

HASSE-ARF FILTRATIONS IN p -ADIC ANALYTIC GEOMETRY

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1. INTRODUCTION

This work is the third in a series dedicated to the study of local monodromy of sheaves defined on the analytic étale site of p -adic analytic curves, where $p > 0$ is any prime number. More precisely, we consider a complete and algebraically closed non-archimedean field K of mixed characteristic $(0, p)$, a noetherian $\mathbb{Z}[1/p]$ -algebra Λ , and we investigate the category of locally constant Λ -modules of finite length on the étale site of a punctured disc $\mathbb{D}(z, \varepsilon)^*$ of radius $\varepsilon > 0$, centered at a given K -rational point $z \in \mathbb{P}_K^1$.

Suppose for a moment that Λ is a finite ring, in which case, any object F of our category is trivialized over a finite Galois étale covering $f : X \rightarrow \mathbb{D}(z, \varepsilon)^*$, and it is known that the restriction of f to some smaller disc $\mathbb{D}(z, \delta)^*$ is a disjoint union of Kummer coverings; since we are interested only in the local behaviour of F around z , we may replace ε by such smaller radius δ , and we conclude that the local monodromy of F is described very simply (and uninterestingly) by a continuous representation of the profinite completion $\mathbb{Z}^\wedge$ of $\mathbb{Z}$.

Here, as well as in [25] and [24], we are rather interested in rings Λ that are not finite, but only *ind-finite*, i.e. are the filtered union of their finite subrings, in which case – owing to the existence of Galois étale coverings of $\mathbb{D}(z, \varepsilon)^*$ of infinite degree – the category of étale Λ -modules on the punctured disc is much richer, and new phenomena appear. The latter are best understood as a manifestation of a new world of “analytic wild ramification” in characteristic zero, closely resembling what one finds in the study of local monodromy of sheaves on the étale site of a smooth (algebraic) curve defined over a perfect field of positive characteristic.

The prototype of such wildly ramified coverings is the logarithm map, given by the usual power series (see the beginning of section 2.3). We may contrast the situation with that of complex analytic geometry, where we have the exponential map $\mathbb{C} \rightarrow \mathbb{C}^*$ which yields the universal covering of $\mathbb{G}_m$; in the p -adic world, the exponential power series has a rather small radius of convergence, and it only defines an isometric transformation of a small open disc. On

the other hand, the logarithm power series $\log(1+x)$ has p -adic radius of convergence equal to one, and it defines an étale map onto the analytic affine line (étaleness, as usual, means that the induced map on tangent spaces is an isomorphism at every K -rational point). Moreover, just as in complex analytic geometry, the logarithm map is a morphism of analytic groups, so a standard argument shows that it is a Galois covering, with Galois group naturally isomorphic to the kernel of the map. This kernel obviously consists of all the roots ζ of 1 such that $|1-\zeta| < 1$, and this is an infinite group isomorphic to $\mathbb{Q}_p/\mathbb{Z}_p$. The restriction of the logarithm map to any punctured disc $\mathbb{D}(\infty, \varepsilon)^*$ is still connected, but it is far from being a universal covering, so we deduce a natural surjection of the étale fundamental group of $\mathbb{D}(\infty, \varepsilon)^*$ onto $\mathbb{Q}_p/\mathbb{Z}_p$, which is not a bijection.

One may also compare our category of local systems to that of connections of finite rank on the field $\mathbb{C}((t))$: as is well known, the role of wild ramification is taken there by the notion of irregularity, that measures the order of the pole of a connection. Somehow, the analogy with differential equations is even more fitting, since a connection on a complex analytic punctured disc may present an essential singularity (which therefore, in some sense, has infinite irregularity), and correspondingly, we find in our category certain Λ -modules that have “infinitely wild monodromy”.

In [25] we have singled out a tensor abelian subcategory of Λ -modules *with bounded ramification*, which, in terms of the foregoing analogy, would correspond to the class of connections with meromorphic poles on the complex analytic punctured disc; these are the Λ -modules F on $\mathbb{D}(z, \varepsilon)^*$ such that the étale cohomology group $H^1(\mathbb{D}(z, \varepsilon)^*, F)$ is a Λ -module of finite length: it is easily seen that this notion is actually local at the point z , *i.e.* it holds for the given F , if and only if it holds for the restriction of F to any smaller punctured disc $\mathbb{D}(z, \delta)^*$.

A Λ -module with bounded ramification on $\mathbb{D}(z, \varepsilon)_{\text{ét}}$ has a well defined *Swan conductor at the point z* , which is an integer (playing the role of the irregularity index of a meromorphic connection), and we have shown that our Swan conductors behave as their namesakes in positive characteristic: for instance, there is an identity analogous to the classical Grothendieck-Ogg-Shafarevich formula, that calculates the Euler characteristic of an affine curve C with coefficients in a locally constant Λ -module which has bounded ramification at every closed point of the compactification of C .

Moreover, we have shown that every Λ -module with bounded ramification admits a *break decomposition*, analogous to the classical one in positive characteristic. For meromorphic connections on $\mathbb{C}((t))$, the break decomposition is a fundamental result of Levelt and Turrin; the latter shows that, in a precise sense, this category of connections is generated by a single object, namely the rank one connection ∇ given by the rule:

$$\nabla(f) = f \cdot dt - df$$

(where d is exterior differentiation). Notice that the horizontal sections of ∇ are the constant multiples of the exponential power series $\exp(t)$; especially, the same rule defines a connection on the structure sheaf of the affine line, which, in the p -adic analytic world, does not admit global horizontal sections (whereas it does admit complex analytic ones); however, if we pull back ∇ via the logarithm map, we obtain the connection $\log^*(\nabla)$ on the p -adic disc of radius one which does admit global horizontal sections (indeed its horizontal sections are multiples of $\log^*(\exp(t)) = t$). We hope that this remark illustrates the central role played by the logarithm étale covering in our theory, and justifies the expectation that, just as for meromorphic connections, the (essentially unique) rank one Λ -module on $\mathbb{D}(\infty, \varepsilon)^*$ trivialized by the logarithm covering, should be (in a precise sense which is made explicit in remark 3.3.1) a generator for our category of Λ -modules with bounded ramification.

Again, the formal properties of our break decomposition parallel closely those of their older prototypes, but one important point was left unresolved in [25]. Namely, the break decomposition was shown to exist only on some complicated (connected) open subset of $\mathbb{D}(z, \varepsilon)^*$, obtained by removing from the punctured disc possibly infinitely many smaller discs not containing the center z : this is unsatisfactory, since it means that we have to leave our tensor category in order to get the decomposition.

The aim of the present work is to show that the break decomposition for a Λ -module with bounded ramification is already defined on a punctured disc of possibly smaller radius (centered at the same point z). The significance of this refinement is better conveyed, if we restate it in terms of the category

$$\Lambda[\pi(z)]\text{-b.Mod}$$

of *germs* of Λ -modules with bounded ramification (around a given closed point $z \in \mathbb{P}^1$: see (2.4.31)); namely, it implies that $\Lambda[\pi(z)]\text{-b.Mod}$ is a tensor category *with a Hasse-Arf filtration*, in the sense of Y.André's paper [1]. Thus, the tannakian machinery of [1] is now available, to complete the classification of germs of modules with bounded ramification: see section 3.3 for some indication about the work that remains to be done, in order to achieve this goal.

The plan of the proof is to chop such a module along its breaks, using certain operators (the local Fourier transforms), whose behaviour can be analyzed very precisely by a local-to-global method (the principle of stationary phase). Of course, all this is directly inspired by the article [22] of G.Laumon, where the same sort of constructions are made, but in the world of k -schemes, where k is a perfect field of positive characteristic.

Chapter 2 implements the first step of this programme; namely we prove the main finiteness results concerning the (global) Fourier transform of modules with bounded ramification everywhere on the affine line (on a non-archimedean field of characteristic zero), then we construct our local Fourier transforms and establish the principle of stationary phase.

Chapter 3 completes the programme: our break decomposition is achieved in theorem 3.2.17; however, this is not the end of the story, since currently we do not even know how to classify the modules of break zero: what we know about this case is gathered in section 3.3.

The global Fourier transform used here is the same that was introduced in [24], and a construction of local Fourier transforms was also already attempted there, with modest success. However, [24] was only concerned with a special class of examples (the modules with meromorphic ramification), defined by an ad hoc condition, whereas now we are dealing with modules with bounded ramification, which form *the* natural class for which a general study of local monodromy can reasonably be carried out.

More generally, one may wonder whether the familiar tools of perverse ℓ -adic sheaves and weight filtrations that Deligne introduced in his proof of the Weil conjectures, can be transplanted in the setting of non-archimedean analytic geometry. We are still several key insights away from being able to answer such kind of questions; however, it is already clear that the condition of bounded ramification must intervene in the definition of any category of perverse sheaves which is stable under the usual cohomological operations, and under Fourier transform. For instance, the cohomology of a Zariski constructible module M on a quasi-compact analytic curve C has finite length, if and only if M has bounded ramification at every point of C .

The article is structured as follows. In section 2.1 we introduce a wide class of étale coverings of analytic (more precisely, adic in the sense of R.Huber) spaces; the definition is actually found already in [24], but the current treatment is more general, systematic and tidier.

Section 2.2 is devoted to the construction and basic properties of the vanishing cycles which later are used to define the local Fourier transforms. Again, this parallels what is done in [24], but many unnecessary complications have been removed, and some inaccuracy has been repaired. Section 2.3 recalls the (global) Fourier transform of [24]; the main new theorem here

(theorem 2.3.30) morally states that the Fourier transform of a sheaf with bounded ramification is a perverse sheaf with bounded ramification everywhere. (Except that we do not really try to define what a perverse sheaf is.)

Finally, the local Fourier transforms are introduced in section 2.4, and our version of the principle of stationary phase is proved (theorem 2.4.10). These functors act on certain categories, whose objects should be thought of as representations of inertia subgroups (of the fundamental group of a punctured disc). In truth, in our situation these inertia subgroups are rather elusive, so the actual definitions are somewhat more complicated than in the algebraic geometric case : see (2.2.15). The basic properties of the local Fourier transforms are then established in section 3.1 : they are wholly analogous to the ones found in [22]. The proofs are similar, but usually more difficult; partly, this is because we do not have a functorial local-to-global extension of representations of the inertia subgroups (in positive characteristic, the existence of such an extension is a theorem of Gabber : see [20]). Instead, we rely on a theorem of Garuti ([13]), which provides a non-functorial extension, on a Zariski-open subset of the affine line (see (2.4.14)). This suffices, up to some extra contorsions, to derive everything we need; however, it would be interesting to know whether Garuti's methods can be strengthened enough, to give a functorial extension which would be truly analogous to Gabber's theorem.

Section 3.2 contains the proof of our break decomposition; with its help we may then also complete the analysis of the local Fourier transforms.

We conclude with an application to the question of the *localization of the determinant of cohomology*. This refers to a problem, first proposed by Deligne in [12], which can be stated as follows. Let C be a quasi-projective curve defined over a subfield K_0 of an algebraically closed field K , and set $G := \text{Gal}(K/K_0)$; let also F be an ℓ -adic *lisse* sheaf on C ; then one wants to decompose the G -representation

$$\det(R\Gamma_c(C \times_{K_0} K, F))$$

as a tensor product of *local ε -factors*. The latter should be certain characters $\varepsilon(x, F)$ of G , attached to the points x of the compactification of C , and depending functorially on the local monodromy of F around x . Deligne was motivated by the related problem of the factorization of the ε -constant appearing in the functional equation of the L -function associated to F when K_0 is a finite field. And indeed, the latter arithmetic problem was later solved by Laumon in [22], by reducing it to the cohomological localization problem, and proving a version of the sought tensor decomposition in case where C is an open subset of the affine line over a finite field.

More recently, there has been a renewed interest in this problem : for instance, the article [8] contains an analogue of Laumon's theorem for $\mathcal{D}$ -modules over a field K_0 of characteristic zero. Also, in Beilinson's paper [6] one finds a real-analytic counterpart, that actually works for manifolds of arbitrary dimension. And the very recent preprint [7], also by Beilinson, sets up a general formalism that allows to compare the deRham ε -factors from [8] with the topological ones of [6].

In section 3.4 we establish a p -adic analytic analogue of Laumon's formula, for the case where K_0 is a local field of zero characteristic. This is, in some sense, an obvious application of our work, since the proof of [22] uses the full apparatus of local Fourier transforms and stationary phase. However, I should stress that only now our theory has matured enough for such an application to lie within our grasp : neither the results of our previous [25], nor the rudimentary local Fourier transforms of [24] would have sufficed in order to adapt the arguments of [22].

We prove our tensor decomposition only for local systems of $\overline{\mathbb{F}}_\ell$ -modules (rather than ℓ -adic sheaves, but see remark 3.4.24); on the other hand, we prove it for general local systems with bounded ramification on the analytic étale site of a Zariski open subset U of the affine

line (as remarked above, heuristically, these sheaves correspond to holonomic $\mathcal{D}$ -modules with arbitrary meromorphic singularities). This includes the category of local systems coming from the algebraic étale site of U (which correspond to $\mathcal{D}$ -modules with regular singularities), and in this case we rejoin Deligne's original problem; our results appear to be new even for this special case (cp. the "Question and hopes" in the introduction of [7]), and even for this restricted class of sheaves, our proof employs essentially all the arsenal amassed in the previous sections.

An unexpected feature of our decomposition, is that it takes place in the category of Galois characters that are *semilinear* with respect to a certain action of the Galois group of K on $\overline{\mathbb{F}}_\ell$. We refer to remark 3.4.23 for some observations concerning this kind of hybrid, partly p -adic, partly ℓ -adic representations.

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2. HARMONIC ANALYSIS FOR MODULES WITH BOUNDED RAMIFICATION

2.1. Locally algebraic coverings. Throughout this work, $(K, |\cdot|)$ is a fixed algebraically closed valued field, complete for a rank one valuation and of residue characteristic $p > 0$, and we denote by K^+ the valuation ring of $|\cdot|$. We shall use freely the language of adic spaces from [17], and we set :

$$S := \mathrm{Spa}(K, K^+).$$

For a K -scheme X locally of finite type, we let X^{ad} be the adic space associated to X (this is denoted $X \times_{\mathrm{Spec} K} S$ in [16, Prop.3.8]).

Definition 2.1.1. A morphism $f : Y \rightarrow X$ of analytic S -adic spaces is said to be a *locally algebraic covering*, if the following holds. For every quasi-compact open subset $U \subset X$, the preimage $f^{-1}U$ decomposes as the (possibly infinite) disjoint union of open and closed subspaces, each of which is a finite étale covering of U . We denote by

$$\mathrm{Cov}^{\mathrm{alg}}(X) \quad (\text{resp. } \mathrm{Cov}^{\mathrm{loc.alg}}(X))$$

the full subcategory of the category of X -adic spaces, whose objects are the finite étale (resp. the locally algebraic) coverings.

Remark 2.1.2. (i) It is immediate from the definition, that every locally algebraic covering is a separated morphism.

(ii) It is easily seen that a fibre product $Y_1 \times_X Y_2$ of locally algebraic covering spaces of X , is also a locally algebraic covering of X . If $f : Y \rightarrow X$ is a locally algebraic covering, and $X' \rightarrow X$ is any morphism of S -adic spaces, then $f \times_X 1_{X'} : Y \times_X X' \rightarrow X'$ is a locally algebraic covering.

(iii) Likewise, if $f : Z \rightarrow Y$ and $g : Y \rightarrow X$ are locally algebraic coverings, then the same holds for $g \circ f : Z \rightarrow X$. And if $(Y_i \rightarrow X \mid i \in I)$ is any family of locally algebraic coverings, then the disjoint union $\coprod_{i \in I} Y_i \rightarrow X$ is a locally algebraic covering as well. (These two properties do not hold for general étale coverings, defined as in [19, Def.2.1] : in *loc. cit.* the language of Berkovich's analytic spaces is used, but the definition and the proofs of several of the preliminary results can be repeated *verbatim* for adic spaces).

(iv) Moreover, let $f_i : Y_i \rightarrow X$, for $i = 1, 2$, be two locally algebraic coverings, and $g : Y_1 \rightarrow Y_2$ a morphism of X -adic spaces; then g is a locally algebraic covering of Y_2 . Indeed, let $U \subset Y_2$ be a quasi-compact open subset; after replacing X by $f_2 U$, and Y_i by $f_i^{-1} f_2 U$ ($i = 1, 2$) we may assume that X is quasi-compact, hence $Y_1 = \bigcup_{\alpha \in I} Z_\alpha$ for a family of finite and étale X -adic spaces Z_α . Denote by $g_\alpha : Z_\alpha \rightarrow Y_2$ the restriction of g ; we may write $g_\alpha = p_\alpha \circ s_\alpha$, where

$p_\alpha : Z_\alpha \times_X Y_2 \rightarrow Y_2$ is the projection (which is a finite étale morphism) and $s_\alpha : Z_\alpha \rightarrow Z_\alpha \times_X Y_2$ is the graph of g_α , which is a closed immersion, in view of (i) and a standard argument. Hence g_α is a finite étale morphism. However, $g^{-1}U = \bigcup_{\alpha \in I} g_\alpha^{-1}U$, and the foregoing shows that each $g_\alpha^{-1}U$ is a finite étale U -adic space, whence the assertion.

(v) If X is quasi-compact, we have a natural equivalence of categories :

$$\mathrm{Ind}(\mathbf{Cov}^{\mathrm{alg}}(X)) \xrightarrow{\sim} \mathbf{Cov}^{\mathrm{loc.alg}}(X)$$

where, for a category $\mathcal{C}$, we denote by $\mathrm{Ind}(\mathcal{C})$ the category of ind-objects of $\mathcal{C}$, defined as in [4, Exp.I, §8.2.4].

(vi) On the other hand, local algebraicity is *not* a local property on X : for instance, let E be an elliptic curve over K with bad reduction over the residue field of K^+ , and consider the analytic uniformization $p : \mathbb{G}_m \rightarrow E$. This is not a locally algebraic covering of E , but there exists a covering $E = X_1 \cup X_2$ by open (quasi-compact) subspaces, such that the two restrictions $p^{-1}X_i \rightarrow X_i$ are both locally algebraic. (I learned of this example from [2, Ch.II, Rem.1.2.4].)

Lemma 2.1.3. *Let $Y \rightarrow X$ be a locally algebraic covering, and R an equivalence relation on Y which is represented by a union of open and closed subspaces of $Y \times_X Y$. Then the (categorical) quotient sheaf Y/R is represented by a locally algebraic covering.*

Proof. Let $U' \subset U \subset X$ be any inclusion of open subsets of X , and suppose that both restrictions $(Y/R)|_U$ and $(Y/R)|_{U'}$ are represented by morphisms $Z \rightarrow U$, respectively $Z' \rightarrow U'$; we deduce easily a natural isomorphism $Z \times_U U' \xrightarrow{\sim} Z'$, compatible with inclusion of smaller open subsets $U'' \subset U'$. It follows that the lemma holds for the datum (X, Y, R) if and only if it holds for every datum of the form $(U, f^{-1}U, R \times_X U)$ where $U \subset X$ is an arbitrary quasi-compact open subset. Hence we may assume that X is quasi-compact. In that case, $Y = \bigcup_{i \in I} Y_i$ for a family $(Y_i \mid i \in I)$ of finite étale X -spaces, such that Y_i is open and closed in Y , for every $i \in I$. For every finite subset $S \subset I$, let $Y_S := \bigcup_{i \in S} Y_i$; it follows easily that Y is the (categorical) colimit of the filtered system of open and closed immersions $Y_{S'} \rightarrow Y_S$, where $S' \subset S$ range over the finite subsets of I . For every such S , let $R_S := R \cap (Y_S \times_X Y_S)$. Then R is likewise the colimit of the open and closed immersions $R_{S'} \rightarrow R_S$ for S, S' as above. Finally, Y/R (resp. Y_S/R_S) is the coequalizer of the two projections $R \rightarrow Y$ (resp. $R_S \rightarrow Y_S$); by general nonsense, we deduce that Y/R is naturally isomorphic to the colimit of the filtered system of open and closed immersions $Y_{S'}/R_{S'} \rightarrow Y_S/R_S$. We are therefore reduced to showing that each quotient Y_S/R_S is representable, hence we may assume from start that $Y \rightarrow X$ is a finite étale covering. In this case, the lemma can be checked locally on X , hence we may further assume that X is affinoid, say $X = \mathrm{Spa}(A, A^+)$; then there exists a finite étale A -algebra B such that $Y = \mathrm{Spec} B \times_{\mathrm{Spec} A} X$. Likewise, $R = Z \times_{\mathrm{Spec} A} X$ for some open and closed equivalence relation $Z \subset \mathrm{Spec} B \otimes_A B$. We may then represent Y/R by the X -adic space $((\mathrm{Spec} B)/Z) \times_{\mathrm{Spec} A} X$. $\square$

2.1.4. Fix a *maximal geometric point* of X , i.e. a morphism $\xi : \mathrm{Spa}(E, E^+) \rightarrow X$ of S -adic spaces, where $(E, |\cdot|)$ is a complete and algebraically closed valued field extension of K , with valuation group of rank one. To ξ we associate a *fibre functor*

$$F_{X, \xi} : \mathbf{Cov}^{\mathrm{loc.alg}}(X) \rightarrow \mathbf{Set} \quad : \quad (Y \rightarrow X) \mapsto \mathrm{Hom}_X(\mathrm{Spa}(E, E^+), Y)$$

(unless we are dealing with more than one adic space, we shall usually drop the subscript X , and write just F_ξ). The *locally algebraic fundamental group* of X , pointed at ξ , is defined as usual, as the automorphism group of F_ξ :

$$\pi_1^{\mathrm{loc.alg}}(X, \xi) := \mathrm{Aut}(F_\xi).$$

It is endowed with a natural topology, as in [19, §2]. Namely, for every pair of the form $(Y, \bar{y})$, with Y a locally algebraic covering of X and $\bar{y} \in F_\xi(Y)$, let $H(Y, \bar{y}) \subset \pi_1^{\text{loc.alg}}(X, \xi)$ be the stabilizer of $\bar{y}$ (for the action on $F_\xi(Y)$); then the family $\mathcal{H}$ of such subgroups is stable under finite intersections (by remark 2.1.2(ii)) and under conjugation by arbitrary elements $\gamma \in \pi_1^{\text{loc.alg}}(X, \xi)$, since $\gamma H(Y, \bar{y}) \gamma^{-1} = H(Y, \gamma \cdot \bar{y})$. Hence, there is a unique topology on $\pi_1^{\text{loc.alg}}(X, \xi)$ for which the family $\mathcal{H}$ forms a fundamental system of open neighborhoods of the identity element.

Arguing as in the proof of [19, Lemma 2.7] we see that $\pi_1^{\text{loc.alg}}(X, \xi)$ is Hausdorff and prodiscrete, and more precisely, the natural map :

$$\pi_1^{\text{loc.alg}}(X, \xi) \rightarrow \lim_{H \in \mathcal{H}} \pi_1^{\text{loc.alg}}(X, \xi)/H$$

is an isomorphism of topological groups (where the target is the (projective) limit computed in the category of topological groups). Also, the fibre functor can be upgraded, as usual, to a functor

$$(2.1.5) \quad F_\xi : \mathbf{Cov}^{\text{loc.alg}}(X) \rightarrow \pi_1^{\text{loc.alg}}(X, \xi)\text{-Set}$$

with values in the category of (discrete) sets with continuous left action of the locally algebraic fundamental group. The continuity condition means that the stabilizer of any point of such a set, is an open subgroup.

Proposition 2.1.6. *Let X be a connected analytic S -adic space, and ξ a maximal geometric point of X . Then the functor (2.1.5) is an equivalence.*

Proof. Taking into account remark 2.1.2(iii) and lemma 2.1.3, the proof of [19, Th.2.10] can be taken over *verbatim* : the crucial point is the following

Claim 2.1.7. In the situation of the theorem, suppose that ξ and ξ' are two maximal geometric points of X . Then there exists an isomorphism of functors $F_\xi \xrightarrow{\sim} F_{\xi'}$.

Proof of the claim. For every point $x \in X$, let $U(x) \subset X$ be the union of all the quasi-compact connected open neighborhoods of x in X . It is easily seen that $U(x) \cap U(x') \neq \emptyset$ if and only if $U(x) = U(x')$; since X is connected, it follows that $U(x) = X$ for every $x \in X$. Thus, we may find a connected quasi-compact open subset $U \subset X$ containing the supports of both ξ and ξ' . The restrictions of $F_{U, \xi}$ and $F_{U, \xi'}$ yield fibre functors

$$F_{U, \xi}^{\text{alg}}, F_{U, \xi'}^{\text{alg}} : \mathbf{Cov}^{\text{alg}}(U) \rightarrow \mathbf{Set}$$

which fulfill the axiomatic conditions for a Galois theory, as in [14, Exp.V, §4]; there follows an isomorphism of functors $F_{U, \xi}^{\text{alg}} \xrightarrow{\sim} F_{U, \xi'}^{\text{alg}}$ ([14, Exp.V, Cor.5.7]). In view of remark 2.1.2(v), we deduce an isomorphism between the two fibre functors

$$F_{U, \xi}, F_{U, \xi'} : \mathbf{Cov}^{\text{loc.alg}}(U) \rightarrow \mathbf{Set}.$$

However, $F_{X, \xi} = F_{U, \xi} \circ j^*$ and $F_{X, \xi'} = F_{U, \xi'} \circ j^*$, where $j^* : \mathbf{Cov}^{\text{loc.alg}}(X) \rightarrow \mathbf{Cov}^{\text{loc.alg}}(U)$ is the restriction functor (for the open immersion $j : U \rightarrow X$). The claim follows. $\square$

2.1.8. Let $(G, \cdot)$ be a group; we denote by G^o the group *opposite* to G , whose elements are the same as those of G , and whose composition law is given by the rule : $(g_1, g_2) \mapsto g_2 \cdot g_1$ for all $g_1, g_2 \in G$. A *right G -torsor* (resp. a *right G -principal homogeneous space*) on X is, as usual, a sheaf F on $X_{\text{ét}}$ (resp. a morphism $Y \rightarrow X$ of analytic adic spaces), with a group homomorphism $G^o \rightarrow \text{Aut}(F)$ (resp. $G^o \rightarrow \text{Aut}_X(Y)$), for which there exists a covering family $(U_i \rightarrow X \mid i \in I)$ in $X_{\text{ét}}$, such that the restriction $F|_{U_i}$ (resp. $(U_i \times_X Y \rightarrow U_i)$), together with the induced right G -actions, is isomorphic to the constant sheaf G_{U_i} (resp. is U_i -isomorphic to $U_i \times G$) for every $i \in I$. As usual, any right G -principal homogeneous space on X represents a right G -torsor, unique up to isomorphism.

Lemma 2.1.9. *Let $f : Y \rightarrow X$ be a G -principal space on X . Then the morphism of Y -adic spaces*

$$(2.1.10) \quad Y \times G \rightarrow Y \times_X Y \quad : \quad (y, g) \mapsto (y, yg)$$

is an isomorphism. Especially, f is separated.

Proof. Let F denote the G -torsor on X represented by Y ; then (2.1.10) represents the morphism of sheaves $\varphi : F \times G \rightarrow F \times F$ on $X_{\text{ét}}$ given by the rule : $(s, g) \mapsto (s, sg)$. By assumption, there exists a covering family $(U_i \rightarrow X \mid i \in I)$ such that $\varphi|_{U_i}$ is an isomorphism for every $i \in I$; hence φ is an isomorphism, and then the same holds for (2.1.10). It follows that the diagonal immersion $\Delta_{Y/X} \rightarrow Y \times_X Y$ can be identified to the closed immersion $Y \rightarrow Y \times G$ which maps Y onto the closed subspace $Y \times \{1\}$ (where $1 \in G$ is the neutral element); hence f is separated. $\square$

One defines *left G -torsors*, and *left G -principal homogeneous spaces* in the same way, except that instead of G^o one has the group G . The set of isomorphism classes of right G -torsors on X is denoted

$$H^1(X_{\text{ét}}, G).$$

If G is abelian, this is a group, naturally isomorphic to the Čech cohomology group $\check{H}^1(X_{\text{ét}}, G)$. By general nonsense, the latter is isomorphic to the cohomology group $H^1(X_{\text{ét}}, G)$ calculated by injective resolutions of $\mathbb{Z}$ -modules ([3, Ch.II, Cor.3.6]).

If $G \rightarrow G'$ is any group homomorphism, and F is a G -torsor on X , we obtain a G' -torsor

$$F \times^G G'$$

such that $F \times^G G'(Z) = (F(Z) \times G')/G$ for any étale X -adic space Z (where G acts on $F(Z) \times G'$ by the rule : $(g, (s, g')) \mapsto (sg, g^{-1}g')$). This operation defines a natural transformation :

$$H^1(X_{\text{ét}}, G) \rightarrow H^1(X_{\text{ét}}, G').$$

Proposition 2.1.11. *Let G be an ind-finite group (i.e. G is the union of the filtered family $\mathcal{F}$ of its finite subgroups), X a S -adic space, and F a right G -torsor on X . Then :*

- (i) *F is represented by a right G -principal homogeneous space $Y \rightarrow X$.*
- (ii) *If X is quasi-separated, $Y \rightarrow X$ is a locally algebraic covering.*
- (iii) *If X is quasi-compact and quasi-separated, the natural map :*

$$\operatorname{colim}_{H \in \mathcal{F}} H^1(X_{\text{ét}}, H) \rightarrow H^1(X_{\text{ét}}, G)$$

is a bijection.

Proof. (iii): This is proved as in [25, Lemma 4.1.2], which is a special case. Since one assumption is missing in *loc.cit.*, we repeat briefly the argument here.

One covers X by finitely many affinoids $U_1, \dots, U_n$ which admit finite étale coverings $V_i \rightarrow U_i$, such that $F|_{V_i}$ is the trivial G -torsor on V_i , for every $i \leq n$. The latter is endowed with a natural descent datum, given by a cocycle consisting of a locally constant function $g_i : V_i \times_{U_i} V_i \rightarrow G$. By quasi-compactness, such a function assumes only finitely many values; we may therefore find a finite subgroup $H \subset G$ containing the image of all maps g_i , and it follows that the isomorphism class of the restriction $F|_{U_i}$ lies in the image of $H^1(U_{i,\text{ét}}, H)$. Lastly, the gluing datum for the G -torsor F , relative to the covering $(U_i \mid i = 1, \dots, n)$ amounts to another cocycle $(g_{ij} : U_i \cap U_j \rightarrow G \mid i, j \leq n)$ of the same type. Since X is quasi-separated, $U_i \cap U_j$ is again quasi-compact, hence again the class of Y is the image of some class in $H^1(X_{\text{ét}}, H')$, for some finite group H' containing H . This shows the surjectivity of our map; the injectivity is proved in a similar way : details left to the reader.

(i): The assertion is local on X , hence we may assume that X is quasi-compact and quasi-separated, in which case, by (iii), $F \simeq F' \times^H G$, for some finite subgroup $H \subset G$, and a H -torsor F' ; according to [17, §2.2.3], F is representable by a H -principal homogeneous space $Y_H \rightarrow X$. It follows easily that F is representable by the G -principal homogeneous space $Y_H \times^H G$.

(ii): If X is quasi-separated, the foregoing proof of (i) works on every quasi-compact open subset of X , and a simple inspection shows that the construction yields a locally algebraic covering in this case. $\square$

2.1.12. Let X be a connected quasi-separated S -adic space, ξ a maximal geometric point of X , G an ind-finite group, and $f : Y \rightarrow X$ a right G -principal homogeneous space. By propositions 2.1.11(ii) and 2.1.6, the isomorphism class of Y is determined by the fibre $f^{-1}(\xi)$ together with its natural left $\pi_1^{\text{loc.alg}}(X, \xi)$ -action and right G -actions, which commute with each other. Conversely, every set S endowed with commuting left $\pi_1^{\text{loc.alg}}(X, \xi)$ -action and right G -action, such that moreover the G -action is free and transitive, arises as the fibre of a right G -torsor.

In this situation, it is easily seen that the stabilizer – for the left $\pi_1^{\text{loc.alg}}(X, \xi)$ -action – of any point $y \in f^{-1}(\xi)$ is an open normal subgroup H , and we obtain a group homomorphism :

$$(2.1.13) \quad \pi_1^{\text{loc.alg}}(X, \xi)/H \rightarrow G$$

by assigning to the class $\bar{\sigma}$ of any $\sigma \in \pi_1^{\text{loc.alg}}(X, \xi)$ the unique $g \in G$ such that $\sigma(y_0) = y_0 \cdot g$.

The effect of replacing y_0 by a different point, is to change (2.1.13) by an inner automorphism of G ; summing up, we deduce a natural bijection :

$$H^1(X_{\text{ét}}, G)/\sim \xrightarrow{\sim} \text{Hom}_{\text{Top.Grp}}(\pi_1^{\text{loc.alg}}(X, \xi), G)/\sim$$

onto the set of equivalence classes of continuous group homomorphisms as in (2.1.13) (where G is endowed with the discrete topology), for the equivalence relation $\sim$ induced by inner conjugation on G . Finally, under this bijection, the isomorphism classes of connected G -torsors correspond precisely to the equivalence classes of continuous surjective group homomorphisms.

2.1.14. Let Λ be an ind-finite ring, X a connected S -adic space, ξ a maximal geometric point of X , and F a locally constant Λ -module of finite type on $X_{\text{ét}}$. Set $M := F_\xi$; denote by

$$\mathcal{I}som_\Lambda(M_X, F)$$

the sheaf on $X_{\text{ét}}$ whose sections on any étale X -adic space Z are the isomorphisms $M_Z \xrightarrow{\sim} F|_Z$ of Λ -modules on $Z_{\text{ét}}$. Clearly $\mathcal{I}som_\Lambda(M_X, F)$ is a right G -torsor, for $G := \text{Aut}_\Lambda(M)$; by proposition 2.1.11(ii) it follows that F is trivialized on a locally algebraic covering $g : Y \rightarrow X$, corresponding to a certain open normal subgroup of $\pi_1^{\text{loc.alg}}(X, \xi)$. The natural map $\Gamma(Y, F) \rightarrow M$ is an isomorphism, therefore the natural action of $\pi_1^{\text{loc.alg}}(X, \xi)$ on $\Gamma(Y, F)$ (cp. (2.2.1)) can be transferred to M . This action is *continuous*, i.e. the stabilizer of every element of M is an open subgroup. In this way we obtain an equivalence :

$$(2.1.15) \quad \Lambda_X\text{-Mod}_{\text{loc}} \xrightarrow{\sim} \Lambda[\pi_1^{\text{loc.alg}}(X, \xi)]\text{-Mod}_{\text{f.cont}} \quad F \mapsto F_\xi$$

from the category of locally constant Λ -modules of finite type on $X_{\text{ét}}$, to the category of Λ -modules of finite type, endowed with a continuous linear action of $\pi_1^{\text{loc.alg}}(X, \xi)$.

2.2. Vanishing cycles. For every $\varepsilon \in |K^\times|$, denote by $\mathbb{D}(z, \varepsilon)$ the disc of radius ε , centered at a K -rational point $z \in (\mathbb{P}_K^1)^{\text{ad}}$. Then $\mathbb{D}(z, \varepsilon)$ is an open analytic subspace of $(\mathbb{P}_K^1)^{\text{ad}}$, and

$$\mathbb{D}(z, \varepsilon)^* := \mathbb{D}(z, \varepsilon) \setminus \{z\}$$

is an open subspace of $\mathbb{D}(z, \varepsilon)$. Choose a maximal geometric point $\xi \in \mathbb{D}(z, \varepsilon)^*$ and set

$$\pi(z, \varepsilon) := \pi_1^{\text{loc.alg}}(\mathbb{D}(z, \varepsilon)^*, \xi).$$

Moreover, for every open subgroup $H \subset \pi(z, \varepsilon)$, choose a locally algebraic covering

$$\varphi_H : C_H \rightarrow \mathbb{D}(z, \varepsilon)^*$$

with an isomorphism $F_\xi(C_H) \xrightarrow{\sim} \pi(z, \varepsilon)/H$ of $\pi(z, \varepsilon)$ -sets (where F_ξ is the fibre functor as in (2.1.5)). The family

$$(C_H \mid H \subset \pi(z, \varepsilon))$$

is a cofiltered system of analytic adic spaces (in the terminology of [14, Exp.V, §5], this is a *fundamental pro-covering* of $\mathbb{D}(z, \varepsilon)^*$).

2.2.1. Let now Λ be an ind-finite ring such that $p \in \Lambda^\times$, and $f : X \rightarrow \mathbb{D}(z, \varepsilon)$ a morphism of analytic S -adic spaces. For every open subgroup $H \subset \pi(z, \varepsilon)$, let $j_H : C_H \times_{\mathbb{D}(z, \varepsilon)} X \rightarrow X$ be the projection; set $X_0 := f^{-1}(z)$, and let $i : X_0 \rightarrow X$ be the closed immersion. To any lower bounded complex $\mathcal{F}^\bullet$ of Λ -modules on $X_{\text{ét}}$, we associate the complex of Λ -modules

$$R\Psi_{\eta_{z, \varepsilon}} \mathcal{F}^\bullet := \operatorname{colim}_{H \subset \pi(z, \varepsilon)} i^* Rj_{H*} j_H^* \mathcal{F}^\bullet \quad \text{on } X_{0, \text{ét}}.$$

We construct as follows an action of $\pi(z, \varepsilon)$ on the cohomology sheaves $R^i \Psi_{\eta_{z, \varepsilon}} \mathcal{F}^\bullet$ of this complex. To begin with, let F be any sheaf on $X_{\text{ét}}$, and set

$$\tilde{F} := \operatorname{colim}_{H \subset \pi(z, \varepsilon)} j_{H*} j_H^* F.$$

Let $g \in \pi(z, \varepsilon)$ be any element, and $H \subset \pi(z, \varepsilon)$ an open subgroup; the right translation action of g on $\pi(z, \varepsilon)$ induces a bijection of pointed sets :

$$\pi(z, \varepsilon)/H \xrightarrow{\sim} \pi(z, \varepsilon)/g^{-1}Hg \quad : \quad [\gamma H] \mapsto [(\gamma g)g^{-1}Hg]$$

which is clearly equivariant for the left action of $\pi(z, \varepsilon)$ on both sets, and therefore corresponds to a unique isomorphism of $\mathbb{D}(z, \varepsilon)^*$ -adic spaces :

$$g_H : C_H \xrightarrow{\sim} C_{g^{-1}Hg}$$

whence an isomorphism

$$(2.2.2) \quad F(U \times_{\mathbb{D}(z, \varepsilon)} g_H) : F(U \times_{\mathbb{D}(z, \varepsilon)} C_{g^{-1}Hg}) \xrightarrow{\sim} F(U \times_{\mathbb{D}(z, \varepsilon)} C_H)$$

for every object $U \rightarrow X$ of $X_{\text{ét}}$. Moreover, if $H \subset H'$ for any other open subgroup $H' \subset \pi(z, \varepsilon)$, we have a commutative diagram of S -adic spaces :

$$\begin{array}{ccc} C_H & \xrightarrow{g_H} & C_{g^{-1}Hg} \\ \downarrow & & \downarrow \\ C_{H'} & \xrightarrow{g_{H'}} & C_{g^{-1}H'g} \end{array}$$

whose vertical arrows are induced by the quotient map $\pi(z, \varepsilon)/H \rightarrow \pi(z, \varepsilon)/H'$. Therefore the automorphisms (2.2.2), for fixed g and varying H , organize into a directed system. Suppose that U is affinoid; then, taking colimits of the maps (2.2.2), we obtain an automorphism :

$$\tilde{g}(U) := \operatorname{colim}_{H \subset \pi(z, \varepsilon)} F(U \times_{\mathbb{D}(z, \varepsilon)} g_H) : \tilde{F}(U) \xrightarrow{\sim} \tilde{F}(U)$$

(since U is quasi-compact and quasi-separated, the functor $\Gamma(U, -)$ commutes with filtered colimits). Clearly the maps $\tilde{g}(U)$ for variable U , assemble to a well defined automorphism

$$\tilde{g} : \tilde{F} \xrightarrow{\sim} \tilde{F}$$

which, for variable g , defines a *left* $\pi(z, \varepsilon)$ -action on $\tilde{F}$ (the contravariance of F transforms the right translation action into a left action).

In case F is a sheaf of groups, it is clear that $\pi(z, \varepsilon)$ acts by group automorphisms on $\widetilde{F}$. We may then apply the above constructions to the terms of an injective resolution of $\mathcal{F}^\bullet$; by functoriality, this yields an action of $\pi(z, \varepsilon)$ on $R^i\Psi_{\eta_{z,\varepsilon}}\mathcal{F}^\bullet$.

Remark 2.2.3. With the notation of (2.2.1), notice that g_H is the identity automorphism of C_H , whenever $g \in H$. It follows that, for every geometric point $\bar{x}$ of X_0 (see [17, Def.2.5.1]), the $\pi(z, \varepsilon)$ -action on the stalk $\widetilde{F}_{\bar{x}}$ is continuous in the sense of (2.1.14). Hence, for every $i \in \mathbb{Z}$, the stalks of $R^i\Psi_{\eta_{z,\varepsilon}}\mathcal{F}^\bullet$ are objects of the category

$$\Lambda[\pi(z, \varepsilon)]\text{-}\mathbf{Mod}_{\text{cont}}$$

whose objects are all the Λ -modules endowed with a continuous action of $\pi(z, \varepsilon)$.

Proposition 2.2.4. *In the situation of (2.2.1), let $g : Y \rightarrow X$ be a morphism of analytic S -adic spaces, set $Y_0 := g^{-1}X_0$, and denote by $g_0 : Y_0 \rightarrow X_0$ the restriction of g . Let also $\mathcal{F}^\bullet$ (resp. $\mathcal{G}^\bullet$) be a lower bounded complex of Λ -modules on $X_{\text{ét}}$ (resp. on $Y_{\text{ét}}$). We have :*

(i) *If g is smooth, there exists a natural isomorphism :*

$$g_0^*R\Psi_{\eta_{z,\varepsilon}}\mathcal{F}^\bullet \xrightarrow{\sim} R\Psi_{\eta_{z,\varepsilon}}g^*\mathcal{F}^\bullet$$

in the derived category $D^+(Y_{0,\text{ét}}, \Lambda)$ of lower bounded complexes of Λ -modules on $Y_{0,\text{ét}}$.

(ii) *If g is weakly of finite type and quasi-separated, there exists a natural isomorphism :*

$$R\Psi_{\eta_{z,\varepsilon}}(Rg_*\mathcal{G}^\bullet) \xrightarrow{\sim} Rg_{0*}R\Psi_{\eta_{z,\varepsilon}}\mathcal{G}^\bullet \quad \text{in } D^+(X_{0,\text{ét}}, \Lambda).$$

(iii) *There is a spectral sequence :*

$$E_2^{pq} := R^p\Psi_{\eta_{z,\varepsilon}}(\mathcal{H}^q\mathcal{F}^\bullet) \Rightarrow R^{p+q}\Psi_{\eta_{z,\varepsilon}}\mathcal{F}^\bullet.$$

(iv) *The isomorphisms in (i) and (ii) induce $\pi(z, \varepsilon)$ -equivariant isomorphisms on the cohomology sheaves of the respective terms, and the spectral sequence of (iii) is $\pi(z, \varepsilon)$ -equivariant.*

Proof. (i): This follows easily from the base change theorem [17, Th.4.1.1(a)], applied to the cartesian diagram :

$$(2.2.5) \quad \begin{array}{ccc} C_H \times_{\mathbb{D}(z,\varepsilon)} Y & \longrightarrow & Y \\ C_H \times_{\mathbb{D}(z,\varepsilon)} g \downarrow & & \downarrow g \\ C_H \times_{\mathbb{D}(z,\varepsilon)} X & \xrightarrow{j_H} & X. \end{array}$$

(ii): Likewise, this follows from (2.2.5) and [17, Th.4.1.1(c)] (since j_H is generalizing, and g is weakly of finite type and quasi-separated), together with the cartesian diagram :

$$\begin{array}{ccc} Y_0 & \longrightarrow & Y \\ g_0 \downarrow & & \downarrow g \\ X_0 & \xrightarrow{i} & X \end{array}$$

and again [17, Th.4.1.1(c)], since g is weakly of finite type ([17, Def.1.2.1(iv)]) and quasi-separated, and i is generalizing. Notice here, that the functor Rg_* commutes with filtered colimits, since g is quasi-compact and quasi-separated ([17, Lemma 2.3.13(ii)]).

Finally, (iii) and (iv) are immediate from the construction. $\square$

Lemma 2.2.6. *In the situation of (2.2), let G be any ind-finite group. We have :*

$$\operatorname{colim}_{H \subset \pi(z,\varepsilon)} H^1(C_{H,\text{ét}}, G) = 0.$$

Proof. Let $c \in H^1(C_{H,\text{ét}}, G)$ for some C_H as above. The class c represents a right G -torsor F on $C_{H,\text{ét}}$, and by proposition 2.1.11(ii) we may find a connected locally algebraic covering $Y \rightarrow C_H$ trivializing F . In view of remark 2.1.2(iii), there exists an open subgroup $L \subset H$ such that C_L dominates Y , which means that the image of c vanishes in $H^1(C_{L,\text{ét}}, G)$. $\square$

2.2.7. Let now $\varepsilon' \in |K^\times|$ be a value with $\varepsilon' < \varepsilon$, choose a maximal geometric point $\xi' \in \mathbb{D}(z, \varepsilon')^*$, and let $\pi(z, \varepsilon')$ be the automorphism group of the corresponding fibre functor $F_{\xi'}$ on the locally algebraic coverings of $\mathbb{D}(z, \varepsilon')^*$.

As usual, the open immersion $j : \mathbb{D}(z, \varepsilon')^* \rightarrow \mathbb{D}(z, \varepsilon)^*$ induces a group homomorphism

$$(2.2.8) \quad \rho : \pi(z, \varepsilon') \rightarrow \pi(z, \varepsilon)$$

well-defined up to inner automorphisms; more precisely, ρ is induced by a choice of isomorphism of fibre functors on $\mathbf{Cov}^{\text{loc.alg}}(\mathbb{D}(z, \varepsilon)^*)$:

$$(2.2.9) \quad F_\xi \xrightarrow{\sim} F_{\xi'} \circ j^*$$

where $j^* : \mathbf{Cov}^{\text{loc.alg}}(\mathbb{D}(z, \varepsilon)^*) \rightarrow \mathbf{Cov}^{\text{loc.alg}}(\mathbb{D}(z, \varepsilon')^*)$ is the restriction functor. Now, pick a fundamental pro-covering of $\mathbb{D}(z, \varepsilon')^*$:

$$(C'_L \mid L \subset \pi(z, \varepsilon'))$$

as in (2.2). For every open subgroup $H \subset \pi(z, \varepsilon)$, the preimage $\rho^{-1}H \subset \pi(z, \varepsilon')$ is an open subgroup, and the induced injective map

$$\pi(z, \varepsilon') / \rho^{-1}H \rightarrow \pi(z, \varepsilon) / H$$

corresponds to an open and closed immersion of $\mathbb{D}(z, \varepsilon')$ -adic spaces :

$$(2.2.10) \quad C'_{\rho^{-1}H} \rightarrow C_H \times_{\mathbb{D}(z, \varepsilon)} \mathbb{D}(z, \varepsilon').$$

If F is any sheaf on $X_{\text{ét}}$, define $\tilde{F}$ as in (2.2.1); let also $X' := f^{-1}\mathbb{D}(z, \varepsilon')$, and for every open subgroup $L \subset \pi(z, \varepsilon')$, denote by $j'_L : C'_L \times_{\mathbb{D}(z, \varepsilon')} X(\varepsilon') \rightarrow X(\varepsilon')$ the projection. Set $F' := F|_{X'}$, and define :

$$\tilde{F}' := \text{colim}_{L \subset \pi(z, \varepsilon')} j'_{L*} j'^*_{L*} F'.$$

Notice that, for every open subgroup $H \subset \pi(z, \varepsilon)$, the morphism $j'_{\rho^{-1}H}$ can be written as the composition of $b_H := X' \times_{\mathbb{D}(z, \varepsilon')} (2.2.10)$ and the restriction $j''_H : C_H \times_{\mathbb{D}(z, \varepsilon)} X' \rightarrow X'$ of j_H . Then the unit of adjunction $\mathbf{1} \rightarrow b_{H*} b_H^*$ yields a natural morphism :

$$\tilde{F}|_{X'} \xrightarrow{\sim} \text{colim}_{H \subset \pi(z, \varepsilon)} j''_{H*} b_{H*} b_H^* j''^*_{H*} F' \rightarrow \tilde{F}'$$

which is equivariant for the left action of $\pi(z, \varepsilon')$ defined above (where $\pi(z, \varepsilon')$ acts on $\tilde{F}|_{\mathbb{D}(z, \varepsilon')}$ by restriction of the $\pi(z, \varepsilon)$ -action, via (2.2.8)). Finally, let $\mathcal{F}^\bullet$ be a complex of Λ -modules on $X_{\text{ét}}$; by applying this construction to the terms of an injective resolution of $\mathcal{F}^\bullet$, we deduce natural morphisms :

$$R^i \Psi_{\eta_{z, \varepsilon}} \mathcal{F}^\bullet \rightarrow R^i \Psi_{\eta_{z, \varepsilon'}} \mathcal{F}^\bullet_{|X'} \quad \text{for every } i \geq 0.$$

2.2.11. We wish now to “take the limit” for $\varepsilon \rightarrow 0$, of the previous construction. To begin with, fix a strictly decreasing sequence $(\varepsilon_n \mid n \in \mathbb{N})$ with $\varepsilon_n \in |K^\times|$ for every $n \in \mathbb{N}$, and such that $\varepsilon_n \rightarrow 0$ for $n \rightarrow +\infty$. For every $n \in \mathbb{N}$, fix a geometric base point ξ_n of $\mathbb{D}(z, \varepsilon_n)^*$ and a group homomorphism

$$(2.2.12) \quad \pi(z, \varepsilon_{n+1}) \rightarrow \pi(z, \varepsilon_n)$$

as in (2.2.8), as well as a fundamental pro-covering of $\mathbb{D}(z, \varepsilon_n)^*$:

$$(C_H^{(n)} \mid H \subset \pi(z, \varepsilon_n)).$$

Suppose that we have a morphism of analytic S -adic spaces $f : X \rightarrow \mathbb{D}(z, \varepsilon_0)$, and a complex $\mathcal{F}^\bullet$ of Λ -modules on $X_{\text{ét}}$; for every $n \in \mathbb{N}$, set

$$X_n := f^{-1}\mathbb{D}(z, \varepsilon_n).$$

In light of the discussion in (2.2.7), the foregoing choices determine a system of morphisms of Λ -modules on $X_{0,\text{ét}}$:

$$(2.2.13) \quad R^i \Psi_{\eta_z, \varepsilon_n} \mathcal{F}_{|X_n}^\bullet \rightarrow R^i \Psi_{\eta_z, \varepsilon_{n+1}} \mathcal{F}_{|X_{n+1}}^\bullet \quad \text{for every } i, n \in \mathbb{N}.$$

For each $n \in \mathbb{N}$, the group $\pi(z, \varepsilon_n)$ acts (on the left) on the n -th term of this system, and the maps are $\pi(z, \varepsilon_{n+1})$ -equivariant (for the action on the n -th term obtained by restriction via (2.2.12)). If we forget the group actions, we have for each $i \in \mathbb{N}$ a direct system of Λ -modules on $X_{0,\text{ét}}$, indexed by $n \in \mathbb{N}$, whose colimit is independent of the choices of ε_n and ξ_n , and shall be denoted simply

$$R^i \Psi_{\eta_z} \mathcal{F}^\bullet.$$

With the notation of (2.2.1), the unit of adjunction $1 \rightarrow j_{H*} j_H^*$ induces, after taking the colimit over the subgroups $H \subset \pi(z, \varepsilon)$, a natural morphism :

$$\mathcal{F}^\bullet \rightarrow R \Psi_{\eta_z} \mathcal{F}^\bullet \quad \text{for every complex } \mathcal{F}^\bullet \text{ of } \Lambda\text{-modules}$$

whose cone shall be denoted :

$$R \Phi_{\eta_z} \mathcal{F}^\bullet.$$

Corollary 2.2.14. *Assertions (i), (ii) and (iii) of proposition 2.2.4 still hold with $R \Psi_{\eta_z, \varepsilon}$ replaced everywhere by either $R \Psi_{\eta_z}$ or $R \Phi_{\eta_z}$.*

Proof. To show assertion (i) for $R \Psi_{\eta_z}$, it suffices to remark that, by general nonsense, the functors g^* and g_0^* commute with arbitrary colimits; for assertion (ii), likewise the functors and Rg_* and Rg_{0*} commute with arbitrary filtered colimits ([17, Lemma 2.3.13(ii)]). Assertion (iii) is immediate. Finally, the assertions concerning $R \Phi_{\eta_z}$ follow from the case of $R \Psi_{\eta_z}$, since g^* and Rg_* are triangulated functors. $\square$

Of course, the limit of the projective system (2.2.12) still acts on $R^i \Psi_{\eta_z} \mathcal{F}^\bullet$; however, not much is known concerning this projective limit. Nevertheless, in certain cases it is possible to salvage a useful residual group action on $R^0 \Psi_{\eta_z} \mathcal{F}^\bullet$ and $R^0 \Phi_{\eta_z} \mathcal{F}^\bullet$, as explained in the following paragraph.

2.2.15. Namely, let F be a Λ -module on $\mathbb{D}(z, \varepsilon_0)_{\text{ét}}$, and suppose that there exists $N \in \mathbb{N}$, such that the restriction F' of F to $\mathbb{D}(z, \varepsilon_N)_{\text{ét}}^*$ is a locally constant Λ -module of finite type. In that case, F' is determined, up to natural isomorphism, by the stalk F_{ξ_N} endowed with its natural continuous $\pi(z, \varepsilon_N)$ -action. The latter is obtained as follows. According to (2.1.14), F' is trivialized by some locally algebraic Galois covering $Y \rightarrow \mathbb{D}(z, \varepsilon_N)^*$, corresponding to a certain open normal subgroup $L \subset \pi(z, \varepsilon_N)$. Then we have a natural isomorphism $\Gamma(Y, F) \xrightarrow{\sim} F_{\xi_N}$, and the quotient group $\pi(z, \varepsilon_N)/L$ acts on $\Gamma(Y, F)$ as explained in (2.2.1).

On the other hand, by inspecting the definitions, we see that the Λ -module $R^0 \Psi_{\eta_z} F[0]$ is computed as the colimit

$$\operatorname{colim}_{n \in \mathbb{N}} \operatorname{colim}_{H \subset \pi(z, \varepsilon_n)} \Gamma(C_{H, \text{ét}}^{(n)}, F).$$

The first colimit does not change if we replace $\mathbb{N}$ by the cofinal subset of all integers $n \geq N$. Likewise, by (2.1.14), we may moreover replace the filtered system of all open subgroups $H \subset \pi(z, \varepsilon_n)$ by the cofinal system of all such H with the property that F restricts to a constant sheaf on $C_{H, \text{ét}}^{(n)}$. After these changes, notice that all the transition maps in the filtered system are isomorphisms, since each $C_H^{(n)}$ is connected. There follows a natural isomorphism :

$$R^0 \Psi_{\eta_z} F[0] \xrightarrow{\sim} F_{\xi_N}$$

especially we obtain a natural continuous $\pi(z, \varepsilon_N)$ -action on the above vanishing cycle. It is then convenient to introduce the notation

$$F_{\eta_z}$$

for the datum of $R^0\Psi_{\eta_z}F[0]$, viewed as an object of $\Lambda[\pi(z, \varepsilon_N)]\text{-}\mathbf{Mod}_{f,\text{cont}}$. Finally, to eliminate the dependence on the choice of N , notice that the group homomorphisms (2.2.12) induce a direct system of categories and functors :

$$(2.2.16) \quad \Lambda[\pi(z, \varepsilon_n)]\text{-}\mathbf{Mod}_{f,\text{cont}} \rightarrow \Lambda[\pi(z, \varepsilon_{n+1})]\text{-}\mathbf{Mod}_{f,\text{cont}} \quad \text{for every } n \in \mathbb{N}.$$

We may therefore consider the 2-colimit

$$\Lambda[\pi(z)]\text{-}\mathbf{Mod} := 2\text{-colim}_{n \in \mathbb{N}} \Lambda[\pi(z, \varepsilon_n)]\text{-}\mathbf{Mod}_{f,\text{cont}}.$$

Then, under the foregoing assumptions on F , we may regard F_{η_z} as an object of this category, which should be thought of as a replacement for the category of continuous representations of the inertia subgroup which one encounters in the study of the local monodromy (around a given closed point) of étale sheaves on algebraic curves.

Likewise, we may represent $R^0\Phi_{\eta_z}F[0]$ as the cokernel of the natural map :

$$F_z \rightarrow F_{\eta_z}$$

which is a morphism in $\Lambda[\pi(z)]\text{-}\mathbf{Mod}$, provided we endow F_z with the trivial $\pi(z, \varepsilon_0)$ -action. These mostly trivial observations shall be amplified later, starting with remark (2.4.12)(i).

Remark 2.2.17. Following standard conventions, we shall use the notation F_{η_z} also for Λ -modules defined only on $\mathbb{D}(z, \varepsilon)_{\text{ét}}^*$: rigorously, this means that we replace F by its extension by zero over the whole of $\mathbb{D}(z, \varepsilon)$.

Proposition 2.2.18. *In the situation of (2.2.11), suppose that $X \rightarrow \mathbb{D}(z, 1)$ is a smooth morphism, and let M be a Λ -module of finite length. Then we have :*

$$R^i\Psi_{\eta_z}M_X = \begin{cases} M_{X_0} & \text{if } i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. In light of corollary 2.2.14, we may assume that $X = \mathbb{D}(z, 1)$, in which case, notice that the open subspaces $\mathbb{D}(z, \varepsilon) \subset \mathbb{D}(z, 1)$ form a cofinal system of étale neighborhoods of $z \in \mathbb{D}(z, 1)$. Since the stalk of a presheaf is isomorphic to the stalk of the associated sheaf, we are thus reduced to showing that the Λ -module

$$\text{colim}_{n \in \mathbb{N}} \text{colim}_{H \subset \pi(z, \varepsilon_n)} H^i(C_{H, \text{ét}}^{(n)}, M)$$

equals M for $i = 0$, and vanishes for $i > 0$. The assertion for $i = 0$ just translates the fact that $C_H^{(n)}$ is a connected adic space, for every $n \in \mathbb{N}$ and every open subgroup $H \subset \pi(z, \varepsilon_n)$. The case where $i = 1$ follows from the more precise lemma 2.2.6.

The assertion for $i \geq 2$ can be deduced from [17, Cor.7.5.6] by a reduction to the case where Λ is a finite ring. However, it is quicker to reduce first to the case where Λ has finite length as a Λ -module, then to the case where Λ is a field (by an easy induction on the length of Λ), and then apply the Poincaré duality of lemma 2.3.10(iii) : the details shall be left to the reader. $\square$

Corollary 2.2.19. *Suppose that $X \rightarrow \mathbb{D}(z, 1)$ is a smooth morphism, and let F be a locally constant Λ -module of finite type on $X_{\text{ét}}$. Then :*

$$R\Phi_{\eta_z}F = 0 \quad \text{in } D^+(X_0, \Lambda).$$

Proof. By (2.1.14), F is trivialized on a locally algebraic covering $g : Y \rightarrow X$. Let $g_0 : g^{-1}X_0 \rightarrow X_0$ be the restriction of g ; clearly it suffices to show that $g_0^*R\Phi_{\eta_z}F$ vanishes. The latter is an immediate consequence of propositions 2.2.4(i) and 2.2.18, and corollary 2.2.14. $\square$

2.3. Fourier transform. We wish to recall briefly the construction and the main properties of the cohomological Fourier transform introduced in [24]. To begin with, we remark that the logarithm power series :

$$\log(1+x) := \sum_{n \in \mathbb{N}} (-1)^n \cdot \frac{x^{n+1}}{n+1}$$

defines a locally algebraic covering

$$\log : \mathbb{D}(0, 1^-) \rightarrow (\mathbb{A}_K^1)^{\text{ad}} \quad \text{where} \quad \mathbb{D}(0, 1^-) := \bigcup_{0 < \varepsilon < 1} \mathbb{D}(0, \varepsilon).$$

Moreover, $\log$ is a morphism of *analytic adic groups*, if we endow $(\mathbb{A}_K^1)^{\text{ad}}$ with the additive group law of $(\mathbb{G}_a)^{\text{ad}}$, and $\mathbb{D}(0, 1^-)$ with the restriction of the multiplicative group law of $(\mathbb{G}_m)^{\text{ad}}$ (more precisely, with the group law given by the rule : $(x, y) \mapsto x + y + xy$, which is the “translation” of the usual multiplication, whose neutral element is $0 \in \mathbb{D}(0, 1^-)$).

The kernel of $\log$ is the group

$$\mu_{p^\infty} := \bigcup_{n \in \mathbb{N}} \mu_{p^n}$$

of p -primary roots of 1 in K ; therefore $\mathbb{D}(0, 1^-)$ represents a μ_{p^∞} -torsor on $(\mathbb{A}_K^1)^{\text{ad}}_{\text{ét}}$, which we denote $\mathcal{L}$.

2.3.1. Suppose now that the multiplicative group $\Lambda^\times$ contains a subgroup isomorphic to μ_{p^∞} (for instance, this holds for $\Lambda := \overline{\mathbb{F}}_\ell$, if ℓ is any prime number different from p). Then, the choice of a non-trivial group homomorphism :

$$\psi : \mu_{p^\infty} \rightarrow \Lambda^\times$$

determines a locally constant Λ -module, free of rank one on $(\mathbb{A}_K^1)^{\text{ad}}_{\text{ét}}$:

$$\mathcal{L}_\psi := \mathcal{L}^{\mu_{p^\infty}} \times \Lambda$$

whose restriction to $\mathbb{D}(0, 1^-)_{\text{ét}}$ is a constant Λ -module.

Let X be any S -adic space, and $f \in \mathcal{O}_X(X)$ be any global section; we may regard f as a morphism of S -adic spaces $f : X \rightarrow (\mathbb{A}_K^1)^{\text{ad}}$, and then we may set :

$$\mathcal{L}_\psi \langle f \rangle := f^* \mathcal{L}_\psi$$

which is a locally constant Λ -module on $X_{\text{ét}}$. If $f, g \in \Gamma(X, \mathcal{O}_X)$ are any two analytic functions, we have a natural isomorphism of Λ -modules :

$$(2.3.2) \quad \mathcal{L}_\psi \langle f + g \rangle \xrightarrow{\sim} \mathcal{L}_\psi \langle f \rangle \otimes_\Lambda \mathcal{L}_\psi \langle g \rangle.$$

2.3.3. Let A and A' be two copies of the affine line $(\mathbb{A}_K^1)^{\text{ad}}$, with global coordinates x and respectively x' . Let also D and D' be two copies of the projective line $(\mathbb{P}_K^1)^{\text{ad}}$, and fix open immersions

$$\alpha : A \rightarrow D \quad \alpha' : A' \rightarrow D'$$

so that $D = A \cup \{\infty\}$ and $D' = A' \cup \{\infty'\}$. We use $\mathcal{L}_\psi$ as the “integral kernel” of our operator; namely, the *Fourier transform* is the triangulated functor

$$\mathcal{F}_\psi : \mathcal{D}^b(A_{\text{ét}}, \Lambda) \rightarrow \mathcal{D}^b(A'_{\text{ét}}, \Lambda) \quad F^\bullet \mapsto R p'_! (p^* F^\bullet \otimes_\Lambda \mathcal{L}_\psi \langle m \rangle)[1]$$

where

$$A \xleftarrow{p} A \times_S A' \xrightarrow{p'} A'$$

are the two projections, and $m : A \times_S A' \rightarrow (\mathbb{A}_K^1)^{\text{ad}}$ is the dual pairing, given by the rule : $(x, x') \mapsto x \cdot x'$. Especially, let $0' \in A'$ be the origin of the coordinate x' (i.e. $x'(0') = 0$ in the residue field of the point $0'$); then notice the natural isomorphism :

$$(2.3.4) \quad \mathcal{F}(F^\bullet)_{0'} \xrightarrow{\sim} R \Gamma_c F^\bullet \quad \text{for every } F^\bullet \in \text{Ob}(\mathcal{D}^b(A_{\text{ét}}, \Lambda)).$$

It is shown in [24, Th.7.1.2] that $\mathcal{F}_\psi$ is an equivalence of categories; more precisely, let A'' be a third copy of the affine line, with global coordinate x'' , and dual pairing

$$m' : A' \times_S A'' \rightarrow (\mathbb{A}_K^1)^{\text{ad}} \quad (x', x'') \mapsto x' \cdot x''.$$

We have a Fourier transform $\mathcal{F}'_\psi : D^b(A'_{\text{ét}}, \Lambda) \rightarrow D^b(A''_{\text{ét}}, \Lambda)$ (namely, the “integral operator” whose kernel is $\mathcal{L}_\psi \langle m' \rangle$), and there is a natural isomorphism of functors :

$$(2.3.5) \quad \mathcal{F}'_\psi \circ \mathcal{F}_\psi F^\bullet \xrightarrow{\sim} a_* F^\bullet(-1)$$

where (-1) denotes Tate twist and $a : A \xrightarrow{\sim} A''$ is (-1) -times the double duality isomorphism, given by the rule : $x \mapsto -x''$. An immediate consequence of (2.3.4) and (2.3.5) is the natural isomorphism :

$$(2.3.6) \quad F_0^\bullet(-1)[-1] \xrightarrow{\sim} R\Gamma_c \mathcal{F}_\psi(F^\bullet) \quad \text{for every } F^\bullet \in \text{Ob}(D^b(A_{\text{ét}}, \Lambda))$$

where $F_0^\bullet$ denotes the stalk of $F^\bullet$ over the origin $0 \in A$ of the coordinate x .

2.3.7. Moreover, $\mathcal{F}$ commutes with Poincaré-Verdier duality; the latter assertion is a consequence of the following. Define a second functor :

$$\mathcal{F}_{\psi,*} : D^b(A_{\text{ét}}, \Lambda) \rightarrow D^b(A'_{\text{ét}}, \Lambda) \quad F^\bullet \mapsto Rp'_*(p^* F^\bullet \otimes_\Lambda \mathcal{L}_\psi \langle m \rangle)[1].$$

Proposition 2.3.8. *The natural transformation*

$$\mathcal{F}_\psi \rightarrow \mathcal{F}_{\psi,*}$$

(deduced from the natural morphism $Rp'_! \rightarrow Rp'_*$) is an isomorphism of functors.

Proof. This is shown in [24, Th.7.1.6]. A “formal” proof can also be given as in a recent preprint by Boyarchenko and Drinfeld ([11, Th.G.6]) : in order to repeat the argument in our context, one needs a version of Poincaré duality for Λ -modules. We sketch the arguments in the following paragraph. $\square$

2.3.9. To begin with, let R be any torsion ring, and $f : X \rightarrow Y$ a morphism of analytic pseudo-adic spaces which is locally of $^+$ weakly finite type, separated, taut (see [17, Def.5.1.9]), and of finite transcendental dimension; by [17, Th.7.1.1], the functor

$$Rf_! : D^+(X_{\text{ét}}, R) \rightarrow D^+(Y_{\text{ét}}, R)$$

admits a right adjoint

$$Rf_{(R)}^! D^+(Y_{\text{ét}}, R) \rightarrow D^+(X_{\text{ét}}, R).$$

On the other hand, for any ring homomorphism $R \rightarrow R'$, denote by $\rho_X : D^+(X_{\text{ét}}, R') \rightarrow D^+(X_{\text{ét}}, R)$ the restriction of scalars, and likewise define ρ_Y ; then we have :

Lemma 2.3.10. *With the notation of (2.3.9), the following holds :*

(i) *There exists a natural isomorphism of functors :*

$$\rho_X \circ Rf_{(R')}^! \xrightarrow{\sim} Rf_{(R)}^! \circ \rho_Y.$$

(ii) *Suppose additionally that f is smooth of pure dimension d , and that R is a $\mathbb{Z}/n\mathbb{Z}$ -algebra, where $n \in \mathbb{N}$ is invertible in $\mathcal{O}_Y^+$; then there is a natural isomorphism :*

$$f^* F^\bullet(d)[2d] \xrightarrow{\sim} Rf_{(R)}^! F^\bullet \quad \text{for every } F^\bullet \in \text{Ob}(D^+(Y_{\text{ét}}, \Lambda)).$$

(iii) *In the situation of (ii), assume that $Y = S$, and that R is an injective R -module (e.g. R is a field). Let F be a constructible locally constant R -module on $X_{\text{ét}}$; then there are natural isomorphisms :*

$$H^{2d-q}(X_{\text{ét}}, \mathcal{H}om_R(F, R_X)(d)) \xrightarrow{\sim} \text{Hom}_R(H_c^q(X_{\text{ét}}, F), R) \quad \text{for every } q \in \mathbb{Z}.$$

Proof. (i): This isomorphism is not explicitly stated in [17], but it can be proved as in [5, Exp.XVIII, Prop.3.1.12.1].

(ii): Such isomorphism is established in [17, Th.7.5.3], for $R = \mathbb{Z}/n\mathbb{Z}$; the general case follows from this and from (i).

(iii) is a standard consequence of (ii) : cp. [17, Cor.7.5.6]. $\square$

2.3.11. Clearly our Λ is a $\mathbb{Z}/n\mathbb{Z}$ -algebra, for a suitable n fulfilling the conditions of lemma 2.3.10(ii) (take $n > 0$ such that $n \cdot 1 = 0$ in Λ). On the other hand, it is not clear to me whether, under our current assumptions, Λ is always an injective Λ -module. In order to apply lemma 2.3.10(iii) to a given Λ -module F , we shall therefore try to find a finite descending filtration $\text{Fil}^\bullet F$ of F , whose graded terms $\text{gr}^i F$ are annihilated by residue fields of Λ . This is always possible, provided Λ is noetherian (since in this case Λ shall be also artinian). For this reason, *henceforth we assume that Λ is a noetherian ind-finite ring such that $p \in \Lambda^\times$.*

2.3.12. Let X be any scheme locally of finite type over $\text{Spec } K$, and F a Λ -module on the étale site of X^{ad} ; recall that F is *Zariski constructible*, if for every $x \in X^{\text{ad}}$ there exists a subset $Z \subset X$ locally closed and constructible for the Zariski topology of X , such that $x \in Z^{\text{ad}}$, and such that the restriction of F is a locally constant Λ -module of finite type on the étale site of Z^{ad} . We are especially interested in the Fourier transform of Zariski constructible Λ -modules with *bounded ramification* on $A_{\text{ét}}$. To explain the latter condition, consider more generally, for a given K -rational point z of $\mathbb{P}_K^1$, a locally constant Λ -module F of finite type on $\mathbb{D}(z, \varepsilon)^*$; in [25, Cor.4.1.16] we attach to F its *Swan conductor*

$$\text{sw}_z^{\mathfrak{h}}(F, 0^+) \in \mathbb{N} \cup \{+\infty\}$$

and following [25, Def.4.2.1], we say that F has *bounded ramification around z* , if $\text{sw}_z^{\mathfrak{h}}(F, 0^+)$ is a (finite) integer. Regrettably, the definition in *loc.cit.* assumes moreover that the stalks of F are free Λ -modules, but by inspecting the proofs, one sees easily that all the results of [25, §4.2] hold when the stalks are only assumed to be Λ -modules of finite length : the point is that the Swan conductor depends only on the class of F in the Grothendieck group of Λ -modules on $D(z, \varepsilon)^*$, hence we may reduce to the case where F is annihilated by a maximal ideal of Λ , by considering a descending filtration $F \supset F_1 \supset F_2 \supset \dots$, where $F_{i+1} = \mathfrak{n}_i F_i$ for every $i \geq 1$, with $\mathfrak{n}_i \subset \Lambda$ a maximal ideal. (Such a filtration is compatible with the break decompositions of *loc.cit.*)

2.3.13. For instance, suppose that F has *finite monodromy*, i.e. F is trivialized on a finite étale covering $\mathbb{D}(z, \varepsilon)^*$; in that case $\text{sw}_z^{\mathfrak{h}}(F, 0^+) = 0$, especially F has bounded ramification. In such situation, the p -adic Riemann existence theorem ([25, Th.2.4.3]) ensures that – after replacing ε by a smaller radius – F is trivialized on a covering of *Kummer type* :

$$\mathbb{D}(z, \varepsilon^{1/d})^* \rightarrow \mathbb{D}(z, \varepsilon)^* \quad : \quad x \mapsto x^d.$$

For this reason, we shall also say that such F is *tamely ramified*.

2.3.14. Let C be a smooth connected projective curve defined over $\text{Spec } K$, and F a Λ -module on the étale site of C^{ad} ; assume that F is Zariski-constructible, so all the stalks of F are Λ -modules of finite length, and there exists an open subset $U \subset C$ such that F restricts to a locally constant Λ -module on the étale site of U^{ad} . For every $x \in C \setminus U$, we may choose an open immersion $\varphi_x : \mathbb{D}(0, 1) \rightarrow C^{\text{ad}}$ of S -adic spaces, such that $\varphi(0) = x$. Then we may define the *Swan conductor of F at x* as the integer

$$\text{sw}_x^{\mathfrak{h}}(F, 0^+) := \text{sw}_0^{\mathfrak{h}}(\varphi_x^* F, 0^+).$$

One verifies easily that this quantity is independent of the choice of φ_x , hence we say that F has bounded ramification around x , if $\text{sw}_x^{\natural}(F, 0^+) \in \mathbb{N}$. One may calculate the Euler-Poincaré characteristic of F , i.e. the integer

$$\chi_c(C, F) := \sum_{i=0}^2 (-1)^i \cdot \text{length}_{\Lambda} H_c^i(C_{\text{ét}}, F)$$

in terms of these local invariants. Indeed, denote by $l(F)$ the *generic length* of F , defined as $\text{length}_{\Lambda}(F_{\xi})$ for any (hence all) geometric point ξ of U^{ad} ; moreover, for every $x \in C \setminus U$, let :

$$a_x(F) := l(F) + \text{sw}_x^{\natural}(F, 0^+) - \text{length}_{\Lambda} F_{\bar{x}}$$

where $\bar{x}$ is any geometric point of C^{ad} localized at x ; this is the *total drop of the rank* of F at the point x . With this notation, we have the following *Grothendieck-Ogg-Shafarevich formula* :

Lemma 2.3.15. *In the situation of (2.3.14), suppose that F has bounded ramification at all the points of $C \setminus U$, and that Λ is a local ring with residue field $\bar{\Lambda}$. Then :*

$$\chi_c(C, F) = \chi_c(C, \bar{\Lambda}_C) \cdot l(F) - \sum_{x \in C \setminus U} a_x(F).$$

Proof. We may easily reduce to the case where $F = j_!(F|_{U^{\text{ad}}})$, where $j : U^{\text{ad}} \rightarrow C^{\text{ad}}$ is the open immersion. Moreover, using the additivity of the Swan conductor and of the Euler-Poincaré characteristic, we may assume that Λ is a field (for a general Λ , one considers the $\mathfrak{m}_{\Lambda}$ -filtration of F , where $\mathfrak{m}_{\Lambda} \subset \Lambda$ is the maximal ideal). Then we may argue as in [18, Rem.10.6] : the details shall be left to the reader. $\square$

2.3.16. In the situation of (2.3.12), suppose that F has bounded ramification around z ; then [25, Th.4.2.40] ensures the existence of a connected open subset $U \subset \mathbb{D}(z, \varepsilon)^*$ such that $U \cap \mathbb{D}(z, \delta)^* \neq \emptyset$ for every $\delta \in |K^{\times}|$, and a canonical *break decomposition* of Λ_U -modules :

$$F|_U = \bigoplus_{\gamma \in \Gamma} F(\gamma)$$

indexed by the group $\Gamma := \mathbb{Q} \times |K^{\times}|$, which we endow with the lexicographic ordering given by the rule :

$$(q, c) \leq (q', c') \quad \Leftrightarrow \quad \text{either } q < q' \text{ or else } q = q' \text{ and } c \geq c'$$

(there is a slight confusion in [25, §4.2.39] concerning this ordering). The projection :

$$\Gamma \rightarrow \mathbb{Q} \quad (q, c) \mapsto (q, c)^{\natural} := q$$

is a homomorphism of ordered groups; we have $F(\gamma) = 0$ unless $\gamma^{\natural} \geq 0$. The break decomposition is functorial : if $\varphi : F \rightarrow G$ is a morphism of Λ -modules with bounded ramification, and U as above is chosen small enough so that both $F|_U$ and $G|_U$ admit break decompositions, then $\varphi|_U$ restricts to morphisms of Λ_U -modules $F(\gamma) \rightarrow G(\gamma)$ for every $\gamma \in \Gamma$.

Moreover, the break decomposition is well behaved with respect to tensor products : namely, if F and G have both bounded ramification, then the same holds for $F \otimes_{\Lambda} G$, and for a sufficiently small U as above, and every $\gamma, \gamma' \in \Gamma$, we have :

$$(2.3.17) \quad F(\gamma) \otimes_{\Lambda} G(\gamma') \subset \begin{cases} (F \otimes_{\Lambda} G)(\max(\gamma, \gamma')) & \text{if } \gamma \neq \gamma' \\ \bigoplus_{\rho \leq \gamma} (F \otimes_{\Lambda} G)(\rho) & \text{otherwise.} \end{cases}$$

Also, the break decomposition of $\text{Hom}_{\Lambda}(F, G)$ is likewise related to those of F and G .

Furthermore, if F is a locally free Λ -module, $\gamma^{\natural} \cdot \text{rk}_{\Lambda} F(\gamma) \in \mathbb{N}$ for every $\gamma \in \Gamma$ (Hasse-Arf property); when F is only constructible, we have the identity :

$$(2.3.18) \quad \text{sw}_z^{\natural}(F, 0^+) = \sum_{\gamma \in \Gamma} \gamma^{\natural} \cdot \text{length}_{\Lambda} F(\gamma).$$

The elements $\gamma \in \Gamma$ such that $F(\gamma) \neq 0$ are called the *breaks* of F around z , and the integer $\text{length}_{\Lambda} F(\gamma)$ is called the *multiplicity* of γ . In view of (2.3.18), it is convenient to define

$$F^{\natural}(r) := \bigoplus_{\gamma^{\natural}=r} F(\gamma) \quad \text{for every } r \in \mathbb{Q}_+.$$

The $r \in \mathbb{Q}_+$ such that $F^{\natural}(r) \neq 0$ are called the $\natural$ -breaks of F .

2.3.19. For instance, the Λ -module $\mathcal{L}_{\psi}$ of (2.3.1) has bounded ramification at the point $\infty \in D$. Since $\mathcal{L}_{\psi}$ has Λ -rank one, obviously it admits a single break γ around ∞ . More generally, let $f \in \Gamma(\mathbb{A}_K^1, \mathcal{O}_{\mathbb{A}_K^1})$ be an algebraic function, and denote by $o_{\infty}(f) \in \mathbb{N}$ the pole order of f at ∞ ; then $\mathcal{L}_{\psi}\langle f \rangle$ has bounded ramification around ∞ , with a unique break of the form

$$(o_{\infty}(f), c) \in \mathbb{N} \times |K^{\times}| \subset \Gamma$$

where the value c depends only on the valuation of the leading coefficient of f , and on the largest power of p dividing $o_{\infty}(f)$. Indeed, this follows from [25, Lemma 4.2.6(i),(ii)] if f is a monomial $f(T) = a \cdot T^n$ (for a chosen coordinate T , so that $\mathbb{A}_K^1 = \text{Spec } K[T]$), and the general case is easily deduced, by writing $f = f_n + g$, where f_n is a monomial and $\deg_T g < n := \deg_T f$, and by applying induction on n , together with (2.3.2) and (2.3.17).

Now, let F be any Λ -module on $\mathbb{D}(\infty, \varepsilon)^*$, with bounded ramification around ∞ ; set

$$e(F) := \sum_{r \geq 1} r \cdot \text{length}_{\Lambda} F^{\natural}(r) + \sum_{r < 1} \text{length}_{\Lambda} F^{\natural}(r).$$

For any $a \in K$, let $F_{[a]} := F \otimes_{\Lambda} \mathcal{L}_{\psi}\langle m_a \rangle|_{\mathbb{D}(\infty, \varepsilon)^*}$, where $m_a(T) := aT \in \Gamma(\mathbb{A}_K^1, \mathcal{O}_{\mathbb{A}_K^1})$; taking (2.3.17) and (2.3.18) into account, we see that :

$$\text{sw}_{\infty}^{\natural}(F_{[a]}, 0^+) \leq e(F) \quad \text{for every } a \in K.$$

We consider the subset

$$d_K(F) := \{a \in K \mid \text{sw}_{\infty}^{\natural}(F_{[a]}, 0^+) < e(F)\}.$$

We shall regard $d_K(F)$ as a subset of $A'(K)$, via the identification $K = A'(K)$ induced by the global coordinate x' .

Remark 2.3.20. Notice that $a \in d_K(F)$ if and only if $F_{[a]}^{\natural}(r) \neq 0$ for some $r < 1$ (on a suitable $U \subset \mathbb{D}(\infty, \varepsilon)^*$ as in (2.3.16)). Especially, $0 \in d_K(F)$ if and only if $F^{\natural}(r) \neq 0$ for some $r < 1$.

Proposition 2.3.21. *With the notation of (2.3.19), the following holds :*

- (i) $d_K(F)$ is a finite set.
- (ii) *There exist :*
 - (a) *a connected open subset $U \subset \mathbb{D}(\infty, \varepsilon)^*$, such that $U \cap \mathbb{D}(\infty, \delta)^* \neq \emptyset$ for every $\delta \in |K^{\times}|$*
 - (b) *and a natural decomposition of Λ_U -modules :*

$$F|_U = \left(\bigoplus_{r > 1} F^{\natural}(r) \right) \oplus \left(\bigoplus_{a \in d_K(F)} \bigoplus_{r < 1} F_{[a]}^{\natural}(r) \otimes_{\Lambda} \mathcal{L}_{\psi}\langle m_{-a} \rangle|_U \right) \oplus G$$

where

$$G := F^{\natural}(1) \cap \bigcap_{a \in K} (F_{[a]}^{\natural}(1) \otimes_{\Lambda} \mathcal{L}_{\psi}\langle m_{-a} \rangle|_U).$$

(iii) If $0 \rightarrow F_1 \rightarrow F \rightarrow F_2 \rightarrow 0$ is any short exact sequence of locally constant Λ -modules on $\mathbb{D}(\infty, \varepsilon)_{\text{ét}}^*$, we have :

$$d_K(F) = d_K(F_1) \cup d_K(F_2).$$

(iv) If F is a locally free Λ -module of finite type on $\mathbb{D}(\infty, \varepsilon)_{\text{ét}}^*$, then :

$$d_K(F^\vee) = \{-a \mid a \in d_K(F)\} \quad \text{where } F^\vee := \mathcal{H}om_\Lambda(F, \Lambda_{\mathbb{D}(\infty, \varepsilon)}).$$

Proof. For any finite subset $\Sigma \subset K$, let

$$G_\Sigma := F^\natural(1) \cap \bigcap_{a \in \Sigma} (F_{[a]}^\natural(1) \otimes_\Lambda \mathcal{L}_\psi \langle m_{-a} \rangle_{|U})$$

where the intersection is taken over a sufficiently small open subset U , fulfilling condition (a).

Claim 2.3.22. For every $b \in K \setminus \Sigma$, we have (on a suitable open subset U) :

$$G_\Sigma = \left(\bigoplus_{r < 1} F_{[b]}^\natural(r) \otimes_\Lambda \mathcal{L}_\psi \langle m_{-b} \rangle_{|U} \right) \oplus G_{\Sigma \cup \{b\}}.$$

Proof of the claim. On the one hand, we may write :

$$(2.3.23) \quad F^\natural(1) = \left(\bigoplus_{r < 1} F_{[b]}^\natural(r) \otimes_\Lambda \mathcal{L}_\psi \langle m_{-b} \rangle_{|U} \right) \oplus \left((F_{[b]}^\natural(1) \otimes_\Lambda \mathcal{L}_\psi \langle m_{-b} \rangle_{|U}) \cap F^\natural(1) \right).$$

On the other hand, notice that :

$$(2.3.24) \quad F_{[b]}^\natural(r) \otimes_\Lambda \mathcal{L}_\psi \langle m_{-b} \rangle_{|U} \subset G_\Sigma \quad \text{for every } r < 1.$$

Indeed, (2.3.24) is equivalent to the family of inclusions :

$$F_{[b]}^\natural(r) \otimes_\Lambda \mathcal{L}_\psi \langle m_{a-b} \rangle_{|U} \subset F_{[a]}^\natural(1) \quad \text{for every } a \in \Sigma \text{ and every } r < 1$$

which hold, since $a - b \neq 0$ for every $a \in \Sigma$. The claim follows straightforwardly from (2.3.23) and (2.3.24). $\diamond$

Using claim 2.3.22 we deduce, by induction on the cardinality of Σ , the decomposition :

$$F_{|U} = \left(\bigoplus_{r > 1} F^\natural(r) \right) \oplus \left(\bigoplus_{a \in \Sigma} \bigoplus_{r < 1} F_{[a]}^\natural(r) \otimes_\Lambda \mathcal{L}_\psi \langle m_{-a} \rangle_{|U} \right) \oplus G_\Sigma$$

for every finite subset $\Sigma \subset K$ such that $0 \in \Sigma$ (on a suitable U , dependent on Σ ; see remark 2.3.20). However, it is easily seen that $a \in d_K(F)$ if and only if there exists $r < 1$ such that $F_{[a]}^\natural(r) \neq 0$. Since F is a locally constant Λ -module of finite type, assertion (i) is an immediate consequence, and we also see that $G = G_{d_K(F)}$.

Finally, (iii) is an immediate consequence of the additivity and non-negativity properties of the Swan conductor and (iv) is straightforward from (2.3.2). $\square$

2.3.25. Let now $K \subset L$ be an extension of algebraically closed non-archimedean fields with rank 1 valuations; denote by

$$\pi_L : \mathbb{D}(\infty, \varepsilon) \times_S \text{Spa}(L, L^+) \rightarrow \mathbb{D}(\infty, \varepsilon)$$

the projection. In view of [25, Lemma 3.3.8] we have $\text{sw}_\infty^\natural(\pi_L^* F, 0^+) = \text{sw}_\infty^\natural(F, 0^+)$, hence $\pi_L^* F$ has bounded ramification, and we may consider the set $d_L(\pi_L^* F)$. With this notation we have :

Corollary 2.3.26. $d_L(\pi_L^* F) = d_K(F)$.

Proof. Let $A'_L := A' \times_S \mathrm{Spa}(L, L^+)$ and denote by $\pi'_L : A'_L \rightarrow A'$ the projection; under the natural identifications $K = A'(K)$ and $L = A'_L(L)$, the inclusion $K \subset L$ corresponds to the map

$$A'(K) \rightarrow A'_L(L) \quad : \quad x \mapsto \pi'^{-1}_L(x)$$

and the assertion means precisely that $d_L(\pi'^*_L F) = \pi'^{-1}_L d_K(F)$. Taking into account proposition 2.3.21(i), it then suffices to show that, for every $a \in L \setminus K$, there exist an infinite subset $\Sigma \subset K$ such that

$$(2.3.27) \quad \mathrm{sw}^{\natural}_{\infty}((\pi'^*_L F)_{[a+x]}) = \mathrm{sw}^{\natural}_{\infty}((\pi'^*_L F)_{[a]}) \quad \text{for every } x \in \Sigma.$$

To this aim, we may assume that L is the completion of the algebraic closure of $K(a)$ ([25, Lemma 3.3.8]); in this case, we will exhibit, more precisely, an infinite subset $\Sigma \subset K$, and for every $x \in \Sigma$, an isometric K -automorphism $\sigma_x : L \rightarrow L$ such that $\sigma_x(a) = a + x$. Then σ_x will induce an automorphism σ'_x of A'_L such that

$$\pi'_L \circ \sigma'_x = \pi'_L \quad \text{and} \quad \sigma'^*_x \mathcal{L}_{\psi} \langle m_a \rangle \simeq \mathcal{L}_{\psi} \langle m_{a+x} \rangle$$

from which (2.3.27) follows straightforwardly.

Claim 2.3.28. Let $(E, |\cdot|_E)$ be any complete non-archimedean valued field, $(E^a, |\cdot|_{E^a})$ a fixed algebraic closure of E , and σ an isometric automorphism of E . Then σ extends to an isometric automorphism of E^a .

Proof of the claim. This is presumably well known : by Zorn's lemma, we may find a maximal subextension $F \subset E^a$ such that σ extends to an isometry $\sigma_F : F \xrightarrow{\sim} F'$ (where $F' \subset E^a$ as well). We claim that $F = E^a$; indeed, if $x \in E^a \setminus F$, let P be the minimal polynomial of x over F , P^σ the polynomial obtained by applying σ_F to all the coefficients of P , and x' any root of P^σ ; we may extend σ_F to an isomorphism $\tilde{\sigma}_F : F[x] \rightarrow F[x']$ by mapping x to x' . Notice now that the spectral norms of $F[x]$ and $F[x']$ agree with the restriction of $|\cdot|_{E^a}$ ([25, Lemma 1.1.17]), and on the other hand, $\tilde{\sigma}_F$ clearly preserves these spectral norms; therefore $\tilde{\sigma}_F$ is an isometry $(F[x], |\cdot|_{E^a}) \rightarrow (F[x'], |\cdot|_{E^a})$. By maximality, we must have $F[x] = F$, a contradiction. We leave to the reader the verification that $\sigma^a(E^a) = E^a$. $\diamond$

Now, the norm $|\cdot|_L$ of L corresponds to a maximal point of $\mathrm{Spec} K[a] \times_{\mathrm{Spec} K} S$. According to the classification in [9, §1.4.4], such point can only be of type (2), (3) or (4). We will show that, in either of these cases, there exists $\rho_a \in |K^\times|$ such that the K -automorphism ω_x of $K[a]$ given by the rule $a \mapsto a + x$ is an isometry for the norm $|\cdot|_L$, whenever $x \in K$ and $|x| < \rho_a$. In view of claim 2.3.28, this ω_x will extend to an automorphism σ_x of L as sought.

Indeed, points of type (2) and (3) arise from the sup norm on a certain disc $\mathbb{D}(x_0, \rho)$: in these cases, it is easily seen that $\rho_a := \rho$ will do. Likewise, a point of type (4) arises from a family $\{\mathbb{D}(x_i, \rho_i) \mid i \in I\}$ of embedded discs; the infimum of the radii ρ_i must be > 0 , otherwise a would be of type (1); then we may take ρ_a equal to this infimum. $\square$

2.3.29. Let $U \subset A$ be a non-empty Zariski open subset (i.e. $U = V^{\mathrm{ad}}$ for some open subset V of the scheme $\mathbb{A}_K^1$), F a constructible and locally constant Λ -module on $U_{\mathrm{\acute{e}t}}$, and $j : U \rightarrow A$ the open immersion. Set $F' := \mathcal{F}_{\psi}(j_* F)$ and $U' := A' \setminus d_K(F)$.

Theorem 2.3.30. *In the situation of (2.3.29), suppose moreover that F has bounded ramification at all the (finitely many) points of $D \setminus U$. Then :*

- (i) $\mathcal{H}^1(F')$ is a constructible Λ -module, and its support is contained in $d_K(F)$.
- (ii) $\mathcal{H}^0(F')$ restricts to a locally constant Λ -module on $U'_{\mathrm{\acute{e}t}}$, and has bounded ramification at every point of $d_K(F) \cup \{\infty'\}$.
- (iii) $\mathcal{H}^i(F') = 0$ for every $i \in \mathbb{Z} \setminus \{0, 1\}$.
- (iv) $R^{-1}\Phi_{\eta_z} F' = 0$ for every $z \in A'(K)$.

Proof. Assertion (iii) is obvious.

(i): For F and F' as in the theorem, let $\Sigma(F)$ denote the support of $\mathcal{H}^1(F')$; by [17, Cor.8.2.4], $\mathcal{H}^i(F')$ is an overconvergent Λ -module for every $i \in \mathbb{Z}$, especially, $\Sigma(F)$ is the set of all specializations (in A') of the subset $\Sigma_0(F)$ of all the maximal points of A' lying in $\Sigma(F)$. Now, let $a \in \Sigma_0(F)$ be any point, and $(L, |\cdot|_L)$ the algebraic closure of the residue field of a ; after base change along the natural morphism $\mathrm{Spa}(L, L^+) \rightarrow S$, we may assume that $L = K$ (corollary 2.3.26), in which case the condition $\mathcal{H}^1(F')_a \neq 0$ means that

$$(2.3.31) \quad H_c^2(U, F \otimes_{\Lambda} \mathcal{L}_{\psi}\langle m_a \rangle|_U) \neq 0.$$

Let $0 \rightarrow F_1 \rightarrow F \rightarrow F_2 \rightarrow 0$ be a short exact sequence of Λ -modules on $U_{\text{ét}}$; from (2.3.31) it is clear that $\Sigma(F) \subset \Sigma(F_1) \cup \Sigma(F_2)$. In view of proposition 2.3.21(iii), we reduce to prove assertion (i) for F_1 and F_2 . Especially, if $\mathfrak{n} \subset \Lambda$ is any maximal ideal, assertion (i) for F follows from the same assertion for $\mathfrak{n}F$ and $F/\mathfrak{n}F$; hence, an easy induction on the length of Λ reduces to the case where Λ is a field. In this case, lemma 2.3.10(iii) says that (2.3.31) is equivalent to :

$$H^0(U, F^{\vee} \otimes_{\Lambda} \mathcal{L}_{\psi}\langle m_{-a} \rangle) \neq 0 \quad \text{where } F^{\vee} := \mathcal{H}om_{\Lambda}(F, \Lambda_U)$$

especially $(F^{\vee})_{[-a]}^{\natural}(0) \neq 0$, and therefore $-a \in d_K(F^{\vee})$ (remark 2.3.20), whence $a \in d_K(F)$, according to proposition 2.3.21(iv). This shows that $\Sigma_0(F) \subset d_K(F)$, whence (i), in view of proposition 2.3.21(i).

(ii): Let $(\rho_n \mid n \in \mathbb{N})$ be an increasing sequence of values in $|K^{\times}|$, such that $\rho_n \rightarrow +\infty$ for $n \rightarrow +\infty$, and such that $A \setminus U \subset \mathbb{D}(0, \rho_0)$. For every $n \in \mathbb{N}$, let $j_n : \mathbb{D}(0, \rho_n) \rightarrow A$ be the open immersion, and set

$$H_n := \mathcal{H}^0(\mathcal{F}_{\psi}(j_{n!}j_n^*F)).$$

Claim 2.3.32. $H_n \subset H_{n+1}$ for every $n \in \mathbb{N}$.

Proof of the claim. The assumption can be checked on the stalks, hence let $a \in A'$ be any point; set

$$V_n := p'^{-1}(a) \cap (\mathbb{D}(0, \rho_n) \times_S A') \quad \text{and} \quad Z_n := V_{n+1} \setminus V_n.$$

The assumption on ρ_0 implies that the restriction of $L := p^*F \otimes_{\Lambda} \mathcal{L}_{\psi}\langle m \rangle$ is locally constant on the étale site of the pseudo-adic space $(A \times_S A', Z_n)$. However, let $i_a : Z_n \rightarrow V_{n+1}$ (resp. $j_a : V_n \rightarrow V_{n+1}$) be the closed (resp. open) immersion of pseudo-adic spaces; we have a short exact sequence of Λ -modules on $(A \times_S A', V_{n+1})_{\text{ét}}$:

$$0 \rightarrow j_{a!}L|_{V_n} \rightarrow L|_{V_{n+1}} \rightarrow i_{a*}L|_{Z_n} \rightarrow 0$$

whence, by [17, Cor.5.4.8], a cohomology exact sequence :

$$H_c^0(Z_n, L) \rightarrow H_{n,a} \rightarrow H_{n+1,a}.$$

Since Z_n is not proper over S , the left-most term vanishes, and the claim follows. $\diamond$

Claim 2.3.33. Let X be an adic space, locally of finite type over S , and $G_{\bullet} := (G_n \mid n \in \mathbb{N})$ a direct system of constructible Λ -modules on $X_{\text{ét}}$, with injective transition maps $G_n \rightarrow G_{n+1}$ (for every $n \in \mathbb{N}$). Suppose that :

- (a) the colimit G of the system $G_{\bullet}$ is overconvergent ([17, Def.8.1.1]);
- (b) there exists $l \in \mathbb{N}$ such that $\text{length}_{\Lambda} G_{\xi} = l$ for every geometric point ξ of X .

Then G is a locally constant Λ -module of finite type on $X_{\text{ét}}$.

Proof of the claim. Let $x \in X$ be any point, and $\bar{x}$ a geometric point of X localized at x ; since $G_{\bar{x}}$ has finite length, we may find $n \in \mathbb{N}$ large enough, so that $G_{n, \bar{x}} = G_{\bar{x}}$. By assumption, there exists a locally closed constructible subset $Z \subset X$ containing x , and such that G_n restricts to a locally constant Λ -module on $Z_{\text{ét}}$; after shrinking Z , we may then assume that the stalks $G_{n, \xi}$ are isomorphic to $G_{n, \bar{x}}$, for every geometric point ξ of Z . Let ξ be any geometric point of Z ; by

assumption $G_{n,\xi} \subset G_{m,\xi}$ for every $m \geq n$, therefore $l = \text{length}_\Lambda G_{n,\xi} \leq \text{length}_\Lambda G_{m,\xi} \leq l$. It follows that $G_{n,\xi} = G_{m,\xi}$, hence $G_{n|Z} = G_{m|Z}$ for every $m \geq n$, so finally $G|Z = G_{n|Z}$. This shows that G is constructible, and then the claim follows from [17, Lemma 2.7.11]. $\diamond$

Clearly $\mathcal{H}^0(F')$ is the colimit of the direct system $(H_n \mid n \in \mathbb{N})$, which consists of constructible Λ -modules, according to [17, Th.6.2.2]. On the other hand, (i), (iii) and lemma 2.3.15 easily imply that the length of the stalk $\mathcal{H}^0(F')_\xi$ is constant for ξ ranging over the geometric points of U' whose support is a maximal point, hence also for ξ ranging over all the geometric points of U' , since – as it has already been noted – $\mathcal{H}^0(F')$ is overconvergent. Then the first assertion of (ii) follows from claims 2.3.32 and 2.3.33. Finally, (i), (iii) and (2.3.6) imply that the Euler-Poincaré characteristic of $\mathcal{H}^0(F')$ is finite, hence its Swan conductor at every point of $d_K(F) \cup \{\infty'\}$ is finite, in view of lemma 2.3.15, therefore $\mathcal{H}^0(F')$ has bounded ramification everywhere.

(iv): In view of (iii) we have $R^{-1-i}\Phi_{\eta_z}\mathcal{H}^i(F') = 0$ for every $z \in A'(K)$ and every $i \neq 0$, hence we are reduced to showing that $R^{-1}\Phi_{\eta_z}\mathcal{H}^0(F') = 0$ (proposition 2.2.4(iii) and corollary 2.2.14). In view of (ii), (and corollary 2.2.19) the assertion is already known for every $z \in U'(K)$. Hence, we may assume that $z \in d_K(F)$, in which case we have to prove that the natural map $\mathcal{H}^0(F')_z \rightarrow \mathcal{H}^0(F')_{\eta_z}$ is injective. The latter will follow, once we have shown that $H_c^0(A'_{\text{ét}}, \mathcal{H}^0(F')) = 0$. Again by (iii), this is the same as showing that $H_c^0(A'_{\text{ét}}, F')$ vanishes, which holds, due to (2.3.6). $\square$

Remark 2.3.34. (i) For $a \in K = A(K)$, let $i_a : S \rightarrow A$ be the closed immersion with image $\{a\}$, and $\pi : A \rightarrow S$ the structure morphism. If $G^\bullet$ is any object of $D^+(S_{\text{ét}}, \Lambda)$, a direct computation yields a natural isomorphism (cp. [22, Th.1.4.2.1(i)]) :

$$\mathcal{F}_\psi(i_{a*}G^\bullet) \xrightarrow{\sim} \pi^*G^\bullet \otimes_\Lambda \mathcal{L}_\psi\langle a \cdot x' \rangle[1].$$

(ii) In the situation of (2.3.29), let H be a Zariski constructible Λ -module on $A_{\text{ét}}$, such that j^*H is locally constant on $U_{\text{ét}}$, and has bounded ramification at the points of $D \setminus U$. Notice that the kernel and cokernel of the unit of adjunction $H \rightarrow j_*j^*H$ are direct sum of sheaves of the form $i_{a*}G$ (for various Λ -modules G , and with a running over the points of $A \setminus U$). Since $\mathcal{F}_\psi$ is a triangulated functor, it follows easily from (i) that theorem 2.3.30(i),(ii) still holds with F' replaced by $\mathcal{F}_\psi(H)$.

(iii) On the other hand, in the situation of (ii), we have in general $\mathcal{H}^i(\mathcal{F}_\psi(H)) = 0$ only for $i \in \mathbb{Z} \setminus \{-1, 0, 1\}$. More precisely, $\mathcal{H}^{-1}(\mathcal{F}_\psi(H)) = 0$ if and only if $\Gamma_c(A_{\text{ét}}, H) = 0$, and in this case, also assertion (iv) of theorem 2.3.30 still holds.

We point out the following result, even though it will not be used in the rest of this work.

Proposition 2.3.35. *Let $j : U \rightarrow A$ be a Zariski open immersion, and F a locally constant Λ -module on $U_{\text{ét}}$, with bounded ramification at all points of $D \setminus U$. Suppose that F does not admit any subquotient of the type $\mathcal{L}_\psi\langle xa \rangle|_U \otimes_\Lambda \overline{\Lambda}$, where $\overline{\Lambda}$ is any residue field of Λ , and $a \in K$ is any element. Then there exists a Zariski open immersion $j' : U' \rightarrow A'$, and a locally constant Λ -module G on $U'_{\text{ét}}$ such that $\mathcal{F}_\psi(j_!F) = j'_*G[0]$.*

Proof. First, we prove that $\mathcal{H}^i\mathcal{F}_\psi(j_!F) = 0$ for $i \neq 0$. Indeed, by virtue of remark 2.3.34(iii), the assertion is already known for all $i \neq 0, 1$; hence it suffices to show :

Claim 2.3.36. Under the assumptions of the lemma, we have $H_c^2(U_{\text{ét}}, F \otimes_\Lambda \mathcal{L}_\psi\langle xa \rangle) = 0$ for every $a \in K$.

Proof of the claim. Notice that F satisfies the assumptions of the lemma, if and only if the same holds for $F \otimes_\Lambda \mathcal{L}_\psi\langle xa \rangle$, for every $a \in K$. Hence, we are reduced to showing that $H_c^2(U_{\text{ét}}, F) = 0$. Since Λ is artinian, we may assume that Λ is local, say with maximal ideal $\mathfrak{m}$. Set $\overline{F} := F/\mathfrak{m}F$; under our assumptions, $\overline{F}$ does not admit constant quotients, hence

$H^0(U_{\text{ét}}, \mathcal{H}om_{\Lambda}(\overline{F}, \Lambda/\mathfrak{m}\Lambda)) = 0$, and therefore $H_c^2(U_{\text{ét}}, \overline{F}) = 0$, by Poincaré duality (lemma 2.3.10(iii)). This easily implies that the natural map $\alpha : H_c^2(U_{\text{ét}}, \mathfrak{m}F) \rightarrow H_c^2(U_{\text{ét}}, F)$ is surjective. Now, let $t_1, \dots, t_k$ be a system of generators $\mathfrak{m}$; we may then define a map of Λ -modules $\beta : F^{\oplus k} \rightarrow F$ in the obvious way, with image equal to $\mathfrak{m}F$. It follows easily that the image of α equals the image of the induced map $H_c^2(\beta) : H_c^2(U_{\text{ét}}, F)^{\oplus k} \rightarrow H_c^2(U_{\text{ét}}, F)$; however, the latter is just $\mathfrak{m}H_c^2(U_{\text{ét}}, F)$. By Nakayama's lemma, the claim follows. $\diamond$

By theorem 2.3.30(ii), there exists a Zariski open immersion $j' : U' \rightarrow A'$, such that $G := j'^* \mathcal{H}^0 \mathcal{F}_{\psi}(j_! F)$ is a locally constant Λ -module, with bounded ramification at all the points of $D' \setminus U'$. Let C denote the cone of the natural morphism $\mathcal{F}_{\psi}(j_! F) \rightarrow j'_* G[0]$. Then $\mathcal{H}^i C = 0$ for every $i \neq -1, 0$, and C is supported on finitely many K -rational points. In view of remark 2.3.34(i), it follows easily that $\mathcal{F}'_{\psi}(C)$ is concentrated in degrees $-2, -1$, and we have an exact sequence :

$$\mathcal{H}^{-2} \mathcal{F}'_{\psi}(j'_* G[0]) \rightarrow \mathcal{H}^{-2} \mathcal{F}'_{\psi}(C) \rightarrow \mathcal{H}^{-1}(\mathcal{F}'_{\psi} \circ \mathcal{F}_{\psi}(j_! F))$$

whose first (resp. third) term vanishes, due to theorem 2.3.30(iii) (resp. due to (2.3.5)); hence the middle term vanishes as well. Likewise, we have an exact sequence :

$$\mathcal{H}^{-1} \mathcal{F}'_{\psi}(j'_* G[0]) \rightarrow \mathcal{H}^{-1} \mathcal{F}'_{\psi}(C) \xrightarrow{\beta} j_! F \rightarrow \mathcal{H}^0 \mathcal{F}'_{\psi}(j'_* G[0]) \rightarrow 0.$$

However, remark 2.3.34(i) also implies that $\mathcal{H}^{-1} \mathcal{F}'_{\psi}(C)$ is a quotient of a sum of Λ -modules of the type $\mathcal{L}_{\psi}\langle xa \rangle$, for various $a \in K$; under our assumptions, it follows that the map β vanishes; furthermore, $\mathcal{H}^{-1} \mathcal{F}'_{\psi}(j'_* G[0]) = 0$, again by theorem 2.3.30(iii). Therefore $\mathcal{H}^{-1} \mathcal{F}'_{\psi}(C)$ must vanish as well, so that $\mathcal{F}'_{\psi}(C) = 0$, and finally $C = 0$, whence the assertion. $\square$

2.4. Stationary phase. Keep the notation of (2.3.3), and let $\varepsilon \in |K^{\times}|$ be any value; set

$$X(\varepsilon) := D \times_S \mathbb{D}(\infty', \varepsilon) \subset D \times_S D'$$

and denote by $\overline{p}_{\varepsilon} : X(\varepsilon) \rightarrow D$, $\overline{p}'_{\varepsilon} : X(\varepsilon) \rightarrow \mathbb{D}(\infty', \varepsilon)$ the projections; here the radius ε is meant relative to the global coordinate x' on A' ; in other words :

$$(2.4.1) \quad \mathbb{D}(\infty', \varepsilon) = \{\infty'\} \cup \{b \in A' \mid |x'(b)| \geq 1/\varepsilon\}.$$

The vanishing cycle construction of (2.2.1) applies to the current situation, and yields functors :

$$R^i \Psi_{\eta_{\infty'}, \varepsilon} : D^+(X(\varepsilon)_{\text{ét}}, \Lambda) \rightarrow \Lambda[\pi(\infty', \varepsilon)]_{D\text{-Mod}} \quad \text{for every } i \in \mathbb{N}.$$

(Where $\Lambda[\pi(\infty', \varepsilon)]_{D\text{-Mod}}$ denotes the category of Λ -modules on $D_{\text{ét}}$, endowed with a linear action of $\pi(\infty', \varepsilon)$; notice that $\overline{p}'_{\varepsilon}{}^{-1}(\infty')$ is naturally isomorphic to D .) Let also

$$\overline{\mathcal{L}}_{\psi}\langle m \rangle := (\alpha \times_S \alpha')_! \mathcal{L}_{\psi}\langle m \rangle$$

which is a Λ -module on the étale site of $D \times_S D'$. The following result is the counterpart of the “universal local acyclicity” of [22, Th.1.3.1.2].

Theorem 2.4.2. *Let F be a Λ -module on $D_{\text{ét}}$, and $j : U \rightarrow D$ an open immersion such that $j(U) \subset A$ and $j^* F$ is a constructible locally constant Λ -module on $U_{\text{ét}}$. Then*

$$j^* R\Psi_{\eta_{\infty'}, \varepsilon}(\overline{p}_{\varepsilon}^* F \otimes_{\Lambda} \overline{\mathcal{L}}_{\psi}\langle m \rangle|_{X(\varepsilon)}) = 0 \quad \text{in } D(U_{\text{ét}}, \Lambda).$$

Proof. Suppose that $0 \rightarrow F_1 \rightarrow F \rightarrow F_2 \rightarrow 0$ is a short exact sequence of Λ -modules on $A_{\text{ét}}$, such that $j^* F_i$ is locally constant, for $i = 1, 2$; since $R\Psi_{\eta_{\infty'}, \varepsilon}$ is a triangulated functor, it is clear that the stated vanishings for F_1 and F_2 imply the same vanishing for F . Since Λ is a Λ -module of finite length, an easy induction allows then to reduce to the case where Λ has length one, *i.e.* we may assume that Λ is a field.

Set $G := \overline{p}_{\varepsilon}^* F \otimes_{\Lambda} \overline{\mathcal{L}}_{\psi}\langle m \rangle|_{X(\varepsilon)}$; choose a fundamental pro-covering $(C_H \mid H \subset \pi(\infty', \varepsilon))$ of $\mathbb{D}(\infty', \varepsilon)$, and for every open subgroup $H \subset \pi(\infty', \varepsilon)$ let $G_H := Rj_{H*} j_H^* G$ (notation of (2.2.1)). We will show more precisely that $j^* G_H = 0$ for every such H .

Notice first that j_H is partially proper, and both $\overline{p}_\varepsilon^* F$ and $\overline{\mathcal{L}}_\psi \langle m \rangle$ restrict to overconvergent Λ -modules on the étale site of $\overline{p}_\varepsilon^{-1} U$, hence $\mathcal{H}^i j^* G_H$ is overconvergent for every $i \in \mathbb{Z}$, and it suffices to show that :

$$\mathcal{H}^i(G_H)_z = 0 \quad \text{for every maximal point } z \in U \text{ and every } i \in \mathbb{N}.$$

Fix such a maximal point $z \in U$; in view of [17, Prop.2.4.4, Lemma 2.5.12, Prop.2.5.13(i)], the point $(z, \infty') \in U \times_S \mathbb{D}(\infty', \varepsilon)$ admits a fundamental system of étale neighborhoods of the form

$$V \times_S \mathbb{D}(\infty', \delta) \quad \text{for every } V \in \mathcal{V} \text{ and every } \delta \in |K^\times| \text{ with } \delta < \varepsilon$$

where $\mathcal{V}$ is a fundamental system of quasi-compact étale neighborhoods of z in $U_{\text{ét}}$. Moreover, since z is maximal in U , we may assume that the structure morphism $q_V : V \rightarrow U$ induces a finite étale covering $V \rightarrow q_V(V)$, for every $V \in \mathcal{V}$ ([17, Prop.1.5.4, Cor.1.7.4]). For every δ as above, let $C_H(\delta) := C_H \times_{\mathbb{D}(\infty', \varepsilon)} \mathbb{D}(\infty', \delta)$. It follows easily that :

$$\mathcal{H}^i(G_H)_z \simeq \operatorname{colim}_{0 < \delta \leq \varepsilon} \operatorname{colim}_{V \in \mathcal{V}} H^i(V \times_S C_H(\delta), j_H^* G).$$

Up to replacing $\mathcal{V}$ by a cofinal system, we may assume that $F|_V$ is a constant Λ -module for every $V \in \mathcal{V}$, in which case – since Λ is a field – $F|_V$ is the direct sum of finitely many copies of Λ_U . We are then easily reduced to the case where $F|_U = \Lambda_U$, therefore $G = \overline{\mathcal{L}}_\psi \langle m \rangle|_{X(\varepsilon)}$.

Fix any $V \in \mathcal{V}$; we shall show more precisely that there exists $\delta_0 > 0$ small enough, so that :

$$H^i(V \times_S C_H(\delta), \mathcal{L}_\psi \langle m \rangle) = 0 \quad \text{for every } \delta \in |K^\times| \text{ with } \delta \leq \delta_0 \text{ and every } i \in \mathbb{N}.$$

For every δ as above, denote by $q_\delta : V \times_S C_H(\delta) \rightarrow C_H(\delta)$ the projection; clearly it suffices to show that $Rq_{\delta*} \mathcal{L}_\psi \langle m \rangle = 0$ for every sufficiently small $\delta > 0$. However, q_δ is a smooth, separated and quasi-compact morphism, therefore, for every $i \in \mathbb{N}$ the Λ -module $R^i q_{\delta*} \mathcal{L}_\psi \langle m \rangle$ is constructible on the étale site of $C_H(\delta)$ ([17, Th.6.2.2]); the latter is locally of finite type over S , hence the subset of its maximal points is everywhere dense for the constructible topology ([15, Cor.4.2]), so we reduce to showing that $(R^i q_{\delta*} \mathcal{L}_\psi \langle m \rangle)_a = 0$ for every sufficiently small $\delta > 0$, every $i \in \mathbb{N}$ and every maximal point $a \in C_H(\delta)$. Given $\delta > 0$ as above, and a maximal point $a \in C_H(\delta)$, let $(L, |\cdot|_L)$ be the algebraic closure of the residue field of a , with its rank one valuation; after base change along the natural morphism $\operatorname{Spa}(L, L^+) \rightarrow S$, we may assume that $L = K$, in which case the image $b \in \mathbb{D}(\infty', \delta)^*$ of a is a K -rational point, which we identify to an element of $K^\times$, via the global coordinate x' , especially, $|b| \geq 1/\delta$. Moreover, let $i_a : q_\delta^{-1}(a) \rightarrow V \times_S C_H(\delta)$ be the immersion; [17, Th.4.1.1(c)] says that

$$(R^i q_{\delta*} \mathcal{L}_\psi \langle m \rangle)_a \simeq H^i(q_\delta^{-1}(a)_{\text{ét}}, i_a^* \mathcal{L}_\psi \langle m \rangle) \simeq H^i(V_{\text{ét}}, \mathcal{L}_\psi \langle m_b \rangle)$$

where $m_b : A \rightarrow (\mathbb{A}_K^1)^{\text{ad}}$ corresponds to the global section $bx \in \mathcal{O}_A(A)$ (recall that x is a fixed global coordinate on A). By lemma 2.3.10(iii), it is clear that

$$(2.4.3) \quad H^i(V_{\text{ét}}, \mathcal{L}_\psi \langle m_b \rangle) = 0 \quad \text{for every } i > 1 \text{ and every } \delta > 0 \text{ as above.}$$

To deal with the cases $i = 0, 1$, we shall compute explicitly the Euler-Poincaré characteristic

$$\chi(V, \mathcal{L}_\psi \langle m_b \rangle) := \sum_{i=0}^2 (-1)^i \cdot \operatorname{length}_\Lambda H^i(V_{\text{ét}}, \mathcal{L}_\psi \langle m_b \rangle).$$

Let us remark that

$$(2.4.4) \quad Rq_{V*} \mathcal{L}_\psi \langle m_b \rangle|_{q_V(V)} \simeq q_{V*} \mathcal{L}_\psi \langle m_b \rangle|_{q_V(V)} \simeq \mathcal{L}_\psi \langle m_b \rangle|_{q_V(V)} \otimes_\Lambda q_{V*} \Lambda_V.$$

We may write $q_V(V) = \mathbb{D}(y_0, \rho_0) - \bigcup_{i=1}^n \mathbb{E}(y_i, \rho_i)$ where $y_0, \dots, y_n \in U$ are distinct K -rational points, and $\rho_0, \dots, \rho_n \in |K^\times|$, with $\rho_i \leq \rho_0$ and :

$$(2.4.5) \quad \mathbb{E}(y_i, \rho_i) := \{y \in A \mid |(x - y_i)(y)| < \rho_i\} \quad \text{for every } i = 1, \dots, n.$$

Clearly we may assume that $\mathbb{E}(y_i, \rho_i) \cap \mathbb{E}(y_j, \rho_j) = \emptyset$ for $i \neq j$, in which case

$$(2.4.6) \quad \chi(q_V(V), \Lambda) = \chi(\mathbb{D}(y_0, \rho_0), \Lambda) - \sum_{i=1}^n \chi(\mathbb{E}(y_i, \rho_i), \Lambda) = 1 - n$$

(where – for any locally closed constructible subset M of A – we denote by $\chi(M, \Lambda)$ the Euler-Poincaré characteristic of the constant Λ -module Λ_M on $M_{\text{ét}}$). On the other hand, let $\bar{V}$ be the topological closure of V in A ; then $\bar{V} \setminus V = \{w_0, \dots, w_n\}$, where $\{w_0\} \cup \mathbb{D}(y_0, \rho_0)$ is the topological closure of $\mathbb{D}(y_0, \rho_0)$, and $\mathbb{E}(y_i, \rho_i) \setminus \{w_i\}$ is the interior of $\mathbb{E}(y_i, \rho_i)$, for every $i = 1, \dots, n$. To the Λ -module $q_{V*}\Lambda_V$ and each point w_i , the theory of [18] associates a sequence of *breaks*

$$\beta_{i1}, \dots, \beta_{im_i} \in \Gamma_{w_i} \otimes_{\mathbb{Z}} \mathbb{Q}$$

where Γ_{w_i} is the valuation group of the residue field of w_i ; likewise, every stalk $\mathcal{L}_{\psi}\langle m_b \rangle_{w_i}$ is a rank one Λ -module, hence it admits a single break $\alpha_i(b)$. Define the group homomorphism

$$\sharp_{\mathbb{Q}} : \Gamma_{w_i} \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow \mathbb{Q}$$

as in [18, §1]. With this notation, we have the following :

Claim 2.4.7. There exists $\delta_0 \in |K^{\times}|$ such that :

- (i) $\beta_{ij} > \alpha_i(b)$ for every $i = 0, \dots, n$, and every $j = 1, \dots, m_i$
- (ii) $\sharp_{\mathbb{Q}}\alpha_0(b) = 1$ and $\sharp_{\mathbb{Q}}\alpha_i(b) = -1$ for every $i = 1, \dots, n$

for every $b \in K^{\times}$ with $|b| \geq 1/\delta_0$.

Proof of the claim. Everything follows easily from [18, Lemma 9.4]. $\diamond$

In light of (2.4.4), claim 2.4.7(i) and [18, §8.5(ii)], we deduce that – whenever $|b| \geq 1/\delta_0$ – the value $\alpha_i(b)$ is the unique break of the stalk $q_{V*}\mathcal{L}_{\psi}\langle m_b \rangle_{w_i}$. Moreover, claim 2.4.7(ii) already ensures that :

$$(2.4.8) \quad H^0(V_{\text{ét}}, \mathcal{L}_{\psi}\langle m_b \rangle) = 0 \quad \text{for every } b \in K^{\times} \text{ with } |b| \geq 1/\delta_0.$$

Furthermore, let d be the degree of the finite morphism q_V ; claim 2.4.7(ii), (2.4.6), and [18, Cor.8.4(ii), Lemma 10.1, Cor.10.3] yield the identity :

$$\begin{aligned} \chi(V, \mathcal{L}_{\psi}\langle m_b \rangle) &= \chi(q_V(V), q_{V*}\mathcal{L}_{\psi}\langle m_b \rangle) \\ &= d \cdot \chi(q_V(V), \Lambda) - d \cdot \sum_{i=0}^n \sharp_{\mathbb{Q}}\alpha_i(b) \\ &= d \cdot (1 - n) - d \cdot (1 - n) = 0 \end{aligned}$$

whenever $|b| \geq 1/\delta_0$. In view of (2.4.3) and (2.4.8), we deduce that $H^1(V_{\text{ét}}, \mathcal{L}_{\psi}\langle m_b \rangle)$ vanishes as well, and the theorem follows. $\square$

2.4.9. For every $z \in D(K) = K \cup \{\infty\}$, as explained in (2.1.14), the choice of a maximal geometric point ξ_z of $\mathbb{D}(z, \varepsilon)^*$ induces an equivalence :

$$\tau_z : \Lambda[\pi(z, \varepsilon)]\text{-}\mathbf{Mod}_{\text{f.cont}} \xrightarrow{\sim} \Lambda_{\mathbb{D}(z, \varepsilon)^*}\text{-}\mathbf{Mod}_{\text{loc}}$$

whose quasi-inverse is the stalk functor $F \mapsto F_{\xi_z}$. For every continuous $\Lambda[\pi(z, \varepsilon)]$ -module of finite type M , denote by $M_{\dagger}$ the Λ -module on $D_{\text{ét}}$ obtained as extension by zero of $\tau_z M$.

Let $i_z : S \rightarrow \bar{p}'^{-1}(\infty')$ be the unique S -morphism whose image is the point (z, ∞') . In view of remark 2.2.3, we may define the ε -local Fourier transform from z to ∞' as the functor :

$$\mathcal{F}_{\psi, \varepsilon}^{(z, \infty')} : \Lambda[\pi(z, \varepsilon)]\text{-}\mathbf{Mod}_{\text{f.cont}} \rightarrow \Lambda[\pi(\infty', \varepsilon)]\text{-}\mathbf{Mod}_{\text{cont}}$$

given by the rule :

$$M \mapsto i_z^* R^1 \Psi_{\eta_{\infty', \varepsilon}}(\bar{p}_{\varepsilon}^*(M_{\dagger}) \otimes_{\Lambda} \overline{\mathcal{L}}_{\psi}\langle m \rangle_{|X(\varepsilon)}).$$

Finally, in the situation of (2.3.29), denote by F'_{ε} the complex of Λ -modules on $\mathbb{D}(\infty', \varepsilon)_{\text{ét}}$ obtained as extension by zero of $\mathcal{F}_{\psi}(j_! F)_{|\mathbb{D}(\infty', \varepsilon)^*}$. With this notation, we have :

Theorem 2.4.10. (Stationary Phase) *In the situation of (2.3.29), suppose moreover that F has bounded ramification at all the (finitely many) points of $D \setminus U$. Then for every $\varepsilon \in |K^\times|$ there is a natural $\pi(\infty', \varepsilon)$ -equivariant decomposition :*

$$R^0\Psi_{\eta_{\infty'}, \varepsilon} \mathcal{H}^0(F'_\varepsilon) \xrightarrow{\sim} \bigoplus_{z \in D \setminus U} \mathcal{F}_{\psi, \varepsilon}^{(z, \infty')}(F_{\xi_z}).$$

Proof. Resume the notation of (2.4), and let as well $\bar{j} : U \rightarrow D$ be the open immersion. By inspecting the definitions we get a natural isomorphism :

$$F'_\varepsilon \xrightarrow{\sim} R\bar{p}'_{\varepsilon*}(\bar{p}_\varepsilon^*(\bar{j}_! F) \otimes_\Lambda \overline{\mathcal{L}}_\psi \langle m \rangle_{|X(\varepsilon)})[1]$$

from which, by proposition 2.2.4(ii),(iv), there follows an equivariant isomorphism :

$$(2.4.11) \quad R\Psi_{\eta_{\infty'}, \varepsilon} F'_\varepsilon \xrightarrow{\sim} R\Gamma(D, R\Psi_{\eta_{\infty'}, \varepsilon}(\bar{p}_\varepsilon^*(\bar{j}_! F) \otimes_\Lambda \overline{\mathcal{L}}_\psi \langle m \rangle_{|X(\varepsilon)}))[1].$$

To evaluate the left-hand side of (2.4.11), we use the spectral sequence of proposition 2.2.4(iii): combining with remark 2.3.34(iii) we obtain the equivariant isomorphism

$$R^0\Psi_{\eta_{\infty'}, \varepsilon} F'_\varepsilon \xrightarrow{\sim} R^0\Psi_{\eta_{\infty'}, \varepsilon} \mathcal{H}^0(F'_\varepsilon).$$

Lastly, by theorem 2.4.2, the vanishing cycles appearing on the right-hand side of (2.4.11) are supported on $D \setminus U$, whence the theorem (details left to the reader). $\square$

Remark 2.4.12. (i) Not much can be said concerning the ε -local Fourier transforms appearing in theorem 2.4.10. The situation improves when we take the limit for $\varepsilon \rightarrow 0$. Namely, choose a sequence $(\varepsilon_n \mid n \in \mathbb{N})$ and geometric points $(\xi_n \mid n \in \mathbb{N})$ as in (2.2.11), and define

$$\mathcal{F}_\psi^{(z, \infty')}(M) := \operatorname{colim}_{n \in \mathbb{N}} \mathcal{F}_{\psi, \varepsilon_n}^{(z, \infty')}(M_n) \quad \text{for every continuous } \Lambda[\pi(z, \varepsilon_0)]\text{-module } M$$

where M_n denotes the $\Lambda[\pi(z, \varepsilon_n)]$ -module which is the image of $M_0 := M$, under the composition of the functors (2.2.16). It is clear that $\mathcal{F}_\psi^{(z, \infty')}(M)$ depends only on the image of M in $\Lambda[\pi(z)]\text{-Mod}$; hence – and in view of theorem 2.4.10 – we deduce a natural decomposition (notation of (2.2.15)) :

$$(2.4.13) \quad \mathcal{H}^0(\mathcal{F}_\psi(j_! F))_{\eta_{\infty'}} \xrightarrow{\sim} \bigoplus_{z \in D \setminus U} \mathcal{F}_\psi^{(z, \infty')}(F_{\eta_z})$$

for every Λ -module with bounded ramification F on $U_{\text{ét}}$ (see remark 2.2.17). However, remark 2.3.34(ii) and the discussion in (2.2.15) imply that the left-hand side of (2.4.13) has finite length, hence the same holds for each of the summands on the right-hand side.

(ii) We shall later need also a localized version of the stationary phase identity. Namely, let $\delta \in |K^\times|$, and suppose that F is a locally constant Λ -module of finite type on $\mathbb{D}(0, \delta)_{\text{ét}}^*$. Let $w \in \bar{p}'_\varepsilon^{-1}(\infty')$ be the unique point such that $(\mathbb{D}(0, \delta) \times \{\infty'\}) \cup \{w\}$ is the topological closure of $\mathbb{D}(0, \delta) \times \{\infty'\}$, and set

$$\mathcal{F}_\psi^{(w, \infty')}(F) := \Gamma(\{w\}, R^1\Psi_{\eta_{\infty'}}(\bar{p}_\varepsilon^*(\bar{j}_! F) \otimes_\Lambda \overline{\mathcal{L}}_\psi \langle m \rangle_{|X(\varepsilon)}))$$

where $\bar{j} : \mathbb{D}(0, \delta)^* \rightarrow D$ is the open immersion ($\{w\}$ is here seen as a pseudo-adic space). Let also F'_ε denote the complex of Λ -modules on $\mathbb{D}(\infty', \varepsilon)_{\text{ét}}$ obtained as extension by zero of $\mathcal{F}_\psi((\bar{j}_! F)_{|A})$, and notice that we still have $\mathcal{H}^i(F'_\varepsilon) = 0$ for $i \notin \{0, 1\}$, hence the proof of theorem 2.4.10 still applies to this Λ -module F , and we get the identity :

$$\mathcal{H}^0(F'_\varepsilon)_{\eta_{\infty'}} \xrightarrow{\sim} \mathcal{F}_\psi^{(0, \infty')}(F_{\eta_0}) \oplus \mathcal{F}_\psi^{(w, \infty')}(F).$$

2.4.14. In order to apply theorem 2.4.10, we shall need a local-to-global extension result for $\Lambda[\pi(z, \varepsilon)]$ -modules, analogous to Gabber's theorem [20, Th.1.4.1]. This is contained in the following :

Proposition 2.4.15. *Let F be a locally constant Λ -module on $\mathbb{D}(z, \varepsilon)_{\text{ét}}^*$ of finite type. Then there exists a Zariski open subset $U \subset D \setminus \{z\}$, and a locally constant Λ -module $\tilde{F}$ on $U_{\text{ét}}$, such that:*

- (a) *the restriction $\tilde{F}|_{\mathbb{D}(z, \varepsilon)^*}$ is isomorphic to F ;*
- (b) *For every sufficiently small $\delta \in |K^\times|$, and every $w \in D \setminus (U \cup \{z\})$, the restriction $\tilde{F}|_{D(w, \delta)}$ is a tamely ramified locally constant Λ -module (see (2.3.13)).*

Proof. Denote by $C \subset \mathbb{D}(z, \varepsilon)^*$ the annulus $\{a \in A \mid |(x - z)(a)|_a = \varepsilon\}$, and let $\mathbb{D}'$ be another disc containing C , such that $\mathbb{D}' \cup \mathbb{D}(z, \varepsilon) = D \setminus \{z\}$. We may find a finite Galois covering $\varphi : C' \rightarrow C$, such that $F|_{C'}$ is a constant Λ -module (see (2.1.14)). Then, by [13, Th.1], φ extends to a finite Galois covering $C'' \rightarrow \mathbb{D}'$, with the same Galois group, and ramified over finitely many points $w_1, \dots, w_n \in \mathbb{D}'(K)$. It follows easily that $F|_C$ extends to a locally constant Λ -module F' on the étale site of $\mathbb{D}' \setminus \{w_1, \dots, w_n\}$. By construction, it is clear that F' has finite monodromy around each of the points $w_1, \dots, w_n$. We may then glue F and F' along their common restriction on C , to obtain the sought $\tilde{F}$. $\square$

2.4.16. From (2.2.15) and theorem 2.3.30(ii), we know that the source of the isomorphism (2.4.13) is an object of $\Lambda[\pi(\infty')]\text{-Mod}$; we wish to show that each direct summand in the target is also naturally an object in the same category, so that (2.4.13) is an isomorphism of $\Lambda[\pi(\infty')]$ -modules.

To this aim, let $z \in D(K)$ be any point, F a Λ -module on $\mathbb{D}(z, \varepsilon_0)_{\text{ét}}^*$. For every $\varepsilon, \delta \in |K^\times|$ with $\varepsilon \leq \varepsilon_0$, choose as usual a fundamental pro-covering $(C_H \mid H \subset \pi(\infty', \delta))$, and consider the commutative diagram :

$$\begin{array}{ccccc}
 \mathbb{D}(\infty', \delta)^* & \xleftarrow{q'_{\varepsilon, \delta}} & \mathbb{D}(z, \varepsilon)^* \times_S \mathbb{D}(\infty', \delta)^* & \xrightarrow{q_{\varepsilon, \delta}} & \mathbb{D}(z, \varepsilon)^* \\
 \parallel & & \downarrow j_{\varepsilon, \delta} & & \downarrow j_\varepsilon \\
 \mathbb{D}(\infty', \delta)^* & \xleftarrow{\bar{q}'_{\varepsilon, \delta}} & \mathbb{D}(z, \varepsilon) \times_S \mathbb{D}(\infty', \delta)^* & \xrightarrow{\bar{q}_{\varepsilon, \delta}} & \mathbb{D}(z, \varepsilon) \\
 & & \uparrow j_H & & \\
 & & \mathbb{D}(z, \varepsilon) \times_S C_H & &
 \end{array}$$

where :

- j_H is obtained by base change from the projection $C_H \rightarrow \mathbb{D}(\infty', \delta)^*$
- $q_{\varepsilon, \delta}, q'_{\varepsilon, \delta}, \bar{q}_{\varepsilon, \delta}$ and $\bar{q}'_{\varepsilon, \delta}$ are the natural projections
- j_ε and $j_{\varepsilon, \delta}$ are the open immersions.

Define

$$(2.4.17) \quad \mathcal{G}(F, \psi, \varepsilon, \delta) := q_{\varepsilon, \delta}^* F \otimes_\Lambda \mathcal{L}_\psi \langle m_{\varepsilon, \delta} \rangle$$

where $m_{\varepsilon, \delta}$ is the restriction of m to $\mathbb{D}(z, \varepsilon)^* \times_S \mathbb{D}(\infty', \delta)^*$. Mostly we shall drop the explicit mention of F and ψ , and simply write $\mathcal{G}(\varepsilon, \delta)$. With this notation, set

$$\begin{aligned}
 \mathcal{G}_!(\varepsilon, \delta) &:= j_{\varepsilon, \delta!} \mathcal{G}(\varepsilon, \delta) \\
 \mathcal{G}_*(\varepsilon, \delta) &:= Rj_{\varepsilon, \delta*} \mathcal{G}(\varepsilon, \delta) \\
 M^i(\varepsilon, \delta) &:= \operatorname{colim}_{H \subset \pi(\infty', \delta)} H^i(\mathbb{D}(z, \varepsilon) \times_S C_{H, \text{ét}}, j_H^* \mathcal{G}_!(\varepsilon, \delta)).
 \end{aligned}$$

Notice that the Λ -module $M^i(\varepsilon, \delta)$ carries a natural continuous action of $\pi(\infty', \delta)$. Moreover, if $\varepsilon' \leq \varepsilon$ and $\delta' \leq \delta$, any choice of a group homomorphism $\pi(\infty', \delta') \rightarrow \pi(\infty', \delta)$ as in (2.2.7),

induces a $\pi(\infty', \delta')$ -equivariant homomorphism of Λ -modules :

$$M^i(\varepsilon, \delta) \rightarrow M^i(\varepsilon', \delta').$$

Lemma 2.4.18. *There is a natural isomorphism of Λ -modules :*

$$(2.4.19) \quad \mathcal{F}_\psi^{(z, \infty')} (F_{\eta_z}) \xrightarrow{\sim} \operatorname{colim}_{n \in \mathbb{N}} \operatorname{colim}_{\delta > 0} M^1(\varepsilon_n, \delta).$$

Proof. For a given $\varepsilon \in |K^\times|$, fix a fundamental pro-covering $(C_H^\varepsilon \mid H \subset \pi(\infty', \varepsilon))$ as usual, and denote by $j_H^\varepsilon : X(\varepsilon) \times_{\mathbb{D}(\infty', \varepsilon)} C_H^\varepsilon \rightarrow X(\varepsilon)$ the natural projection, for every open subgroup $H \subset \pi(\infty', \varepsilon)$. Let G be any Λ -module on $X(\varepsilon)_{\text{ét}}$; since the stalk functors commute with filtered colimits, we have a natural isomorphism of Λ -modules :

$$R^1 \Psi_{\eta_{\infty', \varepsilon}}(G)_z \xrightarrow{\sim} \operatorname{colim}_{H \subset \pi(\infty', \varepsilon)} (R^1 j_{H*}^\varepsilon j_H^{\varepsilon*} G)_{(z, \infty')}$$

and recall that $R^1 j_{H*}^\varepsilon G$ is the sheaf associated to the presheaf on $X(\varepsilon)_{\text{ét}}$ given by the rule : $U \mapsto H^1(U \times_{\mathbb{D}(\infty', \varepsilon)} C_H^\varepsilon, j_H^{\varepsilon*} G)$ for every étale morphism $U \rightarrow X(\varepsilon)$.

Now, on the one hand, the stalks of a presheaf are isomorphic to the stalks of its associated sheaf; on the other hand, the point (z, ∞') admits the cofinal system of étale neighborhoods $(\mathbb{D}(z, \delta) \times_S \mathbb{D}(\infty', \delta) \mid \delta \leq \varepsilon)$. We thus obtain a natural isomorphism of Λ -modules :

$$R^1 \Psi_{\eta_{\infty', \varepsilon}}(G)_z \xrightarrow{\sim} \operatorname{colim}_{H \subset \pi(\infty', \varepsilon)} \operatorname{colim}_{\delta \leq \varepsilon} H^1(\mathbb{D}(z, \delta) \times_S C_{H, \text{ét}}^\varepsilon, j_H^{\varepsilon*} G).$$

Lastly, if $(\varepsilon_n \mid n \in \mathbb{N})$ is our strictly decreasing sequence, we may take the colimit over $n \in \mathbb{N}$, and we arrive at the identity :

$$R^1 \Psi_{\eta_{\infty'}}(G)_z \xrightarrow{\sim} \operatorname{colim}_{n \in \mathbb{N}} \operatorname{colim}_{\delta > 0} \operatorname{colim}_{H \subset \pi(\infty', \delta)} H^1(\mathbb{D}(z, \varepsilon_n) \times_S C_{H, \text{ét}}^\varepsilon, j_H^{\varepsilon*} G|_{\mathbb{D}(z, \varepsilon_n) \times_S \mathbb{D}(\infty', \delta)^*})$$

from which the lemma follows easily. $\square$

2.4.20. For given $\varepsilon, \delta \in |K^\times|$, denote by $Q_{\varepsilon, \delta}$ the cone of the natural morphism $\mathcal{G}_1(\varepsilon, \delta) \rightarrow \mathcal{G}_*(\varepsilon, \delta)$. Clearly the cohomology sheaves $\mathcal{H}^i Q_{\varepsilon, \delta}$ are concentrated on $\{z\} \times_S \mathbb{D}(\infty', \delta)^*$ for every $i \in \mathbb{Z}$, and they agree on this closed subspace with the restriction of $R^i j_{\varepsilon, \delta*} \mathcal{G}(\varepsilon, \delta)$.

After applying the functor $R\bar{q}'_{\varepsilon, \delta*}$ and inspecting the resulting distinguished triangle, we obtain the exact sequence :

$$\mathcal{E} : q'_{\varepsilon, \delta*} \mathcal{G}(\varepsilon, \delta) \rightarrow \bar{q}'_{\varepsilon, \delta*} (\mathcal{H}^0 Q_{\varepsilon, \delta}) \rightarrow R^1 \bar{q}'_{\varepsilon, \delta*} \mathcal{G}_1(\varepsilon, \delta) \rightarrow R^1 q'_{\varepsilon, \delta*} \mathcal{G}(\varepsilon, \delta) \rightarrow \bar{q}'_{\varepsilon, \delta*} (\mathcal{H}^1 Q_{\varepsilon, \delta}).$$

On the other hand, for every maximal point $a \in A'$ and every $\varepsilon \in |K^\times|$, let $w(a, \varepsilon) \in A \times_S A'$ be the unique point such that $(\mathbb{E}(z, \varepsilon) \times \{a\}) \setminus \{w(a, \varepsilon)\}$ is the interior of $\mathbb{E}(z, \varepsilon) \times \{a\}$ in $D \times_S \{a\}$ (notation of (2.4.5)). Set $w(\varepsilon) := p(w(a, \varepsilon))$ (which is independent of a), denote $\Gamma_{w(\varepsilon)}$ and $\Gamma_{w(a, \varepsilon)}$ the value groups of the residue fields of $w(\varepsilon)$, respectively $w(a, \varepsilon)$, and notice that the projection p induces an injective homomorphism of ordered groups $\Gamma_{w(\varepsilon)} \rightarrow \Gamma_{w(a, \varepsilon)}$. Let

$$\beta_{\varepsilon, 1}, \dots, \beta_{\varepsilon, r} \in \Gamma_{w(\varepsilon)} \otimes_{\mathbb{Z}} \mathbb{Q}$$

be the breaks of the stalk $F_{w(\varepsilon)}$ (here r is the Λ -rank of $F_{w(\varepsilon)}$, and each break is repeated with multiplicity equal to the rank of the corresponding direct summand in the break decomposition of this stalk; moreover the break decompositions are invariant under extension from the base field $(K, |\cdot|)$ to the residue field $(L, |\cdot|_L)$ of a : see [25, Lemma 3.3.8]). Denote also by $\alpha(a, \varepsilon) \in \Gamma_{w(a, \varepsilon)} \otimes_{\mathbb{Z}} \mathbb{Q}$ the unique break of the stalk $\mathcal{L}_\psi \langle m \rangle_{w(a, \varepsilon)}$.

Lemma 2.4.21. *For every $\varepsilon \in |K^\times|$ with $\varepsilon \leq \varepsilon_0$ we may find $\delta \in |K^\times|$ such that :*

- (i) $\beta_{\gamma, i} > \alpha(a, \gamma)$
- (ii) $\sharp_{\mathbb{Q}} \alpha(a, \gamma) = 1$
- (iii) $\alpha(a, \gamma)$ is the unique break of the stalk $\mathcal{G}(\varepsilon, \delta)_{w(a, \gamma)}$

for every $\gamma \in |K^\times| \cap [\varepsilon/2, \varepsilon]$, every $i = 1, \dots, r$, and every maximal point $a \in \mathbb{D}(\infty', \delta)^*$.

Proof. (i) and (ii) are analogous to claim 2.4.7, and they follow likewise from [18, Lemma 9.4], together with the continuity properties of the breaks ([25, Lemma 4.2.12]). Assertion (iii) follows immediately from (i) (cp. the proof of theorem 2.4.2). $\square$

Lemma 2.4.21(ii),(iii) implies that

$$(2.4.22) \quad q'_{\varepsilon, \delta*} \mathcal{G}(\varepsilon, \delta) = 0 \quad \text{for } \varepsilon \text{ and } \delta \text{ as in lemma 2.4.21.}$$

Lemma 2.4.23. *In the situation of (2.4.16), suppose Λ is a field, F is a locally constant Λ -module of finite type, with bounded ramification at the point z , and let $\varepsilon \in |K^\times|$ with $\varepsilon \leq \varepsilon_0$. Then there exists $\delta \in |K^\times|$ such that :*

- (i) *The Λ -module $R^i q'_{\varepsilon, \delta!} \mathcal{G}(\varepsilon, \delta)$ vanishes for $i \neq 1$, and is locally constant of finite type on $\mathbb{D}(\infty', \delta)_{\text{ét}}^*$ for $i = 1$.*
- (ii) *The Λ -module $R^i q'_{\varepsilon, \delta*} \mathcal{G}(\varepsilon, \delta)$ vanishes for $i \neq 1$, and is locally constant of finite type on $\mathbb{D}(\infty', \delta)_{\text{ét}}^*$ for $i = 1$.*
- (iii) *For $i \leq 1$, the Λ -module $\mathcal{H}^i Q_{\varepsilon, \delta}$ is locally constant of finite type on $\{z\} \times_S \mathbb{D}(\infty', \delta)_{\text{ét}}^*$.*

Proof. (i): The vanishing assertion for $i = 0$ is clear. Hence, suppose that $i > 0$, and for every $\gamma, \gamma' \in |K^\times|$ with $\gamma < \gamma' \leq \varepsilon$ denote by $\Lambda(\gamma, \gamma')$ the Λ -module on the étale site of $\mathbb{D}(z, \varepsilon)^* \times_S \mathbb{D}(\infty', \delta)^*$ obtained as extension by zero of the constant Λ -module with stalk Λ on the étale site of $A(z, \gamma, \gamma') \times_S \mathbb{D}(\infty', \delta)^*$, where $A(z, \gamma, \gamma') := \mathbb{D}(z, \gamma') \setminus \mathbb{E}(z, \gamma)$ is the annulus centered at z of external radius γ' and internal radius γ . Set

$$G_{\gamma, \gamma'}^i := R^i q'_{\varepsilon, \delta!} (\mathcal{G}(\varepsilon, \delta) \otimes_{\Lambda} \Lambda(\gamma, \gamma')).$$

By [17, Th.6.2.2], the Λ -module $G_{\gamma, \gamma'}^i$ is constructible on $\mathbb{D}(\infty', \delta)_{\text{ét}}^*$, for every γ, γ' as above and every $i \in \mathbb{N}$. Moreover

$$(2.4.24) \quad R^i q'_{\varepsilon, \delta!} \mathcal{G}(\varepsilon, \delta) = \operatorname{colim}_{\gamma > 0} G_{\gamma, \varepsilon}^i$$

by [17, Prop.5.4.5(i)]. Let $a \in \mathbb{D}(\infty', \delta)$ be any maximal point, and $(L, |\cdot|_L)$ the algebraic closure of the residue field of a ; the stalk $(G_{\gamma, \gamma'}^i)_a$ calculates

$$H_c^i(A(z, \gamma, \gamma') \times_S \operatorname{Spa}(L, L^+), \mathcal{G}(\varepsilon, \delta))$$

([17, Th.5.4.6]). Pick $\delta \in |K^\times|$ fulfilling conditions (i),(ii) of lemma 2.4.21; by Poincaré duality (lemma 2.3.10(iii)) and lemma 2.4.21(ii),(iii) we deduce that

$$(2.4.25) \quad (G_{\gamma, \gamma'}^i)_a = 0 \quad \text{for every } i \neq 1 \text{ and every } \gamma' \in |K^\times| \cap [\varepsilon/2, \varepsilon].$$

Since the maximal points are dense in the constructible topology of $\mathbb{D}(\infty', \delta)^*$ ([15, Cor.4.2]), we see that $G_{\gamma, \varepsilon}^i = 0$ for $i > 1$; combining with (2.4.24) we get the sought vanishing for $i > 1$ and every sufficiently small $\delta \in |K^\times|$.

To deal with the remaining case $i = 1$, we remark :

Claim 2.4.26. There exists $\delta \in |K^\times|$ such that $R^1 q'_{\varepsilon, \delta!} \mathcal{G}(\varepsilon, \delta)$ is overconvergent, and all its stalks have the same finite length.

Proof of the claim. Indeed, choose δ fulfilling conditions (i),(ii) of lemma 2.4.21, so that (2.4.25) holds. In view of [18, Cor.10.3], it follows that the natural morphisms

$$(2.4.27) \quad G_{\gamma, \gamma'}^1 \rightarrow G_{\gamma, \varepsilon}^1$$

induce isomorphisms on the stalks over the maximal points of $\mathbb{D}(\infty', \delta)^*$, whenever $\gamma < \varepsilon/2 \leq \gamma' \leq \varepsilon$. Then again, since the maximal points are dense in the constructible topology of

$\mathbb{D}(\infty', \delta)^*$, we deduce that (2.4.27) restrict to isomorphisms on $\mathbb{D}(\infty', \delta)_{\text{ét}}^*$ for the stated range of γ, γ' . Set

$$G^1 := \operatorname{colim}_{\gamma > 0} \operatorname{colim}_{\gamma' < \varepsilon} G_{\gamma, \gamma'}^1.$$

Combining with (2.4.24) we see that the natural map :

$$G^1 \rightarrow R^1 q'_{\varepsilon, \delta!} \mathcal{G}(\varepsilon, \delta)$$

is an isomorphism. However, notice that $G^1 = R^i q'_{\varepsilon, \delta!} (\mathcal{G}(\varepsilon, \delta) \otimes_{\Lambda} \Lambda(0^+, \varepsilon^-))$, where

$$\Lambda(0^+, \varepsilon^-) = \operatorname{colim}_{\gamma > 0} \operatorname{colim}_{\gamma' < \varepsilon} \Lambda(\gamma, \gamma')$$

which is an overconvergent sheaf, whose support is partially proper over $\mathbb{D}(\infty', \delta)^*$; then G^1 is overconvergent ([17, Cor.8.2.4]), so the same holds for $R^1 q'_{\varepsilon, \delta!} \mathcal{G}(\varepsilon, \delta)$.

In order to compute the length of the stalk over a given point $a \in \mathbb{D}(\infty', \delta)^*$, we may therefore assume that a is maximal. Since the vanishing assertion (i) is already known for $i \neq 1$, this length is completely determined – in view of [18, Cor.10.3] and lemma 2.4.21 – by the Swan conductor at the point (z, a) of the restriction of $\mathcal{G}(\varepsilon, \delta)$ to the fibre $q'_{\varepsilon, \delta}{}^{-1}(a)$. Now, if $z \in A(K)$ it is clear that this Swan conductor equals $\operatorname{sw}_z^1(F, 0^+)$, so it does not depend on a , whence the claim. Lastly, if $z = \infty$, proposition 2.3.21(i) implies that the Swan conductor will also be independent of a , provided $\mathbb{D}(\infty', \delta)^* \cap d_K(F) = \emptyset$, which can be arranged by further shrinking δ . $\diamond$

To achieve the proof of (i), it suffices now to invoke claims 2.3.33 and 2.4.26, together with identity (2.4.24) for $i = 1$.

(ii): Set $\mathcal{G}(\varepsilon, \delta)^\vee := \mathcal{H}om_{\Lambda}(\mathcal{G}(\varepsilon, \delta), \Lambda_X)$ and notice that $\mathcal{G}(\varepsilon, \delta)^\vee \simeq \mathcal{G}(F^\vee, \psi^{-1}, \varepsilon, \delta)$ (notation of (2.4.17)) hence assertion (i) holds as well with $\mathcal{G}(\varepsilon, \delta)$ replaced by $\mathcal{G}(\varepsilon, \delta)^\vee$. Moreover, since Λ is a field, the natural morphism

$$\mathcal{G}(\varepsilon, \delta) \rightarrow R\mathcal{H}om_{\Lambda}(\mathcal{G}(\varepsilon, \delta)^\vee, \Lambda)$$

is an isomorphism in $\mathbf{D}(\mathbb{D}(z, \varepsilon)^* \times_S \mathbb{D}(\infty', \delta)_{\text{ét}}^*, \Lambda)$. By Poincaré duality (lemma 2.3.10(ii)), there follows a natural isomorphism

$$Rq'_{\varepsilon, \delta*} \mathcal{G}(\varepsilon, \delta) \xrightarrow{\sim} R\mathcal{H}om_{\Lambda}(Rq'_{\varepsilon, \delta!} \mathcal{G}(\varepsilon, \delta)^\vee, \Lambda(1)[2]) \xrightarrow{\sim} \mathcal{H}om_{\Lambda}(Rq'_{\varepsilon, \delta!} \mathcal{G}(\varepsilon, \delta)^\vee, \Lambda(1)[2])$$

whence the contention.

(iii): Suppose first that $z \in A(K)$; then, by smooth base change ([17, Th.4.1.1(a)]), it is easily seen that $\mathcal{H}^i Q_{\varepsilon, \delta}$ is locally isomorphic (in the étale topology) to the constant Λ -module whose stalk is $(R^i j_{\varepsilon*} F)_z$ (notation of (2.4.16)). It remains therefore only to show that the latter is a Λ -module of finite type, which is clear, since by assumption F has bounded ramification at the point z (details left to the reader).

In case $z = \infty$, proposition 2.3.8 implies that actually $Q_{\varepsilon, \delta} = 0$ (details left to the reader). $\square$

Proposition 2.4.28. *In the situation of (2.4.16), suppose that F is locally constant of finite type, and with bounded ramification at the point z . Then for every $\varepsilon \in |K^\times|$ with $\varepsilon \leq \varepsilon_0$ we may find $\delta_\varepsilon \in |K^\times|$ such that the following holds for every $\delta \leq \delta_\varepsilon$:*

- (i) $M^1(\varepsilon, \delta)$ is a Λ -module of finite length.
- (ii) $M^i(\varepsilon, \delta) = 0$ for every $i \neq 1$.

Proof. By the usual arguments, we may reduce to the case where Λ is a field. We notice :

Claim 2.4.29. In order to prove the proposition, it suffices to find $\delta_\varepsilon \in |K^\times|$, such that the following holds for every $\delta \leq \delta_\varepsilon$:

- (a) The Λ -module $R^1 \bar{q}'_{\varepsilon, \delta*} \mathcal{G}_1(\varepsilon, \delta)$ is locally constant of finite type on $\mathbb{D}(\infty', \delta)_{\text{ét}}^*$.
- (b) $R^i \bar{q}'_{\varepsilon, \delta*} \mathcal{G}_1(\varepsilon, \delta) = 0$ for every $i \neq 1$.

Proof of the claim. Indeed, for every open subgroup $H \subset \pi(\infty', \delta)$ let $q'_H : \mathbb{D}(z, \varepsilon) \times_S C_H \rightarrow C_H$ be the projection; given δ_ε as in the claim, the smooth base change theorem of [17, Th.4.1.1(a)] implies that $R^i q'_{H*} j_H^* \mathcal{G}_!(\varepsilon, \delta)$ vanishes for $i \neq 1$, and is a locally constant Λ -module of finite type on $C_{H, \text{ét}}$, for $i = 1$ and every $\delta \leq \delta_\varepsilon$. Then the Leray spectral sequence for the morphism q'_H yields a natural (equivariant) isomorphism :

$$(2.4.30) \quad \begin{aligned} M^i(\varepsilon, \delta) &\xrightarrow{\sim} \operatorname{colim}_{H \subset \pi(\infty', \delta)} H^{i-1}(C_H, R^1 q'_{H*} j_H^* \mathcal{G}_!(\varepsilon, \delta)) \\ &\xrightarrow{\sim} \operatorname{colim}_{H \subset \pi(\infty', \delta)} H^{i-1}(C_H, R^1 \bar{q}'_{\varepsilon, \delta*} \mathcal{G}_!(\varepsilon, \delta)). \end{aligned}$$

for $i = 1$, the target of (2.4.30) is a Λ -module of finite length, since every C_H is connected, and Λ is noetherian. For $i = 0$, the target of (2.4.30) trivially vanishes. For $i > 1$, we remark that, (up to restricting the colimit to a cofinal system of open subsets H , we may replace $R^1 \bar{q}'_{\varepsilon, \delta*} \mathcal{G}_!(\varepsilon, \delta)$ by a constant Λ -module; then, the sought vanishing for $i = 2$ follows from lemma 2.2.6. Lastly, for $i > 2$, we may argue by Poincaré duality, as in the proof of proposition 2.2.18, to conclude. $\diamond$

Now, to begin with, it is clear that $\bar{q}'_{\varepsilon, \delta*} \mathcal{G}_!(\varepsilon, \delta) = 0$ for every $\varepsilon \leq \varepsilon_0$ and every $\delta \in |K^\times|$. Next, in view of (2.4.22), the exact sequence $\mathcal{E}$ of (2.4.20), and lemma 2.4.23(ii,iii), we see that condition (a) of claim 2.4.29 holds. Similarly, for $i > 1$, we consider the exact sequence

$$0 = R^{i-1} \bar{q}'_{\varepsilon, \delta*} Q_{\varepsilon, \delta} \rightarrow R^i \bar{q}'_{\varepsilon, \delta*} \mathcal{G}_!(\varepsilon, \delta) \rightarrow R^i q'_{\varepsilon, \delta*} \mathcal{G}(\varepsilon, \delta)$$

whose last term vanishes, according to lemma 2.4.23(ii); so also condition (b) of claim 2.4.29 holds, and the proposition follows. $\square$

2.4.31. The significance of proposition 2.4.28(i) is that it allows to endow the local Fourier transform of F_{η_z} with an action of $\pi(\infty', \delta)$, for some sufficiently small $\delta \in |K^\times|$. Indeed, pick a monotonically descending sequence $(\delta_n \mid n \in \mathbb{N})$ of values in $|K^\times|$ such that $M(\varepsilon_n, \delta_n)$ is a Λ -module of finite length for every $n \in \mathbb{N}$. In view of (2.4.19) the Λ -module $\mathcal{F}_\psi^{(z, \infty')}(F_{\eta_z})$ is the colimit of the filtered system $\mathcal{M}$ of such $M(\varepsilon_n, \delta_n)$. However, proposition 2.4.15, together with remark 2.4.12(i) implies that the local Fourier transform has finite length as a Λ -module, hence some $M(\varepsilon_n, \delta_n)$ must surject onto it. Moreover, the image of $M(\varepsilon_n, \delta_n)$ into $\mathcal{F}_\psi^{(z, \infty')}(F_{\eta_z})$ is already isomorphic to the image of the same module into $M(\varepsilon_m, \delta_m)$ for some $m > n$. Since the transition maps in the filtered system $\mathcal{M}$ are equivariant, we may in this way endow $\mathcal{F}_\psi^{(z, \infty')}(F_{\eta_z})$ with an action of $\pi(\infty', \delta_m)$. Let $F \rightarrow F'$ be a morphism of locally constant Λ -modules on $\mathbb{D}(z, \varepsilon_0)^*$ with bounded ramification at z ; by repeating the above procedure on F' , we endow $\mathcal{F}_\psi^{(z, \infty')}(F'_{\eta_z})$ with a $\pi(\infty', \delta_{m'})$ -action for some other $m' \in \mathbb{N}$. A direct inspection of the construction shows that the induced map $\mathcal{F}_\psi^{(z, \infty')}(F_{\eta_z}) \rightarrow \mathcal{F}_\psi^{(z, \infty')}(F'_{\eta_z})$ shall be $\pi(\infty', \delta_{m''})$ -equivariant for some $m'' \geq \max(m, m')$.

Now, for any $z \in D(K)$ (and any $z \in D'(K)$), define the category of $\Lambda[\pi(z)]$ -modules with bounded ramification :

$$\Lambda[\pi(z)]\text{-b.Mod}$$

as the full subcategory of $\Lambda[\pi(z)]\text{-Mod}$ whose objects are represented by Λ -modules on the étale site of $\mathbb{D}(z, \varepsilon)^*$ (for some $\varepsilon \in |K^\times|$) which have bounded ramification at the point z . With this notation, the foregoing means that the datum of $\mathcal{F}_\psi^{(z, \infty')}(F_{\eta_z})$ together with its $\pi(\infty', \delta_m)$ -action, is well defined as a functor

$$\mathcal{F}_\psi^{(z, \infty')} : \Lambda[\pi(z)]\text{-b.Mod} \rightarrow \Lambda[\pi(\infty')]\text{-b.Mod}$$

which we call the *local Fourier transform from z to ∞'* .

2.4.32. Furthermore, (2.4.30) implies that the natural decomposition (2.4.13) is already well defined in $\Lambda[\pi(\infty')]\text{-b.Mod}$. Indeed, a direct inspection shows that the composition of (2.4.13) with the projection onto the direct factor indexed by z , is obtained as the colimit of the filtered system of maps

$$\Gamma(C_H, \mathcal{H}^1(\mathcal{F}_\psi(j_!F))) \rightarrow \Gamma(C_H, R^1\bar{q}'_{\varepsilon, \delta*} \mathcal{G}_!(\varepsilon, \delta))$$

induced by the natural morphism

$$R^1p'_*(p^*(j_!F) \otimes_\Lambda \mathcal{L}_\psi \langle m \rangle)_{|\mathbb{D}(\infty', \delta)} \rightarrow R^1\bar{q}'_{\varepsilon, \delta*} \mathcal{G}_!(\varepsilon, \delta)$$

which is clearly $\pi(\infty', \delta)$ -equivariant.

Remark 2.4.33. We will also use the “dual” functors $\mathcal{F}_\psi^{(z, \infty')}$, for $z \in D'(K)$: they are defined by exchanging everywhere the roles of D and D' in the foregoing (cp. [22, Déf.2.4.2.3]). Especially, given a Zariski open immersion $j' : U' \rightarrow A'$, and a local system F' on $U'_{\text{ét}}$ with bounded ramification everywhere, one has a similar stationary phase decomposition for $b_* \mathcal{F}'_\psi(j'_* F')_{\eta_\infty}$, in terms of such functors (here $b : A'' \rightarrow A$ is the double duality isomorphism). We leave to the reader the task of spelling out this isomorphism.

3. LOCAL ANALYSIS OF MONODROMY

3.1. Study of the local Fourier transforms. To begin with, we notice :

Proposition 3.1.1. *For every $z \in D(K)$, the local Fourier transform $\mathcal{F}_\psi^{(z, \infty')}$ is an exact functor on $\Lambda[\pi(z)]\text{-b.Mod}$.*

Proof. Choose a sequence $(\varepsilon_n \mid n \in \mathbb{N})$ as in (2.2.11). For every continuous $\Lambda[\pi(z, \varepsilon)]$ -module M of finite type, set

$$T^i(M) := \operatorname{colim}_{n \in \mathbb{N}} i_z^* R^i \Psi_{\eta_{\infty', \varepsilon_n}}(\bar{p}_\varepsilon^*(M_l) \otimes_\Lambda \overline{\mathcal{L}}_\psi \langle m \rangle_{|X(\varepsilon)})$$

(notation of (2.4.9)). It suffices to check :

Claim 3.1.2. Suppose that M_l has bounded ramification at the point z . Then $T^i(M) = 0$ for every $i \neq 1$.

Proof of the claim. The assertion follows easily from proposition 2.4.28(ii). $\square$

3.1.3. For any $z \in K$, let $\vartheta_z : A \rightarrow A$ the translation by $z : x \mapsto x + z$ (notation of (2.3.3)). Then ϑ_z induces an isomorphism of categories :

$$\vartheta_{z*} : \Lambda[\pi(0)]\text{-Mod} \xrightarrow{\sim} \Lambda[\pi(z)]\text{-Mod}$$

in an obvious way. With this notation, an easy calculation yields a natural isomorphism :

$$(3.1.4) \quad \mathcal{F}_\psi^{(z, \infty')}(\vartheta_{z*} M) \xrightarrow{\sim} \mathcal{F}_\psi^{(0, \infty')}(M) \otimes_\Lambda \mathcal{L}_\psi \langle zx' \rangle_{\eta_{\infty'}}$$

for all $\Lambda[\pi(0)]$ -modules M with bounded ramification, where $\mathcal{L}_\psi \langle zx' \rangle_{\eta_{\infty'}}$ is the object of $\Lambda[\pi(\infty')]\text{-b.Mod}$ defined as in (2.2.15) (and remark 2.2.17). It follows that the study of $\mathcal{F}_\psi^{(z, \infty')}$ for $z \in K$ is reduced to the case where $z = 0$.

Notice as well that a $\Lambda[\pi(z)]$ -module M with bounded ramification has a well defined length and Swan conductor :

$$\operatorname{length}_\Lambda M \quad \operatorname{sw}(M)$$

namely, the length (resp. the Swan conductor at z) of any $\Lambda[\pi(z, \varepsilon)]$ -module representing M .

Let $U \subset A$ be a Zariski open subset, $j : U \rightarrow A$ the open immersion, and F a Zariski constructible Λ -module on $A_{\text{ét}}$, such that $j^* F$ is locally constant on $U_{\text{ét}}$, with bounded ramification at all the points of $D \setminus U$. Denote by $\lambda_1, \dots, \lambda_l$ the breaks of F_{η_∞} (repeated with their respective

multiplicities). Then, in view of lemma 2.3.15, theorem 2.3.30 and remark 2.3.34, we deduce the following identity :

$$(3.1.5) \quad \text{length}_\Lambda \mathcal{H}^0(\mathcal{F}_\psi(F))_{\eta_{\infty'}} = \sum_{x \in A \setminus U} a_x(F) + \sum_{i=1}^l \max(0, \lambda_i^{\natural} - 1).$$

(Details left to the reader.)

3.1.6. Hence, let $z \in \{0, \infty\}$. We consider first the case of a *tamely ramified* object of $\Lambda[\pi(z)]\text{-Mod}$. Namely, for a given value $\varepsilon \in |K^\times|$, let T be a coordinate on $\mathbb{D}(z, \varepsilon)$, such that $T(z) = 0$. For every integer $n > 0$, the rule $T \mapsto T^n$ determines a finite morphism $\mathbb{D}(z, \varepsilon^{1/n}) \rightarrow \mathbb{D}(z, \varepsilon)$; its restriction to $\mathbb{D}(z, \varepsilon^{1/n})^*$ is a finite connected Galois étale covering of $\mathbb{D}(z, \varepsilon)^*$, whose Galois group is naturally isomorphic to μ_n , the n -torsion in $K^\times$. This covering corresponds, as in (2.1.12), to a surjective continuous group homomorphism $\pi(z, \varepsilon) \rightarrow \mu_n$. For variable $n > 0$, we obtain an inverse system of group homomorphisms, whence a continuous group homomorphism :

$$(3.1.7) \quad \pi(z, \varepsilon) \rightarrow \widehat{\mathbb{Z}}(1) := \lim_{n>0} \mu_n$$

(where the target group is endowed with the profinite topology). The map (3.1.7) is surjective; to see this, we may suppose that the image v of the geometric point ξ chosen in (2.2) corresponds to the Gauss valuation “at the border” of $\mathbb{D}(z, \varepsilon)$, *i.e.* :

$$|f(v)|_v = \max(|f(a)| \text{ for all } a \in K \text{ such that } |a - z| \leq \varepsilon)$$

for every power series $f(T)$ convergent on $\mathbb{D}(z, \varepsilon)$. Let $\kappa(v)$ denote the residue field of v ; then $\pi_1^{\text{loc.alg}}(v, \xi)$ is the Galois group of $\kappa(v)$ -automorphisms of any algebraic closure of $\kappa(v)$. We have a natural group homomorphism :

$$\pi_1^{\text{loc.alg}}(v, \xi) \rightarrow \pi(z, \varepsilon)$$

and it suffices to see that its composition with (3.1.7) is surjective. The latter is a continuous map of profinite groups, so we reduce to showing that the induced maps $\pi_1^{\text{loc.alg}}(v, \xi) \rightarrow \mu_n$ are surjective for every $n > 0$. To this aim, consider the unique rank two specialization of v such that $|T| < \varepsilon$; the value group of this valuation is isomorphic to $|K^\times| \times \mathbb{Z}$, with its lexicographic ordering, and the value of T is primitive in this group, *i.e.* is not divisible by any integer $n > 1$. Consequently, $[\kappa(v)(T^{1/n}) : \kappa(v)] = n$ for every $n > 1$, and the claim follows.

3.1.8. With the notation of (3.1.6), a $\Lambda[\pi(z, \varepsilon)]$ -module M is *tamely ramified* relative to the coordinate T , if the action of $\pi(z, \varepsilon)$ on M factors through (3.1.7). We say that a $\Lambda[\pi(z)]$ -module M is *tamely ramified* if it is represented by a tamely ramified $\Lambda[\pi(z, \varepsilon)]$ -module, for some $\varepsilon \in |K^\times|$. Notice that this notion is independent of the choice of coordinate T . For such a module M , we may find $n > 0$ large enough, so that the action of $\pi(z, \varepsilon)$ factors through μ_n . If the action of $\pi(z, \varepsilon)$ on M is trivial, we say that M is *unramified*.

We may study the local Fourier transform $\mathcal{F}_\psi^{(z, \infty)}(M)$ by a global argument, as in [22]. Namely, suppose now that T is a global coordinate on $\mathbb{A}_K^1$; then the rule $T \mapsto T^n$ defines the *Kummer covering* of $\mathbb{G}_m := \mathbb{A}_K^1 \setminus \{0\}$ with Galois group μ_n . Let $\chi : \mu_n \rightarrow \Lambda^\times$ be any non-trivial character; we denote by $\mathcal{K}_\chi$ the locally constant Λ -module of rank one on $(\mathbb{G}_m)_{\text{ét}}^{\text{ad}}$ associated to this Kummer covering and the character χ . If $f : X \rightarrow (\mathbb{G}_m)_{\text{ét}}^{\text{ad}}$ is any morphism of S -adic spaces, we let as usual $\mathcal{K}_\chi \langle f \rangle := f^* \mathcal{K}_\chi$. Especially, the global coordinate x on A (notation of (2.3.3)) yields a morphism $x : A \setminus \{0\} \rightarrow (\mathbb{G}_m)_{\text{ét}}^{\text{ad}}$, whence the local system $\mathcal{K}_\chi \langle x \rangle$ on $A \setminus \{0\}$. Likewise we have the Λ -module $\mathcal{K}_\chi \langle x' \rangle$ on $A' \setminus \{0'\}$. Let $j : A \setminus \{0\} \rightarrow A$ and $j' : A' \setminus \{0'\} \rightarrow A'$ be the open immersions; the Fourier transform of $j_* \mathcal{K}_\psi \langle x \rangle = j_! \mathcal{K}_\chi \langle x \rangle$ is calculated by the following :

Lemma 3.1.9. *There is a natural isomorphism of Λ -modules :*

$$\mathcal{F}_\psi(j_* \mathcal{K}_\chi \langle x \rangle) \xrightarrow{\sim} j'_* \mathcal{K}_{\chi^{-1}} \langle x' \rangle [0] \otimes_\Lambda G(\chi, \psi)$$

where $G(\chi, \psi) := H_c^1((\mathbb{G}_m)_{\text{ét}}^{\text{ad}}, \mathcal{K}_\chi \otimes_\Lambda \mathcal{L}_\psi)$ is a free Λ -module of rank one.

Proof. *Mutatis mutandi*, the proof of [22, Prop.1.4.3.2] can be taken over. $\square$

As a corollary we obtain :

Proposition 3.1.10. *Let M be a tamely ramified $\Lambda[\pi(z)]$ -module. We have :*

(i) *If $z = 0$, then*

$$\text{length}_\Lambda \mathcal{F}_\psi^{(z, \infty')}(M) = \text{length}_\Lambda M$$

and the $\Lambda[\pi(\infty')]$ -module $\mathcal{F}_\psi^{(z, \infty')}(M)$ admits a filtration

$$0 = F_0 \subset F_1 \subset \cdots \subset F_l = \mathcal{F}_\psi^{(z, \infty')}(M)$$

whose subquotients are tamely ramified $\Lambda[\pi(\infty')]$ -modules.

(ii) *If $z = \infty$, then $\mathcal{F}_\psi^{(z, \infty')}(M) = 0$.*

Proof. The module M corresponds to a representation of μ_n for some $n > 0$; we may then easily reduce to the case of an irreducible representation of μ_n , i.e. Λ is a field, and $\dim_\Lambda M = 1$ (see [27, Partie II, §2.5, 2.6]), so M is given by a character $\chi : \mu_n \rightarrow \Lambda^\times$. For the case of the trivial character χ , we have the more precise :

Claim 3.1.11. For every unramified $\Lambda[\pi(0)]$ -module, there is a natural isomorphism :

$$\mathcal{F}_\psi^{(0, \infty')}(M) \xrightarrow{\sim} M.$$

Proof of the claim. We may take over *verbatim* the proof of [22, Prop.2.5.3.1(i)]. $\diamond$

In case χ is non-trivial and $z = 0$ (resp. $z = \infty$), M is represented by $\mathcal{K}_\chi \langle x \rangle_{\eta_0}$ (resp. by $\mathcal{K}_{\chi^{-1}} \langle x \rangle_{\eta_\infty}$). Using lemma 3.1.9 and the stationary phase argument of [22, Prop.2.5.3.1(ii)], we deduce a natural isomorphism :

$$\mathcal{F}_\psi^{(0, \infty')}(M) \oplus \mathcal{F}_\psi^{(\infty, \infty')}(\mathcal{K}_{\chi^{-1}} \langle x \rangle_{\eta_\infty}) \xrightarrow{\sim} \mathcal{K}_{\chi^{-1}} \langle x' \rangle_{\eta_\infty} \otimes_\Lambda G(\chi, \psi).$$

By comparing ranks, we see that

$$(3.1.12) \quad \dim_\Lambda \mathcal{F}_\psi^{(0, \infty')}(M) \leq 1$$

and both assertions will follow, once we show that $\mathcal{F}_\psi^{(0, \infty')}(M)$ does not vanish. To this aim, let $\overline{\mathcal{K}}_\chi \langle x \rangle$ denote the Λ -module on $D_{\text{ét}}$ which is the extension by zero of $\mathcal{K}_\chi \langle x \rangle$. The rule $x \mapsto x/(x+1)$ defines an automorphism ω of D such that $\omega(0) = 0$, $\omega(\infty) = 1$ and $\omega(-1) = \infty$. Let F denote the restriction to $A_{\text{ét}}$ of $\omega^* \overline{\mathcal{K}}_\chi \langle x \rangle$. Clearly $F|_{\mathbb{D}(\infty, \varepsilon)^*}$ corresponds to the trivial character of $\pi(\infty, \varepsilon)$ (for ε small enough); also F_{η_0} represents M , and $F_{\eta_{-1}}$ is a tamely ramified $\Lambda[\pi(-1)]$ -module of length one. We apply theorem 2.4.10 : by the foregoing, the term $\mathcal{F}_\psi^{(\infty, \infty')}(F_{\eta_\infty})$ vanishes, whence a natural isomorphism :

$$\mathcal{H}^0(\mathcal{F}_\psi(F))_{\eta_\infty} \xrightarrow{\sim} \mathcal{F}_\psi^{(0, \infty')}(M) \oplus \mathcal{F}_\psi^{(-1, \infty')}(F_{\eta_{-1}}).$$

Computing with (3.1.5) we find that $\mathcal{H}^0(\mathcal{F}_\psi(F))_{\eta_\infty}$ has length equal to 2. Taking (3.1.12) and (3.1.4) into account, we conclude that both direct summands on the right hand-side must have length 1, as required. $\square$

Corollary 3.1.13. *Let M be a $\Lambda[\pi(\infty)]$ -module with bounded ramification, and $\lambda_1, \dots, \lambda_l$ the breaks of M . Suppose that $\lambda_i^\natural \leq 1$ for every $i = 1, \dots, l$. Then $\mathcal{F}_\psi^{(\infty, \infty')}(M) = 0$.*

Proof. We may extend M to a locally constant Λ -module F on a Zariski open subset $U \subset A$, tamely ramified at the points of $\Sigma := A \setminus U$, and with $F_{\eta_\infty} = M$ (proposition 2.4.15). Let $j : U \rightarrow A$ be the open immersion, and set $F' := \mathcal{H}^0(\mathcal{F}_\psi(j_! F))$; according to (3.1.5) we have

$$\text{length}_\Lambda F'_{\eta_{\infty'}} = \#\Sigma \cdot l$$

where $\#\Sigma$ denotes the cardinality of Σ , and l is the generic length of F . On the other hand, (2.4.13), (3.1.4) and proposition 3.1.10(i) imply :

$$\text{length}_\Lambda F'_{\eta_{\infty'}} = \#\Sigma \cdot l + \text{length}_\Lambda \mathcal{F}_\psi^{(\infty, \infty')}(M)$$

whence the contention. $\square$

Theorem 3.1.14. *Let M be any $\Lambda[\pi(0)]$ -module with bounded ramification. Then we have :*

- (i) $\text{length}_\Lambda \mathcal{F}_\psi^{(0, \infty')}(M) = l := \text{length}_\Lambda M + \text{sw}(M)$.
- (ii) $\text{sw}(\mathcal{F}_\psi^{(0, \infty')}(M)) = \text{sw}(M)$.
- (iii) *Let $\gamma_1, \dots, \gamma_l$ be the breaks of $\mathcal{F}_\psi^{(0, \infty')}(M)$ (repeated with their respective multiplicities). Then $\gamma_i^{\natural} \leq 1$ for $i = 1, \dots, l$.*

Proof. We shall borrow an argument from the proof of [21, Prop.8.6.2]. By the usual reductions, we may assume that Λ is a field. Moreover, we may assume that M does not admit any quotient which is a *trivial* $\Lambda[\pi(0)]$ -module (meaning, a module represented by a constant Λ -module on $\mathbb{D}(0, \varepsilon)_{\text{ét}}^*$, for some $\varepsilon \in |K^\times|$). Indeed, for such trivial modules, the theorem is already known, in view of proposition 3.1.10(i).

By proposition 2.4.15, we may find a Zariski open subset $U \subset A \setminus \{0\}$ and a locally constant Λ -module F on $U_{\text{ét}}$, with bounded ramification at every point of $\Sigma := D \setminus U$, such that F is tamely ramified at every point of $\Sigma \setminus \{0\}$, and such that $F_{\eta_0} = M$. Set $G := \mathcal{H}^0(\mathcal{F}_\psi(j_! F))$, where $j : U \rightarrow A$ is the open immersion; from (3.1.5) we deduce easily :

$$(3.1.15) \quad \text{length}_\Lambda G_{\eta_{\infty'}} = (\#\Sigma - 1) \cdot \text{length}_\Lambda M + \text{sw}(M)$$

(where $\#\Sigma$ denotes the cardinality of Σ). On the other hand, theorem 2.4.10, together with proposition 3.1.10(ii) yields a natural isomorphism :

$$(3.1.16) \quad G_{\eta_{\infty'}} \xrightarrow{\sim} \bigoplus_{z \in \Sigma \setminus \{\infty\}} \mathcal{F}_\psi^{(z, \infty')}(F_{\eta_z})$$

which, by virtue of proposition 3.1.10(i), leads to the identity :

$$(3.1.17) \quad \text{length}_\Lambda G_{\eta_{\infty'}} = (\#\Sigma - 2) \cdot \text{length}_\Lambda M + \text{length}_\Lambda \mathcal{F}_\psi^{(0, \infty')}(M).$$

Assertion (i) follows by comparing (3.1.15) and (3.1.17). Next we remark :

Claim 3.1.18. $\mathcal{F}_\psi(j_! F) = G[0]$.

Proof of the claim. Indeed, by remark 2.3.34(ii),(iii) we know already that $\mathcal{H}^i(\mathcal{F}_\psi(j_! F)) = 0$ for $i \notin \{0, 1\}$. Moreover, in light of theorem 2.3.30(i) (and again remark 2.3.34(ii)), in order to show the same vanishing for $i = 1$, it suffices to verify that $H_c^2(U_{\text{ét}}, F \otimes_\Lambda \mathcal{L}_\psi \langle ax \rangle) = 0$ for every $a \in K$ (where, as usual, x is our fixed coordinate on A). Since Λ is a field, lemma 2.3.10(iii) reduces to checking that $H^0(U_{\text{ét}}, F^\vee \otimes_\Lambda \mathcal{L}_\psi \langle ax \rangle) = 0$ for every $a \in K$, where $F^\vee := \mathcal{H}om_\Lambda(F, \Lambda_U)$. However, since M does not admit trivial quotients, the $\Lambda[\pi(0)]$ -module $F_{\eta_0}^\vee = (F^\vee \otimes_\Lambda \mathcal{L}_\psi \langle ax \rangle)_{\eta_0}$ does not contain trivial submodules, whence the claim. $\diamond$

Notice as well that $d_K(F) = \{0\}$ (notation of (2.3.19)), hence G is locally constant on $A' \setminus \{0'\}$. From claim 3.1.18 and (2.3.5) we get a natural isomorphism :

$$\mathcal{F}_\psi' G \xrightarrow{\sim} a_* j_! F(-1)$$

where $a : A \rightarrow A''$ is (-1) -times the double duality isomorphism, and (-1) denotes the Tate twist. Then we may apply (3.1.5) to derive the identity :

$$(3.1.19) \quad \text{length}_\Lambda M = a_0(G) + \sum_{i=1}^l \max(0, \gamma_i^{\natural} - 1).$$

Let $\bar{0}'$ be a geometric point localized at $0' \in A'$; from claim 3.1.18 we deduce as well that :

$$(3.1.20) \quad \text{length}_\Lambda G_{\bar{0}'} = -\chi_c(A, F) = (\sharp\Sigma - 2) \cdot \text{length}_\Lambda M + \text{sw}(M).$$

Taking (3.1.15) into account, we get :

$$a_{0'}(G) = \text{length}_\Lambda M + \text{sw}(G_{\eta_{0'}}).$$

Comparing with (3.1.19) we conclude that assertion (iii) holds, and also that

$$(3.1.21) \quad a_{0'}(G) = \text{length}_\Lambda M \quad \text{sw}(G_{\eta_{0'}}) = 0.$$

Next, from (3.1.4) and proposition 3.1.10(i) we deduce that $\text{length}_\Lambda \mathcal{F}_\psi^{(z, \infty)}(F_{\eta_z}) = \text{length}_\Lambda M$ for every $z \in \Sigma \setminus \{0, \infty\}$. Combining with (3.1.16) we obtain :

$$(3.1.22) \quad \text{sw}(\mathcal{F}_\psi^{(0, \infty')}(M)) = \text{sw}(G_{\eta_{\infty'}}) - (\sharp\Sigma - 2) \cdot \text{length}_\Lambda M.$$

On the other hand, (3.1.21) and claim 3.1.18 also imply that

$$0 = \chi_c(A', G) = -2 \cdot \text{length}_\Lambda G_{\eta_{\infty'}} + a_{0'}(G) + a_{\infty'}(G) = \text{length}_\Lambda M + \text{sw}(G_{\eta_{\infty'}}) - \text{length}_\Lambda G_{\eta_{\infty'}}$$

which, in view of (3.1.15) leads to the identity :

$$(3.1.23) \quad \text{sw}(G_{\eta_{\infty'}}) = (\sharp\Sigma - 2) \cdot \text{length}_\Lambda M + \text{sw}(M).$$

Assertion (ii) follows by comparing (3.1.22) and (3.1.23). $\square$

3.1.24. Following [22, §2.4], we wish now to exhibit a left quasi-inverse for the functor $\mathcal{F}_\psi^{(0, \infty')}$. Namely, for any $\varepsilon \in |K^\times|$, set

$$Y(\varepsilon) := \mathbb{D}(0, \varepsilon) \times_S D' \subset D \times_S D'$$

and denote by $\bar{p}_\varepsilon : Y(\varepsilon) \rightarrow \mathbb{D}(0, \varepsilon)$, $\bar{p}'_\varepsilon : Y(\varepsilon) \rightarrow D'$ the projections. Also, for every $z' \in D'(K)$ and every object M' of $\Lambda[\pi(z', \varepsilon)]\text{-Mod}_{f, \text{cont}}$, let $i_{z'} : S \rightarrow \bar{q}'_\varepsilon^{-1}(0)$ be the closed immersion with image $(0, z')$, and denote by $M'_!$ the Λ -module on $D'_{\text{ét}}$ which is the extension by zero of $\tau_{z'} M'$, where $\tau_{z'}$ is the equivalence between continuous $\Lambda[\pi(z', \varepsilon)]$ -modules of finite length and locally constant Λ -modules of finite length on $\mathbb{D}(z', \varepsilon)_{\text{ét}}^*$, as in (2.4.9). We define :

$$\mathcal{F}_\psi^{(z', 0)}(M') := i_{z'}^* R^1 \Phi_{\eta_0}(\bar{p}'_\varepsilon^*(M'_!) \otimes_\Lambda \overline{\mathcal{L}}_\psi \langle m \rangle_{|Y(\varepsilon)})$$

which clearly depends only on the image of M' in $\Lambda[\pi(z')]\text{-Mod}$. In fact, we will see that this definition is interesting only for $z' = \infty'$, since this functor vanishes identically for $z' \in A'(K)$. As a first step, a simple inspection shows that, for $z' \in A'(K)$, the translation map $\vartheta_{z'} : A' \rightarrow A'$ given by $x' \mapsto x' + z'$ induces an isomorphism :

$$(3.1.25) \quad \mathcal{F}_\psi^{(z', 0)}(\vartheta_{z*} M') \xrightarrow{\sim} \mathcal{F}_\psi^{(0', 0)}(M') \quad \text{for all } \Lambda[\pi(0')]\text{-modules } M'$$

(analogous to (3.1.4)). Next, we remark that these functors obey as well a principle of stationary phase. Indeed, let F' be a Λ -module on $D'_{\text{ét}}$, and $j' : U' \rightarrow D'$ an open immersion such that $j'^* F'$ is a locally constant Λ -module of finite length on $U'_{\text{ét}}$. Then corollary 2.2.19 implies that :

$$(3.1.26) \quad j'^* R \Phi_{\eta_0}(\bar{p}'_\varepsilon^* F' \otimes_\Lambda \overline{\mathcal{L}}_\psi \langle m \rangle_{|Y(\varepsilon)}) = 0 \quad \text{in } \text{D}(U'_{\text{ét}}, \Lambda).$$

3.1.27. Suppose now $U' \subset A'$ is a Zariski open subset, and that $F' = j'_! j'^* F'$. Let $b : A'' \xrightarrow{\sim} A$ be the double duality isomorphism, given by the rule $x'' \mapsto x$ (notation of (2.3.3)), and set $F_\varepsilon := b_* \mathcal{F}'_\psi(F')|_{\mathbb{D}(0, \varepsilon)}$, which is a complex of Λ -modules on $\mathbb{D}(0, \varepsilon)_{\text{ét}}$. Combining (3.1.26) with proposition 2.2.4(ii) and corollary 2.2.14, we deduce a natural isomorphism :

$$(3.1.28) \quad R^0 \Phi_{\eta_0}(F_\varepsilon) \xrightarrow{\sim} \bigoplus_{z' \in D' \setminus U'} \mathcal{F}'_\psi^{(z', 0)}(F'_{\eta_{z'}}).$$

Remark 3.1.29. We have as well a localized version of (3.1.28), analogous to remark 2.4.12(ii). Namely, let $\delta \in |K^\times|$, and suppose that F' is a locally constant Λ -module of finite type on $\mathbb{D}(\infty', \delta)_{\text{ét}}^*$. Let $w' \in \bar{p}_\varepsilon^{-1}(0)$ be the unique point such that $(\{0\} \times \mathbb{D}(\infty', \delta)) \cup \{w'\}$ is the topological closure of $\{0\} \times \mathbb{D}(\infty', \delta)$, and set

$$\mathcal{F}'_\psi^{(w', 0)}(F') := \Gamma(\{w'\}, R^1 \Phi_{\eta_0}(\bar{p}_\varepsilon^*(\bar{j}_! F') \otimes_\Lambda \overline{\mathcal{L}}_\psi \langle m \rangle_{|Y(\varepsilon)}))$$

where $\bar{j} : \mathbb{D}(\infty', \delta)^* \rightarrow D'$ is the open immersion. Let also F_ε be the Λ -module on $\mathbb{D}(0, \varepsilon)_{\text{ét}}$ that is the restriction of $b_* \mathcal{F}'_\psi((\bar{j}_! F')|_{A'})$; arguing as in (3.1.27) we get a natural isomorphism :

$$R^0 \Phi_{\eta_0}(F_\varepsilon) \xrightarrow{\sim} \mathcal{F}'_\psi^{(\infty', 0)}(F'_{\eta_{\infty'}}) \oplus \mathcal{F}'_\psi^{(w', 0)}(F').$$

3.1.30. We are now sufficiently equipped to make some basic computations. To begin with, in the situation of (3.1.27), theorem 2.3.30(iv) and remark 2.3.34(iii) imply that the sequence of Λ -modules :

$$(3.1.31) \quad 0 \rightarrow \mathcal{H}^0(F_\varepsilon)_0 \rightarrow \mathcal{H}^0(F_\varepsilon)_{\eta_0} \rightarrow R^0 \Phi_{\eta_0}(F_\varepsilon) \rightarrow \mathcal{H}^1(F_\varepsilon)_0 \rightarrow 0$$

is exact. Let us take for F' the extension by zero of the constant Λ -module $\Lambda_{A'}$ on $A'_{\text{ét}}$. In this case, $F_\varepsilon = i_{0*} \Lambda_S(-1)[-1]$, where $i_0 : S \rightarrow \mathbb{D}(0, \varepsilon)$ is the closed immersion with image 0. From (3.1.31) and (3.1.28) we deduce that :

$$(3.1.32) \quad \text{length}_\Lambda \mathcal{F}'_\psi^{(\infty', 0)}(\Lambda) = \text{length}_\Lambda R^0 \Phi_{\eta_0}(i_{0*} \Lambda_S[-1]) = \text{length}_\Lambda \mathcal{H}^0(i_{0*} \Lambda_S)_0 = l$$

with $l := \text{length}_\Lambda \Lambda$. Next we have :

Lemma 3.1.33. *Let M' be a tamely ramified $\Lambda[\pi(z')]$ -module. We have :*

- (i) *If $z' \in A'(K)$, then $\mathcal{F}'_\psi^{(z', 0)}(M') = 0$.*
- (ii) *If $z' = \infty'$, then $\text{length}_\Lambda \mathcal{F}'_\psi^{(z', 0)}(M') = \text{length}_\Lambda M'$.*

Proof. As usual, we may reduce to the case where Λ is a field, and we may also assume that M' arises from a character $\chi : \mu_n \rightarrow \Lambda^\times$, for some $n > 0$.

(i): In view of (3.1.25) we may assume that $z' = 0'$. Consider first the case where χ is not the trivial character; then M' is represented by $\mathcal{K}_\chi \langle x' \rangle_{\eta_0}$. Denote by $\overline{\mathcal{K}}_\chi \langle x' \rangle$ the Λ -module on $D'_{\text{ét}}$ which is the extension by zero of $\mathcal{K}_\chi \langle x' \rangle$. Let $\omega' : D' \xrightarrow{\sim} D'$ be the automorphism given by the rule $x' \mapsto x'/(x' + 1)$, and set $F' := \omega'^* \overline{\mathcal{K}}_\chi \langle x' \rangle|_{A'}$. By theorem 2.3.30(ii) and remark 2.3.34(ii),(iii) we easily see that $F := b_* \mathcal{F}'_\psi(F')$ (notation of (3.1.27)) is concentrated in degree zero (since χ is not trivial), and is locally constant on $A \setminus \{0\}$. Set $F_\varepsilon := F|_{\mathbb{D}(0, \varepsilon)}$; a simple calculation using lemma 2.3.15 yields the identities:

$$\text{length}_\Lambda \mathcal{H}^0(F_\varepsilon)_0 = 1 \quad \text{length}_\Lambda \mathcal{H}^0(F_\varepsilon)_{\eta_0} = 2$$

whence $\text{length}_\Lambda R^0 \Phi_{\eta_0}(F_\varepsilon) = 1$, by virtue of (3.1.31). Notice that $F'_{\eta_{\infty'}} = \Lambda$ is the trivial $\Lambda[\pi(\infty')]$ -module; hence, combining with (3.1.28) and (3.1.32) we conclude that $\mathcal{F}'_\psi^{(0', 0)}(F'_{\eta_{0'}})$ and $\mathcal{F}'_\psi^{(-1', 0)}(F'_{\eta_{-1'}})$ must both vanish in this case, as stated.

The case where χ is the trivial character is analogous, but easier : we may take F' to be the extension by zero of the trivial Λ -module, with stalk Λ , on $(A' \setminus \{0'\})_{\text{ét}}$. Then the same sort of calculation yield :

$$\text{length}_{\Lambda} \mathcal{H}^0(F_{\varepsilon})_0 = \text{length}_{\Lambda} \mathcal{H}^1(F_{\varepsilon})_0 = \text{length}_{\Lambda} \mathcal{H}^0(F_{\varepsilon})_{\eta_0} = 1$$

and one may then argue as in the foregoing, to conclude.

(ii): The case where χ is trivial is known by (3.1.32). The case where χ is not trivial follows easily from lemma 3.1.9, (3.1.28), and assertion (i) : the details shall be left to the reader. $\square$

Proposition 3.1.34. *For $z' \in A'(K)$, let M' be any $\Lambda[\pi(z')]$ -module with bounded ramification. Then $\mathcal{F}_{\psi}^{(z',0)}(M') = 0$.*

Proof. In view of (3.1.25) we may assume that $z' = 0'$. Moreover, we may assume that M' does not admit any quotient which is a Λ -module with trivial $\pi(0')$ -action, since the case of a trivial module is already covered by lemma 3.1.33(i). By proposition 2.4.15, we may find a Zariski open subset $U' \subset A' \setminus \{0'\}$ and a locally constant Λ -module F' on $U'_{\text{ét}}$, with bounded ramification at every point of $\Sigma' := D' \setminus U'$, such that F' is tamely ramified at every point of $\Sigma' \setminus \{0'\}$, and such that $F'_{\eta_{0'}} = M'$. Set $G := \mathcal{H}^0(\mathcal{F}'_{\psi}(j'_! F'))$, where $j' : U' \rightarrow A'$ is the open immersion. The proof of claim 3.1.18 shows that

$$(3.1.35) \quad F_{\varepsilon} = b_* G[0]_{|\mathbb{D}(0,\varepsilon)}$$

(notation of (3.1.27)). Likewise, (3.1.15) implies that :

$$(3.1.36) \quad \text{length}_{\Lambda} \mathcal{H}^0(F_{\varepsilon})_{\eta_0} = (\sharp \Sigma' - 1) \cdot \text{length}_{\Lambda} M' + \text{sw}(M')$$

whereas (3.1.20) implies that

$$(3.1.37) \quad \text{length}_{\Lambda} \mathcal{H}^0(F_{\varepsilon})_0 = (\sharp \Sigma' - 2) \cdot \text{length}_{\Lambda} M' + \text{sw}(M').$$

Combining (3.1.35), (3.1.36), (3.1.37) with (3.1.31) and lemma 3.1.33(ii), we deduce that :

$$\text{length}_{\Lambda} R^0 \Phi_{\eta_0}(F_{\varepsilon}) = \text{length}_{\Lambda} M' = \text{length}_{\Lambda} \mathcal{F}^{(\infty',0)}(F'_{\eta_{\infty'}}).$$

In view of (3.1.28), the contention follows. $\square$

3.1.38. Let $U' \subset A'$ be a Zariski open subset, F' a locally constant Λ -module of finite length on $U'_{\text{ét}}$, and suppose that F' has bounded ramification at all the points of $D' \setminus U'$. In light of proposition 3.1.34, the (fake) decomposition (3.1.28) is revealed as a natural isomorphism :

$$(3.1.39) \quad R^0 \Phi_{\eta_0}(F_{\varepsilon}) \xrightarrow{\sim} \mathcal{F}_{\psi}^{(\infty',0)}(F'_{\eta_{\infty'}}).$$

Therefore, (3.1.31) is a short exact sequence :

$$(3.1.40) \quad 0 \rightarrow H_c^1(A', j'_! F') \rightarrow \mathcal{H}^0 b_* \mathcal{F}'_{\psi}(j'_! F')_{\eta_0} \rightarrow \mathcal{F}_{\psi}^{(\infty',0)}(F'_{\eta_{\infty'}}) \rightarrow H_c^2(A', j'_! F') \rightarrow 0.$$

By (2.2.15) we know that $R^0 \Phi_{\eta_0}(F_{\varepsilon})$ is a $\Lambda[\pi(0)]$ -module, and we wish now to show that, for every $\Lambda[\pi(\infty')]$ -module M' with bounded ramification, also $\mathcal{F}_{\psi}^{(\infty',0)}(M')$ is naturally a $\Lambda[\pi(0)]$ -module, in such a way that (3.1.39) is actually an isomorphism in $\Lambda[\pi(0)]$ -b.Mod. This is achieved as for the functors $\mathcal{F}_{\psi}^{(z,\infty')}$, up to some minor modification (and simplification).

Namely, say that M' is represented by a locally constant Λ -module of finite length on the étale site of $\mathbb{D}(\infty', \varepsilon_0)^*$, which we may denote again M' . For any $\varepsilon, \delta \in |K^{\times}|$, let

$$j_{\varepsilon,\delta} : \mathbb{D}(0, \delta) \times_S \mathbb{D}(\infty', \varepsilon)^* \rightarrow \mathbb{D}(0, \delta) \times_S \mathbb{D}(\infty', \varepsilon)$$

be the open immersion. Denote also by $\bar{q}_{\varepsilon,\delta}$ (resp. $\bar{q}'_{\varepsilon,\delta}$) the projection of $\mathbb{D}(0, \delta) \times_S \mathbb{D}(\infty', \varepsilon)$ onto $\mathbb{D}(0, \delta)$ (resp. onto $\mathbb{D}(\infty', \varepsilon)$), and set

$$q_{\varepsilon,\delta} := \bar{q}_{\varepsilon,\delta} \circ j_{\varepsilon,\delta} \quad q'_{\varepsilon,\delta} := \bar{q}'_{\varepsilon,\delta} \circ j_{\varepsilon,\delta}.$$

Choose a fundamental pro-covering $(C_H \mid H \subset \pi(0, \delta))$, and for any open subgroup $H \subset \pi(0, \delta)$, denote by $j_H : C_H \times_S \mathbb{D}(\infty', \varepsilon) \rightarrow \mathbb{D}(0, \delta) \times_S \mathbb{D}(\infty', \varepsilon)$ the morphism obtained by base change from the covering $C_H \rightarrow \mathbb{D}(0, \delta)^*$. For $\varepsilon < \varepsilon_0$, define

$$\mathcal{G}'(\varepsilon, \delta) := q_{\varepsilon, \delta}^* M' \otimes_{\Lambda} \mathcal{L}_{\psi} \langle m_{\varepsilon, \delta} \rangle \quad \mathcal{G}'_!(\varepsilon, \delta) := j_{\varepsilon, \delta!} \mathcal{G}'(\varepsilon, \delta) \quad \mathcal{G}'_*(\varepsilon, \delta) := Rj_{\varepsilon, \delta*} \mathcal{G}'(\varepsilon, \delta)$$

where $m_{\varepsilon, \delta}$ is the restriction of m to $\mathbb{D}(0, \delta) \times_S \mathbb{D}(\infty', \varepsilon)^*$. Finally set

$$N^i(\varepsilon, \delta) := \operatorname{colim}_{H \subset \pi(0, \delta)} H^i(C_H \times_S \mathbb{D}(\infty', \varepsilon)_{\text{ét}}, j_H^* \mathcal{G}'_!(\varepsilon, \delta)).$$

Clearly $N^i(\varepsilon, \delta)$ is a $\Lambda[\pi(0, \delta)]$ -module, and just as in (2.4.19), we have a natural isomorphism:

$$\mathcal{F}_{\psi}^{(\infty', 0)}(M') \xrightarrow{\sim} \operatorname{colim}_{n \in \mathbb{N}} \operatorname{colim}_{\delta > 0} N^1(\varepsilon_n, \delta).$$

Lemma 3.1.41. *For every $\varepsilon \leq \varepsilon_0$ there exists $\delta \in |K^\times|$ such that $N^1(\varepsilon, \delta)$ is a Λ -module of finite length, and $N^i(\varepsilon, \delta) = 0$ for $i \neq 1$.*

Proof. This is analogous to the proof of proposition 2.4.28, so we just provide an outline, which the industrious reader is invited to flesh out.

To begin with, the usual dévissage allows to assume that Λ is a field. Next, arguing as in the proof of claim 2.4.29, we reduce to showing that, for a given ε , there exists δ such that $R^i \bar{q}_{\varepsilon, \delta*} \mathcal{G}'_!(\varepsilon, \delta)$ is locally constant of finite type on $\mathbb{D}(0, \delta)_{\text{ét}}^*$, for $i = 1$, and vanishes for $i \neq 1$. However, proposition 2.3.8 implies that the natural map $\mathcal{G}'_!(\varepsilon, \delta) \rightarrow \mathcal{G}'_*(\varepsilon, \delta)$ is an isomorphism, hence it suffices to prove that $R^i q_{\varepsilon, \delta*} \mathcal{G}'(\varepsilon, \delta)$ is locally constant of finite type for $i = 1$, and vanishes for $i \neq 1$. Using Poincaré duality as in the proof of lemma 2.4.23(ii), this will follow from the :

Claim 3.1.42. For every $\varepsilon \leq \varepsilon_0$ there exists $\delta \in |K^\times|$ such that $R^i q_{\varepsilon, \delta!} \mathcal{G}'(\varepsilon, \delta)$ vanishes for $i \neq 1$, and is locally constant of finite type for $i = 1$.

Proof of the claim. For every maximal point $a \in \mathbb{D}(0, \delta)$, let $\mathcal{G}(\varepsilon, \delta, a)$ denote the restriction of $\mathcal{G}(\varepsilon, \delta)$ to $\{\bar{a}\} \times_S \mathbb{D}(\infty', \varepsilon)^*$, where $\bar{a}$ is a geometric point localized at a . Moreover, for every $\gamma \in |K^\times|$ with $\gamma \leq \delta$, let $w(a, \gamma)$ be the unique point of $\{\bar{a}\} \times_S D'$ such that $(\{\bar{a}\} \times_S \mathbb{E}(\infty', \gamma)) \cup \{w(a, \gamma)\}$ is the topological closure of $\{\bar{a}\} \times_S \mathbb{E}(\infty', \gamma)$ in $\{\bar{a}\} \times_S D'$.

By direct inspection we find that, for every $\varepsilon \leq \varepsilon_0$, there exists $\delta \in |K^\times|$ such that :

- (a) the stalk $\mathcal{G}(\varepsilon, \delta, a)_{w'(a, \gamma)}$ has Swan conductor equal to the Swan conductor of M' , for every maximal point $a \in \mathbb{D}(0, \delta)$ and every $\gamma \in |K^\times| \cap [\varepsilon/2, \varepsilon]$
- (b) the Swan conductor of the $\Lambda[\pi(0)]$ -module $\mathcal{G}(\varepsilon, \delta, a)_{\eta_{\infty'}}$ is a constant function on the set of maximal points a of $\mathbb{D}(0, \delta)^*$, and all the breaks of these modules are strictly greater than 0 (more precisely, if λ is such a break, then $\lambda^{\natural} \geq 1$).

Indeed, to ensure (b) it suffices, in view of proposition 2.3.21(i), to take δ sufficiently small, so that $\mathbb{D}(0, \delta)^* \cap d_K(M') = \emptyset$. With these conditions (a) and (b), one may easily redo the proof of lemma 2.4.23(i), and thus obtain the claim. $\square$

3.1.43. Having lemma 3.1.41 at our disposal, we may now repeat the considerations of (2.4.31) and (2.4.32). The upshot is that we have a well defined functor

$$\mathcal{F}_{\psi}^{(\infty', 0)} : \Lambda[\pi(\infty')]\text{-}\mathbf{b.Mod} \rightarrow \Lambda[\pi(0)]\text{-}\mathbf{b.Mod}$$

and (3.1.39) is indeed $\pi(0)$ -equivariant. Moreover, we can argue as in the proof of proposition 3.1.1, to deduce that $\mathcal{F}_{\psi}^{(\infty', 0)}$ is an exact functor. To ease notation, we shall also denote by $\mathcal{F}_{\psi}^{(\infty', 0)}(1)$ the functor on $\Lambda[\pi(\infty')]\text{-}\mathbf{b.Mod}$ given by the rule :

$$M \mapsto \mathcal{F}_{\psi}^{(\infty', 0)}(M)(1)$$

where (1) denotes Tate twist. Next, denote by $c : A \rightarrow A$ the automorphism given by the rule : $x \mapsto -x$; it induces an automorphism c_* of $\Lambda[\pi(0)]\text{-b.Mod}$.

Theorem 3.1.44. *With the notation of (3.1.43), we have :*

- (i) *Let M' be a $\Lambda[\pi(\infty')]$ -module with bounded ramification, $\lambda_1 \leq \dots \leq \lambda_l$ the breaks of M' (repeated with their multiplicities), and set $k := \max\{i \leq l \mid \lambda_i^{\natural} < 1\}$. Then*

$$\text{length}_{\Lambda} \mathcal{F}_{\psi}^{(\infty', 0)}(M') = k - \sum_{i=1}^k \lambda_i^{\natural}.$$

- (ii) *The composition of functors*

$$c_* \circ \mathcal{F}_{\psi}^{(\infty', 0)}(1) \circ \mathcal{F}_{\psi}^{(0, \infty')} : \Lambda[\pi(0)]\text{-b.Mod} \rightarrow \Lambda[\pi(0)]\text{-b.Mod}$$

is naturally isomorphic to the identity functor.

Proof. (i): Pick as usual a Zariski open immersion $j' : U' \subset A'$ and a locally constant Λ -module F' on $U'_{\text{ét}}$, tamely ramified at the points of $\Sigma' := A' \setminus U'$, and such that $F'_{\eta_{\infty'}} = M'$. Set $F := \mathcal{F}'_{\psi}(j'_! F')$; according to (3.1.5) we have :

$$\text{length}_{\Lambda} \mathcal{H}^0(F)_{\eta_0} = \#\Sigma' \cdot l + \sum_{i=1}^l \max(0, \lambda_i - 1)$$

(where l is the generic length of F'). On the other hand, lemma 2.3.15 shows that

$$\text{length}_{\Lambda} \mathcal{H}^0(F)_0 - \text{length}_{\Lambda} \mathcal{H}^1(F)_0 = \#\Sigma' \cdot l + \sum_{i=1}^l (\lambda_i - 1).$$

Combining with the exact sequence (3.1.31) and with (3.1.39), we deduce the assertion.

(ii): The existence of such an isomorphism is shown as in [22, §2.5.6] : for a given $\Lambda[\pi(0)]$ -module M with bounded ramification, find a Zariski open subset $U \subset A \setminus \{0\}$, and a locally constant Λ -module F on $U_{\text{ét}}$, with bounded ramification at all the points of $\Sigma := D \setminus U$, tamely ramified at the points of $\Sigma \setminus \{0\}$ and such that F_{η_0} is isomorphic to M (proposition 2.4.15). Let $j : U \rightarrow A$ be the open immersion, set $F' := \mathcal{F}_{\psi}(j_! F)$ and notice that $\mathcal{H}^0(F')$ is locally constant on $U' := A' \setminus \{0\}$ (theorem 2.3.30(ii)). Let $j' : U' \rightarrow A'$ be the open immersion, and set $G := j'_! j'^* \mathcal{H}^0(F')[0]$; by theorem 2.3.30(i), the cone C of the natural morphism $G \rightarrow F'$ is supported on $\{0'\}$, i.e. $C = i_{0'*} i_{0'}^* C$, where $i_{0'} : S \rightarrow A'$ is the closed immersion with image $0'$.

In view of remark 2.3.34(i) and (2.3.5), we obtain a distinguished triangle :

$$\mathcal{F}'_{\psi}(G) \rightarrow a_* F(-1) \rightarrow \pi^* i_{0'}^* C[1]$$

where $\pi : A' \rightarrow S$ is the structure morphism, and $a = b^{-1} \circ c : A \xrightarrow{\sim} A'$ is (-1) -times the double duality isomorphism. From corollary 2.2.19 and (3.1.39), there follow natural isomorphisms :

$$\mathcal{F}_{\psi}^{(\infty', 0)}(G_{\eta_{\infty'}}) \xrightarrow{\sim} R^0 \Phi_{\eta_0} b_* \mathcal{F}'_{\psi}(G)_{|\mathbb{D}(0, \varepsilon)} \xrightarrow{\sim} c_* F(-1)_{\eta_0} = c_* M(-1).$$

On the other hand, (2.4.13) and proposition 3.1.10(ii) yield a natural decomposition :

$$G_{\eta_{\infty'}} \xrightarrow{\sim} \bigoplus_{z \in \Sigma \setminus \{\infty\}} \mathcal{F}_{\psi}^{(z, \infty')}(F_{\eta_z}).$$

Moreover, by proposition 3.1.10(i) and (3.1.4), for every $z \in \Sigma \setminus \{0, \infty\}$, the $\Lambda[\pi(\infty')]$ -module $\mathcal{F}_{\psi}^{(z, \infty')}(F_{\eta_z})$ has just one break, and this break equals 1. Taking (i) into account, we deduce the isomorphism

$$\mathcal{F}_{\psi}^{(\infty', 0)}(G_{\eta_{\infty'}}) \xrightarrow{\sim} \mathcal{F}_{\psi}^{(\infty', 0)}(\mathcal{F}_{\psi}^{(0, \infty')}(M))$$

as required. However, we still have to worry about the naturality of the isomorphism, *i.e.* we have to check that it does not depend on the choice of F . To this aim, let $\tilde{F}$ be the Λ -module on $A_{\text{ét}}$ obtained as extension by zero of $F|_{\mathbb{D}(0,\varepsilon)^*}$, where ε is chosen small enough so that F restricts to a locally constant Λ -module on $\mathbb{D}(0,\varepsilon)_{\text{ét}}^*$. Set $\tilde{F}' := \mathcal{F}_\psi(\tilde{F})$ and $\tilde{G} := j'_! j'^* \mathcal{H}^0(\tilde{F}')[0]$, and notice that $\mathcal{H}^i(\tilde{F}') = 0$ for every $i < 0$, hence we have again a natural morphism $\tilde{G} \rightarrow \tilde{F}'$, as well as a commutative diagram :

$$\begin{array}{ccc} R^0 \Phi_{\eta_0} b_* \mathcal{F}'_\psi(\tilde{G})|_{\mathbb{D}(0,\varepsilon)} & \xrightarrow{\alpha} & c_* \tilde{F}(-1)_{\eta_0} \\ \downarrow & & \parallel \\ R^0 \Phi_{\eta_0} b_* \mathcal{F}'_\psi(G)|_{\mathbb{D}(0,\varepsilon)} & \xrightarrow{\sim} & c_* F(-1)_{\eta_0} \end{array}$$

whose right vertical arrow is induced by the natural map $\tilde{F} \rightarrow F$. From remark 2.4.12(ii) we get a natural map

$$(3.1.45) \quad \mathcal{F}_\psi^{(0,\infty')}(F_{\eta_0}) \rightarrow \tilde{G}_{\eta_{\infty'}}.$$

We may find $\delta \in |K^\times|$ small enough, and a locally constant Λ -module T of finite type on $\mathbb{D}(\infty', \delta)_{\text{ét}}^*$, with a $\pi(\infty', \delta)$ -equivariant isomorphism $T_{\eta_{\infty'}} \xrightarrow{\sim} \mathcal{F}_\psi^{(0,\infty')}(F_{\eta_0})$, and after shrinking δ , we may also find a morphism $\omega : T \rightarrow \tilde{G}|_{\mathbb{D}(\infty', \delta)_{\text{ét}}^*}$ of Λ -modules on $\mathbb{D}(\infty', \delta)_{\text{ét}}^*$, such that $\omega_{\eta_{\infty'}}$ equals (3.1.45) (since Λ is noetherian, $T_{\eta_{\infty'}}$ is a finitely presented Λ -module). The morphism ω is not unique, but any two such morphisms agree on $\mathbb{D}(\infty', \delta')_{\text{ét}}^*$, for some $\delta' \leq \delta$.

Let $T_!$ be the Λ -module on $A'_{\text{ét}}$ obtained as extension by zero of T ; the morphism ω extends to a map $T_! \rightarrow \tilde{G}$, whence an induced map

$$(3.1.46) \quad R^0 \Phi_{\eta_0} b_* \mathcal{F}'_\psi(T_!)|_{\mathbb{D}(0,\varepsilon)} \rightarrow R^0 \Phi_{\eta_0} b_* \mathcal{F}'_\psi(\tilde{G})|_{\mathbb{D}(0,\varepsilon)}.$$

Furthermore, from remark 3.1.29 we obtain a natural map :

$$(3.1.47) \quad \mathcal{F}_\psi^{(\infty',0)}(G_{\eta_{\infty'}}) \rightarrow R^0 \Phi_{\eta_0} b_* \mathcal{F}'_\psi(T_!)|_{\mathbb{D}(0,\varepsilon)}$$

and by inspecting the construction, it is easily seen that the morphism β obtained as the composition of the map (3.1.46) and the map (3.1.47) does not depend on the choice of ω , and lastly, $\beta \circ \alpha$ is the isomorphism obtained in the foregoing, so the proof of the theorem is complete. $\square$

3.2. Break decomposition. Let $j : U \rightarrow A$ be a Zariski open immersion, such that $0 \in A \setminus U$, and denote by $i : A \setminus U \rightarrow A$ the closed immersion. Let also F be a locally constant Λ -module on $U_{\text{ét}}$, with bounded ramification at all the points of $D \setminus U$. We consider the diagram of $\Lambda[\pi(\infty')]$ -modules :

$$(3.2.1) \quad \begin{array}{ccc} (j_* F)_0 & \longrightarrow & \mathcal{H}^0 \mathcal{F}_\psi(j_* F)_{\eta_{\infty'}} \\ \downarrow & & \downarrow \\ \mathcal{F}_\psi^{(0,\infty')}((j_* F)_0) & \longrightarrow & \mathcal{F}_\psi^{(0,\infty')}(F_{\eta_0}) \end{array}$$

where $(j_* F)_0$ is regarded as an unramified $\Lambda[\pi(\infty')]$ -module. The bottom horizontal arrow is deduced from the natural map $(j_* F)_0 \rightarrow F_{\eta_0}$; the left vertical arrow is the isomorphism of claim 3.1.11; the right vertical arrow is the projection obtained from the stationary phase decomposition (2.4.13), and the top horizontal arrow comes from the boundary map :

$$\mathcal{F}_\psi(i_* i^* j_* F)[-1] \rightarrow \mathcal{F}_\psi(j_* F)$$

associated to the short exact sequence $0 \rightarrow j_! F \rightarrow j_* F \rightarrow i_* i^* j_* F \rightarrow 0$ (see remark 2.3.34(i)).

Lemma 3.2.2. *Diagram (3.2.1) commutes.*

Proof. The detailed verification shall be left as an exercise for the reader. The main point is that also the left vertical arrow is defined in terms of a boundary map, just as for the top horizontal arrow. $\square$

3.2.3. We denote by $\Lambda[\pi(\infty')]\text{-b.Mod}^\dagger$ the full subcategory of $\Lambda[\pi(\infty')]\text{-b.Mod}$ whose objects are all the $\pi(\infty')$ -modules M' such that :

- (a) the $\natural$ -breaks of M' are different from 1
- (b) M' does not admit unramified quotients.

For every object M' of $\Lambda[\pi(\infty')]\text{-b.Mod}^\dagger$, we define :

$$\begin{aligned}\mathcal{G}_<(M') &:= \mathcal{F}_\psi^{(0,\infty')} \circ c_* \mathcal{F}_\psi^{(\infty',0)}(M') \\ \mathcal{G}_>(M') &:= \mathcal{F}_\psi^{(\infty,\infty')} \circ c_* \mathcal{F}_\psi^{(\infty',\infty)}(M')\end{aligned}$$

(where c_* is defined as in (3.1.43)). We will exhibit a natural isomorphism :

$$(3.2.4) \quad M'(-1) \xrightarrow{\sim} \mathcal{G}_<(M') \oplus \mathcal{G}_>(M').$$

To this aim, pick as usual a Zariski open immersion $j' : U' \subset A'$ and a locally constant Λ -module F' on $U'_{\text{ét}}$, tamely ramified at the points of $A' \setminus U'$, and such that $F'_{\eta_{\infty'}} = M'$. Set $F := a^* \mathcal{F}'_\psi(j'_! F')$, where $a : A \rightarrow A'$ is (-1) -times the double duality isomorphism; from condition (a) and remark 2.3.34(ii) we know that F restricts to a locally constant Λ -module on $U := A \setminus \{0\}$, and condition (b) implies that $\mathcal{H}^1 F_0 = 0$ (cp. the proof of claim 2.3.36), hence $\mathcal{H}^1 F = 0$, by theorem 2.3.30(i). In this situation, (3.1.40) reduces to the short exact sequence

$$(3.2.5) \quad 0 \rightarrow H_c^1(A', j'_! F') \rightarrow \mathcal{H}^0 F_{\eta_0} \rightarrow c_* \mathcal{F}_\psi^{(\infty',0)}(M') \rightarrow 0$$

and the stationary phase decomposition (see remark 2.4.33) yields a natural isomorphism :

$$(3.2.6) \quad \mathcal{H}^0 F_{\eta_\infty} \xrightarrow{\sim} c_* \mathcal{F}_\psi^{(\infty',\infty)}(M') \oplus R$$

where R is a $\Lambda[\pi(\infty)]$ -module whose $\natural$ -breaks are all ≤ 1 , by (the dual of) (3.1.4) and proposition 3.1.10(i). Let $j : U \rightarrow A$ be the open immersion, set $G := j_! j^* F[0]$, and let C denote the cone of the natural map $G \rightarrow F$; clearly $C = i_* i^* C$, where $i : \{0\} \rightarrow A$ is the closed immersion. From (2.4.13) we get as well, a natural isomorphism :

$$(3.2.7) \quad \mathcal{H}^0 \mathcal{F}_\psi(G)_{\eta_{\infty'}} \xrightarrow{\sim} \mathcal{F}_\psi^{(0,\infty')}(\mathcal{H}^0 F_{\eta_0}) \oplus \mathcal{F}_\psi^{(\infty,\infty')}(\mathcal{H}^0 F_{\eta_\infty}).$$

Combining with (3.2.5), (3.2.6), and corollary 3.1.13, we deduce a short exact sequence :

$$0 \rightarrow \mathcal{F}_\psi^{(0,\infty')} (H_c^1(A', j'_! F')) \rightarrow \mathcal{H}^0 \mathcal{F}_\psi(G)_{\eta_{\infty'}} \xrightarrow{\beta} \mathcal{G}_<(M') \oplus \mathcal{G}_>(M') \rightarrow 0.$$

On the other hand, in view of (2.3.5) and remark 2.3.34(i) we have a short exact sequence :

$$(3.2.8) \quad 0 \rightarrow (C_0)_A \rightarrow \mathcal{H}^0 \mathcal{F}_\psi(G) \rightarrow j'_! F'(-1) \rightarrow 0$$

and taking into account lemma 3.2.2, we see that β factors through $j'_! F'(-1)_{\eta_{\infty'}} = M'(-1)$, and induces the sought isomorphism (3.2.4) of $\Lambda[\pi(\infty')]$ -modules. The naturality of (3.2.4) can be shown as in the proof of theorem 3.1.44(ii) : the details shall be left to the reader.

3.2.9. Clearly, the breaks of $\mathcal{G}_<(M')$ are also breaks of M' . On the other hand, theorem 3.1.14(iii) implies that the breaks λ of M' such that $\lambda^\natural > 1$ are all found (with their multiplicities) in $\mathcal{G}_>(M')$. Notice also the natural isomorphism :

$$\mathcal{F}_\psi^{(\infty',0)} \circ \mathcal{G}_<(M') \xrightarrow{\sim} \mathcal{F}_\psi^{(\infty',0)}(M')(-1)$$

obtained from theorem 3.1.44(ii). Taking into account theorem 3.1.14(iii), we deduce :

$$\text{length}_\Lambda \mathcal{F}_\psi^{(\infty',0)}(M') = \text{length}_\Lambda \mathcal{G}_<(M') - \text{sw}(\mathcal{G}_<(M')).$$

From this, in view of theorem 3.1.44(i), we easily deduce that the breaks λ of M' such that $\lambda^{\natural} < 1$ are all found (with their multiplicities) in $\mathcal{G}_{<}(M')$. In other words, by means of the natural isomorphism (3.2.4), we have been able to separate the breaks λ such that $\lambda^{\natural} < 1$ from those such that $\lambda^{\natural} > 1$.

Remark 3.2.10. A more detailed analysis might allow to drop condition (b) of (3.2.3) from the above discussion. However, this would probably entail some messy verifications; instead, we will resort to a trick : given any $\Lambda[\pi(\infty')]$ -module, we shall twist it by a tamely ramified character of order high enough; this operation does not alter the distribution of breaks, but it will remove all unramified quotients.

3.2.11. For every $n \in \mathbb{Z}$, let $\varphi_n : A' \rightarrow A'$ be the map given by the rule : $x' \mapsto x'^n$ (where x' is a global coordinate on A). It induces exact endomorphisms φ_n^* and φ_{n*} of the category $\Lambda[\pi(\infty')]\text{-b.Mod}$, in the obvious way. Moreover, the trace map induces a natural transformation

$$\varphi_{n*} \circ \varphi_n^* M' \rightarrow M' \quad \text{for every object } M' \text{ of } \Lambda[\pi(\infty')]\text{-b.Mod}.$$

This trace map is an epimorphism whenever $n \in \Lambda^\times$. For every $s \in \mathbb{Q}_+$, write $s = ab^{-1}$, with $a, b \in \mathbb{Z}$ and $(a, b) = 1$, and set

$$\varphi_s^* := \varphi_{b*} \circ \varphi_a^* \quad \varphi_{s*} := \varphi_{a*} \circ \varphi_b^*.$$

Notice the natural map :

$$(3.2.12) \quad \varphi_{s*} \circ \varphi_s^* M' \rightarrow \varphi_{a*} \circ \varphi_a^* M' \rightarrow M'$$

where the first arrow is given by the counit of the adjoint pair $(\varphi_b^*, \varphi_{b*})$, and the second arrow is the trace map; especially, (3.2.12) is an epimorphism, provided $a \in \Lambda^\times$. Furthermore, since $\varphi_{b*} = \varphi_{b!}$, it is easily seen that φ_s^* is left adjoint to φ_{s*} .

Suppose that M' is represented by a Λ -module on $\mathbb{D}(\infty', \varepsilon)_{\text{ét}}^*$ (for some $\varepsilon \in |K^\times|$), and recall that the break decomposition of (2.3.16) is defined on a connected open subset $U \subset \mathbb{D}(\infty', \varepsilon)^*$ such that $U \cap \mathbb{D}(\infty', \delta)^* \neq \emptyset$ for every $\delta \in |K^\times|$. Then, for every given integer $n > 0$, the subset $\varphi_n^{-1}U \subset \mathbb{D}(\infty', \varepsilon^{1/n})^*$ is also connected, and still intersects every punctured disc centered at ∞' . Likewise, we may find an open connected subset $V \subset \mathbb{D}(\infty', \varepsilon^n)^*$ such that $\varphi_n^{-1}(V) \subset U$, and V intersects every punctured disc centered at ∞' ; thus, $\varphi_{n*}(M'^{\natural}(r))$ can be seen as a Λ -module on $V_{\text{ét}}$, and with this notation we may state the following :

Lemma 3.2.13. *For every object M' of $\Lambda[\pi(\infty')]\text{-b.Mod}$, every $n \in \mathbb{Z}$ with $(p, n) = 1$, and every $r \in \mathbb{Q}_+$, we have natural identifications :*

$$\varphi_n^*(M'^{\natural}(r)) = (\varphi_n^* M')^{\natural}(nr) \quad \varphi_{n*}(M'^{\natural}(r)) = (\varphi_{n*} M')^{\natural}(r/n)$$

(notation of (2.3.16)).

Proof. As in (2.4.5), set

$$\mathbb{E}(0', 1) := \{y \in A' \mid |(x'(y))| < 1\}$$

and recall that $\mathbb{E}(0', 1)$ is a closed subset of A' ; moreover, there exists a unique point, corresponding to a rank two valuation and denoted $\eta(1)$ in [25, §2.2.10], such that $\mathbb{E}(0', 1) \setminus \{\eta(1)\}$ is the interior of $\mathbb{E}(0', 1)$ in A' . Let G be the automorphism group of a finite Galois extension of the residue field of the point $\eta(1) \in A'$, and suppose that $H \subset G$ is a normal subgroup such that G/H is cyclic of order n . Let us first remark :

Claim 3.2.14. Let M (resp. N) be any $\Lambda[G]$ -module (resp. a $\Lambda[H]$ -module), and suppose that M (resp. N) admits a single break. Then the same holds for the $\Lambda[H]$ -module $\text{Res}_H^G(M)$ (resp. for the $\Lambda[G]$ -module $\text{Ind}_H^G(N)$).

Proof of the claim. Since G and H have the same p -Sylow subgroup $G^{(p)}$, and since the break decomposition depends only on the underlying $\Lambda[G^{(p)}]$ -module (see [25, §4.1.17]), we see more generally that, for any $\Lambda[G]$ -module M , the restriction $\text{Res}_H^G(M)$ has the same number of breaks as M . Next, recall that $\text{Res}_H^G(\text{Ind}_H^G(N)) = \bigoplus_{g \in G/H} {}^g N$, so the assertion for the induced module is reduced to the foregoing case. $\diamond$

Now, the lemma is easily reduced to the corresponding assertion for $\Lambda[G]$ -modules, where G is a Galois group as above. For such modules, we may assume that $M' = M'^{\natural}(r)$, in which case – in view of claim 3.2.14 – it suffices to compare the Swan conductors of M' and $\varphi_n^* M'$ (resp. $\varphi_{n*} M'$). The assertion for $\varphi_{n*} M'$ follows straightforwardly from [18, Lemma 8.1(ii)]. To deal with $\varphi_n^* M'$, we are then reduced to computing the Swan conductor of $\varphi_{n*}(\varphi_n^* M')$. The latter is the same as $M' \otimes_{\Lambda} \varphi_{n*}(\varphi_n^* \Lambda) \simeq M' \otimes_{\Lambda} \varphi_{n*}(\Lambda)$, whence the contention (details left to the reader). $\square$

Lemma 3.2.15. *Let M' be any object of $\Lambda[\pi(\infty')]$ -b.Mod. There exists a finite set $\Sigma_{M'} \subset \mathbb{N}$, such that $1 \notin \Sigma_{M'}$, and the following holds :*

- (i) *If $\chi : \mu_n \rightarrow \Lambda^\times$ is a non-trivial character, and $M' \otimes_{\Lambda} \mathcal{K}_{\chi} \langle x' \rangle_{\eta_{\infty'}}$ admits an unramified quotient, then the order of χ lies in $\Sigma_{M'}$. (Notation of (3.1.8).)*
- (ii) *For a given integer $n \neq 0$, suppose that M' does not admit unramified quotients, but $\varphi_n^* M'$ admits an unramified quotient. Then an element of $\Sigma_{M'}$ divides n .*
- (iii) *If M' does not admit unramified quotients, then the same holds for $\varphi_{n*} M'$, for every integer $n \neq 0$.*

Proof. (i): Since the only simple quotient of Λ is its residue field, we may assume from start that Λ is a field. Suppose that we have found non-trivial characters $\chi_1, \dots, \chi_r$, of orders respectively $a_1, \dots, a_r$, and epimorphisms $t_i : M' \otimes_{\Lambda} \mathcal{K}_{\chi_i} \langle x' \rangle_{\eta_{\infty'}} \rightarrow \Lambda$, for $i = 1, \dots, r$, and suppose that $a_i \neq a_j$ for all $i < j \leq r$. Now, t_i is the same as the datum of an epimorphism $\tau_i : M' \rightarrow \mathcal{K}_{\chi_i^{-1}} \langle x' \rangle$, and we claim that $\text{length}_{\Lambda} \bigcap_{i=1}^j \text{Ker } \tau_j \leq \text{length}_{\Lambda} M' - j$ for every $j \leq r$. This is the same as showing that :

$$(3.2.16) \quad \bigcap_{i=1}^{j-1} \text{Ker } \tau_i \not\subset \text{Ker } \tau_j \quad \text{for every } j \leq r.$$

By way of contradiction, let j be the smallest index for which (3.2.16) fails (so $j \geq 2$); then it is easily seen that the induced map

$$\tau : M' \rightarrow \bigoplus_{i=1}^{j-1} \mathcal{K}_{\chi_i^{-1}} \langle x' \rangle$$

is an epimorphism, and τ_j factors through τ , therefore we get a non-zero map $\mathcal{K}_{\chi_i^{-1}} \langle x' \rangle \rightarrow \mathcal{K}_{\chi_j^{-1}}$ for some $i < j$, or what is the same, a non-zero map $\Lambda \rightarrow \mathcal{K}_{\chi_i \chi_j^{-1}}$. Under our assumptions, it is easily seen that no such map can exist, whence the claim. Hence, we must have $r \leq \text{length}_{\Lambda} M'$, and the contention follows easily.

(ii): Again, we may assume that Λ is a field, say of characteristic l , and moreover that Λ is algebraically closed. Suppose that we have found an epimorphism $t : \varphi_n^* M' \rightarrow \Lambda$, for some integer $n > 1$. We may assume that $(l, n) = 1$ (since we can assume that our candidate $\Sigma_{M'}$ contains l). This t is the same as the datum of a map $\tau : M' \rightarrow \varphi_{n*} \Lambda$. Under our assumptions, $\varphi_{n*} \Lambda$ is the direct sum of $\Lambda[\pi(\infty')]$ -modules of the type $\mathcal{K}_{\chi} \langle x' \rangle_{\eta_{\infty'}}$, where χ ranges over the characters $\mu_n \rightarrow \Lambda^\times$. Moreover, τ cannot factor through the trivial character of μ_n , since M' does not have unramified quotients. Hence, after composing with a projection $\varphi_{n*} \Lambda \rightarrow \mathcal{K}_{\chi} \langle x' \rangle_{\eta_{\infty'}}$, we deduce a non-zero map $M' \otimes_{\Lambda} \mathcal{K}_{\chi^{-1}} \langle x' \rangle_{\eta_{\infty'}} \rightarrow \Lambda$, where χ is non-trivial, and its order divides n . Then the assertion follows easily from (i).

(iii): Since $\varphi_{n*} = \varphi_{n!}$, a non-zero map $\varphi_{n*}M' \rightarrow N'$ to an unramified $\Lambda[\pi(\infty')]$ -module corresponds by adjunction, to a non-zero map $M' \rightarrow \varphi_n^*N'$, whence the contention. $\square$

The following is the main result of this paper.

Theorem 3.2.17. *Let M' be any object of $\Lambda[\pi(\infty')]\text{-b.Mod}$. There exists a unique decomposition*

$$M' = \bigoplus_{r \in \mathbb{Q}_+} M'_r$$

such that each M'_r is an object of $\Lambda[\pi(\infty')]\text{-b.Mod}$, whose only $\natural$ -break is r .

Proof. For every $\natural$ -break r of M' , we will exhibit a decomposition $M' = M'_{[0,r]} \oplus M'_{[r,+\infty]}$, such that the $\natural$ -breaks of $M'_{[0,r]}$ (resp. of $M'_{[r,+\infty]}$) are all $\leq r$ (resp. $> r$). If such submodules exist, it is easily seen that they are unique (since their lengths are determined by M'); then the theorem will follow straightforwardly. Moreover, notice that the theorem holds for M' if and only if it holds for $M' \otimes_{\Lambda} \mathcal{K}_{\chi} \langle x' \rangle_{\eta_{\infty'}}$, where $\chi : \mu_n \rightarrow \Lambda^\times$ is any character; by lemma 3.2.15(i), we may then assume that M' does not admit unramified quotients (recall that $\Lambda^\times$ contains a subgroup isomorphic to μ_{p^∞} , hence we may find χ of arbitrarily high order).

Now, pick a subset $\Sigma_{M'} \subset \mathbb{N}$ as in lemma 3.2.15(ii), let $r' \in \mathbb{Q}_+$ be the smallest $\natural$ -break of M' such that $r < r'$, and choose $s = ab^{-1} \in \mathbb{Q}_+$ such that :

$$r < s^{-1} < r' \quad a \in \Lambda^\times \quad (p, a) = (p, b) = (a, b) = 1$$

and moreover, $(c, a) = 1$ for every $c \in \Sigma_{M'}$. Notice that $\varphi_s^*(M')$ does not have any breaks λ with $\lambda^\natural = 1$ (by lemma 3.2.13); moreover, $\varphi_s^*(M')$ does not admit any quotients of pure break zero (by lemma 3.2.15(ii,iii)). Especially, $\varphi_s^*(M')$ is an object of the subcategory $\Lambda[\pi(\infty')]\text{-b.Mod}^\dagger$, hence the discussion of (3.2.3) and (3.2.9), yields a decomposition

$$(3.2.18) \quad \varphi_s^*(M')(-1) = \mathcal{G}_<(\varphi_s^*(M')) \oplus \mathcal{G}_>(\varphi_s^*(M'))$$

such that all the $\natural$ -breaks of the first (resp. the second) summand are < 1 (resp. > 1). Whence, a decomposition $\varphi_{s*}(3.2.18)(1)$ of $\varphi_{s*}\varphi_s^*(M')$, and again by lemma 3.2.13, we see that the first (resp. second) summand of the latter have $\natural$ -breaks in the range $[0, r]$ (resp. $[r', +\infty]$).

Lastly, using (3.2.12) we may project $\varphi_{s*}(3.2.18)(1)$ onto a decomposition of M' with the same separation of $\natural$ -breaks, as required (details left to the reader). $\square$

We may now complete the study of the local Fourier transforms.

Lemma 3.2.19. *Let M' be any unramified $\Lambda[\pi(\infty')]$ -module. There is a natural isomorphism :*

$$\mathcal{F}_{\psi}^{(\infty',0)}(M') \xrightarrow{\sim} M'(-1).$$

Proof. Let F' be the constant Λ -module on A' , such that $F'_{\eta_{\infty'}} = M'$; in this situation, the exact sequence (3.1.40) reduces to the isomorphism :

$$\mathcal{F}_{\psi}^{(\infty',0)}(M') \xrightarrow{\sim} H_c^2(A', F') \xrightarrow{\sim} H_c^2(A', \Lambda) \otimes_{\Lambda} M'$$

whence the lemma. $\square$

Proposition 3.2.20. *Let M' be any object of $\Lambda[\pi(\infty')]\text{-b.Mod}$, whose $\natural$ -breaks are all < 1 . Then $\text{sw}(\mathcal{F}_{\psi}^{(\infty',0)}(M')) = \text{sw}(M')$.*

Proof. In view of lemma 3.2.19, we may assume that M' does not admit unramified quotients, therefore M' is an object of $\Lambda[\pi(\infty')]\text{-b.Mod}^\dagger$. Then, define F' , F and G as in (3.2.3); the short exact sequence (3.2.5) shows that

$$\text{sw}(\mathcal{F}_{\psi}^{(\infty',0)}(M')) = \text{sw}(\mathcal{H}^0 F_{\eta_0}) = \text{sw}(\mathcal{H}^0 G_{\eta_0}) = \text{sw}(\mathcal{F}_{\psi}^{(0,\infty')}(G_{\eta_0}))$$

where the last identity holds by virtue of theorem 3.1.14(ii). Recall that G is locally constant on $A \setminus \{0\}$, whence a natural stationary phase decomposition :

$$\mathcal{H}^0 \mathcal{F}_\psi(G)_{\eta_{\infty'}} \xrightarrow{\sim} \mathcal{F}_\psi^{(0,\infty')}(G_{\eta_0}) \oplus \mathcal{F}_\psi^{(\infty,\infty')}(G_{\eta_{\infty}}).$$

However, $G_{\eta_{\infty}} = F_{\eta_{\infty}}$ admits as well a stationary phase decomposition, whose factors are of the form $\mathcal{F}_\psi^{(z,\infty)}(F'_{\eta_z})$, for various $z \in D'(K)$. Especially, $\mathcal{F}_\psi^{(\infty',\infty)}(F'_{\eta_{\infty'}}) = 0$, by corollary 3.1.13; on the other hand, the contributions with $z \neq 0, \infty'$ (resp. with $z = 0$) are all pure of $\natural$ -break equal to 1 (resp. equal to 0), by (3.1.4) and proposition 3.1.10(i). Hence, applying again corollary 3.1.13, we conclude that $\mathcal{F}_\psi^{(\infty,\infty')}(G_{\eta_{\infty}}) = 0$, and therefore, $\text{sw}(\mathcal{F}_\psi^{(\infty',0)}(M')) = \text{sw}(\mathcal{H}^0 \mathcal{F}_\psi(G)_{\eta_{\infty'}})$. Taking into account (3.2.8) $_{\eta_{\infty'}}$, the assertion follows. $\square$

3.2.21. For every subset $I \subset \mathbb{Q}_+$, denote by $\Lambda[\pi(\infty')]\text{-b.Mod}_I$ the full subcategory of the category $\Lambda[\pi(\infty')]\text{-b.Mod}$, whose objects are the modules that have all their $\natural$ -breaks in I . The following corollary refines theorem 3.1.14(iii).

Corollary 3.2.22. *Let M be any $\Lambda[\pi(0)]$ -module with bounded ramification. We have :*

- (i) *All the $\natural$ -breaks of $\mathcal{F}_\psi^{(0,\infty')}(M)$ are < 1 .*
- (ii) *More precisely, $\mathcal{F}_\psi^{(0,\infty')}$ is an equivalence $\Lambda[\pi(0)]\text{-b.Mod} \xrightarrow{\sim} \Lambda[\pi(\infty')]\text{-b.Mod}_{[0,1[}$, whose quasi-inverse is the restriction of the functor $c_* \mathcal{F}_\psi^{(\infty',0)}(1)$.*
- (iii) *If M has length l , and is pure of $\natural$ -break r , then $\mathcal{F}_\psi^{(0,\infty')}(M)$ has length $(1+r)l$, and is pure of $\natural$ -break $r/(1+r)$.*

Proof. (i): In view of theorem 3.1.14(iii) and theorem 3.2.17, we may write $\mathcal{F}_\psi^{(0,\infty')}(M) = N'_{[0,1[} \oplus N'_1$, where all the $\natural$ -breaks of $N'_{[0,1[}$ (resp. of N'_1) are < 1 (resp. are equal to 1). By theorem 3.1.14(ii) we have $\text{sw}(M) = \text{sw}(N'_{[0,1[}) + \text{length}_\Lambda N'_1$. On the other hand, theorem 3.1.44 and proposition 3.2.20 imply that

$$\text{sw}(M) = \text{sw}(\mathcal{F}_\psi^{(\infty',0)}(N'_{[0,1[})) + \text{sw}(\mathcal{F}_\psi^{(\infty',0)}(N'_1)) = \text{sw}(\mathcal{F}_\psi^{(\infty',0)}(N'_{[0,1[})) = \text{sw}(N'_{[0,1[}).$$

The contention is an immediate consequence.

(ii): This follows directly from (i), in view of theorem 3.1.44(ii) and the discussion of (3.2.9).

(iii): We may assume that M' is irreducible, in which case the same holds for $\mathcal{F}_\psi^{(0,\infty')}(M)$, in view of (ii). Then the latter has a unique $\natural$ -break (by theorem 3.2.17), and the contention follows easily from theorem 3.1.14(i,ii). $\square$

Theorem 3.2.23. *Let M be any object $\Lambda[\pi(\infty)]\text{-b.Mod}_{]1,+\infty[}$. We have :*

- (i) *There exists a natural isomorphism of $\Lambda[\pi(\infty)]$ -modules :*

$$M(-1) \xrightarrow{\sim} c_* \mathcal{F}_\psi^{(\infty',\infty)} \circ \mathcal{F}_\psi^{(\infty,\infty')}(M)$$

(where (-1) denotes the Tate twist and c_ is defined as in (3.1.43)).*

- (ii) *More precisely, $\mathcal{F}_\psi^{(\infty,\infty')}$ restricts to an equivalence of categories*

$$\Lambda[\pi(\infty)]\text{-b.Mod}_{]1,+\infty[} \xrightarrow{\sim} \Lambda[\pi(\infty')]\text{-b.Mod}_{]1,+\infty[}$$

whose quasi-inverse is the restriction of $c_ \mathcal{F}_\psi^{(\infty',\infty)}(1)$.*

- (iii) *If M has length l and is pure of $\natural$ -break r , then $\mathcal{F}_\psi^{(\infty,\infty')}$ has length $(r-1)l$ and is pure of $\natural$ -break $r/(r-1)$.*

Proof. (i): As usual, pick a Zariski open immersion $j : U \rightarrow A$, and a locally constant Λ -module F on $U_{\text{ét}}$, such that $F_{\eta_{\infty}} = M$, and F is tamely ramified at all the points of $A \setminus U$. Set $F' := \mathcal{F}_\psi(j_! F)$; according to remark 2.3.34(ii,iii), the complex F' is concentrated in degree

zero, and the Λ -module $\mathcal{H}^0 F'$ is locally constant on A' . In this situation, the stationary phase decomposition (2.4.13) yields the natural isomorphisms

$$\begin{aligned} \mathcal{H}^0 F'_{\eta_{\infty'}} &\xrightarrow{\sim} \mathcal{F}_{\psi}^{(\infty, \infty')}(M) \oplus R \\ a^* \mathcal{H}^0 \mathcal{F}'_{\psi}(F')_{\eta_{\infty}} &\xrightarrow{\sim} c_* \mathcal{F}_{\psi}^{(\infty', \infty)}(\mathcal{H}^0 F'_{\eta_{\infty'}}) \end{aligned}$$

where R is a $\Lambda[\pi(\infty')]$ -module whose $\natural$ -breaks are all ≤ 1 , and whose length is $\sharp(A \setminus U) \cdot \text{length}_{\Lambda} M$, by virtue of (3.1.4) and proposition 3.1.10(i). The sought isomorphism follows easily, taking into account corollary 3.1.13.

(ii): This is an immediate consequence of (i) and the discussion in (3.2.9).

(iii): Keep the notation of the proof of (i); on the one hand, (3.1.5) tells us that

$$\text{length}_{\Lambda} \mathcal{H}^0 F'_{\eta_{\infty'}} = (\sharp(A \setminus U) - 1) \cdot \text{length}_{\Lambda} M + \text{sw}(M).$$

On the other hand, the foregoing implies that

$$\text{length}_{\Lambda} \mathcal{H}^0 F'_{\eta_{\infty'}} = \text{length}_{\Lambda} \mathcal{F}_{\psi}^{(\infty, \infty')}(M) + \sharp(A \setminus U) \cdot \text{length}_{\Lambda} M.$$

Therefore :

$$(3.2.24) \quad \text{length}_{\Lambda} \mathcal{F}_{\psi}^{(\infty, \infty')}(M) = \text{sw}(M) - \text{length}_{\Lambda} M.$$

Dually, this also proves that $\text{length}_{\Lambda} \mathcal{F}_{\psi}^{(\infty', \infty)}(M') = \text{sw}(M') - \text{length}_{\Lambda} M'$ for every object M' of $\Lambda[\pi(\infty')]\text{-b.Mod}_{1, +\infty[}$. Letting $M' := \mathcal{F}_{\psi}^{(\infty, \infty')}(M)$, we deduce – in view of (i) – that :

$$\text{length}_{\Lambda} M = \text{sw}(\mathcal{F}_{\psi}^{(\infty, \infty')}(M)) - \text{sw}(M) + \text{length}_{\Lambda} M$$

whence :

$$(3.2.25) \quad \text{sw}(\mathcal{F}_{\psi}^{(\infty, \infty')}(M)) = \text{sw}(M).$$

The contention follows easily by combining (3.2.24) and (3.2.25). $\square$

3.3. Modules of break zero. The most glaring omission in this work concerns the $\Lambda[\pi(0)]$ -modules that are pure of break zero, or equivalently, those of Swan conductor equal to zero. The obvious conjecture is that all such modules are tamely ramified (see (3.1.8)); however, I do not know how to show this. If one can prove this conjecture, then one can apply a general tannakian argument due to Y.André, to derive an analogue, for modules with bounded ramification, of the so-called “local monodromy theorems” that one encounters in various situations : see [1].

Remark 3.3.1. The main theorem of [1] (théorème 5.2.1) is stated for tannakian categories over a field of characteristic zero. However, by going through the arguments of *loc.cit.* one can check that the proof of part (a) of that theorem goes through *verbatim* for a base field E of arbitrary characteristic, provided the (pro-algebraic) fundamental group of the tannakian category is reduced : the only extra precaution one must take is in step two of the proof, where one must choose the prime ℓ different from the characteristic of E (and this can always be done, by a simple inspection). In our situation, the upshot is as follows. Take $\Lambda := \overline{\mathbb{F}}_{\ell}$, the algebraic closure of the field with ℓ elements (where $\ell \neq p$); then, by the following lemma 3.3.2, the category $\Lambda[\pi(0)]\text{-b.Mod}$ is tannakian with a reduced pro-algebraic fundamental group G , and G admits a surjective map

$$G \rightarrow \mathbb{Z}^{\wedge}$$

(namely, the G -representations that factor through the quotient $\mathbb{Z}^{\wedge}$ correspond to the tamely ramified Λ -modules on a punctured disc $\mathbb{D}(0, \varepsilon)^*$). Now, if our conjecture is true, then we may apply the above mentioned theorem of André, which says that the kernel of this group homomorphism is the radical of G . Translating back to the language of local systems, this means (by virtue of Lie-Kolchin’s theorem for solvable algebraic groups) that every object F of

$\Lambda[\pi(0)]\text{-b.Mod}$ becomes an iterated extension of one-dimensional objects, after restriction to some smaller disc, and pull back via a (finite) Kummer morphism.

Lemma 3.3.2. *The tensor abelian category $\overline{\mathbb{F}}_\ell[\pi(0)]\text{-b.Mod}$ is tannakian (i.e. it admits a fibre functor ω) and its fundamental group $\pi_1(\overline{\mathbb{F}}_\ell[\pi(0)]\text{-b.Mod}, \omega)$ is reduced.*

Proof. Let us first show how to construct a fibre functor for $\overline{\mathbb{F}}_\ell[\pi(0)]\text{-b.Mod}$ with values in finite dimensional $\overline{\mathbb{F}}_\ell$ -vector spaces. As in (2.2.11), we fix a strictly decreasing sequence $(\varepsilon_n \mid n \in \mathbb{N})$ of radii, and geometric points $\xi_n \in \mathbb{D}(0, \varepsilon_n)$ for every $n \in \mathbb{N}$. Then, for every $n \in \mathbb{N}$ there is an obvious fibre functor

$$\omega_n : \overline{\mathbb{F}}_\ell[\pi(0, \varepsilon_n)]\text{-Mod} \rightarrow \overline{\mathbb{F}}_\ell\text{-Mod}$$

that assigns to each object of $\overline{\mathbb{F}}_\ell[\pi(0, \varepsilon_n)]\text{-Mod}$ its underlying $\overline{\mathbb{F}}_\ell$ -vector space. These functors are compatible (in the obvious manner) with the transition functors in the direct system of categories (2.2.16), and therefore their 2-colimit is a fibre functor for $\overline{\mathbb{F}}_\ell[\pi(0)]\text{-Mod}$. By restriction to the subcategory $\overline{\mathbb{F}}_\ell[\pi(0)]\text{-b.Mod}$, we obtain the sought ω .

For every $n \in \mathbb{N}$, denote by $\overline{\mathbb{F}}_\ell[\pi(0, \varepsilon_n)]\text{-b.Mod}$ the full subcategory of $\overline{\mathbb{F}}_\ell[\pi(0, \varepsilon_n)]\text{-Mod}$ whose objects have bounded ramification at 0; this is a tannakian category, and $\overline{\mathbb{F}}_\ell[\pi(0)]\text{-Mod}$ is also the 2-colimit of these categories. Moreover, the composition of ω with the natural functor

$$\overline{\mathbb{F}}_\ell[\pi(0, \varepsilon_n)]\text{-b.Mod} \rightarrow \overline{\mathbb{F}}_\ell[\pi(0)]\text{-b.Mod}$$

is a fibre functor for $\overline{\mathbb{F}}_\ell[0, \varepsilon_n]\text{-b.Mod}$, naturally isomorphic to the restriction of ω_n . By general nonsense, there follows an isomorphism of pro-algebraic groups

$$\pi_1(\overline{\mathbb{F}}_\ell[\pi(0)]\text{-b.Mod}, \omega) \xrightarrow{\sim} \lim_{n \in \mathbb{N}} \pi_1(\overline{\mathbb{F}}_\ell[\pi(0, \varepsilon_n)]\text{-b.Mod}, \omega_n).$$

Since a (projective) limit of reduced pro-algebraic groups is reduced, we come down to showing that $\pi_1(\overline{\mathbb{F}}_\ell[\pi(0, \varepsilon_n)]\text{-b.Mod}, \omega_n)$ is reduced for every $n \in \mathbb{N}$. More generally, we have :

Claim 3.3.3. Let G be any group, K any field, and $\mathcal{C}$ a full tannakian subcategory of the category of all representations of G into finite dimensional K -vector spaces (with fibre functor given by the forgetful functor ω). Then $\pi_1(\mathcal{C}, \omega)$ is reduced.

Proof of the claim. $\mathcal{C}$ is the 2-colimit of all its finitely generated tannakian subcategories, so by an argument as in the foregoing, we may assume from start that $\mathcal{C}$ is finitely generated, in which case it admits a generator M . Let $\overline{G}$ denote the Zariski closure of the image of G in the K -algebraic group $\text{Aut}_K(M)$; then it is well known that $\pi_1(\mathcal{C}, \omega)$ is naturally isomorphic to $\overline{G}$, and the assertion follows (cp. [26, Ch.VI, §1.1.1.2]). $\square$

In this final section, we present some evidence supporting our conjecture; namely, proposition 3.3.9, a very partial result, which however might suggest a line of attack for the general case.

3.3.4. We need some preliminaries from [25]. Let R be any artinian $\mathbb{Z}[1/p]$ -algebra, F a locally constant, constructible R -module on $\mathbb{D}(0, 1)_{\text{ét}}^*$, say with stalks of length d . To F we have associated in [25, §4.2.10], a set of continuous piecewise linear maps, the *break functions*

$$0 \leq f_1(\rho) \leq f_2(\rho) \leq \cdots \leq f_d(\rho)$$

defined for $\rho \in \mathbb{R}_{\geq 0}$, and with values in $\mathbb{R}_{\geq 0}$ (actually, in *loc.cit.* we assume as well that the stalks of F are free R -modules, and then d is taken to be the common rank of the stalks; but the whole discussion can adapted easily to our current setting). The sum :

$$\delta_F(\rho) := \sum_{i=1}^d f_i(\rho)$$

is therefore a piecewise linear function $\mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$, and we know that the right derivative of δ_F at a given point $\rho := -\log r$ (for $r \in |K^\times| \cap [0, 1]$) is the integer

$$\mathrm{sw}^{\natural}(F, r^+)$$

called the Swan conductor of F at the point $\eta(r)$ (see [25, Prop.4.1.15]; the point $\eta(r)$ is the one introduced in [25, §2.2.10]).

3.3.5. On the other hand, let $f : X \rightarrow \mathbb{D}(0, 1)^*$ be a Galois, locally algebraic étale covering; to f we attach a *normalized discriminant function*, as follows. For every $r \in |K^\times| \cap [0, 1]$, let X_r denote a connected component of $f^{-1}(A(r, 1))$, where $A(r, 1)$ denotes the affinoid annulus of internal radius r , and external radius 1; the restriction $f_r : X_r \rightarrow \mathbb{D}(0, r)^*$ is a finite étale Galois covering, whose Galois group we denote G_r . In such situation, [25, §2.3.12] assigns to f_r a discriminant function

$$\delta_{f_r} : [0, -\log r] \cap |K^\times| \rightarrow \mathbb{R}_{\geq 0}.$$

Let $s \in |K^\times| \cap [0, 1]$, with $s > r$; then we may choose X_s as one of the connected components of $f_r^{-1}(A(s, 1))$, so that $G_s \subset G_r$, and by inspecting the definitions, it is easily seen that the restriction of δ_{f_r} to $[0, -\log s] \cap |K^\times|$ agrees with $\delta_{f_s} \cdot [G_r : G_s]$. Therefore :

$$\delta_f(\rho) := [G_r : 1]^{-1} \delta_{f_r}(\rho) = [G_s : 1]^{-1} \delta_{f_s}(\rho) \quad \text{for every } \rho \in [0, -\log s] \cap |K^\times|.$$

By [25, Th.2.3.25], this function extends to a continuous piecewise linear function $\delta_f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$, which is convex and with rational slopes.

3.3.6. Clearly, the considerations of (3.3.4) (resp. of (3.3.5)) can be repeated for a locally constant R -module F on the étale site of an annulus $A(a, b)$ (resp. for a locally algebraic Galois covering $f : X \rightarrow A(a, b)$). In either case, we obtain a piecewise linear function defined on $[\log 1/b, \log 1/a]$, with values in $\mathbb{R}_{\geq 0}$. Likewise, the whole discussion can be repeated for modules (resp. coverings) on $A \setminus \{0\}$. We shall denote by Y any of these adic spaces, and by I_Y the domain of the functions δ_F and δ_f .

Especially, to a Λ -module F on $Y_{\text{ét}}$, we may associate the locally algebraic Galois covering $f_F : \mathcal{I} \mathrm{som}_{\Lambda}(M_Y, F) \rightarrow Y$ (where M is any fixed stalk of F : see (2.1.14)).

I owe the following simple observation to Ofer Gabber.

Lemma 3.3.7. *With the notation of (3.3.6), the following estimates hold for every $\rho \in I_Y$:*

$$\begin{aligned} \delta_F(\rho)/d &\leq f_d(\rho) \leq \delta_F(\rho) \\ f_d(\rho) \cdot \left(1 - \frac{1}{p}\right) &\leq \delta_{f_F}(\rho) \leq f_d(\rho). \end{aligned}$$

Proof. The inequalities for $\delta_F(\rho)$ are obvious. To show the other two inequalities, we may assume that $Y = A(a, b)$, and we may replace f_F by its restriction $f : X \rightarrow Y$ to a connected component. Now, pick any $x \in f^{-1}(\eta(r))$, and let $\mathrm{St}_x \subset G := \mathrm{Aut}(X/Y)$ be the stabilizer of x ; the group St_x admits an upper numbering ramification filtration :

$$P^{\flat, \gamma_m} \subset P^{\flat, \gamma_{m-1}} \subset \dots \subset P^{\flat, \gamma_1} \subset \mathrm{St}_x$$

by non-zero normal p -subgroups, indexed by certain values $\gamma_1 > \dots > \gamma_{m-1} > \gamma_m \in |K^\times|$, called the $\flat$ -breaks of the filtration (see [25, §4.1.26]). Let $(f_i \mid i = 1, \dots, d)$ be the break functions of F ; by inspecting the definitions, it is easily seen that

$$(3.3.8) \quad f_d(-\log r) = -\log \gamma_m.$$

Moreover, we may decompose

$$\mathbb{C}[G] = \mathrm{Ind}_{\mathrm{St}_x}^G(T) \oplus \mathbb{C}[G/P^{\flat, \gamma_m}]$$

where T is a direct sum of irreducible $\mathbb{C}[\mathrm{St}_x]$ -modules, on each of which $P^{\flat, \gamma_m}$ acts non-trivially; it follows that the break functions of the $\mathbb{C}$ -module H_1 on $Y_{\text{ét}}$ arising from $\mathrm{Ind}_{\mathrm{St}_x}^G(T)$

all take the value $-\log \gamma_m$, when evaluated for $\rho = -\log r$. On the other hand, the break functions of the $\mathbb{C}$ -module H_2 on $Y_{\text{ét}}$ corresponding to $\mathbb{C}[C/P^{b,\gamma_m}]$ all take values strictly less than $-\log \gamma_m$, when evaluated for $\rho = -\log r$. Furthermore, from [25, Lemma 3.3.10] we derive :

$$\delta_f = [G : 1]^{-1} \cdot (\delta_{H_1} + \delta_{H_2}).$$

Lastly, notice that $\dim_{\mathbb{C}} \mathbb{C}[G/P^{b,\gamma_m}] \leq p^{-1} \cdot [G : 1]$, since P^{b,γ_m} is a non-trivial p -group. Taking (3.3.8) into account, the sought inequalities for $\delta_f(\rho)$ follow straightforwardly. $\square$

Proposition 3.3.9. *Let F be a locally constant and constructible Λ -module on the étale site of $A \setminus \{0\}$, such that the $\Lambda[\pi(0)]$ -module F_{η_0} (resp. the $\Lambda[\pi(\infty)]$ -module F_{η_∞}) has Swan conductor equal to 0 (resp. is tamely ramified). Then F_{η_0} is tamely ramified.*

Proof. Let $\varphi_n : A \rightarrow A$ be as in (3.2.11); we may choose $n > 0$ such that $(\varphi^* F)_{\eta_\infty}$ is an unramified module, and clearly it suffices to show that $\varphi_n^* F$ is a constant Λ -module on the étale site of $A \setminus \{0\}$. Thus, we may replace F by $\varphi_n^* F$, and assume from start that F is locally constant on the étale site of $D^* := D \setminus \{0\}$. Let $f_F : \mathcal{S}om_{\Lambda}(M_{D^*}, F) \rightarrow D^*$ (where $M := F_\infty$) be as in (3.3.6). It suffices to show that the restriction of f_F to each connected component, is an isomorphism onto D^* .

To this aim, notice that, since F is unramified around ∞ , the assertion holds for the restriction of f_F to some open subset of the type $f_F^{-1}(\mathbb{D}(\infty, \varepsilon))$. Consequently, the normalized discriminant function δ of X (which is defined on the whole of $\mathbb{R}$) vanishes identically on a half-line $(-\infty, a]$. Moreover, recall that

$$0 = \text{sw}_0^{\natural}(F, 0^+) = \lim_{r \rightarrow 0^+} \text{sw}^{\natural}(F, r^+)$$

([25, Cor.4.1.16]). Therefore all the break functions of F are bounded everywhere on $\mathbb{R}$. In view of lemma 3.3.7, it follows that δ is bounded as well. Since the latter is also convex, we deduce that δ actually vanishes identically on $\mathbb{R}$. Now, for given $r \in |K^\times|$, let X_r be a connected component of $f_F^{-1}(\mathbb{D}(\infty, r))$, and denote by $\mathfrak{d}_r^+$ the discriminant of $B_r^+ := \mathcal{O}_X^+(X_r)$ over the ring $A(r)^+$ (notation as in [25, §2.2.7], and notice that $\mathfrak{d}_r^+$ is well defined, in view of [25, Prop.2.3.5(i)]). Then $\mathfrak{d}_r^+$ is a unit in $A(r) := A(r)^+ \otimes_{K^+} K$, and therefore it is of the form $c \cdot u$, where $c \in K^\times$ and u is a unit in $A(r)^+$. However, since $\delta(-\log r) = 0$, the constant c must actually be a unit of K^+ , i.e. $\mathfrak{d}_r^+$ is a unit of $A(r)^+$, therefore B_r^+ is an étale $A(r)^+$ -algebra. By standard arguments, this implies that, for every $s \in |K^\times|$ with $s < r$, the restriction of f_F to the preimage of $\mathbb{D}(\infty, s)$ in X_r , splits as a disjoint union of discs, each of which maps isomorphically onto $\mathbb{D}(\infty, s)$. The contention is an immediate consequence. $\square$

Remark 3.3.10. We should also mention that, if M is a $\Lambda[\pi(0)]$ -module of length one and of Swan conductor zero, then we do know that M is tamely ramified. Indeed, in this case we may extend M to a locally constant Λ -module of generic length one on some Zariski open subset $U \subset A$; such Λ -modules have been classified completely in [24]. The assertion follows by a direct inspection of this classification.

3.4. Application : the determinant of cohomology. In this section we assume that K is the completion of the algebraic closure of a field K_0 which is a finite extension of $\mathbb{Q}_p$. Also, we shall take $\Lambda := \overline{\mathbb{F}}_\ell$, the algebraic closure of the field with ℓ elements, where ℓ is a prime number different from p .

3.4.1. Notice first that the μ_{p^∞} -torsor $\mathcal{L}$ of (2.3) is already defined over $\mathbb{Q}_p$. The associated Λ -module $\mathcal{L}_\psi$ of (2.3.1) is not necessarily defined on the étale site of $(\mathbb{A}_{\mathbb{Q}_p}^1)^{\text{ad}}$, but we wish to show that there exists an integer $N \in \mathbb{N}$, depending only on ℓ , such that $\mathcal{L}_\psi$ carries a natural action of $\text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p[\mu_{p^N}])$. Indeed, set

$$N := \max(1, v_p(\ell - 1)) \quad K_N := K_0[\mu_{p^N}] \quad S_N := \text{Spa}(K_N, K_N^+)$$

where $v_p : \mathbb{Z} \rightarrow \mathbb{N} \cup \{+\infty\}$ is the p -adic valuation. It is easily seen that there exist natural isomorphisms of Galois groups

$$(3.4.2) \quad G_p := \text{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p[\mu_{p^N}]) \xrightarrow{\sim} \mathbb{Z}_p \xrightarrow{\sim} G_\ell := \text{Gal}(\mathbb{F}_\ell[\mu_{p^\infty}]/\mathbb{F}_\ell[\mu_{p^N}]).$$

Let $\widehat{\mathbb{Z}}$ denote the profinite completion of $\mathbb{Z}$; under (3.4.2), the surjection

$$(3.4.3) \quad \widehat{\mathbb{Z}} \xrightarrow{\sim} \text{Gal}(\Lambda/\mathbb{F}_\ell[\mu_{p^N}]) \rightarrow G_\ell$$

is identified with the natural projection $\widehat{\mathbb{Z}} \rightarrow \mathbb{Z}_p$, and the latter admits a natural (continuous) splitting $\mathbb{Z}_p \rightarrow \widehat{\mathbb{Z}}$. This means that there exists a (continuous) group homomorphism

$$(3.4.4) \quad G_p \rightarrow \text{Gal}(\Lambda/\mathbb{F}_\ell[\mu_{p^N}]) \quad \sigma \mapsto \bar{\sigma}$$

whose composition with (3.4.3) equals (3.4.2). Let us denote by Λ the field Λ , endowed with the G_p -action given by (3.4.4). Then the constant Λ -module Λ_S on the étale site of S is correspondingly endowed with a system of isomorphisms

$$\sigma_\Lambda : \sigma_S^* \Lambda_S \xrightarrow{\sim} \Lambda_S \quad \text{for all } \sigma \in G_p$$

(where $\sigma_S : S \xrightarrow{\sim} S$ is the isomorphism deduced from the automorphism σ of K) that amount to an action of G_p on this sheaf, such that

$$\sigma_\Lambda(a\lambda) = \bar{\sigma}(a) \cdot \sigma_\Lambda(\lambda)$$

for every $\sigma \in G_p$, every étale morphism $U \rightarrow S$, every $\lambda \in \sigma_S^* \Lambda_S(U)$, and every $a \in \Lambda$. In other words, the action of G_p on Λ_S is *semilinear*.

On the other hand, $\mathcal{L}$ is endowed with a corresponding action of $\text{Gal}(\mathbb{Q}_p[\mu_{p^\infty}]/\mathbb{Q}_p)$, i.e. a system of isomorphisms

$$\sigma_{\mathcal{L}} : \sigma_{\mathbb{A}_K^1}^* \mathcal{L} \xrightarrow{\sim} \mathcal{L} \quad \text{for all } \sigma \in G_p$$

(where $\sigma_{\mathbb{A}_K^1} : (\mathbb{A}_K^1)^{\text{ad}} \xrightarrow{\sim} (\mathbb{A}_K^1)^{\text{ad}}$ is induced by σ) compatible with the same action on μ_{p^∞} , i.e. such that

$$\sigma_{\mathcal{L}}(\zeta s) = \sigma(\zeta) \cdot \sigma_{\mathcal{L}}(s)$$

for every $\zeta \in \mu_{p^\infty}$ and every local section s of $\sigma_{\mathbb{A}_K^1}^* \mathcal{L}$. It follows that we may endow $\mathcal{L}_\psi$ with the action

$$\sigma_{\mathcal{L}} \times^{\mu_{p^\infty}} \sigma_\Lambda : \sigma_{\mathbb{A}_K^1}^* \mathcal{L}_\psi \xrightarrow{\sim} \mathcal{L}_\psi \quad \text{for all } \sigma \in G_p$$

which again, shall be semilinear, relative to the action of G_p on Λ .

These considerations lead us to make the following :

Definition 3.4.5. Let X_N be any S_N -adic space, and set $X := X_N \times_{S_N} S$. The group $H_K := \text{Gal}(\overline{\mathbb{Q}_p}/K_0[\mu_{p^N}])$ acts on X , and for every $\sigma \in H_K$, let $\sigma_X : X \xrightarrow{\sim} X$ denote the corresponding automorphism. A Λ -module on $X_{\text{ét}}$ is the datum of a Λ -module F on $X_{\text{ét}}$ together with a *semilinear action* of H_K , i.e. a system of isomorphisms of Λ -modules

$$\sigma_F : \sigma_X^* F \xrightarrow{\sim} F \quad \text{for all } \sigma \in H_K$$

such that

$$\sigma_F(as) = \bar{\sigma}(a) \cdot \sigma_F(s)$$

for every $a \in \Lambda$ and every local section s of $\sigma_X^* F$ (here $\bar{\sigma}$ denotes the image under (3.4.4) of the projection of σ in G_p), and such that

$$\tau_F \circ \tau_X^*(\sigma_F) = (\tau \circ \sigma)_F \quad \text{for every } \tau, \sigma \in H_K.$$

The category of locally constant Λ -modules of finite type on $X_{\text{ét}}$ shall be denoted $\Lambda_X\text{-Mod}_{\text{loc}}$.

Example 3.4.6. In the situation of definition 3.4.5, let F_N be any $\mathbb{F}_\ell$ -module on $X_{N,\text{ét}}$, and denote by $\pi_X : X \rightarrow X_N$ and $\pi_S : X \rightarrow S$ the natural projections. Then the sheaf $\pi_X^* F_N \otimes_{\mathbb{F}_\ell} \pi_S^* \Lambda_S$ carries a natural structure of Λ -module (details left to the reader).

3.4.7. Let X_N and X be as in definition 3.4.5. Since we shall be working with vanishing cycles, we need to add Galois equivariance to our geometric constructions of sections 2.1 and 2.2. We use a method that works under the assumption that X_N admits a K_N -rational point; this is sufficient for our purposes. Namely, for every locally algebraic covering $f : Y \rightarrow X$, and every $\sigma \in H_K$, denote Y^σ the fibre product in the cartesian diagram :

$$\begin{array}{ccc} Y^\sigma & \xrightarrow{\sigma_Y} & Y \\ f^\sigma \downarrow & & \downarrow f \\ X & \xrightarrow{\sigma_X} & X. \end{array}$$

Clearly f^σ is a locally algebraic covering as well, and the rule $f \mapsto f^\sigma$ defines an automorphism of the category $\mathbf{Cov}^{\text{loc.alg}}(X)$. We fix now a geometric point $\xi : S \rightarrow X$, whose image in X_N is a K_N -rational point. From ξ we deduce a cartesian diagram :

$$\begin{array}{ccc} Y^\sigma \times_X S & \xrightarrow{\sigma_Y \times_X \sigma_S} & Y \times_X S \\ \downarrow & & \downarrow \\ S & \xrightarrow{\sigma_S} & S \end{array}$$

whence a natural bijection :

$$\omega_{Y,\sigma} : F_\xi(Y^\sigma) \xrightarrow{\sim} F_\xi(Y)$$

such that

$$\omega_{Y,\tau} \circ \omega_{Y^\tau,\sigma} = \omega_{Y,\tau \circ \sigma} \quad \text{for every } \tau, \sigma \in H_K.$$

Next, for every $g \in \pi_1(X, \xi)$, let $g_Y : F_\xi(Y) \xrightarrow{\sim} F_\xi(Y)$ be the corresponding bijection; we define an automorphism

$$\Omega_\sigma : \pi_1^{\text{loc.alg}}(X, \xi) \xrightarrow{\sim} \pi_1^{\text{loc.alg}}(X, \xi)$$

by the rule :

$$\Omega_\sigma(g)_Y := \omega_{Y,\sigma} \circ g_{Y^\sigma} \circ \omega_{Y,\sigma}^{-1}$$

for every $g \in \pi_1^{\text{loc.alg}}(X, \xi)$ and every locally algebraic covering $Y \rightarrow X$. Notice that

$$\Omega_\sigma^{-1} H(Y, \bar{y}) = H(Y^\sigma, \omega_{Y,\sigma}^{-1}(\bar{y}))$$

for every open subgroup $H(Y, \bar{y})$ as in (2.1.4); especially Ω_σ is a continuous map, for every $\sigma \in H_K$, and the reader may check that the rule $\sigma \mapsto \Omega_\sigma$ defines a group homomorphism :

$$\Omega : H_K \rightarrow \text{Aut}(\pi_1^{\text{loc.alg}}(X, \xi))$$

and we define the arithmetic (locally algebraic) fundamental group of X_N as the semi-direct product of groups arising from Ω :

$$\pi_1^{\text{loc.alg}}(X_N, \xi) := \pi_1^{\text{loc.alg}}(X, \xi) \rtimes_\Omega H_K.$$

3.4.8. Let Σ be a set, and $\rho : \pi_1^{\text{loc.alg}}(X_N, \xi) \rightarrow \text{Aut}(\Sigma)$ a representation. We shall say that ρ is *continuous* if the same holds for the restriction of ρ to the subgroup $\pi_1^{\text{loc.alg}}(X, \xi)$. Likewise we may define continuous representations of $\pi_1^{\text{loc.alg}}(X_N, \xi)$ in Λ -modules : namely, these are continuous in the foregoing sense, and moreover they are semilinear for the action of $\pi_1^{\text{loc.alg}}(X_N, \xi)$ on Λ induced via the projection onto H_K . The category of continuous representations of $\pi_1^{\text{loc.alg}}(X_N, \xi)$ on Λ -modules of finite type shall be denoted

$$\Lambda[\pi_1^{\text{loc.alg}}(X_N, \xi)]\text{-}\mathbf{Mod}_{\text{f.cont}}.$$

Lemma 3.4.9. *Let X_N and X be as in definition 3.4.5, and suppose furthermore, that X is connected. We have :*

- (i) *Let M be any locally constant Λ -module on $X_{\text{ét}}$. The $\pi_1^{\text{loc.alg}}(X, \xi)$ -action on M_ξ extends naturally to a (continuous) semilinear action of $\pi_1^{\text{loc.alg}}(X_N, \xi)$.*
- (ii) *More precisely, the rule $M \mapsto M_\xi$ defines an equivalence of categories :*

$$\Lambda_X\text{-Mod}_{\text{loc}} \xrightarrow{\sim} \Lambda[\pi_1^{\text{loc.alg}}(X_N, \xi)]\text{-Mod}_{\text{f.cont}}.$$

Proof. More generally, let Σ be any (discrete) set, endowed with a continuous left action ρ of $\pi_1^{\text{loc.alg}}(X, \xi)$. By proposition 2.1.6, the set Σ corresponds to a locally algebraic covering $Y \rightarrow X$. Suppose now that Y is endowed with an action of H_K covering the action on X ; this means that we have a system of isomorphisms of X -adic spaces :

$$\sigma_Y : Y^\sigma \xrightarrow{\sim} Y \quad \text{for every } \sigma \in H_K$$

compatible as usual with the composition law of H_K . Under the identification $\omega_{Y, \sigma}$, the left action ρ_σ of $\pi_1^{\text{loc.alg}}(X, \xi)$ on $F_\xi(Y^\sigma)$ is none else than ρ^{Ω_σ} , *i.e.*

$$\rho_\sigma(g) = \rho \circ \Omega_\sigma(g) \quad \text{for every } g \in \pi_1^{\text{loc.alg}}(X, \xi).$$

On the other hand, σ_Y corresponds to an isomorphism $t_\sigma : F_\xi(Y^\sigma) \xrightarrow{\sim} F_\xi(Y)$ of sets with left $\pi_1^{\text{loc.alg}}(X, \xi)$ -action; under the identification $\omega_{Y, \sigma}$, the bijection t_σ becomes an automorphism of the set Σ , such that

$$t_\sigma \circ \rho(g) = \rho_\sigma(g) \circ t_\sigma \quad \text{for every } g \in \pi_1^{\text{loc.alg}}(X, \xi).$$

Therefore, we obtain a continuous action of $\pi_1^{\text{loc.alg}}(X_N, \xi)$ on Σ by the rule :

$$(g, \sigma) \mapsto \rho(g) \circ t_\sigma \quad \text{for every } \sigma \in H_K \text{ and } g \in \pi_1^{\text{loc.alg}}(X, \xi).$$

By reversing the construction, it is clear that from such an action we may recover a locally algebraic covering Y of X , together with a compatible system of isomorphisms σ_Y as above. In view of the equivalence (2.1.15), the same argument applies unchanged to Λ -modules, whence the lemma. $\square$

We need also to check the effect of a change of base point. This is the same as in the algebraic geometric case; namely, we have :

Lemma 3.4.10. *Let X and X_N be as in lemma 3.4.9. Let also $\xi, \xi' : S \rightarrow X$ be two geometric points whose images in X_N are K_N -rational points. Then there exists an isomorphism of arithmetic fundamental groups :*

$$\pi_1^{\text{loc.alg}}(X_N, \xi) \xrightarrow{\sim} \pi_1^{\text{loc.alg}}(X_N, \xi').$$

Proof. In light of claim 2.1.7, we have an isomorphism $b : F_\xi \xrightarrow{\sim} F_{\xi'}$ of fibre functors, inducing an isomorphism

$$\beta : \pi_1^{\text{loc.alg}}(X, \xi) \xrightarrow{\sim} \pi_1^{\text{loc.alg}}(X, \xi').$$

We claim that the rule

$$(g, \sigma) \mapsto (\beta(g), \sigma) \quad \text{for every } g \in \pi_1^{\text{loc.alg}}(X, \xi) \text{ and } \sigma \in H_K$$

defines an isomorphism as sought. Indeed, let

$$\text{Aut}(\pi_1^{\text{loc.alg}}(X, \xi')) \xleftarrow{\Omega'} H_K \xrightarrow{\Omega} \text{Aut}(\pi_1^{\text{loc.alg}}(X, \xi))$$

denote the homomorphisms as in (3.4.7), associated to ξ and respectively ξ' . The assertion comes down to the following :

Claim 3.4.11. $\Omega'_\sigma \circ \beta = \beta \circ \Omega_\sigma$ for every $\sigma \in H_K$.

Proof of the claim. By inspecting the constructions, we get the following (non-commutative!) diagram of sets :

$$\begin{array}{ccccc}
 F_{\xi'}(Y^\sigma) & \xrightarrow{\omega'_{Y,\sigma}} & F_{\xi'}(Y) & & \\
 \downarrow \beta(g_{Y^\sigma}) & \swarrow & \downarrow \beta(g_Y) & \searrow & \\
 & F_\xi(Y^\sigma) \xrightarrow{\omega_{Y,\sigma}} F_\xi(Y) & & & \\
 & \downarrow g_{Y^\sigma} \quad \downarrow g_Y & & & \\
 & F_\xi(Y^\sigma) \xrightarrow{\omega_{Y,\sigma}} F_\xi(Y) & & & \\
 & \swarrow & \downarrow \beta(g_Y) & \searrow & \\
 F_{\xi'}(Y^\sigma) & \xrightarrow{\omega'_{Y,\sigma}} & F_{\xi'}(Y) & &
 \end{array}$$

(where $\omega'_{Y,\sigma}$ is the natural bijection that yields Ω') for every $g \in \pi_1^{\text{loc.alg}}(X, \xi)$, and every locally algebraic covering $Y \rightarrow X$. The diagonal maps are given by the isomorphism b . The right and left subdiagrams commute, and the assertion will follow, once we know that the upper (hence also the the lower) subdiagram commutes as well. However, arguing as in the proof of claim 2.1.7 we may find a quasi-compact connected open subset $U_N \subset X_N$ such that $U := U_N \times_{S_N} S$ contains the images of ξ and ξ' , and we may assume that b comes from an isomorphism of fibre functors $F_{U,\xi} \xrightarrow{\sim} F_{U,\xi'}$ for the categories of locally algebraic coverings of U (notice that U is connected, since U_N contains K_N -rational points). Hence, we may replace X_N by U_N , and therefore assume from start that X_N is quasi-compact, in which case $\text{Cov}^{\text{loc.alg}}(X)$ is the category of ind-objects of $\text{Cov}^{\text{alg}}(X)$ (remark 2.1.2(v)), and under this identification, both F_ξ and $F_{\xi'}$ are natural extensions of fibre functors on $\text{Cov}^{\text{alg}}(X)$.

By the general theory of [14, Exp.V], we may find a fundamental pro-object

$$P_\bullet := (P_H \mid H \subset \pi_1^{\text{alg}}(X, \xi))$$

indexed by the family of open normal subgroups H of $\pi_1^{\text{alg}}(X, \xi)$, and an automorphism $b_\bullet : P_\bullet \xrightarrow{\sim} P_\bullet$ inducing b . The latter condition means that F_ξ and $F_{\xi'}$ are both isomorphic to the functor

$$(Y \rightarrow X) \mapsto \text{Hom}_X(P_\bullet, Y) := \text{colim}_H \text{Hom}_X(P_H, Y)$$

and under this identification, b becomes the automorphism of $\text{Hom}_X(P_\bullet, Y)$ given by the rule :

$$(\varphi : P_H \rightarrow Y) \mapsto \varphi \circ b_H \quad \text{for every } H \subset \pi_1^{\text{alg}}(X, \xi).$$

On the other hand, both $\omega_{Y,\sigma}$ and $\omega'_{Y,\sigma}$ get identified with the mapping :

$$\text{Hom}_X(P_\bullet, Y^\sigma) \xrightarrow{\sim} \text{Hom}_X(P_\bullet, Y) \quad (\varphi : P_H \rightarrow Y^\sigma) \mapsto (\sigma_Y \circ \varphi : P_{\Omega_\sigma(H)} \rightarrow Y).$$

Thus, the commutativity of the upper subdiagram comes down to the assertion that composition on the left commutes with composition on the right, which is obvious. $\square$

Remark 3.4.12. (i) It is also easily seen that any two “geometric” isomorphisms of our arithmetic fundamental groups (*i.e.* those arising from isomorphisms of fibre functors, as in the proof of lemma 3.4.10), differ by an inner automorphism.

(ii) A purist might prefer to define the arithmetic fundamental group more directly, as the automorphism group of a fibre functor for the category of locally algebraic coverings of X_N . This is the approach taken in [19]. However, our method allows to prove rapidly the basic

properties one needs, and avoids certain technical issues. For instance, basically by decree, our fundamental group sits in a short exact sequence

$$1 \rightarrow \pi_1^{\text{loc.alg}}(X, \xi) \rightarrow \pi_1^{\text{loc.alg}}(X_N, \xi) \rightarrow H_K \rightarrow 1.$$

As de Jong points out, the corresponding assertion for his fundamental groups is not clear ([19, Rem.2.15]).

(iii) On the other hand, [19] obtains automatically a topology on the arithmetic fundamental group, whereas we do not try to define a topology for our groups (only the kernel of the projection onto H_K is endowed with its usual topology). Alternatively, one may say that our approach replaces the profinite topology of H_K by the discrete topology.

(iv) Notice that the “geometric” isomorphisms of lemma 3.4.10 are compatible with the projection onto H_K , in the obvious fashion (details left to the reader).

3.4.13. Let A and D be as (2.3.3), and fix as usual a global coordinate x on $\mathbb{A}_{K_N}^1$. If $z \in D(K)$ is any K_N -rational point (i.e. $x(z) \in K_N$), there is also a version of definition 3.4.5 for $\Lambda[\pi(z)]$ -modules : indeed, for $z \in A$, and every $\varepsilon > 0$, we have

$$\mathbb{D}(z, \varepsilon)^* = \mathbb{D}_N(z_0, \varepsilon)^* \times_{S_N} S \quad \text{where} \quad \mathbb{D}_N(z_0, \varepsilon)^* := \{t \in (\mathbb{A}_{K_N}^1)^{\text{ad}} \mid 0 < |x(t) - z_0|_t \leq \varepsilon\}$$

(and $z_0 \in \mathbb{A}_{K_N}^1$ is the projection of z); likewise we get $\mathbb{D}_N(\infty, \varepsilon)^*$. We have then the group

$$\pi(z_0, \varepsilon) := \pi_1^{\text{loc.alg}}(\mathbb{D}_N(z_0, \varepsilon)^*, \xi)$$

(for some choice of K_N -rational geometric point ξ of $\mathbb{D}(z, \varepsilon)$) and the corresponding category $\Lambda[\pi(z_0, \varepsilon)]\text{-Mod}_{\text{f.cont}}$ of continuous semilinear representations of $\pi(z_0, \varepsilon)$ into Λ -modules of finite type. For $\varepsilon' < \varepsilon$ we may pick (thanks to lemma 3.4.10) a group homomorphism $\pi(z_0, \varepsilon') \rightarrow \pi(z_0, \varepsilon)$, well defined up to inner automorphisms, and we may define the category

$$\Lambda[\pi(z_0)]\text{-Mod}$$

as the 2-colimit of the system of categories of $\Lambda[\pi(z_0, \varepsilon)]$ -modules of finite type (where the transition functors are induced by the foregoing group homomorphisms), as well as its subcategory $\Lambda[\pi(z_0)]\text{-b.Mod}$ of $\Lambda[\pi(z_0)]$ -modules with bounded ramification. The usual operations for modules extend to $\Lambda[\pi(z_0)]$ -modules of finite type; especially, we shall denote

$$\det : \Lambda[\pi(z_0)]\text{-Mod} \rightarrow \Lambda[\pi(z_0)]\text{-Mod}$$

the *determinant* functor, that assigns to any object M of $\Lambda[\pi(z_0)]\text{-Mod}$ its highest exterior power $\Lambda_{\mathbb{F}_\ell}^d M$ (where d is the length of M).

3.4.14. We may now add Galois equivariance to our nearby and vanishing cycle functors. Indeed, let X_N and X be as in definition 3.4.5. Also, let $f_N : X_N \rightarrow \mathbb{D}_N(z_0, \varepsilon)$ be a morphism of S_N -adic spaces, and set $f := f_N \times_{S_N} S$. We use the notation of (2.2.1), and set as well $X_H := C_H \times_{\mathbb{D}(z, \varepsilon)} X$ for every open subgroup $H \subset \pi(z, \varepsilon)$. By inspecting the definitions, we find, for every such H , and every $\sigma \in H_K$, a commutative diagram

$$(3.4.15) \quad \begin{array}{ccc} X_{\sigma^{-1}H\sigma} & \longrightarrow & X_H \\ \downarrow & & \downarrow \\ X & \xrightarrow{\sigma_X} & X \end{array}$$

(whose vertical arrows are the projections) as well as a natural isomorphism of X -adic spaces :

$$X_H^\sigma \xrightarrow{\sim} X_{\sigma^{-1}H\sigma}$$

and under this isomorphism, the top horizontal arrow of (3.4.15) is identified with σ_{X_H} (notation of (3.4.7)). Let now F be any sheaf on $X_{\text{ét}}$, and set $F_\sigma := \sigma_X^* F$; we define the sheaf $\tilde{F}$ on $X_{\text{ét}}$ as in (2.2.1), and by repeating the construction with F replaced by F_σ , we obtain likewise $\tilde{F}_\sigma$.

Considering the diagrams (3.4.15) for varying H , and taking the colimit over the system of all open subgroups H of $\pi(z_0, \varepsilon)$, we deduce a natural isomorphism :

$$\tilde{F}_\sigma \xrightarrow{\sim} \sigma_X^* \tilde{F} \quad \text{for every } \sigma \in H_K.$$

Especially, if F is endowed with a system of compatible isomorphisms $(\sigma_F : \sigma_X^* F \xrightarrow{\sim} F \mid \sigma \in H_K)$, then $\tilde{F}$ is endowed with the system of compatible isomorphisms

$$(\tilde{\sigma}_F : \sigma_X^* \tilde{F} \xrightarrow{\sim} \tilde{F} \mid \sigma \in H_K).$$

Next, for any $g \in \pi(z_0, \varepsilon)$, we have a commutative diagram :

$$(3.4.16) \quad \begin{array}{ccc} X_{\sigma^{-1}H\sigma} & \xrightarrow{\sigma_{X_H}} & X_H \\ (\sigma^{-1}g\sigma)_{X_{\sigma^{-1}H\sigma}} \downarrow & & \downarrow g_{X_H} \\ X_{\sigma^{-1}g^{-1}Hg\sigma} & \xrightarrow{\sigma_{X_{g^{-1}Hg}}} & X_{g^{-1}Hg} \end{array}$$

where g_{X_H} is deduced from the morphism g_H as in (2.2.1), and likewise for the left vertical arrow in (3.4.16). Recall that $\tilde{F}$ carries a natural continuous action

$$\rho_F : \pi(z, \varepsilon) \rightarrow \text{Aut}(F).$$

By unwinding the definitions, we deduce from (3.4.16) that the isomorphism $\tilde{\sigma}_F$ is $\pi(z, \varepsilon)$ -equivariant, provided we endow $\sigma_X^* \tilde{F}$ with the action

$$(\sigma_X^* \rho)^{\Omega_\sigma} := (\sigma_X^* \rho) \circ \Omega_\sigma$$

where Ω_σ is the automorphism of $\pi(z, \varepsilon)$ defined as in (3.4.7). Arguing as in the proof of lemma 3.4.9, we deduce that the stalks of $\tilde{F}$ over the K_N -rational points, inherit a natural continuous action of $\pi(z_0, \varepsilon)$. Lastly, if F is any Λ -module on $X_{\text{ét}}$, all the discussion extends to the stalks of $R^i \Psi_{\eta, \varepsilon} F$, for every $i \in \mathbb{Z}$, in the usual way.

3.4.17. Let $U_N \subset \mathbb{A}_{K_N}^1$ be a (Zariski) open subset, and $z_0 \in \Sigma_0 := \mathbb{A}_{K_N}^1 \setminus U_N$ any point. Let also $\pi_A : A \rightarrow (\mathbb{A}_{K_N}^1)^{\text{ad}}$ be the natural projection (where A is as in (2.3.3)), and denote by $\varepsilon(z_0)$ the minimum of $|z - z'|$, where (z, z') ranges over the pair of distinct elements of $\pi_A^{-1}(z_0)$; for every non-zero $\varepsilon < \varepsilon(z_0)$, consider the morphism

$$p_{z_0, \varepsilon} : V(z_0, \varepsilon) := \bigcup_{z \in \pi_A^{-1}(z_0)} \mathbb{D}(z, \varepsilon)^* \rightarrow \mathbb{D}(0, \varepsilon)^*$$

whose restriction to $\mathbb{D}(z, \varepsilon)^*$ is the translation $x \mapsto x - z$, for every $z \in \pi_A^{-1}(z_0)$. It is easily seen that $V(z_0, \varepsilon)$ is invariant under the action of H_K on A , and $p_{z_0, \varepsilon}$ is H_K -equivariant. Let now F be a Λ -module on the étale site of $U := U_N \times_{\text{Spec } K_N} S$; it follows easily that

$$G := p_{z_0, \varepsilon*}(F|_{V(z_0, \varepsilon)})$$

inherits from F a natural Λ -module structure. Suppose furthermore, that F has bounded ramification at every point of $\pi_A^{-1}(z_0)$; then it follows that the $\Lambda[\pi(0)]$ -module

$$p_{z_0*}(F)_{\eta_0} := G_{\eta_0}$$

has also bounded ramification (clearly this module is independent of the choice of ε).

3.4.18. Keep the situation of (3.4.17), and let $j : U \rightarrow A$ be the open immersion. Suppose moreover, that F is locally constant and constructible on U , and with bounded ramification at every point of $D \setminus U$; from (2.4.13) and (3.1.4) we get the natural decomposition

$$\mathcal{H}^0(\mathcal{F}_\psi(j_!F))_{\eta_{\infty'}} \xrightarrow{\sim} \mathcal{F}_\psi^{(\infty, \infty')}(F_{\eta_\infty}) \oplus \bigoplus_{z_0 \in \Sigma_0} \left(\bigoplus_{z \in \pi_A^{-1}(z_0)} \mathcal{F}_\psi^{(0, \infty')}(\vartheta_{z*}F_{\eta_z}) \otimes_\Lambda \mathcal{L}_\psi \langle zx' \rangle_{\eta_{\infty'}} \right).$$

Since $\mathcal{L}_\psi$ is a Λ -module on $A_{\text{ét}}$, the left-hand side is naturally a $\Lambda[\pi(\infty')]$ -module. Likewise, it is easily seen that $\mathcal{F}_\psi^{(\infty, \infty')}(F_{\eta_\infty})$ is an object of $\Lambda[\pi(\infty')]\text{-b.Mod}$, and the same holds for each of the direct sums in parenthesis. Moreover, the decomposition is equivariant. Recall that $\Lambda = \overline{\mathbb{F}}_\ell$ in this section; taking determinants on both sides, we deduce – in view of theorem 3.1.14(i) – the natural isomorphism of rank one $\Lambda[\pi(\infty')]$ -modules :

$$\det \mathcal{H}^0(\mathcal{F}_\psi(j_!F))_{\eta_{\infty'}} \xrightarrow{\sim} \det \mathcal{F}_\psi^{(\infty, \infty')}(F_{\eta_\infty}) \otimes \bigotimes_{z_0 \in \Sigma_0} \left(\det \mathcal{F}_\psi^{(0, \infty')}(p_{z_0*}(F)_{\eta_0}) \otimes \mathcal{L}_\psi \langle \delta_{z_0}^F x' \rangle_{\eta_{\infty'}} \right)$$

where

$$\delta_{z_0}^F := \sum_{z \in \pi_A^{-1}(z_0)} a_z(F) \cdot z \quad \text{for every } z_0 \in \Sigma_0$$

(notation of (2.3.14); notice that $a_z(F)$ depends only on $\pi_A(z)$, hence $\delta_{z_0}^F$ is an integer multiple of the trace $\sum_{z \in \pi_A^{-1}(z_0)} z$).

Remark 3.4.19. (i) Notice now that the (unique) $\natural$ -break of the rank one $\Lambda[\pi(\infty')]$ -module $\det \mathcal{F}_\psi^{(0, \infty')}(p_{z_0*}(F)_{\eta_0})$ is < 1 for each $z_0 \in \Sigma_0$ (corollary 3.2.22(i)), hence this $\natural$ -break must equal 0, by (2.3.18), and therefore all these modules are tamely ramified (remark 3.3.10).

(ii) Let M be any tamely ramified $\Lambda[\pi(\infty')]$ -module. Then M admits a natural extension to a Λ -module on the étale site of $A' \setminus \{0'\}$. Indeed, let x' be a global coordinate on A' as in (2.3.3), and set

$$U'_N := \text{Spec } K_N[x', 1/x'] \quad U' := U'_N \times_{K_N} \overline{\mathbb{Q}}_p.$$

For every $\varepsilon > 0$ we have a commutative diagram whose top (resp. bottom) row is right exact by [19, Prop.2.13] (resp. is exact) :

$$\begin{array}{ccccccc} \pi_1^{\text{alg}}(\mathbb{D}(\infty', \varepsilon)^*, \xi) & \longrightarrow & \pi_1^{\text{alg}}(\mathbb{D}_N(\infty', \varepsilon)^*, \xi) & \longrightarrow & H_K & \longrightarrow & 1 \\ \beta \downarrow & & \downarrow & & \parallel & & \\ 1 & \longrightarrow & \pi_1(U', \xi) & \longrightarrow & \pi_1(U'_N, \xi) & \longrightarrow & H_K \longrightarrow 1 \end{array}$$

where ξ is a chosen K_N -rational point of $\mathbb{D}_N(\infty, \varepsilon)^*$. The closed immersion of $\{\xi\}$ into the S_N -adic space $\mathbb{D}_N(\infty, \varepsilon)^*$ induces a splitting for both exact sequences, and a Λ -module on $\mathbb{D}(\infty', \varepsilon)_{\text{ét}}^*$ with finite geometric monodromy corresponds to a continuous representation ρ of $\pi_1^{\text{alg}}(\mathbb{D}_N(\infty', \varepsilon)^*, \xi)$, semilinear with respect to the action of H_K on Λ . Then it is clear that ρ is determined uniquely by its restrictions to $\pi_1^{\text{alg}}(\mathbb{D}(\infty', \varepsilon)^*, \xi)$ and to the image of H_K under the fixed splitting. Our tamely ramified M is represented by a Λ -module G on $\mathbb{D}(\infty', \varepsilon)_{\text{ét}}^*$ for some $\varepsilon > 0$, and if ε is small enough, the geometric monodromy of G factors through the image of the surjective map β . Hence G yields naturally both a representation of $\pi_1(U', \xi)$ and a semilinear representation of H_K , that assemble into a unique continuous semilinear representation of $\pi_1(U'_N, \xi)$. The latter is the sought canonical extension of M .

(iii) Let M be as in (ii), and denote by G the canonical extension of M to a Λ -module on $(A' \setminus \{0'\})_{\text{ét}}$. Then we may consider the stalk G_t over the K_N -rational point t such that $x'(t) = 1$. We shall denote this Λ -module by $M_{1'}$.

(iv) Also, in the situation of (3.4.18) suppose that, for some K_N -rational point $z \in D \setminus U$, the $\Lambda[\pi(z)]$ -module F_{η_z} is unramified, so it extends to a locally constant Λ -module G on $U' := U \cup \{z\}$; in this case, it is easily seen that G is a Λ -module on U' , and the stalk G_z is a Λ -module. With a slight abuse of notation, we shall denote by F_z this Λ -module. With this notation, we may state the following :

Theorem 3.4.20. *In the situation of (3.4.18), suppose that F_{η_∞} is an unramified $\Lambda[\pi(\infty)]$ -module. Then there exists a natural isomorphism of rank one Λ -modules :*

$$\det(R\Gamma_c(U, F)[1]) \otimes_{\Lambda} \det(F_{\infty}(-1)) \xrightarrow{\sim} \bigotimes_{z_0 \in \Sigma_0} \det(\mathcal{F}_{\psi}^{(0, \infty')}(p_{z_0*}(F)_{\eta_0}))_1.$$

Proof. We proceed as in the proof of [22, Th.3.4.2]. Taking into account (3.1.39) (or better, its dual for Λ -modules F on U) and lemma 3.2.19, we find a short exact sequence of $\Lambda[\pi(0')]$ -modules :

$$(3.4.21) \quad 0 \rightarrow H_c^1(U, F) \rightarrow \mathcal{H}^0 \mathcal{F}_{\psi}(j_! F)_{\eta_{0'}} \rightarrow F_{\infty}(-1) \rightarrow H_c^2(U, F) \rightarrow 0.$$

By theorem 2.3.30(ii), the Λ -module $\mathcal{H}^0 \mathcal{F}_{\psi}(j_! F)$ is locally constant on $U' := A' \setminus \{0'\}$; set $F' := \mathcal{H}^0 \mathcal{F}_{\psi}(j_! F)|_{U'}$. It follows from (3.4.21) that $(\det F')_{\eta_{0'}}$ is unramified, and there is a natural isomorphism of Λ -modules :

$$(\det F')_{0'} \xrightarrow{\sim} \det(R\Gamma_c(U, F)[1]) \otimes_{\Lambda} \det(F_{\infty}(-1))$$

(notation of remark 3.4.19(iv)). On the other hand, in view of (3.4.18) and proposition 3.1.10(ii), we have the isomorphism of $\Lambda[\pi(\infty')]$ -modules :

$$(\det F')_{\eta_{\infty'}} \xrightarrow{\sim} \mathcal{L}_{\psi} \langle \delta^F x' \rangle_{\eta_{\infty'}} \otimes_{\Lambda} \bigotimes_{z_0 \in \Sigma_0} \det \mathcal{F}_{\psi}^{(0, \infty')}(p_{z_0*}(F)_{\eta_0})$$

where $\delta^F := \sum_{z_0 \in \Sigma_0} \delta_{z_0}^F$. Let $j' : U' \rightarrow A'$ be the open immersion; we deduce that $j'_*(\det F') \otimes_{\Lambda} \mathcal{L}_{\psi} \langle -\delta^F x' \rangle$ is locally constant on A' , and tamely ramified at ∞' (remark 3.4.19(i)). Hence the latter is actually (geometrically) constant on A' , and since we have a natural isomorphism of Λ -modules :

$$\mathcal{L}_{\psi} \langle -\delta^F x' \rangle_0 \xrightarrow{\sim} \Lambda$$

the theorem follows. $\square$

Example 3.4.22. Let $U_N \subset \mathbb{A}_{K_N}^1$ be a (Zariski) open subset, F a locally constant $\mathbb{F}_{\ell}$ -module on the étale site of the scheme U_N , and set $U := U_N \times_{K_N} K$. We may then consider the $\mathbb{F}_{\ell}$ -module F^{ad} on $U_{\text{ét}}^{\text{ad}}$ obtained by pullback of F along the morphism of sites $U_{\text{ét}}^{\text{ad}} \rightarrow U_{N, \text{ét}}$, and $F^{\text{ad}} \otimes_{\mathbb{F}_{\ell}} \Lambda$ is a Λ -module on $U_{\text{ét}}^{\text{ad}}$ (example 3.4.6). It is well known that the cohomology of F^{ad} on U^{ad} is naturally isomorphic to that of F on U ; hence theorem 3.4.20 yields an equivariant decomposition of

$$\det(R\Gamma_c(U, F)) \otimes_{\mathbb{F}_{\ell}} \det(F_{\infty}(-1)) \otimes_{\mathbb{F}_{\ell}} \Lambda$$

as a tensor product of rank one Λ -modules attached to the points of $\mathbb{A}_{K_N}^1 \setminus U$, and it is easily seen that each of the factors in this decomposition is determined functorially by the local monodromy of F at the corresponding point.

Remark 3.4.23. (i) The most unusual feature of example 3.4.22 is the appearance of semilinear Galois representations in our decomposition. This might be an artifact of our method, in which case a different method will eventually be discovered, yielding a more standard decomposition in terms of linear characters.

Or else – and more intriguingly – it might be a manifestation of some new hybrid phenomenon, partaking of both the ℓ -adic and p -adic worlds. Indeed, on the one hand, one encounters such semilinear representations in the study of p -adic Hodge theory; on the other hand, both

Hodge theory and Fourier transforms find their common roots in harmonic analysis, and the same can be said of Witten's proof of the Morse inequalities via stationary phase.

(ii) According to this interpretation, our ring Λ (with its H_K -action) should be viewed as a sort of ring of ℓ -adic (or better, ℓ -torsion) periods, analogous perhaps to the field $\mathbb{C}_p$, the simplest of rings of p -adic periods. It also suggests that one should redo the whole theory for p -adic étale coefficients, and then there should be a comparison functor from p -adic étale to deRham ε -factors, in the style of [7]. However, the technical obstacles involved seem to me too daunting, at present.

(iii) If one is not interested in hybridizations of this sort, one can restrict the Galois actions to the closed subgroup $\mathrm{Gal}(K/K_0[\mu_{p^\infty}])$, that acts trivially on Λ , and hence linearly on all our constructions.

Remark 3.4.24. (i) One can generalize example 3.4.22 to the case of a locally constant k -module on the étale site of $U_{\text{ét}}$, where k is a finite extension of $\mathbb{F}_\ell$. The only difference is that, in this case, the tensor decomposition shall be equivariant only for the action of a certain open subgroup of H_K (details left to the reader).

(ii) On the other hand, I expect that theorem 3.4.20 can be generalized by replacing Λ with more general coefficient rings; especially, with truncated Witt vectors $W_n(\Lambda)$, and even with the completion of $\mathbb{Q}_\ell(\mu_{p^\infty})$. This generalization seems to be mostly a technical refinement, and would not bring any new insight, so I prefer to leave it to a more motivated reader.

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