FRAGMENTS OF ALMOST RING THEORY

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CONTENTS

1. INTRODUCTION

Let V be a ring and $\mathfrak{m}$ an ideal of V such that $\mathfrak{m}^2 = \mathfrak{m}$. The V-modules killed by $\mathfrak{m}$ are the object of a (full) Serre subcategory Σ of the category $V - \mathbf{Mod}$ of all V-modules, and the quotient category $V - \mathbf{Mod}/\Sigma$ is an abelian category $V - \mathbf{al.Mod}$ which we call the category of almost V-modules. It is easy to check that the usual tensor product of V-modules descends to a bifunctor $\otimes$ on almost V-modules, so that $V - \mathbf{al.Mod}$ is a monoidal abelian category in a natural way. Then an almost ring is just an almost V-module A endowed with a "multiplication" morphism $A \otimes A \to A$ satisfying certain natural axioms. Together with the obvious morphisms, these gadgets form a category $V - \mathbf{al.Alg}$ and there is a natural localisation functor $V - \mathbf{Alg} \to V - \mathbf{al.Alg}$ which associates to any V-algebra the same object viewed in the localised category.

While the notion of almost V-module had already arisen (in the sixties) in Gabriel's memoir "Des categories abeliennes" [2], the usefulness of almost rings did not become apparent until Faltings' paper [1] on "p-adic Hodge theory". To be accurate, the definition of almost étale extension found in [1] is still given in terms of usual rings and modules, and the idea of passing to the quotient category is not really developed, rather it is scattered around in a series of clues that an honest reader may choose to pursue if so inclined.

About two years ago I decided that, if no one else was interested in writing up this story, I could as well take a stab at it myself. The result of my efforts appeared in the preprint [7]. The main aim of [7] was to provide convenient foundations for the study of deformations of almost rings and their morphisms. In view of applications to p-adic Hodge theory, it is especially interesting to establish suitable almost versions of the standard results on infinitesimal lifting of étale rings and morphisms. The method I chose to tackle this question, was to extend Illusie's theory of the cotangent complex to the almost case. This method worked reasonably well, and yielded more general results than could be achieved with Hochschild cohomology (Faltings' original method). As an added bonus, proofs using the cotangent complex also require far fewer calculations.

Last year, on invitation of Francesco Baldassarri, I presented these results in a talk in Padova. Due to time constraints, I preferred to forgo the detailed introduction of almost terminology, and rather to focus on deformations. That is, I dealt throughout the lecture with usual rings and modules but I gave proofs of the lifting theorems which could be adapted to the almost case with relative ease (to obtain such proofs, one has just to copy the arguments given in [7], omitting everywhere the word "almost"). After the talk, it was convened that I would write up my lecture as a contribution to the Rendiconti. This would have had to be a mainly expository article, and much of the work would have gone into fleshing out the references to homotopical algebra, so that the paper could have also been useful as a first introduction to the theory of the cotangent complex.

However, a string of circumstances kept on delaying this project in the intervening months. In the meanwhile, Ofer Gabber sent me some important remarks about my preprint; these remarks led to extensive discussions and eventually to the decision of collaborating on a radical overhaul of the preprint. Our new approach is based on the systematic exploitation of the left adjoint $V - \mathbf{al}.\mathbf{Alg} \to V - \mathbf{Alg}$ to the localization functor. This approach is in many ways superior to the previous one (which was based on the right adjoint); in particular, rather than extending Illusie's theory to the almost case, one can now *reduce* the study of deformations of almost rings to the case of usual rings.

The net outcome of these developments is that, on one hand the theory of deformations of almost rings is now almost as polished as the theory for usual rings, and on the other hand, much of the industrious work that buttressed the first preprint has been obsoleted.

Since I am still rather fond of the elementary arguments that I devised for the first version, I prefer to think instead that this material has been liberated from the paper and is now free to be included in a revised note for Rendiconti. The following article is a compromise born out of these considerations : it is at the same time more and less than what was originally planned. It is more, because it now consists mostly of original results. It is less, because it is no longer designed as a gentle first introduction to the cotangent complex and to general homotopical algebra; besides, whenever the proof of a result has not significantly changed from the first to the second preprint, I avoid duplication and refer instead to the appropriate statement in the "official" version [3].

2. Homological theory

2.1. **Almost categories.** Unless otherwise stated, every ring is commutative with unit. If C is a category, and X, Y two objects of C, we will usually denote by $\mathrm{Hom}_C(X, Y)$ the set of morphisms in C from X to Y and by $\mathbf{1}_X$ the identity morphism of X. Moreover we denote by C^o the opposite category of C and by $s.C$ the category of simplicial objects over C, that is, functors $\Delta^o \to C$, where Δ is the category whose objects are the ordered sets $[n] = \{0, ..., n\}$ for each integer $n \geq 0$ and where a morphism $\phi : [p] \to [q]$ is a non-decreasing map. A morphism $f : X \to Y$ in $s.C$ is a sequence of morphisms $f_{[n]} : X[n] \to Y[n]$, $n \geq 0$ such that the obvious diagrams commute. We can imbed C in $s.C$ by sending each object X to the "constant" object $s.X$ such that $s.X[n] = X$ for all $n \geq 0$ and $s.X[\phi] = \mathbf{1}_X$ for all morphisms ϕ in Δ. We let $\Delta(n)$ be the simplicial set represented by $[n]$, *i.e* such that $\Delta(n)[i] = \mathrm{Hom}_\Delta([i], [n])$.

If C is an abelian category, $\mathsf{D}(C)$ will denote the derived category of C. As usual we have also the subcategories $\mathsf{D}^+(C)$, $\mathsf{D}^-(C)$ of complexes of objects of C which are exact for sufficiently large negative (resp. positive) degree. If A is a ring, the category of A-modules (resp. A-algebras) will be denoted by $A - \mathbf{Mod}$ (resp. $A - \mathbf{Alg}$). Most of the times we will write $\mathrm{Hom}_A(M, N)$ instead of $\mathrm{Hom}_{A-\mathbf{Mod}}(M, N)$.

We denote by $\mathbf{Set}$ the category of sets and by $\mathbf{Ab}$ the category of abelian groups. The symbol $\mathbb{N}$ denotes the set of non-negative integers; in particular $0 \in \mathbb{N}$.

Our basic setup consists of a fixed base ring V containing an ideal $\mathfrak{m}$ such that $\mathfrak{m}^2 = \mathfrak{m}$. However, pretty soon we will need to introduce further hypotheses on the pair $(V, \mathfrak{m})$; such additional restrictions are conveniently expressed by the following axioms (**S1**) and (**S2**). Before discussing them, let us stress that many interesting assertions can be proved without assuming either of these axioms, and some more require only the validity of (**S1**), which is the weakest of the two. For these reasons, neither of them is part of our basic setup, and they will be explicitly invoked only as the need arises.

(**S1**) The ideal $\mathfrak{m}$ is generated by a multiplicative system $S \subset \mathfrak{m}$ with the following property. For all $x \in S$ there exist $y, z \in S$ such that $x = y^2 \cdot z$.

Suppose that axiom (**S1**) holds. The divisibility relation induces a partial order on S by setting $y \prec x$ if and only if there exists $z \in S \cup \{1\}$ such that $x = y \cdot z$.

(**S2**) The ideal $\mathfrak{m}$ is generated by a multiplicative system $S \subset \mathfrak{m}$ consisting of non-zero divisors and the resulting partially ordered set $(S, \prec)$ is cofiltered.

Let us remark that (**S2**) implies (**S1**). First of all, without loss of generality we can assume that $u \cdot x \in S$ whenever $u \in V$ is a unit and $x \in S$. Next, pick any $x \in S$; if x is smaller (under $\prec$) than every element of S, then $\mathfrak{m} = x \cdot V$. Then $\mathfrak{m} = \mathfrak{m}^2$ implies that x^2 divides x, and the claim follows easily. Otherwise, there exists $s \in S$ which is not greater than x; then (**S2**) implies that there exists $t, t' \in S$ such that $x = t \cdot t'$; again (**S2**) gives an element $y \in S$ which divides both t and t', so y^2 divides x and (**S1**) follows.

Moreover, the second axiom implies that $\mathfrak{m}$ is a flat V-module : indeed, under (**S2**), $\mathfrak{m}$ is a filtered colimit of the free V-modules $s \cdot V$, where s ranges over all the elements of S.

Example 2.1.1. i) The main example is given by a non-discrete valuation ring V with valuation $\nu : V - \{0\} \to \Gamma$ of rank one (where Γ is the totally ordered abelian group of values of ν). Then we can take for S the set of elements $x \in V - \{0\}$ such that $\nu(x) > \nu(1)$ and (**S2**) is satisfied.

ii) Suppose that S contains invertible elements. This is the "classical limit". In this case almost ring theory reduces to usual ring theory. Thus, all the discussion that follows specialises to, and sometimes gives alternative proofs for, statements about rings and their modules.

Let M be a given V-module. We say that M is *almost zero* if $\mathfrak{m} \cdot M = 0$. A morphism ϕ of V-modules is an *almost isomorphism* if both Ker(ϕ) and Coker(ϕ) are almost zero V-modules.

The full subcategory Σ of $V-\mathbf{Mod}$ consisting of all V-modules which are almost isomorphic to 0 is clearly a Serre subcategory and hence we can form the quotient category $V - \mathbf{al.Mod} = V - \mathbf{Mod}/\Sigma$ (after taking some set-theoretic precautions : the interested reader will find in [8] (§10.3) a discussion of these issues). Then $V - \mathbf{al.Mod}$ is an abelian category which will be called the *category of almost V-modules*.

There is a natural localization functor

$$V - \mathbf{Mod} \to V - \mathbf{al.Mod} \qquad M \mapsto M^a$$

which takes a V-module M to the same module, seen as an object of $V - \mathbf{al.Mod}$.

Since the almost isomorphisms form a multiplicative system (see *e.g.* [8] Ex. 10.3.2), it is possible to describe the morphisms in $V - \mathbf{al.Mod}$ via a calculus of fractions, as follows. Let $V - \mathbf{al.Iso}$ be the category which has the same objects as $V - \mathbf{Mod}$, but such that $\mathrm{Hom}_{V-\mathbf{al.Iso}}(M, N)$ consists of all almost isomorphisms $M \to N$. If M is any object of $V - \mathbf{al.Iso}$ we write $(V - \mathbf{al.Iso}/M)$ for the category of objects of $V - \mathbf{al.Iso}$ over M (*i.e.* morphisms $\phi : X \to M$). If $\phi_i : X_i \to M$ $(i = 1, 2)$ are two objects of $(V - \mathbf{al.Iso}/M)$ then $\mathrm{Hom}_{(V-\mathbf{al.Iso}/M)}(\phi_1, \phi_2)$ consists of all morphisms $\psi : X_1 \to X_2$ in $V - \mathbf{al.Iso}$ such that $\phi_1 = \phi_2 \circ \psi$. For any two V-modules M, N we define a functor $F_N : (V - \mathbf{al.Iso}/M)^o \to$

$V-\mathbf{Mod}$ by associating to an object $\phi : P \to M$ the V-module $\mathrm{Hom}_V(P,N)$ and to a morphism $\alpha : P \to Q$ the map

$$\mathrm{Hom}_V(Q,N) \to \mathrm{Hom}_V(P,N) \ : \ \beta \mapsto \beta \circ \alpha.$$

Then we have

$$\mathrm{Hom}_{V-\mathbf{al.Mod}}(M^a, N^a) = \operatorname*{colim}_{(V-\mathbf{al.Iso}/M)^o} F_N. \tag{2.1.2}$$

In particular $\mathrm{Hom}_{V-\mathbf{al.Mod}}(M,N)$ has a natural structure of V-module, for any two almost V-modules M, N, *i.e.* $\mathrm{Hom}_{V-\mathbf{al.Mod}}(-,-)$ is a bifunctor which takes values in the category $V-\mathbf{Mod}$.

The usual tensor product induces a bifunctor $-\otimes_V-$ on almost V-modules, which, in the jargon of [5] makes of $V-\mathbf{al.Mod}$ a *(closed symmetric) monoidal category*. Then an *almost V-algebra* is just a monoid in the monoidal category $V-\mathbf{al.Mod}$. This means (a bit sloppily) the following. For two almost V-modules M and N let $\eta_{M|N} : M \otimes_V N \to N \otimes_V M$ be the automorphism which "switches the two factors" ($m \otimes n \mapsto n \otimes m$) and $\nu_{A/V} : V \otimes_V A \to A$ the "scalar multiplication" ($v \otimes a \mapsto va$). Then an almost V-algebra is an almost V-module A endowed with a morphism of almost V-modules $\mu_{A/V} : A \otimes_V A \to A$ (the "multiplication" of A) and a "unit morphism" $\underline{1}_{A/V} : V \to A$ satisfying the conditions

$$\begin{array}{rll}
\mu_{A/V} \circ (\mathbf{1}_A \otimes_V \mu_{A/V}) = \mu_{A/V} \circ (\mu_{A/V} \otimes_V \mathbf{1}_A) & & \text{(associativity)} \\
\mu_{A/V} = \mu_{A/V} \circ \eta_{A|A} & & \text{(commutativity)} \\
\mu_{A/V} \circ (\underline{1}_{A/V} \otimes_V \mathbf{1}_A) = \nu_{A/V} & & \text{(unit property)}
\end{array}$$

With the morphisms defined in the obvious way, the almost V-algebras form a category $V-\mathbf{al.Alg}$. Clearly the localization functor restricts to a functor $V-\mathbf{Alg} \to V-\mathbf{al.Alg}$. Occasionally we will have to deal with non-commutative or non-unital almost algebras; for these more general monoids, the second, resp. the third of the above axioms fails.

Furthermore, a *left almost A-module* is an almost V-module M endowed with a morphism $\sigma_M : A \otimes_V M \to M$ such that

$$\begin{gathered}
\sigma_M \circ (\mathbf{1}_A \otimes_V \sigma_M) = \sigma_M \circ (\mu_{A/V} \otimes_V \mathbf{1}_M) \\
\sigma_M \circ (\underline{1}_{A/V} \otimes_V \mathbf{1}_M) = \nu_{M/V}.
\end{gathered} \tag{2.1.3}$$

Similarly we define right almost A-modules. The A-linearity of a morphism $\phi : M \to N$ of A-modules is expressed by the condition

$$\sigma_N \circ (\mathbf{1}_A \otimes_V \phi) = \phi \circ \sigma_M. \tag{2.1.4}$$

We denote by $A-\mathbf{al.Mod}$ the category of left almost A-modules and *A-linear morphisms* defined as one expects. Clearly $A-\mathbf{al.Mod}$ is an abelian category. For any V-algebra B we have a localization functor $B-\mathbf{Mod} \to B^a-\mathbf{al.Mod}$.

Next, if A is an almost V-algebra, we can define the category $A-\mathbf{al.Alg}$ of almost A-algebras. It consists of all the morphisms $A \to B$ of almost V-algebras.

Being, as they are, objects in a quotient category, almost A-modules do not possess elements in the same way as usual modules do. However, not everything is lost, as we show in the following definition.

Definition 2.1.5. Let M be an almost A-module. An *almost A-element* of M is just an A-linear morphism $A \to M$. We denote by M_* the set of all almost A-elements of the almost A-module M. If B is an almost A-algebra, we can multiply almost elements as follows. First we remark that the morphism $\mu_{A/A} : A \otimes_A A \to A$ is an isomorphism. Then, given almost elements $b_1 : A \to B$ and $b_2 : A \to B$ set $b_1 \cdot b_2 = \mu_{B/A} \circ (b_1 \otimes_A b_2) \circ \mu^{-1}_{A/A} : A \to B$. In particular, A_* is a V-algebra and B_* is endowed with a natural A_*-algebra structure, whose identity is the structure morphism $\underline{1}_{B/A} : A \to B$.

Moreover, if $m : A \to M_1$ is an almost A-element of M_1 and $\phi : M_1 \to M_2$ is an A-linear morphism, we denote by $\phi_*(m) : A \to M_2$ the almost A-element of M_2 defined as $\phi_*(m) = \phi \circ m$. In this way we obtain an A_*-linear morphism $\phi_* : M_{1*} \to M_{2*}$, *i.e.* the assignment $M \mapsto M_*$ extends to a functor from almost A-modules (resp. almost A-algebras) to A_*-modules (resp. A_*-algebras).

For any two almost A-modules M, N, the set $\mathrm{Hom}_{A-\mathbf{al.Mod}}(M, N)$ has a natural structure of A_*-module and we obtain an internal Hom functor by letting

$$\mathrm{alHom}_A(M, N) = \mathrm{Hom}_{A-\mathbf{al.Mod}}(M, N)^a.$$

This is the functor of *almost homomorphisms* from M to N.

For any almost A-module M we have also a functor of tensor product $M \otimes_A -$ on almost A-modules which, in view of the following proposition 2.1.7 can be shown to be a left adjoint to the functor $\mathrm{alHom}_A(M, -)$. It can be defined as $M \otimes_A N = (M_* \otimes_{A_*} N_*)^a$ but an appropriate almost version of the usual construction would also work.

With this tensor product, $A - \mathbf{al.Mod}$ is a monoidal category as well, and $A - \mathbf{al.Alg}$ could also be described as the category of monoids in the category of almost A-modules. Under this equivalence, a morphism $\phi : A \to B$ of V-algebras becomes the unit morphism $\underline{1}_{B/A} : A \to B$ of the corresponding monoid. We will sometimes drop the subscript and write simply $\underline{1}$.

Suppose that B_1 and B_2 are two almost A-algebras and M is an almost B_1-module. Then we can induce a structure of almost $B_2 \otimes_A B_1$-module on $B_2 \otimes_A M$, by declaring that the scalar multiplication $\sigma_{B_2 \otimes_A M} : (B_2 \otimes_A B_1) \otimes_A (B_2 \otimes_A M) \to B_2 \otimes_A M$ acts by the rule: $(c_1 \otimes b) \otimes (c_2 \otimes m) \mapsto (c_1 \cdot c_2) \otimes (b \cdot m)$ (for all $c_1, c_2 \in B_{2*}$, $b \in B_{1*}$, $m \in M_*$). In categorical notation, this translates as follows. Let $\eta_{B_1|B_2} : B_1 \otimes_A B_2 \to B_2 \otimes_A B_1$ be the A-linear isomorphism which "switches the two factors"; we set

$$\sigma_{B_2 \otimes_A M} = (\mu_{B_2/A} \otimes_A \sigma_M) \circ (\mathbf{1}_{B_2} \otimes_A \eta_{B_1|B_2} \otimes_A \mathbf{1}_M). \tag{2.1.6}$$

Proposition 2.1.7. *i) There is a natural isomorphism $A \simeq A_*^a$ of almost V-algebras.*

ii) The functor $M \mapsto M_*$ from $A - \mathbf{al.Mod}$ to $A_* - \mathbf{Mod}$ (resp. from $A - \mathbf{al.Alg}$ to $A_* - \mathbf{Alg}$) is right adjoint to the localization functor $A_* - \mathbf{Mod} \to A_*^a - \mathbf{al.Mod} = A - \mathbf{al.Mod}$ (resp. $A_* - \mathbf{Alg} \to A - \mathbf{al.Alg}$).

iii) The counit of the adjunction $M_*^a \to M$ is a natural equivalence from the composition of the two functors to the identity functor $\mathbf{1}_{A-\mathbf{al.Mod}}$ (resp. $\mathbf{1}_{A-\mathbf{al.Alg}}$).

Proof. (i) : quite generally, for any almost A-module M we have a standard isomorphism of A_*-modules

$$\mathrm{Hom}_{A-\mathbf{al.Mod}}(A, M) \simeq \mathrm{Hom}_{V-\mathbf{al.Mod}}(V, M). \tag{2.1.8}$$

Then the claim follows easily from the explicit description of the set of morphisms in $V - \mathbf{al.Mod}$ given above.

(ii) : For any given almost A_*^a-module N and any A_*-module M we need to establish a natural bijection

$$\mathrm{Hom}_{A_*^a-\mathbf{al.Mod}}(M^a, N) \simeq \mathrm{Hom}_{A_*}(M, N_*). \tag{2.1.9}$$

For any $m \in M$ define the morphism of A_*-modules $\sigma_m : A_* \to M$ by $a \mapsto a \cdot m$. Then the bijection sends a morphism $\phi : M^a \to N$ of almost A_*^a-modules to the morphism of A_*-modules $\widetilde{\phi} : M \to N_*$ defined as $m \mapsto \phi \circ \sigma_m^a$. Moreover, $\widetilde{\phi}$ is a morphism of A_*^a-algebras whenever ϕ is a morphism of almost A-algebras.

An explicit inverse for the bijection can be obtained as follows. Pick any A_*-linear morphism $\widetilde{\phi} : M \to N_*$. We construct a morphism of V-modules $\phi : \mathfrak{m} \cdot M \to N/(\mathfrak{m} - \text{torsion})$ which represents the required morphism of almost A_*^a-modules $M^a \to N$. If $m \in M$, by definition the morphism $\widetilde{\phi}(m) \circ \underline{1}_{A/V} : V \to N$ is represented by a pair consisting of an almost isomorphism

$\psi : X \to V$ and a V-linear morphism $\phi' : X \to N$. For given $\varepsilon \in \mathfrak{m}$ choose $u \in X$ such that $\varepsilon = \psi(u)$ and define $\phi(\varepsilon \cdot m)$ to be the class modulo $\mathfrak{m}$-torsion of $\phi'(u)$. We leave it to the reader to verify that ϕ thus defined extends to a unique A^a_*-linear map and that the assignment $\widetilde{\phi} \mapsto \phi$ is an inverse for the map (2.1.9).

For (iii) we need to show that the natural morphism $\phi : M^a_* \to M$ corresponding via (2.1.9) to the identity of M_*, is an isomorphism of almost A-modules. Inspecting the proof of (ii), we see that this morphism is represented by a V-linear morphism $\mathfrak{m} \cdot M_* \to M/(\mathfrak{m}-\text{torsion})$ which can be explicitly computed. We indicate an inverse for ϕ and leave the details to the reader. For $m \in M$, let $\omega_m : A \to A \otimes_V M$ be the A-linear morphism $a \mapsto a \otimes m$. Then the morphism $M \to M^a_*$ defined as $m \mapsto \sigma_M \circ \omega_m$ provides the required inverse. □

Remark 2.1.10. (i) Proposition 2.1.7 follows also directly from [2] (chap.III §3 Cor.1).

(ii) It is also easy to check that, for any V-module M, the natural map (unit of the adjunction) $M \to M^a_*$ is an almost isomorphism. Moreover, under **(S2)** we have a natural isomorphism

$$M^a_* \simeq \mathrm{Hom}_V(\mathfrak{m}, M) \simeq \lim_{\varepsilon \in S} M$$

where, for any two elements $\varepsilon \prec \delta$ of the cofiltered set S, the corresponding morphism $\phi_{\varepsilon,\delta} : M \to M$ is defined by $m \mapsto (\delta/\varepsilon) \cdot m$.

Corollary 2.1.11. *The categories $A-\mathbf{al.Mod}$ and $A-\mathbf{al.Alg}$ are both complete and cocomplete.*

Proof. We recall that the categories $A_*-\mathbf{Mod}$ and $A_*-\mathbf{Alg}$ are both complete and cocomplete. Now let I be any small indexing category and $M : I \to A-\mathbf{al.Mod}$ be any functor. Denote by $M_* : I \to A_* - \mathbf{Mod}$ the composed functor $i \mapsto M(i)_*$. We claim that

$$\operatorname*{colim}_I M = (\operatorname*{colim}_I M_*)^a.$$

The proof is an easy application of proposition 2.1.7(iii). A similar argument also works for limits and for the category $A-\mathbf{al.Alg}$. □

Next recall that the forgetful functor $A_* - \mathbf{Alg} \to \mathbf{Set}$ (resp. $A_* - \mathbf{Mod} \to \mathbf{Set}$) has a left adjoint $A_*[-] : \mathbf{Set} \to A_* - \mathbf{Alg}$ (resp. $A^{(-)} : \mathbf{Set} \to A_* - \mathbf{Mod}$) which assigns to a set S the free A_*-algebra $A_*[S]$ (resp. the free A_*-module $A_*^{(S)}$) generated by S. If S is any set, it is natural to write $A[S]$ (resp. $A^{(S)}$) for the almost A-algebra $(A_*[S])^a$ (resp. for the almost A-module $(A_*^{(S)})^a$. This yields a left adjoint, called the *free almost algebra* functor $\mathbf{Set} \to A-\mathbf{al.Alg}$ (resp. the *free almost module* functor $\mathbf{Set} \to A-\mathbf{al.Mod}$) to the "forgetful" functor $A-\mathbf{al.Alg} \to \mathbf{Set}$ (resp. $A-\mathbf{al.Mod} \to \mathbf{Set}$) $B \mapsto B_*$.

Remark 2.1.12. The functor of almost elements commutes with arbitrary limits, because all right adjoints do. It does not in general commute with arbitrary colimits, not even with arbitrary infinite direct sums.

2.2. **Almost homological algebra.** In this section we fix an almost V-algebra A and we consider various constructions in the category of almost A-modules.

Remark 2.2.1. i) Let M_1, M_2 be almost A-modules. By proposition 2.1.7 it is clear that a morphism $\phi : M_1 \to M_2$ of almost A-modules is uniquely determined by the induced morphism $M_{1*} \to M_{2*}$.

ii) It is a bit tricky to deal with preimages of almost elements under morphisms: for instance, if $\phi : M_1 \to M_2$ is an epimorphism (by which we mean that $\mathrm{Coker}(\phi) \simeq 0$) and $m_2 \in M_{2*}$, then it is not true in general that we can find an almost A-element $m_1 \in M_{1*}$ such that $\phi_*(m_1) = m_2$. What remains true is that for arbitrary $\varepsilon \in \mathfrak{m}$ we can find m_1 such that $\phi_*(m_1) = \varepsilon \cdot m_2$.

Suppose that (**S1**) holds. Let $T = \{t_n \mid n \in \mathbb{N}\}$ be an infinite sequence of elements of S. We say that T is a *Cauchy sequence* if, for any $\varepsilon \in S$ there exists $s = s(\varepsilon) \in S$ such that $s \prec t_n \prec s \cdot \varepsilon$ for all but finitely many $n \in \mathbb{N}$. We say that T *converges to* 1 if, for any $s \in S$, we have $t_n \prec s$ for all but finitely many $n \in \mathbb{N}$. We illustrate the use of this language in the proof of the following lemma.

Lemma 2.2.2. *Assume* (**S2**) *and let* $\{M_n\ ;\ \phi_n : M_n \to M_{n+1} \mid n \in \mathbb{N}\}$ *(resp.* $\{N_n\ ;\ \psi_n : N_{n+1} \to N_n \mid n \in \mathbb{N}\}$*) be a direct (resp. inverse) system of almost A-modules and morphisms and* $\{\varepsilon_n \mid n \in \mathbb{N}\}$ *a sequence of S which converges to* 1.

i) If $\varepsilon_n \cdot M_n = 0$ *for all* $n \in \mathbb{N}$ *then* $\operatorname*{colim}_{n\in\mathbb{N}} M_n \simeq 0$.

ii) If $\varepsilon_n \cdot N_n = 0$ *for all* $n \in \mathbb{N}$ *then* $\lim_{n\in\mathbb{N}} N_n \simeq 0 \simeq \lim_{n\in\mathbb{N}}{}^1 N_n$.

iii) If $\varepsilon_n \cdot \mathrm{Coker}(\psi_n) = 0$ *for all* $n \in \mathbb{N}$ *and moreover* $\{\prod_{j=0}^{n} \varepsilon_j \mid n \in \mathbb{N}\}$ *is a Cauchy sequence, then* $\lim_{n\in\mathbb{N}}{}^1 N_n \simeq 0$.

Proof. (i) and (ii) are left as exercises to the reader. We prove (iii). Let ψ be the morphism

$$\psi : \prod_{n=0}^{\infty} N_n \to \prod_{n=0}^{\infty} N_n$$

which assigns to any sequence $(b_n \mid n \in \mathbb{N})$ of almost A-elements $b_n \in N_{n*}$ the sequence

$$\psi_*(b_n \mid n \in \mathbb{N}) = (\psi_{n*}(b_{n+1}) - b_n \mid n \in \mathbb{N}).$$

It is a standard result (see *e.g.* [8] (§3.5)) that $\lim_{n\in\mathbb{N}}{}^1 N_n \simeq \mathrm{Coker}(\psi)$. In other words, we have $\lim_{n\in\mathbb{N}}{}^1 N_n \simeq (\lim_{n\in\mathbb{N}}{}^1 N_{n*})^a$. Let $\delta \in S$ be any element; it suffices therefore to show that $\delta \cdot \lim_{n\in\mathbb{N}}{}^1 N_{n*} = 0$.

After replacing ε_n by ε_n^2 we can assume that $\varepsilon_n \cdot \mathrm{Coker}(\psi_{n*}) = 0$ for all $n \in \mathbb{N}$. Moreover, for any fixed $m \in \mathbb{N}$ we have a natural isomorphism $\lim_{n\in\mathbb{N}}{}^1 N_{n*} \simeq \lim_{n\in\mathbb{N}}{}^1 N_{n+m*}$. Hence, up to omitting the first m modules and renumbering the others, we can assume that

$$\prod_{j=0}^{n} \varepsilon_j \prec \delta \quad \text{for all } n \in \mathbb{N}.$$

Set $\delta_n = \delta / \prod_{j=0}^{n-1} \varepsilon_j$ for all $n \in \mathbb{N}$. Also, for any element $(c_n \mid n \in \mathbb{N})$ of $\prod_{n=0}^{\infty} N_{n*}$ we denote by $(c_n^* \mid n \in \mathbb{N})$ the new element defined by $c_n^* = \delta_n \cdot c_n$.

Now, let $(c_n \mid n \in \mathbb{N})$ be any almost A-element of $\prod_{n=0}^{\infty} N_{n*}$. We will construct inductively a sequence $(a_n \mid n \in \mathbb{N})$ with the property that $c_n^* + a_n^* = \psi_{n*}(a_{n+1}^*)$ for all $n \in \mathbb{N}$, in other words $\psi_*(a_n^* \mid n \in \mathbb{N}) = (c_n^* \mid n \in \mathbb{N})$.

We let $a_0 = 0$. Suppose that we have already found $a_1, ..., a_n$ such that $c_j^* + a_j^* = \psi_{j*}(a_{j+1}^*)$ for all $j < n$. Since $\varepsilon_n \cdot \mathrm{Coker}(\psi_{n*}) = 0$, we can find $a_{n+1} \in N_{n+1*}$ such that $\varepsilon_n \cdot (c_n + a_n) = \psi_{n*}(a_{n+1})$. We multiply both sides of this equation by δ_{n+1} to obtain

$$\delta_n \cdot (c_n + a_n) = \delta_{n+1} \cdot \psi_{n*}(a_{n+1})$$

which is what we need. Finally, for any sequence $(b_n \mid n \in \mathbb{N})$ as above, let $(c_n \mid n \in \mathbb{N})$ be the sequence defined by $c_n = \prod_{j=0}^{n-1} \varepsilon_j \cdot b_n$ for all $n \in \mathbb{N}$. It is clear that $(c_n^* \mid n \in \mathbb{N}) = \delta \cdot (b_n \mid n \in \mathbb{N})$, which means that $\delta \cdot \mathrm{Coker}(\psi) = 0$, as required. □

Definition 2.2.3. Let M be an almost A-module.

i) We say that M is *flat* if the functor $N \mapsto M \otimes_A N$, from the category of almost A-modules to itself is exact. M is *almost projective* if the functor $N \mapsto \mathrm{alHom}_A(M, N)$ is exact.

ii) We say that M is *almost finitely generated* if, for arbitrarily small $\varepsilon \in S$ there exist a positive integer $n = n(\varepsilon)$ and an A-linear morphism $\phi_\varepsilon : A^n \to M$ such that $\varepsilon \cdot \mathrm{Coker}(\phi_\varepsilon) = 0$.

iii) We say that M is *almost finitely presented* if, for arbitrarily small $\varepsilon \in S$ there exist positive integers $n = n(\varepsilon)$, $m = m(\varepsilon)$ and a three term complex $A^m \stackrel{\psi_\varepsilon}{\to} A^n \stackrel{\phi_\varepsilon}{\to} M$ such that $\varepsilon \cdot \mathrm{Coker}(\phi_\varepsilon) = \varepsilon \cdot \mathrm{Ker}(\phi_\varepsilon)/\mathrm{Im}(\psi_\varepsilon) = 0$.

The abelian category $A - \mathbf{al.Mod}$ satisfies axiom (AB5) (see *e.g.* [8] (§A.4)) and it has a generator, namely the object A itself. It then follows by a general result that $A - \mathbf{al.Mod}$ has enough injectives. It is also clear that $A - \mathbf{al.Mod}$ has enough almost projective (resp. flat) objects. Given an almost A-module M, we can derive the functors $M \otimes_A -$ (resp. $\mathrm{alHom}_A(M, -)$, resp. $\mathrm{alHom}_A(-, M)$) by taking flat (resp. injective, resp. almost projective) resolutions (one remarks that bounded above exact complexes of flat (resp. almost projective) almost modules are acyclic for the functor $M \otimes_A -$ (resp. $\mathrm{alHom}_A(-, M)$), and then uses the construction detailed in [8] th. 10.5.9). We denote by $\mathrm{Tor}_i^A(M, -)$ (resp. $\mathrm{alExt}^i_A(M, -)$, resp. $\mathrm{alExt}^i_A(-, M)$) the corresponding derived functors. If $A = B^a$ for some V-algebra B we obtain easily natural isomorphisms

$$\mathrm{Tor}_i^B(M, N)^a \simeq \mathrm{Tor}_i^A(M^a, N^a)$$

for all B-modules M, N. A similar result holds for $\mathrm{Ext}^i_B(M, N)$.

Remark 2.2.4. i) Clearly, an almost A-module M is flat (resp. almost projective) if and only if $\mathrm{Tor}_i^A(M, N) = 0$ (resp. $\mathrm{alExt}^i_A(M, N) = 0$) for all almost A-modules N and all $i > 0$.

ii) Let M, N be two flat (resp. almost projective) almost A-modules. Then $M \otimes_A N$ is a flat (resp. almost projective) A-module and for any almost B-algebra A, the almost B-module $B \otimes_A M$ is flat (resp. almost projective).

Lemma 2.2.5. *(see* [3]*) Let M be an almost finitely generated almost A-module. Consider the following properties:*

i) M is almost projective.

ii) For arbitrary $\varepsilon \in \mathfrak{m}$ there exist $n(\varepsilon) \in \mathbb{N}$ and A-linear morphisms

$$M \xrightarrow{u_\varepsilon} A^{n(\varepsilon)} \xrightarrow{v_\varepsilon} M \tag{2.2.6}$$

such that $v_\varepsilon \circ u_\varepsilon = \varepsilon \cdot \mathbf{1}_M$.

iii) M is flat.

Then (i) ⇔ (ii) ⇒ (iii). □

There is a converse to lemma 2.2.5 in case M is almost finitely presented. To prove it, we need some preparation.

Let R be a ring and M any R-module. Recall (see [8] (3.2.3)) that the *Pontrjagin dual* of M is the R-module $M^\dagger = \mathrm{Hom}_{\mathbf{Ab}}(M, \mathbb{Q}/\mathbb{Z})$. An element r of R acts on $M^\dagger$ via $(r \cdot f)(m) = f(r \cdot m)$ (for $f \in M^\dagger$, $m \in M$).

Proposition 2.2.7. *(cp.* [8] *§3.2)*

i) The Pontrjagin dual is a contravariant exact functor on the category $R - \mathbf{Mod}$.

ii) A sequence $M \to N \to P$ of morphisms of R-modules is exact if and only if the dual sequence $P^\dagger \to N^\dagger \to M^\dagger$ is exact.

iii) For any two R-modules M, N there is a natural R-linear morphism $\sigma : M^\dagger \otimes_R N \to \mathrm{Hom}_R(N, M)^\dagger$, defined by $\sigma(f \otimes n) : h \mapsto f(h(n))$, which is an isomorphism if N is finitely presented. □

Corollary 2.2.8. *Let M be a finitely presented R-module and $r \in R$ an element such that $r \cdot \mathrm{Tor}_i^R(M, N) = 0$ for any R-module N and any integer $i > 0$. Then we have also $r \cdot \mathrm{Ext}^i_R(M, N) = 0$ for any R-module N and any integer $i > 0$.*

Proof. Suppose we are given a surjection $\phi : B \to C$ of R-modules. It suffices to show that r kills the cokernel of the induced morphism $\phi_* : \mathrm{Hom}_R(M, B) \to \mathrm{Hom}_R(M, C)$. By proposition 2.2.7(i) the map $C^\dagger \to B^\dagger$ is injective. We consider the commutative diagram

$$\begin{array}{ccc} C^\dagger \otimes_R M & \longrightarrow & B^\dagger \otimes_R M \\ \downarrow & & \downarrow \\ \mathrm{Hom}_R(M, C)^\dagger & \longrightarrow & \mathrm{Hom}_R(M, B)^\dagger \end{array}$$

whose vertical arrows are isomorphisms according to 2.2.7(iii). By hypothesis, r kills the kernel of the upper horizontal arrows, hence also the kernel of the lower one. By proposition 2.2.7(i) this kernel is the Pontrjagin dual of $D = \mathrm{Coker}(\phi_*)$ and by proposition 2.2.7(ii) it follows easily that r must kill D itself, as required. □

Proposition 2.2.9. *If (***S1***) holds then every almost finitely presented flat almost A-module is almost projective.*

Proof. Let M be such an almost A-module. For $\varepsilon \in S$ pick morphisms $A^m \stackrel{\psi}{\to} A^n \stackrel{\phi}{\to} M$ such that $\varepsilon \cdot \mathrm{Coker}(\phi) = \varepsilon \cdot (\mathrm{Ker}(\phi)/\mathrm{Im}(\psi)) = 0$. Let $C = \mathrm{Coker}(\psi_* : A_*^m \to A_*^n)$. The kernel and cokernel of the natural morphism $C \to M_*$ are killed by ε^2. Moreover, by hypothesis, ε kills $\mathrm{Tor}_i^{A_*}(M_*, N)$ for all A_*-modules N and all $i > 0$. It follows easily that ε^5 kills $\mathrm{Tor}_i^{A_*}(C, N)$ for all A_*-modules N and all $i > 0$. Then it follows from corollary 2.2.8 that ε^5 also kills $\mathrm{Ext}^i_{A_*}(C, N)$ for all A_*-modules N and all $i > 0$. In turns, this implies that ε^9 kills $\mathrm{Ext}^i_{A_*}(M_*, N)$ for all A_*-modules N and all $i > 0$. As ε can be taken arbitrarily small, the claim follows. □

For the proof of the following two lemmata, we refer the reader to [3].

Lemma 2.2.10. *Assume (***S2***) and let $\{M_n \,;\, \phi_n : M_n \to M_{n+1} \mid n \in \mathbb{N}\}$ be a direct system of almost A-modules and suppose there exist sequences $T = \{\varepsilon_n \mid n \in \mathbb{N}\}$ and $\{\delta_n \mid n \in \mathbb{N}\}$ of elements of S such that*

i) T converges to 1 and $\{\prod_{j=0}^n \delta_j \mid n \in \mathbb{N}\}$ is a Cauchy sequence;

ii) for all $n \in \mathbb{N}$ there exist integers $N(n)$ and morphisms of almost A-modules $\psi_n : A^{N(n)} \to M_n$ such that $\varepsilon_n \cdot \mathrm{Coker}(\psi_n) = 0$;

iii) $\delta_n \cdot \mathrm{Coker}(\phi_n) = 0$ for all $n \in \mathbb{N}$.

Then $\operatorname*{colim}_{n\in\mathbb{N}} M_n$ is an almost finitely generated almost A-module. □

Lemma 2.2.11. *Assume (***S2***) and let $\{M_n \,;\, \phi_n : M_n \to M_{n+1} \mid n \in \mathbb{N}\}$ be a direct system of almost A-modules and suppose there exist sequences $T = \{\varepsilon_n \mid n \in \mathbb{N}\}$ and $\{\delta_n \mid n \in \mathbb{N}\}$ of elements of S such that*

i) T converges to 1 and $P = \{\prod_{j=0}^n \delta_j \mid n \in \mathbb{N}\}$ is a Cauchy sequence;

ii) $\varepsilon_n \cdot \mathrm{alExt}^i_A(M_n, N) = \delta_n \cdot \mathrm{alExt}^i_A(\mathrm{Coker}(\phi_n), N) = 0$ for all almost A-modules N, all $i > 0$ and all $n \in \mathbb{N}$;

iii) $\delta_n \cdot \mathrm{Ker}(\phi_n) = 0$ for all $n \in \mathbb{N}$.

Then $\operatorname*{colim}_{n\in\mathbb{N}} M_n$ is an almost projective almost A-module. □

2.3. **Almost homotopical algebra.** Our first task is to extend all of section 2.1 to simplicial almost modules and algebras. This requires no particular effort, so we only sketch how to proceed. A simplicial almost V-algebra is just an object in the category $s.(V - \mathbf{al.Alg})$. Then, for a given simplicial algebra A, we define the category $A - \mathbf{al.Mod}$ of almost A-modules : it consists of all simplicial almost V-modules M such that $M[n]$ is an almost $A[n]$-module and such that the face and degeneracy morphisms $d_i : M[n] \to M[n-1]$ and $s_i : M[n] \to M[n+1]$ $(i = 0, 1, ..., n)$ are $A[n]$-linear. For instance, if A is an almost V-algebra, we can form the constant simplicial almost V-algebra $s.A$ and then the category $s.A - \mathbf{al.Mod}$ is the same thing as the category $s.(A - \mathbf{al.Mod})$ of simplicial almost A-modules. Sometimes we may have to consider, for a given simplicial almost algebra A, the category $s.(A - \mathbf{al.Mod})$ of simplicial almost A-modules. This is the same as the category of all bisimplicial complexes of almost V-modules, with additional $A[n]$-linearity conditions which the reader can easily figure out. The category $A - \mathbf{al.Mod}$ is abelian and it is even a monoidal category with the tensor product formed dimension-wise: $(M \otimes_A N)[n] = M[n] \otimes_{A[n]} M[n]$. The internal hom functor $s.\mathrm{alHom}_A(-,-) : A - \mathbf{al.Mod}^o \times A - \mathbf{al.Mod} \to A - \mathbf{al.Mod}$ can be defined as

$$s.\mathrm{alHom}_A(X,Y)[n] = \mathrm{Hom}_{A-\mathbf{al.Mod}}(X \otimes_V V^{\Delta(n)}, Y)^a$$

with face morphisms induced naturally from those of $\Delta(n)$. Here $V^{\Delta(n)}$ denotes the simplicial almost V-module obtained by applying dimension-wise the free almost module functor to the simplicial set $\Delta(n)$.

Again, an almost A-algebra B is a monoid in the category of almost A-modules, which is the same as a simplicial almost V-algebra with a morphism of simplicial almost V-algebras $A \to B$. We let $A - \mathbf{al.Alg}$ be the category of these objects.

Next, by applying dimension-wise the functor of almost elements we may form the simplicial V-algebra A_*. The functor of almost elements then extends to a functor $A - \mathbf{al.Mod} \to A_* - \mathbf{Mod}$ (and similarly for almost algebras). Then the whole of proposition 2.1.7 and its corollary carries over to this more general situation.

Similarly we have forgetful functors $A_* - \mathbf{Mod} \to s.\mathbf{Set}$ (resp. $A_* - \mathbf{Alg} \to s.\mathbf{Set}$) and their adjoints associate to a simplicial set S the simplicial free module (resp.algebra) generated dimension-wise by $S[n]$. These can be composed with the previous functors to give constructions of free simplicial almost A-modules $A^{(S)}$ and A-algebras $A[S]$.

In the following we let A be some fixed simplicial almost V-algebra. Our next task is to extend parts of [4] to the almost case, leading up to the construction of the almost cotangent complex in the coming section. Most of the work consists in poring over Illusie's manuscript and accepting that the arguments given there carry through *verbatim* to the almost setting. This is usually the case, with few exceptions mostly due to the fact that the functor of almost elements is only left exact (rather than exact). We will point out these potential difficulties as they arise.

Definition 2.3.1. Let B be any almost A-algebra and M any almost B-module. An *almost A-derivation* of B with values in M is an A-linear morphism $\partial : B \to M$ such that

$$\partial_{[n]*}(b_1 \cdot b_2) = b_1 \cdot \partial_{[n]*}(b_2) + b_2 \cdot \partial_{[n]*}(b_1)$$

for all objects $[n]$ of Δ and all $b_1, b_2 \in B_*[n]$. The set of all M-valued almost A-derivations of B forms a V-module $\mathrm{Der}_A(B,M)$ and the corresponding almost V-module $\mathrm{alDer}_A(B,M)$ has a natural structure of almost $B[0]$-module (via the degeneracy morphisms).

Definition 2.3.2. Let again B be an almost A-algebra. An *ideal* of B is a monomorphism $I \to B$ of almost B-modules. We reserve the notation $I_{B/A}$ for the ideal $\mathrm{Ker}(\mu_{B/A} : B \otimes_A B \to B)$. The *almost module of relative differentials* of ϕ is defined as the (left) almost B-module $\Omega_{B/A} = I_{B/A}/I^2_{B/A}$. It is endowed with a natural almost A-derivation

$$\delta : B \to \Omega_{B/A}$$

which is defined by $b \mapsto \underline{1} \otimes b - b \otimes \underline{1}$ for all $b \in B_*$. The assignment $(A \to B) \mapsto \Omega_{B/A}$ defines a functor

$$\Omega : s.\mathbf{al.Alg.Morph} \to s.\mathbf{al.Alg.Mod}$$

from the category of morphisms $A \to B$ of simplicial almost V-algebras to the category denoted $s.\mathbf{al.Alg.Mod}$ consisting of all pairs (B,M) where B is a simplicial almost V-algebra and M is an almost B-module. The morphisms in $s.\mathbf{al.Alg.Morph}$ are all the commutative squares;

the morphisms $(B, M) \to (B', M')$ in $s.\mathbf{al}.\mathbf{Alg}.\mathbf{Mod}$ are all pairs (ϕ, f) where $\phi : B \to B'$ is a morphism of simplicial almost V-algebras and $f : B' \otimes_B M \to M'$ is a morphism of almost B'-modules.

The almost module of relative differentials enjoys the familiar universal properties which one expects. In particular $\Omega_{B/A}$ represents the functor $\mathrm{Der}_A(B, -)$, *i.e.* for any (left) almost B-module M the morphism

$$\mathrm{Hom}_{B-\mathbf{al.Mod}}(\Omega_{B/A}, M) \to \mathrm{Der}_A(B, M) \qquad f \mapsto f \circ \delta \tag{2.3.3}$$

is an isomorphism. As an exercise, the reader can supply the proof for this claim and for the following standard proposition.

Proposition 2.3.4. *i) Let B and C two almost A-algebras. Then there is a natural isomorphism $\Omega_{C\otimes_A B/C} \simeq C \otimes_A \Omega_{B/A}$.*

ii) Let B be an almost A-algebra and C an almost B-algebra. Then there is a natural exact sequence of almost C-modules

$$C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to \Omega_{C/B} \to 0.$$

iii) Let I be an ideal of the almost A-algebra B and let $C = B/I$ be the quotient almost A-algebra. Then there is a natural exact sequence

$$I/I^2 \to C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to 0. \tag{2.3.5}$$

iv) The functor $\Omega : s.\mathbf{al}.\mathbf{Alg}.\mathbf{Morph} \to s.\mathbf{al}.\mathbf{Alg}.\mathbf{Mod}$ commutes with all colimits. □

Definition 2.3.6. An *A-extension* of an almost A-algebra B by an almost B-module I is a short exact sequence of almost A-modules

$$X: \qquad 0 \longrightarrow I \longrightarrow C \xrightarrow{p} B \longrightarrow 0$$

such that C is an almost A-algebra, p is a morphism of almost A-algebras and I is a square zero ideal in C. The A-extensions form a category $\mathbf{Exal}_A$. The morphisms are commutative diagrams with exact rows

$$\begin{tikzcd}
X: \arrow[d] & 0 \arrow[r] & I \arrow[r] \arrow[d, "f"] & E \arrow[r, "p"] \arrow[d, "g"] & B \arrow[r] \arrow[d, "h"] & 0 \\
X': & 0 \arrow[r] & I' \arrow[r] & E' \arrow[r, "p'"] & B' \arrow[r] & 0
\end{tikzcd}$$

such that g and h are morphisms of almost A-algebras. We denote by $\mathbf{Exal}_A(B, I)$ the subcategory of $\mathbf{Exal}_A$ consisting of all A-extensions of B by I, where the morphisms are all short exact sequences as above such that $f = \mathbf{1}_I$ and $h = \mathbf{1}_B$.

For a morphism $\phi : C \to B$ of almost A-algebras, and an A-extension X in $\mathbf{Exal}_A(B, I)$, we can pullback X via ϕ to obtain an A-extension $X * \phi$ in $\mathbf{Exal}_A(C, I)$ with a morphism $X * \phi \to X$ of A-extensions. Similarly, given a B-linear morphism $\psi : I \to J$, we can push out X and obtain an object $\psi * X$ in $\mathbf{Exal}_A(B, J)$ with a morphism $X \to \psi * X$ of $\mathbf{Exal}_A$. These operations can be used to induce a structure of abelian group on the set $\mathrm{Exal}_A(B, I)$ of isomorphism classes of objects of $\mathbf{Exal}_A(B, I)$ as follows. For any two objects X, Y of $\mathbf{Exal}_A(B, I)$ we can form $X \oplus Y$ which is an object of $\mathbf{Exal}_A(B \oplus B, I \oplus I)$. Let $\phi : B \to B \oplus B$ be the diagonal morphism and $\psi : I \oplus I \to I$ the addition morphism of I. Then we set $X + Y = \psi * (X \oplus Y) * \phi$. One can check that $X + Y \simeq Y + X$ for any X, Y and that the trivial split A-extension $B \oplus I$ is a neutral element for $+$. Moreover every isomorphism class has an inverse $-X$. The functors $X \mapsto X * \phi$ and $X \mapsto \psi * X$ commute with the operation thus defined, and induce group homomorphisms

$$*\phi : \mathrm{Exal}_A(B, I) \to \mathrm{Exal}_A(C, I) \text{ and } \psi* : \mathrm{Exal}_A(B, I) \to \mathrm{Exal}_A(B, J).$$

Definition 2.3.7. (cp. [4] (III.1.1.7)) We say that an almost A-algebra C verifies condition (L) if, for all A-extensions $X = (0 \to I \to B \to C \to 0)$, the sequence of C-modules

$$0 \to I \to C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to 0 \tag{2.3.8}$$

obtained by extending (2.3.5) by zero on the left, is exact.

Suppose now that C verifies (L). Then we denote by diff(X) the exact sequence (2.3.8) associated to the A-extension X. This defines a functor from $\mathbf{Exal}_A(C, I)$ to the category of extensions (in the category of almost C-modules) of $\Omega_{C/A}$ by I. Hence we derive a morphism of abelian groups

$$\text{diff} : \text{Exal}_A(C, I) \to \text{Ext}^1_{C-\mathbf{al.Mod}}(\Omega_{C/A}, I)$$

where Ext^1 denotes here the Yoneda Ext functor on the abelian category of almost C-modules.

Conversely, let C be any almost A-algebra and $Y = (0 \to I \to J \to \Omega_{C/A} \to 0)$ an exact sequence of almost C-modules. We deduce an A-extension of $(C \otimes_A C)/I^2_{C/A} \simeq C \oplus \Omega_{C/A}$

$$Y' : \qquad 0 \to I \to C \oplus J \to C \oplus \Omega_{C/A} \to 0$$

where J is also a square zero ideal in $C \oplus J$ and I is an almost $C \oplus \Omega_{C/A}$-module annihilated by the ideal $\Omega_{C/A}$. Let $j_2 : C \to (C \otimes_A C)/I^2_{C/A}$ be the morphism $c \mapsto 1 \otimes c$ ($c \in C_*$). Then we obtain an A-extension of C by I

$$\text{alg}(Y) = Y' * j_2.$$

This defines a group homomorphism

$$\text{alg} : \text{Ext}^1_{C-\mathbf{al.Mod}}(\Omega_{C/A}, I) \to \text{Exal}_A(C, I). \tag{2.3.9}$$

The basic observation is the following:

Proposition 2.3.10. *(cp.* [4] *(III.1.1.9)) Suppose that the almost A-algebra C verifies condition (L). Then the homomorphisms* diff *and* alg *are inverse to each other.* □

Our aim is to extend proposition 2.3.10 to general almost A-algebras. The strategy will be the following: find a suitable (functorial) simplicial resolution $P \to C$ by a system of A-algebras $P[n]$ satisfying condition (L). Then form the simplicial C-module $\mathbb{L}_{C/A} = \Omega_{P/A} \otimes_P C$. This will replace the C-module $\Omega_{C/A}$ in (2.3.9). Moreover the Yoneda Ext will be replaced by a hyperext functor, computed in a suitable derived category. This program will occupy the rest of this section and the following one. To start out, we verify that there are enough A-algebras to play with, which satisfy (L).

Lemma 2.3.11. *Assume that* (**S2**) *holds and suppose that $C = A[S]$ for some simplicial set S. Then C satisfies (L).*

Proof. This is one of the points which require a little extra care. Let $X = (0 \to I \to B \xrightarrow{f} C \to 0)$ be an A-extension. We need to show that the sequence (2.3.8) is exact. By virtue of proposition 2.3.4(iv) we can assume that S is a simplicial finite set (*i.e.* all $S[n]$ are finite sets). By applying termwise the functor of almost elements to X we obtain a sequence X_*. However the morphism f_* will not be in general surjective, but rather only almost surjective. Now pick $\varepsilon \in S$ and consider the morphism $\phi_\varepsilon : A_*[S] \to A[S]_*$ which sends a generator $s \in S[n] \subset A_*[S[n]]$ to the element $\varepsilon \cdot s \in A[S[n]]_*$. Let $X_{\varepsilon *} = X_* * \phi_\varepsilon$. Then $X_{\varepsilon *}$ is an A_*-extension of $A_*[S]$ by I. According to [4] (III.1.1.7.1) the almost A_*-algebra $A_*[S]$ satisfies the analogous condition (L) for A_*-algebras. Hence we can form the exact sequence of $A_*[S]$-modules diff$(X_{\varepsilon *})$. Let $D_\varepsilon = \text{diff}(X_{\varepsilon *})^a$ be the exact sequence of $A[S]$-modules obtained by applying termwise the functor $M \mapsto M^a$. Because of (**S2**) the sequences D_ε form a direct system for varying $\varepsilon \in S$ and moreover the colimit of the system is naturally isomorphic to the sequence (2.3.8). As a filtered colimit of exact sequences of almost modules is again exact, the claim follows. □

Next, we construct a few auxiliary derived categories, for the future complex $\mathbb{L}_{B/A}$ to live in. A bit more generally, let C be any abelian category. Recall (see [4] (I.1.3)) the construction of the *normalized complex* which associates to every object X of $s.C$ a chain complex $N(X)^\bullet$ defined by

$$N(X)_n = \bigcap_{i>0} \mathrm{Ker}(d_i : X[n] \to X[n-1]). \tag{2.3.12}$$

(We have used the standard convention $N(X)_n = N(X)^{-n}$). It is an object of the category $\mathsf{C}_\bullet(C)$ of chain complexes $M^\bullet$ of objects of C such that $M^i = 0$ for all $i > 0$. The theorem of Dold-Kan (see [8] th.8.4.1) states that $X \mapsto N(X)$ induces an equivalence

$$N : s.C \to \mathsf{C}_\bullet(C)$$

and in fact an explicit essential inverse to N can be produced. Now we say that a morphism $X \to Y$ in $s.C$ is a *quasi-isomorphism* if the induced morphism of $N(X) \to N(Y)$ is a quasi-isomorphism of chain complexes.

Definition 2.3.13. Let A be a simplicial almost V-algebra. We say that a morphism $\phi : M \to N$ of almost A-modules (or almost A-algebras) is a *quasi-isomorphism* if the morphism ϕ of underlying simplicial almost V-modules is a quasi-isomorphism. We define the category $\mathsf{D}_\bullet(A)$ (resp. $\mathsf{D}_\bullet(A - \mathbf{al}.\mathbf{Alg})$) as the localization of the category $A - \mathbf{al}.\mathbf{Mod}$ (resp. $A - \mathbf{al}.\mathbf{Alg}$) with respect to the class of quasi-isomorphisms. (We assume that our universe is large enough to accommodate this kind of constructions).

If A is an almost V-algebra, then the functor N induces an equivalence from $\mathsf{D}_\bullet(s.A)$ to the derived category $\mathsf{D}_\bullet(A - \mathbf{al}.\mathbf{Mod})$ which is the localization of $\mathsf{C}_\bullet(A - \mathbf{al}.\mathbf{Mod})$ with respect to the class of quasi-isomorphisms. Many tools that are available for derived categories have adequate simplicial counterparts. For instance we have a *suspension functor* $\Sigma : A - \mathbf{al}.\mathbf{Mod} \to A - \mathbf{al}.\mathbf{Mod}$ (for any simplicial V-algebra A). We recall the definition from [4] (I.3.2.1). Let $d_0, d_1 : \Delta(0) \rightrightarrows \Delta(1)$ be the two natural simplicial maps and set $\sigma = V^{\Delta(1)}/(\mathrm{Im}V^{(d_0)} + \mathrm{Im}V^{(d_1)})$. This is a simplicial almost V-module whose normalised chain complex is

$$N(\sigma) = (0 \to V \to 0)$$

where the component V is placed in homological degree 1. Then set $\Sigma M = \sigma \otimes_V M$ for any almost A-module M. It follows from the Eilenberg-Zilber theorem (see [8] th.8.5.1) that the simplicial almost V-module underlying $N(\Sigma M)$ decomposes as a direct sum $K(M) \oplus N(M)[1]$ where $K(M)$ is a contractible complex depending functorially on M (and where $F \mapsto F[1]$ denotes the usual shift operator on the category $\mathsf{C}_\bullet(V - \mathbf{al}.\mathbf{Mod})$). Similarly we have a *cone functor* on almost A-modules, defined as $E \mapsto C(E) = \gamma \otimes_V E$ where γ is the simplicial V-module $V^{\Delta(1)}/\mathrm{Im}V^{(d_0)}$. The morphism $d_1 : \Delta(0) \to \Delta(1)$ induces a morphism $i_E : E \to C(E)$ and there is a short exact sequence

$$0 \longrightarrow E \xrightarrow{i_E} C(E) \xrightarrow{p_E} \Sigma E \longrightarrow 0.$$

With this notation we can now introduce the hyperext functor which will replace Yoneda Ext. It is defined as

$$\mathbb{E}\mathsf{xt}^p_A(E, F) = \operatorname*{colim}_{n \geq -p} \mathrm{Hom}_{\mathsf{D}_\bullet(A)}(\Sigma^n E, \Sigma^{n+p} F)$$

for any two almost A-modules E, F and for any integer p. It turns out that the computation of $\mathbb{E}\mathsf{xt}^1$ can be reduced to some extent, to the computation of certain related Yoneda Ext^1 groups. We are going to describe how this is accomplished.

First, for a morphism $u : E \to F$ of almost A-modules we define the cone $C(u)$ via the push-out diagram

$$\begin{tikzcd} E \arrow[r, "i_E"] \arrow[d, "u"] & C(E) \arrow[d] \\ F \arrow[r] & C(u). \end{tikzcd}$$

We derive a sequence of morphisms of almost A-modules

$$E \xrightarrow{u} F \longrightarrow C(u) \xrightarrow{p_u} \Sigma E \tag{2.3.14}$$

where the morphism $p_u : C(u) \to \Sigma E$ is induced, via the universal property of the push-out, by the morphism p_E and the zero morphism $F \to \Sigma E$.

Definition 2.3.15. We call a *triangle* of $\mathsf{D}_\bullet(A)$ any sequence of morphisms in $\mathsf{D}_\bullet(A)$

$$L \to M \to N \to \Sigma L.$$

A morphism from a triangle $(L \to M \to N \to \Sigma L)$ to a triangle $(L' \to M' \to N' \to \Sigma L')$ is a sequence of morphisms $f : L \to L'$, $g : M \to M'$ and $h : N \to N'$ such that the diagram

$$\begin{tikzcd} L \arrow[r] \arrow[d, "f"] & M \arrow[r] \arrow[d, "g"] & N \arrow[r] \arrow[d, "h"] & \Sigma L \arrow[d, "\Sigma f"] \\ L' \arrow[r] & M' \arrow[r] & N' \arrow[r] & \Sigma L' \end{tikzcd}$$

commutes. We say that a triangle is *distinguished* if it is isomorphic to a triangle of the kind (2.3.14).

Now let $X = (0 \to E \xrightarrow{u} F \to G \to 0)$ be a short exact sequence of almost A-modules. One shows (see [4] (I.3.2.3)) that X induces a natural quasi-isomorphism $s_X : C(u) \to G$ and moreover the triangle

$$E \xrightarrow{u} F \longrightarrow G \xrightarrow{p_u \circ s_X^{-1}} \Sigma E$$

is distinguished. We set

$$\chi(X) = (p_u \circ s_X^{-1} : G \to \Sigma E) \in \mathrm{Hom}_{\mathsf{D}_\bullet(A)}(G, \Sigma E).$$

It is easy to check that $\chi(X)$ depends only on the class of X in the Yoneda Ext^1 group and moreover, for any morphisms $f : E \to E'$, $g : G' \to G$ we have

$$\chi(f * X) = (\Sigma f) \circ \chi(X) \qquad \chi(X * g) = \chi(X) \circ g$$

which means that χ induces a group homomorphism

$$\chi : \mathrm{Ext}^1_{A-\mathbf{al.Mod}}(G, E) \to \mathbb{E}\mathsf{xt}^1_A(G, E)$$

functorial in both variables E, G. Finally we have the following

Proposition 2.3.16. *(cp.* [4] *(I.3.2.3.8)) Let E, F be two almost A-modules.*

i) for any $u \in \mathbb{E}\mathsf{xt}^1_A(E, F)$ there is a quasi-isomorphism $t : F \to F'$ (resp. $s : E' \to E$) and an extension X (resp. Y) of E by F' (resp. of E' by F) such that $(\Sigma t)^{-1} \circ \chi(X) = u$ (resp. $\chi(Y) \circ s^{-1} = u$).

ii) Let $t' : F \to F'$ (resp; $t'' : F \to F''$) be a quasi-isomorphism and X' (resp. X'') an extension of E by F' (resp. F''). We have $(\Sigma t')^{-1} \circ \chi(X') = (\Sigma t'')^{-1} \circ \chi(X'')$ if and only if there exist quasi-isomorphisms $r' : F' \to G$ and $r'' : F'' \to G$ such that $r' \circ t'$ is homotopic to $r'' \circ t''$ and $r' * X' = r'' * X''$ in $\mathrm{Ext}^1_{A-\mathbf{al.Mod}}(E, G)$.

iii) Let $s' : E' \to E$ (resp. $s'' : E'' \to E$) be a quasi-isomorphism and Y' (resp. Y'') an extension of E' (resp. E'') by F. We have $\chi(Y') \circ (s')^{-1} = \chi(Y'') \circ (s'')^{-1}$ if and only if there *exist quasi-isomorphisms* $r' : D \to E'$ *and* $r'' : D \to E''$ *such that* $s' \circ r'$ *is homotopic to* $s'' \circ r''$ *and* $Y' * r' = Y'' * r''$ *in* $\mathrm{Ext}^1_{A-\mathbf{al.Mod}}(D, F)$. □

2.4. **Almost cotangent complex.** Let $A \to B$ be a morphism of almost V-algebras. In this section we complete the program announced in section 2.3 : first of all we construct a natural simplicial resolution of B by almost A-algebras which satisfy condition (L) of definition 2.3.7. This is just the simplicial almost A-algebra $P_A(B) = P_{A_*}(B_*)^a$ where $P_{A_*}(B_*)$ is the simplicial A_*-algebra associated to the morphism $A_* \to B_*$ as in [4] (II.1.2.1). It comes with a natural augmentation

$$P = P_A(B) \stackrel{p}{\to} B$$

which induces a quasi-isomorphism of simplicial almost A-algebras $P \to s.B$. The components of P are free almost A-algebras, which therefore, by lemma 2.3.11, satisfy condition (L) whenever (**S2**) holds. In this case P itself satisfies (L).

Definition 2.4.1. The *almost cotangent complex* of B over A is the simplicial almost B-module

$$\mathbb{L}_{B/A} = B \otimes_P \Omega_{P/A}.$$

For any morphism $C \to D$ of rings, let $\mathbb{L}_{D/C}$ be the cotangent complex defined by Illusie. Then obviously $\mathbb{L}_{B/A} = \mathbb{L}^a_{B_*/A_*}$. It is clear that $\Omega_{P/A}$ is a dimension-wise free almost P-module, in particular it is flat. The augmentation p induces an augmentation

$$\mathbb{L}_{B/A} \to \Omega_{B/A}$$

and by paraphrasing the argument in [4] we obtain the following

Proposition 2.4.2. *(cp.* [4] *(II.1.2.4.2)) The natural morphism*

$$H_0(\mathbb{L}_{B/A}) \to \Omega_{B/A}$$

is an isomorphism. □

Corollary 2.4.3. *Let M be an almost B-module. There exists a natural isomorphism*

$$\mathbb{E}\mathrm{xt}^0_B(\mathbb{L}_{B/A}, M) \simeq \mathrm{Der}_A(B, M).$$

□

Finally we can return to our chief preoccupation, which is to extend proposition 2.3.10. First we would like to extend the homomorphism diff. This is achieved as follows. Let X be an A-extension of B by an almost B-module M. We deduce an A-extension $X * p$ of P by M, hence, assuming (**S2**), an extension of P-modules diff$(X * p)$ of $\Omega_{P/A}$ by M and finally an element

$$\alpha(X) = \chi \circ \mathrm{diff}(X * p) \in \mathbb{E}\mathrm{xt}^1_P(\Omega_{P/A}, M).$$

On the other hand, as $\Omega_{P/A}$ is flat, [4] (I.3.3.4.4) yields natural isomorphisms

$$\mathbb{E}\mathrm{xt}^n_P(\Omega_{P/A}, M) \simeq \mathbb{E}\mathrm{xt}^n_B(\mathbb{L}_{B/A}, M). \tag{2.4.4}$$

Theorem 2.4.5. *(cp.* [4] *(III.1.2.3) Assume (*__S2__*). Then the natural homomorphism*

$$\mathrm{Exal}_A(B, M) \to \mathbb{E}\mathrm{xt}^1_B(\mathbb{L}_{B/A}, M)$$

obtained by composing α and the isomorphism (2.4.4) *is an isomorphism.*

Proof. The proof is just the transcription of Illusie's argument. Therefore we just outline how to construct an inverse for α and leave it at that. Let $y \in \mathbb{E}\mathrm{xt}^1_P(\Omega_{P/A}, M)$. According to proposition 2.3.16(i) there exists a quasi-isomorphism of P-modules $s : M \to N$ and an extension Y of $\Omega_{P/A}$ by N such that $y = (\Sigma s)^{-1} \circ \chi(Y)$. Thus we obtain an A-extension alg(Y) of P by N. As P is acyclic in degre > 0, we deduce, by applying H_0 termwise, an A-extension $H_0(\mathrm{alg}(Y))$ of $H_0(P)$ by $H_0(N)$, and finally an A-extension

$$\beta(y) = H_0(s)^{-1} * H_0(\mathrm{alg}(Y)) * H_0(p)^{-1} \in \mathrm{Exal}_A(B, M).$$

One shows using proposition 2.3.16(ii) that the result is independent of the choices of s and Y. An explicit calculation using proposition 2.3.10 then shows that β is an inverse for α. □

We proceed now to list the other main properties of the almost cotangent complex.

Theorem 2.4.6. *Let $A \to B \to C$ be a sequence of morphisms of almost V-algebras. There exists a natural distinguished triangle of* $\mathsf{D}_{\bullet}(C)$

$$C \otimes_B \mathbb{L}_{B/A} \xrightarrow{u} \mathbb{L}_{C/A} \xrightarrow{v} \mathbb{L}_{C/B} \longrightarrow C \otimes_B \Sigma\mathbb{L}_{B/A}$$

where the morphisms u and v are obtained by functoriality of $\mathbb{L}$.

Proof. It follows directly from [4] (II.2.1.2). □

Finally we have a fundamental spectral sequence as in [4] (III.3.3.2). It goes as follows. For every integer $q \geq 0$ let $\mathrm{Sym}^q_B : B - \mathbf{al.Mod} \to B - \mathbf{al.Mod}$ be the non-additive functor which sends an almost B-module M to its q-th symmetric tensor power. We can extend Sym^q_B to a functor $s.B - \mathbf{al.Mod} \to s.B - \mathbf{al.Mod}$ by applying it termwise to the components of a simplicial B-module.

Theorem 2.4.7. *(cp.*[4] *(III.3.3.2)) Let $\phi : R \to S$ be a morphism of V-algebras such that $S \otimes_R S \simeq S$ (*e.g. *such that S is a quotient of R). Let $A = R^a$ and $B = S^a$. Then there is a first quadrant homology spectral sequence of bigraded almost algebras*

$$E^2_{pq} = H_{p+q}(\mathrm{Sym}^q_B(\mathbb{L}^a_{S/R})) \Rightarrow \mathrm{Tor}^A_{p+q}(B, B).$$

□

Corollary 2.4.8. *Under the assumptions of theorem 2.4.7 there is a five term exact sequence*

$$\mathrm{Tor}^A_3(B,B) \to H_3(\mathbb{L}^a_{S/R}) \to \Lambda^2_B H_1(\mathbb{L}^a_{S/R}) \to \mathrm{Tor}^A_2(B,B) \to H_2(\mathbb{L}^a_{S/R}) \to 0.$$

□

3. Almost ring theory

3.1. **Flat, unramified and étale morphisms.** Let $A \to B$ be a morphism of almost V-algebras. Using the natural "multiplication" morphism of almost A-algebras $\mu_{B/A} : B \otimes_A B \to B$ we can see B as an almost $B \otimes_A B$-algebra.

Definition 3.1.1. Let $\phi : A \to B$ be a morphism of almost V-algebras.

i) We say that ϕ is a *flat (resp. almost projective) morphism* if B is a flat (resp. almost projective) almost A-module.

ii) We say that ϕ is *faithfully flat* if it is flat and, for any A-module M, the natural morphism of almost A-modules $\phi \otimes_A \mathbf{1}_M : A \otimes_A M \to B \otimes_A M$ is a monomorphism.

iii) We say that ϕ is *almost finite* if B is an almost finitely generated almost A-module.

iv) We say that ϕ is *unramified* if B is an almost projective almost $B \otimes_A B$-module (via the morphism $\mu_{B/A}$ defined above).

v) We say that ϕ is *étale* if it is flat and unramified.

Lemma 3.1.2. *(see* [3]*) Let $\phi : A \to B$ and $\psi : B \to C$ be morphisms of almost V-algebras.*

i) Any base change of a flat (resp. almost projective, resp. faithfully flat, resp. almost finite, resp. unramified, resp. étale) morphism is flat (resp. almost projective, resp. faithfully flat, resp. almost finite, resp. unramified, resp. étale);

ii) if both ϕ and ψ are flat (resp. almost projective, resp. faithfully flat, resp. almost finite, resp. unramified, resp. étale), then so is $\psi \circ \phi$;

iii) if ϕ is flat and $\psi \circ \phi$ is faithfully flat, then ϕ is faithfully flat;

iv) if ϕ is unramified and $\psi \circ \phi$ is flat (resp. étale), then ψ is flat (resp. étale). □

Recall the topological meaning of idempotents: if A is a ring and $e \in A$ satisfies the relation $e^2 = e$, then the ideal eA (resp. $(1-e)A$) is also a ring with the identity given by e (resp. by $1-e$). Then the natural A-linear morphism $eA \oplus (1-e)A \to A$ is in fact an isomorphism of rings, so that Spec(A) decomposes as the union of two disjoint open and closed subspaces.

In almost ring theory we will find useful to study certain "approximate idempotents", as in the following proposition, whose proof can be found in [3].

Proposition 3.1.3. *A morphism $\phi : A \to B$ is unramified if and only if there exists an almost A-element $e_{B/A} \in B \otimes_A B_*$ such that*
i) $e_{B/A}^2 = e_{B/A}$;
ii) $\mu_{B/A}(e_{B/A}) = \underline{1}$;
iii) $x \cdot e_{B/A} = 0$ for all $x \in I_{B/A*}$. □

Corollary 3.1.4. *Under the hypotheses and notation of the proposition the ideal $I = I_{B/A}$ has a natural structure of almost A-algebra, with unit morphism given by $\underline{1}_{I/A} = \underline{1}_{B\otimes_A B/A} - e_{B/A}$ and whose multiplication is the restriction of $\mu_{B\otimes_A B/A}$ to I. Moreover the natural morphism*

$$B \otimes_A B \to I_{B/A} \oplus B \qquad x \mapsto (x \cdot \underline{1}_{I/A} \oplus \mu_{B/A}(x))$$

is an isomorphism of almost A-algebras.

Proof. Left to the reader as an exercise. □

In order to manipulate idempotents we will need the following almost version of a well known lifting trick.

Lemma 3.1.5. *Assume (**S1**) and let $\phi : A \to B$ be an epimorphism of almost V-algebras such that $I = \mathrm{Ker}(\phi)$ is a nilpotent ideal, so that $I^m = 0$ for some positive integer m. Suppose that $\overline{e}$ is an idempotent almost element of B. Then there exists a unique lifting $e \in \phi^{-1}(\overline{e})$ which is also idempotent.*

Proof. Suppose first that $I^2 = 0$. Pick any $\varepsilon \in \mathfrak{m}$ and choose an element $x \in \phi^{-1}(\varepsilon \cdot \overline{e})$. Let $y = x - \varepsilon \cdot \underline{1}$. Clearly $x \cdot y \in I$. We write $\varepsilon^3 \cdot \underline{1} = (x+y)^3 = e_\varepsilon + e'_\varepsilon$ where

$$e_\varepsilon = x^3 + 3 \cdot x^2 \cdot y$$
$$e'_\varepsilon = 3 \cdot x \cdot y^2 + y^3.$$

It is easy to verify that $e_\varepsilon \cdot (\underline{1} - e_\varepsilon) = e_\varepsilon \cdot e'_\varepsilon = 0$, that is $e_\varepsilon^2 = \varepsilon^3 \cdot e_\varepsilon$. Moreover $\phi(e_\varepsilon) = \varepsilon^3 \cdot \overline{e}$. If $\delta \in \mathfrak{m}$ is any other element, we have

$$0 = (\delta^3 \cdot e_\varepsilon - \varepsilon^3 \cdot e_\delta)^3 = \delta^9 \cdot e_\varepsilon^3 - \varepsilon^9 \cdot e_\delta^3 + 3\delta^3 \cdot e_\varepsilon \cdot \varepsilon^6 \cdot e_\delta^2 - 3\delta^6 \cdot e_\varepsilon^2 \cdot \varepsilon^3 \cdot e_\delta$$

whence

$$\delta^9 \cdot e_\varepsilon^3 = \varepsilon^9 \cdot e_\delta^3. \tag{3.1.6}$$

Let us define a morphism $e : \mathfrak{m} \otimes_V \mathfrak{m} \to A_*$ as follows. For $x, y \in \mathfrak{m}$ choose $m_i \in V$ and $s_i \in S$ such that $y = \sum_i m_i \cdot s_i^9$ (under (**S1**) these can always be found). Then we set

$$x \otimes y \mapsto x \cdot \sum_i m_i \cdot e_{s_i}^3.$$

To check that e is well-defined, pick also a similar decomposition $x = \sum_j r_j \cdot q_j^9$ with $r_j \in V$, $q_j \in S$. We derive from (3.1.6)

$$x \cdot \sum_i m_i \cdot e_{s_i}^3 = \sum_{ij} r_j \cdot q_j^9 \cdot m_i \cdot e_{s_i}^3 = \sum_{ij} r_j \cdot m_i \cdot s_i^9 \cdot e_{t_j}^3 = \sum_j r_j \cdot y \cdot e_{q_j}^3$$

which shows that $e(x \otimes y)$ does not depend on the choice of decomposition for y. Similarly one proves that the map is bilinear in x and y. Then e defines an almost element of A such that

$\phi(e) = \overline{e}$ and the same kind of arguments using (3.1.6) shows that e is the unique idempotent of A_* with this property.

For the general case $I^m = 0$, one proceeds by induction, first lifting to A/I^{m-1} and then to A. $\square$

Theorem 3.1.7. *Assume (**S1**) and let $\phi : A \to B$ be an étale morphism of almost algebras. Then* $\mathrm{Exal}_A(B, M) = 0$ *for all almost B-modules M.*

Proof. Let $X = (0 \to M \xrightarrow{\iota} C \xrightarrow{\alpha} B \to 0)$ be an A-extension of B by the almost B-module M. We need to construct a splitting of almost A-algebras $s : B \to C$. As ϕ is flat, the diagram

$$B \otimes_A X = (0 \to B \otimes_A M \to B \otimes_A C \to B \otimes_A B \to 0)$$

is an A-extension of $B \otimes_A B$. Let $\sigma_M : B \otimes_A M \to M$ be the scalar multiplication morphism for the almost B-module M and set

$$Y = \sigma_M * (B \otimes_A X) = (0 \to M \xrightarrow{\iota'} D \xrightarrow{\alpha'} B \otimes_A B \to 0).$$

On $B \otimes_A M$ we have also the structure of almost $B \otimes_A B$-module mandated by (2.1.6). Then we have a well-defined structure of almost $B \otimes_A B$-module on M, characterized by requiring the morphism σ_M to be $B \otimes_A B$-linear. Let $\sigma'_M : (B \otimes_A B) \otimes_A M \to M$ be the corresponding scalar product; in conformity with (2.1.4) we have the equality :

$$\sigma'_M \circ (\mathbf{1}_{B \otimes_A B} \otimes_A \sigma_M) = \sigma_M \circ \sigma_{B \otimes_A M}. \tag{3.1.8}$$

Furthermore, let $e \in B \otimes_A B_*$ be the idempotent almost element provided by proposition 3.1.3. We define a B-linear morphism $j : B \to B \otimes_A B$ by $b \mapsto e \cdot (1 \otimes b)$ for all $b \in B_*$. Notice that j is in fact a morphism of non-unital almost V-algebras. Form the diagram $Y * j = (0 \to M \xrightarrow{\iota''} E \xrightarrow{\alpha''} B \to 0)$; this is an A-extension of non-unital almost V-algebras. We construct a splitting $B \to E$ as follows. Let $f \in D_*$ be the unique lifting of the idempotent e provided by lemma 3.1.5 and denote by $\pi : B \otimes_A C \to D$ the projection induced by σ_M. Define a morphism of non-unital almost V-algebras $\psi_B : B \to D$ by $b \mapsto f \cdot \pi(b \otimes 1)$ for all $b \in B_*$. Since $\alpha' \circ \psi_B = j$, the pair $(\psi_B, \mathbf{1}_B)$ induces a morphism of non-unital almost V-algebras $\omega : B \to E$ which splits α''. Notice that $\omega(1) = f$. We want to show that $Y * j$ is isomorphic to X in the category of A-extensions of non-unital almost V-algebras. To this purpose we need to construct a morphism of non-unital almost V-algebras $\beta : C \to E$ such that $\alpha'' \circ \beta = \alpha$ and $\beta \circ \iota = \iota''$. This is achieved as follows. Define a morphism $\psi_C : C \to D$ of non-unital almost V-algebras by $c \mapsto f \cdot \pi(1 \otimes c)$ ($c \in C_*$). We verify that $\alpha' \circ \psi_C = j \circ \alpha$. Indeed

$$\alpha' \circ \psi_C(c) = \alpha'(f \cdot \pi(1 \otimes c)) = e \cdot (1 \otimes \alpha(c)) = e \cdot (\alpha(c) \otimes 1) = j \circ \alpha(c).$$

Therefore the pair (ψ_C, α) determines a morphism $\beta : C \to E$ of non-unital almost V-algebras such that $\alpha'' \circ \beta = \alpha$. It remains to verify that $\beta \circ \iota = \iota''$. It suffices to show that $\psi_C \circ \iota = \iota'$. However we have $\psi_C \circ \iota(m) = f \cdot \pi(1 \otimes m) = e \cdot \sigma_M(1 \otimes m)$ because M is an almost $B \otimes_A B$-module with the scalar multiplication σ'_M introduced above. Then we compute using (3.1.8), (2.1.3) and (2.1.6) :

$$\begin{aligned}
e\cdot\sigma_M(1\otimes m) &= \sigma'_M(e\otimes\sigma_M(1\otimes m)) = \sigma_M\circ\sigma_{B\otimes_A M}(e\otimes 1\otimes m)\\
&= \sigma_M\circ(\mu_{B/A}\otimes\sigma_M)\circ(\mathbf{1}_B\otimes\eta_{B|B}\otimes\mathbf{1}_M)(e\otimes 1\otimes m)\\
&= \sigma_M\circ(\mu_{B/A}\otimes\mathbf{1}_M)\circ(\mathbf{1}_{B\otimes_A B}\otimes\sigma_M)((\mathbf{1}_B\otimes\eta_{B|B})(e\otimes 1)\otimes m)\\
&= \sigma_M\circ(\mathbf{1}_B\otimes\sigma_M)\circ(\mathbf{1}_{B\otimes_A B}\otimes\sigma_M)((\mathbf{1}_B\otimes\eta_{B|B})(e\otimes 1)\otimes m)\\
&= \sigma_M\circ(\mathbf{1}_B\otimes(\sigma_M\circ(\mathbf{1}_B\otimes\sigma_M)))((\mathbf{1}_B\otimes\eta_{B|B})(e\otimes 1)\otimes m)\\
&= \sigma_M\circ(\mathbf{1}_B\otimes(\sigma_M\circ(\mu_{B/A}\otimes\mathbf{1}_M)))((\mathbf{1}_B\otimes\eta_{B|B})(e\otimes 1)\otimes m)\\
&= \sigma_M\circ(\mathbf{1}_B\otimes\sigma_M)((\mathbf{1}_B\otimes(\mu_{B/A}\circ\eta_{B|B}))(e\otimes 1)\otimes m)\\
&= \sigma_M\circ(\mathbf{1}_B\otimes\sigma_M)((\mathbf{1}_B\otimes\mu_{B/A})(e\otimes 1)\otimes m)\\
&= \sigma_M\circ(\mu_{B/A}\otimes\mathbf{1}_M)((\mathbf{1}_B\otimes\mu_{B/A})(e\otimes 1)\otimes m)\\
&= \sigma_M((\mu_{B/A}\circ(\mathbf{1}_B\otimes\mu_{B/A}))(e\otimes 1)\otimes m)\\
&= \sigma_M((\mu_{B/A}\circ(\mu_{B/A}\otimes\mathbf{1}_B))(e\otimes 1)\otimes m)\\
&= \sigma_M(\mu_{B/A}(1\otimes 1)\otimes m) = \sigma_M(1\otimes m)\\
&= m.
\end{aligned}$$

It follows in particular that β is an isomorphism of non-unital almost algebras. Then the morphism $s=\beta^{-1}\circ\omega : B\to C$ is a splitting for α in the category of non-unital almost V-algebras. But we remark that $\beta(1)=f$ which means $s(1)=1$ so that s is actually a homomorphism of (unital) almost V-algebras. The theorem follows. □

3.2. **Lifting theorems.**

Lemma 3.2.1. *(see* [7]*) Let $A\to B$ be an epimorphism of almost V-algebras with kernel I. Let U be the A-extension $0\to I/I^2\to A/I^2\to B\to 0$. Then the assignment $f\mapsto f*U$ defines a natural isomorphism*

$$\mathrm{Hom}_B(I/I^2,M)\xrightarrow{\sim}\mathrm{Exal}_A(B,M). \tag{3.2.2}$$

□

Now consider any morphism of A-extensions

$$\begin{array}{ccc} \widetilde{B}: & & 0\longrightarrow I\longrightarrow B\longrightarrow B_0\longrightarrow 0\\ \downarrow{\scriptstyle\widetilde{f}} & & \qquad\ \downarrow{\scriptstyle u}\qquad\ \downarrow{\scriptstyle f}\qquad\ \downarrow{\scriptstyle f_0}\\ \widetilde{C}: & & 0\longrightarrow J\longrightarrow C\longrightarrow C_0\longrightarrow 0. \end{array} \tag{3.2.3}$$

The morphism u induces by adjunction a morphism of almost C_0-modules

$$C_0\otimes_{B_0} I\to J \tag{3.2.4}$$

whose image is the ideal $I\cdot C$, so that the square diagram of almost algebras defined by $\widetilde{f}$ is cofibred (*i.e.* $C_0\simeq C\otimes_B B_0$) if and only if (3.2.4) is an epimorphism.

Lemma 3.2.5. *Let $\widetilde{f}:\widetilde{B}\to\widetilde{C}$ be a morphism of A-extensions as above, such that the corresponding square diagram of almost algebras is cofibred. Then the morphism $f:B\to C$ is flat if and only if $f_0:B_0\to C_0$ is flat and* (3.2.4) *is an isomorphism.*

Proof. It follows directly from the (almost version of the) local flatness criterion (see [6] Th. 22.3). □

We are now ready to put together all the work done so far and begin the study of deformations of almost algebras. *From here on, throughout the rest of this section we assume that axiom (***S2***) is verified.*

The morphism $u : I \to J$ is an element in $\mathrm{Hom}_{B_0}(I, J)$; by lemma 3.2.1 the latter group is naturally isomorphic to $\mathrm{Exal}_B(B_0, J)$. By applying transitivity (theorem 2.4.6) to the sequence of morphisms $B \to B_0 \stackrel{f_0}{\to} C_0$ we obtain an exact sequence of abelian groups

$$\mathrm{Exal}_{B_0}(C_0, J) \to \mathrm{Exal}_B(C_0, J) \to \mathrm{Hom}_{B_0}(I, J) \stackrel{\partial}{\to} \mathbb{E}\mathrm{xt}^2_{C_0}(\mathbb{L}_{C_0/B_0}, J).$$

Hence we can form the element

$$\omega(\widetilde{B}, f_0, u) = \partial(u) \in \mathbb{E}\mathrm{xt}^2_{C_0}(\mathbb{L}_{C_0/B_0}, J).$$

The proof of the next result goes exactly as in [4] (III.2.1.2.3).

Proposition 3.2.6. *i) Let the A-extension $\widetilde{B}$, the B_0-linear morphism $u : I \to J$ and the morphism of almost A-algebras $f_0 : B_0 \to C_0$ be given as above. Then there exists an A-extension $\widetilde{C}$ and a morphism $\widetilde{f} : \widetilde{B} \to \widetilde{C}$ completing diagram* (3.2.3) *if and only if $\omega(\widetilde{B}, f_0, u) =$* 0. *(*i.e. *$\omega(\widetilde{B}, f_0, u)$ is the obstruction to the lifting of $\widetilde{B}$ over f_0.)*

ii) Assume that the obstruction $\omega(\widetilde{B}, f_0, u)$ vanishes. Then the set of isomorphism classes of A-extensions $\widetilde{C}$ as in (i) forms a torsor under the group $\mathrm{Exal}_{B_0}(C_0, J)$ $(\simeq \mathbb{E}\mathrm{xt}^1_{C_0}(\mathbb{L}_{C_0/B_0}, J))$.

iii) The group of automorphisms of an A-extension $\widetilde{C}$ as in (i) is naturally isomorphic to $\mathrm{Der}_{B_0}(C_0, J)$ $(\simeq \mathbb{E}\mathrm{xt}^0_{C_0}(\mathbb{L}_{C_0/B_0}, J))$. □

The obstruction $\omega(\widetilde{B}, f_0, u)$ depends functorially on u. More exactly, if we denote by

$$\omega(\widetilde{B}, f_0) \in \mathbb{E}\mathrm{xt}^2_{C_0}(\mathbb{L}_{C_0/B_0}, C_0 \otimes_{B_0} I)$$

the obstruction corresponding to the natural morphism $I \to C_0 \otimes_{B_0} I$, then for any other morphism $u : I \to J$ we have

$$\omega(\widetilde{B}, f_0, u) = v \circ \omega(\widetilde{B}, f_0)$$

where v is the morphism (3.2.4). Taking lemma 3.2.5 into account we deduce

Corollary 3.2.7. *Suppose that $B_0 \to C_0$ is flat. Then*

(i) The class $\omega(\widetilde{B}, f_0)$ is the obstruction to the existence of a flat deformation of C_0 over B, i.e. *of a B-extension $\widetilde{C}$ as in* (3.2.3) *such that C is flat over B and $C \otimes_B B_0 \to C_0$ is an isomorphism.*

(ii) When the obstruction $\omega(\widetilde{B}, f_0)$ vanishes, the set of isomorphism classes of flat deformations of C_0 over B forms a torsor under the group $\mathrm{Exal}_{B_0}(C_0, C_0 \otimes_{B_0} I)$.

(iii) The group of automorphisms of a given flat deformation of C_0 over B is naturally isomorphic to $\mathrm{Der}_{B_0}(C_0, C_0 \otimes_{B_0} I)$. □

Now, suppose we are given two A-extensions $\widetilde{C}^1, \widetilde{C}^2$ with morphisms of A-extensions

$$\begin{array}{cccccccccc}
\widetilde{B}: & & 0 & \longrightarrow & I & \longrightarrow & B & \longrightarrow & B_0 & \longrightarrow 0 \\
\big\downarrow{\scriptstyle \widetilde{f}^i} & & & & \big\downarrow{\scriptstyle u^i} & & \big\downarrow{\scriptstyle f^i} & & \big\downarrow{\scriptstyle f^i_0} & \\
\widetilde{C}^i: & & 0 & \longrightarrow & J^i & \longrightarrow & C^i & \longrightarrow & C^i_0 & \longrightarrow 0
\end{array}$$

and morphisms $v : J^1 \to J^2$, $g_0 : C^1_0 \to C^2_0$ such that

$$u^2 = v \circ u^1 \qquad \text{and} \qquad f^2_0 = g_0 \circ f^1_0. \tag{3.2.8}$$

We consider the problem of finding a morphism of A-extensions

$$\begin{array}{cccccccccc}
\widetilde{C}^1: & & 0 & \longrightarrow & J^1 & \longrightarrow & C^1 & \longrightarrow & C^1_0 & \longrightarrow 0 \\
\big\downarrow{\scriptstyle \widetilde{g}} & & & & \big\downarrow{\scriptstyle v} & & \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle g_0} & \\
\widetilde{C}^2: & & 0 & \longrightarrow & J^2 & \longrightarrow & C^2 & \longrightarrow & C^2_0 & \longrightarrow 0
\end{array} \tag{3.2.9}$$

such that $\widetilde{f}^2 = \widetilde{g} \circ \widetilde{f}^1$. Let us denote by $e(\widetilde{C}^i) \in \mathbb{E}\mathrm{xt}^1_{C^i_0}(\mathbb{L}_{C^i_0/B}, J^i)$ the classes defined by the B-extensions $\widetilde{C}^1, \widetilde{C}^2$ via the isomorphism of theorem 2.4.5 and by

$$v* : \mathbb{E}\mathrm{xt}^1_{C^1_0}(\mathbb{L}_{C^1_0/B}, J^1) \to \mathbb{E}\mathrm{xt}^1_{C^1_0}(\mathbb{L}_{C^1_0/B}, J^2)$$
$$*g_0 : \mathbb{E}\mathrm{xt}^1_{C^2_0}(\mathbb{L}_{C^2_0/B}, J^2) \to \mathbb{E}\mathrm{xt}^1_{C^2_0}(C^2_0 \otimes_{C^1_0} \mathbb{L}_{C^1_0/B}, J^2)$$

the canonical morphisms defined by v and g_0. Using the natural isomorphism

$$\mathbb{E}\mathrm{xt}^1_{C^1_0}(\mathbb{L}_{C^1_0/B}, J^2) \simeq \mathbb{E}\mathrm{xt}^1_{C^2_0}(C^2_0 \otimes_{C^1_0} \mathbb{L}_{C^1_0/B}, J^2)$$

we can identify the target of both $v*$ and $*g$ with $\mathbb{E}\mathrm{xt}^1_{C^1_0}(\mathbb{L}_{C^1_0/B}, J^2)$. It is clear that the problem admits a solution if and only if the A-extensions $v * \widetilde{C}^1$ and $C^2 * g_0$ coincide, *i.e.* if and only if $v * e(C^1) - e(C^2) * g_0 = 0$. By applying transitivity to the sequence of morphisms $B \to B_0 \to C^1_0$ we obtain an exact sequence

$$0 \to \mathbb{E}\mathrm{xt}^1_{C^1_0}(\mathbb{L}_{C^1_0/B_0}, J^2) \to \mathbb{E}\mathrm{xt}^1_{C^1_0}(\mathbb{L}_{C^1_0/B}, J^2) \to \mathrm{Hom}_{C^1_0}(C^1_0 \otimes_{B_0} I, J^2)$$

It follows from (3.2.8) that the image of $v*e(C^1)-e(C^2)*g_0$ in the group $\mathrm{Hom}_{C^1_0}(C^1_0 \otimes_{B_0} I, J^2)$ vanishes, therefore

$$v * e(C^1) - e(C^2) * g_0 \in \mathbb{E}\mathrm{xt}^1_{C^1_0}(\mathbb{L}_{C^1_0/B_0}, J^2). \tag{3.2.10}$$

In conclusion, we derive the following result as in [4] (III.2.2.2).

Proposition 3.2.11. *With the above notations, the class* (3.2.10) *is the obstruction to the existence of a morphism of A-extensions $\widetilde{g} : \widetilde{C}^1 \to \widetilde{C}^2$ as in* (3.2.9) *such that $\widetilde{f}^2 = \widetilde{g} \circ \widetilde{f}^1$. When the obstruction vanishes, the set of such morphisms forms a torsor under the group* $\mathrm{Der}_{B_0}(C^1_0, J^2)$ *(the latter being identified with $\mathbb{E}\mathrm{xt}^0_{C^2_0}(C^2_0 \otimes_{C^1_0} \mathbb{L}_{C^1_0/B_0}, J^2)$).* □

Let B be an almost A-algebra. We can find a set S and an epimorphism of almost A-algebras $\psi : A[S] \to B$. Let $J = \mathrm{Ker}(\psi)$. There is a natural isomorphism $\mathrm{Tor}_1^{A[S]}(B, B) \simeq J/J^2$ which extends to a natural homomorphism of graded anticommutative almost B-algebras

$$\underset{n\geq 0}{\oplus} \Lambda^n_B(J/J^2) \to \underset{n\geq 0}{\oplus} \mathrm{Tor}_n^{A[S]}(B, B) \tag{3.2.12}$$

(for the natural product structures on the exterior algebra and the almost Tor-algebra).

Proposition 3.2.13. *Keep the above hypotheses and suppose moreover that B is an étale almost A-algebra. Then J/J^2 is a free almost B-module and the morphism* (3.2.12) *is an isomorphism of graded anticommutative almost algebras.*

Proof. By hypothesis B is a flat almost A-algebra, hence $B[S]$ is a flat almost $A[S]$-algebra. Moreover the morphism ψ factors as a composition

$$A[S] \longrightarrow B[S] \xrightarrow{\mathbf{1}_B \otimes_A \psi} B \otimes_A B \xrightarrow{\mu_{B/A}} B$$

thereby inducing on B a structure of almost $B[S]$-module which restricts to the given almost $A[S]$-module structure. Hence we deduce, by flat base change (see [8] (3.2.9)), a natural isomorphism of graded almost algebras

$$\underset{n\geq 0}{\oplus} \mathrm{Tor}_n^{A[S]}(B, B) \simeq \underset{n\geq 0}{\oplus} \mathrm{Tor}_n^{B[S]}(B \otimes_A B, B).$$

On the other hand corollary 3.1.4 implies

$$\mathrm{Tor}_n^{B[S]}(B \otimes_A B, B) \simeq \mathrm{Tor}_n^{B[S]}(B, B) \oplus \mathrm{Tor}_n^{B[S]}(I_{B/A}, B)$$

for all integers n. The morphism $\eta = \mu_{B/A} \circ (\mathbf{1}_B \otimes_A \psi) : B[S] \to B$ is determined by a map $S \to B_* : s \mapsto b_s$. Define a B-automorphism $\phi : B[S] \to B[S]$ by $s \mapsto s - b_s$. We derive a commutative diagram with short exact rows

$$\begin{tikzcd} 0 \arrow[r] & S \cdot B[S] \arrow[r] \arrow[d, "\simeq"] & B[S] \arrow[r] \arrow[d, "\phi"] & B \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & J_1 = \mathrm{Ker}(\eta) \arrow[r] & B[S] \arrow[r, "\eta"] & B \arrow[r] & 0 \end{tikzcd}.$$

If we denote by $\pi : F = B[S]^{(S)} \to J_1$ the epimorphism coming from the above map $\phi : S \cdot B[S] \to J_1$, then the Koszul complex $K_\bullet(\pi)$ associated to π is a flat resolution of B as a $B[S]$-module. Recall that $K_n(\pi) = \Lambda^n_{B[S]}(F)$ and the differential of the complex is given by the formula

$$\begin{aligned} d_n : \Lambda^{n+1}_{B[S]}(F) &\to \Lambda^n_{B[S]}(F) \\ x_0 \wedge \ldots \wedge x_n &\mapsto \sum_{j=0}^{n} (-1)^j \pi(x_j) \cdot (x_0 \wedge \ldots \wedge x_{j-1} \wedge x_{j+1} \wedge \ldots \wedge x_n). \end{aligned}$$

We claim that $\mathrm{Tor}_n^{B[S]}(I_{B/A}, B)$ vanishes for all integers n. It suffices to show that the complex $K_\bullet(\pi) \otimes_{B[S]} I_{B/A}$ is acyclic. Let e be the idempotent of $B \otimes_A B$ provided by proposition 3.1.3 and for any $\varepsilon \in S$ choose $f_\varepsilon \in B_*[S]$ such that $(\mathbf{1}_B \otimes_A \psi)(f_\varepsilon) = \varepsilon \cdot (1 - e)$. Clearly $f_\varepsilon \in J_{1*}$, hence we can find $g_\varepsilon \in F_*$ with $\pi(g_\varepsilon) = f_\varepsilon$ and we can define a sequence of morphisms

$$s_n : \Lambda^n_{B[S]}(F) \to \Lambda^{n+1}_{B[S]}(F) \qquad x_0 \wedge \ldots \wedge x_{n-1} \mapsto g_\varepsilon \wedge x_0 \wedge \ldots \wedge x_{n-1}.$$

A straightforward computation yields

$$(d_n \circ s_n + s_{n-1} \circ d_{n-1}) \otimes_{B[S]} \mathbf{1}_{I_{B/A}} = \varepsilon \cdot \mathbf{1}_{K_n(\pi) \otimes_{B[S]} I_{B/A}}.$$

As ε can be chosen arbitrarily small, the claim follows. Summing up we get an isomorphism of graded almost algebras

$$\bigoplus_{n \geq 0} \mathrm{Tor}_n^{A[S]}(B, B) \simeq \bigoplus_{n \geq 0} \mathrm{Tor}_n^{B[S]}(B, B).$$

To determine the right-hand side we use again the Koszul complex. The claim follows. □

Proposition 3.2.14. *Suppose that B is an étale almost A-algebra. Then we have*

$$H_q(\mathbb{L}_{B/A}) \simeq 0 \qquad 0 \leq q \leq 2.$$

Proof. For $q = 0$ the claim follows easily from corollary 2.4.3 and corollary 3.1.4. For $q = 1$ the claim follows from the standard cohomology spectral sequence

$$E_2^{pq} = \mathrm{Ext}^p_B(H_q(\mathbb{L}_{B/A}), M) \Rightarrow \mathbb{E}\mathrm{xt}^{p+q}_B(\mathbb{L}_{B/A}, M) \tag{3.2.15}$$

together with theorem 2.4.5 and theorem 3.1.7. Finally, pick a set S large enough and a morphism $A[S] \to B$ of almost A-algebras such that $A_*[S] \to B_*$ is surjective. Applying transitivity (theorem 2.4.6) to the sequence $A \to A[S] \to B$ we see easily that $H_p(\mathbb{L}_{B/A}) \simeq H_p(\mathbb{L}^a_{B_*/A_*[S]})$ for all $p \geq 2$. Then proposition 3.2.13 and corollary 2.4.8 yield $H_2(\mathbb{L}_{B/A}) \simeq 0$. □

For a given almost V-algebra A, let $\mathbf{\acute{E}t}(A)$ be the category of étale almost A-algebras. Notice that, by lemma 3.1.2(iv) all morphisms in $\mathbf{\acute{E}t}(A)$ are étale.

Theorem 3.2.16. *i) Let $A \to B$ be an étale morphism of almost algebras. Let C be any almost A-algebra and $I \subset C$ a nilpotent ideal. Then the natural morphism*

$$\mathrm{Hom}_{A-\mathbf{al.Alg}}(B, C) \to \mathrm{Hom}_{A-\mathbf{al.Alg}}(B, C/I)$$

is bijective.

ii) Let A be an almost V-algebra and $I \subset A$ a nilpotent almost ideal. Set $A' = A/I$. Then the natural functor

$$\mathbf{\acute{E}t}(A) \to \mathbf{\acute{E}t}(A') \qquad (\phi : A \to B) \mapsto (\phi' = \mathbf{1}_{A'} \otimes_A \phi : A' \to A' \otimes_A B)$$

is an equivalence of categories.

Proof. By induction we can assume that $I^2 \simeq 0$. Then, taking into account proposition 3.2.14 and the spectral sequence (3.2.15), claim (i) follows directly from proposition 3.2.11.

We show (ii) : by corollary 3.2.7 (and again proposition 3.2.14) we know that we can lift an étale morphism $\phi' : A' \to B'$ to a flat morphism $\phi : A \to B$ (such that $B' \simeq A' \otimes_A B$). We need to verify that ϕ is étale. To this purpose we apply proposition 3.1.3 : since ϕ' is étale, there is an idempotent $e' = e_{B'/A'} \in B' \otimes_{A'} B'_*$ such that $\mu_{B'/A'}(e') = \underline{1}$ and $x \cdot e' = 0$ for all $x \in I_{B'/A'*}$. By lemma 3.1.5 there exists a unique lifting of e' to an idempotent $e \in B \otimes_A B_*$. Clearly $\mu_{B/A}(e) \in B_*$ is a lifting of the idempotent $\mu_{B'/A'}(e') = \underline{1} \in B'_*$, hence $\mu_{B/A}(e) = \underline{1}$ by lemma 3.1.5(iii). It remains to verify that $x \cdot e = 0$ for all $x \in I_{B/A*}$. We consider the morphism

$$\partial : B \to B \otimes_A B \qquad b \mapsto (\underline{1} \otimes b - b \otimes \underline{1}) \cdot e.$$

It is easily checked that the image of ∂ is contained in the ideal $I(B \otimes_A B)$. Moreover ∂ factors through the projection $B \to B'$ and we can denote

$$\partial' : B' \to I(B \otimes_A B)$$

the induced morphism. Let J be the $B \otimes_A B$-submodule generated by the image of ∂. We remark that the annihilator of J contains $I_{B/A} = \mathrm{Ker}(\mu_{B/A})$ (so that the $B \otimes_A B$-module structure on J is obtained by restriction of scalars via $\mu_{B/A}$). In fact we have, for all $b, b' \in B_*$

$$(\underline{1} \otimes b - b \otimes \underline{1}) \cdot e \cdot (\underline{1} \otimes b' - b' \otimes \underline{1}) = \{(\underline{1} \otimes b - b \otimes \underline{1}) \cdot e\} \cdot \{e \cdot (\underline{1} \otimes b' - b' \otimes \underline{1})\} = 0$$

because $I^2 = 0$. Pick $b_1, b_2 \in B_*$. We compute

$$\partial(b_1 b_2) = \partial(b_1) \cdot (\underline{1} \otimes b_2) + (b_1 \otimes \underline{1}) \cdot \partial(b_2).$$

In other words, ∂' is an almost A-derivation of B' with values in the almost B'-module J. Now it follows from corollary 2.4.3 and proposition 3.2.14 that $\partial' = 0$ and therefore $\partial = 0$. This proves that ϕ is unramified, hence étale as claimed. $\square$

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