^i\gamma$ for $i = 1, 2$, with a suitable $\gamma \in P^{\mathrm{gp}}$, we may assume that $\beta_1 \in P^{\mathrm{gp}}_{\mathbb{R}}(\rho)$. Now, for every $\gamma \in P^{\mathrm{gp}}_{\mathbb{R}}$, let $\overline{\Omega}(\gamma)$ be the topological closure of the subset $\Omega(P_{\mathbb{R}}, P^{\mathrm{gp}} \cap (P_{\mathbb{R}} - \gamma))$ in $P^{\mathrm{gp}}_{\mathbb{R}}$ (notation of (6.3.34)); in view of condition (i) and of proposition 6.3.35(v), condition (iii) is implied by :

$$\beta'_1 \in \overline{\Omega}(\beta_1) \cap (-\overline{\Omega}(\beta_2)). \tag{17.4.17}$$

Moreover, suppose that

$$\|\beta'_1 - \beta_1\| \leq \delta/2. \tag{17.4.18}$$

Then

$$\|\beta'_2 - \beta_2\| = \|\beta'_2 + \beta_1 - \beta_1 - \beta_2\| \leq \| - \beta'_1 + \beta_1\| + \|\beta_1 + \beta_2\| < \delta/2 + \varepsilon.$$

Hence condition (ii) will hold, provided $\varepsilon \leq \delta/2$. Thus, we have to exhibit $m > 0$ and $\varepsilon > 0$ such that, for every $\beta_1, \beta_2 \in P^{\mathrm{gp}}_{\mathbb{R}}$ with $\|\beta_1\| \leq \rho$ and $\|\beta_1 + \beta_2\| < \varepsilon$, there exists $\beta'_1 \in P^{\mathrm{gp}}_{(m)}$ fulfilling (17.4.17) and (17.4.18). Then, by proposition 6.3.35(i) and lemma 6.3.42, it suffices to find $\beta'_1 \in P^{\mathrm{gp}}_{\mathbb{R}}$ fulfilling these two latter conditions. By way of contradiction, suppose that such β'_1 cannot always be found : this means that there exists a sequence $\underline{\beta} := ((\beta_{1,k}, \beta_{2,k}) \mid k \in \mathbb{N})$ of pairs of elements in $P^{\mathrm{gp}}_{\mathbb{R}}$, with $\beta_{1,k} \in P^{\mathrm{gp}}_{\mathbb{R}}(\rho)$ for every $k \in \mathbb{N}$, and such that

(a) $\|\beta_{1,k} + \beta_{2,k}\| < 2^{-k}$ for every $k \in \mathbb{N}$

(b) $\overline{\Omega}(\beta_{1,k}) \cap (-\overline{\Omega}(\beta_{2,k})) \cap (P^{\mathrm{gp}}_{\mathbb{R}}(\delta/2) + \beta_{1,k}) = \varnothing$ for every $k \in \mathbb{N}$.

However, by proposition 6.3.35(i), after replacing $\underline{\beta}$ by a subsequence we may assume that both $\overline{\Omega}(\beta_{1,k})$ and $\overline{\Omega}(\beta_{2,k})$ are independent of k. Since $P^{\mathrm{gp}}_{\mathbb{R}}(\rho)$ is a compact subset, we may also assume that the sequence $(\beta_{1,k} \mid k \in \mathbb{N})$ converges to an element $\beta'_1 \in P^{\mathrm{gp}}_{\mathbb{R}}(\rho)$; then clearly $(\beta_{2,k} \mid k \in \mathbb{N})$ converges to $\beta'_2 := -\beta'_1$. Since $\beta_{i,k} \in \overline{\Omega}(\beta_{i,k})$ for $i = 1, 2$ and every $k \in \mathbb{N}$, we also have $\beta'_i \in \overline{\Omega}(\beta_{i,k})$ for $i = 1, 2$. Lastly, we have $\|\beta'_1 - \beta_{1,k}\| < \delta/2$ for every sufficiently large $k \in \mathbb{N}$; this contradicts (b), and the claim follows. □

Theorem 17.4.19. *In the situation of* (17.4.5)*, suppose that* (A, P, β) *is regular. We have :*

(i) $e_f \in \vartheta_{\underline{f}}(\mathsf{R}(B, \Gamma_{\mathbb{Q}}))$.

(ii) *Let* $J \subset A$ *be the radical ideal such that* $\mathrm{Spec}\, A/J = X \setminus U_f$*. Then* $Je_f \subset \vartheta_{\underline{f}}(\mathsf{R}(B, \Gamma))$.

Proof. For every datum $\underline{f} := (f, \beta, P)$ as in (17.4.5), let $\mathsf{X}(\underline{f}) := \Gamma(U_f, \mathscr{B} \otimes_{\mathscr{O}_X} \mathscr{B})$; denote also by $\mathsf{Y}(\underline{f}) \subset \mathsf{X}(\underline{f})$ the image of $\vartheta_{\underline{f}}$, and set $\mathsf{Z}(\underline{f}) := Ae_f \subset \mathsf{X}(\underline{f})$. We notice :

Claim 17.4.20. Let A' be a noetherian ring, $g : A \to A'$ a flat ring homomorphism; set $X' := \mathrm{Spec}\, A'$, $f' := A' \otimes_A f$ and $\underline{f}' := (f', P, g \circ \beta)$. Then $U_{f'} = X' \times_X U_f$, and we have a natural isomorphism

$$A' \otimes_A \mathsf{X}(\underline{f}) \xrightarrow{\sim} \mathsf{X}(\underline{f}')$$

that identifies $\mathsf{Y}(\underline{f}')$ and $\mathsf{Z}(\underline{f}')$ respectively with $A' \otimes_A \mathsf{Y}(\underline{f})$ and $A' \otimes_A \mathsf{Z}(\underline{f})$.

Proof of the claim. The first assertion follows from [66, Ch.IV, Prop.17.7.1]. Next, set $B' := A' \otimes_A B$; since B is an A-module of finite type, B' is an A'-module of finite presentation. Hence, the natural map $A' \otimes_A B^\vee \to B'^\vee := \mathrm{Hom}_{A'}(B', A')$ is an isomorphism (details left to the reader). Then the second assertion follows by a direct inspection of the constructions. ◇

Now, assertion (i) comes down to the inclusion $\mathsf{Z}(\underline{f}) \subset \mathsf{Y}(\underline{f})$, and it suffices to check that $\mathsf{Z}(\underline{f})_{\mathfrak{p}} \subset \mathsf{Y}(\underline{f})_{\mathfrak{p}}$ in $\mathsf{X}(\underline{f})_{\mathfrak{p}}$, for every $\mathfrak{p} \in \mathrm{Spec}\, A$. Let $j_{(\mathfrak{p})} : A \to A_{\mathfrak{p}}$ be the localization map, set $f_{\mathfrak{p}} := A_{\mathfrak{p}} \otimes_A f : A_{\mathfrak{p}} \to B_{\mathfrak{p}}$, and $\underline{f}_{\mathfrak{p}} := (f_{\mathfrak{p}}, P, \beta \circ j_{(\mathfrak{p})})$; by claim 17.4.20, we are then reduced to checking that $\mathsf{Z}(\underline{f}_{\mathfrak{p}}) \subset \mathsf{Y}(\underline{f}_{\mathfrak{p}})$ in $\mathsf{X}(\underline{f}_{\mathfrak{p}})$, for every such $\mathfrak{p}$. Thus, in order to show (i), *we may assume that* $(A, \mathfrak{m})$ *is a local noetherian ring*. Likewise we may reduce assertion (ii) to the case where $(A, \mathfrak{m})$ is local. Next, let $A^\wedge$ be the $\mathfrak{m}$-adic completion of A; since the completion map $j : A \to A^\wedge$ is faithfully flat, we are reduced to checking that $A^\wedge \otimes_A \mathsf{Z}(\underline{f}) \subset A^\wedge \otimes_A \mathsf{Y}(\underline{f})$ in $A^\wedge \otimes_A \mathsf{X}(\underline{f})$. Moreover, the log structure attached to the chart $j \circ \beta : P \to A^\wedge$ is still regular (theorems 12.5.10 and 12.5.31); after invoking again claim 17.4.20, we may thus *assume that* $(A, \mathfrak{m})$ *is a complete noetherian local ring* in order to prove (i), and a similar argument reduces the proof of (ii) as well to the complete case.

Next, let $\mathfrak{p} := \beta^{-1}(A^\times)$; then β extends uniquely to a morphism of monoids $\beta' : P_{\mathfrak{p}} \to A$ (notation of remark 6.1.15(i)); moreover, the log structures on the Zariski site of X induced by β and by β' are naturally isomorphic, so the datum $\underline{f}' := (f, P_{\mathfrak{p}}, \beta')$ still fulfills the conditions of (17.4.5) and $(A, P_{\mathfrak{p}}, \beta')$ is regular. Moreover, $(P_{\mathfrak{p}})^{\mathrm{gp}} = P^{\mathrm{gp}}$ and $(P_{\mathfrak{p}})^{\mathrm{gp}}_{\mathbb{Q}} = P^{\mathrm{gp}}_{\mathbb{Q}}$, hence $(P_{\mathfrak{p}})_{\mathbb{Q}}$ is still naturally Γ-graded. Furthermore, for every $\gamma \in P^{\mathrm{gp}}_{\mathbb{Q}}$ we have

$$\mathrm{gr}_{[\gamma]}((P_{\mathfrak{p}})_{\mathbb{Q}}) = (\mathrm{gr}_{[\gamma]}(P_{\mathbb{Q}}))_{\mathfrak{p}} \qquad \text{whence :} \qquad A \otimes_{\mathbb{Z}[P_{\mathfrak{p}}]} \mathbb{Z}[\mathrm{gr}_{[\gamma]}((P_{\mathfrak{p}})_{\mathbb{Q}})] = A_{[\gamma]}.$$

Hence, the A-modules $\mathsf{X}(\underline{f}'), \mathsf{Y}(\underline{f}'), \mathsf{Z}(\underline{f}')$ are naturally identified with $\mathsf{X}(\underline{f}), \mathsf{Y}(\underline{f}), \mathsf{Z}(\underline{f})$. Thus, in order to prove (i), it suffices to show that $\mathsf{Z}(\underline{f}') \subset \mathsf{Y}(\underline{f}')$; after replacing $\underline{f}$ by $\underline{f}'$, we may therefore *assume that the chart* β *is local at the closed point of* X (see definition 12.1.17(vi)). Arguing likewise we reduce as well the proof of (ii) to the case where β is local.

Recall now that P admits a decomposition $P \xrightarrow{\sim} Q \times P^\times$, where Q is a fine, sharp and saturated monoid, and $P^\times \subset P$ is the abelian group of invertible elements of P (lemma 6.2.10). Let $\alpha : Q \to A$ be the restriction of β; by direct inspection, we see that the inclusion $Q \to P$

induces an isomorphism between the log structures on $\mathrm{Spec}\, A$ associated with the charts (P, β) and (Q, α) : see (12.1.6); hence, the datum $\underline{f}'' := (f, Q, \alpha)$ fulfills again the conditions of (17.4), and (A, Q, α) is regular. In light of lemma 17.4.10(ii), we may then replace $\underline{f}$ by $\underline{f}''$, and thus *assume that* β *is a sharp chart at the closed point of* X (see definition 12.1.17(vi)).

After these preliminaries, we may then assume, as in (17.3.4), that there exists $r \in \mathbb{N}$, a coefficient ring $(V, \mathfrak{m}_V)$ of A, and a surjective ring homomorphism

$$\varphi : V[[P \times \mathbb{N}^{\oplus r}]] \to A$$

and β is the composition of φ with the natural inclusion map $P \to V[[P \times \mathbb{N}^{\oplus r}]]$.

Claim 17.4.21. The theorem holds if A contains a field of positive characteristic.

Proof of the claim. In this case, V is the residue field of A, and φ is an isomorphism. The induced morphism of monoids $\beta' : P \times \mathbb{N}^{\oplus r} \to V[[P \times \mathbb{N}^{\oplus r}]] \to A$ is the chart for another regular log structure on the Zariski site of X (theorems 12.5.10 and 12.5.31); in view of lemma 17.4.10(ii) we may then replace β by β', and assume that $A = V[[P]]$ and $\mathfrak{m} = \mathfrak{m}_P \cdot A$ (with $\mathfrak{m}_P := P \setminus \{1\}$). Next, let k be an algebraic closure of V; the induced map $V[P] \to k[P]$ is faithfully flat, so the same holds for its completion $A \to k[[P]]$ ([126, Th.22.4]); moreover, the inclusion map $P \to k[[P]]$ is again a chart for a regular log structure on the Zariski site of $\mathrm{Spec}\, k[[P]]$. In view of claim 17.4.20, we may then replace V by k and B by $k[[P]] \otimes_A B$, and assume that $A = k[[P]]$. In this case, notice that, with the notation of (9.8.11), we have :

$$A^{\mathrm{perf}} = \Gamma \times_{\Gamma_{\mathbb{Q}}} A_\infty.$$

Let now $h \in J$. The isomorphism (9.8.12) yields a natural identification of A^{perf}-algebras :

$$\Gamma \times_{\Gamma_{\mathbb{Q}}} B_\infty[h^{-1}] \xrightarrow{\sim} B^{\mathrm{perf}}[h^{-1}]$$

and then remark 17.4.1 shows that $h^{1/p^n} e_f$ lies in the image of the induced map

$$B^{\mathrm{perf}} \otimes_{A^{\mathrm{perf}}} B^{\mathrm{perf}} = (B \otimes_A B)^{\mathrm{perf}} \to B_\infty \otimes_{A_\infty} B_\infty[h^{-1}] \to \Gamma(U_f, \mathscr{B}_\infty \otimes_{\mathscr{O}_{X_\infty}} \mathscr{B}_\infty) \tag{17.4.22}$$

for every $n \in \mathbb{N}$. On the other hand, B^{perf} is an integral A^{perf}-algebra, and A^{perf} is a normal domain, hence the trace map of the finite étale map $A^{\mathrm{perf}}[h^{-1}] \to B^{\mathrm{perf}}[h^{-1}]$ restricts to an A^{perf}-linear map $B^{\mathrm{perf}} \to A^{\mathrm{perf}}$ (lemma 14.3.15(i)) which in turns induces an A^{perf}-linear map

$$\omega_{B^{\mathrm{perf}}/A^{\mathrm{perf}}} : B^{\mathrm{perf}} \to \mathrm{Hom}_{A^{\mathrm{perf}}}(B^{\mathrm{perf}}, A^{\mathrm{perf}}) \to B_\infty^\vee$$

such that

$$A_\infty[h^{-1}] \otimes_{A^{\mathrm{perf}}} \omega_{B^{\mathrm{perf}}/A^{\mathrm{perf}}} = \Gamma(\mathrm{Spec}\, A[h^{-1}], \omega_{\mathscr{B}_\infty/\mathscr{O}_{X_\infty}}).$$

A simple inspection then shows that (17.4.22) is the composition of $\omega_{B^{\mathrm{perf}}/A^{\mathrm{perf}}} \otimes_{A^{\mathrm{perf}}} \omega_{B^{\mathrm{perf}}/A^{\mathrm{perf}}}$ with (17.4.6). Summing up, this proves that $h^{1/p^n} e_f$ lies in the image of the map (17.4.6) for every $n \in \mathbb{N}$, and especially, $he_f \in \vartheta_{\underline{f}}(\mathsf{R}(B, \Gamma))$, whence assertion (ii).

In order to show (i), say that $h = \sum_{\gamma \in P} h_\gamma \cdot \gamma$ for a system $(h_\gamma \mid \gamma \in P)$ of elements of k, and pick $\gamma_0 \in P$ with $h_{\gamma_0} \neq 0$. According to claim 17.3.7, there exists $n \in \mathbb{N}$ such that $\mathrm{gr}_{[\gamma_0^{1/p^n}]} P_{\mathbb{Q}} = \gamma_0^{1/p^n} P$; then we have an A-linear isomorphism :

$$\mathsf{X}(\underline{f}) \xrightarrow{\sim} \mathrm{gr}_{[\gamma_0^{1/p^n}]} \Gamma(U_f, \mathscr{B}_\infty \otimes_{\mathscr{O}_{X_\infty}} \mathscr{B}_\infty) = \mathsf{X}(\underline{f}) \otimes_A A_{[\gamma_0^{1/p^n}]} \qquad x \mapsto x \otimes \gamma_0^{1/p^n}. \tag{17.4.23}$$

We have already remarked that $h^{1/p^n} e_f$ lies in the image of (17.4.6); let us write $h^{1/p^n} = \sum_{[\gamma] \in \Gamma} h_{[\gamma]}$, with $h_{[\gamma]} \in A_{[\gamma]}$ for every $[\gamma] \in \Gamma$; then $h^{1/p^n} = \sum_{\gamma \in P} h_\gamma^{1/p^n} \cdot \gamma^{1/p^n}$, and $h_{[\gamma_0^{1/p^n}]} = h_{\gamma_0}^{1/p^n} \cdot \gamma_0^{1/p^n} \cdot a$ for some $a \in 1 + \mathfrak{m}$. Hence, $h_{[\gamma_0^{1/p^n}]} e_f$ is in the image of $\mathrm{gr}_{[\gamma_0^{1/p^n}]}(B_\infty^\vee \otimes_{A_\infty} B_\infty^\vee)$, and so the same holds for $e_f \otimes \gamma_0^{1/p^n}$. Thus, there exists a finite subset $\Sigma \subset \Gamma$ and an element of

$\bigoplus_{[\gamma]\in\Sigma} B^\vee_{[\gamma]} \otimes_A B^\vee_{[\gamma_0^{1/p^n}/\gamma]}$ whose image under the map (17.4.6) equals $e_f \otimes \gamma_0^{1/p^n}$. Then, lemma 17.4.16 yields for every $[\gamma] \in \Sigma$ an element $[\gamma'] \in \Sigma_{\mathbb{Q}}$ such that :

$$\gamma^{-1}P_{\mathbb{Q}} \cap P^{\mathrm{gp}} \subset \gamma'^{-1}P_{\mathbb{Q}} \cap P^{\mathrm{gp}} \qquad \text{and} \qquad (\gamma/\gamma_0^{1/p^n})P_{\mathbb{Q}} \cap P^{\mathrm{gp}} \subset \gamma' P_{\mathbb{Q}} \cap P^{\mathrm{gp}}.$$

With $\Sigma' := \{[\gamma'] \mid [\gamma] \in \Sigma\}$, we finally deduce a commutative diagram of A-modules :

$$\begin{array}{ccc} \bigoplus_{[\gamma]\in\Sigma} B^\vee_{[\gamma]} \otimes_A B^\vee_{[\gamma_0^{1/p^n}/\gamma]} & \longrightarrow & \mathsf{X}(\underline{f}) \otimes_A A_{[\gamma_0^{1/p^n}]} \\ \downarrow & & \uparrow \\ \mathsf{R}(B, \Sigma') & \longrightarrow & \mathsf{X}(\underline{f}) \end{array}$$

whose right vertical arrow is the isomorphism (17.4.23) and whose bottom horizontal arrow is the restriction of $\vartheta_{\underline{f}}$; it follows easily that $e_f \in \vartheta_{\underline{f}}(\mathsf{R}(B, \Sigma'))$, as required. ◊

Suppose next that A is a $\mathbb{Q}$-algebra, so that V is a field of characteristic zero, and φ is an isomorphism. We pick a complete discrete valuation ring W with residue field V, and a surjective map of monoids $\pi : \mathbb{N}^{\oplus r'} \to P$; for $s := r + r'$ we get a surjective map of W-algebras $W[\mathbb{N}^{\oplus s}] \to A_0 := V[P \times \mathbb{N}^{\oplus r}]$ whose completion is a surjective map $W[[\mathbb{N}^{\oplus s}]] \to A$. According to [32, §3.6, Th.12], the completion map $W[\mathbb{N}^{\oplus s}] \to W[[\mathbb{N}^{\oplus s}]]$ is the colimit of a filtered system $(R_\lambda \mid \lambda \in \Lambda)$ of smooth $W[\mathbb{N}^{\oplus s}]$-algebras; set $A_\lambda := R_\lambda \otimes_{W[\mathbb{N}^{\oplus s}]} A_0$ for every $\lambda \in \Lambda$. Then A is the colimit of the filtered system of smooth A_0-algebras $A_\bullet := (A_\lambda \mid \lambda \in \Lambda)$. For every $\lambda \in \Lambda$, let X'_λ be the connected component of $\operatorname{Spec} A_\lambda$ containing the image of X, and let A'_λ be the quotient of A_λ with $\operatorname{Spec} A'_\lambda = X'_\lambda$. It is easily seen that the colimit of the induced system $A'_\bullet := (A'_\lambda \mid \lambda \in \Lambda)$ is still A; after replacing $A_\bullet$ by $A'_\bullet$, we may then assume that A_λ is a domain for every $\lambda \in \Lambda$. Next, after replacing Λ by a cofinal subset, we may assume that for every $\lambda \in \Lambda$ there exists an open subset $U_\lambda \subset X_\lambda$ such that $U_f = X \times_{X_\lambda} U_\lambda$, and $U_\mu = X_\mu \times_{X_\lambda} U_\lambda$ for every $\mu \in \Lambda$ with $\mu \geq \lambda$. We may then find $\lambda \in \Lambda$ and a finite morphism of schemes $\varphi_\lambda : Y_\lambda \to X_\lambda$ with an isomorphism of X-schemes $X \times_{X_\lambda} Y_\lambda \xrightarrow{\sim} \operatorname{Spec} B$. Then there exists $\mu \in \Lambda$ with $\mu \geq \lambda$ such that $U_\mu \times_{X_\lambda} Y_\lambda$ is a finite étale U_μ-scheme; set $Y_\mu := X_\mu \times_{X_\lambda} Y_\lambda$, and $B_\mu := \mathscr{O}_{Y_\mu}(Y_\mu)$. By construction, the morphism $Y_\mu \to X_\mu$ is surjective; since A_μ is a domain, it follows easily that the induced ring homomorphism $f_\mu : A_\mu \to B_\mu$ is injective. Let $\beta_\mu : P \to A_0 \to A_\mu$ be the natural map; by example 17.4.13, the datum (A_μ, P, β_μ) is a regular log ring, and $\underline{f}_\mu := (f_\mu, P, \beta_\mu)$ is a datum as in (17.4.5). In light of remark 17.4.7(i), it then suffices to prove assertions (i) and (ii) for the map $\vartheta_{\underline{f}_\mu}$; the latter is covered by the following :

Claim 17.4.24. The theorem holds if there exists a field K of characteristic zero and a smooth ring homomorphism $\psi : K[P] \to A$, such that β is the composition of ψ and the inclusion map $P \to K[P]$ (see example 17.4.13).

Proof of the claim. The field K is the colimit of the filtered system of its smooth $\mathbb{Z}$-subalgebras $(V_\lambda \mid \lambda \in \Lambda)$. Then we may find $\lambda \in \Lambda$, a smooth $V_\lambda[P]$-algebra A_λ, and a finite, injective and generically finite ring homomorphism $f_\lambda : A_\lambda \to B_\lambda$ with isomorphisms

$$K \otimes_{V_\lambda} A_\lambda \xrightarrow{\sim} A \qquad A \otimes_{A_\lambda} B_\lambda \xrightarrow{\sim} B$$

(details left to the reader). Let $\alpha : P \to V_\lambda[P] \to A_\lambda$ be the natural map; since V_λ is a regular noetherian ring, the datum (A_λ, P, α) is a regular log ring, by example 17.4.13, and in light of remark 17.4.7(i), it suffices to show assertions (i) and (ii) for the map $\vartheta_{\underline{f}_\lambda}$ corresponding to the datum $\underline{f}_\lambda := (f_\lambda, P, \alpha)$. Now, for every $\mathfrak{p} \in \operatorname{Spec} V_\lambda$, every V_λ-module M, let $M(\mathfrak{p}) := \kappa(\mathfrak{p}) \otimes_{A_\lambda} M$; also, for every homomorphism of V_λ-modules $h : M \to N$ let likewise $h_{(\mathfrak{p})} := \kappa(\mathfrak{p}) \otimes_{V_\lambda} h : M(\mathfrak{p}) \to N(\mathfrak{p})$; moreover, let $\alpha_{(\mathfrak{p})} : P \to A_\lambda(\mathfrak{p})$ be the composition of α with the natural map $A_\lambda \to A_\lambda(\mathfrak{p})$. Let also $a \in A_\lambda \setminus \{0\}$ such that $\operatorname{Spec} A_\lambda[a^{-1}] \subset U_{f_\lambda}$. According to [65, Ch.IV, Prop.9.5.3], the set $Z := \{\mathfrak{p} \in \operatorname{Spec} V_\lambda \mid \operatorname{Spec} A_\lambda[a^{-1}](\mathfrak{p}) \text{ is dense in } \operatorname{Spec} A_\lambda(\mathfrak{p})\}$

is *constructible* in $\mathrm{Spec}\, V_\lambda$; furthermore, Z contains the generic point of $\mathrm{Spec}\, V_\lambda$, so it is *dense* in $\mathrm{Spec}\, V_\lambda$. For every $\mathfrak{p} \in Z$, the datum $\underline{f}_{\lambda,(\mathfrak{p})} := (f_{\lambda,(\mathfrak{p})}, P, \alpha_{(\mathfrak{p})})$ fulfills therefore the conditions of (17.4), and moreover $(A_\lambda(\mathfrak{p}), P, \alpha_{(\mathfrak{p})})$ is still a regular log ring. Set $X_\lambda := \mathrm{Spec}\, A_\lambda$, and denote by $\mathscr{B}_\lambda$ the quasi-coherent $\mathscr{O}_{X_\lambda}$-algebra arising from B_λ. By remark 17.4.7(i), for Δ either equal to $\Gamma_\mathbb{Q}$ or to Γ we have a commutative diagram :

$$\begin{array}{ccc} \mathsf{R}(B_\lambda, \Delta) & \xrightarrow{\vartheta'_\Delta} & B_\lambda \otimes_{A_\lambda} B_\lambda[a^{-1}] \\ \downarrow & & \downarrow \\ \mathsf{R}(B_\lambda(\mathfrak{p}, \Delta)) & \xrightarrow{\vartheta'^{(\mathfrak{p})}_\Delta} & B_\lambda \otimes_{A_\lambda} B_\lambda(\mathfrak{p})[a^{-1}] \end{array}$$

whose right vertical arrow maps e_{f_λ} to $e_{f_{\lambda,(\mathfrak{p})}}$, and where ϑ'_Δ is the composition of the restriction of $\vartheta_{\underline{f}_\lambda}$ to $\mathsf{R}(B, \Delta)$ with the restriction map $\Gamma(U_{f_\lambda}, \mathscr{B}_\lambda \otimes_{\mathscr{O}_{X_\lambda}} \mathscr{B}_\lambda) \to B_\lambda \otimes_{A_\lambda} B_\lambda[a^{-1}]$, and likewise $\vartheta'^{(\mathfrak{p})}_\Delta$ is deduced from $\vartheta_{\underline{f}_{\lambda,(\mathfrak{p})}}$. Lemma 17.4.10(i) yields a finite subset $\Sigma \subset \Delta$ such that

$$\vartheta_{\underline{f}_{\lambda,(\mathfrak{p})}}(\mathsf{R}(B(\mathfrak{p}), \Sigma)) = \vartheta_{\underline{f}_{\lambda,(\mathfrak{p})}}(\mathsf{R}(B(\mathfrak{p}), \Delta)) \qquad \text{for every } \mathfrak{p} \in Z.$$

On the other hand, in view of lemma 17.4.14(i), after replacing V_λ by a suitable localization $V_\lambda[t^{-1}]$ (with $t \neq 0$), and Z by $Z \cap \mathrm{Spec}\, V[t^{-1}]$, we may assume that the natural maps

$$(B_\lambda^\vee)_{[\gamma]}(\mathfrak{p}) \to (B_\lambda(\mathfrak{p}))^\vee_{[\gamma]} \qquad (B_\lambda^\vee)_{[1/\gamma]}(\mathfrak{p}) \to (B_\lambda(\mathfrak{p}))^\vee_{[1/\gamma]}$$

are isomorphisms, for every $\mathfrak{p} \in Z$ and every $[\gamma] \in \Sigma$. Summing up, we may assume that *the image of* $\vartheta'^{(\mathfrak{p})}_\Delta$ *equals the image of* $\vartheta'_{\Delta,(\mathfrak{p})}$, *for every* $\mathfrak{p} \in Z$. Lastly, recall that the set Z' of all $\mathfrak{p} \in \mathrm{Spec}\, V_\lambda$ such that $\kappa(\mathfrak{p})$ is a finite field is *dense in the constructible topology of* $\mathrm{Spec}\, V_\lambda$, since V_λ is a $\mathbb{Z}$-algebra of finite type ([65, Ch.IV, Cor.10.4.6]). Hence, $Z \cap Z'$ is a dense subset of $\mathrm{Spec}\, V_\lambda$, and by claim 17.4.21, the image of $\vartheta'^{(\mathfrak{p})}_{\Gamma_\mathbb{Q}}$ (resp. of $\vartheta'^{(\mathfrak{p})}_\Gamma$) contains $e_{f_{\lambda,(\mathfrak{p})}} = 1 \otimes e_{f_\lambda}$ (resp. $be_{f_{\lambda,(\mathfrak{p})}} = 1 \otimes be_{f_\lambda}$) for every $\mathfrak{p} \in Z \cap Z'$ (resp. and every $b \in J$). By lemma 17.4.14(iii) it follows that the image of $\vartheta'_{\Gamma_\mathbb{Q}}$ contains e_f (resp. that the image of ϑ'_Γ contains be_f for every $b \in J$). The claim is an immediate consequence. $\diamondsuit$

It remains to deal with the case where A does not contain a field, hence $A = V[[P]]/(h)$, for a discrete valuation ring $(V, \mathfrak{m}_V)$ whose residue field k has positive characteristic, such that $\mathfrak{m}_V = pV$, and with some $h := \sum_{\gamma \in P} h_\gamma \cdot \gamma$ such that $h_0 \in pV \setminus p^2V$. Arguing as in (17.3.4), and taking into account claim 17.4.20, we may then further reduce to the case where k is algebraically closed. In this situation, let $g \in J \setminus \{0\}$, and set

$$A_\Gamma := \Gamma \times_{\Gamma_\mathbb{Q}} A_\infty \qquad \text{and} \qquad A'_\Gamma := A_\Gamma[X^{1/p^\infty}]/(X - g) = A_\Gamma[g^{1/p^\infty}].$$

We get a basic setup $(A_\Gamma, \mathfrak{n})$, where $\mathfrak{n} \subset A_\Gamma$ is the ideal generated by $(g^\delta \mid \delta \in \mathbb{N}[1/p] \setminus \{0\})$. Moreover, let D be the p-root closure of A'_Γ in $A'_\Gamma[1/p]$, and $D^\wedge$ the p-adic completion of D; also, set $D_1^\wedge := (D^\wedge)^a_*$, the ring of almost elements of the $(A_\Gamma, \mathfrak{n})^a$-algebra $(D^\wedge)^a$, and denote by B'_1 the integral closure of $D_1^\wedge$ in $B' := D^\wedge \otimes_A B[1/g]$. Pick $N \in \mathbb{N}$ and $e \in B \otimes_A B$ whose image in $B \otimes_A B[1/g]$ equals $g^N e_f$, and let $e' \in B'_1 \widehat{\otimes}_{D^\wedge} B'_1$ be the image of e.

Claim 17.4.25. For every $\delta \in \mathbb{N}[1/p] \setminus \{0\}$ we have $g^\delta e' \in g^N(B'_1 \widehat{\otimes}_{D^\wedge} B'_1)$.

Proof of the claim. For every $n \in \mathbb{N}$, the $(D/p^nD)^a$-algebra $(B'_1/p^nB'_1)^a$ is étale of finite rank, and we let $\varepsilon_n \in (B'_1/p^nB'_1 \otimes_D B'_1/p^nB')^a_*$ be the corresponding diagonal idempotent. The system $(\varepsilon_n \mid n \in \mathbb{N})$ corresponds to a unique element

$$\varepsilon \in (B'_1 \widehat{\otimes}_{D^\wedge} B'_1)^a_*$$

where $B'_1 \widehat{\otimes}_{D^\wedge} B'_1$ denotes the p-adic completion of $B'_1 \otimes_{D^\wedge} B'_1$. Let $\mu : B'_1 \otimes_{D^\wedge} B'_1 \to B'_1$ be the multiplication law of B'_1, and $\widehat{\mu}$ the p-adic completion of μ; by construction, we see that :

(17.4.26) $\qquad \widehat{\mu}(\varepsilon) = 1 \qquad$ and $\qquad \varepsilon \cdot (1 \otimes x - x \otimes 1) = 0 \quad$ for every $x \in B'_1 = (B'_1)^a_*$.

By corollary 16.9.52(ii), the completion map $B'_1 \otimes_{D^\wedge} B'_1 \to B'_1 \widehat{\otimes}_{D^\wedge} B'_1$ induces an isomorphism

$$B \otimes_A B \otimes_A D^\wedge[1/g] \xrightarrow{\sim} B'_1 \otimes_{D^\wedge} B'_1[1/g] \xrightarrow{\sim} B'_1 \widehat{\otimes}_{D^\wedge} B'_1[1/g]$$

and it follows easily from (17.4.26) that the image of $e_f \otimes 1$ in $B'_1 \widehat{\otimes}_{D^\wedge} B'_1[1/g]$ agrees with the image of ε. But g is a regular element of $(B'_1 \widehat{\otimes}_{D^\wedge} B'_1)^a_*$, due to corollary 16.9.52(ii), hence the image of e in $(B'_1 \widehat{\otimes}_{D^\wedge} B'_1)^a_*$ agrees with $g^N \varepsilon$, whence the claim. ◊

Recall that $D_1^\wedge$ is integrally closed in $D^\wedge[1/g]$ (theorem 16.9.42(iv)), and A is a normal domain; then the trace maps for the étale ring homomorphisms $D^\wedge[1/g] \to B'$ and $A[1/g] \to B[1/g]$ restrict to a $D_1^\wedge$-linear map and respectively an A-linear map

$$\mathrm{tr}_{B'_1/D_1^\wedge} : B'_1 \to D_1^\wedge \qquad \mathrm{tr}_{B/A} : B \to A$$

(lemma 14.3.15(i)) whence an A-linear map and a $D_1^\wedge$-linear map

$$\tau : B \to B^\vee := \mathrm{Hom}_A(B, A) \quad b \mapsto (b' \mapsto \mathrm{tr}_{B/A}(bb'))$$
$$\tau' : B'_1 \to \mathrm{Hom}_A(B, D_1^\wedge) \qquad b \mapsto (b' \mapsto \mathrm{tr}_{B'_1/D_1^\wedge}(bb')).$$

Let also $i : A \to A_\Gamma$ and $i_D : D^\wedge \to D_1^\wedge$ be the natural maps; since i_D^a is an isomorphism of A_Γ^a-modules, we get by adjunction a unique A_Γ-linear map $j_D : \widetilde{\mathfrak{n}} \otimes_{A_\Gamma} D_1^\wedge \to D^\wedge$ such that $i_D \circ j_D = \mu \otimes_{A_\Gamma} D_1^\wedge$, where as usual $\widetilde{\mathfrak{n}} := \mathfrak{n} \otimes_{A_\Gamma} \mathfrak{n}$, and $\mu : \widetilde{\mathfrak{n}} \to A$ is the multiplication map : $a \otimes a' \mapsto aa'$. Then, for every $\delta, \delta' \in \mathbb{N}[1/p] \setminus \{0\}$ we let $j_{\delta+\delta'} : D_1^\wedge \to D^\wedge$ be the A_Γ-linear map such that $j_{\delta+\delta'}(d) := j_D(g^\delta \otimes g^{\delta'} \otimes d)$ for every $d \in D_1^\wedge$; it is easily seen that this map depends only on the sum $\delta + \delta'$. Hence :

$$i_D \circ j_\delta = g^\delta \cdot \mathbf{1}_{D_1^\wedge} \qquad \text{and} \qquad j_{\delta+\delta'} = g^\delta \cdot j_{\delta'} \qquad \text{for every } \delta, \delta' \in \mathbb{N}[1/p] \setminus \{0\}$$

and moreover for every such δ we get a commutative diagram of A_Γ-modules :

$$\begin{array}{ccc} A_\Gamma & \xrightarrow{g^\delta \cdot \mathbf{1}_{A_\Gamma}} & A_\Gamma \\ \downarrow & & \downarrow \\ D_1^\wedge & \xrightarrow{j_\delta} & D^\wedge \end{array}$$

whose vertical arrows are the structure maps of the A_Γ-algebras $D_1^\wedge$ and $D^\wedge$. Set

$$B_\Gamma^\vee := \mathrm{Hom}_A(B, A_\Gamma) \qquad B_1'^\vee := \mathrm{Hom}_A(B, D_1^\wedge) \qquad C := \mathrm{Hom}_A(B, D^\wedge)$$

and let $i_* : B^\vee \to B_\Gamma^\vee$ (resp. $j_{\delta*} : B_1'^\vee \to C$) be the A-linear (resp. A_Γ-linear) map such that $i_*(\varphi) := i \circ \varphi$ for every A-linear map $\varphi : B \to A$ (resp. $j_{\delta*}(\varphi) := j_\delta \circ \varphi$ for every A-linear map $\varphi : B \to D_1^\wedge$). To ease notation, set as well

$$B_{\Gamma,i}^\vee := \mathrm{Hom}_A(B, A_\Gamma/p^i A_\Gamma) \qquad \text{and} \qquad C_i := \mathrm{Hom}_A(B, D/p^i D) \qquad \text{for every } i \in \mathbb{N}.$$

We then obtain a commutative diagram for every $\delta \in \mathbb{N}[1/p] \setminus \{0\}$ and every $i \in \mathbb{N}$:

$$\begin{array}{ccccccc} B \otimes_A B & \xrightarrow{\tau \otimes_A \tau} & B^\vee \otimes_A B^\vee & \xrightarrow{g^\delta \cdot i_* \otimes_A g^\delta \cdot i_*} & B_\Gamma^\vee \otimes_{A_\Gamma} B_\Gamma^\vee & \longrightarrow & B_{\Gamma,i}^\vee \otimes_{A_\Gamma} B_{\Gamma,i}^\vee \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ B'_1 \widehat{\otimes}_{D^\wedge} B'_1 & \xrightarrow{\tau' \widehat{\otimes}_{D^\wedge} \tau'} & B_1'^\vee \widehat{\otimes}_{D^\wedge} B_1'^\vee & \xrightarrow{j_{\delta*} \widehat{\otimes}_{D^\wedge} j_{\delta*}} & C \widehat{\otimes}_{D^\wedge} C & \longrightarrow & C_i \otimes_D C_i. \end{array}$$

For given δ, pick $\delta', \delta'' \in \mathbb{N}[1/p] \setminus \{0\}$ with $\delta = \delta' + \delta''$; then

$$(j_{\delta*} \circ \tau') \widehat{\otimes}_{D^\wedge} (j_{\delta*} \circ \tau')(e') = (j_{\delta'*} \circ \tau') \widehat{\otimes}_{D^\wedge} (j_{\delta'*} \circ \tau')(g^{2\delta''} e').$$

By claim 17.4.25, we deduce that the image of $(j_{\delta *} \circ \tau')\widehat{\otimes}_{D^\wedge}(j_{\delta *} \circ \tau')(e')$ in $C_i \otimes_D C_i$ is divisible by g^N. Recall now that D is a faithfully flat A_Γ-algebra (theorem 16.9.17); since A is noetherian, and B is a finite A-algebra, we deduce a natural isomorphism

$$B^\vee_{\Gamma,i} \otimes_{A_\Gamma} D \stackrel{\sim}{\to} C_i \qquad \text{for every } i \in \mathbb{N}.$$

Set $e'' := (i_* \circ \tau) \otimes_A (i_* \circ \tau)(e)$; it follows that the image of ge'' in $B^\vee_{\Gamma,i} \otimes_{A_\Gamma} B^\vee_{\Gamma,i}$ is divisible by g^N, for every $i \in \mathbb{N}$. Next, for every $i \in \mathbb{N}$ and every subset $\Delta \subset \Gamma_{\mathbb{Q}}$, let :

$$\mathsf{R}(B, \Delta, i) := \bigoplus_{[\gamma]\in\Delta} B^\vee_{[\gamma],i} \otimes_A B^\vee_{[1/\gamma],i} \quad \text{with } B^\vee_{[\gamma],i} := \mathrm{Hom}_A(B, A_{[\gamma]}/p^i A_{[\gamma]}) \text{ for every } [\gamma] \in \Gamma_{\mathbb{Q}}$$

and notice that we have an A-linear surjection $\mathsf{R}(B, \Gamma, i) \to \mathrm{gr}_0(B^\vee_{\Gamma,i} \otimes_{A_\Gamma} B^\vee_{\Gamma,i})$, for the natural Γ-grading on $B^\vee_{\Gamma,i} \otimes_{A_\Gamma} B^\vee_{\Gamma,i}$. Summing up, for every $i \in \mathbb{N}$ there exists $e_i \in \mathsf{R}(B, \Gamma, i)$ such that the image of ge'' in $\mathrm{gr}_0(B^\vee_{\Gamma,i} \otimes_{A_\Gamma} B^\vee_{\Gamma,i})$ equals the image of $g^N e_i$. On the other hand, as in remark 17.4.7(ii) we obtain for every $i \in \mathbb{N}$ a commutative diagram of A-linear maps

$$\begin{array}{ccccc}
A/p^iA \otimes_A \mathsf{R}(B,\Gamma) & \longrightarrow & A/p^iA \otimes_A \mathrm{gr}_0(B^\vee_\Gamma \otimes_{A_\Gamma} B^\vee_\Gamma) & \longrightarrow & A/p^iA \otimes_A (B \otimes_A B)^\vee \\
\downarrow & & \downarrow & & \downarrow \\
\mathsf{R}(B,\Gamma,i) & \longrightarrow & \mathrm{gr}_0(B^\vee_{\Gamma,i} \otimes_{A_\Gamma} B^\vee_{\Gamma,i}) & \longrightarrow & \mathrm{Hom}_A(B \otimes_A B, A/p^iA)
\end{array}$$

whose left vertical arrow is induced by the projections $A_{[\gamma]} \to A_{[\gamma]}/p^i A_{[\gamma]}$ for every $[\gamma] \in \Gamma_{\mathbb{Q}}$, and such that the composition of the top horizontal arrows is $A/p^iA \otimes_A \xi_{\underline{f}}$. Denote by $\xi_{\underline{f},i}$ the composition of the bottom horizontal arrows; arguing as in the proof of lemma 17.4.10(i) we find a finite subset $\Sigma \subset \Gamma$ such that $\xi_{\underline{f}}(\mathsf{R}(B,\Gamma)) = \xi_{\underline{f}}(\mathsf{R}(B,\Sigma))$ and $\xi_{\underline{f},i}(\mathsf{R}(B,\Gamma,i)) = \xi_{\underline{f},i}(\mathsf{R}(B,\Sigma,i))$ for every $i \in \mathbb{N}$. We notice:

Claim 17.4.27. Let R be a noetherian ring, M, N two R-modules of finite type, and $I \subset R$ an ideal. Set $H^i := \mathrm{Hom}_R(M, N/I^iN)$ for every $i \in \mathbb{N}$. Then there exists $c \in \mathbb{N}$ such that

$$\mathrm{Im}(H^{i+c} \to H^i) = \mathrm{Im}(\mathrm{Hom}_R(M,N) \to H^i) \qquad \text{for every } i \in \mathbb{N}.$$

Proof of the claim. Pick a finite presentation $R^{\oplus q} \stackrel{\varphi}{\to} R^{\oplus p} \to M$ of the R-module M; let $\varphi^\vee : R^{\oplus p} \to R^{\oplus q}$ be the transpose of φ, set $\varphi^\vee_N := \varphi^\vee \otimes_A N : N^{\oplus q} \to N^{\oplus p}$ and $N' := \mathrm{Im}\,\varphi^\vee_N$; by the Artin-Rees lemma ([126, Th.8.5]) there exists $c \in \mathbb{N}$ such that $N' \cap I^{i+c}N^{\oplus q} = I^i(N' \cap I^c N^{\oplus q})$ for every $i \in \mathbb{N}$. We consider then the induced commutative diagram :

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Hom}_R(M,N) & \longrightarrow & N^{\oplus p} & \xrightarrow{\varphi^\vee_N} & N^{\oplus q} \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & H^{i+c} & \longrightarrow & N^{\oplus p}/I^{i+c}N^{\oplus p} & \xrightarrow{\varphi^\vee_{N/I^{i+c}N}} & N^{\oplus q}/I^{i+c}N^{\oplus q} \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & H^i & \longrightarrow & N^{\oplus p}/I^iN^{\oplus p} & \xrightarrow{\varphi^\vee_{N/I^iN}} & N^{\oplus q}/I^iN^{\oplus q}
\end{array}$$

whose horizontal rows are left exact sequences. Let then $\overline{h} \in \mathrm{Ker}\,\varphi^\vee_{N/I^{i+c}N}$, and pick a representative $h \in N^{\oplus p}$ for the class $\overline{h}$. Then $\varphi^\vee_N(h) \in N' \cap I^{i+c}N^{\oplus q}$, so there exists $k \in \mathbb{N}$ such that $\varphi^\vee_N(h) = \sum_{j=1}^k a_j \varphi^\vee_N(x_j)$ for certain $a_1, \dots, a_k \in I^i$ and $x_1, \dots, x_k \in N^{\oplus p}$. Set $h' := h - \sum_{j=1}^k a_j x_j$. Then $h' \in \mathrm{Ker}\,\varphi^\vee_N$, and the image of h' in $N^{\oplus p}/I^iN^{\oplus p}$ agrees with the image of $\overline{h}$, whence the claim. ◊

By claim 17.4.27, for every $i \in \mathbb{N}$ there exists $e'_i \in \mathsf{R}(B,\Sigma)$ such that $\xi_{\underline{f},i}(e_i)$ agrees with the image of $1 \otimes \xi_{\underline{f}}(e'_i) \in A/p^iA \otimes_A (B \otimes_A B)^\vee$. Now, let $M \subset (B \otimes_A B)^\vee$ be the A-submodule generated by the system $(\xi_{\underline{f}}(e'_i) \mid i \in \mathbb{N})$; we conclude that the image of ge'' in $(B \otimes_A B)^\vee$ lies in the submodule $g^N M + p^i(B \otimes_A B)^\vee$, for every $i \in \mathbb{N}$; but then it lies already in $g^N M$, since $g^N M$ is a closed subset for the p-adic topology of the finitely generated A-module $(B \otimes_A B)^\vee$ ([126, Th.8.10(i)]). Lastly, this shows that the image of ge'' in $\mathsf{X}(\underline{f})$ lies in $g^N \cdot \vartheta_{\underline{f}}(\mathsf{R}(B,\Gamma))$; but this image equals $g^{N+1}e_f$, whence $ge_f \in \vartheta_{\underline{f}}(\mathsf{R}(B,\Gamma))$, which achieves the proof of (ii).

To show (i), let e''_i be the image of e'' in $B^\vee_{\Gamma,i} \otimes_{A_\Gamma} B^\vee_{\Gamma,i}$ for every $i \in \mathbb{N}$, and set

$$I_i := \{a \in A_\Gamma \mid ae''_i \in g^N(B^\vee_{\Gamma,i} \otimes_{A_\Gamma} B^\vee_{\Gamma,i})\} \qquad \text{and} \qquad I_{D,i} := \{d \in D \mid de''_i \in g^N(C_i \otimes_D C_i)\}.$$

As already observed in the foregoing, $g^\delta \in I_{D,i}$ for every $\delta \in \mathbb{N}[1/p] \setminus \{0\}$; then, since D is a flat A_Γ-algebra, we have $I_{D,i} = I_i D$ for every $i \in \mathbb{N}$ (details left to the reader), so that $I^k_{D,i} = I^k_i D$ for every $i, k \in \mathbb{N}$. Since D is a faithfully flat A_Γ-algebra, it follows that $I^k_i = I^k_{D,i} \cap A_\Gamma$ for every $i, k \in \mathbb{N}$ ([126, Th.7.5(ii)]), whence $g \in I^k_i$ for every $i, k \in \mathbb{N}$, and notice as well that I_i is a Γ-graded ideal of A_Γ. Let $\lambda_1, \dots, \lambda_k \in P^\vee := \mathrm{Hom}_{\mathbf{Mnd}}(P_\mathbb{R}, \mathbb{R}_+)$ such that $\mathbb{R}\lambda_1, \dots, \mathbb{R}\lambda_k$ are the extremal rays of $P^\vee_\mathbb{R}$, and set $\lambda := \lambda_1 + \cdots + \lambda_k$; by lemma 17.3.10 there exists a sequence $(\gamma_n \mid n \in \mathbb{N})$ of elements of $P^{(\infty)}$ with $\lim_{n \to +\infty} \lambda(\gamma_n) = 0$ and such that $\mathrm{gr}_{[\gamma_n]} I_i = A_{[\gamma_n]}$ for every $n \in \mathbb{N}$, and by claim 17.3.7 we may assume that $\mathrm{gr}_{[\gamma_n]} P^{(\infty)} = \gamma_n P$ for every $n \in \mathbb{N}$, so that $A_{[\gamma_n]} = A \cdot (1 \otimes \gamma_n)$, and therefore

$$(1 \otimes \gamma_n) \cdot e''_i \in g^N \cdot \mathrm{gr}_{[\gamma_n]}(B^\vee_{\Gamma,i} \otimes_{A_\Gamma} B^\vee_{\Gamma,i}).$$

On the other hand, lemma 17.4.16 yields for every $[\gamma] \in \Gamma$ an element $[\gamma'] \in \Sigma_\mathbb{Q}$ such that :

$$\gamma^{-1} P_\mathbb{Q} \cap P^{\mathrm{gp}} \subset \gamma'^{-1} P_\mathbb{Q} \cap P^{\mathrm{gp}} \qquad \text{and} \qquad (\gamma/\gamma_n) P_\mathbb{Q} \cap P^{\mathrm{gp}} \subset \gamma' P_\mathbb{Q} \cap P^{\mathrm{gp}}.$$

We then get a commutative diagram of A-modules :

$$\begin{array}{ccccc}
\mathsf{R}(B, \Gamma_\mathbb{Q}, i) & \longrightarrow & \mathrm{gr}_0(B^\vee_{\infty,i} \otimes_{A_\Gamma} B^\vee_{\infty,i}) & \longrightarrow & \mathrm{Hom}_A(B \otimes_A B, A/p^i A) \\
\uparrow & & & & \downarrow \\
\bigoplus_{[\gamma] \in \Gamma} B^\vee_{[\gamma],i} \otimes_A B^\vee_{[\gamma_n/\gamma],i} & \longrightarrow & \mathrm{gr}_{[\gamma_n]}(B^\vee_{\Gamma,i} \otimes_{A_\Gamma} B^\vee_{\Gamma,i}) & \longrightarrow & \mathrm{Hom}_A(B \otimes_A B, A_{[\gamma_n]}/p^i A_{[\gamma_n]})
\end{array}$$

whose right vertical arrow is an isomorphism induced by scalar multiplication by $1 \otimes \gamma_n$. Moreover, the composition of the top horizontal arrows is $\xi_{\underline{f},i}$. With this notation, we have $e''_i \in \mathrm{gr}_0(B^\vee_{\Gamma,i} \otimes_{A_\Gamma} B^\vee_{\Gamma,i})$, and it follows easily that the image of e''_i lies in $g^N \cdot \xi_{\underline{f},i}(\mathsf{R}(B, \Gamma_\mathbb{Q}, i))$, for every $i \in \mathbb{N}$. Arguing as in the proof of lemma 17.4.10(i), we then find a finite subset $\Delta \subset \Gamma_\mathbb{Q}$ such that the image of e''_i in $\mathrm{Hom}_A(B \otimes_A B, A/p^i A)$ lies in the A-submodule $g^N \cdot \xi_{\underline{f},i}(\mathsf{R}(B, \Delta, i))$, for every $i \in \mathbb{N}$. By invoking again claim 17.4.27, we deduce that for every $i \in \mathbb{N}$, the image of e'' in $(B \otimes_A B)^\vee$ lies in the A-submodule $g^N \cdot \xi_{\underline{f}}(\mathsf{R}(B, \Delta)) + p^i \cdot (B \otimes_A B)^\vee$. Arguing as in the foregoing proof of (ii), we conclude that the image of e'' in $(B \otimes_A B)^\vee$ lies already in $g^N \cdot \xi_{\underline{f}}(\mathsf{R}(B, \Delta))$, and finally $e_f \in \vartheta_{\underline{f}}(\mathsf{R}(B, \Gamma_\mathbb{Q}))$, as stated. □

17.4.28. Keep the notation of (17.4.5), and for every $m \in \mathbb{N}$, let $\Gamma_m := \{\gamma \in \Gamma_\mathbb{Q} \mid m\gamma = 0\}$. By theorem 17.4.19, if (A, P, β) is regular, there exists $m \in \mathbb{N}$ such that $e_f \in \vartheta_{\underline{f}}(\mathsf{R}(B, \Gamma_m))$. It turns out that a suitable integer m with this property can be exhibited purely in terms of the combinatorics of the monoid P. Indeed, let $P^\vee$ be the dual of P (see (6.4.13)); since P is fine, $P^\vee$ is fine, sharp and saturated (proposition 6.4.14(i,ii)), and we let $F_1, \dots, F_k$ be the one-dimensional faces of $P^\vee$. Then $F_i \simeq \mathbb{N}$ (theorem 6.4.18(ii)), and we denote by λ_i the unique generator of F_i, for every $i = 1, \dots, k$. Set $\Lambda := \{\lambda_1, \dots, \lambda_k\}$, and denote by $\mathscr{E}$ the set of all subsets $\Sigma \subset \Lambda$ such that $\{\lambda^{\mathrm{gp}} \otimes_\mathbb{Z} \mathbb{Q} : P^{\mathrm{gp}}_\mathbb{Q} \to \mathbb{Q} \mid \lambda \in \Sigma\}$ is a basis of $(P^{\mathrm{gp}}_\mathbb{Q})^\vee$. For every $\Sigma \in \mathscr{E}$, let $L_\Sigma \subset P^{\vee\mathrm{gp}}$ be the subgroup generated by $(\lambda^{\mathrm{gp}} \mid \lambda \in \Sigma)$, and denote by m_Σ the exponent of the finite abelian group $P^{\vee\mathrm{gp}}/L_\Sigma$. Lastly, denote by m_P the least common multiple of the finite system of integers $(m_\Sigma \mid \Sigma \in \mathscr{E})$. We may then state :

Corollary 17.4.29. *In the situation of theorem* 17.4.19*, we have* $e_f \in \vartheta_{\underline{f}}(\mathsf{R}(B, \Gamma_{m_p}))$.

Proof. For every $\Sigma \in \mathscr{E}$, let $\lambda_\Sigma : P^{\mathrm{gp}}_{\mathbb{Q}} \xrightarrow{\sim} \mathbb{Q}^\Sigma$ be the $\mathbb{Q}$-linear isomorphism given by the rule :

$$\gamma \mapsto (\lambda^{\mathrm{gp}}_{\mathbb{Q}}(\gamma) \mid \lambda \in \Sigma) \qquad \text{for every } \gamma \in P^{\mathrm{gp}}_{\mathbb{Q}}.$$

Notice that $P^{\mathrm{gp}} \subset \lambda_\Sigma^{-1}(\mathbb{Z}^\Sigma)$ for every such B, and m_Σ equals the exponent of the finite abelian group $\lambda_\Sigma^{-1}(\mathbb{Z}^\Sigma)/P^{\mathrm{gp}}$. With this notation, we have :

Claim 17.4.30. For every $\beta \in P^{\mathrm{gp}}_{\mathbb{Q}}$ there exist $\Sigma \in \mathscr{E}$ and an element $\beta^\dagger \in \lambda_\Sigma^{-1}(\mathbb{Z}^\Sigma)$ such that

$$\beta P_{\mathbb{Q}} \cap P^{\mathrm{gp}} \subset \beta^\dagger P_{\mathbb{Q}} \cap P^{\mathrm{gp}} \qquad \text{and} \qquad \beta^{-1} P_{\mathbb{Q}} \cap P^{\mathrm{gp}} \subset \beta^{\dagger -1} P_{\mathbb{Q}} \cap P^{\mathrm{gp}}. \tag{17.4.31}$$

Proof of the claim. Notice that, for every $\beta \in P^{\mathrm{gp}}_{\mathbb{R}}$, we have

$$\beta P_{\mathbb{R}} \cap P^{\mathrm{gp}} = \{x \in P^{\mathrm{gp}} \mid \lambda^{\mathrm{gp}}(x) \geq \lambda^{\mathrm{gp}}_{\mathbb{R}}(\beta) \text{ for every } \lambda \in \Lambda\}.$$

For every such β and every $\lambda \in \Lambda$, denote by $N_{\beta,\lambda}$ (resp. $N'_{\beta,\lambda}$) the largest (resp. smallest) integer which is smaller than (resp. larger than) or equal to $\lambda^{\mathrm{gp}}_{\mathbb{R}}(\beta)$. Recalling that $\lambda^{\mathrm{gp}}(P^{\mathrm{gp}}) \subset \mathbb{Z}$ for every $\lambda \in \Lambda$, it is easily seen that (17.4.31) holds if and only if

$$\lambda^{\mathrm{gp}}_{\mathbb{Q}}(\beta^\dagger) \in [N_{\beta,\lambda}, N'_{\beta,\lambda}] \qquad \text{for every } \lambda \in \Lambda. \tag{17.4.32}$$

Set $\Sigma_\beta := \{\lambda \in \Lambda \mid \lambda^{\mathrm{gp}}_{\mathbb{R}}(\beta) \in \mathbb{Z}\}$. Now, if $\beta \in P^{\mathrm{gp}}_{\mathbb{Q}}$ and Σ_β spans $(P^{\mathrm{gp}}_{\mathbb{Q}})^\vee$, obviously the claim holds with $\beta^\dagger := \beta$. Thus, a simple induction argument reduces to showing that, if $\beta \in P^{\mathrm{gp}}_{\mathbb{Q}}$ and Σ_β does not span $(P^{\mathrm{gp}}_{\mathbb{Q}})^\vee$, there exists $\gamma \in P^{\mathrm{gp}}_{\mathbb{Q}}$ such that (17.4.32) holds, and such that Σ_γ strictly contains Σ_β. To this aim, let $l : P^{\mathrm{gp}}_{\mathbb{R}} \to \mathbb{R}^\Lambda$ be the $\mathbb{R}$-linear map such that

$$l(\gamma) := (\lambda^{\mathrm{gp}}_{\mathbb{R}}(\gamma) \mid \lambda \in \Lambda) \qquad \text{for every } \gamma \in P^{\mathrm{gp}}_{\mathbb{R}}$$

and set

$$K := \prod_{\lambda \in \Lambda} [N_{\beta,\gamma}, N'_{\beta,\gamma}] \subset \mathbb{R}^\Lambda \qquad M := \bigcap_{\lambda \in \Sigma_\beta} \mathrm{Ker}\, \lambda^{\mathrm{gp}}_{\mathbb{Q}} \subset P^{\mathrm{gp}}_{\mathbb{Q}}.$$

Notice that l is a closed immersion, K is a compact subset of $\mathbb{R}^\Lambda$, and by assumption, $M \neq 0$. Clearly $\beta \in l^{-1}K$, and we pick any non-zero $\mu \in M$. It follows that the subset

$$\{a \in \mathbb{R} \mid \beta + a\mu \in l^{-1}K\}$$

is non-empty and compact, so it admits a largest element a_0. Set $\gamma := \beta + a_0\mu$; by construction, $\Sigma_\beta \subset \Sigma_\gamma$, and it is easily seen that there must exist some $\lambda \in \Lambda \setminus \Sigma_\beta$ such that $\lambda^{\mathrm{gp}}_{\mathbb{R}}(\gamma) \in \{N_{\beta,\lambda}, N'_{\beta,\lambda}\}$, since otherwise we could find a real number $a > a_0$ with $\beta + a\mu \in l^{-1}K$. Thus Σ_γ strictly contains Σ_β. Lastly, since both β and μ lie in $P^{\mathrm{gp}}_{\mathbb{Q}}$ and $\lambda^{\mathrm{gp}}_{\mathbb{R}}(\beta) \notin \mathbb{Z}$, we must have $\lambda^{\mathrm{gp}}_{\mathbb{R}}(a_0\mu) \in \mathbb{Q} \setminus \mathbb{Z}$, and since $\lambda^{\mathrm{gp}}_{\mathbb{R}}(\mu) \in \mathbb{Q}$, we conclude that $a_0 \in \mathbb{Q}$, hence $\gamma \in P^{\mathrm{gp}}_{\mathbb{Q}}$, whence the claim. ◊

Now, for every $[\beta] \in \Gamma_{\mathbb{Q}}$ pick $\beta^\dagger \in P^{\mathrm{gp}}_{\mathbb{Q}}$ fulfilling the condition of claim 17.4.30, and set $\Delta := \{[\beta^\dagger] \mid [\beta] \in \Gamma_{\mathbb{Q}}\}$. Notice that $\Delta \subset \Gamma_{m_P}$. Arguing as in the proof of lemma 17.4.10(ii) we easily deduce that $\vartheta_{\underline{f}}(\mathsf{R}(B, \Delta)) = \vartheta_{\underline{f}}(\mathsf{R}(B, \Gamma_{\mathbb{Q}}))$, whence the corollary. □

Example 17.4.33. (i) Suppose that $\dim P = 2$. Then we may find a basis (f_1, f_2) of P^{gp} and integers $a, b \in \mathbb{N}$ such that $b > a$, $(a, b) = 1$ and P is the submonoid of P^{gp} generated by f_1 and $af_1 + bf_2$ (example 6.4.19(ii)). With this notation, a simple calculation shows that $m_P = b$ (details left to the reader).

(ii) In the situation of theorem 17.4.19, suppose that $P = \mathbb{N}^{\oplus r} \times \mathbb{Z}^{\oplus s}$ for some $r, s \in \mathbb{N}$; by corollary 12.5.19, the latter holds if and only if A is a regular ring. In this case, the integer m_P of (17.4.28) equals 1, and therefore corollary 17.4.29 says that $e_f \in \vartheta_{\underline{f}}(B^\vee \otimes_A B^\vee)$. However, the latter follows also more directly from lemma 17.4.10(ii). Moreover, in case A is regular, the submodule $\vartheta_{\underline{f}}(B^\vee \otimes_A B^\vee)$ can be interpreted in terms of the *different ideal* of the finite ring extension $f : A \to B$. We conclude this section with a review of different ideals in the more general context of quasi-coherent algebras on schemes; especially, remark 17.4.44 explains the connection with theorem 17.4.19.

17.4.34. Let X be a reduced scheme, $\mathscr{A}$ a quasi-coherent and finite $\mathscr{O}_X$-algebra and $j : U \to X$ a quasi-compact open immersion with dense image, such that $j^*\mathscr{A}$ is a locally free $\mathscr{O}_U$-module. Denote by

$$\mathrm{Max}\, X$$

the set of maximal points of X, and notice that $\mathrm{Max}\, X \subset U$, by proposition 8.1.47(i). In this situation, there is a trace form as in remark 14.3.10(iii) :

$$t_{j^*\mathscr{A}} : j^*\mathscr{A} \otimes_{\mathscr{O}_U} j^*\mathscr{A} \to \mathscr{O}_U$$

whence a pairing

$$t_{\mathscr{A},U} : \mathscr{A} \otimes_{\mathscr{O}_X} j_*j^*\mathscr{A} \to j_*(j^*\mathscr{A} \otimes_{\mathscr{O}_U} j^*\mathscr{A}) \xrightarrow{j_*t_{j^*\mathscr{A}}} j_*\mathscr{O}_U$$

which in turns yields a morphism of $\mathscr{O}_X$-modules

$$\tau_{\mathscr{A},U} : j_*j^*\mathscr{A} \to \mathscr{H}om_{\mathscr{O}_X}(\mathscr{A}, j_*\mathscr{O}_U).$$

Since X is reduced, the natural morphism $\mathscr{O}_X \to j_*\mathscr{O}_U$ is injective, and therefore the same holds for the induced $\mathscr{O}_X$-linear morphism

$$\mathscr{A}^\vee \to \mathscr{H}om_{\mathscr{O}_X}(\mathscr{A}, j_*\mathscr{O}_U)$$

so we may define

$$\mathscr{D}^{-1}_{\mathscr{A},U} := \tau^{-1}_{\mathscr{A},U}(\mathscr{A}^\vee).$$

We call $\mathscr{D}^{-1}_{\mathscr{A},U} \subset j_*j^*\mathscr{A}$ the *inverse different of $\mathscr{A}$ relative to the open subset U*.

Remark 17.4.35. (i) In the situation of (17.4.34), notice that the natural isomorphism

$$\mathscr{H}om_{\mathscr{O}_X}(\mathscr{A}, j_*\mathscr{O}_U) \xrightarrow{\sim} j_*(j^*\mathscr{A})^\vee$$

(see (10.1.19)) identifies $\tau_{\mathscr{A},U}$ with $j_*\tau_{j^*\mathscr{A}}$, where $\tau_{j^*\mathscr{A}} : j^*\mathscr{A} \to j^*\mathscr{A}^\vee$ is the $\mathscr{O}_U$-linear morphism deduced from the trace pairing $t_{j^*\mathscr{A}}$.

(ii) Moreover, set $\mathscr{B} := \mathrm{Im}\,(\mathscr{A} \to j_*j^*\mathscr{A})$. Since the natural map $\mathscr{O}_X \to j_*\mathscr{O}_U$ is a monomorphism, it is easily seen that the the epimorphism $\mathscr{A} \to \mathscr{B}$ induces an isomorphism $\mathscr{B}^\vee \xrightarrow{\sim} \mathscr{A}^\vee$, and taking (i) into account, we deduce that the latter induces an isomorphism

$$\mathscr{D}^{-1}_{\mathscr{A},U} \xrightarrow{\sim} \mathscr{D}^{-1}_{\mathscr{B},U}.$$

(iii) Suppose next that $U' \subset U$ is another open subset, such that the open immersion $j' : U' \to X$ is also quasi-compact with dense image. A direct inspection of the definitions yields a commutative diagram

$$\begin{array}{ccc} j_*j^*\mathscr{A} & \xrightarrow{\tau_{\mathscr{A},U}} & \mathscr{H}om_{\mathscr{O}_X}(\mathscr{A}, j_*\mathscr{O}_U) \\ \downarrow & & \downarrow \\ j'_*j'^*\mathscr{A} & \xrightarrow{\tau_{\mathscr{A},U'}} & \mathscr{H}om_{\mathscr{O}_X}(\mathscr{A}, j'_*\mathscr{O}_{U'}) \end{array}$$

whose vertical arrows are monomorphisms, under the current assumptions. It follows easily that

$$\mathscr{D}^{-1}_{\mathscr{A},U} = \mathscr{D}^{-1}_{\mathscr{A},U'} \cap j_*j^*\mathscr{A}.$$

(iv) In the situation of (iii), suppose that $\mathscr{A}$ is generically étale, and let $U' \subset U$ be the largest open subset such that $\mathscr{A}_{|U'}$ is an étale $\mathscr{O}_{U'}$-algebra. Then the inclusion $U' \to U$ is a dense and quasi-compact open immersion. For the proof, we may assume that U is affine, say $U = \mathrm{Spec}\, R$, and that $j^*\mathscr{A}$ is the quasi-coherent $\mathscr{O}_U$-algebra associated with an R-algebra A

which is a projective R-module of finite rank; then we easily reduce to the case where A is a free R-module of finite rank. In this case, let $(e_i \mid i = 1, \dots, r)$ be a basis of the R-module A; consider the matrix $D := (t_{j^*\mathscr{A}}(e_i, e_j) \mid i, j = 1, \dots, r)$, and set $d := \det(D)$. We have $U' = \operatorname{Spec} R[d^{-1}]$, a quasi-compact dense open subset of U ([75, Th.4.1.14]), as required.

(v) In the situation of (iii), suppose additionally that $j^*\mathscr{A}$ is an étale $\mathscr{O}_U$-algebra. Then $t_{j^*\mathscr{A}}$ and $t_{j'^*\mathscr{A}}$ are perfect pairings ([75, Th.4.1.14]). Taking (i) into account, we conclude that $\tau_{\mathscr{A},U}$ is an isomorphism, and the same applies to $\tau_{\mathscr{A},U'}$, so the inclusion map $j_*j^*\mathscr{A} \to j'_*j'^*\mathscr{A}$ restricts to a natural isomorphism

$$\mathscr{D}^{-1}_{\mathscr{A},U} \xrightarrow{\sim} \mathscr{D}^{-1}_{\mathscr{A},U'}.$$

More generally, if $j : U \to X$ and $j' : U' \to X$ are any two quasi-compact dense open immersions such that $j^*\mathscr{A}$ and $j'^*\mathscr{A}$ are respectively an étale $\mathscr{O}_U$-algebra and an étale $\mathscr{O}_{U'}$-algebra, then both $\mathscr{D}^{-1}_{\mathscr{A},U}$ and $\mathscr{D}^{-1}_{\mathscr{A},U'}$ are naturally identified with $\mathscr{D}^{-1}_{\mathscr{A},U\cap U'}$, and in this sense we may say that the inverse different of a finite and generically étale $\mathscr{O}_X$-algebra is independent of the open subset U. We shall therefore henceforth denote just by $\mathscr{D}^{-1}_{\mathscr{A}}$ the inverse different of $\mathscr{A}$.

Lemma 17.4.36. *In the situation of* (17.4.34)*, denote by $\mathscr{A}^\nu$ the normalization of $j^*\mathscr{A}$ over X (see remark* 14.4.3(ii)*). The following holds :*

(i) *If X is normal, $\mathscr{A} \subset \mathscr{D}^{-1}_{\mathscr{A},U}$.*

(ii) *Suppose that $\mathscr{A}$ is a generically étale $\mathscr{O}_X$-algebra. Then :*

 (a) *There exists a dense quasi-compact open subset $U' \subset U$ such that the map $\tau_{\mathscr{A},U'}$ restricts to an $\mathscr{O}_X$-linear isomorphism $\mathscr{D}^{-1}_{\mathscr{A}} \xrightarrow{\sim} \mathscr{A}^\vee$.*

 (b) *If X is locally coherent, $\mathscr{D}^{-1}_{\mathscr{A}}$ is a reflexive $\mathscr{O}_X$-module of finite type.*

 (c) *If X is normal, $\mathscr{A}^\nu \subset \mathscr{D}^{-1}_{\mathscr{A}}$.*

(iii) *Suppose that X is normal, and $j^*\mathscr{A}$ is an étale $\mathscr{O}_U$-algebra. Let also $j' : U' \to X$ be a quasi-compact open immersion with dense image, with $U' \subset U$, and $\mathscr{A}'^\nu$ the integral closure of $\mathscr{O}_X$ in $j'^*\mathscr{A}$. Then the natural map $\mathscr{A}^\nu \to \mathscr{A}'^\nu$ is an isomorphism.*

Proof. (i) follows immediately from lemma 14.3.15(i).

(ii.a) follows immediately from the discussion of remark (17.4.35)(iv,v), and (ii.b) follows from (ii.a) and lemma 11.3.3.

(ii.c): Taking into account remark 17.4.35(ii), we may assume without loss of generality that $\mathscr{A} = \operatorname{Im}(\mathscr{A} \to j_*j^*\mathscr{A})$, in which case the natural map $\mathscr{A} \to \mathscr{A}^\nu$ is a monomorphism. Moreover, the assertion is clearly local on X, hence we may assume that X is affine, in which case $\mathscr{A}^\nu$ is the union of the filtered family of its $\mathscr{A}$-subalgebras $\mathscr{B}$ that are quasi-coherent and finite $\mathscr{O}_X$-algebras. For any such $\mathscr{B}$, the $\mathscr{O}_X$-module $\mathscr{B}/\mathscr{A}$ is supported on $X \setminus U$. It follows that $(\mathscr{B}/\mathscr{A})^\vee = 0$, so the transpose map $\mathscr{B}^\vee \to \mathscr{A}^\vee$ is a monomorphism as well; by the same token, we see that $\tau_{\mathscr{A},U} = \tau_{\mathscr{B},U}$ (notation of (17.4.34)), so $\mathscr{D}^{-1}_{\mathscr{B}} \subset \mathscr{D}^{-1}_{\mathscr{A}}$, whence $\mathscr{B} \subset \mathscr{D}^{-1}_{\mathscr{A}}$, by virtue of (i). Since $\mathscr{B}$ is arbitrary, the assertion follows.

(iii): We may assume that X is local, say $X = \operatorname{Spec} R$ for a local and normal domain R, and let K be the field of fractions of R; then $A^\nu := \mathscr{A}^\nu(X)$ (resp. A'^ν) is the integral closure of R in $\mathscr{A}(U)$ (resp. in $\mathscr{A}(U')$); but under the stated assumptions, both A^ν and A'^ν are also the integral closure of R in $K \otimes_R \mathscr{A}(X)$, whence the assertion. □

Remark 17.4.37. Let X be a reduced and quasi-separated scheme, such that $\operatorname{Max} X$ is finite.

(i) Let also $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module of finite type. Then we claim that there exists a dense and quasi-compact open immersion $j : U \to X$ such that $j^*\mathscr{F}$ is a locally free $\mathscr{O}_U$-module. Indeed, for any maximal point $\eta \in X$, pick an affine open subset $V_\eta \subset X$ such that η is the unique maximal point of V_η; hence $V_\eta = \operatorname{Spec} A$ for an integral domain A and $\mathscr{F}_{|V} = M^\sim$ for an A-module M of finite type. Set $K := \operatorname{Frac} A$; then $P := M \otimes_A K$ is a finite dimensional K-vector space, and we may find a element $f \in A \setminus \{0\}$, a free A_f-module Q of finite rank and an A_f-linear map $\varphi : Q \to M \otimes_A A_f$ such that $\operatorname{Ker} \varphi \otimes_A K = 0 = \operatorname{Coker} \varphi \otimes_A K$. However, the natural map $Q \to Q \otimes_{A_f} K$ is injective, so we see that $\mathrm{Ker}\,\varphi = 0$, and on the other hand $\mathrm{Coker}\,\varphi$ is an A_f-module of finite type, so we may find $g \in A \backslash \{0\}$ such that φ_g is surjective. Set $U_\eta := \mathrm{Spec}\,A_{fg}$; by construction $\mathscr{F}_{|U_\eta}$ is a locally free $\mathscr{O}_{U_\eta}$-module, and since η is arbitrary, the assertion follows easily.

(ii) Let $\mathscr{A}$ be a quasi-coherent and finite $\mathscr{O}_X$-algebra, and suppose that $\mathscr{A}_\eta$ is an étale $\mathscr{O}_{X,\eta}$-algebra, for every maximal point η of X. Then we claim that there exists an dense quasi-compact open immersion $j : U \to X$ such that $j^*\mathscr{A}$ is an étale $\mathscr{O}_U$-algebra. Indeed, by (i), we may already assume that $\mathscr{A}$ is a locally free $\mathscr{O}_X$-module, and then we may argue as in remark 17.4.35(iv).

17.4.38. Keep the situation of (17.4.34), let X' be another reduced scheme, and $g : X' \to X$ a quasi-compact and quasi-separated morphism of schemes which restricts to a map

$$\mathrm{Max}\,X' \to \mathrm{Max}\,X.$$

Set

$$U' := g^{-1}U \qquad \mathscr{A}' := g^*\mathscr{A}$$

and denote by $j' : U' \to X'$ the resulting open immersion. It is easily seen that j' is quasi-compact and has dense image, and clearly $j'^*\mathscr{A}'$ is a locally free $\mathscr{O}_{U'}$-module, so there is a well defined inverse different $\mathscr{D}^{-1}_{\mathscr{A}',U'}$ relative to U'. Moreover, we have a commutative diagram

(17.4.39)
$$\begin{array}{ccccc} g^*j_*j^*\mathscr{A} & \xrightarrow{g^*\tau_{\mathscr{A},U}} & g^*\mathscr{H}om_{\mathscr{O}_X}(\mathscr{A}, j_*\mathscr{O}_U) & \longleftarrow & g^*(\mathscr{A}^\vee) \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} \\ j'_*j'^*\mathscr{A}' & \xrightarrow{\tau_{\mathscr{A}',U'}} & \mathscr{H}om_{\mathscr{O}_{X'}}(\mathscr{A}', j'_*\mathscr{O}_{U'}) & \longleftarrow & \mathscr{A}'^\vee \end{array}$$

([75, Lemma 4.1.13(iii)]). It follows that the left vertical arrow restricts to an $\mathscr{O}_{X'}$-linear map

(17.4.40)
$$g^*\mathscr{D}^{-1}_{\mathscr{A},U} \to \mathscr{D}^{-1}_{\mathscr{A}',U'}.$$

Lemma 17.4.41. *The map* (17.4.40) *is an isomorphism in the following cases :*

(a) *if $\mathscr{A}$ is a locally free $\mathscr{O}_X$-module of finite type and $j^*\mathscr{A}$ is an étale $\mathscr{O}_U$-algebra*

(b) *or if g is a flat morphism.*

Proof. For case (a), notice that when $\mathscr{A}$ is locally free, the map c is an isomorphism, and if $j^*\mathscr{A}$ is an étale $\mathscr{O}_U$-algebra, the discussion of remark 17.4.35(v) identifies (17.4.40) with the map c.

If g is flat, the map a in (17.4.39) is an isomorphism (corollary 10.3.8). The isomorphism of remark 17.4.35(i) identifies b with the composition

$$g^*j_*\mathscr{H}om_{\mathscr{O}_X}(j^*\mathscr{A}, \mathscr{O}_U) \xrightarrow{d} j'_*g^*\mathscr{H}om_{\mathscr{O}_X}(j^*\mathscr{A}, \mathscr{O}_U) \xrightarrow{e} j'_*\mathscr{H}om_{\mathscr{O}_{U'}}(j'^*\mathscr{A}', \mathscr{O}_{U'})$$

where d is an isomorphism, since g is flat (corollary 10.3.8), and the same holds for e, since $j^*\mathscr{A}$ is locally free of finite type (proposition 10.3.3(ii)). To conclude, it then suffices to show that c is an isomorphism as well. The assertion is local on X, hence we may assume that X is affine; in this case, we may find a finitely presented $\mathscr{O}_X$-module $\mathscr{G}$ and an $\mathscr{O}_X$-linear epimorphism $\varphi : \mathscr{G} \to \mathscr{A}$ such that $j^*\varphi$ is an isomorphism. Set $\mathscr{G}' := g^*\mathscr{G}$; there follows a commutative diagram

$$\begin{array}{ccc} g^*(\mathscr{A}^\vee) & \xrightarrow{g^*(\varphi^\vee)} & g^*(\mathscr{G}^\vee) \\ \downarrow & & \downarrow \\ \mathscr{A}'^\vee & \xrightarrow{(\varphi^*g)^\vee} & \mathscr{G}'^\vee \end{array}$$

whose right vertical arrows is an isomorphism, by virtue of [75, Lemma 2.4.29(i)]. Moreover $\mathrm{Ker}\,(\varphi^\vee) = 0$ and $\mathrm{Coker}\,(\varphi^\vee) \subset (\mathrm{Ker}\,\varphi)^\vee$, and since $\mathrm{Max}\,X \subset U$, we see that $\mathrm{Ker}\,\varphi_\eta = 0$ for every $\eta \in \mathrm{Max}\,X$, whence $(\mathrm{Ker}\,\varphi)^\vee = 0$, since X is reduced. Thus, the top horizontal arrow of the foregoing diagram is also an isomorphism; the same argument applies as well to the bottom horizontal arrow, and the assertion follows. □

17.4.42. In the situation of (17.4.38), suppose now that X and X' are locally coherent and $\mathscr{A}$ is generically étale. Taking into account lemma 17.4.36(ii.b), we see that (17.4.40) factors as a composition

$$g^*\mathscr{D}^{-1}_{\mathscr{A}} \xrightarrow{\beta} (g^*\mathscr{D}^{-1}_{\mathscr{A}})^{\vee\vee} \xrightarrow{\gamma} \mathscr{D}^{-1}_{\mathscr{A}'}$$

where β is the natural map as in (11.3).

Proposition 17.4.43. *With the notation of* (17.4.42)*, suppose moreover that :*

(a) *both X and X' are locally noetherian normal schemes*
(b) *g maps the points of codimension one of X' to points of codimension ≤ 1 of X.*

Then γ is an isomorphism.

Proof. Set $\mathscr{B} := \mathrm{Im}\,(\mathscr{A} \to j_*j^*\mathscr{A})$ and $\mathscr{B}' := g^*\mathscr{B}$. It is easily seen that the image of $\mathscr{B}'$ in $j'_*j'^*\mathscr{B}'$ agrees with the image of $\mathscr{A}'$; in view of remark 17.4.35(ii) we deduce that the natural maps $\mathscr{A} \to \mathscr{B}$ and $\mathscr{A}' \to \mathscr{B}'$ induce isomorphisms

$$\mathscr{D}^{-1}_{\mathscr{A}} \xrightarrow{\sim} \mathscr{D}^{-1}_{\mathscr{B}} \qquad \mathscr{D}^{-1}_{\mathscr{A}'} \xrightarrow{\sim} \mathscr{D}^{-1}_{\mathscr{B}'}.$$

We may then replace $\mathscr{A}$ by $\mathscr{B}$, and assume that the natural map $\mathscr{A} \to j_*j^*\mathscr{A}$ is a monomorphism. By lemmata 11.3.3 and 17.4.36(ii.b), both the source and target of γ are reflexive $\mathscr{O}_{X'}$-modules. Thus, denote by $V \subset X'$ the subset of all points such that γ_x is an isomorphism; then V is an open subset of X' ([126, Th.4.10]) and the inclusion map $V \to X'$ is quasi-compact, since X' is locally noetherian. In view of proposition 11.3.8(ii), it suffices to check that V contains all the points of codimension one of X'. Thus, let x' be such a point, and set $x := g(x')$; by assumption (b), the point x has codimension ≤ 1 in X, so $\mathscr{O}_{X,x}$ is either a field or a discrete valuation ring and $(j_*j^*\mathscr{A})_x$ is a torsion-free $\mathscr{O}_{X,x}$-module of finite type. Then the same holds for $\mathscr{A}_x$, so the latter is a free $\mathscr{O}_{X,x}$-module of finite rank. By lemma 17.4.41, it follows that $(17.4.40)_{x'}$ is an isomorphism. It is also clear that $\beta_{x'}$ is an isomorphism, and since $\gamma_{x'} = (17.4.40)_{x'} \circ \beta^{-1}_{x'}$, the contention follows. □

Remark 17.4.44. (i) Let us now consider a domain A, and a finite, injective, and generically étale ring homomorphism $f : A \to B$. As in (17.4), we let $X := \mathrm{Spec}\,A$, and denote by $\mathscr{B}$ the quasi-coherent $\mathscr{O}_X$-algebra arising from B. Let also K be the field of fractions of A; set $B_K := K \otimes_A B$, and let $\overline{B}$ be the image of B in B_K. We may then regard the diagonal idempotent e_f of the map f as an element in $B_K \otimes_K B_K$, and the associated trace form as a K-bilinear map $B_K \otimes_K B_K \to K$, inducing an isomorphism $\omega_f : B_K \xrightarrow{\sim} B^\vee_K$. With this notation, the inverse different $\mathscr{D}^{-1}_{\mathscr{B}}$ corresponds to $\omega_f^{-1}(\overline{B}^\vee) \subset B_K$, and the subset

$$\mathscr{D}_{\mathscr{B}} := \{b \in B \mid b\mathscr{D}^{-1}_{\mathscr{B}} \subset \overline{B}\}$$

is an ideal of B called the *different of f*. (If A and B are noetherian normal domains, then $\mathscr{D}^{-1}_{\mathscr{B}}$ is the inverse of $\mathscr{D}_{\mathscr{B}}$ in the group of reflexive fractional ideals of B : see example 6.4.38.)

(ii) Suppose moreover that B is a *projective A-module*. Then we have a natural isomorphism $B \otimes_A B^\vee \xrightarrow{\sim} \mathrm{Hom}_A(B, B)$, and we denote by ζ_f the unique element of $B \otimes_A B^\vee$ corresponding to $\mathbf{1}_B : B \to B$ under this isomorphism. According to [75, Rem.4.1.17], we have the identity :

$$(B_K \otimes_K \omega_f)(e_f) = \zeta_f$$

and it follows easily that :

$$(B \otimes_A \mathscr{D}_{\mathscr{B}}) \cdot e_f \subset \mathrm{Im}(B \otimes_A B \to B_K \otimes_K B_K).$$

(iii) If B is not a projective A-module, the inclusion of (ii) can fail; however, if A is a regular and noetherian domain, theorem 17.4.19 tells us that we have at least :

$$(\mathscr{D}_{\mathscr{B}} \otimes_A \mathscr{D}_{\mathscr{B}}) \cdot e_f \subset \mathrm{Im}(B \otimes_A B \to B_K \otimes_K B_K).$$

17.5. **Big Cohen-Macaulay algebras.** We shall use the following auxiliary construction. For every abelian group $(G, 0, +)$, set

$$G^\diamond := G^{\mathbb{N}}/G^{(\mathbb{N})}.$$

Hence, the elements of $G^\diamond$ are represented by sequences $(g_n \mid n \in \mathbb{N})$ of elements of G, and such a sequence represents the zero element of the quotient group $G^\diamond \Leftrightarrow$ the set $\{n \in \mathbb{N} \mid g_n \neq 0\}$ is finite. Especially, $G = 0 \Leftrightarrow G^\diamond = 0$. Moreover, for every group homomorphism $\varphi : G \to H$, we have $\varphi^{\mathbb{N}}(G^{(\mathbb{N})}) \subset H^{(\mathbb{N})}$, hence φ induces a group homomorphism

$$\varphi^\diamond : G^\diamond \to H^\diamond$$

and the rules : $G \mapsto G^\diamond$ and $\varphi :\mapsto \varphi^\diamond$ define an endofunctor of the category of abelian groups. Notice as well that $(\varphi + \psi)^\diamond = \varphi^\diamond + \psi^\diamond$ for every pair of group homomorphisms $\varphi, \psi : G \to H$. Lastly, we have an injective natural transformation that maps G diagonally into $G^\diamond$

$$\Delta_G : G \to G^\diamond \quad : \quad g \mapsto (g, g, \dots).$$

• If $(R, +, \cdot, 0, 1)$ is a ring, then $R^{(\mathbb{N})}$ is an ideal of the R-algebra $R^{\mathbb{N}}$ (the product in the category of rings of copies of R indexed by $\mathbb{N}$), hence $R^\diamond$ is naturally an R-algebra via the map Δ_R, and for every ring homomorphism $\varphi : R \to S$, the map $\varphi^\diamond : R^\diamond \to S^\diamond$ is a ring homomorphism such that

$$\Delta_S \circ \varphi = \varphi^\diamond \circ \Delta_R.$$

• Moreover, if M is any R-module, then $M^{\mathbb{N}}$ is obviously an $R^{\mathbb{N}}$-module, and since $R^{(\mathbb{N})} \cdot M^{\mathbb{N}} = M^{(\mathbb{N})}$, it follows that $M^\diamond$ is naturally an $R^\diamond$-module, and for every R-linear map $f : M \to N$, the map $f^\diamond : M^\diamond \to N^\diamond$ is $R^\diamond$-linear.

Lemma 17.5.1. (i) *The functor* $(-)^\diamond : R\text{-}\mathbf{Mod} \to R^\diamond\text{-}\mathbf{Mod}$ *is exact and faithful.*

(ii) *We have* $(IM)^\diamond = IM^\diamond$ *for every ideal* $I \subset R$ *of finite type, and every* R*-module* M.

Proof. (i): Since we have $M = 0 \Leftrightarrow M^\diamond = 0$, it is clear that $(-)^\diamond$ is faithful. Next, let $f : M \to N$ be an R-linear map, and $g_\bullet := (g_n \mid n \in \mathbb{N}) \in M^{\mathbb{N}}$; denote by $\overline{g}_\bullet \in M^\diamond$ the class of $g_\bullet$, and suppose that $f^\diamond(\overline{g}_\bullet) = 0$. This means that there exists $k \in \mathbb{N}$ such that $f(g_n) = 0$ for every $n \geq k$; then clearly $\overline{g}_\bullet$ is in the image of the natural map $(\mathrm{Ker}\, f)^\diamond \to M^\diamond$. In particular, if φ is injective, the same holds for $\varphi^\diamond$, and it is also clear that if φ is surjective, the same holds for $\varphi^\diamond$. It follows that the natural surjection $N \to \mathrm{Coker}\, f$ induces a surjection $N^\diamond \to (\mathrm{Coker}\, f)^\diamond$, whose kernel is $(\mathrm{Im}\, f)^\diamond$, and it is easily seen that $(\mathrm{Im}\, f)^\diamond = \mathrm{Im}(f^\diamond)$; hence $(\mathrm{Coker}\, f)^\diamond$ represents $\mathrm{Coker}(f^\diamond)$, whence the exactness of $(-)^\diamond$.

(ii): Let $a_1, \dots, a_k$ a finite system of generators of I; there follows an R-linear surjection $f : M^{\oplus k} \to IM$ whose composition with the inclusion map $j : IM \to M$ is given by the rule : $(m_1, \dots, m_k) \mapsto a_1 m_1 + \cdots + a_k m_k$. Hence $j^\diamond \circ f^\diamond = (j \circ f)^\diamond : (M^\diamond)^{\oplus k} \to M^\diamond$ is given by the same rule, and by the foregoing $f^\diamond : (M^\diamond)^{\oplus k} \to (IM)^\diamond$ is a surjection, whence the claim. □

17.5.2. Let now $(V, \mathfrak{m})$ be a basic setup (in the sense of [75, §2.1.1]), such that $\mathfrak{m}$ is the filtered union of a countable system of principal subideals. We consider the subset

$$\mathsf{S} \subset V^{\mathbb{N}}$$

of all $(a_n \mid n \in \mathbb{N})$ such that $Va_n \subset Va_{n+1}$ for every $n \in \mathbb{N}$, and $\mathfrak{m} \subset \bigcup_{n \in \mathbb{N}} Va_n$.

Remark 17.5.3. (i) Since $\mathfrak{m} = \mathfrak{m}^2$, it is easily seen that S is a multiplicative system.

(ii) Also, for every V-module M we have

$$\mathrm{Ker}(j_M : M^\diamond \to \mathsf{S}^{-1}M^\diamond) = \{x \in M^\diamond \mid \mathfrak{m}x = 0\} \qquad \text{and} \qquad \mathfrak{m} \cdot \mathrm{Coker}(j_M) = 0.$$

Especially, $j^a_M : (M^\diamond)^a \to (\mathsf{S}^{-1}M^\diamond)^a$ is an isomorphism of $(V,\mathfrak{m})^a$-modules.

(iii) Notice that S contains the multiplicative subset $\mathsf{T} := (e_{k,\bullet} \mid k \in \mathbb{N})$, where $e_{k,\bullet} := (e_{k,n} \mid n \in \mathbb{N})$, with $e_{k,n} := 0$ for every $n = 0, \dots, k-1$ and $e_{k,n} := 1$ for every $n \geq k$. Notice moreover that $e_{k,\bullet}$ is invertible in $V^\diamond$ for every $k \in \mathbb{N}$, and for every V-module M the projection $M^{\mathbb{N}} \to M^\diamond$ factors through the localization $M^{\mathbb{N}} \to \mathsf{T}^{-1}M^{\mathbb{N}}$ and an isomorphism

$$\mathsf{T}^{-1}M^{\mathbb{N}} \xrightarrow{\sim} M^\diamond.$$

Hence, we also get an isomorphism of $V^\diamond$-modules

$$\mathsf{S}^{-1}M^\diamond \xrightarrow{\sim} \mathsf{S}^{-1}M^{\mathbb{N}} \qquad \text{for every } V\text{-module } M.$$

Lemma 17.5.4. *In the situation of* (17.5.2)*, let $A \to V$ be a ring homomorphism. We have :*

(i) *$M^a = 0$ if and only if $\mathsf{S}^{-1}M^\diamond = 0$.*

(ii) *The functor V-$\mathbf{Mod} \to V^\diamond$-$\mathbf{Mod}$: $M \mapsto \mathsf{S}^{-1}M^\diamond$ factors via an exact faithful functor*

$$\mathsf{S}^{-1}(-)^\diamond : V^a\text{-}\mathbf{Mod} \to V^\diamond\text{-}\mathbf{Mod} \qquad M^a \mapsto \mathsf{S}^{-1}M^\diamond.$$

(iii) *The unit of adjunction $\mathsf{S}^{-1}M^\diamond \to (\mathsf{S}^{-1}M^\diamond)^a_*$ is an isomorphism of V-modules.*

(iv) *If A is a coherent ring, then $\mathsf{S}^{-1}M^\diamond$ is a flat A-module if and only if M^a is flat over A in the sense of definition* 14.2.40.

(v) *If A is noetherian, then $\mathsf{S}^{-1}M^\diamond$ is a faithfully flat A-module if and only if M^a is faithfully flat over A in the sense of definition* 14.2.40.

Proof. The proof of (i) shall be left to the reader.

(ii) is an immediate consequence of (i) and lemma 17.5.1(i).

(iii): Let $x \in \mathsf{S}^{-1}M^\diamond$ be an element such that $\mathfrak{m} \cdot x = 0$, and write $x = s_\bullet^{-1}m_\bullet$ for some $s_\bullet := (s_n \mid n \in \mathbb{N}) \in \mathsf{S}$ and $m_\bullet := (m_n \mid n \in \mathbb{N}) \in M^\diamond$. Let us also write $\mathfrak{m} = \bigcup_{i\in\mathbb{N}} Va_n$ for a sequence $(a_i \mid i \in \mathbb{N})$ of elements of V such that $Va_i \subset \mathfrak{m} \cdot a_{i+1}$ for every $i \in \mathbb{N}$. Then, for every $i \in \mathbb{N}$ there exist $\rho(i) \in \mathbb{N}$, $c_i \in \mathfrak{m}$, and $t_{i,\bullet} := (t_{i,n} \mid n \in \mathbb{N}) \in \mathsf{S}$ such that

$$a_i = c_i a_{i+1} \qquad \text{and} \qquad a_i m_n t_{i,n} = 0 \qquad \text{for every } n \geq \rho(i)$$

and we may assume that $\rho(i+1) > \rho(i)$ for every $i \in \mathbb{N}$. Moreover, since $\mathfrak{m} \subset \bigcup_{n\in\mathbb{N}} Vt_{i,n}$, we may also assume that $a_i \in Vt_{i,n}$ for every $n \geq \rho(i)$, in which case we get :

$$a_i^2 m_n = 0 \qquad \text{for every } i \in \mathbb{N} \text{ and } n \geq \rho(i).$$

Hence, let $b_\bullet := (b_n \mid n \in \mathbb{N}) \in V^{\mathbb{N}}$ be the element such that $b_n := 0$ for every $n < \rho(0)$, and $b_n := a_i^2$ for every $i \in \mathbb{N}$ and every $n = \rho(i), \dots, \rho(i+1)-1$. It follows easily that $b_\bullet \in \mathsf{S}$ and $b_\bullet \cdot m_\bullet = 0$ in $M^\diamond$, so finally $x = 0$. This shows that the unit of adjunction $\eta_{\mathsf{S}^{-1}M^\diamond} : \mathsf{S}^{-1}M^\diamond \to (\mathsf{S}^{-1}M^\diamond)^a_*$ is injective. Next, let $y \in (\mathsf{S}^{-1}M^\diamond)^a_*$. By remark 16.9.32(ii), the V-module $\widetilde{\mathfrak{m}}$ is the inductive limit of a system $(L_n \mid n \in \mathbb{N})$ with $L := V$ for every $n \in \mathbb{N}$, and with V-linear transition maps $\varphi_n : L_n \to L_{n+1}$ such that $\varphi_n(1) = c_n$ for every $n \in \mathbb{N}$; hence y is represented by a system $(y^{(n)} \mid n \in \mathbb{N})$ of elements of $\mathsf{S}^{-1}M^\diamond$ such that

$$(17.5.5) \qquad c_n y^{(n+1)} = y^{(n)} \qquad \text{for every } n \in \mathbb{N}.$$

Claim 17.5.6. Let $\Sigma \subset \mathsf{S}$ be any countable subset. Then $\mathsf{S} \cap \bigcap_{s_\bullet\in\Sigma} V^\diamond s_\bullet \neq \varnothing$.

Proof of the claim. Say that $\Sigma = \{s_{n,\bullet} \mid n \in \mathbb{N}\}$. Let the sequence $(a_n \mid n \in \mathbb{N})$ of generators of $\mathfrak{m}$ be as in the foregoing; then, for every $n, k \in \mathbb{N}$ there exists $\rho(n,k) \in \mathbb{N}$ such that

$$a_k \in Vs_{n,j} \qquad \text{for every } j \geq \rho(n,k).$$

Hence, for every $k \in \mathbb{N}$, we can find $\rho'(k) \in \mathbb{N}$ such that

$$a_k \in Vs_{n,j} \qquad \text{for every } n = 0, \dots, k \text{ and every } j \geq \rho'(k)$$

and moreover we may assume that $\rho'(k+1) > \rho'(k)$ for every $k \in \mathbb{N}$. Then, let $x_\bullet := (x_n \mid n \in \mathbb{N}) \in V^{\mathbb{N}}$ be the sequence such that $x_n := 0$ for every $n = 0, \dots, \rho'(0)-1$, and $x_n := a_k$

whenever $\rho'(k) \leq n < \rho'(k+1)$, for every $k \in \mathbb{N}$. It is easily seen that $x_\bullet \in \mathsf{S} \cap V^\diamond s_{n,\bullet}$ for every $n \in \mathbb{N}$, whence the claim. ◊

Now, say that $y^{(n)} = s_{n,\bullet}^{-1} m_{n,\bullet}$, with $s_{n,\bullet} \in \mathsf{S}$ and $m_{n,\bullet} \in M^\diamond$ for every $n \in \mathbb{N}$; by claim 17.5.6, we may assume that all the elements $s_{n,\bullet}$ are equal, in which case (17.5.5) amounts to the identities :

$$m_{n,\bullet} = c_n m_{n+1,\bullet} \qquad \text{in } \mathsf{S}^{-1}M^\diamond \text{ for every } n \in \mathbb{N}.$$

The latter in turns means that for every $n \in \mathbb{N}$ there exists $t_{n,\bullet} \in \mathsf{S}$ such that $t_{n,\bullet} m_{n,\bullet} = c_n t_{n,\bullet} m_{n+1,\bullet}$ in $M^\diamond$. By invoking again claim 17.5.6, we may also assume that all the $t_{n,\bullet}$ are equal; then set $m'_{n,\bullet} := t_{0,\bullet} m_{n,\bullet}$ for every $n \in \mathbb{N}$, so that $m'_{n,\bullet} = c_n m'_{n+1,\bullet}$ in $M^\diamond$ for every $n \in \mathbb{N}$. The latter means that for every $n \in \mathbb{N}$ there exists $\rho(n) \in \mathbb{N}$ such that

$$m_{n,j} = c_n m_{n+1,j} \qquad \text{for every } j \geq \rho(n)$$

and we may assume that $\rho(n+1) > \rho(n)$ for every $n \in \mathbb{N}$. Then, let $t_\bullet := (t_n \mid n \in \mathbb{N}) \in V^{\mathbb{N}}$ (resp. $l_\bullet := (l_n \mid n \in \mathbb{N}) \in M^{\mathbb{N}}$) be the element with $t_j := 0$ (resp. $l_j := 0$) for $j = 0, \dots, \rho(0) - 1$ and $t_j := a_n$ (resp. $l_j := m_{n,j}$) whenever $\rho(n) \leq j < \rho(n+1)$, for every $n \in \mathbb{N}$.

Claim 17.5.7. $t_\bullet m_{n,\bullet} = a_n l_\bullet$ in $M^\diamond$ for every $n \in \mathbb{N}$.

Proof of the claim. It suffices to show that for every $k \geq n$ and every $j = \rho(k), \dots, \rho(k+1) - 1$ we have $a_k m_{n,j} = a_n m_{k,j}$. The latter follows by a simple inspection. ◊

Set $z := t_\bullet^{-1} l_\bullet$; claim 17.5.7 implies that $m_{n,\bullet} = a_n y$ in $\mathsf{S}^{-1}M^\diamond$ for every $n \in \mathbb{N}$; hence $\eta_{\mathsf{S}^{-1}M^\diamond}(z) = y$, whence the assertion.

(iv): Suppose that M^a is flat over A, and let $I \subset A$ be an ideal of finite type; in light of lemma 17.5.1(ii), it suffices to check that the natural map

$$f : I \otimes_A \mathsf{S}^{-1}M^\diamond \to \mathsf{S}^{-1}(IM)^\diamond$$

is an isomorphism. Now, it is easily seen that for every A-module N of finite type, the natural map $N \otimes_A \mathsf{S}^{-1}M^\diamond \to \mathsf{S}^{-1}(N \otimes_A M)^\diamond$ is surjective; on the other hand, since A is coherent, there exists $k \in \mathbb{N}$ and a surjective A-linear map $A^{\oplus k} \to I$, whose kernel is an A-module N of finite type. Thus, we have the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N \otimes_A \mathsf{S}^{-1}M^\diamond & \longrightarrow & A^{\oplus k} \otimes_A \mathsf{S}^{-1}M^\diamond & \longrightarrow & I \otimes_A \mathsf{S}^{-1}M^\diamond & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & \mathsf{S}^{-1}(N \otimes_A M)^\diamond & \longrightarrow & \mathsf{S}^{-1}(M^{\oplus k})^\diamond & \longrightarrow & \mathsf{S}^{-1}(IM)^\diamond & \longrightarrow & 0
\end{array}$$

whose first and third vertical arrows are surjections, and whose central vertical arrow is an isomorphism. Moreover, the top horizontal sequence is right exact, and since M^a is flat over A, combining (i) and lemma 17.5.1(i) it is easily seen that the bottom horizontal arrow is a short exact sequence. Then, by the snake lemma, $\operatorname{Ker} f = 0$, whence the contention.

Conversely, suppose that $\mathsf{S}^{-1}M^\diamond$ is a flat A-module, and consider any short exact sequence $\Sigma := (0 \to N' \to N \to N'' \to 0)$ of A-modules; we need to show that the induced sequence $M^a \otimes_A \Sigma$ is still exact. To this aim, notice that Σ is the filtered colimit of a system of short exact sequences $(0 \to N'_\lambda \to N_\lambda \to N''_\lambda \to 0 \mid \lambda \in \Lambda)$ such that N_λ and N''_λ are finitely presented A-modules, for every $\lambda \in \Lambda$. Thus, we may assume that N and N'' is finitely presented, in which case the same holds for N', since A is coherent, and we consider the diagram :

$$\begin{array}{ccccccccccc}
\Sigma \otimes_A \mathsf{S}^{-1}M^\diamond & \quad & 0 & \longrightarrow & N' \otimes_A \mathsf{S}^{-1}M^\diamond & \longrightarrow & N \otimes_A \mathsf{S}^{-1}M^\diamond & \longrightarrow & N'' \otimes_A \mathsf{S}^{-1}M^\diamond & \longrightarrow & 0 \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
\mathsf{S}^{-1}((\Sigma \otimes_A M)^\diamond) & & 0 & \longrightarrow & \mathsf{S}^{-1}(N' \otimes_A M)^\diamond & \longrightarrow & \mathsf{S}^{-1}(N \otimes_A M)^\diamond & \longrightarrow & \mathsf{S}^{-1}(N'' \otimes_A M)^\diamond & \longrightarrow & 0.
\end{array}$$

Since N, N' and N'' are finitely presented, lemma 17.5.1(i) easily implies that all the vertical arrows are isomorphisms. By assumption, the top horizontal row is short exact, hence the same holds for the bottom one, and therefore the sequence of $(V, \mathfrak{m})^a$-modules $((\Sigma \otimes_A M)^\diamond)^a$ is short exact; then we conclude by invoking again lemma 17.5.1(i).

(v): Suppose that M^a is faithfully flat over A; in view of (iv), it suffices to show that for every A-module $N \neq 0$ of finite type, we have $\mathsf{S}^{-1}M^\diamond \otimes_A N \neq 0$. But then N is finitely presented, since A is noetherian, hence the natural map $\mathsf{S}^{-1}M^\diamond \otimes_A N \to \mathsf{S}^{-1}(M \otimes_A N)^\diamond$ is an isomorphism, and by assumption $M^a \otimes_A N \neq 0$; then the assertion follows from (i). One argues similarly for the converse : details left to the reader. □

Proposition 17.5.8. *In the situation of* (17.5.2)*, let $I \subset V$ be any ideal of finite type; then both $M^\diamond$ and $\mathsf{S}^{-1}M^\diamond$ are I-adically complete V-modules.*

Proof. Let $x_{\bullet\bullet} := (x_{k,\bullet} \mid k \in \mathbb{N})$ be a Cauchy sequence for the I-adic topology in $M^\diamond$. Hence, $x_{k,\bullet}$ is the class of a sequence $(x_{k,i} \mid i \in \mathbb{N})$, for every $i \in \mathbb{N}$, and after replacing $x_{\bullet\bullet}$ by a subsequence, we may assume that $x_{k,\bullet} - x_{k+1,\bullet} \in I^k M^\diamond$ for every $k \in \mathbb{N}$. In light of lemma 17.5.1(ii), the latter means that for every $k \in \mathbb{N}$ there exists $n(k) \in \mathbb{N}$ such that $x_{k,i} - x_{k+1,i} \in I^k M$ for every $i \geq n(k)$, and clearly we may assume that $n(k+1) > n(k)$ for every $k \in \mathbb{N}$. Now, let $y_\bullet := (y_i \mid i \in \mathbb{N})$ be the following element of $M^\diamond$. For every $i \leq n(0)$ we let $y_i := x_{0,i}$; then, for every $k \in \mathbb{N}$ and $n(k) < i \leq n(k+1)$ we let $y_i := x_{k+1,i}$. For every $k \geq j$ and $n(k) < i \leq n(k+1)$, we get :

$$y_i - x_{j,i} = (x_{k+1,i} - x_{k,i}) + \cdots + (x_{j+1,i} - x_{j,i}) \in I^k M + \cdots + I^j M = I^j M.$$

Hence, $y_\bullet - x_{j,\bullet} \in I^j M^\diamond$ for every $j \in \mathbb{N}$, so $y_\bullet$ is a limit of $x_{\bullet\bullet}$.

Next, let $x_{\bullet\bullet} := (x_{k,\bullet} \mid k \in \mathbb{N})$ be a Cauchy sequence for the I-adic topology in $\mathsf{S}^{-1}M^\diamond$; again, we may assume that $y_{k,\bullet} := x_{k+1,\bullet} - x_{k,\bullet} \in I^k\mathsf{S}^{-1}M^\diamond$ for every $k \in \mathbb{N}$, and then there exist $s_{k,\bullet} \in \mathsf{S}$ and $z_{k,\bullet} \in I^k M^\diamond$ with $y_{k,\bullet} = s_{k,\bullet}^{-1} \cdot z_{k,\bullet}$ for every $k \in \mathbb{N}$. Also, we may assume :

$$s_{k+1,\bullet} \in V^{\mathbb{N}} s_{k,\bullet} \qquad \text{for every } k \in \mathbb{N}. \tag{17.5.9}$$

Now, by assumption there exists a sequence $(a_n \mid n \in \mathbb{N})$ of elements of V such that $Va_n \subset Va_{n+1}$ for every $n \in \mathbb{N}$, and $\bigcup_{n \in \mathbb{N}} Va_n = \mathfrak{m}$. For every $k \in \mathbb{N}$ we define inductively a sequence of integers $(n(k,i) \mid i \in \mathbb{N})$ as follows. For $i = 0$, let $n(k,0)$ be the smallest of the integers $n \in \mathbb{N}$ such that $a_0 \in Vs_{k,n}$; then, for every $i > 0$ let $n(k,i) \in \mathbb{N}$ be the smallest of the integers $n > n(k,i-1)$ such that $a_i \in Vs_{k,n}$. Notice that (17.5.9) implies

$$n(k+1,i) \geq n(k,i) \qquad \text{for every } k,i \in \mathbb{N}.$$

Lastly, let $t_\bullet$ be the sequence such that $t_n := 0$ for every $n < n(0,0)$, and $t_n := a_i$ for every $n, i \in \mathbb{N}$ with $n(i,i) \leq n < n(i+1,i+1)$. We have $t_\bullet \in \mathsf{S}$, and $t_i \in Vs_{k,i}$ for every $k \in \mathbb{N}$ and every $i \geq n(k,k)$; thus, the class of $s_{k,\bullet}$ in $V^\diamond$ divides the class of $t_\bullet$, and we may replace $s_{k,\bullet}$ by $t_\bullet$ for every $k \in \mathbb{N}$. By the foregoing case, the series $\sum_{k \in \mathbb{N}} z_{k,\bullet}$ converges in $M^\diamond$, and let $z_\bullet$ be a limit; then $t_\bullet^{-1} z_\bullet$ is a limit for $x_{\bullet\bullet}$ in $\mathsf{S}^{-1}M^\diamond$. □

Lemma 17.5.10. *In the situation of* (16.9.34)*, let us endow $\mathsf{S}^{-1}A_0^\diamond$ with its b-adic topology, and let $A_0^\wedge$ be the b-adic completion of A_0; then the basic setup $(A_0^\wedge, \mathfrak{m}A_0^\wedge)$ is almost perfectoid if and only if the maximal separated quotient $(\mathsf{S}^{-1}A_0^\diamond)^{\mathrm{sep}}$ of $\mathsf{S}^{-1}A_0^\diamond$ is perfectoid.*

Proof. Clearly $\mathsf{S}^{-1}\overline{\Phi}_{A_0}^\diamond : \mathsf{S}^{-1}(A_0/bA_0)^\diamond \to \mathsf{S}^{-1}(A_0/b^pA_0)^\diamond$ is the ring endomorphism induced by Frobenius endomorphism of $\mathsf{S}^{-1}(A_0/b^pA_0)^\diamond$. On the other hand, by lemma 17.5.4(ii), $\overline{\Phi}_{A_0}^a$ is an isomorphism if and only if the same holds for $\mathsf{S}^{-1}\overline{\Phi}_{A_0}^\diamond$. Notice also that $C := (\mathsf{S}^{-1}A_0^\diamond)^{\mathrm{sep}}$ is b-adically complete and separated, by virtue of propositions 17.5.8 and 8.2.13(v). Then let $\overline{\Phi}_C : C/bC \to C/b^pC$ be the ring homomorphism induced by the Frobenius endomorphism of C/b^pC; in view of theorem 16.4.1, we are reduced to showing :

Claim 17.5.11. $\overline{\Phi}_C$ is an isomorphism if and only if the same holds for $\mathsf{S}^{-1}\overline{\Phi}^\diamond_{A_0}$.

Proof of the claim. By lemma 17.5.1(ii), the projection $A_0 \to A_0/bA_0$ induces a ring isomorphism $\mathsf{S}^{-1}A_0^\diamond/(b) \xrightarrow{\sim} \mathsf{S}^{-1}(A_0/bA_0)^\diamond$, and likewise $\mathsf{S}^{-1}A_0^\diamond/(b^p) \xrightarrow{\sim} \mathsf{S}^{-1}(A_0/b^pA_0)^\diamond$. Therefore, the projections $\mathsf{S}^{-1}A_0^\diamond \to C/bC$ and $\mathsf{S}^{-1}A_0^\diamond \to C/b^pC$ factor through ring isomorphisms

$$\mathsf{S}^{-1}(A_0/bA_0)^\diamond \xrightarrow{\sim} C/bC \qquad \mathsf{S}^{-1}(A_0/b^pA_0)^\diamond \xrightarrow{\sim} C/b^pC$$

whence the claim. ☐

Remark 17.5.12. As an alternative to our constructions of $G^\diamond$ and $\mathsf{S}^{-1}M^\diamond$, one could use an ultraproduct with respect to a non-principal ultrafilter of $\mathbb{N}$: this would lead to analogous results. In a similar vein, see remark 17.5.47.

17.5.13. Let A be a Tate ring, $A_0 \subset A$ a subring of definition, and $p \in \mathbb{N}$ a prime integer; suppose that there exists $b \in A_0^{\circ\circ} \cap A^\times$ with $pA_0 \subset b^pA_0$, and let $A_0^\wedge$ be the b-adic completion of A_0. Let also $g_\bullet := (g_n \mid n \in \mathbb{N}) \in \mathbf{E}(A_0)$, and set $g := g_0 \in A_0$, so that g^γ is well defined in A_0 for every $\gamma \in \mathbb{N}[1/p]$. We let $\mathfrak{m} := \bigcup_{\gamma\in\mathbb{N}[1/p]\setminus\{0\}} A_0g^\gamma$, and we suppose that the basic setup $(A_0^\wedge, \mathfrak{m}A_0^\wedge)$ is almost perfectoid (see definition 16.9.35).

Proposition 17.5.14. *In the situation of* (17.5.13)*, suppose moreover that A_0 is complete and separated, and let A_0^ν be the integral closure of the image of A_0 in $A_0[1/g]$. Then we have :*

(i) *The induced morphism $A_0^a \to (A_0^\nu)^a$ is an isomorphism of $(A_0, \mathfrak{m})^a$-algebras.*
(ii) *If moreover, the unit of adjunction $A_0 \to A_{0*}^a$ is an isomorphism, the same holds for the natural map $A_0 \to A_0^\nu$.*

Proof. (i): Let us check first that the kernel of the localization $A_0 \to A_0[1/g]$ is annihilated by $\mathfrak{m}$. Hence, let $x \in A_0$ such that $g^nx = 0$ for some $n \in \mathbb{N}$, and denote by $\overline{x}_\bullet \in B := (\mathsf{S}^{-1}A_0^\diamond)^{\mathrm{sep}}$ the image of the constant sequence $(x, x, x, \dots) \in A_0^{\mathbb{N}}$ (notation of lemma 17.5.10). Since B is reduced (corollary 16.3.63), we have $g^\gamma \cdot \overline{x} = 0$ in B for every $\gamma \in \mathbb{N}[1/p] \setminus \{0\}$. The latter means that for all $i \in \mathbb{N}$ and $\gamma \in \mathbb{N}[1/p] \setminus \{0\}$ there exists $s_\bullet := (s_n \mid n \in \mathbb{N}) \in \mathsf{S}$ such that $s_\bullet \cdot g^\gamma \cdot \overline{x} \in b^iA_0^\diamond$. In turn, this means that for every such i and γ there exists $s_\bullet$ and $n \in \mathbb{N}$ such that $s_mg^\gamma x \in b^iA_0$ for every $m \geq n$; equivalently, for every $i \in \mathbb{N}$ and $\gamma \in \mathbb{N}[1/p] \setminus \{0\}$ we have $g^\gamma x \in b^iA_0$, and then $g^\gamma x = 0$ in A_0 for every such γ, as required.

Next, we show that the cokernel of the map $A_0 \to A_0^\nu$ is annihilated by $\mathfrak{m}$. Indeed, let $a/g^n \in A_0^\nu$ for some $a \in A_0$ and $n \in \mathbb{N}$, and denote by $\overline{a}_\bullet \in B$ the image of the constant sequence $(a, a, a, \dots) \in A_0^{\mathbb{N}}$. Hence $\overline{a}_\bullet/g^n \in B[g^{-1}]$ lies in the integral closure C of the image $\overline{B}$ of B in $B[1/g]$. Since B is perfectoid, the induced map $B^a \to C^a$ is an isomorphism of $(A_0, \mathfrak{m})^a$-algebras (theorem 16.9.42(iv)), hence $g^\gamma \cdot (\overline{a}_\bullet/g^n) \in \overline{B}$ for every $\gamma \in \mathbb{N}[1/p] \setminus \{0\}$; then there exists $y_\bullet := (y_n \mid n \in \mathbb{N}) \in B$ such that

$$g^{\gamma+\gamma'} \cdot \overline{a}_\bullet = g^{n+\gamma'} \cdot y_\bullet \qquad \text{for every } \gamma, \gamma' \in \mathbb{N}[1/p].$$

Hence, for every $i \in \mathbb{N}$ there exists $s_\bullet \in \mathsf{S}$ such that $s_\bullet(g^{\gamma+\gamma'} \cdot \overline{a}_\bullet - g^{n+\gamma'} \cdot y_\bullet) \in b^iA_0^\diamond$. The latter implies that for every $i \in \mathbb{N}$ and every $\gamma \in \mathbb{N}[1/p] \setminus \{0\}$ the image $z_{\gamma,i}$ of $g^\gamma a$ in A_0/b^iA_0 lies in $J_i := g^n(A_0/b^iA_0)$. For every $i \in \mathbb{N}$, let $K_i := \mathrm{Ker}\, g^n\mathbf{1}_{A_0/b^iA_0}$; we get an exact sequence

$$K := \lim_{i\in\mathbb{N}} K_i \to A_0 = \lim_{i\in\mathbb{N}} A_0/b^iA_0 \xrightarrow{\alpha} J := \lim_{i\in\mathbb{N}} J_i \to \lim_{n\in\mathbb{N}}{}^1 K_i$$

and notice that $K = \mathrm{Ker}\, g^n\mathbf{1}_{A_0}$ lies in the kernel of the localization $A_0 \to A_0[1/g]$, hence $K^a = 0$, by the foregoing. Moreover, the system of inclusions $(J_i \to A_0/b^iA_0 \mid i \in \mathbb{N})$ induces a map $\beta : J \to A_0$ such that $\beta \circ \alpha = g^n\mathbf{1}_{A_0}$.

Claim 17.5.15. The inverse system $(K_i \mid i \in \mathbb{N})$ is almost essentially zero.

Proof of the claim. Since A_0 is almost perfectoid, the Frobenius endomorphism of A_0/b^pA_0 induces a map $A_0/bA_0 \to A_0/b^pA_0$ whose kernel is almost zero. By a simple induction on j, it follows that for every $j \in \mathbb{N}$ and every $x \in A_0$ such that $x^p \in b^{pj}A_0$, we have $g^\gamma x \in b^jA_0$ for every $\gamma \in \mathbb{N}[1/p] \setminus \{0\}$. Now, fix $i, k \in \mathbb{N}$ and let $x \in A_0$ whose class in $A_0/b^{p^ki}A_0$ lies in K_{p^ki}, *i.e.* $g^nx \in b^{p^ki}A_0$. Then $(g^{n/p^k}x)^{p^k} \in b^{p^ki}A_0$, and an easy induction on j yields

$$(g^{(j+1)n/p^k}x)^{p^{k-j}i} \in b^{p^{k-j}i}A_0 \qquad \text{for every } j = 0, \dots, k.$$

For $j = k$, it follows that $g^{(k+1)n/p^k}x \in b^iA_0$. This shows that for every $i \in \mathbb{N}$ and every $\beta \in \mathbb{N}[1/p] \setminus \{0\}$ there exists $k \in \mathbb{N}$ such that the image of K_{p^ki} in K_i is annhilated by g^β, whence the contention. ♢

From proposition 14.2.25(iii) and claim 17.5.15 we get $\lim^1_{i\in\mathbb{N}} K_i^a = 0$, hence α is almost surjective. By construction, $z_{\gamma,\bullet} := (z_{\gamma,i} \mid i \in \mathbb{N}) \in \lim_{i\in\mathbb{N}} J_i$; so we find for every $\gamma \in \mathbb{N}[1/p] \setminus \{0\}$ an element $a' \in A$ such that $\alpha(a') = g^\gamma z_{\gamma,\bullet}$, whence $g^na' = \beta \circ (a') = g^\gamma\beta(z_{\gamma,\bullet}) = g^{2\gamma}a$. The image of a' in A_0^ν therefore equals $g^{2\gamma}(a/g^n)$, and the proof of (i) is complete.

(ii): We have already seen that $\mathfrak{m}$ annihilates the kernel of the localization $j : A_0 \to A_0[1/g]$; on the other hand, A^a_{0*} has no $\mathfrak{m}$-torsion elements, so j is injective under the assumptions of (ii). By (i), the inclusion map $A_0 \to A_0^\nu$ induces an isomorphism $A_0 \xrightarrow{\sim} A^a_{0*} \xrightarrow{\sim} (A_0^\nu)^a_*$, and notice that $(A_0^\nu)^a_*$ is a subring of $A_0[1/g]$ containing A_0^ν; the assertion follows immediately. □

Proposition 17.5.16. *In the situation of* (17.5.13)*, let $h \in A_0$, and set $B := A_0[h^{1/p^\infty}] = A_0[X^{1/p^\infty}]/(X - h)$. Let moreover C be the p-root closure of B in $B[1/b]$, and $C^\wedge$ the b-adic completion of C. Then C^a is a faithfully flat $(A_0, \mathfrak{m})^a$-algebra, and the basic setup $(C^\wedge, \mathfrak{m}C^\wedge)$ is almost perfectoid for the b-adic topology of $C^\wedge$.*

Proof. For every A_0-module M, let $\Delta_M : M \to M^{\mathbb{N}}$ be the diagonal map, and for every $n \in \mathbb{N}$, let $\pi_{M,n} : M^{\mathbb{N}} \to M$ be the projection on the n-th factor. Let us regard $A_0^{\mathbb{N}}$ as an A_0-algebra via $\Delta_{A_0} : A_0 \to A_0^{\mathbb{N}}$, and denote by C' the p-root closure of $B' := B \otimes_{A_0} A_0^{\mathbb{N}}$ in $B'[1/b]$. By remark 9.8.25(ii), the induced map $B[1/b] \to B'[1/b]$ restricts to a ring homomorphism $j_C : C \to C'$. By the same token, $B[1/p] \otimes_{A_0} \pi_{A_0,n} : B'[1/b] \to B[1/b]$ restricts to a map $\varphi_{C,n} : C' \to C$ such that $\varphi_{C,n} \circ j_C = \mathbf{1}_C$, for every $n \in \mathbb{N}$. For every such M and every $n \in \mathbb{N}$ we then get maps of A_0-modules :

$$T_0 := \mathrm{Tor}_1^{A_0}(M, C) \xrightarrow{\alpha} T := \mathrm{Tor}_1^{A_0^{\mathbb{N}}}(M^{\mathbb{N}}, C') \xrightarrow{\beta_n} T_0$$

where α is induced by $(\Delta_{A_0}, \Delta_M, j_C)$, and β_n is induced by $(\pi_{A_0,n}, \pi_{M,n}, \varphi_{C,n})$. On the other hand, $\mathsf{S}^{-1}A_0^{\mathbb{N}}$ is a formally perfectoid ring for its b-adic topology, by lemma 17.5.10 and remark 17.5.3(iii), hence $\mathsf{S}^{-1}C'$ is a faithfully flat $\mathsf{S}^{-1}A_0^{\mathbb{N}}$-algebra, by theorem 16.9.17(i) and lemma 9.8.26(iv). Hence, $\mathsf{S}^{-1}T = 0$. Now, let $x \in T_0$; then there exists $s_\bullet := (s_n \mid n \in \mathbb{N}) \in \mathsf{S}$ with $s_\bullet \cdot \alpha(x) = 0$ in T, whence $s_n \cdot \beta_n \circ \alpha(x) = s_nx = 0$ in T_0 for every $n \in \mathbb{N}$, so $T_0^a = 0$. This shows that C^a is a flat A_0^a-algebra. Next, consider, for every $n \in \mathbb{N}$, the commutative diagram :

$$\begin{array}{ccc} M & \xrightarrow{\quad\Delta_M\quad} & M^{\mathbb{N}} \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\delta} \\ C \otimes_{A_0} M & \xrightarrow{j_C \otimes_{A_0} \Delta_M} & C' \otimes_{A_0^{\mathbb{N}}} M^{\mathbb{N}} \end{array}$$

and let $y \in \mathrm{Ker}\,\gamma$; then $\Delta_M(y) \in \mathrm{Ker}\,\delta$, and since $\mathsf{S}^{-1}C'$ is a faithfully flat $\mathsf{S}^{-1}A_0^{\mathbb{N}}$-algebra, it follows that there exists $s_\bullet \in \mathsf{S}$ with $s_\bullet \cdot \Delta_M(y) = 0$, *i.e.* $s_ny = 0$ for every $n \in \mathbb{N}$, so that $(\mathrm{Ker}\,\gamma)^a = 0$, which proves that C^a is a faithfully flat A_0^a-algebra.

Next, since b is a regular element of C, it is also regular in $C^\wedge$; also, we already know that the Frobenius endomorphism of $C^\wedge$ induces an injection $\overline{\Phi}_C : C/bC \to C/b^pC$ (lemma 9.8.26(i));

on the other hand, for any $n \in \mathbb{N}$, consider the commutative diagram :

$$\begin{array}{ccccc} C/bC & \xrightarrow{A_0/bA_0 \otimes_{A_0} j_C} & C'/bC' & \xrightarrow{A_0/bA_0 \otimes_{A_0} \varphi_{C,n}} & C/bC \\ \downarrow{\scriptstyle \overline{\Phi}_C} & & \downarrow{\scriptstyle \overline{\Phi}_{C'}} & & \downarrow{\scriptstyle \overline{\Phi}_C} \\ C/b^pC & \xrightarrow{A_0/b^pA_0 \otimes_{A_0} j_C} & C'/b^pC' & \xrightarrow{A_0/b^pA_0 \otimes_{A_0} \varphi_{C,n}} & C/b^pC. \end{array}$$

By theorem 16.9.17(ii), the map $\mathsf{S}^{-1}\overline{\Phi}_{C'}$ is an isomorphism. Now, let $x \in C/b^pC$, and pick $s_\bullet := (s_n \mid n \in \mathbb{N}) \in \mathsf{S}$ and $y_\bullet \in C'/bC'$ such that $\overline{\Phi}_{C'}(y_\bullet) = s_\bullet \cdot (A_0/b^pA_0 \otimes_{A_0} j_C)(x)$. Set $z_n := (A_0/bA_0 \otimes_{A_0} \varphi_{C,n})(y_\bullet)$ for every $n \in \mathbb{N}$; then $\overline{\Phi}_C(z_n) = s_n x$ for every $n \in \mathbb{N}$. This shows that $\overline{\Phi}_C$ is an almost isomorphism, so $(C^\wedge, \mathfrak{m}C^\wedge)$ is an almost perfectoid basic setup. □

Remark 17.5.17. (i) In the situation of (17.5.13), suppose moreover that there exists a perfectoid Tate ring R, with a perfectoid subring of definition R_0, and an adic ring homomorphism $R_0 \to A_0$, such that $g_\bullet$ is the image of an element $g'_\bullet := (g'_n \mid n \in \mathbb{N})$ of $\mathbf{E}(R_0)$. Then we get a basic setup $(R_0, \mathfrak{m}_R)$ with $\mathfrak{m}_R := \bigcup_{\gamma \in \mathbb{N}[1/p]\setminus\{0\}} g_0^{\prime\gamma} R_0$, and we can form as in (16.9.59) the topological R-algebra $A_0^\natural := R \otimes_{W(\mathbf{E}_0(R_0))} W(\mathbf{E}(A_0/pA_0))$, which is perfectoid and admits a natural adic map $A_0^\natural \to A_0$; the latter is an almost isomorphism relative to the basic setup $(R_0, \mathfrak{m}_R)$, by theorem 16.9.63(iii.a). Let A_0^ν (resp. $(A_0^\natural)^\nu$) be the integral closure of A_0 in $A_0[1/g]$ (resp. of $A_0^\natural$ in $A_0^\natural[1/g'_0]$). Then, on the one hand, the induced morphism $((A_0^\natural)^\nu)^a \to (A_0^\nu)^a$ is an isomorphism of $(R_0, \mathfrak{m}_R)^a$-algebras ([75, Lemma 8.2.28]); on the other hand, the same holds for the morphism $(A_0^\natural)^a \to ((A_0^\natural)^\nu)^a$, due to theorem 16.9.42(iv). In this way we thus obtain a shorter proof of proposition 17.5.14, under the stated further assumptions.

(ii) We can also give an alternative proof of the first assertion of proposition 17.5.16 along the same lines, under the assumptions of (i). Indeed, first let us remark that, since the map $A_0^\natural \to A_0$ is an almost isomorphism, there exists an almost element $l \in (A_0^\natural)^a_*$ whose image in $(A_0)^a_*$ agrees with h; by definition, $l : \mathfrak{m}_R \otimes_R \mathfrak{m}_R \to A_0^\natural$ is an R-linear map, and we set $l_{\gamma+\delta} := l(g_0^{\prime\gamma} \otimes g_0^{\prime\delta})$ for every $\gamma, \delta \in \mathbb{N}[1/p] \setminus \{0\}$. It is easily seen that $l_{\gamma+\delta}$ depends only on the sum $\gamma + \delta$, hence we get a compatible system $(l_\gamma \mid \gamma \in \mathbb{N}[1/p] \setminus \{0\})$ of elements of $A_0^\natural$ such that $g_0^{\prime\delta} \cdot l_\gamma = l_{\gamma+\delta}$ for every such γ and δ. Let D_γ (resp. E_γ) be the p-root closure of $A_{0,\gamma}^\natural := A_0^\natural[l_\gamma^{1/p^\infty}]$ in $A_{0,\gamma}^\natural[1/b]$ (resp. of $A_{0,\gamma} := A_0[(g^\gamma h)^{1/p^\infty}]$ in $A_{0,\gamma}[1/b]$) for every $\gamma \in \mathbb{N}[1/p] \setminus \{0\}$; by theorem 16.9.17(i), D_γ is a faithfully flat $A_0^\natural$-algebra; moreover, for every $\gamma, \delta \in \mathbb{N}[1/p] \setminus \{0\}$, the natural map $A_{0,\gamma+\delta}^\natural \to A_{0,\gamma}^\natural$ induces a map of $A_0^\natural$-algebras $D_{\gamma+\delta} \to D_\gamma$ (remark 9.8.25(ii)). Furthermore, for every such γ we have a natural map $i_\gamma : A_{0,\gamma}^\natural \to A_{0,\gamma}$, whence an induced map $j_\gamma : D_\gamma \to E_\gamma$. Since $p \in b^pA_0$, the localization $A_{0,\gamma} \to A_{0,\gamma}[1/b]$ induces an isomorphism $A_{0,\gamma}[1/p] \xrightarrow{\sim} A_{0,\gamma}[1/b, 1/p]$, and therefore E_γ is also the weak normalization of $A_{0,\gamma}$ in $A_{0,\gamma}[1/b]$ (proposition 9.8.28); likewise we see that D_γ is the weak normalization of $A_{0,\gamma}^\natural$ in $A_{0,\gamma}^\natural[1/b]$. On the other hand, clearly i_γ^a is an isomorphism; taking into account 14.3.12(iii), we conclude that the same holds for j_γ^a. Lastly, the compatible system of ring homomorphisms $(A_0[(g^\gamma h)^{1/p^\infty}] \to A_0[h^{1/p^\infty}] \mid \gamma \in \mathbb{N}[1/p] \setminus \{0\})$ yields a compatible system of maps $(E_\gamma \to B \mid \gamma \in \mathbb{N}[1/p] \setminus \{0\})$, and it is easily seen that the induced co-cone of A_0^a-algebras $(E_\gamma^a \to B^a \mid \gamma \in \mathbb{N}[1/p] \setminus \{0\})$ is universal. Summing up, B^a is the colimit of the filtered system of faithfully flat A_0^a-algebras $(D_\gamma^a \mid \gamma \in \mathbb{N}[1/p] \setminus \{0\})$, so it is a faithfully flat A_0^a-algebra (lemma 14.3.7(ii)).

(iii) Likewise, by applying the functor $(-)^\natural$ in a similar fashion, one may show that the basic setup $(C^\wedge, \mathfrak{m}C^\wedge)$ is almost perfectoid : we leave the details as an exercise for the reader.

17.5.18. Let $(A, \mathfrak{n})$ be a noetherian local ring. The aim of this section is to exhibit an A-algebra B such that $\mathrm{depth}_A B = \dim A$ (notation of remark 10.4.29). Quite generally, we say that an

A-module M is *big Cohen-Macaulay* (or briefly, that M is *big CM*) if $\mathrm{depth}_A M = \dim A$; hence our sought B is a *big CM A-algebra*.

Remark 17.5.19. Set $d := \dim A$, and let $\mathbf{f} := (f_1, \dots, f_d)$ a *system of parameters* for A, *i.e.* a system of generators for an $\mathfrak{n}$-primary ideal of A. Let also M be an A-module.

(i) By proposition 10.4.32(i), M is a big CM A-module if and only if $H^i(\mathbf{f}, M) = 0$ for every $i = 0, \dots, d-1$, and $H^d(\mathbf{f}, M) \neq 0$, or equivalently, if and only if $\mathbf{f}$ is completely secant on M (see definition 7.8.6), and $H_0(\mathbf{f}, M) = M/\mathbf{f}M \neq 0$ (lemma 7.8.2(v)). It follows easily that M is a big CM A-module if and only if $H^i(\mathbf{f}, M) = 0$ for $i = 0, \dots, d-1$, and $M/\mathfrak{n}M \neq 0$.

(ii) Notice also that a big CM A-module is the same as a Cohen-Macaulay module whose support contains an irreducible component of $\mathrm{Spec}\, A$ of dimension d (see definition 11.3.60(ii)). The additional *big* adjective can then be somewhat justified, by remarking that the condition on the support often forces an A-algebra B to be of non-finite type : the question is discussed in B.Bhatt's article [21], which exhibits a non-trivial cohomological obstruction to the existence of *finite* CM A-algebras, when A has positive characteristic. On the other hand, it is conjectured that every complete local noetherian domain admits a faithful Cohen-Macaulay *module* of finite type : see [142, Conj.12]; moreover, sometimes a big CM-algebra can be of finite type : *e.g.* this is the case for a Nagata universally catenary local domain $(A, \mathfrak{n})$ of dimension 2. Indeed, the normalization B of A in its field of fractions is a finite A-algebra, hence noetherian, catenary and of dimension 2; so it is a Cohen-Macaulay ring, by Serre's criterion for normality ([126, Th.23.8]), and the support of B is $\mathrm{Spec}\, A$.

(iii) Let $f : (A', \mathfrak{n}') \to (A, \mathfrak{n})$ be a local ring homomorphism of local noetherian rings with $\dim A' = \dim A$, and such that $\mathfrak{n}$ is the radical of $\mathfrak{n}'A$. Let B be an A-algebra, and B' an A'-algebra. We have :

(a) B is a big CM A-algebra if and only if, by restriction of scalars, B is a big CM A'-algebra (lemma 10.4.17(ii)).

(b) If f is faithfully flat, then B' is a big CM A'-algebra if and only if $A \otimes_{A'} B'$ is a big CM A-algebra (corollary 10.4.37).

(iv) Let $B_\bullet := (B_\lambda \mid \lambda \in \Lambda)$ be a filtered system of big CM A-algebras. Then the colimit B of $B_\bullet$ is also a big CM A-algebra. Indeed, since the functor $H_i(\mathbf{f}, -)$ commutes with filtered colimits, we have $H_i(\mathbf{f}, B) = 0$ for $i = 0, \dots, d-1$, and if we had $B/\mathfrak{n}B = 0$, we would have $1 = 0$ in $B/\mathfrak{n}B$, so $1 \in \mathfrak{n}B_\lambda$ for some $\lambda \in \Lambda$, but we know from (i) that $B_\lambda/\mathfrak{n}B_\lambda \neq 0$, whence the assertion.

Lemma 17.5.20. *Let $(A, \mathfrak{m}_A)$ and $(B, \mathfrak{m}_B)$ be local noetherian rings, $f : A \to B$ a local ring homomorphism of complete intersection (see definition* 9.7.30(ii)*), and M a big CM A-module. Then the $\mathfrak{m}_B$-adic completion $B\widehat{\otimes}_A M$ of $B \otimes_A M$ is a big CM B-module.*

Proof. We establish first the following special cases :

Claim 17.5.21. (i) The $\mathfrak{m}_A$-adic completion $M^\wedge$ of M is a big CM A-module.

(ii) If f is surjective, with kernel generated by a regular sequence $\mathbf{g} := (g_1, \dots, g_r)$ of A, and M is $\mathfrak{m}_A$-adically complete and separated, then $B \otimes_A M$ is a big CM B-module.

Proof of the claim. (i): Let $\mathbf{f}$ be any system of parameters for A. Clearly the sequence $\mathbf{f}$ is M-quasi-regular if and only if it is $M^\wedge$-quasi-regular (see definition 7.8.13). Then the assertion follows from proposition 7.8.15 and remark 17.5.19(i). (The assertion is already essentially contained in [15, Th.1.7], which shows, by a different argument, that if a given system of parameters for A is M-regular, then every system of parameters for A is $M^\wedge$-regular.)

(ii): By [63, Ch.0, Prop.16.4.1] we may find a sequence $\mathbf{g}' := (g_{r+1}, \dots, g_d)$ such that $\mathbf{g}'' := (g_1, \dots, g_d)$ is a system of parameters for A, and then $H_i(\mathbf{g}'', M) = 0$ for every $i > 0$, by remark 17.5.19(i). Consequently, $\mathbf{g}''$ is a regular sequence on M (proposition 7.8.15), so that

$\mathbf{g}'$ is regular on $M/\mathbf{g}M$, and thus $H_i(\mathbf{g}', B \otimes_A M) = 0$ for every $i > 0$, again by proposition 7.8.15; in view of remark 17.5.19(i), we are then reduced to checking that $\mathbf{g}'$ is a system of parameters for B. But the latter follows immediately from [126, Th.14.1]. ◊

Now, let us pick a Cohen factorization $A \xrightarrow{g} (C, \mathfrak{m}_C) \xrightarrow{h} B^\wedge$ of f, such that $\mathrm{Ker}\, h$ is generated by a regular sequence of $\mathfrak{m}_C$. Since g is flat, and $\overline{C} := C/\mathfrak{m}_A C$ is regular, we have :

$$\dim C = \dim A + \dim \overline{C} = \mathrm{depth}_A M + \mathrm{depth}_{\overline{C}} \overline{C} = \mathrm{depth}_C (C \otimes_A M)$$

by [126, Th.15.1] and theorem 10.4.38. Combining with claim 17.5.21(i), we see that the $\mathfrak{m}_C$-adic completion $C\widehat{\otimes}_A M$ of $C \otimes_A M$ is a CM C-module. Then, claim 17.5.21(ii) says that $N := B^\wedge \otimes_C (C\widehat{\otimes}_A M)$ is a CM $B^\wedge$-module, so also a CM B-module, by remark 17.5.19(iii.a). Lastly, it is easily seen that the $\mathfrak{m}_B$-adic completion of N is naturally isomorphic to $B\widehat{\otimes}_A M$, so the latter is a big CM B-module, again by claim 17.5.21(i). □

To explain how we shall produce big CM-algebras, let us make the following :

Definition 17.5.22. Let $(A, \mathfrak{n})$ be a noetherian local ring, V an A-algebra, $\mathfrak{m} \subset V$ an ideal such that $\mathfrak{m} = \mathfrak{m}^2$, so that $(V, \mathfrak{m})$ is a basic setup, in the sense of [75, §2.1.1], and suppose moreover that $\widetilde{\mathfrak{m}} := \mathfrak{m} \otimes_V \mathfrak{m}$ is a flat V-module. Let also M be a V-module; we say that M is *big almost CM over* A if $M^a_!$ is a big Cohen-Macaulay A-module.

With this terminology, our strategy relies on the following observation, which is a replacement of Hochster's techniques of *algebra modifications*, that have been employed in previous papers, to turn almost CM algebras into CM algebras (see *e.g.* [94], [5], [7], [153]).

Lemma 17.5.23. *In the situation of definition* 17.5.22*, set* $d := \dim A$*, and suppose that the basic setup* $(V, \mathfrak{m})$ *fulfills the condition of* (17.5.2)*. The following conditions are equivalent :*

(a) *M is big almost CM over A.*
(b) $(R^i\Gamma_{\{\mathfrak{n}\}}M)^a = 0$ *for every* $i = 0, \dots, d-1$*, and* $(R^d\Gamma_{\{\mathfrak{n}\}}M)^a \neq 0$.
(c) *$M^\diamond$ is big almost CM over A.*
(d) $\mathsf{S}^{-1}M^\diamond$ *is a big Cohen-Macaulay A-module.*

Proof. (Here, $\mathsf{S} \subset V^{\mathbb{N}}$ is the multiplicative subset associated with $\mathfrak{m}$, as in (17.5.2).)
Let $Z := \mathrm{Spec}\, V/\mathfrak{n}V$; ; in view of lemma 10.4.13(v), we have a natural identification

$$R^i\Gamma_{\{\mathfrak{n}\}}M \xrightarrow{\sim} R^i\Gamma_Z M \qquad \text{for every } i \in \mathbb{N}$$

whence a natural V-module structure on $R^i\Gamma_{\{\mathfrak{n}\}}M$, so the $(V, \mathfrak{m})^a$-module $(R^i\Gamma_{\{\mathfrak{n}\}}M)^a$ is well defined, and the lemma will follow from the more general :

Claim 17.5.24. Let $I \subset V$ be a finitely generated ideal, $d \in \mathbb{N}$, and $Z := \mathrm{Spec}\, V/I \subset \mathrm{Spec}\, V$. The following conditions are equivalent :

(a) $\mathrm{depth}_I(M^a_!) = d$ (notation of (10.4.29)).
(b) $(R^i\Gamma_Z M)^a = 0$ for every $i < d$ and $(R^d\Gamma_Z M)^a \neq 0$.
(c) $\mathrm{depth}_I((M^\diamond)^a_!) = d$.
(d) $\mathrm{depth}_I(\mathsf{S}^{-1}M^\diamond) = d$.

Proof of the claim. (a)⇔(b): Since $\widetilde{\mathfrak{m}}$ is a flat V-module, we have natural identifications :

$$(R^i\Gamma_Z M)^a_! \xrightarrow{\sim} \widetilde{\mathfrak{m}} \otimes_V R^i\Gamma_Z M \xrightarrow{\sim} R^i\Gamma_Z(\widetilde{\mathfrak{m}} \otimes_V M) \xrightarrow{\sim} R^i\Gamma_Z(M^a_!) \qquad \text{for every } i \in \mathbb{N}$$

provided by lemma 10.4.17(iv), whence the contention.

(a)⇔(c)⇔(d): Let $\mathbf{f}$ be a finite system of generators of I, and $\mathbf{K}^\bullet(\mathbf{f}, M)$ the Koszul complex of M relative to the sequence $\mathbf{f}$. Let $\mathsf{S}^{-1}\mathbf{K}^\bullet(\mathbf{f}, M_!)^\diamond$ be the complex obtained by applying the functor $\mathsf{S}^{-1}(-)^\diamond$ termwise to $\mathbf{K}^\bullet(\mathbf{f}, M_!)$, and define likewise $\mathbf{K}^\bullet(M)^\diamond$ and $\mathbf{K}^\bullet(\mathbf{f}, M)^a_!$. By direct inspection (and since $\widetilde{\mathfrak{m}}$ is a flat V-module), we easily obtain natural identifications :

$$\mathbf{K}^\bullet(\mathbf{f}, M)^a_! \xrightarrow{\sim} \mathbf{K}^\bullet(\mathbf{f}, M^a_!) \quad \mathbf{K}^\bullet(\mathbf{f}, M)^\diamond \xrightarrow{\sim} \mathbf{K}^\bullet(\mathbf{f}, M^\diamond) \quad \mathsf{S}^{-1}\mathbf{K}^\bullet(\mathbf{f}, M)^\diamond \xrightarrow{\sim} \mathbf{K}^\bullet(\mathbf{f}, \mathsf{S}^{-1}M^\diamond).$$

In view of lemmata 17.5.1(i) and 17.5.4(ii), we deduce for every $i \in \mathbb{N}$ natural isomorphisms :

$$H^i(\mathbf{f}, M)^a_! \xrightarrow{\sim} H^i(\mathbf{f}, M^a_!) \qquad H^i(\mathbf{f}, M)^\diamond \xrightarrow{\sim} H^i(\mathbf{f}, M^\diamond) \qquad \mathsf{S}^{-1}H^i(\mathbf{f}, M)^\diamond \xrightarrow{\sim} H^i(\mathbf{f}, \mathsf{S}^{-1}M^\diamond)$$

and moreover, $H^i(\mathbf{f}, M)^a = 0$ if and only if $\mathsf{S}^{-1}H^i(\mathbf{f}, M)^\diamond = 0$. Combining with proposition 10.4.32(i), the assertion follows. □

Hence, we shall seek a suitable basic setup $(V, \mathfrak{m})$ as in definition 17.5.22, fulfilling the additional condition of (17.5.2), together with a V-algebra C whose underlying V-module is big almost CM over A; in this situation, lemma 17.5.23 will ensure that $\mathsf{S}^{-1}C^\diamond$ is a big CM A-algebra. Proposition 17.5.25 and theorem 17.5.28 below illustrate the method in the easier case where A is an $\mathbb{F}_p$-algebra, for some prime integer p.

Proposition 17.5.25. *Let p be a prime integer, $(A, \mathfrak{n})$ a noetherian local $\mathbb{F}_p$-algebra, and $a \in A$ a non-nilpotent element. Let also A^{perf} be the perfection of A, defined as in* (9.8.11), *and $J \subset A^{\mathrm{perf}}$ the ideal generated by $(a^{1/p^n} \mid n \in \mathbb{N})$. We have :*

(i) $J(A^{\mathrm{perf}}/\mathfrak{n}A^{\mathrm{perf}}) \neq 0$.

(ii) *If A is excellent and equidimensional, and $A[a^{-1}]$ is a Cohen-Macaulay ring, then :*

$$J \cdot R^i\Gamma_{\{\mathfrak{n}\}}A^{\mathrm{perf}} = 0 \qquad \textit{for } i = 0, \dots, \dim A - 1.$$

Proof. (i): Let $\mathfrak{q} \subset A$ be a minimal prime ideal that does not contain a, and set $B := A/\mathfrak{q}$; clearly B^{perf} is a quotient of A^{perf}, hence it suffices to check that $J(B^{\mathrm{perf}}/\mathfrak{n}B^{\mathrm{perf}}) \neq 0$. Thus, we may replace A by B, and assume that A is a domain. If $a \in A^\times$, the assertion is trivial; suppose then that $a \notin A^\times$, in which case a lies in a prime ideal $\mathfrak{p} \subset A$ of height one ([126, Th.13.5]). Set $C := A_\mathfrak{p}$; clearly C^{perf} is a localization of A^{perf}, hence it suffices to check that $J(C^{\mathrm{perf}}/\mathfrak{n}C^{\mathrm{perf}}) \neq 0$. Thus, we may replace A by C, and assume that A is a one-dimensional domain, in which case there exists a one-dimensional valuation ring V of $\mathrm{Frac}\,A$ that dominates A (claim 15.3.27). It is easily seen that V^{perf} is a one-dimensional valuation ring of $\mathrm{Frac}\,(A^{\mathrm{perf}})$ dominating A^{perf} (the details shall be left to the reader). We have to show that $JA^{\mathrm{perf}} \not\subset \mathfrak{n}A^{\mathrm{perf}}$, and it suffices to check that $JV^{\mathrm{perf}} \not\subset \mathfrak{n}V^{\mathrm{perf}}$. However, since $\mathfrak{n}$ is an ideal of finite type, $\mathfrak{n}V^{\mathrm{perf}}$ is a principal ideal lying in the maximal ideal $\mathfrak{m}_V^{\mathrm{perf}}$ of V^{perf}; on the other hand, the value group of V^{perf} is of rank one and p-divisible, and $a \in \mathfrak{m}_V^{\mathrm{perf}}$, so $JV^{\mathrm{perf}} = \mathfrak{m}_V^{\mathrm{perf}}$, whence the assertion.

(ii): Let us first notice :

Claim 17.5.26. Let $i \in \mathbb{N}$ such that $a^n R^i\Gamma_{\{\mathfrak{n}\}}A = 0$ for some $n \in \mathbb{N}$; then $J \cdot R^i\Gamma_{\{\mathfrak{n}\}}A^{\mathrm{perf}} = 0$.

Proof of the claim. Let $\Phi_A : A \to A$ be the Frobenius endomorphism, and for every $k \in \mathbb{N}$ and every A-module M, let $M_{(k)}$ be the A-module obtained from M by restriction of scalars along the map $\Phi_A^k : A \to A$. Clearly $\mathfrak{n}$ is the radical of the ideal $\Phi_A^k(\mathfrak{n})$, for every $k \in \mathbb{N}$, hence we have a natural identification of A-modules :

$$H^i_k := R^i\Gamma_{\{\mathfrak{n}\}}(A_{(k)}) = (R^i\Gamma_{\{\mathfrak{n}\}}A)_{(k)} \qquad \text{for every } i, k \in \mathbb{N}$$

(lemma 10.4.13(v)). It follows that $a^{n/k}H^i_k = 0$ for every $k \in \mathbb{N}$. But by definition, A^{perf} is the direct limit of the system of A-algebras $(A_{(k)} \mid k \in \mathbb{N})$, hence $R^i\Gamma_{\{\mathfrak{n}\}}A^{\mathrm{perf}}$ is the direct limit of the system of A-modules $(H^i_k \mid k \in \mathbb{N})$ (proposition 10.4.9), whence the claim. ◊

In light of claim 17.5.26, it suffices to check that for every $i = 0, \dots, \dim A - 1$ we have $a^n R^i\Gamma_{\{\mathfrak{n}\}}A = 0$ for some $n \in \mathbb{N}$. Let $(A^\wedge, \mathfrak{n}^\wedge)$ be the completion of A; by assumption, the induced morphism of schemes $\mathrm{Spec}\,A^\wedge \to \mathrm{Spec}\,A$ is faithfully flat and regular, hence $A^\wedge[a^{-1}]$ is again a Cohen-Macaulay ring ([126, Th.32.2(i)]), and $\dim A^\wedge = \dim A$; moreover, $A^\wedge$ is equidimensional, by [126, Th.31.7]. Furthermore, since $\mathfrak{n}^\wedge = \mathfrak{n}A^\wedge$, we have a natural identification

$$R^i\Gamma_{\{\mathfrak{n}^\wedge\}}A^\wedge = A^\wedge \otimes_A R^i\Gamma_{\{\mathfrak{n}\}}A \qquad \text{for every } i \in \mathbb{N}$$

(lemma 10.4.17(iii)). Thus, it suffices to check that for every $i = 0, \dots, \dim A^\wedge - 1$ there exists $n \in \mathbb{N}$ such that $a^n R^i\Gamma_{\{\mathfrak{n}^\wedge\}}A^\wedge = 0$; *i.e.* we may replace A by $A^\wedge$, and assume from start that A is complete, and especially, A admits a dualizing complex $\omega^\bullet$ (remark 11.3.45(iv)). Set $X := \operatorname{Spec} A$, and define the map $c_X : |X| \to \mathbb{Z}$ as in (11.3.49); since A is equidimensional, we have $c_X(\mathfrak{n}) - c_X(\mathfrak{p}) = \dim A$ for every minimal prime ideal $\mathfrak{p} \subset A$ (lemma 11.3.50). Thus, after replacing $\omega^\bullet$ by a suitable shift $\omega^\bullet[c]$, we may assume that $c_X(\mathfrak{p}) = 0$ for every minimal prime ideal $\mathfrak{p} \subset A$, and consequently $c_X(x) = \dim \mathscr{O}_{X,x}$ for every $x \in X$.

Claim 17.5.27. $H^i(\omega^\bullet) = 0$ for every $i < 0$, and $A[a^{-1}] \otimes_A H^i(\omega^\bullet) = 0$ for every $i > 0$.

Proof of the claim. Let $x \in X$ be any point; by proposition 11.3.37, the complex $\omega^\bullet(x)$ is dualizing on $X(x)$, and denote by $\mathscr{D}_x$ the corresponding dualizing functor. Recall that $\mathscr{D}_x(\mathscr{O}_{X(x)}) = \omega^\bullet_x$, and that the natural map

$$\mathscr{O}_{X(x)} \to \mathscr{D}_x \circ \mathscr{D}_x(\mathscr{O}_{X(x)}) = \mathscr{D}_x(\omega^\bullet(x))$$

is an isomorphism. Let also $A^\wedge_x$ be the completion of the local ring $A_x := \mathscr{O}_{X,x}$; by corollary 11.5.17(iii) there follow natural isomorphisms :

$$D(H^{c(x)-i}_{\{x\}}\mathscr{O}_{X(x)}) \xrightarrow{\sim} D(H^{c(x)-i}_{\{x\}}\mathscr{D}_x(\omega^\bullet(x))) \xrightarrow{\sim} A^\wedge_x \otimes_A H^i(\omega^\bullet(x)) \qquad \text{for every } i \in \mathbb{Z}$$

where the functor D is defined as in (11.5.16). By assumption, if $x \in \operatorname{Spec} A[a^{-1}]$, we have $H^{c(x)-i}_{\{x\}}\mathscr{O}_{X(x)} = 0$ for every $i > 0$, so that $H^i(\omega^\bullet(x)) = 0$ for every such i and every such x, since $A^\wedge_x$ is a faithfully flat A_x-algebra. Lastly, we have $H^{c(x)-i}_{\{x\}}\mathscr{O}_{X(x)} = 0$ for every $i < 0$ and every $x \in X$, by proposition 10.4.7(ii.b), hence $H^i(\omega^\bullet(x)) = 0$ for every such i and x. ◊

By corollary 11.5.17(ii) we have a natural isomorphism :

$$H^i_{\{\mathfrak{n}\}}(A) \xrightarrow{\sim} H^i_{\{\mathfrak{n}\}}(\mathscr{D}(\omega^\bullet)) \xrightarrow{\sim} D(H^{c(\mathfrak{n})-i}(\omega^\bullet)) \qquad \text{for every } i \in \mathbb{Z}$$

and $c(\mathfrak{n}) = \dim A$. Since $H^j(\omega^\bullet)$ is a finitely generated A-module for every $j \in \mathbb{Z}$, claim 17.5.27 implies that there exists $n \in \mathbb{N}$ such that $a^n \cdot H^j(\omega^\bullet) = 0$ for every $j > 0$. Since D is an A-linear functor, it follows that $a^n \cdot H^i_{\{\mathfrak{n}\}}(A) = 0$ for every $i < \dim A$, as required. □

Theorem 17.5.28. *Let p be a prime integer, and $(A, \mathfrak{n})$ a local noetherian $\mathbb{F}_p$-algebra. Then there exists a perfect big CM A-algebra.*

Proof. We first notice :

Claim 17.5.29. We may assume that A is an excellent and equidimensional local ring.

Proof of the claim. Let $\mathfrak{p}$ be a minimal prime ideal of the completion $A^\wedge$ of A, such that $A' := A^\wedge/\mathfrak{p}$ has dimension equal to $\dim A$; then the natural map $A \to A'$ fulfills the conditions of remark 17.5.19(ii); thus it suffices to exhibit a big CM A'-algebra, and A' is an excellent local domain ([64, Ch.IV, Sch.7.8.3(iii)]). ◊

Thus, we assume now that A is as in claim 17.5.29. Then, by [64, Ch.IV, Sch.7.8.3(iv)], the subset U of points $\mathfrak{p} \in X := \operatorname{Spec} A$ such that $A_\mathfrak{p}$ is Cohen-Macaulay, is open in X, and trivially contains all the maximal points of X, hence it is dense in X. We may therefore find $a \in A$ such that $\varnothing \neq \operatorname{Spec} A[a^{-1}] \subset U$, so that a is not nilpotent, and $A[a^{-1}]$ is a Cohen-Macaulay ring. Define A^{perf} and J as in proposition 17.5.25. Clearly $J^2 = J$, and the basic setup (A^{perf}, J) fulfills the condition of (17.5.2). By proposition 17.5.25(ii), we then have

$$(A^{\mathrm{perf}}/\mathfrak{n}A^{\mathrm{perf}})^a \neq 0 \qquad \text{and} \qquad (R^i\Gamma_{\{\mathfrak{n}\}}A^{\mathrm{perf}})^a = 0 \qquad \text{for every } i = 0, \dots, \dim A - 1.$$

It follows that $(A^{\mathrm{perf}})^a_!/\mathfrak{n}(A^{\mathrm{perf}})^a_! \neq 0$. Recall that $J^\sim := J \otimes_{A^{\mathrm{perf}}} J$ is a flat A^{perf}-module ([75, Prop.2.1.7(i)]), and let $\mathsf{S}_a \subset (A^{\mathrm{perf}})^{\mathbb{N}}$ be the multiplicative subset associated with J, as in (17.5.2); then by lemmata 17.5.23 and 17.5.4(i,ii) we conclude that $\mathsf{S}_a^{-1}(A^{\mathrm{perf}})^\diamond$ is a big CM A-algebra, and since A^{perf} is perfect, it is easily seen that the same holds for $\mathsf{S}_a^{-1}(A^{\mathrm{perf}})^\diamond$. □

It is useful to generalize theorem 17.5.28 to non-local noetherian $\mathbb{F}_p$-algebras. To this aim, let us make the following :

Definition 17.5.30. Let A be a noetherian ring, M an A-module, and B an A-algebra.

(i) We say that A is *locally equidimensional* if $A_\mathfrak{p}$ is equidimensional for every $\mathfrak{p} \in \mathrm{Spec}\, A$.

(ii) We say that M is a *big locally CM A-module*, if $M_\mathfrak{p}$ is a big CM $A_\mathfrak{p}$-module for every $\mathfrak{p} \in \mathrm{Spec}\, A$. Likewise, B is a *big locally CM A-algebra*, if it is big locally CM as an A-module.

(iii) We say that a ring homomorphism $f : A \to B$ is *maximizing*, if the continuous map $\mathrm{Spec}\, f$ is maximing (see definition 8.1.46(ii)).

Lemma 17.5.31. *Let A be a catenary noetherian ring (see definition* 9.7.3(iii)*). The following conditions are equivalent :*

(a) *A is locally equidimensional.*

(b) *$A_\mathfrak{m}$ is equidimensional for every maximal ideal $\mathfrak{m} \subset A$.*

(c) *For every two irreducible components Z and Z' of* $\mathrm{Spec}\, A$*, and every maximal point $\mathfrak{p}$ of the topological space $Z \cap Z'$, the ring $A_\mathfrak{p}$ is equidimensional.*

Proof. Trivially (a)⇒(b), and (a)⇒(c).

(b)⇒(a): Indeed, say that $\mathfrak{p} \in \mathrm{Spec}\, A$, and let $\mathfrak{q}$ and $\mathfrak{q}'$ be two minimal prime ideal contained in $\mathfrak{p}$; we need to check that two every saturated chain of prime ideals

$$\mathfrak{q} \subset \cdots \subset \mathfrak{p} \qquad \text{and} \qquad \mathfrak{q}' \subset \cdots \subset \mathfrak{p} \tag{17.5.32}$$

have the same length. To this aim, pick any maximal ideal $\mathfrak{m}$ of A containing $\mathfrak{p}$, and complete the two chains with the same saturated chain $\mathfrak{p} \subset \cdots \subset \mathfrak{m}$; the resulting chains $\mathfrak{q} \subset \cdots \subset \mathfrak{m}$ and $\mathfrak{q}' \subset \cdots \subset \mathfrak{m}$ are saturated, so by assumption they both have length equal to $\dim A_\mathfrak{m}$, whence the assertion.

(c)⇒(b): Let $\mathfrak{m} \subset A$ be a maximal ideal, and $\mathfrak{q}, \mathfrak{q}'$ two minimal prime ideals of A contained in $\mathfrak{m}$; set $Z := \mathrm{Spec}\, A/\mathfrak{q}$ and $Z' := \mathrm{Spec}\, A/\mathfrak{q}'$. Then, $\mathfrak{m} \in Z \cap Z'$, so let $\mathfrak{p}$ be any maximal generization of $\mathfrak{m}$ in $Z \cap Z'$; pick any two saturated chains as in (17.5.32), and complete them again with the same saturated chain $\mathfrak{p} \subset \cdots \subset \mathfrak{m}$, say of length l_0. Since $A_\mathfrak{p}$ is equidimensional, the chains (17.5.32) have the same length l_1; now, let $\mathfrak{q} \subset \cdots \mathfrak{m}$ and $\mathfrak{q}' \subset \cdots \mathfrak{m}$ be any other pair of saturated chains, of lengths l and respectively l'; since A is catenary, we must have $l = l_0 + l_1 = l'$, so $A_\mathfrak{m}$ is equidimensional. □

Corollary 17.5.33. *Let A be an excellent and locally equidimensional $\mathbb{F}_p$-algebra. Then there exists a big locally CM A-algebra B.*

Proof. The subset U of points $\mathfrak{p} \in X := \mathrm{Spec}\, A$ such that $A_\mathfrak{p}$ is Cohen-Macaulay is open ([64, Ch.IV, Sch.7.8.3(iv)]), and trivially contains the subset $\mathrm{Min}(A)$ of all minimal prime ideals of A, hence it is dense in X. Say that $X \setminus U = \mathrm{Spec}\, A/I$ for some ideal $I \subset A$; then I is not contained in any minimal ideal of A, and so there exists $a \in I \setminus \bigcup_{\mathfrak{p} \in \mathrm{Min}(A)} \mathfrak{p}$ ([12, Prop.1.11(i)]). Let A^{perf} be the perfection of A (see (9.8.11)), and $J \subset A^{\mathrm{perf}}$ the ideal generated by $(a^{1/p^n} \mid n \in \mathbb{N})$. Let moreover $\mathsf{S}_a \subset (A^{\mathrm{perf}})^{\mathbb{N}}$ be the multiplicative subset associated with J, as in (17.5.2). We claim that $B := \mathsf{S}_a^{-1}(A^{\mathrm{perf}})^\diamond$ will do. Indeed, let $\mathfrak{p} \subset A$ be any prime ideal, $k(\mathfrak{p})$ its residue field, $j : A \to A_\mathfrak{p}$ the localization, and $\mathsf{S}_{j(a)} \subset (A_\mathfrak{p}^{\mathrm{perf}})^{\mathbb{N}}$ the corresponding multiplicative subset, associated with the ideal $JA_\mathfrak{p}$.

Claim 17.5.34. $\mathsf{S}_{j(a)}^{-1}(A_\mathfrak{p}^{\mathrm{perf}})^\diamond$ is a big CM $A_\mathfrak{p}$-algebra; especially, $\mathsf{S}_{j(a)}^{-1}(A_\mathfrak{p}^{\mathrm{perf}})^\diamond \otimes_{A_\mathfrak{p}} k(\mathfrak{p}) \neq 0$.

Proof of the claim. By [64, Ch.IV, Sch.7.8.3(ii)], $A_{\mathfrak{p}}$ is excellent and equidimensional. Then the assertion follows from the proof of theorem 17.5.28. $\diamondsuit$

The map j induces a ring homomorphism $B \to \mathsf{S}_{j(a)}^{-1}(A_{\mathfrak{p}}^{\mathrm{perf}})^{\diamond}$; with claim 17.5.34 we then get $B \otimes_A k(\mathfrak{p}) \neq 0$ for every $\mathfrak{p} \in \operatorname{Spec} A$. In view of (10.4.30), we are then reducing to showing :

Claim 17.5.35. $\operatorname{depth}_{\mathfrak{p}} B = \dim A_{\mathfrak{p}}$ for every $\mathfrak{p} \in \operatorname{Spec} A$.

Proof of the claim. Set $I := \mathfrak{p}A^{\mathrm{perf}}$, $Z := \operatorname{Spec} A/\mathfrak{p}$ and $Z' := \operatorname{Spec} A^{\mathrm{perf}}/I$; by lemma 10.4.13(v), we have a natural identification :

$$R^i\Gamma_Z B \xrightarrow{\sim} R^i\Gamma_{Z'} B \qquad \text{for every } i \in \mathbb{N}.$$

In view of claim 17.5.24, we are then reduced to checking that :

$$\operatorname{depth}_I((A^{\mathrm{perf}})^a_!) = \dim A_{\mathfrak{p}}$$

for the almost structure attached to the basic setup (A^{perf}, J). However, recall that $(A^{\mathrm{perf}})_{\mathfrak{q}} = (A_{\mathfrak{q}})^{\mathrm{perf}}$ for every $\mathfrak{q} \in \operatorname{Spec} A$ (see (9.8.11)); also, $((A^{\mathrm{perf}})^a_!)_{\mathfrak{q}}$ agrees with $(A_{\mathfrak{q}}^{\mathrm{perf}})^a_!$, for every such $\mathfrak{q}$, where now $(A_{\mathfrak{q}}^{\mathrm{perf}})^a$ is regarded as an $(A_{\mathfrak{q}}^{\mathrm{perf}}, JA_{\mathfrak{q}}^{\mathrm{perf}})^a$-module, and $(-)_!$ refers to the left adjoint of the localization $A_{\mathfrak{q}}^{\mathrm{perf}}\text{-}\mathbf{Mod} \to (A_{\mathfrak{q}}^{\mathrm{perf}}, JA_{\mathfrak{q}}^{\mathrm{perf}})^a\text{-}\mathbf{Mod}$. It follows that :

$$\operatorname{depth}_{A_{\mathfrak{q}}}(A_{\mathfrak{q}}^{\mathrm{perf}})^a_! = \dim A_{\mathfrak{q}} \qquad \text{for every } \mathfrak{q} \in \operatorname{Spec} A$$

by virtue of lemma 17.5.23 and claim 17.5.34. We conclude by invoking again (10.4.30). $\square$

Remark 17.5.36. (i) Theorem 17.5.28 was first proved by Hochster and Huneke in [95]; more precisely, they showed that for every excellent local domain A of equicharacteristic $p > 0$, the integral closure of A in the algebraic closure of $\operatorname{Frac}(A)$ is a big CM A-algebra; this result has been recently extended to excellent local domains of mixed characteristic, in [26]. Our construction yields a different big CM A-algebra.

(ii) Our construction can be easily upgraded to a functor on the category

$$\mathscr{B}_p$$

whose objects are all the excellent and locally equidimensional $\mathbb{F}_p$-algebras, and whose morphisms are the maximizing ring homomorphisms (see definition 17.5.30(iii)). Namely, for every $A \in \operatorname{Ob}(\mathscr{B}_p)$, let $\operatorname{Min}(A)$ be the set of minimal primes of A, and endow $A^* := A \setminus \bigcup_{\mathfrak{p} \in \operatorname{Min}(A)} \mathfrak{p}$ with the partial preordering such that $a \leq b$ if and only if $bA \subset aA$, and for every $a \in A^*$, define the multiplicative subset $\mathsf{S}_a \subset (A^{\mathrm{perf}})^{\mathbb{N}}$ as in the proof of theorem 17.5.28. Clearly $(A^*, \leq)$ is a filtered category (see example 1.1.6(iii)), and we have $\mathsf{S}_a \subset \mathsf{S}_b$ for every $a, b \in A^*$ with $a \leq b$. Then, denote by $E_p(A)$ the colimit of the induced filtered system of A-algebras $(\mathsf{S}_a^{-1}(A^{\mathrm{perf}})^{\diamond} \mid a \in A^*)$. The proof of corollary 17.5.33 shows that A^* admits a cofinal subset $\Sigma \subset A^*$ such that $\mathsf{S}_a^{-1}(A^{\mathrm{perf}})^{\diamond}$ is a big locally CM A-algebra for every $a \in \Sigma$; then the same holds for $E_p(A)$ (remark 17.5.19(iv)). Now, let $f : A \to B$ be a morphism of $\mathscr{B}_p$; clearly f extends uniquely to a ring homomorphism $f^{\mathrm{perf}} : A^{\mathrm{perf}} \to B^{\mathrm{perf}}$, and restricts to a map $f^* : A^* \to B^*$, as f is maximizing. Moreover, $(f^{\mathrm{perf}})^{\mathbb{N}}(\mathsf{S}_a) \subset \mathsf{S}_{f(a)}$ for every $a \in A^*$. Hence, we obtain an induced map $\mathsf{S}^{-1}(A^{\mathrm{perf}})^{\diamond} \to \mathsf{S}_{f(a)}^{-1}(B^{\mathrm{perf}})^{\diamond} \to E_p(B)$ for every such a, and the colimit of these maps is a ring homomorphism $E_p(f) : E_p(A) \to E_p(B)$. It is easily seen that the rules : $A \mapsto E_p(A)$ and $f \mapsto E_p(f)$ yield the sought functor $E_p : \mathscr{B}_p \to \mathbb{F}_p\text{-}\mathsf{Alg}$.

(iii) Moreover, consider the functor

$$(-)^{\nu} : \mathscr{B}_p \to \mathscr{B}_p \qquad A \mapsto A^{\nu} := \prod_{\mathfrak{p} \in \operatorname{Min} A} (A/\mathfrak{p})^{\nu}$$

where, for every $\mathfrak{p} \in \operatorname{Min} A$, we denote by $(A/\mathfrak{p})^{\nu}$ the integral closure of $A/\mathfrak{p}$ in its field of fractions. Clearly, the natural map $j_A : A \to A^{\nu}$ is a morphism of $\mathscr{B}_0$. It is easily seen that every morphism $f : A \to B$ in $\mathscr{B}_p$ induces a well defined morphism $f^\nu : A^\nu \to B^\nu$, such that $f^\nu \circ j_A = j_B \circ f$; this completes the construction of the functor $(-)^\nu$, and shows that the rule : $A \mapsto j_A$ yields a natural transformation $j_\bullet : \mathbf{1}_{\mathscr{B}_p} \Rightarrow (-)^\nu$. With this notation, we have :

Proposition 17.5.37. (i) *The functor E_p factors through the functor $(-)^\nu$.*
(ii) *The functor E_p commutes with finite products.*

Proof. (i): We need to check that $E_p(j_A)$ is an isomorphism for every $A \in \mathrm{Ob}(\mathscr{B}_p)$; to this aim, let A_{red} be the maximal reduced quotient of A, and notice that j_A factors through the quotient map $j'_A : A \to A_{\mathrm{red}}$ and a unique map $j''_A : A_{\mathrm{red}} \to A^\nu$, and both maps are morphisms in $\mathscr{B}_p$. Now, clearly j'_A induces an isomorphism $(j'_A)^{\mathrm{perf}} : A^{\mathrm{perf}} \xrightarrow{\sim} (A_{\mathrm{red}})^{\mathrm{perf}}$ that identifies S_a with $\mathsf{S}_{j'_A(a)}$ for every $a \in A^*$ (notation of remark 17.5.36(ii)). Then, under these identifications, $E_p(A_{\mathrm{red}})$ is the colimit of the functor

$$F : ((A_{\mathrm{red}})^*, \leq) \to \mathbb{F}_p\text{-}\mathsf{Alg} \qquad a \mapsto \mathsf{S}_a^{-1}(A^{\mathrm{perf}})^\diamond$$

and $E_p(A)$ is the colimit of the functor $F \circ (j'_A)^* : (A^*, \leq) \to \mathbb{F}_p\text{-}\mathsf{Alg}$. But notice that the map $(j'_A)^* : A^* \to (A_{\mathrm{red}})^*$ is surjective, so it is a cofinal functor of filtered categories (lemma 1.5.7(i)), and thus $E_p(j'_A)$ is an isomorphism (proposition 1.5.2). Next, we remark :

Claim 17.5.38. (i) There exists $c \in A^*_{\mathrm{red}}$ such that $j''_A(c) \cdot A^\nu \subset j''_A(A_{\mathrm{red}})$.
(ii) For every c as in (i), and every $n \in \mathbb{N}$, we have $c^{1/p^n}(A^\nu)^{\mathrm{perf}} \subset A^{\mathrm{perf}}$.

Proof of the claim. (i): We further write j''_A as the composition of injective maps

$$A_{\mathrm{red}} \xrightarrow{i_1} B := \prod_{\mathfrak{p} \in \mathrm{Min}\, A} A/\mathfrak{p} \xrightarrow{i_2} A^\nu.$$

It then suffices to find $c_1 \in A^*_{\mathrm{red}}$ and $c_2 \in B^*$ such that $c_1 B \subset i_1(A_{\mathrm{red}})$ and $c_2 A^\nu \subset i_2(B)$, since in that case $c_3 := i_1(c_1)c_2 \in i_1(A_{\mathrm{red}}) \cap B^* = i_1(A^*_{\mathrm{red}})$, and $i_2(c_3)A^\nu \subset j''_A(A_{\mathrm{red}})$. Now, let $Z_1, \dots, Z_k$ be the irreducible components of $\mathrm{Spec}\, A_{\mathrm{red}}$, and set $Z := \bigcup_{1 \leq i < j \leq k} Z_i \cap Z_j$; pick any ideal $J \subset A_{\mathrm{red}}$ such that $Z = \mathrm{Spec}\, A/J$. Hence, J does not lie in any minimal prime ideal of A_{red}, so $J \cap A^*_{\mathrm{red}} \neq \varnothing$ ([12, Prop.1.11(i)]); let then $d \in J \cap A^*_{\mathrm{red}}$. We have $Z \cap \mathrm{Spec}\, A_{\mathrm{red}}[d^{-1}] = \varnothing$, so i_1 induces an isomorphism $A_{\mathrm{red}}[d^{-1}] \xrightarrow{\sim} B[d^{-1}]$. Since B is a finite A_{red}-algebra, it follows that $d^n B \subset i_1(A_{\mathrm{red}})$, for some $n \in \mathbb{N}$, so $c_1 := d^n$ will do. Next, notice that $B^* = \prod_{\mathfrak{p} \in \mathrm{Min}\, A}(A/\mathfrak{p})^*$ and $(A^\nu)^* = \prod_{\mathfrak{p} \in \mathrm{Min}\, A}((A/\mathfrak{p})^\nu)^*$, so in order to exhibit c_2 it suffices to find for every $\mathfrak{p} \in \mathrm{Min}\, A$ some $c_\mathfrak{p} \in (A/\mathfrak{p})^* = (A/\mathfrak{p}) \setminus \{0\}$ with $c_\mathfrak{p}(A/\mathfrak{p})^\nu \subset A/\mathfrak{p}$. To this aim, it suffices to notice that, as A is excellent, $(A/\mathfrak{p})^\nu$ is a finite $A/\mathfrak{p}$-subalgebra of $\mathrm{Frac}(A/\mathfrak{p})$.
(ii) shall be left to the reader. ◊

Pick $c \in A^*_{\mathrm{red}}$ as in claim 17.5.38(i); set $\Sigma := \{ca \mid a \in A^*_{\mathrm{red}}\} \subset A^*_{\mathrm{red}}$, and let $c_\bullet \in (A_{\mathrm{red}})^\diamond$ be the class of $(c^{1/p^n} \mid n \in \mathbb{N})$. With this notation, claim 17.5.38(i) states that the image of Σ is cofinal in the partially ordered set $((A^\nu)^*, \leq)$. Moreover, j''_A induces an injection $A^{\mathrm{perf}}_{\mathrm{red}} = A^{\mathrm{perf}} \to (A^\nu)^{\mathrm{perf}}$; combining with claim 17.5.38(ii) we deduce that j''_A induces an isomorphism

$$(A^{\mathrm{perf}})^\diamond[c_\bullet^{-1}] \xrightarrow{\sim} ((A^\nu)^{\mathrm{perf}})^\diamond[c_\bullet^{-1}]$$

and on the other hand, $c_\bullet \in \mathsf{S}_b$ for every $b \in \Sigma$. Summing up, we conclude that $E_p(j''_A)$ is naturally identified with the colimit of the system of maps

$$\mathsf{S}_b^{-1}((A^\nu)^{\mathrm{perf}})^\diamond \to \mathsf{S}_{j''_A(b)}^{-1}((A^\nu)^{\mathrm{perf}})^\diamond \qquad \text{for every } b \in \Sigma \tag{17.5.39}$$

induced by the inclusion $\mathsf{S}_b \subset \mathsf{S}_{j''_A(b)}$, for every such b (here, S_b lies in $(A^{\mathrm{perf}})^\mathbb{N}$, whereas $\mathsf{S}_{j''_A(b)}$ lies in $((A^\nu)^{\mathrm{perf}})^\mathbb{N}$). In order to check that $E_p(j''_A)$ is an isomorphism, it then suffices to prove that the same holds for the maps (17.5.39); the latter follows at once from :

Claim 17.5.40. $c_\bullet \cdot s_\bullet \in \mathsf{S}_b$, for every $s_\bullet \in \mathsf{S}_{j_A(b)}$.

Proof of the claim. Say that $s_\bullet = (s_n \mid n \in \mathbb{N}) \in ((A^\nu)^{\mathrm{perf}})^{\mathbb{N}}$ and $b = ac$ for some $a \in (A^{\mathrm{perf}})^*$; we need to check that $b^{1/p^k} \in \bigcup_{n\in\mathbb{N}} c^{1/p^n} s_n A^{\mathrm{perf}}$ for every $k \in \mathbb{N}$. However, by assumption $j''_A(b^{1/p^k}) \in \bigcup_{n\in\mathbb{N}} s_n (A^\nu)^{\mathrm{perf}}$, so say that $j''_A(b^{1/p^k}) = s_n x$ for some $n \in \mathbb{N}$ and $x \in (A^\nu)^{\mathrm{perf}}$; also, we may assume that $n > k$. By claim 17.5.38(ii), we get $c^{1/p^n} x \in A^{\mathrm{perf}}$, and it follows easily that $(ac^2)^{1/p^k} = (bc)^{1/p^k} \in c^{1/p^n} s_n A^{\mathrm{perf}}$. Then we get $b^{2/p^k} = (ac)^{2/p^k} \in \bigcup_{n\in\mathbb{N}} c^{1/p^n} s_n A^{\mathrm{perf}}$ for every $k \in \mathbb{N}$, whence the claim. ◊

(ii): First, it is easily seen that the finite products in $\mathscr{B}_p$ are represented by the usual (cartesian) products of rings (details left to the reader). Hence, say that $A = A_1 \times \cdots \times A_n$, for some $A_1, \dots, A_n \in \mathrm{Ob}(\mathscr{B}_p)$. It is easily seen that $A^* = A_1^* \times \cdots \times A_n^*$. It is also clear that $(A^{\mathrm{perf}})^\diamond = \prod_{i=1}^n (A_i^{\mathrm{perf}})^\diamond$ (lemma 17.5.1(i)). Moreover, for each $a_\bullet := (a_1, \dots, a_n) \in A^*$, we have $\mathsf{S}_{a_\bullet} = \mathsf{S}_{a_1} \times \cdots \times \mathsf{S}_{a_n}$, and thus $\mathsf{S}_{a_\bullet}^{-1} = \prod_{i=1}^n \mathsf{S}_{a_i}^{-1}(A_i^{\mathrm{perf}})\diamond$. Since finite products commute with filtered colimits in the category of rings, the assertion follows. □

17.5.41. It turns out that the construction of big CM algebras over $\mathbb{Q}$-algebras can be reduced to the case of algebras of positive characteristic. Indeed, consider first a locally equidimensional $\mathbb{Q}$-algebra A of finite type. Let $\Delta(A)$ be the set of all $\mathbb{Z}$-subalgebras $B \subset A$ of finite type with $\mathbb{Q} \otimes_{\mathbb{Z}} B = A$. Let also $\mathrm{Max}\,\mathbb{Z}$ be the maximal spectrum of $\mathbb{Z}$, *i.e.* the set of prime integers.

Lemma 17.5.42. *With the notation of* (17.5.41)*, the following holds :*

(i) *Every connected component of* $\mathrm{Spec}\,A$ *is equidimensional.*

(ii) *For every* $B \in \Delta(A)$ *there exists a finite subset* $S_B \subset \mathrm{Max}\,\mathbb{Z}$ *such that* B/pB *is excellent and locally equidimensional, for every* $p \in \mathrm{Max}\,(\mathbb{Z}) \setminus S_B$.

Proof. (i): It suffices to check that if Z and Z' are two irreducible components ot $\mathrm{Spec}\,A$ with $Z \cap Z' \neq \varnothing$, then $\dim Z = \dim Z'$. Hence, pick any maximal ideal $\mathfrak{m}$ of A that lies in $Z \cap Z'$; by assumption, $A_{\mathfrak{m}}$ is equidimensional, and $Z(\mathfrak{m}), Z'(\mathfrak{m})$ are two irreducible components of $\mathrm{Spec}\,A_{\mathfrak{m}}$ (notation of definition 8.1.44(iii)), so $\dim Z(\mathfrak{m}) = \dim Z'(\mathfrak{m})$. On the other hand, $\dim Z = \dim Z(\mathfrak{m})$, and likewise for Z' ([64, Ch.IV, Prop.5.2.1]), whence the assertion.

(ii): Since the localization $B \to \mathbb{Q} \otimes_{\mathbb{Z}} B$ is injective, the unique morphism of schemes $\varphi : \mathrm{Spec}\,B \to \mathrm{Spec}\,\mathbb{Z}$ is flat. Let $T_1, \dots, T_l$ be the connected components of $\mathrm{Spec}\,A$, and for $i = 1, \dots, l$, denote by $\overline{T}_i$ the topological closure of T_i in $\mathrm{Spec}\,B$; since φ is generizing, it follows easily that $\mathrm{Spec}\,B = \overline{T}_1 \cup \cdots \cup \overline{T}_l$. On the other hand, for every $i = 1, \dots, l$, we have also $T_i = \mathrm{Spec}\,A/e_iA$, for a unique idempotent element $e_i \in A$; we may then find $d \in \mathbb{Z} \setminus \{0\}$ such that $e_1, \dots, e_l \in B[d^{-1}]$. Set $U := \mathrm{Spec}\,B[d^{-1}]$; it follows easily that the connected components of U are $\overline{T}_1 \cap U, \dots, \overline{T}_l \cap U$. After replacing B by $B[d^{-1}]$, we may then assume that $B = B_1 \times \cdots \times B_l$, and $\mathrm{Spec}\,\mathbb{Q} \otimes_{\mathbb{Z}} B_i$ is connected for every $i = 1, \dots, l$. Thus, it suffices to prove the assertion for A replaced by $\mathbb{Q} \otimes_{\mathbb{Z}} B_i$ for every $i = 1, \dots, l$, and we may therefore assume from start that $\mathrm{Spec}\,A$ is connected, so A is equidimensional, by (i).

Let $Z_1, \dots, Z_k$ be the irreducible components of $\mathrm{Spec}\,A$, and denote by $\overline{Z}_i$ the topological closure of Z_i in $\mathrm{Spec}\,B$, for every $i = 1, \dots, k$; it follows easily that $\overline{Z}_1, \dots, \overline{Z}_k$ are the irreducible components of $\mathrm{Spec}\,B$, and we endow them with their reduced closed subscheme structures. Then the restriction $\varphi_i : \overline{Z}_i \to \mathrm{Spec}\,\mathbb{Z}$ of φ is a flat morphism of schemes, for every $i = 1, \dots, k$. According to lemma 11.4.1(i), for every $p \in \mathrm{Max}\,\mathbb{Z}$ and every $i = 1, \dots, k$, every irreducible component of $\varphi_i^{-1}(p)$ has dimension equal to $\dim Z_i$, and we know already that the latter equals $d := \dim A$. Summing up, for every $p \in \mathrm{Max}\,\mathbb{Z}$, every irreducible component of $\mathrm{Spec}\,B/pB$ has dimension d; combining with [64, Ch.IV, Prop.5.2.1], we deduce that for every maximal ideal $\mathfrak{m} \subset B/pB$, the localization $(B/pB)_{\mathfrak{m}}$ is equidimensional of dimension d. Lastly, B/pB is an excellent ring, by [64, Ch.IV, Sch.7.8.3(ii)], so we may invoke lemma 17.5.31 to conclude. □

For every $B \in \Delta(A)$, pick a finite subset $S_B \subset \mathrm{Max}\,\mathbb{Z}$ fulfilling the condition of lemma 17.5.42(ii), and set

$$F_0(A,B) := \prod_{p\in\mathrm{Max}(\mathbb{Z})\setminus S_B} E_p(B/pB) \qquad \text{and} \qquad I_0(A,B) := \bigoplus_{p\in\mathrm{Max}(\mathbb{Z})\setminus S_B} E_p(B/pB)$$

where E_p is the functor exhibited in remark 17.5.36(ii). Notice then that $F_0(A,B)$ is naturally a B-algebra, and $I_0(A,B)$ is an ideal of $F_0(A,B)$; then the quotient

$$E_0(A,B) := F_0(A,B)/I_0(A,B)$$

is again endowed with a natural B-algebra structure. Moreover, if $B, B' \in \Delta(A)$, we may find $d \in \mathbb{Z} \setminus \{0\}$ such that $B[d^{-1}] = B'[d^{-1}]$, and hence $B/pB = B'/pB'$ for every $p \in \mathrm{Max}(\mathbb{Z}) \setminus \mathrm{Spec}\,\mathbb{Z}/d\mathbb{Z}$. There follows a natural identification

$$E_0(A,B) \xrightarrow{\sim} E_0(A,B')$$

in which of which, we shall denote this ring simply by $E_0(A)$. Since $A = \bigcup_{B\in\Delta(A)} B$, we conclude that $E_0(A)$ carries a natural A-algebra structure.

17.5.43. Next, let $f : A \to A'$ be a maximizing ring homomorphism of locally equidimensional $\mathbb{Q}$-algebras of finite type. Clearly, we may find $B \in \Delta(A)$ and $B' \in \Delta(A')$ such that f restricts to a ring homomorphism $g : B \to B'$. If $h : C \to C'$ is another such restriction with $C \in \Delta(A)$ and $C' \in \Delta'$, then for any $d \in \mathbb{Z} \setminus \{0\}$ such that $B[d^{-1}] = C[d^{-1}]$ and $B'[d^{-1}] = C'[d^{-1}]$ we have $\mathbb{Z}[d^{-1}] \otimes_\mathbb{Z} g = \mathbb{Z}[d^{-1}] \otimes_\mathbb{Z} h$, so the induced maps $\mathbb{F}_p \otimes_\mathbb{Z} g : B/pB \to B'/pB'$ and $\mathbb{F}_p \otimes_\mathbb{Z} h : C/pC \to C'/pC'$ are naturally identified for every $p \in \mathrm{Max}(\mathbb{Z}) \setminus \mathrm{Spec}\,\mathbb{Z}/d\mathbb{Z}$. Moreover, we have :

Lemma 17.5.44. *With the notation of* (17.5.43)*, there exists a finite subset $S_g \subset \mathrm{Max}\,\mathbb{Z}$ such that $\mathbb{F}_p \otimes_\mathbb{Z} g$ is maximizing for every $p \in \mathrm{Max}(\mathbb{Z}) \setminus S_g$.*

Proof. Notice first that B and B' are flat $\mathbb{Z}$-algebra, so all the maximal points of $X := \mathrm{Spec}\,B$ and $X' := \mathrm{Spec}\,B'$ lie in $\mathrm{Spec}\,A$ and respectively $\mathrm{Spec}\,A'$, so $\varphi := \mathrm{Spec}\,g$ is maximizing. Let $Z_1, \dots, Z_k$ (resp. $Z'_1, \dots, Z'_l$) be the irreducible components of X (resp. of X'), which we endow with their reduced closed subscheme structures; it follows that for every $i = 1, \dots, l$ there exists a unique $t(i) \in \{1, \dots, k\}$ such that φ restricts to a dominant morphism of schemes $\varphi_i : Z'_i \to Z_{t(i)}$. By [64, Ch.IV, Lemme 6.9.2], we may then find $h \in B \setminus \{0\}$ such that, with $U := \mathrm{Spec}\,B[h^{-1}]$, the morphism φ_i restricts to a flat morphism

$$\varphi_{i|U} : \varphi_i^{-1}(U \cap Z_{t(i)}) \to U \cap Z_{t(i)} \qquad \text{for every } i = 1, \dots, l.$$

Endow likewise the irreducible components $T_1, \dots, T_n$ of $\mathrm{Spec}\,B/hB$ with their reduced closed subscheme structures; by the same token, we may find $d \in \mathbb{Z} \setminus \{0\}$ such that, with $W := \mathrm{Spec}\,\mathbb{Z}[d^{-1}]$, the unique morphism of schemes $\psi_i : T_i \to \mathrm{Spec}\,\mathbb{Z}$ restricts to a flat morphism

$$\psi_i^{-1}(W) \to W \qquad \text{for every } i = 1, \dots, n.$$

Let $\eta \in \mathrm{Spec}\,\mathbb{Z}$ be the generic point; on the one hand, we have

$$\dim \psi_i^{-1}(\eta) < \dim A \qquad \text{for every } i = 1, \dots, n$$

and on the other hand, lemma 11.4.1(i) says that for every $p \in W$ we have :

$$\dim \psi_i^{-1}(p) = \dim \psi_i^{-1}(\eta) \quad \text{for every } i = 1, \dots, n \quad \text{and} \quad \dim B/pB = \dim A.$$

Summing up, we conclude that for every $p \in W \cap \mathrm{Max}(\mathbb{Z})$, all the maximal points of $\mathrm{Spec}\,B/pB$ lie in U. Arguing likewise with the irreducible components of $\mathrm{Spec}\,B'/hB'$, we get an open subset $W' \subset \mathrm{Spec}\,\mathbb{Z}$ such that, for every $p \in W \cap \mathrm{Max}(\mathbb{Z})$, all the maximal points of $\mathrm{Spec}\,B'/pB'$ lie in $\varphi^{-1}U$. But since each $\varphi_{i|U}$ is flat, for every $p \in \mathrm{Max}(\mathbb{Z})$ the restriction of φ :

$$\varphi^{-1}(U) \cap \mathrm{Spec}\,B'/pB' \to U \cap \mathrm{Spec}\,B/pB$$

is maximizing, so the lemma holds with $S_g := \mathrm{Max}(\mathbb{Z}) \setminus (W \cap W')$. □

Using lemma 17.5.44, we may attach to any $f : A \to A'$ as in (17.5.43) a well defined map

$$E_0(f) : E_0(A) \to E_0(A').$$

Namely, pick B, B' and $g : B \to B'$ as in (17.5.43), and finite subsets $S_B, S_{B'}, S_g \subset \mathrm{Max}\,\mathbb{Z}$ as in lemmata 17.5.42 and 17.5.44; set $M := \mathrm{Max}(\mathbb{Z}) \setminus (S_B \cup S_{B'} \cup S_g)$ and

$$F_0(f, g) := \prod_{p\in M} E_p(\mathbb{F}_p \otimes_{\mathbb{Z}} g) : \prod_{p\in M} E_p(B/pB) \to \prod_{p\in M} E_p(B'/pB').$$

Clearly $F_0(f, g)$ maps the ideal $\bigoplus_{p\in M} E_p(B/pB)$ into $\bigoplus_{p\in M} E_p(B'/pB')$, so it induces a unique map $E_0(f)$ as sought. It is also easily seen that if A'' is any other locally equidimensional $\mathbb{Q}$-algebra of finite type, and $f' : A' \to A''$ is any maximizing ring homomorphism, then $E_0(f' \circ f) = E_0(f') \circ E_0(f)$: details left to the reader. Summing up, we have a well defined functor

$$E_0 : \mathscr{B}_0 \to \mathbb{Z}\text{-}\mathsf{Alg}$$

where $\mathscr{B}_0$ is the category whose objects are the locally equidimensional $\mathbb{Q}$-algebras of finite type, and whose morphisms are the maximizing ring homomorphisms. Furthermore, just as in remark 17.5.36(iii), we have a normalization functor

$$(-)^\nu : \mathscr{B}_0 \to \mathscr{B}_0 \qquad A \mapsto \prod_{\mathfrak{p}\in\mathrm{Min}\,A} (A/\mathfrak{p})^\nu$$

(where $(A/\mathfrak{p})^\nu$ denotes the integral closure of $A/\mathfrak{p}$ in $\mathrm{Frac}(A/\mathfrak{p})$, for every $\mathfrak{p} \in \mathrm{Min}\,A$), as well as a natural transformation $j_\bullet : \mathbf{1}_{\mathscr{B}_0} \Rightarrow (-)^\nu$. With this notation, we may now state :

Theorem 17.5.45. (i) *$E_0(A)$ is a big locally CM A-algebra, for every $A \in \mathrm{Ob}(\mathscr{B}_0)$.*
(ii) *The functor E_0 commutes with finite products.*
(iii) *The functor E_0 factors through the functor $(-)^\nu$.*

Proof. (ii): It is easily seen that the products in $\mathscr{B}_0$ are represented by the usual (cartesian) products of rings; then, say that $A = A_1 \times \cdots \times A_k$ for some $A_1, \dots, A_k \in \mathrm{Ob}(\mathscr{B}_0)$, and pick $B_i \in \Delta(A_i)$ for $i = 1, \dots, k$. Clearly $B := B_1 \times \cdots \times B_k \in \Delta(A)$, and in light of proposition 17.5.37(ii), we easily deduce that $E_0(A, B) = \prod_{i=1}^k E_0(A_i, B_i)$, whence the assertion.

(i): In view of (ii) and lemma 17.5.42, we may assume that A is equidimensional. Then, let $\mathfrak{p} \subset A$ be any prime ideal; set $Z := \mathrm{Spec}\,A/\mathfrak{p}$, and $c := \dim\, A_\mathfrak{p}$. We show :

Claim 17.5.46. $\mathrm{depth}_\mathfrak{p}(E_0(A)) = c$.

Proof of the claim. Pick any $B \in \Delta(A)$, and let $\overline{Z}$ be the topological closure of Z in $\mathrm{Spec}\,B$, which we endow with its reduced closed subscheme structure. Hence, $\overline{Z} = \mathrm{Spec}\,B/\mathfrak{q}$, with $\mathfrak{q} = B \cap \mathfrak{p}$, and let $\mathbf{f} := (f_1, \dots, f_r)$ be any finite system of generators of $\mathfrak{q}$. In view of (10.4.30) and proposition 10.4.32(i), it suffices to check that $H^i(\mathbf{f}, E_0(A, B)) = 0$ for every $i < c$ and $H^c(\mathbf{f}, E_0(A, B)) \neq 0$. However, pick also a finite subset $S_B \subset \mathrm{Max}\,\mathbb{Z}$ as in lemma 17.5.42(ii); moreover, notice that the image of $\overline{Z}$ in $\mathrm{Spec}\,\mathbb{Z}$ is constructible and dense ([63, Ch.IV, Th.1.8.4]), so its complement S_Z is a finite set of closed points. Set $S := S_B \cup S_Z$; a direct inspection yields a natural short exact sequence of complexes of B-modules :

$$0 \to \bigoplus_{p\in\mathrm{Max}(\mathbb{Z})\setminus S} \mathbf{K}^\bullet(\mathbf{f}, E_p(B/pB)) \to \prod_{p\in\mathrm{Max}(\mathbb{Z})\setminus S} \mathbf{K}^\bullet(\mathbf{f}, E_p(B/pB)) \to \mathbf{K}^\bullet(\mathbf{f}, E_0(A, B)) \to 0$$

so we are reduced to checking that for every $p \in \mathrm{Max}(\mathbb{Z}) \setminus S$ we have :

$$H^i(\mathbf{f}, E_p(B/pB)) = 0 \qquad \text{for every } i < c, \text{ and} \qquad H^c(\mathbf{f}, E_p(B/pB)) \neq 0.$$

But since $E_p(B/pB)$ is a big locally CM B/pB-algebra, invoking again (10.4.30) and proposition 10.4.32(i), we are further reduced to showing that $\overline{Z} \cap \mathrm{Spec}\, B/pB$ has codimension c in $\mathrm{Spec}\, B/pB$, for every $p \in \mathrm{Max}(\mathbb{Z}) \setminus S$. But the unique morphisms of schemes $\varphi : \mathrm{Spec}\, B \to \mathrm{Spec}\, \mathbb{Z}$ and $\psi : \overline{Z} \to \mathrm{Spec}\, \mathbb{Z}$ are flat, and the generic fibre of φ (resp. ψ) is the equidimensional scheme $\mathrm{Spec}\, A$ (resp. Z), so $\varphi^{-1}(p)$ is as well equidimensional of dimension $\dim A$, for every $p \in \mathrm{Max}(\mathbb{Z}) \setminus S$, by lemma 11.4.1(i). By the same token, $\psi^{-1}(p)$ is equidimensional of dimension $\dim Z$, for every such p, whence the assertion. ◇

From claim 17.5.46 and (10.4.30), we already see that $\mathrm{depth}_{A_\mathfrak{p}} E_0(A)_\mathfrak{p} \geq \dim A_\mathfrak{p}$ for every prime ideal $\mathfrak{p} \subset A$. By the same token, if $\mathfrak{p} \subsetneq \mathfrak{q}$ are two such prime ideals, we have $\mathrm{depth}_{A_\mathfrak{q}} E_0(A)_\mathfrak{q} > \dim A_\mathfrak{p}$, so, again by virtue of claim 17.5.46 and (10.4.30), we must have $\mathrm{depth}_{A_\mathfrak{p}} E_0(A)_\mathfrak{p} = \dim A_\mathfrak{p}$ for every $\mathfrak{p} \in \mathrm{Spec}\, A$, as required.

(iii): Arguing as in the proof of proposition 17.5.37(i), for every $A \in \mathrm{Ob}(\mathscr{B}_0)$ we write $j_A : A \to A^\nu$ as the composition of the projection $j'_A : A \to A_{\mathrm{red}}$ onto the maximal reduced quotient, and the natural injective map $j''_A : A_{\mathrm{red}} \to A^\nu$. It is easily seen that both maps are morphisms of $\mathscr{B}_0$, and we then need to check that $E_0(j'_A)$ and $E_0(j''_A)$ are isomorphisms. However, clearly $B_{\mathrm{red}} \in \Delta(A_{\mathrm{red}})$, for every $B \in \Delta(A)$; for such B, pick a finite subset $S_B \subset \mathrm{Max}\, \mathbb{Z}$ as in lemma 17.5.42(ii). In order to prove that $E_0(j'_A)$ is an isomorphism, it suffices to check that the induced map $j'_{B,p} : B/pB \to B_{\mathrm{red}}/pB_{\mathrm{red}}$ induces an isomorphism $E_p(j'_{B,p}) : E_p(B/pB) \xrightarrow{\sim} E_p(B_{\mathrm{red}}/pB_{\mathrm{red}})$ for every $p \in \mathrm{Max}(\mathbb{Z}) \setminus S_B$; but $j'_{B,p}$ induces an isomorphism

$$(B/pB)_{\mathrm{red}} \xrightarrow{\sim} (B_{\mathrm{red}}/pB_{\mathrm{red}})_{\mathrm{red}}$$

so the assertion follows from proposition 17.5.37(i). Next, set $B^\nu := \prod_{\mathfrak{p} \in \mathrm{Max}\, B} (B/\mathfrak{p})^\nu$, and consider the commutative diagram of schemes :

$$\begin{array}{ccc} \mathrm{Spec}\, B^\nu & \xrightarrow{\pi} & \mathrm{Spec}\, B_{\mathrm{red}} \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ \mathrm{Spec}\, \mathbb{Z} & = & \mathrm{Spec}\, \mathbb{Z}. \end{array}$$

We have a maximal open subset $U \subset \mathrm{Spec}\, B_{\mathrm{red}}$ such that π restricts to an isomorphism $\pi^{-1}U \xrightarrow{\sim} U$, and since B_{red} is reduced, U is dense in $\mathrm{Spec}\, B_{\mathrm{red}}$. Let $\eta \in \mathrm{Spec}\, \mathbb{Z}$ be the generic point; it follows that $U \cap \varphi^{-1}(\eta)$ is a dense open subset of $\varphi^{-1}(\eta)$, and then there exists an open subset $V \subset \mathrm{Spec}\, \mathbb{Z}$ such that $U \cap \varphi^{-1}(x)$ is dense in $\varphi^{-1}(x)$ for every $x \in V$ ([65, Ch.IV, Prop.9.5.3]). Therefore, for every $x \in V$, the subset $\pi^{-1}U \cap \psi^{-1}(x)$ is open and dense in $\psi^{-1}(x)$, and π induces an isomorphism

$$\pi^{-1}U \cap \psi^{-1}(x) \xrightarrow{\sim} U \cap \varphi^{-1}(x) \qquad \text{for every } x \in V.$$

On the other hand, since the $\mathbb{Q}$-schemes $\varphi^{-1}(\eta)$ and $\psi^{-1}(\eta)$ are respectively geometrically reduced and geometrically normal, there exists a dense open subset $W \subset \mathrm{Spec}\, \mathbb{Z}$ such that $\varphi^{-1}(x)$ and $\psi^{-1}(x)$ are respectively a geometrically reduced and a geometrically normal $\kappa(x)$-scheme, for every $x \in W$ ([65, Ch.IV, Prop.9.9.4]). Summing up, we conclude that, for every $x \in V \cap W$, the fibre $\psi^{-1}(x)$ is the normalization of the affine reduced scheme $\varphi^{-1}(x)$. By proposition 17.5.37(ii), we conclude that the natural map $j''_{B,p} : B_{\mathrm{red}}/pB_{\mathrm{red}} \to B^\nu/pB^\nu$ induces an isomorphism $E_p(j''_{B,p})$ for every $p \in \mathrm{Max}(\mathbb{Z}) \cap V \cap W$. Therefore, j''_A induces an isomorphism $E_0(A_{\mathrm{red}}, B_{\mathrm{red}}) \xrightarrow{\sim} E_0(A^\nu, B^\nu)$, as required. □

Remark 17.5.47. For the definition of $E_0(A)$, we could have used instead an ultraproduct with respect to a non-principal ultrafilter of $\mathbb{N}$: this would have amounted to replacing $I_0(A, B)$ by a larger ideal (and thus, $E_0(A)$ by a smaller quotient); the resulting functor would have similar properties (and some others as well). The use of ultraproducts in commutative algebra is expounded in Schoutens's monograph [152], and our construction can be viewed as a simplification of the construction of big CM algebras in characteristic zero via ultraproducts, presented in [152, (6.4.2)].

17.5.48. The functor E_0 can be extended naturally to the category

$$\mathscr{B}'_0$$

whose objects are all locally equidimensional $\mathbb{Q}$-algebras of essentially finite type, and whose morphisms are again the maximizing ring homomorphisms. Indeed, for $A \in \mathrm{Ob}(\mathscr{B}'_0)$, let $\Delta'(A)$ be the set of $\mathbb{Q}$-subalgebras of finite type $B \subset A$, such that A is a localization of B. Notice that $\Delta'(A)$ is a filtered set, for the partial order given by inclusion of $\mathbb{Q}$-subalgebras; moreover, $A = \bigcup_{B\in\Delta'(A)} B$. Let also $\Delta''(A) := \{B \in \Delta'(A) \mid B \text{ is locally equidimensional}\}$. We notice:

Lemma 17.5.49. *With the notation of* (17.5.48)*, we have :*

(i) *For every* $B \in \Delta'(A)$*, the natural map* $\mathrm{Spec}\, A \to \mathrm{Spec}\, B$ *restricts to a bijection* $\mathrm{Min}\, A \xrightarrow{\sim} \mathrm{Min}\, B$ *on the respective sets of minimal prime ideals.*

(ii) $\Delta''(A)$ *is a cofinal subset of* $\Delta'(A)$*.*

Proof. (i): This is clear, since the natural morphism of schemes $\mathrm{Spec}\, A \to \mathrm{Spec}\, B$ is flat, injective on the underlying topological spaces, and with dense image.

(ii): Let $B \in \Delta'(A)$, so that $A = S^{-1}B$ for some multiplicative subset $S \subset B$, and say that $\mathrm{Min}\, A = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$. Set $\mathfrak{q}_i := B \cap \mathfrak{p}_i$ for every $i = 1, \dots, n$. Also, for $1 \le i < j \le n$, set $B_{ij} := B/(\mathfrak{q}_i + \mathfrak{q}_j)$ and $A_{ij} := A/(\mathfrak{p}_i + \mathfrak{p}_j) = S^{-1}B_{ij}$. For every such i, j and every $s \in S$, the induced map $\mathrm{Spec}\, A_{ij} \to \mathrm{Spec}\, B_{ij}[s^{-1}]$ restricts to an injective map of finite sets $\mathrm{Min}\, A_{ij} \to \mathrm{Min}\, B_{ij}[s^{-1}]$, and the system of such maps yields a natural identification : $\mathrm{Min}\, A_{ij} \xrightarrow{\sim} \lim_{s\in S} \mathrm{Min}\, B_{ij}[s^{-1}]$. Hence, for a suitable $s \in S$ we get bijections

$$\mathrm{Min}\, A_{ij} \xrightarrow{\sim} \mathrm{Min}\, B_{ij}[s^{-1}] \qquad \text{for every } 1 \le i < j \le n. \tag{17.5.50}$$

Now, both A and B are excellent rings ([64, Ch.IV, Sch.7.8.3(ii)]), and A fulfills condition (c) of lemma 17.5.31, since it is locally equidimensional; but in light of (17.5.50), we see that $B[s^{-1}]$ fulfills the same condition as well, so $B[s^{-1}] \in \Delta''(A)$, by lemma 17.5.31. □

In view of lemma 17.5.49(i), for every pair $B_1 \subset B_2$ of elements of $\Delta'(A)$, the inclusion map $B_1 \to B_2$ is trivially maximizing; with lemma 17.5.49(ii), we then get a well defined ring

$$E'_0(A) := \operatorname*{colim}_{B\in\Delta''(A)} E_0(B)$$

and the B-algebra structures on each $E_0(B)$ assemble into a natural A-algebra structure for $E'_0(A)$. Then, with theorem 17.5.45 and remark 17.5.19(iv), we see that $E'_0(A)$ is a big locally CM A-algebra. If $A \in \mathrm{Ob}(\mathscr{B}_0)$, clearly A is a final object of $\Delta''(A)$, so $E'_0(A) = E_0(A)$ in this case. Furthermore, let $f : A \to A'$ be any morphism of $\mathscr{B}'_0$; for every $B \in \Delta''(A)$, we may pick $B' \in \Delta''(A')$ such that f restricts to a map of $\mathbb{Q}$-algebras $g : B \to B'$. Since f is maximizing, from lemma 17.5.49(i) it follows easily that the same holds for g. We may therefore compose $E_0(g) : E_0(B) \to E_0(B')$ with the natural map $E_0(B') \to E_0(A')$, to get a ring homomorphism $h_B : E_0(B) \to E'_0(A)$, and it is easily seen that h_B depends only on the restriction $B \to A'$ of f (and is independent of the choice of B' : details left to the reader). The colimit of the system of maps $(h_B \mid B \in \Delta''(A))$ is a well defined ring homomorphism

$$E'_0(f) : E'_0(A) \to E'_0(A')$$

and the rules : $A \mapsto E'_0(A)$ and $f \mapsto E'_0(f)$ yield the sought functor $E' : \mathscr{B}'_0 \to \mathbb{Z}\text{-}\mathsf{Alg}$. Lastly, from theorem 17.5.45 we easily deduce that E'_0 commutes with finite products, and factors through the corresponding normalization functor $(-)^\nu : \mathscr{B}'_0 \to \mathscr{B}'_0$: details left to the reader.

17.5.51. We turn now to consider the case of a noetherian local ring $(A, \mathfrak{n})$ with residue field $k := A/\mathfrak{n}$ of characteristic $p > 0$. Arguing as in the proof of theorem 17.5.28, we can assume that A is a complete domain, with field of fractions of zero characteristic. By [126, Th.29.4], there exists a complete discrete valuation ring $R \subset A$ whose maximal ideal is pR, and such that the induced map $R/pR \to k$ is an isomorphism; moreover, if $p, x_1, \dots, x_n$ is any system of parameters of A, then the inclusion of R into A extends to a finite injective map :

$$R[[X_1, \dots, X_n]] \to A \qquad X_i \mapsto x_i \qquad \text{for } i = 1, \dots, n$$

whose image we call A_0. Let $\mathfrak{n}_0 \subset A_0$ be the maximal ideal; clearly the natural map $A_0/\mathfrak{n}_0 \to k$ is an isomorphism. Pick also $g \in A_0 \setminus \{0\}$ such that $A[g^{-1}]$ is faithfully flat étale over $A_0[g^{-1}]$.

• *We consider first the case where* $[k : k^p] < +\infty$, and we pick $x_{n+1}, \dots, x_m \in A_0$ whose classes in k form a p-basis (*i.e.* a basis of the k^p-vector space k). Set $x_0 := p$, and

$$A_\infty := A_0[T_0^{1/p^\infty}, \dots, T_m^{1/p^\infty}]/(x_0 - T_0, \dots, x_m - T_m)$$

(notation as in example 4.8.55(iii)). Notice that the sequence $(p, x_{n+1}, \dots, x_m)$ is maximal in the sense of remark 9.6.35(iii); hence A_∞ is formally perfectoid for its p-adic topology, according to theorem 17.2.14(iv). Next, if $g \in \{p, x_1, \dots, x_m\}$, set $B_\infty := A_\infty$, and otherwise, let

$$B_\infty := A_\infty[T_{m+1}^{1/p^\infty}]/(g - T_{m+1}).$$

In either case, let C be the p-root closure of B_∞ in $B_\infty[1/p]$, and $C^\wedge$ the p-adic completion of C; then $C^\wedge$ is perfectoid for its p-adic topology, and C is a faithfully flat A_∞-algebra (theorem 16.9.17). Let D be the integral closure of $C^\wedge$ in $C^\wedge \otimes_{A_0} A[1/g]$, and $D^\wedge$ the p-adic completion of D. Denote also by $\mathfrak{m} \subset C$ the ideal generated by $(g^{1/p^n} \mid n \in \mathbb{N})$; clearly $(C, \mathfrak{m})$ is a basic setup (in the sense of [75, §2.1.1]), and we let $(C^\wedge)^a$ and D^a be the $(C, \mathfrak{m})^a$-algebras arising from $C^\wedge$ and respectively D.

Lemma 17.5.52. (i) *D^a is faithfully flat over A_0, in the sense of definition* 14.2.40.
(ii) *The basic setup $(D^\wedge, \mathfrak{m}D^\wedge)$ is almost perfectoid.*

Proof. Denote as usual by $(C^\wedge)^a_*$ the ring of almost elements of $(C^\wedge)^a$, and let D_1 be the integral closure of $(C^\wedge)^a_*$ in $C^\wedge \otimes_{A_0} A[1/g]$. Recall that the natural morphism $D^a \to D_1^a$ of $(C^\wedge, \mathfrak{m})^a$-algebras is an isomorphism ([75, Lemma 8.2.28]); taking into account theorem 16.9.42(i,iii), it follows that D^a is a p-adically complete C^a-algebra, that $(D/p^nD)^a$ is a faithfully flat $(C/p^nC)^a$-algebra, for every $n \in \mathbb{N}$, and that (ii) holds. By virtue of proposition 14.2.43, in order to prove that D^a is flat over A_0, it then suffices to check that $(D/p^nD)^a$ is flat over A_0/p^nA_0, for every $n \in \mathbb{N}$. But it is easily seen that A_∞ is a flat A_0-algebra, whence the assertion. By lemma 14.2.41(ii), we then come down to checking that $(D/\mathfrak{n}_0D)^a \neq 0$. Since we already know that $(D/pD)^a$ is a faithfully flat $(C/pC)^a$-algebra, and since $p \in \mathfrak{n}_0$, we are further reduced to checking that $(C/\mathfrak{n}_0C)^a \neq 0$. In turns, this amounts to showing that $g^{1/p^m} \notin \mathfrak{n}_0C$ for some $m \in \mathbb{N}$. If the latter fails, we get :

$$g \in \mathfrak{n}_0^n C \qquad \text{for every } n \in \mathbb{N}. \tag{17.5.53}$$

Now, recall that C is a faithfully flat A_∞-algebra, and A_∞ is a faithfully flat A_0-algebra, so C is a faithfully flat A_0-algebra. Then, from (17.5.53) we deduce that $g \in \bigcap_{n \in \mathbb{N}} \mathfrak{n}_0^n$. But the $\mathfrak{n}_0$-adic topology of A_0 is separated ([126, Th.8.10(i)]), hence $g = 0$, a contradiction. □

Denote by $\mathsf{S} \subset C^{\mathbb{N}}$ the multiplicative subset associated with $\mathfrak{m}$, as in (17.5.2); we endow $E := \mathsf{S}^{-1}D^\diamond$ with its p-adic topology, and denote by E^{sep} the maximal separated quotient of E. Notice that D contains the integral closure of A_0 in $A[1/g]$, which contains A, hence D is naturally an A-algebra, and therefore the same holds for E and E^{sep}. We may then state :

Theorem 17.5.54. *E and E^{sep} are big CM A-algebras, and E^{sep} is perfectoid.*

Proof. In order to show that $E^{\rm sep}$ is perfectoid, it suffices to invoke lemmata 17.5.52(ii) and 17.5.10. Next, in light of remark 17.5.19(iii), it suffices to show that E and $E^{\rm sep}$ are big CM A_0-algebras. By lemmata 17.5.52(i) and 17.5.4(v), E is a faithfully flat A_0-algebra; then, by proposition 14.2.43(i), the same holds for the A_0-algebra $E^{\rm sep}$, since the latter is the p-adic completion of E (proposition 17.5.8). Especially, $E/\mathfrak{n}_0 E \stackrel{\sim}{\to} E^{\rm sep}/\mathfrak{n}_0 E^{\rm sep} \neq 0$. Now the theorem follows from lemma 10.4.20, proposition 10.4.32(i), and remark 17.5.19(i). □

• *Next, we consider the case where $[k : k^p]$ can be infinite.* Define the regular local ring $(A_0, \mathfrak{n}_0)$, and pick $g \in A_0$ as in the foregoing case; choose also a family $(b_\lambda \mid \lambda \in \Lambda)$ of elements of A_0 whose image $(\bar{b}_\lambda \mid \lambda \in \Lambda)$ in k is a p-basis. Again, we set $x_0 := p$, and

$$A_\infty := A_0[T_0^{1/p^\infty}, \dots, T_n^{1/p^\infty}][T_\lambda^{1/p^\infty} \mid \lambda \in \Lambda]/(x_0 - T_0, \dots, x_n - T_n, b_\lambda - T_\lambda \mid \lambda \in \Lambda).$$

We endow A_∞ with its $\mathfrak{n}_0$-adic topology, and let $A_\infty^\wedge$ be the completion of A_∞.

Lemma 17.5.55. *$A_\infty^\wedge$ is perfectoid for its p-adic topology.*

Proof. With the notation of (9.8.11), set

$$B_0 := k[[T_0, \dots, T_n]] \otimes_k k^{\rm perf} \qquad B_\infty := k[[T_0^{1/p^\infty}, \dots, T_n^{1/p^\infty}]] \otimes_k k^{\rm perf}$$

and let $\bar{\mathfrak{n}}_0 \subset B_0$ be the ideal generated by $T_0, \dots, T_n$. Endow B_∞ with its $\bar{\mathfrak{n}}_0$-adic topology, and let $B_\infty^\wedge$ be the completion of B_∞. Then B_∞ is naturally an A_0-algebra, via the projection $A_0 \to k$, and by simple inspection we get an isomorphism of topological A_0-algebras

$$A_\infty/pA_\infty \stackrel{\sim}{\to} B_\infty/T_0 B_\infty \quad T_i^{1/p^k} \mapsto T_i^{1/p^k} \ (i = 0, \dots, n) \quad T_\lambda^{1/p^k} \mapsto 1 \otimes \bar{b}_\lambda^{1/p^k} \ \text{ for every } \lambda \in \Lambda$$

for the respective quotient topologies. Notice that the sequence $(p, x_1, \dots, x_n)$ is A_0-regular, hence also A_∞-regular, since A_∞ is a faithfully flat A_0-algebra; by proposition 7.8.15, we deduce that the $\mathfrak{n}_0$-topology of A_∞ induces the $\mathfrak{n}_0$-topology on the ideal pA_∞, and therefore the natural map $A_\infty^\wedge/pA_\infty^\wedge \to (A_\infty/pA_\infty)^\wedge$ is an isomorphism of topological rings (proposition 8.2.13(i,v)). Likewise, the sequence $(T_0, \dots, T_n)$ is B_0-regular, hence also B_∞-regular, since B_∞ is a faithfully flat B_0-algebra; arguing as in the foregoing, we deduce that the natural map $B_\infty^\wedge/T_0 B_\infty^\wedge \to (B_\infty/T_0 B_\infty)^\wedge$ is as well an isomorphism of topological rings. However, by direct inspection we see that $B_0^{\rm perf}$ is a dense subring of $B_\infty^\wedge$ (here it is important to work with the $\bar{\mathfrak{n}}_0$-adic topology), and the $\bar{\mathfrak{n}}_0$-adic topology of $B_\infty^\wedge$ induces the $\bar{\mathfrak{n}}_0$-adic topology on $B_0^{\rm perf}$; hence $B_\infty^\wedge$ is also the completion C of $B_0^{\rm perf}$ (theorem 8.2.8(iii)), and therefore it is a perfect topological $\mathbb{F}_p$-algebra (remark 9.4.9(v) and corollary 9.4.14). Summing up, we see that $A_\infty^\wedge/pA_\infty^\wedge$ is isomorphic to $C/T_0 C$, and clearly the Frobenius endomorphism of C induces a bijection $C/T_0^{1/p} C \stackrel{\sim}{\to} C/T_0 C$; therefore the Frobenius endomorphism of $A_\infty^\wedge$ induces a bijection $A_\infty^\wedge/p^{1/p} A_\infty^\wedge \stackrel{\sim}{\to} A_\infty^\wedge/pA_\infty^\wedge$. Moreover, $A_\infty^\wedge$ is complete and separated for its p-adic topology (lemma 8.3.12); then the lemma follows from corollary 16.3.75. □

17.5.56. We may now proceed as in the previous case : if $g \in \{x_0, \dots, x_n\} \cup \{b_\lambda \mid \lambda \in \Lambda\}$, set $B_\infty := A_\infty$, and otherwise set $B_\infty := A_\infty^\wedge[X^{1/p^\infty}]/(g - X)$. Let also C be the p-root closure of B_∞ in $B_\infty[1/p]$, and $C^\wedge$ the p-adic completion of C. Then, again by theorem 16.9.17, $C^\wedge$ is perfectoid and C is a faithfully flat $A_\infty^\wedge$-algebra. Denote by $\mathfrak{m} \subset C^\wedge$ the ideal generated by $(g^{1/p^n} \mid n \in \mathbb{N})$, by D be the integral closure of $C^\wedge$ in $C^\wedge \otimes_{A_0} A[1/g]$, and by $D^\wedge$ the p-adic completion of D. Arguing as in (17.5.51) we see that $(D/p^n D)^a$ is a faithfully flat $(C/p^n C)^a$-algebra for every $n \in \mathbb{N}$ (relative to the basic setup $(C^\wedge, \mathfrak{m})$), and $(D^\wedge, \mathfrak{m}D^\wedge)$ is an almost perfectoid basic setup. Moreover, $A_\infty^\wedge$ is the $\mathfrak{n}_0$-adic completion of a faithfully flat A_0-algebra, hence it is a faithfully flat A_0-algebra (proposition 14.2.43(i)). Thus, C is a faithfully flat A_0-algebra. After these observations, the proof of lemma 17.5.52 can be repeated in our current situation, so that D^a is still faithfully flat over A_0 (in the sense of definition 14.2.40). Lastly, we define again $E := \mathsf{S}^{-1} D^\diamond$, and let $E^{\rm sep}$ be the maximal p-adically separated quotient of E; now the proof of theorem 17.5.54 applies again *verbatim*, and we finally conclude that E and E^{sep} are big CM A-algebras, and that E^{sep} is perfectoid for its p-adic topology.

17.5.57. Next, we would like to upgrade the constructions of the previous paragraphs to a functor on a suitable category of local noetherian domains. Hence, let $(A, \mathfrak{n})$ be a local noetherian domain, complete for its $\mathfrak{n}$-adic topology, with field of fractions of zero characteristic, and residue field k of characteristic $p > 0$, and suppose first that $[k : k^p] < +\infty$. Let $\Sigma \subset A \setminus \{0\}$ be a finite subset with $p \in \Sigma$, and $\Sigma' \subset \Sigma$ a non-empty subset; we consider the category

$$\mathscr{C}^A_{\Sigma,\Sigma'}$$

whose objects are all pairs (f, Δ), where $f : A \to B$ is a ring homomorphism, and $\Delta := (\beta_{a,\bullet} \mid a \in \Sigma)$ is a family of elements of $\mathbf{E}(B)$ (notation of remark 9.4.12(ii)), fulfilling the following conditions :

(a) the projection $\beta_{a,0} \in B$ of $\beta_{a,\bullet}$ equals $f(a)$, for every $a \in \Sigma$ (especially, $\beta^p_{p,1} = p$)
(b) let $\mathfrak{m} \subset B$ be the ideal generated by the system $(\delta_n := \prod_{a\in\Sigma'} \beta_{a,n} \mid n \in \mathbb{N})$; we call $(B, \mathfrak{m})$ the *basic setup attached to* (f, Δ), and denote by $(B, \mathfrak{m})^a$ the $(B, \mathfrak{m})^a$-algebra arising from B; then the unit of adjunction $B \to (B, \mathfrak{m})^a_*$ is an isomorphism
(c) B is p-torsion free, and complete and separated for its p-adic topology
(d) B is δ_0-torsion free, integrally closed in $B[\delta_0^{-1}]$, and p-root closed in $B[p^{-1}]$.

For two such pairs $(f : A \to B, \Delta)$ and $(f' : A \to B', \Delta' := (\beta'_{a,\bullet} \mid a \in \Sigma))$, the morphisms $(f, \Delta) \to (f', \Delta')$ in $\mathscr{C}^A_{\Sigma,\Sigma'}$ are the ring homomorphisms $h : B \to B'$ with :

$$f' = h \circ f \qquad \text{and} \qquad \mathbf{E}(h)(\beta_{a,\bullet}) = \beta'_{a,\bullet} \quad \text{for every } a \in \Sigma.$$

We shall say that the object $(f : A \to B, \Delta)$ of $\mathscr{C}^A_{\Sigma,\Sigma'}$ is *almost perfectoid*, if the basic setup $(B, \mathfrak{m})$ attached to (f, Δ) is almost perfectoid for the p-adic topology of B.

Remark 17.5.58. Let $\Sigma_1 \subset \Sigma_2$ be two finite subsets of $A \setminus \{0\}$ with $p \in \Sigma_1$, and $\Sigma'_1 \subset \Sigma_1$, $\Sigma'_2 \subset \Sigma_2$ two subsets, with $\varnothing \neq \Sigma'_1 \subset \Sigma'_2$. For every object $(f : A \to B, \Delta := (\beta_{a,\bullet} \mid a \in \Sigma_2))$ of $\mathscr{C}^A_{\Sigma_2,\Sigma'_2}$ set $\Delta_{|\Sigma_1} := (\beta_{a,\bullet} \mid a \in \Sigma_2)$. By remark 14.2.7(ii), we get a well defined functor :

$$\mathscr{C}^A_{\Sigma_2,\Sigma'_2} \to \mathscr{C}^A_{\Sigma_1,\Sigma'_1} \qquad (f, \Delta) \mapsto (f, \Delta)_{|\Sigma_1,\Sigma'_1} := (f, \Delta_{|\Sigma_1}).$$

Proposition 17.5.59. *Let $\Sigma_1 \subset \Sigma_2$ be an inclusion of finite subsets of $A \setminus \{0\}$ with $p \in \Sigma_1$, and $\Sigma'_1 \subset \Sigma_1$, $\Sigma'_2 \subset \Sigma_2$ two non-empty subsets, with $\Sigma'_1 \subset \Sigma'_2$. If the category $\mathscr{C}^A_{\Sigma_1,\Sigma'_1}$ admits an almost perfectoid initial object, the same holds for the category $\mathscr{C}^A_{\Sigma_2,\Sigma'_2}$.*

Proof. Let $(f_1 : A \to D_1, \Delta_1)$ be a given almost perfectoid initial object of $\mathscr{C}^A_{\Sigma_1,\Sigma'}$, with $\Delta_1 := (\beta_{x,\bullet} \mid x \in \Sigma_1)$, and let $\mathfrak{m}_1 \subset D_1$ be the ideal generated by $(\delta_n := \prod_{x\in\Sigma'_1} \beta_{x,n} \mid n \in \mathbb{N})$.

We suppose first that $\Sigma' := \Sigma'_2 = \Sigma'_1$, and we are easily reduced to the case where $\Sigma_2 = \Sigma_1 \cup \{a\}$ for some $a \in A$. We set

$$D'_2 := D_1[T^{1/p^\infty}]/(a - T).$$

Let D''_2 be the p-root closure of D'_2 in $D'_2[1/p]$, and $D''^{\wedge}_2$ the p-adic completion of D''_2. For every $x \in \Sigma_1$, denote by $\beta'_{x,\bullet} \in \mathbf{E}(D'_2)$ the image of $\beta_{x,\bullet}$, and let likewise

$$\beta'_{a,\bullet} := (\overline{T}^{1/p^n} \mid n \in \mathbb{N}) \in \mathbf{E}(D'_2)$$

where $\overline{T}^{1/p^n} \in D'_2$ denotes the image of T^{1/p^n}, for every $n \in \mathbb{N}$. Set $D_2 := (D''^{\wedge}_2, \mathfrak{m}_1 D''^{\wedge}_2)^a_*$, and let $f_2 : A \to D_2$ be the structure map of the A-algebra D_2. Lastly, let $\beta''_{x,\bullet} \in \mathbf{E}(D_2)$ be the image of $\beta'_{x,\bullet}$, for every $x \in \Sigma_2$, and set $\Delta_2 := (\beta''_{x,\bullet} \mid x \in \Sigma_2)$. We show more precisely :

Claim 17.5.60. (f_2, Δ_2) is an almost perfectoid initial object of the category $\mathscr{C}^A_{\Sigma_2,\Sigma'}$.

Proof of the claim. By proposition 17.5.16, the basic setup $(D_2''^{\wedge}, \mathfrak{m}_1 D_2''^{\wedge})$ is almost perfectoid, hence the same holds for $(D_2, \mathfrak{m}_1 D_2)$, by proposition 16.9.37(iii); then D_2 is p-root closed in $D_2[1/p]$, by proposition 16.9.37(iv), and is integrally closed in $D_2[1/\delta_0]$, by virtue of proposition 17.5.14(ii). This shows that (f_2, Δ_2) is an almost perfectoid object of $\mathscr{C}^A_{\Sigma_2,\Sigma'}$.

Next, let $(f : A \to D, \Delta := (\delta_{x,\bullet} \mid x \in \Sigma_2))$ be any other object of $\mathscr{C}^A_{\Sigma_2,\Sigma'}$; by assumption, there exists a unique morphism $h_0 : (f_1, \Delta_1) \to (f, \Delta')_{|\Sigma_1,\Sigma'}$ in $\mathscr{C}^A_{\Sigma_1,\Sigma'}$ (notation of remark 17.5.58). Clearly h_0 extends to a unique map $h_1 : D_2' \to D$ such that $\mathbf{E}(h_1)(\beta'_{x,\bullet}) = \delta_{x,\bullet}$ for every $x \in \Sigma_2$, and arguing as in the proof of proposition 17.5.62, we see that $A_0[1/p] \otimes_{1_0} h_1 : D_2'[1/p] \to D[1/p]$ restricts to a map of A-algebras $h_2 : D_2'' \to D$. Since D is p-adically complete and separated, the completion of h_2 is a map $h_2^{\wedge} : D_2''^{\wedge} \to D$, and since $D = D^a_*$, the latter induces a map of A-algebras $h := (h_2^{\wedge})^a_* : D_2 \to D$; it is then easily seen that h is the unique morphism $(f_2, \Delta_2) \to (f, \Delta)$ in $\mathscr{C}^A_{\Sigma_2,\Sigma'}$. ◊

It remains to consider the case where $\Sigma := \Sigma_1 = \Sigma_2$. Then, let $\mathfrak{m}_2 \subset D_1$ be the ideal generated by $(\gamma_n := \prod_{x\in\Sigma_2'} \beta_{x,n} \mid n \in \mathbb{N})$; endow $D_2 := (D_1, \mathfrak{m}_2)^a_*$ with its p-adic topology, and let $\eta : D_1 \to D_2$ be the unit of adjunction. Since the basic setup $(D_1, \mathfrak{m}_1)$ is almost perfectoid, and since $\mathfrak{m}_2 \subset \mathfrak{m}_1$, it is clear that the basic setup $(D_1, \mathfrak{m}_2)$ is almost perfectoid. Then the same holds for the basic setup $(D_2, \mathfrak{m}_2 D_2)$, according to proposition 16.9.37(iii). By proposition 17.5.14(ii), it follows that D_2 is γ_0-torsion free and integrally closed in $D_2[\gamma_0^{-1}]$. For every $x \in \Sigma$, let $\beta'_{x,\bullet} \in \mathbf{E}(D_2)$ be the image of $\beta_{x,\bullet}$; set $\Delta_2 := (\beta'_{x,\bullet} \mid x \in \Sigma)$ and $f_2 := \eta \circ f_1$. The foregoing already shows that (f_2, Δ_2) is an object of $\mathscr{C}^A_{\Sigma,\Sigma_2'}$. Lastly, let $(f' : A \to D', \Delta')$ be any object of $\mathscr{C}^A_{\Sigma,\Sigma_2'}$; by assumption, there exists a unique morphism $h : (f_1, \Delta_1) \to (f', \Delta')_{|\Sigma,\Sigma_1'}$ in $\mathscr{C}^A_{\Sigma,\Sigma_1'}$, and since $D' = (D', \mathfrak{m}_2)^a_*$, it follows easily that $(h, \mathfrak{m}_2)^a_* : D_2 \to D'$ is the unique morphism $(f_2, \Delta_2) \to (f', \Delta')$ in $\mathscr{C}^A_{\Sigma,\Sigma_2'}$. □

17.5.61. Resume the notation of (17.5.51), set $x_0 := p$, and define either $t := m$ if $g \in \{x_0, \dots, x_m\}$, or else $t := m + 1$ and $x_t := g$; we take

$$\Sigma_0 := \{x_0, \dots, x_t\} \qquad \Sigma_0' := \{x_t\}.$$

Let also $f_0 : A \to D^a_* := (D, \mathfrak{m}D)^a_*$ be the structure map of the A-algebra D^a_*, and for $i = 0, \dots, t$, set $\beta_{x_i,\bullet} := (\overline{T}_i^{1/p^n} \mid n \in \mathbb{N}) \in \mathbf{E}(D^a_*)$, where $\overline{T}_i^{1/p^n} \in D^a_*$ denotes the image of T_i^{1/p^n}, for every such i and every $n \in \mathbb{N}$. Set $\Delta_0 := (\beta_{a,\bullet} \mid a \in \Sigma_0)$. Lastly, let $\Sigma_1' \subset \Sigma_1 \subset A \setminus \{0\}$ be two finite subsets with $\Sigma_0 \subset \Sigma_1$ and $\Sigma_0' \subset \Sigma_1'$.

Proposition 17.5.62. *In the situation of* (17.5.61)*, we have :*

(i) *The pair (f_0, Δ_0) is an almost perfectoid initial object of the category $\mathscr{C}^A_{\Sigma_0,\Sigma_0'}$.*

(ii) *Let $(f_1 : A \to D_1, \Delta_1)$ be an initial object of $\mathscr{C}^A_{\Sigma_1,\Sigma_1'}$, and $(D_1, \mathfrak{m}_1)$ the basic setup attached to (f_1, Δ_1). Then $(D_1, \mathfrak{m}_1)^a$ is faithfully flat over A_0, in the sense of definition* 14.2.40.

Proof. (i): From theorem 16.9.42(i,v) and lemma 16.9.41 we see that (f_0, Δ_0) is an almost perfectoid object of $\mathscr{C}^A_{\Sigma_0,\Sigma_0'}$. Now, define B_∞, C and $C^{\wedge}$ as in (17.5.51), and let

$$(f' : A \to D', \Delta' := (\beta'_{a,\bullet} \mid a \in \Sigma_0))$$

be any other object of $\mathscr{C}^A_{\Sigma_0,\Sigma'}$; clearly there exists a unique homomorphism $h_0 : B_\infty \to D'$ of A_0-algebras such that $\mathbf{E}(h_0)(\beta_{a,\bullet}) = \beta'_{a,\bullet}$ for every $a \in \Sigma_0$. Since D' is p-root closed in $D'[1/p]$, by proposition 9.8.28(i,ii) and remark 9.8.16(i), the map

$$A_0[1/p] \otimes_{A_0} h_0 : B_\infty[1/p] \to D'[1/p]$$

restricts to a map of A_0-algebras $h_1 : C \to D'$, and clearly h_1 is the unique map of A_0-algebras extending h_0. Next, h_1 extends uniquely to a map of A_0-algebras $h_1^{\wedge} : C^{\wedge} \to D'$, since by assumption D' is p-adically complete and separated. Set $h_2 := A[1/g] \otimes_{A_0} h_1^{\wedge} :$

$A[1/g] \otimes_{A_0} C^\wedge \to D'[1/g]$; by assumption, the localization $D' \to D'[1/g]$ is injective, and D' is integrally closed in $D'[1/g]$, so h_2 restricts to a map of A-algebras $h_3 : D \to D'$. Lastly, since $D' = (D')^a_*$, we get an induced map of A-algebras $h := h^a_{3*} : D^a_* \to D'$, and by construction it is easily seen that h is the unique morphism $(f_0, \Delta_0) \to (f', \Delta')$ in $\mathscr{C}^A_{\Sigma_0,\Sigma'_0}$.

(ii): Suppose first that $\Sigma'_1 = \Sigma'_0$. Recall that D_1 is p-adically complete and separated; taking into account proposition 14.2.43, in order to show that $(D_1, \mathfrak{m}_1)^a$ is flat over A_0, it then suffices to check that $(D_1/p^n D_1)^a$ is flat over $A_0/p^n A_0$ for every $n \in \mathbb{N}$. But lemma 17.5.52(i) implies that $(D/p^n D)^a$ is flat over $A_0/p^n A_0$ for every such n; hence we come down to the following :

Claim 17.5.63. In the situation of proposition 17.5.59, suppose that $\Sigma' := \Sigma'_1 = \Sigma'_2$, and let $(f_1 : A \to D_1, \Delta_1)$ (resp. $(f_2 : A \to D_2, \Delta_2)$) be an initial object of $\mathscr{C}^A_{\Sigma_1,\Sigma'}$ (resp. of $\mathscr{C}^A_{\Sigma_2,\sigma'}$). Let also $(D_1, \mathfrak{m}_1)$ be the basic setup attached to (f_1, Δ_1). Then the unique morphism $(f_1, \Delta_1) \to (f_2, \Delta_2)_{\Sigma_1,\Sigma'}$ in $\mathscr{C}^A_{\Sigma_1,\Sigma'}$ induces a faithfully flat morphism of $(D_1, \mathfrak{m}_1)^a$-algebras

$$(D_1/p^n D_1)^a \to (D_2/p^n D_2)^a \qquad \text{for every } n \in \mathbb{N}.$$

Proof of the claim. We are easily reduced to the case where $\Sigma_2 = \Sigma_1 \cup \{a\}$ for some $a \in A\setminus\{0\}$, and then the assertion follows from proposition 17.5.16, after inspecting the proof of proposition 17.5.59. ◊

By lemma 14.2.41(ii), there remains only to show that $(D_1/\mathfrak{n}_0 D_1)^a \neq 0$. Since $p \in \mathfrak{n}_0$, it suffices to check that $(D_1/pD_1)^a$ is faithfully flat over A_0/pA_0; since the morphism $(D/pD)^a \to (D_1/pD_1)^a$ is faithfully flat (claim 17.5.63), lemma 14.2.41(iii) further reduces to checking that $(D/pD)^a$ is faithfully flat over A_0/pA_0, which follows from lemma 17.5.52(i).

To deal with the general case, let $(f'_1 : A \to D'_1, \Delta'_1 := (\beta'_{a,\bullet} \mid a \in \Sigma_1))$ be an initial object of $\mathscr{C}^A_{\Sigma_1,\Sigma'_0}$, and $\mathfrak{m}'_1 \subset D'_1$ the ideal generated by $(\beta'_n := \prod_{a\in\Sigma'_1} \beta'_{a,n} \mid n \in \mathbb{N})$. By the foregoing, the $(C, \mathfrak{m})^a$-algebra $(D'_1, \mathfrak{m}D'_1)^a$ is faithfully flat over A_0, and by inspecting the proof of proposition 17.5.59, we see that the A-algebra D_1 is isomorphic to $(D'_1, \mathfrak{m}'_1)^a_*$. Hence, the two functors

$$A_0\text{-}\mathbf{Mod} \to (D_1, \mathfrak{m}_1)^a\text{-}\mathbf{Mod} \qquad N \mapsto (D_1, \mathfrak{m}_1)^a \otimes_{A_0} N \qquad N \mapsto (D'_1, \mathfrak{m}'_1)^a \otimes_{A_0} N$$

are isomorphic (notation of definition 14.2.40). However, the functor $(D'_1, \mathfrak{m}'_1)^a \otimes_{A_0} -$ is the composition of the exact functor

$$A_0\text{-}\mathbf{Mod} \to (D'_1, \mathfrak{m}D'_1)^a\text{-}\mathbf{Mod} \qquad N \mapsto (D'_1, \mathfrak{m}D'_1)^a \otimes_{A_0} N$$

and the natural functor $(D'_1, \mathfrak{m}D'_1)^a\text{-}\mathbf{Mod} \to (D'_1, \mathfrak{m}'_1)^a\text{-}\mathbf{Mod} \xrightarrow{\sim} (D_1, \mathfrak{m}_1)^a\text{-}\mathbf{Mod}$, which is obviously exact, so the same holds for the functor $(D'_1, \mathfrak{m}'_1)^a \otimes_{A_0} -$, and hence also for $(D_1, \mathfrak{m}_1)^a \otimes_{A_0} -$, *i.e.* $(D_1, \mathfrak{m}_1)^a$ is flat over A_0. Again, there remains to show that the $(D_1, \mathfrak{m}_1)^a$-module $(D_1/\mathfrak{n}_0 D_1)^a$ is not zero, or equivalently, that the same holds for the $(D'_1, \mathfrak{m}'_1)^a$-module $(D'_1/\mathfrak{n}_0 D'_1)^a$. Suppose that the latter fails; then we have $\beta'_n \in \mathfrak{n}_0 D'_1$ for every $n \in \mathbb{N}$. Set $b := \prod_{a\in\Sigma'_1} a \in A \setminus \{0\}$; it follows that $f'_1(b) \in \mathfrak{n}_0^k D'_1$ for every $k \in \mathbb{N}$.

Claim 17.5.64. (i) There exists $c \in A \setminus \{0\}$ such that $bc \in A_0$.
(ii) f'_1 induces an injective map $A_0/\mathfrak{n}_0^k \to D'_1/\mathfrak{n}_0^k D'_1$ for every $k \in \mathbb{N}$.

Proof of the claim. (i): Since A is a finite A_0-algebra, we may find a monic polynomial $P \in A_0[X]$ such that $P(b) = 0$, and since A is a domain, we may assume that $P(0) \neq 0$. Then it is easily seen that $P(0) = bc$ for some $c \in A \setminus \{0\}$.

(ii): We know already that $(D'_1, \mathfrak{m}'_1)^a$ is faithfully flat over A_0, hence $(D'_1/\mathfrak{n}_0^k D'_1)^a$ is faithfully flat over $A_0/\mathfrak{n}_0^k$ for every $k \in \mathbb{N}$. Then the assertion follows from lemma 14.2.41(i). ◊

Let c be as in claim 17.5.64(i); hence $bc \in A_0 \setminus \{0\}$, and $f'_1(bc) \in \mathfrak{n}_0^k D'_1$ for every $k \in \mathbb{N}$. By claim 17.5.64(ii), it follows that $bc \in \bigcap_{k\in\mathbb{N}} \mathfrak{n}_0^k$; but the $\mathfrak{n}_0$-adic topology of A_0 is separated ([126, Th.8.10(i)]), so $bc = 0$, a contradiction. □

17.5.65. Next, let $(A, \mathfrak{n})$ be any local noetherian domain, complete for its $\mathfrak{n}$-adic topology, with $\mathrm{Frac}\, A$ of zero characteristic, and residue field of characteristic $p > 0$. Let $\Sigma' \subset \Sigma$ be two subsets of $A \setminus \{0\}$ such that Σ' is finite and non-empty, and $p \in \Sigma$; we consider the category

$$\mathscr{D}^A_{\Sigma,\Sigma'}$$

whose objects are all pairs (f, Δ), where $f : A \to B$ is a ring homomorphism, and $\Delta := (\beta_{a,\bullet} \mid a \in \Sigma)$ is a family of elements of $\mathbf{E}(B)$, fulfilling conditions (a)–(d) of (17.5.57), as well as the following :

(e) B is complete and separated for its $\mathfrak{n}$-adic topology.

If $\Sigma_1 \subset \Sigma_2$ is an inclusion of subsets of $A \setminus \{0\}$, and $\Sigma'_1 \subset \Sigma_1$, $\Sigma'_2 \subset \Sigma_2$ are two non-empty finite subsets with $\Sigma'_1 \subset \Sigma'_2$, then clearly, remark 17.5.58 holds *verbatim* in the present situation, *i.e.* we get a well defined restriction functor :

$$\mathscr{D}^A_{\Sigma_2,\Sigma'_2} \to \mathscr{D}^A_{\Sigma_1,\Sigma'_1} \qquad (f, \Delta) \mapsto (f, \Delta)_{|\Sigma_1,\Sigma'_1} := (f, \Delta_{|\Sigma_1}).$$

Let moreover the subring $A_0 \subset A$ be as in (17.5.51); we shall say that an object $(f : A \to B, \Delta)$ of $\mathscr{D}^A_{\Sigma,\Sigma'}$ is almost flat (resp. almost faithfully flat) over A_0, if B^a is flat (resp. faithfully flat) over A_0 (as usual, we refer here to the almost structure associated with the basic setup $(B, \mathfrak{m})$ attached to (f, Δ)). We have the following variant of proposition 17.5.59 :

Proposition 17.5.66. *Let $\Sigma_1 \subset \Sigma_2$ be two subsets of $A \setminus \{0\}$ with $p \in \Sigma_1$, and $\Sigma'_1 \subset \Sigma_1$, $\Sigma'_2 \subset \Sigma_2$ two finite subsets, with $\varnothing \neq \Sigma'_1 \subset \Sigma'_2$. If the category $\mathscr{D}^A_{\Sigma_1,\Sigma'_1}$ admits an almost perfectoid initial object that is almost flat (resp. almost faithfully flat) over A_0, the same holds for $\mathscr{D}^A_{\Sigma_2,\Sigma'_2}$.*

Proof. Let $(f_1 : A \to D_1, \Delta_1 := (\beta_{a,\bullet} \mid a \in \Sigma_1))$ be an almost perfectoid initial object of $\mathscr{D}^A_{\Sigma_1,\Sigma'_1}$. Consider first the case where $\Sigma' := \Sigma'_2 = \Sigma'_1$. Say that $\Sigma_2 \setminus \Sigma_1 = (a_\lambda \mid \lambda \in \Lambda)$; we set :

$$D'_2 := D_1[T_\lambda^{1/p^\infty} \mid \lambda \in \Lambda]/(a_\lambda - T_\lambda \mid \lambda \in \Lambda).$$

Let D''_2 be the p-root closure of D'_2 in $D'_2[1/p]$; endow D''_2 with its p-adic topology, and let $D''^{\wedge}_2$ be the completion of D''_2. Let moreover $(D_1, \mathfrak{m}_1)$ be the basic setup attached to (f_1, Δ_1).

Claim 17.5.67. (i) The basic setup $(D''^{\wedge}_2, \mathfrak{m}_1 D''^{\wedge}_2)$ is almost perfectoid.

(ii) $(D''_2)^a$ is a faithfully flat D_1^a-algebra.

Proof of the claim. Let $\mathscr{P}$ be the set of all finite subsets of Λ, and for every $\Omega \in \mathscr{P}$, set

$$D'_{2,\Omega} := D_1[T_\lambda^{1/p^\infty} \mid \lambda \in \Omega]/(a_\lambda - T_\lambda \mid \lambda \in \Omega)$$

and let $D''_{2,\Omega}$ be the p-root closure of $D'_{2,\Omega}$ in $D'_{2,\Omega}[1/p]$; endow $D''_{2,\Omega}$ of its p-adic topology, and denote by $D''^{\wedge}_{2,\Omega}$ the completion of $D''_{2,\Omega}$. Clearly, every inclusion $\Omega \subset \Omega'$ of elements of $\mathscr{P}$ induces an injective map of D_1-algebras $D'_{2,\Omega} \to D'_{2,\Omega'}$, which induces in turn a map of D_1-algebras $D''_{2,\Omega} \to D''_{2,\Omega'}$ (remark 9.8.16(i) and proposition 9.8.28). Clearly D'_2 is the colimit of the filtered system of D_1-algebras $(D'_{2,\Omega} \mid \Omega \in \mathscr{P})$, and by lemma 9.8.17, D''_2 is the colimit of the filtered system $(D''_{2,\Omega} \mid \Omega \in \mathscr{P})$. Recall that, by construction, p admits a p-th root $p^{1/p}$ in D_1. Let Φ be the Frobenius endomorphism of D_1/pD_1, and suppose now that the Frobenius endomorphism of $D''_{2,\Omega}$ induces an isomorphism $(D''_{2,\Omega}/p^{1/p}D''_{2,\Omega})^a \xrightarrow{\sim} (D''_{2,\Omega}/pD''_{2,\Omega})^a_{(\Phi)}$, for every $\Omega \in \mathscr{P}$ (notation as in definition 16.9.35); then the Frobenius endomorphism of D''_2 induces an isomorphism $(D''_2/p^{1/p}D''_2)^a \xrightarrow{\sim} (D''_2/pD''_2)^a_{(\Phi)}$. We are thus reduced to checking that the basic setup $(D''^{\wedge}_{2,\Omega}, \mathfrak{m}_1 D''^{\wedge}_{2,\Omega})$ is almost perfectoid, and that $(D''_{2,\Omega})^a$ is a faithfully flat D_1^a-algebra, for every $\Omega \in \mathscr{P}$. The latter follows from proposition 17.5.16, by an easy induction on the cardinality of Ω : the details shall be left to the reader. ◊

Set $D_2 := (D''^{\wedge}_2, \mathfrak{m}_1 D''^{\wedge}_2)^a_*$; the basic setup $(D_2, \mathfrak{m}_1 D_2)$ is almost perfectoid for the p-adic topology of D_2, by claim 17.5.67(i) and proposition 16.9.37(iii). Moreover, if D_1^a is flat (resp.

faithfully flat) over A_0, the same holds for $(D''_2)^a$, by claim 17.5.67(ii) and lemma 14.2.41(iii); consequently, the same holds as well for both $(D''^{\wedge}_2)^a$ and D^a_2, by proposition 14.2.43(i).

Endow D_2 with its $\mathfrak{n}$-adic topology, let $D^{\wedge}_2$ be the completion of D_2, and $E_2 := (D^{\wedge}_2, \mathfrak{m}_1 D^{\wedge}_2)^a_*$. Notice that the $\mathfrak{n}$-adic topology agrees with the $\mathfrak{n}_0$-adic topology, on every A-module M.

Claim 17.5.68. (i) E_2 has no p-torsion and is $\mathfrak{n}$-adically complete and separated.

(ii) Let R be any D_1/pD_1-algebra, and denote by $R^{\wedge}$ and $(R_{(\Phi)})^{\wedge}$ the $\mathfrak{n}$-adic completions of R and of $R_{(\Phi)}$. Then $(R^{\wedge})_{(\Phi)} = (R_{(\Phi)})^{\wedge}$.

(iii) The basic setup $(E_2, \mathfrak{m}_1 E_2)$ is almost perfectoid for the p-adic topology of E_2.

Proof of the claim. (i): For the first assertion, it suffices to check that the endomorphism $p\cdot\mathbf{1}_{(D^{\wedge}_2)^a}$ is a monomorphism, since the functor $(-)_*$ is left exact. The latter will follow from lemma 14.2.45, after we check that the $\mathfrak{n}_0$-adic topology of D^a_2 induces the $\mathfrak{n}_0$-adic topology on the ideal pD^a_2 (see [75, §5.3] for generalities on topological almost algebras and modules). However, by the Artin-Rees lemma, there exists $c \in \mathbb{N}$ such that $\mathfrak{n}^{n+c}_0 A_0 \cap pA_0 \subset \mathfrak{n}^n_0 pA_0$ for every $n \in \mathbb{N}$. But we have already noticed that D^a_2 is flat over A_0, hence $\mathfrak{n}^{n+c}_0 D^a_2 \cap pD^a_2 \subset \mathfrak{n}^n_0 pD^a_2$ for every $n \in \mathbb{N}$, as required. The second assertion follows from claim 16.9.39.

(ii): By definition, $\mathfrak{n} \cdot R_{(\Phi)} = J \cdot R$, where $J \subset A$ is the ideal generated by $(a^p \mid a \in \mathfrak{n})$; since $\mathfrak{n}$ is finitely generated, there exists $N \in \mathbb{N}$ such that $\mathfrak{n}^N \subset J \subset \mathfrak{n}$, so the J-adic topology agrees with the $\mathfrak{n}$-adic topology on R, whence the assertion.

(iii): First, E_2 is p-adically complete and separated, by (i) and lemma 8.3.12. Next, pick $c \in \mathbb{N}$ as in the proof of (i); for every $n \in \mathbb{N}$ and every $x \in \mathfrak{n}^{n+c}_0 D_2 \cap p^{1/p} D_2$ we have $p^{(p-1)/p}x \in \mathfrak{n}^{n+c}_0 D_2 \cap pD_2$, whence $\mathfrak{m}_1 \cdot p^{(p-1)/p}x \subset \mathfrak{n}^n_0 pD_2$, and therefore $\mathfrak{m}_1 \cdot x \subset \mathfrak{n}^n_0 p^{1/p} D_2$. This shows that $\mathfrak{n}^{n+c}_0 D^a_2 \cap p^{1/p} D^a_2 \subset \mathfrak{n}^n_0 p^{1/p} D^a_2$ for every $n \in \mathbb{N}$, so the $\mathfrak{n}_0$-adic topology of D_2 induces the $\mathfrak{n}_0$-adic topology on the ideal $p^{1/p}D^a_2$. By assumption, the Frobenius endomorphism of D_2/pD_2 induces an isomorphism $D^a_2/p^{1/p}D^a_2 \xrightarrow{\sim} (D^a_2/pD^a_2)_{(\Phi)}$; the latter persists after taking $\mathfrak{n}_0$-adic completions, and in view of (ii) and lemma 14.2.45, we conclude that the Frobenius endomorphism of E_2/pE_2 induces an isomorphism $E^a_2/p^{1/p}E^a_2 \xrightarrow{\sim} (E^a_2/pE^a_2)_{(\Phi)}$, as needed. $\diamondsuit$

For every $a \in \Sigma_1$, let $\beta'_{a,\bullet} \in \mathbf{E}(D_2)$ be the image of $\beta_{a,\bullet}$, and set likewise

$$\beta'_{a_\lambda,\bullet} := (\overline{T}^{1/p^n}_\lambda \mid n \in \mathbb{N}) \in \mathbf{E}(D_2) \qquad \text{for every } \lambda \in \Lambda$$

where $\overline{T}^{1/p^n}_\lambda \in D_2$ denotes the image of T^{1/p^n}_λ, for every such λ and every $n \in \mathbb{N}$. Moreover, for every $a \in \Sigma_2$, let $\beta''_{a,\bullet} \in \mathbf{E}(E_2)$ be the image of $\beta'_{a,\bullet}$, and set $\Delta_2 := (\beta''_{a,\bullet} \mid a \in \Sigma_2)$. Let also $f_2 : A \to E_2$ be the structure map of the A-algebra E_2; by claim 17.5.68(i) the pair $(f_2 : A \to E_2, \Delta_2)$ fulfills conditions (a),(b),(c) of (17.5.57) and condition (e) of (17.5.65). Combining with propositions 16.9.37(iv) and 17.5.14(ii) we deduce that also condition (d) of (17.5.57) holds for this pair, *i.e.* (f_2, Δ_2) is an almost perfectoid object of $\mathscr{D}^A_{\Sigma_2,\Sigma'}$; moreover, if D^a_2 is flat (resp. faithfully flat) over A_0, the same holds for E^a_2, by virtue of proposition 14.2.43.

Lastly, let $(g : A \to F, \Delta := (\delta_{a,\bullet} \mid a \in \Sigma_2))$ be any other object of $\mathscr{D}^A_{\Sigma_2,\Sigma'}$; arguing as in the proof of claim 17.5.60, we see that there exists a unique map $h : D_2 \to F$ of A-algebras such that $\mathbf{E}(h')(\beta'_{a,\bullet}) = \delta_{a,\bullet}$ for every $a \in \Sigma_2$. Let $h^{\wedge} : D^{\wedge}_2 \to F$ be the $\mathfrak{n}$-adic completion of h'; it follows easily that $h := (h^{\wedge})^a_* : E_2 \to F$ is the unique morphism $(f_2, \Delta_2) \to (g, \Delta)$ in $\mathscr{D}^A_{\Sigma_2,\Sigma'}$.

It remains consider the case where $\Sigma := \Sigma_1 = \Sigma_2$. Then, let $\mathfrak{m}_2 \subset D_1$ be the ideal generated by $(\prod_{a\in\Sigma'_2} \beta_{a,n} \mid n \in \mathbb{N})$; let $D^{\wedge}_2$ be the $\mathfrak{n}$-adic completion of $D_2 := (D_1, \mathfrak{m}_2)^a_*$, and set $E_2 := (D^{\wedge}_2, \mathfrak{m}_2 D^{\wedge}_2)^a_*$. Arguing as in the proof of proposition 17.5.59, we see that the basic setup $(D_2, \mathfrak{m}_2 D_2)$ is almost perfectoid for the p-adic topology of D_2. Moreover, arguing as in the proof of proposition 17.5.62(ii), we see that if $(D_1, \mathfrak{m}_1)^a$ is flat over A_0, the same holds for $(D_1, \mathfrak{m}_2)^a$, so also for $(D_2, \mathfrak{m}_2 D_2)^a$, and then also for $(D^{\wedge}_2, \mathfrak{m}_2 D^{\wedge}_2)^a$, by proposition 16.9.37. Furthermore, if $(D_1, \mathfrak{m}_1)^a$ is faithfully flat over A_0, then arguing again as in the proof of proposition 17.5.62(ii), we check that the same holds for $(D_2, \mathfrak{m}_2 D_2)^a$, and thus, also for

$(D_2^\wedge, \mathfrak{m}_2 D_2^\wedge)^a$. Summing up, we conclude that if $(D_1, \mathfrak{m}_1)^a$ is flat (resp. faithfully flat) over A_0, the same holds for $(E_2, \mathfrak{m}_2 E_2)^a$. Then, arguing as in the proof of claim 17.5.68 we see that E_2 has no p-torsion and is $\mathfrak{n}$-adically complete and separated, and the basic setup $(E_2, \mathfrak{m}_1 E_2)$ is almost perfectoid for the p-adic topology of E_2. For every $a \in \Sigma$, let $\beta'_{a,\bullet} \in \mathbf{E}(E_2)$ be the image of $\beta_{a,\bullet}$, and set $\Delta_2 := (\beta'_{a,\bullet} \mid a \in \Sigma)$; let also $f_2 : A \to E_2$ be the structure map of the A-algebra E_2; as in the previous case, we conclude that (f_2, Δ_2) is an almost perfectoid object of $\mathscr{D}^A_{\Sigma,\Sigma'_2}$, and it is easily seen that it is the sought initial object : details left to the reader. □

17.5.69. Resume the notation of (17.5.56), and let $D^\wedge$ be the $\mathfrak{n}$-adic completion of D, and $D' := (D^\wedge, \mathfrak{m}D^\wedge)^a_*$. Set also $x_0 := p$, and let either $t := n$ if $g \in \{x_0, \dots, x_n\} \cup \{b_\lambda \mid \lambda \in \Lambda\}$, or else $t := n+1$ and $x_t := g$; we take

$$\Sigma'_0 := \{x_t\} \qquad \Sigma_0 := \{x_0, \dots, x_t\} \cup \{b_\lambda \mid \lambda \in \Lambda\}.$$

For $i = 0, \dots, t$ and for every $\lambda \in \Lambda$, let $\beta_{x_i,\bullet} := (\overline{T}_i^{1/p^k} \mid k \in \mathbb{N}) \in \mathbf{E}(D')$ and $\beta_{b_\lambda,\bullet} := (\overline{T}_\lambda^{1/p^k} \mid k \in \mathbb{N}) \in \mathbf{E}(D')$, where as usual $\overline{T}_i^{1/p^k}, \overline{T}_\lambda^{1/p^k} \in D'$ denote the image of T_i^{1/p^k} and T_λ^{1/p^k}, for every such i, every λ, and every $k \in \mathbb{N}$. Lastly, set $\Delta_0 := (\beta_{a,\bullet} \mid a \in \Sigma_0)$, and let $f_0 : A \to D'$ be the structure map of the A-algebra D'.

Proposition 17.5.70. *The pair (f_0, Δ_0) is an almost perfectoid initial object of $\mathscr{D}^A_{\Sigma_0,\Sigma'_0}$, and is almost faithfully flat over A_0.*

Proof. Let D_1 be the integral closure of $(C^\wedge, \mathfrak{m}C^\wedge)^a_*$ in $C^\wedge \otimes_{A_0} A[1/g]$, and recall that the inclusion map $j : D \to D_1$ induces an isomorphism $D^a \xrightarrow{\sim} D_1^a$ of $(C, \mathfrak{m})^a$-algebras ([75, Lemma 8.2.28]), so its $\mathfrak{n}$-adic completion $j^\wedge : D^\wedge \to D_1^\wedge$ induces an isomorphism of C-algebras $(j^\wedge)^a_* : D' \xrightarrow{\sim} (D_1^\wedge)^a_*$. By theorem 16.9.42(i,iii), $(D_1, \mathfrak{m}D_1)$ is an almost perfectoid basic setup, for the p-adic topology of D_1, and $D_1^a/p^n D_1^a$ is a faithfully flat $C^a/p^n C^a$-algebra, for every $n \in \mathbb{N}$; hence $D_1^a/p^n D_1^a$ is faithfully flat over $A_0/p^n A_0$, for every $n \in \mathbb{N}$, and therefore D_1^a is faithfully flat over A_0 (proposition 14.2.43(i)). We may now argue as in the proof of claim 17.5.68(iii), to conclude that $((D_1^\wedge)^a_*, \mathfrak{m}(D_1^\wedge)^a_*)$ is an almost perfectoid basic setup, for the p-adic topology of D_1, so the same holds for $(D', \mathfrak{m}D')$. This already shows that the pair (f_0, Δ_0) fulfills conditions (a),(b),(c) of (17.5.57) and condition (e) of (17.5.65). Combining with propositions 16.9.37(iv) and 17.5.14(ii) we deduce that also condition (d) of (17.5.57) holds for this pair, *i.e.* (f_0, Δ_0) is an almost perfectoid object of $\mathscr{D}^A_{\Sigma_0,\Sigma'_0}$. Moreover, since D_1^a is faithfully flat over A_0, the same holds for $(D_1^\wedge)^a$, again by proposition 14.2.43(i), so also for D'^a. Lastly, one verifies as in the proof of proposition 17.5.62(i) that (f_0, Δ_0) is an initial object of $\mathscr{D}^A_{\Sigma_0,\Sigma'_0}$: details left to the reader. □

17.5.71. Consider now the category

$$\mathscr{A}$$

whose objects are all local noetherian domains $(A, \mathfrak{n})$, complete for their $\mathfrak{n}$-adic topology, with $\mathrm{Frac}\, A$ of zero characteristic, residue field k of characteristic $p > 0$ and with $[k : k^p] < +\infty$. The morphisms $(A, \mathfrak{n}) \to (A', \mathfrak{n}')$ of $\mathscr{A}$ are all the *injective* ring homomorphisms $A \to A'$.

For every object $(A, \mathfrak{n})$ of $\mathscr{A}$, let $\Omega(A)$ be the set of all pairs (Σ, Σ') of finite non-empty subsets of $A \setminus \{0\}$ with $\Sigma' \subset \Sigma$ and $p \in \Sigma$, such that the corresponding category $\mathscr{C}^A_{\Sigma,\Sigma'}$ admits an initial object (notation of (17.5.57)), and for every $(\Sigma, \Sigma') \in \Omega(A)$, fix such an initial object

$$(f_{\Sigma,\Sigma'} : A \to D_{\Sigma,\Sigma'}, \Delta_{\Sigma,\Sigma'}).$$

We let $\leq$ be the partial ordering on $\Omega(A)$ such that $(\Sigma_1, \Sigma'_1) \leq (\Sigma_2, \Sigma'_2)$ if and only if $\Sigma_1 \subset \Sigma_2$ and $\Sigma'_1 \subset \Sigma'_2$. Clearly, $(\Omega(A), \leq)$ is a filtered partially ordered set, and with the notation of remark 17.5.58, we have a unique morphism

$$(f_{\Sigma_1,\Sigma'_1},\Delta_{\Sigma_1,\Sigma'_1}) \to (f_{\Sigma_2,\Sigma'_2},\Delta_{\Sigma_2,\Sigma'_2})_{|\Sigma_1,\Sigma'_1} \qquad \text{in } \mathscr{C}^A_{\Sigma_1,\Sigma'_2}, \text{ whenever } (\Sigma_1,\Sigma'_1) \leq (\Sigma_2,\Sigma'_2).$$

In other words, we get a well defined filtered system of A-algebras $(D_{\Sigma,\Sigma'} \mid (\Sigma,\Sigma') \in \Omega(A))$.

For every $(\Sigma,\Sigma') \in \Omega(A)$, let $(D_{\Sigma,\Sigma'},\mathfrak{m}_{\Sigma,\Sigma'})$ be the basic setup attached to $(f_{\Sigma,\Sigma'},\Delta_{\Sigma,\Sigma'})$, and consider the subset $\mathsf{S}_{\Sigma,\Sigma'} \subset D^{\mathbb{N}}_{\Sigma,\Sigma'}$ of all $(a_n \mid n \in \mathbb{N})$ such that $a_n \in D_{\Sigma,\Sigma'}a_{n+1}$ for every $n \in \mathbb{N}$, and $\mathfrak{m}_{\Sigma,\Sigma'} \subset \bigcup_{n\in\mathbb{N}} D_{\Sigma,\Sigma'}a_n$. We let $\Theta(A) \subset \Omega(A)$ be the subset of all (Σ,Σ') such that

$$E_{\Sigma,\Sigma'} := \mathsf{S}^{-1}_{\Sigma,\Sigma'} D^{\diamond}_{\Sigma,\Sigma'}$$

is a big CM A-algebra (see (17.5.2)).

Lemma 17.5.72. *$\Theta(A)$ is a cofinal subset of $\Omega(A)$, for every $(A,\mathfrak{n}) \in \mathrm{Ob}(\mathscr{A})$.*

Proof. Choose a subring $(A_0,\mathfrak{n}_0) \subset (A,\mathfrak{n})$ as in (17.5.51); then propositions 17.5.59 and 17.5.62 show that there exists $(\Sigma_0,\Sigma'_0) \in \Omega(A)$ such that $(D_{\Sigma,\Sigma'},\mathfrak{m}_{\Sigma,\Sigma'})^a$ is faithfully flat over A_0, for every $(\Sigma,\Sigma') \geq (\Sigma_0,\Sigma'_0)$. Then, arguing as in the proof of theorem 17.5.54 we see that $E_{\Sigma,\Sigma'}$ is a big CM A-algebra for every such (Σ,Σ'). □

Notice that whenever $(\Sigma_1,\Sigma'_1) \leq (\Sigma_2,\Sigma'_2)$, the induced map $D^{\mathbb{N}}_{\Sigma_1,\Sigma'_1} \to D^{\mathbb{N}}_{\Sigma_2,\Sigma'_2}$ sends $\mathsf{S}_{\Sigma_1,\Sigma_2}$ into $\mathsf{S}_{\Sigma_2,\Sigma'_2}$. Let then $E(A)$ be the colimit of the induced filtered system $(E_{\Sigma,\Sigma'} \mid (\Sigma,\Sigma') \in \Omega(A))$. By lemma 17.5.72 and remark 17.5.19(iv), $E(A)$ is a big CM A-algebra. Next, let $g : (A,\mathfrak{n}) \to (A',\mathfrak{n}')$ be any morphism of $\mathscr{A}$, and $(\Sigma_1,\Sigma'_1) \in \Omega(A)$, $(\Sigma_2,\Sigma'_2) \in \Omega(A')$ with

$$g(\Sigma_1) \subset \Sigma_2 \qquad \text{and} \qquad g(\Sigma'_1) \subset \Sigma'_2.$$

For every object $(f,\Delta := (\beta_{a,\bullet} \mid a \in \Sigma_2))$ of $\mathscr{C}^{A'}_{\Sigma_2,\Sigma'_2}$, set $\Delta_{|\Sigma_1} := (\beta_{g(a),\bullet} \mid a \in \Sigma_1)$; since g is injective, the pair $(f \circ g,\Delta_{|\Sigma_1})$ fulfills conditions (a),(c) and (d) of (17.5.57), defining the objects of $\mathscr{C}^A_{\Sigma_1,\Sigma'_1}$, and with remark 14.2.7(ii), we easily check that also condition (b) holds for this pair. We get therefore a well defined functor :

$$\mathscr{C}^{A'}_{\Sigma_2,\Sigma'_2} \to \mathscr{C}^A_{\Sigma_1,\Sigma'_1} \qquad (f,\Delta) \mapsto (f,\Delta)_{|\Sigma_1,\Sigma'_1} := (f \circ g,\Delta_{|\Sigma_1}).$$

Especially, notice that for every $(\Sigma,\Sigma') \in \mathrm{Ob}(A)$ we have $(g\Sigma,g\Sigma') \in \mathrm{Ob}(A')$, whence a natural morphism in $\mathscr{C}^A_{\Sigma,\Sigma'}$:

$$(f_{\Sigma,\Sigma'},\Delta_{\Sigma,\Sigma'}) \to (f_{g\Sigma,g\Sigma'},\Delta_{g\Sigma,g\Sigma'})_{|\Sigma,\Sigma'}.$$

Moreover, the induced map $D^{\mathbb{N}}_{\Sigma,\Sigma'} \to D^{\mathbb{N}}_{g\Sigma,g\Sigma'}$ sends $\mathsf{S}_{\Sigma,\Sigma'}$ into $\mathsf{S}_{g\Sigma,g\Sigma'}$, so we get a map of A-algebras :

$$E_{\Sigma,\Sigma'} \to E_{g\Sigma,g\Sigma'} \qquad \text{for every } (\Sigma,\Sigma') \in \Omega(A).$$

Finally, the colimit of this system of maps yields a well defined map $E(g) : E(A) \to E(A')$, and a simple inspection shows that the rules : $(A,\mathfrak{n}) \mapsto E(A)$ and $g \mapsto E(g)$ define a functor

$$E : \mathscr{A} \to \mathbb{Z}\text{-}\mathsf{Alg}.$$

17.5.73. Using the categories $\mathscr{D}^A_{\Sigma,\Sigma'}$ of (17.5.65) we may obtain the following variant of the functor E. Consider the category

$$\mathscr{A}'$$

whose objects are all local noetherian domains $(A,\mathfrak{n})$, complete for their $\mathfrak{n}$-adic topology, with $\mathrm{Frac}\,A$ of zero characteristic, and residue field k of characteristic $p > 0$. The morphisms $(A,\mathfrak{n}) \to (A',\mathfrak{n}')$ of $\mathscr{A}'$ are all the *injective and local* ring homomorphisms $A \to A'$.

For every object $(A,\mathfrak{n})$ of $\mathscr{A}$, let $\Omega'(A)$ be the set of all pairs (Σ,Σ') of non-empty subsets of $A \setminus \{0\}$ where Σ' is finite, and $\Sigma' \cup \{p\} \subset \Sigma$, and such that the corresponding category $\mathscr{D}^A_{\Sigma,\Sigma'}$ admits an initial object. For every $(\Sigma,\Sigma') \in \Omega(A)$, fix such an initial object

$$(f_{\Sigma,\Sigma'} : A \to D_{\Sigma,\Sigma'},\Delta_{\Sigma,\Sigma'}).$$

We define a partial ordering $\leq$ on $\Omega'(A)$ just as for $\Omega(A)$ in (17.5.71). Clearly, $(\Omega'(A), \leq)$ is a filtered partially ordered set, and with the notation of (17.5.65), we have a unique morphism

$$(f_{\Sigma_1,\Sigma'_1}, \Delta_{\Sigma_1,\Sigma'_1}) \to (f_{\Sigma_2,\Sigma'_2}, \Delta_{\Sigma_2,\Sigma'_2})_{|\Sigma_1,\Sigma'_1} \qquad \text{in } \mathscr{D}^A_{\Sigma_1,\Sigma'_2}, \text{ whenever } (\Sigma_1, \Sigma'_1) \leq (\Sigma_2, \Sigma'_2).$$

In other words, we get a well defined filtered system of A-algebras $(D_{\Sigma,\Sigma'} \mid (\Sigma, \Sigma') \in \Omega'(A))$.

For every $(\Sigma, \Sigma') \in \Omega'(A)$, let $(D_{\Sigma,\Sigma'}, \mathfrak{m}_{\Sigma,\Sigma'})$ be the basic setup attached to $(f_{\Sigma,\Sigma'}, \Delta_{\Sigma,\Sigma'})$, and define the multiplicative subset $\mathsf{S}_{\Sigma,\Sigma'} \subset D^{\mathbb{N}}_{\Sigma,\Sigma'}$ as in (17.5.71). We let $\Theta'(A) \subset \Omega'(A)$ be the subset of all (Σ, Σ') such that $E'_{\Sigma,\Sigma'} := \mathsf{S}^{-1}_{\Sigma,\Sigma'} D^{\diamond}_{\Sigma,\Sigma'}$ is a big CM A-algebra. Using propositions 17.5.66 and 17.5.70, and arguing as in the proof of lemma 17.5.72, we see that $\Theta'(A)$ is a cofinal subset of $\Omega'(A)$, for every $(A, \mathfrak{n}) \in \mathrm{Ob}(\mathscr{A}')$. Then, the colimit $E'(A)$ of the induced filtered system $(E'_{\Sigma,\Sigma'} \mid (\Sigma, \Sigma') \in \Omega'(A))$ is a big CM A-algebra (remark 17.5.19(iv)). Next, let $g : (A, \mathfrak{n}) \to (A', \mathfrak{n}')$ be any morphism of $\mathscr{A}'$, and $(\Sigma_1, \Sigma'_1) \in \Omega'(A)$, $(\Sigma_2, \Sigma'_2) \in \Omega'(A')$ with

$$g(\Sigma_1) \subset \Sigma_2 \qquad \text{and} \qquad g(\Sigma'_1) \subset \Sigma'_2.$$

For every object $(f, \Delta := (\beta_{a,\bullet} \mid a \in \Sigma_2))$ of $\mathscr{D}^{A'}_{\Sigma_2,\Sigma'_2}$, set again $\Delta_{|\Sigma_1} := (\beta_{g(a),\bullet} \mid a \in \Sigma_1)$; since g is injective, arguing as in the foregoing we see that the pair $(f \circ g, \Delta_{|\Sigma_1})$ fulfills conditions (a),(b), (c),(d) of (17.5.57). Moreover, since g is local, lemma 8.3.12 implies that also condition (e) of (17.5.65) is fulfilled. We get therefore a well defined functor :

$$\mathscr{D}^{A'}_{\Sigma_2,\Sigma'_2} \to \mathscr{D}^{A}_{\Sigma_1,\Sigma'_1} \qquad (f, \Delta) \mapsto (f, \Delta)_{|\Sigma_1,\Sigma'_1} := (f \circ g, \Delta_{|\Sigma_1}).$$

From this, arguing as in the foregoing case, we obtain a natural map of A-algebras $E'(g) : E'(A) \to E'(A')$, whence a well defined functor :

$$E' : \mathscr{A}' \to \mathbb{Z}\text{-}\mathsf{Alg}.$$

17.5.74. Next, we wish to establish a *weak funtoriality* property of big CM algebras, for not necessarily injective maps of local rings. The first step is the following :

Proposition 17.5.75. *Let A be noetherian ring, $I \subset A$ an ideal, $f_1, \dots, f_r \in I$ a finite sequence such that the following holds for every $\mathfrak{p} \in \Sigma := \mathrm{Ass}_A(A/I)$ (see definition* 10.5.1(iii)*) :*

(a) *$I_\mathfrak{p}$ is generated by the image of the sequence $f_\bullet$.*
(b) *The image of the sequence $f_\bullet$ is regular in $I_\mathfrak{p}$.*

Then we have :

(i) *There exists $g \in A \setminus \bigcup_{\mathfrak{p}\in\Sigma} \mathfrak{p}$ such that $gI \subset J := \sum_{i=1}^r f_i A$.*
(ii) *For every g as in* (i)*, the sequence $f'_\bullet := (f_1/g, \dots, f_r/g)$ is quasi-regular in the subring $A' := A[f_1/g, \dots, f_r/g]$ of $A[1/g]$.*
(iii) *With the notation of* (ii)*, let $J' \subset A'$ be the ideal generated by $f'_\bullet$. The natural map $A \to A'$ induces an isomorphism $A/I \xrightarrow{\sim} A'/J'$.*

Proof. (i): Let $L := \{x \in A \mid xI \subset J\}$; since I is finitely generated, it is easily seen that $L_\mathfrak{p} = \{x \in A_\mathfrak{p} \mid xI_\mathfrak{p} \subset J_\mathfrak{p}\}$ for every $\mathfrak{p} \in \mathrm{Spec}\, A$. Especially, $L_\mathfrak{p} = A_\mathfrak{p}$ for every $\mathfrak{p} \in \Sigma$, by assumption (a), *i.e.* $L \not\subset \mathfrak{p}$ for every such $\mathfrak{p}$; then the assertion follows from [12, Prop.1.11(i)].

(iii): Clearly the natural map $A \to A'$ induces a surjection $\varphi : A \to A'/J'$, and we need to check that $\mathrm{Ker}\,\varphi = I$. However, if $x \in I$, then $gx = \sum_{i=1} a_i f_i$ for some $a_1, \dots, a_r \in A$, whence $x = \sum_{i=1}^r a_i f_i/g \in J'$ in A', so $I \subset \mathrm{Ker}\,\varphi$. Let then $\overline{\varphi} : A/I \to A'/J'$, be the map induced by φ; notice that $A[1/g] = A'[1/g]$, and $IA[1/g] = J'A'[1/g]$, so that the localization $\overline{\varphi}_g$ of $\overline{\varphi}$ is an isomorphism. Especially, $\overline{\varphi}_\mathfrak{p}$ is an isomorphism for every $\mathfrak{p} \in \Sigma$, and it suffices then to invoke the following :

Claim 17.5.76. Let M and N be two A-modules, and $h : M \to N$ an A-linear map. Then h is injective if and only if $h_\mathfrak{p} : M_\mathfrak{p} \to N_\mathfrak{p}$ is injective for every $\mathfrak{p} \in \mathrm{Ass}_A(M)$.

Proof of the claim. To ease notation, set $\Omega := \mathrm{Ass}_A(M)$, and consider the commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\quad h \quad} & N \\ \downarrow & & \downarrow \\ \prod_{\mathfrak{p}\in\Omega} M_{\mathfrak{p}} & \xrightarrow{\prod_{\mathfrak{p}\in\Omega} h_{\mathfrak{p}}} & \prod_{i\in\Omega} N_{\mathfrak{p}} \end{array}$$

whose left vertical arrow is injective, by lemma 10.5.3(iii), and whose bottom horizontal arrow is injective if and only if $h_{\mathfrak{p}}$ is injective for every $\mathfrak{p} \in \Omega$. The assertion is an immediate consequence. $\Diamond$

(ii): Set $A_r := A/I[T_1, \dots, T_r]$, let $I_r \subset A_r$ be the ideal generated by $T_1, \dots, T_r$, and denote by $\mathrm{gr}_\bullet A_r$ the graded A/I-algebra associated with the I_r-adic filtration of A_r. Also, for every $i = 1, \dots, r$, let $\overline{T}_i \in \mathrm{gr}_1 A_r$ (resp. $x_i \in J'/J'^2$) be the image of T_i (resp. of f_i/g); we consider the map of graded A/I-algebras

$$\psi_\bullet : \mathrm{gr}_\bullet A_r \to \bigoplus_{n\in\mathbb{N}} J'^n/J'^{n+1} \qquad \overline{T}_1 \mapsto x_1, \dots, \overline{T}_r \mapsto x_r.$$

In view of (iii), we need to check that ψ is an isomorphism, and since ψ is surjective, it suffices to show that its restriction $\psi_n : \mathrm{gr}_n A_r \to J'^n/J'^{n+1}$ is injective for every $n \in \mathbb{N}$. However, by assumption (b), the image of the sequence $f_\bullet$ is regular in $I_{\mathfrak{p}} = J'_{\mathfrak{p}}$, for every $\mathfrak{p} \in \Sigma$, hence $\psi_{n,\mathfrak{p}}$ is an isomorphism for every $n \in \mathbb{N}$ and every $\mathfrak{p} \in \Sigma$ (proposition 7.8.15). Then the assertion follows from claim 17.5.76. $\square$

Remark 17.5.77. Let A be a noetherian ring, and $\mathfrak{p} \subset A$ a prime ideal such that $A_{\mathfrak{p}}$ is a regular local ring. Then $\mathrm{Ass}_A(A/\mathfrak{p}) = \{\mathfrak{p}\}$, and we may find $g_1, \dots, g_r \in \mathfrak{p}$ whose images in $A_{\mathfrak{p}}$ are a regular system of parameters (just take any regular system of parameters for $A_{\mathfrak{p}}$, and multiply by a suitable element of $A\backslash\mathfrak{p}$ to clear denominators : details left to the reader). Hence, assumptions (a) and (b) of proposition 17.5.75 hold for $I := \mathfrak{p}$ and $f_\bullet := g_\bullet$.

17.5.78. Consider now a sequence of local ring homomorphisms

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} A_n$$

of some finite length n, where $A_0, \dots, A_n$ are complete, local noetherian domains.

Proposition 17.5.79. *In the situation of* (17.5.78)*, there exists a commutative diagram of complete local noetherian domains, and local ring homomorphisms :*

$$\begin{array}{ccccccc} A_0 & \xrightarrow{f_0} & A_1 & \xrightarrow{f_1} & \cdots & \xrightarrow{f_{n-1}} & A_n \\ & \searrow^{h_0} & \downarrow{\scriptstyle j_1} & & & & \downarrow{\scriptstyle j_n} \\ & & B_1 & \xrightarrow{g_1} & \cdots & \xrightarrow{g_{n-1}} & B_n \\ & & \uparrow{\scriptstyle \pi_1} & & & & \uparrow{\scriptstyle \pi_n} \\ & & C_1 & \xrightarrow{h_1} & \cdots & \xrightarrow{h_{n-1}} & C_n \end{array} \qquad (17.5.80)$$

such that $h_0, \dots, h_{n-1}$ are injective maps, and the following holds for every $i = 1, \dots, n$:

(i) *the map j_i is finite and injective*
(ii) *the map π_i is surjective, and* $\mathrm{Ker}\, \pi_i$ *is generated by a regular sequence of C_i.*

Proof. Let us show first how to reduce to the case where $n = 1$. Indeed, suppose that $n > 1$, and B_1 and C_1 are given as required, with local ring homomorphisms $h_0 : A_0 \to C_0$, $j_1 : A_1 \to B_1$ and $\pi_1 : C_1 \to B_1$ such that h_0 is injective, and conditions (i) and (ii) hold for $i = 1$. By induction on n, we further suppose that the proposition is already known for every sequence $A_0 \to \cdots \to A_{n-1}$ of local ring homomorphisms of complete local noetherian domains.

• We construct by induction on $i = 1, \dots, n$, a commutative diagram :

$$\begin{array}{ccc} A_1 & \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} & A_n \\ \downarrow{\scriptstyle j'_1} & & \downarrow{\scriptstyle j'_n} \\ A'_1 & \xrightarrow{f'_1} \cdots \xrightarrow{f'_{n-1}} & A'_n \end{array} \qquad (17.5.81)$$

where $A'_1, \dots, A'_n$ are complete local noetherian domains, $j'_1, \dots, j'_n$ are local, finite and injective ring homomorphisms, and $f'_1, \dots, f'_{n-1}$ are local ring homomorphisms. For $i = 1$, we take $A'_1 := B_1$ and $j'_1 := j_1$; suppose next that $i > 1$, and A'_j has already been given for every $j < i$, together with the required commutative diagram of maps $j'_1, \dots, j'_{i-1}$ and $f'_1, \dots, f'_{i-2}$. Since j'_{i-1} is finite and injective, the induced map $\mathrm{Spec}\, A'_{i-1} \to \mathrm{Spec}\, A_{i-1}$ is surjective, so the same holds for its base change $\mathrm{Spec}\, (A_i \otimes_{A_{i-1}} A'_{i-1}) \to \mathrm{Spec}\, A_i$, and hence we may pick $\mathfrak{p} \in \mathrm{Spec}(A_i \otimes_{A_{i-1}} A'_{i-1})$ whose image in $\mathrm{Spec}\, A_i$ is the generic point; we set $A'_i := (A_i \otimes_{A_{i-1}} A'_{i-1})/\mathfrak{p}$, and let $j_i : A_i \to A'_i$ be the induced map. By construction, $\mathrm{Spec}(j_i)$ is dominant with closed image, hence it is surjective, so $\mathrm{Ker}\, j_i = 0$, since A_i is a domain. Moreover, since A_i is complete and separated, j_i is local, and A'_i is a finite A_i-algebra, so it is a complete local noetherian domain, as required. Lastly, let k_{i-1}, k_i and k'_{i-1} be the residue fields of A_{i-1}, A_i and A'_{i-1}; to see that the natural map $f'_{i-1} : A'_{i-1} \to A'_i$ is local, it suffices to remark that the residue field of A'_i is a quotient of $k_i \otimes_{k_{i-1}} k'_{i-1}$.

• Next, we apply the inductive assumption to the sequence of local ring homomorphisms

$$C_1 \xrightarrow{f'_1 \circ \pi_1} A'_2 \xrightarrow{f'_2} \cdots \xrightarrow{f'_{n-1}} A'_n. \qquad (17.5.82)$$

Thus, we find a commutative diagram of complete local noetherian domains and local ring homomorphisms :

$$\begin{array}{ccccc} C_1 & \xrightarrow{f'_1 \circ \pi_1} & A'_2 & \xrightarrow{f'_2} \cdots \xrightarrow{f'_{n-1}} & A'_n \\ & \searrow^{h_1} & \downarrow{\scriptstyle j''_2} & & \downarrow{\scriptstyle j''_n} \\ & & B_2 & \xrightarrow{g_2} \cdots \xrightarrow{g_{n-1}} & B_n \\ & & \uparrow{\scriptstyle \pi_2} & & \uparrow{\scriptstyle \pi_n} \\ & & C_2 & \xrightarrow{h_2} \cdots \xrightarrow{h_{n-1}} & C_n \end{array}$$

where $h_1, \dots, h_{n-1}$ are injective, and the map j''_i (resp. π_i) fulfills the corresponding condition (i) (resp. (ii)) of the proposition, for every $i = 2, \dots, n$. Then we obtain the sought diagram with $g_1 := j''_2 \circ f'_1$ and $j_i := j''_i \circ j'_i$ for every $i = 2, \dots, n$.

• We consider now the case $n = 1$, so we are given a single local ring homomorphism $f_0 : A_0 \to A_1$ of complete local noetherian domains. We notice :

Claim 17.5.83. There exist a complete local noetherian domain D_0, and local ring homomorphisms $\varphi_0 : A_0 \to D_0$, $\psi_0 : D_0 \to A_1$ such that φ_0 is injective, ψ_0 is surjective, and $\psi_0 \circ \varphi_0 = f_0$.

Proof of the claim. By proposition 9.7.31(i) there exists a complete local noetherian ring D'_0, a faithfully flat ring homomorphism $\varphi'_0 : A_0 \to D'_0$, and a surjective map $\psi'_0 : D'_0 \to A_1$ with $\psi'_0 \circ \varphi'_0 = f_0$. Then, let $\mathfrak{p}$ be any minimal prime ideal of D'_0 contained in $\mathrm{Ker}\, \psi'_0$, and set $D_0 := D'_0/\mathfrak{p}$; since φ'_0 is flat, we have $\varphi'^{-1}_0 \mathfrak{p} = 0$, so the induced map $\varphi_0 : A_0 \to D_0$ is still injective, and ψ'_0 factors through a surjection $\psi_0 : D_0 \to A_1$, whence the claim. ◊

Let $\varphi_0 : A_0 \to D_0$ and $\psi_0 : D_0 \to A_1$ be as in claim 17.5.83, and set $\mathfrak{p} := \mathrm{Ker}\, \psi_0$; lemma 9.7.4(v) says that D_0 is excellent, so $D_{0,\mathfrak{p}}$ is quasi-excellent, by lemma 9.7.4(ii,iv).

Especially, $D_{0,\mathfrak{p}}$ is a Nagata ring, and the regular locus of $\mathrm{Spec}\, D_{0,\mathfrak{p}}$ is an open subset ([66, Ch.IV, Th.6.12.4]); by corollary 9.6.56, we may then find a regular local domain $(D_1, \mathfrak{m}_1)$, and an injective local ring homomorphism $D_{0,\mathfrak{p}} \to D_1$ of essentially finite type, inducing an isomorphism $\mathrm{Frac}(D_0) \xrightarrow{\sim} \mathrm{Frac}(D_1)$ and a finite field extension $\mathrm{Frac}(D_0/\mathfrak{p}) \to D_1/\mathfrak{m}_1$. Pick $y_1, \dots, y_n \in D_1$ and a prime ideal $\mathfrak{q} \subset D_2 := D_0[y_1, \dots, y_n]$ such that $D_1 = D_{2,\mathfrak{q}}$; we remark :

Claim 17.5.84. We can assume that the induced injective map $A_1 \xrightarrow{\sim} D_0/\mathfrak{p} \to D_2/\mathfrak{q}$ is finite.

Proof of the claim. The image $\overline{y}_i \in D_1/\mathfrak{m}_1 = \mathrm{Frac}(D_2/\mathfrak{q})$ of y_i is algebraic over $\mathrm{Frac}(D_0/\mathfrak{p})$ for every $i = 1, \dots, n$, so we may find $x \in D_0 \setminus \mathfrak{p}$ such that the images of $xy_1, \dots, xy_n$ in $\mathrm{Frac}(D_2/\mathfrak{q})$ are integral over $D_0/\mathfrak{p}$. Set $D'_2 := D_0[xy_1, \dots, xy_n]$ and $\mathfrak{q}' := \mathfrak{q} \cap D'_2$; then clearly $D'_{2,\mathfrak{q}'} = D_1$, so we may replace y_i by xy_i for $i = 1, \dots, n$, and assume that $\overline{y}_i$ is integral over $D_0/\mathfrak{p}$, for every $i = 1, \dots, n$, whence the claim. ◊

• Now we apply proposition 17.5.75 and remark 17.5.77 to $A := D_2$, $I := \mathfrak{q}$, and to a finite sequence $x_1, \dots, x_d$ of elements of $\mathfrak{q}$ whose images in the regular local ring D_1 are a regular system of parameters : we obtain $g \in D_2 \setminus \mathfrak{q}$ with $g \cdot \mathfrak{q} \subset \sum_{i=1}^d x_i D_2$, and such that the sequence $t_1 := x_1/g, \dots, t_d := x_d/g$ is quasi-regular in the subring $D'_2 := D_2[t_1, \dots, t_d] \subset D_2[1/g]$; moreover, if $\mathfrak{q}' \subset D'_2$ denotes the ideal generated by $t_1, \dots, t_d$, the inclusion map $D_2 \to D'_2$ induces an isomorphism

$$D_2/\mathfrak{q} \xrightarrow{\sim} B_1 := D'_2/\mathfrak{q}'.$$

Since A_1 is complete, claim 17.5.84 implies that $D_2/\mathfrak{q}$ is a complete local noetherian domain, so the same holds for B_1, and the resulting map $g_1 : A_1 \to B_1$ is injective and finite; then, let $\mathfrak{m} \subset B_1$ be the maximal ideal, $\mathfrak{m}' \subset D'_2$ its preimage, and C_1 the $\mathfrak{m}'$-adic completion of D'_2. Since C_1 is a flat D'_2-algebras, the image $\mathbf{t}$ in C_1 of the sequence $t_1, \dots, t_d$ is still quasi-regular, hence it is regular, by proposition 7.8.15 and [126, Th.8.10]. Also, the localization $D'_2 \to D'_{2,\mathfrak{m}'}$ is injective, since D_2 is a domain, and the same holds for the completion map $D'_{2,\mathfrak{m}'} \to C_1$, since the latter is faithfully flat; summing up, we deduce that the structure map $h_0 : A_0 \to C_1$ of the resulting A_0-algebra C_1 is injective. To conclude the proof, it now suffices to remark :

Claim 17.5.85. C_1 is a domain, and h_0 is a local ring homomorphism.

Proof of the claim. In order to show that h_0 is local, it suffices to check that the same holds for the natural map $A_0 \to D'_{2,\mathfrak{m}'}$. The latter comes down to verifying that the preimage of $\mathfrak{m}'$ under the natural map $A_0 \to D'_2$ is the maximal ideal of A_0. But in turns, this is the same as the preimage of $\mathfrak{m}$ under the composition $A_0 \to A_1 \to D_2/\mathfrak{q} \xrightarrow{\sim} B_1$, which is local, whence the assertion. Next, let $\mathrm{gr}_\bullet C_1$ be the graded ring associated with the $\mathfrak{q}'$-adic filtration on C_1; hence $\mathrm{gr}_0 C_1$ is the $\mathfrak{m}'$-adic completion of B_1. But B_1 is $\mathfrak{m}'$-adically complete and separated, so $\mathrm{gr}_0 C_1 = B_1$ and since $\mathbf{t}$ is quasi-regular, the natural map of graded B_1-algebras

$$B_1[T_1, \dots, T_d] \to \mathrm{gr}_\bullet C_1 \qquad T_1 \mapsto t_1, \dots, T_d \mapsto t_d$$

is an isomorphism. Since the $\mathfrak{q}'$-adic filtration is separated on C_1, the assertion follows. □

17.5.86. We will also need the following variant of the situation contemplated in (17.5.78) : we consider a sequence, of some finite length n, of ring homomorphisms

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} A_n$$

where $A_0, \dots, A_n$ are quasi-excellent domains, and $f_0, \dots, f_{n-1}$ are of finite type.

Proposition 17.5.87. *In the situation of* (17.5.86)*, there exists a commutative diagram* (17.5.80) *of quasi-excellent domains, such that $h_0, \dots, h_{n-1}$ are injective maps of finite type, and for every $i = 1, \dots, n$ the following holds :*

(i) *the map j_i is finite and injective*

(ii) *the map* π_i *is surjective, and* $\operatorname{Ker}\pi_i$ *is generated by a quasi-regular sequence of* C_i.

Proof. We argue as in the proof of proposition 17.5.79 : first, we show how to reduce to the case where $n = 1$. Hence, suppose that $n > 1$, and B_1 and C_1 are given as required, with local ring homomorphisms $h_0 : A_0 \to C_0$, $j_1 : A_1 \to B_1$ and $\pi_1 : C_1 \to B_1$ such that h_0 is injective, and conditions (i) and (ii) of proposition 17.5.79 hold for $i = 1$. By induction on n, we further suppose that the proposition is already known for every sequence $A_0 \to \cdots \to A_{n-1}$ consisting of maps of finite type, and quasi-excellent domains $A_0, \dots, A_{n-1}$.

• We construct by induction on $i = 1, \dots, n$, quasi-excellent domains $A'_1, \dots, A'_n$, and a commutative diagram (17.5.81), where $j'_1, \dots, j'_n$ are injective and finite ring homomorphisms, and $f'_1, \dots, f'_{n-1}$ are of finite type. For $i = 1$, we take $A'_1 := B_1$ and $j'_1 := j_1$; suppose next that $i > 1$, and A'_j has already been given for every $j < i$, together with the required commutative diagram of maps $j'_1, \dots, j'_{i-1}$ and $f'_1, \dots, f'_{i-2}$. Just as in the proof of proposition 17.5.79, we may then find $\mathfrak{p} \in \operatorname{Spec}(A_i \otimes_{A_{i-1}} A'_{i-1})$ whose image in $\operatorname{Spec} A_i$ is the generic point; we set $A'_i := (A_i \otimes_{A_{i-1}} A'_{i-1})/\mathfrak{p}$, and let $j_i : A_i \to A'_i$ and $f'_{i-1} : A'_{i-1} \to A'_i$ be the induced maps.

• Next, we apply the inductive assumption to the sequence (17.5.82), to construct the sought commutative diagram, as in the proof of proposition 17.5.79.

• We consider now the case $n = 1$, so we are given a single map of finite type $f_0 : A_0 \to A_1$ of quasi-excellent domains. Then we pick a polynomial A_0-algebra $D_0 := A_0[T_1, \dots, T_k]$ with a surjection of A_0-algebras $\psi_0 : D_0 \to A_1$. Set $\mathfrak{p} := \operatorname{Ker}\psi_0$; then $D_{0,\mathfrak{p}}$ is quasi-excellent, by lemma 9.7.4(ii,iv). Especially, $D_{0,\mathfrak{p}}$ is a Nagata ring (lemma 9.7.4(i)), and the regular locus of $\operatorname{Spec} D_{0,\mathfrak{p}}$ is an open subset ([66, Ch.IV, Th.6.12.4]); by corollary 9.6.56, we may then find a regular local domain $(D_1, \mathfrak{m}_1)$, and an injective local ring homomorphism $D_{0,\mathfrak{p}} \to D_1$ of essentially finite type, inducing an isomorphism $\operatorname{Frac}(D_0) \xrightarrow{\sim} \operatorname{Frac}(D_1)$ and a finite field extension $\operatorname{Frac}(D_0/\mathfrak{p}) \to D_1/\mathfrak{m}_1$. Pick $y_1, \dots, y_n \in D_1$ and a prime ideal $\mathfrak{q} \subset D_2 := D_0[y_1, \dots, y_n]$ such that $D_1 = D_{2,\mathfrak{q}}$. Arguing as in the proof of claim 17.5.84, we can assume that the induced injective map $A_1 \xrightarrow{\sim} D_0/\mathfrak{p} \to D_2/\mathfrak{q}$ is finite.

• Now we apply proposition 17.5.75 and remark 17.5.77 to $A := D_2$, $I := \mathfrak{q}$, and to a finite sequence $x_1, \dots, x_d$ of elements of $\mathfrak{q}$ whose images in the regular local ring D_1 are a regular system of parameters : we obtain $g \in D_2 \setminus \mathfrak{q}$ with $g \cdot \mathfrak{q} \subset \sum_{i=1}^d x_i D_2$, and such that the sequence $t_1 := x_1/g, \dots, t_d := x_d/g$ is quasi-regular in the subring $C_1 := D_2[t_1, \dots, t_d] \subset D_2[1/g]$; moreover, if $\mathfrak{q}' \subset C_1$ denotes the ideal generated by $t_1, \dots, t_d$, the inclusion map $D_2 \to C_1$ induces an isomorphism

$$D_2/\mathfrak{q} \xrightarrow{\sim} B_1 := C_1/\mathfrak{q}'.$$

By construction, the resulting map $g_1 : A_1 \to B_1$ is injective and finite, and the natural map $A_0 \to C_1$ is injective of finite type, as required. □

Corollary 17.5.88. *In the situation of* (17.5.86)*, suppose moreover that* A_0 *is a* $\mathbb{Q}$*-algebra of essentially finite type. Then there exists a commutative diagram of ring homomorphisms :*

$$\begin{array}{ccccccc}
A_0 & \xrightarrow{f_0} & A_1 & \xrightarrow{f_1} & \cdots & \xrightarrow{f_{n-1}} & A_n \\
\downarrow & & \downarrow & & & & \downarrow \\
D_0 & \longrightarrow & D_1 & \longrightarrow & \cdots & \longrightarrow & D_n
\end{array}$$

such that D_i *is a big locally CM* A_i*-algebra for every* $i = 0, \dots, n$.

Proof. Pick a commutative diagram (17.5.80) fulfilling the conditions of proposition 17.5.87. Especially, $h_0, \dots, h_n$ are maximizing, since they are injective maps of domains. Then, with the functor E'_0 of (17.5.48), we get a commutative diagram of ring homomorphisms :

$$\begin{array}{ccccccc} A_0 & \xrightarrow{h_0} & C_1 & \xrightarrow{h_1} & \cdots & \xrightarrow{h_{n-1}} & C_n \\ \downarrow & & \downarrow & & & & \downarrow \\ D'_0 & \longrightarrow & D'_1 & \longrightarrow & \cdots & \longrightarrow & D'_n \end{array} \tag{17.5.89}$$

such that D'_0 is a big locally CM A_0-algebra, and D'_i is a big locally CM C_i-algebra, for $i = 1, \dots, n$. Next, by construction, for every $i = 1, \dots, n$ the kernel $\mathfrak{p}_i$ of π_i is a prime ideal generated by a quasi-regular sequence $\mathbf{f}_i := (f_{i,1}, \dots, f_{i,r_i})$ of C_i; then, the image of the sequence $\mathbf{f}_i$ is regular in $C_{i,\mathfrak{p}_i}$ (proposition 7.8.15), so it can be completed to a system of parameters for $C_{i,\mathfrak{p}_i}$ ([63, Ch.0, Prop.16.4.1]) and hence $0 = \dim B_{i,\mathfrak{p}_i} = \dim C_{i,\mathfrak{p}_i} - r_i$ ([126, Th.14.1(i)]), *i.e.* $\mathfrak{p}_i$ is a prime of height r_i, for every $i = 1, \dots, n$. Then, by corollary 10.4.36, the ring $D_i := B_i \otimes_{C_i} D'_i$ is a big locally CM B_i-algebra, for $i = 1, \dots, n$; set as well $D_0 := D'_0$. It remains to check that D_i is a big CM A_i-algebra, for $i = 1, \dots, n$, after restriction of scalars along the map j_i. Thus, for any such i, let $\mathfrak{q}$ be any prime ideal of A_i; clearly $D_{i,\mathfrak{q}}$ is a locally CM $B_{i,\mathfrak{q}}$-algebra, and we need to show that $\mathrm{depth}_{A_{i,\mathfrak{q}}} D_{i,\mathfrak{q}} = \dim A_{i,\mathfrak{q}}$. Since $B_{i,\mathfrak{q}}$ is a finite $A_{i,\mathfrak{q}}$-algebra, taking into account [126, §9, Lemma 2], lemma 10.4.17(ii), and (10.4.30), we come down to checking that the height of every maximal ideal of $B_{i,\mathfrak{q}}$ equals the height of $\mathfrak{q}$. The latter holds by [64, Ch.IV, Prop.5.6.1]. □

We are now ready to state :

Theorem 17.5.90. *In the situation of* (17.5.78)*, there exists a commutative diagram of ring homomorphisms :*

$$\begin{array}{ccccccc} A_0 & \xrightarrow{f_0} & A_1 & \xrightarrow{f_1} & \cdots & \xrightarrow{f_{n-1}} & A_n \\ \downarrow & & \downarrow & & & & \downarrow \\ D_0 & \longrightarrow & D_1 & \longrightarrow & \cdots & \longrightarrow & D_n \end{array}$$

such that D_i is a big CM A_i-algebra for every $i = 0, \dots, n$.

Proof. Suppose first that the residue field k_i of A_i is of characteristic $p > 0$, for $i = 0, \dots, n$. Then, we pick a commutative diagram as in proposition 17.5.79. Especially, $h_0, \dots, h_{n-1}$ are injective and local maps, so if $\mathrm{Frac}\, A_0$ has characteristic zero, the same holds for $\mathrm{Frac}\, C_i$, for every $i = 1, \dots, n$, in which case the discussion of (17.5.73) yields a commutative diagram (17.5.89) such that D'_0 is a big CM A_0-algebra, and D'_i is a big CM C_i-algebra for every $i = 1, \dots, n$. Let $\mathfrak{m}_0, \mathfrak{n}_1, \dots, \mathfrak{n}_n$ be the maximal ideals of $A_0, C_1, \dots, C_n$, and endow D'_0 (resp. D'_i for $i = 1, \dots, n$) with its $\mathfrak{m}_0$-adic (resp. $\mathfrak{n}_i$-adic) topology; by claim 17.5.21(i), the completion D_0 of D'_0 (resp. $D'^{\wedge}_i$ of D'_i, for $i = 1, \dots, n$) is still a CM D_0-algebra (resp. a CM C_i-algebra). Moreover, since $f_0, \dots, f_{n-1}$ are local maps, the maps $D'_0 \to D'_1 \to \cdots \to D'_n$ are continuous, so they induce a sequence of ring homomorphisms $D_0 \to D'^{\wedge}_1 \to \cdots \to D'^{\wedge}_n$. After setting $D_i := B_i \otimes_{C_i} D'^{\wedge}_i$ for $i = 1, \dots, n$, we get a commutative diagram as sought, such that D_i is a big CM B_i-algebra for every $i = 1, \dots, n$, by claim 17.5.21(ii). Next, by restriction of scalars along the finite injective map j_i, the ring D_i is a big CM A_i-algebra for $i = 1, \dots, n$ (remark 17.5.19(iii)), whence the theorem, in this case.

Next, if A_0 is an $\mathbb{F}_p$-algebra, the same holds for $C_1, \dots, C_n$, and then we may argue likewise, using the functor E_p from remark 17.5.36(ii) (recall that all complete noetherian local rings are excellent : see lemma 9.7.4(v)).

Lastly, we consider the case where $k_0, \dots, k_n$ are of zero characteristic, in which case, notice that $H_1 \mathbb{L}_{k_i/\mathbb{Q}} = 0$ for $i = 0, \dots, n$, by proposition 9.6.1(ii). Then, invoking theorem 9.7.41, we obtain a commutative diagram of rings :

$$\begin{array}{ccccc} A'_0 & \xrightarrow{f'_0} & \cdots & \xrightarrow{f'_{n-1}} & A'_n \\ \downarrow{\scriptstyle i_0} & & & & \downarrow{\scriptstyle i_n} \\ A_0 & \xrightarrow{f_0} & \cdots & \xrightarrow{f_{n-1}} & A_n \end{array}$$

such that $A'_0, \dots, A'_n$ are local $\mathbb{Q}$-algebras of essentially finite type, the maps $f'_0, \dots, f'_{n-1}$ are local, and the maps $i_0, \dots, i_n$ are local and of complete intersection. We show :

Claim 17.5.91. There exists a commutative diagram of rings :

$$\begin{array}{ccccccc} A'_0 & \xrightarrow{f'_0} & A'_1 & \xrightarrow{f'_1} & \cdots & \xrightarrow{f'_{n-1}} & A'_n \\ \downarrow & & \downarrow & & & & \downarrow \\ D'_0 & \longrightarrow & D'_1 & \longrightarrow & \cdots & \longrightarrow & D'_n \end{array}$$

such that D'_i is a big CM A'_i-algebra for every $i = 0, \dots, n$.

Proof of the claim. We may find a further commutative diagram of rings :

$$\begin{array}{ccccc} A''_0 & \xrightarrow{f''_0} & \cdots & \xrightarrow{f''_{n-1}} & A''_n \\ \downarrow{\scriptstyle i'_0} & & & & \downarrow{\scriptstyle i'_n} \\ A'_0 & \xrightarrow{f'_0} & \cdots & \xrightarrow{f'_{n-1}} & A'_n \end{array}$$

such that A''_j is a $\mathbb{Q}$-algebra of finite type for $j = 0, \dots, n$, and i'_j is a localization map, so that $A'_j = (A''_j)_{\mathfrak{q}_j}$ for some prime ideal $\mathfrak{q}_j \subset A''_j$. Then we must have $\mathfrak{q}_j = f''^{-1}_j(\mathfrak{q}_{j+1})$ for every $j = 0, \dots, n-1$, since the maps $f'_0, \dots, f'_n$ are local. Notice that A''_0 is excellent ([64, Ch.IV, Sch.7.8.3(ii)]); then, by corollary 17.5.88 we have a commutative diagram

$$\begin{array}{ccccccc} A''_0 & \xrightarrow{f''_0} & A''_1 & \xrightarrow{f''_1} & \cdots & \xrightarrow{f''_{n-1}} & A''_n \\ \downarrow & & \downarrow & & & & \downarrow \\ D''_0 & \longrightarrow & D''_1 & \longrightarrow & \cdots & \longrightarrow & D''_n \end{array}$$

such that D''_i is a big locally CM A''_i-algebra for every $i = 0, \dots, n$. Therefore, $D'_i := (D''_i)_{\mathfrak{q}_i}$ is a big CM A'_i-algebra for $i = 0, \dots, n$, and the map $D''_i \to D''_{i+1}$ induces a map $D'_i \to D'_{i+1}$ for every $i = 0, \dots, n-1$. ◊

Let then $D'_0, \dots, D'_n$ be as in claim 17.5.91; according to lemma 17.5.20, the ring $D_i := A_i \widehat{\otimes}_{A'_i} D'_i$ is a big CM A_i-algebra for $i = 0, \dots, n$, and since each map f_i is local, we may argue as in the foregoing, to obtain the sought sequence of maps $D_0 \to \cdots \to D_n$. □

Corollary 17.5.92. *Every local noetherian ring* $(A, \mathfrak{m})$ *admits a big CM algebra.*

Proof. If the residue field k of A has characteristic $p > 0$, the assertion just summarizes theorems 17.5.28 and 17.5.54. If k has characteristic zero, we may argue as in the proof of claim 17.5.29, to reduce to the case where A is a complete local domain, and the assertion is then the special case of theorem 17.5.90 where $n = 0$. □

Remark 17.5.93. The existence of big CM algebras over rings of equicharacteristic zero was first established by Hochster and Huneke in [95, Th.8.1]; moreover, a later paper by the same authors proves for such algebras a weak functoriality property which is stronger than our theorem 17.5.90 ([96, Th.3.9]). A similar result was proved by Aschenbrenner and Schoutens in

[11, §7], using a technique that has been further developed in [57]. Theorem 17.5.90 for local rings of residue characteristic $p > 0$ has also been proved by Y.André via a different method which yields weakly functorial perfectoid big CM algebras ([7]); the recent preprint [26] gives another proof of this using p-adic completions of absolute integral closures.

17.5.94. *Further refinements.* We conclude by explaining a variant of our constructions, which has some further useful properties, and especially, it generalizes to mixed characteristic [56, Th.7.8]. Consider a complete local noetherian ring $(A, \mathfrak{n})$ of dimension d, with residue field $k := A/\mathfrak{n}$ of characteristic $p > 0$. Suppose moreover that B is a big CM A-algebra, perfectoid for its $\mathfrak{n}$-adic topology; let $\beta_\bullet := (\beta_n \mid n \in \mathbb{N})$ be any element of $\mathbf{E}(B) \setminus \{0\}$, and denote by $\mathfrak{m}_\beta \subset B$ the ideal generated by the system $(\beta_n \mid n \in \mathbb{N})$. Then $(B, \mathfrak{m}_\beta)$ is a basic setup in the sense of [75, §2.1.1], and we denote as usual by $\mathsf{S}_\beta \subset B^{\mathbb{N}}$ the multiplicative subset associated with $(B, \mathfrak{m}_\beta)$ as in (17.5.2).

Lemma 17.5.95. *In the situation of* (17.5.94)*, endow* $\mathsf{S}_\beta^{-1}B^\diamond$ *with its* $\mathfrak{n}$*-adic topology. Then the maximal separated quotient* $(\mathsf{S}_\beta^{-1}B^\diamond)^{\mathrm{sep}}$ *of* $\mathsf{S}_\beta^{-1}B^\diamond$ *is a perfectoid big CM* A*-algebra.*

Proof. Recall that $C := (\mathsf{S}_\beta^{-1}B^\diamond)^{\mathrm{sep}}$ is the $\mathfrak{n}$-adic completion of $\mathsf{S}_\beta^{-1}B^\diamond$ (proposition 17.5.8). In order to show that C is a big CM A-algebra, it then suffices to check that the same holds for $\mathsf{S}_\beta^{-1}B^\diamond$ (lemma 17.5.20), and since B is a big CM A-algebra, lemma 17.5.23 reduces to verify that $(R\Gamma^d_{\{\mathfrak{n}\}}B)^a \neq 0$, for the almost structure relative to the basic setup $(B, \mathfrak{m}_\beta)$. Arguing as in remark 17.5.19(i), we are then further reduced to checking that $(B/\mathfrak{n}B)^a \neq 0$, *i.e.* that $\mathfrak{m}_\beta B \not\subset \mathfrak{n}B$. If the latter fails, we have $\beta_k \in \mathfrak{n}B$ for every $k \in \mathbb{N}$, whence $\beta_0^k \in \mathfrak{n}^k$ for every such k; since the $\mathfrak{n}$-adic topology is separated on B, we then get $\beta_0 = 0$, so $\beta_\bullet = 0$ in $\mathbf{E}(B)$, a contradiction. Next, since B is a P-ring for its $\mathfrak{n}$-adic topology, it is easily seen that the same holds for $B^{\mathbb{N}}$, and then also for its quotient $B^\diamond$, and for C. Now, by remark 16.3.58 we may find an ideal $I \subset B$ of adic definition, fulfilling the conditions of (16.3.55), and for any given finite sequence $b_1, \dots, b_k$ of generators of I, we let J be the ideal generated by $b_1^p, \dots, b_k^p$; let moreover $\mathrm{gr}^\bullet_I B$ and $\mathrm{gr}^\bullet_J B$ be the graded rings associated with the I-adic and respectively the J-adic filtrations of B, and define likewise $\mathrm{gr}^\bullet_I C$ and $\mathrm{gr}^\bullet_J C$. According to proposition 16.3.59, the p-th power map on B (resp. on C) induces an isomorphism $\Phi_{B,I} : \mathrm{gr}^\bullet_I B \xrightarrow{\sim} \mathrm{gr}^\bullet_J B$ (resp. a ring homomorphism $\Phi_{C,I} : \mathrm{gr}^\bullet_I C \to \mathrm{gr}^\bullet_J C$); on the other hand, by lemma 17.5.1 we get natural identifications :

$$\mathrm{gr}^\bullet_I C \xrightarrow{\sim} \mathrm{gr}^\bullet_I(\mathsf{S}_\beta^{-1}B^\diamond) \xrightarrow{\sim} \mathsf{S}_\beta^{-1}(\mathrm{gr}^\bullet_I B)^\diamond \qquad \mathrm{gr}^\bullet_J C \xrightarrow{\sim} \mathrm{gr}^\bullet_J(\mathsf{S}_\beta^{-1}B^\diamond) \xrightarrow{\sim} \mathsf{S}_\beta^{-1}(\mathrm{gr}^\bullet_J B)^\diamond$$

and $\Phi_{C,I}$ corresponds to $\mathsf{S}_\beta^{-1}\Phi^\diamond_{I,B}$, under these identifications. Hence, $\Phi_{C,I}$ is an isomorphism, so C is perfectoid, by theorem 16.3.64. □

Theorem 17.5.96. *For every complete local noetherian ring* $(A, \mathfrak{n})$ *with residue field* $A/\mathfrak{n}$ *of characteristic* $p > 0$*, and every perfectoid big CM* A*-algebra* B_0 *there exists a ring homomorphism* $B_0 \to B$ *such that the following holds :*

(i) B *is a big CM* A*-algebra, and is perfectoid for its* $\mathfrak{n}$*-adic topology.*

(ii) B *is a local domain, whose maximal ideal is the radical of* $\mathfrak{n}B$*.*

(iii) *Set* $\mathbf{E} := \mathbf{E}(B)$*. Then, for every non-constant monic polynomial* $P \in W(\mathbf{E})[X]$ *and every distinguished ideal* $\mathscr{I} \subset W(\mathbf{E})$ *there exists* $\underline{w} \in W(\mathbf{E})$ *such that* $P(\underline{w}) \in \mathscr{I}$*.*

(iv) *Especially,* B *is normal, and* $\mathrm{Frac}(B)$ *is an algebraically closed field.*

(v) *For every* $\beta_\bullet \in \mathbf{E} \setminus \{0\}$*, define the basic setup* $(B, \mathfrak{m}_\beta)$ *as in* (17.5.94)*. Then the unit of adjunction* $\eta_B : B \to (B, \mathfrak{m}_\beta)^a_*$ *is an isomorphism.*

Proof. By proposition 16.3.8(ii), the $\mathfrak{n}$-adic completion of B_0 is perfectoid for its $\mathfrak{n}$-adic topology, and is still a big CM A-algebra (lemma 17.5.20); hence we may assume from start that B_0 is perfectoid for its $\mathfrak{n}$-adic topology. Set $\mathbf{E}_0 := \mathbf{E}(B_0)$. By proposition 16.9.29, we have an adic, adically faithfully flat, and perfectoid $\mathbf{E}_0$-algebra $\mathbf{E}_1$ such that for every non-constant monic polynomial $P \in W(\mathbf{E}_1)[X]$ and every distinguished ideal $\mathscr{I} \subset W(\mathbf{E}_1)$ there exists $\underline{w} \in W(\mathbf{E}_1)$ with $P(\underline{w}) \in \mathscr{I}$. Set $B_1 := B_0 \otimes_{W(\mathbf{E}_0)} W(\mathbf{E}_1)$; the induced map $B_0 \to B_1$ is then adically faithfully flat (proposition 16.4.18), so $B_1/\mathfrak{n}B_1$ is a faithfully flat $B_0/\mathfrak{n}B_0$-algebra, and especially $B_1/\mathfrak{n}B_1 \neq 0$. Moreover $\mathrm{depth}_A B_1 \geq \mathrm{depth}_A B_0$ (corollary 14.2.50(ii) and proposition 16.4.10(ii)). By remark 17.5.19(i), this shows that B_1 is a big CM A-algebra. Notice that, due to the foregoing property of $\mathbf{E}_1$, the map $\overline{u}_{B_1} : \mathbf{E}_1 \to B_1$ is surjective. Choose any minimal prime ideal $\mathfrak{p} \subset B_1$, and endow $S := \overline{u}_{B_1}^{-1}(B_1 \setminus \mathfrak{p})$ with the preordering such that $\beta_\bullet \leq \beta'_\bullet$ if and only if $\beta'_\bullet \mathbf{E}_1 \subset \beta_\bullet \mathbf{E}_1$, for every $\beta_\bullet, \beta'_\bullet \in S$. Then clearly $(S, \leq)$ is a filtered category (see example 1.1.6(iii)), and for every $\beta_\bullet := (\beta_n \mid n \in \mathbb{N})$, define the basic setup $(B_1, \mathfrak{m}_\beta)$ and the multiplicative subset $\mathsf{S}_\beta \subset B_1^{\mathbb{N}}$ as in (17.5.94), and the perfectoid big CM A-algebra $(\mathsf{S}_\beta^{-1} B_1^\diamond)^{\mathrm{sep}}$ as in lemma 17.5.95. Let B_2 be the colimit of the filtered system of B_1-algebras $((\mathsf{S}_\beta^{-1} B_1^\diamond)^{\mathrm{sep}} \mid \beta_\bullet \in S)$. Then the $\mathfrak{n}$-adic completion $B_2^\wedge$ of B_2 is again a perfectoid big CM A-algebra, by remark 17.5.19(iv), lemma 17.5.20 and corollary 16.3.46.

Claim 17.5.97. (i) The kernel of the induced map $j : B_1 \to B_2$ contains $\mathfrak{p}$.

(ii) For every $\beta \in S$ we have a commutative diagram of B_1-modules :

$$\begin{array}{ccc} B_1 & \xrightarrow{\ j\ } & B_2 \\ {\scriptstyle \eta_{B_1}} \downarrow & \nearrow & \downarrow {\scriptstyle \eta_{B_2}} \\ (B_1, \mathfrak{m}_\beta)^a_* & \xrightarrow{\ j^a_*\ } & (B_2, \mathfrak{m}_\beta)^a_* \end{array}$$

whose vertical arrows are the units of adjunctions.

Proof of the claim. (i): Recall that B_1 is reduced (corollary 16.3.63(i)), hence $B_{1,\mathfrak{p}}$ is a field, and in particular, the kernel of the localization map $B_1 \to B_{1,\mathfrak{p}}$ equals $\mathfrak{p}$. The latter means that for every $b \in \mathfrak{p}$ there exists $\beta_0 \in B_1 \setminus \mathfrak{p}$ such that $b\beta_0 = 0$ in B_1. We have already observed that $\overline{u}_{B_1}$ is surjective; then pick $\beta_\bullet \in S$ such that $\overline{u}_{B_1}(\beta_\bullet) = \beta_0$. We regard $\beta_\bullet := (\beta_n \mid n \in \mathbb{N})$ as an element of $B_1^{\mathbb{N}}$, in which case $\beta_\bullet \in \mathsf{S}_\beta$; but notice that $\beta_\bullet \cdot j(b)$ is the class of $(\beta_n b \mid n \in \mathbb{N})$, and we have $\beta_n b = 0$ for every $n \in \mathbb{N}$, since $(\beta_n b)^{p^n} = \beta_0 g^{p^n} = 0$, and since B_1 is reduced. Thus, $j(b) = 0$, as stated.

(ii): According to lemma 17.5.4(iii), the unit of adjunction $\mathsf{S}_\beta^{-1} B_1^\diamond \to (\mathsf{S}_\beta^{-1} B_1^\diamond)^a_*$ is an isomorphism (relative to the basic setup $(B_1, \mathfrak{m}_\beta)$), hence the structure map $B_1 \to \mathsf{S}_\beta^{-1} B_1^\diamond$ factors through η_{B_1}, and then the same holds for j. ◊

Next, let $\overline{\mathfrak{q}}$ be any minimal prime ideal of $B_2^\wedge/\mathfrak{n}B_2^\wedge$, and $\mathfrak{q} \subset B_2^\wedge$ the preimage of $\overline{\mathfrak{q}}$. Let B_3 be the $\mathfrak{n}$-adic completion of the localization $B_{2,\mathfrak{q}}^\wedge$. Clearly B_3 is a P-ring, for its $\mathfrak{n}$-adic topology. By remark 16.3.58 we may find an ideal $I \subset B_2^\wedge$ of adic definition, fulfilling the conditions of (16.3.55), and for any given finite sequence $b_1, \dots, b_k$ of generators of I, we let J be the ideal generated by $b_1^p, \dots, b_k^p$. Let moreover $\mathrm{gr}_I^\bullet B_2^\wedge$ and $\mathrm{gr}_J^\bullet B_2^\wedge$ be the graded rings associated with the I-adic and respectively the J-adic filtrations of B, and define likewise $\mathrm{gr}_I^\bullet B_3$ and $\mathrm{gr}_J^\bullet B_3$. According to proposition 16.3.59, the p-th power map on B induces an isomorphism $\Phi_I : \mathrm{gr}_I^\bullet B_2^\wedge \xrightarrow{\sim} \mathrm{gr}_J^\bullet B_2^\wedge$ and a ring homomorphism $\Phi_{IB_3} : \mathrm{gr}_I^\bullet B_3 \to \mathrm{gr}_J^\bullet B_3$. Clearly $\mathrm{gr}_I^j B_3 = (\mathrm{gr}_I^j B_2^\wedge)_\mathfrak{q}$ and $\mathrm{gr}_J^j B_3 = (\mathrm{gr}_J^j B_2^\wedge)_\mathfrak{q}$, for every $j \in \mathbb{N}$, and Φ_{IB_3} corresponds to the localization $(\Phi_I)_\mathfrak{q}$ under these identifications. Hence Φ_{IB_3} is an isomorphism, so B_3 is perfectoid, by theorem 16.3.64. Let us check that B_3 is a big CM A-algebra. By lemma 17.5.20, it suffices to show that the same holds for $B_{2,\mathfrak{q}}^\wedge$. But clearly $\mathrm{depth}_A B_{2,\mathfrak{q}}^\wedge \geq \mathrm{depth}_A B_2^\wedge$, and moreover $B_{2,\mathfrak{q}}^\wedge/\mathfrak{n}B_{2,\mathfrak{q}}^\wedge \neq 0$, since $B_2^\wedge/\mathfrak{n}B_2^\wedge \neq 0$, and since the preimage of $\mathfrak{q}$ in A equals $\mathfrak{n}$. Then the assertion follows from remark 17.5.19(i). Summing up, to every big CM A-algebra B_0 that is perfectoid for its $\mathfrak{n}$-adic topology, we have attached three more such algebras; we introduce the notation :

$$\mathscr{B}(B_0)_1 := B_1 \qquad \mathscr{B}(B_0)_2 := B_2^\wedge \qquad \mathscr{B}(B_0) := B_3.$$

Let also $\mathfrak{p}(B_0) \subset \mathscr{B}_0(B_0)_1$ be the minimal ideal $\mathfrak{p}$. Then we define by transfinite induction, for every ordinal α an A-algebra $\mathscr{B}_\alpha(B_0)$, as follows. First, we set $\mathscr{B}_0 := B_0$; next, $\mathscr{B}_{\alpha+1} := \mathscr{B}(\mathscr{B}_\alpha)$ for every successor ordinal $\alpha+1$, and $\mathscr{B}_\alpha$ is the $\mathfrak{n}$-adic completion of $\mathrm{colim}_{\beta<\alpha}\mathscr{B}_\beta$, for every limit ordinal α. Taking into account remark 17.5.19(iv) and lemma 17.5.20, we see easily that $\mathscr{B}_\alpha$ is a perfectoid big CM A-algebra, for every ordinal α. Lastly, let ω_1 be the smallest uncountable ordinal; we claim that $B := \mathscr{B}_{\omega_1}$ will do. Indeed, arguing as in the proof of proposition 16.9.29 we see that $B = \mathrm{colim}_{\beta<\alpha}\mathscr{B}_\beta$, and that (iii) holds. Next, let $x, y \in B$ with $xy = 0$; we find $\beta < \omega_1$ such that x and y are the images of elements x' and y' of $\mathscr{B}_\beta$, and we may assume that $x'y' = 0$. But then the image of either x' or y' in $\mathscr{B}(\mathscr{B}_\beta)_1$ lies in $\mathfrak{p}(\mathscr{B}_\beta)_1$, and then the image of either x' or y' vanishes in $\mathscr{B}(\mathscr{B}_\beta)_2$, by claim 17.5.97(i), so either $x = 0$ or $y = 0$; this shows that B is a domain. Moreover, by construction $\mathscr{B}(\mathscr{B}_\alpha)$ is a local ring, and its maximal ideal is the radical of $\mathfrak{n}\mathscr{B}(\mathscr{B}_\alpha)$, for every ordinal α; assertion (ii) follows easily. Assertion (iv) follows easily from (ii) and (iii) : the details shall be left to the reader.

(v): From corollary 16.3.46 and claim 16.9.30, we deduce that $\mathbf{E}(B)$ is the colimit of the system $(\mathbf{E}(\mathscr{B}_\alpha) \mid \alpha < \omega_1)$. Let now $\beta_\bullet \in \mathbf{E}$, so that $\beta_\bullet$ is the image of an element $\beta'_\bullet \in \mathbf{E}(\mathscr{B}_\alpha)$ for some ordinal $\alpha < \omega_1$, and the ideal $\mathfrak{m}_\beta \subset B$ equals $\mathfrak{m}B$, where $\mathfrak{m} \subset \mathscr{B}_\alpha$ is the ideal generated by $(\beta'_n \mid n \in \mathbb{N})$; let $x \in (B, \mathfrak{m}_\beta)^a_*$. We notice :

Claim 17.5.98. $(B, \mathfrak{m}_\beta)^a_* = \mathrm{colim}_{\alpha\leq\gamma<\omega_1}(\mathscr{B}_\gamma, \mathfrak{m}\mathscr{B}_\gamma)^a_*$.

Proof of the claim. Recall that for any basic setup $(V, \mathfrak{m})$ and any V-module M we have $M^a_* = \mathrm{Hom}_V(\widetilde{\mathfrak{m}}, M)$, with $\widetilde{\mathfrak{m}} := \mathfrak{m} \otimes_V \mathfrak{m}$. Moreover, for every ring homomorphism $V \to W$ we have $\mathfrak{m}W \otimes_W \mathfrak{m}W = W \otimes_V \widetilde{\mathfrak{m}}$ ([75, Rem.2.1.4(ii)]). Thus, it suffices to check that $B^a_* = \mathrm{colim}_{\alpha\leq\gamma<\omega_1}\mathscr{B}^a_{\gamma,*}$ for the almost structure given by the basic setup $(\mathscr{B}_\alpha, \mathfrak{m})$. However, according to [75, Th.2.1.12(ii.a)] we have a $\mathscr{B}_\alpha$-linear presentation

$$\mathscr{B}_\alpha^{(\mathbb{N})} \xrightarrow{f} \mathscr{B}_\alpha^{(\mathbb{N})} \to \widetilde{\mathfrak{m}} \to 0.$$

Let $(e_i \mid i \in \mathbb{N})$ be the canonical basis of the free $\mathscr{B}_\alpha$-module $\mathscr{B}_\alpha^{(\mathbb{N})}$, so that for every $i \in \mathbb{N}$ we have a finite subset $S_i \subset \mathbb{N}$ and a system $(a_{ij} \mid j \in S_i)$ of elements of $\mathscr{B}_\alpha$ such that $f(e_i) = \sum_{j\in S_i} a_{ij}e_j$. With this notation, the datum of a $\mathscr{B}_\alpha$-linear map $\varphi : \widetilde{\mathfrak{m}} \to B$ is the same as that of a system $b_\bullet := (b_i \mid i \in \mathbb{N})$ of elements of B with $\sum_{j\in S_i} a_{ij}b_j = 0$ for every $i \in \mathbb{N}$. Since $B = \mathrm{colim}_{\gamma<\omega_1}\mathscr{B}_\gamma$, for every $i \in \mathbb{N}$ the element b_i is the class $[c_i, \gamma_i]$ of a pair consisting of an ordinal $\gamma_i < \omega_1$ and some $c_i \in \mathscr{B}_{\gamma_i}$; pick any ordinal $\gamma < \omega_1$ such that $\gamma \geq \gamma_i$ for every $i \in \mathbb{N}$, and $\gamma \geq \alpha$. After replacing each c_i by its image in $\mathscr{B}_\gamma$, we may assume that $\gamma_i = \gamma$ for every $i \in \mathbb{N}$. Likewise, for every $i \in \mathbb{N}$ there exists an ordinal $\delta_i < \omega_1$ with $\delta_i \geq \gamma$, and such that the image of $\sum_{j\in S_i} a_{ij}c_j$ vanishes in $\mathscr{B}_{\delta_i}$. We pick again $\delta < \omega_1$ such that $\delta \geq \delta_i$ for every $i \in \mathbb{N}$, and let $d_i \in \mathscr{B}_\delta$ be the image of c_i in $\mathscr{B}_\delta$, for every $i \in \mathbb{N}$; then the system $(d_i \mid i \in \mathbb{N})$ corresponds to a $\mathscr{B}_\alpha$-linear map $\psi : \widetilde{\mathfrak{m}} \to \mathscr{B}^a_{\delta,*}$ whose composition with the natural map $\mathscr{B}^a_{\delta,*} \to B^a_*$ agrees with φ. This shows that the natural map $t : \mathrm{colim}_{\alpha\leq\gamma<\omega_1}\mathscr{B}^a_{\gamma,*} \to B^a_*$ is surjective. Lastly, notice that φ is the zero map if and only if $b_i = 0$ for every $i \in \mathbb{N}$; if $b_i = [c_i, \gamma_i]$ as in the foregoing, we may then find $\delta < \omega_1$ with $\delta \geq \gamma_i$ and $\delta \geq \alpha$, and such that the image of c_i vanishes in $\mathscr{B}_\delta$ for every $i \in \mathbb{N}$. This shows that t is injective, and concludes the proof. ◊

From claim 17.5.98, the almost element x is the image of some $y \in (\mathscr{B}_\gamma, \mathfrak{m}\mathscr{B}_\gamma)^a_*$, for some $\gamma < \omega_1$ with $\gamma \geq \alpha$, in which case the image of y in $(\mathscr{B}_{\gamma+1}, \mathfrak{m}\mathscr{B}_{\gamma+1})^a_*$ is also the image of some $z \in \mathscr{B}(\mathscr{B}_\gamma)_2$, by claim 17.5.97(ii); if $z' \in B$ is the image of z, we see that $\eta_B(z') = x$. This shows that η_B is surjective, and we also know that its kernel is annihilated by $\mathfrak{m}_\beta$; since B is a domain, it then follows that η_B is an isomorphism. □

Remark 17.5.99. (i) With the notation of theorem 17.5.96, let $\beta_\bullet := (\beta_n \mid n \in \mathbb{N}) \in \mathbf{E}(B)$ be any non-zero element; notice that every principal ideal of B is a free B-module, since B is a domain (theorem 17.5.96(iv)), hence $\mathfrak{m}_\beta$ is a flat B-module, since it is the filtered union of principal ideals, and therefore $\mathfrak{m}_\beta = \mathfrak{m}_\beta \otimes_B \mathfrak{m}_\beta$ ([75, Rem.2.14(i)]). It follows that

$$(B, \mathfrak{m}_\beta)^a_* = \mathrm{Hom}_B(\mathfrak{m}_\beta, B) = \bigcap_{n\in\mathbb{N}} \{x \in \mathrm{Frac}(B) \mid \beta_n x \in B\}. \tag{17.5.100}$$

(ii) Recall that an integral domain A is said to be *completely integrally closed* if the following holds for every $x \in K := \mathrm{Frac}(A)$. The subring $A[x] \subset K$ lies in a finitely generated A-submodule of K if and only if $x \in A$. Then, we claim that condition (v) of theorem 17.5.96 is equivalent to the assertion that B is completely integrally closed. Indeed, let $\beta_\bullet$ be as in (i), and $x \in (B, \mathfrak{m}_\beta)^a_*$; in light of (17.5.100), we see that $B[x] \subset \beta_0^{-1} B$. Hence, if B is completely integrally closed, we have $B = (B, \mathfrak{m}_\beta)^a_*$. Conversely, suppose the latter condition holds, and let $x \in \mathrm{Frac}(B)$ be an element such that $B[x]$ lies in a B-module of finite type; this means that $B[x] \subset b^{-1}B$ for some $b \in B \setminus \{0\}$, so that $bx^{p^n} \in B$ for every $n \in \mathbb{N}$. But by virtue of theorem 17.5.96(iii), there exists $\beta_\bullet \in \mathbf{E}(B)$ with $\beta_0 = b$; then, with (17.5.100), we deduce that $x \in (B, \mathfrak{m}_\beta)^a_*$, so $x \in B$, by assumption.

17.6. **Monomial conjectures.** The existence of big CM modules and algebras can be applied to settle some other long-standing conjectures concerning systems of parameters in local noetherian rings. In this section we prove some of these conjectures. The first result is the following:

Theorem 17.6.1 (Monomial conjecture). *Let $(A, \mathfrak{m})$ be a local noetherian ring, and $x_1, \dots, x_d$ a system of parameters for A. Then we have :*

$$x_1^t \cdots x_d^t \notin Ax_1^{t+1} + \cdots + Ax_d^{t+1} \qquad \textit{for every } t \in \mathbb{N}.$$

Proof. We reproduce the argument from [92, Prop.3.5]. Let $R := \mathbb{Z}[T_1, \dots, T_d]$, and for every $t \in \mathbb{N}$, denote by $I_t \subset R$ the ideal generated by $T_1^t, \dots, T_d^t$. We have a short exact sequence of R-modules :

$$0 \to R/I_1 \xrightarrow{\varphi_t} R/I_{t+1} \to Q := R/(I_{t+1} + Rp_t) \to 0$$

where φ_t is induced by the endomorphism $R \to R$ such that $q \mapsto qT_1^t \cdots T_d^t$ for every $q \in R$. Let also $f : R \to A$ be the ring homomorphism such that $f(T_i) = x_i$ for $i = 1, \dots, d$. Then, every A-module becomes an R-module, after restriction of scalars along f. We notice :

Claim 17.6.2. For every big CM A-module M we have $\mathrm{Tor}_1^R(Q, M) = 0$.

Proof of the claim. Notice the natural ring isomorphism $R/I_1 \xrightarrow{\sim} \mathbb{Z}$, and let $\mathrm{gr}_\bullet Q$ be the graded $\mathbb{Z}$-module associated with the I_1-adic filtration of Q; we easily see that $I^k Q = 0$ for every sufficiently large $k \in \mathbb{N}$; then the long Tor exact sequences induced by the short exact sequences of R-modules :

$$0 \to I_1^{k+1} Q \to I^k Q \to \mathrm{gr}_k Q \to 0 \qquad \text{for every } k \in \mathbb{N} \tag{17.6.3}$$

easily reduce to checking that $\mathrm{Tor}_i^R(\mathrm{gr}_k Q, M) = 0$ for every $k \in \mathbb{N}$ and every $i > 0$. Moreover, it is easily seen that $\mathrm{gr}_k Q$ is a free $\mathbb{Z}$-module, for every $k \in \mathbb{N}$ (details left to the reader), hence we are further reduced to checking that $\mathrm{Tor}_1^R(R/I_1, Q) = 0$ for every $i > 0$. The latter follows from remark 17.5.19(i) and lemma 7.8.14. ◊

By corollary 17.5.92, we may find a big CM A-module M. Then it suffices to check :

Claim 17.6.4. For every big CM A-module M, the map φ_t induces an injective map

$$\varphi_t \otimes_A M : M/I_1 M \to M/I_{t+1} M \qquad \overline{m} \mapsto \overline{x_1^t \cdots x_d^t m} \qquad \text{for every } t \in \mathbb{N}.$$

Proof of the claim. It follows immediately by combining claim 17.6.2 with the long Tor exact sequence attached to (17.6.3). □

Remark 17.6.5. (i) In the situation of theorem 17.6.1, set $X := \mathrm{Spec}\, A$ and $U_i := \mathrm{Spec}\, A[x_i^{-1}]$ for $i = 1, \dots, d$. Then, $U_\bullet := (U_i \mid i = 1, \dots, d)$ is an affine covering of $X \setminus \{\mathfrak{m}\}$, and proposition 10.4.18 yields a natural isomorphism

$$R\Gamma_{\{\mathfrak{m}\}}\mathscr{O}_X \xrightarrow{\sim} C^\bullet_{\mathrm{alt}}(U_\bullet, \mathscr{O}_X)[-1] \qquad \text{in } \mathsf{D}(A\text{-}\mathbf{Mod})$$

where $C^\bullet_{\mathrm{alt}}(U_\bullet, \mathscr{O}_X)$ denotes the alternating augmented Čech complex of the structure sheaf $\mathscr{O}_X$, relative to the covering $U_\bullet$. Set as well $y := x_1 \cdots x_d$, and $y_i := x_1 \cdots x_{i-1}x_{i+1} \cdots x_d$ for $i = 1, \dots, d$; there follows an A-linear isomorphism

$$H^d_{\{\mathfrak{m}\}}(A) \xrightarrow{\sim} \mathrm{Coker}\Big(\partial : \bigoplus_{i=1,\dots,d} A[y_i^{-1}] \to A[y^{-1}]\Big)$$

where ∂ is the sum of the localization maps $A[y_i^{-1}] \to A[y^{-1}]$. Especially, the class of y^{-1} in $\mathrm{Coker}\,\partial$ corresponds to an element $\beta \in H^d_{\{\mathfrak{m}\}}(A)$, and a simple inspection shows that theorem 17.6.1 is equivalent to the assertion that $\beta \neq 0$: the details shall be left to the reader.

(ii) Moreover, it was already known, by some early work of Hochster, that the monomial conjecture is equivalent to the direct summand conjecture (for finite extensions of regular local rings) : see [93, Th.6.1].

Theorem 17.6.6 (Strong monomial conjecture)**.** *Let $(A, \mathfrak{m})$ be a noetherian local ring, $\mathfrak{p} \subset A$ a prime ideal, and $x_1, \dots, x_d$ a system of parameters for A, such that :*

(a) $\mathfrak{p}$ *has height one, and* $\dim A/\mathfrak{p} = d - 1$.
(b) $x_1 \in \mathfrak{p}$.

Then we have : $\qquad x_1^{t+1}x_2^t \cdots x_d^t \notin \mathfrak{p} \cdot (Ax_1^{t+1} + \cdots + Ax_d^{t+1}) \qquad$ *for every $t \in \mathbb{N}$.*

Proof. We first notice :

Claim 17.6.7. We may assume that $(A, \mathfrak{m})$ is a complete local domain.

Proof of the claim. Let $A^\wedge$ and $(A/\mathfrak{p})^\wedge$ be the $\mathfrak{m}$-adic completions of A and $A/\mathfrak{p}$; recall that the $\mathfrak{m}$-adic topology of A induces the $\mathfrak{m}$-adic topology on $\mathfrak{p}A$ ([126, Th.8.6]), and $\mathfrak{p}A^\wedge$ is a closed ideal for the $\mathfrak{m}$-adic topology of $A^\wedge$ ([126, Th.8.10(i)]), hence the natural map $A^\wedge/\mathfrak{p}A^\wedge \to (A/\mathfrak{p})^\wedge$ is a ring isomorphism (proposition 8.2.13(i,v)). Especially, $\dim A^\wedge/\mathfrak{p}A^\wedge = d - 1$, *i.e.*, there exists a maximal point $\mathfrak{p}'$ of the closed subset $\mathrm{Spec}\, A^\wedge/\mathfrak{p}A^\wedge \subset \mathrm{Spec}\, A^\wedge$, such that $\dim A^\wedge/\mathfrak{q} = d - 1$. Moreover, by the going-down theorem for flat extensions, for every such $\mathfrak{p}'$ we have $\mathfrak{p} = \mathfrak{p}' \cap A$ ([126, Th.9.5]), and furthermore, $\mathfrak{p}'$ has height one ([126, Th.15.1(ii)]). Pick any minimal prime ideal $\mathfrak{q} \subset A^\wedge$ contained in $\mathfrak{p}'$. By the same token, $\mathfrak{q}$ is a closed ideal for the $\mathfrak{m}$-adic topology of $A^\wedge$, and therefore $A' := A^\wedge/\mathfrak{q}$ is still a complete local ring; since $\dim A^\wedge/\mathfrak{q} = d - 1$, we must have $\dim A' = d$, and $\mathfrak{p}'A'$ has height one. Clearly it suffices to prove the theorem for the ring A', its prime ideal $\mathfrak{p}'A'$, and the image in A' of the sequence $x_1, \dots, x_d$, which is still a system of parameters for A'. ◊

Henceforth, we assume that $(A, \mathfrak{m})$ is a complete local domain, so the same holds for $A/\mathfrak{p}$, and we may find a commutative diagram of rings :

$$\begin{array}{ccc} A & \longrightarrow & A/\mathfrak{p} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ B_0 & \longrightarrow & B_1 \end{array}$$

such that B_0 is a big CM A-algebra, and B_1 is a big CM $A/\mathfrak{p}$-algebra (theorem 17.5.90). For a given $t \in \mathbb{N}$, set $y := x_1^{t+1}x_2^t \cdots x_d^t$, and $I := Ax_1^{t+1} + \cdots + Ax_d^{t+1}$. Clearly it suffices to check :

Claim 17.6.8. $f(y) \notin (I\mathfrak{p} + I^2)B_0$.

Proof of the claim. The sequence $x_1, \dots, x_d$ is completely secant in B_0 (remark 17.5.19(i)), so the same holds for the sequence $x_1^{t+1}, \dots, x_d^{t+1}$ (corollary 7.8.8(ii)), and then the latter is also B_0-quasi-regular (proposition 7.8.15). The latter means that the A-linear map

$$\omega : \bigoplus_{i=1}^{d} (B_0/IB_0) \to M := IB_0/I^2B_0 \qquad (\bar{b}_1, \dots, \bar{b}_d) \mapsto \overline{x_1^{t+1}b_1 + \cdots + x_d^{t+1}b_d}$$

is an isomorphism. We have $f(y) \in IB_0$, and clearly $\omega(\overline{f(x_2^t \cdots x_d^t)}, 0, \dots, 0)$ is the class $\overline{f(y)}$ of $f(y)$ in M. Now, if the claim fails, we have $\overline{f(y)} \in \mathfrak{p}M$, and therefore

$$(17.6.9) \qquad (\overline{f(x_2^t \cdots x_d^t)}, 0, \dots, 0) \in \bigoplus_{i=1}^{d} \mathfrak{p}(B_0/IB_0) \qquad i.e. \quad f(x_2^t \cdots x_d^t) \in (\mathfrak{p} + I)B_0.$$

Now, notice that the image $\overline{x}_2, \dots, \overline{x}_d$ of the sequence $x_2, \dots, x_d$ is a system of parameters for $A/\mathfrak{p}$, since $\dim A/\mathfrak{p} = d - 1$. However, from (17.6.9) we get

$$g(\overline{x}_2^t \cdots \overline{x}_d^t) \in IB_1$$

which contradicts claim 17.6.4, since B_1 is a big CM $A/\mathfrak{p}$-algebra. □

Remark 17.6.10. (i) In the situation of theorem 17.6.6, notice that $\dim A/x_1A = d - 1$ ([126, Th.14.1]), hence $\mathfrak{p}$ must be a maximal point of the closed subset $\operatorname{Spec} A/x_1A \subset \operatorname{Spec} A$.

(ii) Conversely, if A is equidimensional and catenary, and $\mathfrak{p}$ is a maximal point of the closed subset $\operatorname{Spec} A/x_1A$, then conditions (a) and (b) of theorem 17.6.6 are clearly fulfilled.

(iii) The hypothesis on $\dim A/\mathfrak{p}$ in condition (a) of theorem 17.6.6 is incorrectly omitted from the statement of the strong monomial conjecture in the survey article [142], where it is listed as conjecture 19. To see that this hypothesis cannot be omitted, consider a local noetherian domain $(A, \mathfrak{m})$ of dimension d, with a prime ideal $\mathfrak{p}$ of height one, such that $\dim A/\mathfrak{p} \leq d - 2$. Pick any $x_1 \in \mathfrak{p} \setminus \{0\}$; then let $\Sigma \subset \operatorname{Spec} A$ be the subset of all prime ideals $\mathfrak{q}$ of A such that $x_1 \in \mathfrak{q}$ and $\dim A/\mathfrak{q} = d - 1$. Clearly every element of Σ must have height one, so Σ lies in the finite set of maximal points of $\operatorname{Spec} A/x_1A$. Moreover, for every $\mathfrak{q} \in \Sigma$ we have $\mathfrak{p} \not\subseteq \mathfrak{q}$, since $\dim A/\mathfrak{p} < d - 1$. Hence, there exists $x_2 \in \mathfrak{p} \setminus \bigcup_{\mathfrak{q} \in \Sigma} \mathfrak{q}$ ([12, Prop.1.11(i)]). Let $I := Ax_1 + Ax_2$; we have $\dim A/I \geq d - 2$ by [126, Th.13.6(ii)], and on the other hand, by construction there exists no prime ideal $\mathfrak{q} \subset A$ such that $I \subset \mathfrak{q}$ and $\dim A/\mathfrak{q} = d - 1$, so $\dim A/I = d - 2$. Now, pick $x_3, \dots, x_d \in \mathfrak{m}$ whose images in A/I are a system of parameters for the latter local ring. It follows easily that $x_1, \dots, x_d$ is a system of parameters for A, and it is easily seen that $x_1^2 x_2 \cdots x_d \in \mathfrak{p} \cdot (Ax_1^2 + \cdots + Ax_d^2)$.

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